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Broken Symmetries at High Temperature as a Solution to the Domain Wall and Monopole Problems

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Candidate:
ALEJANDRA MELFO

Supervisors:
GORAN SENJANOVIĆ
DENNIS W. SCIAMA

SISSA - SCUOLA
INTERNAZIONALE
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Abstrax

The year is 10^{10} B.C. All the symmetries of Nature broken at low temperatures are completely restored. All of them?
No!
A tiny space of parameters, near the nonperturbative region, is there to resist now and ever to the invading forces of symmetry restoration. And life is not easy for the thermally produced strings, monopoles and domain walls...

Acknowledgments

Esta tesis, el trabajo de estos años, mi presencia aquí tan lejos de casa, no hubieran sido posibles sin el apoyo constante e impermeable de mis amigos y mi familia, desde el otro lado del Atlántico. La gente del grupo de Física Teórica que no necesito nombrar, sabe que aprendí todo lo que pude aquí en gran parte por el gusto de volver y contarles. No se trata entonces de agradecerles su amistad y su ayuda. Simplemente, mis panas, esta tesis es de ustedes.

Ma l'oceano é troppo grande, e non si vive solo delle lettere. Alla SISSA, tra compagni e professori, e specialmente nella stanza 101, ho trovatto l'amicizia indispensabile per sopravivvere questi inverni. E un ambiente di studio e lavoro stimulante e aperto, che tanto ha contribuito a questa tesi. Non ho bisogno ne anche qua di dire nomi, ma vorrei che si sappia ufficialmente che scherzavo, che mai mi avete fatto sentire una "extracomunitaria".

Studying at SISSA is much more than studying in Italy, and one of the best things here was always to cross the tunnel into the rest of the world, at ICTP. Finding people from every remote corner of the planet, doing and sharing physics, was a beautiful experience. In a sense it all started there, at the Diploma Course, and it was there later that I've found the friends and collaborators that made this thesis possible. Again, I will not say the names, but just thank you.

But then of course all of this was not enough to produce a thesis, and I still have a language left to say it. Hvala ljepa Gorane, tebi a ne svima, da si mi pokazao da se dobra fizika pravi sa ljubavlju i poštovanjem.

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Introduction

More than 30 years after it was suggested that topological defects may be produced in a cosmological phase transition, the monopole and the domain wall problems remain some of the most interesting open issues in the field of astroparticle physics. This is particularly true for the monopole problem, a consequence of an idea as fundamental as grand unification in particle physics. Although several solution have been suggested, among them the far-reaching proposal of an inflationary period during the evolution of the Universe, the problems are still far from solved. On the contrary, in the domain wall case it is frequently proposed to altogether abandon the possibility of spontaneous breakdown of discrete symmetries in general. The idea of this Thesis is to investigate how fundamental the incompatibility of the standard cosmological model is with theories that admit domain wall or monopole solutions.

Topological defect production in cosmological phase transitions has been my main interest during these years at SISSA. Having studied the formation of strings in first-order phase transitions in collaboration first with Leandros Perivolaropoulos, and later with Antonio Ferrera, it was natural to turn to the more "dangerous" defects -domain walls and monopoles. Research in this direction was carried out mainly with Goran Senjanović, and in collaboration with Gia Dvali and Borut Bajc. It is this later work which will be presented in this Thesis.

The first chapter concerns the generalities of topological defect production in a cosmological context. After a brief review of phase transitions in theories with spontaneous symmetry breaking, topological defects are introduced, and the mechanism of its production is described. Some specific calculations in thermal field theory are left for the Appendix.

The second chapter is the central one, where the monopole and domain wall problems are described, together with a review of the solutions proposed in the literature. It is in this chapter that the proposal of this Thesis is presented; namely, that phase transitions are not unavoidable in theories of symmetry breaking with more than one scalar field. A general discussion is offered on how this can eliminate the domain wall and monopole problem.

In the last three chapters the original results are presented. Chapter 3 is devoted to discrete symmetries and domain walls, Chapter 4 to gauge symmetries and monopoles. In Chapter 5 the high-temperature behavior of non-renormalizable theories is studied, with results on supersymmetric theories.

1 Of Phase Transitions in Cosmology

1.1 Symmetry restoration at high temperature

In 1972, when the Standard Model of electroweak interactions started attracting wide interest, Kirzhnits [1] observed that field theories with spontaneous symmetry breaking could have an extremely interesting behavior at high temperature. The observation was based on the analogy with solid state physics systems. Kirzhnits pointed out that the effect of a thermal bath in equilibrium with the Higgs field was to restore the symmetry at sufficiently high temperatures. In the context of a "hot" Big Bang Universe, his observation is of fundamental importance, since symmetry restoration would lead to phase transitions in the Universe. Much of the modern day early Universe cosmology has its root in Kirzhnits's paper.

The idea was further explored by Kirzhnits and Linde [2]. Later, Weinberg carefully studied the issue in his classic paper [3], providing the fundamental tools for calculations at finite temperature in theories with spontaneously broken symmetries. At the same time, Dolan and Jackiw [4] performed similar calculations using functional integral methods, which resulted in the second classic paper on the subject.

The conclusion was that the existence of a thermal bath fundamentally affects the vacuum structure of the theory. Since it will be essential for this Thesis, we will give a short discussion on how this comes about. Some of the details of the calculations are left for an Appendix, as it will be of use to establish a notation for the following chapters.

In theories with spontaneous symmetry breaking, the Higgs field has a vacuum expectation value (VEV) which is non zero. To find it, one minimizes the energy of the system or, in other words, finds the minimum of the potential. We take the example of a real scalar field φ, with a Lagrangian invariant under a discrete symmetry $\mathbf{D} : \varphi \to -\varphi$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \tag{1.1}$$

With $m^2 > 0$, the symmetry is broken. The minimum of the potential is at $\langle\varphi\rangle \equiv v = m^2/\lambda$.

At finite temperature, on the other hand, the relevant quantity to be minimized is not the energy but the free energy, F. It is defined in thermodynamics in terms of the partition function, Z, and the temperature, T, of the system

$$F \equiv -T \ln Z \tag{1.2}$$

To calculate the free energy is an easy task once we know the partition function

$$Z = \mathrm{Tr} e^{-H/T} \tag{1.3}$$

with H as the Hamiltonian. For a bosonic (fermionic) field, the free Hamiltonian is a collection of bosonic oscillators with energy $\omega \equiv \sqrt{\vec{k}^2 + m^2}$.

$$H^{B(F)} = \sum_i H_i^{B(F)} = \sum_i \omega_i(\hat{N}_i \overset{+}{(-)} \frac{1}{2}) \tag{1.4}$$

where B stands for bosonic and F for fermionic systems. The sum represents a continuous integral over momenta, and $\hat{N}$ is the number operator. The partition function of the system is found by integrating over momenta the individual functions $Z(w)$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \ln Z(\omega) \tag{1.5}$$

(with V as the volume) and we have

$$Z_B(\omega) = \sum_n e^{-(n+\frac{1}{2})\omega/T} = e^{-\omega/2T}(1 - e^{-\omega/T})^{-1} \tag{1.6}$$

$$Z_F(\omega) = \sum_n e^{-(n-\frac{1}{2})\omega/T} = e^{\omega/2T}(1 + e^{-\omega/T}) \tag{1.7}$$

We will be interested in the free energy density, $f = F/V$. From (1.2) and (1.7)

$$f_B = \int \frac{d^3p}{(2\pi)^3} \left[\frac{\omega}{2} + T \ln(1 - e^{-\omega/T})\right] \tag{1.8}$$

$$f_F = \int \frac{d^3p}{(2\pi)^3} \left[-\frac{\omega}{2} - T \ln(1 + e^{-\omega/T})\right] \tag{1.9}$$

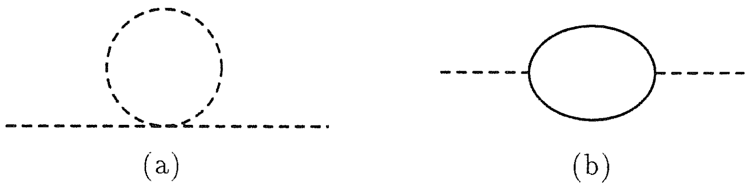


Figure 1.1: Feynman diagrams for the thermal correction to the mass. Dashed lines are the boson φ, continuous lines are fermionic

The expression (1.8) is precisely the effective potential at one loop and at finite temperature for a scalar field ϕ with a mass m, including the zero-temperature term. It was derived by Dolan and Jackiw [4] using functional integral methods, for a scalar field with a Lagrangian (1.1) The mass of the field is defined as

$$M^2 = \frac{\partial^2 V}{\partial \varphi^2} = -m^2 + 3\lambda\varphi^2 \tag{1.10}$$

We use $\omega^2 = \vec{k}^2 + M^2$ to write the temperature dependent part of the free energy, or effective potential at one loop, as

$$V_1(T) = \frac{T^4}{2\pi^2} \int_0^\infty dx x^2 \ln(1 + e^{-\sqrt{x^2+M^2/T^2}}) \tag{1.11}$$

Now we can expand it for $T >> M$,

$$\begin{aligned} V_1(T) &\sim \frac{\pi^2}{90}T^4 + \frac{T^2}{24}M^2 - \frac{T}{12\pi}M^3 \\ &\sim \frac{\pi^2}{90}T^4 + \frac{T^2}{24}(-m^2 + 3\lambda\varphi^2) - \frac{T}{12\pi}m(-m^2 + 3\lambda\varphi^2) \end{aligned} \tag{1.12}$$

At high temperature and to highest order in T/M, we will write the complete effective potential for φ as

$$V(T) = -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\phi^4 + \frac{T^2}{24}3\lambda\varphi^2 \tag{1.13}$$

where we have ignored constant terms such as the T^4 contribution from the radiation energy.

The result (1.13) is nothing but the contribution from the self-energy of φ at one loop, coming from the bosonic graph of figure 1.1(a) . This contribution was first calculated by Weinberg [3] (details are given in the Appendix). The fermionic graph (b) gives a similar result, and, more importantly, with the same sign. For a Yukawa coupling of the type

$$h\bar{\psi}\psi\varphi \tag{1.14}$$

the contribution to the effective potential at high temperature is

$$\Delta V(T) = \frac{1}{24}T^2[3\lambda + |h|^2]\varphi^2 \tag{1.15}$$

But what is of interest to us is the fact that, even when considering only the self-interactions of φ, the thermal bath induces a temperature-dependent mass for φ

$$\Delta m(T)^2 = 3\lambda\frac{T^2}{24} \tag{1.16}$$

which is positive definite. This means that there will be critical temperature

$$T_c^2 = \frac{4m^2}{\lambda} \tag{1.17}$$

above which the effective mass of φ is positive. In other words, the minimum of the theory is no longer at $\langle\phi\rangle \equiv v = m^2/\lambda$, but at $\langle\phi\rangle = 0$. The symmetry is restored at high temperatures.

Weinberg [3] gives a very useful general formula for calculating the contribution to the thermal mass for a theory with more complicated group structure, and including the gauge field contribution. In a theory invariant under a group G, with fields ϕ_i transforming under a representation of G with generators T_i, he finds

$$\Delta V(T) = \frac{T^2}{24}\left[\left(\frac{\partial^2 V}{\partial\varphi_i\partial\varphi^i}\right) + 3(T_aT_a)_{ij}\,\varphi^i\varphi^j\right] \tag{1.18}$$

where sum over repeated indices is understood.

In the simple theory of (1.1), the pint $\langle\varphi\rangle = 0$ is not a minimum at high temperature: it is a saddle point. The potential at very high and very low T is depicted in figure 1.2. The system undergoes a phase transition at $T = T_c$. If we consider a system like the Universe, going from high to low temperatures, after the phase transition the field will take up values in the vacuum manifold. This is called a second order phase transition, since it can be shown that in this case both the free energy and the entropy are continuous at the transition. Only the second derivative of f with respect to T becomes discontinuous. The field smoothly rolls down to its new values in the vacuum, once the temperature falls below the critical one.

A phase transition will be of first order if the entropy has a discontinuity. This is what happens, for example, in theories with a potential of the Coleman-Weinberg type

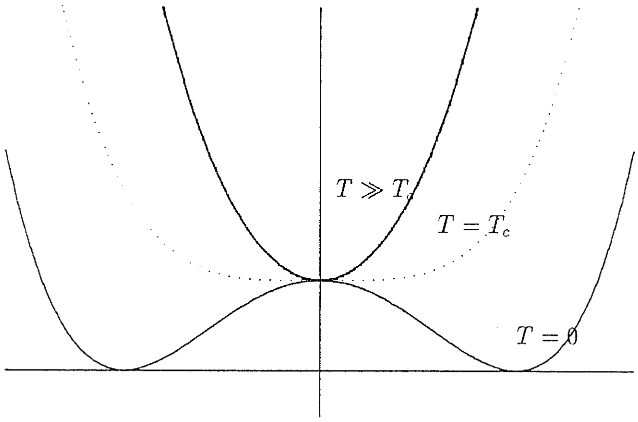


Figure 1.2: Phase diagram for a second-order phase transition.

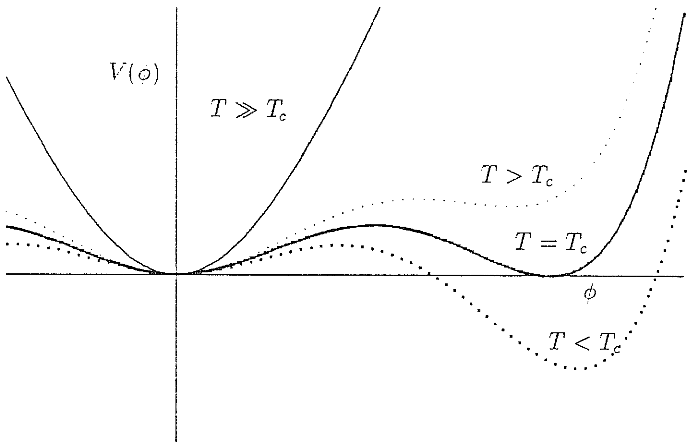


Figure 1.3: Phase diagram for a first-order phase transition.

$$V(\phi) = \frac{a}{4}\varphi^4(\ln\frac{\varphi^2}{\sigma^2} - \frac{1}{2}) \tag{1.19}$$

In this case, the point $\langle\varphi\rangle = 0$ is always a minimum, although not a global one. The situation is that of figure 1.3

To get to its true minimum the field has to tunnel through the barrier, and it does so by nucleating "bubbles" of the true vacuum, as demonstrated by Coleman [5]. These configurations will rapidly expand, collide, and merge, and the final result is that the field acquires a VEVall over spacetime.

1.2 Topological defects

It is well known that some systems in field theory admit classical solutions to the equations of motion that are both non-dissipative and have a finite energy. Coleman [6], in his excellent review, calls such solutions "lumps," to distinguish them from the usual wave solutions that dissipate with time. The term "soliton" is applied to these solutions if in addition they satisfy some requirements on the particular dependence of their energy with space and time (see, for example, the book by Rajaraman [7] for a complete discussion). Roughly speaking, solitons have the characteristic of keeping their shape after collisions. Some of the topological defects we will discuss below are true solitons, and some not; but they all share the important feature of being time independent solutions to the classical equations of motion whose energy is finite.

As so often happens in particle physics, it was Nambu [8] who first pointed out that topological defects are intimately connected with spontaneous symmetry breaking, and that they might be produced in a cosmological phase transition. He was referring specifically to one-dimensional lumps, the so-called domain walls.

1.2.1 One dimension

Domain walls are solutions to the classical equations of motion in one dimension. The simplest such solution, the "kink" [9, 10, 11] is found in the theory we considered in the last section, namely a real scalar field with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{1.20}$$

The potential is minimized by $\langle \phi^2 \rangle = v^2$, and when ϕ takes this value in vacuum, the symmetry is spontaneously broken. There are, however, two possible configurations for $\langle \phi^2 \rangle$ that minimize the energy, $\langle \phi \rangle = +v$ or $\langle \phi \rangle = -v$. The vacuum manifold is discrete, consisting of two disconnected points. This is just a manifestation of the fact that the spontaneously broken symmetry is a discrete symmetry. The space of minima is therefore degenerate, and the two possible configurations are completely equivalent.

We can give an intuitive argument as to why there should be a non trivial solution to the equations of motion in this case. Consider a situation in which the field takes a VEVin one of the minima (say $+v$) for large positive values of one of the coordinates ($x \to +\infty$), and another in the second minimum ($-v$) for large negative values of that coordinate ($x \to -\infty$). Since the field must be a *continuous* quantity, it cannot

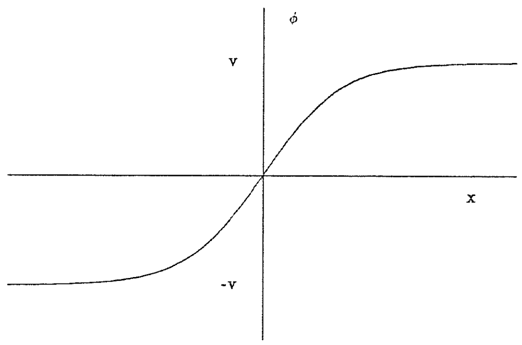


Figure 1.4: The kink solution

jump in between the two minima–it has to acquire a configuration that smoothly interpolates between them at some point. The exact shape of such a configuration is to be determined by solving the equations of motion, but it is clear that the boundary conditions that we have imposed will force $\langle\phi\rangle$ to take non-constant values in some region of space.

These conditions are a map between the manifold of the points at infinity

$$\mathcal{M}_\infty = \{x \to -\infty \; ; \; x \to +\infty\} \tag{1.21}$$

and the vacuum manifold

$$\mathcal{M}_0 = \{\phi_0 \therefore V(\phi_o) = 0 \Rightarrow \phi_0 = \pm v\} \tag{1.22}$$

Each manifold consist of two points, and our map is single-valued. The problem then is one-dimensional. One can picture the field taking VEVs in the minimum of the potential in two regions of three dimensional space, and then interpolating in the two dimensional boundary between them, forming a domain wall.

The kink solution is easily found by integrating the equations of motion

$$\phi_0 = v \tanh \frac{x}{\delta} \qquad \delta \equiv \frac{2}{\sqrt{\lambda} v} \tag{1.23}$$

Figure 1.4 is a plot of this function (with $\delta = 1$), In its effort to interpolate between the two minima, the field has to leave $\mathcal{M}_0$ and take the value zero inside the domain wall. This means that its potential energy is *maximal* at the origin. The only way for the field to do that is to simultaneously try to minimize its gradient energy. We expect then the wall to be very thin, and the solution (1.23) tells us precisely this; the thickness of the wall is given by δ which is

$$\delta = \frac{2}{\sqrt{\lambda}} \frac{10^{-14}}{v} cm \tag{1.24}$$

where v is in GeV. Even walls formed at a low energy are extremely thin. The walls we are considering here are infinite in the other two spatial directions. Their energy per unit area is

$$\sigma \equiv \frac{E}{A} = \frac{2}{3}\sqrt{\lambda} v^3 \tag{1.25}$$

a result that could be expected on purely dimensional grounds.

The energy is indeed finite. It remains to determine whether this solution is stable. Note that we can define a conserved current

$$J^{\mu} = \epsilon^{\mu\nu}\partial_{\nu}\phi \tag{1.26}$$

with an associated charge

$$Q = \int_{-\infty}^{+\infty} dx J_o(z) = \int_{-\infty}^{+\infty} dx \frac{d\phi}{dx} = \phi(+\infty) - \phi(-\infty) \tag{1.27}$$

Now, Q is a conserved quantity, a *topological charge.* Although it is conserved in the same sense as any familiar Noether charge, it is not of the same nature: it does not correspond to an invariance of the action (since the symmetry $\phi \rightarrow -\phi$ is actually broken). The domain wall cannot be "unwinded" into a trivial, constant configuration for ϕ, since for the kink above, the topological charge is

$$Q = 2v \tag{1.28}$$

A constant solution, on the other hand, can have $\phi(+\infty) = \phi(-\infty) = +v$, for instance. Its topological charge is zero. The kink cannot decay and become a constant solution–it is protected by a conservation law. This is the origin of the term "topological" defects.

Another interesting example of domain wall appears in theories with a potential of the sine-Gordon type

$$V(\phi) = \lambda v^4[1 - cos(\phi/v)] \tag{1.29}$$

The corresponding solution is

$$\phi(x) = 4v \tan^{-} 1 \left(e^{\sqrt{\frac{\lambda}{2}} vx}\right) \tag{1.30}$$

The domain wall separates the degenerate vacua $\langle\phi\rangle = 2\pi n v$. In this case, the domain wall is a real soliton, and it can be proven [7] that it does not change its shape during collision. A potential similar to (1.29) appears in the Peccei-Quinn [12] axion model, and Sikivie [13] first suggested the existence of domain walls in this theories. We will be dealing with them in chapter 3.

1.2.2 Two dimensions

The domain wall is a two-dimensional object which results from the mapping of one of the spatial coordinates to the only internal degree of freedom of the real scalar field. We are immediately led to a generalization, when the internal degrees of freedom of the field are two, and expect to find a one-dimensional topological defect. Namely, one has to consider a theory where the vacuum manifold is two-dimensional, and find a nontrivial map between this manifold and spacetime. Take the simplest example of a complex field with an $U(1)$ symmetry, with a potential

$$V = \frac{\lambda}{4}(\phi^*\phi - v^2)^2 \tag{1.31}$$

The vacuum manifold $\mathcal{M}_o$ is a circle, since the minimum of V is at

$$\langle|\phi|\rangle^2 = v^2, \quad \langle\phi\rangle \;=\; ve^{i\alpha} \tag{1.32}$$

In cylindrical coordinates, ρ, θ, z, we look for a static solution of the equations of motion that provides a map between the vacuum manifold $\mathcal{M}_o = S^1$ and the spacetime manifold at infinity, $\mathcal{M}_\infty = \{\rho = R,\ R \to \infty\}$, or

$$\phi(R \to \infty) \to ve^{in\theta} \tag{1.33}$$

where n is an integer. We have mapped the field's $U(1)$ phase to the angular coordinate θ, for any value of ρ and z. But the map is singular at $\rho = 0$, where θ is undefined. The field is not able to pick up a phase there, and therefore is forced to go out of the vacuum manifold, and take the value $\langle\phi\rangle = 0$ at the origin. It leaves the vacuum in a finite cylindrical region around the origin, with radius δ determined by the minimization of energy, just as in the domain wall case.

However, the energy per unit length of such object is easily seen to diverge. Inside the string, for $\rho < \delta$, the main contribution to the energy comes from the potential, while for $\delta < \rho < R$, it comes from the gradient energy

$$\mu \equiv \frac{E}{L} \sim \int_0^\delta \rho d\rho d\theta V(\phi = 0) + \int_\delta^R \rho d\rho d\theta \frac{1}{2\rho^2}|\partial_\theta\phi|^2 \sim v^2 + \pi v^2 \ln\frac{R}{\delta} \tag{1.34}$$

This is just an example of a general theorem due to Derrick [14]. It states that there are no stable, finite energy solutions for scalar field theories in more than one dimension. The kink is then the only stable topological defect that one can form with a scalar field.

One way to avoid this is to consider that the string solution does not have to be valid up to infinity, *i.e.* some cut-off is placed at a finite distance. One can consider that there is an anti-string or, equivalently, that strings are only formed in closed loops; or one can use the fact that the Robertson-Walker Universe has horizons, and therefore a natural cut-off . There is, however, a much more interesting way of getting beyond the validity of Derrick's theorem–gauge the theory. This is what Nielsen and Olesen [15] did. obtaining the so-called vortex solutions in two dimensions.

Local strings are perfectly well-defined topological defects, with a finite energy. They manage to do so by satisfying

$$D_\mu\phi = (\partial_\mu - igA_\mu)\phi \rightarrow 0, \quad r \rightarrow \infty \tag{1.35}$$

The string solution is found to be

$$\phi = vf(r)e^{in\theta} \tag{1.36}$$

where $f(r)$ vanishes at the origin and becomes one at infinity. The gauge field behavior at infinity

$$A_\mu \rightarrow \frac{n}{g}\partial_\mu\theta \tag{1.37}$$

guarantees the finiteness of the energy per unit length, since now there will be a magnetic flux inside the string

$$\int \vec{B}\cdot d\vec{S} = \oint_{R\rightarrow\infty} A_\mu dx^\mu = \frac{n}{g}\oint_{R\rightarrow\infty} \partial_\mu\theta dx^\mu = \frac{2\pi n}{g} \tag{1.38}$$

If we calculate the magnetic field inside a string of radius δ

$$B = \frac{2n}{g\delta^2} \tag{1.39}$$

(assuming that it is constant) we see that while the energy in the Higgs field is minimized for $\delta \rightarrow 0$, that of the magnetic field prefers $\delta \rightarrow \infty$. It is the interplay between the two factors that gives the string its stability. The energy per unit length and thickness are

$$\mu \propto \sqrt{\frac{\lambda}{g^2}} v^2 \quad ; \quad \delta \propto \left(\frac{1}{g^2\lambda}\right)^{1/4} v^{-1} \tag{1.40}$$

where the precise numerical factor will depend on $f(r)$.

There are other types of strings one can consider, apart from the $U(1)$ we have referred to. Z_N strings are formed when the symmetry group breaks down to a subgroup with a Z_n factor [16]. In any case, the stability of the solution is guaranteed by a topological law similar to the one we encountered for domain walls: the winding number of the string, n, is a conserved quantity. The string cannot "unwind" to the trivial vacua, and the magnetic flux is conserved and quantized.

1.2.3 Three dimensions

Going up one dimension means allowing the field to have three internal degrees of freedom, all of which can be mapped to the spacetime coordinates. The defect formed will be one-dimensional, the monopole. It is precisely the fact that one-dimensional topological defects can be identified with monopoles, and in particular electromagnetic monopoles if the symmetry is broken to $U(1)_{em}$, that makes them so interesting. Monopoles were among the first topological defects to be studied, and the first monopole solution was obtained by 't Hooft [17] and Polyakov [10].

Derrick's theorem precludes the existence of finite energy global monopoles, prompting us to discuss the gauge case only. The simplest example comes from a theory with a $SO(3)$ gauge symmetry and a scalar field $\vec{\phi}$ transforming under the fundamental representation of the group. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) + V(\phi) \tag{1.41}$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \\ D_\mu\phi^a &= \partial_\mu\phi^a + \epsilon_{abc}A^b_\mu\phi^c \\ V(\phi) &= \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2 \qquad a = 1, 2, 3 \end{aligned} \tag{1.42}$$

We will identify the diagonal generator with the generator of an electromagnetic charge, ensuring charge quantization. At the minimum

$$\langle\phi^a\phi^a\rangle = v^2 \tag{1.43}$$

which tells us that the vacuum manifold is the surface of a three-dimensional sphere

$$\mathcal{M}_0 = \left\{(\phi_0^1)^2 + (\phi_0^2)^2 + (\phi_0^3)^2 = v^2\right\} = S^2 \tag{1.44}$$

The symmetry is broken down to $U(1)$ when $\langle\phi^a\rangle$ takes values in this vacuum. It can, for example, be

$$\langle\phi^a\rangle = v\delta_3^a \tag{1.45}$$

homogeneously over spacetime. Then the massless gauge boson A_μ^3 is identified with the photon. On the other hand, ϕ can take non-trivial vacuum expectation values, leading to a topological defect configuration.

The analogy with the string case is evident. We want to make a map from $\mathcal{M}_0$ to the manifold of spacetime points at infinity, $\mathcal{M}_\infty$. A non-trivial map is provided in spherical coordinates r, θ, φ as

$$\langle\phi^1\rangle = \sin\theta\cos n\varphi, \quad \langle\phi^2\rangle = \sin\theta\sin n\varphi, \quad \langle\phi^3\rangle = \cos\theta \tag{1.46}$$

where n is the winding number. For $n = 1$, we can write

$$\langle\phi^a\rangle(r \to \infty) = \frac{x^a}{r}v \tag{1.47}$$

The energy is guaranteed to converge at infinity by asking that the covariant derivative vanishes there, which tells us that the gauge field takes the asymptotic form

$$A_i^a \stackrel{r\to\infty}{\longrightarrow} \epsilon^{aij}\frac{x_j}{gr^2} \tag{1.48}$$

As for the local string, we expect the magnetic field to be nonvanishing inside the monopole. To find it, however, we need a gauge invariant definition of the electromagnetic stress tensor, $\mathcal{F}_{\mu\nu}$, in terms of the $SO(3)$ gauge fields. In the absence of the monopole our definition should reduce to the usual electromagnetic one, written in terms of the photon field. 't Hooft [17] found the correct form of $\mathcal{F}_{\mu\nu}$

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu}^a\frac{\phi^a}{|\phi|} - \frac{1}{g}\epsilon^{abc}\frac{(D_\mu\phi)^a(D_\nu\phi)^b\phi^c}{|\phi|^3} \tag{1.49}$$

It is easy to see that in the vacuum given by (1.45), we have the usual form $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$. The asymptotic magnetic field can be obtained from (1.48) and (1.49)

$$B_k \stackrel{r\to\infty}{\longrightarrow} -\frac{x_a}{gr^3} \tag{1.50}$$

This indeed corresponds to the electromagnetic field produced by a pure magnetic charge at the origin or, in other words, a monopole with magnetic charge $g_m = 4\pi/g$. Already in 1931, Dirac [18] proved that a magnetic monopole would imply, in quantum mechanics, the quantization of the electric charge. The topological defect that we call a monopole is an example of the inverse reasoning –a theory with a quantized charge admits monopole solutions. Of course, one has to identify the $U(1)$ group remaining from the breaking of $SO(3)$ by the VEVof a vector field as the symmetry group of electromagnetism.

It can be shown that, just as with domain walls and strings, there is a conserved current of the form

$$J_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial^\nu \left(\frac{\phi}{|\phi|}\right)^a \partial^\rho \left(\frac{\phi}{|\phi|}\right)^b \partial^\sigma \left(\frac{\phi}{|\phi|}\right)^c \tag{1.51}$$

corresponding to a conserved charge

$$Q = \int d^3x J_0 = n \tag{1.52}$$

which is just the winding number. The magnetic charge of the monopole can be related to this charge, and it is obtained

$$g_M = \frac{4\pi}{g}Q = \frac{4\pi}{g}n \tag{1.53}$$

which is the topological conservation law that we expected, guaranteeing the stability of the monopole solution, and in this case, also the charge quantization.

The 't Hooft-Polyakov monopole, also called the hedgehog solution, takes the form

$$\langle\phi^a\rangle = H(r)\frac{x^a}{r}v, \quad A_i^a = [1 - K(r)]\epsilon^{aij}\frac{x_j}{gr^2} \tag{1.54}$$

We know the asymptotic forms for $H(r)$ and $K(r)$, but their exact dependence cannot be found in general in exact form. Prasad and Sommerfeld [19] showed that it is possible to do it in the limit when the ratio of couplings $\sqrt{\lambda}/g \to 0$. In this case

$$H(r) = \coth(g\,v\,r) - \frac{1}{g\,v\,r}\ , \qquad K(r) = \frac{g\,v\,r}{\sinh(g\,v\,r)} \tag{1.55}$$

Moreover, Bogomol'nyi [20] has found that the hedgehog mass is bounded from below, so that

$$m_M \geq \frac{4\pi}{g}v \tag{1.56}$$

and that the inequality is saturated in the Prasad-Sommerfeld limit.

Monopoles can be formed in more complicated theories, and solutions similar to the 't Hooft-Polyakov can be found when the vacuum manifold allows for them (see below). Dokos and Tomaras [21], and Schellekens and Zachos [22], have found a number of such solutions in the $SU(5)$ theory.

1.3 Phase transitions and the formation of topological defects

Topological defects would be little more than curiosities for a cosmologist, were it not for the possibility that the Universe underwent one or several phase transitions. It is during these phase transitions, taking place when the temperature of the Universe reaches the critical one, that the defects may actually be produced. The possibility was pointed out, as we already mention, by Nambu [8], who suggested that the Universe may consist of distinct domains, each one with a different (but equivalent) value of the VEVof the Higgs field, and having correspondingly different physical properties. In Weinberg's classical paper [3] on symmetry breaking at high temperature, the possibility of having domain walls formed is already mentioned. Everett [23] and Zel'dovich, Kobzarev and Okun [24] took the possibility of domain walls seriously and first calculated their effects in the evolution of the Universe. It was Kibble [25] who later made a thorough study of the possibility of formation of such structures, and in fact showed that the causal structure of our Universe *implies* their appearance in a phase transition. Topological defects where no longer a possibility –they became an unavoidable consequence of the phase transitions, if the topology of the vacuum manifold admits them. For this reason, the process by which the topological defects are produced is often called the Kibble mechanism.

The way in which this actually takes place depends, in the first place, on the nature of the phase transition, namely on whether they are of first or second order. In a second order phase transition, the false vacuum becomes unstable near $T = T_c$. The field begins taking values in the vacuum manifold, and the scales on which it can do so coherently are dictated by its correlation length. In a real scalar field theory with a double-well potential

$$V = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{1.57}$$

the thermal mass is

$$m(T)^2 = \frac{\lambda}{4}[T_c^2 - T^2] \tag{1.58}$$

for temperatures below the critical, $T_c = v^2$. The correlation length (ξ) is of the order of the inverse Higgs mass, or

$$\xi \sim \frac{2}{\sqrt{\lambda}T_c}\left[1 - \frac{T^2}{T_c^2}\right]^{-1/2} \tag{1.59}$$

Of course, the correlation length thus defined becomes infinite at T_c, but this is just a reflection of the fact that we cannot trust (1.59) there.

After the phase transition has taken place, the field still has enough thermal energy to jump over the potential. In regions of radius ξ, $\langle\phi\rangle$ fluctuates wildly from negative to positive values. It will do so until its thermal energy (T) becomes too small to jump over the potential barrier ΔV. The energy required for a region of radius ξ to overcome the potential barrier is proportional to

$$\frac{8\pi}{3}\xi^3\Delta V \simeq 8\left(\frac{8}{\lambda^{3/2}T_c^3}\left[1 - \frac{T^2}{T_c^2}\right]^{-3/2}\right)\left(\frac{\lambda}{32}T_c^4\left[1 - \frac{T^2}{T_c^2}\right]^2\right) = \frac{1}{\sqrt{\lambda}}T_c\left[1 - \frac{T^2}{T_c^2}\right]^{1/2} \tag{1.60}$$

and it will become of order T when the Ginzburg temperature T_G is reached, namely

$$T_G^2 = \frac{1}{1+\lambda}T_c^2 \tag{1.61}$$

which does not differ much from T_c for small λ. When this "freeze out" temperature is reached, the field takes its definitive values in regions of size

$$\xi_G \sim \left(\frac{\lambda}{2}\sqrt{1-\lambda}T_c\right)^{-1} \sim (\lambda T_c)^{-1} \tag{1.62}$$

for $\lambda \ll 1$. This is much smaller than the horizon size

$$H^{-1} \sim \frac{M_P}{T_G}\frac{1}{T_G} \gg \frac{1}{T_c} \tag{1.63}$$

The field takes inhomogeneous VEV's over spacetime. Domain walls will be formed in the boundaries of regions with a different sign of the vacua.

If the transition is first order, the process occurs by quantum tunneling of the field to its new values in the vacuum. Bubbles of true vacuum with the critical size are formed inside the false vacuum, and certainly the values of the VEVof the field in two different bubbles are uncorrelated. In the discrete symmetry case, the VEVcan take positive or negative values. The bubbles expand, rapidly reaching their terminal velocities (close to the speed of light), and start colliding. If two bubbles with the same sign of the VEVmake contact, their walls simply dissolve (the vacuum configuration at both sides being trivial), and coalesce. If the value of the VEVis opposite, the walls of the bubbles will evolve into a domain wall, which will later reach its equilibrium thickness.

Kibble also provided the general tools for the study of the formation of defects in a field theory. As can be seen from the discussion in the previous section, the nature of the defect solution will essentially depend on the map between the manifold of the vacua and the spacetime manifold at infinity. In other words, it will depend on the topology of the vacuum manifold.

A spontaneously broken discrete symmetry will produce domain walls because the vacuum manifold consists of two or more disconnected pieces. We will map regions of spacetime at infinity with different such pieces, and one or more domain walls will be formed in between. This property of the vacuum manifold is expressed in group theory by the homotopy groups.

Homotopy groups π_n classify the possible mappings of the manifold with a sphere S^n. A manifold consisting of disconnected points has a zeroth homotopic group, π_0. non trivial. In the example of a real scalar field given above, the manifold is Z_2, and in the axionic walls case that we will see in chapter 3 it is Z_N.

Strings are formed when the first homotopy group, π_1 of the vacuum manifold is not trivial. π_1 determines the mappings with a circle S^1, and count the number of non equivalent closed loops one can form in the manifold. In the string of the previous section, a $U(1)$ symmetry was completely broken, and the vacuum manifold is a circle. There are n ways to map a circle into a circle (one can wind n times around it), and in this case π_1 is Z_n. In a completely analogous way, monopoles are formed whenever the vacuum manifold has a non trivial π_2, which counts the number of inequivalent closed surfaces that can be formed in the manifold. In the example of $SO(3)$ broken to $U(1)$, this manifold was S^2 and again, there are n closed surfaces one can form in the surface of a three dimensional sphere, depending on how many times one winds around it.

These results, that domain walls, strings or monopoles will be formed according to which homotopy group is non trivial, is absolutely general. For more complicated groups and symmetry breaking patterns, the task of determining the homotopy groups of the vacuum manifold is facilitated by a useful theorem. If G is a connected and simply-connected Lie group, having a subgroup H with a component H_0 connected to the identity, then

$$\pi_{n+1}(G/H) \simeq \pi_n(H) \tag{1.64}$$

This means that if we have G breaking down to H, then the $n+1$th homotopy group of the vacuum manifold $\mathcal{M} = G/H$ is the nth homotopy group of the unbroken piece H. To determine if strings are formed, then, it is enough to see if the group surviving symmetry breaking is simply connected. For monopoles, the unbroken group should have $\pi_1 \neq 1$, or more than one family of closed loops.

In particular, consider the case of monopoles. Any simple group G that breaks down to a group that has a $U(1)$ factor will form monopoles, since $\pi_1(U(1)) \neq 1$. This is a very important statement, since we know that the Standard Model of particle physics has a $U(1)$ factor, that of electromagnetism. So we are led to conclude that any grand unified theory (GUT) that incorporates electromagnetism in a larger symmetry (and in doing so, implies quantization of electromagnetic charge), when breaking spontaneously will produce monopoles, much to the amusement of Dirac.

An interesting application of these ideas comes about if one considers the breakdown of G to H and further breaking to J. Depending on the topology, one can form the so-called hybrid defects. One can have, for example, the $U(1)$ group of the string's example above, and two complex fields with different charges under this symmetry. One can have

$$V = -\frac{m_i^2}{2}\phi_i^*\phi_i + \frac{\lambda_i}{4}(\phi_i^*\phi_i)^2 + \frac{\alpha}{2}\,\phi_1^*\phi_1\,\phi_2^*\phi_2 - \frac{\mu}{2}[\phi_1^*\phi_2^2 + {\phi_2^*}^2\phi_1] \tag{1.65}$$

namely, the $U(1)$ charge of ϕ_1 doubles that of ϕ_2. When ϕ_1 acquires a vev at a scale v_1, $U(1)$ is broken and strings are formed

$$\langle\phi_1\rangle = ve^{i\theta} \tag{1.66}$$

in cylindrical coordinates. $U(1)$ is broken to Z_2, since the potential is still invariant under $\phi_2 \rightarrow -\phi_2$. Later, at a scale v_2, ϕ_2 also gets values in the vacuum. Writing it as $\phi_2 = v_2 e^{i\delta}$, we see that the μ term becomes

$$\mu v_1 v_2^2 \cos(2\delta - \theta) \tag{1.67}$$

If μ is negative, $\delta = \theta/2$ will be preferred. At $\theta = 2\pi$, the value of δ is still π, so in order for $\langle\phi_2\rangle$ to be continuous it has to jump over to 2π in a narrow region, for all values of r and z. This is nothing but a wall attached to the string.

What happened then is that $U(1)$ first broke down to Z_2, whose π_0 is non trivial, and the theorem dictates that strings be formed. Afterwards the discrete symmetry Z_2 got broken, and walls had to form. However, the final result is that $U(1)$ broke down to nothing, and the walls were just a consequence of the pattern of symmetry breaking. They are not protected by topology, and they are unstable. We will in the next chapter come back to the question of this instability.

In a similar way, strings can be attached to monopoles, if we first break a group G to $U(1)$ and then $U(1)$ to nothing, for example. In the first step, π_1 of the unbroken group is non trivial and we get monopoles. Strings get formed attached to the monopoles in the second stage. In general, one can have complicated sequences of symmetry breaking, and in each case the only sure way to know the nature of the topological defects formed is to investigate the topological properties of the vacuum manifold.

2 Of the Problems Caused by Domain Walls and Monopoles

2.1 Introduction

Since it was proposed that topological defects can be formed, intense research has been carried out on their properties and in particular, on their interest to cosmology. Strings are by far the most studied by cosmologists. They may provide the necessary perturbations in the primordial Universe to account for the large scale structure. and in this sense are a rival theory to inflation. We are not going to discuss in strings in this thesis, and refer for example to [26] for an extensive review.

The main reason why strings are so interesting, however, is a simple fact–they can be formed in our Universe without spoiling the standard Big Bang model. This is not the case with domain walls and monopoles, as we note below. We give a review on the domain wall and monopole problem and its proposed solutions. and then state the proposal of this thesis.

2.2 The domain wall problem

The problem with domain walls is that they carry a large amount of energy. Roughly, their energy per unit area is proportional to the scale of symmetry breaking

$$\sigma \simeq v^3 \tag{2.1}$$

so that even for a domain wall formed at low scales, say 100 GeV, we will have

$$\sigma \simeq 10^6 GeV = 10^{10} \frac{gr}{cm^2} \tag{2.2}$$

We have then an object with a surface tension of 10,000 tons per square centimeter. Depending on its size, this energy can be significant in comparison to the energy carried by the rest of the matter and radiation in the Universe. It becomes important to determine the size of the domain walls created during the phase transition.

A first estimate was done in [27] and [28], with a simple Monte Carlo simulation. The idea is to take a three-dimensional lattice, with each cell representing a causally connected region at the moment of the phase transition. The VEVof the field is taken to be $+v$ or $-v$ in each cell, at random. Suppose that we call p the number of cells in which the positive value is assigned, divided by the total number. Now, a basic result of percolation theory is that every lattice has a characteristic critical value for p, call it p_c, for which the system forms a large cluster of cells with the same (in this case positive) value. That is, for $p > p_c$, the lattice will contain a region that extends up to the boundaries, with the same value of the field's VEV. This is equivalent to saying that an infinite domain wall will form. The value of p_c depends on the specific form of the lattice. For a cubic lattice, for example, $p_c = 0.31$, but in general it is smaller than 0.5 for any regular three-dimensional lattice.

Clearly, the probability of having a cell with a positive value of the VEVis 0.5. We will then have $p = 0.5$, and so $p > p_c$ always. Thus, in general we expect one infinite domain wall to be formed as a consequence of the phase transition. The distribution of smaller, closed walls depends on the characteristics of the lattice. Given the large surface tension, one expects this closed system to evolve rapidly, contracting and disappearing.

But a domain wall that extends all over the horizon length cannot contract over itself and disappear. Rather, its tension will smoothen it out in larger and larger regions, until we are left with a unique, "infinite" (in the sense that is larger than the horizon) almost plane domain wall. It is not difficult to convince oneself that this object is much too heavy to live in our Universe. We can compare its energy density with the critical energy density of the Universe. We have

$$\rho_c = H^2 M_P^2 \frac{3}{8\pi}, \quad \rho_{\text{wall}} = \frac{\sigma}{R} \tag{2.3}$$

where H is the Hubble constant and R is the radius of the present-day Universe, or H^{-1}. Thus,

$$\Omega_{\text{wall}} \equiv \frac{\rho_{\text{wall}}}{\rho_c} = \frac{8\pi}{3g_*} \frac{\sigma}{H M_P^2} \tag{2.4}$$

where g_* counts the relativistic degrees of freedom. With $H \simeq 10^{42}$ GeV, and for walls formed at the electroweak scale this gives

$$\Omega \simeq 10^{11} \tag{2.5}$$

The Universe is "overclosed" by the wall's energy density. This is the domain wall problem. It was first stated by Zel'dovich, Kobzarev and Okun in 1975 [29], who calculated the evolution of a domain-wall dominated Universe, starting form the energy-density tensor for a domain wall. They show that such a Universe will expand as t^2, much faster than the radiation-dominated Universe. Later, Kibble [25] studied the evolution of the domain walls created during the phase transition, concluding once more that they were not compatible with our Universe.

In fact, with an infinite domain wall the Universe will never evolve to present-day dimension, but will become wall-dominated at early times. A wall's evolution is dominated by the interplay between the tension force, $f_T \sim \sigma/R$ (where R is the radius over which the wall is smooth), and the friction force produced by its movement in the radiation background, $f_F \sim vT^4$. The wall's velocity v is determined by the balance between these two forces

$$v \sim \frac{\sigma}{RT^4} \sim \frac{\sigma t^2}{M_P R} \tag{2.6}$$

for a radiation-dominated Universe. The region over which the wall is smooth at time t will be vt, thus it will grow with time as

$$R(t)^2 \sim \frac{\sigma t^3}{M_P^2} \tag{2.7}$$

Now we can calculate how much a region of wall with radius $R(t)$ will contribute to the energy density. We have

$$\rho_{\text{wall}}(t) \sim \sqrt{\frac{M_P^2 \sigma}{t^3}} \tag{2.8}$$

and

$$\Omega_{\text{wall}} \sim \frac{\sqrt{\sigma t}}{M_P} \tag{2.9}$$

The domain wall energy will start dominating the Universe at times $t_W \sim \sigma/M_P^2$. For our low energy domain wall, with a scale of symmetry breaking 100 GeV, this happens at $t \sim 10^7 sec \sim$ one year after the Big Bang.

In general, one expects walls to be formed at much higher energies, so that the problem gets worse. In order not to interfere with the evolution of the Universe, the scale of the discrete symmetry breaking must be smaller than about $1MeV$ [29].

There have been several attempts to cure the domain wall problem of theories with spontaneous breakdown of discrete symmetries. One of them is inflation [30]. If the Universe went through an inflationary regime, expanding exponentially during a vacuum-dominated era, the wall's number density can be made exponentially small. In other words, our Universe today is made up of a region where there were no domain walls, and one says that the walls (and any other defect produced during the phase transition) are inflated away. For this to work, it is necessary that inflation takes place after the walls are formed. At present, no successful model of inflation that requires no fine-tuning of parameters and is justified by the particle physics phenomenology has been found.

Another way of getting rid of the domain walls is to actually break the discrete symmetry. If the two vacua are not exactly degenerate, but some mechanism has lifted the degeneracy by some small amount ϵ, the evolution of the walls is different. Basically, there will be a pressure difference between the two vacua, proportional to ϵ. Suppose that the pressure becomes bigger than the tension of the walls before they can start dominating the energy density, *i.e.* for $t < \sigma/M_P^2$. We require [31]

$$\epsilon > f_T > \frac{\sigma^2}{M_P^2} \sim \frac{v^6}{M_P^2} \tag{2.10}$$

If this is satisfied, the infinite walls will be pushed away from our Universe by the pressure of the true vacuum. Of course, it may be regarded as unnatural to break the symmetry explicitly by hand, even if it were by a small amount. Rai and Senjanović [32] have proposed a way to achieve this. They suggest that gravity may violate the fundamental global symmetries. through black-hole physics process. Because of no-hair theorems, the quantum number associated with a global symmetry will be lost in black-hole collapse. Moreover, there are reasons to believe that time reversal is violated by black holes. If gravity does not respect the symmetry, this will be manifest in the Lagrangian by the presence of higher dimensional operators cut-off by powers of the Planck scale. But a term like

$$\frac{\phi^5}{M_P} \tag{2.11}$$

will give an asymmetric contribution to the potential of order v^5/M_P, which easily satisfies the bound of (2.10). This will provide then a solution to the domain wall problem, but of course to get into details of the mechanism, one should have a theory of quantum gravity.

Another mechanism that may be used to get rid of the walls is to have the discrete symmetry embedded in a larger group. Thus, one can have the domain walls attached to strings, in the way we discussed in the previous chapter. If a string forms the boundary of a domain wall, the vacuum manifold is in fact trivial, and the wall can decay. Kibble, Lazarides and Shafi [16] have studied the possibility of having the domain wall-string system decaying fast enough to avoid conflicting with cosmology. Essentially, quantum tunneling can create a hole in the wall, made up from a loop of string. However, in order to create a loop of string, one has to spend some energy, which should be compensated by the energy gained in removing a corresponding piece of wall inside the loop. The probability per unit area per unit time of nucleating a loop of string in the wall will be

$$P \sim Ae^{-S_0} \tag{2.12}$$

where S_0 is the Euclidean action corresponding to this tunneling process, and we will not be concerned with the pre-exponential factor A. The difference in the Euclidean action produced by the nucleation of a string of radius R and the disappearance of the corresponding wall hole will be

$$S_0 = 4\pi R^2\mu - \frac{4\pi}{3}R^3\sigma \tag{2.13}$$

where μ is the energy per unit length of the string, and we are calculating the tunneling in four dimensional Euclidean space [5]. The minimum radius for which the hole will not collapse over itself is found by minimizing this action respect to R, giving $R = 2\mu/\sigma$. So we have

$$S_0 = \frac{16\pi}{3}\frac{\mu^3}{\sigma^2} \sim \left(\frac{v_{\text{string}}}{v_{\text{wall}}}\right)^6 \tag{2.14}$$

We see that the probability is strongly suppressed for $v_{\text{string}} \gg v_{\text{wall}}$. In concrete examples as in $SO(10)$ symmetry breaking (which can produce such hybrid defects), the walls are practically stable.

Vilenkin and Everett [33] have considered a particular class of domain walls attached to strings, that of the Peccei-Quinn model, or axionic domain walls. We will give more details on these kind of models in chapter 3. Essentially, the Peccei-Quinn model has a $U(1)$ symmetry that gets broken by the VEVof a Higgs field ϕ, producing strings. The field takes values in the vacuum

$$\langle\phi\rangle = ve^{i\theta} \tag{2.15}$$

However, the potential for the field θ below a certain scale Λ is

$$V_\theta = \Lambda(1 - \cos N\theta) \tag{2.16}$$

Therefore, θ takes also values in the vacuum, $\langle\theta\rangle = 2N\pi$. The strings get attached to domain walls, which separate vacua with different values for θ. When $N = 1$, Vilenkin and Everett find that the evolution of the string network affects the domain walls. While the strings collide and interconnect, interchanging ends, the walls get successively "chopped" into smaller and smaller pieces. The final result is that the domain wall-string system decays very rapidly. This should not be surprising, since for $N = 1$ the vacua separated by the walls are the same, and the total topology if the domain wall-string system is trivial. However, in the Peccei-Quinn model N is not arbitrary and it is not 1–it is the number of quark species. For $N \neq 1$, the walls are stable, as shown by Sikivie [13].

2.3 The monopole problem

The problem with the monopoles is that they are produced in too large quantities for an object of such a huge mass. With the scale of grand unification at 10^{16} GeV, the monopole's mass is 16 orders of magnitude bigger than a typical baryon. In the case of the monopoles, therefore, we will be interested in calculating their number density.

During the phase transition at $T \sim T_c$, monopoles are created whenever the expectation value of the Higgs field takes a non-trivial configuration in vacuum. A simple estimate on the number of monopoles is obtained by supposing that approximately one monopole per horizon is created. Kibble [25] estimates (with techniques similar to the ones in the previous section) that the number of monopoles per horizon is roughly equal to 0.1, but our conclusions will be little affected by one or two orders of magnitude. If d_H is the horizon size at $T = T_c$, during the radiation-dominated era, we will have for the number density of monopoles

$$n_M \sim \frac{1}{d_H^3} = \left(g_* \frac{8\pi}{3}\right)^{3/2} \frac{T_c^6}{M_P^3} \tag{2.17}$$

where g_* counts all the massless degrees of freedom at T_c. Now, in a Universe expanding adiabatically, the number density over the entropy is a constant number. So we have

$$\frac{n_m}{n_\gamma} \sim \frac{n_m}{g_* T^3} = \sqrt{g_*} \left(\frac{8\pi}{3}\right)^{3/2} \left(\frac{T_c}{M_P}\right)^3 \tag{2.18}$$

For monopoles produced during the GUT phase transition, we get

$$\frac{n_m}{n_\gamma} \sim 10^{-10} \tag{2.19}$$

and this number has remained fixed until the present time. In other words, monopoles produced at the GUT phase transitions should be almost as abundant as baryons today, and 16 orders of magnitude heavier, which is clearly impossible in the context of standard cosmology.

This simple estimate, however, does not take into account that monopoles are in general produced together with antimonopoles. A monopole-antimonopole ($M\bar{M}$) pair can annihilate, and their present day abundance could be in principle significantly reduced, depending on the efficiency of the process. Zel'dovich and Khlopov [34] calculated the annihilation rate and the monopole number density assuming that their initial distribution corresponds to thermal equilibrium, and concluded that the problem of the overabundance of monopoles persists. We now illustrate how this happens following the analysis of Preskill [35], who demonstrated that the assumption of thermal equilibrium can be dropped and the same conclusion reached.

If the annihilation process is characterized by D, in an expanding Universe the number density will obey

$$\frac{dn_M}{dt} = -Dn_M^2 - 3Hn_M \tag{2.20}$$

Preskill [35] assumes a general form for D

$$D = \frac{A}{m_M^2}\left(\frac{m_M}{T}\right)^p \tag{2.21}$$

Given that in a radiation-dominated Universe $H \sim \sqrt{g_*}T^2/(2M_p)$, we can already see that depending on the value of p, the annihilation process can dominate or not dominate the expansion of the Universe in (2.20). The solution of (2.20) can be found in terms of p. For temperatures much smaller than the initial one, the density becomes independent of its initial value, and the solution reduces to [35]

$$\frac{n_M}{T^3} \sim \frac{p-1}{A}\frac{\sqrt{g_*}m}{2M_P}\left(\frac{m_M}{T}\right)^{p-1} \tag{2.22}$$

The specific form of D depends essentially on the ratio between the monopole's mean free path ℓ and the range over which the Coulomb attraction becomes important, r_o. In a thermal bath at temperature T, we have $r_o = g^2/(4\pi T)$, where g is the monopole's magnetic charge. When $\ell < r_o$, the monopoles move through the plasma in a difussion regime. The monopoles are scattered by the charged particles in the plasma, and we have

$$D = \frac{g^2}{\beta T^2} \tag{2.23}$$

where $1/\beta T^2$ gives the sum over all the spin states of charged particles of the scattering cross sections at large angles with each particle. For a particle of charge q, this cross section is $\sigma \sim g^2 q^2/(4\pi T)^2$, hence the temperature dependence in (2.23). With D increasing linearly with time (as is the case in a radiation-dominated Universe), it can be seen that the first term in the right-hand-side of (2.20) will start to dominate over the second term as the Universe evolves. The annihilations will become important then for

$$T_d \sim \frac{\sqrt{g_*}g^2}{6} M_P \left(\frac{n_M}{T^3}\right) \tag{2.24}$$

On the other hand, the diffusion regime will last until $\ell \sim r_o$. We can estimate the mean free path as $\ell \sim v\tau$, with $v \sim \sqrt{T/m_M}$, the average velocity of the monopoles in the thermal bath; and $\tau \sim m_M/(\beta T^2)$ the mean collision time of monopoles in the plasma. So we will have a diffusion regime until

$$T_f \sim \frac{(4\pi)^2 m_M}{g^4 \beta^2} \tag{2.25}$$

If annihilations become important before the diffusion regime ends, that is, if $T_d < T_f$, the abundance of monopoles at the end of this regime will be given by (2.22), evaluated at $T = T_f$ and with D given by (2.23), that is

$$\frac{n_M}{T^3} \sim \frac{(4\pi)^2}{\beta g^6} \frac{\sqrt{g_*}m}{2M_P} \sim 10^{-10} \tag{2.26}$$

for $g^2/(4\pi) \sim 100$, $\beta \sim 10$, $g_* \sim 100$ and $m_M \sim 10^{16}$.

When the Universe cools below T_f, one can begin to consider the monopoles as free particles. But then their annihilation function can be calculated as

$$D \sim \frac{g^2}{(4\pi)^2 m^2} \left(\frac{m}{T}\right)^{\frac{9}{10}} \tag{2.27}$$

With p so close to one, from (2.22) it is evident that the annihilation becomes unimportant. Alternatively, with this D in (2.20), it is easily seen that the annihilation rate never approaches the expansion rate.

Thus, all the important $M\bar{M}$ annihilation takes place while $\ell < r_o$. If $T_d < T_f$, or

$$\frac{n_M}{T^3} < 10^{-7}\frac{m_M}{M_P} \tag{2.28}$$

the diffusion regime ends before annihilation can become important. The abundance of monopoles remains to be the initial one. If, on the other hand, we have initially $n_M/T^3 > 10^{-7}(m_M/M_P)$, the diffusive regime will last until the annihilations can become efficient, but they will reduce the abundance only until it reaches its final value (2.26). We conclude that annihilations are not efficient enough to significantly change our crude estimate of the present-day monopole abundance: they still should be as abundant as baryons, or roughly so.

The domain wall problem represents a threat to models with spontaneously broken discrete symmetries, but the monopole problem is, in general, considered more serious. If one wishes to have charge quantization, the $U(1)$ factor group of the Standard Model has to be embedded in a larger group, and the spontaneous breakdown of this larger symmetry necessarily will produce monopoles. It is remarkable that Dirac [18] has shown that the requisite to have charge quantized is the existence of an electromagnetic monopole. While domain walls are altogether undesirable, monopoles provide us with interesting physics, and it is hard to believe that a unified theory does not include them. The search for a solution to the monopole problem has been very intense.

The first solution proposed is again inflation. As with domain walls, inflation can dissipate the monopole number density until it is practically zero. In fact, the solution to the monopole problem was one of the main motivations of the original paper by Guth [30]. Inflation has to take place at the GUT scale, and care must be taken that monopoles are not produced at the reheating time. Again, we must admit that no successful model for inflation is at present available.

Even if inflated away, monopoles may be produced later by particle collisions in $M\bar{M}$ pairs, although their abundance can be easily kept at safe values [36, 37]. We give more details on this mechanism of monopole production in chapter 4.

An interesting suggestion was made by Langacker and Pi [38]. They propose that after the GUT phase transition had taken place and monopoles were formed, the Universe went temporarily into a phase where $U(1)_{em}$ was broken, to be restored again at lower temperatures. How a symmetry can be broken at high temperatures and restored at $T = 0$ will be the main subject of this thesis, and we will discuss the mechanism in detail in the next session. Accepting for now that this can be done, it is easy to picture how the monopole problem can be solved this way.

When $U(1)_{em}$ gets broken, strings are formed according to our general topological arguments. Since electromagnetic monopoles were formed in a previous phase transition, these strings will form attached to them, connecting monopole-antimonopole pairs. One can view this process as the same one that occurs in a superconductor: with $U(1)$broken, the Universe enters a superconducting phase, and the monopoles get connected by flux tubes. Now, if a string connects them, it should be much easier for the pairs to annihilate. Note first that a string can be cut into pieces by quantum creation of a monopole-antimonopole pair. Following the same arguments used in the case of quantum creation of a loop of string in a domain wall, we easily find that the probability is exponentially suppressed by

$$S_0 = \pi \left(\frac{v_M}{v_s}\right)^2 \tag{2.29}$$

so that it is negligible when the symmetry breaking scales are widely separated.

The $M\bar{M}$ pairs will be subject to the string's tension μ, and therefore accelerate at $a = \mu/m_M$. The time for annihilation is roughly the same as the time required for the energy of the string to be dissipated. For monopoles formed at a temperature T_M at an average density of one per horizon, the distance between them at the moment of string formation when $T = T_S$ will be [39] (see [26] for a complete discussion and references on the subject)

$$d = \frac{M_P}{T_M T_S} \tag{2.30}$$

So we have to dissipate an energy

$$E \sim \mu d \sim M_P \frac{T_S}{T_M} \tag{2.31}$$

The mechanism of dissipation is the frictional force $F \sim T^2 v$ acting on the monopoles in a plasma. The characteristic monopole velocity under the action of the string's tension is $v \sim \sqrt{\mu d/m_M}$, and we have for the energy loss

$$\dot{E} \sim -T^2 v^2 \tag{2.32}$$

so that the characteristic dissipation time will be

$$\tau \sim \frac{E}{\dot{E}} \sim \frac{m_M}{T^2} \ll \frac{M_P}{T^2} \tag{2.33}$$

This is much smaller than a Hubble time. Of course, in the above, one has assumed that monopoles are connected by straight strings and that they do not radiate, and that the dynamics of the string network was not taken into account.

The efficiency of the annihilation mechanism will ultimately depend on the details of the model. The main criticism of the Langacker-Pi mechanism is that it has to assume a very complicated form for the Higgs potential, with at least three Higgs fields. The $M\bar{M}$ annihilation process is extremely sensitive to the details of the potential, which is considered a serious drawback.

2.4 The problem of this Thesis

The fact that monopoles and domain walls are incompatible with cosmology can be viewed in two ways. One is to consider it as a useful piece of experimental information, namely, to decide to discard extensions to the Standard Model on the basis of their incompatibility with the Big Bang Universe. The idea of having input for particle physics model building which comes from cosmology is certainly an extremely appealing one, although given the large amount of model dependence on cosmological predictions, it can be a difficult task to convince particle physicists of the strength of this argument. And although it may be possible to renounce to the advantages of having the discrete symmetries of nature spontaneously broken, so that domain walls are avoided, it is less plausible that the search for a Grand Unified Theory is detered by the fear of producing electromagnetic monopoles in catastrophically large numbers. Electromagnetic charge *is* observed to be quantized, and the existence of monopoles remains inevitably linked to this fundamental fact.

On the other hand, one can view domain walls and monopoles as problems, and devise mechanisms to either prevent their formation altogether, or ensure that they decay or are unobservable in the present-day Universe. This is what inflation does, for example. But, of course, inflation does something even more fundamental than diluting monopoles–it is the simplest and more elegant way to avoid the horizon problem. If there was not an inflationary era, some other mechanism surely was at work in the early Universe to ensure homogeneity on scales larger than the causal horizon. To use inflation to solve the problem of the overproduction of topological defects, however, is a delicate question. In a theory like $SO(10)$ grand unification, for example, one can have more than one phase transition, and form all kinds of topological defects and hybrid topological defects at diverse energy scales. Invoking inflation at each stage poses at least technical problems, and may be altogether incompatible with the scale of the fluctuations detected in the microwave background.

There is a crucial question hidden in the previous paragraph: are the monopole and domain wall problems as fundamental as the horizon problem? Whatever view one wishes to take on the meaning of the incompatibility of these defects with cosmology, the first essential thing to establish is whether they *do* constitute a generic problem of the particle physics theories, an unavoidable consequence of the marrying of particle physics with cosmology. This question is what we address in this section, and the answer is on the negative.

The point is not whether the theory admits topological solitons, but whether they get produced. We will show later that production of defects by thermal fluctuation of the field configuration can be easily suppressed. For example, monopoles can be produced in monopole-antimonopole pairs by particle-antiparticle collisions, but their number density depends on many factors, and can be controlled, as we shall see. The real danger is the Kibble mechanism, by which the defects get produced as a consequence of the phase transitions. This is a familiar enough process in solid state physics, for example.

The Kibble mechanism is triggered by the phase transition following spontaneous symmetry breaking, and is based on the fact that our Universe has finite causally connected regions. We cannot question this last fact and, as we said before, we do not want to renounce to the benefits of spontaneous symmetry breaking. But are phase transitions really unavoidable? In other words, do symmetries really get restored in the early Universe? One would say yes, intuitively. The idea behind spontaneously broken symmetries is that there is a critical energy scale above which the hidden symmetry is manifest, and adding thermal energy to the system is likely to have the same effect. This is what common experience tells us: on heating up a system, more disorder is produced, therefore we go to a less symmetric phase.

However, clearly everyday intuition cannot be solid ground on which to found the concept of a monopole or domain wall problem. Interestingly enough, there is at least one system that functions in an inverse way. Weinberg [3] cites the example of the Rochelle salt, that crystallizes more with increasing temperature.

Let us go back to the discussion on symmetry restoration of the previous chapter. In a simple theory with a real scalar field ϕ and a discrete symmetry $\phi \to -\phi$, the effect of the thermal bath at very high temperature (*i.e.* , at temperatures much higher than the field's mass) is to induce a temperature-dependent mass term proportional to the quartic self-coupling. In such a simple theory symmetry breaking is triggered by a negative mass term, and it is easily concluded, by calculating the self-energy graphs at finite temperature, that the induced thermal mass has the sign of the coupling constant. But this coupling constant has to be positive, so the thermal mass squared induced by the self interactions of the field in a thermal bath is positive. When the temperature-dependent part of the mass term becomes dominant, symmetry is restored.

We have repeated the argument that leads to symmetry restoration in order to make manifest the assumptions taken. Namely, we have seen how symmetries get restored in a very simple model, and it is clearly necessary to extend the analysis to more realistic theories before one can claim that phase transitions producing topological defects are unavoidable. We have seen that given a potential $V[\varphi]$ for a set of fields φ_i, the thermal contribution to the effective potential from the self-interaction and gauge terms is

$$\Delta V(T) = \frac{T^2}{24}\left[\left(\frac{\partial^2 V}{\partial\varphi_i\partial\varphi^i}\right) + 3(T_aT_a)_{ij}\,\varphi^i\varphi^j\right] \tag{2.34}$$

where a sum over repeated indices is assumed. Both terms are positive definite. Contributions from the Yukawa interactions with fermions are found to be positive too, depending on the squared Yukawa coupling.

It remains, naturally, to extend the theory by adding more scalar fields. After all, almost every extension to the Standard Model includes more than one Higgs field, and, in particular, Grand Unified theories require at least two of them. This was done already in the original paper by Weinberg [3], where the analysis of a two-fields model with an $O(N_1)\times O(N_2)$ symmetry is carried out. According to Weinberg, it was Coleman who suggested that the sign of the coupling (or couplings) governing the interaction of the two fields can be negative. This in turn would give a negative contribution to the thermal mass. Can this induce a negative sign for the whole mass term, even at very high temperatures? Weinberg's answer is that it can, under certain conditions.

Let us illustrate the point with the simple example of a theory with a $U(1)\times U(1)$ global symmetry, two complex Higgs fields ϕ and χ and a potential

$$V = -\frac{m_\phi^2}{2}\phi^*\phi + \frac{\lambda_\phi}{4}(\phi^*\phi)^2 - \frac{m_\chi^2}{2}\chi^*\chi + \frac{\lambda_\chi}{4}(\chi^*\chi)^2 + \frac{\alpha}{2}\phi^*\phi\chi^*\chi \tag{2.35}$$

We calculate the effective masses at in the high temperature limit using (2.34)

$$\begin{aligned} m_\phi^2(T) &= -m_\phi^2 + \tfrac{T^2}{12}(2\lambda_\phi+\alpha) \equiv -m_\phi^2 + T^2\nu_\phi^2 \\ m_\chi^2(T) &= -m_\chi^2 + \tfrac{T^2}{12}(2\lambda_\chi+\alpha) \equiv -m_\chi^2 + T^2\nu_\chi^2 \end{aligned} \tag{2.36}$$

The crucial point is that the coupling constant alpha enters the mass terms at high temperature. Nothing forces α to be positive, all that is required is that the potential (2.35) is bounded from below, which implies

$$\lambda_\phi \lambda_\chi > \alpha^2 \tag{2.37}$$

One can have $\alpha < 0$, and for example $\lambda_\phi > 2|\alpha| > 4\lambda_\chi$. Then ν_χ in (2.36) is negative, and $m_\chi(T)$ is negative for all temperatures. Note that (2.37) prevents us from taking *both* ν_χ and ν_ϕ negative. Then one of the $U(1)$ groups is broken for any value of T.

What has happened, essentially, is that in a theory with more than one field, symmetry restoration at high temperature has become a question of dynamics. It all depends on the range of the couplings.

This remarkable conclusion was, so to speak, lost among the many valuable results of Weinberg's paper. It was completely ignored until 1979, when it was re-discovered by Mohapatra and Senjanović [40, 41]. They were trying to find a mechanism for keeping CP violation at high temperatures, to allow for successful baryogenesis in left-right symmetric models. In order to keep CP broken at any temperature, the authors add new Higgs fields and found the range of couplings for which the symmetry is never restored. Maybe the best proof that the idea is so contrary to intuition, is that the authors themselves (not to mention the physics community!) were reluctant to believe their results, until they found out that the possibility had already been mentioned in [3].

Mohapatra and Senjanović also point out in [40, 41] that this may be a solution to the formation of CP domain walls. Later on, Langacker and Pi [38] put forward the model we described in the previous section, including three Higgs scalars whose couplings would ensure the non-restoration of $U(1)_{em}$ during an intermediate regime.

However, the efforts to implement the idea of non-restoration to avoid domain wall or monopoles quoted above, have a common feature: they try to solve the problem. As we said before, what we are interested in is determining whether or not the problem is there in the first place. We are going to be concerned with whether a given theory, motivated by particle physics and relying on spontaneous breakdown of symmetries, would or would not lead to a phase transition at high temperature. To do so, we will examine different, well-known models from the literature that are considered relevant for particle physics, and determine under which conditions they have a domain wall or a monopole problem.

The fundamental point in this approach will be a "minimality" requirement. Namely, all the theories considered have more than one Higgs field, and we will try to determine if this can be of use in order to avoid symmetry restoration, and in what range of parameters of the potential. We will not modify any model in order to allow for a solution of the problems caused by topological defects.

2.5 Non-restoration as a solution

With the fields that transform under a symmetry group always have a VEV, the symmetry is always spontaneously broken, and the phase transition does not happen. The field in the example above has a temperature-dependent VEV

$$\langle\phi\rangle^2 = \frac{m_\phi^2\lambda_\chi + m_\chi^2\alpha}{\lambda_\chi\lambda_\phi - \alpha^2} \quad \text{for } T^2 \ll m_\phi^2 \tag{2.38}$$

$$\langle\phi\rangle^2 = \frac{T^2}{12}\frac{(\alpha - 2\lambda_\phi)}{\lambda_\phi} \quad \text{for } T^2 \gg m_\phi^2 \tag{2.39}$$

Of course, this is valid as long as the theory is valid. We cannot say that the symmetry is "always" broken, since completely new physics may take over at very high temperatures. At most we can trust our results up to the Planck scale. This is only natural, even the Rochelle salt that crystallizes as we increase the temperature, will burn at some point. But as long as the theory is in its range of validity, we can say that the symmetry remains broken.

How does this solve the domain wall or monopole problems? Avoiding the Kibble mechanism is not enough. One has to ensure that the field has taken values in the vacuum since early times, and it has to do so even at scales larger than the causal horizon. It should start near Planck scale having a unique fixed VEV, and then it is guaranteed by the equations of motion that it will remain there, at the minimum of the potential. We will have to solve the horizon problem in order not to have topological defects produced. Any mechanism that guarantees homogeneity at large scales will do, such as inflation, for example. This means that we do need inflation to get rid of monopoles or domain walls, but the crucial difference is that the epoch of inflation need not be related to the scale of symmetry breaking: it only has to happen before. And if we have theories with many phase transitions, only one very early period of inflation is enough. The so-called primordial or chaotic inflation suggested by Linde [42] is particularly suited for this task.

Of course, one has to resort to more realistic theories and this we will do in the next three chapters. Before, however, it will be useful to have a general discussion on how non-restoration can work in those cases.

2.5.1 Global symmetries: $O(N_1) \times O(N_2)$ model

It is not difficult to convince oneself that non-restoration of symmetries is also possible when the fields transform under more complicated groups. Take the same example of two fields, now transforming under an arbitrary symmetry group. One can have a large number of coupling constants, depending on the group's structure and the field's transformations properties. The conditions of boundedness of the potential analogous to (2.37) can be many, and very complicated. However, it is enough that non-restoration occurs for a reasonable range of parameters, so it is perfectly natural to ask for some of the couplings to be small. Then one can consider only those couplings analogous to the ones of the simple model. That is, for fields $(\Phi, \Xi\ )$ transforming under the representations R_1, R_2 of some group G containing N_1, N_2 real fields (ϕ, χ), write the Higgs potential as

$$V = \sum_{a=1}^{N_1}\sum_{b=1}^{N_2}\left\{-\frac{m_\phi^2}{2}\phi^a\phi_a + \frac{\lambda_\phi}{4}(\phi^a\phi_a)^2 - \frac{m_\chi^2}{2}\chi^b\chi_b + \frac{\lambda_\chi}{4}(\chi^b\chi_b)^2 - \frac{\alpha}{2}\phi^a\phi_a\chi^b\chi_b\right\} + V_s \tag{2.40}$$

where V_s contains terms whose coupling constants are assumed to be much smaller than λ_ϕ, λ_χ and α. Thus in this case the symmetry is $O(N_1)\times O(N_2)$. We will use the $O(N_1) \times O(N_2)$ models as a prototype that can effectively mimic more complicated groups.

Taking $\alpha < 0$, the condition for the boundedness of the potential is again (2.37). The high temperature contributions to the masses are

$$\begin{aligned} \Delta m_\phi^2(T) &= T^2\nu_\phi^2 = T^2\left[\lambda_\phi\left(\tfrac{2+N_1}{12}\right) - \tfrac{N_2}{12}\alpha\right] \\ \Delta m_\chi^2(T) &= T^2\nu_\chi^2 = T^2\left[\lambda_\chi\left(\tfrac{2+N_2}{12}\right) - \tfrac{N_1}{12}\alpha\right] \end{aligned} \tag{2.41}$$

and the G symmetry will not be restored if the couplings lie in the range

$$\lambda_\phi > \left(\frac{2+N_2}{N_1}\right)\alpha > \left(\frac{2+N_2}{N_1}\right)^2\lambda_\chi \tag{2.42}$$

Note that there is no lower bound on the smallest coupling, so one can always take it small enough to avoid the danger of the couplings getting too large and in conflict with perturbation theory. This is not the case if G is a gauge symmetry, since the gauge coupling will have to enter into the discussion, as we will see later.

Also, it is important to point out that the conditions are weaker if the ratio N_2/N_1 is big, that is, it will be easier for the representation with fewer real fields to maintain its VEVat high temperature.

In the simple example considered, only one of the fields can have a VEV. This means that any subgroup of G preserved by its VEVwill be restored. But condition (2.37) only prevents us from taking both mass terms negative, and with an adequate coupling one can have both VEVs non zero even if one of the masses is positive. For example, imagine that ϕ and χ are real scalar fields transforming under D:$\phi \to -\phi$, $\chi \to -\chi$. Write the potential

$$\begin{aligned} V &= -\frac{m_\phi^2}{2}\phi^2 - \frac{m_\chi^2}{2}\chi^2 + \frac{\lambda_\phi}{4}\phi^4 + \frac{\lambda_\chi}{4}\chi^4 \\ &\quad -\frac{\alpha}{2}\phi^2\chi^2 + \beta_1\phi^3\chi + \beta_2\phi\chi^3 \end{aligned} \tag{2.43}$$

With $\beta_1, \beta_2 > 0$, the potential is bounded from below if

$$\lambda_\phi\lambda_\chi > \alpha^2 \tag{2.44}$$

The extrema of the potential are at $\langle\phi\rangle = v$, $\langle\chi\rangle = u$, satisfying

$$[-m_\phi^2 + \lambda_\phi v^2 - \alpha u^2 + 3\beta_1 uv]v + \beta_2 u^3 = 0 \tag{2.45}$$

$$[-m_\chi^2 + \lambda_\chi u^2 - \alpha v^2 + 3\beta_1 uv]u + \beta_2 v^3 = 0 \tag{2.46}$$

And if $m_\phi^2, m_\chi^2 > 0$, both VEVs are non zero. At high temperature, the effective potential acquires the temperature-dependent terms

$$\Delta V(T) = \frac{T^2}{24}\left[(3\lambda_\phi - \alpha)\phi^2 + (3\lambda_\chi - \alpha)\chi^2 + 6(\beta_1 + \beta_2)\phi\chi\right] \tag{2.47}$$

By asking $\alpha > 3\lambda_\phi$, one can keep one of the mass terms negative at high temperature. the other forced to be positive by the boundedness conditions. However, the last terms in (2.45), (2.46) prevent us from taking only one of the VEVs non zero. In other words, the field with a negative mass term acquires a VEVand "forces" the other to get one also, via the linear terms in the potential. Clearly, one can redefine the fields at high temperature so that only one of them has a VEV. However. note that the same holds true at zero temperature; the point is that the symmetry breaking patterns at high and low T are equal.

2.5.2 Gauged case

As we have already mentioned, when the symmetry is gauged non-restoration is not straightforward. The gauge coupling provides a lower bound on the coupling constants and (depending on the particular gauge group chosen) one may have to require the coupling constants to be of order one, away from the perturbative regime.

To see it explicitly, consider a simplified model as the one of section 2.5.1, that is, one where only the relevant coupling constants are taken into account, and now the group G is gauged. The two fields Φ and Ξ transform under the representations R_i $(i = 1, 2)$ whose generators satisfy

$$Tr(T_i^a T_i^b) = c_i \delta^{ab} \tag{2.48}$$

Then the high-temperature masses are

$$\begin{aligned} \Delta m_\phi^2(T) &= T^2 \nu_\phi^2 = T^2 \left[\lambda_\phi \left(\tfrac{2+N_1}{12}\right) - \tfrac{N_2}{12}\alpha + \tfrac{1}{4} g^2 \tfrac{Dim(G)}{N_1} r_1 c_1\right] \\ \Delta m_\chi^2(T) &= T^2 \nu_\chi^2 = T^2 \left[\lambda_\chi \left(\tfrac{2+N_2}{12}\right) - \tfrac{N_1}{12}\alpha + \tfrac{1}{4} g^2 \tfrac{Dim(G)}{N_2} r_2 c_2\right] \end{aligned} \tag{2.49}$$

where g is the gauge coupling, $Dim(G)$ is the dimension of the group and r_i is 1 when the representation contains real fields, 2 when it is complex. Asking ν_χ to be negative and at the same time the fulfillment of the bound (2.37) now implies

$$\lambda_\phi > \frac{\alpha^2}{\lambda_\chi} > \frac{1}{\lambda_\chi}\left[\left(\frac{N_2+2}{N_1}\right)\lambda_1 + 3g^2 \frac{Dim(G)}{N_1 N_2} r_2 c_2\right]^2 \tag{2.50}$$

As a function of λ_χ, λ_ϕ has a minimum at

$$\lambda_\chi = 3g^2 \frac{Dim(G)}{N_2(2+N_2)} r_2 c_2 \tag{2.51}$$

So λ_ϕ is bounded from below as

$$\lambda_\phi > 12 g^2 \frac{(2+N_2) Dim(G) r_2 c_2}{N_1^2 N_2} \tag{2.52}$$

The dimension of the representation R_1 (under which the fields that looses its VEVtransforms) now plays an even more fundamental role: it has to be big enough, if we want perturbation theory to be valid.

Up to now we have used the high-temperature expansion of the effective potential at finite temperature calculated up to one loop order. However, when the coupling constants can get relatively large, it is not evident that next-to-leading order effects are negligible. In the next chapters, we will carefully examine this issue.

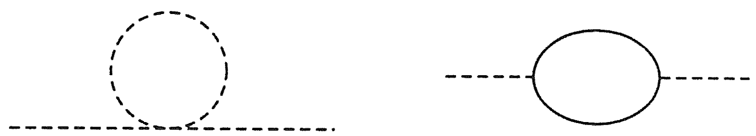


Figure 2.1: Feynman diagrams for the thermal mass correction in the supersymmetric example. Dashed lines represent the scalar boson ϕ, continuous lines represent its fermion counterpart $\tilde{\phi}$

2.5.3 Supersymmetric theories

In supersymmetric theories, one writes the scalar potential in terms of the superpotential, W,

$$V = \left|\frac{\partial W}{\partial \varphi_i}\right|^2 \tag{2.53}$$

where W is an holomorphic function of the superfields φ_i, and renormalizability requires that it is at most cubic in them.

Now, in finding the high-temperature effective potential, one has to take into account the contribution of the fermions (we will for simplicity discuss global symmetries here). That means, in a theory with a $\lambda\phi^4$ interaction, calculating the usual graphs of figure 2.1

The quadratically divergent parts of these graphs exactly cancel, since they are made up with superpartner fields, given supersymmetry its best-known characteristic of providing a solution to the hierarchy problem. The contributions to the thermal mass coming form this graphs, however, do not cancel, since different boundary conditions are applied in thermal field theory for bosons and fermions, to account for the different statistics. We have calculated this contributions in the Appendix. The fermion contribution is exactly double of that of its boson superpartners. It is straightforward to generalize the formula for the thermal contribution to the effective potential to the supersymmetric case

$$\Delta V(T) = \frac{T^2}{8}\left|\frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j}\right|^2 \tag{2.54}$$

where now a sum over all i and j is understood.

Note in the first place that a non zero temperature immediately implies supersymmetry breaking, since there will be a constant contribution T^4 from the thermal bath.

At very high temperature, the minimum of the potential will be found simply at $\Delta V(T) = 0$, the potential being a sum of squares. This will make restoration of the symmetry a straightforward matter. The superpotential being a cubic function of the fields at most, its second derivatives cannot be too complicated. If on top of that we impose that the fields transform under a given symmetry, W will be further restricted. In fact, it has been shown that it is so restricted, that restoration *always* follows [43, 44].

As an example, consider a simple theory with two chiral superfields Φ, Ξ, and a superpotential with a discrete symmetry $\mathbf{D}$:$\Phi \rightarrow -\Phi$

$$W = m\Phi^2 + (\mu^2 - \Phi^2)\Xi \tag{2.55}$$

The high-temperature effective potential for the corresponding scalar fields ϕ, χ is

$$\begin{aligned} V &= |\mu^2 - \phi^2|^2 + 4|\phi|^2|m - \chi|^2 \\ &\quad + \frac{T^2}{8}\left\{8|\phi|^2 + 4|m - \chi|^2\right\} \end{aligned} \tag{2.56}$$

At T=0, the terms in the first line above forces (the real part of) ϕ to take a VEV, breaking the symmetry. The fact that χ also takes a VEVis irrelevant, since it is invariant under such symmetry. At high temperature, however, while χ keeps the same VEV, we see that the minimum of the potential occurs now for $\langle\phi\rangle = 0$, and the $\mathbf{D}$ symmetry gets restored.

The results of Haber [43] and Mangano [44] have been formulated as a theorem, stating the impossibility of avoiding symmetry restoration in supersymmetric models. We come back to this point in Chapter 5.

3 Where we solve the Domain Wall problem in several theories

3.1 Introduction

Two of the fundamental symmetries in nature, time reversal and parity, are discrete. Both are explicitly broken in the Standard Model. Having them spontaneously broken is an old dream, ever since the work by T.D. Lee [45] on spontaneous breakdown of CP and the construction by Senjanović and Mohapatra of left-right symmetric extensions to the Standard Model where parity is broken spontaneously [46].

Having these symmetries spontaneously broken has the notoriuos advantange of rendering the P or CP phases calculable as physical quantities, and is particularly desirable in the case of the strong CP phase. It can actually provide a solution to the strong CP problem. But the search for realistic models has been in some way obscured by the menace of a domain wall problem. It becomes important to investigate in which of these theories the problem is really unavoidable, along the lines of the previous chapter's discussion.

Mohapatra and Senjanović [41] suggested for the first time to use non-restoration to cure the problem in spontaneously broken Left-Right theories, although they had to modify the model in order to do so. Years later the idea was revived by Dvali and Senjanović [47], for the T.D. Lee model and the Peccei-Quinn [12] solution to the strong CP problem.

In this Chapter we present results from a collaboration with Gia Dvali and Goran Senjanović, published in [48]. We study the possibility of avoiding the domain wall problem with symmetry non-restoration in some theories of spontaneously broken CP, P and strong CP. Three kinds of models of spontaneously broken CP are investigated: the T.D. Lee model with two Higgs doublet; the three doublet model that allows for natural flavor conservation [49, 50, 51]; and models in which the fermion sector is enlarged [52], including a new version [53]. Later, we turn to parity and show that the proposal of [41] is not applicable, although the domain wall problem can be avoided in models with singlets as in [54, 55]. Finally, we show how the axionic domain wall problem can be solved in the invisible axion version [56] of the Peccei-Quinn model.

3.2 Spontaneous CP Violation

There are many ways to break CP spontaneously. One can identify the CP violating phase with the relative phase of the VEV's of doublet Higgs fields. This is the idea behind the original model of T.D. Lee [45]. Another way is to extend the particle sector of the theory, as well as the Higgs sector, so that couplings with the new fermions introduce phases in the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Whatever the mechanism, however, the important point is that the discrete time-reversal symmetry gets spontaneously broken, producing CP domain walls. We discuss in the following how to implement the ideas of non-restoration in order to avoid domain wall production in some models with spontaneously broken CP.

Furthermore, CP is a particularly interesting discrete symmetry. We need some amount of CP violation to make baryogenesis work [57]. It is desirable not to restore it in the early Universe, at least not until the time of baryogenesis. This was actually the original motivation of the first application in particle physics of the phenomenon of non-restoration of symmetries at high temperature [58]. The model presented in [58], however, is not minimal, in the sense that additional fields where included just in order to have non-restoration working.

3.2.1 $\not{C}\not{P}$ with doublets

To have the CP phase as a doublet's phase, we obviously require at least two Higgses. The original model is due to T. D. Lee [45], and consists in an extension of the Standard Model with two complex Higgs doublets, with

$$\mathcal{L}_H = \sum_{i=1}^{2} \frac{1}{2}(D_\mu\Phi_i)^\dagger(D^\mu\Phi_i) - V(\Phi_1, \Phi_2) \tag{3.1}$$

where

$$\begin{aligned} V(\Phi_i, \Phi_2) &= \sum_{i=1}^{2}\left(-\frac{m_i^2}{2}\Phi_i^\dagger\Phi_i + \frac{\lambda_i}{4}(\Phi_i^\dagger\Phi_i)^2\right) - \frac{\alpha}{4}\Phi_1^\dagger\Phi_1\,\Phi_2^\dagger\Phi_2 - \frac{\beta}{4}\Phi_1^\dagger\Phi_2\,\Phi_2^\dagger\Phi_1 \\ &+ \frac{1}{8}\left[\Phi_1^\dagger\Phi_2\left(a\Phi_1^\dagger\Phi_2 + b\Phi_1^\dagger\Phi_1 + c\Phi_2^\dagger\Phi_2\right) + h.c.\right] \end{aligned} \tag{3.2}$$

Choosing the parameter $\beta > 0$, one can prove that the minimum of the potential is achieved when the fields acquire VEVs

$$\Phi_1 = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad ; \quad \Phi_2 = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} e^{i\theta} \tag{3.3}$$

The terms in brackets in the potential will force the CP-violating phase θ to be non-zero. This can be readily seen by writing (3.2) at the minimum (3.3), and wisely rearranging terms so that

$$V(\langle\Phi_1\rangle, \langle\Phi_2\rangle) = \sum_{i=1}^{2} \left(-\frac{m_i^2}{2} v_i^2 + \frac{p_i}{4} v_i^4 \right) + \frac{\rho}{4} v_i^2 v_2^2 + \frac{a}{2} v_i^2 v_2^2 [\cos\theta - \delta]^2 \tag{3.4}$$

where

$$p_1 = \lambda_1 - \frac{b^2}{8a} \quad ; \quad p_2 = \lambda_2 - \frac{c^2}{8a}$$
$$\rho = \alpha + \beta + a + \frac{cb}{4a} \quad ; \quad \delta = \frac{-(bv_1^2 + cv_2^2)}{4av_1v_2} \tag{3.5}$$

Obviously, for $a > 0$ the minimum will be at $\cos\theta = \delta$, and CP is broken spontaneously. Moreover, it will be broken for any value of $\theta = \cos^{-1}(\delta + 2n\pi)$, with n an integer. In general, when the phase transition occurs, θ will take different values in causally disconnected regions. Whenever two regions with different θ values will come into contact, a domain wall (of the sine-Gordon type [7]) will form between them.

We therefore explore the possibility that CP remains broken at arbitrarily high temperature. For this to happen in T.D. Lee's model, we need to have the VEV's of *both* Φ_1 and Φ_2 nonzero at high temmperature. In a potential with terms cubic in one of the fields (such as $\Phi_1^\dagger\Phi_1\Phi_1^\dagger\Phi_2$ in (3.2)), it is not necessary to have both masses negative, as we have seen in the previous Chapter. The interaction terms will force even the field with a positive mass squared to have a VEValso. We require, however, more than that: the relative phase between the doublets also has to be kept non-zero at any temperature, for CP to indeed be broken.

The high temperature corrections to the effective potential for a model with N Higgs doublets can be found by generalizing Weinberg's formula [3] for complex doublets. The most general potential for N complex doublets can be written as

$$V = -\sum_{i=1}^{N} m_i^2 \Phi_i^\dagger \Phi_i + \sum_{i,j,k,l=1}^{N} \lambda_{ijkl} \Phi_i^\dagger \Phi_j \Phi_k^\dagger \Phi_l \tag{3.6}$$

Then the high temperature correction is

$$\Delta V(T) = \sum_{i,j,k=1}^{N} \frac{T^2}{6} \left(2\lambda_{ijkk} + \lambda_{kijk}\right) \Phi_i^\dagger \Phi_j \tag{3.7}$$

For the two doublet model (3.2), this gives

$$\Delta V(T) = \frac{T^2}{6} \left[(6\lambda_1 - 2\alpha - \beta)\Phi_1^\dagger\Phi_1 + (6\lambda_2 - 2\alpha - \beta)\Phi_2^\dagger\Phi_2 + \frac{3}{2}(b+c)(\Phi_1^\dagger\Phi_2 + h.c) \right] \tag{3.8}$$

The potential at high temperature can then be cast in the same form (3.4), where now the masses m_i^2 are replaced by $m_i^2(T)$

$$\begin{aligned} m_1^2(T) &= -m_1^2 + 2T^2\left(\lambda_1 - \frac{\alpha}{3} - \frac{\beta}{6} - \frac{b(b+c)}{16a}\right) \simeq 2T^2\nu_1^2 \\ m_2^2(T) &= -m_2^2 + 2T^2\left(\lambda_2 - \frac{\alpha}{3} - \frac{\beta}{6} - \frac{c(b+c)}{16a}\right) \simeq 2T^2\nu_2^2 \end{aligned} \tag{3.9}$$

for $T >> m$; and δ becomes $\delta(T)$:

$$\delta(T) = -\left[\frac{bv_1^2 + cv_2^2 + T^2(b+c)}{4av_1v_2}\right] \tag{3.10}$$

Again, as in the simpler model, one can have one and only one mass negative at high temperature, due to the condition that the potential be bounded from below

$$p_1p_2 > \frac{\rho^2}{4} \tag{3.11}$$

since now

$$\nu_1^2 = p_1 - \sigma\ ; \quad \nu_2^2 = p_2 - \sigma\ ; \quad \text{with} \quad \sigma = \frac{\rho}{2} - \frac{\alpha}{6} - \frac{\beta}{3} - \frac{a}{2} < \frac{\rho}{2} \tag{3.12}$$

Requiring $\nu_1^2, \nu_2^2 < 0$ will give $p_1p_2 < \sigma^2 < \rho^2/4$, which contradicts (3.11).

Considering only the θ-dependent part, we see as before that there is a minimum for $\theta = \delta(T)$. However, it is not difficult to see that with only one mass term negative, both VEVs cannot be nonzero at high temperature, due to the fact that the mass terms now depend on the coupling constants. Taking $\nu_2^2 < 0$, the requirement that v_1 be real gives

$$|\nu_2^2|\frac{\rho}{2} > \nu_1^2 p_2 \tag{3.13}$$

together with (3.11), this is also enough to ensure that v_2 is real. Substituting for ν_1^2 and ν_2^2 one gets

$$\frac{\rho}{2}\left(\frac{\rho}{2} - p_2\right) > \frac{\rho}{2}(\sigma - p_2) > (p_1 - \sigma)p_2 > \left(p_1 - \frac{\rho}{2}\right)p_2 \tag{3.14}$$

which again implies $p_1 p_2 < \rho^2/4$, contradicting (3.11).

We conclude that the only way to have both fields with a nonvanishing VEVat high temperatureemperature is to set the phase θ to zero. In other words, the field with a negative mass term can "force" the other to acquire a VEV, but it drags it in the same direction in $U(1)$ space.

Notice that in [58] the fact that both VEVs can be nonzero was overlooked, but it was still concluded correctly that with only two doublets, CP would become a good symmetry at high temperature.

There is however a good reason for taking more than two doublets in order to break CP. A common feature of models with two Higgs doublets is that they allow for flavor-violating interactions in neutral current phenomena. As shown in [49, 50, 51], the minimal model for spontaneous CP violation involving doublets only that conserves flavor, requires three of them.

To see why, consider a Lagrangian with two complex Higgs as in(3.1), (3.2), and an extra symmetry D_1

$$\Phi_1 \longrightarrow -\Phi_1 \qquad u_{iR} \longrightarrow -u_{iR} \tag{3.15}$$

(where u_{aR} are up quarks and hereafter $a, b, ..$ are flavor indices). The Yukawa interactions are written now

$$\mathcal{L}_Y = (\bar{u}\bar{d})^a_L h^1_{ab} \Phi_1 d^b_R + (\bar{u}\bar{d})^a_L h^2_{ab}(i\tau_2)\Phi^*_2 u^b_R \tag{3.16}$$

so that flavor violation through neutral Higgs exchange is avoided. However, now the symmetry prohibits the terms of the type $\Phi_1^\dagger\Phi_1\Phi_1^\dagger\Phi_2$ in the Higgs potential, and therefore at the minimum we have the phase $\theta = 0$ or $\pi/2$, both leading to CP conservation.

The way out is to have three doublets, and an additional symmetry D_2 that prevents it from coupling to the quarks: $\Phi_3 \rightarrow -\Phi_3$, with other fields unchanged. The most general potential invariant under $SU(2) \times U(1) \times D_1 \times D_2$ is

$$V = \sum_{i=1}^{3}\left[-m_i^2 \Phi_i^\dagger \Phi_i + \lambda_i (\Phi_i^\dagger \Phi_i)^2\right] +$$

$$+ \sum_{i<j} \left[-\alpha_{ij}(\Phi_i^\dagger \Phi_i)(\Phi_j^\dagger \Phi_j) - \beta_{ij}(\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) + \gamma_{ij}(\Phi_i^\dagger \Phi_j \Phi_i^\dagger \Phi_j + h.c.)\right] \quad (3.17)$$

It can be shown [49, 50, 51] that choosing $\beta_{ij}, \gamma_{ij} > 0$, the above potential has a minimum at

$$\Phi_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_i e^{i\theta_i} \end{pmatrix} \quad (3.18)$$

where only two of the θ_i (say, θ_1 and θ_3) are relevant. Extremization with respect to θ yields [50]

$$\gamma_{12} v_2^2 \sin 2\theta_1 + \gamma_{13} v_3^2 \sin 2(\theta_1 - \theta_3) = 0 \quad (3.19)$$

$$\gamma_{13} v_1^2 \sin 2(\theta_1 - \theta_3) + \gamma_{23} v_2^2 \sin 2\theta_3 = 0 \quad (3.20)$$

Notice that to have CP violation, we need all three v_i and both θ_1, θ_3 to be nonzero.

It can be shown [51] that the CP violating solution of (3.19) is indeed a minimum. When the phases take this value, the remaining potential is

$$V(v_i) = \sum_{i=1}^{3} \left(-\frac{m_i}{2} v_i^2 + \frac{p_i}{4} v_i^4 \right) - \sum_{i<j} \frac{(\alpha_{ij} + \beta_{ij})}{4} v_i^2 v_j^2 \quad (3.21)$$

where

$$p_1 = \lambda_1 - \frac{\gamma_{12}\gamma_{13}}{\gamma_{23}} \quad (3.22)$$

and analogous expressions for p_2, p_3.

Once again, we are interested in whether the CP symmetry can remain broken at high temperatures. It is straightforward using (3.7) to calculate the masses at high temperature

$$m_i^2(T) = -m_i^2 + \frac{T^2}{6} \left[6p_i - \sum_{j \neq i} (2\alpha_{ij} + \beta_{ij}) \right] \simeq \frac{T^2}{3} \nu_i^2 \quad (3.23)$$

Due to the high degree of symmetry of the potential, temperature contributions are independent of the phases, so equations (3.19) are the same.

For the potential to be bounded from below, a set of constraints has to be imposed on the couplings, in this case

$$p_i > 0 \qquad p_i p_j > a_{ij} \qquad \text{for each } i < j \quad (3.24)$$

$$p_1 p_2 p_3 - p_1 a_{23}^2 - p_2 a_{13}^2 - p_3 a_{12}^2 - 2a_{12}a_{13}a_{23} > 0 \quad (3.25)$$

with $a_{ij} \equiv \alpha_{ij} + \beta_{ij}$, and we choose $\alpha_{ij} > 0$, so $a_{ij} > 0$.

It is easy to prove that (3.24) prevents us from taking all three of the mass terms negative at high temperature, as we could have expected. Necessary conditions would be

$$\sum_{j \neq i} a_{ij} > 3p_i \tag{3.26}$$

Multiplying these equations by pairs and adding them results in a contradiction with eq. (3.24). But it turns out that with only two negative mass terms, all three VEVs cannot be nonzero at arbitrarily high temperature. Take for example $\nu_1^2 > 0$. $\nu_2^2, \nu_3^2 < 0$. We need v_1 to be real, that is, minimizing (3.21)

$$v_1^2 = \left(\frac{T^2}{3}\right) \frac{-\nu_1^2(p_2p_3 - a_{23}^2) + \nu_2^2(p_3a_{12} + a_{23}a_{13}) + \nu_3^2(p_2a_{13} + a_{23}a_{12})}{p_1p_2p_3 - p_1a_{23}^2 - p_2a_{13}^2 - p_3a_{12}^2 - 2a_{12}a_{13}a_{23}} > 0 \tag{3.27}$$

We have already required the denominator to be positive. For the numerator to be positive also, necessary (though not sufficient) conditions are

$$\bar{\nu_2}^2(p_3a_{12} + a_{23}a_{13}) + \bar{\nu_3}^2(p_2a_{13} + a_{23}a_{12}) > -\bar{\nu_1}^2(p_2p_3 - a_{23}^2) \tag{3.28}$$

where

$$\begin{aligned} \bar{\nu_1}^2 &= 3p_1 - a_{12} - a_{13} < \nu_1^2 \\ \bar{\nu_2}^2 &= a_{12} + a_{23} - 3p_2 > \nu_2^2 \\ \bar{\nu_3}^2 &= a_{13} + a_{23} - 3p_3 > \nu_3^2 \end{aligned} \tag{3.29}$$

Inserting (3.28) in (3.29), one gets

$$\begin{aligned} &-2p_2p_3(a_{12} + a_{13}) - a_{23}(p_2a_{13} + p_3a_{12}) > \\ &\qquad p_1p_2p_3 - p_1a_{23}^2 - p_2a_{13}^2 - p_3a_{12}^2 - 2a_{12}a_{13}a_{23} + 2p_1(p_2p_3 - a_{23}^2) \end{aligned} \tag{3.30}$$

which in view of (3.24) and (3.25) cannot be satisfied.

Thus, once again, the CP violating phase disappears at high temperature. As in the two-doublet case, here too the problem is that CP violation is achieved through the relative phase of the VEVs of the doublets.

3.2.2 $\not{C}\not{P}$ with a singlet field

It should be clear from the previous examples that when the CP phase is related to the relative phases of doublet fields, high temperature effects will make it vanish. We therefore look for models in which CP violation is broken spontaneously by the VEVof just one field, which may be easier to keep at high temperature.

The simplest such model is a minimal extension of the Standard Model with

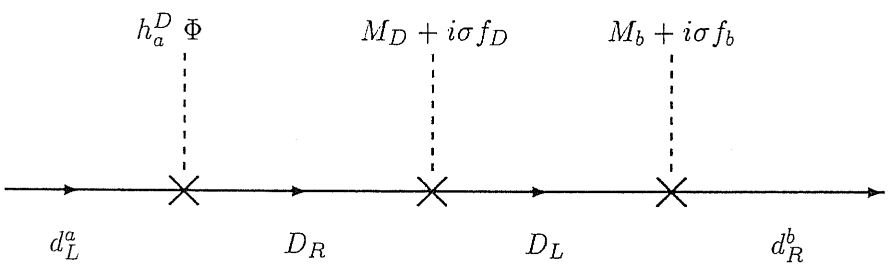
a) a real singlet field S which transforms under CP as $S \rightarrow -S$; and

b) an additional down quark, with both left and right components D^a_L and D^a_R singlets under $SU(2)$.

The interaction Lagrangian for the down quarks, symmetric under CP, contains the terms

$$\begin{aligned}\mathcal{L}_Y &= (\bar{u}\bar{d})^a_L h_a \Phi D_R + (\bar{u}\bar{d})^a_L h_{ab} \Phi d^b_R \\ &+ M_D \bar{D_L} D_R + M_a(\bar{D_L} d^a_R + h.c.) \\ &+ i f_D S(\bar{D_L} D_R - \bar{D_R} D_L) + i f_a S(\bar{D_L} d^a_R - \bar{d^a_R} D_L) \qquad (3.31)\end{aligned}$$

Clearly, when S gets a VEV(at a scale σ much bigger than the weak scale M_W) CP is spontaneously broken by the terms in the last line. A model of this kind was developed by Bento and Branco [52], in the version where the singlet is a complex field and gets a complex VEV, and with an additional symmetry under which S and D_R are odd, with all other fields even. We will, for the sake of simplicity, keep S real (and impose no further symmetries), noting that the analysis goes over the same lines as in [52]. CP violation is achieved by complex phases appearing in the CKM matrix through the mixing of d and D quarks. Through a diagram like



the down quark mass matrix will be roughly

$$m_{ab} \simeq h_a b\langle\Phi\rangle + \frac{h^D_a\langle\Phi\rangle(M_D + i\sigma f_D)(M_b + i\sigma f_b)}{M^2_D} \qquad (3.32)$$

$$\simeq \left[h_a b\langle\Phi\rangle + h_a^D\langle\Phi\rangle\frac{M_b}{M_D} - h_a^D f_a f_b \frac{\sigma^2}{M_D^2}\right] + i\left[h_a^D\langle\Phi\rangle f_b \frac{\sigma}{M_D} + h_a^D\langle\Phi\rangle f_D \frac{M_b}{M_D}\frac{\sigma}{M_D}\right]$$

giving a CP violating phases of order σ/M_D These phases remain in the limit $M_D, \sigma \to \infty$ when the heavy quarks decouple. This should not come as a surprise, since in the decoupling limit the theory reduces to the minimal Standard Model, which in general has complex Yukawa couplings and a complex CKM matrix. Also, flavor-violating currents are suppressed by powers of M_W/σ, disappearing in the decoupling limit. Thus the measure of the departure from the standard model is the dimensionless parameter M_W/M_D, and for the theory to be experimentally testable M_D should not be much bigger than 1 TeV.

To leading order, the high-temperature behavior of the $\Phi - \sigma$ system is very simple. The most general potential can be written as

$$\begin{aligned} V(\Phi, S) &= -m_\Phi^2\Phi^\dagger\Phi + \lambda_\Phi(\Phi^\dagger\Phi)^2 \\ &\quad -\frac{m_S^2}{2}S^2 + \frac{\lambda_S}{4}S^4 - \frac{\alpha}{2}\Phi^\dagger\Phi S^2 \end{aligned} \tag{3.33}$$

and it has a minimum at

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \quad ; \quad \langle S\rangle = +\sigma \tag{3.34}$$

At high temperature, the masses are replaced by

$$\begin{aligned} m_\Phi^2(T) &= -m_\Phi^2 + \frac{T^2}{24}(12\lambda_\Phi - \alpha) \\ \frac{m_S^2(T)}{2} &= -\frac{m_S^2}{2} + \frac{T^2}{24}(3\lambda_S - 2\alpha) \end{aligned} \tag{3.35}$$

We can have $m_S^2 < 0$ always by requiring $2\alpha > 3\lambda_S$, and thus $\sigma \neq 0$ at any temperature . The only further restriction is the usual $\lambda_\Phi > \alpha^2/\lambda_S$.

It seems then that in this model, one can have CP broken at any temperature. Remember, however, that up to now we have only considered the leading order contributions to the effective potential in calculating the masses (3.35). A complete analysis should include the next-to leading order corrections, as we already mentioned at the end of Chapter 2. We can anticipate that for a singlet field these effects will not change the picture much, but we leave a detailed analysis for a separate section.

3.3 Spontaneous P violation

Parity is the other fundamental discrete space-time symmetry, and the fact that it is broken is one of the most intriguing in particle physics. The Standard Model simply incorporates this observational fact, giving no hints of its origin. Having parity spontaneously broken, on the other hand, is an appealing idea. The Left-Right symmetric models where suggested by Pati and Salam [59] and Mohapatra and Pati [60], but the possibility of the spontaneous breakdown of parity in these theories was demonstrated by Senjanović and Mohapatra [46]. Left-Right models have a number of very interesting features (best reviewed in a detailed form in [61]), including the fundamental characteristic of having an elegant and simple way of explaining the smallness of neutrino mass [62, 63].

Left-Right models are based on the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge group, with an additional discrete parity symmetry (P). Breaking $SU(2)_R \times U(1)_{B-L}$ at a high scale M_R not only gives the gauge right-handed filed a large mass (thus hiding the right-handed interactions), but also breaks parity spontaneously. This however has to lead to domain walls. Interestingly enough, the first intent to use non-restoration was precisely in Left-Right models [41], mostly in connection with strong CP violation. It was concluded then that in the minimal models of spontaneous P violation, left-right asymmetry may persist at high temperature. The analysis however was carried out without considering carefully the role of the gauge couplings, which is now known to be fundamental, and which as we will show may invalidate that conclusion.

Let us recall the salient features of the minimal left-right symmetric theories [61] based on a $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge symmetry. The fermions are in doublet representations

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad ; \quad \begin{pmatrix} u \\ d \end{pmatrix}_R$$
$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad ; \quad \begin{pmatrix} \nu \\ e \end{pmatrix}_R \tag{3.36}$$

The minimal Higgs sector of the theory consists of

- the bi-doublets (one or more) Φ needed to provide Yukawa couplings and fermion masses
- two multiplets Δ_L and Δ_R which may be either doublets or triplets under $SU(2)_L$ and $SU(2)_R$, and which are in charge of breaking P spontaneously.

We will not give here a review of howLeft-Right theories work, it will be enough for our purposes to consider a simplified toy example which has all the relevant features of the theory. More precisely, we take Δ_L and Δ_R as real scalar fields and assume a left-right symmetric potential

$$\begin{aligned} V &= -\frac{m^2}{2}(\Delta_L^2+\Delta_R^2)+\frac{\lambda}{4}(\Delta_L^4+\Delta_R^4)+\frac{\lambda'}{2}\Delta_L^2\Delta_R^2 \\ &= -\frac{m^2}{2}(\Delta_L^2+\Delta_R^2)+\frac{\lambda}{4}(\Delta_L^2+\Delta_R^2)^2+\frac{\lambda'-\lambda}{2}\Delta_L^2\Delta_R^2 \end{aligned} \quad (3.37)$$

A simple inspection of V is enough to convince oneself that for $m^2 > 0$ and $\lambda' - \lambda > 0$, the global minimum of the theory is obtained for

$$\langle\Delta_L\rangle^2 = 0 \quad ; \quad \langle\Delta_R\rangle^2 = \frac{m^2}{\lambda} \quad (3.38)$$

or vice versa. Thus the left-right symmetry is broken spontaneously. With $\lambda > \lambda'$, the global minimum is found for both VEVs vanishing, and partity is preserved in this case.

Of course in realistic models, besides Δ's being non-trivial representations under the gauge group, we do need a field Φ to break the Standard Model gauge group. One can then try to take one or more of the coupling constants between Φ and the Δ's negative, thus achieving a negative mass term for the Δ's at all temperatures.

Let us concentrate in the version of the theory which incorporates the see-saw mechanism with Δ_L and Δ_R being triplets [63]. Since we wish to keep $\langle\Delta_R\rangle$ nonzero at high temperature, it is enough to look at the $\Delta_R - \Phi$ system and, as in [58], consider a simplified model in which the potential is written

$$\begin{aligned} V &= -m_\Delta^2\Delta_R^\dagger\Delta_R+\lambda_\Delta(\Delta_R^\dagger\Delta_R)^2 \\ &\quad -m_\Phi^2\mathrm{Tr}\Phi^\dagger\Phi+\lambda_\Phi(\mathrm{Tr}\Phi^\dagger\Phi)^2-2\alpha\mathrm{Tr}\Phi^\dagger\Phi\Delta_R^\dagger\Delta_R \end{aligned} \quad (3.39)$$

where Δ_R is a triplet under $SU(2)_R$, has $B-L$ number 2, and other couplings are taken to be small. The high temperature masses are[1]

$$m_\Phi^2(T) = -m_\Phi^2+T^2\left\{\frac{5}{6}\lambda_\Phi-\frac{1}{3}\alpha+\frac{3}{16}g^2\right\} \quad (3.40)$$

$$m_\Delta^2(T) = -m_\Delta^2+T^2\left\{\frac{1}{2}\lambda_\Delta-\frac{2}{3}\alpha+\frac{3}{8}(g'^2+2g^2)\right\} \quad (3.41)$$

[1] We use the normalization $\mathrm{Tr}\Phi^\dagger\Phi = \Phi_a\Phi_a/2$; $\Delta_R^\dagger\Delta_R = \Delta_R^a\Delta_R^a$, where a sums over six real fields.

where g'^2 is the $U(1)$ gauge coupling, g^2 the $SU(2)_R$ one. We have to keep $m^2_\Delta(T)$ negative at high temperature while preserving the boundedness condition $\lambda_\Phi \lambda_\Delta > \alpha^2$, thus we arrive at

$$\lambda_\Phi > \frac{\alpha^2}{\lambda_\Delta} > \frac{9}{4}\left[\frac{1}{2}\lambda_\Delta + \frac{3}{8}(g'^2 + 2g^2)\right] \tag{3.42}$$

Then the largest coupling of the theory is bounded from below by the gauge coupling, as we have seen already in the general discussion of the previous Chapter. Using (2.52), we find the minimum value for the largest coupling λ_Φ

$$\lambda_\Phi > \frac{27}{16}(g'^2 + 2g^2) \tag{3.43}$$

If we now use $g'^2 = g^2/2$ and take $g^2 = 1/4$, we see that non-restoration of P requires $\lambda_\Phi > 1$ in conflict with perturbation theory. Including other couplings does not help, since new conditions on the couplings coming from the mass matrices have to be imposed.

Although physically less attractive, one can in principle use doublets to break P spontaneously. This is actually the case studied in [58]. It is easily found that with doublets the condition equivalent to (3.43) is down by a factor of half. Thus this case may be considered borderline.

Now, for the implementation of the see-saw mechanism in its minimal form, it turns out that a parity odd singlet field is needed [55]. The singlet field S will couple to the Δ fields with a left-right symmetric term

$$MS(\Delta_L^\dagger \Delta_L - \Delta_R^\dagger \Delta_R) \tag{3.44}$$

Without the lower bound imposed by the gauge couplings, the situation in this case goes along the same lines as that of Section 3.2.2–the VEVof the singlet can be kept nonzero at high temperatures with the aid of the bi-doublet field Φ, or even of the Δ's. Exactly as it worked with CP, now P may remain broken at high temperature, and the presence of more fields coupled to S than in the CP case only makes it easier.

3.4 Strong CP Problem and High temperature

The strong CP problem arises in QCD when nonperturbative effects, resulting from the existence of instanton solutions, induce effective terms in the Lagrangian that violate CP. The resulting CP violating phase is

$$\bar{\Theta} = \Theta + \arg\det(M) \tag{3.45}$$

where Θ is the coefficient of the $\epsilon_{\alpha\beta\mu\nu}F_a^{\alpha\beta}F_a^{\mu\nu}$ term, and M is the quark's mass matrix. $\bar{\Theta}$ is constrained experimentally to be zero to a very high precision ($\bar{\Theta} < 10^{-9}$), giving rise to a "naturalness" problem [56].

3.4.1 The invisible axion solution

The most popular solution to the strong CP problem is the Peccei-Quinn mechanism [12], in which the phase $\bar{\Theta}$ is identified with the pseudo-Goldstone boson resulting from the spontaneous breakdown of a global symmetry $U(1)_{PQ}$. Observational constraints require this breakdown to occur at a scale M_{PQ} much bigger than the electroweak scale, making the axion "invisible" [64, 65]. Besides the axion field a, the breaking of $U(1)_{PQ}$ produces a network of global strings [13]. As we go around each minimal string, the phase $\bar{\Theta} = a/M_{PQ}$ winds by 2π. Instanton effects appear later, when the temperature has reached the QCD scale Λ_{QCD}. Their effects in the Higgs sector can be mimicked by an effective term

$$\Delta V = \Lambda_{QCD}^4(1 - cos N\bar{\Theta}) \tag{3.46}$$

where N is the number of quark flavors. It becomes energetically favorable for $\bar{\Theta}$ to choose one out of the discrete set of values $2\pi k/N$ ($k = 1, 2, ..N$). But since we must have $\Delta\bar{\Theta} = 2\pi$ around a string, this results in the formation of N domain walls attached to each string [33]. For $N > 1$, these domain walls are stable and therefore in conflict with standard cosmology, as we argued in Chapter 2.

Clearly, without the global strings no walls will be formed: above $T \simeq \Lambda_{QCD}$, $\bar{\Theta}$ would be aligned having some typical value $\bar{\Theta}_0$ which after the QCD phase transition would relax to the nearest minimum. We wish then to study in detail the high temperature behavior of the invisible axion mechanism, well above the scale M_{PQ}.

For concreteness we concentrate on the minimal extension of the original Peccei-Quinn model [65]. The potential for the PQ model with the doublets ϕ_i (i=1,2) both having $Y = 1$ and a $SU(2) \times U(1)$ singlet S may be written as

$$\begin{aligned} V_{PQ} &= \sum_i \left[-\frac{m_i^2}{2}\phi_i^\dagger\phi_i + \frac{\lambda_i}{4}(\phi_i^\dagger\phi_i)^2 \right] - \frac{\alpha}{2}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - \frac{\beta}{2}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\ &- \frac{m_s^2}{2}S^*S + \frac{\lambda_s}{4}(S^*S)^2 - \sum_i(\frac{\gamma_i}{2}\phi_i^\dagger\phi_i)S^*S - M(\phi_1^\dagger\phi_2 S + \phi_2^\dagger\phi_1 S^*) \end{aligned} \tag{3.47}$$

Besides the $SU(2)_L \times U(1)_Y$ local gauge symmetry, V_{PQ} has a chiral $U(1)_{PQ}$ symmetry (ϕ_1 couples to say down quarks, and ϕ_2 to up quarks)

$$\phi_1 \to e^{i\alpha}\phi_1 \;\; ; \;\; \phi_2 \to e^{-i\alpha}\phi_2 \;\; ; \;\; S \to e^{2i\alpha}S \tag{3.48}$$

For $\beta > 0$, the minimum is found at

$$\langle \Phi_i \rangle = \begin{pmatrix} 0 \\ v_i \end{pmatrix} \quad ; \quad \langle S \rangle = v_S \tag{3.49}$$

To have $U(1)_{PQ}$ broken at any temperature, it is enough to keep the VEVof the singlet nonzero for all T. From our analysis of the previous section for a potential with three doublets, one can already expect that keeping the VEVof only one field nonzero will not be difficult. In this model then the conditions on the potential parameters cannot be an obstacle for non-restoration, but we present them here for the sake of completeness. Taking $v_S \gg v_i$, the conditions over the couplings are, to leading order

$$\lambda_i > 0 \;\; , \;\; \lambda_S > 0 \;\; ; \;\; \lambda_i\lambda_S > \gamma_i^2 \;\; ; \;\; \lambda_1\lambda_2 > (\alpha+\beta)^2 \tag{3.50}$$

$$\begin{aligned} & M v_s^3 \left[\frac{v_1^3}{v_2}(\lambda_1\lambda_S - \gamma_1^2) + \frac{v_2^3}{v_1}(\lambda_2\lambda_S - \gamma_2^2) - 2v_1v_2(\lambda_S(\alpha+\beta) + \gamma_1\gamma_2 \right] \\ + \;\; & v_S^2 v_1^2 v_2^2 \left[\lambda_1\lambda_2\lambda_S - \lambda_1\gamma_2^2 - \lambda_2\gamma_1^2 - \lambda_S(\alpha+\beta)^2 - 2\gamma_1\gamma_2(\alpha+\beta) \right] > 0 \end{aligned} \tag{3.51}$$

It is easily proven that (3.50) imply that the first line of eq. (3.51) is positive. A sufficient condition for boundedness will then require (3.50) and the second line of (3.51) to be positive, the same conditions that were required in the three-doublet model of Section 3.2.1 (equations (3.24),(3.25)).

The mass term of the singlet at high temperature will be

$$m_S^2(T) = -m_S^2 + \frac{T^2}{3}(\lambda_S - \gamma_1 - \gamma_2) \tag{3.52}$$

so that imposing $\gamma_1 + \gamma_2 > \lambda_S$, we get the $U(1)_{PQ}$ symmetry broken at all temperatures. We already know that at high temperature one cannot have all three VEVs nonzero, and notice that because of the linear terms in (3.47), having $v_S \neq 0$ forces v_1, v_2 to vanish.

Up to this order then, it seems quite natural to keep the VEVof S nonzero at high temperature. It should be evident that the same holds true for Kim's version [64] of the invisible axion idea.

3.4.2 Spontaneous P or CP violation

Another well-known solution to the strong CP problem is based on the idea of spontaneous CP or P violation [66]. Here, the symmetries can be used to set $\bar{\Theta}_{\text{tree}} = 0$ and the effective $\bar{\Theta}$ is then finite and calculable in perturbation theory, and in many models small enough. The high temperature behavior of these theories is completely analogous to the one discussed in Section 3.2 and 3.3, and thus we can conclude that the solution of the domain wall problem favors models with singlets. However, before *the* model is found we find it fruitless to study this question in detail.

3.5 Thermal production of domain walls and strings

If the symmetry never gets restored at high temperature, the system will not undergo a phase transition when the temperature of the Universe becomes of the order of the symmetry breaking scale. Without phase transition, the Kibble mechanism is never activated. For this to work, of course, it is necessary to set the initial conditions appropriately. Namely, one has to demand that the field takes the same VEVover comoving distances at least as big as the present-day horizon, which for earlier times means requiring homogeneity beyond the causal limit. There is nothing surprising about this: it is yet another manifestation of the fact that in order to have a consistent cosmology, one has to solve the horizon problem first. One can, for example, have a period of primordial inflation near the Planck scale, forcing the field to go to one and only one point in the vacuum manifold once and for all.

This, however, is not enough. The problem is that the theory still admits topological defects as a solution to its classical equations of motion, and they can therefore be created by other means. In particular, the field may be able to use its thermal energy to nucleate a defect in vacuum. All that is required is that the field's configuration is trivial at infinity. For a domain wall this means creating a spherical one, for strings a loop is allowed, and for the monopoles (that we will study in the next chapter) a monopole-antimonopole pair.

The thermal production of defects at finite temperature has been studied by Linde [67, 68]. The production rate per unit time per unit volume at a temperature T will be given by [67]

$$\Gamma = T^4 \left(\frac{S_3}{2\pi T}\right)^{3/2} e^{-S_3/T} \tag{3.53}$$

where S_3 is the energy of the closed defect. For a spherical domain wall with radius R,

$$\frac{S_3}{T} = 4\pi R^2 \frac{\sigma}{T} \tag{3.54}$$

where

$$\sigma = \frac{2}{3}\sqrt{\lambda}v^3 \tag{3.55}$$

is the domain wall energy per unit area. We cannot estimate the radius, but surely we must have

$$R \gg \delta_w = \frac{2}{\sqrt{\lambda}v} \tag{3.56}$$

where δ_w is the domain wall's thickness. Then

$$\frac{S_3}{T} \gg 4\pi\delta_W^2\frac{\sigma}{T} \sim \frac{v}{\sqrt{\lambda}T} \tag{3.57}$$

When the symmetry is not restored, the VEVof the relevant field is directly proportional to T. The exponential factor S_3/T turns out to be a constant in these models, which makes it very easy to have the required suppression in the domain wall's thermal production. We have for the case of spontaneously broken CP with a singlet (section 3.2.2)

$$\frac{S_3}{T} \gg \frac{16\pi}{3\sqrt{6}}\frac{\sqrt{2\alpha - 3\lambda_S}}{\lambda_S} \tag{3.58}$$

The production of domain walls is sufficiently suppressed just by taking λ_S small enough.

A similar estimate can be made for the thermally produced loops of strings, that could be formed in the Peccei-Quinn model of Section 3.4.1. A loop of string with radius R will have

$$\frac{S_3}{T} = 2\pi R\frac{\mu}{T} \tag{3.59}$$

where this time

$$\mu = \pi v^2 K \tag{3.60}$$

is the string's energy per unit length. K is a factor depending on the specific shape of the string solution, and is of order one. Again one must take the radius of the loop much bigger than the string's radius δ_s,

$$\delta_s = \sqrt{\frac{\lambda}{6}}\frac{1}{v} \tag{3.61}$$

so that

$$\frac{S_3}{T} \gg \frac{2\pi^2\sqrt{6}}{\sqrt{\lambda}}\frac{v}{T} \tag{3.62}$$

For the Peccei-Quinn case we get

$$\frac{S_3}{T} \gg 2\pi^2 \frac{\sqrt{\gamma_1 + \gamma_2 - \lambda_S}}{\lambda_S} \tag{3.63}$$

It is not necessary to repeat the argument for the model of spontaneous P violation with a singlet. We see that in all these cases, it suffices to take the singlet's self-coupling λ_S small to avoid significant thermal production of defects.

3.6 Next-to-Leading Order Corrections

In a series of recent papers, Bimonte and Lozano [69, 70] have addressed the issue of next-to-leading order contributions to the effective potential. As was already pointed out in [4], in a theory with a $\lambda\phi^4$ potential, the next-to-leading order contributions to the squared mass are of order

$$m^2(T) \propto \lambda^{3/2} T^2 \tag{3.64}$$

while higher loop corrections do not contribute significantly. We derive this result in Appendix. The point is that in a theory with two fields where one of the self-coupling constants is required to be larger than the other (as we did to avoid symmetry restoration), the larger constant will enter in corrections to the other field's mass. Thus one has to make sure that the results to leading order are maintained when including such terms.

In fact, in the case of gauge symmetries, it was concluded [70] that the inclusion of these effects can alter significantly the phase diagram of the theory. This is mainly due to the fact that in the gauged case the coupling constants cannot be as small as one wishes, but are bounded from below by the value of the gauge coupling. In the case of singlets [69], although the effects are not so dramatic, they do alter the parameter space for symmetry non-restoration.

We begin by reviewing briefly the contributions of next-to-leading corrections in the effective potential of a $O(N_1) \times O(N_2)$-symmetric model. Take two real fields ϕ_1, ϕ_2, transforming as vectors under $O(N_1), O(N_2)$ respectively, and write the potential

$$V(\phi_1, \phi_2) = \sum_i \left(-\frac{m_i^2}{2}|\phi_i|^2 + \frac{\lambda_i}{4}|\phi_i|^4 \right) - \frac{\alpha}{2}|\phi_1|^2|\phi_2|^2 \tag{3.65}$$

The temperature contributions to the effective masses to leading order are

$$\Delta m_1^2(T) = T^2\nu_1^2 = T^2\left[\lambda_i\left(\frac{2+N_1}{12}\right) - \frac{N_2}{12}\alpha + \frac{g^2}{4}Dim(G)\frac{c_1r_1}{N_1}\right] \tag{3.66}$$

(and a similar expression for Δm_2) while to next-to-leading, $\Delta m_i \equiv Tx_i$ is found by solving the coupled pair of equations

$$\begin{aligned} x_1^2 &= \nu_1^2 - \left(\frac{2+N_1}{4\pi}\right)\lambda_1 x_1 + \frac{N_2}{4\pi}\alpha x_2 - \frac{g^2}{4\pi}Dim(G)\frac{c_1r_1}{N_1}\nu_g \\ x_2^2 &= \nu_2^2 - \left(\frac{2+N_2}{4\pi}\right)\lambda_2 x_2 + \frac{N_1}{4\pi}\alpha x_1 \frac{g^2}{4\pi}Dim(G)\frac{c_2r_2}{N_2}\nu_g \end{aligned} \tag{3.67}$$

where

$$\nu_g = g^2\left(\frac{N}{3} + \sum_{F_i}\frac{c_i}{6} + \sum_{H_i}\frac{c_ir_i}{6}\right) \tag{3.68}$$

for a group $SU(N)$, where the sums are over the fermions and Higgs fields, respectively.

Symmetry is restored when solutions of (3.67) are real and positive. The conditions under which those solutions do not exist, and therefore the $O(N_2)$ symmetry is *not* restored can be found to be [69]

$$\alpha\left(\frac{N_1}{2+N_2}\right)[1 - f(\lambda_1,\alpha)] > \lambda_2 + 3g^2 Dim(G)\frac{c_2r_2}{N_2(2+N_2)}\left(1 - \frac{\nu_g}{\pi}\right) \tag{3.69}$$

$$\lambda_1\lambda_2 > \alpha^2 \tag{3.70}$$

where

$$f(\lambda_1,\alpha) = \frac{3(2+N_1)}{8\pi^2}\left(\sqrt{\lambda_1^2 + \frac{16\pi^2}{3(2+N_1)}\left[\lambda_1 - \frac{N_2}{2+N_1}\alpha + \frac{3g^2 Dim(G)c_1r_1}{N_1(2+N_1)}\left(1 - \frac{\nu_g}{\pi}\right)\right]} - \lambda_1\right) \tag{3.71}$$

is a function that can take values from 0 to 1. The leading order conditions are (3.69) with $f = 0$. One can see then why the parameter space is reduced: it gets more difficult to fulfill (3.69). The behavior with the number of fields also becomes nontrivial, since $(1 - f)$ is a decreasing function of N_1, and the two factors of α in (3.69) compete (up to leading order, it is always preferable to keep nonzero the VEVof the field in the smallest representation).

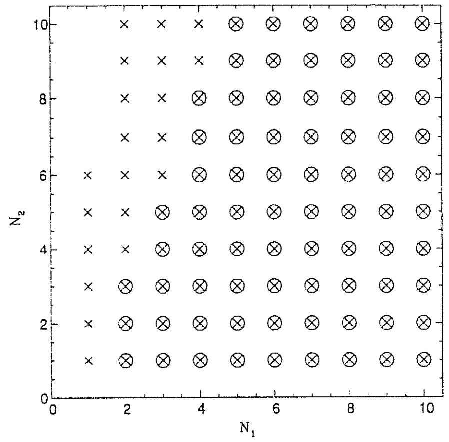


Figure 3.1: Symmetry non-restoration in a model with $O(N_1) \times O(N_2)$ symmetry. Points indicate the values of N_1, N_2 for which the VEVof the $O(N_2)$ vector can be kept nonzero at high temperature, for fixed values of the potential's parameters: circles correspond to $\lambda_1 = 0.1$, $\alpha = 0.03$, $\lambda_2 = 0.01$, crosses to $\lambda_1 = 0.1$, $\alpha = 0.01$, $\lambda_2 = 0.001$

The $O(N_1) \times O(N_2)$ toy model can mimic models with more complicated symmetries involving two fields with N_1 and N_2 real components, in the approximation where their interaction is just of the type $\alpha|\phi_1|^2|\phi_2|^2$. In particular, no approximation needs to be done in the doublet+singlet case. Also, when the field to be non-restored is a singlet, the gauge coupling enters (3.70) only through f, and its effect is very small.

In Figure 3.1 we show how symmetry non-restoration depends in the number of fields when the next-to-leading order effects are included, i.e., we find the values of N_1 and N_2 for which the conditions (3.69),(3.70) are satisfied when the parameters of the potential are fixed. The plot shows the situation for two sets of ratios of the couplings: $\lambda_1 : \alpha : \lambda_2 = 1 : 1/3 : 1/9$ and $1 : 1/10 : 1/100$, in the global case. Notice that $N_2 < N_1$ is still preferred. As the ratio N_2/N_1 increases, it becomes necessary for non-restoration to take smaller ratio λ_2/λ_1.

The cases of $N_1 = 4, N_2 = 1$ (a complex doublet plus a real singlet, as required for CP violation in Section 3.2.2), that of $N_1 = 8, N_2 = 2$ (two doublets and one complex singlet, as in the invisible axion model of Section 3.4) and that of $N_1 = 8, N_2 = 1$ (two doublets and a singlet, as in the parity-violating model of Section 3.3) lie in the non-restoration region.

The relevant question is how big is the region in parameter space where non-restoration occurs. In Figure 3.2 we show that region for the case of the CP violation

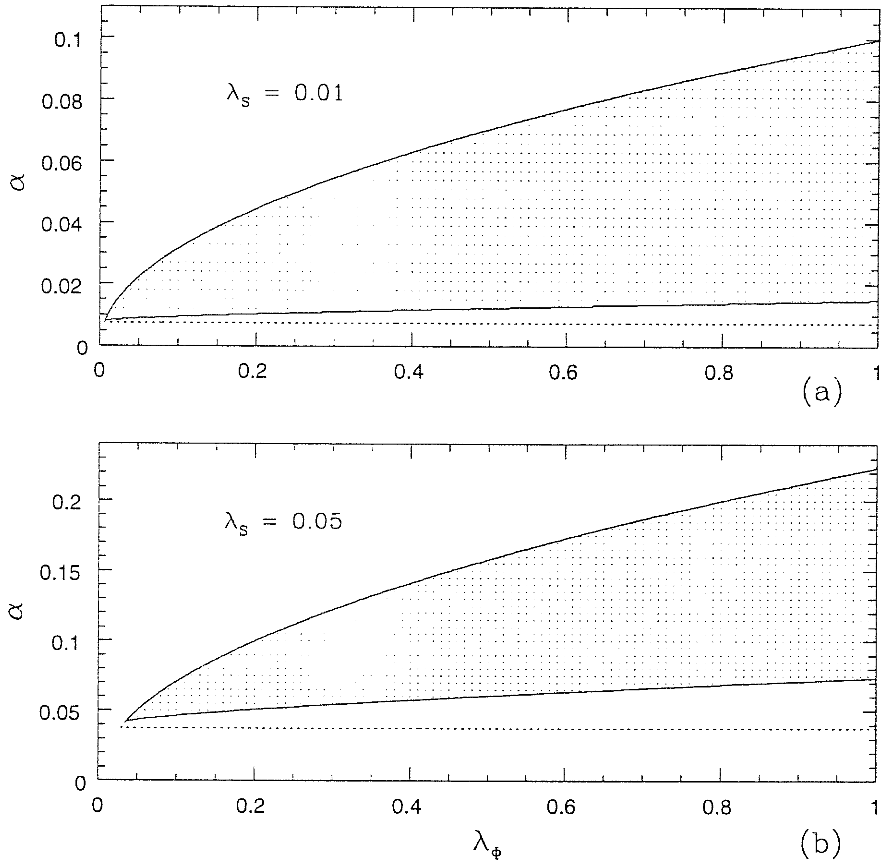


Figure 3.2: The region of symmetry non-restoration for the model of$\not{C}\not{P}$ with a real, CP odd singlet, for two values of the singlet's self coupling constant λ_s as indicated. When only leading order effects are taken into account, the region extends up to the dotted line

with a real singlet, in λ_Φ, α space. when λ_S is kept at a fixed value, and for $g^2 = 1/4$. Varying λ_S basically 'rescales' the whole picture in the α axis. The corresponding region with only leading-order effects is also shown. Although the parameter space is reduced by higher order corrections, the difference with the leading order case is not dramatic.

For the Peccei-Quinn model, the next-to-leading order calculations are only approximated by an $O(8) \times O(2)$ model, in the limit where in (3.47), $\lambda_1 = \lambda_2 = 2\alpha \equiv \lambda_\Phi$, $\beta = 0$, and $\gamma_1 = \gamma_2 \equiv \gamma$. Under such approximation, the region where non-restoration is allowed is presented in Figure 3.3, for the same range of parameters as in Figure 2. It is evident comparing both figures that non-restoration does not depend only on the ratio N_2/N_1.

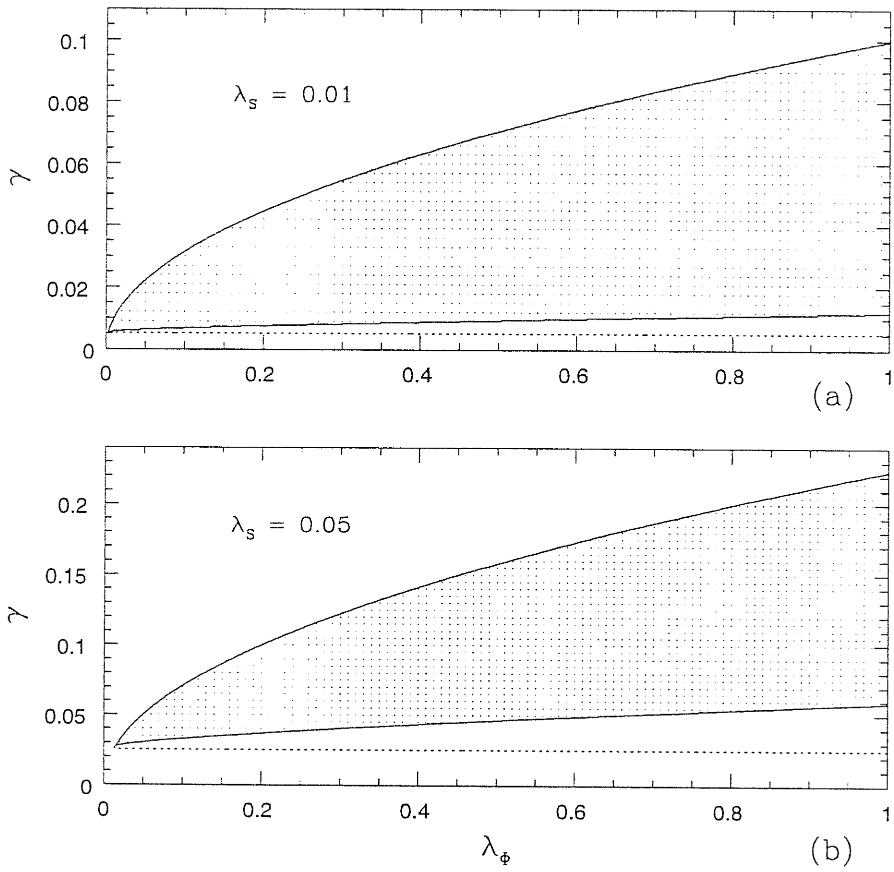


Figure 3.3: The region of symmetry non-restoration for a $O(8) \times O(2)$ model, an approximation of the Peccei-Quinn model

As for the model of P violation with a singlet of Section 3.3, it can be imitated by a $O(8) \times O(1)$ model if the quartic coupling with the two doublet fields is taken negative. One can also choose the couplings with the bi-doublet negative, and then consider an approximated model with some of the self and mixed couplings small. The non-restoration region is clearly bigger than in the weak or strong CP cases.

3.7 Summary

It seems then that it is possible, in some models, to have CP, P or strong CP broken at high temperature. It is clear, however, that in realistic theories it is not enough to have the VEVof a scalar field nonzero at high temperature. Care must be taken that the desired characteristics of the models are maintained in the range of couplings that allow for a nonvanishing VEV. Thus in the T.D. Lee model and in the three-doublet CP violating one, the CP phase disappeared at high temperature.

In particular, models with singlets are favored, since the gauge coupling works in favor of restoration, and its value cannot be adjusted to satisfy our purposes. It is also clear that the non-restoration of a field is facilitated if it belongs to the smaller (in the sense of having less components) representation of the group.

The next-to-leading effects, although affecting the parameter space where non-restoration is allowed, are not so large as to prevent non-restoration. This is only true for singlet fields, since if they are coupled with the gauge sector, the gauge constant provides a lower limit for the biggest self-coupling in the theory, with the risk of invalidating the perturbative expansion. We have presented some examples from the literature where the spontaneous breakdown of the discrete symmetries is achieved by such singlet fields, and thus non-restoration is a possibility.

In addition to avoid the domain wall problem, having CP broken at high temperatures may be relevant for baryogenesis, as first discussed in [62]. This will require embedding the models discussed here in a GUT. In the next chapter we turn to the discussion of how non-restoration can be applied in the context of GUTs.

4 Where we attempt to solve the monopole problem

4.1 Introduction

As we pointed out in Chapter 2, any Grand Unified Theory based upon a simple group will admit monopole solutions. This is just a consequence of the fact that the symmetry group of electromagnetism is $U(1)$, and is not broken today. In this chapter we will explore the possibility of using the phenomenon of non-restoration of a gauge symmetry to solve the monopole problem in GUTs. To do so, we have chosen the prototype of Grand Unified Theories, namely the one based on unification under the $SU(5)$ gauge symmetry. $SU(5)$ is the minimal GUT, in the sense that it is the smallest simple group that can accommodate all the fermions of the Standard Model, and only those. The symmetry breaking pattern of $SU(5)$ is simple and elegant, providing us with a suitable frame in which to apply the non-restoration scenario.

We present results from a collaboration with Gia Dvali and Goran Senjanović, published in [53]. The first attempt to solve the monopole problem with non-restoration in $SU(5)$ is due to Salomonson, Skagerstarm and Stern [71]. The analysis of [71] did not include the gauge symmetry, however, and thus was not realistic–their conclusions are radically different from ours.

We will present the theory at high and low temperature, and find the conditions under which the symmetry is not restored in the early universe. As was the case for domain walls, monopoles can be thermally produced, and we will therefore have to find which further restrictions arise from this. In gauge theories, the next-to-leading order contributions can become important. We will see how they actually do so in the case of $SU(5)$ non-restoration, posing serious problems to the possibility of solving the monopole problem.

Finally, we will briefly discuss the application of non-restoration to the other best-studied GUT, $SO(10)$. Breaking of $SO(10)$ to the Standard Model can be achieved in a large variety of ways. In such a rich pattern of possible phase transitions, almost any kind of topological defect can be formed. Much in the same way as $SU(5)$ provides us the simplest scenario for discussing phase transitions in GUTs, $SO(10)$ offers the most baroque one, and we will profit from this to give some examples. As in $SU(5)$, however, next-to-leading order corrections conspire against non-restoration. We will carefully discuss its consequences at the end of the chapter.

4.2 Grand Unification with SU(5)

Unification based on $SU(5)$ was first suggested by Georgi and Glashow [72]. The minimal model contains all the fermions of the Standard Model, in the fundamental ($\underline{5}$) and antisymmetric ($\underline{10}$) representations. We will normalize the generators of $SU(5)$ in the fundamental representation as usual

$$TrT_aT_b = \frac{1}{2}\delta_{ab} \tag{4.1}$$

So that if T_3 is the diagonal generator of the SU(2) subgroup, and T_0 the one of the $U(1)$ subgroup, the charge is defined as

$$Q = T_3 - \sqrt{\frac{5}{3}}T_0 \equiv \mathrm{diag}(-1/3, -1/3, -1/3, 1, 0) \tag{4.2}$$

Therefore the fermion fields will be

$$\psi = \begin{pmatrix} d^r \\ d^g \\ d^b \\ e^+ \\ -\nu^C \end{pmatrix}_R \tag{4.3}$$

and

$$\chi = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & u_b^C & -u_g^C & -u^r & -d^r \\ -u_b^C & 0 & u_r^C & -u^g & -d^g \\ u_g^C & -u_r^C & 0 & -u^b & -d^b \\ u^r & u^g & u^b & 0 & e^+ \\ d^r & d^g & d^b & -e^+ & 0 \end{bmatrix}_L \tag{4.4}$$

where the superscript C denotes the antiparticles.

In order to break $SU(5)$ to $SU(3)_c \times U(1)_{em}$, two Higgs fields are required. First, the field H in the adjoint ($\underline{24}$) representation breaks $SU(5)$ to $SU(3)_c \times SU(2)_L \times$

$U(1)_Y$, at a high scale $M_X \simeq 10^{16} GeV$. Later at the Standard Model scale, the field Φ in the fundamental ($\underline{5}$) representation does the further breaking. The potential for the Higgs fields takes the form

$$\begin{aligned} V &= -m_H^2 TrH^2 + \lambda_1 (TrH^2)^2 + \lambda_2 TrH^4 \\ &- m_\Phi^2 \Phi^\dagger \Phi + \lambda_\Phi (\Phi^\dagger \Phi)^2 - \alpha \Phi^\dagger \Phi TrH^2 - \beta \Phi^\dagger H^2 \Phi \end{aligned} \tag{4.5}$$

where we have not included cubic terms in H just for the sake of simplicity, since they are irrelevant to our discussion. The scales being widely separated, one can consider the first breaking to take place independently of the field Φ. It is easily shown that if the conditions

$$\lambda_2 > 0 \quad 7\lambda_2 + 30\lambda_1 > 0 \tag{4.6}$$

are fulfilled, the absolute minimum of the theory is found at $\langle H \rangle = v_H \,\mathrm{diag}(1, 1, 1, -3/2, -3/2)$. This ensures that $SU(5)$ is broken to $SU(3)_c \times SU(2)_L \times U(1)_Y$. However, when in the next stage Φ gets a VEV, the solution for H gets slightly perturbed. Although the perturbation is very small, it should be taken into account when considering the minimum conditions to be imposed over the coupling constants, and it will also be relevant when constructing the monopole solutions in the next section. After all symmetry breaking has taken place, we found the VEV's to be

$$< H >= v_H \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2}+\epsilon & \\ & & & & -\frac{3}{2}-\epsilon \end{pmatrix} \qquad < \Phi >= v_\Phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.7}$$

with $\epsilon \sim (v_\Phi/v_H)^2$. These will be the global minima of the theory if, in addition to (4.6), the following conditions are also satisfied

$$\lambda_\Phi > 0, \quad \beta > 0, \quad (30\lambda_1 + 7\lambda_2)(40\lambda_2\lambda_\Phi - \frac{9}{2}\beta^2) - 3(10\alpha + 3\beta)^2 > 0 \tag{4.8}$$

Thus this Higgs content is sufficient to ensure the desired symmetry breaking.

It is well known however that minimal $SU(5)$ has serious problems, in particular concerning proton decay and the fermion masses. Consider the Yukawa couplings that will give rise to the d quark masses

$$\mathcal{L}_\mathcal{Y} = h\bar{\psi}_\alpha \Phi^\dagger_\beta \chi_{\alpha\beta} + h.c. \tag{4.9}$$

With the VEVof Φ in the $\alpha = 5$ direction, we see immediately from (4.3) and (4.4) that $m(d) = m(e) = hv_\Phi$, and similar relations for the second and third generations. Although this relation is valid only while $SU(5)$ is a good symmetry, one can construct ratios m_e/m_μ, for example, which will be weakly dependent on the energy. Then one has a clear prediction on the mass ratios

$$\frac{m_e}{m_\mu} \simeq \frac{m_d}{m_s} \tag{4.10}$$

and similar relations including the third generation. Obviously this contradicts experimental results. To cure this problem, it was suggested [73, 74] to include an extra Higgs field (Σ) in the $\underline{45}$ representation. It can be written as a three index tensor with the properties

$$\Sigma^\alpha_{\beta\gamma} = -\Sigma^\alpha_{\gamma\beta} \; ; \qquad \Sigma^\alpha_{\alpha\gamma} = 0 \tag{4.11}$$

It can be shown [75] that Σ acquires vacuum expectation values that leave invariant $SU(3)_c \times U(1)_{em}$ much in the same way as Φ in the $\underline{5}$ representation did. More specifically, it is found

$$\Sigma^i_{i5} = v_\Sigma, \quad \Sigma^4_{45} = -3v_\Sigma \tag{4.12}$$

which gives a new relation between the masses of the d-quark and the electron, namely $m_d = 3m_e$. This in turn will imply

$$\frac{m_e}{m_\mu} = \frac{1}{9}\frac{m_d}{m_s} \tag{4.13}$$

which agrees reasonably with experiment.

4.3 Non-restoration in SU(5)

To compute the effective potential at high temperature in the $SU(5)$ model, we again use the techniques of chapter 1. Consider in the first place the minimal model with Φ and H, and the potential (4.5). We calculate the effective potential at high temperatures, *i.e.* for $T \gg m_\Phi, m_H$, to be

$$\begin{aligned} V(T) &= V_0 + \frac{T^2}{24}\left\{(12\lambda_\Phi - 24\alpha - \frac{12}{5}\beta + \frac{36}{5}g^2)\Phi^\dagger\Phi \right. \\ &+ \left. (52\lambda_1 + \frac{94}{5}\lambda_2 - 20\alpha - 2\beta + 30g^2)TrH^2\right\} \end{aligned} \tag{4.14}$$

The above form has already been given in [76]. To achieve non-restoration of the $SU(5)$ symmetry, we would like to keep the VEVof H nonzero at any temperature. That is, we want to have

$$52\lambda_1 + \frac{94}{5}\lambda_2 - 20\alpha - 2\beta + 30g^2 < 0 \tag{4.15}$$

As in the cases before, the signs of some of the couplings are not determined: they are subject only to the conditions (4.6) and (4.8). The question is then whether the set of inequalities (4.6), (4.8) and (4.15) can be simultaneously satisfied. We will first analyze the approximate model in which some of the constants are small, in this case λ_2 and β. In this limit, the potential becomes

$$\begin{aligned} V &= -m_\Phi^2 \Phi^\dagger\Phi + \lambda_\Phi(\Phi^\dagger\Phi)^2 \\ &= -m_H^2 TrH^2 + \lambda_1(TrH^2)^2 - \alpha\Phi^\dagger\Phi TrH^2 \end{aligned} \tag{4.16}$$

and possesses now a $O(24) \times O(10)$ symmetry. We can now use the results of Chapter2 regarding non-restoration of gauge symmetries in this kind of simple models.

The conditions (4.6) and (4.8) now read (we will call $\lambda_1 = \lambda_H$ for clearness)

$$\lambda_H > 0\ ,\ \ \lambda_\Phi > 0\ ,\ \ \lambda_H\lambda_\Phi > \alpha^2 \tag{4.17}$$

and $m_H^2(T) < 0$ requires

$$\alpha > \frac{13}{5}\lambda_H + \frac{3}{2}g^2 \tag{4.18}$$

We have already found the condition over the biggest of the couplings in a $O(N_1) \times O(N_2)$ theory, when asking non-restoration of the field in R_2, namely

$$\lambda_1 > 12g^2\frac{(N_2+2)}{N_1^2 N_2}Dim(G)r_2c_2 \tag{4.19}$$

Thus we have a lower limit for λ_Φ

$$\lambda_\Phi \geq \frac{78}{5}g^2 \tag{4.20}$$

Here it becomes evident that the mechanism cannot work. Taking a typical value $g^2/(4\pi) \simeq 1/50$, we get $\lambda_\Phi/4$ dangerously close to 1, invalidating the small coupling limit. Requiring non-restoration seems to be in open conflict with perturbation theory. Figure 4.1 shows the region in parameter space where non-restoration is possible, namely the one above the curve representing equation (4.18) and below the one of (4.17) For any value of λ_H, α, it is seen that λ_Φ has to become too large.

One can hope that this is a consequence of the simplifications we have taken. It is an easy task to repeat the analysis for the full set of constants. We want to satisfy (4.8), therefore from (4.15) we must have at least

$$(10\lambda_2\lambda_\Phi - \frac{9}{2}\beta^2) > 3\lambda_2\frac{(10\alpha+3\beta)^2}{(30\lambda_1+7\lambda_2)} > 3\lambda_2\frac{(26\lambda_1+\frac{47}{5}\lambda_2+2\beta+15g^2)^2}{(30\lambda_1+7\lambda_2)} \qquad (4.21)$$

Again finding the minimum respect to λ_1 we get

$$\lambda_\Phi > \frac{78}{5}g^2 + \frac{52}{15}\lambda_2 + \frac{52}{25}\beta + \frac{9}{20}\frac{\beta^2}{\lambda_2} \qquad (4.22)$$

Taking into account that $\lambda_2, \beta > 0$, we see that the minimum value is found for $\lambda_2 = \beta = 0$, as in our approximation.

The inclusion of Σ in the 45 representation solves the problem. The most general potential including H, Φ and Σ is rather involved. However, it is enough to look at the simplified version, namely a theory where the only relevant terms are those in the fields H and Σ that posses a $O(24) \times O(90)$ symmetry (the 45 representation of $SU(5)$ is complex). Then one can once again find a lower limit for the biggest of the couplings, in this case λ_Σ. 45 contains 9 times as much fields as 5. From (4.19), we see that the equivalent to (4.20) is smaller by a factor of 9^2.

$$\lambda_\Sigma > \frac{26}{135}g^2 \qquad (4.23)$$

which is enough to keep perturbation theory valid. The new region is shown in figure 4.2, and now the values of the coupling constants lay well below the limits imposed by perturbation theory.

We see then that the model with a Higgs field in 45 seems to allow for non-restoration, at least to leading order in the expansion at high temperature. Again, the phase transition is avoided and monopoles do not get formed via the Kibble mechanism. They can however be thermally produced, as we will see in the next section.

4.4 Thermal production of monopoles

Monopoles can be produced in monopole-antimonopole pairs ($M - \bar{M}$) in particle collisions, *i.e.*

$$q + \bar{q} \to M + \bar{M}, \quad 2\gamma \to M + \bar{M}, \quad W^+ + W^- \to M + \bar{M}, \quad tec. \qquad (4.24)$$

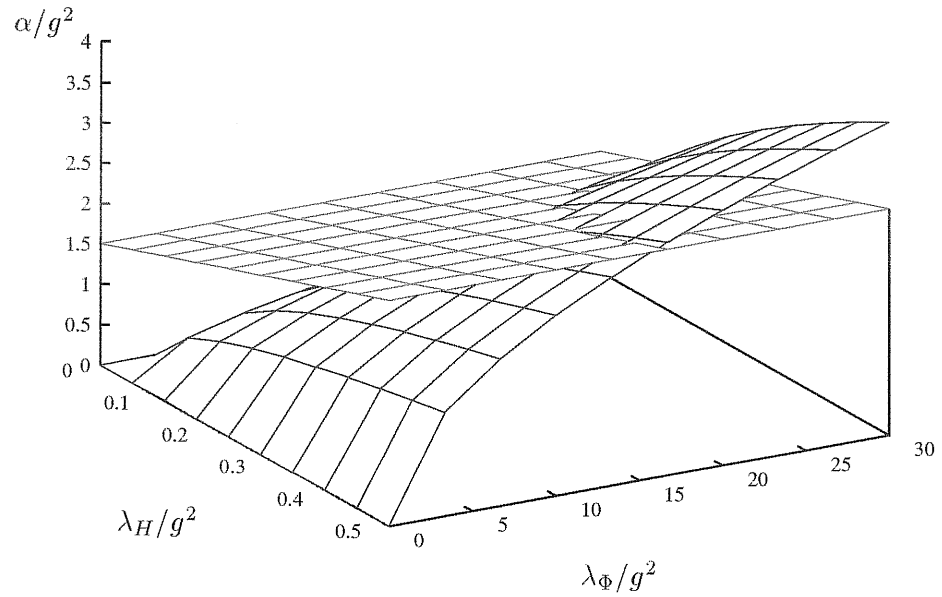


Figure 4.1: The non-restoration region in the model with Φ in the $\underline{5}$ representation. For non-restoration to occur, the parameters α, $\lambda\Phi$ and λ_H have to take values below the red surface and above the green one.

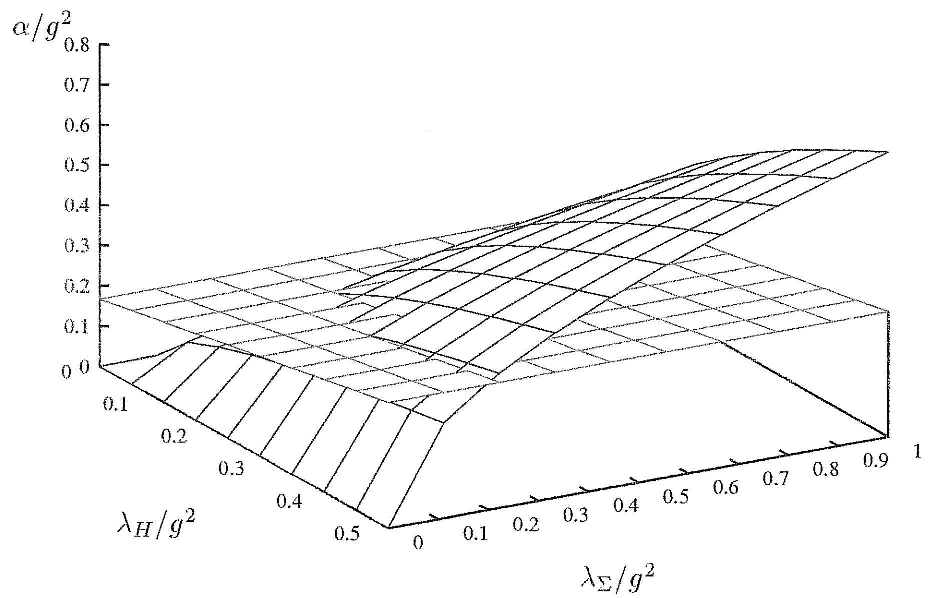


Figure 4.2: Same as in the previous figure, now for the Σ field in the $\underline{\mathbf{45}}$ representation.

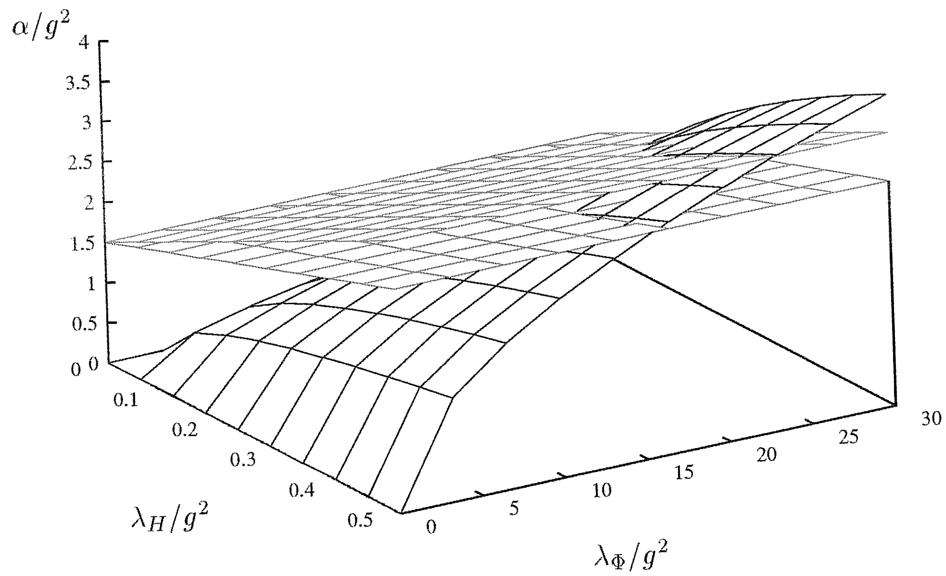


Figure 4.3: Same as in figure 5.1 , but now requiring that monopoles are not thermally produced. The allowed region now lays below the red curve and above the blue one

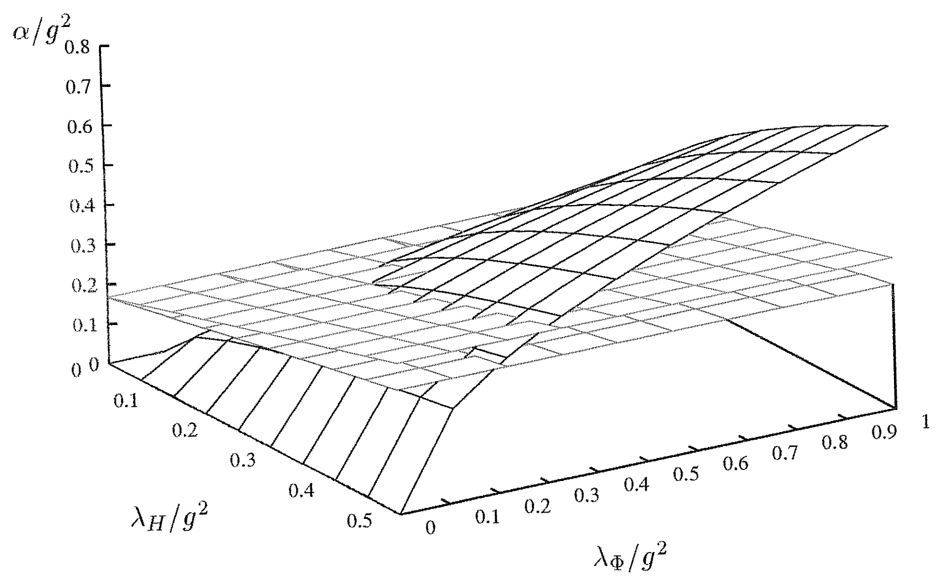


Figure 4.4: Same as in figure 5.3, for non-restoration with the Σ field

The monopole abundance is governed by the Boltzmann equation

$$\dot{n}_M + 3Hn_M = -\langle\sigma v\rangle[n_M^2 - f(T)n_\gamma^2] \tag{4.25}$$

where n_M, n_γ denote monopole's and photon's number densities respectively, H is the Hubble's constant, and $\langle\sigma v\rangle$ is the annihilation cross-section times the relative velocity. The first term in the right hand side of (4.25) takes into account the monopole annihilation, and the second its production on all the possible processes. To determine $f(T)$, Turner uses the fact that in thermal equilibrium the right hand side has to vanish. Then $f(T)$ must equal $(n_M/n_\gamma)^2$ at equilibrium, that is

$$f(T) \simeq (\frac{n_M}{n_\gamma})^2 \simeq (\frac{m}{T})^3 e^{-2m/T} \tag{4.26}$$

Preskill [35] has estimated $\langle\sigma v\rangle \sim 10^4/T$. The abundance will depend on the ratio of the annihilation rate Γ and the universe's expansion, H. If $\Gamma > H$, the monopole abundance will reach equilibrium, and remain at its equilibrium values until H equals Γ ("freeze out"). Afterwards, the monopole density is simply diluted by the expansion. On the other hand, if $\Gamma < H$ always, equilibrium is never reached, but one can easily integrate (4.25) by neglecting the annihilation term.

The ratio m/T determines whether we are in the first or second case. If $m/T < 20$, freeze out takes place for $m/T_f \sim 18 - 23$. The ratio of n_m to n_γ at freeze out,

$$\frac{n_M}{n_\gamma} = \frac{4}{g*(T_f)}\left(\frac{m}{T_f}\right)^3 e^{-2m/T_f} \sim 5 \times 10^{-9} \tag{4.27}$$

will remain constant in a Universe expanding adiabatically. This gives us a value of Ω_M which exceeds the critical one by 15 orders of magnitude.

On the other hand, if $m/T > 20$, we are in the second case above, equilibrium is never reached. Integrating Boltzmann's equation, we find

$$\frac{n_M}{n_\gamma} \simeq 3 \times 10^-3 \left(\frac{m}{T}\right)^3 e^{-2m/T} \tag{4.28}$$

The present-day monopole density will be acceptable if

$$m/T \gtrsim 35 \tag{4.29}$$

Now, the interesting point is that the monopole mass is directly proportional to the VEVof the Higgs field, and in the non-restoration scenario this VEVturns out to be proportional to the temperature. The ratio m/T is then a fixed number, and all we have to do is to find the region in parameter space where (4.29) is satisfied.

In $SU(5)$, the lightest monopoles weigh [21],

$$m_M = \frac{10\pi}{\sqrt{2}g} v_H \tag{4.30}$$

For $g^2/(4\pi) \simeq 1/50$ or $g \simeq 1/2$, $m_M \simeq 40 v_H$, and thus the consistency with the cosmological bound (4.29) implies

$$\frac{v_H}{T} \geq 1 \tag{4.31}$$

From (4.5) and (4.14), we get for $T \gg m_H$

$$\frac{v_H^2}{T^2} = -\frac{208\lambda_1 + \frac{376}{5}\lambda_2 - 20\alpha - 4\beta + \frac{15}{2}g^2}{12(30\lambda_1 + 7\lambda_2)} \tag{4.32}$$

Obviously (4.31) and (4.32) will put even more restrictive conditions on the parameters of the theory than just (4.20) or (4.23). The situation is similar to the thermal production of domain walls and strings that we encountered before. For the general case with two fields and a symmetry $O(N_1) \times O(N_2)$, we can rewrite the condition over the largest coupling, (4.19), to fulfill the new condition (4.31). The VEVof the non-restored field will be

$$v_2 = -\frac{T^2}{12}\frac{1}{\lambda_2}\left[(N_2+2)\lambda_2 - N_1\alpha + 3\frac{Dim(G) r_2 c_2}{N_2} g^2\right] \tag{4.33}$$

from where the the bound becomes

$$\lambda_1 > 12 g^2 \frac{(N_2+14)}{N_1^2 N_2} Dim(G) r_2 c_2 \tag{4.34}$$

Let us see first what happens for the minimal model with Φ in the <u>5</u> representation. For $\lambda_1 = \lambda_H$, we get

$$\lambda_\Phi > \frac{213}{20} g^2 \tag{4.35}$$

For $g^2 \simeq 1/4$, $\lambda_\Phi \geq 2.7$ and the perturbation theory clearly fails. Figure 4.3 shows the new allowed region in parameter space, including the surface represented by eq. (4.35).

We repeat the same for the more realistic version with the field Σ in the <u>45</u> representation. As before, the condition (4.35) relaxes by a factor of 1/81, and we get

$$\lambda_\Sigma > \frac{213}{1620} g^2 \tag{4.36}$$

which for $g^2 \simeq 1/4$ would give $\lambda_\Sigma > 1/30$. Thus, the largest coupling of the theory λ_Σ is still quite small and the perturbation theory is operative. Figure 4.4 shows the situation.

4.5 Next-to-leading order corrections

As we announced in Chapter 3, when the symmetry is gauged the next-to-leading order corrections are much more important. We have to satisfy

$$\alpha\left(\frac{N_1}{2+N_2}\right)[1-f(\lambda_1,\alpha)] > \lambda_2 + 3g^2 Dim(G)\frac{c_2 r_2}{N_2(2+N_2)}\left(1-\frac{\nu_g}{\pi}\right) \tag{4.37}$$

$$\lambda_1\lambda_2 > \alpha^2 \tag{4.38}$$

where

$$f(\lambda_1,\alpha) = \frac{3(2+N_1)}{8\pi^2}\left(\sqrt{\lambda_1^2 + \frac{16\pi^2}{3(2+N_1)}\left[\lambda_1 - \frac{N_2}{2+N_1}\alpha + \frac{3g^2 Dim(G)c_1 r_1}{N_1(2+N_1)}\left(1-\frac{\nu_g}{\pi}\right)\right]} - \lambda_1\right) \tag{4.39}$$

as described in section 3.6. But there the singlet field was asked to break the symmetry, and from the gap equations one could see that g entered only in the function f. This is not the case now, and as shown in [70], the consequences are dramatic.

Since the model with Σ in the $\underline{\mathbf{45}}$ representation was the one that allowed for non-restoration, let us consider only this one. In figures 4.5 and 4.6 we have plotted (4.38) for two values of the largest constant λ_Σ, in α, λ_H space.

The condition that the potential be bounded from below is the dashed line, and non-restoration is allowed above it. The leading case conditions for non-restoration is given for comparison as the continuous line, and the region lies below it. The dotted line gives the corresponding upper bound when next-to-leading order corrections are considered. There is no region where the symmetry can be kept broken, even for values of λ_Σ/g^2 as large as 5.

One could try to increase the dimension of the representation that looses the VEV, since for leading effects it is much easier to non-restore this way. A quick look at the gap equation can convince us that this is not the case when next-to-leading effects are considered (we go back to this point in the next section).

It seems then that one should carry out a more complete analysis, including all the couplings in the gap equations, to be able to determine if the monopole problem can be solved in this theories.

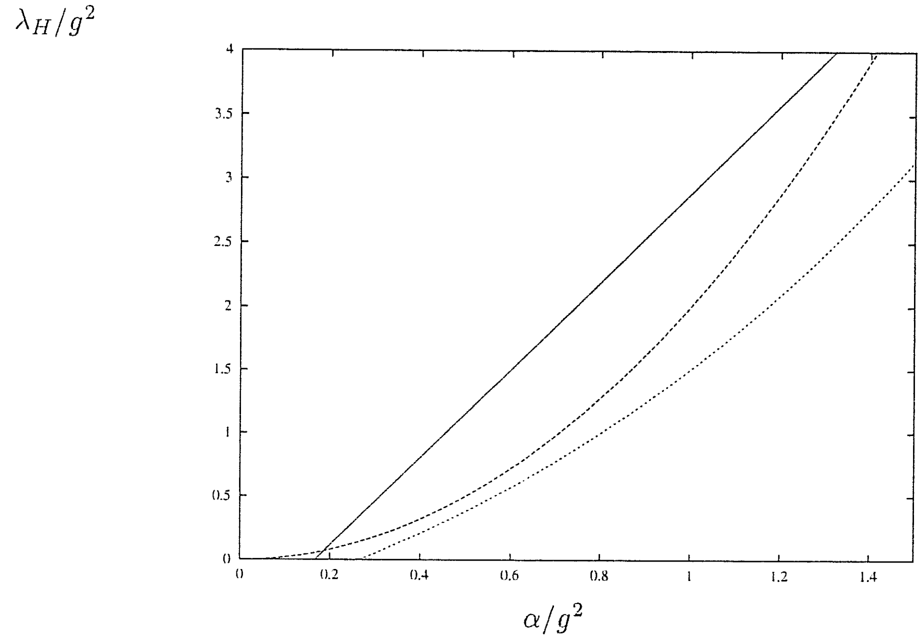


Figure 4.5: Non-restoration region with and without the next-to-leading order effects in the model with Σ in the $\underline{\mathbf{45}}$ representation, with $g^2 = 1/4$ and $\lambda_\Sigma/g^2 = 0.5$. The region lies above the dashed curve. When leading order effects are considered, it is bounded from above by the continuous line; with higher order corrections included it should be bounded from above by the dotted line

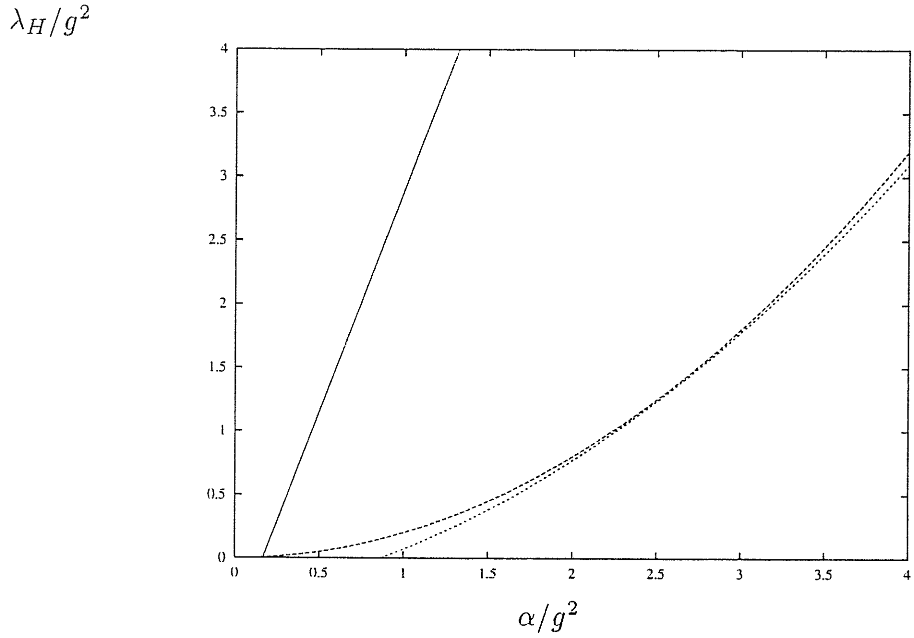


Figure 4.6: Same as the previous figures, now with $\lambda_\Sigma/g^2 = 5$.

4.6 $SO(10)$ and topological defects

Perhaps the most elegant way of unifying the known interactions is under the $SO(10)$ group. All the known fermions can fit in a single representation, the spinorial $\underline{16}$, which in addition includes a right-handed neutrino an consequently the possibility of a neutrino mass. In addition, $SO(10)$ has as maximal subgroups

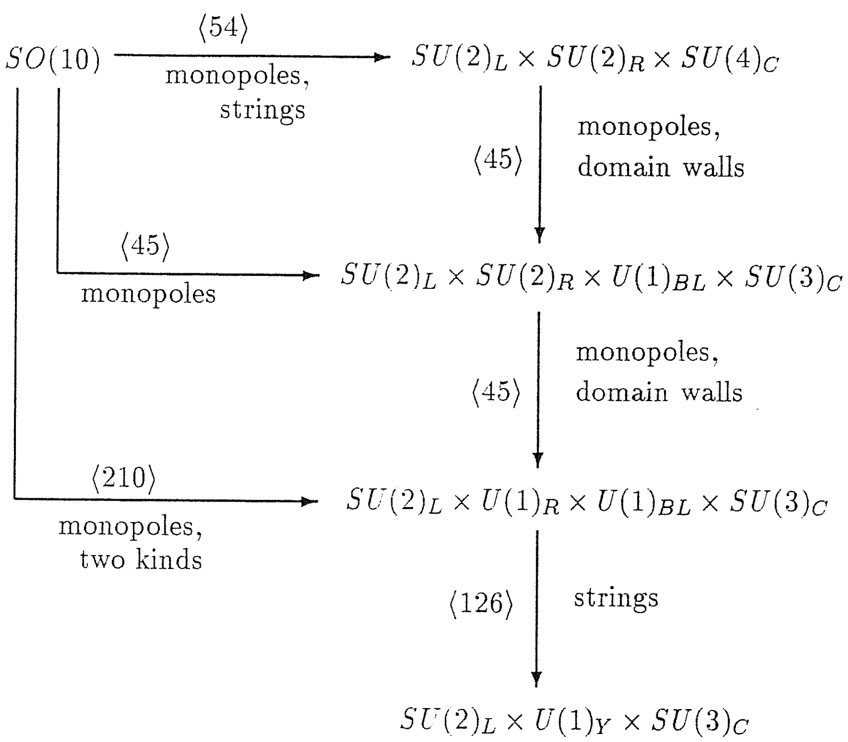
$$SU(5) \times U(1), \quad SO(4) \times SO(6) \tag{4.40}$$

so that $SO(10)$ contains not only the full $SU(5)$, but also the extremely important group $SO(4) \times SO(6)$, homeomorphic to $SU(2)_L \times SU(3)_R \times SU(4)_C$. $SU(4)_C$, the Pati-Salam theory that unifies the three colors with a fourth "leptonic" one, contains as a subgroup $SU(3)_C \times U(1)_{B-L}$. Thus $SO(4) \times SO(6)$ contains the complete Left-Right model with all its interesting features, and more.

Symmetry breaking in $SO(10)$ can go through this two main channels. The first one, breaking through $SU(5)$, does not produce topological defects in the first phase transition. Afterwards, $SU(5)$ breaks down to the Standard Model in the manner we studied in the first part of this chapter. We already know how non-restoration behaves in this case.

We turn our attention then to the second kind of breaking, through $SU(2)_L \times SU(2)_R \times SU(4)_C$. There are a large number of ways of getting to $SU(3)_C \times U(1)_{em}$ from there (18 of them, for example, are quoted in ref. [77]). Depending on the chain followed, many kinds of topological defects may be formed. In any case, electromagnetic monopoles must remain after all the breaking has taken place, since the end product will contain the Standard Model.

To have an example of the topological defect production in $SO(10)$, we have chosen the three chains represented in the following diagram



In each case, the representation whose VEVproduces the breaking is given, and we have specified which type of defects are formed.

The first chain starts by breaking $SO(10)$ with a 54, a symmetric, traceless second rank tensor, taking a VEVin the (1,1,1) direction (where here and in the following the parenthesis denotes the ($SU(2)_L$, $SU(2)_R$, $SU(4)_C$) contents). At first sight one may think that no defects are produced, since the zeroth and first homotopical groups of $SU(2)_L \times SU(2)_R \times SU(4)_C$ are trivial. However, the diagram is inexact, $SO(10)$ does not simply break to a direct product of $SO(4) \times SO(6)$, but to a more complicated group H. The problem is that $SO(4)$ and $SO(6)$ have a non trivial intersection, they share an element [78], so that the connected component of H is $[SO(4) \times SO(6)]/Z_2$. This means that its first homotopic group is nontrivial, and as a consequence Z_2 monopoles are produced. In addition, 54 leaves invariant more than just $SU(2)_L \times SU(2)_R \times SU(4)_C$. Its VEVpreserves also a generator not in $SO(4)$ or $SO(6)$, that can be identified by its action over the fermion mutliplet in 16 as charge conjugation. Thus there is also a disconnected component of H, and as we know this leads to the existence of Z_2-strings.

The symmetry breaking proceeds further when a field in the 45 representation acquires a VEVin the direction (1,1,15), breaking up $SU(4)_C$. Another tipe of monopoles, with $B-L$ charge is cretaed. Through further breaking these monopoles will evolve to become the familiar electromagnetic monopoles. The 45 VEVbreaks also the charge conjugation symmetry we mentioned above, which results in a domain wall attached to each of the Z_2-strings produced in the previous phase transition [16]. Now, notice that these domain walls emerge as a result of the particular chain of breaking. In fact, if we had gone from $SO(10)$ directly to $SU(2)_L \times SU(2)_R \times U(1)_{BL} \times SU(3)_C$, only the monopoles would have been produced. Thus one may argue that the domain walls are not stable, since they are not topologically protected. However, just as with the case of the Peccei-Quinn walls, one can show that the decaying rate for the walls is negligible [16].

A similar process occurs later when yet another **45** field takes a VEV, this time in the (1,3,1) direction, braking $SU(2)_R$ to $U(1)_R$. The third kind of monopoles. R-monopoles, are produced. Additionally, parity is broken, which gives raise to domain walls. $U(1)_R$ is broken further to nothing by the VEVof a **126** field, which contains a triplet of $SU(2)_R$ with a $B-L$ charge. In Left-Right models, this breaks down both $U(1)$ groups to produce the $U(1)_Y$ of the Standard Model. The R-monopoles get then connected by R-strings, much in the same way suggested in the Langacker-Pi mechanism. They can then annihilate, although whether they can do so at a reasonable rate is a delicate question (that we will not address here). Still supposing that unstable defects can decay, the first chain has left us with Z_2-monopoles, electromagnetic monopoles and Parity domain walls.

The second chain starts by breaking directly with a **45**, bypassing the phase transition that produced the Z_2 monopoles and strings. Only $B-L$ monopoles are produced. The end result is again electromagnetic monopoles and Parity domain walls.

One can avoid the formation of these domain walls by breaking directly $SO(10)$ with a **210** as in the third chain. Both kind of monopoles are produced in the first phase transition, so even if R-monopoles are not effciciently annihilated by connecting them to R-strings, all one has to do is to avoid the restoration of **210** to get rid of the monopole problem. As a bonus, strings get formed in the next phase transition, and this may offer interesting possibilities for the formation of large-scale structure. Thus chain three provides us with a good excuse to try to use non-restoration in $SO(10)$.

It is straightforard to use previous results here. We want to use the 252 fields in the **126** complex representation to help non-restore the VEVof the **210** real representation. We resort to the simplified model with a $O(210) \times O(252)$ symmetry in the Higgs potential, and apply (4.19) to get

$$\lambda_{126} > \frac{1}{2}g^2 \tag{4.41}$$

for the **126** self-coupling. If we require additionally that monopoles are not produced thermically in too large numbers, according to (4.34) the above bound has to be increased by a factor $(N_2 + 14)/N_2$, in this case $224/210 \sim 1$. The largest coupling is still safely smaller than the gauge coupling. To leading order then the phase transition can be avoided. However, also in this case next-to-leading order effects are far from negligible.

There is a simple way to estimate the consequence of including next-to-leading order effects. From the discussion in section 4.5 one can see that the high temperature effects do not come as a perturbative expansion in the couplings, but rather in the couplings times the number of fields. This is only natural, since the number of fields count the number of graphs contributing to the thermal masses. For example, the thermal mass of the field ϕ_i will not be an expansion in powers of λ_i but rather of

$$\lambda_i' = \frac{N_i + 2}{12}\lambda_i \tag{4.42}$$

Thus we will encounter problems in the validity of the expansion when dealing with large representations, as is the case of $SO(10)$. In particular for a field in the **126** representation, $\lambda_1' \sim 20\lambda_1 \gg g^2/2$. The same is true for the gauge coupling, the expansion is around

$$g'^2 = \frac{Dim(G)c_i r_i}{4N_i}g^2 \tag{4.43}$$

and for the **210** representation, $g'^2 = 3g^2$. We are, once again, beyond the validity of perturbation theory.

4.7 Summary

The question of whether one can solve the monopole problem by avoiding symmetry restoration remains an open one. One certainly has to go near the non-perturbative region, and then higher order effects start to become important, until they force restoration. The problem gets worse in considering large representations, which are normally in charge of symmetry breaking in GUTs. Since the study of the next-to-leading order effects has to be done near the limits of validity of perturbation theory, it is desirable to check these results in a non-perturbative framework.

Recently, various authors have tried to confirm the result of Bimonte and Lozano [69, 70] using non-perturbative methods [79, 80, 81, 82, 83, 84]. The results are not conclusive yet.

There is, however, a different and perhaps more interesting way of achieving non-restoration in GUTs. The idea suggested by Riotto and Senjanović [85] and pursued actively at the moment of writing this thesis by the same authors in collaboration with Bajc, is to consider the effects of the chemical potential, as originally suggested by Linde [87]. If viable, it could be the solution to the monopole problem in GUT, and we eagerly await for their conclusions.

5 Where we explore the non-renormalizable lands

5.1 Introduction

It is clear, from the general discussion of Chapter 2 and the examples given up to now, that in theories with only one Higgs field the symmetry is restored at high temperatures. This just arises from the fact that the potential must be bounded from below, which forces the coupling of the self-interaction term to be positive. The thermal masses, we have seen, have the same sign as the coupling that induces them.

Including extra scalar fields is the natural way out. However, one can find in diverse examples in the literature [88, 89] a different attempt to induce a negative mass term: allow for higher dimension, non-renormalizable interactions. This is a very interesting idea, especially if one considers that higher dimensional operators *must* be a part of the low-energy Lagrangian if there is to be new physics at a higher scale. At least one believes in the existence of higher dimensional operators suppressed by inverse powers of the Planck mass, since one should accept that still unknown physics takes place at the Planck scale and above it.

We will give details on how these non-renormalizable terms may allow for a negative self coupling, giving rise to a negative contribution to the thermal mass at arbitrarily large temperatures. Essentially, non-renormalizable terms are introduced in such a way that they stabilize the potential even in the presence of a negative self coupling in the renormalizable part of the potential. The aim of this chapter, however, is to show why this idea cannot work.

We will present results from a collaboration with Borut Bajc and Goran Senjanović, which have been published in [90]. The point is that when introducing non-renormalizable terms that are large enough to be relevant, one must be very careful when performing a loop expansion. It turns out that a careful analysis shows that two-loop effects at high temperature become relevant in this situations, and that they have the effect of always restoring the symmetry. We show this in the first two sections of this chapter.

The addition of non-renormalizable terms in the hope of avoiding symmetry restoration becomes especially interesting when considering supersymmetric models. As we mentioned in Chapter 2, it was first suggested by Haber [43], and then shown by Mangano [44] that internal symmetries in a supersymmetric theory *always* get restored, no matter how many fields one includes. One way out of this no-go theorem could be to include non-renormalizable interactions, as suggested by Dvali and Tamvakis [89]. Not surprisingly, here one can also demonstrate that the proposal cannot work. Indeed, the work of ref [89] was the original motivation to start a study on restoration in models with higher order operators, that turned out to be very general. We show how the attempt to avoid restoration via non-renormalizable terms in this cases also fails.

5.2 Non-renormalizable terms and high temperature

Consider a theory with a real scalar fieldφ and a Lagrangian invariant under the discrete transformation $\varphi \to -\varphi$. The potential

$$V = -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 \tag{5.1}$$

requires $\lambda > 0$ for boundedness as $\varphi \to \infty$. A positive λ, as we know, produces a positive contribution to the thermal mass as high temperature, and this leads to symmetry restoration.

Imagine now the situation in which our theory is an effective one, coming from a more complete (and perhaps unknown) theory, valid at a higher scale M. This in general will produce higher order terms in the potential, with strength suppressed by inverse powers of the high scale. We can have for example

$$V = \frac{m^2}{2}\varphi^2 - \frac{\epsilon}{4}\varphi^4 + \frac{1}{M^2}\varphi^6 \tag{5.2}$$

where $M^2 >> m^2$. This potential could be the product of a different theory, valid above the scale M, involving in addition a heavy field χ, for example

$$V = \frac{m^2}{2}\varphi^2 - \frac{\epsilon}{4}\varphi^4 - \frac{M_\chi^2}{2}\chi^2 + \alpha\chi\varphi^3 \tag{5.3}$$

with M_χ of order M. χ transforms just as φ under the discrete symmetry, and other terms can be taken to have small coupling constants for simplicity. The theory is

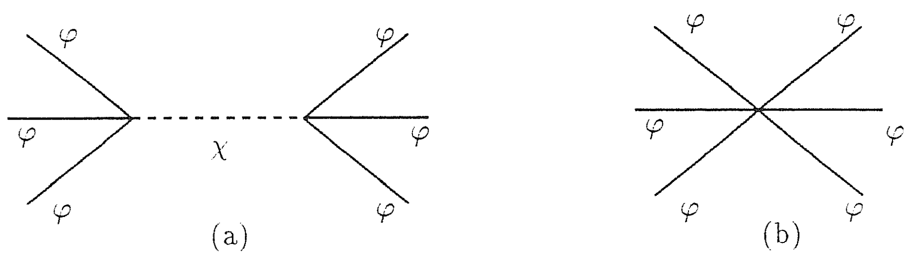


Figure 5.1: Feynman diagrams for the interaction of φ (a) with χ, for scales above M_ξ and (b) with itself, for scales below M_χ, when the heavy field is integrated out. Solid lines are φ propagators, dashed lines correspond to χ

perfectly renormalizable, but below the scale M_χ the field becomes "too heavy" to propagate. The result of this fact is depicted in the graphs of figure 5.1. Below the scale M, the propagator for the χ field becomes simply $1/M_\chi$, and the lines representing it disappear. The resulting diagram gives a self-interaction for φ of order

$$\frac{\alpha^2}{M_\chi^2}\varphi^6 \qquad (5.4)$$

which is just the corresponding term in (5.3) with $M_\chi/\alpha \equiv M$. Formally, one can solve the equations of motion for the heavy field χ, assuming its kinetic term has become negligible, *i.e.*

$$\frac{\partial V}{\partial \chi} = 0 \quad \Rightarrow \chi = \frac{\alpha}{M_\chi^2}\varphi^3 \qquad (5.5)$$

Upon substitution of this solution in (5.3), and defining $M = \sqrt{2}M_\chi/\alpha$, we arrive at the effective potential (5.2).

Now, the non-renormalizable term will not be completely negligible if ϵ is sufficiently small. In particular, it can play a role in symmetry breaking, as we will see in a moment. The important point is that the sign of the self coupling ϵ has now become arbitrary. It can be safely taken to be negative, and the boundedness of the potential as $\varphi \to \infty$ is ensured by the presence of the sixth order term. This is the crucial point for symmetry non-restoration.

At $T = 0$, the minimum of the potential (5.2) will be at

$$\langle\varphi\rangle^2 = \frac{\epsilon}{2}M^2\left(1 \pm \sqrt{1 - \frac{4m^2}{\epsilon^2 M^2}}\right) \qquad (5.6)$$

when M is finite. Thus with $m \to 0$, we have a VEVfor φ of order $\sqrt{\epsilon}M$. As M

increases, the VEVgoes to infinity, and we are in the familiar case of an unbounded quartic potential.

The question then is what happens in these kind of models at high temperature. The contribution is calculated as usual

$$\Delta V(T) = \frac{T^2}{24}\left(\frac{\partial^2 V}{\partial\varphi} = \frac{T^2}{24}m^2 - 3\epsilon\phi^2 + \frac{30}{M^2}\phi^4\right) \tag{5.7}$$

By "high" temperature, we mean here as usual $T^2 \gg \langle\varphi\rangle^2 \sim \epsilon M^2$, but we must be careful to remember that M is the largest scale of this theory, otherwise our integrating out of χ would not be valid. Thus we are in the regime

$$M^2 \gg T^2 \gg \epsilon M^2 \tag{5.8}$$

The non-renormalizable term has provided for us a peculiar behavior at high temperature. At sufficiently high T, the temperature-dependent part of the potential will have the relevant role. Note that the mass term is negative at any T, and that the temperature dependent quartic self-coupling now is dominant over the non-renormalizable term. In the range (5.8), and in the limit $m = 0$, we have a VEVfor φ

$$\langle\varphi\rangle^2 = \frac{\epsilon}{10}M^2 \tag{5.9}$$

which is *independent* of the temperature. This is in sharp contrast with the usual non-restoration mechanism, whose main characteristic is a VEVdirectly proportional to the temperature. Here, on the other hand, the field basically keeps the same VEVas for $T = 0$.

Up to now, it seems that non-renormalizable interactions can induce symmetry breaking at large temperatures. It is only required that the temperature is bigger than the scale that suppressed the higher order operators.

Is it then possible to have symmetry broken at high temperatures, even in theories with only one field? The answer is no, as we will now show.

5.3 The role of the two-loop corrections

The reason why the idea above does not work is best understood by looking at the Feynman diagrams that give contributions to the high temperature effective potential. While using the formula

$$\Delta V(T) \propto \frac{\partial^2 V}{\partial\varphi} \tag{5.10}$$

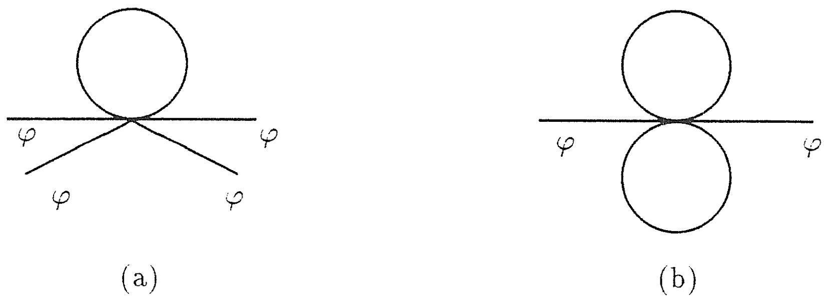


Figure 5.2: Feynman diagrams for the dominant temperature contributions from the non-renormalizable sixth order term, at (a) one loop and (b) two loops

we were taking into account also the contributions coming from the sixth order term at one loop. This contribution is the one represented in figure 5.2 (a).

It is clear that the loop will give the usual integral in thermal field theory

$$\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - m^2} \equiv I_B \tag{5.11}$$

(see the Appendix), which in turn gives a contribution $I_B(T) = T^2/12$; while we will have four external legs. This is the only quadratically divergent one-loop diagram coming from this term. We do not have contributions to the mass coming from it: to get them, obviously, we have to go to two loops, as in the diagram in figure 5.2 (b). It is evident that each loop will give a quadratically divergent integral of the type I_B. The contribution to the temperature-dependent part of the effective potential coming form this diagram will be

$$\Delta V_{(b)} = 45 \left(\frac{1}{M^2}\right) (I_B)^2 \varphi^2 = 45 \left(\frac{1}{M^2}\right) \left(\frac{T^2}{12}\right)^2 \varphi^2 \tag{5.12}$$

with the symmetry factors taken into account. But now we have obtained a new mass term, which is *not* negligible compared to the one-loop one coming from the quartic coupling ϵ. On the contrary,

$$|m(T)_{1\,\mathrm{loop}}|^2 = \frac{T^2}{4}\epsilon < \frac{5}{16}\frac{T^4}{M^2} = |m(T)_{2\,\mathrm{loops}}|^2 \tag{5.13}$$

since we have required that $T^2 \gg M^2 \epsilon$ in order for the high temperature expansion to be valid in the first place.

We have thus generated at two-loops a thermal mass that is at the same time positive and bigger than the one-loop mass. The symmetry gets, therefore, inevitably restored.

One may wonder how this is possible: if perturbation theory is valid, the two-loops effects should never overcome the one-loop ones. One must be careful, however, when dealing with theories with more than one coupling constant. The same behavior was noted by Coleman and Weinberg in their classic paper [91]. Calculating the effective potential for a $\lambda\phi^4$ a gauge theory, they found that the gauge coupling induces a quartic term proportional to g^4 (with g the gauge coupling). Depending on the values of the coupling, then, the one-loop effects can become important. In our case, we have explicitly required that ϵ is small compared to $1/M^2$, so it is not surprising that two-order effects affect the one-loop potential.

Note that perturbation theory is still valid after the two-loops. The following term in the expansion will be proportional to T^6/M^4, and is smaller than the one in (5.12). Perturbation theory "starts later," and this is just a result of our asking that the higher-dimensional operator has a large coupling constant.

With a positive mass term, the symmetry is of course restored. The mechanism for non-restoration is not applicable. One can include even higher order interactions, in the hopes of inducing negative terms (even if they are not mass terms, this can potentially induce symmetry breaking). In general, one may have

$$V = \frac{\mu^2}{2}\phi^2 - \epsilon\phi^4 + \frac{\phi^{2n+4}}{M^{2n}} , \tag{5.14}$$

where we include the first important non-renormalizable term. The power n varies from model to model ($n = 1$ in the case discussed above). At one loop level, one gets for $T << M$ the corrections

$$\Delta V_{1-\text{loop}}(T) = \frac{T^2}{24}\left[-12\epsilon\phi^2 + \frac{(2n+4)(2n+3)}{M^{2n}}\phi^{2n+2}\right] . \tag{5.15}$$

The idea is then that the temperature-induced non-renormalizable term is to combine with the one coming from the negative self-coupling to induce a VEVwhen $\Delta V(T)$ starts to dominate, *i.e.* for $T^2 \gg \mu^2$. But of course, for this to happen one has to assume that the non-renormalizable term is not negligible, i.e. ϵ very small. This means that the expansion cannot end at one loop, but has to be pursued up to $n+1$ loops. At that level, the "butterfly" diagrams with $n+1$ loops and two external legs of which Fig. 5.2 (b) is the $n = 1$ example, will induce the high temperature contribution

$$\Delta V_{\text{n+1-loops}}|_{\text{mass term}}(T) = \frac{1}{2}\left(\frac{T^2}{12}\right)^{n+1} \frac{(2n+4)!}{2^{n+1}(n+1)!}\frac{1}{M^{2n}}\phi^2 . \tag{5.16}$$

Any other term in the expansion of the couplings $1/M^{2n}$ and ϵ will be suppressed. Each loop in the diagram will provide a positive contribution $T^2/12$, so the sign of (5.16) is the sign of the coupling. A positive mass term already indicates that the symmetry will be restored, however one should look at all the temperature-dependent interactions that follow from the non-renormalizable terms. The diagrams that give the dominant $1/M^{2n}$ contribution to the ϕ^{2m} interaction terms are again the "butterflies" with $2m$ external legs, and they are readily calculated

$$\Delta V(T) = \sum_{m=1}^{n+1} \frac{(2n+4)!}{(2m)!(n-m+2)!2^{n-m+2}} \left(\frac{T^2}{12}\right)^{n-m+2} \frac{\phi^{2m}}{M^{2n}} . \tag{5.17}$$

All the terms of the series have a positive sign, not surprisingly, as we mentioned before the high-T contributions carry the sign of the coupling constant. Symmetry restoration then follows.

We can easily generalize (5.17) to get the "butterfly" contribution to the high temperature effective potential of an arbitrary $V(\phi)$:

$$V(\phi, T) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{T^2}{24}\right)^m \left\{ \frac{d^{2m}V}{d\phi^{2m}}(\phi) - \frac{d^{2m}V}{d\phi^{2m}}(\phi = 0) \right\} . \tag{5.18}$$

5.4 Restoration in supersymmetric models

The possibility of avoiding restoration of symmetries via non-renormalizable terms becomes particularly interesting in the context of supersymmetric theories. Although we have already shown that it cannot work in general, we believe it is useful to give a further example of the ideas above. Furthermore, it was the paper of ref. [89] on the possibility of non-restoration in supersymmetric theories which started the investigation quoted on the last section.

We have discussed in Chapter 2 the issue of non-restoration in supersymmetric models, and quoted a theorem [43, 44] stating that internal symmetries in a supersymmetric theory always get restored. However, the Mangano-Haber theorem was formulated for supersymmetric theories with renormalizable interactions only, and this stimulated Dvali and Tamvakis [89] to find a way out using non-renormalizable interactions. They take a superpotential for a chiral superfield Φ with the same symmetry as above, but including non-renormalizable interactions

$$W = -\frac{\mu}{2}\Phi^2 + \frac{1}{4M}\Phi^4 \tag{5.19}$$

where $M \gg \mu$. This leads to the scalar potential

$$\begin{aligned} V &= |\phi|^2| - \mu + \frac{\phi^2}{M}|^2 \\ &= \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\mu}{2M}(\phi_1^4 - \phi_2^4) + \frac{1}{8M^2}(\phi_1^2 + \phi_2^2)^3 \,, \end{aligned} \tag{5.20}$$

where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ is the scalar component of the chiral Wess-Zumino superfield Φ. Note that ϕ_1 has a negative quartic self coupling. At $T = 0$, as usual, one finds a set of two degenerate minima: $\langle\phi\rangle = 0$ and $\langle\phi\rangle^2 = \mu M$. To see what happens at high T, in Ref. [89] the usual 1-loop induced correction to the

effective potential is computed

$$\Delta V_{1-\text{loop}}(T) = \frac{T^2}{8}|\frac{\partial^2 W}{\partial\phi^2}|^2 = \frac{T^2}{8}| - \mu + \frac{3\phi^2}{M}|^2 \tag{5.21}$$

or

$$\Delta V_{1-\text{loop}}(T) = \frac{T^2}{8}\left[\mu^2 - 3\frac{\mu}{M}(\phi_1^2 - \phi_2^2) + \frac{9}{4M^2}(\phi_1^2 + \phi_2^2)^2\right] \,. \tag{5.22}$$

If this was the complete potential for $M^2 \gg T^2 \gg \mu M$, one would get $\langle\phi\rangle^2 \neq 0$, as concluded in [89]

As we did in the general example of section 5.2, it is required that $M^2 \gg T^2 \gg \mu M$. This amounts, once again, to the assumption that the non-renormalizable terms are not negligible compared to the renormalizable ones. Note that this does not bring into question the validity of perturbation theory, since the ϕ^4 terms are suppressed by the small parameter μ/M. This is the analogy with the Coleman-Weinberg case that we drew before. Perturbation theory is perfectly safe, since the next term in the series would be of order ϕ^8/M^4, or $T^2\phi^6/M^4$, which are strongly suppressed by $T/M \ll 1$ or $\phi/M \ll 1$.

Following the lines of the discussion in the non-supersymmetric case, one has to go one step forward in the loop expansion. The leading contribution to the field's mass will come from the two-loop diagrams of figure 5.3. It is straightforward to calculate the boson contribution from the scalar potential (5.20), since for bosonic fields one can use the general formula (5.18). The bosonic graph in figure 5.3 is proportional to I_B^2. We can immediately guess that the fermionic counterpart has to be proportional to $I_F^2 - 2I_BI_F$, otherwise the quartic divergences would not cancel at two loops, as we know they do in a supersymmetric theory. The Yukawa Lagrangian and fermion mass terms in the non-renormalizable model are

$$\mathcal{L}_F = \mu\bar{\tilde{\phi}}\tilde{\phi} - \frac{3}{2M}(\phi_1^2 - \phi_2^2)\bar{\tilde{\phi}}\tilde{\phi} - \frac{3}{M}(\phi_1\phi_2)\bar{\tilde{\phi}}i\gamma_5\tilde{\phi} \tag{5.23}$$

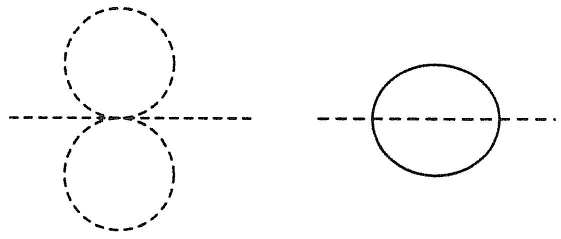


Figure 5.3: Feynman diagrams for the thermal mass correction from the non-renormalizable term in the supersymmetric example. Dashed lines represent the scalar boson ϕ, continuous lines represent its fermion counterpart $\tilde{\phi}$

from where we see that indeed 5.3 (b) is the only graph that provides a T^4 mass contribution at two loops. When the internal boson is ϕ_1, the integral coming from 5.3 (b) is

$$J_F = \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{1}{(k+p)^2-\mu^2}\mathrm{Tr}\left[\frac{1}{\not p-\mu}\frac{1}{\not k-\mu}\right] \tag{5.24}$$

while with ϕ_2 inside we get

$$J'_F = \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{1}{(k+p)^2-\mu^2}\mathrm{Tr}\left[i\gamma_5\frac{1}{\not p-\mu}i\gamma_5\frac{1}{\not k-\mu}\right] \tag{5.25}$$

so that

$$J_F + J'_F = 4\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{1}{(k+p)^2-\mu^2}\frac{2pk}{(p^2-\mu^2)(k^2-\mu^2)} \tag{5.26}$$

and taking into account that the propagator with momentum $k-p$ is the bosonic one, it is not difficult to show that the integral gives exactly the result we expected. Indeed, for $p, k \gg \mu$, we have

$$J_F + J'_F \simeq 4\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\left[\frac{1}{(k+p)^2p^2}+\frac{1}{(k+p)^2k^2}-\frac{1}{k^2p^2}\right] \simeq I_F^2 - 2I_BI_F \tag{5.27}$$

Combining the contributions to both graphs with the adequate symmetry factors we get a result proportional to $(I_B^2 - I_F^2)^2$, more precisely

$$\Delta V_{2-\mathrm{loops}}(T) = \frac{9T^4}{32M^2}|\phi|^2 = \frac{9T^4}{64M^2}(\phi_1^2+\phi_2^2) \; . \tag{5.28}$$

This term, in the range of parameters considered ($M^2 \gg T^2 \gg \mu M$), dominates over the mass term in (5.22), and therefore must be taken into account. Since it is positive, the conclusion is contrary to the one in Ref. [89]: the discrete symmetry is restored at high temperature.

5.5 Effective potential from renormalizable one

Up until now, we have used the field theory methods of [3, 4] to calculate the temperature dependent effective potential. However, these methods were developed supposing that the theory is renormalizable. One may wonder if non renormalizable interactions can spoil the consistency of the method.

There is a straightforward, if tedious, way to check the results obtained in the non-renormalizable model: resort to the original renormalizable theory. We have discussed how a non-renormalizable potential can be obtained from a normal theory, with an additional heavy field integrated out. We can take the original theory and find all the relevant Feynman diagrams, *i.e.* those that reduce to the ones in figure 5.3 when the auxiliary field is "pinched out". The graphs can be safely calculated as usual, taking the limit $M \gg T$ at the end of the calculation.

The superpotential (2.55) can be considered the effective, low energy theory resulting from integrating out a heavy field X from

$$W = \frac{\mu}{2}\phi^2 + \frac{M_X}{2}X^2 + \lambda X\Phi^2 \tag{5.29}$$

This will produce a scalar potential

$$V = |\mu\phi + 2\lambda\chi\phi|^2 - \left|M\chi + \lambda\phi^2\right| \tag{5.30}$$

which we can now write in terms of real fields, with $\phi = (\phi_1 + i\phi 2)/\sqrt{2}$ and $\chi = (\chi_1 + i\chi_2)/\sqrt{2}$

$$\begin{aligned} V &= \frac{\mu^2}{2}(\phi_1^2+\phi_2^2) + \frac{M_\chi^2}{2}(\chi_1^2+\chi_2^2) - \lambda^2(\phi_1^2+\phi_2^2)(\chi_1^2+\chi_2^2) + \frac{\lambda^2}{4}(\phi_1^2+\phi_2^2)^2 \\ &+ \frac{\lambda}{\sqrt{2}}M[\chi_1(\phi_1^2-\phi_2^2) + 2\chi_2\phi_1\phi_2] + \sqrt{2}\lambda\mu(\phi_1^2+\phi_2^2)\chi_1 \end{aligned} \tag{5.31}$$

and the fermion and Yukawa parts of the Lagrangian

$$\begin{aligned} \mathcal{L}_Y &= \mu\bar{\tilde{\phi}}\tilde{\phi} - M_\chi\bar{\tilde{\chi}}\tilde{\chi} - \sqrt{2}\lambda\bar{\tilde{\phi}}\tilde{\phi}\chi_1 - \sqrt{2}\lambda\bar{\tilde{\phi}}i\gamma_5\tilde{\phi}\chi_1 \\ &- \sqrt{2}\lambda(\bar{\tilde{\phi}}\tilde{\chi} + \bar{\tilde{\chi}}\tilde{\phi})\phi_1 - \sqrt{2}\lambda(\bar{\tilde{\phi}}i\gamma_5\tilde{\chi} + \bar{\tilde{\chi}}i\gamma_5\tilde{\phi})\phi_2 \end{aligned} \tag{5.32}$$

The two-loops graphs with two external legs that reduce to the ones in 5.3 (a) and (b) when $M \gg T$ are the ones depicted in figures 5.4 and 5.5, respectively.

Let us first calculate the ones giving the bosonic graph 5.3 (a), those of figure 5.4. Each one has two χ propagators, approximated by $1/M_\chi^2$ in this limit, and two χ propagators. Integrating over the two loops will give in each case

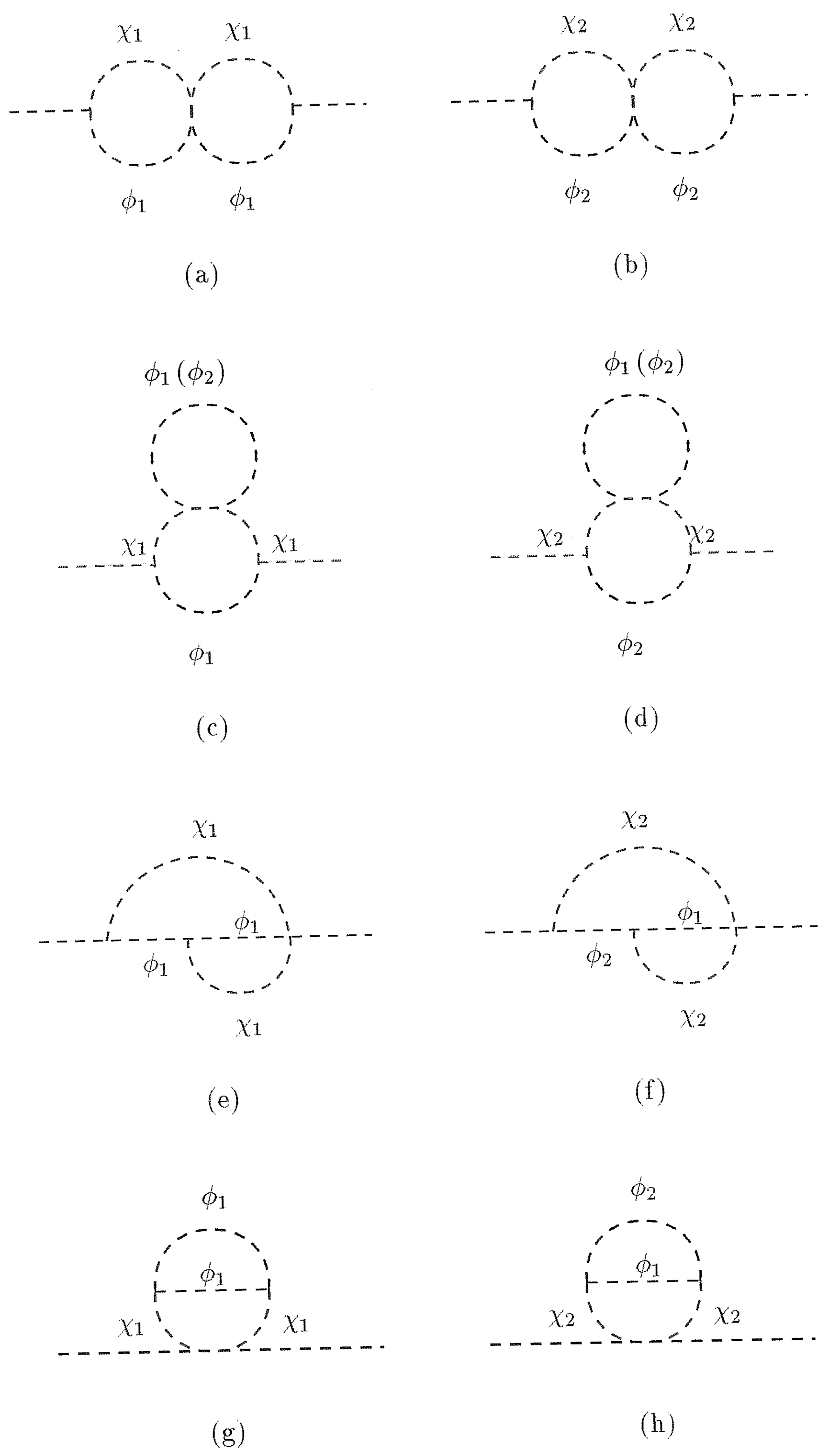


Figure 5.4: Bosonic graphs contributing to the $T^4|\phi|^2$ terms. External legs can be ϕ_1 or ϕ_2.

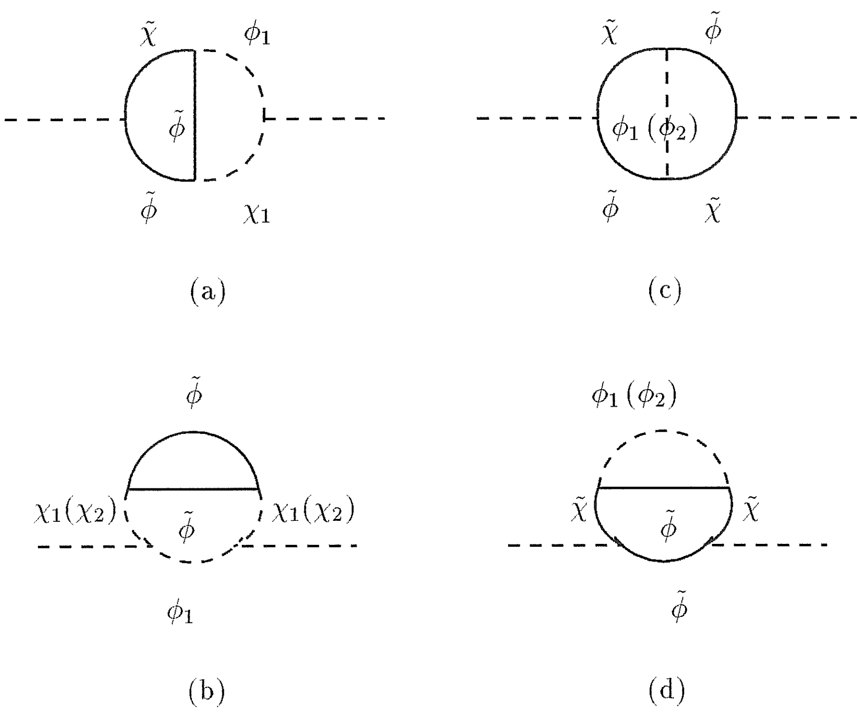


Figure 5.5: Fermionic graphs contributing to the $T^4|\phi|^2$ terms. External legs can be ϕ_1 or ϕ_2.

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\mu^2} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2-\mu^2} \frac{1}{M_\chi^2} \frac{1}{M_\chi^2} = \frac{1}{M_\chi^4}(I_B)^2 \tag{5.33}$$

In each graph we have two three-legs vertex (either $\chi_1\phi_1^2$ or $\chi_1\phi_2^2$ or $\chi_2\phi_1\phi_2$). each proportional to λM_χ; and one four-legs vertex ($\chi_1^2\phi_1^2$, $\chi_1^2\phi_2^2$, etc.) giving a λ^2 factor. All the graphs will give then a contribution proportional to

$$\frac{\lambda^4}{M_\chi^2}(I_B)^2 \tag{5.34}$$

Taking into account symmetry factors, the contributions add up to give

$$\text{all bosonic graphs} = 18\frac{1}{M^2}I_B^2 \tag{5.35}$$

where $M = M_\chi/(\lambda^2\sqrt{2})$. The fermionic graphs of 5.5 give contributions similar to their non-renormalizable counterpart. In each graph, one can have either ϕ_1 and χ_1 or ϕ_2 and χ_2 boson propagators, and the integration will give $J_F - J'_F = I_F^2 - 2I_BI_F$, as before. Graph (a) has two χ boson propagators, giving a factor $1/M_\chi^4$; two three-boson vertices giving $\lambda^2 M_\chi^2$; and two fermion-boson vertices contributing a factor of λ^2. Graph (b) has one fermion and one boson χ propagator, giving $1/M_\chi^3$; one three-boson vertex giving λ and 3 boson-fermion vertices giving λ^3. The two remaining graphs have only 2 fermionic χ propagators, contributing $1/M_\chi^2$; and 4 fermion-boson vertices contributing λ^4. We have then a total contribution proportional to $\lambda^4/M_\chi^4(I_F^2 - 2I_BI_F)$. Considering all the symmetry and numerical factors carefully,

$$\text{all fermionic graphs} = 18\frac{1}{M^2}(I_F^2 - 2I_BI_F) \tag{5.36}$$

Adding all the graphs, then

$$\Delta V(T) = 9\frac{1}{M^2}(I_F - I_B)^2(\phi_1^2 + \phi_2^2) \tag{5.37}$$

The (infinite) temperature independent contributions to I_F and I_B cancel out, while the temperature-dependent part adds up as usual. We arrive this way to

$$\Delta V(T) = \frac{9}{32}\frac{T^4}{M^2}|\phi|^2 \tag{5.38}$$

the same result obtained with calculations in the non-renormalizable model.

5.6 Summary

According to our investigations, the idea of using non-renormalizable terms to produce non-restoration cannot work in general. In particular, it seems that the theorem that forbids non-restoration in supersymmetric models holds true even if higher order, non-renormalizable interactions are included.

Admittedly, we have not given a rigorous proof of the theorem, nor being completely general in our calculations, restricted to the case of a single chiral field. For example, derivative couplings coming from higher order terms in the Khaler potential can change the picture, as suggested by Dvali.

There is however a more interesting way to evade the Mangano-Haber theorem by going away from its limits of validity. We are referring to the idea put forward recently by Riotto and Senjanović [85], based on the observation by Linde [87] that a non-vanishing chemical potential can induce a negative mass term, already mentioned in the previous chapter. In [85], the issue is studied for supersymmetric abelian models, and the conclusion is that the presence of a global charge in the Universe can prevent symmetries from restoring, even in the supersymmetric case.

6 Summary and Conclusions

In this Thesis we explored the possibility of avoiding the domain wall and the monopole problem in a number of theories. We used the fact that the cosmological phase transition associated with symmetry breakdown is not an unavoidable phenomenon; rather that its existence is a dynamical question that depends upon the parameters of the theory. We have shown that in some of the theories under study, the phase transition does not take place for a set of values of the couplings of the fields involved. We have shown how, in such theories, the thermal production of defects is naturally suppressed.

The mechanism of non-restoration works well when discrete symmetries are considered. We have studied spontaneous breakdown of CP, P and the strong CP problem in some of the best known theories in the literature. In all cases, we have restricted ourselves to the original, "minimal" version. All the theories considered share the essential characteristic of having more than one scalar field in charge of symmetry breaking, which is the first requirement for symmetry non-restoration. The second has been found to be that the field whose VEVdoes not vanish at arbitrarily high temperature has the smaller number of independent components. The third and most important requirement, is that the field that survives symmetry restoration must be a singlet under all the gauge groups of the theory.

Thus, we have found that non-restoration is possible in models of spontaneously broken CP where an $SU(2)_L$ singlet down quark is added [52, 53], since those models require just a singlet in order to break the symmetry and generate a CP phase. It is also possible to have non-restoration in Left-Right models where the scale of parity breaking is higher than that of $SU(2)_R$ breaking [54, 55]. Finally, the invisible axion version of the Peccei-Quinn model [64, 65] also allows naturally for the symmetry to be broken at high temperature. Our results cannot clearly be valid for arbitrarily large temperature, and they can surely not be trusted near the Planck scale.

When the spontaneously broken symmetry is gauged, non-restoration becomes more difficult. This is basically due to the contribution of the gauge fields to the thermal mass of the scalar fields. The gauge coupling sets a lower bound to the value of the largest of the scalar self-coupling, and this can be disastrous even when only leading order effects are considered. We have found that, when only the one-loop effective potential at finite temperature is considered, it is possible to have non-restoration in $SU(5)$ GUTs, when the field that breaks the Standard Model symmetry is in the $\underline{\mathbf{45}}$ representation, that is, in theories where realistic fermion masses are predicted. In the minimal model where the Standard Model Higgs is in the $\underline{\mathbf{5}}$ representation, the symmetry is restored for all values of the couplings allowed by perturbation theory. Once again, the result is that the field that keeps the VEVat high temperature must be in the smaller representation. We have also given an example in one of the symmetry breaking channels of $SO(10)$ GUTs where the symmetry is not restored, avoiding the phase transition that could give rise to magnetic monopoles and allowing for the one that would produce cosmic strings, which are not only not dangerous but welcomed by cosmologists.

Unfortunately, it is not enough to take into account the one-loop effects, since due to the lower bound introduced by the gauge coupling, the scalar self-couplings are dangerously close to one. When considering the next-to-leading order effects, it is seen that non-restoration would require going beyond perturbation theory in all the gauge symmetries discussed. In particular, since the expansion of the thermal mass for the scalar field is not in the couplings but in a product of the couplings times the number of fields, having large representations for the Higgs fields works against non-restoration.

The calculation of the thermal masses near the non-perturbative regime is however a delicate task. Work is in progress by a number of authors, in order to check the conclusions reached above on the importance of the next-to-leading order effects. It is also important to remark that the higher order calculations have been carried out in simplified $O(N_1) \times O(N_2)$ models only, not in realistic $SU(5)$ or $SO(10)$ models. Clearly more study is necessary, but for the moment it seems that one cannot have non-restoration in gauged theories.

We have also explored the possibility, suggested in [89], of using non-renormalizable terms to have symmetry non-restoration with just one field. We have shown how this is not possible in general, by carefully calculating the effective potential at high temperature in this theories up to two-loops. It was found that when non-renormalizable terms are important enough to ply a role in symmetry breaking, perturbation theory does not start at one loop, and higher-loop effects can be more important. This is a process analogous to the well-known case of the symmetry breaking with a Coleman-Weinberg potential, where one-loop effects fundamentally affect the tree-level ones.

In particular, we have proved that using higher-order non-renormalizable terms cannot help in avoiding symmetry restoration in supersymmetric theories. In this sense, we have extended the theorem by Haber and Mangano [43, 44], on the impossibility of having non-restoration in supersymmetric theories, to the case in which non-renormalizable terms are included.

Having proved how in some instances the phase transition that produces topological defects can be avoided, we have investigated the rate of thermal production of such defects. It was found, in the case of discrete symmetries, that the thermal production of domain walls can be kept at a low enough rate, in a region of parameter space perfectly compatible with the requirements for non-restoration. This is due to the fact that when the symmetry does not get restored, the VEVof the field becomes directly proportional to the temperature. The exponential suppression factor for the thermal production of domain walls (or strings, in the axion case) is then a constant, becoming easy to control.

Although non-restoration in gauge theories is still under discussion, we have also calculated the thermal production of monopoles in the version of $SU(5)$ where restoration can be avoided up to leading-order effects. Also in this case, due to the temperature dependence of the VEV, thermal production can be kept at low values, even allowing for an observationally interesting monopole abundance.

In the discrete symmetry case, therefore, the domain wall problem can be solved in some very interesting examples. All that is required is that the field that breaks the symmetry has homogeneous values in the vacuum to start with, which is equivalent to solve the horizon problem. The solution of the horizon problem however is not related to the critical temperature for symmetry breaking, and therefore a single, primordial epoch of inflation could suffice. The gauge case, as we said, requires more investigation. This could be one of the future directions of the work presented in this thesis. Another interesting possibility is one we already mentioned, of including a chemical potential. In any case, we believe that the possibility of using this mechanism to avoid the formation of the "dangerous" topological defects has been demonstrated to exist and to be worthy of further investigation.

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Appendix: Effective potential at high temperature

Finite temperature field theory

We consider a scalar field with a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{A.1}$$

At finite temperature, on defines the finite-temperature 2-point function

$$D_T(x-y) = \frac{Tre^{-H/T}P\phi(x)\phi(y)}{Z} \tag{A.2}$$

where P stands for time-ordering. For noninteracting bosonic fields it satisfies

$$(\Box + m^2)D_T(x-y) \equiv -i\delta^(k)(x-y) \tag{A.3}$$

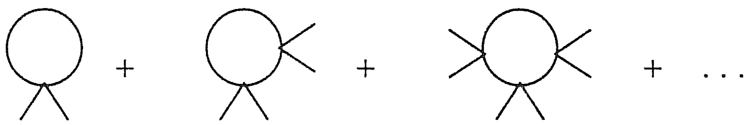
The boundary conditions are found by analytically continuing t to the imaginary interval $0 \le ix_0, iy_0 \le 1/T$ and the "time" ordering is defined as

$$\begin{aligned} \langle P\phi(x)\phi(y)\rangle &= \langle\phi(x)\phi(y)\rangle \quad ix_0 > iy_0 \\ &= \langle\phi(y)\phi(x)\rangle \quad iy_0 > ix_0 \end{aligned} \tag{A.4}$$

This implies periodic boundary conditions

$$D_T(x-y)|_{x_0=0} = D_T(x-y)|_{x_0=-i/T} \tag{A.5}$$

The usual formulas



$$D_T(x-y) = \int \frac{d^4p}{(2\pi^4)} e^{ip(x-y)} D_T(p) \tag{A.6}$$

$$D_T(p) = \int d^4x e^{ipx} D_T(x) \tag{A.7}$$

go through, but due to the periodicity now

$$\int \frac{d^4p}{(2\pi^4)} = iT \sum_{n=0}^{\infty} \frac{d^3p}{(2\pi^3)} \tag{A.8}$$

$$\int d^4x = \int_0^{-i/T} dx_0 \int d^3x \tag{A.9}$$

and now p_0 is quantized

$$p_0 = i2\pi nT \tag{A.10}$$

Thus

$$D_T(p) = \frac{i}{p^2 - m^2} = \frac{-i}{4\pi^2 n^2 T^2 + \vec{p}^2 + m^2} \tag{A.11}$$

For fermions the corresponding expression is

$$S_T(x-y) = \frac{Tre^{-H/T} P\psi(x)\bar{\psi}(y)}{Z} \tag{A.12}$$

but now the boundary conditions are anti-periodic

$$S_T(x-y)|_{x_0=0} = -S_T(x-y)|_{x_0=-i/T} \tag{A.13}$$

Effective potential at one loop

At one loop, for the bosonic contribution one needs to calculate the graphs at finite temperature. We will calculate only the first one.

$$-i\phi^2 \frac{\lambda}{4}(12) \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + M^2} \equiv -3i\lambda\phi^2 I_B(T) \tag{A.14}$$

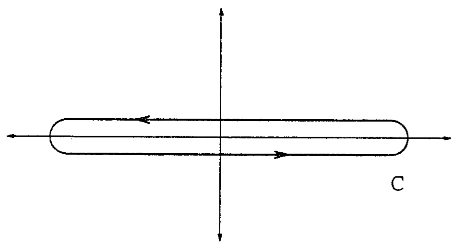


Figure A.1: Contour C

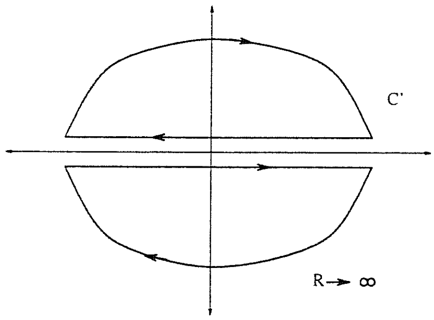


Figure A.2: Contour C'

which in at finite temperature becomes

$$- i\phi^2\frac{\lambda}{4}(12)iT\sum_{n=0}^{\infty}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{-i}{4\pi^2n^2T^2+\vec{p}^2+m^2} \tag{A.15}$$

$I_B(T)$ can be written as

$$I_B(T)=\int\frac{d^3\vec{p}}{(2\pi)^3}\oint_C dz\frac{\cot(z/2T)}{z^2+\vec{p}^2+m^2} \tag{A.16}$$

where the contour is given by figure A.1. Obviously the poles of $\cot(z/2T)$ are for $\sin(z/2T) = 0$, or $z/2T = \pi n$, $n = 0, 1, ...$, which in turn by the theorem of residues produces (A.15).

Now, by changing the contour C into C' (Fig. A.2)
one picks up poles at $z = \pm i\sqrt{\vec{p}^2+m^2} = \pm i\omega$, and we get

$$\begin{aligned} I_B(T) &= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\cot(\omega/2T)}{2\omega} \\ &= I_B(0)+\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\omega(e^{\omega/T}-1)} \end{aligned} \tag{A.17}$$

where $I_B(0)\equiv\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega}$ is the usual (infinite) correction at $T = 0$ which we did not bother to keep before. We get finally (for $T \gg m$)

$$I_B(T) - I_B(0) = T^2 \frac{4\pi}{(2\pi)^3} \int_0^\infty x dx \frac{1}{e^x - 1} = \frac{T^2}{12} \tag{A.18}$$

Thus the diagram of figure 1.1(a) gives from (A.15)

$$-i\frac{3\lambda}{12} T^2 \phi^2 \tag{A.19}$$

which produces the finite temperature correction for the potential

$$\frac{3\lambda}{24} T^2 \phi^2 = \frac{\lambda}{8} T^2 \phi^2 \tag{A.20}$$

In the same manner, one can compute the complete one-loop effective potential

$$V_{1B} = \frac{-i}{2} \int \frac{d^4k}{(2\pi)^4} \ln(iD^{-1}) \tag{A.21}$$

which at $T \neq 0$ becomes

$$V_{1B} = \frac{T}{2} \sum_{n=0}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln(4\pi^2 n^2 T^2 + \vec{k}^2 + m^2) \tag{A.22}$$

which gives

$$V_1(T) = \frac{T^4}{2\pi^2} \int_0^\infty dx x^2 \ln(1 + e^{-\sqrt{x^2 + M^2/T^2}}) \tag{A.23}$$

and in the high T limit, substituting

$$m^2 = \frac{\partial^2 V}{\partial \phi^2} \tag{A.24}$$

one gets

$$V(T) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{T^2}{24} 3\lambda\phi^2 \tag{A.25}$$

For the fermionic graph in figure 1.1(b), we can define

$$I_F \equiv \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + M^2} \tag{A.26}$$

and now fermionic boundary condition must be imposed. The calculation follows the lines of the bosonic one, but now because of the different statistics one gets

$$I_F(T) - I_F(0) = T^2 \frac{4\pi}{(2\pi)^3} \int_0^\infty x dx \frac{1}{e^x + 1} = -\frac{T^2}{24} \tag{A.27}$$

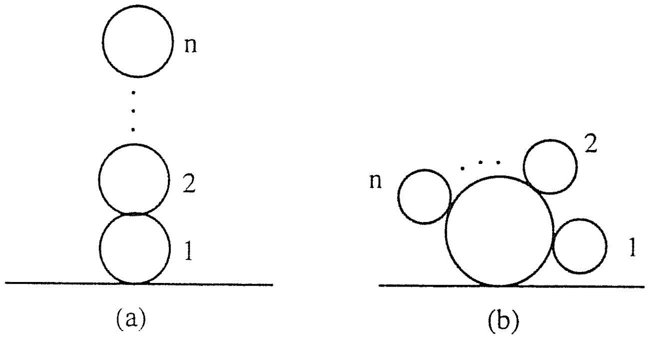


Figure A.3: Feynman graphs contributing to the n-th loop order terms of the high temperature effective potential.

Next-to-Leading Order Corrections

One can get a next-order result by substituting m by m_T in the high temperature expansion (A.25)

$$\begin{aligned} V_{\text{eff}} &= \frac{1}{2}\left[m^2+(N+2)\lambda\frac{T^2}{12}-(N+2)\lambda\left(m^2+(N+2)\lambda\frac{T^2}{12}-(N+2)\lambda\frac{mT}{4\pi}\right)^{1/2}\frac{T}{4\pi}\right] \\ &+ \frac{\lambda}{4}(\varphi_a\varphi^a)^2 \\ &\simeq \frac{1}{2}[m^2+(N+2)\lambda\frac{T^2}{12}-(N+2)\lambda\frac{T^2}{(4\pi)^2}\sqrt{\frac{N\lambda}{12}}+\frac{\lambda}{4}(\varphi_a\varphi^a)^2+O(\lambda^2) \end{aligned} \tag{A.28}$$

This iteration procedure is in fact justified by considering the N-loop graphs of figure A.3. The graphs of the kind of figure A.3 a have a contribution of order T^2 for the top bubble, plus for each bubble with to vertices an integral

$$\int\left(\frac{idk}{k^2-m^2}\right)^2 \propto \frac{\partial}{\partial m^2}\left(\int\frac{idk}{k^2-m^2}\right) \propto \frac{\partial}{\partial m^2}(Tm) \propto \frac{T}{m} \tag{A.29}$$

The ones of A.3b, on the other hand contribute a factor of T^2 for each small bubble, plus an integral for the big one of the form

$$\int\left(\frac{idk}{k^2-m^2}\right)^n \propto \frac{\partial^{(n-1)}}{\partial m^2}\left(\int\frac{idk}{k^2-m^2}\right) \propto \frac{\partial^{(n-1)}}{\partial m^2}(Tm) \propto \frac{T}{m^n} \tag{A.30}$$

so we will have in the end

$$\text{a graphs} : \frac{T^{n+1}}{m^{n-1}} \quad \text{b graphs} : \frac{T^{2n-1}}{m^n} \tag{A.31}$$

which tells us that for each term in the loop expansion, the second type of graphs will be bigger by a factor of T^{n-2}/m, and we only need to consider these, so called "daisy" diagrams. Carefully taking into account the symmetry factor, it is obtained for the daisy diagram with $n-1$ small bubbles

$$V^{(n)} = [-i(N+2)\lambda]^n \left(\frac{T^2}{12}\right)^{n-1} \frac{1}{(n-1)!} \left(\frac{\partial}{\partial m^2}\right)^{n-1} \left[\frac{T^2}{12} - \frac{mT}{4\pi}\right] \tag{A.32}$$

And the total contribution, *to all orders*, is obtained by summing over all the diagrams, from $n=1$ to ∞. This is nothing but a Taylor expansion

$$f(x+a) = \sum_0 \frac{x^n}{n!} \left(\frac{\partial}{\partial a}\right)^n f(a) \tag{A.33}$$

where for us a is m^2, x is the dominant temperature contribution to the mass at one loop, $(N+2)\lambda T^2/12$, and f is the thermal mass m_T I We have derived the so-called "gap" equation for one field

$$m_T^2 = m^2 + (N+2)\lambda \left[\frac{T^2}{12} - m_T \frac{T}{4\pi}\right] \tag{A.34}$$

whose solution gives the improved thermal mass in equation (A.28).

The generalization to the two-field case is straightforward. For a potential with fields φ_1, φ_2 transforming as vectors under $O(N_1)$, $O(N_2)$ respectively, the gap equations will get contribution to the thermal masses from similar graphs, involving bubbles of both fields.