ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Homoclinic and heteroclinic orbits for some classes of second order Hamiltonian systems

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE
Paolo Caldiroli

SUPERVISOR
Prof. Vittorio Coti Zelati

November 1995

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

TRIESTE

Scuola Internazionale Superiore di Studi Avanzati
International School for Advanced Studies

Homoclinic and heteroclinic orbits for some classes of second order Hamiltonian systems

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE
Paolo Caldiroli

SUPERVISOR
Prof. Vittorio Coti Zelati

November 1995

Contents

Acknowledgments

To Prof. Vittorio Coti Zelati, who introduced me to the subject developed in this thesis, guiding me with useful advice.

To Piero Montecchiari, Stefano Luzzatto, Simonetta Abenda, Marta Nolasco, Colette De Coster, Giulia Treu and Louis Jeanjean, for the stimulating conversations and the fruitful collaboration.

To the professors, students, and to all the staff of SISSA, for providing a supportive and warm environment in which to work.

To all the friends I met during these years at SISSA, for the nice time spent together.

To my family, to which this thesis is dedicated.

Introduction

1. Homoclinics, heteroclinics and dynamical systems

Let us consider a Hamiltonian system in $\mathbb{R}^{2N}$ given by

$$\text{(H)} \qquad \dot{z} = J\nabla_z H(t,z) \qquad t \in \mathbb{R}$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the symplectic matrix, $H \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is the Hamiltonian function, periodic in time with period 1, that is $H(t+1,z) = H(t,z)$ for any $(t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$.

Let $\bar{z}(t)$ be a periodic orbit to (H), with period $T \in \mathbb{N}$. Denoted by $M(t)$ the fundamental matrix solution of the linearized system, defined by

$$\begin{cases} \frac{d}{dt}M = JH''(t,\bar{z}(t))M \\ M(0) = I, \end{cases}$$

we say that the periodic orbit $\bar{z}(t)$ is hyperbolic if $M(T)$ has no eigenvalue on the unit circle.

A solution $z(t)$ to the Hamiltonian system (H) is called homoclinic orbit to $\bar{z}(t)$ if $z(t) \neq \bar{z}(t)$ and $|z(t) - \bar{z}(t)| \to 0$ as $t \to \pm\infty$.

Given two distinct periodic orbits $z_-(t)$ and $z_+(t)$, a solution $z(t)$ to the system (H) such that $|z(t) - z_-(t)| \to 0$ as $t \to -\infty$ and $|z(t) - z_+(t)| \to 0$ as $t \to +\infty$ is called heteroclinic orbit between $z_-(t)$ and $z_+(t)$.

Let us consider the special case in which the system (H) admits an equilibrium at 0, i.e., $\nabla_z H(t,0) = 0$ for every $t \in \mathbb{R}$. Denoting by ϕ^t the flow of the system (H), defined by

$$\begin{cases} \frac{\partial}{\partial t}\phi^t(z) = J\nabla_z H(t,\phi^t(z)) \\ \phi^0(z) = z \end{cases}$$

we set

$$W^s(0) = \{z \in \mathbb{R}^{2N} : \phi^t(z) \to 0 \text{ as } t \to +\infty\}$$
$$W^u(0) = \{z \in \mathbb{R}^{2N} : \phi^t(z) \to 0 \text{ as } t \to -\infty\}.$$

We remark that a solution $z(t)$ to (H) is a homoclinic orbit to 0 if and only if $z(0) \in W^s(0) \cap W^u(0) \setminus \{0\}$.

If 0 is a hyperbolic equilibrium, that is $M(1)$ has no eigenvalue on the unit circle, then $W^s(0)$ and $W^u(0)$ are two immersed manifolds of dimension N, called respectively stable and unstable manifolds for ϕ at 0.

When 0 is a hyperbolic equilibrium and there is a homoclinic orbit $z(t)$ to 0 such that the intersection of $W^s(0)$ and $W^u(0)$ at $z(0)$ is transversal, i.e., $T_{z(0)}W^s(0) \oplus T_{z(0)}W^s(0) = \mathbb{R}^{2N}$, we say that the homoclinic orbit $z(t)$ is transversal and that $z(0)$ is a transversal homoclinic point to 0 for the time–one map ϕ^1 associated to (H). Similar definitions can be given in the case of heteroclinic orbit between two distinct equilibria.

For an autonomous system $\dot{z} = J\nabla H(z)$, 0 is a hyperbolic equilibrium if and only if $\nabla H(0) = 0$ and $JH''(0)$ has no eigenvalue on the imaginary axis. Moreover, if $z(t)$ is a homoclinic orbit to 0, it is not tranversal. Indeed the Hamiltonian H does not depend on time and then $\dot{z}(0) \in T_{z(0)}W^s(0) \cap T_{z(0)}W^s(0)$ whenever $z(0) \in W^s(0) \cap W^u(0)$. In particular, in the two-dimensional case, this implies that $W^s(0) = W^u(0)$.

The existence of a transversal homoclinic orbit to a hyperbolic equilibrium gives rise to a very complicated dynamics. In fact, assuming that 0 is a hyperbolic equilibrium for (H), if there is one point $z_0 \in W^s(0) \cap W^u(0) \setminus \{0\}$, then, since $\phi^n(z_0) \to 0$ as $n \to \pm\infty$ also $\phi^n(z_0)$ are homoclinic points.

Moreover, since the map ϕ^1 preserves orientation, the two homoclinic points z_0 and $\phi^1(z_0)$ must be separated by at least one further point $\bar{z}_0 \in W^s(0) \cap W^u(0) \setminus \{0\}$ that does not belong to the orbit of z_0.

In addition, assuming transversal intersection between $W^s(0)$ and $W^u(0)$, the fact that the time–one map ϕ^1 is area preserving produces a violent winding of the stable and unstable manifolds $W^s(0)$ and $W^u(0)$ in a neighborhood of 0. This leads to a sensitive dependence of orbits $\phi^t(z)$ on the initial condition z for z belonging to a Cantor–like set near 0 and $z(0)$. Hence the presence of homoclinic transversal orbits is responsible for a chaotic behavior for the system (H).

This important fact was first noticed by Poincaré [P] in his studies of the restricted three body problem. Then Birkhoff, Smale and others continued Poincaré's studies giving a more precise description of the chaotic behavior near a transversal intersection and relating the dynamics of a map with a transversal homoclinic point to the dynamics of the Bernoulli shift (for a review of these results see [GH], [KS], [Mo], [W]).

The problem of detecting transversal homoclinic and heteroclinc points can be tackled using perturbation techniques. The main idea behind this approach is developped in a seminal work by Melnikov [Me]. As a first step, one considers a two–dimensional autonomous Hamiltonian system, as for instance the unperturbed Duffing equation $\ddot{q} = q - aq^3$, having a hyperbolic equilibrium connected to itself by a homoclinic orbit. Such a system turns out to be completely integrable. Then, adding a time periodic perturbation, the hyperbolic equilibrium becomes a hyperbolic periodic orbit whose stable and unstable manifolds split. A computable formula for the distance between these manifolds can be found and used to show the existence of a transversal homoclinic point for the Poincaré map associated to the equation.

This result relies heavily on the two–dimensional geometry of the phase portrait of the unperturbed system. However, using a more functional analytic approach based on the notion of exponential dichotomy [Co], Melnikov type techniques have been developed for time periodic perturbations of N–dimensional autonomous systems having a hyperbolic equilibrium and a corresponding homoclinic orbit (see [Pa]).

2. The variational approach

In recent years, starting with [Bo], [CZES], [R], [BG], [HW], variational methods have been successfully applied to study the existence and the multiplicity of homoclinic solutions to Hamiltonian systems having a hyperbolic equilibrium.

The variational approach has the advantage of providing global conditions on the Hamiltonian for the existence of homoclinic orbits and their multiplicity. It also gives criteria to detect a chaotic behavior, which are different from the transversality condition, and actually more general.

A first existence result of homoclinic solutions of (H) was established by Coti Zelati, Ekeland and Séré [CZES], for a Hamiltonian

$$H(t,z) = \tfrac{1}{2} z \cdot Az + R(t,z)\,, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$$

being A a symmetric constant matrix such that JA is hyperbolic (i.e., has no imaginary eigenvalue) and $R(t,z) = o(|z|^2)$ as $z \to 0$, uniformly in t, so that 0 is a hyperbolic equilibrium for (H). Assuming R positive, convex with respect to z, 1–periodic in t and globally superquadratic (i.e. satisfying $\nabla_z R(t,z) \cdot z \ge \alpha R(t,z)$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$, with $\alpha > 2$), in [CZES] a homoclinic solution is obtained as critical point of the dual action functional, using the mountain–pass lemma [AR]. The lack of compactness due to the invariance of the functional under the non compact translation group $\mathbb{Z}$ is overcome by the concentration–compactness principle by P.L. Lions [L].

Then Hofer and Wysocki [HW] dropped the convexity assumption and found a homoclinic orbit applying an infinite dimensional linking theorem (firsty stated by Benci and Rabinowitz to deal with periodic problems [BR]) to the direct action functional defined on $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$.

The same result was achieved by Tanaka [T] who obtained a homoclinic orbit as limit in the C^1_{loc} topology of n–periodic solutions of (H) as $n \to \infty$. This method was introduced by Rabinowitz [R] to study a second order Hamiltonian system in $\mathbb{R}^N$

$$\text{(HS)} \qquad \ddot{q} = L(t)q - \nabla_q V(t,q)$$

being $L(t)$ a positive definite symmetric matrix, continuous and 1–periodic with respect to t, and V positive, globally superquadratic in q and 1–periodic in t.

A similar approach, already introduced in [DN], and consisting in solving a sequence of Dirichelet problems on $[-n,n]$ and passing to the limit $n \to \infty$, has been subsequently used in [GJT] to find a homoclinic solution for second order Hamiltonian systems on a non–compact Riemannian manifold with a potential having a non degenerate local maximum and satisfying a weaker condition than the global superquadraticity.

In the autonomous case the existence of a homoclinic orbit can be proved under less restrictive assumptions on the Hamiltonian function.

In the framework of conservative systems the first work using variational methods to detect this kind of solutions is due to Bolotin [Bo]. In particular he considered a second order system constrained on a compact non contractible Riemannian manifold and, solving suitable approximating minimizing problems, with a limit process, he found a solution homoclinic to a point which is assumed to be the unique maximum point for the potential.

Independently, Benci and Giannoni [BG] studied the same problem and obtained a homoclinic, applying a Maupertuis variational principle and using Lyusternik Schnirelman category theory.

Then, Ambrosetti and Bertotti [AB] and Rabinowitz and Tanaka [RT], with different techniques, proved the existence of a homoclinic solution for a second order Hamiltonian system $\ddot{q} = -\nabla U(q)$ in $\mathbb{R}^N$ without the superquadraticity condition, but assuming that the component Ω of $\{q \in \mathbb{R}^N : U(q) < 0\} \cup \{0\}$ containing 0 is bounded and $\nabla U(q) \neq 0$ for any $q \in \partial\Omega$. Precisely in [RT] the solution is found as a minimum of the Lagrangian functional in the class of functions connecting 0 to $\partial\Omega$, while in [AB] the homoclinic is obtained as a limit of solutions of approximating Dirichelet problems. For a further different approach, using a refined version of the mountain–pass Lemma due to Ghoussoub and Preiss [GP], see [C2].

Some extensions to the case of unbounded domain Ω have been subsequently developed in [C1] and then in [J], where it is also allowed ∇U to be 0 on $\partial\Omega$, with a suitable behavior of $U(q)$ as $\operatorname{dist}(q, \partial\Omega) \to 0$.

The argument followed in [AB] was later adapted to study a similar autonomous system on a compact Riemannian manifold [B].

The existence of homoclinics for first order systems $\dot{z} = J\,\nabla H(z)$ in $\mathbb{R}^{2N}$ was proved by Séré [S3], supposing that the zero–energy surface $\{z \in \mathbb{R}^{2N} : H(z) = 0,\ z \neq 0\}$ is compact and of restricted contact type.

Concerning the question of multiplicity of homoclinic solutions, we firstly mention again [CZES], where, in addition to the homoclinic orbit $z_1(t)$ obtained as mountain–pass critical point, introducing another minimax class, the authors found a second homoclinic $z_2(t)$ which is geometrically distinct from $z_1(t)$, in the sense that $z_2(t) \neq z_1(t+n)$ for every $n \in \mathbb{Z}$.

Actually the pioneering works about multiplicity of homoclinics for periodic Hamiltonian systems are to ascribe to Séré, [S1] and [S2].

In [S1], under the same assumptions of [CZES], infinitely many geometrically distinct homoclinic solutions to (H) were found, without any transversality–like hypothesis. This paper inspired a later work [CZR1] by Coti Zelati and Rabinowitz, who proved an analogous result for second order Hamiltonian systems (HS) possessing superquadratic potentials, as in [R].

In [S2] a more precise description of the set of homoclinic solutions to (H) was given. Precisely it was shown that if this set is countable then there is a primary homoclinic orbit $z(t)$ with the following property: for every $\epsilon > 0$ there is $M(\epsilon) \in \mathbb{N}$ such that for any finite sequence of integers $\overline{p} = (p_1, \dots, p_k)$ with $k \geq 2$ and $p_{j+1} - p_j \geq M(\epsilon)$ $(j = 1, \dots, k-1)$ there is a homoclinic orbit $z_{\bar{p}}$ with

$$\left|z_{\bar{p}}(t) - \sum_{j=1}^{k} z(t - p_j)\right| \leq \epsilon \quad \text{for any } t \in \mathbb{R}.$$

These homoclinic orbits $z_{\bar{p}}(t)$ are called k–bump solutions because they follow k times suitable translations of the primary homoclinic $z(t)$. The value $M(\epsilon)$ represents the minimal distance at which two consecutive bumps can be arranged.

Any k–bump solution $z_{\bar{p}}(t)$ corresponds to a critical point of the dual action functional, belonging to an ϵ–neighborhood $B_{\epsilon,\bar{p}}$ of $\sum_{j=1}^{k} u(\cdot - p_j)$ in the variational space, being u the mountain–pass critical point corresponding to the primary solution $z(t)$. A homological argument reduces the topology of the functional in $B_{\epsilon,\bar{p}}$ to k one–dimensional local minimax classes, for any $k \geq 2$, each of them corresponding to the mountain–pass topology near u. For this reason, the value $M(\epsilon)$ does not depend on k. Then one can consider the limit $k \to \infty$ and get solutions to (H) with infinitely many bumps, which clearly are not homoclinics any more.

As a consequence, the system (H) exhibits sensitive dependence on inital conditions and the dynamics of the time–one map ϕ^1 associated to (H) is chaotic, in the precise sense that it has positive topological entropy. This does not occur in the autonomous case, where indeed the assumption of countability of the critical set is never verified, because of the invariance under the translation group $\mathbb{R}$.

Hence, in [S2] the variational condition regarding the cardinality of the set of homoclinics plays essentially the same role of the assumption of transversal intersection between the stable and unstable manifolds $W^s(0)$ and $W^u(0)$, but it is weaker than the transversality one. (About the problem of weakening the tranversality condition we also mention a work by Bessi [Be1], concerning a second order Hamiltonian system of the type $\ddot{q} = -D_q(V_1(q) + \epsilon V_2(t,q))$ with $q \in S$, being S the unit circle in $\mathbb{R}^2$.)

We remark that already in [CZR1] k–bump homoclinic solutions were detected for second order systems like (HS), but the limit $k \to \infty$ could not be taken, because the minimax argument used in [CZR1] to get k–bump solutions depends on k and then the minimal distance between two consecutive bumps also depends on k and becomes larger and larger as $k \to \infty$.

Multiplicity theorems in the same spirit of [S1] and [CZR1] has been later obtained in different situations: by Giannoni and Rabinowitz [GR] for a class of second order Hamiltonian systems on a Riemannian manifold (possibly non compact) with a time–periodic potential having a global maximum; by Cielibak and Séré [CS1] for a class of first order time–periodic Hamiltonian systems on a compact Riemannian manifold, having a hyperbolic equilibrium.

The results given in [S2] about multibump solutions and chaotic dynamics for first order systems were extended in [CM] to the case of time periodic second order Hamiltonian systems in $\mathbb{R}^N$ of the form (HS) where the superquadratic term $V(t,q)$ is allowed to change sign. Here the construction of multibump solutions is obtained as in [S2] starting from a mountain–pass critical point of the Lagrangian functional associated to (HS), under the assumption that the critical set is countable.

We also mention a recent paper [CS2] which continues and completes the results of [CS1], getting multibump solutions for first order systems on a compact Riemannian manifold.

In [M1] Montecchiari studied a second order system in $\mathbb{R}^N$ like (HS), globally superqadratic in q and with an asymptotically periodic time dependence, and proved the existence of multibump solutions for it, assuming that the periodic system at infinity admits a finite set (up to translations) of homoclinic orbits.

Then, the same result was obtained under weaker conditions in [ACM], where the system (HS) is merely assumed to have a hyperbolic equilibrium at the origin, uniformly in t, and it is asymptotic as $t \to \pm\infty$ to two possibly distinct time periodic systems $\ddot{q} = L_{\pm}(t)q - \nabla_q V_{\pm}(t,q)$. If the systems at infinity admit a countable set of homoclinic solutions, then also (HS) has infinitely many homoclinic orbits, and actually multibump solutions, which are generated, as in [S2], by the primary homoclinics $q_{\pm}(t)$ of the periodic systems $\ddot{q} = L_{\pm}(t)q - \nabla_q V_{\pm}(t,q)$.

In a subsequent paper [CMN], for the same system considered in [ACM], always assuming countability conditions for the periodic systems at infinity, the authors showed that there is an uncountable set of solutions to (HS) which are asymptotic to $q_{\pm}(t)$ as $t \to \pm\infty$.

These results [CM], [ACM] and [CMN] are the content of the second chapter of this thesis and will be presented in a more detailed form in the next section.

Second order Hamiltonian systems in $\mathbb{R}^N$ with other time dependence, more general than the periodic one, were studied, too. In particular we mention [MN] where a perturbation with arbitrary time dependence of a time periodic potential was treated. In [STT] and [CZMN] almost periodically second order Hamiltonian systems possessing superquadratic potentials were considered. Concerning other Hamiltonian systems almost periodic in time, see also [BB1], [BB2], [R4] and [R5].

We point out that the construction of multibump solutions was done by Bessi in [Be2] and [Be3] for a class of second order damped systems in $\mathbb{R}^N$ of the form $\ddot{q} + \epsilon\dot{q} = q - \nabla_q V(t,q)$ with $\epsilon > 0$ small enough.

All the above mentioned multiplicity results rely on the key assumption that the set of homoclinic solutions is countable. Since this condition fails when the Hamiltonian is independent of time, in this case the multibump construction is not possible any more and, to get multiple homoclinic orbits, a different approach strictly related to the shape of the Hamiltonian has to be followed.

The multiplicity problem for conservative systems was firstly studied by Ambrosetti and Coti Zelati in [ACZ], and by Tanaka in [T2]. In both these works two geometrically distinct homoclinics (i.e., that cannot be obtained one from the other by time translation or reflection) were found for a second order Hamiltonian system in $\mathbb{R}^N$ when the potential is a perturbation of a radially symmetric function.

Other multiplicity results for the autonomous case concern a class of second order Hamiltonian systems in $\mathbb{R}^N$ ruled by a potential V with an absolute maximum point at 0 and a singularity at some point $\xi \neq 0$, i.e., $V(q) \to -\infty$ as $q \to \xi$.

Assuming the strong–force condition at the singularity and other conditions about the behavior of V near the origin and at infinity, Tanaka [T1] proved the existence of a first homoclinic solution to 0, through a minimax argument exploiting the non trivial topology of the sublevel sets of the Lagrangian functional, due to the singularity. Then, assuming a pinching condition on the potential V, Bessi [Be] was able to find $N-1$ distinct homoclinics.

In the case $N = 2$, Rabinowitz [R1] proved that a first homoclinic solution can be characterized as minimizer of the Lagrangian functional with respect to the curves starting and ending at the origin and winding once around the singularity. Then, supposing an additional condition about the ratio between the cost to wind the singularity passing or not through the origin, he showed the existence of a second homoclinic with a winding number sufficiently large.

This work inspired a study of these planar singular systems, with the aim of finding homoclinics with an arbitrary winding number. In fact, in [CN] the authors proved that when the potential satisfies further simmetry conditions then the above fact holds true and infinitely many geometrically distinct homoclinics are found.

A deeper analysis of this situation was subsequently developed in [CJ]. In particular it was proved that the condition introduced by Rabinowitz in [R1] to get a second homoclinic is actually sufficient to guarantee the existence of infinitely many homoclinic orbits with any winding number sufficiently large. This result, as well as [CN], will be proved in the first chapter of this thesis.

We notice that in a recent work by Buffoni and Séré [BS] multibump solutions were detected also for an autonomous four–dimensional Hamiltonian system having a saddle–focus equilibrium (see also [Bu]).

Other multiplicity results were obtained for autonomous systems on manifolds having a large enough fundamental group π_1 ([CS1] and [BJ]).

We conclude this section giving some references about the study of heteroclinic orbits via variational methods.

The first results are due to Rabinowitz, [R2] and [R3], who found heteroclinic solutions between absolute isolated maxima of an autonomous smooth potential, periodic in the space variable, using a minimization procedure.

This result was generalized by Felmer [F] for first order spatially periodic Hamiltonian systems. In a first step, approximated solutions joining two different equilibria in a finite time T were obtained using the Rabinowitz saddle point theorem. Then, getting suitable estimates independent of T, a limit process as $T \to \infty$ can be carried out to get a heteroclinic solution.

In a later work [R6] Rabinowitz studied a class of second order Hamiltonian systems in $\mathbb{R}^N$ with a smooth potential of the type $V(t,q) = W(t,q) + f(t) \cdot q$ where W is periodic both in t and in q, f is periodic with the same time-period T of W and satisfies $\int_0^T f = 0$ and V is time reversible, i.e., $V(t,q) = V(-t,q)$. Firstly, T-periodic solutions are determinated as global minima of the Lagrangian functional associated to the periodic problem. Then, assuming that these minima are isolated, connecting orbits between two of them were found. See also [R7] for complementary results.

In this thesis we will prove the existence of a heteroclinic orbit between an unstable equilibrium point and a periodic solution of a planar singular system like that one considered in [R2].

Even if this result will be discussed in a more detailed way in the next section, we just point out that, differently from [R6] and [R7], in our setting we do not make any requirement of isolateness about the periodic orbit.

3. About this thesis

This thesis deals with existence and multiplicity of homoclinic and heteroclinic orbits for some classes of second order Hamiltonian systems in $\mathbb{R}^N$

having an unstable equilibrium at the origin.

We consider both time dependent potentials and autonomous potentials. As explained in the previous section, different techniques are developed to study these two cases. Therefore we will present our results in two distinguished subsections.

3A. Conservative singular second order Hamiltonian systems in $\mathbb{R}^2$

We consider an autonomous second order Hamiltonian system in $\mathbb{R}^2$ of the type

$$\ddot{q} + \nabla V(q) = 0 \tag{3.1}$$

where the potential V has a strict global maximum at the origin and a singularity at some point $\xi \neq 0$. More precisely on V we assume that

(V1) $V \in C^{1,1}(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$ with $\xi \in \mathbb{R}^2 \setminus \{0\}$;
(V2) $V(0) = 0 > V(q)$ for every $q \in \mathbb{R}^2 \setminus \{0, \xi\}$;
(V3) $\lim_{q \to \xi} V(q) = -\infty$ and there is a neighborhood N_ξ of ξ and a function $U \in C^1(N_\xi \setminus \{\xi\}, \mathbb{R})$ such that $\lim_{q \to \xi} |U(q)| = \infty$ and $V(q) \leq -|\nabla U(q)|^2$ for every $q \in N_\xi \setminus \{\xi\}$;
(V4) $\lim_{|q| \to \infty} |q|^2 V(x) = -\infty$.

Assumptions (V3) and (V4) regard the local behavior of V respectively near the singularity and at infinity. In particular (V3) is the strong force condition introduced by Gordon [G] and governs the rate at which $V(q) \to -\infty$ as $q \to \xi$. It is satisfied for instance if $V(q) \leq -a|q - \xi|^{-\alpha}$ for q in a neighborhood of ξ, $a > 0$ and $\alpha \geq 2$. The same role is played by (V4) as $|q| \to \infty$.

Let us introduce the Lagrangian functional associated to (3.1)

$$\varphi(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - V(u))\, dt$$

defined on the set

$$\Lambda = \{\, u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \ : \ u(t) \neq \xi \ \forall t \in \mathbb{R}\,,\ \lim_{t \to \pm\infty} u(t) = 0 \,\}.$$

Any function $u \in \Lambda$ describes a closed curve in $\mathbb{R}^2$ which starts and ends at the origin without entering the singularity. Hence we can associate to u an integer, denoted $\mathrm{ind}_\xi u$, giving the winding number of u around ξ.

Then, for every $k \in \mathbb{Z}$ we set

$$\Lambda_k = \{ u \in \Lambda \ : \ \mathrm{ind}_\xi u = k \},$$
$$\lambda_k = \inf \{ \varphi(u) \ : \ u \in \Lambda_k \}$$

and we study the problem of existence of a minimizer for φ in Λ_k. If there exists $v \in \Lambda_k$ such that $\varphi(v) = \lambda_k$ then v is a homoclinic solution of (3.1).

By the translational invariance under the action of the non compact group $\mathbb{R}$, a lack of compactness may occur at the levels λ_k with $|k| > 1$. More precisely, according to the concentration–compactness principle [L], the minimizing sequences in Λ_k may exhibit a dichotomy behavior.

This fact was already observed by Rabinowitz in [R1] who studied the system (3.1) under the same assumptions (V1)–(V4) and proposed a condition (labelled by $(*)$ in Theorem 3.1 below) which implies compactness (up to translations) of the minimizing sequences at least at one level λ_k with $k > 1$ sufficiently large.

We can improve this result showing that $(*)$ is actually a sufficient condition to get infinitely many geometrically distinct homoclinics. Precisely we state the following result.

Theorem 3.1. *Let $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy $(V1)$-$(V4)$. Assume that:*

$(*)$ *there is $T \in (0,\infty)$ and $u \in H^1([0,T],\mathbb{R}^2)$ such that $u(0) = u(T)$, $u(t) \neq \xi$ for any $t \in [0,T]$, $\mathrm{ind}_\xi u = 1$ and $\int_0^T (\frac{1}{2}|\dot{u}|^2 - V(u))\, dt < \lambda_1$.*

Then there is $\bar{k} \in \mathbb{N}$ such that for every $k > \bar{k}$ there exists $v_k \in \Lambda_k$ for which $\varphi(v_k) = \lambda_k$. Moreover v_k is a homoclinic solution of (3.1).

As next step, we point out that any homoclinic v_k given by Theorem 3.1 admits a subloop u_k, defined as restriction of v_k to some compact interval $[s_k, t_k]$, such that the sequence (u_k), up to translations, converges in the C^1 topology to a periodic solution $\bar{u}$ with energy zero and winding once about the singularity. This observation suggests to look for heteroclinic orbits between

0 and $\bar{u}$, i.e., solutions to (HS) whose α–limit set (in the phase space) is 0 and whose ω–limit set is $\{(\bar{u}(t),\dot{\bar{u}}(t)) \,:\, t \in \mathbb{R}\}$.

In fact the following result holds.

Theorem 3.2. *Let $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy* (V1)–(V4). *Assume also that* $(*)$ *holds. Let $(v_k)_{k>\bar{k}} \subset \Lambda$ be a sequence of homoclinics such that $v_k \in \Lambda_k$ and $\varphi(v_k) = \lambda_k$, as given by Theorem 3.1. Then there are:*

- *$\bar{T} \in (0,\infty)$ and a $\bar{T}$-periodic orbit $\bar{u}$ with energy 0 and* $\mathrm{ind}_\xi\, \bar{u} = 1$*;*
- *a subsequence (v_{k_j}) of (v_k) and a corresponding sequence of intervals $[s_{k_j}, t_{k_j}] \subset \mathbb{R}$ such that $t_{k_j} - s_{k_j} \to \bar{T}$ and $\|u_j - \bar{u}\|_{C^1([0,\bar{T}])} \to 0$, where $u_j(t) = v_{k_j}(\frac{t_{k_j}-s_{k_j}}{\bar{T}}t - s_{k_j})$ for $t \in [0,\bar{T}]$ and $j \in \mathbb{N}$;*
- *a heteroclinic orbit $\bar{v}$ between 0 and $\bar{u}$ such that $v_{k_j}(\cdot - \tau_{k_j}) \to \bar{v}$ strongly in $C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$ for a suitable sequence $(\tau_{k_j}) \subset \mathbb{R}$.*

Finally we give a geometric condition on the system (3.1) which guarantees that $(*)$ holds true and $\bar{k} = 1$.

Theorem 3.3. *Let $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy $(V1)$-$(V4)$. Suppose also that*

$(V5)$ $\limsup_{x\to 0} -V(x)|x|^{-2} = a < \infty$ *and* $\liminf_{x\to 0} -V(x)|x|^{-2} = b > 0$*;*
$(V6)$ *there exists $v_1 \in \Lambda_1$ such that $\varphi(v_1) = \lambda_1$ and*

$$\liminf_{\substack{s\to-\infty \\ t\to+\infty}} \frac{v_1(s)}{|v_1(s)|} \cdot \frac{v_1(t)}{|v_1(t)|} > \sqrt{1 - \frac{b}{a}}.$$

Then $(*)$ holds and for any $k \in \mathbb{N}$ there is a homoclinic solution $v_k \in \Lambda_k$ such that $\varphi(v_k) = \lambda_k$.

We point out that (V6) is a condition regarding the angle θ_1 formed by the directions at which a homoclinic orbit $v_1 \in \Lambda_1$ at level λ_1 leaves and enters the origin. In particular, if $V(x) \sim -a|x|^2$ as $x \to 0$ for some $a > 0$ then in (V6) we ask that $\theta_1 \in [0, \frac{\pi}{2})$.

Theorems 3.1 and 3.2 are contained in a forthcoming paper in collaboration with L. Jeanjean. These results improve and generalize a previous joint work with M. Nolasco [CN], containing a first version of Theorem 3.3.

3B. Asymptotically periodic Duffing–like systems in $\mathbb{R}^N$

We study a second order Hamiltonian system in $\mathbb{R}^N$

$$\ddot{q} = q - \nabla_q V(t,q) \tag{3.2}$$

where $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies:

(V1) $V \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ with $\nabla_q V(t,\cdot)$ locally Lipschitz continuous, uniformly with respect to $t \in \mathbb{R}$;

(V2) $\nabla_q V(t,q)/|q| \to 0$ as $q \to 0$ uniformly with respect to $t \in \mathbb{R}$.

The assumption (V2) implies that the origin is a hyperbolic equilibrium for the system (3.2), uniformly in time. The regularity (V1) and the hyperbolicity condition (V2) are the only properties assumed on the potential at finite times.

Then we put conditions on the shape of V with respect to to the space variable only asymptotically, as $t \to \pm\infty$. In particular we require that there are two functions $V_\pm : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satysfying (V1), (V2),

(V3) there are $T_\pm > 0$ such that $V_\pm(t,q) = V_\pm(t + T_\pm, q)$ for every (t,q);

(V4) (i) there is $(t_\pm, q_\pm) \in \mathbb{R} \times \mathbb{R}^N$ such that $V_\pm(t_\pm, q_\pm) \geq \frac{1}{2}|q_\pm|^2$;

(ii) there are two pairs of constants $\beta_\pm > 2$ and $\alpha_\pm < \frac{\beta_\pm}{2} - 1$ such that:
$$\beta_\pm V_\pm(t,q) - \nabla_q V_\pm(t,q) \cdot q \leq \alpha_\pm |q|^2 \text{ for every } (t,q).$$

The assumption (V4) is a generalized version of the global superquadraticity, allowing $V_\pm$ to change sign.

Moreover (V4) is satisfied by functions which are α–homogeneous in q with $\alpha > 2$. Therefore the systems that we study include for instance the forced Duffing equation $\ddot{q} = q - (1 + \epsilon a(t))q^3$ with a forcing depending on time in an asymptotically periodic way.

The functions $V_\pm$ give the asymptotic behavior of V as $t \to \pm\infty$ according to the following condition:

(V5) $\nabla_q V(t,q) - \nabla_q V_\pm(t,q) \to 0$ as $t \to \pm\infty$ uniformly on the compact sets of $\mathbb{R}^N$.

Therefore the dynamics of (3.2) will be related to the dynamics of the periodic systems at infinity:

$$\ddot{q} = q - \nabla_q V_\pm(t,q). \tag{3.3}$$

In general the fact that the systems (3.3) admit homoclinic solutions does not imply necessarily that the same holds true for (3.2). This can be seen considering the following example. Let us take $V(t,q) = a(t)|q|^4$, where $a(t)$ is a smooth bounded monotone positive function. The systems at infinity correspond to $V_\pm(t,q) = a_\pm |q|^4$, where $a_\pm = \lim_{t\to\pm\infty} a(t) > 0$. If $q(t)$ is a solution to (3.2) satisfying $q(t) \to 0$ and $\dot q(t) \to 0$ as $t \to \pm\infty$ then, setting $H(t) = \frac{1}{2}|\dot q(t)|^2 - \frac{1}{2}|q(t)|^2 + \frac{1}{4}a(t)|q(t)|^4$ we get $0 = \int_{\mathbb{R}} \frac{d}{dt} H(t)\, dt = \int_{\mathbb{R}} \frac{1}{4}\dot a(t)|q(t)|^4\, dt$ and then $q(t) \equiv 0$.

Hence, differently from the periodic case, defined by one of the systems (3.3), in which a homoclinic orbit always exists, the asymptotically peridic case can exhibit situations with no homoclinic solution.

We prove that if the set of homoclinics of (3.3) is countable, then the asymptotically periodic system (3.2) itself admits infinitely many homoclinic solutions. More precisely if the construction of multibump solutions for (3.3) can be done, then this kind of solutions persists also for the system (3.2).

This result is obtained using variational techniques. In fact the homoclinic solutions of (3.2) are exactly the critical points in $H^1(\mathbb{R}, \mathbb{R}^N)$ of the functional

$$\varphi(u) = \tfrac{1}{2}\|u\|^2_{H^1} - \int_{\mathbb{R}} V(t,u)\, dt.$$

Since $V_\pm$ satisfies (V1)–(V4), the functionals $\varphi_\pm$ associated to (3.3) verify the geometrical properties of the mountain pass lemma.

Let us denote by $c_\pm$ the corresponding minimax levels and by $K_\pm$ the sets of critical points of $\varphi_\pm$.

Now we can state a first result.

Theorem 3.4. *Let $V, V_\pm$ satisfy* (V1)–(V5). *Let us assume that the sets $K_\pm \cap \{\varphi_\pm \le c^*_\pm\}$ are countable, for some $c^*_\pm > c_\pm$.*
Then there are $v_\pm$ homoclinic solutions of (3.3) having the following property: for any $r > 0$ there are $m(r), m_1(r) > 0$ such that for every biinfinite sequence $(p_j)_{j\in\mathbb{Z}}$ with $(p_j)_{j>0} \subset P_+ = T_+\mathbb{Z}$ and $(p_j)_{j<0} \subset P_- = T_-\mathbb{Z}$ satisfying $p_1 \ge m_1(r)$, $p_{-1} \le -m_1(r)$, $p_{j+1} - p_j \ge m(r)$ $(j \in \mathbb{Z})$ and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there is a solution v_σ to (3.2) such that

$$\|v_\sigma - \sigma_j v_+(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} \le r \quad \textit{for any } j > 0$$

$$\|v_\sigma - \sigma_j v_-(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} \le r \quad \textit{for any } j < 0$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$.
In addition, if $\sigma_j = 0$ *for all* $j \geq j_0$ *(respectively* $j \leq j_0$*) then the solution* v_σ *also satisfies* $v_\sigma(t) \to 0$ *and* $\dot{v}_\sigma(t) \to 0$ *as* $t \to +\infty$ *(respectively* $t \to -\infty$*).*

We remark that Theorem 3.4 can be read as a shadowing lemma for an asymptotically periodic system. The value $m(r)$ represents the minimal distance at which two consecutive bumps can be arranged. This value $m(r)$, as well as $m_1(r)$, becomes larger and larger as $r \to 0$. According to this remark, instead of fixing $r > 0$, it is possible to take a sequence $(r_j) \subset (0, \infty)$ such that $r_j \to 0$. Thus we get the following result, concerning connecting orbits between 0 and the basic homoclinics $v_\pm$ of (3.3).

Theorem 3.5. *Under the same assumptions of Theorem 3.4, the system (3.2) admits an uncountable set of multibump solutions whose* α*-limit set is 0 or* $\{(v_-(t), \dot{v}_-(t)) : t \in \mathbb{R}\} \cup \{0\}$ *and whose* ω*-limit set is 0 or* $\{(v_+(t), \dot{v}_+(t)) : t \in \mathbb{R}\} \cup \{0\}$.

If we specialize Theorem 3.5 to the case V periodic in time, we get the existence of a homoclinic solution v of (3.2) and an uncountable set of connecting orbits between 0 and v and between v and itself.

Theorem 3.4 has been proved firstly in the periodic case in a joint work with P. Montecchiari [CM]. Its generalization to an asymptotically periodic system and related results have been developed in collaboration with S. Abenda and P. Montecchiari [ACM]. Finally Theorem 3.5 is a result obtained with P. Montecchiari and M. Nolasco [CMN].

Chapter 1
Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems in $\mathbb{R}^2$

1. Introduction

In this chapter we describe some features of the dynamics of an autonomous second order Hamiltonian system in $\mathbb{R}^2$ of the type

$$\text{(HS)} \qquad \ddot{x} + \nabla V(x) = 0$$

ruled by a potential V with a strict global maximum at the origin and a singularity at some point $\xi \neq 0$. More precisely on V we assume that

(V1) $V \in C^{1,1}(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$ with $\xi \in \mathbb{R}^2 \setminus \{0\}$;

(V2) $V(0) = 0 > V(x)$ for every $x \in \mathbb{R}^2 \setminus \{0, \xi\}$;

(V3) $\lim_{x \to \xi} V(x) = -\infty$ and there is a neighborhood N_ξ of ξ and a function $U \in C^1(N_\xi \setminus \{\xi\}, \mathbb{R})$ such that $\lim_{x \to \xi} |U(x)| = \infty$ and $V(x) \leq -|\nabla U(x)|^2$ for every $x \in N_\xi \setminus \{\xi\}$;

(V4) $\lim_{|x| \to \infty} |x|^2 V(x) = -\infty$.

Assumptions (V3) and (V4) concern the local behavior of the potential respectively at the singularity and at infinity. In particular (V3) is the strong force condition, introduced by Gordon [G]. It governs the rate at which $V(x) \to -\infty$ as $x \to \xi$ and it is satisfied for instance in the case $\limsup_{x \to \xi} |x - \xi|^\alpha V(x) < 0$ for some $\alpha \geq 2$. Similarly (V4) says that $V(x)$ can go to 0 as $|x| \to \infty$ but not too fast.

By (V2) the origin is an unstable equilibrium but we note that no hypothesis is made on the behavior of the potential in a neighborhood of 0.

In the first part of this chapter we study the problem of multiplicity of homoclinics, namely solutions to (HS) doubly asymptotic to the origin, i.e., such that $x(t) \to 0$ and $\dot{x}(t) \to 0$ as $t \to \pm\infty$.

In particular, given an integer $k \neq 0$, we look for a homoclinic orbit which turns k times around the singularity ξ. This is done using a minimization argument, similar to [R1]. Precisely we introduce the Lagrangian functional associated to (HS), given by

$$\varphi(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - V(u))\, dt$$

defined on the Hilbert space

$$E = \{\, u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \;:\; \int_{\mathbb{R}} |\dot{u}|^2\, dt < \infty \,\}$$

endowed with the norm

$$\|u\|^2 = |u(0)|^2 + \int_{\mathbb{R}} |\dot{u}|^2\, dt.$$

We notice that $E \subset C(\mathbb{R}, \mathbb{R}^2)$ and, by (V2), $\varphi(u) \in [0, +\infty]$ for $u \in E$.

Then we consider the set

$$\Lambda = \{\, u \in E \;:\; u(t) \neq \xi \;\forall t \in \mathbb{R}\,,\; \lim_{t \to \pm\infty} u(t) = 0 \,\}.$$

Any function $u \in \Lambda$ describes a closed curve in $\mathbb{R}^2$ which starts and ends at the origin and never touches the point ξ. Hence we can associate to u an integer, denoted $\mathrm{ind}_\xi\, u$, giving the winding number of u about ξ.

For every $k \in \mathbb{Z}$ we set

$$\Lambda_k = \{\, u \in \Lambda \;:\; \mathrm{ind}_\xi\, u = k \,\},$$
$$\lambda_k = \inf \{\, \varphi(u) \;:\; u \in \Lambda_k \,\}$$

and we study the problem of existence of a minimizer for φ in Λ_k. In fact, as we will see in the sequel, if there exists $v \in \Lambda_k$ such that $\varphi(v) = \lambda_k$ then v is a homoclinic solution of (HS) (Lemma 4.4).

We remark that, since the potential V is time independent, $\varphi(u(\cdot + s)) = \varphi(u_-) = \varphi(u)$ for any $u \in E$ and $s \in \mathbb{R}$, being $u_-(t) = u(-t)$. This implies that $\lambda_k = \lambda_{-k}$ and if $v \in \Lambda_k$ satisfies $\varphi(v) = \lambda_k$, then $v_- \in \Lambda_{-k}$ and $\varphi(v_-) = \lambda_{-k}$. Therefore we can restrict ourselves to consider $k \in \mathbb{N}$.

The main result concerning multiple homoclinic solutions is the following.

Theorem 1.1. *Let* $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ *satisfy* $(V1)$*-*$(V4)$ *and let* $\bar{k} = \sup\{ k \in \mathbb{N} : \lambda_k = k\lambda_1 \}$. *Then:*

(*i*) $\bar{k} < \infty$ *if and only if*

(∗) *there is* $T \in (0, \infty)$ *and* $u \in H^1([0,T], \mathbb{R}^2)$ *such that* $u(0) = u(T)$, $u(t) \neq \xi$ *for any* $t \in [0,T]$, $\mathrm{ind}_\xi u = 1$ *and* $\int_0^T (\frac{1}{2}|\dot{u}|^2 - V(u))\, dt < \lambda_1$.

(*ii*) *If* $\bar{k} < \infty$ *then for any* $k > \bar{k}$ *there exists* $v_k \in \Lambda_k$ *for which* $\varphi(v_k) = \lambda_k$. *Moreover* v_k *is a homoclinic solution of* (HS).

(*iii*) *If* $\bar{k} > 2$ *then for* $1 < k < \bar{k}$ *the value* λ_k *is never achieved in* Λ_k, *i.e.,* $\varphi(u) > \lambda_k$ *for any* $u \in \Lambda_k$.

As we will see in the next sections, Theorem 1.1 holds true under somewhat weaker assumptions. In particular, instead of (V4), we can assume the following more general condition:

(V4)' there are $\bar{R} > 0$ and $U_\infty \in C^1(\mathbb{R}^2 \setminus B_{\bar{R}}, \mathbb{R})$ such that $\lim_{|x|\to\infty} |U_\infty(x)| = \infty$ and $V(x) \leq -|\nabla U_\infty(x)|^2$ for every $x \in \mathbb{R}^2 \setminus B_{\bar{R}}$, being $B_{\bar{R}} = \{x \in \mathbb{R}^2 : |x| < \bar{R}\}$.

We note that, contrary to (V4), condition (V4)' is satisfied also when $\limsup_{|x|\to\infty} |x|^2 V(x) < \infty$.
Moreover the local Lipschitz continuity of ∇V, which guarantees the uniqueness of solution to the Cauchy problem associated to (HS), is used only to prove Part (*iii*) (see Lemma 4.6). Whereas, Parts (*i*) and (*ii*) hold true more generally assuming $V \in C^1(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$ instead of (V1).

In [R1], under the same assumptions of Theorem 1.1, Rabinowitz shows that the value λ_1 corresponds to a minimum for φ in Λ_1, namely there exists a homoclinic solution $v_1 \in \Lambda_1$ of (HS) such that $\varphi(v_1) = \lambda_1$. Moreover in [R1] condition (∗) is introduced to prove that if (∗) holds then, in addition to the solution v_1, there is a second homoclinic orbit $v \in \Lambda$ found as minimizer for φ with respect to $\Lambda_{\bar{k}+1}$.

Part (*iii*) in the statement of Theorem 1.1 says that for $1 < k < \bar{k}$ there is no precompact minimizing sequence for φ in Λ_k. As we will see in Lemma 3.3, all the minimizing sequences at these levels λ_k $(1 < k < \bar{k})$ exhibit a dichotomy behavior. Instead, for $k = \bar{k}$, we can say nothing about the existence of a minimizer for φ in $\Lambda_{\bar{k}}$. Theorem 1.1 constitues a remarkable improvement of this result presented in [R1]. Indeed it states that $(*)$ is actually a sufficient condition to get infinitely many geometrically distinct homoclinics.

The proof of Theorem 1.1 relies on a careful analysis of the minimizing sequences (Section 3). We start in Section 2 by studying more generally the sequences contained in the sublevel sets of the functional φ. In particular in Lemma 2.4 it is shown that a sequence $(u_n) \subset E$ such that $\sup \varphi(u_n) < \infty$ can exhibit at the same time both vanishing and dichotomy behavior. In fact, only finite dichotomies can occur for the minimizing sequences for φ with respect to some class Λ_k (see Lemma 3.3) and, thanks to $(*)$, for $k \in \mathbb{N}$ sufficiently large (and precisely for $k > \bar{k}$), actually compactness holds, up to translations (Lemmas 4.1 and 4.2).

As auxiliary result, we also get that for $k > \bar{k}$ any trajectory v_k given by Theorem 1.1 contains a subloop u_k winding once about ξ and, up to a subsequence, u_k converges in the C^1–norm to a periodic solution $\bar{u}$ with energy zero.

This remark suggests to look for connecting orbits between 0 and $\bar{u}$, i.e., solutions to (HS) whose α–limit set (in the phase space) is 0 and whose ω–limit set is $\{(\bar{u}(t), \dot{\bar{u}}(t)) \, : \, t \in \mathbb{R}\}$.

We recall that the α–limit set and the ω–limit set of a solution v to (HS) are defined respectively by

$$L_\alpha(v) = \{ \zeta \in \mathbb{R}^4 \, : \, \exists \, (t_n) \subset \mathbb{R} \text{ s.t. } t_n \to -\infty \text{ and } (v(t_n), \dot{v}(t_n)) \to \zeta \}$$
$$L_\omega(v) = \{ \zeta \in \mathbb{R}^4 \, : \, \exists \, (t_n) \subset \mathbb{R} \text{ s.t. } t_n \to +\infty \text{ and } (v(t_n), \dot{v}(t_n)) \to \zeta \}.$$

Hence, the second part of this chapter is devoted to investigate the existence of these trajectories, asymptotic to two different unstable orbits and known as heteroclinic solutions.

Precisely, we look for heteroclinics as limit in the C^1_{loc}–topology of the sequence of homoclinic solutions v_k found in Theorem 1.1. Therefore, to guarantee the existence of these homoclinics v_k as well as the existence of the periodic orbit $\bar{u}$, hereinafter we assume that condition $(*)$ holds.

Before stating Theorem 1.2, concerning the existence of a heteroclinic orbit, we introduce some notation. For $T \in (0,\infty)$ let

$$E_T = \{\, u \in H^1([0,T],\mathbb{R}^2) \;:\; u(0) = u(T)\,\}$$
$$\Lambda_{1,T} = \{\, u \in E_T \;:\; u(t) \neq \xi \;\forall t \in [0,T]\,,\; \mathrm{ind}_\xi\, u = 1\,\}$$
$$\varphi_T(u) = \int_0^T (\tfrac{1}{2}|\dot u|^2 - V(u))\,dt \;\text{ for } u \in E_T$$
$$\bar\lambda = \inf\{\, \varphi_T(u) \;:\; T \in (0,\infty)\,,\; u \in \Lambda_{1,T}\,\}.$$

We point out that condition $(*)$ is equivalent to say $\bar\lambda < \lambda_1$.

Theorem 1.2. *Let $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy (V1)–(V4). Assume also that $(*)$ holds. Let $(v_k)_{k>\bar k} \subset \Lambda$ be a sequence of homoclinics such that $v_k \in \Lambda_k$ and $\varphi(v_k) = \lambda_k$, as given by Theorem 1.1. Then there are:*

- *$\bar T \in (0,\infty)$ and a $\bar T$-periodic orbit $\bar u \in \Lambda_{1,\bar T}$ with energy 0 and such that $\varphi_{\bar T}(\bar u) = \bar\lambda$;*
- *a subsequence (v_{k_j}) of (v_k) and a corresponding sequence of intervals $[s_{k_j}, t_{k_j}] \subset \mathbb{R}$ such that $t_{k_j} - s_{k_j} \to \bar T$ and $\|u_j - \bar u\|_{C^1([0,\bar T])} \to 0$, where $u_j(t) = v_{k_j}(\frac{t_{k_j}-s_{k_j}}{\bar T}t - s_{k_j})$ for $t \in [0,\bar T]$ and $j \in \mathbb{N}$;*
- *a heteroclinic orbit $\bar v$ between 0 and $\bar u$ such that $v_{k_j}(\cdot - \tau_{k_j}) \to \bar v$ strongly in $C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$ for a suitable sequence $(\tau_{k_j}) \subset \mathbb{R}$.*

Remark 1.3. Since the system (HS) is conservative, the function $\bar u_-(t) = \bar u(-t)$ is a $\bar T$-periodic orbit and $\bar v_-(t) = \bar v(-t)$ is a solution such that $L_\alpha(\bar v_-) = \{(\bar u_-(t), \dot{\bar u}_-(t)) \,:\, t \in \mathbb{R}\}$ and $L_\omega(\bar v_-) = 0$. Moreover, considering the sequence $(v_{-k})_{k>\bar k} \subset \Lambda$, we also get that, up to a subsequence, it converges strongly in $C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$ to a solution $\tilde v$ to (HS) such that $L_\alpha(\tilde v) = 0$ and $L_\omega(\tilde v) = \{(\bar u_-(t), \dot{\bar u}_-(t)) \,:\, t \in \mathbb{R}\}$.

The proof of Theorem 1.2 consists in three main steps. In the first one, given a sequence $(v_k) \subset \Lambda$ of homoclinics according to Theorem 1.1, we consider the restriction of v_k to a suitable interval I_k such that $v_k|_{I_k}$ draws a closed curve winding once the singularity and we prove that the limit curve defines a periodic orbit $\bar u$ which is a minimizer for $\bar\lambda$ (Section 5). In the second step (Section 6) we get uniform estimates on the sequence (v_k). With an application of Ascoli–Arzelà Theorem, this allows us to find a solution to (HS) that, in Section 7, is proved to be a heteroclinic orbit between 0 and $\bar{u}$.

We point out that in our setting, unlike [R6]–[R7], we do not make any requirement of isolateness about the periodic orbit $\bar{u}$.

In the last section of this chapter we discuss the assumption $(*)$ and we exhibit a condition which guarantees that $(*)$ holds and in particular that $\bar{k} = 1$. This condition involves an estimate on the angle formed by the directions at which a homoclinic orbit $v_1 \in \Lambda_1$ at level λ_1 leaves and enters the origin (Theorem 8.2). Then, in Theorems 8.4 and 8.6 some examples of systems satisfying this property are presented.

2. Preliminary results

First of all we point out that no L^p space is contained in E for any $p \in [1, \infty]$. On the other hand the space E can be characterized as the completion of $C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ endowed with the norm $(\int_{\mathbb{R}} |\dot{u}|^2 \, dt)^{1/2}$, usually denoted $D^{1,2}(\mathbb{R}, \mathbb{R}^2)$.

Moreover any bounded sequence of E admits a subsequence which converges weakly in E and strongly in $L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) = L^\infty_{\text{loc}}$ to some $u \in E$.

This plainly implies that the functional φ is weakly lower sequentially semicontinuous on E (briefly w.l.s.s.). Indeed if $(u_n) \subset E$ converges weakly in E to some $u \in E$ then $\int_{\mathbb{R}} |\dot{u}|^2 \, dt \leq \liminf \int_{\mathbb{R}} |\dot{u}_n|^2 \, dt$. Moreover, up to a subsequence, $u_n \to u$ uniformly on compact sets and then, fixed $T > 0$, $\int_{-T}^{T} -V(u) \, dt = \lim \int_{-T}^{T} -V(u_n) \, dt$. Hence $\frac{1}{2} \int_{\mathbb{R}} |\dot{u}|^2 \, dt - \int_{-T}^{T} V(u) \, dt \leq \liminf \varphi(u_n)$ and, by the arbitrariness of $T > 0$, we conclude that $\varphi(u) \leq \liminf \varphi(u_n)$.

We notice that φ is not continuous. In fact for any sequence $(a_n) \subset \mathbb{R}_+$ we can easily construct a sequence $(u_n) \subset E$ such that $\|u_n\| \to 0$ and $\varphi(u_n) = a_n$ for any $n \in \mathbb{N}$.

Remark 2.1. We point out that if one makes the further assumption that V is twice differentiable at 0 and $V''(0)$ is negative definite (i.e., the origin is a non degenerate strict maximum for V), then the homoclinic solutions of (HS) belong to $H^1(\mathbb{R}, \mathbb{R}^2)$ (see Chapter 2, Lemma 2.3). Therefore in this case

$H^1(\mathbb{R},\mathbb{R}^2)$ is the more natural variational space for the considered problem. In addition, the functional φ turns out to be of class C^1 on $H^1(\mathbb{R},\mathbb{R}^2)$, the notion of differential is now meaningful and the homoclinic orbits are exactly the critical points of φ (see Chapter 2, Lemmas 2.1 and 2.3). But under the only hypotheses (V1)–(V2), we have not such a regularity and we have to proceed more carefully.

To study the minimizing sequences or more generally the sublevel sets of φ we will often apply the next key lemma, that we state in a quite general form, using the following notation.

We denote by $\mathcal{I}$ the class of all intervals of $\mathbb{R}$. If $I, J \in \mathcal{I}$, $I \subset J$, and $u \in H^1_{\rm loc}(J,\mathbb{R}^2)$, we write $u|_I$ to mean the restriction of u to I and we set $\varphi_I(u) = \varphi(u|_I) = \int_I(\frac{1}{2}|\dot u|^2 - V(u))\,dt$. For $\delta > 0$ and $u \in H^1_{\rm loc}(I,\mathbb{R}^2)$ we set $S_\delta(u) = \{\, t \in I : |u(t)| \geq \delta \,\}$. Moreover, given $u \in H^1_{\rm loc}([t_1,t_2],\mathbb{R}^2)$ such that $u(t) \neq \xi$ for any $t \in [t_1,t_2]$ and $u(t_1) = u(t_2)$ we denote by $\operatorname{ind}_\xi u$ the winding number of the curve defined by u about ξ. We recall that if $u_1, u_2 \in H^1_{\rm loc}([t_1,t_2],\mathbb{R}^2)$ satisfy $u_i(t) \neq \xi$ for any $t \in [t_1,t_2]$, $u_i(t_1) = u_i(t_2)$ $(i = 1,2)$ and $|u_1(t) - u_2(t)| < |u_1(t) - \xi|$ for every $t \in [t_1,t_2]$, then $\operatorname{ind}_\xi u_1 = \operatorname{ind}_\xi u_2$. Finally, given a measurable set $A \subseteq \mathbb{R}$ we denote by $|A|$ the Lebesgue measure of A.

From now on, in all this Section, as well as in Sections 3 and 4, we assume (V1)–(V3) and (V4)'.

Lemma 2.2. *Given* $a, b > 0$ *let* $X_{a,b} = \{u \in H^1_{\rm loc}(I,\mathbb{R}^2) : I \in \mathcal{I},\ \varphi_I(u) \leq a,\ \exists\, t \in I \text{ s.t. } |u(t)| \leq b\}$. *Then:*

(*i*) *there is* $R = R(a,b) > 0$ *such that* $\|u\|_{L^\infty} \leq R$ *for any* $u \in X_{a,b}$*;*

(*ii*) *there is* $\rho = \rho(a,b) > 0$ *such that* $\operatorname{dist}(\xi, \operatorname{range} u) \geq \rho$ *for any* $u \in X_{a,b}$*;*

(*iii*) *for any* $\delta > 0$ *there is* $\tau_\delta = \tau_\delta(a,b) > 0$ *such that* $|S_\delta(u)| \leq \tau_\delta$ *for all* $u \in X_{a,b}$.

Proof. (*i*) By the contrary, assume that $\|u_n\|_{L^\infty} \to \infty$ holds for some sequence $(u_n) \subset X_{a,b}$. Since by assumption for any $n \in \mathbb{N}$ there are $t_n \in I_n = \operatorname{dom} u_n$ such that $|u_n(t_n)| \leq b$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$ there are $\bar t_n, s_n \in I_n$ (without restriction we can assume that $\bar t_n \leq s_n$) such that

$|u_n(\bar t_n)| = \bar r$ for a suitable $\bar r \geq \bar R$, $|u_n(s_n)| \to \infty$ and $|u_n(t)| \geq \bar r$ for $t \in (\bar t_n, s_n)$, where $\bar R$ is given by (V4)'. Then (V4)' yields

$$\begin{aligned}|U_\infty(u_n(s_n))| &\leq |U_\infty(u_n(\bar t_n))| + \int_{\bar t_n}^{s_n} |\nabla U_\infty(u_n)\cdot \dot u_n|\,dt \\ &\leq \max_{|x|=\bar r} |U_\infty(x)| + \left(\int_{\bar t_n}^{s_n} -V(u_n)\,dt\right)^{1/2} \left(\int_{\bar t_n}^{s_n} |\dot u_n|^2\,dt\right)^{1/2} \\ &\leq \max_{|x|=\bar r} |U_\infty(x)| + 2\,\varphi(u_n).\end{aligned}$$

But $|U_\infty(u_n(s_n))| \to \infty$, while $\varphi(u_n)$ is bounded. Thus we get a contradiction.

A similar argument can be followed to prove (ii), using $(V3)$.

(iii) Fixed $\delta > 0$, we set $\beta_\delta = \inf\{\,|V(x)| \;:\; |x-\xi| \geq \rho,\ \delta \leq |x| \leq R\}$ where $R > 0$ and $\rho > 0$ are given by (i) and (ii) respectively. Then for any $u \in X_{a,b}$ it holds that

$$\beta_\delta\,|S_\delta(u)| \leq \int_{S_\delta(u)} -V(u)\,dt \leq \varphi_I(u) \leq a.$$

□

Remark 2.3. By Lemma 2.2, if $I \in \mathcal{I}$ and $u \in H^1_{\mathrm{loc}}(I,\mathbb{R}^2)$ is such that $\varphi_I(u) < \infty$ then $\mathrm{dist}\,(\xi, \mathrm{range}\,u) > 0$. Moreover if $I = (-\infty, T)$, with $T \in \mathbb{R}$ and $\varphi_I(u) < \infty$ then $\lim_{t\to-\infty} u(t) = 0$. This is easily obtained using the fact that for any compact $J \subset I$ we have that $\varphi_J(u) \geq (2\inf_{t\in J} |V(u(t))|\,)^{1/2} \int_J |\dot u|\,dt$ (see [R2, Lemma 3.6]). An analogous result holds if $I = (T, +\infty)$. Therefore, in particular, if $u \in E$ is such that $\varphi(u) < \infty$ then $u \in \Lambda$.

The sequences $(u_n) \subset E$ such that $\sup \varphi(u_n) < \infty$ can exhibit different behaviors. For instance it may happen that $\int_{\mathbb{R}} |\dot u_n|^2\,dt \to a > 0$, $\varphi(u_n) \to b > a$ but $\|u_n\|_{L^\infty} \to 0$. Nonetheless, if we assume that $\limsup \|u_n\|_{L^\infty} > 0$, then we can better describe these sequences, giving a characterization in the spirit of the concentration–compactness principle by P.L. Lions [L].

Lemma 2.4. *Let $(u_n) \subset E$ be such that $\varphi(u_n) \to a > 0$ and $\limsup \|u_n\|_{L^\infty} = \delta_0 > 0$. Then for any $\delta \in (0,\delta_0)$ there are a subsequence of (u_n), denoted* *again by* (u_n), *functions* $w_1, \ldots, w_l \in \Lambda$, *with* $S_\delta(w_i) \neq \emptyset$ *for any* $i = 1, \ldots, l$, *and corresponding sequences* $(t_n^1), \ldots, (t_n^l) \subset \mathbb{R}$ *such that:*

(2.1) $$\lim_{n\to\infty}(t_n^{i+1} - t_n^i) = \infty \text{ for any } i = 1, \ldots, l-1;$$

(2.2)
$$\lim_{n\to\infty} u_n(\cdot + t_n^i) = w_i \text{ weakly in } E \text{ and strongly in } L^\infty_{\text{loc}} \text{ for any } i = 1, \ldots, l;$$

(2.3) $$\sum_{i=1}^{l} \varphi(w_i) \leq a;$$

for any $\epsilon \in (0, \delta)$ *there exists* $n_\epsilon \in \mathbb{N}$ *such that for* $n \geq n_\epsilon$*:*

(2.4) $$\bigcup_{i=1}^{l} S_{\delta+\epsilon}(w_i(\cdot - t_n^i)) \subseteq S_\delta(u_n) \subseteq \bigcup_{i=1}^{l} S_{\delta-\epsilon}(w_i(\cdot - t_n^i))$$
$$S_{\delta-\epsilon}(w_i(\cdot - t_n^i)) \cap S_{\delta-\epsilon}(w_j(\cdot - t_n^j)) = \emptyset \quad \text{for } i \neq j.$$

Moreover $1 \leq l \leq l_{\delta,a}$ *with* $l_{\delta,a} \in \mathbb{N}$ *independent of the sequence* (u_n).

Remark 2.5. Property (2.4) says that, up to a subsequence, the δ–support of u_n, given by $S_\delta(u_n)$, decomposes asymptotically in a finite union of δ–supports $S_\delta(w_i(\cdot - t_n^i))$ $(i = 1, \ldots, l)$ which are bounded sets whose distance becomes larger and larger as $n \to \infty$.

Remark 2.6. To better illustrate Lemma 2.4 we consider the following example. Let $(y_n) \subset E$ be a sequence such that $\|y_n\|_{L^\infty} \to 0$, $\varphi(y_n) \to \frac{a}{2} > 0$ and $\operatorname{supp} y_n \subset (-\infty, 0)$ for any $n \in \mathbb{N}$. Let $(w_j) \subset E$ be a sequence such that $\sum_{j=1}^{\infty} \varphi(w_j) = \frac{a}{2}$, $\operatorname{supp} w_j \subset [0,1]$ and $\|w_j\|_{L^\infty} \geq \|w_{j+1}\|_{L^\infty} > 0$ for any $j \in \mathbb{N}$. For $n > 1$ set $u_n = y_n + \sum_{j=1}^{\infty} w_j(\cdot - n^j)$. Then the sequence (u_n) satisfies the hypotheses of Lemma 2.4. In fact for any $\delta > 0$ sufficiently small there is $l_\delta \in \mathbb{N}$ such that $S_\delta(w_j) \neq \emptyset$ if and only if $1 \leq j \leq l_\delta$ and clearly $\sum_{j=1}^{l_\delta} \varphi(w_j) < a$. We point out that $l_\delta \to \infty$ as $\delta \to 0$ and even if we take all the w_j's the strict inequality $\sum_{j=1}^{\infty} \varphi(w_j) < a$ persists.
This example shows that in general a sequence $(u_n) \subset E$ such that $\sup \varphi(u_n) < \infty$ can exhibit at the same time both vanishing and dichotomy behavior with

possibly infinitely many dichotomies. We recall that in the more regular case mentioned in Remark 2.1, a similar characterization holds for the Palais Smale sequences of φ (see Chapter 2, Lemma 3.3). But in that case the vanishing behavior cannot occur and the dycothomies are always finite.

Proof. Let $(u_n) \subset E$ be such that $\varphi(u_n) \to a \in (0, \infty)$ and $\limsup \|u_n\|_{L^\infty} = \delta_0 > 0$. Firstly we observe that by Lemma 2.2 and Remark 2.3 $(u_n) \subset \Lambda$. Fixed $\delta \in (0, \delta_0)$, passing to a subsequence if necessary, for any $n \in \mathbb{N}$ there is $t_n^1 \in \mathbb{R}$ such that $|u_n(t_n^1)| = \delta$ and $|u_n(t)| < \delta$ for $t < t_n^1$. Setting $u_n^1 = u_n(\cdot + t_n^1)$ for any $n \in \mathbb{N}$, we have that $(u_n^1) \subset \Lambda$, $\varphi(u_n^1) \to a$ and $(-\infty, 0] \cap S_\delta(u_n^1) = \{0\}$. In particular for any $n \in \mathbb{N}$

$$S_\delta(u_n) \cap (-\infty, t_n^1) = \emptyset. \tag{2.5}$$

Since in addition the sequence (u_n^1) is bounded in E, up to a subsequence, it converges weakly in E and strongly in L^∞_{loc} to some $w_1 \in E$. Since φ is w.l.s.s., we get that $\varphi(w_1) \leq a$ and then, by remark 1.5, $w_1 \in \Lambda$. Moreover $|w_1(0)| = \delta$, so that $S_\delta(w_1) \neq \emptyset$. Since $S_\delta(w_1)$ is bounded, $u_n^1 \to w_1$ uniformly on $S_\delta(w_1)$ and then, fixed $\epsilon \in (0, \delta)$ there exists $n_\epsilon^1 \in \mathbb{N}$ such that for any $n \geq n_\epsilon^1$ $S_{\delta+\epsilon}(w_1) \subseteq S_\delta(u_n^1)$. Moreover, there is $t_1 = t_1(\epsilon) > 0$ such that $|w_1(t_1)| = \delta - \epsilon$ and $|w_1(t)| < \delta - \epsilon$ for $t > t_1$. Hence, again by the uniform convergence on the compact sets, there is $m_\epsilon^1 \in \mathbb{N}$ such that for $n \geq m_\epsilon^1$ $S_\delta(u_n^1) \cap [0, t_1] \subseteq S_{\delta-\epsilon}(w_1)$, that is

$$S_\delta(u_n) \cap [t_n^1, t_n^1 + t_1] \subseteq S_{\delta-\epsilon}(w_1(\cdot - t_n^1)). \tag{2.6}$$

Now we distinguish two alternative cases.

Case A_1: For $n \in \mathbb{N}$ sufficiently large $S_\delta(u_n^1) \cap (t_1, +\infty) = \emptyset$.

Case B_1: For a subsequence of (u_n^1), denoted again by (u_n^1), $S_\delta(u_n^1) \cap (t_1, +\infty) \neq \emptyset$ for any $n \in \mathbb{N}$.

If case A_1 occurs then there is $n_\epsilon \geq \max\{n_\epsilon^1, m_\epsilon^1\}$ such that for $n \geq n_\epsilon$ $S_{\delta+\epsilon}(w_1) \subseteq S_\delta(u_n^1) \subseteq S_{\delta-\epsilon}(w_1)$. Therefore $S_{\delta+\epsilon}(w_1(\cdot - t_n^1)) \subseteq S_\delta(u_n) \subseteq S_{\delta-\epsilon}(w_1(\cdot - t_n^1))$ and the lemma is proved with $l = 1$.

Suppose now that case B_1 holds. Then, up to a subsequence, there exists $(s_n^1) \subset \mathbb{R}$ such that $s_n^1 > t_1$, $|u_n^1(s_n^1)| = \delta$ and $|u_n^1(t)| < \delta$ for $t \in (t_1, s_n^1)$. In this case we get that for $n \in \mathbb{N}$ sufficiently large

$$S_\delta(u_n) \cap (t_n^1 + t_1, t_n^1 + s_n^1) = \emptyset. \tag{2.7}$$

Moreover $s_n^1 \to \infty$ holds. Otherwise, up to a subsequence, $s_n^1 \to s \in [t_1, +\infty)$ and then

$$\begin{aligned} \big|\, |w_1(s)| - \delta \,\big| &\le |w_1(s) - u_n^1(s_n^1)| \\ &\le |w_1(s) - w_1(s_n^1)| + |w_1(s_n^1) - u_n^1(s_n^1)| \\ &\le |w_1(s) - w_1(s_n^1)| + \sup_{|t| \le s+1} |w_1(t) - u_n^1(t)| \end{aligned}$$

which implies that $|w_1(s)| = \delta$ contrary to the fact that $s > t_1 = \sup\{\, t \in \mathbb{R} : |w_1(t)| = \delta - \epsilon \,\}$.
For any $n \in \mathbb{N}$ set $t_n^2 = t_n^1 + s_n^1$ and $u_n^2 = u_n^1(\cdot + s_n^1) = u_n(\cdot + t_n^2)$. Then $(u_n^2) \subset \Lambda$, $\varphi(u_n^2) \to a$, $|u_n^2(0)| = \delta$ for any $n \in \mathbb{N}$, (u_n^2) is bounded in E and, up to a subsequence, it converges weakly in E and strongly in $L^\infty_{\rm loc}$ to some $w_2 \in E$. As before, we get that $\varphi(w_2) \le a$, $w_2 \in \Lambda$, $|w_2(0)| = \delta$ and, for the same $\epsilon \in (0, \delta)$ fixed above, there is $n_\epsilon^2 \ge n_\epsilon^1$ such that for $n \ge n_\epsilon^2$

$$S_{\delta+\epsilon}(w_2) \subseteq S_\delta(u_n^2) \tag{2.8}$$

and then $\bigcup_{i=1}^2 S_{\delta+\epsilon}(w_i(\cdot - t_n^i)) \subseteq S_\delta(u_n)$. Moreover, since $S_{\delta-\epsilon}(w_i)$ is bounded and $t_n^2 - t_n^1 = s_n^1 \to \infty$, we can choose $n_\epsilon^2 \in \mathbb{N}$ so large that for any $n \ge n_\epsilon^2$

$$S_{\delta-\epsilon}(w_1(\cdot - t_n^1)) \cap S_{\delta-\epsilon}(w_2(\cdot - t_n^2)) = \emptyset. \tag{2.9}$$

Now we prove that $\sum_{i=1}^2 \varphi(w_i) \le a$. Fixing $T > 0$ and setting $t_i^- = \inf S_\delta(w_i)$ and $t_i^+ = \sup S_\delta(w_i)$ we have that, for (2.9), for $n \in \mathbb{N}$ large enough

$$\sum_{i=1}^2 \int_{t_i^- - T}^{t_i^+ + T} -V(u_n(t + t_n^i))\, dt \le \int_{\mathbb{R}} -V(u_n)\, dt. \tag{2.10}$$

Moreover we also have

$$\int_{\mathbb{R}} |\dot u_n|^2\, dt \ge \sum_{i=1}^2 \Big(2 \int_{\mathbb{R}} \dot u_n(t + t_n^i)\, \dot w_i(t)\, dt - \int_{\mathbb{R}} |\dot w_i|^2\, dt \Big). \tag{2.11}$$

Thus, from (2.10)–(2.11) we get
(2.12)

$$\sum_{i=1}^{2}\Big(\int_{\mathbb{R}} \dot{u}_n(t+t_n^i)\,\dot{w}_i(t)\,dt - \frac{1}{2}\int_{\mathbb{R}} |\dot{w}_i|^2\,dt - \int_{t_i^- -T}^{t_i^+ +T} V(u_n(t+t_n^i))\,dt\Big) \le \varphi(u_n).$$

Since $u_n(\cdot + t_n^i) = u_n^i \to w_i$ weakly in E and strongly in $L^\infty_{\rm loc}$, from (2.12) we infer that

$$\sum_{i=1}^{2}\Big(\frac{1}{2}\int_{\mathbb{R}} |\dot{w}_i|^2\,dt - \int_{t_i^- -T}^{t_i^+ +T} V(u_n(t+t_n^i))\,dt\Big) \le a$$

and then, for the arbitrariness of $T > 0$ we obtain

$$\sum_{i=1}^{2}\varphi(w_i) \le a.$$

Let now $t_2 = t_2(\epsilon) > 0$ be such that $|w_2(t_2)| = \delta - \epsilon$ and $|w_2(t)| < \delta - \epsilon$ for $t > t_2$. Since $u_n^2 \to w_2$ uniformly on $[0, t_2]$ we have that for $n \in \mathbb{N}$ large enough $S_\delta(u_n^2) \cap [0, t_2] \subseteq S_{\delta-\epsilon}(w_2)$ that is

$$S_\delta(u_n) \cap [t_n^2, t_n^2 + t_2] \subseteq S_{\delta-\epsilon}(w_2(\cdot - t_n^2)). \tag{2.13}$$

Thus we arrive to the same alternative as before with (u_n^2), w_2 and t_2 instead of (u_n^1), w_1 and t_1, respectively. Therefore, either:

Case A_2: For $n \in \mathbb{N}$ sufficiently large $S_\delta(u_n^2) \cap (t_2, +\infty) = \emptyset$.

or

Case B_2: For a subsequence of (u_n^2), denoted again by (u_n^2), $S_\delta(u_n^2) \cap (t_2, +\infty) \neq \emptyset$ for any $n \in \mathbb{N}$.

If case A_2 occurs then

$$S_\delta(u_n) \cap (t_n^2 + t_2, +\infty) = \emptyset \tag{2.14}$$

for $n \in \mathbb{N}$ large enough. Consequently, from (2.5)–(2.9) and (2.13)–(2.14), it follows that $S_\delta(u_n) \subseteq \bigcup_{i=1}^2 S_{\delta-\epsilon}(w_i(\cdot - t_n^i))$ and the lemma is proved with $l = 2$.

Otherwise Case B_2 holds and we repeat the same argument as before.

Finally we point out that this procedure must end in a finite number of steps. Indeed, after l steps we have $w_1, \ldots, w_l \in \Lambda$ with $S_\delta(w_i) \neq \emptyset$ $(i = 1, \ldots, l)$ and sequences $(t_n^1), \ldots, (t_n^l) \subset \mathbb{R}$ such that for any $n \in \mathbb{N}$ large enough $\bigcup_{i=1}^{l} S_{\delta/2}(w_i(\cdot - t_n^i)) \subseteq S_\delta(u_n)$ and $S_{\delta/2}(w_i(\cdot - t_n^i)) \cap S_{\delta/2}(w_j(\cdot - t_n^j)) = \emptyset$ if $i \neq j$. Then

$$|S_\delta(u_n)| \geq \sum_{i=1}^{l} |S_{\delta/2}(w_i(\cdot - t_n^i))| = \sum_{i=1}^{l} |S_{\delta/2}(w_i)| \tag{2.15}$$

On one hand we have that

$$\delta^2 \leq \Big(\int_{S_{\delta/2}(w_i)} |\dot{w}_i|\, dt\Big)^2 \leq |S_{\delta/2}(w_i)| \int_{S_{\delta/2}(w_i)} |\dot{w}_i|^2\, dt \leq 2\,\varphi(w_i)\, |S_{\delta/2}(w_i)|$$

and thus

$$|S_{\delta/2}(w_i)| \geq \frac{\delta^2}{2\,\varphi(w_i)} \geq \frac{\delta^2}{2a}. \tag{2.16}$$

On the other hand, by Lemma 2.2, there is $\tau_\delta = \tau_\delta(a) > 0$ such that

$$|S_\delta(u_n)| \leq \tau_\delta \quad \text{for any } n \in \mathbb{N}. \tag{2.17}$$

Therefore, from (2.15)–(2.17), we infer that $l \leq 2\, a\, \tau_\delta\, \delta^{-2} = l_{\delta,a}$. $\square$

3. Minimizing sequences

To study the minimizing sequences of φ in Λ_k, being $k \in \mathbb{N}$ fixed, it is convenient to introduce the sets

$$\widehat{\Lambda}_k = \{\, u \in \Lambda \ :\ \operatorname{ind}_\xi u \geq k \,\}$$

and the corresponding values

$$\widehat{\lambda}_k = \inf\{\, \varphi(u) \ :\ u \in \widehat{\Lambda}_k \,\}.$$

We notice that $\lambda_k \geq \widehat{\lambda}_k > 0$ for any $k \in \mathbb{N}$.

Lemma 3.1. *Let $k \in \mathbb{N}$, $(u_n) \in \widehat{\Lambda}_k$ and $u \in E \setminus \{0\}$ be such that $\varphi(u_n) \to \widehat{\lambda}_k$ and $u_n \to u$ weakly in E. Then $u \in \Lambda$, $\mathrm{ind}_\xi\, u > 0$ and $\varphi(u) = \widehat{\lambda}_{k_0}$ where $k_0 = \mathrm{ind}_\xi\, u$.*

Proof. Without loss of generality we can assume that $u_n \to u$ weakly in E and strongly in L^∞_{loc}. Since φ is w.l.s.s., we have $\varphi(u) \le \widehat{\lambda}_k$ and then, by Remark 2.3, $u \in \Lambda$. Hence for any $\delta \in (0, |\xi|)$ there is $T_\delta > 0$ such that $|u(t)| < \delta$ for $|t| > T_\delta$. We set

$$v_n(t) = \begin{cases} u_n(t) & \text{for } |t| > T_\delta + 1 \\ 0 & \text{for } |t| \le T_\delta \\ (t - T_\delta)\, u_n(T_\delta + 1) & \text{for } T_\delta < t \le T_\delta + 1 \\ (-t - T_\delta)\, u_n(-T_\delta - 1) & \text{for } -T_\delta - 1 \le t < -T_\delta \end{cases}$$

and we evaluate

$$\varphi(v_n) = \varphi(u_n) - \varphi_{I_\delta}(u_n) + \varphi_{I_\delta^-}(v_n) + \varphi_{I_\delta^+}(v_n) \tag{3.1}$$

where $I_\delta = [-T_\delta - 1, T_\delta + 1]$, $I_\delta^- = [-T_\delta - 1, -T_\delta]$ and $I_\delta^+ = [T_\delta, T_\delta + 1]$. For the pointwise convergence, we have $|u_n(|T_\delta + 1|)| \le \delta$. Hence

$$\varphi_{I_\delta^-}(v_n) + \varphi_{I_\delta^+}(v_n) \le \delta^2 + M_\delta \tag{3.2}$$

where $M_\delta = 2 \max_{|x| \le \delta} |V(x)|$. In addition, since $u_n \to u$ weakly in E and strongly in L^∞_{loc} we also have

$$\varphi_{I_\delta}(u) \le \liminf \varphi_{I_\delta}(u_n). \tag{3.3}$$

Then (3.1)–(3.3) imply

$$\limsup \varphi(v_n) \le \widehat{\lambda}_k + \delta^2 + M_\delta - \varphi_{I_\delta}(u). \tag{3.4}$$

By (V1)–(V2) and since $u \ne 0$, taking $\delta \in (0, |\xi|)$ sufficiently small, we can insure that $\delta^2 + M_\delta - \varphi_{I_\delta}(u) < 0$ and thus

$$\limsup \varphi(v_n) < \widehat{\lambda}_k. \tag{3.5}$$

If it were $\mathrm{ind}_\xi\, u \le 0$, then, for $\delta > 0$ small we would have $\mathrm{ind}_\xi\, v_n = \mathrm{ind}_\xi\, u_n \ge k$, that is $v_n \in \widehat{\Lambda}_k$, which, together with (3.5) leads to a contradiction with the definition of $\widehat{\lambda}_k$. Therefore $\mathrm{ind}_\xi\, u = k_0 > 0$.

To prove that $\varphi(u) = \widehat{\lambda}_{k_0}$ we argue by contradiction assuming that $\varphi(u) > \widehat{\lambda}_{k_0}$. Then there is $v \in \widehat{\Lambda}_{k_0}$ such that $\operatorname{supp} v \subset [-R, R]$ for some $R > 0$ and $\varphi(v) \leq \widehat{\lambda}_{k_0} + \frac{\epsilon}{2}$ where $\epsilon = \varphi(u) - \widehat{\lambda}_{k_0}$. We choose $\delta = \delta(\epsilon)$ such that $\delta^2 + M_\delta < \frac{\epsilon}{8}$ and $T_\delta > R$ such that $|u(t)| < \delta$ for $|t| > T_\delta$ and $\int_{\mathbb{R}\setminus I_\delta}(\frac{1}{2}|\dot{u}|^2 - V(u))\,dt < \frac{\epsilon}{8}$. Setting $y_n = v_n + v$, we have that $y_n \in \widehat{\Lambda}_k$ and

$$\varphi(y_n) = \varphi(v_n) + \varphi(v) \leq \varphi(v_n) + \varphi(u) - \frac{\epsilon}{2}$$

which, together with (3.4) and by the choice of δ and T_δ gives

$$\limsup \varphi(y_n) \leq \widehat{\lambda}_k - \frac{\epsilon}{4},$$

contrary to the definition of $\widehat{\lambda}_k$. □

Now we recall a result, already proved by Rabinowitz [R1, Proposition 3.41].

Lemma 3.2. *Let $k \in \mathbb{N}$ and $v \in \widehat{\Lambda}_k$ such that $\varphi(v) = \widehat{\lambda}_k$. If $v(\bar{s}) = v(\bar{t})$ for some $\bar{s}, \bar{t} \in \mathbb{R}$ with $\bar{s} < \bar{t}$, then there are $s, t \in \mathbb{R}$ such that $\bar{s} \leq s < t \leq \bar{t}$, $v(s) = v(t)$ and $\operatorname{ind}_\xi v|_{[s,t]} = 1$.*
In particular, for any $k \in \mathbb{N}\setminus\{1\}$ and $v \in \widehat{\Lambda}_k$ such that $\varphi(v) = \widehat{\lambda}_k$ there exist $s, t \in \mathbb{R}$ such that $s < t$, $v(s) = v(t)$ and $\operatorname{ind}_\xi v|_{[s,t]} = 1$.

In the next lemma we give the characterization of the minimizing sequences for φ with respect to $\widehat{\Lambda}_k$.

Lemma 3.3. *Let $k \in \mathbb{N}$ and $(u_n) \subset \widehat{\Lambda}_k$ such that $\varphi(u_n) \to \widehat{\lambda}_k$. Then there are a subsequence of (u_n), denoted again by (u_n), functions $w_1, \ldots, w_l \in \Lambda$ and corresponding sequences $(t_n^1), \ldots, (t_n^l) \subset \mathbb{R}$ with $\lim_{n\to\infty}(t_n^{i+1} - t_n^i) = \infty$ for any $i = 1, \ldots, l-1$, such that:*

(3.6)

$$\lim_{n\to\infty} u_n(\cdot + t_n^i) = w_i \text{ weakly in } E \text{ and strongly in } L^\infty_{\text{loc}} \text{ for any } i = 1, \ldots, l;$$

(3.7) $\operatorname{ind}_\xi w_i = k_i \in \{1, \ldots, k\}$ *and* $\varphi(w_i) = \widehat{\lambda}_{k_i}$ *for any* $i = 1, \ldots, l;$

$$\sum_{i=1}^{l} \operatorname{ind}_\xi w_i = k; \tag{3.8}$$

$$\sum_{i=1}^{l} \varphi(w_i) = \widehat{\lambda}_k. \tag{3.9}$$

Proof. Let $k \in \mathbb{N}$ and $(u_n) \subset \widehat{\Lambda}_k$ such that $\varphi(u_n) \to \widehat{\lambda}_k$. Noting that $\limsup \|u_n\|_{L^\infty} \geq |\xi|$ and fixing $\delta = \frac{3}{4}|\xi|$, by Lemma 2.4, there are a subsequence of (u_n), denoted again by (u_n), functions $w_1, \dots, w_l \in \Lambda \setminus \{0\}$ and sequences $(t_n^1), \dots, (t_n^l) \subset \mathbb{R}$ satisfying (2.1)–(2.4). By Lemma 3.1 $\operatorname{ind}_\xi w_i = k_i > 0$ and $\varphi(w_i) = \widehat{\lambda}_{k_i}$ for any $i = 1, \dots, l$. Taking $\epsilon = \frac{1}{2}|\xi|$ in (2.4), we get that for $n \in \mathbb{N}$ large enough $\sum_{i=1}^l \operatorname{ind}_\xi w_i = \operatorname{ind}_\xi u_n$. Moreover, defining $y_n = \sum_{i=1}^l w_i(\cdot - t_n^i)$, by (2.1), we can easily check that $\varphi(y_n) \to \sum_{i=1}^l \varphi(w_i)$ and $\operatorname{ind}_\xi y_n = \sum_{i=1}^l \operatorname{ind}_\xi w_i = \operatorname{ind}_\xi u_n$. Hence $(y_n) \subset \widehat{\Lambda}_k$ and thus $\widehat{\lambda}_k \leq \sum_{i=1}^l \varphi(w_i)$. By (2.3) we infer that $\widehat{\lambda}_k = \sum_{i=1}^l \varphi(w_i)$.
If we suppose that $\sum_{i=1}^l \operatorname{ind}_\xi w_i > k$ then either $\operatorname{ind}_\xi w_j > 1$ for some $j \in \{1, \dots, l\}$, or $\operatorname{ind}_\xi w_i = 1$ for every $i = 1, \dots, l$. In the first case, we set

$$\bar{w}_j(t) = \begin{cases} w_j(t) & \text{for } t \leq s_j \\ w_j(t - s_j + t_j) & \text{for } t > s_j \end{cases}$$

where $s_j, t_j \in \mathbb{R}$ are given by applying Lemma 3.2 to w_j. Then, for any $n \in \mathbb{N}$ we put $\bar{y}_n = y_n - w_j(\cdot - t_n^j) + \bar{w}_j(\cdot - t_n^j)$, getting a contradiction because $\operatorname{ind}_\xi \bar{y}_n = \operatorname{ind}_\xi y_n - 1 \geq k$ and

$$\lim \varphi(\bar{y}_n) = \sum_{i=1}^{l} \varphi(w_i) + \varphi(\bar{w}_j) - \varphi(w_j) < \sum_{i=1}^{l} \varphi(w_i) = \widehat{\lambda}_k.$$

In the second case we consider the sequence $\bar{y}_n = \sum_{i=1}^{l-1} w_i(\cdot - t_n^i)$ which lies in $\widehat{\Lambda}_k$ but satisfies $\lim \varphi(\bar{y}_n) = \sum_{i=1}^{l-1} \varphi(w_i) < \widehat{\lambda}_k$, contrary to the definition of $\widehat{\lambda}_k$. □

Remark 3.4. In the proof of Lemma 3.3 we have seen that $\mathrm{ind}_\xi u_n = \sum_{i=1}^{l} \mathrm{ind}_\xi w_i = k$, that is, for $n \in \mathbb{N}$ sufficiently large $u_n \in \Lambda_k$. Since $\widehat{\lambda}_k \leq \lambda_k$ and $\varphi(u_n) \to \widehat{\lambda}_k$, we get $\widehat{\lambda}_k = \lambda_k$. Hence any minimizing sequence (u_n) for φ with respect to Λ_k is also minimizing with respect to $\widehat{\Lambda}_k$ and then (3.6)–(3.9) hold for (u_n), with λ_k instead of $\widehat{\lambda}_k$.

Remark 3.5. As immediate consequence of Lemma 3.3, we get that there exists $v_1 \in \Lambda_1$ such that $\varphi(v_1) = \lambda_1$. Moreover, fixed $k \in \mathbb{N}$, setting $u_n = \sum_{j=1}^{k} v_1(\cdot - jn)$, we have that for $n \in \mathbb{N}$ large enough $u \in \Lambda_k$ and $\lim \varphi(u_n) = k\lambda_1$. Therefore $\lambda_k \leq k\lambda_1$. In addition, since $\widehat{\lambda}_k = \lambda_k$ we also have that $\lambda_k \leq \lambda_{k+1}$ for every $k \in \mathbb{N}$.

4. Proof of Theorem 1.1

To begin, we state a sufficient condition for the existence of a mimimizer for φ in Λ_k.

Lemma 4.1. *Let $k \in \mathbb{N} \setminus \{1\}$. If $\lambda_k < \lambda_{k_1} + \cdots + \lambda_{k_l}$ whenever $l > 1$ and $k_1, \ldots, k_l \in \mathbb{N}$ satisfy $k_1 + \cdots + k_l = k$, then there is $v \in \Lambda_k$ such that $\varphi(v) = \lambda_k$.*

Proof. Let $k \in \mathbb{N}\setminus\{1\}$ and let $(u_n) \in \Lambda_k$ be a sequence such that $\varphi(u_n) \to \lambda_k$. Then by Lemma 3.3 and Remark 3.4, there are a subsequence of (u_n), denoted always (u_n), functions $w_1, \ldots, w_l \in \Lambda$ and corresponding sequences $(t_n^1), \ldots, (t_n^l) \subset \mathbb{R}$ satisfying (3.6)–(3.9). If $l > 1$, then by (3.7)–(3.9), $\sum_{i=1}^{l} k_i = k$ and $\sum_{i=1}^{l} \lambda_{k_i} = \lambda_k$, contradicting the hypothesis. Therefore $l = 1$ and thus, using (3.8)–(3.9), we get that $w_1 \in \Lambda_k$ with $\varphi(w_1) = \lambda_k$. □

Firstly we prove Part *(ii)* of Theorem 1.1. By Remark 3.5, we know that $\lambda_k \leq k\lambda_1$ for any $k \in \mathbb{N}$. As we shall see in the next lemma, the value $\bar{k} = \sup\{\, k \in \mathbb{N} \ : \ \lambda_k = k\lambda_1 \,\}$ plays an important role in the problem of existence of a minimum point for φ in Λ_k.

Lemma 4.2. *Suppose $\bar{k} < \infty$. Then for any $k > \bar{k}$:*

$(i)_k$ $\lambda_k k^{-1} < \lambda_{k-1}(k-1)^{-1} < \lambda_1$*;*

$(ii)_k$ *there is $v_k \in \Lambda_k$ such that $\varphi(v_k) = \lambda_k$;*

$(iii)_k$ *there are* $s_k, t_k \in \mathbb{R}$ *such that* $s_k < t_k$, $v_k(s_k) = v_k(t_k)$, $\mathrm{ind}_\xi\, v_k|_{[s_k,t_k]} = 1$ *and* $\lambda_{k+1} - \lambda_k \le \varphi(v_k|_{[s_k,t_k]}) \le \lambda_k - \lambda_{k-1}$.

Proof. Since $\bar{k} < \infty$ the value $k_0 = \inf\{\, k \in \mathbb{N} \;:\; \lambda_k < k\lambda_1 \,\}$ is well defined. We will prove $(i)_k$–$(iii)_k$ by induction on $k \in \mathbb{N}$, $k \ge k_0$.
Let us start with $k = k_0$. Firstly $(i)_{k_0}$ is true by definition of k_0. Secondly, $(ii)_{k_0}$ follows from Lemma 4.1. Indeed if $l > 1$ and $k_1, \dots, k_l \in \mathbb{N}$ satisfy $k_1 + \dots + k_l = k_0$, then $k_i \le k_0$ for any $i = 1, \dots, l$ and hence $\sum_{i=1}^{l} \lambda_{k_i} = \sum_{i=1}^{l} k_i\lambda_1 = k_0\lambda_1 > \lambda_{k_0}$. Therefore, by Lemma 4.1, there is $v_{k_0} \in \Lambda_{k_0}$ satisfying $\varphi(v_{k_0}) = \lambda_{k_0}$. To prove $(iii)_{k_0}$ we apply Lemma 3.2 to v_{k_0}. Thus there exist $s_{k_0}, t_{k_0} \in \mathbb{R}$ such that $s_{k_0} < t_{k_0}$, $v_{k_0}(s_{k_0}) = v_{k_0}(t_{k_0})$ and $\mathrm{ind}_\xi\, v_{k_0}|_{[s_{k_0},t_{k_0}]} = 1$. Then, setting

$$u_{k_0-1}(t) = \begin{cases} v_{k_0}(t) & \text{for } t \le s_{k_0} \\ v_{k_0}(t - s_{k_0} + t_{k_0}) & \text{for } t > s_{k_0} \end{cases} \tag{4.1}$$

and

$$u_{k_0+1}(t) = \begin{cases} v_{k_0}(t) & \text{for } t \le t_{k_0} \\ v_k(t + s_{k_0} - t_{k_0}) & \text{for } t > t_{k_0} \end{cases}$$

we have that $u_{k_0\pm1} \in \Lambda_{k_0\pm1}$ and, by $(i)_{k_0}$,

$$\lambda_{k_0\pm1} \le \varphi(u_{k_0\pm1}) = \varphi(v_{k_0}) \pm \varphi(v_{k_0}|_{[s_{k_0},t_{k_0}]}) = \lambda_{k_0} \pm \varphi(v_{k_0}|_{[s_{k_0},t_{k_0}]})$$

that is $(iii)_{k_0}$.
Now let us assume that $(i)_k$–$(iii)_k$ hold and let us prove $(i)_{k+1}$–$(iii)_{k+1}$. By $(ii)_k$ there is $v_k \in \Lambda_k$ such that $\varphi(v_k) = \lambda_k$. Then, let $s_k, t_k \in \mathbb{R}$ be given by $(iii)_k$. Setting

$$u_{k+1}(t) = \begin{cases} v_k(t) & \text{for } t \le t_k \\ v_k(t + s_k - t_k) & \text{for } t > t_k \end{cases} \tag{4.2}$$

we have that $u_{k+1} \in \Lambda_{k+1}$ and then, by $(i)_k$ and $(iii)_k$

$$\lambda_{k+1} \le \varphi(u_{k+1}) = \varphi(v_k) + \varphi(v_k|_{[s_k,t_k]}) \le \lambda_k + (\lambda_k - \lambda_{k-1}) < \lambda_k + \frac{\lambda_k}{k},$$

that is $\lambda_{k+1}(k+1)^{-1} < \lambda_k k^{-1}$. Consequently we also have

$$\lambda_1 = \dots = \frac{\lambda_{k_0-1}}{k_0 - 1} > \frac{\lambda_{k_0}}{k_0} > \dots > \frac{\lambda_{k+1}}{k+1}. \tag{4.3}$$

To prove $(ii)_{k+1}$ we use again Lemma 4.1. Indeed let $l>1$ and $k_1,\ldots,k_l \in \mathbb{N}$ such that $k_1+\cdots+k_l=k+1$. Since $k_i<k+1$ for any $i=1,\ldots,l$, by (4.3) $\sum_{i=1}^{l}\lambda_{k_i} > \sum_{i=1}^{l}\frac{k_i}{k+1}\lambda_{k+1}=\lambda_{k+1}$. Therefore Lemma 4.1 applies, giving the existence of $v_{k+1}\in\Lambda_{k+1}$ such that $\varphi(v_{k+1})=\lambda_{k+1}$. The proof of $(iii)_{k+1}$ is the same as for $(iii)_{k_0}$. Finally we point out that, by (4.3), $k_0=\bar{k}+1$ and then the lemma is proved. □

Lemma 4.3. *Suppose $\bar{k}<\infty$. If $k>\bar{k}$, $l>1$ and $k_1,\ldots,k_l\in\mathbb{N}$ satisfy $k_1+\cdots+k_l=k$, then*

$$\lambda_{k_1}+\cdots+\lambda_{k_l}-\lambda_k \geq \lambda_1-\frac{\lambda_{\bar{k}+1}}{\bar{k}+1}.$$

Proof. For any $k>\bar{k}$ let $v_k\in\Lambda_k$ and $s_k,t_k\in\mathbb{R}$ be given according to Lemma 4.2 $(ii)_k$–$(iii)_k$. Always by $(iii)_k$ we remark that

$$\varphi(v_{k+1}|_{[s_{k+1},t_{k+1}]}) \leq \varphi(v_k|_{[s_k,t_k]}) \quad \text{for any } k>\bar{k}. \tag{4.4}$$

For $k>\bar{k}$, by (4.4) we have

$$\sum_{j=\bar{k}+1}^{k}\varphi(v_j|_{[s_j,t_j]}) \geq \sum_{j=\bar{k}+1}^{k}\varphi(v_k|_{[s_k,t_k]}) = (k-\bar{k})\,\varphi(v_k|_{[s_k,t_k]}). \tag{4.5}$$

On the other hand, by Lemma 4.2 $(iii)_k$ we get

$$\sum_{j=\bar{k}+1}^{k}\varphi(v_j|_{[s_j,t_j]}) \leq \sum_{j=\bar{k}+1}^{k}(\lambda_j-\lambda_{j-1}) = \lambda_k-\lambda_{\bar{k}} = \lambda_k-\bar{k}\lambda_1. \tag{4.6}$$

Now (4.5)–(4.6) and Lemma 4.2 imply

$$(k-\bar{k})\,\varphi(v_k|_{[s_k,t_k]}) \leq \lambda_k-\lambda_1-(\bar{k}-1)\lambda_1 < \lambda_k-\lambda_1-(\bar{k}-1)\,\varphi(v_k|_{[s_k,t_k]})$$

and then

$$\lambda_k > \lambda_1+(k-1)\,\varphi(v_k|_{[s_k,t_k]}). \tag{4.7}$$

Taking now $k, k' \in \mathbb{N}$ such that $\bar{k} < k' \leq k$, by (4.4) we have
(4.8)
$$\sum_{j=k'+1}^{k} \varphi(v_{j-1}|_{[s_{j-1},t_{j-1}]}) \leq \sum_{j=k'+1}^{k} \varphi(v_{k'}|_{[s_{k'},t_{k'}]}) = (k-k')\varphi(v_{k'}|_{[s_{k'},t_{k'}]}).$$

In addition, by Lemma 4.2 $(iii)_k$

$$\sum_{j=k'+1}^{k} \varphi(v_{j-1}|_{[s_{j-1},t_{j-1}]}) \geq \sum_{j=k'+1}^{k} (\lambda_j - \lambda_{j-1}) = \lambda_k - \lambda_{k'}. \tag{4.9}$$

Hence (4.8) and (4.9) give

$$\lambda_k - \lambda_{k'} \leq (k-k')\varphi(v_{k'}|_{[s_{k'},t_{k'}]}). \tag{4.10}$$

Let now $k > \bar{k}$, $l > 1$ and $k_1, \ldots, k_l \in \mathbb{N}$ such that $k_1 + \cdots + k_l = k$. Since $\lambda_{k_i} + \lambda_{k_j} \geq \lambda_{k_i+k_j}$, it is sufficient to prove the thesis for $l = 2$. We can assume that $k_1 \geq k_2$. If $k_2 \leq \bar{k}$ then, by Lemma 4.2 $(i)_k$

$$\lambda_{k_1} + \lambda_{k_2} - \lambda_k \geq \tfrac{k_1}{k}\lambda_k + k_2\lambda_1 - \lambda_k = k_2\left(\lambda_1 - \tfrac{\lambda_k}{k}\right) \geq k_2\left(\lambda_1 - \tfrac{\lambda_{\bar{k}+1}}{\bar{k}+1}\right) \geq \lambda_1 - \tfrac{\lambda_{\bar{k}+1}}{\bar{k}+1}.$$

Let us assume now $k_1 \geq k_2 > \bar{k}$ so that, by (4.4),

$$\varphi(v_{k_1}|_{[s_{k_1},t_{k_1}]}) \leq \varphi(v_{k_2}|_{[s_{k_2},t_{k_2}]}) \leq \frac{\lambda_{\bar{k}+1}}{\bar{k}+1}. \tag{4.11}$$

Then, applying (4.7) with $k = k_2$, (4.10) with $k' = k_1$, and using (4.11) we get

$$\lambda_{k_1} + \lambda_{k_2} - \lambda_k \geq \lambda_1 - \varphi(v_{k_2}|_{[s_{k_2},t_{k_2}]}) \geq \lambda_1 - \frac{\lambda_{\bar{k}+1}}{\bar{k}+1}.$$

□

Lemma 4.4. *If $v \in \Lambda_k$ satisfies $\varphi(v) = \lambda_k$ then v is a homoclinic solution of (HS).*

Proof. Standard arguments apply to show that the function v is a classical solution to (HS) (see for instance [R1] or [R2]). Moreover $v(t) \to 0$ as $t \to \pm\infty$ because $v \in \Lambda$. Finally, since the system (HS) is autonomous, i.e. V does not depend on t, the energy $h = \frac{1}{2}|\dot{v}(t)|^2 + V(v(t))$ is constant and then, by (V2), $|\dot{v}(t)|^2 \to 2h$ as $t \to \pm\infty$. Since $v \in E$, we get $h = 0$. □

Now we prove Part (i) of Theorem 1.1.

Lemma 4.5. *The value $\bar{k}$ is finite if and only if the following condition holds:*

$(*)$ *there exist $T \in (0,\infty)$ and $u \in H^1([0,T],\mathbb{R}^2)$ such that $u(0) = u(T)$, $u(t) \neq \xi$ for any $t \in [0,T]$, $\mathrm{ind}_\xi\, u = 1$ and $\int_0^T (\frac{1}{2}|\dot{u}|^2 - V(u))\, dt < \lambda_1$.*

Proof. If $\bar{k}$ is finite, then, taking an integer $k > \bar{k}$, we consider the function $v_k \in \Lambda_k$ and the values $s_k, t_k \in \mathbb{R}$ given by Lemma 4.2. We set $T = t_k - s_k$ and $u(t) = v_k(t - s_k)$ for $t \in [0,T]$. Then, by $(iii)_k$, $\mathrm{ind}_\xi\, u = \mathrm{ind}_\xi\, v_k|_{[s_k,t_k]} = 1$ and $\int_0^T (\frac{1}{2}|\dot{u}|^2 - V(u))\, dt = \varphi(v_k|_{[s_k,t_k]}) < \lambda_k k^{-1} < \lambda_1$, that is $(*)$.
Conversely, let us assume $(*)$. We can always suppose that $\theta u(0) \neq \xi$ for every $\theta \in [0,1]$. Defining for any $k \in \mathbb{N}$

$$u_k(t) = \begin{cases} 0 & \text{for } t \leq -1 \text{ and } t > kT+1 \\ (1+t)\,u(0) & \text{for } -1 < t \leq 0 \\ u(t - jT) & \text{for } (j-1)T < t \leq jT \text{ and } j = 1,\dots,k \\ (kT+1-t)\,u(0) & \text{for } kT < t \leq kT+1 \end{cases}$$

we have that $u_k \in \Lambda_k$ and $\varphi(u_k) = c + k\,\varphi_{[0,T]}(u)$ where c is a positive constant independent from k. Since $\varphi_{[0,T]}(u) < \lambda_1$, for $k \in \mathbb{N}$ large enough $c + k\,(\varphi_{[0,T]}(u) - \lambda_1) < 0$ and then $\lambda_k \leq \varphi(u_k) < k\lambda_1$, that is $\bar{k} < \infty$. □

The proof of Theorem 1.1 is completed with the following result.

Lemma 4.6. *If $\bar{k} > 2$, then, for $1 < k < \bar{k}$, $\varphi(u) > \lambda_k$ for every $u \in \Lambda_k$.*

Proof. By contradiction, let $k \in \mathbb{N} \cap \{2,\dots,\bar{k}-1\}$ be such that $\varphi(v_k) = \lambda_k$ for some $v_k \in \Lambda_k$. Then, by Remark 3.4 and Lemma 3.2, there are $s_k, t_k \in \mathbb{R}$ such that $s_k < t_k$, $v_k(s_k) = v_k(t_k)$ and $\mathrm{ind}_\xi\, v_k|_{[s_k,t_k]} = 1$. Let $u_{k+1} \in \Lambda_{k+1}$ be defined by (4.2) and $u_{k-1} \in \Lambda_{k-1}$ be given by (4.1) with k instead of k_0. If it were $\varphi(u_{k-1}) = \lambda_{k-1}$, then, by Lemma 4.4, u_{k-1} would be a solution of (HS) and in particular $\dot{v}_k(s_k) = \dot{v}_k(t_k)$. Then $v(\cdot - s_k)$ and $v(\cdot - t_k)$ are two distinct solutions to the Cauchy problem defined by (HS) with initial data $x(0) = v_k(s_k)$, $\dot{x}(0) = \dot{v}_k(s_k)$, contradicting the uniqueness of solution, given by the locally Lipschitz continuity of ∇V. Then $\varphi(u_{k-1}) > \lambda_{k-1}$ and $\varphi(v_k|_{[s_k,t_k]}) = \varphi(v_k) - \varphi(u_{k-1}) < \lambda_k - \lambda_{k-1} = \lambda_1$, because $k < \bar{k}$. But now we get $\lambda_{k+1} \leq \varphi(u_{k+1}) = \varphi(v_k) + \varphi(v_k|_{[s_k,t_k]}) < (k+1)\lambda_1$ and then $k+1 > \bar{k}$, a contradiction. □

5. Construction of a periodic orbit at level $\bar{\lambda}$

With this Section we tackle the problem of the existence of heteroclinic solutions to (HS). Here and in the following Sections 6 and 7 we assume that the potential V satisfies (V1)–(V4) and that condition $(*)$ holds.

Lemma 5.1. $\lim_{k\to\infty} \lambda_k k^{-1} = \inf_{k>\bar{k}} \lambda_k k^{-1} = \bar{\lambda}$.

Proof. By Lemma 4.2 (i) and (iii), there exists $\lim \lambda_k k^{-1} = \inf \lambda_k k^{-1} \geq \bar{\lambda}$. Fixed $\epsilon > 0$ let $T \in (0,\infty)$ and $u \in \Lambda_{1,T}$ be such that $\varphi_T(u) \leq \bar{\lambda} + \epsilon$ and, for $k \in \mathbb{N}$, let $u_k \in \Lambda_k$ be defined as in the proof of Lemma 4.5. Then for any $k \in \mathbb{N}$ $\lambda_k \leq \varphi(u_k) = c + k\,\varphi_T(u) \leq c + k\,(\bar{\lambda} + \epsilon)$ being $c > 0$ a constant independent of k. Then $\lim \lambda_k k^{-1} \leq \bar{\lambda} + \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $\lim \lambda_k k^{-1} = \bar{\lambda}$. □

By Lemmas 4.1 to 4.5, there is $\bar{k} \in \mathbb{N}$ such that for any $k > \bar{k}$ there are a homoclinic solution to (HS) $v_k \in \Lambda_k$ and a pair $s_k, t_k \in \mathbb{R}$ satisfying $\varphi(v_k) = \lambda_k$, $s_k < t_k$, $v_k(s_k) = v_k(t_k)$ $\mathrm{ind}_\xi\, v_k|_{[s_k,t_k]} = 1$ and $\varphi(v_k|_{[s_k,t_k]}) < \frac{\lambda_k}{k}$. For any $k > \bar{k}$ let $T_k = t_k - s_k$ and $I_k = [s_k, t_k]$.

Lemma 5.2. (i) *There exist* $R, \rho > 0$ *such that for any* $k > \bar{k}$ *and* $t \in I_k$ $|v_k(t) - \xi| \geq \rho$ *and* $|v_k(t)| \leq R$.
(ii) $0 < \inf T_k \leq \sup T_k < \infty$.

Proof. For any $k > \bar{k}$ let $\delta_k = \inf_{t\in I_k} |v_k(t)|$. We claim that $\sup_{k>\bar{k}} \delta_k < \infty$. Indeed, by (V4), for any $j \in \mathbb{N}$ there is $r_j > 0$ such that $-V(x) \geq j|x|^{-2}$ for $|x| \geq r_j$. If, by contradiction, $\sup_{k>\bar{k}} \delta_k = \infty$, there is $k_j > \bar{k}$ such that $\delta_{k_j} \geq r_j$ and then

$$\begin{aligned}
\varphi(v_{k_j}|_{I_{k_j}}) &\geq \int_{s_{k_j}}^{t_{k_j}} (\tfrac{1}{2}|\dot{v}_{k_j}|^2 + j|v_{k_j}|^{-2})\,dt \\
&\geq \inf \{ \int_0^{T_{k_j}} (\tfrac{1}{2}|\dot{u}|^2 + j|u|^{-2})\,dt \;:\; u \in \Lambda_{1,T_{k_j}},\ |u(t)| \geq \delta_{k_j}\ \forall t \in [0, T_{k_j}] \} \\
&= 2\pi^2 \rho_j^2 T_{k_j}^{-1} + j\rho_j^{-2} T_{k_j} \\
&\geq 2\pi\sqrt{2j}.
\end{aligned}$$

(We remark that the infimum in the above estimate is attained by the function $u_j(t) = (\rho_j \cos\omega_j t, \rho_j \sin\omega_j t)$, with $\omega_j = 2\pi/T_{k_j}$ and a suitable $\rho_j \geq \delta_{k_j}$). For $j \to \infty$ we get $\varphi(v_{k_j}|_{I_{k_j}}) \to \infty$, contrary to the fact that $\varphi(v_k|_{I_k}) \leq \lambda_1$ for every $k > \bar{k}$. Therefore the claim is proved and we can apply Lemma 2.2 getting (i). Hence we obtain

$$2\pi\rho \leq \int_{I_k} |\dot{v}_k|\,dt \leq \sqrt{T_k}\Big(\int_{I_k} |\dot{v}_k|^2\,dt\Big)^{1/2} \leq \sqrt{\lambda_1 T_k}$$

which implies $\inf T_k > 0$.
To prove that $\sup T_k < \infty$, firstly we verify that $\inf_{t\in I_k} |v_k(t)| \geq \delta$ for some $\delta > 0$ independent from k. Indeed, if not, then, up to a subsequence, for any $k \in \mathbb{N}$ there is $\tau_k \in I_k$ such that $v_k(\tau_k) \to 0$. Then, setting

$$u_k(t) = \begin{cases} 0 & \text{for } t \leq \tau_k - 1 \text{ and } t > \tau_k + T_k + 1 \\ (t - \tau_k + 1)\, v_k(\tau_k) & \text{for } \tau_k - 1 < t \leq \tau_k \\ v_k(t) & \text{for } \tau_k < t \leq t_k \\ v_k(t - T_k) & \text{for } t_k < t \leq \tau_k + T_k \\ (t - \tau_k - T_k + 1)\, v_k(\tau_k) & \text{for } \tau_k + T_k < t \leq \tau_k + T_k + 1 \end{cases}$$

we easily check that $u_k \in \Lambda_1$ and $\varphi(u_k) = \epsilon_k + \varphi(v_k|_{I_k})$ with $\epsilon_k \to 0$. Hence, taking the limit $k \to \infty$, we get $\lambda_1 \leq \bar{\lambda}$, contrary to $(*)$. Therefore, by (V2) and since $\|v_k\|_{L^\infty(I_k)} \leq R$, we also have $\inf_{t\in I_k} |V(v_k(t))| \geq b$ for some constant $b > 0$ independent from k. Consequently

$$\lambda_1 \geq \varphi(v_k|_{I_k}) = 2\int_{I_k} |V(v_k)|\,dt \geq 2T_k b$$

which gives $\sup T_k < \infty$. □

Remark 5.3. A careful analysis of the proof of Lemma 5.2 shows that instead of (V4) it is enough to assume that $\limsup_{|x|\to\infty} |x|^2 V(x) < -\frac{\bar{\lambda}^2}{8\pi}$.

By Lemma 5.2, passing to a subsequence if necessary, we can assume that

$$\lim_{k\to\infty} T_k = \overline{T} \tag{5.1}$$

for some $\overline{T} \in (0,\infty)$. For any $k > \bar{k}$ we define $\theta_k = T_k \overline{T}^{-1}$ and

$$u_k(t) = v_k(\theta_k t - s_k)\,,\ t \in [0, \overline{T}].$$

We point out that for any $k > \bar{k}$ $u_k \in \Lambda_{1,\overline{T}} \cap C^2([0,\overline{T}],\mathbb{R}^2)$ and u_k solves

$$\ddot{u}_k + \theta_k^2 \nabla V(u_k) = 0 \text{ and } \frac{1}{2}|\dot{u}_k|^2 + \theta_k^2 V(u_k) = 0 \text{ in } (0,\overline{T}). \tag{5.2}$$

Lemma 5.4. *There is $\bar{u} \in \Lambda_{1,\overline{T}} \cap C^1([0,\overline{T}],\mathbb{R}^2)$ and a subsequence of (u_k), denoted again by (u_k), such that $\|u_k - \bar{u}\|_{C^1([0,\overline{T}])} \to 0$. Moreover $\varphi_{\overline{T}}(\bar{u}) = \bar{\lambda}$ and $\bar{u}$ is a $\overline{T}$-periodic solution to (HS) with zero energy.*

Proof. For any $k > \bar{k}$ it holds that

$$\min\{\theta_k, \theta_k^{-1}\}\,\varphi(v_k|_{I_k}) \le \varphi_{\overline{T}}(u_k) \le \max\{\theta_k, \theta_k^{-1}\}\,\varphi(v_k|_{I_k}).$$

Then, by Lemma 5.1 and (5.1), $\lim \varphi_{\overline{T}}(u_k) = \bar{\lambda}$. Therefore, in particular, $\bar{\lambda} = \inf\{\varphi_{\overline{T}}(u) \,:\, u \in \Lambda_{1,\overline{T}}\}$ and thus (u_k) is a minimizing sequence for $\varphi_{\overline{T}}$ in $\Lambda_{1,\overline{T}}$. Since (u_k) is bounded in $E_{\overline{T}}$, it admits a subsequence which converges weakly in $E_{\overline{T}}$ and uniformly on $[0,\overline{T}]$ to some $\bar{u} \in \Lambda_{1,\overline{T}}$ satisfying $\varphi_{\overline{T}}(\bar{u}) = \bar{\lambda}$. By standard arguments $\bar{u}$ turns out to be a $\overline{T}$-periodic solution to (HS). Moreover, by Lemma 5.2 (i) and by (5.1)–(5.2), $\sup_{k>\bar{k}} \|u_k\|_{C^2([0,\overline{T}])} < \infty$. Then, by Ascoli–Arzelà Theorem, up to a subsequence $\|u_k - \bar{u}\|_{C^1([0,\overline{T}])} \to 0$ and $\frac{1}{2}|\dot{\bar{u}}|^2 + V(\bar{u}) = 0$. □

6. Uniform properties of the homoclinics

Let us introduce the sets

$$K_{\text{per}} = \{\, u \in \Lambda_{1,T} \,:\, T \in (0,\infty),\ \varphi_T(u) = \bar{\lambda} \,\}$$
$$K_k = \{\, v \in \Lambda_k \,:\, \varphi(v) = \lambda_k \,\} \quad (k \in \mathbb{N})$$
$$K_{\text{hom}} = \bigcup_{k \in \mathbb{N}} K_k.$$

From Sections 3 and 4, we know that $K_1 \neq \emptyset$ and $K_k \neq \emptyset$ for $k > \bar{k}$. Moreover, by Lemma 5.4, $K_{\text{per}} \neq \emptyset$ and K_{per} consists of periodic solutions with energy 0. This last property can be easily proved noting that given any $u \in K_{\text{per}}$ defines a peridic solution to (HS) with a period $T \in (0,\infty)$ and an energy $h = \frac{1}{2}|\dot{u}|^2 + V(u)$. Moreover, setting $u_s(t) = u(\frac{t}{s})$ for $t \in [0, sT]$

and $s > 0$, it holds that $u_s \in \Lambda_{1,sT}$ and the map $s \mapsto \varphi_{sT}(u_s)$ attains its minimum at $s = 1$. Since $\varphi_{sT}(u_s) = \int_0^T (\frac{1}{2}s^{-1}|\dot{u}|^2 - sV(u))\,dt$, we get $0 = \frac{d}{ds}\varphi_{sT}(u_s)|_{s=1} = -\int_0^T (\frac{1}{2}|\dot{u}|^2 + V(u))\,dt = -hT$ and then $h = 0$.

Lemma 6.1. *For any* $v \in K_{\text{hom}}$ *and any* $u \in K_{\text{per}}$ *it holds that* $\operatorname{range} v \cap \operatorname{range} u = \emptyset$.

Proof. By contradiction, let us assume that there are $k \in \mathbb{N}$, $v \in K_k$, $T \in (0,\infty)$ and $u \in K_{\text{per}} \cap \Lambda_{1,T}$ such that $v(t_0) = u(0)$ for some $t_0 \in \mathbb{R}$. Then the function

$$\bar{v}(t) = \begin{cases} v(t) & \text{for } t \le t_0 \\ u(t-t_0) & \text{for } t_0 < t \le t_0 + T \\ v(t-T) & \text{for } t > t_0 + T \end{cases}$$

lies in Λ_{k+1} and $\varphi(\bar{v}) = \lambda_k + \bar{\lambda}$. In particular $\lambda_{k+1} < (k+1)\lambda_1$ and thus $k \ge \bar{k}$. Then there are $v_{k+1} \in K_{k+1}$ and $s_{k+1}, t_{k+1} \in \mathbb{R}$ satisfying $s_{k+1} < t_{k+1}$, $v_{k+1}(s_{k+1}) = v_{k+1}(t_{k+1})$ and $\operatorname{ind}_\xi v_{k+1}|_{[s_{k+1},t_{k+1}]} = 1$. Setting

$$u_k(t) = \begin{cases} v_{k+1}(t) & \text{for } t \le s_{k+1} \\ v_{k+1}(t - s_{k+1} + t_{k+1}) & \text{for } t > s_{k+1} \end{cases}$$

we have that $u_k \in \Lambda_k$ and $\lambda_{k+1} = \varphi(v_{k+1}) = \varphi(u_k) + \varphi(v_{k+1}|_{[s_{k+1},t_{k+1}]}) \ge \lambda_k + \bar{\lambda}$. Therefore $\bar{v} \in K_{k+1}$ and, by Lemma 4.4, $\bar{v}$ is a solution to (HS). This implies in particular that $\dot{v}(t_0) = \dot{u}(0)$. Hence v and $\bar{v}$ are two different solutions to the same Cauchy problem defined by (HS) with initial conditions $x(t_0) = u(0)$ and $\dot{x}(t_0) = \dot{u}(0)$, contradicting the uniqueness of solution due to the locally Lipschitz continuity of ∇V. □

We point out that, since $\bar{\lambda} < \lambda_1$, $0 \notin \operatorname{range} u$ for any $u \in K_{\text{per}}$. Fixed $u \in K_{\text{per}}$ let $C_u(0)$ be the component of $\mathbb{R}^2 \setminus \operatorname{range} u$ containing 0. By Lemma 6.1, it holds that

(6.1) $$\overline{\operatorname{range} v} \subset C_u(0) \text{ for every } v \in K_{\text{hom}} \text{ and } u \in K_{\text{per}}.$$

Moreover, always by Lemma 6.1, fixing $v_1 \in K_1$ and $u \in K_{\text{per}}$, $\operatorname{range} u$ is contained in a component $C_{v_1}(u)$ of $\mathbb{R}^2 \setminus \overline{\operatorname{range} v_1}$.

Lemma 6.2. *Let* $v_1 \in K_1$ *and* $u \in K_{\text{per}}$. *If* $v \in K_{\text{hom}}$ *is such that* $\operatorname{range} v \cap C_{v_1}(u) \ne \emptyset$, *then* $\operatorname{range} v \subset C_{v_1}(u)$.

The proof of Lemma 6.2 relies on the following observations.

Remark 6.3. If $v \in K_{\text{hom}}$ and there are $\sigma, \tau, \sigma', \tau' \in \mathbb{R}$ such that $\sigma < \tau$, $\sigma' < \tau'$, $v(\sigma) = v(\tau)$, $v(\sigma') = v(\tau')$ and $\text{ind}_\xi v|_{[\sigma,\tau]} = \text{ind}_\xi v|_{[\sigma',\tau']} = 1$ then $(\sigma,\tau) \cap (\sigma',\tau') \neq \emptyset$.

Indeed, otherwise, assuming for instance that $\tau \le \sigma'$, we consider the functions:

$$u(t) = \begin{cases} v(t) & \text{for } t \le \tau \\ v(t-\tau+\sigma) & \text{for } \tau < t \le \tau - \sigma + \sigma' \\ v(t-\tau+\sigma-\sigma'+\tau') & \text{for } t > \tau - \sigma + \sigma' \end{cases}$$

$$u'(t) = \begin{cases} v(t) & \text{for } t \le \sigma \\ v(t-\sigma+\tau) & \text{for } \sigma < t \le \sigma - \tau + \tau' \\ v(t-\sigma+\tau-\tau'+\sigma') & \text{for } t > \sigma - \tau + \tau'. \end{cases}$$

Then $u, u' \in \Lambda$, $\text{ind}_\xi u = \text{ind}_\xi u' = \text{ind}_\xi v$, $\varphi(u) = \varphi(v) + \varphi_I(v) - \varphi_{I'}(v)$ and $\varphi(u') = \varphi(v) - \varphi_I(v) + \varphi_{I'}(v)$ where $I = [\sigma,\tau]$ and $I' = [\sigma',\tau']$. Therefore, since $\varphi(u) \ge \varphi(v)$ and $\varphi(u') \ge \varphi(v)$, we get $\varphi_I(v) = \varphi_{I'}(v)$ and consequently $\varphi(u) = \varphi(u') = \varphi(v)$. Then, by 5, u and u' are homoclinics orbits. In particular u and u' are different solutions to (HS) coinciding on $(-\infty,\sigma)$. This contradicts the uniqueness property of the Cauchy problem, given by (V1).

Remark 6.4. If $v \in K_{\text{hom}}$, $v_1 \in K_1$ and there is $s \in \mathbb{R}$ such that $v(s) \in \text{range}\, v_1$ and $v(t) \notin \text{range}\, v_1$ for every $t \in (-\infty, s)$ (respectively for $t \in (s, +\infty)$), then there are $\sigma, \tau \in (-\infty, s]$ (respectively $\sigma, \tau \in [s, +\infty)$) such that $\sigma < \tau$ and $v(\sigma) = v(\tau)$.

The proof is by contradiction. Let us assume that $v|_{(-\infty,s]}$ is 1-1. Since $v(s) \in \text{range}\, v_1$ there is $s_1 \in \mathbb{R}$ such that $v(s) = v_1(s_1)$. Setting $I = (-\infty, s]$, $I_- = (-\infty, s_1]$, $I_+ = [s_1, +\infty)$, $\Gamma = \{v(t) : t \in I\}$, $\Gamma_- = \{v_1(t) : t \in I_-\}$ and $\Gamma_+ = \{v_1(t) : t \in I_+\}$, there are $u_+, u_- \in \Lambda$ such that $u_\pm(t) = v(t)$ fot $t \in I$, $\text{range}\, u_\pm = \Gamma \cup \Gamma_\pm$ and $\varphi(u_\pm) = \varphi_I(v) + \varphi_{I_\pm}(v_1)$. Moreover either $|\text{ind}_\xi u_+| = 1$ and $\text{ind}_\xi u_- = 0$, or $|\text{ind}_\xi u_-| = 1$ and $\text{ind}_\xi u_+ = 0$. Let us assume that the first case holds (in the other case the same argument applies). Since $\varphi(u_+) \ge \lambda_1 = \varphi(v_1)$ we get that $\varphi_I(v) \ge \varphi_{I_-}(v_1)$. Then, setting

$$u(t) = \begin{cases} u_-(2s-t) & \text{for } t \le s \\ v(t) & \text{for } t > s \end{cases}$$

we have that $\{u(t) : t \leq s\} = \Gamma_-$, $u \in \Lambda$, $\mathrm{ind}_\xi u = \mathrm{ind}_\xi v$ and $\varphi(u) = \varphi_{I_-}(v_1) - \varphi_I(v) + \varphi(v)$. Then $\varphi(u) \geq \varphi(v)$, that is $\varphi_I(v) \leq \varphi_{I_-}(v_1)$. Therefore $\varphi_I(v) = \varphi_{I_-}(v_1)$ and thus $\varphi(u) = \varphi(v)$. Hence $u \in K_{\mathrm{hom}}$ and, by Lemma 4.4, is a solution to (HS) different from v but coinciding with v in $(s, +\infty)$. This is in contrast to the uniqueness of solution, due to (V1).

Proof of Lemma 6.2. Arguing by contradiction, let us assume that $v(t_0) \notin C_{v_1}(u)$ for some $t_0 \in \mathbb{R}$. Thus the values

$$t_- = \inf \{t \in \mathbb{R} : v(t) \in \mathrm{range}\, v_1 \}$$
$$t_+ = \sup \{t \in \mathbb{R} : v(t) \in \mathrm{range}\, v_1 \}$$

are well defined. If $t_- = -\infty$, then, fixed $\delta \in (0, |\xi|)$ there are $s, s_1 \in \mathbb{R}$ such that $v(s) = v_1(s_1)$, $|v(t)| \leq \delta$ for any $t \leq s$ and one of the following cases occurs: either $|v_1(t)| \leq \delta$ for $t \leq s_1$ or $|v_1(t)| \leq \delta$ for $t \geq s_1$. If the first case holds, we define

$$\bar{v}(t) = \begin{cases} v_1(t - s + s_1) & \text{for } t \leq s \\ v(t) & \text{for } t > s \end{cases}$$

and

$$\bar{v}_1(t) = \begin{cases} v(t - s_1 + s) & \text{for } t \leq s_1 \\ v_1(t) & \text{for } t > s_1. \end{cases}$$

Instead, in the second case we set

$$\bar{v}(t) = \begin{cases} v_1(-t + s + s_1) & \text{for } t \leq s \\ v(t) & \text{for } t > s \end{cases}$$

and

$$\bar{v}_1(t) = \begin{cases} v(t) & \text{for } t \leq s_1 \\ v_1(-t + s + s_1) & \text{for } t > s_1. \end{cases}$$

In both the cases $\bar{v}_1, \bar{v} \in \Lambda$, $\mathrm{ind}_\xi \bar{v}_1 = 1$ and $\mathrm{ind}_\xi \bar{v} = \mathrm{ind}_\xi v$. From $\varphi(\bar{v}) \geq \varphi(v)$ and $\varphi(\bar{v}_1) \geq \varphi(v_1)$ it follows that $\varphi_I(v) = \varphi_{I_1}(v_1)$ where $I = (-\infty, s)$ and $I_1 = (-\infty, s_1)$ in the first case, while $I_1 = (s_1, +\infty)$ in the second case. In both the cases we get that $\varphi(\bar{v}_1) = \varphi(v_1)$ and consequently $\bar{v}_1 \in K_1$ is a solution to (HS) different from v_1 but coinciding with v_1 in I_1, contrary to the uniqueness of the solution to the Cauchy problem for (HS). Thus we have proved that $t_- \in \mathbb{R}$. Analogously we can check that $t_+ \in \mathbb{R}$. Hence $v(t_\pm) \in \mathrm{range}\, v_1$ and $v(t) \notin \mathrm{range}\, v_1$ for $t < t_-$ and for $t > t_+$. Using

Remark 6.4 we deduce that there are $\sigma_\pm, \tau_\pm \in \mathbb{R}$ such that $\sigma_- < \tau_- \le t_-$, $t_+ \le \sigma_+ < \tau_+$ and $v(\sigma_\pm) = v(\tau_\pm)$. By Lemma 3.2 we can also assume that $\mathrm{ind}_\xi\, v|_{[\sigma_\pm,\tau_\pm]} = 1$. Since $t_- \le t_+$ we have that $(\sigma_-,\tau_-)\cap(\sigma_+,\tau_+) = \emptyset$, in contradiction with Remark 6.3. This concludes the proof. □

Lemma 6.5. *For any $\epsilon \in (0,|\xi|)$ there is $\delta_\epsilon \in (0,\epsilon)$ such that for every $v \in K_{\mathrm{hom}} \setminus K_1$, if $|v(\tau)| = \delta_\epsilon$ then $|v(t)| < \epsilon$ for any $t \le \tau$ or any $t \ge \tau$.*

Proof. By contradiction, let us suppose that there is $\epsilon \in (0,|\xi|)$ such that for any $\delta \in (0,\epsilon)$ there exist $v_\delta \in K_{\mathrm{hom}} \setminus K_1$ and $t_\delta^-, \tau_\delta, t_\delta^+ \in \mathbb{R}$ such that $t_\delta^- < \tau_\delta < t_\delta^+$ and $|v_\delta(t_\delta^\pm)| \ge \epsilon$ and $|v_\delta(\tau_\delta)| = \delta$. We define

$$v_\delta^-(t) = \begin{cases} v_\delta(t) & \text{for } t \le \tau_\delta \\ (\tau_\delta + 1 - t)\, v_\delta(\tau_\delta) & \text{for } \tau_\delta \le t \le \tau_\delta + 1 \\ 0 & \text{for } t \ge \tau_\delta + 1 \end{cases}$$

$$v_\delta^+(t) = \begin{cases} 0 & \text{for } t \le \tau_\delta - 1 \\ (t - \tau_\delta + 1)\, v_\delta(\tau_\delta) & \text{for } \tau_\delta - 1 \le t \le \tau_\delta \\ v_\delta(t) & \text{for } t \ge \tau_\delta. \end{cases}$$

Clearly $v_\delta^-, v_\delta^+ \in \Lambda$ and since $|v_\delta(t_\delta^\pm)| \ge \epsilon$ we get that

$$\varphi(v_\delta^-) \ge \mu_\epsilon \quad \text{and} \quad \varphi(v_\delta^-) \ge \mu_\epsilon \tag{6.2}$$

for some $\mu_\epsilon > 0$ independent of δ. Setting $k_\delta^- = \mathrm{ind}_\xi\, v_\delta^-$ and $k_\delta^+ = \mathrm{ind}_\xi\, v_\delta^+$ we have

$$k_\delta^- \ge 0,\ k_\delta^+ \ge 0 \ \text{ and } \ k_\delta^- + k_\delta^+ = k_\delta.$$

Now standard estimates show that

$$0 \le \varphi(v_\delta^-) + \varphi(v_\delta^+) - \varphi(v_\delta) = \omega(\delta) \to 0 \tag{6.3}$$

as $\delta \to 0$. Since $\varphi(v_\delta) = \lambda_{k_\delta}$, combining (6.2)–(6.3) we see that for $\delta > 0$ sufficiently small both $k_\delta^- \ge 1$ and $k_\delta^+ \ge 1$. Indeed if we assume for example that $k_\delta^- = 0$ then $k_\delta^+ = k_\delta$ and, by (6.2), we have

$$\varphi(v_\delta^+) = \varphi(v_\delta) - \varphi(v_\delta^-) + \omega(\delta) < \varphi(v_\delta) = \lambda_{k_\delta}$$

for $\delta > 0$ sufficiently small, contradicting the definition of λ_{k_δ}. On the other hand (6.3) implies that

$$\lambda_{k_\delta^-} + \lambda_{k_\delta^+} - \lambda_{k_\delta} \le \omega(\delta).$$

Since $\omega(\delta) \to 0$ we get a contradiction with Lemma 4.3. □

7. Limit process and conclusion of the proof of Theorem 1.2

Let $(v_k) \subset \Lambda$ be the sequence of homoclinics given by Theorem 1.1. Let us fix $\bar{\epsilon} \in (0, \min\{\mathrm{dist}\,(0, \mathrm{range}\,\bar{u}), |\xi|)$ and for any $k > \bar{k}$ let us put

$$\tau_k = \inf\{\, t \in \mathbb{R} \,:\, |v_k(t)| = \bar{\epsilon}\,\}.$$

By the translational invariance of (HS), we can assume that $\tau_k = 0$, i.e.,

$$|v_k(0)| = \bar{\epsilon} \ \text{ and } \ |v_k(t)| < \bar{\epsilon} \ \text{ for any } t < 0. \tag{7.1}$$

Let $\bar{u} \in K_{\mathrm{per}}$ be given by Lemma 5.4. According to the same lemma, up to a subsequence, $\mathrm{dist}\,(\mathrm{range}\,v_k, \mathrm{range}\,\bar{u}) \to 0$. Thus $\mathrm{range}\,v_k \cap C_{v_1}(\bar{u}) \neq \emptyset$ and by Lemma 6.2, this implies that $\mathrm{range}\,v_k \subset C_{v_1}(\bar{u})$. Taking into account (6.1) we get

$$\mathrm{range}\,v_k \subset C_{v_1}(\bar{u}) \cap C_{\bar{u}}(0). \tag{7.2}$$

In particular (7.2) implies that there are $\rho, R > 0$ such that

$$|v_k(t) - \xi| \geq \rho \ \text{ and } \ |v_k(t)| \leq R, \ \text{for any } \ t \in \mathbb{R}. \tag{7.3}$$

Now combining (7.3), (HS) and the energy conservation

$$\tfrac{1}{2}|\dot{v}_k(t)|^2 + V(v_k(t)) = 0 \ \text{ for any } \ t \in \mathbb{R}$$

we get that $\sup_k \|v_k\|_{C^2(\mathbb{R})} < \infty$. An application of Ascoli–Arzelà Theorem shows that, there is $\bar{v} \in C^1(\mathbb{R}, \mathbb{R}^2)$ and a subsequence of (v_k), still denoted by (v_k), such that

$$v_k \to \bar{v} \ \text{ in the } C^1_{\mathrm{loc}}\text{–topology}. \tag{7.4}$$

We note that by (7.2) and (7.4)

$$\mathrm{range}\,\bar{v} \subset \overline{C_{v_1}(\bar{u}) \cap C_{\bar{u}}(0)} \tag{7.5}$$

and by (7.3) and (7.4)

$$\mathrm{dist}\,(\mathrm{range}\,\bar{v}, \xi) \geq \rho \ \text{ and } \ \|\bar{v}\|_{L^\infty} \leq R. \tag{7.6}$$

Lemma 7.1. *The function* $\bar{v}$ *is a non zero classical solution to (HS) with energy zero and such that* $\bar{v}(t) \to 0$ *and* $\dot{\bar{v}}(t) \to 0$ *as* $t \to -\infty$.

Proof. Standard arguments show that $\bar{v}$ is a classical solution of (HS) with energy zero. By (7.1) and (7.4), $|\bar{v}(0)| = \bar{\epsilon}$ and thus $\bar{v} \neq 0$. Defining

$$u_k(t) = \begin{cases} 0 & \text{for } t \leq -1 \\ (t+1)\, v_k(0) & \text{for } -1 \leq t \leq 0 \\ v_k(t) & \text{for } t > 0 \end{cases}$$

we see that $u_k \in \Lambda_k$ and then $\varphi|_{\mathbb{R}_-}(v_k) \leq \varphi|_{\mathbb{R}_-}(u_k) \leq \frac{1}{2}\bar{\epsilon}^2 + \max_{|x| \leq \bar{\epsilon}} |V(x)| = C_{\bar{\epsilon}}$. By (7.4) we get that for any $T < 0$

$$\int_T^0 (\tfrac{1}{2}|\dot{\bar{v}}|^2 - V(\bar{v}))\, dt = \lim_{k \to \infty} \int_T^0 (\tfrac{1}{2}|\dot{v}_k|^2 - V(v_k))\, dt \leq C_{\bar{\epsilon}}$$

and then $\varphi|_{\mathbb{R}_-}(\bar{v}) < \infty$. By Remark 2.3, this implies that $\bar{v}(t) \to 0$ as $t \to -\infty$ and, since

$$\tfrac{1}{2}|\dot{\bar{v}}(t)|^2 + V(\bar{v}(t)) = 0 \quad \text{for any } t \in \mathbb{R} \tag{7.7}$$

also $\dot{\bar{v}}(t) \to 0$ as $t \to -\infty$. □

To conclude the proof of Theorem 1.2 it remains to check that $L_\omega(\bar{v}) = \Gamma$ where $\Gamma = \{(\bar{u}(t), \dot{\bar{u}}(t)) : t \in \mathbb{R}\}$.

By (7.6) and (7.7), $\|\bar{v}\|_{C^1(\mathbb{R})} < \infty$ and then, by well known theorems on the ω–limit set (see, e.g. [BhS]) $L_\omega(\bar{v})$ is a non empty, compact, connected, positively invariant subset of $\mathbb{R}^4$.

Lemma 7.2. *There is* $\bar{\delta} > 0$ *such that* $|\bar{v}(t)| \geq \bar{\delta}$ *for any* $t > 0$.

Proof. Let $\bar{\delta} = \delta_{\bar{\epsilon}} > 0$ be given by Lemma 6.5. Since $\bar{\epsilon} \in (0, |\xi|)$ and $\text{ind}_\xi\, v_k = k$, any homoclinic v_k must turn $k-1$ times around ξ without coming closer than $\bar{\delta}$ to the origin. Then, defining $\bar{t}_k = \inf\{t > 0 : |v_k(t)| = \bar{\delta}\}$ and taking into account (7.3), we have

$$(k-1)\pi\rho \leq \int_0^{\bar{t}_k} |\dot{v}_k|\, dt = \int_0^{\bar{t}_k} \sqrt{2|V(v_k)|}\, dt \leq \bar{t}_k\, M$$

where $M = \max\{\sqrt{2|V(x)|} : |x - \xi| \geq \rho,\ |x| \leq R\} < \infty$. Hence $\bar{t}_k \to \infty$. Therefore for any $t > 0$ there is $k_t > \bar{k}$ such that $|v_k(t)| \geq \bar{\delta}$ for every $k \geq k_t$. By (7.4), we conclude that $|\bar{v}(t)| \geq \bar{\delta}$. □

By (7.5) and Lemma 7.2 we infer that

$$\bar{v}(t) \in D = \overline{C_{v_1}(\bar{u}) \cap C_{\bar{u}}(0)} \setminus B_{\bar{\delta}}(0) \quad \text{for any } t \geq 0. \tag{7.8}$$

By the positively invariance of $L_\omega(\bar{v})$, it is enough to prove that $L_\omega(\bar{v}) \subseteq \Gamma$. From now on we assume by contradiction there is $\zeta = (x, y) \in L_\omega(\bar{v}) \setminus \Gamma$.

Lemma 7.3. *If $\zeta = (x, y) \in L_\omega(\bar{v}) \setminus \Gamma$ then $x \notin \operatorname{range} \bar{u}$.*

Proof. By contradiction, let $\zeta = (x, y) \in L_\omega(\bar{v}) \setminus \Gamma$ such that $x = \bar{u}(t_0)$ for some $t_0 \in \mathbb{R}$. Then $y \neq \dot{\bar{u}}(t_0)$. Since $\zeta \in L_\omega(\bar{v})$, there is a sequence $(t_n) \subset \mathbb{R}$ such that $t_n \to +\infty$, $\bar{v}(t_n) \to x$ and $\dot{\bar{v}}(t_n) \to y$. By (7.4), (7.7) and (7.8) $|y|^2 = -2V(x) > 0$. Hence for $n \in \mathbb{N}$ sufficiently large, there is $\epsilon_n > 0$ such that $\operatorname{range} \bar{v}|_{[t_n - \epsilon_n, t_n + \epsilon_n]}$ crosses transversally $\operatorname{range} \bar{u}$. Observing that $\operatorname{range} \bar{u} \subset \partial D$, we get a contradiction with (7.8). □

Let now $\epsilon_0 = \frac{1}{2} \min\{ |x - \xi|, \operatorname{dist}(x, \operatorname{range} \bar{u}) \}$ and $M_0 = (2 \max\{ |V(x')| : |x' - x| \leq \epsilon_0 \})^{1/2}$. Let $(t_n) \subset \mathbb{R}$ be such that $t_{n+1} - t_n \geq 1$ for every $n \in \mathbb{N}$ and $\bar{v}(t_n) \to x$. Set $I_n = [t_n, t_{n+1}]$.

Fixed $\epsilon \in (0, \min\{\epsilon_0, \bar{\lambda}/4M_0\})$ there is $n_\epsilon \in \mathbb{N}$ such that

$$|\bar{v}(t_n) - x| \leq \epsilon \quad \text{for any } n \geq n_\epsilon. \tag{7.9}$$

For $n \geq n_\epsilon$ we define

$$u_n(t) = \begin{cases} \bar{v}(t) & \text{for } t_n \leq t < t_{n+1} \\ (t_{n+1} + 1 - t)\, \bar{v}(t_{n+1}) + (t - t_{n+1})\, \bar{v}(t_n) & \text{for } t_{n+1} \leq t \leq t_{n+1} + 1 \end{cases}$$

and $\nu_n = \operatorname{ind}_\xi u_n$.

Then, by (7.4), for every $n \geq n_\epsilon$ there is $k_{\epsilon,n} > \bar{k}$ such that

$$|v_k(t) - \bar{v}(t)| \leq \epsilon \quad \text{for any } k \geq k_{\epsilon,n} \text{ and for any } t \in I_n. \tag{7.10}$$

For $n \geq n_\epsilon$ and $k \geq k_{\epsilon,n}$ we set

$$u_{k,n}(t) = \begin{cases} v_k(t) & \text{for } t \leq t_n \\ \dfrac{t_n + \delta_{k,n} - t}{\delta_{k,n}}\, v_k(t_n) + \dfrac{t - t_n}{\delta_{k,n}}\, v_k(t_{n+1}) & \text{for } t_n < t \leq t_n + \delta_{k,n} \\ v_k(t + t_{n+1} - t_n + \delta_{k,n}) & \text{for } t > t_n + \delta_{k,n} \end{cases}$$

with suitable $\delta_{k,n} > 0$. It holds that $u_{k,n} \in \Lambda$, $\mathrm{ind}_\xi\, u_{k,n} = k - \nu_n$ and

$$\varphi(u_{k,n}) = \varphi(v_k) - \varphi_{I_n}(v_k) + \varphi_{I_{k,n}}(u_{k,n}) \tag{7.11}$$

where $I_{k,n} = [t_n, t_n + \delta_{k,n}]$. We can estimate

$$\varphi_{I_{k,n}}(u_{k,n}) \le \frac{1}{2\delta_{k,n}} |v_k(t_{n+1}) - v_k(t_n)|^2 + \delta_{k,n} \max_{|x'-x|\le\epsilon_0} |V(x')| = \omega_{k,n}. \tag{7.12}$$

To guarantee that $\lim\limits_{n\to\infty} \lim\limits_{k\to\infty} \omega_{k,n} = 0$ we take $\delta_{k,n} = |v_k(t_{n+1}) - v_k(t_n)|/M_0$ so that, by (7.9) and (7.10)

$$\omega_{k,n} = |v_k(t_{n+1}) - v_k(t_n)|\, M_0 \le 4\epsilon M_0 < \bar\lambda. \tag{7.13}$$

Then, by (7.11)-(7.13), we get that $\varphi(u_{k,n}) < \lambda_k + \bar\lambda$. Since $\widehat\lambda_{k+1} = \lambda_{k+1} \ge \lambda_k + \bar\lambda$ we obtain that $\nu_n \ge 0$ for any $n \ge n_\epsilon$. Therefore, by (4.3), $\lambda_k \le \lambda_{k-\nu_n} + \nu_n \frac{\lambda_{k-\nu_n}}{k-\nu_n}$, that, together with (7.11)-(7.12), implies

$$\varphi_{I_n}(v_k) \le \nu_n \frac{\lambda_{k-\nu_n}}{k-\nu_n} + \omega_{k,n}. \tag{7.14}$$

As $k \to \infty$ in (7.14), by Lemma 5.1, (7.4), and by the definition of $\omega_{k,n}$, we infer that for any $n \ge n_\epsilon$

$$\varphi_{I_n}(\bar v) \le \nu_n \bar\lambda + \omega_n \tag{7.15}$$

where $\omega_n = |\bar v(t_{n+1}) - \bar v(t_n)|\, M_0$. Since $\omega_n \to 0$ we get a contradiction if we prove that there is $\eta > 0$ independent of n such that $\varphi_{I_n}(\bar v) \ge \nu_n \bar\lambda + \eta$ for $n \ge n_\epsilon$ sufficiently large.

To this aim we introduce the following values. For any $\nu \in \mathbb{N} \cup \{0\}$ and $T \ge 1$ we set

$$\lambda_\nu(T, x, \epsilon) = \inf\{\varphi_T(u) \,:\, u \in \Lambda_{\nu,T}(x,\epsilon)\}$$

where $\Lambda_{\nu,T}(x,\epsilon) = \{u \in H^1([0,T],\mathbb{R}^2) \,:\, u(t) \in D\ \forall t \in [0,T],\ |u(0) - x| \le \epsilon,\ |u(T) - x| \le \epsilon,\ \mathrm{ind}_\xi\, u = \nu\}$. Here $\mathrm{ind}_\xi\, u$ denotes the winding number of the closed curve defined by u and by the segment line connecting $u(T)$ to $u(0)$.

Lemma 7.4. *There are* $\epsilon_1 \in (0, \epsilon_0)$ *and* $\eta > 0$ *such that for any* $\epsilon \in (0, \epsilon_1)$, $\nu \in \mathbb{N} \cup \{0\}$ *and* $T \geq 1$ *it holds that* $\lambda_\nu(T, x, \epsilon) \geq \nu\bar{\lambda} + \eta$.

Proof. Let $\lambda_\nu(x, \epsilon) = \inf_{T\geq 1} \lambda_\nu(T, x, \epsilon)$. Firstly we show that $\lambda_\nu(x)$ is attained. Indeed let $(T_j) \subset [1, \infty)$ and $u_j \in \Lambda_{\nu,T_j}(x, \epsilon)$ such that $\varphi_{T_j}(u_j) \to \lambda_\nu(x, \epsilon)$. Since $\operatorname{range} u_j \subset D$ and $\inf_{x' \in D} |V(x')| = \eta_0 > 0$, by Lemma 2.2, there are $R_0 > 0$ and $T_0 > 1$ such that $\|u_j\|_{L^\infty(I_j)} \leq R_0$ and $1 \leq T_j \leq T_0$ for any $j \in \mathbb{N}$, where $I_j = [0, T_j]$. Thus, passing to a subsequence if necessary, we have that $T_j \to T \in [1, \infty)$. Setting $\bar{u}_j(t) = u_j(tT_j/T)$ for $t \in [0, T]$, we get that the sequence $(\bar{u}_j) \subset \Lambda_\nu(T, x, \epsilon)$ is bounded in $H^1([0, T], \mathbb{R}^2)$ and, up to a subsequence, converges weakly in $H^1([0, T], \mathbb{R}^2)$ and strongly in $L^\infty([0, T])$ to some $u \in \Lambda_\nu(T, x, \epsilon)$. Then $\varphi_T(u) = \lambda_\nu(x, \epsilon)$.
Now, if $\nu = 0$ we remark that $\lambda_0(x, \epsilon) \geq \eta_0$. If $\nu > 1$ and $u \in \Lambda_\nu(T, x, \epsilon)$ is such that $\varphi_T(u) = \lambda_\nu(x, \epsilon)$, then, by a result similar to Lemma 3.2, there are $s, t \in [0, T]$ such that $s < t$, $u(s) = u(t)$ and $\operatorname{ind}_\xi u|_{[s,t]} = 1$. Then

$$\lambda_\nu(x, \epsilon) = \varphi_T(u) \geq \lambda_{\nu-1}(x, \epsilon) + \varphi(u|_{[s,t]}) \geq \lambda_{\nu-1}(x, \epsilon) + \bar{\lambda}$$

and, by recurrence

$$\lambda_\nu(x, \epsilon) \geq \lambda_1(x, \epsilon) + (\nu - 1)\bar{\lambda}. \tag{7.16}$$

Since the function $\epsilon \mapsto \lambda_1(x, \epsilon)$ is continuous at $\epsilon = 0$, it is enough to prove that $\lambda_1(x, 0) > \bar{\lambda}$. Because of the variational characterization of $\bar{\lambda}$ we clearly have $\lambda_1(x, 0) \geq \bar{\lambda}$. Assume that $\lambda_1(x, 0) = \bar{\lambda}$. Since $\lambda_1(x, 0)$ is attained, there is a $u_0 \in K_{\text{per}}$ such that $x \in \operatorname{range} u_0$ and $\operatorname{range} u_0 \subset D$. By Lemma 7.3 $x \notin \operatorname{range} \bar{u}$ and then, by Lemma 5.4, there is $k > \bar{k}$ such that v_k crosses $\operatorname{range} u_0$ and this is in contradiction with Lemma 6.1. Hence, setting $\eta_1 = \frac{1}{2}(\lambda_1(x, 0) - \bar{\lambda})$, we get that $\eta_1 > 0$ and, by (7.16), for $\epsilon > 0$ sufficiently small $\lambda_\nu(x, \epsilon) \geq \nu\bar{\lambda} + \eta_1$ for any $\nu \in \mathbb{N}$.
Then the conclusion follows taking $\eta = \min\{\eta_0, \eta_1\}$. □

Now we can conclude the proof of Theorem 1.2 observing that, since $t_{n+1} - t_n \geq 1$, by Lemma 7.4,

$$\varphi_{I_n}(\bar{v}) \geq \lambda_{\nu_n}(t_{n+1} - t_n, x, \epsilon) \geq \nu_n\bar{\lambda} + \eta$$

in contradiction with (7.15).

8. About Condition $(*)$

In this section we discuss the assumption $(*)$. We recall that $(*)$ is equivalent to the strict inequality $\bar{\lambda} < \lambda_1$, whose meaning is clearly explained by the following result.

Proposition 8.1. *Let* $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ *satisfy* $(V1)$–$(V4)$ *and for any* $T \in (0, \infty)$ *let* $\lambda_1(T) = \inf\{\varphi_T(u) : u \in \Lambda_{1,T}\}$. *Then:*

(i) *the function* $T \mapsto \lambda_1(T)$ *is continuous on* $(0, \infty)$;

(ii) $\bar{\lambda} = \inf_{T>0} \lambda_1(T)$;

(iii) $\lim_{T\to 0} \lambda_1(T) = +\infty$;

(iv) $\lim_{T\to+\infty} \lambda_1(T) = \lambda_1$.

Proof. If $(u_n) \subset \Lambda_{1,T}$ is a minimizing sequence for φ_T in $\Lambda_{1,T}$, then, by Lemma 2.2, (u_n) is bounded in E_T and then admits a subsequence converging weakly in E_T and uniformly on $[0, T]$ to some $u_T \in \Lambda_{1,T}$ satisfying $\varphi_T(u_T) = \lambda_1(T)$. Such a function u_T can be extended periodically on $\mathbb{R}$, with period T, to a solution to (HS), still denoted by u_T.

(i) Taking any $T_1, T_2 \in (0, \infty)$ let $u_{T_1} \in \Lambda_{1,T_1}$ and $u_{T_2} \in \Lambda_{1,T_2}$ be such that $\varphi_{T_i}(u_{T_i}) = \lambda_1(T_i)$ $(i = 1, 2)$. We set $u_1(t) = u_{T_2}(\frac{T_2}{T_1}t)$ for $t \in [0, T_1]$. Then

$$\varphi_{T_1}(u_1) = \frac{T_2}{T_1}\int_0^{T_2} \tfrac{1}{2}|\dot{u}_{T_2}|^2\,dt - \frac{T_1}{T_2}\int_0^{T_2} V(u_{T_2})\,dt \le \max\{\frac{T_1}{T_2}, \frac{T_2}{T_1}\}\varphi_{T_2}(u_{T_2})$$

and consequently

$$\frac{\lambda_1(T_1)}{\lambda_1(T_2)} \le \max\{\frac{T_1}{T_2}, \frac{T_2}{T_1}\}.$$

Changing T_1 with T_2 we get

$$\min\{\frac{T_1}{T_2}, \frac{T_2}{T_1}\} \le \frac{\lambda_1(T_1)}{\lambda_1(T_2)} \le \max\{\frac{T_1}{T_2}, \frac{T_2}{T_1}\}$$

that implies the continuity of the mapping $T \mapsto \lambda_1(T)$.

The part (ii) is nothing but the definition of $\bar{\lambda}$.

(iii) Let $\rho_T = \min_{t\in[0,T]} |u_T(t)-\xi|$ where $u_T \in \Lambda_{1,T}$ satisfies $\varphi_T(u_T) = \lambda_{1,T}$. Then $2\pi\rho_T \le \int_0^T |\dot{u}_T|\,dt \le (2T\varphi_T(u_T))^{1/2}$. If there is a sequence $T_n \to 0$ such that $\sup\varphi_{T_n}(u_{T_n}) < \infty$, then $\rho_{T_n} \to 0$ and in particular there is $b > 0$ such that $\operatorname{dist}(\operatorname{range} u_{T_n}, 0) \le b$ for any $n \in \mathbb{N}$. Thus, by Lemma 2.2, there is $\rho > 0$ such that $\rho_{T_n} \ge \rho$, a contradiction. Hence $\lambda_1(T) = \varphi_T(u_T) \to +\infty$ as $T \to 0$.

(iv) Let $v \in \Lambda_1$ be such that $\varphi(v) = \lambda_1$. For any $\epsilon > 0$ there are $\delta_\epsilon > 0$ such that $|V(x)| \le \epsilon$ for $|x| \le \delta_\epsilon$ and $T_\epsilon > 0$ such that $|v(t)| \le \epsilon$ for $|t| \ge T_\epsilon$. For any $T \ge 2T_\epsilon + 2$ we define

$$v_T(t) = \begin{cases} t\,v(-T_\epsilon) & \text{for } 0 \le t \le 1 \\ v(t - T_\epsilon - 1) & \text{for } 1 < t \le 2T_\epsilon + 1 \\ (2T_\epsilon + 2 - t)\,v(T_\epsilon) & \text{for } 2T_\epsilon + 1 < t \le 2T_\epsilon + 2 \\ 0 & \text{for } 2T_\epsilon + 2 \le t \le T \end{cases}$$

Then $v_T \in \Lambda_{1,T}$ and $\varphi_T(v_T) \le \lambda_1 + \omega_\epsilon$ with $\omega_\epsilon \to 0$ as $\epsilon \to 0$. Hence

$$\limsup_{T\to\infty} \lambda_1(T) \le \lambda_1. \tag{8.1}$$

For any $T \ge 1$ let $u_T \in \Lambda_{1,T}$ be such that $\varphi_T(u_T) = \lambda_1(T)$. By (8.1), there is $a > 0$ such that $\varphi_T(u_T) \le a$ for every $T \ge 1$. Moreover with an argument similar to the proof of Lemma 5.2, we also get that there is $b > 0$ such that $\operatorname{dist}(0, \operatorname{range} u_T) \le b$ for any $T \ge 1$. Then, by Lemma 2.2, for any $\delta > 0$ there is $\tau_\delta > 0$ such that $|S_\delta(u_T)| \le \tau_\delta$ for every $T \ge 1$. This means that $\operatorname{dist}(0, \operatorname{range} u_T) \to 0$ as $T \to \infty$. Let $x_T \in \operatorname{range} u_T$ be such that $|x_T| = \operatorname{dist}(0, \operatorname{range} u_T)$ and let $\bar{u}_T \in \Lambda_1$ be obtained by extending u_T on $\mathbb{R}$ in order to connect $\operatorname{range} u_T$ to 0, following the segment line that joins 0 to x_T. Then $\lambda_1 \le \varphi(\bar{u}_T) \le \lambda_1(T) + \epsilon_T$ with $\epsilon_T \to 0$ as $T \to \infty$. Consequently $\lambda_1 \le \liminf_{T\to\infty} \lambda_1(T)$, that, together with (8.1), implies (iv). □

Now we present a condition which assures that $(*)$ holds, with $\bar{k} = 1$.

Theorem 8.2. *Let $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy $(V1)$-$(V3)$ and (V4)'. Suppose also that*

$(V5)$ $\limsup_{x\to 0} -V(x)|x|^{-2} = a < \infty$ *and* $\liminf_{x\to 0} -V(x)|x|^{-2} = b > 0$*;*

$(V6)$ *there exists $v_1 \in \Lambda_1$ such that $\varphi(v_1) = \lambda_1$ and*

$$\liminf_{\substack{s\to-\infty \\ t\to+\infty}} \frac{v_1(s)}{|v_1(s)|} \cdot \frac{v_1(t)}{|v_1(t)|} > \sqrt{1 - \frac{b}{a}}.$$

Then $(*)$ *holds and for any* $k \in \mathbb{N}$ *there is a homoclinic solution* $v_k \in \Lambda_k$ *such that* $\varphi(v_k) = \lambda_k$.

Remark 8.3. We point out that (V6) is a condition regarding the angle θ_1 formed by the directions at which a homoclinic orbit $v_1 \in \Lambda_1$ at level λ_1 leaves and enters the origin. In particular, if $V(x) \sim -a|x|^2$ as $x \to 0$ for some $a > 0$ then in (V6) we ask that $\theta_1 \in [0, \frac{\pi}{2})$.

Proof. Proving that $\lambda_2 < 2\lambda_1$ we get $\bar{k} = 1$ and then the thesis, by Theorem 1.1. We will construct a function $u \in \Lambda_2$ in this way. Let $v_1 \in \Lambda_1$ satisfying (V6). Fixing $\delta > 0$ sufficiently small, let $s_\delta, t_\delta \in \mathbb{R}$ such that $|v_1(s_\delta)| = |v_1(t_\delta)| = \delta$ and $|v_1(t)| < \delta$ for $t < s_\delta$ and $t > t_\delta$. Choosing a suitable $T > 0$ let $u \in \Lambda_2$ be such that

$$u(t) = \begin{cases} v_1(t + T + t_\delta) & \text{for } t < -T \\ v_1(t - T + s_\delta) & \text{for } t > T. \end{cases}$$

Then $\varphi(u) = 2\lambda_1 - \varphi_{I_\delta^-}(v_1) - \varphi_{I_\delta^+}(v_1) + \varphi_{[-T,T]}(u)$, where $I_\delta^- = (-\infty, s_\delta)$ and $I_\delta^+ = (t_\delta, \infty)$. We have to fix $\delta > 0$ and $T > 0$ and to define $u \in H^1([-T,T];\mathbb{R}^2)$ in such a way that $u(T) = v_1(s_\delta)$, $u(-T) = v_1(t_\delta)$, $|u(t)| \le \delta$ for $|t| \le T$ and $\varphi_{[-T,T]}(u) < \varphi_{I_\delta^-}(v_1) + \varphi_{I_\delta^+}(v_1)$. To this extent we introduce the following values. For $r, T, \gamma > 0$ and $x_-, x_+ \in \mathbb{R}^2$ with $|x_-| = |x_+| \le r$ let

$$P_T(x_-, x_+) = \{\, u \in H^1([-T,T];\mathbb{R}^2) \;:\; u(\pm T) = x_\pm\,,\ \|u\|_{L^\infty} \le r \,\}$$

$$m_T(x_-, x_+) = \inf\{ \int_{-T}^{T} (\tfrac{1}{2}|\dot{u}|^2 - V(u))\,dt \;:\; u \in P_T(x_-, x_+) \}$$

$$m_T(\gamma; x_-, x_+) = \inf\{ \int_{-T}^{T} (\tfrac{1}{2}|\dot{u}|^2 + \tfrac{1}{2}\gamma|u|^2)\,dt \;:\; u \in P_T(x_-, x_+) \}$$

$$m_\infty(x_-, x_+) = \liminf_{T\to\infty} m_T(x_-, x_+)$$

$$m_\infty(\gamma; x_-, x_+) = \liminf_{T\to\infty} m_T(\gamma; x_-, x_+).$$

One can easily calculate that

$$\begin{aligned} m_T(\gamma; x_-, x_+) &= \frac{\sqrt{\gamma}}{2}\Big(\frac{|x_-|^2 + |x_+|^2}{\tanh 2\sqrt{\gamma}T} - \frac{2x_- \cdot x_+}{\sinh 2\sqrt{\gamma}T} \Big) \\ m_\infty(\gamma; x_-, x_+) &= \frac{\sqrt{\gamma}}{2}(\, |x_-|^2 + |x_+|^2 \,). \end{aligned} \tag{8.2}$$

By (V5), for any $\epsilon \in (0,1)$ there is $r_\epsilon \in (0,|\xi|)$ such that

$$-a(1+\epsilon)|x|^2 \le V(x) \le -b(1-\epsilon)|x|^2 \text{ for } |x| \le r_\epsilon .$$

Let us set $a_\epsilon = 2a(1+\epsilon)$ and $b_\epsilon = 2b(1-\epsilon)$. Then, by (8.2) for any $T \in (0,\infty)$ and for any $x_-, x_+ \in \mathbb{R}^2$ with $|x_-| = |x_+| \le r_\epsilon$ it holds that

(8.3)

$$m_T(x_-,x_+) \le m_\infty(x_-,x_+) - \frac{2\delta^2\sqrt{a_\epsilon}}{\sinh 2\sqrt{a_\epsilon}T}\left(\sqrt{\frac{b_\epsilon}{a_\epsilon}}\sinh 2\sqrt{a_\epsilon}T - \cosh 2\sqrt{a_\epsilon}T + \cos\theta\right)$$

where $|x_-| = |x_+| = \delta$ and $x_- \cdot x_+ = \delta^2 \cos\theta$. Observing that for $c \in (0,1)$ $\sup_{t>0}(c\sinh t - \cosh t) = -\sqrt{1-c^2}$ is attained at some $\bar{t} > 0$, we infer that there is $T_\epsilon \in (0,\infty)$ such that

$$\sqrt{\frac{b_\epsilon}{a_\epsilon}}\sinh 2\sqrt{a_\epsilon}T_\epsilon - \cosh 2\sqrt{a_\epsilon}T_\epsilon = \sup_{t>0}\left(\sqrt{\frac{b_\epsilon}{a_\epsilon}}\sinh t - \cosh t\right) = -\sqrt{1-\frac{b_\epsilon}{a_\epsilon}}.$$

Then (8.3) becomes:

$$\text{(8.4)}\quad m_{T_\epsilon}(x_-,x_+) \le m_\infty(x_-,x_+) - \frac{2\delta^2\sqrt{a_\epsilon}}{\sinh 2\sqrt{a_\epsilon}T_\epsilon}\left(x_-\cdot x_+ - \delta^2\sqrt{1-\frac{b_\epsilon}{a_\epsilon}}\right)$$

for any $x_-, x_+ \in \mathbb{R}^2$ with $|x_-| = |x_+| = \delta \le r_\epsilon$.

By (V6) there are $\bar{s}_\epsilon, \bar{t}_\epsilon \in \mathbb{R}$ such that $|v_1(t)| \le r_\epsilon$ for any $t \le \bar{s}_\epsilon$ and $t \ge \bar{t}_\epsilon$ and

$$\text{(8.5)}\qquad \frac{v_1(s)}{|v_1(s)|}\cdot\frac{v_1(t)}{|v_1(t)|} \ge \alpha_1 - \epsilon \quad \text{for any } s \le \bar{s}_\epsilon \text{ and } t \ge \bar{t}_\epsilon$$

where $\alpha_1 = \liminf_{\substack{s\to-\infty\\ t\to+\infty}} \frac{v_1(s)}{|v_1(s)|}\cdot\frac{v_1(t)}{|v_1(t)|}$.

Moreover, for any $\delta \in (0, r_\epsilon]$ there are $s_\delta \in (-\infty, \bar{s}_\epsilon]$ and $t_\delta \in [\bar{t}_\epsilon, +\infty)$ such that $|v_1(s_\delta)| = |v_1(t_\delta)| = \delta$.

Then, using (8.4)–(8.5), we get:

(8.6)

$$m_{T_\epsilon}(v_1(t_\delta), v_1(s_\delta)) \le m_\infty(v_1(t_\delta), v_1(s_\delta)) - \frac{2\delta^2\sqrt{a_\epsilon}}{\sinh 2\sqrt{a_\epsilon}T_\epsilon}\left(\alpha_1 - \epsilon - \delta^2\sqrt{1-\frac{b_\epsilon}{a_\epsilon}}\right).$$

By (V6) we can fix $\epsilon \in (0,1)$ such that

$$\epsilon + \sqrt{1-\frac{b_\epsilon}{a_\epsilon}} < \alpha_1.$$

Then we fix $\delta \in (0, r_\epsilon]$, so that, by (8.6)

$$m_{T_\epsilon}(v_1(t_\delta), v_1(s_\delta)) \leq m_\infty(v_1(t_\delta), v_1(s_\delta)) - 2C$$

for some $C = C(\epsilon, \delta) > 0$. Finally, taking $u \in P_{T_\epsilon}(v_1(t_\delta), v_1(s_\delta))$ such that $\varphi_{[-T_\epsilon, T_\epsilon]}(u) \leq m_{T_\epsilon}(v_1(t_\delta), v_1(s_\delta)) + C$ we get that

$$\varphi_{[-T_\epsilon, T_\epsilon]}(u) \leq m_\infty(v_1(t_\delta), v_1(s_\delta)) - C < \varphi_{I_\delta^-}(v_1) + \varphi_{I_\delta^+}(v_1),$$

as we wanted. □

Finally we give some examples of systems for which (V6) holds.

Theorem 8.4. *Let* $V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ *satisfy* (V1)-(V3), (V4)',

(V5)' $\nabla V(x) = -ax + o(x)$ *as* $x \to 0$*, for some* $a > 0$*,*

and one of the following conditions: either

(V6)' $V(x) = V_0(|x|) + V_s(x)$ *where*
$V_0 \in C^{1,1}(\mathbb{R}_+, \mathbb{R})$;
$V_s \in C^{1,1}(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$ *and* $\lim_{x\to\xi} V_s(x) = -\infty$
$\operatorname{supp} V_s \subseteq \operatorname{span}_+\{x_1, x_2\}$ *for some* $x_1, x_2 \in \mathbb{R}^2$ *with* $x_1 \cdot x_2 > 0$*;*

or

(V6)" *there are* $x_1, x_2 \in \mathbb{R}^2$ *with* $x_1 \cdot x_2 > 0$ *such that* $\xi \in \operatorname{span}_+\{x_1, x_2\}$ *and* $V(p(x)) \geq V(x)$ *for every* $x \in \mathbb{R}^2 \setminus \{\xi\}$.

Then $(*)$ *holds true and for any* $k \in \mathbb{N}$ *there exists a homoclinic solution* $v_k \in \Lambda_k$ *such that* $\varphi(v_k) = \lambda_k$*. (We denote* $\operatorname{span}_+\{x_1, x_2\} = \{\lambda_1 x_1 + \lambda_2 x_2 : \lambda_1, \lambda_2 \geq 0\}$ *and* p *the projection on* $\operatorname{span}_+\{x_1, x_2\}$*).*

Proof. Let $v_1 \in \Lambda_1$ be a homoclinic orbit such that $\varphi(v_1) = \lambda_1$. Then, by (V5)', $|v_1(t)| \sim C\, e^{-|t|}$ and $\frac{v_1(t)}{|v_1(t)|} \wedge \frac{\dot v_1(t)}{|\dot v_1(t)|} \to 0$ as $t \to \pm\infty$, being $C > 0$ a constant. This implies that there exists $\lim_{t\to\pm\infty} \frac{v_1(t)}{|v_1(t)|} = x_\pm$. Proving that $x_\pm \in \operatorname{span}_+\{x_1, x_2\}$ we get the thesis. Arguing by contradiction, let us suppose that $x_- \notin \operatorname{span}_+\{x_1, x_2\}$. Then there is $T \in \mathbb{R}$ such that $v_1(t) \notin \operatorname{span}_+\{x_1, x_2\}$ for any $t \leq T$. If (V6)' holds, since $V(x) = V_0(|x|)$ for $x \in \mathbb{R}^2 \setminus \operatorname{span}_+\{x_1, x_2\}$, by the conservation of the angular momentum and of the energy we infer that $v_1(t) = |v_1(t)|x_-$ for any $t \in \mathbb{R}$, contrary to the fact that $v_1 \in \Lambda$.

If (V6)" holds, then, setting $\bar{v}_1(t) = p(v_1(t))$, we get that $\bar{v}_1 \in \Lambda_1$, $\varphi(\bar{v}_1) = \lambda_1$ and $\operatorname{range} \bar{v}_1 \subset \operatorname{span}_+\{x_1, x_2\}$. Hence (V6) is satisfied by $\bar{v}_1$. □

Remark 8.5. All the theorems of this chapter can be stated also when the singularity is a compact non empty set $S \subset \mathbb{R}^2 \setminus \{0\}$. In this case the hypotheses (V1)–(V3) have to be modified in an obvious way, substituting S to ξ. In particular the presence of a discrete rotational symmetry can be treated and exploited, as stated in the following result.

Theorem 8.6. *Let $V : \mathbb{R}^2 \setminus \{\xi_1, \ldots, \xi_m\} \to \mathbb{R}$ satisfy* (V1)–(V3), (V4)', (V5)' *(with $\{\xi_1, \ldots, \xi_m\}$ instead of ξ) and*

(V7) *$V(Rx) = V(x)$ for every $x \in \mathbb{R}^2 \setminus S$, where R is the rotation around the origin of an angle $2\pi/m$ with $m \geq 5$.*

Then for any $k \in \mathbb{N}$ and for any $i = 1, \ldots, m$ there exists a homoclinic solution v_k such that $\operatorname{ind}_{\xi_i} v_k = k$.

Proof. By Remark 3.5, there is $v \in \Lambda$ such that $\operatorname{ind}_{\xi_1} v = 1$ and $\varphi(v) = \inf\{\varphi(u) : u \in \Lambda,\ \operatorname{ind}_{\xi_1} u = 1\}$. Let C_v be the unbounded component of $\mathbb{R}^2 \setminus \overline{\operatorname{range} v}$. We claim that $\operatorname{range} Rv \subset \overline{C}_v$ where $Rv(t) = R(v(t))$. Otherwise there are at least two intervals $I = (s_1, t_1)$ and $J = (s_2, t_2)$ with $-\infty \leq s_i < t_i \leq +\infty$, such that the closure of $\{v(t) : t \in J\} \cup \{Rv(t) : t \in I\}$ defines a closed curve in $\mathbb{R}^2 \setminus C_v$ and $\int_I (\frac{1}{2}|\dot{Rv}|^2 - V(Rv))\, dt > \int_J (\frac{1}{2}|\dot{v}|^2 - V(v))\, dt$. We consider the function $w \in \Lambda$ defined by

$$w(t) = \begin{cases} v(t - t_1 + t_2 - s_2 + s_1) & \text{for } t \leq t_1 - t_2 + s_2 \\ R^{-1} v(t - t_1 + t_2) & \text{for } t_1 - t_2 + s_2 < t < t_1 \\ v(t) & \text{for } t \geq t_1. \end{cases}$$

We note that w is obtained substituting $R^{-1}v|_J$ to $v|_I$, up to reparametrizations of the time. By the definition of Rv, $\int_{\mathbb{R}\setminus J} (\frac{1}{2}|\dot{v}|^2 - V(v))\, dt \geq \int_I (\frac{1}{2}|\dot{Rv}|^2 - V(Rv))\, dt$ and it holds that $w \in \Lambda_1(\xi)$ and $\varphi(w) < \varphi(v)$, a contradiction. Then

$$\liminf_{\substack{s \to -\infty \\ t \to +\infty}} \frac{v(s)}{|v(s)|} \cdot \frac{v(t)}{|v(t)|} \geq \cos(2\pi/m)$$

that, for $m \geq 5$, implies (V6). □

Chapter 2

Multibump solutions for asymptotically periodic Duffing–like systems

1. Introduction

This chapter is devoted to the study of a class of periodic and asymptotically periodic Hamiltonian systems.

To present our results we start by describing them in a very particular case which we think interesting in its own. We consider the following Duffing-like equation:

$$\ddot{q} = q - a(t)\,(1 + \epsilon\cos(\omega(t)t))\,q^3 \tag{1.1}$$

where $\epsilon \in \mathbb{R}$, $a(t)$ and $\omega(t)$ are smooth real functions. The dynamics of the equation (1.1) is well known in the periodic case, i.e., when $a(t) \equiv a_0 > 0$ and $\omega(t) \equiv \omega_0 \neq 0$, and can be exhaustively described using a perturbative approach based on the Melnikov theory and on the Smale–Birkhoff homoclinic theorem (see [GH], [Me], [W]).

The results contained in this chapter apply in the asymptotically periodic case, when $a(t)$ is bounded and $a(t) \to a_+ > 0$, $\omega(t) \to \omega_+ \neq 0$ as $t \to +\infty$. In particular we get existence of infinitely many homoclinic orbits of (1.1), namely non zero classical solutions to (1.1) satisfying the further conditions

$$q(t) \to 0 \quad \text{and} \quad \dot{q}(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.2}$$

Indeed we prove the following result.

Theorem 1.1. *Let $a \in C^1(\mathbb{R},\mathbb{R})$ be a bounded function such that $\lim_{t\to+\infty} a(t) = a_+ > 0$ and let $\omega \in C^1(\mathbb{R},\mathbb{R})$ be such that $\lim_{t\to+\infty} \omega(t) = \omega_+ \neq 0$. Then there* *exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *there is a homoclinic orbit* v_+ *for the system at infinity:*

$$\ddot{q} = q - a_+(1 + \epsilon \cos(\omega_+ t))\, q^3 \tag{1.3}$$

for which the following holds: for any $r > 0$ *there are* $m(r), m_1(r) \in \mathbb{R}$ *such that for every sequence* $(p_j) \subset \{2\pi k/\omega_+ \,:\, k \in \mathbb{Z}\}$ *satisfying* $p_1 \geq m_1(r)$ *and* $p_{j+1} - p_j \geq m(r)$ $(j \in \mathbb{N})$, *and for every sequence* $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ *there is a solution* v_σ *to (1.1) such that*

$$\|v_\sigma - \sigma_j v_+(\cdot - p_j)\|_{C^1(I_j, \mathbb{R}^N)} \leq r \quad \textit{for any } j \in \mathbb{N}$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$ *and* $p_0 = -\infty$. *In addition any* v_σ *also satisfies* $v_\sigma(t) \to 0$ *and* $\dot{v}_\sigma(t) \to 0$ *as* $t \to -\infty$ *and it is actually a homoclinic orbit if* $\sigma_j = 0$ *definitively.*

The solutions given by Theorem 1.1 are known as multibump solutions because they behave in this way: they remain in a small neighbourhood of the origin for a suitable large time and then leave it a finite or infinite number of times staying near translates of the basic homoclinic v_+ of (1.3).

The value $m(r)$ represents the minimal distance at which two consecutive bumps can be arranged. This value $m(r)$, as well as $m_1(r)$, becomes larger and larger as $r \to 0$. According to this remark, instead of fixing $r > 0$, it is possible to take a sequence $(r_j) \subset (0, \infty)$ such that $r_j \to 0$. Thus the following result concerning connecting orbits between 0 and the basic homoclinic v_+ of (1.3) holds true.

Theorem 1.2. *Under the same assumptions of Theorem 1.1, there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *there is a homoclinic orbit* v_+ *for the system (1.3) and an uncountable set of multibump solutions to (1.1) whose* α*-limit is 0 and whose* ω*-limit set is* $\{(v_+(t), \dot{v}_+(t)) \,:\, t \in \mathbb{R}\} \cup \{0\}$. *(We recall that the* ω*-limit set of a solution* v *to (1.1) is defined by:*

$\{(x,y) \in \mathbb{R}^{2N} \,:\, \exists\, (t_j) \subset \mathbb{R} \text{ s.t. } t_j \to +\infty\,,\ (v(t_j), \dot{v}(t_j)) \to (x,y)\,\}$.*)*

Remark 1.3. In the case $\epsilon = 0$ and $a(t)$ smooth, bounded and strictly monotone, the equation (1.1) does not have non zero homoclinic orbits. In fact if $q(t)$ satisfies (1.1) and (1.2) and $H(q(t)) = \frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{2}|q(t)|^2 + \frac{1}{4}a(t)|q(t)|^4$ denotes the energy of $q(t)$, then

$$0 = \int_{\mathbb{R}} \frac{dH(q(t))}{dt}\, dt = \int_{\mathbb{R}} \dot{a}(t)\frac{|q|^4}{4}\, dt$$

and this implies $q(t) \equiv 0$. We also see that for $\epsilon = 0$ the stable and unstable manifolds for (1.3) coincide and so their intersection is uncountable. As we will see in the sequel, this is the reason for which the argument used to prove Theorem 1.1 fails in this case.

The class of systems studied here is shaped on (1.1). In fact we deal with second order Hamiltonian systems in $\mathbb{R}^N$

$$\text{(HS)} \qquad \ddot{q} = -U'(t,q)$$

where $U'(t,q)$ denotes the gradient with respect to q of a smooth potential $U : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ having a non degenerate local maximum at the origin.

Precisely we assume:

(U1) $U \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ with $U'(t,\cdot)$ locally Lipschitz continuous, uniformly with respect to $t \in \mathbb{R}$;

(U2) $U(t,0) = 0$ and $U'(t,q) = L(t)q + o(|q|)$ as $q \to 0$ uniformly with respect to $t \in \mathbb{R}$ where $L(t)$ is a symmetric matrix such that $L_0|q|^2 \le q\cdot L(t)q \le L_1|q|^2$ for any $(t,q) \in \mathbb{R}\times\mathbb{R}^N$ with $0 < L_0 \le L_1 < \infty$.

Condition (U2) implies that in the phase space the origin is a hyperbolic equilibrium for the system (HS). We look for homoclinic orbits to (HS) as critical points of the Lagrangian functional

$$\varphi(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - U(t,u))\, dt$$

defined on $X = H^1(\mathbb{R},\mathbb{R}^N)$ and of class C^1, by (U1)–(U2) (see Lemmas 2.1 and 2.3).

Here, as pointed out with the model case, we consider asymptotically periodic potentials. By this we mean that there is a function $U_+(t,q) = -\frac{1}{2}\, q\cdot L_+(t)\, q + V_+(t,q)$ satisfying (U1), (U2) and

(U3) there is $T_+ > 0$ such that $U_+(t,q) = U_+(t+T_+,q)$ for any $(t,q) \in \mathbb{R}\times\mathbb{R}^N$;

(U4) (i) there is $(t_+,q_+) \in \mathbb{R}\times\mathbb{R}^N$ such that $U_+(t_+,q_+) > 0$;

(ii) there are two constants $\beta_+ > 2$ and $\alpha_+ < \frac{\beta_+}{2} - 1$ such that: $\beta_+ V_+(t,q) - V_+'(t,q)\cdot q \le \alpha_+ q \cdot L_+(t)q$ for all $(t,q) \in \mathbb{R}\times\mathbb{R}^N$;

(U5) $U'(t,q) - U_+'(t,q) \to 0$ as $t \to +\infty$ uniformly on the compact sets of $\mathbb{R}^N$.

As we have seen in Remark 1.3, these assumptions are not sufficient in order that (HS) admits homoclinic solutions. In that example the potential U_+ is time independent and hence the corresponding functional

$$\varphi_+(u) = \int_{\mathbb{R}} \left(\tfrac{1}{2}|\dot u|^2 - U_+(t,u)\right) dt$$

and $\|\varphi_+'(u)\|$ are invariant under the action of the translations group $\mathbb{R}$. In particular, if u is a non zero critical point of φ_+ (which always exists, as proved in Section 3) then also $u(\cdot - t)$ is a critical point of φ_+ for any $t \in \mathbb{R}$. Therefore the set of critical points of φ_+ is uncountable.

To avoid this situation, we make an assumption on the cardinality of the critical set of φ_+.

As we will see in Section 3, the functional φ_+ satisfies the geometrical properties of the mountain pass lemma. Denoting by c_+ the mountain pass level of φ_+ and $K_+ = \{u \in X : u \neq 0,\ \varphi_+'(u) = 0\}$, we assume that

(*) there exists $c_+^* > c_+$ such that the set $K_+ \cap \{u \in X : \varphi_+(u) \le c_+^*\}$ is countable.

On one hand, as seen above, condition (*) excludes the class of asymptotically autonomous systems.

On the other hand, (*) holds when the system at infinity exhibits countable intersection between the stable and unstable manifolds relative to the origin and then is a weaker condition than the transversality one, as noted in [S2].

The condition (*) is a key to find a local mountain pass critical point for φ_+ and to develop a minimax argument as in [S2]. The local character of such a procedure allows us to show existence of critical points also for the functional φ.

We can now state a first general result.

Theorem 1.4. *Let U and U_+ satisfy (U1)-(U5) and assume that $(*)$ holds. Then (HS) admits infinitely many homoclinic solutions.*
Precisely there is $v_+ \in K_+$ with the following property: for any $r > 0$ there are $m(r), m_1(r) \in \mathbb{R}$ such that for every $k \in \mathbb{N}$ and $(p_1, \ldots, p_k) \in P_+ = T_+\mathbb{Z}$ with $p_1 \geq m_1(r)$ and $p_{j+1} - p_j \geq m(r)$, for $j = 1, \ldots, k-1$, there exists a homoclinic solution v of (HS) which verifies

$$\|v - v_+(\cdot - p_j)\|_{C^1(I_j, \mathbb{R}^N)} \leq r \quad \textit{for any } j = 1, \ldots, k$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$, $p_0 = -\infty$ and $p_{k+1} = +\infty$.

An immediate consequence of Theorem 1.4 is the existence of infinitely many geometrically distinct homoclinic solutions for the periodic system $(\mathrm{HS})_+$ (we say that two solutions v_1 and v_2 to $(\mathrm{HS})_+$ are geometrically distinct if $v_1 \neq v_2(\cdot + nT_+)$ for every $n \in \mathbb{Z}$). Precisely we get the following alternative.

Corollary 1.5. *Let U_+ satisfy (U1)-(U4). Then $(HS)_+$ admits infinitely many homoclinic solutions. In particular either the set of homoclinics is uncountable or there is $v_+ \in K_+$ with the following property: for any $r > 0$ there is $m(r) > 0$ such that for every $k \in \mathbb{N}$ and $(p_1, \ldots, p_k) \in P_+ = T_+\mathbb{Z}$ with $p_{j+1} - p_j \geq m(r)$, for $j = 1, \ldots, k-1$, there exists a homoclinic solution v of $(HS)_+$ which verifies*

$$\|v - v_+(\cdot - p_j)\|_{C^1(I_j, \mathbb{R}^N)} \leq r \quad \textit{for any } j = 1, \ldots, k$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$, $p_0 = -\infty$ and $p_{k+1} = +\infty$.

Fixing $k = 1$, for any $r > 0$ Theorem 1.4 assures the existence of a value $m_1(r) \in \mathbb{N}$ and of a sequence v_j of homoclinic solutions of (HS) each of them belongs to an r–neighborhood of $v_+(\cdot - (m_1 + j)T_+)$ in $C^1(\mathbb{R}, \mathbb{R}^N)$. In general, unlike the periodic case, these solutions are geometrically distinct.

For an integer $k > 1$ Theorem 1.4 provides homoclinic orbits of (HS) having k bumps, whose positions are defined by the values $p_1, \ldots, p_k \in P_+$. More precisely, for any $j = 1, \ldots, k$ there is an interval I_j centered on p_j where the k-bump solution v of (HS) is not farther from $v_+(\cdot - p_j)$ than r in the norm of $C^1(I_j, \mathbb{R}^N)$. The value $\delta_j = p_{j+1} - p_j$ represents the distance between the corresponding bumps. Fixed r, we can find a solution of this kind for any choice of $k \in \mathbb{N}$ and of the sequence $p_1, \ldots, p_k$ provided that p_1 is sufficiently large, depending on r, and that the distances δ_j are greater than a certain value $m(r) > 0$.

As noticed in [S2], since the number $m(r)$ does not depend on k, one can consider the C^1_{loc}–closure of the set of the multibump homoclinic orbits, which contains solutions with possibly infinitely many bumps. Thus, by Ascoli Arzelà Theorem, the previous result can be generalized in the following way.

Theorem 1.6. *Under the same assumptions of Theorem 1.4, it holds that for any $r > 0$ there are $m(r), m_1(r) \in \mathbb{R}$ such that for every sequence $(p_j)_{j\in\mathbb{N}} \subset P_+ = T_+\mathbb{N}$ satisfying $p_1 \geq m_1(r)$ and $p_{j+1} - p_j \geq m(r)$ $(j \in \mathbb{N})$, and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ there is a solution v_σ to (HS) such that*

$$\|v_\sigma - \sigma_j v_+(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} \leq r \quad for \;\; any \;\; j \in \mathbb{N}$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$, $p_0 = -\infty$ and $v_+ \in K_+$ is the same of Theorem 1.4. In addition any v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot v_\sigma(t) \to 0$ as $t \to -\infty$ and it is actually a homoclinic orbit if $\sigma_j = 0$ definitively.

We can use the classical results on equation (1.3), and in particular the Melnikov theorem, to deduce Theorem 1.1 from Theorem 1.6.The Melnikov function of (1.3) is given by

$$M(s) = \sin(\omega_+ s) \int_{\mathbb{R}} \tfrac{\omega_+}{4} \cos(\omega_+ t)|q_0(t)|^4 \, dt = \sin(\omega_+ s)\, C_{\omega_+}$$

where $q_0(t) = (2/a_+)^{\frac{1}{2}}(\cosh t)^{-1}$ is a homoclinic orbit of the unperturbed system $\ddot q = q - a_+ q^3$ and $C_{\omega_+} \in (0,\infty)$ for any $\omega_+ \neq 0$ (see [GH], [Me], [W]). Since the zeros of $M(s)$ are simple, by the Melnikov Theorem [Me], for $\epsilon \neq 0$ small enough, the stable and unstable manifolds of the perturbed system (1.3) intersect transversally and so countably. Thus $(*)$ is verified.

We notice that Theorem 1.6 can be seen as a version of the shadowing lemma (see [L]). In addition the correspondence $\sigma \mapsto v_\sigma$ permits to define an approximate Bernoulli shift for the system (HS) (see [S2]). The presence of this structure implies sensitive dependence on initial data.

We point out that in the previous Theorems 1.4 and 1.6 no assumption is made on the behaviour of U as $t \to -\infty$, but the regularity and hyperbolicity hypotheses (U1) and (U2).

If the system (HS) is doubly asymptotic as $t \to \pm\infty$ to two, possibly different, periodic systems

$$(\mathrm{HS})_\pm \qquad \ddot{q} = -U'_\pm(t,q)$$

then, by Theorem 1.6, we have two different sets of multibump solutions, that, at $\pm\infty$ are near to solutions of $(\mathrm{HS})_\pm$. Here and in the sequel, with $(\mathrm{HS})_-$ we denote a system ruled by a potential $U_-(t,q) = -\frac{1}{2}\,q \cdot L_-(t)\,q + V_-(t,q)$ satisfying (U1)-(U4).

In fact, we prove that there are also multibump solutions of (HS) of mixed type, as said in the following theorem.

Theorem 1.7. *Let U, U_+ and U_- satisfy (U1)-(U5) and assume that $(*)$ holds both for $(HS)_+$ and $(HS)_-$. Then there are v_+ and v_- homoclinic solutions respectively of $(HS)_+$ and $(HS)_-$ having the following property: for any $r > 0$ there are $m(r), m_1(r) > 0$ such that for every biinfinite sequence $(p_j)_{j\in\mathbb{Z}}$ with $(p_j)_{j>0} \subset P_+ = T_+\mathbb{Z}$ and $(p_j)_{j<0} \subset P_- = T_-\mathbb{Z}$ satisfying $p_1 \geq m_1(r)$, $p_{-1} \leq -m_1(r)$, $p_{j+1} - p_j \geq m(r)$ $(j \in \mathbb{Z})$ and for every sequence $\sigma = (\sigma_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there is a solution v_σ to (HS) such that*

$$\|v_\sigma - \sigma_j v_+(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} \leq r \quad \text{for any } j > 0$$
$$\|v_\sigma - \sigma_j v_-(\cdot - p_j)\|_{C^1(I_j,\mathbb{R}^N)} \leq r \quad \text{for any } j < 0$$

where $I_j = [\frac{p_{j-1}+p_j}{2}, \frac{p_j+p_{j+1}}{2}]$.
In addition, if $\sigma_j = 0$ for all $j \geq j_0$ (respectively $j \leq j_0$) then the solution v_σ also satisfies $v_\sigma(t) \to 0$ and $\dot{v}_\sigma(t) \to 0$ as $t \to +\infty$ (respectively $t \to -\infty$).

Clearly, in the previous statement, when we say that U, U_+ and U_- satisfy (U5) we mean that $U'(t,q) - U'_+(t,q) \to 0$ as $t \to +\infty$ and $U'(t,q) - U'_-(t,q) \to 0$ as $t \to -\infty$ uniformly on the compact sets of $\mathbb{R}^N$.

Finally, as for the model case discussed at the beginning, taking a sequence $(r_j) \subset (0,\infty)$ with $r_j \to 0$, one can also get existence of connecting orbits between the primary homoclinics $v_\pm$ of $(HS)_\pm$.

Theorem 1.8. *Under the same assumptions of Theorem 1.7, the system (HS) admits an uncountable set of multibump solutions whose α-limit set is 0 or $\{(v_-(t),\dot v_-(t)) : t\in\mathbb{R}\}\cup\{0\}$ and whose ω-limit set is 0 or $\{(v_+(t),\dot v_+(t)) : t\in\mathbb{R}\}\cup\{0\}$.*

Coming back to the model equation (1.1) with $a(t)$ bounded and strictly increasing we see that while for $\epsilon=0$ the system has no homoclinic solutions, there exists $\epsilon_0>0$ such that $(*)$ is satisfied for $0<|\epsilon|<\epsilon_0$ and so, by Theorem 1.4, the equation (1.1) has infinitely many homoclinic orbits. Geometrically this means that while the stable and unstable manifolds in the extended phase space do not intersect when $\epsilon=0$ (apart from in the origin), as soon as $\epsilon\neq 0$ they intersect in an infinite set. This suggests that the stable and unstable manifolds for (1.1) accumulate one on the other for $t\to+\infty$.

2. Preliminary results

In this section we introduce the variational setting for the homoclinic problem and we discuss some basic general facts which depend only on the hyperbolicity assumption and therefore are true both for the periodic and the asymptotically periodic case. Hence during this section we assume only (U1)–(U2), without any hypothesis on the time dependence of the potential.

We denote by X the Sobolev space $H^1(\mathbb{R},\mathbb{R}^N)$ endowed with the inner product $\langle u,v\rangle=\int_{\mathbb{R}}(\dot u\cdot\dot v+u\cdot L(t)v)\,dt$, whose corresponding norm $\|u\|=\langle u,u\rangle^{\frac12}$ is equivalent to the standard H^1-norm. We recall that X is continuously imbedded in the space of continuous functions converging to 0 at infinity. Moreover any bounded sequence $(u_n)\subset X$ admits a subsequence which converges weakly in X and strongly in $L^\infty_{\rm loc}(\mathbb{R},\mathbb{R}^N)=L^\infty_{\rm loc}$ to some $u\in X$.

Then, for any measurable set $A\subset\mathbb{R}$ we put $\langle u,v\rangle_A=\int_A(\dot u\cdot\dot v+u\cdot L(t)v)\,dt$ and $\|u\|_A=\langle u,u\rangle_A^{1/2}$ for every $u,v\in X$.

Then we set

$$\varphi(u)=\int_{\mathbb{R}}(\tfrac12|\dot u|^2-U(t,u))\,dt=\tfrac12\|u\|^2-\int_{\mathbb{R}}V(t,u)\,dt$$

for any $u\in X$, where $U(t,q)=-\frac12 q\cdot L(t)q+V(t,q)$ for $(t,q)\in\mathbb{R}\times\mathbb{R}^N$.

Lemma 2.1. *$\varphi \in C^1(X,\mathbb{R})$ and $\varphi'(u)h = \langle u,h\rangle - \int_{\mathbb{R}} V'(t,u)\cdot h\,dt$ for any $u,h \in X$.*

Proof. The functional $u \mapsto \|u\|^2$ is of class C^∞ on X and the Frechet differential of $\|u\|^2$ at $u \in X$ is the functional $2\langle u,\cdot\rangle$.
The functional $u \mapsto \int_{\mathbb{R}} V(t,u)\,dt = \psi(u)$ is well defined on X. Indeed, by (U2), there is $\delta_1 > 0$ such that $|V(t,q)| \le |q|^2$ and $|V'(t,q)| \le |q|$ for $|q| \le \delta_1$. Then, given any $u \in X$ there is $T_u > 0$ such that $|u(t)| \le \delta_1$ for $|t| \ge T_u$. Hence $|V(t,u(t))| \le M_u|u(t)|^2$ for any $t \in \mathbb{R}$, where $M_u = \max\{1, \frac{m_u}{\delta_1}\}$ and $m_u = \max\{|V(t,q)| : t \in \mathbb{R},\ |q| \le \|u\|_{L^\infty}\}$. Hence $\psi(u) \le M_u\|u\|_{L^2}^2 \le \frac{M_u}{L_0}\|u\|^2$. Similarly $|V'(t,u)| \le M_u'|u(t)|$ for any $t \in \mathbb{R}$, for a suitable constant $M_u' \ge 0$. Then given $u,h \in X$

$$\int_{\mathbb{R}} |V'(t,u)\cdot h|\,dt \le L_0^{-1}\left(\int_{\mathbb{R}} |V'(t,u)|^2\,dt\right)^{\frac{1}{2}} \|h\|$$

which implies that the linear operator $h \mapsto \int_{\mathbb{R}} V'(t,u)\cdot h\,dt$ is continuous. Moreover, fixed $s \in [-1,1]$, for any $t \in \mathbb{R}$, by the mean value Theorem, there is $\theta(t) \in [0,1]$ such that

$$\begin{aligned}
&\left|\tfrac{1}{s}(V(t,u(t)+sh(t)) - V(t,u(t)) - V'(t,u(t))\cdot h(t)\right| \\
&= |V'(t,u(t)+s\theta(t)h(t))\cdot h(t) - V'(t,u(t))\cdot h(t)| \\
&\le C_{u,h}|s\theta(t)h(t)|\,|h(t)| \le C_{u,h}|s|\,|h(t)|^2
\end{aligned}$$

where $C_{u,h}$ is a positive constant independent of $t \in \mathbb{R}$. Since $h \in L^2$, we can apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{s\to 0} \tfrac{1}{s}(\psi(u+sh) - \psi(u)) = \int_{\mathbb{R}} V'(t,u)\cdot h\,dt.$$

Hence the functional ψ is Gateaux–differentiable at u and

$$d_G\psi(u)h = \int_{\mathbb{R}} V'(t,u)\cdot h\,dt \quad (h \in X).$$

To conclude the proof of Lemma, it is enough to show that the mapping $u \mapsto d_G\psi(u)$ is continuous. Fixed $u \in X$ let $(u_n) \subset X$ such that $u_n \to u$ in X.

Then the sequence (u_n) is bounded in X and, by the continuous imbedding of X into L^∞, there exists $R \geq 0$ such that $|u_n(t)-u(t)| \leq R$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Since V' is locally Lipschitz continuous with respect to q uniformly in $t \in \mathbb{R}$, there is $C_R > 0$ such that $|V'(t,u_n(t)) - V'(t,u(t))| \leq C_R|u_n(t)-u(t)|$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$ and then

$$|(d_G\psi(u_n) - d_G\psi(u))h| \leq \int_{\mathbb{R}} C_R|u_n - u|\,|h|\,dt \leq C_R L_0^{-2}\|u_n - u\|\,\|h\|.$$

Thus $\|d_G\psi(u_n) - d_G\psi(u)\| \to 0$. This completes the proof. □

Remark 2.2. The functional φ sends bounded sets into bounded sets. Indeed in the proof of Lemma 2.1 we have seen that $\varphi(u) \leq \frac{1}{2}\|u\|^2 + M_u\|u\|_{L^2}^2$ where M_u depends only on $\|u\|_{L^\infty}$. Hence, by the continuous imbedding of X into L^2 and L^∞ we get that if $\|u\| \leq r$ then $|\varphi(u)| \leq C_r$ for some $C_r \geq 0$.

Lemma 2.3. *A function* $u : \mathbb{R} \to \mathbb{R}^N$ *is a homoclinic solution to* (HS) *if and only if* $u \in X$ *and* $\varphi'(u) = 0$. *Moreover, in this case, there are* $t_0, a > 0$ *such that* $|u(t)| \leq |u(t_0)|e^{a\,(t_0-|t|)}$ *for* $|t| \geq t_0$.

Proof. Let $u \in C^2(\mathbb{R},\mathbb{R}^N)$ be a homoclinic solution to (HS). By (U2), since $u(t) \to 0$ as $t \to \pm\infty$, there is $t_0 > 0$ such that $|V'(t,u(t))| \leq L_0|u(t)|$ for $|t| \geq t_0$. Hence, setting $r(t) = |u(t)|^2$, by (HS), we get that $\ddot{r}(t) \geq L_0 r(t)$ for $|t| \geq t_0$. Let us define $h(t) = r(t_0)e^{\sqrt{L_0}(t_0-|t|)} - r(t)$ and $f(t) = \ddot{r}(t) - L_0 r(t)$. For $|t| \geq t_0$ we have that $-\ddot{h} + L_0 h = f(t)$ and then, by the maximum principle, $h(t) \geq 0$ that is $|u(t)| \leq |u(t_0)|e^{a\,(t_0-|t|)}$, with $a = \frac{1}{2}\sqrt{L_0}$. Moreover $|\ddot{u}(t)| \leq (L_1 + L_0)|u(t)| \leq A_0 e^{a\,(t_0-|t|)}$ for $|t| \geq t_0$ and, by a standard interpolation inequality, a similar estimate holds true for $|\dot{u}(t)|$. Therefore, in particular, a homoclinic solution to (HS) belongs to $H^2(\mathbb{R},\mathbb{R}^N)$ and, given any $h \in C_c^1(\mathbb{R},\mathbb{R}^N)$, with an integration by parts, we get that $\varphi'(u)h = 0$. By a density argument we conclude that $\varphi'(u) = 0$.
Conversely, if $u \in X$ is such that $\varphi'(u) = 0$ then $u(t) \to 0$ as $t \to \pm\infty$ (any element of X verifies this property), by standard arguments u is a classical solution to (HS) and $\int_{\mathbb{R}} |\ddot{u}|^2\,dt \leq \int_{\mathbb{R}}(L_1|u|^2 + V'(t,u)\cdot u)\,dt < \infty$, that is $u \in H^2(\mathbb{R},\mathbb{R}^N)$. Hence $\dot{u} \in H^1(\mathbb{R},\mathbb{R}^N)$, that implies $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

Lemma 2.4. $\varphi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *and* $\varphi'(u) = \langle u, \cdot\rangle + o(\|u\|)$ *as* $u \to 0$.

Proof. Let $C>0$ be the imbedding constant of X in L^∞. By (U2), for any $\epsilon>0$ there is $\delta>0$ such that if $|q|\le\delta C$ then $|V(t,q)|\le\epsilon L_0|q|^2$ and $|V'(t,q)|\le\epsilon L_0|q|$ for every $t\in\mathbb{R}$. Hence for any $u\in X$ with $\|u\|\le\delta$ it holds that $|u(t)|\le\delta C$ for every $t\in\mathbb{R}$ and consequently $\int_{\mathbb{R}}|V(t,u)|\,dt\le\epsilon\int_{\mathbb{R}}L_0|u|^2\,dt\le\epsilon\|u\|^2$, that is $\varphi(u)=\frac{1}{2}\|u\|^2+o(\|u\|^2)$ for $\|u\|\to 0$. In a similar way, fixed $h\in X$, $|\varphi'(u)h-\langle u,h\rangle|\le\int_{\mathbb{R}}|V'(t,u)\cdot h|\,dt\le\epsilon\int_{\mathbb{R}}L_0|u|\,|h|\,dt\le\epsilon\|u\|\,\|h\|$, that is $\|\varphi'(u)-\langle u,\cdot\rangle\|\to 0$ as $\|u\|\to 0$. □

Remark 2.5. We recall that given an interval I of $\mathbb{R}$, the imbedding constant C_I of $H^1(I,\mathbb{R}^N)$ into $L^\infty(I,\mathbb{R}^N)$ depends on I like $1+\frac{1}{|I|}$. Therefore, repeating the proof of Lemma 2.4, we get the following property.
For every $\epsilon>0$ there exists $\delta>0$ such that for any given interval $I\subseteq\mathbb{R}$, with $|I|\ge 1$ and for any $u\in X$ with $\|u\|_I\le\delta$ it holds that $\int_I V(t,u)\,dt\le\epsilon\|u\|_I^2$ and $\int_I V'(t,u)\cdot h\,dt\le\epsilon\|u\|_I\|h\|_I$, for every $h\in X$.

Lemma 2.6. *(i) There is $\delta>0$ such that if $(u_n)\subset X$ satisfies $\varphi'(u_n)\to 0$ and $\|u_n\|_{L^\infty}\le\delta$ then $\|u_n\|\to 0$. (ii) There is $\rho>0$ such that if $(u_n)\subset X$ satisfies $\varphi'(u_n)\to 0$ and $\limsup\|u_n\|>0$ then $\limsup\|u_n\|\ge 2\rho$.*

Proof. Part (i) plainly follows from the fact that, by (U2), there is $\delta>0$ such that $|V'(t,q)\cdot q|\le\frac{1}{2}L_0|q|^2$ for $|q|\le\delta$ and for any $t\in\mathbb{R}$. Therefore, if $u\in X$ is such that $\|u\|_{L^\infty}\le\delta$ then $\int_{\mathbb{R}}V'(t,u)\cdot u\,dt\le\frac{1}{2}L_0\int_{\mathbb{R}}|u|^2\,dt\le\frac{1}{2}\|u\|^2$ and consequently $\varphi'(u)u=\|u\|^2-\int_{\mathbb{R}}V'(t,u)\cdot u\,dt\ge\frac{1}{2}\|u\|^2$, that implies $\|u\|\le 2\|\varphi'(u)\|$.
Part (ii) follows from (i) and from the continuous imbedding of X into L^∞.
□

Remark 2.7. There is $\rho>0$ such that $\|u\|\ge 2\rho$ for any $u\in K$, where $K=\{u\in X\,:\,\varphi'(u)=0,\ u\ne 0\}$.

Now we give some properties of the Palais Smale (briefly PS) sequences of φ, namely sequences $(u_n)\subset X$ such that $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n)\to 0$. In general (U1)–(U2) are not sufficient to guarantee the boundedness of these sequences. Anyhow we state the following results, concerning the bounded PS sequences.

Lemma 2.8. *Let $(u_n) \subset X$ be a PS sequence at the level b (namely $\varphi(u_n) \to b$ and $\|\varphi'(u_n)\| \to 0$) weakly converging to some $u \in X$. Then*

(i) $\varphi'(u) = 0$;

(ii) $(u_n - u)$ *is a PS sequence at the level* $b - \varphi(u)$;

(iii) $u_n \to u$ *strongly in* $H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$;

(iv) *either $u_n \to u$ strongly in X, or there is a subsequence (u_{n_k}) and a corresponding sequence $(t_k) \subset \mathbb{R}$ such that $|t_k| \to \infty$ and $\limsup |u_{n_k}(t_k)| \geq \delta$ where $\delta > 0$ is given by Lemma 2.6.*

Proof. (i) Without loss of generality we can take a sequence $(u_n) \subset X$ which converges weakly in X and strongly in L^∞_{loc} to some $u \in X$. Then, for any $h \in C^\infty_c(\mathbb{R}, \mathbb{R}^N)$ we have:

$$\begin{aligned} \varphi'(u)h &= \langle u, h\rangle - \textstyle\int_{\operatorname{supp} h} V'(t,u) \cdot h\, dt \\ &= \lim \langle u_n, h\rangle - \lim \textstyle\int_{\operatorname{supp} h} V'(t,u_n) \cdot h\, dt \\ &= \lim \varphi'(u_n)h. \end{aligned}$$

Therefore, since $\varphi'(u_n) \to 0$, $\varphi'(u) = 0$ follows.

(ii) For any $h \in X$ and for any $T > 0$:

$$\begin{aligned} |\varphi'(u_n - u)h - \varphi'(u_n)h| &= \left|\textstyle\int_{\mathbb{R}} (V'(t,u_n - u) - V'(t,u_n) + V'(t,u)) \cdot h\, dt\right| \\ &\leq \left|\textstyle\int_{|t|\leq T} (V'(t,u_n - u) - V'(t,u_n) + V'(t,u)) \cdot h\, dt\right| \\ &\quad + \textstyle\int_{|t|>T} |V'(t,u_n - u) - V'(t,u_n)|\, |h|\, dt + \int_{|t|>T} |V'(t,u)|\, |h|\, dt \\ &\leq \delta_n(T) \left(\textstyle\int_{|t|\leq T} |h|^2\, dt\right)^{\frac{1}{2}} + \int_{|t|>T} C_R |u|\, |h|\, dt \\ &\quad + \left(\textstyle\int_{|t|>T} |V'(t,u)|^2\, dt\right)^{\frac{1}{2}} \left(\int_{|t|>T} |h|^2\, dt\right)^{\frac{1}{2}} \end{aligned}$$

where:

$$\begin{aligned} &\delta_n(T) = \left(\textstyle\int_{|t|\leq T} |V'(t,u_n - u) - V'(t,u_n) + V'(t,u)|^2\, dt\right)^{\frac{1}{2}} \\ &C_R = \sup\left\{ |V'(t,q) - V'(t,\bar q)|/|q - \bar q| \;:\; t \in \mathbb{R},\ |q|, |\bar q| \leq R,\ q \neq \bar q \right\} \end{aligned}$$

and $R > 0$ is such that $|u_n(t)| + |u(t)| \leq R$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$. We note that $R < \infty$ because (u_n) is bounded in X and so in $L^\infty(\mathbb{R}, \mathbb{R}^N)$. Then, by (U1), $C_R < \infty$ too. Hence we get:

$$\begin{aligned}|\varphi'(u_n-u)h - \varphi'(u_n)h| &\le \delta_n(T)\left(\int_{\mathbb{R}}|h|^2\,dt\right)^{\frac{1}{2}}\\ &+ C_R\left(\int_{|t|>T}|u|^2 dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|h|^2\,dt\right)^{\frac{1}{2}} + \left(\int_{|t|>T}|V'(t,u)|^2\,dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|h|^2\,dt\right)^{\frac{1}{2}}\end{aligned}$$

which implies:

$$L_0^{\frac{1}{2}}\|\varphi'(u_n-u)-\varphi'(u_n)\| \le \delta_n(T)+C_R\left(\int_{|t|>T}|u|^2 dt\right)^{\frac{1}{2}}+\left(\int_{|t|>T}|V'(t,u)|^2\,dt\right)^{\frac{1}{2}}.$$

Now, for any $\epsilon > 0$ we can choose $T > 0$ such that

$$C_R\left(\int_{|t|>T}|u|^2 dt\right)^{\frac{1}{2}} + \left(\int_{|t|>T}|V'(t,u)|^2\,dt\right)^{\frac{1}{2}} < \epsilon L_0^{\frac{1}{2}}.$$

By the Lebesgue dominated convergence Theorem, $\delta_n(T) \to 0$ as $n \to \infty$. Therefore $\limsup \|\varphi'(u_n - u)\| \le \epsilon$ and, for the arbitrariness of $\epsilon > 0$, we get that $\lim \|\varphi'(u_n - u)\| = 0$.
Now we prove that if $b = \lim \varphi(u_n)$ then $\varphi(u_n - u) \to b - \varphi(u)$. Taking $R > 0$ as before and setting

$$C'_R = \sup\left\{\, |V'(t,q)|/|q| \;:\; t \in \mathbb{R},\ |q| \le R,\ q \ne 0 \,\right\},$$

by the mean value Theorem we get that for any $t \in \mathbb{R}$:

$$\begin{aligned}|V(t,u_n(t)-u(t)) - V(t,u_n(t))| &= |V'(t,u_n(t)-\theta u(t))\cdot u(t)|\\ &\le C'_R|u_n(t)-\theta u(t)|\,|u(t)| \le C'_R|u_n(t)|\,|u(t)| + C'_R|u(t)|^2\end{aligned}$$

where $\theta = \theta(t) \in [0,1]$ and then

$$\begin{aligned}|\phi(u_n-u)-\phi(u_n)+\phi(u)| &\le \big|\,\|u\|^2 - \langle u_n,u\rangle\big|\\ &\quad+ \int_{|t|\le T}|V(t,u_n-u)-V(t,u_n)+V(t,u)|\,dt\\ &\quad+ \int_{|t|>T}|V(t,u_n-u)-V(t,u_n)|\,dt + \int_{|t|>T}|V(t,u)|\,dt\\ &\le \big|\,\|u\|^2 - \langle u_n,u\rangle\big|\\ &\quad+ \int_{|t|\le T}|V(t,u_n-u)-V(t,u_n)+V(t,u)|\,dt\\ &\quad+ C'_R\int_{|t|>T}|u_n|\,|u|\,dt + C'_R\int_{|t|>T}|u|^2 dt + \int_{|t|>T}|V(t,u)|\,dt.\end{aligned}$$

Fixing $\epsilon > 0$ we can find $T > 0$ independent of $n \in \mathbb{N}$ such that

$$C'_R \int_{|t|>T} |u_n|\,|u|\,dt + C'_R \int_{|t|>T} |u|^2 dt + \int_{|t|>T} |V(t,u)|\,dt < \epsilon.$$

Since $\int_{|t|\le T} |V(t,u_n-u) - V(t,u_n) + V(t,u)|\,dt \to 0$ as $n\to\infty$ and $u_n \to u$ weakly, we infer that $\limsup |\varphi(u_n-u) - \varphi(u_n) + \varphi(u)| \le \epsilon$ which implies that $\lim \varphi(u_n-u) = b - \varphi(u)$.

(iii) Fixed $T > 0$ let

$$\chi(t) = \begin{cases} 1 & \text{for } |t| \le T \\ -|t| + T + 1 & \text{for } T < |t| \le T+1 \\ 0 & \text{for } |t| > T+1. \end{cases}$$

For any $u \in X$ it holds that

$$\begin{aligned} &\int_{-T}^{T} (|\dot u|^2 + |u|^2)\,dt \le \langle u, \chi u\rangle_{H^1} + \tfrac{1}{2}|u(-T-1)|^2 + \tfrac{1}{2}|u(T+1)|^2 \\ &\varphi'(u)\chi u = \langle u, \chi u\rangle - \int_{\mathbb{R}} V'(t,u)\cdot \chi u\,dt \\ &\langle u, \chi u\rangle_{H^1} \le C_0 \langle u, \chi u\rangle \end{aligned}$$

where $C_0 = \max\{1, L_0^{-1}\}$. Then

$$\begin{aligned} \int_{-T}^{T} (|\dot u|^2 + |u|^2)\,dt \le & C_0 \|\varphi'(u)\|\,\|\chi u\| + C_0 \int_{\mathbb{R}} V'(t,u)\cdot \chi u\,dt \\ & + \frac{1}{2}|u(-T-1)|^2 + \frac{1}{2}|u(T+1)|^2. \end{aligned} \tag{2.1}$$

Hence, applying (2.1) to $u_n - u$, noting that the function $u \mapsto \chi u$ is a bounded linear operator in X and using the fact that $\|\varphi'(u_n-u)\| \to 0$ and $u_n \to u$ uniformly on the compact sets we get that $\|u_n - u\|_{H^1([-T,T])} \to 0$.

(iv) To prove the alternative, we point out that if $\|u_n - u\|_{L^\infty} \le \delta$, then, since $\|\varphi'(u_n-u)\| \to 0$, by Lemma 2.6, $u_n \to u$ strongly in X. Otherwise $\|u_{n_k} - u\|_{L^\infty} > \delta$ for a subsequence (u_{n_k}). Let $t_k \in \mathbb{R}$ be such that $|u_{n_k}(t_k) - u(t_k)| = \|u_{n_k} - u\|_{L^\infty}$. It holds that $|t_k| \to \infty$ because of (iii), and, since $|u(t_k)| \to 0$ $\limsup |u_{n_k}(t_k)| \ge \delta$. □

Lemma 2.9. *Let* $(u_n) \subset X$ *be such that* $\varphi'(u_n) \to 0$ *and let* $\rho > 0$ *be given according to Remark 2.7.*

(*i*) *If* $\limsup \|u_n\|_{|t|>T} < \rho$ *for some* $T > 0$, *then there is* $u \in K \cup \{0\}$ *such that* $u_n \to u$.

(*ii*) *If* $\operatorname{diam}\{u_n\} < \rho$, *then* (u_n) *admits a subsequence strongly converging to some* $u \in K \cup \{0\}$.

Proof. (*i*) Fix $R > 0$ such that $\|u\|_{|t|\geq R} \leq \frac{1}{2}\rho$. Putting $M = \max\{R, T\}$, by Lemma 2.8, we have that $\|u_n - u\|_{|t|\leq M} \to 0$. Therefore $\|u_n - u\|^2 = o(1) + \|u_n - u\|^2_{|t|>M} \leq o(1) + \frac{1}{4}\rho^2 + \rho\,\|u_n\|_{|t|>M} + \|u_n\|^2_{|t|>M}$, from which we get $\limsup \|u_n - u\| < 2\rho$. Since $\varphi'(u_n - u) \to 0$ we infer by Lemma 2.6 (*iii*) that $u_n \to u$ strongly in X.

Let $\delta < \rho - \operatorname{diam}\{u_n\}$ and $T > 0$ such that $\|u_1\|_{|t|>T} < \delta$. Then $\|u_n\|_{|t|>T} \leq \|u_n - u_1\|_{|t|>T} + \delta < \operatorname{diam}\{u_n\} + \delta = \tilde{\rho} \leq \rho$. Hence $\limsup \|u_n\|_{|t|>T} < \rho$ and the conclusion follows from (*i*). □

3. A mountain pass–type critical point for the periodic functional

Here we firstly state some properties satisfied by the functional

$$\varphi_+(u) = \int_{\mathbb{R}} (\tfrac{1}{2}|\dot{u}|^2 - U_+(t,u))\,dt$$

by the periodicity and superquadraticity assumptions (U3) and (U4). Then, using the hypothesis (∗), we get further compactness properties which, together with Lemma 2.9, give the existence of a local mountain pass-type critical point for φ_+.

Lemma 3.1. *If* $(u_n) \subset X$ *is a sequence such that* $\lim \varphi'_+(u_n) = 0$ *and* $\limsup \varphi_+(u_n) < +\infty$, *then* (u_n) *is bounded in* X *and* $\liminf \varphi_+(u_n) \geq 0$. *In particular any PS sequence for* φ_+ *is bounded in* X.

Proof. Setting $\|u\|^2_+ = \int_{\mathbb{R}}(|\dot{u}|^2 + u \cdot L_+(t)u)\,dt$, by (U4)-(*ii*), we have that for any $u \in X$ $(\frac{1}{2} - \frac{1}{\beta_+})\|u\|^2_+ = \varphi_+(u) + \int_{\mathbb{R}} V_+(t,u)\,dt - \frac{1}{\beta_+}\varphi'_+(u)u - \frac{1}{\beta_+}\int_{\mathbb{R}} V'_+(t,u)\cdot u \leq \varphi_+(u) + \frac{1}{\beta_+}\|\varphi'_+(u)\|\,\|u\|_+ + \frac{\alpha_+}{\beta_+}\int_{\mathbb{R}} u \cdot L_+(t)u\,dt$ and so

$$(\tfrac{1}{2} - \tfrac{1}{\beta_+} - \tfrac{\alpha_+}{\beta_+})\|u\|^2_+ - \tfrac{1}{\beta_+}\|\varphi'_+(u)\|\,\|u\|_+ \leq \varphi_+(u). \tag{3.1}$$

Now, given a sequence $(u_n) \subset X$ such that $\varphi'_+(u_n) \to 0$ and $\limsup \varphi_+(u_n) < +\infty$, since $\|\varphi'_+(u_n)\|$ and $\varphi_+(u_n)$ are bounded from above, from (3.1) we get that $\|u_n\| \le C$ for all $n \in \mathbb{N}$, C being a positive constant. Consequently we have that $\varphi_+(u_n) \ge -C\|\varphi'(u_n)\|$ and this implies that $\liminf \varphi_+(u_n) \ge 0$. □

Remark 3.2. By Remark 2.7 and by (3.1) we get that there is $c_0 > 0$ such that $\varphi_+(u) \ge c_0$ for any $u \in K_+$ where $K_+ = \{u \in X : \varphi'_+(u) = 0, \ u \ne 0\}$.

The following result presents a characterization of the PS sequences for φ_+, in the spirit of the concentration-compactness principle [L].

Lemma 3.3. *Let $(u_n) \subset X$ be a PS sequence for φ_+ at the level b. Then there are $v_0 \in K_+ \cup \{0\}$, $v_1, \dots, v_k \in K_+$, a subsequence of (u_n), denoted again (u_n), and corresponding sequences $(p_n^1), \dots, (p_n^k) \in P_+$ such that, as $n \to \infty$:*

$$\|u_n - (v_0 + \tau_{p_n^1} v_1 + \ldots + \tau_{p_n^k} v_k)\| \to 0$$

$$\varphi_+(v_0) + \cdots + \varphi_+(v_k) = b$$

$$p_n^{j+1} - p_n^j \to +\infty \quad (j = 1, \dots, k-1)$$

where we denote $\tau_s u(t) = u(t-s)$.

Proof. Let $(u_n) \subset X$ be such that $\varphi_+(u_n) \to b$ and $\varphi'_+(u_n) \to 0$. By Lemma 3.1, $b \ge 0$ and (u_n) is bounded in X and so, up to a subsequence, converges weakly to some $v_0 \in K \cup \{0\}$. By (3.1), if $b = 0$ then $u_n \to 0$. Let us suppose now $b > 0$. By (U3) there is a sequence $(p_n^1) \subset P_+$ such that $\|u_n\|_{L^\infty} = \max_{t \in [0,T_+]} |u_n(t - p_n^1)|$ and the sequence (u_n^1), given by $u_n^1 = \tau_{p_n^1} u_n$, is a PS sequence for φ_+ at level b and – up to a subsequence – converges to some $v_1 \in K$, weakly in E and uniformly on compact subsets of $\mathbb{R}$. Then we define $u_n^2 = u_n^1 - v_1$. By Lemma 2.8 (u_n^2) is a PS sequence for φ_+ at level $b_1 = b - \varphi(v_1)$. Thus, if $b_1 = 0$ then, as proved before, $u_n^2 \to 0$ and the thesis holds with $k = 1$. If $b_1 > 0$ then we are in the above case $b > 0$ and we repeat the argument. By Remark 3.2, this process must end in at most $[b/c_0]$ steps. □

The hypothesis (U4) gives information about the behaviour of the potential at infinity with respect to q along the direction of q_+ in a neighborhood of t_+. In fact, from (U4), one can infer that

$$V_+(t, sq_+) \geq \delta s^{\beta_+} \quad \forall s \geq 1, \ \forall t \in [t_+ - \epsilon, t_+ + \epsilon] \tag{3.2}$$

where $\delta = 2(V_+(t_+, q_+) - \frac{\alpha_+}{\beta_+ - 2} q_+ \cdot L_+(t_+) q_+) > 0$ and $\epsilon > 0$ small enough. Hence, choosing a function $\chi \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\operatorname{supp} \chi = [t_+ - \epsilon, t_+ + \epsilon]$, and setting $u_0(t) = \chi(t) q_+$ we have that $\varphi_+(s\, u_0) \to -\infty$ as $s \to \infty$.

Together with Lemma 2.4, this says that the functional φ_+ verifies the geometrical hypotheses of the mountain pass theorem [AR].

Then, if we define

$$\Gamma = \{ \gamma \in C([0,1], X) \, : \, \gamma(0) = 0 \, , \ \varphi_+(\gamma(1)) < 0 \}$$

and

$$c_+ = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \varphi_+(\gamma(s))$$

we infer that, by Lemma 2.4, $c_+ > 0$ and there is a PS sequence for φ_+ at the level c_+. Hence, by Lemma 3.3, we get that $K_+ \neq \emptyset$, and thus, by Lemma 2.3, there is a non zero homoclinic solution to $(\mathrm{HS})_+$.

As said in the Introduction, we need a non zero critical point of φ_+ that is well characterized from a variational viewpoint.

To this extent we recall here the definition of mountain pass–type critical point given firstly by Hofer in [H], and then stated by Pucci and Serrin [PS] in a slightly different formulation, more useful in our context.

Definition 3.4. Let f be a functional of class C^1 on a Banach space X and let Ω be a non empty open subset of X. Two points $u_0, u_1 \in \Omega$ are said connectible in $A \subseteq \Omega$ if there is a path $\gamma \in C([0,1], X)$ joining u_0 and u_1, with range $\gamma \subset A$.
A critical point $\bar{u} \in X$ for f is called of *local mountain pass–type* for f on Ω if $\bar{u} \in \Omega$ and for any neighborhood $\mathcal{N}$ of $\bar{u}$ subset of Ω the set $\{u : f(u) < f(\bar{u})\} \cap \mathcal{N}$ contains two points not connectible in $\Omega \cap \{u : f(u) < f(\bar{u})\}$.

To prove the existence of a local mountain pass–type critical point for φ_+ it is useful to introduce the following sets.

Given $b \geq 0$ and letting

$$\mathcal{S}^b_{\rm PS}(\varphi_+) = \{ (u_n) \subset X \,:\, \lim \varphi'_+(u_n) = 0\,,\ \limsup \varphi_+(u_n) \leq b \}$$

we define

$$\Phi^b_+ = \{ l \in \mathbb{R} \,:\, \exists\, (u_n) \in \mathcal{S}^b_{\rm PS}(\varphi_+) \text{ such that } \varphi_+(u_n) \to l \}$$

the set of the asymptotic critical values lower than b and

$$D^b_+ = \{ r \in \mathbb{R} \,:\, \exists\, (u_n), (\bar{u}_n) \in \mathcal{S}^b_{\rm PS}(\varphi_+) \text{ such that } \|u_n - \bar{u}_n\| \to r \}$$

the set of the asymptotic distances between two PS sequences for φ_+ under b.

Lemma 3.5. *For $b \geq 0$, Φ^b_+ and D^b_+ are closed subsets of $\mathbb{R}$.*

Proof. Arguing by contradiction, we assume that there is $r \in \mathbb{R} \setminus D^b_+$ and a sequence $r_n \in (r - \frac{1}{n}, r + \frac{1}{n}) \cap D^b_+$. Then there exist $u_n, \bar{u}_n \in X$ such that $\|\varphi'_+(u_n)\| < \frac{1}{n}$, $\|\varphi'_+(\bar{u}_n)\| < \frac{1}{n}$, $\varphi_+(u_n)$, $\varphi_+(\bar{u}_n) \leq b + \frac{1}{n}$ and $|\,\|u_n - \bar{u}_n\| - r_n| < \frac{1}{n}$. Then $(u_n), (\bar{u}_n) \in \mathcal{S}^b_{\rm PS}(\varphi_+)$ and $\|u_n - \bar{u}_n\| \to r$, i.e. $r \in D^b_+$, a contradiction. In a similar and simpler way one proves that Φ^b_+ is closed. □

We introduce the following notation. Given $b \geq 0$ we set $K^b_+ = \{u \in X : \varphi'_+(u) = 0\,,\ \varphi_+(u) \leq b\}$. For $v \in X$ and $r > 0$ we denote $B_r(v) = \{u \in X : \|u - v\| < r\}$. For $r > r' \geq 0$ and $S \subset X$ we put $B_r(S) = \bigcup\{B_r(u) \,:\, u \in S\}$ and $A_{r',r}(S) = \bigcup\{B_r(u) \setminus \overline{B_{r'}(u)} \,:\, u \in S\}$.

Next result is a straight consequence of Lemma 3.5.

Lemma 3.6. *(i) For any $r \in \mathbb{R}^+ \setminus D^b_+$, there exists $d_r > 0$ such that*

$$\inf \{\|\varphi'_+(u)\| : u \in A_{r-3d_r, r+3d_r}(K^b_+) \cap \{\varphi_+ \leq b\} \} > 0.$$

(ii) For any interval $[a_1, a_2] \subset \mathbb{R}_+ \setminus \Phi^b_+$ it holds that

$$\inf \{\|\varphi'_+(u)\| : a_1 \leq \varphi_+(u) \leq a_2 \} > 0 \ .$$

By Lemma 3.3, the sets Φ^b_+ and D^b_+ can be characterized by means of the set K_+.

Lemma 3.7. $\Phi_+^b = \{\sum_{j=1}^k \varphi_+(v_j) \,:\, k \in \mathbb{N},\ v_j \in K_+\} \cap [0,b]$.
$D_+^b = \{(\sum_{j=1}^k \|v_j - \bar{v}_j\|^2)^{1/2} \,:\, k \in \mathbb{N}, v_j, \bar{v}_j \in K_+ \cup \{0\}, \sum_{j=1}^k \varphi_+(v_j) \leq b, \sum_{j=1}^k \varphi_+(\bar{v}_j) \leq b\}$.

Proof. If $r \in D_+^b$ then there are two sequences $(u_n), (\bar{u}_n) \in \mathcal{S}_{\mathrm{PS}}^b(\varphi_+)$ such that $\|u_n - \bar{u}_n\| \to r$. By Lemma 3.3 we can assume that $u_n = v_0 + \sum_1^k v_j(\cdot - t_n^j)$ and $\bar{u}_n = \sum_{k+1}^{h+k} v_j(\cdot - t_n^j) + v_{h+k+1}$ where $h, k \geq 0$, $|t_n^j| \to \infty$ and $|t_n^i - t_n^j| \to \infty$ for $n \to \infty$ if $1 \leq i < j \leq k$ or $k+1 \leq i < j \leq k+h$. We point out that for any label $j \in \{1, \dots, k\}$ there is at most one label $l = l(j) \in \{k+1, \dots, k+h\}$ such that $\sup_n |t_n^j - t_n^l| < \infty$. If this is the case, we pass to a subsequence (that we write with the same labels) so that $t_n^j - t_n^{l(j)} = \text{const} = t^j$ and we set $v_j^1 = v_j$ and $v_j^2 = v_{l(j)}$. Otherwise we put $l(j) = -1$, $v_j^1 = v_j$, $v_j^2 = 0$ and $t^j = 0$. Now it is clear that for all labels $j \in \{k+1, \dots, k+h\} \setminus \{l(1), \dots, l(k)\}$ it holds that $|t_n^i - t_n^j| \to \infty$ if $i \in \{1, \dots, k+h\} \setminus \{j\}$. For these j's we set $v_j^1 = 0$ and and $v_j^2 = v_j$. For the remaining labels $j \in \{k+1, \dots, k+h\} \cap \{l(1), \dots, l(k)\}$ we define $v_j^1 = v_j^2 = 0$. Moreover we call $v_0^1 = v_0$ and $v_0^2 = v_{h+k+1}$. Therefore $\sum_{j=0}^{k+h} \varphi_+(v_j^1) = \sum_0^k \varphi_+(v_j) = \lim \varphi_+(u_n) \in [0,b]$ and $\sum_{j=0}^{k+h} \varphi_+(v_j^2) = \sum_{j=k+1}^{k+h+1} \varphi_+(v_j) = \lim \varphi_+(\bar{u}_n) \in [0,b]$. In addition $u_n - \bar{u}_n = v_0^1 - v_0^2 + \sum_{j=1}^{k+h} [v_j^1(\cdot - t_n^j - t^j) - v_j^2(\cdot - t_n^j)]$. Using the fact that, given $u, v \in X$, if $(t_n) \subset \mathbb{R}$ is such that $|t_n| \to \infty$ then $\|u(\cdot - t_n) - v\|^2 \to \|u\|^2 + \|v\|^2$, we deduce that $r^2 = \lim \|u_n - \bar{u}_n\|^2 = \sum_{j=0}^{k+h} \|v_j^1 - v_j^2\|^2$.
The inverse inclusion is easier to prove. In fact if $r = (\sum_1^k \|v_j - \bar{v}_j\|^2)^{1/2}$ we define $u_n = \sum_1^k v_j(\cdot - jn)$ and $\bar{u}_n = \sum_1^k \bar{v}_j(\cdot - jn)$. We observe that $(u_n), (\bar{u}_n) \in \mathcal{S}_{\mathrm{PS}}^b(\varphi_+)$. Moreover, by the previous remark, $\|u_n - \bar{u}_n\| \to r$. □

Now we remark that under the assumption $(*)$, by Lemma 3.7, both the sets $D_+^* = D_+^{c_+^*}$ and $\Phi_+^* = \Phi_+^{c_+^*}$ are countable. This fact, together with Lemma 3.5 plainly implies the following.

Corollary 3.8. *(i)* $[0, c_+^*] \setminus \Phi_+^*$ *is open and dense in* $[0, c_+^*]$.
(ii) There is a sequence $(r_n) \subset \mathbb{R}^+ \setminus D_+^*$ *such that* $r_n \to 0$.

Therefore, by Lemma 3.6 and Corollary 3.8, near any level set $\{\varphi_+ = l\}$ at a critical value $l \in (0, c_+^*)$ there is a sequence of slices $\{l_n^1 \leq \varphi_+ \leq l_n^2\}$ with $l_n^2 - l_n^1$ smaller and smaller on which there are neither critical points or Palais

Smale sequences for φ_+. Analogously, around any critical point $u \in K_+^{c_+^*}$ there is a sequence of annuli of radii smaller and smaller (independently of u) on which, as above, there are neither critical points or Palais Smale sequences.

As first step to get a critical point of local mountain pass type for φ_+, we will construct a flow, defined as solution of a Cauchy problem ruled by a suitable vector field.

Lemma 3.9. *Under the assumption $(*)$, if $F : X \to X$ is a locally Lipschitz continuous function such that $\varphi_+'(u)F(u) \le 0$ for all $u \in X$, $\|F(u)\| \le \frac{2}{\|\varphi_+'(u)\|}$ for all $u \in X \setminus (K_+ \cup \{0\})$ and $F(v) = 0$ for every $v \in K_+^b$, with $b \le c^*$, then the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial s}\eta(s,u) = F(\eta(s,u)) \\ \eta(0,u) = u \end{cases}$$

admits a unique solution $\eta(\cdot,u)$ for any $u \in X$, depending continuously on u and defined on $\mathbb{R}^+$ for all $u \in \{\varphi_+ \le b\}$. Moreover the function $s \mapsto \varphi_+(\eta(s,u))$ is nonincreasing.

Proof. The existence, uniqueness and continuity in u of the solution $\eta(\cdot,u)$ of the Cauchy problem is a standard result obtained using the local Lipschitz continuity of F. The function $s \mapsto \varphi_+(\eta(s,u))$ is non increasing because $\frac{d}{ds}\varphi_+(\eta(s,u)) = \varphi_+'(\eta(s,u))F(\eta(s,u)) \le 0$. Now, we have to show that for $u \in \{\varphi_+ \le b\}$ the solution $\eta(\cdot,u)$ is globally defined. We argue by contradiction, assuming that for some $u \in \{\varphi_+ \le b\}$ the maximal right domain of $\eta(\cdot,u)$ is $[0,\bar{s})$, with $\bar{s} < \infty$. Then there is an increasing sequence $(s_n) \subset [0,\bar{s})$ such that $s_n \to \bar{s}$ and $\|F(\eta(s_n,u))\| \to \infty$. We set $u_n = \eta(s_n,u)$ and we observe that, by the properties of F, $\varphi_+'(u_n) \to 0$ and, since $0 < s_n < s_{n+1}$, $\varphi_+(u_{n+1}) \le \varphi_+(u_n) \le \varphi_+(u)$. Therefore $(u_n) \in \mathcal{S}_{\text{PS}}^b(\varphi_+)$. But (u_n) cannot admit any Cauchy subsequence because, otherwise it should have a limit point $v \in \{u \in X : \varphi_+(u) \le b,\ \varphi_+'(u) = 0\}$, where, by hypothesis, $F(v) = 0$, contradicting the fact that $\|F(u_n)\| \to \infty$. Then there are $\delta > 0$ and two sequences $(i_n),(j_n) \subset \mathbb{N}$ such that $i_n < j_n < i_{n+1}$ and $\|u_{i_n} - u_{j_n}\| \ge \delta$ for all $n \in \mathbb{N}$. Hence, by the assumption $(*)$, there is an interval $[r_1,r_2] \subseteq \mathbb{R}^+ \setminus D_+^b$ with $0 < r_1 < r_2 < \delta$ and, consequently, there are two sequences $(\sigma_{i_n}),(\sigma_{j_n}) \subset [0,\bar{s})$ with $s_{i_n} \le \sigma_{i_n} < \sigma_{j_n} \le s_{j_n}$ such that $\|\eta(\sigma_{i_n},u) - u_{i_n}\| = r_1$, $\|\eta(\sigma_{j_n},u) - u_{i_n}\| = r_2$ and $\eta(s,u) \in A_{r_1,r_2}(u_{i_n})$ for any $s \in (\sigma_{i_n},\sigma_{j_n})$. Then, for any $n \in \mathbb{N}$, it holds that $r_2 - r_1 \le \int_{\sigma_{i_n}}^{\sigma_{j_n}} \|F(\eta(s,u))\|\,ds = (\sigma_{j_n} - \sigma_{i_n})\|F(\eta(\bar{s}_n,u))\|$ for a suitable $\bar{s}_n \in [\sigma_{i_n},\sigma_{j_n}]$. Now we call $\bar{u}_n = \eta(\bar{s}_n,u)$ and we notice that $\|F(\bar{u}_n)\| \ge \frac{r_2-r_1}{s_{j_n}-s_{i_n}}$. Since $s_{i_n} \to \bar{s}$ and $s_{j_n} \to \bar{s}$ we obtain that $\|F(\bar{u}_n)\| \to \infty$ and so $(\bar{u}_n) \in \mathcal{S}^b_{\rm PS}$ and $r_1 \le \|\bar{u}_n - u_n\| \le r_2$, that implies $[r_1,r_2] \cap D^b \ne \emptyset$, a contradiction. □

Let us fix $b \in [c,c^*)$. For any $r \in \mathbb{R}^+ \setminus D^*_+$ let $d_r \in (0,r)$ be given by Lemma 3.6. Let

$$\mu_r = \inf\{\|\varphi'_+(u)\| : u \in A_{r-3d_r,r+3d_r}(K^b_+) \cap \{\varphi_+ \le c^*_+\}\} > 0$$

and $h_r = \frac{1}{4}d_r\mu_r$. Moreover we define $\hat{h} = \min\{c^* - b, h_r\}$.

Lemma 3.10. *For any $h \in (0,\hat{h})$ there exists a continuous function η : $\{\varphi_+ \le b+h\} \to \{\varphi_+ \le b+h\}$ such that:*

(η_1) $\varphi_+(\eta(u)) \le \varphi_+(u)$ *for all* $u \in \{\varphi_+ \le b+h\}$;

(η_2) $\varphi_+(\eta(u)) < b-h$ *if* $\eta(u) \notin B_r(K_+ \cap \{b-h \le \varphi_+ \le b+h\})$;

(η_3) $\varphi_+(\eta(u)) < b-h_r$ *if* $\eta(u) \in A_{r-d_r,r+d_r}(K_+ \cap \{b-h \le \varphi_+ \le b+h\})$.

Proof. Let us set $\mathcal{B}_r = B_r(K_+ \cap \{b-h \le \varphi_+ \le b+h\})$ and $\mathcal{A}_{r_1,r_2} = A_{r_1,r_2}(K_+ \cap \{b-h \le \varphi_+ \le b+h\}) \cap \{\varphi_+ \le c^*_+\}$. By definition of μ_r, $\|\varphi'_+(u)\| \ge \mu_r$ for every $u \in \mathcal{A}_{r-3d_r,r+3d_r}$. Then we can build a vector field F on X with the properties of Lemma 3.9 and such that

(3.3) $\varphi'_+(u)F(u) \le -1$ for $u \in (\{b-h \le \varphi_+ \le b+h\}\setminus\mathcal{B}_{r-2d_r})\cup\mathcal{A}_{r-2d_r,r+2d_r}$.

By Lemma 3.9, there is a continuous function $\eta : \mathbb{R}^+ \times \{\varphi_+ \le b+h\} \to \{\varphi_+ \le b+h\}$ solving the Cauchy problem corresponding to F. By abuse of notation, we define $\eta(u) = \eta(3h_r,u)$ for all $u \in \{\varphi_+ \le b+h\}$. Again by Lemma 3.9, $\varphi_+(\eta(u)) \le \varphi_+(u)$ for any $u \in \{\varphi_+ \le b+h\}$.

To prove that η verifies the property (η_2), we argue by contradiction, assuming that there is some $u \in \{\varphi_+ \le b+h\}$ such that $\eta(u) \in \{\varphi_+ \ge b-h\} \setminus \mathcal{B}_r$. We distinguish two alternative cases:

(a) $\eta(s,u) \notin \mathcal{B}_{r-2d_r}$ for all $s \in [0,3h_r]$;
(b) there is $\bar{s} \in [0,3h_r]$ such that $\eta(\bar{s},u) \in \mathcal{B}_{r-2d_r}$.

If (a) holds, then, by (3.3), $\varphi_+(\eta(u)) - \varphi_+(u) = \int_0^{3h_r} \varphi'_+(\eta)F(\eta) \leq -3h_r$ and so $\varphi_+(u) \geq \varphi_+(\eta(u)) + 3h_r > b - h + 3h > b + h$ whereas $\varphi_+(u) \leq b + h$. If (b) occurs, for $\eta(u) \notin \mathcal{B}_r$, there exist $0 \leq s_1 < s_2 \leq 3h_r$ such that $\eta(s_1,u) \in \partial\mathcal{B}_{r-2d_r}$, $\eta(s_2,u) \in \partial\mathcal{B}_r$ and $\eta(s,u) \in \mathcal{A}_{r-2d_r,r}$ for all $s \in (s_1,s_2)$. Hence, we have that

$$2d_r \leq \|\eta(s_2,u) - \eta(s_1,u)\| \leq \int_{s_1}^{s_2} \|F(\eta)\|\, ds \leq \int_{s_1}^{s_2} \frac{2}{\|\varphi'_+(\eta)\|}\, ds \leq \frac{2}{\mu_r}(s_2 - s_1)$$

that is $s_2 - s_1 \geq 4h_r$, in contrast with the fact that $[s_1,s_2] \subseteq [0,3h_r]$.

Let us prove (η_3). By contradiction, we suppose that (η_3) fails, i.e., there exists $u \in \{\varphi_+ \leq b+h\}$ such that $\eta(u) \in \mathcal{A}_{r-d_r,r+d_r} \cap \{\varphi_+ \geq b - h_r\}$. Then we distinguish the two following cases:

(a') $\eta(s,u) \in \mathcal{A}_{r-2d_r,r+2d_r}$ for all $s \in [0,3h_r]$;
(b') there is some $\bar{s} \in [0,3h_r]$ for which $\eta(\bar{s},u) \notin \mathcal{A}_{r-2d_r,r+2d_r}$.

If (a') occurs, since $\eta(s,u) \in \{b - h \leq \varphi_+ \leq b+h\}$ for any $s \in [0,3h_r]$, thanks to (3.3), we infer that $\varphi_+(\eta(u)) = \varphi_+(u) + \int_0^{3h_r} \varphi'_+(\eta)F(\eta) \leq \varphi_+(u) - 3h_r$ and then $\varphi_+(u) \geq \varphi_+(\eta(u)) + 3h_r \geq b - h_r + 3h_r > b + h$, whereas $\varphi_+(u) \leq b + h$. If (b') holds, then there are $0 \leq s_1 < s_2 \leq 3h_r$ such that $\|\eta(s_2,u) - \eta(s_1,u)\| = d_r$ and $\eta(s,u) \in \mathcal{A}_{r-2d_r,r+2d_r}$ for all $s \in [s_1,s_2]$. Since

$$\|\eta(s_2,u) - \eta(s_1,u)\| \leq \int_{s_1}^{s_2} \frac{2}{\|\varphi'_+(\eta)\|}\, ds \leq \frac{2}{\mu_r}(s_2 - s_1),$$

we get $s_2 - s_1 \geq 2h_r$. Then, by (3.3), $\varphi_+(\eta(u)) \leq \varphi_+(\eta(s_2,u)) = \varphi_+(\eta(s_1,u)) + \int_{s_1}^{s_2} \varphi'_+(\eta)F(\eta) \leq \varphi_+(u) - (s_2 - s_1)$ and so $\varphi_+(u) \geq b - h_r + 2h_r > b + h$, in contrast with the fact that $\varphi_+(u) \leq b + h$. □

Corollary 3.11. *For any $h \in (0,\hat{h})$ there exists a path $\gamma \in \Gamma$ and a finite set of critical points $v_1,\ldots,v_k \in K_+ \cap \{c - h \leq \varphi_+ \leq c + h\}$, depending on h and γ, such that:*

(γ_1) $\max_\gamma \varphi_+ < c + h$;
(γ_2) *if $\varphi_+(\gamma(s)) \geq c - h$, then $\gamma(s) \in \bigcup_{j=1}^k B_r(v_j)$;*

(γ_3) *if* $\gamma(s) \in \bigcup_{j=1}^k A_{r-d_r,r+d_r}(v_j)$ *then* $\varphi_+(\gamma(s)) < c_+ - h_r$.

Proof. Taken $\gamma \in \Gamma$ such that $\max_\gamma \varphi_+ < c_+ + h$ we define $\bar\gamma = \eta \circ \gamma$, η being given by Lemma 3.10 with $b = c$. Clearly $\bar\gamma \in \Gamma$ and $\bar\gamma$ satisfies (γ_1), because of (η_1). Moreover, if $\varphi_+(\bar\gamma(s)) \geq c_+ - h$, then, by ($\eta_2$), $\bar\gamma(s) \in B_r(K_+ \cap \{c_+ - h \leq \varphi_+ \leq c_+ + h\})$. But the family $\{ B_r(v) : v \in K_+ \cap \{c_+ - h \leq \varphi_+ \leq c_+ + h\} \}$ is an open cover of the compact set range $\bar\gamma \cap \{\varphi_+ \geq c - h\}$. Hence there are $v_1, \dots, v_k \in K_+ \cap \{c_+ - h \leq \varphi_+ \leq c_+ + h\}$ such that range $\bar\gamma \cap \{\varphi_+ \leq c_+ - h\} \subset \bigcup_{j=1}^k B_r(v_j)$ and so (γ_2) follows. Finally, if $\bar\gamma(s) \in \bigcup_{j=1}^k A_{r-d_r,r+d_r}(v_j)$ then, by (η_3), $\varphi_+(\bar\gamma(s)) < c - h_r$. □

We fix $\bar r \in (0, \rho) \setminus D_+^*$ and $\bar h \in (0, \frac{1}{2} \min\{h_{\bar r}, c_+^* - c_+\})$. By Corollary 3.11, there is a path $\bar\gamma \in \Gamma$ and some critical points $v_1, \dots, v_k \in K_+ \cap \{c - \bar h \leq \varphi_+ \leq c + \bar h\}$ satisfying (γ_1)–(γ_3) with $\bar h$ instead of h. Then, by definition of c_+, there must be $\bar v \in \{v_1, \dots, v_k\}$ and $[s_0, s_1] \subseteq [0,1]$ such that $\bar\gamma(s) \in B_{\bar r}(\bar v)$ for $s \in (s_1, s_2)$ and $\bar\gamma(s_0)$ and $\bar\gamma(s_1)$ lie on $\partial B_{\bar r}(\bar v)$ and are not connectible in $\{\varphi_+ < c_+\}$. Moreover, by (γ_3), $\bar\gamma(s_0)$, $\bar\gamma(s_1) \in \{\varphi_+ \leq c_+ - h_{\bar r}\}$. Hence we put $u_0 = \bar\gamma(s_0)$, $u_1 = \bar\gamma(s_1)$ and we consider the class of paths $\bar\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1,\ \text{range}\,\gamma \subset B_{\bar r}(\bar v) \cup \{\varphi_+ \leq c_+ - \frac{1}{2} h_{\bar r}\}$. Since $\bar\Gamma \neq \emptyset$ we can define the corresponding minimax value $\bar c = \inf_{\bar\Gamma} \sup_\gamma \varphi_+$ that satisfies: $c_+ \leq \bar c < c_+ + \bar h < c_+^*$.

Lemma 3.12. *For any* $r \in (0, \frac{1}{2} d_{\bar r}) \setminus D_+^*$ *and for any* $h \in (0, c + \bar h - \bar c)$ *there exist* $v_{r,h} \in B_{\bar r}(\bar v) \cap K_+ \cap \{\bar c - h \leq \varphi_+ \leq \bar c + h\}$, $u_{r,h}^0, u_{r,h}^1 \in B_{\bar r}(\bar v)$ *and a path* $\gamma_{r,h} \in C([0,1], X)$ *joining* $u_{r,h}^0$ *with* $u_{r,h}^1$ *such that:*

(*i*) $u_{r,h}^0, u_{r,h}^1 \in \partial B_{r+d_r}(v_{r,h}) \cap \{\varphi_+ \leq \bar c - h_r\}$;

(*ii*) $u_{r,h}^0$ *and* $u_{r,h}^1$ *are not connectible in* $B_{\bar r}(\bar v) \cap \{\varphi_+ < \bar c\}$;

(*iii*) range $\gamma_{r,h} \subset \bar B_{r+d_r}(v_{r,h}) \cap \{\varphi_+ \leq \bar c + h\}$

(*iv*) range $\gamma_{r,h} \cap \bar A_{r-d_r,r+d_r}(v_{r,h}) \subset \{\varphi_+ \leq \bar c - h_r\}$.

Proof. We can take $\delta \in (0, d_{\bar r})$ such that $B_\delta(u_0) \cup B_\delta(u_1) \subset \{\varphi_+ \leq c_+ - \frac{1}{2} h_{\bar r}\}$ and we consider a cut-off function $\chi \in C^1(X, \mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(u) = 0$ if $u \in B_{\delta/2}(u_0) \cup B_{\delta/2}(u_1)$ and $\chi(u) = 1$ if $u \notin B_\delta(u_0) \cup B_\delta(u_1)$. Now, given $r \in (0, \frac{1}{2} d_{\bar r}) \setminus D_+^*$ and $h \in (0, c + \bar h - \bar c)$ we can build a vector field

$F_{r,h}$ on X such that $\varphi_+'(u)F_{r,h}(u) \le 0$ for all $u \in X$, $\|F_{r,h}(u)\| \le \frac{2}{\|\varphi_+'(u)\|}$ for all $u \in X \setminus (K_+ \cup \{0\})$, $F_{r,h}(v) = 0$ for any $v \in K_+^{\bar{c}+h}$ and

$$\begin{aligned}\varphi_+'(u)F_{r,h}(u) \le -1 \text{ for } u \in (\{\bar{c} - h \le \varphi_+ \le \bar{c} + h\} \setminus \mathcal{B}_{r-2d_r}) \\ \cup \mathcal{A}_{r-2d_r,r+2d_r}) \cup (\{\varphi_+ \le \bar{c} + h\} \cap A_{\bar{r}-2d_{\bar{r}},\bar{r}+2d_{\bar{r}}}(\bar{v}))\end{aligned} \tag{3.4}$$

where $\mathcal{B}_r = B_r(K_+ \cap \{\bar{c} - h \le \varphi_+ \le \bar{c} + h\})$ and $\mathcal{A}_{r_1,r_2} = A_{r_1,r_2}(K_+ \cap \{\bar{c} - h \le \varphi_+ \le \bar{c} + h\}) \cap \{\varphi_+ \le \bar{c} + h\}$.

Then we consider the function $G_{r,h} = \chi\, F_{r,h}$ and we observe that $G_{r,h}$ is again a vector field on X satisfying the properties of Lemma 3.9. Therefore there is a continuous function $\eta_{r,h} : \mathbb{R}^+ \times \{\varphi_+ \le \bar{c} + h\} \to \{\varphi_+ \le \bar{c} + h\}$ solving the Cauchy problem corresponding to $G_{r,h}$. Thus, we set $\bar{s} = \max\{3h_r, 3h_{\bar{r}}\}$ and $\eta_{r,h}(u) = \eta_{r,h}(\bar{s}, u)$. Since $h < \bar{h} < \frac{1}{2}h_{\bar{r}}$, we have that $(B_\delta(u_0) \cup B_\delta(u_1)) \cap \{\varphi_+ \ge \bar{c} - h\} = \emptyset$ and then $G_{r,h}(u) = F_{r,h}(u)$ for any $u \in \{\bar{c} - h \le \varphi_+ \le \bar{c} - h\}$. Moreover, since $u_0, u_1 \in \partial B_{\bar{r}}(\bar{v})$ and $\delta < d_{\bar{r}}$, we have that $B_\delta(u_0) \cup B_\delta(u_1) \subseteq A_{\bar{r}-d_{\bar{r}},\bar{r}+d_{\bar{r}}}(\bar{v})$. In addition, since $K_+^* \cap A_{\bar{r}-3d_{\bar{r}},\bar{r}+3d_{\bar{r}}}(\bar{v}) = \emptyset$ and $r + 2d_r < d_{\bar{r}}$, it follows that $B_{r+2d_r}(K_+^*) \cap A_{\bar{r}-2d_{\bar{r}},\bar{r}+2d_{\bar{r}}}(\bar{v}) = \emptyset$. Then $(B_\delta(u_0) \cup B_\delta(u_1)) \cap A_{r-2d_r,r+2d_r}(K_+^{c^*}) = \emptyset$ and so $G_{r,h}(u) = F_{r,h}(u)$ for any $u \in \mathcal{A}_{r-2d_r,r+2d_r}$. Then we are in the same situation of the proof of Lemma 3.10, where $G_{r,h}$ satisfies the condition (3.3), with $\bar{c}$ instead of b. Hence we deduce that $\eta_{r,h}$ is a continuous function on $\{\varphi_+ \le \bar{c} + h\}$ verifying the properties (η_1)–(η_3), always with $b = \bar{c}$. Now we take a path $\gamma \in \bar{\Gamma}$ such that $\max_\gamma \varphi_+ \le \bar{c} + h$ and we put $\gamma_{r,h} = \eta_{r,h} \circ \gamma$. We claim that $\gamma_{r,h} \in \bar{\Gamma}$ and $\gamma_{r,h}$ satisfies the properties (γ_1)–(γ_3) of Corollary 3.11, but with $\bar{c}$ instead of c_+. Then, assuming the claim, by definition of $\bar{c}$, there is at least one critical point $v_{r,h} \in K_+ \cap \{\bar{c} - h \le \varphi_+ \le \bar{c} + h\}$ and an interval $[\theta_0, \theta_1] \subseteq [0,1]$ such that $\gamma_{r,h}(\theta) \in B_{r+d_r}(v_{r,h})$ for $\theta \in (\theta_1, \theta_2)$ and $\gamma_{r,h}(\theta_0), \gamma_{r,h}(\theta_1) \in \partial B_{r+d_r}(v_{r,h})$ are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi_+ < \bar{c}\}$. Moreover, by (γ_3), range $\gamma_{r,h} \cap A_{r-d_r,r+d_r}(v_{r,h}) \subset \{\varphi_+ \le \bar{c} - h_r\}$.

We conclude the proof showing the previous claims. First, we prove that $\gamma_{r,h} \in \bar{\Gamma}$. Clearly, $\gamma_{r,h} \in C([0,1], X)$, range $\gamma_{r,h} \subset \{\varphi_+ \le \bar{c} + h\}$ and $\gamma_{r,h}(0) = \eta_{r,h}(u_0) = u_0$ because $G_{r,h}(u_0) = 0$. For the same reason $\gamma_{r,h}(1) = u_1$. Now we show that range $\gamma_{r,h} \subset B_{\bar{r}}(\bar{v}) \cup \{\varphi_+ \le c_+ - \frac{1}{2}h_{\bar{r}}\}$. Fixed $\theta \in [0,1]$ we call $u = \gamma(\theta)$ and $\bar{u} = \gamma_{r,h}(\theta)$. If $\varphi_+(u) \le c_+ - \frac{1}{2}h_{\bar{r}}$ then, by (η_1), also

$\varphi_+(\bar{u}) \le c_+ - \frac{1}{2}h_{\bar{r}}$. Let us suppose that $u \in B_{\bar{r}}(\bar{v}) \cap \{\varphi_+ > c_+ - \frac{1}{2}h_{\bar{r}}\}$ and $\bar{u} \notin B_{\bar{r}}(\bar{v})$. We have to deduce that $\varphi_+(\bar{u}) \le c_+ - \frac{1}{2}h_{\bar{r}}$. In fact, if $\eta_{r,h}(s,u) \in B_\delta(u_0) \cup B_\delta(u_1)$ for some $s \in [0,\bar{s}]$, then $\varphi_+(\bar{u}) \le \varphi_+(\eta_{r,h}(s,u)) \le c_+ - \frac{1}{2}h_{\bar{r}}$, because $B_\delta(u_0) \cup B_\delta(u_1) \subset \{\varphi_+ \le c_+ - \frac{1}{2}h_{\bar{r}}\}$. Alternatively $\eta_{r,h}(s,u) \notin B_\delta(u_0) \cup B_\delta(u_1)$ for any $s \in [0,\bar{s}]$. We distinguish two cases:

(a) $\eta_{r,h}(s,u) \in A_{\bar{r}-2d_{\bar{r}},\bar{r}+2d_{\bar{r}}}(\bar{v})$ for any $s \in [0,\bar{s}]$;

(b) there is some $s \in [0,\bar{s}]$ for which $\eta_{r,h}(s,u) \notin A_{\bar{r}-2d_{\bar{r}},\bar{r}+2d_{\bar{r}}}(\bar{v})$.

In the case (a), with calculations similar to those of case (a') in the proof of Lemma 3.10, we get that $\varphi_+(\bar{u}) \le \varphi_+(u) - \bar{s} \le \bar{c} + h - 3h_{\bar{r}} < c_+ - \frac{1}{2}h_{\bar{r}}$. Instead, the case (b) is similar to the part (b') in the proof of Lemma 3.10. Indeed we see that the trajectory $\eta_{r,h}(\cdot,u)$ crosses an annulus of thickness $d_{\bar{r}}$. Then, there is $[s_1,s_2] \subseteq [0,\bar{s}]$ such that $d_{\bar{r}} = \|\eta_{r,h}(s_2,u) - \eta_{r,h}(s_1,u)\| \le \int_{s_1}^{s_2} \|G_{r,h}\| \le \int_{s_1}^{s_2} \frac{2}{\|\varphi'_+(\eta_{r,h})\|} \le \frac{2}{\mu_{\bar{r}}}(s_2 - s_1)$ and so $s_2 - s_1 \ge \frac{1}{2}\mu_{\bar{r}} d_{\bar{r}} = 2h_{\bar{r}}$. On the other hand $\varphi_+(\bar{u}) \le \varphi_+(\eta_{r,h}(s_2,u)) = \varphi_+(\eta_{r,h}(s_1,u)) + \int_{s_1}^{s_2} \varphi'_+ G_{r,h} \le \varphi_+(u) - (s_2 - s_1)$, because of (3.4). Then $\varphi_+(\bar{u}) < c_+ + \bar{h} - 2h_{\bar{r}} \le c_+ - \frac{1}{2}h_{\bar{r}}$. Finally the properties (γ_1)–(γ_3) can be proved as in Corollary 3.11. □

In the following lemma we construct a convergent sequence of critical points v_n such that $\varphi_+(v_n) = \bar{c}$. This gives the topological structure of a local mountain pass.

Lemma 3.13. *The functional φ_+ admits a critical point of mountain pass type in $B_{\bar{r}}(\bar{v})$. In particular given a sequence $(r_n) \subset (0,\bar{r}) \setminus D^*_+$ such that $r_n \to 0$, there is a convergent sequence $(v_n) \subset K_+$, such that for any $n \in \mathbb{N}$ $\varphi_+(v_n) = \bar{c}$, $\bar{B}_{r_n}(v_n) \subset B_{\bar{r}}(\bar{v})$, and for any $h > 0$ there is a path $\gamma \in C([0,1],X)$, depending on n and h, satisfying the following properties:*

(i) $\gamma(0), \gamma(1) \in \partial B_{r_n}(v_n) \cap \{\varphi_+ \le \bar{c} - \frac{1}{2}h_{r_n}\}$;

(ii) $\gamma(0)$ *and* $\gamma(1)$ *are not connectible in* $B_{\bar{r}}(\bar{v}) \cap \{\varphi_+ < \bar{c}\}$;

(iii) $\operatorname{range}\gamma \subset \bar{B}_{r_n}(v_n) \cap \{\varphi_+ \le \bar{c} + h\}$;

(iv) $\operatorname{range}\gamma \cap A_{r_n - \frac{1}{2}d_{r_n}, r_n}(v_n) \subset \{\varphi_+ \le \bar{c} - \frac{1}{2}h_{r_n}\}$;

(v) $\operatorname{supp}\gamma(\theta) \subset [-R_n, R_n]$ *for any* $\theta \in [0,1]$, *being* R_n *a positive constant independent of* θ.

Proof. Fixed $r \in (0, \frac{1}{2}d_{\bar{r}}) \setminus D^*_+$, we take a sequence $(h_n) \subset (0, c + \bar{h} - \bar{c})$ such that $h_n \to 0$. Let $v_{r,h_n} \in K_+ \cap \{\bar{c}+h_n \le \varphi_+ \le \bar{c}+h_n\}$, $u^0_{r,h_n}, u^1_{r,h_n} \in \partial B_{r+d_r}(v_{r,h_n}) \cap \{\varphi_+ \le \bar{c}-h_r\}$ and $\gamma_{r,h_n} \in C([0,1],X)$ be given by Lemma 3.12. We notice that $(v_{r,h_n})_n \subset B_{\bar{r}}(\bar{v})$ is a PS sequence for φ_+ at level $\bar{c}$ and $\operatorname{diam}\{v_{r,h_n}\} \le 2\bar{r} < \rho$, so that, by Lemma 2.9, up to a subsequence, $v_{r,h_n} \to v_r \in K_+ \cap B_{\bar{r}}(\bar{v})$ and $\varphi_+(\bar{v}) = \bar{c}$. Taken $h > 0$, we choose n large enough so that $A_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \supset A_{r-\frac{1}{2}d_r, r+\frac{1}{2}d_r}(v_r)$ and $h_n < h$. Now for $R > 0$ we define a function $\chi_R \in C(\mathbb{R},\mathbb{R})$ by setting

$$\chi_R(t) = \begin{cases} 1 & \text{for } |t| < R-1 \\ R-|t| & \text{for } R-1 \le |t| \le R \\ 0 & \text{for } |t| > R. \end{cases}$$

Then we put $\bar{\gamma}_{r,h_n} = \chi_R \gamma_{r,h_n}$ and we observe that for $R > 0$ sufficiently large, $\bar{\gamma}_{r,h_n}$ is a path in X such that

$\bar{\gamma}_{r,h_n}(0), \bar{\gamma}_{r,h_n}(1) \in A_{r+\frac{3}{4}d_r, r+\frac{5}{4}d_r}(v_{r,h_n}) \cap \{\varphi_+ \le \bar{c} - \frac{1}{2}h_{r_n}\}$,
$\operatorname{range}\bar{\gamma}_{r,h_n} \subset B_{\bar{r}}(\bar{v}) \cap \{\varphi_+ \le \bar{c} + h_n\}$
$\operatorname{range}\bar{\gamma}_{r,h_n} \cap A_{r-\frac{3}{4}d_r, r+\frac{3}{4}d_r}(v_{r,h_n}) \subset \{\varphi_+ \le \bar{c} - \frac{1}{2}h_r\}$.

We also notice that the two points $\bar{\gamma}_{r,h_n}(0), \bar{\gamma}_{r,h_n}(1)$ are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi < \bar{c}\}$, because otherwise, since $\max\{\varphi_+(u) : u \in [\bar{\gamma}_{r,h_n}(i), \gamma_{r,h_n}(i)]\} < \bar{c}$ $(i = 0,1)$, we contradict the property (i) of Lemma 3.12. Finally we remark that there is a component of $\operatorname{range}\bar{\gamma}_{r,h_n} \cap \bar{B}_r(v_r)$ whose extreme points are not connectible in $B_{\bar{r}}(\bar{v}) \cap \{\varphi_+ < \bar{c}\}$. If we reparametrize this part of $\bar{\gamma}_{r,h_n}$, we obtain a path satisfying the properties (i)–(v). To conclude we have to show that for a sequence (r_n) convergent to 0, $v_{r_n} \to v_\infty$. This follows immediately from the fact that $v_r \in B_{\bar{r}}(\bar{v})$ for any r and from Lemma 2.9. □

Remark 3.14. Since D^*_+ and D^*_- are countable closed subsets of $\mathbb{R}^+$, also $D^*_+ \cap D^*_-$ is so. Therefore Lemma 3.13 holds true at the same time both for φ_+ and for φ_-, provided that we take a sequence $(r_n) \subset (0,\bar{r}) \setminus (D^*_+ \cap D^*_-)$.

4. Construction of a pseudogradient vector field

From now on, we study the system (HS) governed by a potential U asymptotic to periodic potentials $U_\pm$, as $t \to \pm\infty$, according to the assumptions (U1)–(U5) given in the introduction. Moreover, as in the statement of

Theorem 1.7, we assume that the condition $(*)$ holds true both for φ_+ and for φ_-.

We start by introducing some notation. Given $h, k \in \mathbb{N}$ and $M, p_0 > 0$ we set

$$\begin{aligned} P_-^h(M,p_0) &= \{ (p_{-h},\dots,p_{-1}) \in P_-^h \ : p_{-1} \leq -M - p_0\,, \\ &\qquad\qquad p_{j+1} - p_j \geq M \ (-h \leq j \leq -2) \} \\ P_+^k(M,p_0) &= \{ (p_1,\dots,p_k) \in P_+^k \ : p_1 \geq M + p_0\,, \\ &\qquad\qquad p_{j+1} - p_j \geq M \ (1 \leq j \leq k-1) \} \\ P^{h,k}(M,p_0) &= (P_-^h(M,p_0) \times P_+^k(M,p_0)) \cup P_-^h(M,p_0) \cup P_+^k(M,p_0) \end{aligned}$$

To any finite sequence $(p_{-h},\dots,p_{-1},p_1,\dots,p_k) \in P^{h,k}(M,p_0)$ we associate a family of intervals $\{I_{-h},\dots,I_{-1},I_1,\dots,I_k\}$ defined by

$$\begin{aligned} I_j &= [\,\tfrac{1}{2}(p_j + p_{j-1}), \tfrac{1}{2}(p_j + p_{j+1})\,] \quad \text{for } -h \leq j \leq k\,,\ j \neq 0,-1 \\ I_{-1} &= [\,\tfrac{1}{2}(p_{-1} + p_{-2}), \tfrac{1}{2}(p_{-1} - p_0)\,] \\ I_0 &= [\,\tfrac{1}{2}(p_{-1} - p_0), \tfrac{1}{2}(p_0 + p_1)\,] \end{aligned}$$

where $p_{-h-1} = -\infty$ and $p_{k+1} = +\infty$. We define an other family of intervals given by

$$\begin{aligned} M_j &= [p_j + m(m+1), p_{j+1} - m(m+1)] \quad (1 \leq j \leq k)\,, \\ M_j &= [p_{j-1} + m(m+1), p_j - m(m+1)] \quad (-h \leq j \leq -1)\,, \\ M_0 &= [p_{-1} + m(m+1), p_1 - m(m+1)] \\ J &= M_{-h} \cup \dots \cup M_k \end{aligned}$$

for some $m > 0$ such that $M \geq 2m^2 + 3m$.

Moreover we introduce a corresponding family of functionals $\varphi_j : X \to \mathbb{R}$ $(-h \leq j \leq k\,,\ j \neq 0)$ defined by

$$\begin{aligned} \varphi_j(u) &= \int_{I_j} (\tfrac{1}{2}|\dot{u}|^2 - U_-(t,u))\,dt \quad (-h \leq j \leq -1) \\ \varphi_j(u) &= \int_{I_j} (\tfrac{1}{2}|\dot{u}|^2 - U_+(t,u))\,dt \quad (1 \leq j \leq k)\,. \end{aligned}$$

We notice that $\varphi_j \in C^1(X,\mathbb{R})$ and for any $u,h \in X$ $\varphi_j'(u)h = \int_{I_j}(\dot u \cdot \dot h + u \cdot L_\pm(t)h - V_\pm'(t,u)\cdot h)\,dt$ according to the sign of j.

Given $(p_{-h},\dots,p_{-1}) \in P_-^h(M,p_0)$, $r>0$ and $v \in X$ we set

$$B_r(v;p_{-h},\dots,p_{-1}) = \{\, u \in X \ :\ \|u-\tau_{p_j}v\|_{I_j} < r\ (-h \le j \le -1)\,\}.$$

Similarly, for $(p_1,\dots,p_k) \in P_+^k(M,p_0)$ we put

$$B_r(v;p_1,\dots,p_k) = \{\, u \in X \ :\ \|u-\tau_{p_j}v\|_{I_j} < r\ (1 \le j \le k)\,\}.$$

More generally, given $p=(p_{-h},\dots,p_{-1},p_1,\dots,p_k) \in P^{h,k}(M,p_0)$, $r>0$ and $v^-,v^+ \in X$ we set

$$B_r(v^-,v^+;p) = B_r(v^-;p_{-h},\dots,p_{-1}) \cap B_r^0 \cap B_r(v^+;p_1,\dots,p_k)$$

where $B_r^0 = \{u \in X\ :\ \|u\|_{I_0} < r\}$. Finally we set

$$Y_\epsilon = \{u \in X\ :\ \|u\|_{M_j}^2 \le \epsilon\ \ (-h \le j \le k)\}\,.$$

Let $\rho_0 > 0$ such that for any interval $I \subseteq \mathbb{R}$ with $|I| \ge 1$ and for any $u \in X$ with $\|u\|_I \le \rho_0$ it holds that $|V'(t,u(t))|, |V_\pm'(t,u(t))| \le \frac{1}{2}L_0|u(t)|$ for every $t \in \mathbb{R}$.

Now we state the main result of this section.

Lemma 4.1. *Let $(r_n) \subset \mathbb{R}^+$, $(v_n^\pm) \subset K_\pm$, $v_\pm \in K_\pm$ be given by Lemma 3.13 for the functionals $\varphi_\pm$. Then for any $r_n \in (0,\frac{1}{4}\rho_0)$ there is $\nu = \nu(r_n) > 0$ such that for any $a_-,a_+,b_-,b_+ \in \mathbb{R}$ and $\delta > 0$ with*

$$(4.1)\quad \begin{matrix} [a_- -\delta, a_- + 2\delta] \subset (0,\bar c_-)\setminus \Phi_-^* & [b_- -\delta, b_- + 2\delta] \subset (\bar c_-, c_-^*)\setminus \Phi_-^* \\ [a_+ -\delta, a_+ + 2\delta] \subset (0,\bar c_+)\setminus \Phi_+^* & [b_+ -\delta, b_+ + 2\delta] \subset (\bar c_+, c_+^*)\setminus \Phi_+^* \end{matrix}$$

and for any r_1,r_2,r_3 with $r_n - d_{r_n} < r_1 < r_2 < r_3 < r_n$ there exist $p_0 \ge 1$ and $\epsilon_1 > 0$ for which the following holds:
for any $\epsilon \in (0,\epsilon_1)$ there is $m \ge 2$ such that for every $h,k \in \mathbb{N}$ and $p \in P^{h,k}(2m^2+3m,p_0)$ there exists a locally Lipschitz continuous vector field $\mathcal{V} : X \to X$ satisfying:

$(\mathcal{V}_1)$ $\varphi'(u)\mathcal{V}(u) \ge 0\quad \forall\, u \in X$, $\|\mathcal{V}(u)\| \le 1\quad \forall\, u \in X$,

$\mathcal{V}(u) = 0 \quad \forall u \in X \setminus B_{r_3}(v_n^-, v_n^+; p)$;

$(\mathcal{V}_2)$ $\varphi_j'(u)\mathcal{V}(u) \geq \nu \quad \forall u \in \bigcap_j \{\varphi_j \leq b_\pm\} \cap B_{r_3}(v_n^-, v_n^+; p)$ *such that* $r_1 \leq \|u - \tau_{p_j} v_n^\pm\| \leq r_2$;

$(\mathcal{V}_3)$ $\varphi_j'(u)\mathcal{V}(u) \geq 0 \quad \forall u \in \{a_\pm \leq \varphi_j \leq a_\pm + \delta\} \cup \{b_\pm \leq \varphi_j \leq b_\pm + \delta\}$;

$(\mathcal{V}_4)$ $\langle u, \mathcal{V}(u)\rangle_{M_j} \geq 0 \quad \forall u \in X \setminus Y_\epsilon,\ \forall j = -h, \dots, k$.

Moreover, if $B_{r_3}(v_n^-, v_n^+; p) \cap K = \emptyset$ *then there is* $\nu' > 0$ *such that*

$(\mathcal{V}_5)$ $\varphi'(u)\mathcal{V}(u) \geq \nu'$ *for any* $u \in B_{r_2}(v_n^-, v_n^+; p)$.

Proof. We set $v^\pm = v_n^\pm$, $r = r_n$. According to Lemma 3.6, there are $d_r \in (0, \frac{1}{4}r)$, $\mu_r > 0$ and $\lambda > 0$ such that

(4.2) $$\|\varphi_\pm'(u)\| \geq \mu_r \quad \text{for } u \in A_{r-3d_r, r+3d_r}(v^\pm) \cap \{\varphi_\pm \leq c_\pm^*\}.$$

(4.3) $$\|\varphi_\pm'(u)\| \geq \lambda \quad \text{for } u \in \{\bar{a}_\pm \leq \varphi_\pm \leq \bar{a}_\pm'\} \cup \{\bar{b}_\pm \leq \varphi_\pm \leq \bar{b}_\pm'\}$$

where $\bar{a}_\pm = a_\pm - \delta$, $\bar{a}_\pm' = a_\pm + 2\delta$, $\bar{b}_\pm = b_\pm - \delta$, $\bar{b}_\pm' = b_\pm + 2\delta$. Let $N_r \in \mathbb{N}$ be such that, fixed $h, k \in \mathbb{N}$, whenever $p \in P^{h,k}(M, p_0)$ with $M = 2m^2 + 3m$, $m \geq N_r$ and $p_0 \geq 1$, then $\forall u \in B_r(v^-, v^+; p)$ and $\forall i \in \{-h, \dots, k\}$ there exists $j \in \{1, \dots, M\}$ such that

(4.4) $$\|u\|^2_{jm \leq |t - p_i| \leq (j+1)m} \leq \frac{4r^2}{m}.$$

Let $j_{u,i}$ the smallest index in $\{1, \dots, m\}$ which verifies (4.4).
For any $\epsilon \in (0, r)$ there exists $N_\epsilon \in \mathbb{N}$, $N_\epsilon \geq N_r$ such that

(4.5) $$\max\{\|v^-\|^2_{|t| > N_\epsilon}, \|v^+\|^2_{|t| > N_\epsilon}, \tfrac{4r^2}{N_\epsilon}\} < \tfrac{\epsilon}{16}.$$

So if $m > N_\epsilon$ and $p \in P^{h,k}(M, p_0)$, then $\forall u \in B_r(v^-, v^+; p)$ and $\forall i \in \{1, \dots, k\}$ we get that

(4.6) $$\|u\|^2_{j_{u,i} m \leq |t - p_i| \leq (j_{u,i}+1)m} < \tfrac{\epsilon}{16}.$$

Now, for any $u \in B_r(v^-, v^+; p)$ we define the following subsets of $\mathbb{R}$:

$$A_{u,i} = [p_i + (j_{u,i} + 1)m,\ p_{i+1} - (j_{u,i+1} + 1)m] \quad (1 \leq i \leq k)$$
$$A_{u,0} = [p_{-1} + (j_{u,-1} + 1)m,\ p_1 - (j_{u,1} + 1)m]$$
$$A_{u,i} = [p_{i-1} + (j_{u,i-1} + 1)m,\ p_i - (j_{u,i} + 1)m] \quad (-h \leq i \leq -1)$$
$$B_{u,i} = \{t \in \mathbb{R} \ /\ \operatorname{dist}(t, A_{u,i}) < m\} \quad (-h \leq i \leq k)$$
$$A_u = \textstyle\bigcup_{i=-h}^k A_{u,i}, \quad B_u = \bigcup_{i=-h}^k B_{u,i}$$

with the agreement that $j_{u,-h-1} = j_{u,k+1} = 0$. With this notation (4.6) becomes

$$\|u\|^2_{I_i\cap(B_u\setminus A_u)} \leq \tfrac{\epsilon}{16}, \quad \forall\, u \in B_r(v^-,v^+;p),\ \forall\, i \in \{-h,\dots,k\}.$$

Moreover

$$\|u\|^2_{B_{u,i}\setminus A_{u,i}} \leq \tfrac{\epsilon}{8}\ ,\forall\, u \in B_r(v^-,v^+;p)\ \ \forall\, i \in \{-h,\dots,k\}.$$

We remark that, by construction, we always have that $M_i \subset A_{u,i}$, so that $|A_{u,i}| \geq |M_i| \geq m$, $\forall i \in \{-h,\dots,k\}$, $\forall u \in B_r(v^-,v^+;p)$. Moreover $|I_i \cap (B_u \setminus A_u)| = 2m$ and $|B_{u,i} \setminus A_{u,i}| = 2m$.
For $i \in \{-h,\dots,k\}$, we define piecewise linear cut–off functions $\beta_{u,i} : \mathbb{R} \to [0,1]$ such that $\beta_{u,i}(t) = 1$ for $t \in A_{u,i}$ and $\beta_{u,i}(t) = 0$ for $t \notin B_{u,i}$. Then, for $i \in \{-h,\dots,k\}$, we set:

$$\overline{\beta}_{u,i}(t) = \begin{cases} 0 & t \notin I_i \\ 1-\beta_{u,i-1}(t)-\beta_{u,i}(t) & t \in I_i. \end{cases}$$

We note that if $\beta = \beta_{u,i}$ or $\beta = \overline{\beta}_{u,i}$ and if A is a measurable subset of $\mathbb{R}$ we have $\|\beta u\|^2_A \leq 3\|u\|^2_A$, $\forall u \in X$. Moreover, if $u \in B_r(v^-,v^+;p)$ and $i \in \{-h,\dots,k\}$, we get

$$\langle u\,,\,\beta_{u,i}u\rangle \geq \|u\|^2_{A_{u,i}} - \tfrac{1}{16}\epsilon. \tag{4.7}$$

Now we define, for $i \in \{-h,\dots,k\}$, the functions

$$f_i(u) = \begin{cases} 1 & \|u\|^2_{A_{u,i}} \geq \frac{\epsilon}{4} \\ \frac{1}{k+1} & \text{otherwise} \end{cases}$$

and finally, we set $W_u = \sum_{i=-h}^{k} f_i(u)\beta_{u,I}u$.

Lemma 4.2. *Let $\epsilon \in (0,r^2)$. Then for any $u \in B_r(v^-,v^+;p)$ and for any $j \in \{-h,\dots,k\}$ we have*

$$\varphi'(u)W_u \geq \tfrac{1}{2}\sum_{i=-h}^{k} f_i(u)(\|u\|^2_{A_{u,i}} - \tfrac{\epsilon}{4}),\ \ \varphi_j'(u)W_u \geq \tfrac{1}{2}\sum_{i=-h}^{k} f_i(u)(\|u\|^2_{I_j\cap A_{u,i}} - \tfrac{\epsilon}{4}).$$

Proof. By construction $\|u\|_{A_{u,i}} \le 4r \le \rho_0$ and $\|u\|_{B_{u,i}\setminus A_{u,i}} < \rho_0$. Therefore, by the choice of ρ_0 and (4.6), we get

$$\varphi'(u)W_u \ge$$
$$\textstyle\sum_{i=-h}^{k} f_i(u)(\|u\|^2_{A_{u,i}} - \frac{\epsilon}{16} - \displaystyle\int_{A_{u,i}} V'(t,u)u\,dt - \int_{B_{u,i}\setminus A_{u,i}} V'(t,u)\beta_{u,i}u\,dt) \ge$$
$$\textstyle\sum_{i=-h}^{k} f_i(u)(\frac{1}{2}\|u\|^2_{A_{u,i}} - \frac{1}{16}\epsilon - \frac{1}{16}\epsilon) \ge \frac{1}{2}\sum_{i=-h}^{k} f_i(u)(\|u\|^2_{A_{u,i}} - \frac{\epsilon}{4}).$$

The computation is exactly the same for φ_j'. □

Remark 4.3. We remark that, by Lemma 4.2, we always have

$$\varphi'(u)W_u \ge \tfrac{1}{2}\textstyle\sum_{\{i\,:\,\|u\|^2_{A_{u,i}}<\epsilon/4\}} f_i(u)(\|u\|^2_{A_{u,i}} - \frac{\epsilon}{4}) \ge -\frac{\epsilon}{8}$$

and analogously

$$\varphi_i'(u)W_u \ge -\tfrac{\epsilon}{8} \quad (-h \le i \le k)$$

for all $u \in B_r(v^-, v^+; p)$.
Moreover if $\|u\|^2_{I_i\cap A_{u,l}} > \epsilon$ for some pair of indices (i,l), then W_u indicates an increasing direction both for φ and φ_i. Indeed, if for instance $i > 0$, by lemma 4.2, we get

$$\begin{aligned} \varphi'(u)W_u &\ge \tfrac{1}{2}(\|u\|^2_{A_{u,i-1}} + \|u\|^2_{A_{u,i}} - \tfrac{\epsilon}{2}) - \tfrac{\epsilon}{8}\textstyle\sum_{\{l\,:\,\|u\|^2_{A_{u,l}}<\frac{\epsilon}{4}\}} f_l(u) \\ &\ge \tfrac{1}{2}(\|u\|^2_{I_i\cap A_u} - \tfrac{\epsilon}{2}) - \tfrac{\epsilon}{8}\textstyle\sum_{\{l\,:\,\|u\|^2_{A_{u,l}}<\frac{\epsilon}{4}\}} f_l(u) \ge \frac{1}{2}\|u\|^2_{I_i\cap A_u} - \frac{3\epsilon}{8}, \end{aligned} \tag{4.8}$$

and analogously

$$\varphi_i'(u)W_u \ge \tfrac{1}{2}\|u\|^2_{I_i\cap A_u} - \tfrac{3\epsilon}{8}. \tag{4.9}$$

Let $r_1, r_3 > 0$ be such that $r - d_r < r_1 < r_3 < r$. We put $\xi_1 = \frac{1}{4}d_r$, $\xi_2 = \min\{\delta^{1/2}, \rho_0\}$, $\epsilon_1^{1/2} = \frac{1}{8}\min\{\xi_1, \xi_2, \mu_r, \lambda, \bar{h}_\pm\}$. Let us fix $\epsilon \in (0, \epsilon_1)$, $m \ge N_\epsilon$.

By (4.8) and (4.9) if $\xi \in \{\xi_1, \xi_2\}$, we have:
(4.10)
if $u \in B_{r_3}(v^-, v^+; p)$ and $\|u\|_{I_i\cap A_u} \ge \xi$ then $\varphi'(u)W_u \ge \dfrac{\xi^2}{4}$ and $\varphi_i'(u)W_u \ge \dfrac{\xi^2}{4}$.

We now study the case in which $\|u\|_{I_i\cap A_u} < \xi$ for some $i \in \{-h,\dots,k\}$. To this aim we state a result which is a consequence of the asymptoticity assumption (U5).

Lemma 4.4. *For any $\epsilon > 0$ and $C > 0$ there exists $T > 0$ such that every $u \in X$ with $\|u\| \le C$ verifies:*

$$\|\varphi'(u) - \varphi'_-(u)\| \le \epsilon \quad \textit{if } \operatorname{supp} u \subset (-\infty, -T),$$
$$\|\varphi'(u) - \varphi'_+(u)\| \le \epsilon \quad \textit{if } \operatorname{supp} u \subset (T, +\infty).$$

Proof. For any $u, h \in X$ and $\delta > 0$ it holds that:

$$\begin{aligned}
|(\varphi'(u)-\varphi'_+(u))h| &= \left|\int_{\mathbb{R}}(u\cdot(L(t)-L_+(t))h - (V'(t,u)-V'_+(t,u))\cdot h)\,dt\right| \\
&\le \int_{\mathbb{R}}|L(t)-L_+(t)|\,|u|\,|h|\,dt + \int_{|u(t)|>\delta}|V'(t,u)-V'_+(t,u)|\,|h|\,dt \\
&\quad + \int_{|u(t)|\le\delta}|V'(t,u)|\,|h|\,dt + \int_{|u(t)|\le\delta}|V'_+(t,u)|\,|h|\,dt \\
&\le L_0^{-\frac12}\big(\sup\nolimits_{t\in\operatorname{supp} u}|L(t)-L_+(t)|\,\|u\| \\
&\quad + \big(\int_{|u(t)|>\delta}|V'(t,u)-V'_+(t,u)|^2\,dt\big)^{\frac12} \\
&\quad + \big(\int_{|u(t)|\le\delta}|V'(t,u)|^2\,dt\big)^{\frac12} + \big(\int_{|u(t)|\le\delta}|V'_+(t,u)|^2\,dt\big)^{\frac12}\big)\|h\|.
\end{aligned}$$

Then, taking $u \in X$ with $\|u\| \le C$ and fixing $\epsilon > 0$, by (U2) we can find $\delta > 0$ such that $|V'(t,q)| \le \frac{\epsilon}{4C}|q|$ and $|V'_+(t,q)| \le \frac{\epsilon}{4C}|q|$ for any $t \in \mathbb{R}$ and $|q| \le \delta$. Therefore

$$\big(\int_{|u(t)|\le\delta}|V'(t,u)|^2dt\big)^{\frac12} + \big(\int_{|u(t)|\le\delta}|V'_+(t,u)|^2dt\big)^{\frac12} \le \tfrac{\epsilon}{2}.$$

Moreover we have that:

$$\begin{aligned}
\big(\int_{|u(t)|>\delta}|V'(t,u)-V'_+(t,u)|^2\,dt\big) &\le \tfrac{1}{\delta^2}\sup_{t\in\operatorname{supp} u}|V'(t,u)-V'_+(t,u)|^2\int_{\mathbb{R}}|u|^2\,dt \\
&\le \tfrac{C^2}{\delta^2}\sup_{t\in\operatorname{supp} u,\ |q|\le R}|V'(t,q)-V'_+(t,q)|^2
\end{aligned}$$

for a suitable $R > 0$. Finally, by (U2) and (U5) we can take $T > 0$ so large that, if $\operatorname{supp} u \subset (T, +\infty)$, then

$$L_0^{-\frac12}C\sup_{t\in\operatorname{supp} u}|L(t)-L_+(t)| + \tfrac{C}{\delta}\big(\sup_{t\in\operatorname{supp} u,\ |q|\le R}|V'(t,q)-V'_+(t,q)|^2\big)^{\frac12} \le \tfrac{\epsilon}{2}.$$

Then $\|\varphi'(u) - \varphi'_+(u)\| \le \epsilon$. Changing φ_+, L_+ and V_+ respectively with φ_-, L_- and V_- and considering the case $\operatorname{supp} u \subset (-\infty, -T)$ for a possibly larger $T > 0$ we get the same estimate for $\|\varphi'(u) - \varphi'_-(u)\|$. □

Let us state first a consequence of property (4.2).

Lemma 4.5. *There exists* $\mu = \mu(r) > 0$ *and* $p_0 \geq 1$ *such that if* $u \in B_{r_3}(v^-, v^+; p) \cap \{\varphi_i \leq c^*_\pm - \bar{h}_\pm\}$ *with* $p \in P^{h,k}(M, p_0)$, $\|u\|_{I_i \cap A_u} < \xi_1$ *and* $\|u - v^\pm(\cdot - p_i)\|_{I_i} \geq r_1$, *for some* $i \in \{-h, \dots, k\}$, *then there exists* $\mathcal{W}_{u,i} \in X$, $\|\mathcal{W}_{u,i}\| \leq 1$ *such that*

(i) $\varphi_i'(u)\mathcal{W}_{u,i} \geq \mu$;

(ii) $\varphi'(u)\mathcal{W}_{u,i} \geq \mu$;

(iii) *supp* $\mathcal{W}_{u,i} \subset I_i \setminus J$.

Proof. Let μ_r be given by (4.2). Let $u \in B_{r_3}(v^-, v^+; p) \cap \{\varphi_i \leq c^*_\pm - \bar{h}_\pm\}$ and $\|u - v_\pm(\cdot - p_i)\|_{I_i} \geq r_1$, for some $i \in \{-h, \dots, k\}$. If $\|u\|_{I_i \cap A_u} < \xi_1$ we claim that $\overline{\beta}_{u,i} u \in A_{r-d_r, r+d_r}(v_\pm(\cdot - p_i)) \cap \{\varphi_\pm \leq c^*_\pm\}$. First of all, since $\|v^\pm(\cdot - p_i)\|^2_{|t-p_i| \geq m} \leq \frac{\epsilon}{16}$, using (4.7) , we get

$$\begin{aligned}
&\|\overline{\beta}_{u,i} u - v^\pm(\cdot - p_i)\|^2 \geq \|u - v^\pm(\cdot - p_i)\|^2_{I_i \setminus B_u} \\
&\geq \|u - v^\pm(\cdot - p_i)\|^2_{I_i} - (\|v^\pm(\cdot - p_i)\|_{I_i \cap B_u} + \|u\|_{J_{u,i}} + \|u\|_{I_i \cap A_u})^2 \\
&\geq r_1^2 - (\tfrac{\epsilon^{\frac{1}{2}}}{2} + \xi_1)^2 \geq (r - d_r)^2
\end{aligned}$$

where we set $J_{u,i} = I_i \cap (B_u \setminus A_u)$. Moreover

$$\begin{aligned}
&\|\overline{\beta}_{u,i} u - v^\pm(\cdot - p_i)\|^2 \leq \|\overline{\beta}_{u,i} u - v^\pm(\cdot - p_i)\|^2_{I_i} + \tfrac{\epsilon}{16} \\
&\leq (\|\overline{\beta}_{u,i} u - u\|_{I_i} + \|u - v^\pm(\cdot - p_i)\|_{I_i})^2 + \tfrac{\epsilon}{16} \\
&\leq (\|\beta_{u,i} u\|_{I_i} + r_3)^2 + \tfrac{\epsilon}{16} \qquad (4.11)\\
&\leq (\|u\|_{I_i \cap A_u} + \|\beta_{u,i} u\|_{J_{u,i}} + r_3)^2 + \tfrac{\epsilon}{16} \\
&\leq (\xi_1 + \tfrac{\epsilon^{\frac{1}{2}}}{2} + r_3)^2 + \tfrac{\epsilon}{16} \leq (\xi_1 + \epsilon^{\frac{1}{2}} + r_3)^2 \leq (r + d_r)^2 .
\end{aligned}$$

Finally we note that since $\|u\|_{I_i \cap B_u} \leq \frac{1}{4}\rho_0$, by the choice of ρ_0 we have that $\frac{1}{2}\|u\|^2_{I_i \cap B_u} - \int_{I_i \cap B_u} V(t, u)\, dt \geq 0$ and $\int_{J_{u,i}} V(t, \overline{\beta}_{u,i} u)\, dt \leq \frac{1}{4}\|\overline{\beta}_{u,i} u\|^2_{J_{u,i}}$, therefore

$$\begin{aligned}
\varphi_\pm(\overline{\beta}_{u,i} u) = \varphi_i(\overline{\beta}_{u,i} u) &\leq \varphi_i(u) - (\tfrac{1}{2}\|u\|^2_{I_i \cap B_u} - \int_{I_i \cap B_u} V(t, u)\, dt) + \\
&+ \tfrac{1}{4}\|\overline{\beta}_{u,i} u\|^2_{J_{u,i}} \leq \varphi_i(u) + \tfrac{\epsilon}{16} \leq c^*_\pm.
\end{aligned}$$

By (4.2), there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \leq 1$, such that

$$\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} = \varphi_\pm'(\overline{\beta}_{u,i}u)Z_{u,i} \geq \frac{\mu_r}{2}.$$

Let $p_0 = T$ being $T \geq 1$ given by Lemma 4.4 with $C = \|v^\pm\| + \rho_0$ and $\epsilon = \frac{1}{4}\mu_r$. Then

$$\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} \geq \frac{\mu_r}{4}.$$

By the choice of ρ_0, using (4.7) we get

$$\begin{aligned}|\varphi_i'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi_i'(u)\overline{\beta}_{u,i}Z_{u,i}| = |\langle\overline{\beta}_{u,i}u\,,\,Z_{u,i}\rangle_{J_{u,i}} - \langle u\,,\,\overline{\beta}_{u,i}Z_{u,i}\rangle_{J_{u,i}} + \\ - \int_{J_{u,i}}(V'(t,\overline{\beta}_{u,i}u) - V'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| = \\ = |\int_{J_{u,i}}\dot{\overline{\beta}}_{u,i}(u\dot{Z}_{u,i} - \dot{u}Z_{u,i})\,dt - \int_{J_{u,i}}(V'(t,\overline{\beta}_{u,i}u) - V'(t,u)\overline{\beta}_{u,i})Z_{u,i}\,dt| \leq \\ \leq 2\|u\|_{J_{u,i}} + \|u\|_{J_{u,i}} \leq \epsilon^{\frac{1}{2}} \leq \frac{\mu_r}{8}.\end{aligned}$$

The same argument gives also

$$|\varphi'(\overline{\beta}_{u,i}u)Z_{u,i} - \varphi'(u)\overline{\beta}_{u,i}Z_{u,i}| \leq \frac{\mu_r}{8}.$$

We put $\mathcal{W}_{u,i} = \frac{1}{2}\overline{\beta}_{u,i}Z_{u,i}$, observing that $\min\{\varphi_i'(u)\mathcal{W}_{u,i}\,,\,\varphi'(u)\mathcal{W}_{u,i}\} \geq \frac{\mu_r}{8}$. The lemma follows setting $\mu = \frac{\mu_r}{16}$. □

If $u \in B_{r_3}(v^-, v^+; p)$ does not satisfy the assumptions of lemma 4.5 we set $\mathcal{W}_{u,i} = 0$.

Now we state a consequence of the property (4.3).

Lemma 4.6. *There exist $\nu > 0$ and $\bar{p}_0 \geq 1$ such that if $u \in B_{r_3}(v^-, v^+; p)$ with $p \in P^{h,k}(M, p_0)$, $\|u\|_{I_i \cap A_u} < \xi_2$ and $u \in \{a_\pm \leq \varphi_i \leq a_\pm + \delta\} \cup \{b_\pm \leq \varphi_i \leq b_\pm + \delta\}$, for some $i \in \{-h, \ldots, k\}$, then there exists $\mathcal{V}_{u,i} \in X$, $\|\mathcal{V}_{u,i}\| \leq 1$ such that*

(i) $\varphi_i'(u)\mathcal{V}_{u,i} \geq \nu$;

(ii) $\varphi'(u)\mathcal{V}_{u,i} \geq \nu$;

(iii) *supp* $\mathcal{V}_{u,i} \subset I_i \setminus J$.

Proof. Let $u \in B_{r_3}(v^-, v^+; p) \cap (\{a_\pm \le \varphi_i \le a_\pm + \delta\} \cup \{b_\pm \le \varphi_i \le b_\pm + \delta\})$, for some $i \in \{-h, \dots, k\}$ and $\|u\|_{I_i \cap A_u} < \xi_2$. We claim that $\overline{\beta}_{u,i} u \in \{\bar{a}_\pm \le \varphi_\pm \le \bar{a}'_\pm\} \cup \{\bar{b}_\pm \le \varphi_\pm \le \bar{b}'_\pm\}$.
Indeed, we observe that

$$\|u\|^2_{I_i} - \|\overline{\beta}_{u,i} u\|^2_{I_i} \le \|u\|^2_{I_i \cap A_u} + 3\|u\|^2_{J_{u,i}} \le \|u\|^2_{I_i \cap A_u} + \tfrac{\epsilon}{2} \le \xi_2^2 + \tfrac{\epsilon}{2}$$

and that

$$\begin{aligned} &\int_{I_i} (V(t,u) - V(t, \overline{\beta}_{u,i} u))\, dt \\ &\qquad = \int_{I_i \cap A_u} V(t,u)\, dt + \int_{J_{u,i}} (V(t,u) - V(t, \overline{\beta}_{u,i} u))\, dt \\ &\qquad \le \tfrac{1}{4}\|u\|^2_{I_i \cap A_u} + \tfrac{1}{2}\|u\|^2_{J_{u,i}} < \tfrac{1}{2}(\xi_2^2 + \epsilon). \end{aligned}$$

Then, we have

$$\begin{aligned} &|\varphi_i(u) - \varphi_i(\overline{\beta}_{u,i} u)| \\ &= |\tfrac{1}{2}(\|u\|^2_{I_i} - \|\overline{\beta}_{u,i} u\|^2_{I_i}) - \int_{I_i} (V(t,u) - V(t, \overline{\beta}_{u,i} u))\, dt| \le \xi_2^2 + \epsilon < \delta, \end{aligned}$$

which implies $\overline{\beta}_{u,i} u \in \{\bar{a}_\pm \le \varphi_\pm \le \bar{a}'_\pm\} \cup \{\bar{b}_\pm \le \varphi_\pm \le \bar{b}'_\pm\}$.

Therefore by property (4.3), there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \le 1$, such that $\varphi'(\overline{\beta}_{u,i} u) Z_{u,i} = \varphi'_i(\overline{\beta}_{u,i} u) Z_{u,i} \ge \frac{\lambda}{2}$. Let $\bar{p}_0 = T$ being $T \ge 1$ given by Lemma 4.4 with $C = \|v^\pm\| + \rho_0$ and $\epsilon = \frac{\lambda}{4}$. Then

$$\varphi'(\overline{\beta}_{u,i} u) Z_{u,i} \ge \frac{\lambda}{4}.$$

As in lemma 4.5 we have $|\varphi'(\overline{\beta}_{u,i} u) Z_{u,i} - \varphi'(u) \overline{\beta}_{u,i} Z_{u,i}| \le \epsilon^{\frac{1}{2}} \le \frac{\lambda}{8}$ and that $|\varphi'_i(\overline{\beta}_{u,i} u) Z_{u,i} - \varphi'_i(u) \overline{\beta}_{u,i} Z_{u,i}| \le \epsilon^{\frac{1}{2}} \le \frac{\lambda}{8}$.

Therefore $\varphi'(u) \overline{\beta}_{u,i} Z_{u,i} \ge \frac{\lambda}{4}$ and $\varphi'_i(u) \overline{\beta}_{u,i} Z_{u,i} \ge \frac{\lambda}{4}$. We put $\mathcal{V}_{u,i} = \frac{1}{2} \overline{\beta}_{u,i} Z_{u,i}$ and setting $\nu = \frac{\lambda}{16}$ the lemma follows. □

If $u \in B_{r_3}(v^-, v^+; p)$ does not satisfies the assumptions of lemma 4.6 we set $\mathcal{V}_{u,i} = 0$.

Now, collecting the results obtained above we can complete the proof of Lemma 4.1.

Given $u \in B_{r_3}(v^-, v^+; p)$ we define

$\mathcal{I}_1 = \{i \in \{-h, \dots, k\} : \|u - v^\pm(\cdot - p_i)\|_{I_i} \ge r_1,\ \varphi_i(u) \le c^*_\pm - \bar{h}_\pm\}$,

$\mathcal{I}_2 = \{i \in \{-h, \dots, k\} : b_\pm \le \varphi_i(u) \le b_\pm + \delta \text{ or } a_\pm \le \varphi_i(u) \le a_\pm + \delta\}$.

Now let us consider the case $\mathcal{I}_1 = \mathcal{I}_2 = \emptyset$.

We distinguish the two following subcases:

$$\max_{0 \le l \le k} \|u\|^2_{M_l} < \epsilon \quad or \quad \max_{0 \le l \le k} \|u\|^2_{M_l} \ge \epsilon .$$

In the first case, by lemma 2.9, we obtain that if $\mathcal{K} \cap B_{r_3}(v^-, v^+; p) = \emptyset$ then there exists $\mathcal{Z}_u \in X$, $\|\mathcal{Z}_u\| \le 1$ and there exists $\tilde{\mu}_p > 0$, independent of u, such that $\varphi'(u)\mathcal{Z}_u \ge \frac{\tilde{\mu}_p}{2}$.

In the other case if we have $\|u\|_{M_{\bar{l}}} = \max_{-h \le l \le k} \|u\|^2_{M_l} \ge \epsilon$, we get by (4.8) that

$$\varphi'(u)W_u \ge \tfrac{1}{2}\|u\|^2_{M_{\bar{l}}} - \tfrac{3\epsilon}{8} \ge \tfrac{\epsilon}{8}.$$

We set

$$\tilde{\mathcal{W}}_u = \begin{cases} \mathcal{Z}_u & \text{if } \mathcal{I}_1 = \mathcal{I}_2 = \emptyset \text{ and } \max_{0 \le l \le k} \|u\|^2_{M_l} < \epsilon \\ \frac{1}{3}(W_u + \sum_{i \in \mathcal{I}_1} \mathcal{W}_{u,i} + \sum_{i \in \mathcal{I}_2} \mathcal{V}_{u,i}) & \text{otherwise} \end{cases}$$

where $\mathcal{W}_{u,i}$ is given by lemma 4.5 and $\mathcal{V}_{u,i}$ by lemma 4.6.

Then we note that

$$\|\tilde{\mathcal{W}}_u\|_{I_i} \le \max\{\|\mathcal{Z}_u\|_{I_i}, \tfrac{1}{3}(\|W_u\|_{I_i} + \|\mathcal{W}_{u,i}\|_{I_i} + \|\mathcal{V}_{u,i}\|_{I_i})\} \le 1.$$

Moreover by using lemmas 4.5, 4.6, remark 4.3, (4.8) and (4.9) we have the following properties:

i) if $\max_{0 \le l \le k} \|u\|^2_{M_l} \ge \epsilon$ then

$$\langle u, \tilde{\mathcal{W}}_u \rangle_{M_l} = \tfrac{1}{3}\langle u, W_u \rangle_{M_l} \ge \tfrac{1}{3(k+1)}\|u\|^2_{M_l}$$

and

$$\varphi'(u)\tilde{\mathcal{W}}_u \ge \tfrac{1}{3}\varphi'(u)W_u \ge \tfrac{\epsilon}{24};$$

ii) if $i \in \mathcal{I}_1$ and $\|u\|_{I_i \cap A_u} < \xi_1$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \ge \tfrac{1}{3}\varphi'(u)W_u + \tfrac{1}{3}\varphi'(u)\mathcal{W}_{u,i} \ge \tfrac{\mu}{3} - \tfrac{\epsilon}{24} \ge \tfrac{\mu}{6}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \ge \tfrac{1}{3}\varphi_i'(u)W_u + \tfrac{1}{3}\varphi_i'(u)\mathcal{W}_{u,i} \ge \tfrac{\mu}{3} - \tfrac{\epsilon}{24} \ge \tfrac{\mu}{6};$$

iii) if $i \in \mathcal{I}_1$ and $\|u\|_{I_i \cap A_u} \geq \xi_1$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u \geq \tfrac{\xi_1^2}{6} - \tfrac{3\epsilon}{24} \geq \tfrac{\xi_1^2}{12}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u \geq \tfrac{\xi_1^2}{6} - \tfrac{3\epsilon}{24} \geq \tfrac{\xi_1^2}{12};$$

iv) if $i \in \mathcal{I}_2$ and $\|u\|_{I_i \cap A_u} < \xi_2$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u + \tfrac{1}{3}\varphi'(u)\mathcal{V}_{u,i} \geq \tfrac{\nu}{3} - \tfrac{\epsilon}{24} \geq \tfrac{\nu}{6}$$
$$\varphi_i'(u)\tilde{W}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u + \tfrac{1}{3}\varphi_i'(u)\mathcal{V}_{u,i} \geq \tfrac{\nu}{3} - \tfrac{\epsilon}{24} \geq \tfrac{\nu}{6};$$

v) if $i \in \mathcal{I}_2$ and $\|u\|_{I_i \cap A_u} \geq \xi_2$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi'(u)W_u \geq \tfrac{\xi_2^2}{6} - \tfrac{3\epsilon}{24} \geq \tfrac{\xi_2^2}{12}$$
$$\varphi_i'(u)\tilde{\mathcal{W}}_u \geq \tfrac{1}{3}\varphi_i'(u)W_u \geq \tfrac{\xi_2^2}{6} - \tfrac{3\epsilon}{24} \geq \tfrac{\xi_2^2}{12};$$

vi) if $\mathcal{I}_1 = \mathcal{I}_2 = \emptyset$ and $\max_{0 \leq l \leq k} \|u\|_{M_l}^2 < \epsilon$ then

$$\varphi'(u)\tilde{\mathcal{W}}_u = \varphi'(u)\mathcal{Z}_u \geq \tfrac{\mu_p}{2}.$$

By (i)-(vi), the lemma follows with a classical pseudogradient construction setting $\tilde{\mu}_r = \min\{\frac{\xi_1}{24}, \frac{\mu}{12}\}$ and $\mu_p = \min\{\tilde{\mu}_r, \frac{\epsilon}{24}, \frac{\xi_2}{24}, \frac{\nu}{12}\}$. □

5. Multibump solutions to (HS)

This final section is devoted to prove the theorems presented in the introduction.

We start with the following result concerning homoclinic solutions of multibump type for a doubly asymptotic potential.

Theorem 5.1. *Let U, U_- and U_+ verify (U1)-(U5). Assume that the condition $(*)$ holds for the functionals φ_- and φ_+. Let $\bar{v}_- \in K_-$ and $\bar{v}_+ \in K_+$ be the mountain pass critical points for φ_- and φ_+ given by Lemma 3.13. Then for any $r > 0$ there are $M, p_0 \geq 1$ such that $B_r(\bar{v}_-, \bar{v}_+; p) \cap K \neq \emptyset$ for every $p \in \bigcup_{h,k \in \mathbb{N}} P^{h,k}(M, p_0)$.*

Proof. Let $\bar{r}_\pm \in (0, \rho)$, $(r_n) \subset \mathbb{R}^+ \setminus (D_+^* \cup D_-^*)$, $v_n^\pm, \bar{v}_\pm \in K_\pm$ as in Lemma 4.1. Arguing by contradiction, suppose that the conclusion of the theorem is false.

Then there exists $r_0 > 0$ such that for any $M, p_0 \geq 1$ there are $h, k \in \mathbb{N}$ and $p = (p_{-h}, \dots, p_{-1}, p_1, \dots, p_k) \in P^{h,k}(M, p_0)$ for which $B_{r_0}(\bar{v}_-, \bar{v}_+; p) \cap K = \emptyset$.
Let us fix $n \in \mathbb{N}$ in order that $\|v_n^\pm - \bar{v}_\pm\| < \frac{\rho}{2}$, $r_n < \min\{\frac{\rho}{2}, r_0\}$ and $B_{2r_n}(v_n^\pm) \subset B_{\bar{r}_\pm}(\bar{v}_\pm)$. We note that $B_{r_n}(v_n^-, v_n^+; p) \subset B_\rho(\bar{v}_-, \bar{v}_+; p)$ for any $p \in \bigcup_{h,k \in \mathbb{N}} P^{h,k}(M, p_0)$ and for any $M, p_0 \geq 1$. Let $\nu = \nu(r_n) > 0$ be given by Lemma 4.1 with $r_1 = r_n - \frac{1}{2} d_{r_n}$, $r_2 = r_n - \frac{5}{12} d_{r_n}$ and $r_3 = r_n - \frac{1}{3} d_{r_n}$. Thanks to Lemma 3.6, we can choose $a_\pm > \bar{c}_\pm - \min\{h_n, \frac{1}{24}\nu d_{r_n}\}$, $b_\pm < \min\{c_\pm^*, \bar{c}_\pm + \frac{1}{24}\nu d_{r_n}\}$ and $\delta > 0$ satisfying (4.1).

Then Lemma 4.1 assigns two values $p_0 \geq 1$ and $\epsilon_1 > 0$. Now let us take $\epsilon_2 > 0$ such that for any interval $I \subseteq \mathbb{R}$ with $|I| \geq 1$ and for any $u \in X$ with $\|u\|_I^2 \leq \epsilon_2$ it holds that $\int_I |V_\pm(t, u)|\, dt \leq \|u\|_I^2$ (see Remark 2.5). Then we fix $\epsilon \in (0, \min\{\epsilon_1, \frac{1}{2}\epsilon_2, \frac{1}{4}(\bar{c}_+ - a_+), \frac{1}{4}(\bar{c}_- - a_-), \frac{1}{3}r_n^2, \frac{1}{2}d_{r_n}^2\})$. By Lemma 4.1 there exists $m_0 \in \mathbb{N}$ such that for any $p \in P(2m_0^2 + 3m_0, p_0)$ there is a vector field $\mathcal{V}_p : X \to X$ satisfying $(\mathcal{V}1)$-$(\mathcal{V}5)$. Now we apply Lemma 3.13 fixing $h = \min\{b_- - \bar{c}_-, b_+ - \bar{c}_+\}$ and finding two paths $\gamma_n^\pm$ with $\operatorname{supp} \gamma_n^\pm(s) \subset [-R, R]$ for any $s \in [0,1]$, where $R > 0$ depends only on n. Moreover, enlarging R if necessary, we can always assume that $\|v_n^\pm\|_{|t| \geq R}^2 < \epsilon$. Then we choose $m > \max\{m_0, R, T_-^{-1}, T_+^{-1}\}$ and we use the contradiction assumption, for which there is $p = (p_{-h}, \dots, p_{-1}, p_1, \dots, p_k) \in P(2m^2 + 3m, p_0)$ such that $B_{r_n}(v_n^-, v_n^+; p) \cap K = \emptyset$. Consequently, there is a vector field $\mathcal{V}_p = \mathcal{V} : X \to X$ that satisfies $(\mathcal{V}_1)$–$(\mathcal{V}_5)$.

Finally, for any $s \geq 0$ we define a continuous function $G_s : [0,1]^{h+k} \to X$ given by

$$G_0(\theta) = \sum_{-h \leq j \leq -1} \tau_{p_j} \gamma_n^-(\theta_j) + \sum_{1 \leq j \leq k} \tau_{p_j} \gamma_n^+(\theta_j)$$
$$G_s(\theta) = \eta(s, G_0(\theta))$$

where $\theta = (\theta_{-h}, \dots, \theta_{-1}, \theta_1, \dots, \theta_k) \in [0,1]^{h+k}$ and η is the flow generated by $-\mathcal{V}$.

Lemma 5.2. (*i*) *For any* $s \geq 0$ $G_s = G_0$ *on the boundary of* $[0,1]^{h+k}$.
(*ii*) *For any* $s \geq 0$ $\operatorname{range} G_s \subseteq Y_\epsilon$.
(*iii*) *There exists* $\bar{s} > 0$ *such that* $\operatorname{range} G_{\bar{s}} \subseteq \bigcup_j \{\varphi_j \leq a_\pm\}$.

Before proving Lemma 5.2 we continue the proof of the theorem showing that:

(5.1) there is an index $j \in \{-h,\dots,-1,1,\dots,k\}$ and a path $\xi : [0,1] \to [0,1]^{h+k})$ such that $\xi(0) \in \{\theta_j = 0\}$, $\xi(1) \in \{\theta_j = 1\}$ and $\varphi_j(G_{\bar{s}}(\theta)) < a_\pm + \epsilon$ for any $\theta \in \operatorname{range}\xi$.

Indeed, if (5.1) were false, for any $i \in \{-h,\dots,-1,1,\dots,k\}$ the set $D_i = \{\theta \in [0,1]^{h+k} : \varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon\}$ should separate the faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$. Then, from a Miranda fixed point theorem ([Mi]), it follows that $\bigcap_i D_i \neq \emptyset$, that is there exists $\theta \in [0,1]^{h+k}$ such that $\varphi_i(G_{\bar{s}}(\theta)) \geq a_\pm + \epsilon$ for any i, in contrast with Lemma 5.2 (*iii*).

From now on, let j be the index for which (5.1) holds. Let us assume that $j > 0$. Clearly the same argument works if $j < 0$. Let $\chi : \mathbb{R} \to [0,1]$ be a piecewise linear, cut–off function such that $\chi(r) = 1$ if $r \in I_j \setminus J$ and $\chi(r) = 0$ if $r \in \mathbb{R} \setminus I_j$. Notice that, since $m \geq 2$, for any $u \in X$

$$\|\chi\, u\|^2_{I_j \cap J} \leq 2\|u\|^2_{I_j \cap J} \quad \text{and} \quad \|(1-\chi)\, u\|^2_{I_j \cap J} \leq 2\|u\|^2_{I_j \cap J} \tag{5.2}$$

and for any $s \in [0,1]$

$$\operatorname{supp} \tau_{p_j} \gamma_n^+(s) \subseteq [p_j - R, p_j + R] \subseteq I_j \setminus J. \tag{5.3}$$

Then we define a path $\gamma : [0,1] \to X$ by setting

$$\gamma(s) = \tau_{-p_j} \chi G_{\bar{s}}(\xi(s)) \qquad (s \in [0,1]).$$

By Lemma 5.2 and by (5.3), we have that

$$\gamma(0) = \gamma_n^+(0) \ \text{ and } \ \gamma(1) = \gamma_n^+(1). \tag{5.4}$$

Now we will prove that

$$\operatorname{range}\gamma \subset B_{\bar{r}_+}(\bar{v}_+). \tag{5.5}$$

Indeed, if we set $u = G_{\bar{s}}(\xi(s))$ we have that

$$\begin{aligned} \|\gamma(s) - v_n^+\|^2 &= \|\chi u - \tau_{p_j} v_n^+\|^2 \\ &= \|\tau_{p_j} v_n^+\|^2_{\mathbb{R}\setminus I_j} + \|u - \tau_{p_j} v_n^+\|^2_{I_j \setminus J} + \|\chi u - \tau_{p_j} v_n^+\|^2_{I_j \cap J}. \end{aligned} \tag{5.6}$$

By (5.2) and (5.3) it holds that $\|\tau_{p_j} v_n^+\|^2_{\mathbb{R}\setminus I_j} \le \|v_n^+\|^2_{|t|\ge R} \le \epsilon$ and analogously we also get $\|(1-\chi)\,\tau_{p_j} v_n^+\|^2_{I_j\cap J} \le 2\|v_n^+\|^2_{|t|\ge R} \le 2\epsilon$. Consequently from (5.6) we infer that

$$\|\gamma(s) - v_n^+\|^2 \le 3\epsilon + 3\|u - \tau_{p_j} v_n^+\|^2_{I_j}. \tag{5.7}$$

Since, by $(\mathcal{V}_1)$, $B_{r_n - \frac{1}{3}d_{r_n}}(v_n^-, v_n^+; p)$ is positively invariant for η and, by Lemma 3.13, $\operatorname{range}\gamma_n^+ \subseteq \overline{B}_{r_n}(v_n^+)$, we deduce that $\|u - \tau_{p_j} v_n^+\|_{I_j} \le r_n$. Thus, from (5.7), we get that $\|\gamma(s) - v_n^+\|^2 < 4r_n^2$, because $\epsilon < \frac{1}{3}r_n^2$ and, since $B_{2r_n}(v_n^+) \subset B_{\bar r_+}(\bar v_+)$, (5.5) follows.

Now, we show that for any $s \in [0,1]$

$$\varphi_+(\gamma(s)) < \bar c_+. \tag{5.8}$$

As before, we set $u = G_{\bar s}(\xi(s))$. It holds that

$$\begin{aligned}&\varphi_+(\gamma(s)) = \varphi_+(\chi\, u) = \varphi_j(\chi\, u)\\ &= \varphi_j(u) + \tfrac{1}{2}\|\chi\, u\|^2_{I_j\cap J} - \tfrac{1}{2}\|u\|^2_{I_j\cap J} + \textstyle\int_{I_j\cap J}(V_+(t,u) - V_+(t,\chi\, u))\,dt.\end{aligned}$$

By Lemma 5.2 *(iii)*, we know that $\varphi_j(u) \le a_+$. Using again Lemma 5.2 *(iii)* and (5.1) we estimate $\frac{1}{2}\|\chi\, u\|^2_{I_j\cap J} \le \|u\|^2_{I_j\cap J} \le \epsilon$ and $\int_{I_j\cap J}|V_+(t,u)|\,dt \le \|u\|^2_{I_j\cap J}$, for $\epsilon < \frac{1}{2}\epsilon_2$. Hence $\varphi_+(\gamma(s)) \le a_+ + 4\epsilon$ and (5.8) follows, because $\epsilon < \frac{1}{4}(\bar c_+ - a_+)$.

In conclusion, from (5.4), (5.5) and (5.8), γ is a path joining $\gamma_n^+(0)$ with $\gamma_n^+(1)$ inside $B_{\bar r_+}(\bar v_+) \cap \{\varphi_+ < \bar c_+\}$ and, keeping into account Lemma 3.13, this gives the contradiction and concludes the proof of the theorem. □

Before proving Lemma 5.2, we state the following auxiliary result.

Lemma 5.3. *(i) For any $j = -h, \dots, -1$ the sets $\{\varphi_j \le a_-\}$ and $\{\varphi_j \le b_-\}$ are positively invariant with respect to the flow η, i.e., if $\varphi_j(u) \le a_-$ then $\varphi_j(\eta(s,u)) \le a_-$ for every $s \ge 0$ and if $\varphi_j(u) \le b_-$ then $\varphi_j(\eta(s,u)) \le b_-$ for every $s \ge 0$. Similarly, for any $j = 1, \dots, k$ the sets $\{\varphi_j \le a_+\}$ and $\{\varphi_j \le b_+\}$ are positively invariant with respect to η.*

(ii) For any $u \in Y_\epsilon$ and for any $s \ge 0$ $\eta(s,u) \in Y_\epsilon$, namely also the set Y_ϵ is positively invariant with respect to η.

Proof. (i) Let us fix $j \in \{-h, \dots, -1\}$. If, by contradiction, $\eta(\bar{s}, u) \notin \{\varphi_j \leq a_-\}$ for some $u \in \{\varphi_j \leq a_-\}$ and for some $\bar{s} > 0$, then there is an interval $[s_1, s_2] \subset [0, \bar{s}]$ such that $\varphi_j(\eta(s_1, u)) = a_-$, $\varphi_j(\eta(s_2, u)) \in (a_-, a_- + \delta)$ and $a_- < \varphi_j(\eta(s, u)) \leq a_- + \delta$ for any $s \in (s_1, s_2)$. Then, by $(\mathcal{V}_1)$ and $(\mathcal{V}_3)$, we get that $\varphi_j(\eta(s_2, u)) - a_- = -\int_{s_1}^{s_2} \varphi_j'(\eta(s, u))\mathcal{V}(\eta(s, u))ds \leq 0$, a contradiction. The same argument works for the sets $\{\varphi_j \leq a_+\}$ and $\{\varphi_j \leq b_\pm\}$.

(ii) By the contrary, let us suppose that there exist $u \in Y_\epsilon$, an interval (s_1, s_2) and an index $j \in \{-h, \dots, -1, 1, \dots, k\}$ for which $\|\eta(s_1, u)\|_{M_j}^2 = \epsilon$ and $\|\eta(s, u)\|_{M_j}^2 > \epsilon$ for any $s \in (s_1, s_2)$. Then $\frac{d}{ds}\|\eta(s, u)\|_{M_j}^2 = -2\langle \mathcal{V}(\eta(s, u)), \eta(s, u)\rangle_{M_j}$. By $(\mathcal{V}_4)$, we obtain that $\|\eta(s_2, u)\|_{M_j}^2 \leq \|\eta(s_1, u)\|_{M_j}^2 = \epsilon$, a contradiction. □

Proof of Lemma 5.2. (i) If θ belongs to the boundary of $[0,1]^{h+k}$ then $\theta_i = 0$ or $\theta_i = 1$ for some $i \in \{-h, \dots, -1, 1, \dots, k\}$. Let us suppose for instance that $i > 0$ and $\theta_i = 0$. From (5.3) and Lemma 3.13 (i), we deduce that $\|G_0(\theta) - \tau_{p_i} v_n^+\|_{I_i}^2 = \|\gamma_n^+(0) - v_n^+\|^2 - \|\tau_{p_i} v_n^+\|_{\mathbb{R}\setminus I_i}^2 \geq r_n^2 - \epsilon \geq \bar{r}^2$ because $\epsilon < \frac{1}{3} r_n^2$. Then $G_0(\theta) \in X \setminus B_{r_n - \frac{1}{3} d_{r_n}}(v_n^-, v_n^+; p)$ and consequently, by $(\mathcal{V}_1)$, $\eta(s, G_0(\theta)) = G_0(\theta)$.

(ii) By (5.3), we have that $\|G_0(\theta)\|_{M_j} = 0$ for any j and so $G_0(\theta) \in Y_\epsilon$. Hence, by Lemma 5.3 (ii), $\eta(s, G_0(\theta)) \in Y_\epsilon$ for all $\theta \in [0,1]^{h+k}$ and for all $s \geq 0$.

(iii) Let us fix $\theta \in [0,1]^{h+k}$. If $G_0(\theta) \notin B_{r_n - \frac{1}{2} d_{r_n}}(v_n^-, v_n^+; p)$ then there is an index i, for example positive, for which $\|G_0(\theta) - \tau_{p_i} v_n^+\|_{I_i} \geq r_n - \frac{1}{2} d_{r_n}$. But, using (5.3) and Lemma 3.13 (iii), we have also $\|G_0(\theta) - \tau_{p_i} v_n^+\|_{I_i} \leq \|\gamma_n^+(\theta_i) - v_n^+\| \leq r_n$. Therefore $\gamma_n^+(\theta_i) \in A_{r_n - \frac{1}{2} d_{r_n}, r_n}(v_n^+)$ and, by Lemma 3.13 (iv), $\varphi_+(\gamma_n^+(\theta_i)) \leq \bar{c}_+ - h_n$. Thus, since $a_+ \geq \bar{c}_+ - h_n$ we have that $G_0(\theta) \in \{\varphi_i \leq a_+\}$, and, by Lemma 5.3 (i), also $G_{\bar{s}}(\theta) \in \{\varphi_i \leq a_+\}$.

Suppose now that $G_0(\theta) \in B_{r_n - \frac{1}{2} d_{r_n}}$. First, we notice that, by (5.3), Lemma 3.13 (iii) and by the definition of φ_i and the choice of h, $G_0(\theta) \in \bigcap_i \{\varphi_i \leq b_\pm\}$. Hence, on one hand, by Lemma 5.3 (i), all the positive trajectory $s \mapsto G_s(\theta)$ remains in $\bigcap_i \{\varphi_i \leq b_\pm\}$. On the other hand, we claim that as $s \geq 0$ increases, the curve $s \mapsto G_s(\theta)$ must go out from $B_{r_n - \frac{5}{12} d_{r_n}}$ in a finite time

$\bar{s} \geq 0$ independent of θ, that is:

(5.9)

there is $\bar{s} \geq 0$ such that $G_{\bar{s}}(\theta) \notin B_{r_n - \frac{5}{12}d_{r_n}}(v_n^-, v_n^+; p)$ for any $\theta \in [0,1]^{h+k}$.

During this amount of time $\bar{s}$, $G_s(\theta)$ crosses the annular region $\{u \in B : r_n - \frac{1}{2}d_{r_n} \leq \|u - \tau_{p_i} v_n^{\pm}\| \leq r_n - \frac{5}{12}d_{r_n}\} \cap \bigcap_j \{\varphi_j \leq b_{\pm}\}$. In fact, there exists an index i, let us say positive, such that $\|G_{s_\theta}(\theta) - \tau_{p_i} v_n^+\|_{I_i} = r_n - \frac{1}{2}d_{r_n}$, $\|G_{s'_\theta}(\theta) - \tau_{p_i} v_n^+\|_{I_i} = r_n - \frac{5}{12}d_{r_n}$ and $r_n - \frac{1}{2}d_{r_n} \leq \|G_s(\theta) - \tau_{p_i} v_n^+\|_{I_i} \leq r_n - \frac{5}{12}d_{r_n}$ for $s \in (s_\theta, s'_\theta)$. Then, by $(\mathcal{V}_2)$, $\varphi_i'\mathcal{V} \geq \nu$ along the curve described by $G_s(\theta)$ as s goes from s_θ to s'_θ and consequently, $\varphi_i(G_s(\theta))$ decreases. Precisely $\varphi_i(G_{s'_\theta}(\theta)) \leq \varphi_i(G_{s_\theta}(\theta)) - \nu(s'_\theta - s_\theta)$. But, using $(\mathcal{V}_1)$ it holds that $\frac{1}{12}d_{r_n} \leq \|G_{s'_\theta}(\theta) - G_{s_\theta}(\theta)\| \leq \int_{s_\theta}^{s'_\theta} \|\mathcal{V}(\eta(s, G_0(\theta)))\|_{I_i} ds \leq s'_\theta - s_\theta$ and so $\varphi_i(G_{s'_\theta}(\theta)) \leq b_+ - \frac{1}{12}d_{r_n}\nu \leq \bar{c}_+ - \frac{1}{24}d_{r_n}\nu \leq a_+$. Then Lemma 5.3 (i) implies that $\varphi_i(G_{\bar{s}}(\theta)) \leq a_+$.

Now, it remains to prove the claim (5.9). Arguing by contradiction, if (5.9) is false, then there are sequences $(s_i) \subset \mathbb{R}_+$ and $(\theta_i) \subset [0,1]^{h+k}$ such that $s_i \to +\infty$, $\theta_i \to \theta$ and, for any $i \in \mathbb{N}$, $G_s(\theta_i) \in B_{r_n - \frac{5}{12}d_{r_n}}(v_n^-, v_n^+; p)$ for $s \in [0, s_n]$. Then, from $(\mathcal{V}_5)$ $\varphi(G_{s_i}(\theta_i)) \leq \varphi(G_0(\theta_i)) - s_i\nu'$ and thus $\varphi(G_{s_i}(\theta_i)) \to -\infty$. This is in contrast with the fact that, by Remark 2.2, $\varphi(B_{r_n - \frac{5}{12}d_{r_n}}(v_n^-, v_n^+; p))$ is bounded. □

Remark 5.4. The k-bump homoclinic solutions v found by Theorem 5.1 lie in H^1-neighborhoods of $\tau_{p_j}\bar{v}_{\pm}$, and then in C^0-neighborhoods. By the equation (HS) we recover similar estimates for the second derivatives and, by standard interpolation inequalities, for the first derivatives, too, as stated in Theorem 1.4.

References

[ACM] S. ABENDA, P. CALDIROLI & P. MONTECCHIARI, Multibump solutions for Duffing–like systems, Preprint SISSA, (1994).

[AL] S. ALAMA & Y.Y. LI, On "Multibump" Bound States for Certain Semilinear Elliptic Equations, Research Report No. 92-NA-012. Carnegie Mellon University, (1992).

[AT] S. ALAMA & G. TARANTELLO, On semilinear elliptic equations with indefinite nonlinearities, *Calc. Var.*, **1** (1993), 439-475.

[A] A. AMBROSETTI, *Critical points and nonlinear variational problems*, Bul. Soc. Math. France, 120, 1992.

[AB] A. AMBROSETTI & M.L. BERTOTTI, Homoclinics for second order conservative systems, in *Partial Differential Equations and Related Subjects*, ed. M. Miranda, Pitman Research Notes in Math. Ser. (London, Pitman Press), (1992).

[ACZ] A. AMBROSETTI & V. COTI ZELATI, Multiple Homoclinic Orbits for a Class of Conservative Systems, *Rend. Sem. Mat. Univ. Padova*, **89** (1993), 177-194.

[AR] A. AMBROSETTI & P.H. RABINOWITZ, Dual Variational Methods in Critical Point Theory and Applications, *J. Funct. Anal.*, **14** (1973), 349-381.

[An] S. ANGENENT, *The Shadowing Lemma for Elliptic PDE* , Dynamics of Infinite Dimensional Systems, (S.N. Chow and J.K. Hale eds.) , F37, 1987.

[BC] A. BAHRI & J.M. CORON, On a Nonlinear Elliptic Equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.*, **41** (1988), 253-294.

[BG] V. BENCI & F. GIANNONI, Homoclinic orbits on compact manifolds, *J. Math. Anal. Appl.*, **157** (1991), 568-576.

[BR] V. BENCI & P.H. RABINOWITZ, Critical point theorems for indefinite functionals, *Invent. Math.*, **52** (1979), 241-273.

[B] M.L. BERTOTTI, Homoclinics for Lagrangian Systems on Riemannian manifolds, *Dyn. Syst. and Appl.*, **1** (1992), 341-368.

[BB1] M.L. BERTOTTI & S. BOLOTIN, Homoclinic Solutions of Quasiperiodic Lagrangian Systems, Preprint, (1994).

[BB2] M.L. BERTOTTI & S. BOLOTIN, A variational approach for homoclinics in almost periodic Hamiltonian systems, Preprint, (1995).

[BJ] M.L. BERTOTTI & L. JEANJEAN, Multiplicity of homoclinic solutions for singular second order conservative systems, Preprint, (1995).

[Be] U. BESSI, Multiple homoclinic orbits for autonomous singular potentials, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), 785-802.

[Be1] U. BESSI, A Variational Proof of a Sitnikov-like Theorem, *Nonlinear Anal.*, **20** (1993), 1303-1318.

[Be2] U. BESSI, Global Homoclinic Bifurcation for Damped Systems, *Math. Zeit.*, to appear.

[Be3] U. BESSI, Homoclinic and Period-doubling Bifurcations for Damped Systems, Preprint, (1993).

[BhS] N.P. BHATIA & G.P. SZEGO, *Dynamical Systems: Stability Theory and Applications*, Lecture Notes in Mathematics 35, Springer–Verlag, 1967.

[Bo] S.V. BOLOTIN, Existence of homoclinic motions, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, **6** (1980), 98-103.

[BrC] H. BREZIS & J.M. CORON, Large solutions for harmonic maps in two dimensions, *Comm. Math. Phys.*, **92** (1983), 203-215.

[Bu] B. BUFFONI, Infinitely many large amplitude homoclinic orbits for a class of autonomous Hamiltonian systems, *J. Diff. Eq.*, to appear.

[BS] B. BUFFONI & E. SÉRÉ, A global condition for quasi random behavior in a class of conservative systems, Preprint, (1995).

[C1] P. CALDIROLI, Existence and multiplicity of homoclinic orbits for potentials on unbounded domains, *Proc. Roy. Soc. Edinburgh*, **124A** (1994), 317-339.

[C2] P. CALDIROLI, A new proof of the existence of homoclinic orbits for a class of autonomous second order Hamiltonian systems in $\mathbf{R}^n$, *Math. Nachr.*, to appear.

[CJ] P. CALDIROLI & L. JEANJEAN, Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems, in preparation, (1995).

[CM] P. CALDIROLI & P. MONTECCHIARI, Homoclinic orbits for second order Hamiltonian systems with potential changing sign, *Comm. Applied Nonlinear Analysis*, **1** (1994), 97-129.

[CMN] P. CALDIROLI, P. MONTECCHIARI & M. NOLASCO, Asymptotic behaviour for a class of multibump solutions to Duffing–like systems, *Proc. of the Workshop on Variational and Local Methods in the study of Hamiltonian systems*, (1995), World Scientific.

[CN] P. CALDIROLI & M. NOLASCO, Multiple homoclinic solutions for a class of autonomous singular systems in $\mathbf{R}^2$, Preprint S.I.S.S.A., (1995).

[CS1] K. CIELIEBAK & E. SÉRÉ, Pseudo-holomorphic curves and multiplicity of homoclinic orbits, Preprint, (1993).

[CS2] K. CIELIEBAK & E. SÉRÉ, Pseudo-holomorphic curves and the shadowing Lemma, Preprint, (1995).

[Co] W.A. COPPEL, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics 629, Springer–Verlag, 1978.

[CZES] V. COTI ZELATI, I. EKELAND & E. SÉRÉ, A Variational approach to homoclinic orbits in Hamiltonian systems, *Math.Ann.*, **288** (1990), 133-160.

[CZMN] V. COTI ZELATI, P. MONTECCHIARI & M. NOLASCO, Multibump homoclinic solutions for a class of second order, almost periodic Hamiltonian systems, Preprint S.I.S.S.A., (1995).

[CZR1] V. COTI ZELATI & P.H. RABINOWITZ, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.*, 4 (1991), 693-727.

[CZR2] V. COTI ZELATI & P.H. RABINOWITZ, Multibump periodic solutions of a family of Hamiltonian systems, *Top. Meth. in Nonlinear Analysis*, 3 (1994), .

[DN] W.Y. DING & W.M. NI, On the existence of positive entire solutions of a semilinear elliptic equation, *Arch. Rat. Mech. Anal.*, **91** (1986), 283-308.

[F] P.L. FELMER, Heteroclinic orbits for spatially periodic Hamiltonian systems, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, 8 (1991), 477-497.

[GP] N. GHOUSSOUB & D. PREISS, A general mountain pass principle for locating and classifying critical points, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, 6 (1989), 321-330.

[GJT] F. GIANNONI, L. JEANJEAN & K. TANAKA, Homoclinic orbits on non-compact Riemannian manifolds for second order Hamiltonian systems, Preprint S.N.S., (1993).

[GR] F. GIANNONI & P.H. RABINOWITZ, On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of Hamiltonian systems, *Nonlinear Diff. Eq. Appl.*, 1 (1994), 1-46.

[GM] M. GIRARDI & M. MATZEU, Existence and Multiplicity results for periodic solutions of superquadratic Hamiltonian systems where the potential changes sign, Preprint, (1993).

[GY] M. GIRARDI & D. YANHENG, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potential changing sign, *Dyn. Syst. and Appl.*, 2 (1993), 131-145.

[G] W.B. GORDON, Conservative dynamical systems involving strong forces, *Trans. Amer. Math. Soc.*, **204** (1975), 113-135.

[GH] J. GUCKENEIMER & P. HOLMES, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer–Verlag, 42, 1983.

[H] H.HOFER, A geometric description of the neighbourhood of a critical point given by the mountain-pass theorem, *J. London Math. Soc.*, **31** (1985), 566-570.

[HW] H. HOFER & K. WYSOCKI, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, *Math. Ann.*, **288** (1990), 483-503.

[J] L. JEANJEAN, Existence of connecting orbits in a potential well, *Dyn. Sys. Appl.*, 3 (1994), 537-562.

[KS] U. KIRCHGRABER & D. STOFFER, Chaotic behavior in simple dynamical systems, *SIAM Review*, **32** (1990), 424-452.

[La] L. LASSOUED, Periodic solution of a second order superquadratic system with change of sign of potential, *J. Diff. Eq.*, **93** (1991), 1-18.

[L] P.L. LIONS, The concentration–compactness principle in the calculus of variations, *Rev. Mat. Iberoamericana*, 1 (1985), 145-201.

[Me] V.K. MELNIKOV, On the stability of the center for periodic perturbations, *Trans. Moscow Math. Soc.*, **12** (1963), 1-57.

[Mi] C. MIRANDA, Un'osservazione su un teorema di Brouwer, *Boll. Unione Mat. Ital.*, 3 (1940), 5-7.

[M1] P. MONTECCHIARI, Existence and multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems, *Ann. Mat. Pura Appl. (IV)*, **168** (1995), 317-354.

[M2] P. MONTECCHIARI, Multiplicity results for a class of semilinear elliptic equations on $\mathbf{R}^m$, *Rend. Sem. Mat. Univ. Padova*, to appear.

[MN] P. MONTECCHIARI & M. NOLASCO, Multibump solutions for perturbations of periodic second order Hamiltonian systems, *Nonlinear Anal.*, to appear.

[Mo] J. MOSER, *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, 1973.

[Pa] K.J. PALMER, Exponential dichotomies and transversal homoclinic points, *J. Diff. Eq.*, **55** (1984), 225-256.

[P] H. POINCARÉ, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier–Villars, Paris, 1897-1899.

[PS] P. PUCCI & J. SERRIN, The structure of the critical set in the mountain pass theorem, *Tran. Am. Math. Soc.*, 299 (1987), 115-132.

[R] P.H. RABINOWITZ, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh*, **114 A** (1990), 33-38.

[R1] P.H. RABINOWITZ, Homoclinics for a singular Hamiltonian system, to appear in *Geometric Analysis and the Calculus of Variations*, (J. Jost ed.), (International Press).

[R2] P.H. RABINOWITZ, Periodic and heteroclinic orbits for a periodic Hamiltonian system, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **6** (1989), 331-346.

[R3] P.H. RABINOWITZ, A variational approach to heteroclinic orbits for a class of Hamiltonian systems, Frontiers in Pure and Applied Math. (R. Dautray, ed.), 267-278, (1991).

[R4] P.H. RABINOWITZ, Homoclinics for an almost periodically forced Hamiltonian system, *Top. Methods in Nonlinear Analysis*, to appear.

[R5] P.H. RABINOWITZ, Multibump solutions for an almost periodically forced singular Hamiltonian system, Preprint, (1995).

[R6] P.H. RABINOWITZ, Heteroclinics for a reversible Hamiltonian system, *Ergodic Theory and Dynamical systems*, **14** (1994), 817-829.

[R7] P.H. RABINOWITZ, Heteroclinics for a reversible Hamiltonian system, 2, *Differential and Integral Equations*, **7** (1994), 1557-1572.

[RT] P.H. RABINOWITZ & K. TANAKA, Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.*, **206** (1991), 473-479.

[S1] E. SÉRÉ, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.*, **209** (1992), 27-42.

[S2] E. SÉRÉ, Looking for the Bernoulli shift, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **10** (1993), 561-590.

[S3] E. SÉRÉ, Homoclinic orbits on compact hypersurfaces in $\mathbb{R}^{2N}$, of restricted contact type, Preprint, (1992).

[STT] E. SERRA, M. TARALLO & S. TERRACINI, On the existence of homoclinic solutions for almost periodic second order systems, Preprint, (1994).

[T] K. TANAKA, Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits, *J. Diff. Eq.*, **94** (1991), 315-339.

[T1] K. TANAKA, Homoclinic orbits for a singular second order Hamiltonian system, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **7** (1990), 427-438.

[T2] K. TANAKA, A note on the existence of multiple homoclinic orbits for a perturbed radial potential, *Nonlinear Diff. Eq. Appl.*, **1** (1994), 149-162.

[W] S. WIGGINS, *Global bifurcations and chaos*, Applied Mathematical Sciences, Springer–Verlag, 73, 1988.