ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Form Factors and Correlation Functions

Thesis submitted for the degree of
"Doctor Philosophiæ"

Candidate
Angelo Valleriani

Supervisor
Dr. Giuseppe Mussardo

October 1996

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

TRIESTE

Form Factors and Correlation Functions

Thesis submitted for the degree of
"Doctor Philosophiæ"

Candidate
Angelo Valleriani

Supervisor
Dr. Giuseppe Mussardo

Contents

List of Figures

Chapter 1

Introduction

There was a young lady named Bright
Whose speed was faster than light;
She set out one day
In a relative way
And returned home the previous night.
(A. Buller, *Punch*, 19th December 1923)

In this thesis we will describe some of the most recent results of $1+1$ dimensional Quantum Field Theory. The historical developments of this field can be essentially traced back to the pioneering work of Belavin, Polyakov and Zamolodchikov [1]. In this paper the authors have been able to provide the main tools for solving i.e. computing all the correlation functions, and classifying certain (called *minimal*) two–dimensional Euclidean field theories which are (at least) conformal invariant. The Conformal Field Theories (CFT) that in general are interacting theories from the particle physics point of view, are believed to describe [1, 2] all the possible 2nd order phase transitions in two dimensions. Many examples can be given such as the Ising model, the Tricritical Ising and Tricritical 3–states Potts models, the phase transition of the Lee–Yang singularity and many others that is impossible to list here. Moreover quite famous application have been provided in connection with the Kondo effect, the Fractional Quantum Hall effect and phase transitions on surfaces and of systems with impurities. We have also to mention the relevance of these results, and of their extension to the case with boundary (about which we will talk at the beginning of chapter 6), for the string theory. Their importance extends also outside particle physics and statistical mechanics. Indeed the algebraic description of the field contents of the CFT has stimulated a quite deep interchange of informations between

physics and mathematics in fields like for instance the infinite dimensional representations of Kač–Moody and Virasoro algebras, W algebras, modular groups and classification of modular invariant partition functions, graph theory, braid groups and quantum groups...

The possibility to completely solve the CFT is due to the fact that in two dimensions the conformal group is infinite dimensional. This means that the theories are characterized by an infinite dimensional internal symmetry. Indeed the whole field content is given by representations of two independent Virasoro algebras $V(c)\otimes\bar{V}(c)$ whose generators L and $\bar{L}$ satisfy the commutation relations

$$[L_n, L_m] = (m-n)\,L_{m+n} + \frac{c}{12}\,m(m^2-1)\delta_{m+n,0}\,,$$

where c is the central charge, a number which uniquely characterizes the CFT. The highest weight of each representation is a field which has the lowest L_0 eigenvalue

$$L_0\,\phi = \Delta_\phi\,\phi\,,$$

where Δ_ϕ and $\bar{\Delta}_\phi$ are the conformal dimensions of the so–called primary field o. The conformal dimensions encode the properties of transformation of the field under a conformal mapping and are thus related to the scaling dimension d and Lorentz spin s of any field as

$$d = \Delta + \bar{\Delta} \quad \text{and} \quad s = \Delta - \bar{\Delta}\,,$$

where for spinless operators, d enters into the two point connected correlation function as

$$\langle\phi(x)\phi(0)\rangle_c = |x|^{-2d}.$$

The exact knowledge of the conformal dimensions now leads easily to the exact computations of all the critical indices characterizing the critical systems. Moreover the fields satisfy an associative algebra under their Operator Product Expansion (OPE) [3]

$$[A]\,[B] = [C]$$

where $[A]$ is a symbolic notation to mean the field A and its descendents under the action of the operators L_{-n}. The *minimal models* are precisely those CFT in which the operator algebra closes with a finite number of conformal families i.e. there are only a finite number of primary fields. The minimality condition imposes further restrictions on the field contents of any Virasoro representation. Indeed it leads to the so–called null vectors i.e. certain descendents are actually linear combinations of others, which provide

differential equations through which it is possible at least in principle to compute *all the correlation functions*. In other words we can completely solve the theory.

Although they may have been considered as sort of toy models, the number and quality of the results both in the particle physics and in the statistical mechanics framework have put the CFT among the most advanced and fruitful results in theoretical physics. The same attitude should also be taken with one of its closest development namely the one concerning the Integrable Quantum Field Theory (IQFT). These kind of theories are obtained as suitable perturbation of the CFT: the perturbation should be chosen in order to still have an infinite number of conserved charges [4]. Actually, since most of the results will be obtained by means of non–perturbative methods, instead of perturbation we will often use the term deformation, deserving the word perturbation to those circumstances in which a perturbative development for small coupling constants is performed. This distinction, that may sound academic has instead its own rights and can be justified by looking at the procedures used to study these systems. Indeed, the perturbative approach has been very useful in understanding the existence of non–trivial conserved charges (and computing their Lorentz spins) with certain perturbations [4] and to see that any IQFT interpolates between two CFT [5]. However with the non–perturbative techniques, by exploiting the kinematic constraints coming from the infinite conserved currents it has been possible to describe also the non–trivial dynamics in a way which is independent of the strength of the coupling constant.

In this sense the term into the action responsible for breaking the conformal invariance is no more considered as small and it is no longer a perturbation. Thanks to the fact that the conserved charges have non–trivial spin, the scattering theory in the simplified two–dimensional dynamics takes a factorized form, namely the scattering among n particles may be described as the product of $\frac{n(n-1)}{2}$ independent two–particle scattering [6]. Another important consequence is moreover given by the stability of the particles i.e. the spectrum of the theory is stable.

Among the integrable theories considered here we will treat unitary as well as non–unitary cases. They are basically distinguished for the fact that in the unitary case the Hamiltonian is hermitian while in the non–unitary case it should satisfy the relation

$$H^\dagger = \hat{C}\, H\, \hat{C}\,,$$

where the operator $\hat{C}$ is diagonal on the n–particle states [7]. The main consequence is that "left" n–particle states $\langle n_l|$ are not the conjugated of the "right" states $|n_r\rangle$ but

satisfy

$$\langle n_l| = \langle n_r| c\,,$$

where c, the eigenvalue of $\hat{C}$, is a phase factor. Although non–unitarity does not affect the unitarity relation of the S–matrix, which simply states the completeness of IN and OUT states, it will result in wrong signs of the on–shell coupling constants. Moreover a particular care will be taken when dealing with Form Factors, as will be explained in the relative chapter.

Depending on the kind of deformation, the spectrum may be made of massive or massless excitations. While in this last case the theory usually interpolates between two non–trivial CFT, in the massive case, about which we will discuss until the end of the thesis, the IR fixed point is given by the trivial CFT with central charge $c = 0$. Thus a number of fully developed S–matrix theories have been obtained and by means of the bootstrap approach it has been possible to read off all the masses of the particles [8]. Thus now differently from the algebraic approach of the CFT, we will deal with sets of in–coming and out–going particles which are related by the S–matrix and which form what we will call a complete set of asymptotic particles states. Methods like for instance the Thermodynamic Bethe Ansatz (TBA), allow one to extract from the S–matrix informations concerning the ultraviolet CFT and the conformal dimension of the deforming field [9].

The Integrable Quantum Field Theories while being very interesting examples of completely solved massive or massless models, have found their best application to statistical mechanics. Indeed in that context these are interpreted as off–critical systems in the scaling region i.e. near criticality and the most interesting objects to compute are the correlation functions. By using the Form Factors (FF) technique [10, 11] many progresses have been made recently on this direction and many relevant problems have found an analytic solution as for instance the two point correlation function of the magnetization and energy operators of the Ising model in a magnetic field at $T = T_c$ [12] and those of the energy operators in the high temperature Tricritical Ising and Tricritical 3–states Potts models [13], just to mention a few examples.

The FF approach can be considered the most effective method up to now available to compute them. It is essentially based on the computation of the matrix elements of the fields on the asymptotic particle states. Through the solution of a set of monodromy and residue equations they are given as complex functions of the momenta of the particles. Easily integrated, they can give informations with any required accuracy. We will show,

for certain class of integrable theories without internal symmetries, how to obtain the FF of the scaling fields and to compute through the so–called sum rules what are the corresponding conformal dimensions and central charge in the ultraviolet limit.

As we shall see the sum–rules connect off–critical data to parameters of the critical CFT through integrals over all length scales of two–point connected correlation functions [14, 15]. With the insertion of a complete set of asymptotic states in between the fields, the integral turns then to an infinite sum of integrals of the FF. Thus, the knowledge of the exact value of e.g. the central charge provide a test to evaluate the truncation of the sum to the first few coefficients i.e. to the first form factors. All the tests made so far have led to the same picture of a very fast convergent behavior [12, 13, 16].

The form factors provide also the better tool for studying the case of non–integrable systems [17, 16]. Indeed, adding to the CFT action two independent deformations, both of them leading separately to integrability, gives instead a non–integrable system. Thus one may take the attitude of considering one of the terms as a deformation and the other as a small perturbation that lead to a soft breaking of integrability. Looking at the system from this frame of reference the unperturbed theory is now an integrable (massive in our case) one. Perturbative expansion allows us then to compute corrections to the S–matrix and to the masses and vacuum energy by using the FF of the unperturbed model. The comparison of the theoretical results with the numerical ones obtained by using the Truncated Conformal Space Approach (TCSA) [18] may then be used as a further way to identify the form factors of the perturbing field.

In this thesis we will not enter into a detailed discussion of the conformal field theories taking the attitude of explaining certain results only when needed. We will instead try to give a brief review of the main properties of the massive IQFT taking the most "pedagogical" way as possible. Indeed we will try to explain mainly through detailed examples how to find and utilize the form factors. In Chapter 2 we will look at the sum–rules [14, 15] and their derivation together with a brief description of the theory of factorized S–matrix [6] and the method of the TBA to extract informations from it [9]. In Chapter 3 we will then proceed entering into the analysis of the general equations for the determination of the Form Factors [10, 11]: we will study the analyticity requirements, the bound–state and kinematic recursive relations and the cluster hypothesis. In the same chapter we will show a few problems among the simplest ones that can help in seeing all the features of integrable systems and the way to find the FF. Chapter 4 is instead devoted to more involved systems such as the thermal Tricritical Ising and Potts models [13]. Moreover an analytic re-derivation of the Form Factors of the models $M_{2,2n+1} + \varphi_{13}$ in the light of

the recent results concerning the cluster limit is also given. Chapter 5 is dedicated to the two non–integrable perturbations [33] of the integrable model $M_{2,9}+\varphi_{13}$ [16] by using the results of the previous chapter for its Form Factors. Finally in Chapter 6 we will show how to extend the formalism of IQFT to the case with boundary [19, 20]. Through the example of the free massive boson we will find the boundary S–matrix and apply it to the case of the Random Walk with a boundary and attractive potential [21, 22].

Chapter 2

General Properties of Integrable Quantum Field Theories

In this chapter we will set up the general framework that will accompany us until the end of this thesis. We will explain here some general features of the integrable deformations of minimal models [1, 2]. The aim is that of providing all the basic informations that will be useful when dealing with the Form Factors. In particular we will explain how to obtain some *sum–rules*, that provide a connection between properties of the deformed theory and the underlying Conformal Field Theory (CFT). Moreover a section is dedicated to the properties of the S–matrix which completely characterizes the IQFT and one more section to the TBA. In this last we will show how to obtain the ultraviolet central charge and the vacuum bulk energy from the scattering amplitude.

2.1 Integrable Deformation

A deformation is given formally from the action density

$$\mathcal{A}_{qft} = \mathcal{A}_{cft} + g \int \mathrm{d}^2 x \, \varphi \,, \tag{2.1.1}$$

where $\mathcal{A}_{cft}$ is the action that describes the CFT we are starting from, and φ is the unspecified relevant field that breaks conformal invariance. In general we should allow also for more than one field in the rhs of (2.1.1), but we can keep general enough also with only one perturbing field. Concrete examples with two deforming fields will be given in the sections discussing the features of non–integrable deformation.

The field theory described by the action (2.1.1) is in general non–integrable. However if the CFT belongs to the minimal series it is always possible to choose a field in the Kač

table of the CFT that leave the theory integrable [4]. In particular if the field φ appearing in (2.1.1) is given by one of the following

$$\varphi_{12}\,, \quad \varphi_{13}\,, \quad \varphi_{21} \tag{2.1.2}$$

the theory possesses an infinite number of conserved charges with spins higher than one. The conserved charges P_s (and $\bar{P}_s = P_{-s}$, for parity invariant theories), where s denotes the Lorentz spin, are deformations of the charges in the CFT case. Indeed for $g = 0$, there are infinite conserved charges given by integrals like

$$P_s = \oint \frac{\mathrm{d}z}{2\pi i} T_{s+1}(z, \bar{z}) \tag{2.1.3}$$

where T_{s+1} are descendent of the unit operator $\varphi_{11} = 1$ at level $s + 1$. In particular for $s = 1$ we have the stress-energy tensor. Since the deformation breaks conformal invariance, the trace Θ of the stress–energy tensor and the traces over two indices of all the other remaining tensors Θ_{s-1} become different from zero. However, in order to have still a conserved charge of generic spin s, the trace Θ_{s-1} must satisfy the conservation equation (see Appendix A for the conventions used in deriving the conservation equation for the stress–energy tensor)

$$\bar{\partial}\, T_{s+1} + \frac{1}{4}\, \partial\, \Theta_{s-1} \;=\; 0\;. \tag{2.1.4}$$

While for the stress–energy tensor (2.1.4) always hold, for currents with higher spin it can be given only in the case of the minimal models perturbed by one of the fields in (2.1.2) in which case one can use the null–vector equations [4]. The form of $\Theta = T^{\mu}_{\mu}$ in terms of the fields in the CFT can be given by looking at the variation of the action under an infinitesimal scale transformation and reads [14]

$$\Theta \;=\; 4\pi\, g\,(1 - \Delta_{\varphi})\,\varphi\,, \tag{2.1.5}$$

where φ is the perturbing field and Δ_{φ} is its conformal dimension. As we shall see the relation (2.1.5) will play a crucial role in the following sections.

2.2 The sum rules

Here we will derive three sum rules that will allow us to connect off–critical data with properties at the conformal point. To understand them deeply it is worth recalling the whole philosophy behind. Indeed, as we shall mention briefly, several results like for

instance the Zamolodchikov *c–theorem* [5], show that an integrable deformation of a minimal model may be understood as a theory interpolating between two CFT. In particular if the integrable model possesses massless excitations then both the UV and the IR limits will be some non–trivial theory in the minimal series. Instead, in those cases in which the theory is massive the IR conformal field theory is the trivial one characterized by its central charge $c = 0$. Although in this work we will be interested only in the massive case, we will try to keep general enough as to include also the massless case. With this remark, it is clear the importance of having a tool that tells us what CFT we are interpolating. Actually, since the integrable theory is completely determined once the S–matrix and the spectrum are known, a tool that provides as well the information on the central charge (which encodes all the informations on the CFT) is also given by the Thermodynamic Bethe Ansatz (TBA) procedure. However, the sum rules involve integrals over all length scale of the two–point connected correlation functions of some fields. Thus they become an unavoidable tool for checking and finding the matrix elements over the Integrable Quantum Field Theory (IQFT) Hilbert space of the fields involved. For their relevance, it is worth describing them in detail.

The first sum–rule gives the value of the central charge of the CFT from the connected two point correlation function of the trace of the stress–energy tensor. It can be derived from the well known Zamolodchikov's *c–theorem* [5]. The starting point comes from the observation that the conservation equations (2.1.4) applied to the following correlation functions

$$\begin{aligned} \langle\, T(z,\bar{z})T(0,0)\,\rangle &= \frac{F(\tau)}{z^4} \\ \langle\, T(z,\bar{z})\Theta(0,0)\,\rangle &= \frac{G(\tau)}{z^3\bar{z}} \\ \langle\, \Theta(z,\bar{z})\Theta(0,0)\,\rangle &= \frac{H(\tau)}{z^2\bar{z}^2}\,, \end{aligned} \tag{2.2.6}$$

where $\tau = \log(m^2\, z\bar{z})$ and m is a mass scale, lead to the differential equation

$$\dot{C} = -\frac{3}{4}H(\tau)\,, \tag{2.2.7}$$

where $C(\tau) = 2F - G - 3H/8$ and the differentiation is along τ. Actually, since we are deforming the theory according to eq. (2.1.1) the function C depends also on the coupling constant g

$$C = C(\tau, g)\,, \tag{2.2.8}$$

and reduces to the central charge c of the ultraviolet CFT when $g = 0$. The above mentioned theorem proceeds then showing that eq. (2.2.7) can be re-interpreted by using the Callan–Symanzik equation as the variation of the function C along the renormalization flow. In particular, for unitary theories the function H is always non–negative and this implies that along the renormalization flow the central charge can only decrease. Equation (2.2.7) is however also the starting point for our first sum rule obtained by Cardy in ref. [14]. Indeed at fixed value of the coupling constant, eq. (2.2.7) can be integrated over all length scales. From this point of view, we can think at the function C as the one that reaches the values of the UV and IR central charges as far as $R \to 0$ or $R \to \infty$ respectively. Taking into account now that

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = \frac{R}{2}\frac{\mathrm{d}C}{\mathrm{d}R}, \tag{2.2.9}$$

one gets the following equation

$$c_{uv} - c_{ir} = \frac{3}{4\pi}\int \mathrm{d}^2x\, x^2 \langle \Theta(|x|)\Theta(0)\rangle_c\,, \tag{2.2.10}$$

where $|x|$ is the direction of the Euclidean time.

The second sum–rule we are interested in provides the bulk vacuum energy $\mathcal{E}_0$ that can also be obtained by means of the TBA technique, as we shall see in section (2.4). This sum–rule comes from the fact that the singular part of the free energy per correlation volume, $f_s\xi^2$ goes to a universal constant as far as $g \to 0$ [14, 23]. It is useful to rewrite the behavior of the free energy as follows

$$f_s \sim -U\, m_1^2(g)\,, \tag{2.2.11}$$

where m_1 is the lowest massive excitation of the theory whose value depends on the coupling constant. By simple dimensional arguments it is possible to see that

$$g \propto m_1^y\,, \tag{2.2.12}$$

where $y = 2 - 2\Delta_\varphi$ and Δ_φ is the conformal dimension of the perturbing field. The proportionality constant in (2.2.12) can be determined in many different ways [24, 9], the exact value of which will interest us only later. Consider now the singular part of the specific heat (or susceptibility) per unit volume. Since it can be obtained from the second derivative with respect to g of the free energy and as zero moment of the connected correlation function of the perturbing field φ, we have the following identity

$$\frac{\partial^2 f_s}{\partial g^2} = -\int \mathrm{d}^2x \langle \varphi(x)\varphi(0)\rangle_c\,. \tag{2.2.13}$$

By using eqs. (2.1.5), (2.2.11) and (2.2.12) we finally get

$$U = \frac{1}{16\Delta_\varphi}\frac{1}{\pi^2 m_1^2}\int \mathrm{d}^2x\,\langle\Theta(|x|)\Theta(0)\rangle_c\,. \tag{2.2.14}$$

Notice that although the appearance of a mass into eq. (2.2.14), both sides are dimensionless. This can be proved by taking the massless limit $g \to 0$ and observing that all the dimensionful constants cancel.

The third and newest sum rule concerns instead the conformal dimensions of the primary fields in the CFT and it has been derived in [15]. Although its derivation for the perturbing field can be given by using some algebra on the previous sum–rule (we will present this derivation at the end of section (2.4)), it is worth reproducing here the whole argument presented in the above cited paper. Consider indeed the following two–point connected correlation functions for a generic primary field ϕ

$$\begin{aligned}\langle T(z,\bar z)\phi(0,0)\rangle_c &= \frac{M(\tau)}{z^2}\\ \langle\Theta(z,\bar z)\phi(0,0)\rangle_c &= \frac{N(\tau)}{z\bar z}\,,\end{aligned} \tag{2.2.15}$$

where $\tau = \log(m^2 z\bar z)$, and m is a mass scale. Plugging them into the conservation equation (2.1.4), one gets the differential equation

$$\dot D = \frac{N(\tau)}{4}\,, \tag{2.2.16}$$

where $D = M + N/4$. For a fixed value of the coupling constant g, one can look at the conformal limits of eq. (2.2.15). This limit heavily depends on the form of the OPE between the perturbing field and Φ. Indeed in the perturbative framework around the conformal fixed point we require that the correlation functions between ϕ and any other combination of fields be finite at any order in the coupling constant g. Thus if the OPE

$$\varphi(x)\phi(0) = \sum_k C^k_{\varphi\phi}\,|x|^{2(\Delta_k-\Delta_\varphi-\Delta_\phi)}\,A_k \tag{2.2.17}$$

leads to ultraviolet divergences this means that the definition of ϕ near the conformal point needs counterterms to cure them. If this does not happen i.e. if the field ϕ does not mix under renormalization, then we have the following UV limits

$$\begin{cases} M \to \Delta_\phi\langle\phi\rangle \\ N \to 0 \end{cases} \quad \text{as } R \to 0\,, \tag{2.2.18}$$

where $\langle\varphi\rangle$ is the vacuum expectation value of the field ϕ on the Hilbert space of the integrable theory. Its value, that will interest us also later, is different from zero if there are no internal symmetries that constraint it to vanish and for dimensional reasons it is expected to behave as

$$\langle\phi\rangle = v_\phi\, m_1^{2\Delta_\phi}. \tag{2.2.19}$$

Of course, a similar result should hold also in the IR limit $R \to \infty$. Thus, provided that the Operator Product Expansion (OPE) between φ and ϕ behave smoothly as $R \to 0$. we can integrate over all length scales eq. (2.2.16) finally giving [15]

$$\Delta_\phi^{uv} - \Delta_\phi^{ir} = -\frac{1}{4\pi\langle\phi\rangle}\int \mathrm{d}^2x\,\langle\Theta(|x|)\phi(0)\rangle_c\,. \tag{2.2.20}$$

However if the OPE (2.2.17) leads to divergences the second of eqs. (2.2.18) does not hold and as a consequence also the first is incorrect. In this case then also eq. (2.2.20) cannot be used. As mentioned at the beginning of this section, the sum rules provide checks for the exactness of the matrix elements of the operators involved. However, this last one (2.2.20) gives something more. Indeed once a general solution for the matrix elements of a generic field is known, it is not always easy to identify what is the corresponding field in the Kač table (only for the perturbing field the task is trivial, thanks to eq. (2.1.5)). Instead, eq. (2.2.20) allows us to uniquely identify the ultraviolet and infrared fields to the matrix elements in the IQFT case. The nature and properties of these matrix elements will be described in the chapters relative to the Form Factors.

To end this section let us make some more comments on the vacuum expectation values (2.2.19). Indeed consider two fields e.g. ϕ_1 and ϕ_2, which are proportional (here, as will be stressed often in this thesis, for proportionality we mean in the sense of their matrix elements on a given Hilbert space)

$$\phi_1 = A\,\phi_2\,,$$

where A is the proportionality constant. We would like to determine A. This can be easily done by taking the vacuum expectation value of both sides thus giving

$$A = \frac{\langle\phi_1\rangle}{\langle\phi_2\rangle}. \tag{2.2.21}$$

Let us apply this reasoning to our case in considering the relation (2.1.5). Indeed according to (2.2.21) we have also

$$\Theta = \frac{\langle\Theta\rangle}{\langle\varphi\rangle}\varphi\,, \tag{2.2.22}$$

thus leading to the identity

$$4\pi\, g\,(1-\Delta_\varphi) \;=\; \frac{\langle\Theta\rangle}{\langle\varphi\rangle}\,. \tag{2.2.23}$$

This relation will allows to determine the vacuum expectation value (2.2.19) of the perturbing field φ once $\langle\Theta\rangle$ is known, as we shall see in section (2.4).

2.3 Factorized S–matrices

One of the main consequences of integrability is the existence of factorized S–matrices [6]. The factorization relies on the possibility to express multi-particle scattering as products of two–particle ones. If the 2–momenta is parameterized according to

$$(E,p) \;=\; (m\cosh\theta,\, m\sinh\theta)\,, \tag{2.3.24}$$

the two–particle S–matrix is defined as

$$|A_a^\dagger(\theta_a)\,A_b^\dagger(\theta_b)\rangle_{out} \;=\; S_{ab}^{cd}(\theta_{ab})\,|A_d^\dagger(\theta_b)\,A_c^\dagger(\theta_a)\rangle_{in}\,. \tag{2.3.25}$$

However it is more convenient to think at the S–matrix to as a braiding operator between creation and annihilation operators. This formalism is known under the name of Faddeev–Zamolodchikov algebra. Let $A^\dagger_{\alpha_1}(\theta_1)$ be the operator that creates a particle with rapidity θ_1 and internal degrees of freedom labeled by α_1. If $A^{\alpha_1}(\theta_1)$ is the corresponding annihilation operator, then they are supposed to satisfy the following algebra

$$\begin{aligned}
A^{\alpha_1}(\theta_1)\,A^{\alpha_2}(\theta_2) \;&=\; S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\theta_{12})\,A^{\beta_2}(\theta_2)\,A^{\beta_1}(\theta_1)\\
A^\dagger_{\alpha_1}(\theta_1)\,A^\dagger_{\alpha_2}(\theta_2) \;&=\; S^{\beta_1\beta_2}_{\alpha_1\alpha_2}(\theta_{12})\,A^\dagger_{\beta_2}(\theta_2)\,A^\dagger_{\beta_1}(\theta_1)\\
A^{\alpha_1}(\theta_1)\,A^\dagger_{\alpha_2}(\theta_2) \;&=\; S^{\beta_2\alpha_1}_{\alpha_2\beta_1}(\theta_{21})\,A^\dagger_{\beta_2}(\theta_2)\,A^{\beta_1}(\theta_1) + \delta^{\alpha_1}_{\alpha_2}\delta(\theta_{12})\,,
\end{aligned} \tag{2.3.26}$$

where for brevity we have written $\theta_{ab}=\theta_a-\theta_b$.

The validity of this algebra goes beyond the case of the massive theory. Indeed the S–matrix can be considered from the abstract point of view as that operator that governs the algebra of creation and annihilation operators also in the massless case.

The factorization equation we are going to impose on our S–matrix, can be pictorially drawn in a graph in Figure (2.1).

It gives rise to the following equation [6. 25]

$$S^{\beta_1\beta_2}_{\alpha_1\alpha_2}(\theta_{12})S^{\gamma_1\beta_3}_{\beta_1\alpha_3}(\theta_{13})S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(\theta_{23}) \;=\; S^{\beta_2\beta_3}_{\alpha_2\alpha_3}(\theta_{23})S^{\beta_1\gamma_3}_{\alpha_1\beta_3}(\theta_{13})S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(\theta_{12})\,, \tag{2.3.27}$$

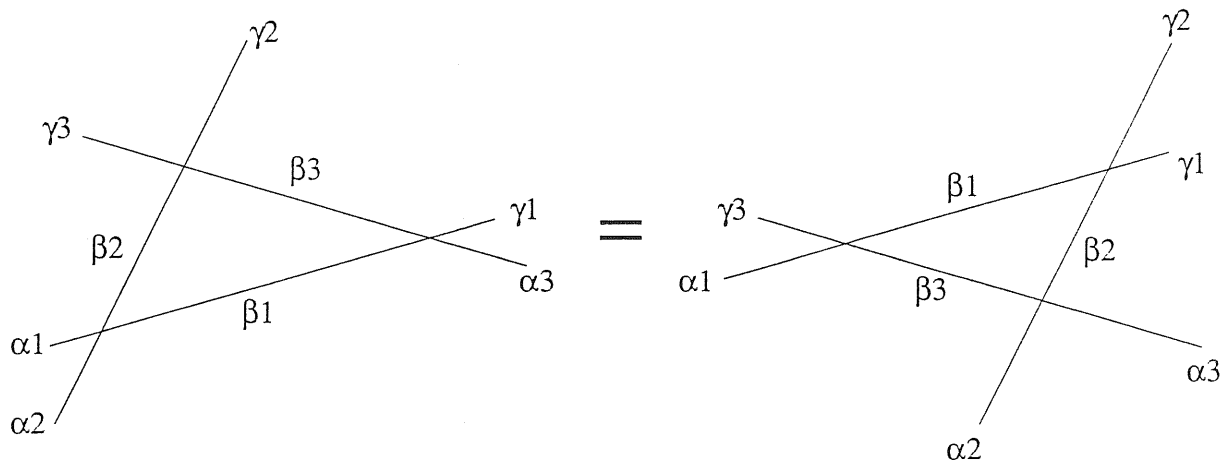

Figure 2.1: Diagrammatic representation of the Yang–Baxter equation for a scattering process.

which is known as the Yang–Baxter equation. The general solution of this equation is unknown, but for those theories we will discuss here things simplifies. Indeed, we will consider only theories with massive diagonal scattering, in which case the particles can be completely distinguished. However also in these cases there are general analyticity requirements that must be satisfied. Since in the following we will discuss only this kind of theories, all the equations will be specified only for this case. While the Yang–Baxter equation is trivially satisfied, the two main monodromy equations left are given by the requirements of crossing symmetry and unitarity expressed respectively by

$$\begin{aligned} S_{a\bar{b}}(\theta) &= S_{\bar{a}b}(\theta) = S_{ab}(i\pi - \theta) \\ S_{ab}(\theta)S_{ab}(-\theta) &= 1\,. \end{aligned} \tag{2.3.28}$$

These two equations constraint the S–matrix to be a $2\pi i$ periodic phase

$$S_{ab}(\theta) = \mathrm{e}^{i\delta_{ab}(\theta)}\,, \tag{2.3.29}$$

where $\delta_{ab}(\theta) = -\delta_{ab}(-\theta)$. The general solution of (2.3.28) can be given in terms of product of functions

$$s_\alpha(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\pi\alpha)}{\sinh\frac{1}{2}(\theta - i\pi\alpha)}\,, \tag{2.3.30}$$

or, in those cases in which the particles are all self–conjugated, in terms of

$$f_\alpha(\theta) = \frac{\tanh\frac{1}{2}(\theta + i\pi\alpha)}{\tanh\frac{1}{2}(\theta - i\pi\alpha)}\,. \tag{2.3.31}$$

The function s_α shows a simple pole at $\theta = i\pi\alpha$. Corresponding to this pole we may have a bound state in $S_{ab}(\theta)$ whose mass is given by

$$m_{\bar{c}}^2 = m_a^2 + m_b^2 + 2m_a m_b \cos\pi\alpha\,. \tag{2.3.32}$$

in the s channel. In this case we should call $i\pi\alpha = iu^c_{ab}$. For what concerns the functions f_α, they instead show a simple pole in $i\pi\alpha$ and at the crossed position $i\pi(1-\alpha)$.

The general form of the two–particle S–matrices is then given by

$$S_{ab} = \prod_{\alpha\in A_{ab}} f_\alpha^{p_\alpha}(\theta)\,, \tag{2.3.33}$$

or

$$S_{ab} = \prod_{\alpha\in A_{ab}} s_\alpha^{p_\alpha}(\theta)\,, \tag{2.3.34}$$

depending on whether the spectrum of the theory is made only of self–conjugated particles or not. In the last equations we have denoted as A_{ab} the set of rational numbers that label the position of the poles in the scattering amplitude, while p_α denotes the multiplicity of the corresponding pole. The sets A_{ab} moreover satisfy the following relation that will be useful in section (2.4)

$$\frac{\sum_{\alpha\in A_{ab}} \sin\pi\alpha}{\sum_{\alpha\in A_{ac}} \sin\pi\alpha} = \frac{m_b}{m_c}\,, \tag{2.3.35}$$

thus leading to the equation

$$\sum_{\alpha\in A_{ab}} \sin\pi\alpha = \hat{m}_a\hat{m}_b \sum_{\alpha\in A_{11}} \sin\pi\alpha\,, \tag{2.3.36}$$

where $\hat{m}_a = m_a/m_1$. The S–matrices depend on the Mandelstam variable

$$s = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta$$

and possesses two cuts, such that the physical strip is given by

$$(m_a - m_b)^2 < s < (m_a + m_b)^2\,.$$

In the rapidity parameterization the physical strip is realized for

$$0 < \theta < i\pi\,. \tag{2.3.37}$$

into which we may find the poles. When a pole corresponds to a bound state it gives rise to an equation that provides the mass of the bound state, given by eq. (2.3.32) and depicted in Figure (2.2). In the case of a simple pole that gives a bound state, we can compute the on–shell three–particle coupling

$$(\Gamma^c_{ab})^2 = -i \lim_{\theta\to iu^c_{ab}} (\theta - iu^c_{ab})\, S_{ab}(\theta)\,. \tag{2.3.38}$$

The important point to mention here is that a sort of 'nuclear democracy' applies. This means, we promote all the bound states to be also asymptotic or fundamental particles.

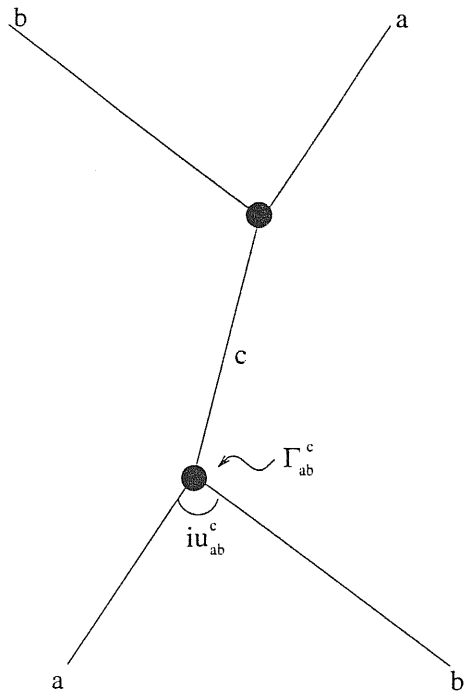

Figure 2.2: Diagrammatic interpretation of the process responsible for a single–pole in the S–matrix.

Thus the particle c that appears as bound state of a and b will also appear as one of the fundamental particles.

The construction of the whole set of two–particle amplitudes relies on a bootstrap procedure. By starting with only one amplitude, let say $S_{11}(\theta)$, one can in principle proceed in deriving the whole set of two–particle S–matrices. As a consequence one obtains also the whole spectra of the theory. One has to say however that this procedure is not always trivial, indeed there is at least one intriguing example, the model $M_{3,5}+\varphi_{1,2}$, in which the amplitude S_{11} is known but it has not yet been possible to derive the other matrix elements (for a complete account of results as well as properties of the functions (2.3.30) and (2.3.31) and a description of the bootstrap procedure see the review article [8]).

After, and during, the derivation of the complete set of consistent two–particle amplitudes, one faces the problem of interpreting the higher order poles appearing in S_{ab}. Indeed, as we shall see in concrete examples, there may be a quite rich zoology of poles and zeroes.

There is a basic principle that must be used in understanding what these poles really mean. In integrable theories which are deformations of minimal models their spectra is completely known and is given by the asymptotic particles. The general pattern, obeyed

by unitary theories with self–conjugated particles, is that as far as the multiplicity of the pole is odd, the pole corresponds to a single particle channel process (see Figure (2.3) for a possible third order pole) while even–order poles do not show such single particle channel

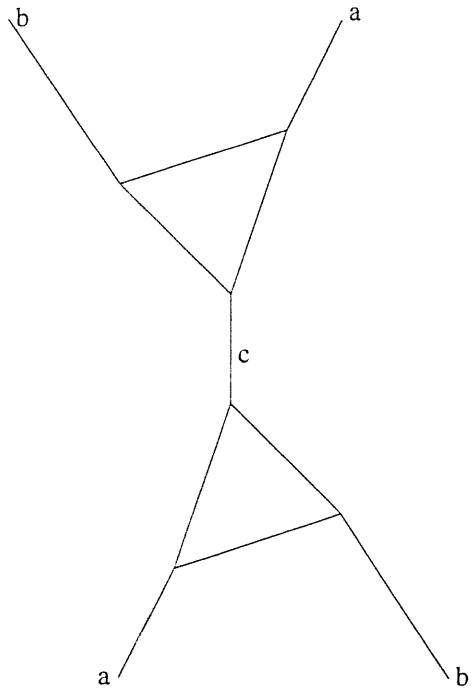

Figure 2.3: Diagrammatic interpretation of the process responsible for a third–order–pole in the S–matrix.

(Figure (2.4)). If the unitary theory has particles which are not all self–conjugated, as for instance the model E_6, then some poles may occur that correspond to poles in the t–channel. We shall see this feature more in detail in the section in which we will discuss this model. A little bit more complicated may be the situation in the non–unitary case. Let us consider only theories with all self–conjugated particles. If the two–particle S–matrices show zeroes, there may be simple (or odd) order poles that do not correspond to any single particle channel. This is due to the fact that these poles are actually even order poles, like the one in Figure (2.4), where in the internal lines there is a zero because $S_{ce}(i\gamma) = 0$. Thus, in order to show that a simple pole does not correspond to a bound state one has to find a graph in which these poles appear as being e.g. of second order but coupled to a zero. Graphs and poles of this kind have been discussed for instance in [16].

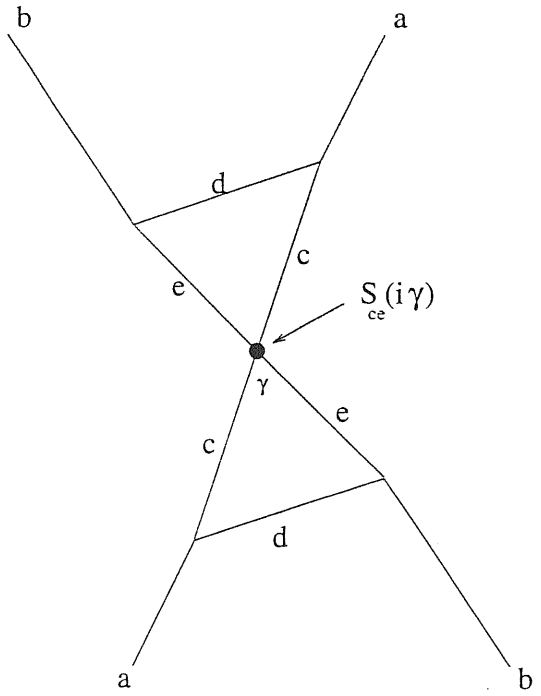

Figure 2.4: Diagrammatic interpretation of the process responsible for a double–pole in the S–matrix.

2.4 Thermodynamic Bethe Ansatz

To give support to the conjectured S–matrices discussed in the previous section we need a method to extract informations on the UV and IR limits from it. The method is provided by the *Thermodynamic Bethe Ansatz* (TBA), introduced in the relativistic quantum field theory in ref. [9], suitably modifying the non–relativistic version presented for the first time in [26]. This method can also provide non–perturbative informations on the ground state of the theory along the whole renormalization flow, as we shall see[1]. Let us consider the system as placed in a periodic cylinder of circumferences R and L. Since in Euclidean field theory both directions can be considered as Euclidean time, the system may be described by both the following partition functions

$$Z(R,L) = \mathrm{Tr}\, \mathrm{e}^{-L\mathcal{H}_R} \tag{2.4.39}$$

or

$$Z(R,L) = \mathrm{Tr}\, \mathrm{e}^{-R\mathcal{H}_L} \tag{2.4.40}$$

[1]For more recent developments on the analytic study of the excited states and connections with quantum group symmetries see [27, 28].

where $\mathcal{H}_R$ and $\mathcal{H}_L$ are the Hamiltonian for the system quantized along the axis R and L respectively (see Appendix A for a derivation of (2.4.39) and (2.4.40)).

In the limit $L \to \infty$ eq. (2.4.39) is dominated by the ground state $E_0(R)$ of $\mathcal{H}_R$ and reads

$$Z(R, L) \sim \mathrm{e}^{-LE_0(R)}. \tag{2.4.41}$$

The same limit in (2.4.40) has the meaning of a thermodynamic limit on a one dimensional quantum system defined to live on the circumference L, at temperature $1/R$. The partition function (2.4.40) can then be rewritten as

$$Z(R, L) \sim \mathrm{e}^{-LRf(R)} \tag{2.4.42}$$

where $f(R)$ is the free energy per unit volume of the system at temperature $1/R$. The above mentioned symmetry under the exchange of the axes (or modular invariance) can be fixed comparing eqs. (2.4.41) and (2.4.42) into the following relation

$$E_0(R) = Rf(R). \tag{2.4.43}$$

Denoting by m_1 the lowest mass of the theory, one defines the a-dimensional variable $r = m_1 R$ in terms of which one can parameterize the ground state energy according to

$$E_0(R) = -\frac{\pi \tilde{c}(r)}{6R}. \tag{2.4.44}$$

where $\tilde{c}$ is called the finite size scaling function. Since $1/m_1$ is the natural correlation length of the theory, the limit $r \to 0$ corresponds to the UV limit. This means that $\tilde{c}(0)$ is nothing but the finite size scaling function of the corresponding CFT. However in this limit the finite size ground state energy takes the value

$$E_0(R) = \frac{2\pi}{R}(\Delta_{\min} + \bar{\Delta}_{\min} - \frac{c}{12}). \tag{2.4.45}$$

This implies that $\tilde{c}(r)$ results to satisfy

$$\tilde{c}(0) = c - 24\Delta_{\min}, \tag{2.4.46}$$

where $\Delta_{\min}$ is the conformal dimension of the most relevant field of the theory (i.e. that field with the lowest conformal dimensions). Notice that if the theory is unitary the most relevant field is the unit operator, which has conformal dimensions zero. Thus in this case we get exactly

$$\tilde{c}(0) = c \tag{2.4.47}$$

where c is the central charge and $\tilde{c}$ is called the effective central charge. The task we have in mind now is the following. We want to find a method to compute the central charge of a certain integrable field theory given the S–matrix two–particle elements. In order to obtain this information we have to specify the variables and the physical quantities we are dealing with. In particular we have to define what is the relativistic Bethe wave function.

Let us suppose that the theory possesses n different particles with masses given by m_a for $a = 1, \cdots, n$. Moreover let us suppose that the scattering is purely elastic and the S–matrix is of the kind described in the previous section. We define an IN–coming asymptotic state as the one in which the distances between the particles are much bigger than the correlation length $|x_i - x_j| \gg \xi$. In configuration space there are $n!$ such regions into which the interactions are absent. The interaction process, when a particle reach the others is mediated by the S–matrix. After all the particles have interacted the state belongs to some other of the $n!$ regions. While in one of these non–interacting domains, the particle positions can be given as $x_{i_1} \ll x_{i_2} \ll \cdots \ll x_{i_n}$, and the system may be described by the plane wave function

$$\Psi(x_1, \ldots, x_n) = \prod_{i=1}^{n} e^{ip_i x_i} \,. \tag{2.4.48}$$

In order to obtain now the quantization conditions for the momentum, we have to take into account that at every diffusion process the wave function must be multiplied by the corresponding diffusion matrix according to the Fateev–Zamolodchikov algebra (2.3.26). By imposing periodic boundary conditions on the wave functions we get the quantization condition

$$e^{ip_i L} \prod_{j \neq i} S(\theta_i - \theta_j) \;=\; 1 \quad i = 1, \ldots, n \,, \tag{2.4.49}$$

that can be set into the following form by taking the logarithm of both sides

$$m_i L \sinh \theta_i + \sum_{j \neq i}^{n} \delta_{ij}(\theta_i - \theta_j) \;=\; 2\pi q_i \tag{2.4.50}$$

where we have used the identity (2.3.24). Once the set of all the q_i has been fixed we can solve eq. (2.4.50) with respect to the rapidities, thus obtaining the Bethe Ansatz states

$$|q_1, \theta_1; q_2, \theta_2; \ldots; q_n, \theta_n\rangle \,, \tag{2.4.51}$$

and giving the total energy and momentum the following values

$$E = \sum_{i=1}^{n} m_i \cosh \theta_i \quad p = \sum_{i=1}^{n} m_i \sinh \theta_i \,. \tag{2.4.52}$$

Moreover, since we will only consider theories in which the particles are bosons and their statistic is of fermionic type $S_{aa}(0) = -1$. the numbers q_i in the rhs of (2.4.50) will be only integers. In particular since the lhs of eq. (2.4.50) increases with θ_i, the values of q_i for a given kind of particles must increase strictly if the particles are bosons. In other words, two identical bosons cannot have the same rapidity.

Now, we would like to take the thermodynamic limit. This means that we let the size of the system $L \to \infty$ while keeping the density of particles fixed. Thus the total number N of particles in the IN or OUT states must also go to infinity. In this limit, we are interested in seeing what distribution of quantum numbers in (2.4.51) dominates the system.

Let now be n^a the number of particles of type a. These numbers of course satisfy $N = \sum_a n^a$. Moreover let q_i^a the subset of all the q_i's, that correspond to the n^a particles. On the light of the above mentioned exclusion principle, also the q_i^a must increase strictly with i. One can rephrase this conclusion by saying that not all the possible sets of q_i^a correspond to physical states. Once a physical set of q_i^a has been fixed, the solutions θ_i^a will be called *roots*. Solutions corresponding instead to some unphysical set will be called *holes*.

In the continuum (thermodynamic) limit, the number of roots and holes in the interval $[\theta, \theta + \delta\theta]$ will be denoted respectively by $\rho_r^a(\theta)$ and $\rho_h^a(\theta)$. In this limit then we have the TBA equation

$$m_a \cosh\theta + \sum_{b=1}^{n} (\phi_{ab} * \rho_r^b)(\theta) = 2\pi \rho^a(\theta) \tag{2.4.53}$$

where $\rho^a = \rho_r^a + \rho_h^a$. In eq. (2.4.53) we have also used the convolution

$$(\phi_{ab} * \rho_r^b)(\theta) = \int_{-\infty}^{+\infty} d\theta' \phi_{ab}(\theta - \theta') \rho_r^b(\theta') \,, \tag{2.4.54}$$

where

$$\phi_{ab}(\theta) = \frac{\partial \delta_{ab}(\theta)}{\partial \theta} \,, \tag{2.4.55}$$

and δ_{ab} has been defined in (2.3.29). The total energy of the system can be rewritten as

$$E = L \sum_{i=a}^{n} \int_{-\infty}^{+\infty} \mathrm{d}\theta \, m_a \rho_r^a(\theta) \cosh\theta \tag{2.4.56}$$

and the entropy of the system is given instead by taking into account that we are dealing with particles obeying a fermionic statistic

$$S[\rho, \rho_r] = L \sum_{a=1}^{n} \int_{-\infty}^{+\infty} \mathrm{d}\theta \, [\rho^a \log \rho^a - \rho_r^a \log \rho_r^a - (\rho^a - \rho_r^a) \log(\rho^a - \rho_r^a)] \,. \tag{2.4.57}$$

Finally, we can come to the equilibrium distributions. We have to find the distributions ρ_r and ρ_h that minimize the free energy

$$Lf[\rho,\rho_r] = E[\rho] - \frac{1}{R}S[\rho,\rho_r] \tag{2.4.58}$$

of the system at finite temperature $T = \frac{1}{R}$. By using now (2.4.56) and (2.4.57) together with the condition (2.4.53), one can derive (2.4.58) with respect to ρ_r thus obtaining

$$m_a R\cosh\theta = \epsilon_a(\theta) + \frac{1}{2\pi}\sum_{b=1}^{n}(\phi_{ab} * L_b)(\theta)\,. \tag{2.4.59}$$

In eq. (2.4.59) we have used the fact that $\phi_{ab} = \phi_{ba}$ and that unitarity of the S-matrix implies $\phi_{ab}(-\theta) = \phi_{ab}(\theta)$. Moreover we have also used the following definitions

$$\begin{aligned}\frac{\rho^a(\theta) - \rho_r^a(\theta)}{\rho_r(\theta)} &= \mathrm{e}^{\epsilon_a(\theta)} \\ L_a(\theta) &= \log\left(1+\mathrm{e}^{-\epsilon_a(\theta)}\right),\end{aligned} \tag{2.4.60}$$

where ϵ_a is called the pseudo–energy. Once the minimum condition has been computed. it is easy to get the free–energy (2.4.61) at the thermodynamic equilibrium

$$f(R) = -\frac{1}{2\pi R}\sum_{a=1}^{n}\int_{-\infty}^{+\infty}\mathrm{d}\theta\, m_a\, L_a(\theta)\cosh\theta \tag{2.4.61}$$

and, thanks to eqs. (2.4.43) and (2.4.44), the corresponding value for the effective central charge

$$\tilde{c}(r) = \frac{3}{\pi^2}\sum_{a=1}^{n}\hat{m}_a r\int_{-\infty}^{+\infty}\mathrm{d}\theta\, L_a(\theta)\cosh\theta\,, \tag{2.4.62}$$

where $\hat{m}_a = m_a/m_1$.

The next step will concern the conformal limit $r \to 0$ of (2.4.62). Observe that for $r \to 0$ the derivative of the pseudo–energies $\epsilon_a(\theta)$ goes to zero as far as $\theta \to 0$. This means that $\epsilon_a(\theta)$ are nearly constant in the following interval

$$-\log(2/r) \ll \theta \ll \log(2/r)\,. \tag{2.4.63}$$

It is then easy to determine the value of these constants ϵ_a by using eq. (2.4.59). Indeed for $r \to 0$ and θ satisfying (2.4.63) we have the transcendental equation

$$\epsilon_a = \sum_{a=1}^{n} N_{ab}\,\log(1+\epsilon^{-\epsilon_b})\,, \tag{2.4.64}$$

where

$$N_{ab} = -\int_{-\infty}^{+\infty}\phi_{ab}(\theta)\mathrm{d}\theta\,. \tag{2.4.65}$$

We have now to perform the UV limit by letting $\theta \sim \log \frac{2}{r}$. This limit transforms eq. (2.4.59) into an equation that provides the so–called kink solution

$$\hat{m}_a e^{\theta} = \hat{\epsilon}_a(\theta) + \frac{1}{2\pi} \sum_{b=1}^{n} \left(\phi_{ab} * \hat{L}_b\right)(\theta) \tag{2.4.66}$$

where the functions $\hat{\epsilon}_a(\theta)$ and $\hat{L}_b(\theta)$, are called *kink* functions [9]. These functions take constant values according to eq. (2.4.64) for $\theta \to -\infty$, while for $\theta \to \infty$ it shows that $\hat{\epsilon}_a(\theta)$ grows exponentially. Finally one can see what happens to the effective central charge. Indeed eq. (2.4.62) get transformed into

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \hat{m}_a \int_{-\infty}^{+\infty} \hat{L}_a(\theta) e^{\theta} \mathrm{d}\theta \tag{2.4.67}$$

that can be solved by using (2.4.66) [9], giving

$$\tilde{c}(0) = \sum_{a=1}^{n} \frac{6}{\pi^2} \mathcal{L}\left(\frac{1}{1+e^{\epsilon_a}}\right) \tag{2.4.68}$$

where $\mathcal{L}(x)$ is the Rogers dilogarithm defined as

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dt \left[\frac{\log t}{1-t} + \frac{\log(1-t)}{t}\right]. \tag{2.4.69}$$

To summarize, the computation of the UV effective central charge reduces to solve the transcendental equation (2.4.64) and put this solution into eq. (2.4.68).

Among the other relevant quantities that can be computed by means of the TBA, there are the vacuum expectation value of the trace of the stress–energy tensor and the universal amplitude $\mathcal{E}_0$. Let us begin with the first one, since the second can be obtained from it. As shown in Appendix A, the trace satisfies the following equation

$$\langle \Theta \rangle = 2\pi \frac{\mathrm{d}(R E_0(R))}{R\,\mathrm{d}R}, \tag{2.4.70}$$

where $E_0(R)$ is given by eqs. (2.4.43) and (2.4.61) and reads

$$E_0(R) = -\frac{m_a}{\pi} \int_0^{\infty} \mathrm{d}\theta \, \cosh\theta \, L_a(\theta). \tag{2.4.71}$$

Let us define the functions

$$\begin{aligned} \frac{1}{m_1 R} \frac{\partial \epsilon_a}{\partial \theta} &= \frac{1}{2}\left(\psi_a^+ - \psi_a^-\right) \\ \frac{1}{m_1} \frac{\partial \epsilon_a}{\partial R} &= \frac{1}{2}\left(\psi_a^+ + \psi_a^-\right) \end{aligned} \tag{2.4.72}$$

with $\psi_a^+(\theta) = \psi_a^-(-\theta)$ and satisfying

$$\psi_a^\pm(\theta) \;=\; \hat{m}_a \mathrm{e}^{\pm\theta} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\beta\, \phi_{ab}(\theta-\beta)\frac{\mathrm{e}^{-\epsilon_b(\beta)}}{1+\mathrm{e}^{-\epsilon_b(\beta)}}\psi_b^\pm(\beta)\,. \tag{2.4.73}$$

With these functions eq. (2.4.70) can be rewritten as

$$\langle\Theta\rangle \;=\; m_1 \sum_a m_a \int_{-\infty}^{+\infty} \mathrm{d}\theta \frac{\mathrm{e}^{-\epsilon_a(\theta)}}{1+\mathrm{e}^{-\epsilon_a(\theta)}}\psi_a^+(\theta)\mathrm{e}^{-\theta}\,, \tag{2.4.74}$$

where now we have to evaluate the rhs in the UV limit $r \to 0$. The key observation is that in this limit the function ψ_a^+ satisfies the same equation as the derivative of the kink pseudo–energy, namely

$$\psi_a^+ \to \frac{\partial\hat{\epsilon}_a}{\partial\theta}\,. \tag{2.4.75}$$

Thus our vacuum expectation value can be evaluated as

$$\langle\Theta\rangle \to -m_1^2 \sum_a \hat{m}_a T_a\,, \tag{2.4.76}$$

where

$$T_a \;=\; \int_{-\infty}^{+\infty} \mathrm{d}\theta\, \frac{\partial\hat{L}_a}{\partial\theta}\,\mathrm{e}^{-\theta}\,. \tag{2.4.77}$$

The sum in the lhs of (2.4.76) is finally given by expanding for $\theta \to -\infty$ the convolution

$$-\frac{1}{2\pi}\int \mathrm{d}\theta'\,\phi_{ab}(\theta'-\theta)\hat{L}_b(\theta') \;=\; \frac{2}{\pi}\mathrm{e}^{\theta}\sum_b\Big(\sum_{\alpha\in A_{ab}}\sin\pi\alpha\Big)T_b + \mathcal{O}(\mathrm{e}^{s\theta}) = \tag{2.4.78}$$

$$= \frac{2}{\pi}\mathrm{e}^{\theta}\,\hat{m}_a\Big(\sum_{\alpha\in A_{11}}\sin\pi\alpha\Big)\sum_b \hat{m}_b T_b + \mathcal{O}(\mathrm{e}^{s\theta})\,,$$

where $s > 1$. In deriving the second equality we have also used eqs. (2.3.35) and (2.3.36).

However the same convolution appears as first term in the expansion for $\theta \to -\infty$ of the derivative of the kink pseudo–energy. Since numerical calculations show that [9, 29]

$$\frac{\mathrm{d}\hat{\epsilon}}{\mathrm{d}\theta} \sim \begin{cases} \mathrm{e}^{(1-\Delta_\varphi)\theta} & \text{non–unitary theories} \\ \mathrm{e}^{(2-2\Delta_\varphi)\theta} & \text{unitary theories} \end{cases} \tag{2.4.79}$$

as a consequence the coefficients of the terms behaving as e^θ must vanish. This finally implies that

$$\langle\Theta\rangle \equiv T_0 \;=\; \frac{\pi m_1^2}{2\sum_{\alpha\in A_{11}}\sin\pi\alpha}\,. \tag{2.4.80}$$

Now we are ready to compute the amplitude U appearing in eq. (2.2.11). Indeed for thermodynamic reasons we expect that the ground state energy behaves as [9]

$$E_0(R) \sim -\frac{\pi\tilde{c}}{6R} - \mathcal{E}_0 R \quad \text{as } g \to 0,\ R \to \infty\,, \tag{2.4.81}$$

where $\mathcal{E}_0 = -Um_1^2$. The determination of U is now easy. Observe indeed that by putting the expansion (2.4.81) into the definition (2.4.70) one obtains the relation

$$T_0 = -4\pi\mathcal{E}_0\,, \tag{2.4.82}$$

from which we finally get

$$\mathcal{E}_0 = -Um_1^2 = -\frac{m_1^2}{8\sum_{\alpha\in A_{11}}\sin\pi\alpha}\,. \tag{2.4.83}$$

Let us go back now to the sum rules. First of all let us show that in the case of the deforming field, the sum rules (2.2.14) and (2.2.20) are actually the same. Indeed taking into account eq. (2.2.22) the sum rule for the bulk energy becomes

$$U = \frac{T_0}{16\pi^2 m_1^2 \Delta_\varphi \langle\varphi\rangle}\int \mathrm{d}^2x\,\langle\Theta(x)\varphi(0)\rangle_c\,, \tag{2.4.84}$$

where φ is the deforming field. Inserting now the fact that

$$U = \frac{T_0}{4\pi m_1^2}\,, \tag{2.4.85}$$

we get the expected eq. (2.2.20). One can notice that there is an apparent "wrong" sign in (2.4.84) with respect to (2.2.20) which however does not matter when dealing with the deforming field. However a particular care must be taken when turning from cylinder to plane: indeed as we shall see in the models $M_{2.2n+1} + \varphi_{13}$ in which (2.2.20) is used for all the relevant primaries, eq. (2.2.22) must be taken with the opposite sign in (2.4.80).

Chapter 3

Form Factors

In the previous chapter we have seen three sum rules that allows us to connect data in the off–critical integrable theory to parameters corresponding to the ultraviolet CFT. In particular this connection is made through the correlation functions. Thus we need a method to compute them and this is provided by the Form Factors (FF). Indeed while in the CFT the fields belong to some infinite dimensional representation of the Virasoro algebra and constitute also the Hilbert space, in the Integrable models the Hilbert space is made of asymptotic particle space. Thus in order to represent the fields of the theory we need their matrix elements on the IN or OUT states of the Hilbert space. This matrix elements are precisely the Form Factors. However we have to stress that the fields in the deformed theory are not the same as in the CFT but are in one-to-one correspondence when the perturbing coupling constant goes to zero. For this reason, with a certain abuse of language, we will often refer to the fields in the IQFT with their names in the CFT i.e. we will call primary fields those fields that reduce to the primaries in the conformal limit.

3.1 General properties of the Form Factors

Suppose we want to compute the following correlation function

$$\langle 0|\phi_b(x,t)\phi_c(0,0)|0\rangle\,, \tag{3.1.1}$$

where ϕ_b and ϕ_c are certain scalar and mutually local fields of the IQFT. This can be done by inserting in between the fields a complete set of asymptotic particle states

$$\sum_n |n\rangle_{in\ in}\langle n| \;=\; 1 \;=\; \sum_n |n\rangle_{out\ out}\langle n| \tag{3.1.2}$$

as to give

$$\langle 0|\phi_b(x,t)\phi_c(0,0)|0\rangle \;=\; \sum_n \sum_{a_i} \int_{\theta_1>\cdots>\theta_n} \frac{\mathrm{d}^n\theta}{(2\pi)^n}\, \mathrm{e}^{-ixP}\, F^b_{a_1,\ldots,a_n}(\theta) F^c_{a_1,\ldots,a_n}(\hat{\theta})\,, \qquad (3.1.3)$$

where $\hat{\theta}_i = i\pi - \theta_i$ and we have inserted the Form Factors defined as

$$\begin{aligned} F^b_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \;&=\; \langle 0|\phi_b(0,0)|\theta_1,\ldots,\theta_n\rangle\,, \\ \langle \theta_1,\ldots,\theta_n|\phi_b(0,0)|0\rangle \;&=\; F^b_{a_1,\ldots,a_n}(i\pi-\theta_1,\ldots,i\pi-\theta_n)\,. \end{aligned} \qquad (3.1.4)$$

Notice that for non–unitary theories the second equation of (3.1.4) is not the complex conjugate of the first. This is the reason why we have inserted both into the spectral representation (3.1.3). The matrix element (3.1.4) is intended to be on an IN asymptotic state i.e. on a state in which the rapidities are ordered as $\theta_1 > \cdots > \theta_n$. This ordering influences also the integration paths in eq. (3.1.3) that must be taken accordingly. However, as we are going to show, any integration in (3.1.3) can be safely taken over all the real plane. Moreover, we have to note that the fields entering into the spectral expansion (3.1.3) are understood to be in Heisenberg representation, from which come the exponential factor. In the following we will often abandon the relativistic notation for the Euclidean one (as for instance in the sum–rules). This will result in a change in eq. (3.1.3), where instead of the exponential written we will have

$$\exp\left(-|x|E\right) \;=\; \exp\left(-|x|\sum_k m_k \cosh\theta_k\right), \qquad (3.1.5)$$

according to the representation of the energy in terms of rapidity.

Relativistic invariance imposes that the FF (3.1.4) for scalar operators depend only on the rapidity differences. If the field possesses instead Lorentz spin s, its matrix elements should also show the same property under Lorentz transformations thus leading to the general following behavior

$$F^b_{a_1,..,a_n}(\theta_1+\Lambda,..,\theta_n+\Lambda) \;=\; \mathrm{e}^{s\Lambda}\, F^b_{a_1,..,a_n}(\theta_1,..,\theta_n)\,, \qquad (3.1.6)$$

for any s. Since in this thesis we will consider only fields which are scalar ones all the following formula will be given for this specific case in order to render more manageable their use. Moreover one should take them carefully if ϕ_b in (3.1.4) is not local with respect to the field that creates the asymptotic particles (as it happens for instance in the thermal Ising between the order or disorder operators and the fermion). Indeed in those cases some of the monodromy equations discussed below may not hold (on the issue of

mutual locality and form factors see [30]). The FF satisfy the so–called Watson equations [31], given by

$$F^{\varphi}_{a_1,\dots,a_i,a_{i+1},\dots,a_n}(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n) =$$

$$= S_{a_i,a_{i+1}}(\theta_i-\theta_{i+1})\, F^{\varphi}_{a_1,\dots,a_{i+1},a_i,\dots,a_n}(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n)\,.$$

$$F^{\varphi}_{a_1,a_2,\dots,a_n}(\theta_1+2\pi i,\theta_2,\dots,\theta_n) = F^{\varphi}_{a_2,\dots,a_n,a_1}(\theta_2,\dots,\theta_n,\theta_1)\,. \tag{3.1.7}$$

The first of (3.1.7) is a direct consequence of the Faddeev–Zamolodchikov algebra (2.3.26) and allows us, thanks to the unitarity equation (2.3.28), to extend the integration paths in (3.1.3) to all the n–dimensional real space. The general solution of eqs. (3.1.7) constitutes a mathematical problem of formidable complexity if one considers also non–diagonal S–matrices like for instance in the Sine–Gordon soliton FF [10, 11]. However in the two–particle case it is always possible [10] to find a solution which is power bounded for F_{ab} as

$$F_{ab}(\theta) \;=\; K(\theta)\, F^{min}_{ab}(\theta)\,, \tag{3.1.8}$$

where F^{min}_{ab} is the *minimal solution* which has no poles and zeroes in the physical strip [1] $Im\,\theta_{ij} \in (0,2\pi)$ and satisfies eq. (3.1.7) while $K(\theta)$ contains all the informations on the poles and satisfy

$$K(\theta) \;=\; K(-\theta) \;=\; K(2\pi i+\theta)\,. \tag{3.1.9}$$

This result allows us to express also the n–particle minimal solution in terms of products of the two–particle ones for diagonal scattering. Indeed, by using the factorization properties of the underlying scattering theory, the minimal solution associated to a generic n–particle FF may be easily expressed in terms of the minimal two–particle FF $F^{min}_{ab}(\theta)$ by

$$F^{min}_{a_1,a_2,\dots,a_n}(\theta_1,\theta_2,\dots,\theta_n) = \prod_{1\le i<j\le n} F^{min}_{a_i a_j}(\theta_i-\theta_j)\;. \tag{3.1.10}$$

where the explicit expressions of $F^{min}_{ab}(\theta)$ are given for theories with degenerate mass spectrum by

$$F^{min}_{ab}(\theta) = \left(-i\sinh\frac{\theta}{2}\right)^{\delta_{a,b}} \prod_{\alpha\in\mathcal{A}_{ab}} h_\alpha(\theta)^{p_\alpha}\;, \tag{3.1.11}$$

while for theories with non–degenerate mass spectrum by

$$F^{min}_{ab}(\theta) = \left(-i\sinh\frac{\theta}{2}\right)^{\delta_{a,b}} \prod_{\alpha\in\mathcal{A}_{ab}} g_\alpha(\theta)^{p_\alpha}\;. \tag{3.1.12}$$

[1] A zero in $\theta=0$ is necessary if $S_{aa}(0)=-1$.

The definition and the properties of the functions $h_\alpha(\theta)$ and $g_\alpha(\theta)$ are collected in Appendix B. Note that the exponents p_α are the order of the poles as given in the corresponding two–particle S–matrices (2.3.33-2.3.34). As mentioned at the end of section (2.3) there might also be zeroes. These zeroes then appear into the S–matrix and into the minimal FF as carrying a *negative* multiplicity i.e. $p_\alpha = -n$ where n is the order of the corresponding zero.

The minimal expression of the FF does not carry any dependence on the specific operator we are considering, as it must be, since the monodromy properties derive from the S–matrix alone. To characterize the different operators and to take into account the dynamical pattern of bound states of the theory, let us consider in more detail the analytic structure of the FF, starting our discussion from the occurrence of their poles. Their pattern may be very complicated for the multi–scattering processes of the theory. There are however two classes of simple order poles in the FF which have a simple and natural origin [11]. The first class is that of kinematic poles relative to particle–antiparticle singularities at the relative rapidity $\theta = i\pi$ with the corresponding residue given by

$$-i \lim_{\tilde{\theta}\to\theta} (\tilde{\theta}-\theta) F_{\bar{a},a,a_1,\ldots,a_n}(\tilde{\theta}+i\pi,\theta,\theta_1,\ldots,\theta_n) = \tag{3.1.13}$$
$$= \left(1 - \prod_1^n S_{aa_i}(\theta-\theta_i)\right) F_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \ .$$

The second class of simple order poles of the FF which admit a simple explanation is that associated to bound state singularities. Namely, whenever $A_a(\theta_a)$ and $A_b(\theta_b)$ form a bound state $A_c(\theta)$ for the value $\theta_{ab} = iu^c_{ab}$ of their relative rapidity, then all the matrix elements $F^\varphi_{a,b,a_1\ldots,a_n}(\theta_a,\theta_b,\theta_1,\ldots,\theta_n)$ involving the two particles $A_a(\theta_a)$ and $A_b(\theta_b)$ will have as well a simple order pole at the same position, with the residue ruled by the on–shell three–point coupling constant Γ^c_{ab}, i.e. (see Figure (3.1))

$$-i \lim_{\theta_{ab}\to iu^c_{ab}} (\theta_{ab} - iu^c_{ab})\, F^\varphi_{a,b,a_1,..,a_n}(\theta_a,\theta_b,\theta_1,..,\theta_n) = \Gamma^c_{ab}\, F^\varphi_{c,a_1,..,a_n}(\theta_c,\theta_1,..,\theta_n) \ . \tag{3.1.14}$$

In addition to the above classes of simple poles, the FF may present poles of higher order relative to the underlying multi–scattering processes [12, 13]. The above two equations (3.1.13) and (3.1.14) form the basis for recursive relations that allow us to completely determine the FF. Moreover, as will be explained in the next subsection, in those theories without internal symmetries one can also apply the cluster equations that will be extremely useful in selecting out the FF of all the primary fields.

A key point to understand the rich analytic structure of the matrix elements is to

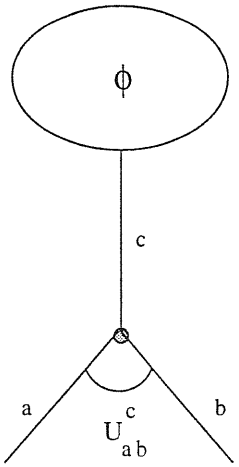

Figure 3.1: Diagrammatic interpretation of the process responsible for a single–pole in a Form Factor.

initially[2] analyze the two–particle FF. Following the analysis of [12] (see also [13. 32]), the two–particle FF can be conveniently written as

$$F^{\varphi}_{ab}(\theta) = Q^{\varphi}_{ab}(\theta)\frac{F^{min}_{ab}(\theta)}{D_{ab}(\theta)} \ , \tag{3.1.15}$$

where D_{ab} takes into account its poles structure and Q^{φ}_{ab} is a polynomial in $\cosh\theta$ which carries the dependence on the operator φ.

The polynomials $D_{ab}(\theta)$ are determined from the poles of the S–matrix. The analysis of ref. [12] gives the following simple rules for determining them in the case of non–degenerate theories:

$$D_{ab}(\theta) = \prod_{\alpha\in A_{ab}} \Big(\mathcal{P}_{\alpha}(\theta)\Big)^{i_\alpha}\Big(\mathcal{P}_{1-\alpha}(\theta)\Big)^{j_\alpha} \ , \tag{3.1.16}$$

$$\begin{aligned} &i_\alpha = n+1 \ , \quad j_\alpha = n \ , \quad \text{if} \qquad p_\alpha = 2n+1 \ ; \\ &i_\alpha = n \ , \qquad j_\alpha = n \ , \quad \text{if} \qquad p_\alpha = 2n \ , \end{aligned} \tag{3.1.17}$$

where A_{ab} and p_α are defined in eqs. (2.3.33) and (2.3.34). The functions

$$\mathcal{P}_{\alpha}(\theta) \equiv \frac{\cos\pi\alpha - \cosh\theta}{2\cos^2\frac{\pi\alpha}{2}} \tag{3.1.18}$$

give a suitable parameterization of the pole at $\theta = i\pi\alpha$. The above prescription can be also generalized to degenerate theories [13]. In fact, referring to equation (B.1.19), one

[2]The reason is that, by factorization, FF with higher number of particles inherit their pole structure from the analytic structure of the two–particle channels. Moreover, the two–particle FF play an important role in the theory since they provide the "initial conditions" needed for solving the recursive functional equations (3.1.13) and (3.1.14).

can write

$$D_{ab}(\theta) = \prod_{\alpha \in \mathcal{A}_{ab}} \left(\mathcal{P}_\alpha(\theta)\right)^{i_\alpha} \ , \tag{3.1.19}$$

$$\begin{aligned} i_\alpha &= n+1 \ , \quad \text{if} \quad p_\alpha = 2n+1 \quad s\text{–channel pole;} \\ i_\alpha &= n \ , \quad \text{if} \quad p_\alpha = 2n+1 \quad t\text{–channel pole;} \\ i_\alpha &= n \ , \quad \text{if} \quad p_\alpha = 2n \ . \end{aligned} \tag{3.1.20}$$

where it is convenient to distinguish between poles associated to the direct s–channel and those relative to the crossed t–channel. Moreover, in addition to the rules (3.1.16) and (3.1.19) we have to discuss the case of the anomalous poles. Indeed these poles appear in the S–matrix as poles with the wrong multiplicity. This effect is due to the presence of zeroes. Indeed, corresponding to any pole one can try to draw a graph in which the internal lines are given by some suitably chosen asymptotic particles (see Figures (2.2–3.3)). Thus, it may happen that these particles meet at certain rapidity differences

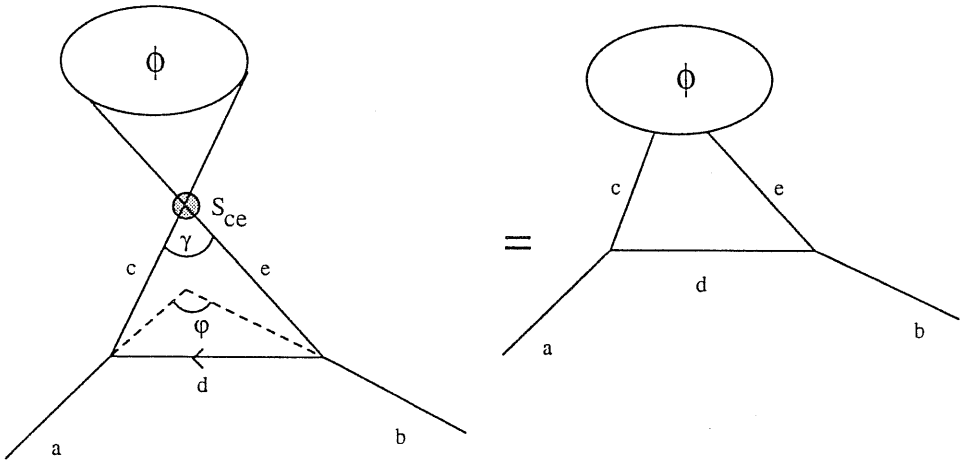

Figure 3.2: Diagrammatic interpretation of the process responsible for a double–pole in a Form Factor.

corresponding to a zero as for instance in Figure (2.4) if $S_{ce}(i\gamma) = 0$. This occurrence of course change the overall multiplicity of the pole that appears then lowered. In order to translate these poles into the FF, one has however to find their corresponding rule which in the case of simple (anomalous) poles is given by (3.1.16) as for "true" simple poles. Moreover they also give rise to equations obtained by cutting the diagrams as in Figures (3.1–3.3) and discussed below [16].

Let us quote at this point the equations which will be often employed in the next sections. Those are: (a) the residue equations at a simple order pole that corresponds to

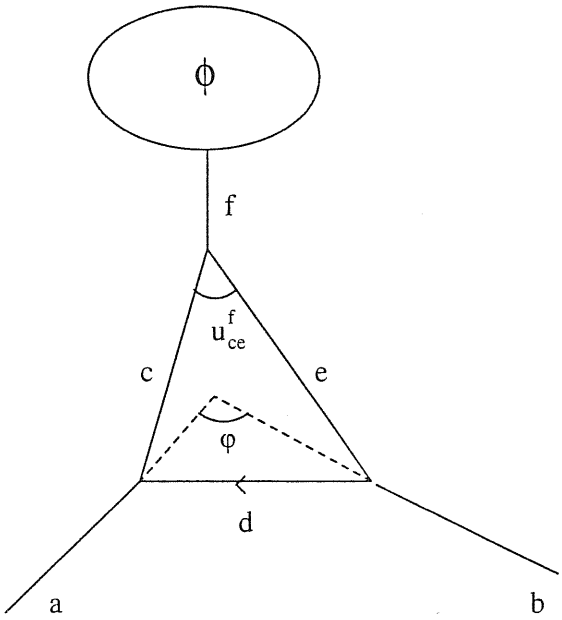

Figure 3.3: Diagrammatic interpretation of the process responsible for a triple–pole in a Form Factor (here $\varphi = u^f_{ab}$).

a bound state

$$-i \lim_{\theta \to i u^c_{ab}} (\theta - i u^c_{ab}) F^{\varphi}_{ab}(\theta) = \Gamma^c_{ab} F^{\varphi}_c \ , \tag{3.1.21}$$

(see Figure (3.1)); (b) the residue equations relative to a simple order pole induced by a double pole in the S–matrix

$$-i \lim_{\theta_{ab} \to i\varphi} (\theta_{ab} - i\varphi) F_{ab}(\theta_{ab}) \ = \ \Gamma^c_{ad} \Gamma^{\bar{e}}_{\bar{d}b} F_{ce}(i\gamma) \ . \tag{3.1.22}$$

where $\gamma = \pi - u^{\bar{a}}_{\bar{c}d} - u^{\bar{b}}_{d\bar{e}}$ (see Figure (3.2)) and finally, (c) the residue equations relative to a double order pole induced by a third order pole in the corresponding S–matrix (see Figure (3.3) where $\varphi = u^f_{ab}$)

$$\lim_{\theta_{ab} \to i u^f_{ab}} (\theta_{ab} - i u^f_{ab})^2 F_{ab}(\theta_{ab}) \ = \ i \Gamma^c_{ad} \Gamma^{\bar{e}}_{b\bar{d}} \lim_{\theta_{ce} \to i u^f_{ce}} (\theta_{ce} - i u^f_{ce}) F_{c\bar{e}}(\theta_{c\bar{e}}) = \tag{3.1.23}$$

$$= \ -\Gamma^c_{ad} \Gamma^{\bar{e}}_{b\bar{d}} \Gamma^f_{ce} F_f \ .$$

After having considered the pole structure of the two–particle FF, let us concentrate our attention on the polynomial $Q^{\varphi}_{ab}(\theta)$ in the numerator of (3.1.15). In contrast to $D_{ab}(\theta)$, which is only fixed by the S–matrix singularities, the polynomials $Q^{\varphi}_{ab}(\theta)$ depend, on the contrary, on the operator $\varphi(x)$ and may be used to characterize it. An upper bound on the maximal degree of the polynomials $Q^{\varphi}_{ab}(\theta)$ has been derived in [12]. Let us describe

it in detail. Consider the theory in the Euclidean continuation $it = y$, the p–moment of the two point correlation function of a given field Φ has to satisfy

$$M_p = \int \mathrm{d}^2x \, |x|^p \, \langle \Phi(|x|)\Phi(0)\rangle_c \, < \infty \,, \tag{3.1.24}$$

for those p which are compatible with the ultraviolet behavior

$$\langle \Phi(|x|)\Phi(0)\rangle \sim |x|^{-2\alpha} \text{ as } |x| \to 0 \,, \tag{3.1.25}$$

where α is the scale dimension $\alpha = 2\Delta$ of the field Φ. This sets a lower bound for the possible values of p to be

$$p \, > \, \max(-1, 4\Delta - 2) \,, \tag{3.1.26}$$

which for $\Delta < 1/2$, which fills all the cases considered here, restricts the choice to $p \geq 0$. Let us take then the "worst" case and insert the spectral expansion (3.1.3) into equation (3.1.24) for $p = 0$. This gives the following equation

$$2\pi \sum_{n=1}^{\infty} \sum_{a_i} \int_{\theta_1 > \ldots > \theta_n} \frac{\mathrm{d}^n\theta}{(2\pi)^n} \, \frac{F^{\Phi}_{a_1,\ldots,a_n}(\theta) F^{\Phi}_{a_1,\ldots,a_n}(\hat{\theta})}{(\sum_{i=1}^n m_i \cosh\theta_i)^2} < \infty \,. \tag{3.1.27}$$

The convergence of the series is of course guaranteed by eqs. (3.1.24) and (3.1.26). What remains to be fixed is the convergence of any single integral in (3.1.27). Denoting by y_Φ the real quantity defined by

$$\lim_{|\theta_i| \to \infty} F^{\Phi}_{a_1,\ldots,a_n}(\theta_1, \ldots, \theta_n) \sim e^{y_\Phi |\theta_i|} \tag{3.1.28}$$

we have [12]

$$y_\Phi \, < \, 1 \, . \tag{3.1.29}$$

Taking into account the degree of the factor $F^{min}_{ab}/D_{ab}(\theta)$ in the two–particle FF (3.1.15) by also using eq. (B.1.11), it is easy to translate the inequality (3.1.29) into an upper bound on the degree of the polynomial $Q^{\Phi}_{ab}(\theta)$.

3.1.1 The Form Factors of the Stress Tensor

In what follows we will often consider the FF of the trace $\Theta(x)$ of the stress–energy tensor, thus it is worth making a brief digression on the properties that they must satisfy. In this case indeed we have additional constraints for the corresponding polynomial $Q_{ab}(\theta)$. As we shall see, the conservation law $\partial_\mu T^{\mu\nu}(x)$ satisfied by the stress–energy tensor implies that the FF of the trace $\Theta(x)$ must contain the kinematic polynomial $P^2 = (p_1 + \cdots + p_n)^2$, with

the exception of the FF with two identical particles and must also satisfy a normalization condition. Let us consider the operator P^0 defined as

$$P^0 = \frac{1}{2\pi}\int \mathrm{d}x\, T^{00}(x)\,, \tag{3.1.30}$$

and let us take its expectation value $\langle\beta|P^0|\theta\rangle$ where for simplicity we will consider a theory with only one particle of mass m. Taking into account that

$$P^0|\theta\rangle = m\cosh\theta|\theta\rangle \quad \text{and} \quad P^1|\theta\rangle = m\sinh\theta|\theta\rangle\,, \tag{3.1.31}$$

we have as first result that

$$\langle\theta|T^{00}(0)|\theta\rangle = 2\pi\, m^2\,\cosh^2\theta\,. \tag{3.1.32}$$

Since $\Theta = T^{00} - T^{11}$ we have now to compute the same matrix element for T^{11} that can be done by evaluating $\langle\beta|\partial_\mu T^{\mu\nu}(0)|\theta\rangle$. A simple algebra based on the substitutions

$$\partial^0\varphi \to -i[P^0,\varphi] \quad \text{and} \quad \partial^1\varphi \to -i[P^1,\varphi]\,, \tag{3.1.33}$$

leads then to the following relation between the two–particle Form Factors

$$\langle\beta|T^{11}(0)|\theta\rangle = \left(\frac{m\cosh\beta - m\cosh\theta}{m\sinh\beta - m\sinh\theta}\right)^2 \langle\beta|T^{00}(0)|\theta\rangle\,. \tag{3.1.34}$$

Taking the limit $\beta \to \theta$ of the last relation finally gives the normalization condition

$$F^{\Theta}_{a\bar{a}}(i\pi) = 2\pi m_a^2\,, \tag{3.1.35}$$

where we have generalized it to the case with more particles and the second of (3.1.4) has been used. The same procedure can be applied to compute the matrix element

$$\langle 0|\partial_\mu T^{\mu\nu}(0)|\theta_1,\ldots,\theta_n\rangle\,, \tag{3.1.36}$$

thus leading to the relation

$$\langle 0|T^{10}(0)|\theta_1,\ldots,\theta_n\rangle = -\frac{E_n\cdot p_n}{E_n^2 - p_n^2}\,\langle 0|\Theta(0)|\theta_1,\ldots,\theta_n\rangle\,, \tag{3.1.37}$$

where we have used the short notations

$$E_n = \sum_i m_i\cosh\theta_i \quad \text{and} \quad p_n = \sum_i m_i\sinh\theta_i\,. \tag{3.1.38}$$

Imposing that all the components of $T^{\mu\nu}$ have the same pole structure means that the FF of Θ factorized the polynomial

$$E_n^2 - p_n^2 = P^2\,, \tag{3.1.39}$$

where P^2 is the total energy associated to the state $|\theta_1,\dots,\theta_n\rangle$. Notice however that this factorization does not hold for two-particle FF with the same particle. Indeed in that case the polynomial in the rhs of (3.1.37) is never singular because the zero in the denominator is canceled by a zero at the same rapidity configuration $\theta_1 = i\pi - \theta_2$ in the numerator. For this last case of two-particles FF, the factorization property can be expressed by means of the following equation

$$Q^{\ominus}_{ab}(\theta) = \left(\cosh\theta + \frac{m_a^2+m_b^2}{2m_a m_b}\right)^{1-\delta_{ab}} P_{ab}(\theta)\ , \tag{3.1.40}$$

where

$$P_{ab}(\theta) \equiv \sum_{k=0}^{N_{ab}} a^k_{ab}\cosh^k\theta\ , \tag{3.1.41}$$

The degree N_{ab} in (3.1.41) may be determined by implementing the inequality (3.1.29). In this way, the problem is reduced to determine the coefficients a^k_{ab} of the polynomials P_{ab}. This goal can be achieved by applying the residue equations together with the normalization condition (3.1.35). The above conditions prove in general sufficient or even redundant in number, to fix all the coefficients of the polynomial $P_{ab}(\theta)$. Once the FF of the stress energy tensor have been computed one can verify their exactness by plugging them into the sum rules (2.2.10) and (2.2.14). They can also be considered as a test for the convergence of the truncated series. Indeed the sum rules can be verified inserting into the integrals the spectral series (3.1.3). Since it is an infinite series we need to truncate it at some point, thus we need a method that tells how good the truncation is. In this sense the two sum-rules (2.2.10) and (2.2.14) provide a quite effective way to measure the relevance of any single added term. As a matter of fact, in all the models considered here, the first few FF ordered according to their total energy conspire in fulfilling the numerical values with a very high precision.

3.2 The cluster limit

One interesting question concerning the recursive relations (3.1.13) and (3.1.14), is that relative to the space of their solution. As shown in many papers [54, 55, 11], the recursive relations admit solutions that should correspond to all the possible fields of the theory. However a deep problem remains that of the identification of the fields corresponding to a given solution. An help in this direction has been given recently in [15], where a new light into the so-called cluster limit has been put. As has been observed many times

[55, 35], certain solutions of the recursive relations with asymptotic behavior characterized by $y_\phi = 0$ in eq. (3.1.28), were known to satisfy the following factorization

$$\lim_{\Lambda\to\infty} F^{\phi}_{a_1,..,a_n}(\theta_1+\Lambda,..,\theta_j+\Lambda,\theta_{j+1},..,\theta_n) = \\ = \frac{1}{\langle\phi\rangle} F^{\phi}_{a_1,..,a_j}(\theta_1,..,\theta_j) F^{\phi}_{a_{j+1},..,a_n}(\theta_{j+1},..,\theta_n)\,, \tag{3.2.42}$$

where the Form Factors in both sides of the equation belong to the same field. However the above equation (3.2.42) does not hold in general. It is indeed strictly true only for the relevant fields in systems without internal symmetries while for theories with some internal symmetry its generalization is still an open problem [15].

Now, in a reversed logic, one can use eq. (3.2.42) together with the recursive relations to select out certain solutions. Before entering into this, let us sketch here the heuristic proof of the above equation following ref. [15]. Let us consider a theory without internal symmetries (in order to avoid vanishing of FF for symmetry reasons) and FF of operators with asymptotic behavior with $y_\phi = 0$. Observe that the limit in eq. (3.2.42) can be rewritten, by using Lorentz invariance as

$$\lim_{\Lambda\to\infty} F^{\phi}_{a_1,..,a_n}(\theta_1+\Lambda,..,\theta_j+\Lambda,\theta_{j+1},..,\theta_n) = \\ = \lim_{\Lambda\to\infty} F^{\phi}_{a_1,..,a_n}(\theta_1+\frac{\Lambda}{2},..,\theta_j+\frac{\Lambda}{2},\theta_{j+1}-\frac{\Lambda}{2},..,\theta_n-\frac{\Lambda}{2})\,, \tag{3.2.43}$$

which is the way in which one can obtain the massless Form Factors [33]. In the massless case we have a decoupling of the particles into left and right movers, according to the following limits for the Faddeev–Zamolodchikov operators [34]

$$A^{\dagger}_R(\theta) = \lim_{\Lambda\to+\infty} A^{\dagger}(\theta+\frac{\Lambda}{2}) \\ A^{\dagger}_L(\theta) = \lim_{\Lambda\to+\infty} A^{\dagger}(\theta-\frac{\Lambda}{2})\,, \tag{3.2.44}$$

where in the lhs we have the creation operators of right or left mover particles respectively. In order now to handle the conformal limit of the FF, we have to avoid trivial vanishing. To this aim let us consider the rescaled (dimensionless) operator $\hat{\phi} = \phi/m_1^{2\Delta}$. The last limit in eq. (3.2.43) thus becomes

$$\lim_{\Lambda\to\infty} F^{\hat{\phi}}_{a_1,..,a_n}(\theta_1+\frac{\Lambda}{2},..,\theta_j+\frac{\Lambda}{2},\theta_{j+1}-\frac{\Lambda}{2},..,\theta_n-\frac{\Lambda}{2}) = \\ = R^{\hat{\phi}}_{a_1,..,a_j}(\theta_1,..,\theta_j)\, L^{\hat{\phi}}_{a_{j+1},..,a_n}(\theta_{j+1},..,\theta_n)\,, \tag{3.2.45}$$

where R and L are the massless FF of $\hat{\phi}$ on right and left movers respectively. On the other hand, one can perform the massless limit by shifting all the rapidities (since the FF does depend only on differences there will not remain memory of the shifting parameter) as e.g.

$$F^{\hat{\phi}}_{a_1,..,a_j}(\theta_1,..,\theta_j) = R^{\hat{\phi}}_{a_1,..,a_j}(\theta_1,..,\theta_j)\, L^{\hat{\phi}}_0\,, \tag{3.2.46}$$

and similar for the left movers. It is also clear from this last equation that $R^{\hat{o}}_0 \cdot L^{\hat{o}}_0 = \langle \hat{o} \rangle$. Thus, substituting (3.2.46) into (3.2.45) and back to (3.2.43) one gets the cluster limit (3.2.42) for the rescaled operator. Restoring the old dimensions gives then eq. (3.2.42).

As mentioned at the beginning, the cluster equation used together with the recursive relations selects a subclass of solutions. These solutions are believed (and actual computations show the exactness of this hypothesis) to correspond to the relevant primary operators of the ultraviolet CFT. In particular, as will be shown in many examples, the application of it results in a system of linear and non–linear coupled equations for the one–particle FF. The solution of these equations will be given for F^o/v_o thus fixing the ratio over the vacuum expectation value (2.2.19) and leaving the correct dimensions in mass [3]. In the cases studied here there are always a finite number of solutions, the number of which exactly corresponds to the number of relevant primary operators of the perturbed CFT. Finally, the sum–rule (2.2.20) allows us to associate each solution to its own operator.

3.3 Examples

In this section we will apply the theoretical framework outlined in the two preceding chapters. In particular, through some examples, we will show how to compute the correlation functions with the Form Factors technique. Going through these examples, one should also realize the effectiveness of this method: the tests on the sum rules indeed give a quite strong support to the fact that they can be safely confronted with the best approaches borrowed from statistical mechanics. Indeed, for example in the case of the Ising model with magnetic field at $T = T_c$ this test has gone until the comparison of the analytic solutions provided by the FF with the most accredited simulations on the lattice [12]. The models shown here are among the simplest one could find and can be seen as a sort of warm up for the more technically involved models of the next chapter.

[3] This statement will be often referred to as putting $v_\phi \equiv F^\phi_0 = 1$.

3.3.1 The Free Massive Boson

Consider the 1+1 dimensional free massive boson, with a diffusivity coefficient κ, described by the relativistic action

$$S_\kappa[\varphi] = 2\pi\kappa \int \mathrm{d}x\, \mathrm{d}t\, \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - \hat{m}^2\varphi^2\right) , \tag{3.3.47}$$

where the mass of the boson has been rescaled as

$$\hat{m}^2 = \frac{m^2}{\kappa} . \tag{3.3.48}$$

The 2π factor is conventional and has been introduced only to give the usual result for the stress-energy tensor, as we shall see. The action (3.3.47) is minimized by the following condition on the field

$$\left(\partial_\mu\partial^\mu + \hat{m}^2\right)\varphi = 0 , \tag{3.3.49}$$

while the Green's function

$$G(x - x_0, t - t_0) = \langle 0|\mathcal{T}\varphi(x,t)\varphi(x_0,t_0)|0\rangle \tag{3.3.50}$$

satisfies the equation

$$\left(\partial_\mu\partial^\mu + \hat{m}^2\right) G(x,t) = -\frac{i}{k}\delta(x - x_0)\delta(t - t_0) \tag{3.3.51}$$

together with the constraint that it has to vanish at infinity.

Instead of solving directly the differential equation (3.3.51), we will compute the Green's functions by using Form Factors technique. Let us insert into the correlation functions, a complete set of asymptotic particles states defined as

$$|\theta_1 \cdots \theta_n\rangle = A^\dagger(\theta_1)\cdots A^\dagger(\theta_n)|0\rangle , \tag{3.3.52}$$

where creation and annihilation operators satisfy the usual bosonic commutation relations

$$\left[A(\theta), A^\dagger(\beta)\right] = 2\pi\delta(\theta - \beta) , \tag{3.3.53}$$

and β is the rapidity defined through eq. (2.3.24). By using the set of states defined in eq. (3.3.52), the two-points time-ordered product

$$G^\phi(x - x_0, t - t_0) = \langle 0|\mathcal{T}\phi(x,t)\phi(x_0,t_0)|0\rangle , \tag{3.3.54}$$

of some field ϕ of the theory, can be expanded according to the spectral series (3.1.3) with the use of the Form Factors (3.1.4). Let us now discuss briefly those few basic properties sufficient to treat in a complete way the model under exam.

Since for the free massive boson the two–particle S–matrix reads

$$S(\theta_{ij}) = 1\,, \tag{3.3.55}$$

we see that there is only one kind of particles and no bound–state pole. Thus all the equations relative to the FF will be specified in this simplest case. Moreover, putting (3.3.55) into eqs. (3.1.7), it implies that $F_n(\theta_1, .., \theta_n)$ must be even and $2\pi i$ periodic in the rapidity differences θ_{ij}, i.e. it can be any function of $\cosh\theta_{ij}$. The absence of poles in the S–matrix implies that also the FF do not have dynamical poles, i.e. eq. (3.1.14) does not exist. For what concerns the kinematic poles, equation (3.3.55) implies that the rhs of (3.1.13) is identically zero and these poles are also absent. The conclusion is that the FF of this theory do not have poles (as expected, since it a free theory). All these conditions restrict the form of $F_n(\theta_1, .., \theta_n)$ to a polynomial

$$F_n(\theta_1, \ldots, \theta_n) = \sum_{m=0}^{N} a_m \left(\sum_{i<j} \cosh\theta_{ij} \right)^m \tag{3.3.56}$$

with the unknowns a_n and N. The determination of these constants depends on the specific operator ϕ we would like to study. We can now determine an upper bound for the degree N, by applying eq. (3.1.27). For instance when $\phi = \varphi$, the scale dimension in eq. (3.1.25) is $\alpha = 0$ and the two–points correlation function has a logarithmic divergence, thus the zero-moment M_0 in (3.1.24) should converge. By inserting now the spectral expansion (3.1.3) into equation (3.1.24) for $p = 0$, one obtains

$$2\pi \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n}{(2\pi)^n} \frac{|F_n^{\varphi}(\theta_1, \ldots, \theta_n)|^2}{(\hat{m} \sum_{i=1}^{n} \cosh\theta_i)^2} < \infty\,. \tag{3.3.57}$$

This equation immediately fixes to zero the degree of the polynomial (3.3.56): this means that all the Form Factors of the field φ are constants. However by inserting the spectral expansion into equation (3.3.51), one sees that it can be satisfied only if the two–particle FF reads

$$F_2^{\varphi} = \frac{1}{\sqrt{2\kappa}} \quad \text{and} \quad F_{n\neq 2}^{\varphi} = 0\,. \tag{3.3.58}$$

With this result the solution of the Klein–Gordon equation (3.3.49) can be written as

$$\varphi(x,t) = \frac{1}{\sqrt{2\kappa}} \int \frac{\mathrm{d}\theta}{2\pi} \left[A(\theta) e^{-i\hat{m}(t\cosh\theta - x\sinh\theta)} + A^{\dagger}(\theta) e^{i\hat{m}(t\cosh\theta - x\sinh\theta)} \right]\,, \tag{3.3.59}$$

and the Green's function (3.3.50) can be given the following expression

$$\langle 0|\mathcal{T}\varphi(x,t)\varphi(0,0)|0\rangle = \frac{1}{\kappa} \int_0^{\infty} \frac{\mathrm{d}\theta}{2\pi} e^{i\hat{m}x\sinh\theta} e^{-i\hat{m}|t|\cosh\theta} = \frac{1}{2\pi k} K_0(\hat{m}R)\,, \tag{3.3.60}$$

where we have expressed it with the Bessel function K_0 in terms of the Euclidean distance R. Together with the field φ we can study also the Form Factors of the components of the stress–energy tensor $T_{\mu\nu}$. With the diffusion term they read

$$T^{01} = T^{10} = 2\pi\kappa \, : \partial^0\varphi\partial^1\varphi :$$
$$\Theta = T^\mu_\mu = 2\pi m^2 \, : \varphi^2 : , \tag{3.3.61}$$

where the products of fields are understood to be normal ordered[4]. Notice that the stress–energy tensor of this model could have been defined differently, by adding a charge at infinity, without changing the equations of motion (3.3.49), as shown in the last reference in [35]. In this paper however we will not consider the effects of such deformations and will work out the consequences of the definition given above.

Since the scale dimension on the space variable of Θ is zero, also for this field the zero-moment (3.1.24) should exist[5]. This implies that all the Form Factors F_n^Θ are constants and can be determined by using eq. (3.3.58). Indeed one can see that the only FF different from zero is given by

$$\langle 0|\Theta(0,0)|\theta_1\theta_2\rangle = 2\pi\hat{m}^2 , \tag{3.3.62}$$

according to the normalization condition for this field (3.1.35).

The two–point Green's function (3.3.54) with $\Phi = \Theta$, can now be computed and is given by

$$\langle 0|\mathcal{T}\Theta(x,t)\Theta(0,0)|0\rangle = 2\,m^4\,K_0^2(\hat{m}R) . \tag{3.3.63}$$

The component T^{01} of the stress-energy tensor is not a scalar under Lorentz transformations. Thus we expect for this field a dependence on the rapidities that takes into account the Lorentz spin, i.e. it should depend not only on the differences. The result (3.3.58) together with the definition (3.3.61) will allow us very easily to compute its Form Factors as

$$F_n^{01}(\theta_1,\ldots,\theta_n) =$$
$$= -2\pi\kappa \sum_{q=1}^{\infty} \frac{1}{q!} \int \frac{\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_q}{(2\pi)^q} \langle 0|\,[\hat{H},\varphi]\,|\beta_1,..,\beta_q\rangle\langle\beta_1,..,\beta_q|\,[\hat{p},\varphi]\,|\theta_1,..,\theta_n\rangle , \tag{3.3.64}$$

[4]From now on, however, we will drop out the normal ordering ":" symbols.

[5]The scale dimension of any scalar component Θ_{s+1} of a conserved current with spin $s+1$ is $s+1$. Since for the trace of the stress-energy tensor (3.3.61) these dimensions are taken into account by the multiplicative mass term, the spatial dependence of G^Θ in eq. (3.3.54) as $|x| \to 0$, can only be of logarithmic type.

where the substitutions (3.1.33) between quantum operators have been done. Thus the only Form Factor different from zero in (3.3.64) is that with $n = 2$, given by

$$F_2^{01}(\theta_1, \theta_2) = \frac{\pi \hat{m}^2}{2} \left(\sinh(2\theta_1) + \sinh(2\theta_2)\right) . \tag{3.3.65}$$

This result agrees with the quantum expectation value between two asymptotic one–particle states of the identity

$$\hat{p}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}x \, T^{10}(x,t) , \tag{3.3.66}$$

where $\hat{p}(t)$ is the momentum operator that satisfies

$$\hat{p}(0)|\theta\rangle = \hat{m} \sinh\theta |\theta\rangle . \tag{3.3.67}$$

The two points Green's function of this field is then given by

$$\begin{aligned} &\langle 0|\mathcal{T}\, T^{01}(x,t) T^{01}(0,0)|0\rangle = \\ &= \frac{1}{2} \int \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2}{(2\pi)^2} \, |F_2^{01}(\theta_1,\theta_2)|^2 \, e^{i\hat{m}x(\sinh\theta_1 + \sinh\theta_2)} e^{-i\hat{m}|t|(\cosh\theta_1 + \cosh\theta_2)} , \end{aligned} \tag{3.3.68}$$

where $F_2^{01}(\theta_1, \theta_2)$ is given in eq. (3.3.65). By using the same methods together with the conservation equation one can also compute the FF of the other components of the stress–energy tensor.

With the aid of eqs. (3.3.63) and (3.3.60) we can check out the sum–rule (2.2.10). Indeed an easy computation shows that it gives exactly the expected value $c = 1$ i.e. this theory has as its ultraviolet limit the CFT with central charge $c = 1$. Since this theory is not minimal, it is expected to have an infinite number of primary fields, as the discussion at the end of this section will outline. From equation (3.3.58) one can compute the Form Factors of the vertex operator $e^{\alpha\varphi}$. Indeed by using Wick theorem one sees that

$$\langle 0|\varphi^k(0)|\theta_1, \ldots, \theta_n\rangle = k! \left(\frac{1}{\sqrt{2k}}\right)^k \delta_{n,k} , \tag{3.3.69}$$

thus giving

$$\langle 0|e^{\alpha\varphi}|\theta_1, \ldots, \theta_n\rangle = \left(\frac{\alpha}{\sqrt{2\kappa}}\right)^n , \tag{3.3.70}$$

while for the correlation function one has

$$\langle 0|e^{\alpha\varphi}(x,t) e^{\beta\varphi}(0,0)|0\rangle = \exp\left(\frac{\alpha\beta}{2\pi k} K_0(\hat{m}R)\right) . \tag{3.3.71}$$

As we shall see all these FF will be useful when dealing with the same model with boundaries and its applications to statistical mechanics. To end, let us note that because of eq. (3.3.58), the FF of the fundamental field φ do not satisfy the cluster equation, while those of the vertex operators (3.3.70) do indeed. This shows that the cluster equation for this model has an infinite number of solutions, all corresponding to the infinite vertex operators of the ultraviolet CFT.

3.3.2 The Model $M_{2,5} + \varphi_{12}$.

The aim of this subsection is to illustrate a simple but non–trivial example of an integrable field theory in which all the features described in the previous chapter may be seen without too much effort. The CFT $M_{2,5}$ is known to describe the universality class of the Lee–Yang singularity of the Ising model in a pure imaginary magnetic field [36, 37].

The conformal field theory is characterized by central charge

$$c = -\tfrac{22}{5} \quad \text{and} \quad \tilde{c} = \tfrac{2}{5}\,. \tag{3.3.72}$$

and only one conformal family (other than that of the unit operator) of the primary field

$$\varphi \equiv \varphi_{12} = \varphi_{13}\,, \tag{3.3.73}$$

with conformal dimensions $\Delta = -1/5$.

The integrable quantum field theory corresponding to the deformation through the field φ is a massive, non–unitary, QFT in which the two–particle scattering amplitude is given by [7]

$$S(\theta) = f_{2/3}(\theta)\,, \tag{3.3.74}$$

where f_α is defined in eq. (2.3.31). This S–matrix, which is self–consistent under bootstrap, shows that there is only one kind of particle with the property of being bound state of itself. Indeed (3.3.74) has a pole at $\theta = 2\pi i/3$, and if m is the mass of the particle, the bound state (2.3.32) has the same mass. From the residue at this pole one can read the on–shell three particle coupling given by eq. (2.3.38) as

$$\Gamma = i\sqrt{2\sqrt{3}}\,. \tag{3.3.75}$$

From the TBA one can easily verify that the field theory with S–matrix (3.3.74) has as ultraviolet limit the model $M_{2,5}$. Indeed, eq. (2.4.64) gives [9]

$$e^{\epsilon} = \frac{\sqrt{5}+1}{2}\,, \tag{3.3.76}$$

that put into equation (2.4.68) gives the effective central charge $\tilde{c}$ shown in (3.3.72).

Now we are ready to compute some Form Factors [35. 38]. Since we have only one field we would like to determine its FF. To this purpose we could use more different strategies: in the following we will indeed show all of them just to make comparisons and examples of the possible techniques. The last one will make use of a combination of the cluster limit and recursion relations. The reason we show it here is due to the fact that it will be used extensively when dealing with the integrable deformations of the class of models $M_{2,2n+1}$. The first strategy consists in considering the FF of the trace $\Theta = T^{\mu}_{\mu}$ of the stress-energy tensor, to obtain also those of φ by means of eq. (2.1.5) or (2.2.23). Let us parameterize the n-particle for a generic field ϕ as

$$F^{\phi}_{n} = H_n \, \sigma_n^{n-1} \, Q^{\phi}_{n}(x_1, .., x_n) \prod_{i<j=1}^{n} \frac{F^{min}(\theta_{ij})}{(x_i + x_j)(x_i - \omega x_j)(x_i - \omega^{-1} x_j)}, \tag{3.3.77}$$

where $\omega = e^{2\pi i/3}$. The above parameterization requires some comment. One can easily see that it is like the one in (3.1.15) but the polynomials encoding the bound state poles are different from those shown in (3.1.18). Indeed in eq. (3.3.77) we have substituted equivalently

$$\mathcal{P}_{\alpha}(\theta_{ij}) \; = \; -\frac{(x_i - \omega x_j)(x_i - \omega^{-1} x_j)}{4 \, x_i x_j \, \cos^2(\frac{\pi\alpha}{2})}, \tag{3.3.78}$$

where $\omega = e^{i\pi\alpha}$ and $x_i = e^{\theta_i}$. The kinematic poles (3.1.13) are instead taken into account by the terms $x_i + x_j$ in the denominator of eq. (3.3.77). By using this parameterization it is possible to rewrite everything in terms of elementary symmetric polynomials $\sigma^{(n)}_{k}(x_1, \ldots, x_n)$ generated by [39]

$$\prod_{i=1}^{n} (x + x_i) \; = \; \sum_{k=0}^{n} x^{n-k} \sigma^{(n)}_{k}(x_1, \ldots, x_n) \, . \tag{3.3.79}$$

The last remark concerns the degree of Q^{ϕ}_{n}. Indeed taking into account eq. (3.1.12), the asymptotic behavior (B.1.11), the limit $y_{\phi} = 0$ in (3.1.28) and Lorentz invariance we get for the total degree t and the partial degree p in any variable

$$t \; = \; \frac{n(n-1)}{2} \quad \text{and} \quad p \; = \; n - 1 \, . \tag{3.3.80}$$

Taking into account all the above observations we get that for any field such that its asymptotic behavior in (3.1.28) is given by $y_{\phi} = 0$, the two-particle FF reads

$$F^{\phi}_{11}(\theta) \; = \; H_{11} \, \sigma_2 \, \frac{F^{min}(\theta)}{(x_1 - \omega x_2)(x_1 - \omega^{-1} x_2)} \, . \tag{3.3.81}$$

According to our first strategy, we put $\phi = \Theta$. For this field we have three equations: the normalization condition (3.1.35) that gives $H_{11} = -2\pi m^2$, the bound-state relation (3.1.14) that provides the one-particle FF

$$\left(F_1^{\Theta}\right)^2 = -\frac{2\pi^2 m^4}{3\sqrt{3}} \left(F^{min}(i\frac{2\pi}{3})\right)^2 , \tag{3.3.82}$$

and the cluster limit (3.2.42) that gives the vacuum expectation value of the trace[6]

$$\langle\Theta\rangle = \frac{\pi m^2}{\sqrt{3}} \frac{(F^{min}(i2\pi/3))^2}{3N} = -\frac{\pi m^2}{\sqrt{3}} , \tag{3.3.83}$$

according to (2.4.80), where in the last equality we have used the relation

$$N \equiv \lim_{\theta\to-\infty} \frac{F^{min}(\theta)}{e^{\theta}} = -\frac{1}{3} \left(F^{min}(i\frac{2\pi}{3})\right)^2 . \tag{3.3.84}$$

Thus the matrix elements of the perturbing field φ can be given now by simply applying eq. (2.1.5). Indeed, putting $F_0^{\varphi} = 1$ we have

$$\begin{aligned} F_{11}^{\varphi}(\theta) &= 2\sqrt{3}\,\sigma_2 \frac{F^{min}(\theta)}{(x_1 - \omega x_2)(x_1 - \omega^{-1} x_2)} \\ (F_1^{\varphi})^2 &= -\frac{2}{\sqrt{3}} \left(F^{min}(i\frac{2\pi}{3})\right)^2 . \end{aligned} \tag{3.3.85}$$

As mentioned, we can present a second strategy that will be often exploited in the following sections. In the present case, the equivalence of these strategies is somewhat trivial, but in order to illustrate in a simple way how it works, it is worth describing it shortly. Let us consider the general case (3.3.81) and let us see where we can go. We have at our disposal two equations: the cluster limit (3.2.42) that tells us that

$$\left(F_1^{\phi}\right)^2 = H_{11}\, N , \tag{3.3.86}$$

where N has been defined in (3.3.84), and the bound-state residue (3.1.14) that gives

$$\left(F_1^{\phi}\right)^2 = -\frac{H_{11}^2}{6\sqrt{3}} \left(F^{min}(i\frac{2\pi}{3})\right)^2 . \tag{3.3.87}$$

Equating the rhs of both equations we get two solutions. The first one is $H_{11} = 0$ that must be put in correspondence with the FF of the scaling unit operator. The second is instead $H_{11} = 2\sqrt{3}$: it can be readily seen that it leads to the FF of the perturbing field (3.3.85). Thus, this second method not only gives the FF of the previous one, but is

[6]Notice that this result fixes without ambiguity the sign of $\langle\Theta\rangle$.

expected to provide also the solutions corresponding to all the primary fields of the CFT. One could now continue trying to obtain the FF with more particles by using the cluster and bound-state equations already seen and the kinematic one given in eq. (3.1.13) if needed. Before entering into this last strategy, let us check the results thus obtained against the sum rules (2.2.10) and (2.2.20). Indeed only with the first two contributions we get

$$\begin{aligned} \Delta &= -0.199971 \quad \text{vs} \quad \Delta = -0.2 \\ c &= -4.39999 \quad \text{vs} \quad c = -4.4\,, \end{aligned} \tag{3.3.88}$$

where the numbers in the column on the left are computed with the sum rules, while those on the right are the exact values. Although the sums are made of contributions with different signs due to the non-unitarity of the model, one can see the good precision reached even with so few contributions. As we shall see this is a quite general pattern in all the models that will be examined here.

Let us consider the bound-state residue equation (3.1.14) and let us put inside it the general form factor (3.3.77). A few simple algebraic passages will soon lead to the following recursive relations

$$H^{\phi}_{n+1} = \left(\frac{2}{g_{2/3}(\frac{2\pi i}{3})}\right)(-1)^n\,\Gamma\,H^{\phi}_n\,, \tag{3.3.89}$$

for the overall normalization constant, and

$$Q^{\phi}_{n+1}(x_1\omega^{1/2}, x_1\omega^{-1/2}, x_2, .., x_n) = x_1\prod_{j=2}^{n}(x_1+x_j)\,Q^{\phi}_n(x_1, .., x_n)\,, \tag{3.3.90}$$

for the characteristic polynomial. We need starting values that are provided by all[7] the solutions obtained before (3.3.85) with the cluster limit i.e.

$$H_1 = F^{\varphi}_1 = i\sqrt{\frac{\sqrt{3}}{2}}\,g_{2/3}(i\frac{2\pi}{3})\,.$$

Putting H_1 into (3.3.89) gives then $H_2 = H_{11}$ as shown in (3.3.85) and so on with all the other constants. Putting instead $Q_1 = 1$ into eq. (3.3.90) and taking into account the degree as determined in (3.3.80) it is easy to see that the first few polynomials are given

[7]The cluster limit selects out all the solutions corresponding to the relevant primary fields (if there are not internal symmetries). In this model we have seen that there is only one non-trivial solution.

by

$$\begin{aligned} Q_2 &= \sigma_1 \\ Q_3 &= \sigma_1\sigma_2 \\ Q_4 &= \sigma_1\sigma_2\sigma_3 \\ &\cdots \end{aligned} \tag{3.3.91}$$

In relation to this solution, it has been shown in [38] that the polynomial Q_n can be written as an $(n-1)\otimes(n-1)$ determinant

$$Q_n = |M_{ij}(k)| \,, \quad \text{for} \quad k = 1 \tag{3.3.92}$$

where the matrix elements are defined as

$$M_{ij}(k) = \sigma_{2i-j}\,\frac{\sin\left((i-j+k)\alpha\right)}{\sin\left(k\alpha\right)}\,, \tag{3.3.93}$$

and $\alpha = 2\pi/3$.

Chapter 4

Form Factors and Applications

This chapter is devoted to the explicit solution of three integrable models. We shall use the techniques seen in the previous chapter applied to more involved computations. The first two cases correspond to the thermal deformation of the Tricritical Ising and Tricritical 3–states Potts models respectively. Since they have some internal symmetry the method used goes through the form factors of the trace Θ of the stress tensor: we will then compute those of the perturbing field and together with a check of the sum rules (2.2.10) and (2.2.14) we provide also a plot of their correlation functions. The third class of models considered in this chapter is the φ_{13} deformation of the non–unitary minimal models $M_{2,2q+1}$. Since an exact solution for the FF on the lightest particle exists, we can compute analytic solutions for certain matrix elements, that will be useful in the next chapter, of all the relevant primaries. Moreover we shall also see the behavior of the sum rules in the limit $q \to \infty$.

4.1 Form Factors of the Energy Operator for the Thermal Perturbation of the Tricritical Ising Model

The Tricritical Ising model is the second model in the minimal unitary conformal series with central charge $c = 7/10$ and four relevant fields [1, 2]. The microscopic formulation of the model, its conformal properties and its scaling region nearby the critical point have been discussed in several papers (see, for instance [40, 41, 42, 43, 44, 45]). In the following we give a short review of the features of the TIM which are most relevant to the FF approach to integrable massive models.

4.1.1 Generalities of the TIM

The Tricritical Ising model may be regarded as the universality class of the Landau-Ginzburg Φ^6–theory

$$L = (\nabla\Phi)^2 + g_6\Phi^6 + g_4\Phi^4 + g_3\Phi^3 + g_2\Phi^2 + g_1\Phi \tag{4.1.1}$$

at its critical point $g_1 = g_2 = g_3 = g_4 = 0$ [46]. This Lagrangian describes the continuum limit of microscopic models with a tricritical point, among them the Ising model with annealed vacancies, with an Hamiltonian given by [42, 43]

$$\mathcal{H} = -\beta \sum_{<ij>} \sigma_i\sigma_j t_i t_j - \mu \sum_i t_i \ . \tag{4.1.2}$$

β is the inverse temperature, μ the chemical potential, $\sigma_i = \pm 1$ the Ising spins and $t_i = 0, 1$ is the vacancy variable. The model has a tricritical point (β_0, μ_0) related to the spontaneous symmetry breaking of the Z_2 symmetry. At the critical point (β_0, μ_0), the TIM can be described by the following scaling fields: the energy density $\epsilon(z, \overline{z})$ with anomalous dimensions $(\Delta, \overline{\Delta}) = (\frac{1}{10}, \frac{1}{10})$, the vacancy operator or sub–leading energy operator $t(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (\frac{3}{5}, \frac{3}{5})$, the irrelevant field ϵ'' with $(\Delta, \overline{\Delta}) = (\frac{3}{2}, \frac{3}{2})$, the magnetization field (or order–parameter) $\sigma(z, \overline{z})$ with $(\Delta, \overline{\Delta}) = (\frac{3}{80}, \frac{3}{80})$, and the so–called sub–leading magnetization operator $\alpha(z, \overline{z})$ with anomalous dimensions $(\frac{7}{16}, \frac{7}{16})$. With respect to the Z_2 symmetry of the spin model, the spin operators are odd while the energy operator, the vacancy operator and the irrelevant field ϵ'' are even.

A peculiar feature of the TIM is the presence of another infinite dimensional symmetry in addition to the Virasoro algebra, i.e. a hidden W-algebra based on the E_7 root system. This is related to the equivalent construction of the TIM in terms of the coset model $(\hat{E}_7)_1 \otimes (\hat{E}_7)_1/(\hat{E}_7)_2$. Let us briefly recall the coset formulation at the critical point [47]. From the theory of Kac–Moody algebras, the central charge of a CFT constructed on an affine Lie algebra G at level k is given by

$$c_G = \frac{k \mid G \mid}{k + h_G} \ , \tag{4.1.3}$$

where $\mid G \mid$ is the dimension of the algebra and h_G the dual Coxeter number. The unitarity condition for the CFT restricts the highest weight representations $\mid \lambda \rangle$ which can appear at the level k. Denoting with ω the highest root, the allowed representations $\mid \lambda \rangle$ at the level k must satisfy

$$\frac{2\,\omega \cdot \lambda}{\omega^2} \leq k \ , \tag{4.1.4}$$

and their dimension is given by

$$\Delta_\lambda = \frac{C_\lambda/\omega^2}{k+h_G} \ , \tag{4.1.5}$$

where C_λ is the quadratic Casimir in the representation $\{\lambda\}$. Using a subgroup $H \subset G$, one can construct a CFT on the coset group G/H, with a central charge equal to

$$c_{G/H} = c_G - c_H = \frac{k_G \mid G \mid}{k_G + h_G} - \frac{k_H \mid H \mid}{k_H + h_H} \ . \tag{4.1.6}$$

Its representations ψ^k are simply obtained by the decomposition of the Hilbert space

$$\mid c_G, \lambda_G \rangle = \ominus_k \ \left[\ \mid c_{G/H}, \psi^k_{G/H} \rangle \otimes \mid c_H, \lambda^k_H \rangle \ \right] \ . \tag{4.1.7}$$

In the case of the TIM, $h = 18$ and eq. (4.1.6) gives $c = \frac{7}{10}$. At level $k = 1$, the possible representations are the identity 1 and the representation Π_6 with scaling dimension 0 and $\frac{3}{4}$ respectively

$$(E_7)_1 \ \rightarrow \ \ \{1, \Pi_6\} = \{0, \tfrac{3}{4}\} \ . \tag{4.1.8}$$

Their components $(n_1, n_2, \cdots, n_7)$ (n_i integer) with respect to the simple roots of E_7 are given by [48]

$$\begin{aligned} 1 \ &\rightarrow \ (0,0,0,0,0,0,0) \\ \Pi_6 \ &\rightarrow \ (0,0,0,0,0,1,0) \ . \end{aligned} \tag{4.1.9}$$

At the level $k = 2$, the representations are given by

$$(E_7)_2 \ \rightarrow \ \ \{1, \Pi_1, \Pi_2, \Pi_5, \Pi_6\} = \{0, \tfrac{9}{10}, \tfrac{21}{16}, \tfrac{7}{5}, \tfrac{57}{80}\} \ , \tag{4.1.10}$$

with the corresponding fundamental weights

$$\begin{aligned} \Pi_1 \ &\rightarrow \ (1,0,0,0,0,0,0) \ , \\ \Pi_2 \ &\rightarrow \ (0,1,0,0,0,0,0) \ , \\ \Pi_5 \ &\rightarrow \ (0,0,0,0,1,0,0) \ . \end{aligned} \tag{4.1.11}$$

Π_1 is the adjoint representation. Using eq. (4.1.7), the scaling dimensions of the TIM are recovered by the decomposition

$$\begin{aligned} (0)_1 \times (0)_1 \ &= \ [(0)_{TIM} \otimes (0)_2] + [(\tfrac{1}{10})_{TIM} \otimes (\Pi_1)_2] + [(\tfrac{6}{10})_{TIM} \otimes (\Pi_5)_2] \ , \\ (0)_1 \times (\tfrac{3}{4})_1 \ &= \ [(\tfrac{7}{16})_{TIM} \otimes (\Pi_2)_2] + [(\tfrac{3}{80})_{TIM} \otimes (\Pi_6)_2] \ , \\ (\tfrac{3}{4})_1 \times (\tfrac{3}{4})_1 \ &= \ (\tfrac{3}{2})_{TIM} \otimes (0)_2 \ . \end{aligned} \tag{4.1.12}$$

The off-critical perturbation considered here is the one given by the leading energy operator $\epsilon(z, \bar{z})$ of conformal weights $\left(\frac{1}{10}, \frac{1}{10}\right)$. Note that this operator is associated to the

adjoint of E_7. According to the analysis of [51], this leads to a structure of the off–critical system deeply related to the root system of E_7, as we briefly recall in the following.

First of all, the off–critical massive model shares the same grading of conserved currents as the Affine Toda Field Theory constructed on the root system of E_7, *i.e.* the spins of the higher conserved currents are equal to the exponents of the E_7 algebra modulo its Coxeter number $h = 18$, i.e.

$$s = 1, 5, 7, 9, 11, 13, 17 \pmod{18} \ . \tag{4.1.13}$$

The presence of these higher conserved currents implies the elasticity of the scattering processes of the massive excitations. To compute the mass spectrum and the scattering amplitudes, it is important to observe that, according to the sign of the coupling constant g in (2.1.1), this perturbation drives the system either in its high–temperature phase or in its low–temperature phase. While in the latter phase we have a spontaneously symmetry breaking of the Z_2 symmetry of the underlying microscopic spin system, in the former phase the Z_2 symmetry is a good quantum number and therefore can be used to label the states. In the low–temperature phase, the massive excitations are given by kink states and bound state thereof, in the high–temperature phase we have instead ordinary particle excitations. The two phases are related by a duality transformation and therefore we can restrict our attention to only one of them, which we choose to be the high–temperature phase. In this phase, the massive excitations are given by seven self–conjugated particles $A_1, \ldots, A_7$ with mass

$$\begin{aligned}
m_1 &= M(g) \,, \\
m_2 &= 2\, m_1 \cos\frac{5\pi}{18} = (1.28557..)\, m_1 \,. \\
m_3 &= 2\, m_1 \cos\frac{\pi}{9} = (1.87938..)\, m_1 \,. \\
m_4 &= 2\, m_1 \cos\frac{\pi}{18} = (1.96961..)\, m_1 \,. \\
m_5 &= 2\, m_2 \cos\frac{\pi}{18} = (2.53208..)\, m_1 \,. \\
m_6 &= 2\, m_3 \cos\frac{2\pi}{9} = (2.87938..)\, m_1 \,. \\
m_7 &= 4\, m_3 \cos\frac{\pi}{18} = (3.70166..)\, m_1 \ .
\end{aligned} \tag{4.1.14}$$

The dependence of the mass scale M on the coupling constant g has been computed in [24]

$$M(g) = C\, g^{\frac{5}{9}}, \tag{4.1.15}$$

where

$$\mathcal{C} = \left[4\,\pi^2\,\gamma(\tfrac{4}{5})\,\gamma(\tfrac{3}{5})\,\gamma^2(\tfrac{7}{10})\right]^{\frac{5}{18}}\,\frac{2\,\Gamma(\frac{2}{9})\,\Gamma(\frac{19}{18})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})\,\Gamma(\frac{10}{9})} = 3.745372836\ldots, \tag{4.1.16}$$

where $\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. The mass ratios are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra E_7 [49] (Figure (4.1)) and therefore the particles A_i may be put in correspondence with the following

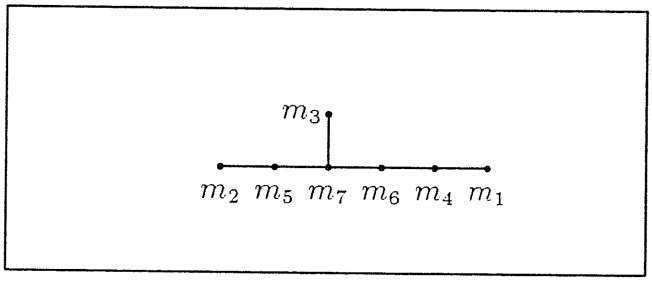

Figure 4.1: Dynkin diagram of E_7 and assignment of the masses to the corresponding dots.

representations of E_7 (here identified by their dimensions)

$$\begin{aligned} A_1 &\rightarrow 56\ , \\ A_2 &\rightarrow 133\ , \\ A_3 &\rightarrow 912\ , \\ A_4 &\rightarrow 1539\ , \\ A_5 &\rightarrow 8645\ , \\ A_6 &\rightarrow 27664\ , \\ A_7 &\rightarrow 365750\ . \end{aligned} \tag{4.1.17}$$

The exact S–matrix of the model is given by the minimal S–matrix of the Affine Toda Field Theory based on the root system of E_7. It has been calculated in [40, 41] and is listed for convenience in Table 2. The structure of the bound states may be written in a concise way by grouping the particle states into two triplets and one singlet states [41]

$$\begin{aligned} (Q_1, Q_2, Q_3) &\equiv (A_6, A_3, A_1)\ , \\ (K_1, K_2, K_3) &\equiv (A_2, A_4, A_7)\ , \\ (N) &\equiv (A_5)\ . \end{aligned} \tag{4.1.18}$$

The first triplet consists of the Z_2 odd particles whereas the other triplet and the singlet are made of Z_2 even particles. The "bootstrap fusions" involving $[N]$ and $[N, K_i]$ form closed subsets

$$\begin{aligned} N \cdot N &= N\,, & N \cdot K_A &= K_1 + K_2 + K_3\,, \\ K_A \cdot K_{A+1} &= K_A + N\,, & K_A \cdot K_A &= K_A + K_{A+1} + N\ . \end{aligned} \tag{4.1.19}$$

Including the first triplet, we obtain the following algebra

$$\begin{aligned} K_A \cdot Q_A &= Q_{A+1}\,, & K_A \cdot Q_{A+1} &= Q_1 + Q_2 + Q_3\,. \\ K_A \cdot Q_{A-1} &= Q_{A-1} + Q_{A+1}\,, & Q_A \cdot Q_A &= K_{A-1} + K_{A+1}\,. \\ Q_A \cdot Q_{A+1} &= K_A + K_{A-1} + N\,, & N \cdot Q_A &= Q_{A-1} + Q_{A+1}\,. \end{aligned} \tag{4.1.20}$$

It has been observed that these bootstrap fusions are a subset of the tensor product decomposition of the associate representations of E_7 [49].

4.1.2 Form Factors of the TIM

After the discussion on the general features of the model, let us consider now the problem of computing the FF of the operator $\epsilon(x)$ or, equivalently, of the trace $\Theta(x)$ of the stress-energy tensor. To this aim, the Z_2 parity of the model is extremely helpful. In fact, because of the even parity of the energy operator, we can immediately conclude that its FF with a Z_2–odd (multi–particle) state must vanish. In particular, the one–particle FF of Θ for the odd particles are all zero.

To start with the bootstrap procedure, let us consider the two–particle FF relative to the fundamental excitation A_1

$$F_{11}^{\Theta}(\theta) = \frac{F_{11}^{min}(\theta)}{D_{11}(\theta)}\, Q_{11}^{\Theta}(\theta)\,, \tag{4.1.21}$$

where

$$F_{11}^{min}(\theta) = -i \sinh(\theta/2)\; g_{5/9}(\theta)\; g_{1/9}(\theta)\,, \tag{4.1.22}$$

and

$$D_{11}(\theta) = \mathcal{P}_{5/9}(\theta)\; \mathcal{P}_{1/9}(\theta)\,. \tag{4.1.23}$$

By using the bound (3.1.29), we see that the polynomial $Q_{11}^{\Theta}(\theta)$ reduces just to a constant, which can be easily determined by means of the normalization condition (3.1.35), i.e. $a_{11}^0 = 2\pi m_1^2$. Thus $F_{11}(\theta)$ is now completely determined and its expression can be used to derive the one–particle FF F_2 and F_4. Indeed, the particles A_2 and A_4 appear as bound state of the particle A_1 with itself, the coupling Γ_{11}^2 and Γ_{11}^4 being easily determined by the residue equation (2.3.38). By using then the equation for the bound state poles of the Form Factors (3.1.21), one gets the desired result (see Table 4).

To proceed further, it is convenient to list the Z_2 even states (the only ones giving non–vanishing FF of the stress–energy tensor) in order of increasing energy, as in Table 3. After computing F_{22}^{Θ} , F_5^{Θ} and F_{13}^{Θ}, which are obtained by means of the same technique, (i.e.

fixing the unknown coefficients of FF by using the simple pole residue equations), a more interesting computation is represented by the two-particle FF $F_{24}(\theta)$. The corresponding S-matrix element displays a double pole and therefore, according to eq. (3.1.16), we have

$$F_{24}^{\Theta}(\theta) = \frac{F_{24}^{min}(\theta)}{D_{24}(\theta)}\, Q_{24}^{\Theta}(\theta)\ , \tag{4.1.24}$$

where

$$F_{24}^{min}(\theta) = g_{7/9}(\theta)\ g_{4/9}(\theta)\ g_{1/3}^2(\theta)\ , \tag{4.1.25}$$

and

$$D_{24}(\theta) = \mathcal{P}_{7/9}(\theta)\ \mathcal{P}_{4/9}(\theta)\ \mathcal{P}_{1/3}(\theta)\ \mathcal{P}_{2/3}(\theta)\ . \tag{4.1.26}$$

Taking into account the asymptotic behavior of the FF and eqs. (3.1.40) and (3.1.41), we conclude that in this case the polynomial P_{24} has degree $N_{24} = 1$ and therefore $Q_{24}(\theta)$ reads

$$Q_{24}^{\Theta}(\theta) \ = \ \left(\cosh\theta + \frac{m_2^2 + m_4^2}{2m_2m_4}\right)(a_{24}^0 + a_{24}^1 \cosh\theta)\,. \tag{4.1.27}$$

To determine the constants a_{24}^0 and a_{24}^1, we need at least two linearly independent equations, which are provided by eq. (3.1.21) on the fusions

$$(A_2\,,\,A_4) \to A_2 \quad \text{and} \quad (A_2\,,\,A_4) \to A_5\,. \tag{4.1.28}$$

Both F_2 and F_5 are known, of course, from previous computations. In this case, the double pole in the S-matrix provides a non-trivial check for the computation. In fact, we have the process drawn in Figure (3.2), with the identification

$$a = 2, \qquad b = 4, \qquad d = e = 1\ ,$$

and respectively

$$c = 1, \qquad \varphi = 2\pi/3, \qquad \gamma = \pi/3\ ,$$

or

$$c = 3, \qquad \varphi = \pi/3, \qquad \gamma = \pi/9\ .$$

These processes give rise to the corresponding residue equations

$$\begin{aligned} -i \lim_{\theta \to i2\pi/3} (\theta - i2\pi/3) F_{24}^{\Theta}(\theta) &= \Gamma_{21}^1 \Gamma_{41}^1\, F_{11}^{\Theta}(i\pi/3)\ , \\ -i \lim_{\theta \to i\pi/3} (\theta - i\pi/3) F_{24}^{\Theta}(\theta) &= \Gamma_{21}^3 \Gamma_{41}^1\, F_{31}^{\Theta}(i\pi/9)\ . \end{aligned} \tag{4.1.29}$$

which are indeed fulfilled. This example clearly shows the over–determined nature of the bootstrap equations and their internal consistency.

The next FF in order of increasing value of the energy of the asymptotic state is given by the lightest Z_2 even three–particle state $|\, A_1 A_1 A_2 \,\rangle$. The FF may be parameterized in the following way

$$F^{\Theta}_{112}(\theta_a,\theta_b,\theta_c) = \frac{F^{min}_{11}(\theta_{ab})\, F^{min}_{12}(\theta_{ac})\, F^{min}_{12}(\theta_{bc})}{D_{11}(\theta_{ab})\, D_{12}(\theta_{ac})\, D_{12}(\theta_{bc})}\, \frac{Q^{\Theta}_{112}}{\cosh\theta_{ac}+\cosh\theta_{bc}}\ , \tag{4.1.30}$$

where F^{min}_{11} and D^{min}_{11} are given by equations (4.1.22) and (4.1.23), while

$$F^{min}_{12}(\theta) = g_{13/18}(\theta)\; g_{7/18}(\theta)\ , \tag{4.1.31}$$

and

$$D_{12}(\theta) = \mathcal{P}_{13/18}(\theta)\; \mathcal{P}_{7/18}(\theta)\ . \tag{4.1.32}$$

We have introduced into (4.1.30) the term

$$\frac{1}{\cosh\theta_{ac}+\cosh\theta_{bc}}\ ,$$

to take into account the kinematic pole of this FF at $\theta_a = \theta_b + i\pi$. The polynomial Q_{112} in the numerator can be further decomposed as

$$Q^{\Theta}_{112}(\theta_a,\theta_b,\theta_c) \;=\; P^2\, P^{\Theta}_{112}\ , \tag{4.1.33}$$

where P^2 is the kinematic polynomial expressed by

$$P^2 = 2m_1^2 + m_2^2 + 2m_1^2\cosh\theta_{ab} + 2m_1 m_2(\cosh\theta_{ac}+\cosh\theta_{bc})\,. \tag{4.1.34}$$

The degree of P^{Θ}_{112} can be computed by means of the asymptotic behavior in the three variables $\theta_{a,b,c}$ separately. This gives the following results for $Q \sim \exp[x_i\theta_i]$:

$$x_a = x_b = 1 \ \text{ and } \ x_c = 2\,. \tag{4.1.35}$$

Hence, a useful parameterization of the polynomial P_{112} is given by

$$P^{\Theta}_{112}(\theta_a,\theta_b,\theta_c) \;=\; p_0 + p_1\cosh\theta_{ab} + p_2(\cosh\theta_{ac}+\cosh\theta_{bc}) + p_3\cosh\theta_{ac}\cosh\theta_{bc}\ , \tag{4.1.36}$$

where four unknown constants have to be determined through the poles of F^{Θ}_{112}. By using the kinematic pole at $\theta_{ab} = i\pi$ and the bound state poles at $\theta_{ab} = i\frac{5\pi}{9}$, $i\frac{\pi}{9}$ and $\theta_{ac} = i\frac{13\pi}{18}$, $i\frac{7\pi}{18}$, one obtains a redundant but nevertheless consistent system of five equations in the four unknown p_i whose solution is given by

$$p_0 = -p_1 = \frac{p_3}{2} = -39.74991118...\ \ , \quad p_2 = -198.2424080... \tag{4.1.37}$$

The other FF which we have computed correspond to the states listed in Table 3. The values of the one–particle FF are collected in Table 4, while the results concerning the two–particle computations are encoded in Table 5 via the coefficients a_{ab}^k of the polynomials $P_{ab}(\theta)$.

4.1.3 Recursive Equations of Form Factors in the TIM

For sake of completeness, we now illustrate an efficient technique to compute multi-particle FF. This is based on recursive identities which relate FF of the type $F_{1,1,\ldots,1}$ with different (even) numbers of fundamental particles. Once these FF are known, those relative to Z_2 even multi–particle state involving heavier particles may be obtained through bootstrap procedure. In general this way of proceeding is the simplest one as far as FF with three or more particles are concerned. In order to write down these recursive equations, we can adopt the following parameterization for the $2n$–particles FF $F_{1,1,\ldots,1}$:

$$F_{1,1\ldots,1}(\theta_1,\ldots,\theta_{2n}) \equiv \mathcal{F}_{2n}(\theta_1,\ldots,\theta_{2n}) = \frac{H_{2n}Q_{2n}(x_1,\ldots,x_{2n})}{\sigma_{2n}^{n-1}} \prod_{i<k} \frac{F_{11}^{min}(\theta_{ik})}{D_{11}(\theta_{ik})} \frac{1}{x_i + x_k} . \tag{4.1.38}$$

Here and in the following $\sigma_k(x_1,\ldots,x_{2n})$ represents the symmetric polynomials of degree k in the variables $x_i = e^{\theta_i}$ defined through their generating function

$$\prod_{k=1}^{m}(x + x_k) = \sum_{j=0}^{m} x^{m-j}\sigma_j(x_1,\ldots,x_m) . \tag{4.1.39}$$

F_{11} and D_{11} are defined by (4.1.22) and (4.1.23) while H_n is an overall multiplicative constant and Q_n is a symmetric polynomial in its variables. The factors $(x_i + x_k)^{-1}$ give a suitable parameterization of the kinematic poles, while the dynamical poles are taken into account by the functions D_{11}'s.

The polynomial Q_{2n} in the numerator can be factorized as

$$Q_{2n}(x_1,\ldots,x_{2n}) \doteq \sigma_1\sigma_{2n-1}P_{2n}(x_1,\ldots,x_{2n}) , \tag{4.1.40}$$

since the FF will be proportional to the kinematic term P^2 relative to the total momentum which can be conveniently written as

$$P^2 = m_1^2 \frac{\sigma_1\,\sigma_{2n-1}}{\sigma_{2n}} . \tag{4.1.41}$$

The Lorentz invariance of the FF requires P_{2n} to be an homogeneous polynomial with respect to all the x_i's of total degree

$$\deg P_{2n} = 4n^2 - 5n , \tag{4.1.42}$$

while the condition (3.1.29), knowing that $\Delta_\epsilon = 1/10$, imposes an upper bound to the degree in a single x_i, given by

$$\deg_{x_i} P_{2n} < 4n - 22/5 \ . \tag{4.1.43}$$

Writing down the most general expression of P_{2n} as a symmetrical polynomial in the basis of the σ_k's and taking into account the above conditions, one can determine the relative coefficients by means of the recursive equations. A first set of recursive relations is obtained by plugging the parameterization of $\mathcal{F}_{2n}$ into the equation of kinematic poles (3.1.13); the polynomial Q_n are then solution of the recursive equation

$$Q_{2n+2}(-x, x, x_1, \ldots, x_{2n}) = -i\, Q_{2n}(x_1, \ldots, x_{2n})\, U_{2n}(x|x_i), \tag{4.1.44}$$

where the polynomial U_{2n} is given by

$$\begin{aligned} U_{2n}(x|x_i) \ = \ & \prod_{i=1}^{n} \prod_{\alpha \in \mathcal{A}_{11}} (x + e^{-i\pi\alpha} x_i)(x - e^{i\pi\alpha} x_i) + \\ & - \prod_{i=1}^{n} \prod_{\alpha \in \mathcal{A}_{11}} (x - e^{-i\pi\alpha} x_i)(x + e^{i\pi\alpha} x_i) \, . \end{aligned} \tag{4.1.45}$$

The overall constants H_n have been fixed to be

$$H_{2n} = 2\,\pi\, m_1^2 \left(16 \prod_{\alpha \in \mathcal{A}_{11}} g_\alpha(0) \frac{\cos^4(\pi\alpha/2)}{\sin(\pi\alpha)} \right)^{-n(n-1)} \ . \tag{4.1.46}$$

with $H_2 = 2\pi m_1^2$. Given Q_{2n}, eq. (4.1.44) restricts the form of the polynomial Q_{2n+2}, although these equations cannot determine uniquely all its coefficients. In fact, polynomials containing the kernel factor $\prod_{i,j=1}^{2n}(x_i + x_j)$ can be added to a given solution Q_{2n+2} with an arbitrary multiplicative factor, without affecting the validity of eq. (4.1.44). In order to have a more restrictive set of equations for the coefficients of the polynomials Q_{2n}, we employ the recursive equations (3.1.21). To relate $\mathcal{F}_{2n+2}$ and $\mathcal{F}_{2n}$, we consider two successive fusions $A_1 A_1 \to A_2$ and $A_2 A_1 \to A_1$, obtaining the following equations

$$Q_{2n+2}(-\varphi x, x, \varphi x, x_2, \ldots, x_{2n}) = \phi_n\, \mathcal{M}\, (\Gamma_{11}^2)^2\, x^5\, Q_{2n}(x, x_2, \ldots, x_{2n})\, P_{2n}(x|x_i) \tag{4.1.47}$$

where

$$\begin{aligned} \mathcal{M} &= 4\cos(5\pi/18)\cos(8\pi/18), \\ \phi_n &= (-1)^{n+1} \exp\Big(-i\pi(10n+1)/18\Big), \\ \varphi &= \exp(-i4\pi/9), \end{aligned}$$

and

$$P_{2n}(x|x_i) = \prod_{i=2}^{2n}(x - e^{i8\pi/9}x_i)(x - e^{i5\pi/9}x_i)(x + e^{i\pi/3}x_i)(x + x_i) \ . \qquad (4.1.48)$$

As an application of the above equations, let us consider the determination of the FF $\mathcal{F}_4$. Taking into account eqs. (4.1.42) and (4.1.43), we can write the following general parameterization for P_4 as

$$P_4(x_1, \ldots, x_4) = c_1\, \sigma_1^2\sigma_4 + c_2\, \sigma_2\sigma_4 + c_3\, \sigma_1\sigma_2\sigma_3 + c_4\, \sigma_3^2 + c_5\, \sigma_2^3 \ . \qquad (4.1.49)$$

From (4.1.44), knowing $Q_2 = \sigma_1$, one gets a first set of equations on the c_i's

$$\begin{aligned} c_2 &= 4\Big(2\,\sin(\pi/9) + \sin(\pi/3) + 2\,\sin(4\pi/9)\Big), \\ c_5 &= -4\Big(\sin(\pi/9) + \sin(4\pi/9)\Big), \\ c_4 &= c_1, \\ c_3 &= c_5 - c_1. \end{aligned} \qquad (4.1.50)$$

The residual freedom in the parameters reflects the presence of kernels of eq. (4.1.44). Given any solution Q_4^*, the space of solutions is spanned by

$$Q_4^\alpha = Q_4^* + \alpha\, \sigma_1\, \sigma_3 \prod_{i,j=1}^{4}(x_i + x_j), \qquad \alpha \in \mathbf{C} \qquad (4.1.51)$$

Eq. (4.1.47) solves this ambiguity giving the last needed equation

$$c_1 = 2\,\frac{4\,\cos(\pi/18) - 11\,\cos(\pi/6) + 12\,\cos(5\pi/18) - 8\,\cos(7\pi/18)}{3 + 5\,\cos(5\pi/9) + \cos(\pi/3) - 3\,\cos(\pi/9)} \ . \qquad (4.1.52)$$

Finally one directly computes H_4 from (4.1.46).

The knowledge of $\mathcal{F}_4 = F_{1111}$ allows us to compute through successive applications of (3.1.21) almost all the FF we needed in order to reach the required precision of the FF expansion of the correlation function. We have used the obtained FF to compute the two–point correlation function of Θ by means of the truncated spectral representation (3.1.3). A plot of $\langle\, \Theta(x)\, \Theta(0)\, \rangle$ as a function of $|x|$ is drawn in Figure (4.2). To control the accuracy of this result we have tested the fast convergence of the spectral series on the checks relative to the first two moments of the correlation function eqs. (2.2.14) and (2.2.10); the single contributions of each multi-particle state in the two series are listed in Table 3 and the partial sum is compared to the exact known values of the central charge c and of the free energy amplitude U. A fast convergence behavior of the spectral sum is indeed observed and therefore the leading dominant role of the first multi-particle states in eq.(3.1.3) is established.

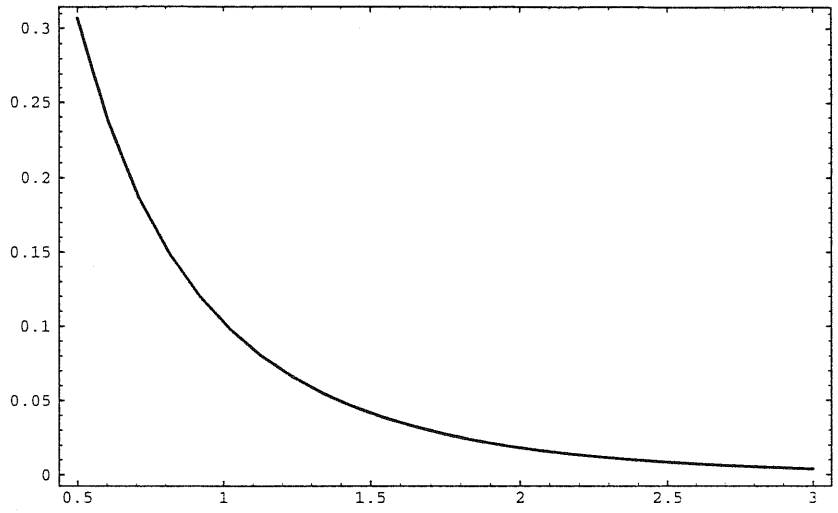

Figure 4.2: Plot of the correlation function $\langle\, \Theta(x)\Theta(0)\,\rangle/m_1^4$ versus the scaling variable $m_1\,|x|$ in the TIM. The spectral series (3.1.3) includes the FF contributions relative to the multi-particle states in Table 3.

4.2 Form Factors of the Energy Operator in the Thermal Deformed Tricritical Potts Model

In this section we will consider the FF computation for the Quantum Field Theory defined by the leading thermal deformation of the Tricritical 3-state Potts Model (TPM). Our strategy will resemble the one already applied to the TIM, with suitable generalizations in order to deal with this theory of degenerate mass spectrum.

4.2.1 Generalities of the TPM

The 3-state Potts Model at its tricritical point may be identified with the universality class of a subset of the minimal conformal model $\mathcal{M}_{6,7}$ [1]. Its central charge is $c = 6/7$. The model is invariant under the permutation group S_3. The group S_3 is the semi-direct product of the two Abelian groups Z_2 and Z_3, where the Z_2 group may be regarded as a charge conjugation symmetry implemented by the generator $\mathcal{C}$. For the generator Ω of the Z_3 symmetry, we have $\Omega^3 = 1$ and $\Omega\mathcal{C} = -\mathcal{C}\,\Omega$. The irreducible representations of S_3 could be either singlets, invariant with respect to Ω ($\mathcal{C}$ even or $\mathcal{C}$ odd) or Z_3 charged doublets.

The off-critical model we are interested in, is obtained by perturbing the fixed point action by means of the leading thermal operator $\epsilon(x)$ with conformal dimension $\Delta = 1/7$. This is a singlet field under both symmetries, $\mathcal{C}$ and Ω. Hence, the discrete S_3

symmetry of the fixed point is still preserved away from criticality and correspondingly the particle states organize into singlets or doublets. The scattering amplitudes of the massive excitations produced by the thermal deformation of the Tricritical Potts Model are nothing but the minimal S–matrix elements of the Affine Toda Field Theory based on the root system of E_6 (they have been determined and discussed in references [40, 50] and can be found in Table 7). Poles occur at values $i\alpha\pi$ with α a multiple of $1/12$, 12 being the Coxeter number of the algebra E_6. The reason of the E_6 structure in the massive model is due both to the equivalent realization of the critical model in terms of the coset $(E_6)_1 \otimes (E_6)_1/(E_6)_2$ and to the fact that the leading energy operator $\epsilon(x)$ is associated to the adjoint representation in the decomposition of the fields [47]. Then, once again, one may apply the argument of references [51] to conclude that the massive theory inherits the E_6 symmetry of the fixed point.

The exact mass spectrum consists in two doublets $(A_l, A_{\bar{l}})$ and $(A_h, A_{\bar{h}})$, together with two singlet particle states A_L and A_H [40, 50]. Their mass ratios are given by

$$\begin{aligned} m_l &= m_{\bar{l}} = M(g)\,, \\ m_L &= 2\, m_l \cos\frac{\pi}{4} = (1.41421..)\, m_l\,, \\ m_h &= m_{\bar{h}} = 2\, m_l \cos\frac{\pi}{12} = (1.93185..)\, m_l\,, \\ m_H &= 2\, m_L \cos\frac{\pi}{12} = (2.73205..)\, m_l\,, \end{aligned} \tag{4.2.53}$$

where the mass scale depends on g as [24]

$$M(g) = \mathcal{C}\, g^{\frac{7}{12}}, \tag{4.2.54}$$

and

$$\mathcal{C} = \left[4\,\pi^2\,\gamma(\tfrac{4}{7})\,\gamma(\tfrac{9}{14})\,\gamma(\tfrac{5}{7})\,\gamma(\tfrac{11}{14})\right]^{\frac{7}{24}} \frac{2\,\Gamma(\frac{1}{4})\,\Gamma(\frac{13}{12})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{2}{3})\,\Gamma(\frac{7}{6})} = 3.746559718\ldots. \tag{4.2.55}$$

The above values of the masses are proportional to the components of the Perron–Frobenius eigenvector of the Cartan matrix of the exceptional algebra E_6 and therefore the particles may be associated to the dots of the Dynkin diagram (see Figure (4.3)). Hence, they may be put in correspondence with the following representations of E_6 (identified by

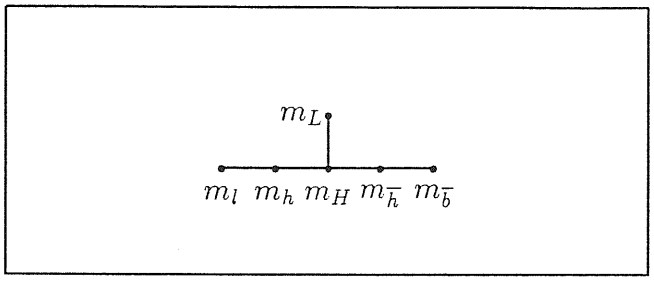

Figure 4.3: Dynkin diagram of E_6 and assignment of the masses to the corresponding dots.

their dimensions)

$$\begin{aligned} A_l &\rightarrow 27 \ , \\ A_{\overline{l}} &\rightarrow \overline{27} \ , \\ A_L &\rightarrow 78 \ , \\ A_h &\rightarrow 351 \ , \\ A_{\overline{h}} &\rightarrow \overline{351} \ , \\ A_H &\rightarrow 2925 \ . \end{aligned} \tag{4.2.56}$$

By introducing the alternative notation

$$\begin{aligned} A_l &\rightarrow A_1 \ , \\ A_{\overline{l}} &\rightarrow \overline{A}_1 \ , \\ A_h &\rightarrow A_2 \ , \\ A_{\overline{h}} &\rightarrow \overline{A}_2 \ , \\ A_L &\rightarrow B_1 \ , \\ A_H &\rightarrow B_2 \ , \end{aligned}$$

the bootstrap fusions of this model can be written in the following compact way [8]

$$\begin{aligned} A_i \times A_i &= \overline{A}_1 + \overline{A}_2 \\ A_i \times A_{i+1} &= \overline{A}_1 + \overline{A}_2 \\ A_i \times \overline{A}_i &= B_i \\ A_i \times \overline{A}_{i+1} &= B_1 + B_2 \\ A_i \times B_i &= A_1 + A_2 \\ \overline{A}_i \times B_i &= A_1 + A_2 \\ A_i \times B_{i+1} &= A_{i+1} \\ B_i \times B_i &= B_1 + B_2 \\ B_i \times B_{i+1} &= B_1 + B_2 \ . \end{aligned} \tag{4.2.57}$$

It is easy to check that the above fusion rules are a subset of the tensor product decompositions of the above representations of E_6 [49].

4.2.2 Form Factors of the TPM

After a brief description of the model, let us turn our attention to the determination of the matrix elements of the leading energy operator $\epsilon(x)$. Our strategy will be similar to that employed in the case of the TIM. For the TPM, however, we have a more stringent selection rule coming from the Z_3 symmetry. Given the even parity of the operator $\epsilon(x)$ and its neutrality under the Z_3 symmetry, the only matrix elements which are different from zero are those of singlet (multi-particle) states and they are the only contributions which enter the spectral representation series (3.1.3). For convenience, the first such states ordered according to the increasing value of the s–variable are listed in Table 8. Because of the selection rules, one very soon encounters three– and four–particle states among the first contributions, and therefore, the computation of FF becomes in general quite involved.

Let us briefly illustrate the most interesting FF computations of this model. As far as one– and two–particles FF are concerned, we just quote the result of the computations since they are quite straightforward and can be obtained by following the same strategy already adopted for the TIM; the one–particle FF are given in Table 9, while the coefficients a^k_{ab} of the polynomials $P_{ab}(\theta)$ of eq. (3.1.41) are listed in Table 10. The need to compute several three–particle FF suggests however to adopt a more systematic technique based on the recursive structure of the FF. The lowest neutral mass state is given in this model by a doublet of conjugated particles l and $\bar{l}$. Hence, in order to build useful "fundamental" singlet multi-particle FF we have to consider recursive equations relating FF of the kind $\mathcal{F}_{n(l\bar{l})} \equiv F_{l\bar{l}l\bar{l}\ldots l\bar{l}}$, with an arbitrary number of particle–antiparticle pairs. From the knowledge of $F_{l\bar{l}l\bar{l}}$ obtained as solutions of the recursive equations, we can next derive (by bootstrap fusion) all the three–particle FF we need in our determination of the correlation function. To write these recursive equations, let us parameterize the FF as

$$\mathcal{F}_{n(l\bar{l})}(\beta_1,\overline{\beta}_1,\ldots,\beta_n,\overline{\beta}_n) = \frac{H_n\, Q_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n)}{(\sigma_n\overline{\sigma}_n)^{n-1}} \cdot \left(\prod_{1\le i<k\le n} \frac{F^{min}_{ll}(\beta_{ik})\; F^{min}_{\bar{l}\bar{l}}(\overline{\beta}_{ik})}{D_{ll}(\beta_{ik})\;\; D_{\bar{l}\bar{l}}(\overline{\beta}_{ik})}\right)\left(\prod_{r,s=1}^{n} \frac{\widehat{F^{min}_{l\bar{l}}}(\beta_r-\overline{\beta}_s)}{(x_r+\overline{x}_s)\; D_{l\bar{l}}(\beta_r-\overline{\beta}_s)}\right), \tag{4.2.58}$$

where

$$\widehat{F^{min}_{l\bar{l}}}(\beta_r-\overline{\beta}_s) \equiv \begin{cases} F^{min}_{l\bar{l}}(\beta_r-\overline{\beta}_s) & \text{if } r\le s\ , \\ \\ F^{min}_{l\bar{l}}(\overline{\beta}_s-\beta_r) & \text{otherwise}\ . \end{cases} \tag{4.2.59}$$

In these expressions $x_i = e^{\beta_i}$ and σ_m is the symmetrical polynomial of degree m in the x_i's

(the quantities $\overline{x}_i$ and $\overline{\sigma}_m$ are analogously defined in terms of the $\overline{\beta}_i$'s). The two–particle minimal FF are given by (see eqs. (3.1.11) and (3.1.19))

$$\frac{F_{ll}^{min}(\beta)}{D_{ll}(\beta)} = \frac{F_{\bar{l}\bar{l}}^{min}(\beta)}{D_{\bar{l}\bar{l}}(\beta)} = \frac{-i\sinh(\beta/2)\; h_{1/6}(\beta)\; h_{2/3}(\beta)\; h_{1/2}(\beta)}{p_{1/6}(\beta)\; p_{2/3}(\beta)} \;, \tag{4.2.60}$$

$$\frac{F_{l\bar{l}}^{min}(\beta)}{D_{l\bar{l}}(\beta)} = \frac{F_{\bar{l}l}^{min}(\beta)}{D_{\bar{l}l}(\beta)} = \frac{h_{5/6}(\beta)\; h_{1/3}(\beta)\; h_{1/2}(\beta)}{p_{1/2}(\beta)} \;. \tag{4.2.61}$$

In (4.2.58), H_n is just a multiplicative overall factor and Q_n is a polynomial in its arguments. The latter is the only unknown quantity which can be computed through the recursive equations. The function Q_n must be a symmetrical polynomial both in the x_i's and in the $\overline{x}_i$'s separately. Furthermore, it must be symmetric under charge conjugation, i.e. under the simultaneous exchange $x_i \leftrightarrow \overline{x}_i$ $(\forall i = 1 \ldots n)$. Hence, it can be parameterized in terms of products of σ's and $\overline{\sigma}$'s with suitable coefficients in order to guarantee the self–conjugacy. The factor P^2 for this set of particles takes the form

$$P^2 = \frac{(\overline{\sigma}_{n-1}\sigma_n + \sigma_{n-1}\overline{\sigma}_n)(\sigma_1 + \overline{\sigma}_1)}{\sigma_n\overline{\sigma}_n}\, m_l^2 \,, \tag{4.2.62}$$

and, correspondingly Q_n will be factorized as

$$Q_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) = (\overline{\sigma}_{n-1}\sigma_n + \sigma_{n-1}\overline{\sigma}_n)(\sigma_1 + \overline{\sigma}_1) P_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) \,. \tag{4.2.63}$$

The Lorentz invariance of the FF requires P_n to be an homogeneous polynomial with respect to all the x's and $\overline{x}$'s of total degree

$$\deg P_n = 3n^2 - 4n \;, \tag{4.2.64}$$

while the condition (3.1.29), knowing that $\Delta_\varphi = 1/7$, imposes the following upper bound for the degree in a single x_i $(\overline{x}_i)$

$$\deg_{x_i} P_n < 3n - 74/21 \;. \tag{4.2.65}$$

These conditions drastically restrict the possible form of the polynomials Q_n.

Let us write down the form assumed by the kinematic recursive equations by using the parameterization (4.2.58)

$$Q_{n+1}(-x, x, x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) = i\; x\; U_n(x|x_i,)\; Q_n(x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n) \;, \tag{4.2.66}$$

where (here $\mathcal{A}_{ll} = \{1/6, 2/3, 1/2\}$)

$$\begin{aligned} U_n(x|x_i, \overline{x}_i) \;=\; & \prod_{i=1}^{n} \prod_{\alpha \in \mathcal{A}_{ll}} (x - e^{i\pi\alpha}\overline{x}_i)(x - e^{i\pi(1-\alpha)}x_i) - \\ & \prod_{i=1}^{n} \prod_{\alpha \in \mathcal{A}_{ll}} (x - e^{-i\pi\alpha}\overline{x}_i)(x - e^{-i\pi(1-\alpha)}x_i) \,. \end{aligned} \tag{4.2.67}$$

The overall constant is explicitly given by:

$$H_n = 2\,\pi\,m_l^2\,\left(2\tan^2(\pi/6)\tan^2(5\pi/12)\prod_{\alpha\in\mathcal{A}_{ll}} g_\alpha(0)\sin(\pi\alpha)\right)^{-\frac{n(n-1)}{2}} . \qquad (4.2.68)$$

However, the equations (4.2.66) are not in general sufficient to fix all the coefficients of Q_{n+1}. A more stringent constraint is obtained by using twice eq. (3.1.21) in relation with the processes $l\,l \to \bar{l}$ and $\bar{l}\,\bar{l} \to l$. The final equations take a very simple form:

$$Q_{n+1}(\eta\,\overline{y},\eta\,y,\overline{\eta}\,\overline{y},\overline{\eta}\,y,x_2,\overline{x}_2\ldots,x_n,\overline{x}_n) \;=\; \qquad (4.2.69)$$

$$= -(\Gamma^{\bar{l}}_{ll})^2\;y\,\overline{y}\;W_n(y,\overline{y},x_2,\overline{x}_2,\ldots,x_n,\overline{x}_n)\;Q_n(y,\overline{y},x_2,\overline{x}_2,\ldots,x_n,\overline{x}_n)\;,$$

where $\eta = e^{i\pi/3}$ and

$$W_n(x_1,\overline{x}_1,\ldots,x_n,\overline{x}_n) \;=\; \qquad (4.2.70)$$

$$= (x_1+\overline{x}_1)(\overline{x}_1 - e^{\frac{7\pi i}{6}}x_1)(\overline{x}_1 - e^{\frac{-7\pi i}{6}}x_1)(\overline{x}_1 - e^{\frac{\pi i}{2}}x_1)(\overline{x}_1 - e^{\frac{-\pi i}{2}}x_1)\cdot$$

$$\cdot\prod_{i=2}^{n}(\overline{x}_1+x_i)(x_1+\overline{x}_i)(\overline{x}_1 - e^{\frac{5\pi i}{6}}\overline{x}_i)(\overline{x}_1 - e^{\frac{-5\pi i}{6}}\overline{x}_i)(x_1 - e^{\frac{5\pi i}{6}}x_i)(x_1 - e^{\frac{-5\pi i}{6}}x_i)\;.$$

Let us now illustrate how this procedure works in the case of $\mathcal{F}_{2(l\bar{l})}$. Let us start from $F_{l\bar{l}}$; using eq. (3.1.35) we easily obtain $Q_1 = 1$ and $H_1 = 2\pi\,m_l^2$. From eqs. (4.2.64) and (4.2.65), the general parameterization for P_2 is given by

$$P_2(x_1,\overline{x}_1,x_2,\overline{x}_2) \;=\; c_1\,(\sigma_2^2+\overline{\sigma}_2^2) + c_2\,(\sigma_1\sigma_2\overline{\sigma}_1 + \overline{\sigma}_1\overline{\sigma}_2\sigma_1) + \qquad (4.2.71)$$

$$+c_3\,(\sigma_1^2\overline{\sigma}_2 + \overline{\sigma}_1^2\sigma_2) + c_4\,\sigma_1^2\overline{\sigma}_1^2 + c_5\sigma_2\overline{\sigma}_2\;.$$

Equation (4.2.66) gives four equations for the five parameters

$$\begin{aligned} c_4 &= -(3+\sqrt{3}),\\ c_2 - c_3 &= -3\,(2+\sqrt{3}),\\ c_1 - c_2 &= 3+2\sqrt{3},\\ 2\,c_2 + c_5 &= -18-10\sqrt{3}, \end{aligned} \qquad (4.2.72)$$

while eq. (4.2.69) solve the residual freedom yielding

$$c_1 = -\frac{9+5\sqrt{3}}{2},$$

$$
\begin{aligned}
c_2 &= -\frac{3(5+3\sqrt{3})}{2}, \\
c_3 &= -\frac{3(1+\sqrt{3})}{2}, \\
c_4 &= c_5 = -(3+\sqrt{3})\ .
\end{aligned}
\tag{4.2.73}
$$

Once we have determined H_1 and P_2, we can obtain $\mathcal{F}_{2(l\bar{l})}$ from eqs. (4.2.58) and (4.2.63). From this four–particles FF it is also easy to obtain the three–particles FF F_{lll}, $F_{l\bar{l}L}$, F_{llh} applying the residue equation (3.1.14) at the fusion angles $u^l_{l\bar{l}}$, $u^L_{l\bar{l}}$ and $u^h_{l\bar{l}}$ respectively. The explicit expressions of these three–particle FF are given in Appendix B.

The FF calculated for the TPM can be used to estimate the two–point function of the stress–energy tensor whose plot is shown in Figure (4.4). The convergence of the series

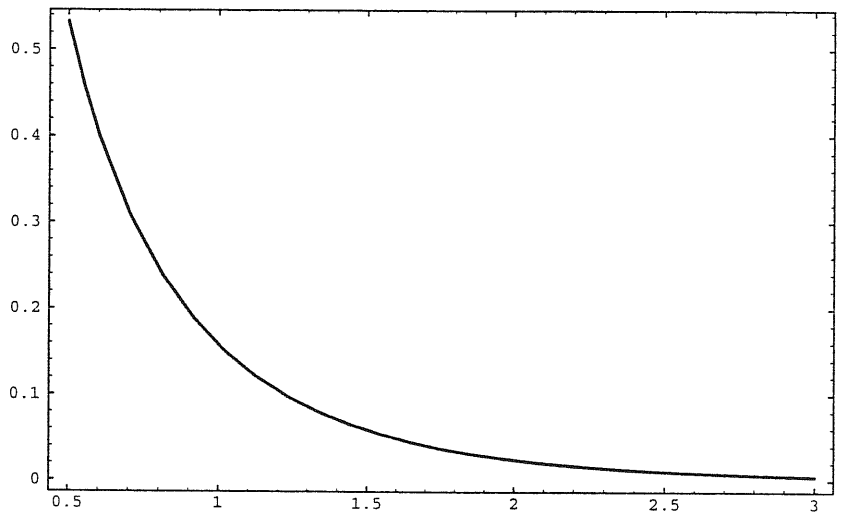

Figure 4.4: Plot of the correlation function $\langle\, \Theta(x)\Theta(0)\,\rangle/m_l^4$ versus the scaling variable $m_l\,|x|$ in the TPM. The spectral series (3.1.3) includes the FF contributions relative to the multi-particle states in Table 8.

may be checked through the sum–rule tests: the contributions of each multi-particle state are listed in Table 8 where the exact and computed values of c and U are compared. A very fast convergence behavior is indeed observed which supports the validity of the spectral approach to correlations functions in integrable massive models.

4.3 The Models $M_{2,2q+1} + \varphi_{13}$.

In this section we will present some computations concerning the whole class of models $M_{2,2q+1}$ perturbed by their field φ_{13} The aim of this section is threefold. First of all we

would like to describe a set of models in which lot of results can be given at once for any element belonging to it, thanks simply to a generalization of the solutions given in section (3.3.2). The second reason, connected to the first, is that we can show the behavior of certain sum rules as $q \to \infty$ and third because we will apply these results in the next section to computations related to non-integrable deformations.

The CFT's $M_{2,2q+1}$ have central charges given by

$$c = -\frac{2(6q^2-7q+1)}{2q+1} \quad \text{and} \quad \tilde{c} = \frac{2(q-1)}{2q+1} \tag{4.3.74}$$

and possesses q primary fields

$$\varphi_{1,j} \quad \text{for} \quad j = 1, \ldots, q \tag{4.3.75}$$

whose conformal dimensions are given by

$$\Delta_{1,j} = -\frac{(j-1)(2q-j)}{2(2q+1)}. \tag{4.3.76}$$

from which one can read off the conformal dimensions of the most relevant field is $\varphi_{1,q}$ by

$$\Delta_{1,q} = -\frac{q(q-1)}{2(2q+1)}. \tag{4.3.77}$$

Any of these models when perturbed by the field φ_{13} is integrable, non-unitary and massive. It has $q-1$ particles with masses given by

$$m_a = \sin(a\frac{\pi\alpha}{2}) \quad \text{for} \quad a = 1, \cdots, q-1\,, \tag{4.3.78}$$

where $\alpha = 2/(2q-1)$. The two-particle S-matrices of this model are given by [52]

$$S_{ab}(\theta) = f_{|a-b|\alpha/2}(\theta) f_{(a+b)\alpha/2}(\theta) \prod_{j=1}^{min(a,b)-1} \left(f_{(|a-b|+2j)\alpha/2}(\theta)\right)^2, \tag{4.3.79}$$

where $f_a(\theta) \equiv (a)$ has been defined in (2.3.31). That these amplitudes actually lead to the models $M_{2,2q+1}$ in the ultraviolet limit has been proved by means of the TBA in [53] and the first few of them are listed below

$$\begin{aligned} S_{11} &= \overset{2}{(\alpha)} \\ S_{12} &= (1-\overset{1}{\alpha/2})\,(\overset{3}{3\alpha/2}) \\ S_{13} &= (1-\overset{2}{\alpha})\,(\overset{4}{2\alpha}) \\ S_{22} &= \overset{4}{(2\alpha)}\ (\alpha)^2 \\ &\cdots\ . \end{aligned} \tag{4.3.80}$$

We have to observe that for $q = 2$ we get back the model $M_{2,5} + \varphi_{12}$ discussed in section (3.3.2). Moreover observe that there is a periodicity in the values of the masses and the primary fields

$$m_a = m_{2q-a-1} \quad \text{and} \quad \varphi_{1,j} = \varphi_{1,2q-j+1} \,. \tag{4.3.81}$$

In the following we will often use this periodicity indeed making reference to a mass that in some model may be absent but its "reflected" is instead in the spectrum. In certain cases we will also consider negative values of a in their labels i.e. we will take m_a as a sort of abbreviation to write $\sin(a\pi\alpha/2)$. The Form Factors on the fundamental particle for these models have been obtained in [55] for any of the operators

$$\varphi_{1.2k+1} \quad \text{for} \quad k = 1, \ldots, q \,. \tag{4.3.82}$$

The general form of them is given by eq. (3.3.77) where $\omega = \exp(i\pi\alpha)$, and the index n means that we are looking at FF with n fundamental particles. The polynomials Q_n^k are given in terms of a determinant (3.3.92) where k now labels the fields as in (4.3.82). This solution can be obtained as in section (3.3.2). Indeed (3.3.92) solves, for any k, the kinematic recursive relations and is then restricted to the values of k shown in (4.3.82) by the bound state equation. Notice however that for $q > 2$ the bound state equation does not connect form factors of the fundamental particle in a single step. We need indeed two "bunching"

$$F_{1\cdots 111} \to F_{1\cdots 12} \to F_{1\cdots 1} \,,$$

to relate the $n+2$ to the n particle FF. The normalization constants H_n^k can then be given by simply imposing e.g. the cluster equations with $F_0^k = 1$. Moreover one gets also the FF for the other particles $2, 3, \ldots$ through the fusion procedure implied by the form of the S–matrix.

Before coming to the explicit expressions of some form factors, we prefer to express the matrix elements in (3.3.92) in a way more closely related to the order of the fields as in (4.3.75). A brief look at eqs. (4.3.75) and (4.3.82) will lead to

$$Q_n^j = \left| M_{ab}^j \right| \tag{4.3.83}$$

where M_{ab}^j is given by

$$M_{ab}^j = \sigma_{2a-b} \begin{cases} \dfrac{m_{2a-2b+j-1}}{m_{j-1}} & j \text{ odd} \\ \\ \dfrac{m_{2b-2a+j-1}}{m_{j-1}} & j \text{ even} \end{cases} \,, \tag{4.3.84}$$

where the upper index j refers to $\varphi_{1,j}$. Notice that the matrix M_{ab} for j odd is the transpose of the one for j even. Since the determinant is invariant under transposition one can safely take only the definition for j odd and use it also for j even. With the determinant (4.3.83) the first characteristic polynomials read for instance

$$\begin{aligned} Q^j_{11} &= \sigma_1 \\ Q^j_{111} &= \sigma_1\sigma_2 - \sigma_3 \frac{m_{j+1}m_{j-3}}{m^2_{j-1}} \\ &\dots \end{aligned} \tag{4.3.85}$$

where it is worth recalling that all the constants depend also on α. Let us list here the form of those polynomials and normalization constants that will be useful later. Let us begin with the FF on the lightest particle

$$F^j_1 = 2\frac{\sqrt{N_{11}\sin(\pi\alpha)}}{m_1} m_{j-1}\,, \tag{4.3.86}$$

where

$$N_{ab} = \lim_{\theta\to-\infty}\left(\frac{F^{min}_{ab}(\theta)}{\exp(\theta)}\right). \tag{4.3.87}$$

Then we have the normalization constant for the two-particle F^j_{11}

$$H^j_{11} = 4\frac{m^2_{j-1}}{m^2_1}\sin\pi\alpha\,, \tag{4.3.88}$$

where one can notice the extremely simplified solution with respect to the cumbersome one that can be found in [55]. From these results we can indeed get a nice recursive relation from the cluster limit (3.2.42). Observing that the polynomials involving n particles of the same kind obey

$$\lim_{\theta_1\to\infty} Q^j_n(x_1,..,x_n) = x_1^p\, Q^j_{n-1}(x_2,..,x_n)\,, \tag{4.3.89}$$

where p is the maximal partial degree of Q_n, the recursive solution for the normalization constants is

$$H^j_n = \left(2\frac{m_{j-1}}{m_1}\sqrt{\sin(\pi\alpha)}\right)^n (N_{11})^{-\frac{n(n-2)}{2}}\,, \tag{4.3.90}$$

where one can see that for the scaling Lee–Yang model one recovers the values shown in eq. (3.3.85).

By using now the fusions it is possible to get the one–particle matrix elements involving heavier particles. However as we shall see, for the two–particle FF a more efficient method can be exploited which does not require the use of the residue equation (3.1.14). For our

purposes, and for reasons that will be clear later, we have computed only a view of them listed in the following. The first computation concerns the one-particle FF

$$F_2^j = -\frac{g_\alpha\,(i\pi\alpha)\,\sqrt{\sin(2\pi\alpha)}}{m_1 m_2}\,m_{j-1}^2\,, \tag{4.3.91}$$

and next the two-particle Form Factor F_{22}. The parameterization for the FF with n particles of type 2 is given by

$$F_{n(2)}^j = H_{n(2)}^j\,\sigma_n^{2(n-1)}\,Q_{2(n)}^j(x_1,..,x_n)\prod_{i<j=1}^{n}\frac{F_{22}^{min}(\theta_{ij})}{(x_i+x_j)\,D_{22}(\theta_{ij})}\,, \tag{4.3.92}$$

where the normalization constants satisfy the relation

$$H_{2(n)}^j = \left(-\frac{g_\alpha(i\pi\alpha)\,\sqrt{\sin(2\pi\alpha)}}{m_1\,m_2}\,m_{j-1}^2\right)^n\,N_{22}^{-\frac{n(n-1)}{2}}\,, \tag{4.3.93}$$

and the polynomial D_{22} is given following the rule (3.1.16) and reads

$$D_{22}(\theta_{ij}) = (x_i-\omega^2x_j)(x_i-\omega^{-2}x_j)(x_i-\omega x_j)(x_i-\omega^{-1}x_j)(x_i+\omega x_j)(x_i+\omega^{-1}x_j)\,. \tag{4.3.94}$$

The characteristic polynomial has degrees

$$t = 3n(n-1)/2 \quad \text{and} \quad p = 3(n-1)\,, \tag{4.3.95}$$

and satisfies the same factorization (4.3.89). Thus for $n=2$ we have

$$Q_{22}^j = (\sigma_1^2 + a_{22}^j\,\sigma_2)\,\sigma_1\,, \tag{4.3.96}$$

and the constant a_{22}^j can be determined as follows from the determinant (4.3.83). Indeed consider the characteristic polynomial Q_4^j given by

$$Q_{1111}^j = \det\begin{vmatrix} \sigma_1 & \frac{m_{j-3}}{m_{j-1}} & 0 \\ \sigma_3\frac{m_{j+1}}{m_{j-1}} & \sigma_2 & \sigma_1\frac{m_{j-3}}{m_{j-1}} \\ 0 & \sigma_4\frac{m_{j+1}}{m_{j-1}} & \sigma_3 \end{vmatrix}\,, \tag{4.3.97}$$

where we have written the matrix elements in (4.3.83) for j odd. To obtain now Q_{22}^j we should apply the residue equations (3.1.14) twice as shown pictorially in the following equation

$$F_{1111}(x_1,..,x_4) \to F_{211}(x,x_3,x_4) \to F_{22}(x,y)\,, \tag{4.3.98}$$

where the residue must be taken for $\theta = i\pi\alpha$. However we shall show that this is not necessary. Indeed it will suffice to follow the transformation of each element in the determinant as far as

$$x_1 = x\omega^{1/2},\ x_2 = x\omega^{-1/2} \quad \text{and} \quad x_3 = y\omega^{1/2},\ x_4 = y\omega^{-1/2}\,, \tag{4.3.99}$$

where we consider the transformations as following one the other. The matrix elements in (4.3.97) then become

$$\begin{aligned} \sigma_1 &\to \tfrac{m_2}{m_1}\,\sigma_1 \\ \sigma_2 &\to \sigma_1^2 + (\tfrac{m_2}{m_1})^2\,\sigma_2 \\ \sigma_3 &\to \tfrac{m_2}{m_1}\,\sigma_1\sigma_2 \\ \sigma_4 &\to \sigma_2^2\,, \end{aligned} \tag{4.3.100}$$

and substituted into (4.3.97) lead to a determinant for Q^j_{22}. Actually we should expect the polynomial Q^j_{22} multiplied by some function that may have been canceled in the process of taking the residue. What is important is the possibility to actually determine a^j_{22} which is indeed given by

$$a^j_{22} = (\frac{m_2}{m_1})^2 - 2\,\frac{m_{j+1}\,m_{j-3}}{m^2_{j-1}} - 2\,. \tag{4.3.101}$$

Since we know exactly the form of H^j_{22} it will be possible to derive also F_4 by using the residue equations. Before that, we can apply this mechanism to compute the coefficient a^j_{12}. Consider indeed the following two–particle FF

$$F^j_{12}(\theta) = \frac{H^j_{12}\,\sigma_2\,Q^j_{12}(x_1,x_2)\,F^{min}_{12}(\theta)}{(x_1+\omega^{1/2}x_2)(x_1+\omega^{-1/2}x_2)(x_1-\omega^{3/2}x_2)(x_1-\omega^{-3/2}x_2)}\,, \tag{4.3.102}$$

where

$$H^j_{12} = -2i\,\frac{m^3_{j-1}}{m^2_1}\,g_\alpha(i\pi\alpha)\,\frac{\sqrt{2N_{11}\cos(\pi\alpha)}}{N_{12}} \tag{4.3.103}$$

with N_{12} defined in (4.3.87) and the characteristic polynomial is given by

$$Q^j_{12} = \sigma_1^2 + a^j_{12}\,\sigma_2\,. \tag{4.3.104}$$

By using the determinant for Q^j_{111} given by the 2×2 minor in (4.3.97) and using the analogous of (4.3.100) one gets

$$a^j_{12} = \frac{m_1}{m_2}\left(\left(\frac{m_2}{m_1}\right)^2 - \frac{m_{j-3}\,m_{j+1}}{m^2_{j-1}} - 2\,\frac{m_2}{m_1} + 1\right). \tag{4.3.105}$$

One could now proceed on the same path computing other two particle FF (we have listed a few of them in Table 13 for the model $M_{29}+\varphi_{13}$). However a few comments

are in order. Indeed the above two–particle FF could have been obtained in a different way by using the equation (3.1.14) but the equivalent results would have been written in a quite cumbersome form, instead of as ratios of sines as for a_{22} and a_{12}. The way we actually followed to obtain them however already contains the information on the bound state poles, although it refers to the fundamental particle only, but expressed in a simpler compact form. For later uses we compute now the one–particle matrix elements F_3^j and F_4^j. They can easily obtained from F_{12}^j and F_{22}^j respectively and are given by

$$F_3^j = -\frac{m_1 H_{12}^j (4\cos^2(3\pi\alpha/4) + a_{12}^j)}{2m_3 m_4 \Gamma_{12}^3} F_{12}^{min}(\frac{3i\pi\alpha}{2})$$
$$F_4^j = \frac{H_{22}^j (4\cos^2(\pi\alpha) + a_{22}^j)}{8m_4 m_2 m_6 \Gamma_{22}^4} F_{22}^{min}(2i\pi\alpha)\,. \tag{4.3.106}$$

where we have used the definition of Γ as in eq (2.3.38). A closer look at the one–particle form factors computed so far reveals a nice behavior that is worth describing here. Indeed it is possible to see that the dependence on j is contained in the single mass term m_{j-1} to some power which depends on the asymptotic particle. By suitably using the cluster limit one can guess that they have a general form as

$$F_a^j = A_a^{(a)}(\alpha)\, m_{j-1}^a + A_a^{(a-2)}\, m_{j-1}^{a-2} + \cdots + \begin{cases} A_a^{(2)}\, m_{j-1}^2 & a \text{ even} \\ A_a^{(1)}\, m_{j-1} & a \text{ odd} \end{cases}\,, \tag{4.3.107}$$

where the coefficients depends only on the ultraviolet model i.e. on the parameter q.

In the next section we will compute the contributions to the sum rules for the central charge and the conformal dimensions of the fields $\varphi_{1,j}$ in the limit $q \to \infty$. To this aim, we will write here the (truncated) sum rules in which we explicitly show only those contributions actually computed.

Let us begin with the conformal dimension. The one-particle FF contribute as

$$\Delta_{1,j}^{(one-part)} = \frac{m_1^2}{4\sin(\pi\alpha)} \left(\frac{F_1^3 F_1^j}{m_1^2} + \frac{F_2^3 F_2^j}{m_2^2} + \cdots \right)\,, \tag{4.3.108}$$

where the coefficient in front of the sum comes from the vacuum normalization (2.4.80) of the trace of the stress–energy tensor. The contributions from the two–particle FF considered here are instead given by

$$\Delta_{1,j}^{(11)} = \frac{1}{16\sin(\pi\alpha)} \int \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2}{2\pi} \frac{F_{11}^3(\theta_{12}) F_{11}^j(\theta_{21})}{(\cosh\theta_1 + \cosh\theta_2)^2}$$
$$\Delta_{1,j}^{(12)} = \frac{m_1^2}{8\sin(\pi\alpha)} \int \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2}{2\pi} \frac{F_{12}^3(\theta_{12}) F_{12}^j(\theta_{21})}{(m_1\cosh\theta_1 + m_2\cosh\theta_2)^2}\,. \tag{4.3.109}$$

The three–particle Form Factor F_{111} gives instead

$$\Delta_{1,j}^{(111)} = \frac{\pi}{24\sin(\pi\alpha)} \int \frac{\mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3}{(2\pi)^2} \frac{F_{111}^3(\theta_1,\theta_2.\theta_3)F_{111}^j(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)}{(\cosh\theta_1+\cosh\theta_2+\cosh\theta_3)^2}. \tag{4.3.110}$$

As we shall see also for the central charge of the UV theory the sum rule gives interesting results. For reasons that will be clarified later let us write here the contribution coming from the FF F_1 and F_{11}. With an obvious notation they are given by

$$c^{(1)} = \frac{3}{4\pi}\left(\frac{\pi}{2\sin(\pi\alpha)}\right)^2 8\,(F_1^3)^2 \tag{4.3.111}$$

where both integration over x and the rapidity have been done, and

$$c^{(11)} = \frac{3}{4\pi}\left(\frac{\pi}{2\sin(\pi\alpha)}\right)^2 \frac{3}{\pi}\int \mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\frac{F_{11}^3(\theta_{12})\,F_{11}^3(\theta_{21})}{(\cosh(\theta_1)+\cosh(\theta_2))^4}. \tag{4.3.112}$$

4.3.1 The limit $q \to \infty$.

Before evaluating with the aid of the form factors the sum rules (4.3.108–4.3.112), let us write down what are the expected values for the central charge and the conformal dimensions. Indeed expanding eqs (4.3.74–4.3.77) we get the limiting values as $q \to \infty$

$$\begin{aligned} c &\sim -6\,q + 10 + \mathcal{O}(1/q) \\ \Delta_{1,j} &\sim -(j-1)/2 + \mathcal{O}(1/q) \\ \Delta_{1,q} &\sim -q/4 + 3/8 + \mathcal{O}(1/q) \end{aligned} \tag{4.3.113}$$

As we shall see there is a general *criteria* that will allows us to determine what FF contribute and with which power of n. Before explaining this general rule it is worth showing the leading behavior for $q \to \infty$ of the one-particle FF considered in the previous section and, as a beginning, their contribution to the conformal dimension of $\varphi_{1,q}$. Since in the computation of $\Delta_{1,q}$ enter the FF of the fields $\varphi_{1,3}$ and $\varphi_{1,q}$, we will consider the aforementioned limit for the label $j = 3$ and $j = q$ respectively. These are given by (in this limit we have also $\alpha \sim 1/q \to 0$ which is always considered in the following equations)

$$\begin{aligned} F_1^3 &\sim 2i\sqrt{\pi\alpha} \\ F_1^q &\sim 2i/\sqrt{\pi\alpha}, \end{aligned} \tag{4.3.114}$$

where the limit

$$N_{11} \to -1/4$$

has been taken into account. By plugging these two values of F_1 into the sum rule for the conformal dimension (4.3.108) we obtain the first contribution

$$\Delta^{(1)}_{1,q} = -\frac{q}{\pi}. \tag{4.3.115}$$

Let us consider now the one-particle FF F^q_2 in this limit. The leading behaviors are

$$\begin{aligned} F^3_2 &\sim -2\sqrt{2}\,(\pi\alpha)^{3/2} \\ F^q_2 &\sim -2\sqrt{2}/\sqrt{\pi\alpha}\,. \end{aligned} \tag{4.3.116}$$

The contribution in the sum rule is thus given by

$$\Delta^{(2)}_{1,q} = \frac{1}{2}. \tag{4.3.117}$$

It is also possible to compute the contribution of the one-particle F_3 that reads

$$F^3_3 = -i\frac{9}{\sqrt{3}}\,(\pi\alpha)^{5/2}, \tag{4.3.118}$$

while for the most relevant field one get

$$F^q_3 = -i\frac{8}{\sqrt{3}}\,\frac{1}{\sqrt{\pi\alpha}}. \tag{4.3.119}$$

These two limiting values lead then to the conclusion that

$$\Delta^{(3)}_{1,q} \to 0. \tag{4.3.120}$$

A few comments are in order now. First of all note that there is a regular behavior for these one–particle FF, i.e. they follow the general pattern

$$F^3_a \sim (\pi\alpha)^{a/2}, \quad F^q_a \sim \frac{1}{\sqrt{\pi\alpha}}. \tag{4.3.121}$$

Second, let us note that the one particle FF F^3_1 allows us for constant term contribution to the sum rule. Indeed the expansion for large q reads

$$F^3_1 \sim 2i\sqrt{\pi\alpha} - \frac{i}{\pi}\,(\pi\alpha)^{3/2}, \tag{4.3.122}$$

in such a way that the matching with F^q_1 in eq.(4.3.114) exactly cancel the $1/\alpha$ term in the sum–rule. These FF are now sufficient to predict what are the other Form Factors that contribute and whether their leading contribution[1] is proportional to q or constant or

[1] As we have seen in the case of the one-particle contribution, those terms contributing with a term proportional to n give also a constant contribution.

zero. The *criteria* comes from the observation that the cluster limit and the limit $q \to \infty$ must commute. This means that the leading term(s) for $\theta_i \to \infty$ in the polynomials Q must coincide with the most singular term(s) in the limit $q \to \infty$. It is a trivial matter now to apply this rule to see that all the FF involving only the particle 1 contribute to the sum rule for $\Delta_{1,q}$ with leading terms proportional to q. Instead the FF with only one particle 2 and all the other of type 1 gives constants contributions while all the other contributions vanish in this limit. To show how this works, let us apply this rule to few simple cases.

- F_{11} has leading contribution proportional to q.

  Indeed by looking at the cluster limit according to the above *criteria* we get

$$\begin{aligned} F_{11}^3 &\to (F_1^3)^2 \sim (\pi\alpha) \\ F_{11}^q &\to (F_1^q)^2 \sim \frac{1}{\pi\alpha}, \end{aligned} \tag{4.3.123}$$

  so that their product cancel the dependence on q. A look now at the first of eq. (4.3.109) reveals that $\Delta_{1,q}^{(11)} \sim q$. It is easy to see that the same applies for all the FF of type $F_{1\ldots1}$. Their exact contributions must however be computed directly and give for example

$$\begin{aligned} \Delta_{1,q}^{(11)} &\sim \frac{q}{\pi^2} \\ \Delta_{1,q}^{(111)} &\sim -\frac{q}{6\pi}. \end{aligned} \tag{4.3.124}$$

  Putting all the contributions together gives finally

$$\Delta_{1,q} \sim q\left(-\frac{1}{\pi} + \frac{1}{\pi^2} - \frac{1}{6\pi} + \cdots\right), \tag{4.3.125}$$

  in which we can recognize the first three terms in the expansion of

$$g(z) = \frac{1}{\pi^2}\arcsin^2(2\pi z) - \frac{1}{\pi}\arcsin(2\pi z) \tag{4.3.126}$$

  around $z = 0$ computed in $z = 1/2\pi$, which gives exactly the wanted value $-1/4$ in eq (4.3.113).

- F_{12} has leading contribution constant.

  The cluster limit give

$$\begin{aligned} F_{12}^3 &\to F_1^3 F_2^3 \sim (\pi\alpha)^2 \\ F_{12}^q &\to F_1^q F_2^q \sim \frac{1}{\pi\alpha}, \end{aligned} \tag{4.3.127}$$

thus their product is proportional to α, whence $\Delta^{(12)}_{1,q} \sim const.$ The same reasoning leads then to the conclusion that all the FF like $F_{1\ldots12}$ have leading contribution constant. In this case there is no reason to worry about these constants, because also all the FF containing only particles of type 1 give constant contributions and their sum is no more under control.

- The contributions from F_{1a} with $a \geq 3$ and from all the other FF go to zero.

  When at least one of the particles is of type 3 it is easy to see, applying the above reasoning on the result shown in eq. (4.3.121), that these FF give vanishing contributions for $q \to \infty$. Moreover by simply using the one-particle FF just discussed one can also prove that all the others go to zero.

- The limit for the generic field $\varphi_{1,j}$.

  In the case we want to see how things go for any other field, whose position remains fixed while doing the limit, things simplify. Indeed in the case of the field $\varphi_{1,j}$ the conformal dimensions in the limit $q \to \infty$ are given by (4.3.113). For this computation we need, as before, the behavior of the one–particle FF. They can be easily computed and are given by[2]

$$\begin{aligned} F_1^j &\sim i\sqrt{\pi/n}\,(j-1) \\ F_2^j &\sim -(\pi/n)^{3/2}\,(j-1)^2/\sqrt{2}\,. \end{aligned} \tag{4.3.128}$$

  Plugging now this limits into equation (4.3.108), one sees that the one–particle contribution of the particle 1 gives

$$\Delta^{(1)}_{1,j} \sim -\frac{j-1}{2}, \tag{4.3.129}$$

  thus fulfilling the limit (4.3.113). For what concerns the contribution of particle 2, it is easy to see that its contribution goes instead to zero. Moreover, applying the same reasoning as in the case of $\Delta_{1,n}$ one readily sees that all the two particle FF give a contribution that goes to zero.

As last application let us show how all this applies to the case of the central charge. By looking at the corresponding sum rule and at the limit values for the one–particle FF, one can see that only F_1 and F_{11} give non-vanishing contributions. In particular one can readily see that F_1 alone must give the term proportional to q plus a constant, and that

[2] One should take care of whether j is odd or even. However one can readily see that the corresponding limit is the one shown here in both cases.

F_{11} gives another constant. The term proportional to q must match the analogous term in the expansion (4.3.74), while the two constants must provide the constant in the same expansion. These contributions can be computed analytically from eq. (4.3.111) and (4.3.112) and give

$$c^{(1)} \sim -6q + 6 + \mathcal{O}(1/q)$$
$$c^{(11)} \sim 4 + \mathcal{O}(1/q)\,. \tag{4.3.130}$$

Their sum then gives

$$c = c^{(1)} + c^{(11)} + \cdots = -6q + 10 + \mathcal{O}(1/q)\,, \tag{4.3.131}$$

as in the expansion (4.3.113).

4.4 Application to non–integrable deformation

In the previous chapters we have seen applications of the Form Factors techniques to theories deformed as in (2.1.1) which remain integrable. An open and interesting problem would be that one concerning theories which are deformed along non–integrable direction. Being this general problem in large part beyond the present possibilities let us look at all the possible ways we have to get in touch with it. This will allow us to understand whether there is some problem that can be treated with the present knowledges. There are basically two situations that can lead to some non–integrable system. The first and more obvious is the one in which into the action (2.1.1) there is a field φ which is not among those in (2.1.2) or in other words such that the only conserved current in (2.1.4) is the stress–energy tensor. This case is up to now among those that can be considered only through some numerical investigation [56] and (or) standard conformal perturbation theory and as such will not be considered here. The second possible path to non–integrability consists in considering instead of the action density (2.1.1) the one in which there are two deforming fields both of them leading separately to some integrable theory. Indeed this approach leads then to consider the non–integrability as a breaking of integrability. In other words we consider an integrable theory and add to the action density a small perturbation that breaks it as in the following equation

$$\mathcal{A}_{qft} = \mathcal{A}_{int} - \epsilon \int \mathrm{d}^2x\, \Phi(x)\,, \tag{4.4.132}$$

where $\mathcal{A}_{int}$ is the action density for the integrable theory and ϵ is the small coupling constant. With this small perturbation we will consider first–order corrections that we

will be able to compute by using the form factors of the integrable theory we are starting from. The following section is dedicated to this problem: we will describe briefly the main consequences of this approach following ref. [17] and then we will pursue an explicit computation on the model $M_{2,9} + \varphi_{13}$ and its two possible non–integrable deformation.

4.4.1 First order corrections

Suppose to take the field theory formally defined by the action density (4.4.132) and let us make the hypothesis that the perturbation is turned off at $t \to \pm\infty$. Thus the asymptotic states are those of the integrable theory

$$|p_1, \dots, p_n\rangle^{in(out)} = |p_1, \dots, p_n\rangle_0^{in(out)}, \tag{4.4.133}$$

where the index 0 in the lhs refers to the integrable theory. The S–matrix of the theory is instead modified due to the presence of the perturbation and is given by the product $S(s) = S_0(s)S_I(s)$ where S_0 is the (known) scattering amplitude of the integrable theory while S_I encodes the dependence on the perturbing field and can be formally written as

$$S_I(s) = T\left(\mathrm{e}^{-i\epsilon\int\Phi}\right). \tag{4.4.134}$$

The S–matrix is then defined as

$$\begin{aligned} {}^{out}\langle q_1, .., q_m | p_1, .., p_n\rangle^{in} &= {}_0^{out}\langle q_1, .., q_m | T\left(\mathrm{e}^{-i\epsilon\int\Phi}\right) | p_1, .., p_n\rangle_0^{in} = \\ &= \sum_{k=0}^{+\infty} \frac{(-i\epsilon)^k}{k!} \int \mathrm{d}^2x_1 \cdots \mathrm{d}^2x_k \; {}_0^{out}\langle q_1, .., q_m | T\left(\Phi(x_1)..\Phi(x_k)\right) | p_1, .., p_n\rangle_0^{in}, \end{aligned} \tag{4.4.135}$$

where by inserting now a complete set of asymptotic states in between the fields and using the known FF of the field Φ one could in principle compute all the contributions order by order. Note that in the non–integrable theory features like inelastic collisions and decay of particles are no longer forbidden and for this reason the S–matrix cannot be factorized into the product of two–particle amplitudes. Moreover as shown in ref. [17] counterterms must be added to the action in order to take properly into account the correct normalization of states. Indeed while the new interaction changes both the vacuum energy $\mathcal{E}_0$ and the masses of the particles at the same time we require that the vacuum be normalized as

$$\langle 0|0\rangle = {}_0\langle 0|0\rangle_0 = 1, \tag{4.4.136}$$

and the one–particle states obey

$${}^{out}\langle q|p\rangle^{in} = {}^{out}\langle q|p\rangle^{in} = 2\pi\, E\, \delta(q^1 - p^1). \tag{4.4.137}$$

Thus the needed counterterms can be obtained with the prescription of adding to the action their appropriate operators. We can formally take this into account by means of the following substitution into equation (4.4.135)

$$\epsilon\,\Phi(0) \to \epsilon\,\Phi(0) - \frac{1}{2}\delta m^2\, O^{(2)}(0) - \delta\mathcal{E}_0\,, \tag{4.4.138}$$

where $O^{(2)}$ is an operator defined in terms of its unperturbed Form Factors as

$$F_n^{O^{(2)}} = {}_0\langle 0|O^{(2)}(0)|p_1,..,p_n\rangle_0^{in} = \delta_{n,2}\,, \tag{4.4.139}$$

and $\delta\mathcal{E}_0$ is the variation of the vacuum energy under the perturbation. With these definitions one can compute the corrections by imposing (4.4.136) and (4.4.137) order by order. In particular it is easy to see that the first order corrections are given by

$$\begin{aligned} \delta m_a^2 &= 2\epsilon\cdot {}_0\langle p_a|\Phi(0)|p_a\rangle_0 \\ \delta\mathcal{E}_0 &= \epsilon\cdot\langle\Phi\rangle_0\,, \end{aligned} \tag{4.4.140}$$

where we have generalized them to the case with many particles and the Form Factors are those of the unperturbed theory. Equation (4.4.140) can be rewritten in a coupling–independent way by taking the ratios over the variations thus leading to [17]

$$\begin{aligned} \frac{\delta m_a}{\delta m_b} &= \frac{m_b\, F_{\bar{a}a}^{\Phi}(i\pi)}{m_a\, F_{\bar{b}b}^{\Phi}(i\pi)} \\ \frac{\delta\mathcal{E}_0}{\delta m_a} &= \frac{m_a\,\langle\Phi\rangle_0}{F_{\bar{a}a}^{\Phi}(i\pi)}\,, \end{aligned} \tag{4.4.141}$$

where all the masses and FF are taken in the unperturbed theory. Notice that eqs. (4.4.141) possesses a double interest. Indeed one may want now to check their behavior on some model versus for instance numerical perturbative computations. On the other hand instead can be interpreted as a way to identify the form factors of the perturbing field entering in the lhs of them.

Many examples could be made to this purpose (see for instance the original reference [17] and [16] where some models have been solved) and in the following we will choose two of them from this last reference, namely the non–integrable models $(M_{29}+\phi_{13})+\epsilon\Phi$ where Φ are the fields ϕ_{12} and ϕ_{14}, just because we have their analytic solution in section (4.3). For these models the identification of the fields has been already obtained through the sum rule (2.2.20) and in the following we will then show the validity of eqs. (4.4.141) comparing the predictions with a numerical computation. In tables (11–13) we have listed the two–particle S–matrices, one–particle FF and the coefficients entering into the

definitions of the two–particle FF for this model. The numerical study will be performed by using the Truncated Conformal Space Approach (TCSA) developed in ref. [18]. By using eqs. (4.3.88,4.3.92) we can indeed obtain the theoretical values to plug into eqs. (4.4.141) that can be compared with the numerical ones as shown in Table (14). One may notice that the values are in quite good agreement. To the purpose of better illustrating this result we have also plotted various values of the ratios taken at different values of the non–integrable coupling in Figures (4.5–4.8).

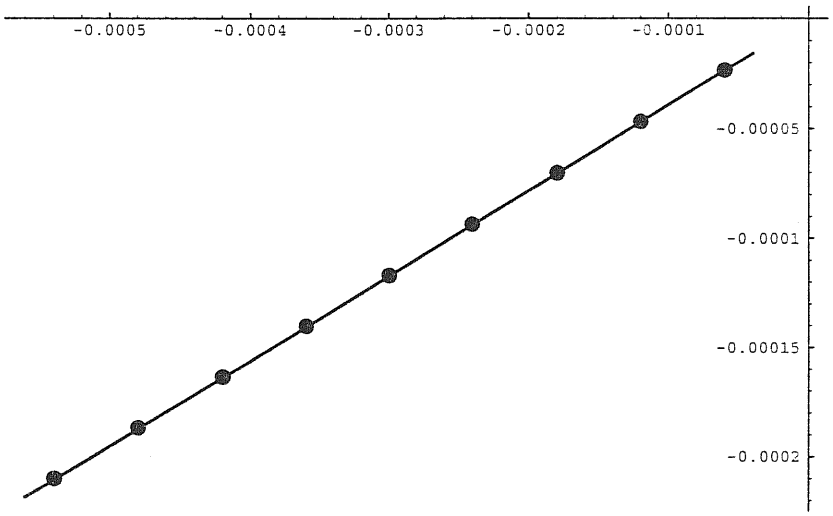

Figure 4.5: Numerical TCSA estimates of δm_1 versus δm_2 for different values of the "non–integrable" coupling in the model $(M_{29} + \varphi_{13}) + \varphi_{12}$. The continuous line represents the theoretical prediction.

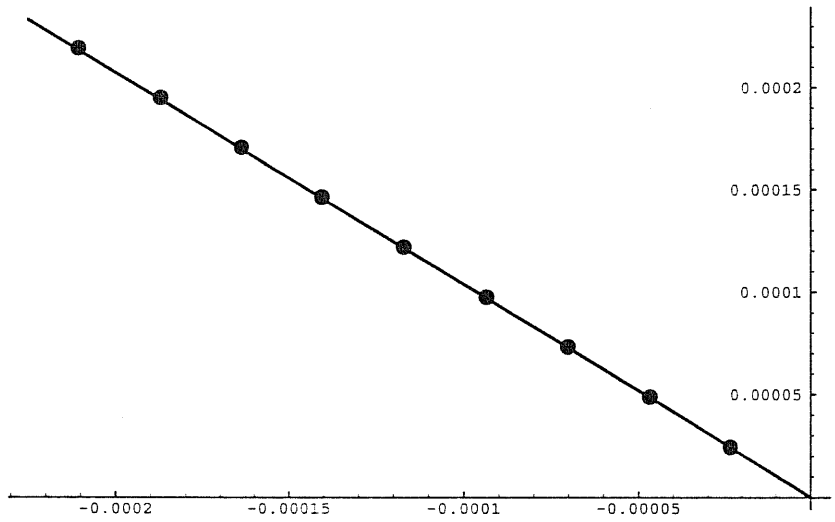

Figure 4.6: Numerical TCSA estimates of $\delta\mathcal{E}_0$ versus $m_1\delta m_1$ for different values of the "non-integrable" coupling in the model $(M_{29}+\varphi_{13})+\varphi_{12}$. The continuous line represents the theoretical prediction.

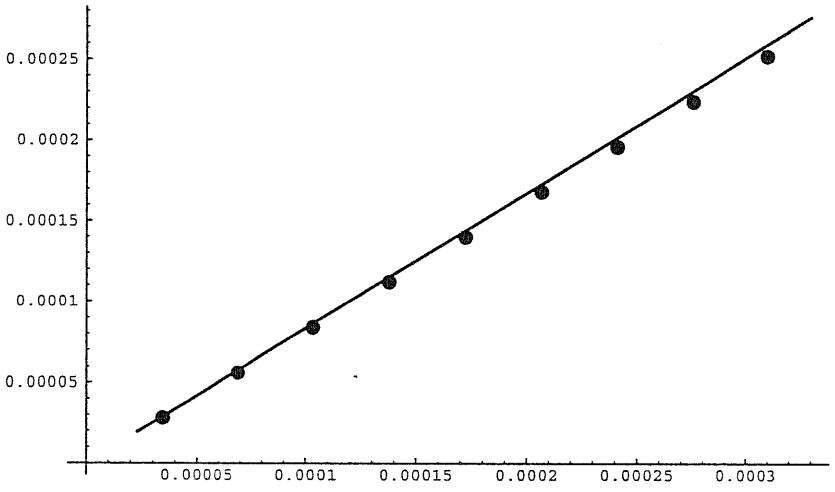

Figure 4.7: Numerical TCSA estimates of δm_1 versus δm_2 for different values of the "non-integrable" coupling in the model $(M_{29}+\varphi_{13})+\varphi_{14}$. The continuous line represents the theoretical prediction.

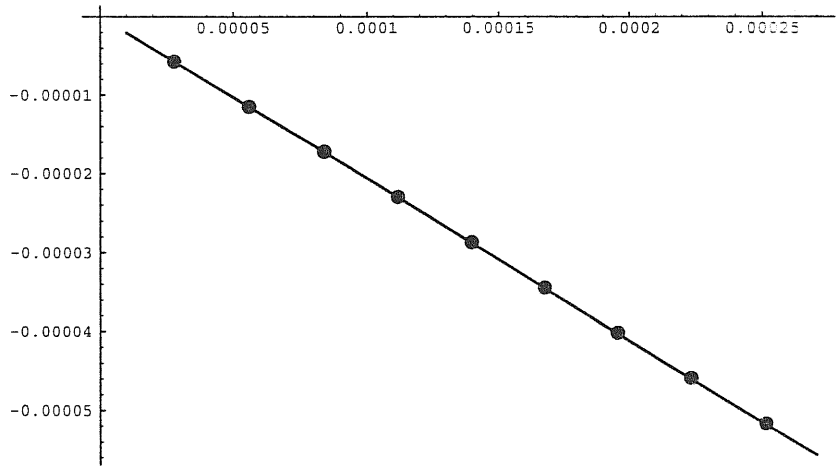

Figure 4.8: Numerical TCSA estimates of $\delta\mathcal{E}_0$ versus $m_1\delta m_1$ for different values of the "non-integrable" coupling in the model $(M_{29}+\varphi_{13})+\varphi_{14}$. The continuous line represents the theoretical prediction.

Chapter 5

The free massive boson with boundary

As discussed in the Introduction, we have seen that at the critical point, conformal invariance of the fixed point action implies that the correlation functions transform covariantly under any conformal map. While this condition in $d = 2$ allows to classify the operators of the theory according to the infinite dimensional Virasoro algebra [1, 2] at the same time it provides a powerful method to compare results at criticality obtained from different geometries. For instance, the results on the plane can be compared to those on the strip (or the cylinder), easier to obtain by using numerical computations. By means of a suitable conformal transformation one can also map the half–plane onto any possible geometry with boundary such as for example a "semi–infinite" cylinder cut on a circle with given boundary conditions [57]. Once the geometry of the system is restricted to half plane one can ask what happens to the Virasoro algebra and the classification of the fields and what are the possible boundary conditions that do not break the conformal invariance. The first question has been answered in [58] by showing that thanks to the conformal covariance of the correlation functions and by imposing that there is no flow of energy through the boundary, "half" of the Virasoro algebra survives. This result is obtained by considering the system as mirrored in the prohibited half–plane. This means that the n–point correlation functions at the bulk critical point satisfy a differential equation like that of the $2n$–point correlation functions in the bulk, but with appropriate boundary conditions. Thus a theory with a conformal boundary retains its infinite dimensional symmetry and the consequent classification of the fields according to the irreducible representations of the Virasoro algebra.

The other question, concerning what kind of boundary fields one can add without

breaking conformal invariance, has been answered in [59]. By translating into operatorial language the absence of flow of energy through the boundary it is possible to show that there is a one–to–one correspondence between the allowed boundary states and the Verma module of the irreducible representations of the Virasoro algebra. However the physical boundary states are obtained by taking advantage of the possibility to express in two different pictures the physics on the strip with two boundary conditions on the opposite edges. Namely, in a cylinder with boundary conditions α and β on the opposite edges, one can either consider the "time" direction *along* the cylinder with the time evolution governed by the boundary Hamiltonian $H_{\alpha\beta}$, or *across* the cylinder, with the bulk Hamiltonian H as evolution operator from the state $|\alpha\rangle$ to the state $|\beta\rangle$. These two pictures should describe the same physics, but in the first case the Hilbert space is made of boundary states while in the second case it is the bulk one. By equating the modular transformed partition function of the first case to the partition function of the second case, one obtains a relation that provides the physical boundary states.

Non–critical theories with boundaries are then obtained by deforming the CFT with conformal boundary conditions by adding a term that drives the theory outside criticality. Among the various deformation, of particular interest are those that leave the theory integrable i.e. allow the existence of an infinite number of conserved charges. The reason for this interest is due to the possibility to apply the methods of factorized scattering theory [6]. In the following we will consider only those deformations that lead to models with massive excitations.

For the theory with boundary outside criticality, one has to add a boundary term that does not break integrability. Although a general answer to the problem of classifying all the integrable boundary conditions has not yet been furnished (for a discussion see for example [60]), the methods developed in [19, 20] allow to translate and solve the problem in terms of boundary scattering theory. Starting from the Euclidean theory in the half–plane, one can go to the relativistic model with the boundary placed parallel to the time direction and compute the amplitude R, that depends on the momentum of the incoming particle, for the boundary–particle scattering (see Figure (5.1)). The bulk bootstrap and the factorization property are sufficient to fix a set of functional equations for the amplitude R. In this picture however one has to deal with the Hilbert space of the theory defined on a half–line and this clearly introduces big difficulties when one deals with the correlation functions. On the other hand, one could consider the boundary as placed in the time direction i.e. it becomes an initial state $|B\rangle$ at $t = 0$ that takes the place of the vacuum in the correlation functions (see Figure (5.2)). Since the particles

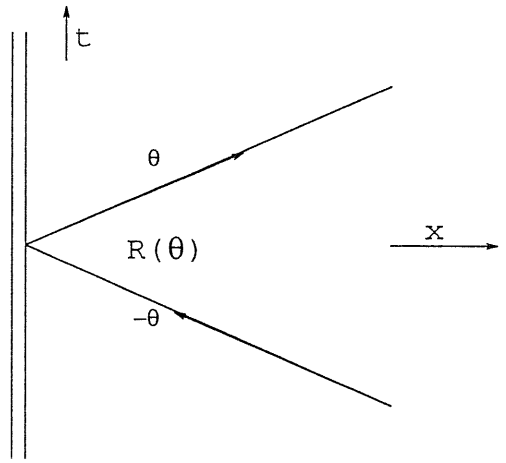

Figure 5.1: The particle hits the boundary coming from the right. $R(\theta)$ is the boundary–particle scattering amplitude.

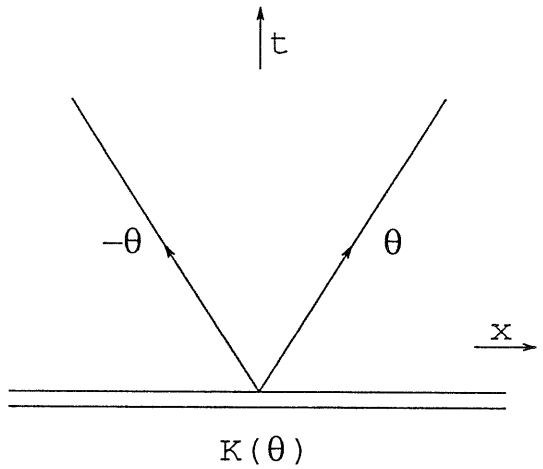

Figure 5.2: In the time direction the boundary is made of couples of particles emitted with opposite rapidities.

can move on the whole space axis, the Hilbert space is the same as in the bulk and the computation of the correlation functions is not more difficult than in absence of boundary. The only difficulty remains then the determination of the state $|B\rangle$. As we shall see the knowledge of R together with the constraint

$$\hat{p}|B\rangle = 0\,, \tag{5.0.1}$$

where $\hat{p}$ is the momentum operator, can solve the problem. Thus, once the correlation functions have been computed in this way, a simple Wick rotation gives also the solution for the Euclidean case.

All the same reasoning apply also when we have two boundaries, so that the Euclidean

system is closed between two parallel walls at say, $x = 0$ and $x = a$. With the boundary placed in the time direction then, the correlation functions become nothing more than the expectation value between the states $|B_0\rangle$ and $\langle B_a|$ (see Figure (5.3)). In this paper

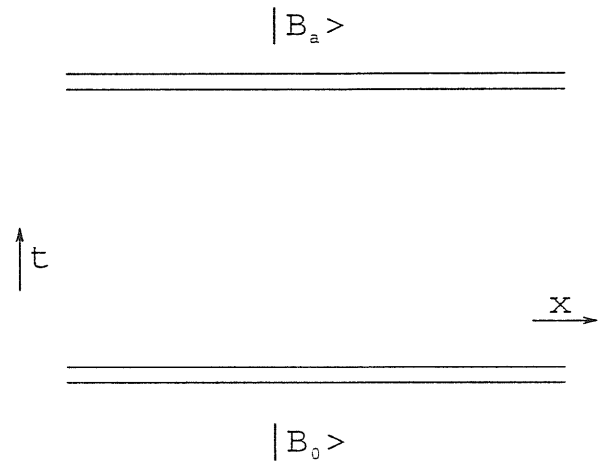

Figure 5.3: The case with two boundaries placed along the time direction.

we will review some physics systems that may be described as high–temperature Landau-Ginzburg models with boundaries. We will show how the Form Factors method can be used to provide some of the most relevant quantities.

5.1 Inserting the boundary

All the Green's functions computed in section (3.3.1) can be given also in the Euclidean case: it is sufficient to continue analytically to imaginary time $it = y$. After this rotation the coordinates are at all effects equivalent. We introduce then a boundary line, in the sense that we will restrict the system to half–plane and introduce into the Euclidean action a boundary operator such that it becomes

$$S_\kappa^E[\varphi] = 2\pi\kappa \int \mathrm{d}y \int_0^\infty \mathrm{d}x \frac{1}{2}\left((\nabla\varphi)^2 + \hat{m}^2\varphi^2 + \hat{h}\delta(x)\varphi^2\right) , \qquad (5.1.2)$$

where h is the strength of the external field coupled to φ^2 through a δ–function potential. The constants have been rescaled as in eq. (3.3.48) and as $\hat{h} = h/\kappa$. The boundary line is now parallel to the y axis, thus restricting the system to live in the half–plane $x > 0$. When one goes back to the Minkowsky space, one can freely decide which of the two directions will be the time. Thus the boundary can either be a true boundary in space or a sort of initial condition. This interpretation will allow us to solve the problem by using the powerful methods of boundary S–matrix [19, 20].

Let us suppose that the boundary is along the space direction so that the particles can only move in the half–line $x > 0$ while $-\infty < t < \infty$ i.e. we consider the y axis as the Euclidean time in the action (5.1.2) (Figure (5.2)). The main consequence of this choice is that the Hilbert space of states is different from the bulk case. Indeed, the Hamiltonian operator now is the integral over half space

$$\hat{H}^b = \frac{1}{2\pi}\int_0^\infty \mathrm{d}x\, T^b_{00}(x,t)\,, \tag{5.1.3}$$

where $T^b_{\mu\nu}$ is the stress-energy tensor of the theory with boundary, given by

$$T^b_{\mu\nu}(x,t) = T_{\mu\nu}(x,t) + \frac{1}{2}\, g_{\mu\nu}\,(2\pi h)\,\delta(x)\varphi^2(x,t)\,, \tag{5.1.4}$$

where $T_{\mu\nu}(x,t)$ is the bulk tensor. This means that the boson is no longer free and that the expansion (3.3.59) cannot be used. The same conclusion could have been inferred by deriving the equations that minimize the relativistic action, given by

$$\begin{gathered}\left(\partial_\mu\partial^\mu + \hat{m}^2\right)\varphi = 0 \\ \partial_x\varphi|_{x=0} = \hat{h}\,\varphi|_{x=0}\,,\end{gathered} \tag{5.1.5}$$

and verifying that the field (3.3.59) is not a solution of these equations.

Even if the Hamiltonian picture tells us that the Hilbert space of states and the vacuum $|0\rangle_b$ is changed with respect to the *bulk*, we can solve the problem by using scattering theory. Indeed the creation operators of the asymptotic particle defined in (3.3.52), can be used to define the incoming state

$$|\theta\rangle_b^{in} = A^\dagger(\theta)|0\rangle_b \quad \text{with} \quad \theta < 0 \tag{5.1.6}$$

of a particle moving towards the boundary in the negative direction of the x axis[1]. For an integrable theory with only one kind of self-conjugated particle, the scattering with the boundary is governed by the amplitude $R(\theta)$ defined by

$$\mathcal{B}\,A^\dagger(-\theta) = \mathcal{B}\,R(\theta)A^\dagger(\theta)\,, \tag{5.1.7}$$

where at the rhs we find the particle moving outwards the boundary. The symbol $\mathcal{B}$ stands for the boundary operator that project into the boundary Hilbert space

$$|0\rangle_b = \mathcal{B}|0\rangle\,. \tag{5.1.8}$$

[1]This is the reason for the negative sign of the rapidity.

Integrability implies also the factorization of the multi-particles boundary scattering amplitude into products of two–particles ones defined in eq. (5.1.7). This factorization can be expressed by means of the Yang–Baxter equation, that involves the bulk S–matrix. However in this case, since the bulk theory is free, the factorization condition is trivially satisfied. The amplitude $R(\theta)$ has to satisfy the unitarity condition that come from applying eq. (5.1.7) twice

$$R(\theta)R(-\theta) = 1\,. \tag{5.1.9}$$

With the vacuum defined as in (5.1.8) the Green's functions are given by

$$G_b(x,t) = \frac{{}_b\langle 0|\Phi(x,t)\Phi(0.0)|0\rangle_b}{{}_b\langle 0|0\rangle_b}\,. \tag{5.1.10}$$

Let us suppose now that the boundary line is placed in such a way that it becomes a sort of initial condition in the Minkowsky space: we choose the x axis of the action (5.1.2) as the Euclidean time (see Figure (5.3)). This means that the Hamiltonian is given by the integral over all space of $T^{00}(x,t)$. Thus the Hilbert space of states and the vacuum are the same as in the bulk theory. The initial boundary condition is given by applying an operator $\hat{B}$ to the vacuum, such that the Green's function in this picture is given by

$$G(x,t) = \frac{\langle 0|\Phi(x,t)\Phi(0,0)|B\rangle}{\langle 0|B\rangle}\,, \tag{5.1.11}$$

where

$$|B\rangle = \hat{B}|0\rangle\,. \tag{5.1.12}$$

Into (5.1.11) it is now possible to use the Form Factors computed for the bulk case, the only difficulty being the determination of the operator $\hat{B}$. The properties that this operator has to satisfy in order to preserve integrability of the system have been determined in [19, 20] and will be reviewed here briefly following the notation of the last reference.

Since the state (5.1.12) belongs to the bulk Hilbert space, it should be expressed by means of a superposition of the states defined in eq. (3.3.52), the coefficient of such a superposition being certain functions $K_n(\theta_1,\ldots,\theta_n)$. We have now to translate into the operatorial language the requirement that there cannot be flow of momentum through the boundary. This condition can indeed be expressed as

$$\hat{p}|B\rangle = 0\,, \tag{5.1.13}$$

where the momentum operator is defined in eq. (3.3.66). This constraint together with the action of $\hat{p}$ on the particle state defined in eq. (3.3.67), imply that the superposition

that gives rise to the boundary state should be made of couples of identical particles with opposite rapidities

$$|B\rangle \propto \sum_{n=1}^{\infty} \int \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \, K_n(\theta_1, \ldots, \theta_n) \, A^\dagger(\theta_1) A^\dagger(-\theta_1) \cdots A^\dagger(\theta_n) A^\dagger(-\theta_n) |0\rangle \,, \qquad (5.1.14)$$

where the fact that the bulk theory is free has been used in the ordering of the creation operators. The trivial factorization properties of this theory implies also that K_n can be written as products of two (identical) particles amplitudes $K(\theta)$. The relation between $R(\theta)$ and $K(\theta)$ is finally given by relating the two pictures with time and space interchanged. In terms of the rapidity this is equivalent to the transformation

$$\theta \to i\frac{\pi}{2} - \theta \,, \qquad (5.1.15)$$

thus leading to the following identity

$$K(\theta) = R(i\frac{\pi}{2} - \theta) \,. \qquad (5.1.16)$$

The boundary state can finally be written as

$$|B\rangle = \exp\left[\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} K(\theta) A^\dagger(-\theta) A^\dagger(\theta)\right] |0\rangle \,, \qquad (5.1.17)$$

provided that the boundary cross–unitary condition [20]

$$K(\theta) = K(-\theta) \,, \qquad (5.1.18)$$

is satisfied.

The amplitude $R(\theta)$ is given by inserting eq. (3.3.59) into the second of (5.1.5) and comparing with (5.1.7). The solution reads [21]

$$R(\theta) = \frac{i\hat{m}\sinh\theta + \hat{h}}{i\hat{m}\sinh\theta - \hat{h}} \,. \qquad (5.1.19)$$

It satisfies the unitarity condition (5.1.9) and through (5.1.16) gives

$$K(\theta) = \frac{\hat{m}\cosh\theta - \hat{h}}{\hat{m}\cosh\theta + \hat{h}} \,, \qquad (5.1.20)$$

thus defining completely the state (5.1.17).

We can analyze now the analytic structure of $R(\theta)$. In the following we will actually analyze only the pole structure, the positions of the zeros being determined by means of eq. (5.1.9). For $\hat{h}/\hat{m} > 1$ this function shows two single poles outside the physical strip

for $\mathrm{Im}(\theta) = -i\pi/2$ and $\mathrm{Re}(\theta) = \beta_+, \beta_-$ where $0 \leq \beta_+ < 2$ and $-\infty < \beta_- \leq 0$ are real numbers. The equality $\beta_+ = \beta_- = 0$ comes for $\hat{h}/\hat{m} = 1$ in which case $R(\theta)$ has only one single pole at $\theta = -i\pi/2$. For $\hat{h}/\hat{m} \to 0^+$ the poles move on the imaginary axis towards $-i\pi$ and 0. If $-\hat{m} < \hat{h} < 0$ we have poles on the physical strip $0 < \theta < i\pi$, for $\theta = iv$ and at the crossing symmetric position $\theta = i\pi - iv \equiv i\bar{v}$. The values of v and $\bar{v}$ are the two solutions of the following equation

$$\sin\alpha = |\hat{h}/\hat{m}|\,, \tag{5.1.21}$$

thus giving rise to the residue equation

$$-i \lim_{\theta \to i\alpha} R(\theta) = \frac{1}{2} g^2\,, \tag{5.1.22}$$

where g is given by

$$g^2 = \frac{4|\hat{h}|}{\sqrt{\hat{m}^2 - \hat{h}^2}}\,. \tag{5.1.23}$$

This pole corresponds to a boundary bound state in the direct channel (see Figure (5.4)), the energy of which is given by

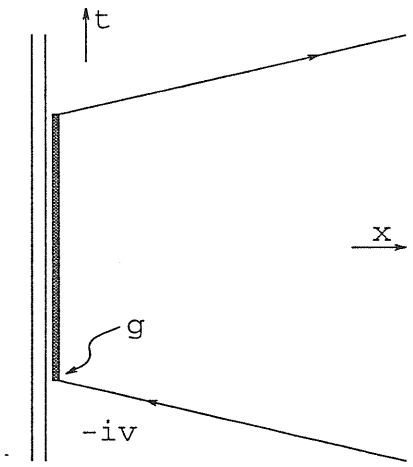

Figure 5.4: For $\theta = iv$ there is a boundary bound state. The constant g is the boundary–particle coupling.

$$e_1 = e_0 + \hat{m}\cos v\,, \tag{5.1.24}$$

where e_0 is the energy of the vacuum, determined by

$$\hat{H}^b|0\rangle_b = e_0|0\rangle_b\,, \tag{5.1.25}$$

where $\hat{H}^b$ is given by eq. (5.1.3). The "boundary–particle coupling" is instead given by g. One can notice that as far as h goes to zero the coupling (5.1.23) becomes smaller and vanish for $h = 0$ whereas the energy (5.1.24) is maximized to $e_1 = e_0 + \hat{m}$. For $\hat{h} \to -\hat{m}^+$ instead the coupling in eq. (5.1.23) diverges while the energy of the bound state becomes equal to that of the boundary ground state. Finally, for $\hat{h} < -\hat{m}$ the poles leave the physical strip moving on the axis $\mathrm{Im}(\theta) = i\pi/2$. One can notice however that for $\hat{h} \leq -\hat{m}$ we have a divergence also in the amplitude $K(\theta)$ in eq. (5.1.20) for real values of the rapidity, thus spoiling the stability of the bound state (5.1.17). As first noticed in [61], this values of $\hat{h}$ have to be considered as unphysical for this model. We will see in the next section the meaning of this limit in various statistical mechanical contexts.

Before going on with the computation of the correlation functions, let us introduce a second boundary at $x = a$. The Euclidean action that we are going to consider, is given by

$$S_\kappa^E[\varphi] = 2\pi\kappa \int \mathrm{d}y \int_0^a \mathrm{d}x \frac{1}{2} \left((\nabla\varphi)^2 + \hat{m}^2\varphi^2 + \hat{h}_0\delta(x)\varphi^2 + \hat{h}_a\delta(a-x)\varphi^2 \right) . \tag{5.1.26}$$

We have now to solve the equations

$$\begin{aligned} \partial_x\varphi|_{x=0} = \hat{h}_0\,\varphi|_{x=0} \qquad & \mathcal{B}_0\,A^\dagger(-\theta) = \mathcal{B}_0\,R_0(\theta)A^\dagger(\theta) \\ \partial_x\varphi|_{x=a} = -\hat{h}_a\,\varphi|_{x=a} \quad & A^\dagger(\theta)\mathcal{B}_a = R_a(\theta)A^\dagger(-\theta)\mathcal{B}_a\,. \end{aligned} \tag{5.1.27}$$

The solution of the first of (5.1.27) is given by (5.1.19) and (5.1.20) with $\hat{h}$ substituted by $\hat{h}_0$. The second of eq. (5.1.27), can be given by using the same method and reads

$$R_a(\theta) = \frac{i\hat{m}\sinh\theta + \hat{h}_a}{i\hat{m}\sinh\theta - \hat{h}_a} e^{2i\hat{m}a\sinh\theta} . \tag{5.1.28}$$

It satisfies the unitarity condition (5.1.9) and through (5.1.16) gives then

$$K_a(\theta) = \frac{\hat{m}\cosh\theta - \hat{h}_a}{\hat{m}\cosh\theta + \hat{h}_a} e^{-2\hat{m}a\cosh\theta} \tag{5.1.29}$$

thus defining completely the state (5.1.17) at $it = a$ (see Figure (5.3)). Once the states (5.1.17) at $it = 0$ and $it = a$ have been computed, one can proceed to the computation of the correlation functions with one and two boundaries. Consider for the time being the two–points correlation function of the field φ with one boundary

$$G_0(x - x_0, t - t_0) = \frac{\langle 0|\mathcal{T}\,\varphi(x,t)\varphi(x_0,t_0)|B_0\rangle}{\langle 0|B_0\rangle} . \tag{5.1.30}$$

By inserting the form factors computed before into the expansion (3.1.3)

$$G_0 =< 0|\varphi|1 >< 1|\varphi|0 >< 0|B_0 > + < 0|\varphi|1 >< 1|\varphi|2 >< 2|B_0 > , \tag{5.1.31}$$

where a simplified notations has been used, one obtains [21]

$$\begin{aligned} G_0(x-x_0,t-t_0) = \frac{1}{\kappa}\int_0^\infty \frac{\mathrm{d}\theta}{2\pi} e^{i\hat{m}(x-x_0)\sinh\theta}\Big(e^{-i\hat{m}|t-t_0|\cosh\theta} + \\ +F(\hat{m},\hat{h}_0,\theta)\, e^{-i\hat{m}(t+t_0)\cosh\theta}\Big) , \end{aligned} \tag{5.1.32}$$

where

$$F(\hat{m},\hat{h}_0,\theta) = \frac{\hat{m}\cosh\theta - \hat{h}_0}{\hat{m}\cosh\theta + \hat{h}_0} . \tag{5.1.33}$$

The case with two boundaries

$$G_0^a(x-x_0,t-t_0) = \frac{\langle B_a|\mathcal{T}\,\varphi(x,t)\varphi(x_0,t_0)|B_0\rangle}{\langle B_a|B_0\rangle} \tag{5.1.34}$$

requires a little more work. The use of the Form Factors allows us to expand it in a infinite sum that can be written as $G_0^a = G_{bu} + G_{bd}$, where (using a simplified notation as before)

$$\begin{aligned} G_{bu}(x-x_0,t-t_0) =< B_a|0 >< 0|\varphi|1 >< 1|\varphi|0 >< 0|B_0 >= \\ = \frac{1}{\kappa}\int_0^\infty \frac{\mathrm{d}\theta}{2\pi} e^{im(x-x_0)\sinh\theta} e^{-im|t-t_0|\cosh\theta} \end{aligned} \tag{5.1.35}$$

is nothing but the *bulk* correlation function, and

$$\begin{aligned} G_{bd} = \textstyle\sum_1^\infty < B_a|2n >< 2n|\varphi|2n-1 >< 2n-1|\varphi|2n-2 >< 2n-2|B_0 > + \\ + < B_a|2n >< 2n|\varphi|2n-1 >< 2n-1|\varphi|2n >< 2n|B_0 > + \\ + < B_a|2n >< 2n|\varphi|2n+1 >< 2n+1|\varphi|2n >< 2n|B_0 > + \\ + < B_a|2n-2 >< 2n-2|\varphi|2n-1 >< 2n-1|\varphi|2n >< 2n|B_0 > . \end{aligned} \tag{5.1.36}$$

This last can be easily computed and gives [22]

$$\begin{aligned} \frac{1}{\kappa}\sum_{n=0}^\infty \int_0^\infty \frac{\mathrm{d}\theta}{2\pi} e^{im(x-x_0)\sinh\theta}\Big[e^{-im(t+t_0)\cosh\theta} K_a^n K_0^{n+1} + e^{im(t+t_0)\cosh\theta} K_a^{n+1} K_0^n + \\ + \left(e^{im(t-t_0)\cosh\theta} + e^{im(t_0-t)\cosh\theta}\right) K_a^{n+1} K_0^{n+1}\Big] , \end{aligned} \tag{5.1.37}$$

such that G_0^a is given by the sum of (5.1.35) and (5.1.37). A few remarks about this result are in order. First of all one has to say that the two–boundary correlation function G_0^a in

the Euclidean continuation, has to satisfy the differential equations

$$\begin{gathered}(-\kappa\,\Delta\,+\,m^2)\;G(\mathbf{r})=\delta^{(2)}(\mathbf{r})\\ \partial_x G(0,y)=\hat{h}_0\,G(0,y)\,,\quad\text{and}\quad\partial_x G(a,y)=-\hat{h}_a\,G(0,y)\,,\end{gathered}\tag{5.1.38}$$

and it does indeed, as can be verified by direct calculation.

The second remark concerns the appearance of terms proportional to products of $\delta(0)$ in the contraction (5.1.36). This feature has been noticed also in [28], where the TBA equations for massive theories with two boundaries have been determined. In that case however, the theory was defined on a finite cylinder with the boundaries at the ends and the authors have been able to regularize the $\delta(0)$'s. In fact, the interpretation in [28] was that they were coming from the infinite volume definition of the boundary state (5.1.17). In our case instead we have found that the solution G_0^a computed above, satisfy (5.1.38) if one *throws away* all the terms containing the $\delta(0)$'s. The direct computation seems to suggest that they come when particles entering the FF expansion do not actually interact with the inserted field. This happens for example in presence of terms like

$$\langle\theta_1\dots\theta_n|\varphi|\beta_1\dots\beta_n\beta_{n+1}\rangle\sim\langle 0|\varphi|\beta_{n+1}\rangle\cdot\langle\theta_1\dots\theta_n|\beta_1\dots\beta_n\rangle\,,\tag{5.1.39}$$

where the particles $1,\dots,n$ behave as if the field was absent. This kinematic feature induces a direct interaction between the boundaries which is responsible for the δ's. However, we have noticed from the direct derivation that a re-summation takes place. In other words, attached to each of the terms in the sum (5.1.36), the final computation shows the multiplicative factor $\langle B_a|B_0\rangle$. Thus, the presence of the denominator in the definition of G_0^a cancels out this term together with all the $\delta(0)$ and gives the claimed results (5.1.35) and (5.1.37).

To end this section we give the FF with a boundary of the trace Θ and of the vertex operator $e^{\alpha\varphi}$. They can be given once the bulk FF (3.3.62) and (3.3.70) are known. For the trace we get

$$\langle 0|\Theta(x,t)|B_0\rangle=2\pi\hat{m}^2\int_0^\infty\frac{\mathrm{d}\theta}{2\pi}\,K_0(\theta)\,e^{-2i\hat{m}t\cosh\theta}\,,\tag{5.1.40}$$

where $K_0(\theta)$ is defined in eq. (5.1.20), and for the vertex operator one gets instead

$$\langle 0|e^{\alpha\varphi}(x,t)|B_0\rangle=\exp\left(\frac{\alpha^2}{2\kappa}\int_0^\infty\frac{\mathrm{d}\theta}{2\pi}\,K_0(\theta)\,e^{-2i\hat{m}t\cosh\theta}\right)\,.\tag{5.1.41}$$

5.2 Applications

In this section we would like to show how these computations turn out to be useful in some context. In the following we will briefly illustrate two problems of mathematical physics in $d = 2$: conduction of heat in a uniform solid with radiation boundary conditions and the Random Walk in presence of a boundary line with a potential. Finally we will discuss the Landau–Ginzburg approach to surface phase transitions.

5.2.1 Conduction of Heat in a Uniform Solid

Consider a two–dimensional sample of uniform conducting material with diffusivity κ without boundaries. At time $t = 0$ we have an instantaneous source of heat at the point $\mathbf{r}_0$. Let $v(\mathbf{r}, t)$ be the temperature of the sample at point $\mathbf{r}$ at time t, this function is solution of the equation

$$(-\kappa\Delta_{\mathbf{r}} + \partial_t)\, v(\mathbf{r}, \mathbf{r}_0; t) = 0 \tag{5.2.42}$$

with the additional condition

$$v(\mathbf{r}, \mathbf{r}_0; t = 0) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{5.2.43}$$

The Laplace transform of the temperature

$$\bar{v}(\mathbf{r}, \mathbf{r}_0; m^2) = \int_0^\infty \mathrm{d}t\, e^{-m^2 t} v(\mathbf{r}, \mathbf{r}_0; t)\,, \tag{5.2.44}$$

satisfies instead the equation

$$\left(-\kappa\Delta_{\mathbf{r}} + m^2\right)\, \bar{v} = \delta(\mathbf{r} - \mathbf{r}_0) \tag{5.2.45}$$

with the constraint that it has to vanish at infinity. This condition is nothing but the Euclidean version of eq. (3.3.51): this means that the problem of conduction of heat in the bulk can be solved by using the Euclidean field theory

$$S_\kappa^E[\varphi] = \kappa \int \mathrm{d}x\mathrm{d}y \frac{1}{2}\left((\nabla\varphi)^2 + \hat{m}^2\varphi^2\right)\,, \tag{5.2.46}$$

with $\bar{v}(x, t)$ given by the Euclidean continuation $it = y$ of equation (3.3.60) and $\hat{m}^2$ given by (3.3.48).

Let us put now a boundary surface at the point $x = 0$ so that the sample is placed in the half plane $x \geq 0$. The boundary conditions are chosen as follows: in the half plane $x < 0$ we suppose to have some other material at temperature v_0 into which our sample loses energy. This condition is expressed by the equation

$$\partial_x v|_{x=0} = \hat{h}\,(v - v_0)|_{x=0}\,, \tag{5.2.47}$$

which is called *radiation* boundary condition[2]. This is a gradient of temperature which is proportional to the difference of temperature between the surface and the surrounding medium. The proportionality constant is h : for $h = 0$ the gradient is zero and this means that there is no flow of energy, while for $h \to \infty$ the surface is taken at constant temperature $v = v_0$.

Suppose now that $v_0 = 0$. The above boundary equation (5.2.47) has to be satisfied also by the Laplace transform $\bar{v}$ which is solution of eq. (5.1.38). The solution is indeed given by eqs. (5.1.32, 5.1.33), where the change of coordinates

$$x \to y \quad \text{and} \quad it \to x \tag{5.2.48}$$

is required.

Notice that the range $h < 0$, the one for which the boundary amplitude $R(\theta)$ in eq. (5.1.19) shows a bound state pole, is unphysical in this context. Indeed it would mean that the boundary is radiating inside the sample, as if it would be excited [62].

Let us introduce now a second boundary parallel to the first in the position $x = a$. The sample is confined in the strip $0 \leq x \leq a$ with radiation boundary conditions on both surfaces. This conditions are encoded into the Euclidean action (5.1.26) and equations (5.1.27).

The Laplace transform of the temperature

$$\bar{v} = \frac{\langle B_a | \varphi(x,y)\varphi(x_0,y_0) | B_0 \rangle}{\langle B_a | B_0 \rangle}, \tag{5.2.49}$$

is then given by $\bar{v} = \bar{u} + \bar{w}$, where $\bar{u}$ is given by $G_{bu}(y - y_0, i(x - x_0))$ computed in eq. (5.1.35) and $\bar{w}$ is given by $G_{bd}(y - y_0, i(x - x_0))$, the Euclidean continuation of eq. (5.1.37). This general solution gives all the other possible solutions in the limits $h_0 = h_a = 0$ (free boundary conditions) or $h_0, h_a \to \infty$ (fixed boundary conditions) or by taking $a \to \infty$ [62].

5.2.2 Ideal Polymer with a Boundary Line

Ideal polymers are chains without self-avoidance, in other words they can be considered as the configuration of a random walk paths of a certain length N. The problem of ideal polymers near a surface in two dimensions has been treated extensively in [63, 64] and the application of the Form Factor formalism to this problem can be found in [21].

[2]For a detailed discussion see chap. 1 of [62]

Let us consider the two–dimensional random walk in the bulk with diffusion coefficient $\kappa = 1$. Its paths are possible ideal chains configurations described by the partition function

$$Z(\mathbf{r}, \mathbf{r}_0, N)\,, \tag{5.2.50}$$

solution of the equation

$$(\partial_t - \Delta_{\mathbf{r}})\, Z(\mathbf{r}, \mathbf{r}_0, N) = 0 \tag{5.2.51}$$

together with the boundary condition

$$Z(\mathbf{r}, \mathbf{r}_0, 0) = \delta^{(2)}(\mathbf{r} - \mathbf{r}_0)\,. \tag{5.2.52}$$

In fact this is the sum over all the configurations of a chain of length N starting at $\mathbf{r}$ and ending at $\mathbf{r}_0$. The Laplace transform $G(\mathbf{r}, \mathbf{r}_0, m^2)$ of Z is solution of (3.3.51) in the Euclidean continuation.

Suppose now to put a boundary line that constraints the walker into the half–plane $x \geq 0$. An elementary study of the configurations of the polymer, can show that the most probable ending point of a chain starting at the boundary is far from it: this means that the configurations near boundary are dominated by the entropic behavior. In order then to compensate it, we can put an attractive potential near the surface. One sees then that once the form of the potential has been fixed, the only remaining variable is the temperature: there exists a critical temperature T_c such that for $T < T_c$ we have the *adsorbed phase*, for $T > T_c$ we have the *non–adsorbed phase*. At the critical temperature the energetic attraction and the entropic repulsion compensate, this is called the *compensation point*. The potential can be chosen as

$$W = \begin{cases} \infty & \text{if } x \leq 0 \\ < 0 & \text{if } 0 < x < b \\ 0 & \text{if } x \geq b \end{cases} \tag{5.2.53}$$

and independent from the coordinate parallel to the boundary line, say y. The Green's function will be solution of the following equation

$$\left(-\Delta_{\mathbf{r}} + m^2 + W(x)\right)\, G(\mathbf{r}, \mathbf{r}_0; m^2) = \delta(\mathbf{r} - \mathbf{r}_0)\,, \tag{5.2.54}$$

with the additional constraint that it has to vanish at infinity, in the allowed half–plane. If one concentrates on the solution of (5.2.54) near the compensation point, for x, $x_0 > b$ and then let $b \to 0$, it can be cast in the universal form given by eq. (5.1.32, 5.1.33), where $\kappa = 1$ and the change (5.2.48) has been done.

The limit $b \to 0$ means that this solution does not take into account the microscopic analysis of the interaction, much like in the spirit of S–matrix approach for the particle models.

The form of the potential and the dependence on the temperature is now described by the function F of eq. (5.1.33). In particular the coupling h_0 is proportional to the temperature deviation from T_c: for $h \to \infty$ we describe the non–adsorbed phase while for $h = 0$ we are at the compensation point.

The regime $h < 0$ describes instead the adsorption phase, where the potential is deep enough to win the entropic repulsion of the hard–wall. However, we must have $h > -m$ in order to ensure the validity of the result. This can be explained starting from the direct computation of the limits into which the result of (5.2.54) can be cast in the universal form (5.1.32). These limits are indeed given by $-m < h < \infty$ [63, 64].

5.2.3 Landau–Ginzburg approach to surface phase transitions

The LG approach to phase transitions in presence of surfaces has been developed in the early 70's by the authors in ref. [66]. Here we will review a few aspects of the theory in order to establish a connection with the features described in the previous section. In particular we will see the meaning of the "coupling" h attached to the boundary and what happens when $h = -m$.

Let us begin with the bulk system described by the Euclidean action

$$S[\varphi] = \int \mathrm{d}x\mathrm{d}y \left[\frac{C}{2}(\partial_i\varphi)^2 + \frac{A}{2}\varphi^2 + \frac{B}{4}\varphi^4\right], \tag{5.2.55}$$

where the constants A, B, C are phenomenological constants with A depending linearly on the temperature as

$$A = A' t \quad \text{where} \quad t = \frac{T - T_c}{T_c} \tag{5.2.56}$$

and T_c is the bulk critical temperature. In presence of a constant external magnetic field H one has to add the term $-H\varphi$ inside the integral. Let us put now a boundary parallel to the y axis at $x = 0$, with an external boundary magnetic field H_1. From the molecular–field theory, the appropriate boundary conditions on the filed φ have been found by D.L. Mills [66] and are given by

$$\partial_x\varphi|_{x=0} = \lambda^{-1}\varphi|_{x=0} - \frac{H_1}{C}, \tag{5.2.57}$$

where one makes the hypothesis that the field φ at equilibrium does not depend on y. The constant λ, which has dimension of a length, is called the extrapolation length: this

name comes from the observation that for $\lambda > 0$ and $t \to 0^+$ the magnetization profile φ vanishes at a distance off the boundary given by λ. At criticality however one can consider the magnetization as vanishing on the boundary since $\lambda << \xi_b$, where $\xi_b = \sqrt{C/A}$ is the bulk correlation length that diverges for $t \to 0$.

The system with boundary is then described by the action

$$S[\varphi] = \int \mathrm{d}y \int_0^\infty \mathrm{d}x \left[\frac{C}{2}(\partial_i \varphi)^2 + \frac{A}{2}\varphi^2 + \frac{B}{4}\varphi^4 - H\varphi + \frac{C\lambda^{-1}}{2}\delta(x)\left(\varphi^2 - H_1\varphi\right) \right], \tag{5.2.58}$$

where one can notice that the integration over x is extended only to the positive half-line. The above action is minimized by the solution of the following equations

$$\begin{aligned} -C\,\Delta\varphi + A\,\varphi + B\varphi^3 &= H \\ \partial_x \varphi|_{x=0} &= \lambda^{-1}\varphi|_{x=0} - \frac{H_1}{C}\,. \end{aligned} \tag{5.2.59}$$

Before describing briefly the solution of the above system, one has to notice that it is the Euclidean version of the system (3.3.49), in presence of external fields H and H_1. However their presence will not disturb us very much because they will be used only in the derivation of the singular behavior of the susceptibilities and then put equal to zero.

Let us discuss the different kind of solutions, near criticality. For $t > 0$ we are in the disordered phase. Even if $H \neq 0$ but small, we can drop out the φ^3 term in the first equation of (5.2.59) while holding the same boundary condition. Moreover, thanks to the hypothesis that the solution of (5.2.59) does not depend on y, we can substitute $\Delta \to \partial_x^2$.

Let us consider first the case $\lambda > 0$. For $t \to 0^+$ one gets the so-called ordinary transition, in which the bulk orders while the magnetization profile vanishes off the boundary. This phase transition induces the following behavior on the susceptibilities

$$\begin{aligned} \chi_b &= \frac{\partial \varphi_b}{\partial H} \propto t^{-1}\,, \\ \chi_1 &= \frac{\partial \varphi_1}{\partial H} \propto \lambda\, t^{-1/2}\,, \\ \chi_{11} &= \frac{\partial \varphi_1}{\partial H_1} \sim \lambda + \mathcal{O}(t^{1/2})\,, \end{aligned}$$

where φ_b is the magnetization in the bulk, while φ_1 is its value on the boundary.

If $\lambda < 0$ instead one can observe a new feature: the susceptibilities show a divergence for a temperature higher than the bulk critical temperature. Indeed one gets

$$\chi_1,\, \chi_{11} \sim (\xi_b^{-1} - |\lambda^{-1}|)^{-1}\,, \tag{5.2.60}$$

where $\xi_b = \sqrt{C/A}$ is the bulk correlation length and its inverse can be thought of as the mass of the lighter particle (in our case the only particle) of the relativistic theory. It is clear that for a certain finite temperature there is a divergence in (5.2.60), this divergence signals the appearance of the surface transition i.e. the boundary orders at a temperature higher than T_c. However one can read this relation from another point of view: after having fixed the bulk temperature and as a consequence the value of the mass of the boson, one sees that for $\lambda^{-1} = -m$ we are in presence of an ordered surface, a situation for which the action (5.1.2) is not adequate. By comparing eq. (5.1.5) with (5.2.59) one can recognize that we have recovered the limit $h > -m$ shown before.

Conclusions

Going through many models we have shown in this thesis the powerful method of the Form Factors for the computation of the correlation functions applied to massive Integrable Quantum Field Theories. We have been able to show that although the off-shell physics is extremely complicated all the relevant informations can be given with very high accuracy and with a relatively easy computational effort. Indeed we have shown that the moment of the correlation functions (i.e. the sum–rules) can be safely truncated to the first few contributions by comparing the reached values with the exact ones when known. Moreover we have also written recursive relations that would allows us to compute any n–particle Form Factors and thus to reach any required precision in the computations. In particular in those integrable models without internal symmetries with the use of the cluster property together with the bound state equations we have been able to compute the Form Factors of all the relevant primary fields. These have then been checked in computing first order corrections of non–integrable perturbations.

The results shown in this thesis are also quite interesting in the statistical mechanics framework. Indeed we have computed the two point correlation function of the trace of the stress–energy tensor for the thermal deformation of the Tricritical Ising and Tricritical 3-states Potts models, re-derived the Form Factors of the series of non–unitary models $M_{2,2n+1} + \varphi_{13}$ showing a few limit cases and we have applied them to the same models with a soft breaking of integrability.

Moreover we have described how to apply the same technique to integrable massive models with boundaries. In this context we have described the connection with the Landau–Ginzburg theory for surface phase transition and shown, as an application, how to derive the Green's function of the Random Walk with boundary and attractive potential.

Through the problems solved while writing this thesis, we have seen many others that seem suited for future investigations. Among them, quite relevant is the one connected to the extension of the Form Factors technique to integrable models with massless excitations. In this case indeed it is possible to circumvent the obvious difficulties in defining

a consistent scattering theory by continuing analytically the massive Form Factors [67]. However this is not the most general case since it needs a massive theory to start from and for this reason it is still an open interesting problem as is shown by a recent result of P. Mejeal and F.A. Smirnov on the WZNW model [68].

Acknowledgements

I wish to acknowledge my supervisor G. Mussardo thanks to whom I have learned most of the topics described here and C. Acerbi for the fruitful collaboration during this last year. Many thanks go also to all my collegues the friendship of whom has been very important for me during my stay in SISSA. I would also like to thank the staff and the elementary particle sector.

Appendix A

Notations and conventions

In this appendix we will collect some of the most common and useful notations and conventions used throughout the thesis. It is worth writing them here because they usually require a lot of time when needed. It will be divided into two parts. In the first part we will set the notations for the transformations from Minkowsky space to the Euclidean and to the complex space. In the second we will derive in some detail the transformations that lead to the partition functions on the cylinder used in the TBA section.

A.1 Minkowsky, Euclidean and complex spaces

In this thesis we consider the Euclidean metric tensor is given by

$$g_E^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g^E_{\mu\nu}\,, \tag{A.1.1}$$

and stress–energy tensor components defined as

$$T_E^{\mu\nu} = T^E_{\mu\nu} = \begin{pmatrix} T_E^{11} & T_E^{12} \\ T_E^{12} & T_E^{22} \end{pmatrix}, \tag{A.1.2}$$

thus showing that $\Theta \equiv t^\mu_\mu = T_E^{11} + T^{22}$. The complex coordinates are defined as

$$z = x^1 + ix^2 \quad \bar{z} = x^1 - ix^2\,. \tag{A.1.3}$$

In terms of these the transformation matrix reads

$$\frac{\mathrm{d}x^\nu_C}{\mathrm{d}x^\mu_E} = \Gamma^\nu_\mu = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{A.1.4}$$

In this case we compute the metric tensors and the stress–energy tensor. For the metric tensors we have

$$g_C^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} , \tag{A.1.5}$$

and

$$g^C_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} . \tag{A.1.6}$$

The stress–energy tensor reads instead

$$T_C^{\mu\nu} = \begin{pmatrix} T^{zz} & T^{z\bar{z}} \\ \\ T^{z\bar{z}} & T^{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} T_E^{11} - T_E^{22} + 2iT_E^{12} & T_E^{11} + T_E^{22} \\ \\ T_E^{11} + T_E^{22} & T_E^{11} - T_E^{22} - 2iT_E^{12} \end{pmatrix} , \tag{A.1.7}$$

and

$$T^C_{\mu\nu} = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ \\ T_{z\bar{z}} & T_{\bar{z}\bar{z}} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} T_{11}^E - T_{22}^E - 2iT_{12}^E & T_{11}^E + T_{22}^E \\ \\ T_{11}^E + T_{22}^E & T_{11}^E - T_{22}^E + 2iT_{12}^E \end{pmatrix} , \tag{A.1.8}$$

thus now

$$\begin{gathered} \Theta = T_{11}^E + T_{22}^E = 4T_{z\bar{z}} \\ \\ \bar{T} \equiv T^{zz} = \tfrac{T_{\bar{z}\bar{z}}}{4} . \end{gathered} \tag{A.1.9}$$

Finally we give the conservation equation in complex coordinates

$$\begin{aligned} \partial_z \bar{T} + \tfrac{1}{4}\partial_{\bar{z}}\Theta &= 0 \\ \\ \partial_{\bar{z}} T + \tfrac{1}{4}\partial_z \Theta &= 0 . \end{aligned} \tag{A.1.10}$$

A.2 The system on a cylinder

The Euclidean two–momentum reads

$$\begin{aligned} H_y &= \frac{1}{2\pi} \int \mathrm{d}x \, T_E^{11}(x,y) \\ \\ P_x &= \frac{1}{2\pi} \int \mathrm{d}x \, T_E^{12}(x,y) , \end{aligned} \tag{A.2.11}$$

where H_y evolves the system along the positive y direction and P_x is responsible for translations in the positive x direction. However one could rotate the frame and consider

x as the Euclidean time, in which case the direction of y would be reversed leading to a sign change in the momentum as

$$\begin{aligned} H_x &= \frac{1}{2\pi}\int \mathrm{d}y\, T_E^{22}(x,y) \\ P_y &= -\frac{1}{2\pi}\int \mathrm{d}y\, T_E^{12}(x,y)\,. \end{aligned} \tag{A.2.12}$$

Let us compactify the space and consider the x direction as a circle of length L and the y direction a circle of length R. With an obvious notation the partition functions are given by

$$\begin{aligned} Z_x &= \mathrm{tr}_X\, \mathrm{e}^{-H_y R} \to \mathrm{e}^{-RLf(R)} \\ Z_y &= \mathrm{tr}_Y\, \mathrm{e}^{-H_x L} \to \mathrm{e}^{-E(R)L}\,, \end{aligned} \tag{A.2.13}$$

where we have written their form in the thermodynamic limit. This limit then lead to the equation (2.4.43). We are now going to derive an equation for the vacuum expectation value of the trace of the stress–energy tensor. Indeed observe that

$$\begin{aligned} \langle T_E^{11}\rangle &= -\frac{2\pi}{L}\frac{\mathrm{d}}{\mathrm{d}R}\log Z_x \to 2\pi\frac{\mathrm{d}E(R)}{\mathrm{d}R} \\ \langle T_E^{22}\rangle &= -\frac{2\pi}{R}\frac{\mathrm{d}}{\mathrm{d}L}\log Z_y \to 2\pi\frac{E(R)}{R}\,, \end{aligned} \tag{A.2.14}$$

that glue together give

$$\langle \Theta\rangle = 2\pi\frac{\mathrm{d}(RE(R))}{R\,\mathrm{d}R}\,. \tag{A.2.15}$$

Appendix B

Building Minimal Form Factors

In this appendix we collect some different explicit representations of the functions $g_\alpha(\theta)$ and $h_\alpha(\theta)$ together with some useful functional relations.

B.1 The functions $g_\alpha(\theta)$ and $h_\alpha(\theta)$

Let us start by considering the non-degenerate field theories. In this case, the basic functions g_α needed to build the minimal form factors are obtained as solution of the equations

$$\begin{aligned} g_\alpha(\theta) &= -f_\alpha(\theta)\, g_\alpha(-\theta)\ , \\ g_\alpha(i\pi+\theta) &= g_\alpha(i\pi-\theta)\ , \end{aligned} \tag{B.1.1}$$

where

$$f_\alpha(\theta) = \frac{\tanh\frac{1}{2}(\theta+i\pi\alpha)}{\tanh\frac{1}{2}(\theta-i\pi\alpha)}\ . \tag{B.1.2}$$

They are called minimal solutions because they do not present neither poles nor zeros in the strip $Im\theta \in (0, 2\pi)$. They admit several equivalent representations. The first is the integral representation given by

$$g_\alpha = \exp\left[2\int_0^\infty \frac{\mathrm{d}t}{t}\frac{\cosh\left[(\alpha-1/2)t\right]}{\cosh t/2\sinh t}\sin^2(\hat{\theta}t/2\pi)\right]\ , \tag{B.1.3}$$

where $\hat{\theta} = i\pi-\theta$. Equation (B.1.3) can be obtained as follows [10]. Consider the derivative of the logarithm of f_α as

$$\frac{1}{f_\alpha}\frac{\mathrm{d}f_\alpha(th)}{\mathrm{d}\theta} \equiv i\int_{-\infty}^{+\infty}\mathrm{d}t\, L(t)\mathrm{e}^{i\theta t}\ , \tag{B.1.4}$$

where we want to determine $L(t)$. It can be done by inverse Fourier transform giving

$$L(t) = \frac{\cosh[\pi t(1-2\alpha)/2]}{\cosh\frac{\pi t}{2}}, \tag{B.1.5}$$

thus letting the function (B.1.2) takes the following form

$$f_\alpha(\theta) = \exp\left[2\int_0^{+\infty}\frac{dt}{t}\frac{\cosh[t(1-2\alpha)/2]}{\cosh t/2}\sinh(\frac{\theta t}{i\pi})\right]. \tag{B.1.6}$$

Inserting this last result into (B.1.1) gives then (B.1.3). The analytic continuation of the (B.1.3) is provided by the infinite product representation

$$g_\alpha(\theta) = \prod_{k=0}^{\infty}\left[\frac{\left[1+\left(\frac{\hat\theta/2\pi}{k+1-\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat\theta/2\pi}{k+\frac{1}{2}+\frac{\alpha}{2}}\right)^2\right]}{\left[1+\left(\frac{\hat\theta/2\pi}{k+1+\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat\theta/2\pi}{k+\frac{3}{2}-\frac{\alpha}{2}}\right)^2\right]}\right]^{k+1}, \tag{B.1.7}$$

which explicitly shows the position of the infinite number of poles outside the strip $Im\theta \in (0, 2\pi)$. Another useful representation particularly suitable for deriving functional equations is the following:

$$g_\alpha(\theta) = \prod_{k=0}^{\infty}\frac{\Gamma^2\left(\frac{1}{2}+k+\frac{\alpha}{2}\right)\Gamma^2\left(1+k-\frac{\alpha}{2}\right)}{\Gamma^2\left(\frac{3}{2}+k-\frac{\alpha}{2}\right)\Gamma^2\left(1+k+\frac{\alpha}{2}\right)}\left|\frac{\Gamma\left(1+k+\frac{\alpha}{2}+i\frac{\hat\theta}{2\pi}\right)\Gamma\left(\frac{3}{2}+k-\frac{\alpha}{2}+i\frac{\hat\theta}{2\pi}\right)}{\Gamma\left(1+k-\frac{\alpha}{2}+i\frac{\hat\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+k+\frac{\alpha}{2}+i\frac{\hat\theta}{2\pi}\right)}\right|^2, \tag{B.1.8}$$

where we have used the notation

$$\left|\Gamma(a+i\hat\theta/2\pi)\right|^2 \equiv \Gamma(a+i\hat\theta/2\pi)\,\Gamma(a-i\hat\theta/2\pi). \tag{B.1.9}$$

A representation that is particularly suitable for numerical evaluations is the mixed one

$$g_\alpha(\theta) = \prod_{k=0}^{N-1}\left[\frac{\left[1+\left(\frac{\hat\theta/2\pi}{k+1-\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat\theta/2\pi}{k+\frac{1}{2}+\frac{\alpha}{2}}\right)^2\right]}{\left[1+\left(\frac{\hat\theta/2\pi}{k+1+\frac{\alpha}{2}}\right)^2\right]\left[1+\left(\frac{\hat\theta/2\pi}{k+\frac{3}{2}-\frac{\alpha}{2}}\right)^2\right]}\right]^{k+1} \times \tag{B.1.10}$$

$$\times \exp\left[2\int_0^{\infty}\frac{dt}{t}\frac{\cosh\left[\frac{t}{2}(1-2\alpha)\right]}{\cosh\frac{t}{2}\sinh t}(N+1-Ne^{-2t})e^{-2Nt}\sin^2\frac{\hat\theta t}{2\pi}\right].$$

In this formula N is an arbitrary integer number which may be adopted to obtain a fast convergence of the integral.

Using the integral representation (B.1.3), it is easy to establish the asymptotic behavior of g_α

$$g_\alpha(\theta) \sim e^{|\theta|/2} \qquad \text{for} \qquad \theta \to \infty. \tag{B.1.11}$$

The function g_α is normalized according to

$$g_\alpha(i\pi) = 1 \ , \tag{B.1.12}$$

and satisfies

$$g_\alpha(\theta) = g_{1-\alpha}(\theta) \ , \tag{B.1.13}$$

with

$$g_0(\theta) = g_1(\theta) = -i \sinh\frac{\theta}{2} \ . \tag{B.1.14}$$

The above functions satisfy the following set of functional equations

$$g_\alpha(\theta + i\pi) g_\alpha(\theta) = -i \frac{g_\alpha(0)}{\sin\pi\alpha} (\sinh\theta + i \sin\pi\alpha) \ . \tag{B.1.15}$$

$$g_\alpha(\theta + i\pi\gamma) g_\alpha(\theta - i\pi\gamma) = \left(\frac{g_\alpha(i\pi\gamma) g_\alpha(-i\pi\gamma)}{g_{\alpha+\gamma}(0) g_{\alpha-\gamma}(0)} \right) g_{\alpha+\gamma}(\theta) g_{\alpha-\gamma}(\theta) \ , \tag{B.1.16}$$

$$g_\alpha(\theta)\, g_{-\alpha}(\theta) = \mathcal{P}_\alpha(\theta) . \tag{B.1.17}$$

Let us turn our attention to the field theories with a degenerate mass spectrum. In complete analogy with the previous case, we start our analysis from the minimal solutions of the equations

$$\begin{aligned} h_\alpha(\theta) &= -s_\alpha(\theta)\, h_\alpha(-\theta) \\ h_\alpha(i\pi + \theta) &= h_\alpha(i\pi - \theta), \end{aligned} \tag{B.1.18}$$

where

$$s_\alpha(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\pi\alpha)}{\sinh\frac{1}{2}(\theta - i\pi\alpha)} \ . \tag{B.1.19}$$

The function $h_\alpha(\theta)$ is explicitly given in terms of the following equivalent representations

$$h_\alpha(\theta) = \exp\left[2 \int_0^\infty \frac{\mathrm{d}t}{t} \, \frac{\sinh[(1-\alpha)t]}{\sinh^2 t} \sin^2(\hat{\theta} t/2\pi) \right] , \tag{B.1.20}$$

$$h_\alpha(\theta) = \prod_{k=0}^{\infty} \left(\frac{1 + \left(\frac{\hat{\theta}}{2\pi} \right)^2}{1 + \left(\frac{\hat{\theta}}{n + \frac{3}{2} - \frac{\alpha}{2}} \right)^2} \right)^{k+1} \ , \tag{B.1.21}$$

$$h_\alpha(\theta) = \prod_{k=0}^{\infty} \frac{\Gamma^2(k + \frac{1}{2} + \frac{\alpha}{2}) \Gamma(k + 1 - \frac{\alpha}{2} - \frac{i\theta}{2\pi}) \Gamma(k + 2 - \frac{\alpha}{2} + \frac{i\theta}{2\pi})}{\Gamma^2(k + \frac{3}{2} - \frac{\alpha}{2}) \Gamma(k + \frac{\alpha}{2} - \frac{i\theta}{2\pi}) \Gamma(k + 1 + \frac{\alpha}{2} + \frac{i\theta}{2\pi})} \ . \tag{B.1.22}$$

The mixed representation is in this case

$$h_\alpha(\theta) = \prod_{k=0}^{N+1} \left(\frac{1 + \left(\frac{\frac{\hat{\theta}}{2\pi}}{n+\frac{1}{2}+\frac{\alpha}{2}} \right)^2}{1 + \left(\frac{\frac{\hat{\theta}}{2\pi}}{n+\frac{3}{2}-\frac{\alpha}{2}} \right)^2} \right)^{k+1} \times \tag{B.1.23}$$

$$\times \exp\left[2 \int_0^\infty \frac{\mathrm{d}t}{t} \left(N + 1 - N\, e^{-2t}\right) e^{-2Nt} \, \frac{\sinh\left[(1-\alpha)t\right]}{\sinh^2 t} \, \sin^2(\hat{\theta}t/2\pi) \right] .$$

and the asymptotic behavior depends on the value of α

$$h_\alpha(\theta) \sim e^{\frac{(1-\alpha)|\theta|}{2}} \text{ for } \theta \to \infty . \tag{B.1.24}$$

The function h_α is normalized according to

$$h_\alpha(i\pi) = 1 \tag{B.1.25}$$

and satisfies the following functional equations:

$$\begin{aligned} h_\alpha(2\pi i - \theta) &= h_\alpha(\theta), \\ h_0(\theta) &= -i \sinh(\theta/2), \\ h_1(\theta) &= 1, \\ h_{1+\alpha}(\theta) &= h_{1-\alpha}^{-1}(\theta), \end{aligned} \tag{B.1.26}$$

The basic "composition rules" for products of h_α's are:

$$\begin{aligned} h_\alpha(\theta)\, h_{-\alpha}(\theta) &= \mathcal{P}_\alpha(\theta), \\ h_\alpha(\theta + i\pi\gamma)\, h_\alpha(\theta - i\pi\gamma) &= \frac{h_\alpha(i\pi\gamma)\, h_\alpha(-i\pi\gamma)}{h_{\alpha+\gamma}(0)\, h_{\alpha-\gamma}(0)}\, h_{\alpha+\gamma}(\theta)\, h_{\alpha-\gamma}(\theta) \\ h_\alpha(\theta + i\pi)\, h_{1-\alpha}(\theta) &= \frac{h_{1-\alpha}(0)}{\cosh(\frac{i\pi\alpha}{2})} \cosh\frac{\theta - i\pi\alpha}{2} \end{aligned} \tag{B.1.27}$$

where the polynomial $\mathcal{P}$ is defined in (3.1.18) of Section 2.

Finally, since $f_\alpha(\theta) = s_\alpha(\theta) s_{1-\alpha}(\theta)$, the function g_α can be obtained from the h_α's simply through:

$$g_\alpha(\theta) = h_\alpha(\theta)\, h_{1-\alpha}(\theta). \tag{B.1.28}$$

Appendix C

Tricritical Potts Model

In this appendix we briefly report the results of the three–particle FF relevant for our computation in the TPM.

C.1 Three–particle Form Factors

These FF have been derived by applying the residue equations (3.1.14) to the four–particles FF $F^{\Theta}_{l\bar{l}l\bar{l}}$, as explained in section 4.2. In writing their final form, we have extensively used the formulas reported in Appendix A. The two–particle minimal FF F^{min}_{ab} appearing in the expressions which follow are defined by eq. (3.1.11) while the D_{ab} factors parameterizing the dynamical poles are defined by eq. (3.1.19).

The FF F^{Θ}_{lll} is obtained from $F^{\Theta}_{l\bar{l}l\bar{l}}$ through the residue equation at $u^{l}_{l\bar{l}} = 2i\pi/3$

$$F^{\Theta}_{lll}(\theta_1,\theta_2,\theta_3) = \left(\prod_{i<j}\frac{F^{min}_{ll}(\theta_{ij})}{D_{ll}(\theta_{ij})}\right)\left(3\,m_l^2 + 2\,m_l^2\sum_{i<j}\cosh(\theta_{ij})\right)\,a^0_{lll}\,. \qquad \text{(C.1.1)}$$

In this expression one immediately recognizes the "minimal" part, the dynamical poles and the P^2 polynomial, while the only remaining polynomial in the $\cosh(\theta_{ij})$'s allowed by eq. (3.1.29) is simply a constant given by

$$a^0_{lll} = -102.3375342\ldots\,.$$

The FF $F^{\Theta}_{l\bar{l}L}$, is obtained from $F^{\Theta}_{l\bar{l}l\bar{l}}$ by using eq.(3.1.14), with $u^{L}_{l\bar{l}} = i\pi/2$. Its final expression is given by

$$F^{\Theta}_{l\bar{l}L}(\theta_1,\theta_2,\theta_3) = \frac{F^{min}_{l\bar{l}}(\theta_{12})\,F^{min}_{lL}(\theta_{13})\,F^{min}_{\bar{l}L}(\theta_{23})}{D_{l\bar{l}}(\theta_{12})\,D_{l\,L}(\theta_{13})\,D_{\bar{l}\,L}(\theta_{23})}\,.$$

$$\cdot\frac{2\,m_l^2+m_L^2+2m_l^2\cosh(\theta_{12})+2m_l\,m_L\Big(\cosh(\theta_{13})+\cosh(\theta_{23})\Big)}{\cosh(\theta_{13})+\cosh(\theta_{23})}\cdot \tag{C.1.2}$$

$$\cdot\Big(a^0_{l\bar{l}L}\left(1-\cosh(\theta_{12})+2\,\cosh(\theta_{13})\,\cosh(\theta_{23})\right)+a^1_{l\bar{l}L}\Big(\cosh(\theta_{13})+\cosh(\theta_{23})\Big)\Big)\,.$$

This expression also exhibits a kinematic pole due to the presence of a particle–antiparticle pair $l\,\bar{l}$. Moreover there is a nontrivial polynomial in the $\cosh(\theta_{ij})$'s with coefficients given by

$$a^0_{l\bar{l}L}=-70.50661963\ldots\,,$$

$$a^1_{l\bar{l}L}=-235.9197474\ldots\,.$$

Finally, applying eq.(3.1.14) to $F^{\Theta}_{l\bar{l}l\bar{l}}$ at $u^h_{l\bar{l}}=i\pi/6$ one obtains

$$F^{\Theta}_{l\,l\,h}(\theta_1,\theta_2,\theta_3)=\frac{F^{min}_{ll}(\theta_{12})\,F^{min}_{lh}(\theta_{13})\,F^{min}_{lh}(\theta_{23})}{D_{ll}(\theta_{12})\,D_{l\,h}(\theta_{13})\,D_{l\,h}(\theta_{23})}\,. \tag{C.1.3}$$

$$\cdot\Big(2\,m_l^2+m_h^2+2m_l^2\cosh(\theta_{12})+2m_l\,m_h\Big(\cosh(\theta_{13})+\cosh(\theta_{23})\Big)\Big)\cdot$$

$$\cdot\Big(a^0_{l\,l\,h}+a^1_{l\,l\,h}\left(\cosh(\theta_{13})+\cosh(\theta_{23})\right)+a^2_{l\,l\,h}\,\cosh(\theta_{12})+a^3_{l\,l\,h}\,\cosh(\theta_{13})\,\cosh(\theta_{23})\Big)$$

where the coefficients $a^k_{l\,l\,h}$ are given by

$$a^0_{l\,l\,h}=78134.00044\ldots\,,$$

$$a^1_{l\,l\,h}=72661.45729\ldots\,,$$

$$a^2_{l\,l\,h}=31793.68905\ldots\,,$$

$$a^3_{l\,l\,h}=43430.98692\ldots\,.$$

Bibliography

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B 241** (1984). 333

[2] D. Friedan. Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **52** (1984), 1575; C. Itzykson. H. Saleur and J.B. Zuber, *Conformal Invariance and Applications to Statistical Mechanics*, (World Scientific, Singapore 1988) and references therein.

[3] K.G. Wilson and J. Kogut, *Phys. Rep.* **12** (1974) 200

[4] A.B. Zamolodchikov, *Advanced Studies in Pure Mathematics* **19** (1989) 641

[5] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* **46(6)** (1987) 1090

[6] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann. Phys.* **120** (1979) 253

[7] J.L. Cardy and G. Mussardo, *Phys. Lett.* **B225** (1989) 275

[8] G. Mussardo, *Phys. Rep.* **218** (1992) 215

[9] Al.B. Zamolodchikov, *Nucl. Phys.* **B342** (1990) 695

[10] B. Berg, M. Karowsky and P. Weisz, *Phys. Rev.* **D19** (1979) 2477; M. Karowsky and P. Weisz, *Nucl. Phys.* **B139** (1978) 445; M. Karowsky, *Phys. Rep.* **49** (1979) 229.

[11] F.A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theories* (World Scientific) 1992, and references therein.

[12] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 455** (1995), 724.

[13] C. Acerbi, G. Mussardo and A. Valleriani, *Form Factors and Correlation Functions of the Stress–Energy Tensor in Massive Deformation of the Minimal Models* $(E_n)_1 \otimes (E_n)_1/(E_n)_2$ ISAS/EP/95/161, hep-th/9601113 (to appear on *Int. J. Mod. Phys.* **A**).

[14] J.L. Cardy, *Phys. Rev. Lett.* **60** (1988) 2709

[15] G. Delfino, P. Simonetti and J.L. Cardy, *Asymptotic Factorisation of Form Factors in Two–Dimensional Quantum Field Theory*, OUTP-96-25S, SWAT/95-96/123

[16] C. Acerbi, G. Mussardo and A. Valleriani, *On the Form Factors of Relevant Operators and their Cluster Property*. ISAS/EP/96/105, IC/96/168, hep-th/9609080

[17] G. Delfino, G. Mussardo and P. Simonetti, *Nucl. Phys.* **B473** (1996) 429

[18] V.P. Yurov and Al.B. Zamolodchikov, *Int. J. Mod. Phys.* **A 6** (1991), 4557: M. Lässig, G. Mussardo and J.L. Cardy, *Nucl. Phys.* **B348** (1991), 591: M. Lässig and G. Mussardo, *Computer Phys. Comm.* **66** (1991), 71.

[19] A. Fring and R. Koberle, *Nucl. Phys.* **B421** (1994) 159

[20] S. Ghoshal and A. Zamolodchikov, *Int. J. Mod. Phys.* **A9**, 21 (1994) 3841

[21] A. Valleriani, *J. Phys.* **A29 (16)** (1996) 4885

[22] A. Valleriani, *Applications of the Form Factors Approach to Free Massive Boson with Boundaries*

[23] D. Stauffer, M. Ferrer and M. Wortis, *Phys. Rev. Lett.* **29** (1972) 345

[24] V.A. Fateev, *Phys. Lett.* **B324** (1994) 45

[25] P.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982

[26] R. Dashen, S.K. Ma and H.J. Bernstein, *Phys. Rev.* **187** (1969) 345; C.N. Yang, Phys. Rev. Lett. **19** (1967) 1312; C.N. Yang and C.P. Yang, Journ. Math. Phys. **10** (1969) 1115

[27] V.V. Bazhanov, S.L. Lukyanov and Al.B. Zamolodchikov, *Commun.Math.Phys.* **177** (1996) 381; D. Fioravanti, F. Ravanini and M. Stanishkov, *Phys.Lett.* **B367** (1996) 113; V.V. Bazhanov, S.L. Lukyanov and Al.B. Zamolodchikov, *Integrable Structure of Conformal Field Theory 2. Q Operator and DDV Equation* hep-th/9604044; *Integrable Quantum Field Theories in Finite Volume: Excited State Energies* hep-th/9607099

[28] A. LeClair, G. Mussardo, H. Saleur and S. Skorik, *Nucl. Phys.* **B453** (1995) 581

[29] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B350** (1991) 635

[30] F.A. Smirnov, *Nucl. Phys.* **B453** (1995) 807

[31] K.M. Watson, *Phys. Rev.* **95** (1954) 228

[32] T. Oota, *Nucl.Phys.* **B466** (1996) 361

[33] G. Delfino, G. Mussardo and P. Simonetti, *Phys. Rev.* **D51** (1995) R6620

[34] Al.B. Zamolodchikov, *Nucl. Phys.* **B358** (1991) 524; Al.B. Zamolodchikov and A.B. Zamolodchikov, *Nucl. Phys.* **B379** (1992) 602

[35] F.A. Smirnov, *Nucl. Phys.* **B337** (1990) 156; A. Koubek and G. Mussardo, *Phys. Lett.* **B311** (1993) 193; G. Mussardo and P. Simonetti, *Int. J. Mod. Phys.* **A9** (1994) 3307

[36] C.N. Yang and T.D.Lee, *Phys. Rev.* **87** (1952) 404; T.D. Lee and C.N. Yang, *Phys. Rev.* **87** (1952) 410

[37] J.L. Cardy, *Phys. Rev. Lett.* **54** (1985) 1354

[38] Al.B. Zamolodchikov, *Nucl. Phys.* **B348** (1991) 619

[39] I.G. MacDonald, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxfod, 1979)

[40] V.A. Fateev and A.B. Zamolodchikov *Int. J. Mod. Phys.* **A5** (1990) 1025

[41] P. Christe and G. Mussardo, *Nucl. Phys.* **B330** (1990) 465.

[42] M. Blume, *Phys. Rev.* **141** (1966), 517; H.W. Capel, *Physica* **32** (1966), 866; R.B. Griffiths, *Physica* **33** (1967), 689; M. Blume, V.J. Emery and R.B. Griffiths, *Phys. Rev.* **A4** (1971), 1071; M.T. Tejwani, O. Ferreira and O.E. Vilches, *Phys. Rev. Lett.* **44** (1980), 152.

[43] I.D. Lawrie and S. Sarbach, in *Phase Transitions and critical phenomena*, vol. 9, ed. C. Domb and J. Lebowitz (Academic Press, New York, 1984).

[44] G. von Gehlen, *Nucl. Phys.* **B330** (1990), 741; M. Henkel, *Phys. Lett.* **B247** (1990), 567; M. Lässig, G. Mussardo and J.L. Cardy, *Nucl. Phys.* **B348** (1991), 591.

[45] D. Friedan, Z. Qiu and S.H. Shenker, *Phys. Lett.* **B151** (1985), 37; Z. Qiu, *Nucl. Phys.* **B270** [FS16] (1986), 205; G. Mussardo, G. Sotkov and M. Stanishov, *Nucl. Phys.* **B305** [FS23] (1988), 69; D.A. Kastor, E.J. Martinec and S.H. Shenker, *Nucl. Phys.* **B316** (1989), 590.

[46] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* **44** (1986), 529.

[47] P. Goddard, A. Kent and D. Olive, *Comm. Math. Phys.* **103** (1986), 105; P. Goddard and D. Olive, *Int. J. Mod. Phys.* **A1** (1986), 304.

[48] R. Slansky, *Physics Reports* **79** (1981), 1; W. G. McKay and J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Lecture Note in Pure and Applied Mathematics (1981).

[49] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689; *Nucl. Phys.* **B356** (1991) 469

[50] G. Sotkov and C.J. Zhu, *Phys. Lett.* B229 (1989),391.

[51] T. Eguchi and S.K. Yang, *Phys. Lett.* **224B** (1989), 373; T.J. Hollowood and P. Mansfield, *Phys. Lett.* **226B** (1989), 73.

[52] P.G. Freund, T.R. Klassen and E. Melzer, *Phys. Lett.* **B229** (1989) 243

[53] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B338** (1990) 485

[54] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B340** (1990) 387

[55] A. Koubek, *Nucl. Phys.* **B435** (1995) 807; A. Koubek, *Nucl. Phys.* **B428** (1994) 655

[56] M.J. Martins, *Int. J. Mod. Phys.* **A7** (1992) 7753

[57] J.L. Cardy and D.C. Lewellen, *Phys. Lett.* **B259** (1991) 274

[58] J.L. Cardy, *Nucl. Phys.* **B240** (1984) 514

[59] J.L. Cardy, *Nucl. Phys.* **B324** (1989) 581

[60] E. Corrigan, *Reflections*, DTP-96/1, hep-th/9601055, Talk given at 7th Regional Conference on Mathematical Physics, Bandar e Anzali, Iran, 15-22 Oct 1995.

[61] G. Delfino, G. Mussardo and P. Simonetti, *Nucl. Phys.* **B432** [FS] (1994) 518

[62] H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids* (Oxford University Press. 1959)

[63] E. Eisenriegler, K. Kremer and K. Binder, *J. Chem. Phys.* **77**, (1982), 6269

[64] E. Eisenriegler, *Polymers Near Surfaces* (World Scientific Publishing, 1993)

[65] P.G. de Gennes , *Phys. Lett.*, **38A** (1972) 339

[66] D.L. Mills, *Phys. Rev.* **B3**, 11 (1971) 3887;
K. Binder and P.C. Hohenberg, *Phys. Rev.* **B6**, 9 (1972) 3461;
T.C. Lubensky and M.H. Rubin, *Phys. Rev.* **B12**(1975) 3885;
A.J. Bray and M.A. Moore, *J. Phys.* **A10** (1977) 1927;
K. Binder, in *Phase Transitions and Critical Phenomena*, Vol. 8 (C. Domb and J.L. Lebowitz,eds), Academic Press, London (1983) p. 1

[67] G. Delfino, G. Mussardo and P. Simonetti, *Phys. Rev.* **D51** (1995) 51

[68] P. Mejean and F.A. Smirnov, *Form Factors for principal chiral field model with Wess-Zumino-Novikov-Witten term.* LPTHE-96-36, hep-th/9609068

List of Tables

Table 12 One–particle Form Factors of the model $M_{29} + \varphi_{13}$.

Table 13 Coefficients entering the definitions of the two–particle Form Factors for $M_{29} + \varphi_{13}$.

Table 14 Comparison between theoretical and numerical estimates of data obtained in different non–integrable deformations of $M_{29} + \varphi_{13}$.

particle	$mass/m_1$	Z_2 *charge*
A_1	1.00000	-1
A_2	1.28558	1
A_3	1.87939	-1
A_4	1.96962	1
A_5	2.53209	1
A_6	2.87939	-1
A_7	3.70167	1

Table 1

a b	S_{ab}
1 1	$-\overset{2}{(10)}\,\overset{4}{(2)}$
1 2	$\overset{1}{(13)}\,\overset{3}{(7)}$
1 3	$-\overset{2}{(14)}\,\overset{4}{(10)}\,\overset{5}{(6)}$
1 4	$\overset{1}{(17)}\,\overset{3}{(11)}\,\overset{6}{(3)}\ (9)$
1 5	$\overset{3}{(14)}\,\overset{6}{(8)}\ (6)^2$
1 6	$-\overset{4}{(16)}\,\overset{5}{(12)}\,\overset{7}{(4)}\ (10)^2$
1 7	$\overset{6}{(15)}\,(9)\,(5)^2\,(7)^2$
2 2	$\overset{2}{(12)}\,\overset{4}{(8)}\,\overset{5}{(2)}$
2 3	$\overset{1}{(15)}\,\overset{3}{(11)}\,\overset{6}{(5)}\ (9)$
2 4	$\overset{2}{(14)}\,\overset{5}{(8)}\ (6)^2$
2 5	$\overset{2}{(17)}\,\overset{4}{(13)}\,\overset{7}{(3)}\ (7)^2\,(9)$
2 6	$\overset{3}{(15)}\ (7)^2\,(5)^2\,(9)$
2 7	$\overset{5}{(16)}\,\overset{7}{(10)^3}\ (4)^2\,(6)^2$
3 3	$-\overset{2}{(14)}\,\overset{7}{(2)}\ (8)^2\,(12)^2$
3 4	$\overset{1}{(15)}\ (5)^2\,(7)^2\,(9)$
3 5	$\overset{1}{(16)}\,\overset{6}{(10)^3}\ (4)^2\,(6)^2$
3 6	$-\overset{2}{(16)}\,\overset{5}{(12)^3}\,\overset{7}{(8)^3}\ (4)^2$
3 7	$\overset{3}{(17)}\,\overset{6}{(13)^3}\ (3)^2\,(7)^4\,(9)^2$

Table 2 (Continued)

a	b	S_{ab}
4	4	$\overset{4}{(12)}\,\overset{5}{(10)^3}\,\overset{4}{(7)}\;(2)^2$
4	5	$\overset{2}{(15)}\,\overset{4}{(13)^3}\,\overset{7}{(7)^3}\;(9)$
4	6	$\overset{1}{(17)}\,\overset{6}{(11)^3}\;(3)^2\,(5)^2\,(9)^2$
4	7	$\overset{4}{(16)}\,\overset{5}{(14)^3}\;(6)^4\,(8)^4$
5	5	$\overset{5}{(12)^3}\;(2)^2\,(4)^2\,(8)^4$
5	6	$\overset{1}{(16)}\,\overset{3}{(14)^3}\;(6)^4\,(8)^4$
5	7	$\overset{2}{(17)}\,\overset{4}{(15)^3}\,\overset{7}{(11)^5}\;(5)^4\,(9)^3$
6	6	$-\,\overset{4}{(14)^3}\,\overset{7}{(10)^5}\;(12)^4\,(16)^2$
6	7	$\overset{1}{(17)}\,\overset{3}{(15)^3}\,\overset{6}{(13)^5}\;(5)^6\,(9)^3$
7	7	$\overset{2}{(16)^3}\,\overset{5}{(14)^5}\,\overset{7}{(12)^7}\;(8)^8$

Table 2 (Continuation)

state	s/m_1^2	*c–series*	*U–series*
A_2	1.28558	0.6450605	0.0706975
A_4	1.96962	0.0256997	0.0066115
$A_1\ A_1$	$\geq$ 2.00000	0.0182735	0.0071135
A_5	2.53209	0.0032417	0.0013783
$A_2\ A_2$	$\geq$ 2.57115	0.0032549	0.0025194
$A_1\ A_3$	$\geq$ 2.87939	0.0012782	0.0020630
$A_2\ A_4$	$\geq$ 3.25519	0.0003010	0.0007277
$A_1\ A_1\ A_2$	$\geq$ 3.28558	0.0007139	0.001184
A_7	3.70167	0.0000316	0.0000287
$A_3\ A_3$	$\geq$ 3.75877	0.0000700	0.0001173
$A_2\ A_5$	$\geq$ 3.81766	0.0000860	0.0001581
partial sum		0.6980109	0.0914150
exact value		0.7000000	0.0942097

Table 3

F_2^Θ	0.9604936853
F_4^Θ	−0.4500141924
F_5^Θ	0.2641467199
F_7^Θ	−0.0556906385

Table 4

a_{11}^0	6.283185307
a_{13}^0	30.70767637
a_{22}^0	15.09207695
a_{22}^1	4.707833688
a_{24}^0	79.32168252
a_{24}^1	16.15028004
a_{33}^0	295.3281130
a_{33}^1	396.9648559
a_{33}^2	123.8295119
a_{25}^0	3534.798444
a_{25}^1	4062.255130
a_{25}^2	556.5589101

Table 5

particle	$mass/m_l$	Z_3 *charge*
A_l	1.00000	$e^{2\pi i/3}$
$A_{\bar{l}}$	1.00000	$e^{-2\pi i/3}$
A_L	1.41421	1
A_h	1.93185	$e^{2\pi i/3}$
$A_{\bar{h}}$	1.93185	$e^{-2\pi i/3}$
A_H	2.73205	1

Table 6

a b	S_{ab}
l l	$\overset{\bar{l}}{[8]}\ [6]\ \overset{\bar{h}}{[2]}$
$\bar{l}$ $\bar{l}$	$\overset{l}{[8]}\ [6]\ \overset{h}{[2]}$
l $\bar{l}$	$-\overset{L}{[10]}\ [6]\ [4]$
l L	$\overset{l}{[9]}\ [7]\ \overset{h}{[5]}\ [3]$
$\bar{l}$ L	$\overset{\bar{l}}{[9]}\ [7]\ \overset{\bar{h}}{[5]}\ [3]$
l h	$[9]\ \overset{\bar{h}}{[7]}\ [5]^2\ [3]\ \overset{\bar{l}}{[11]}$
$\bar{l}$ $\bar{h}$	$[9]\ \overset{h}{[7]}\ [5]^2\ [3]\ \overset{l}{[11]}$
l $\bar{h}$	$\overset{L}{[9]}\ [7]^2\ [5]\ \overset{H}{[3]}\ [1]$
$\bar{l}$ h	$\overset{L}{[9]}\ [7]^2\ [5]\ \overset{H}{[3]}\ [1]$
l H	$\overset{h}{[10]}\ [8]^2\ [6]^2\ [4]^2\ [2]$
$\bar{l}$ H	$\overset{\bar{h}}{[10]}\ [8]^2\ [6]^2\ [4]^2\ [2]$
L L	$-\overset{L}{[10]}\ \overset{L}{[8]}\ [6]^2\ [4]\ \overset{H}{[2]}$
L h	$\overset{l}{[10]}\ [8]^2\ [6]^2\ [4]^2\ [2]$
L $\bar{h}$	$\overset{\bar{l}}{[10]}\ [8]^2\ [6]^2\ [4]^2\ [2]$
L H	$\overset{L}{[11]}\ [9]^2\ \overset{H}{[7]}^3\ [5]^3\ [3]^2\ [1]$
h h	$\overset{\bar{l}}{[10]}\ \overset{\bar{h}}{[8]}^3\ [6]^3\ [4]^2\ [2]^2$
$\bar{h}$ $\bar{h}$	$\overset{l}{[10]}\ \overset{h}{[8]}^3\ [6]^3\ [4]^2\ [2]^2$
h $\bar{h}$	$-[10]^2\ [8]^2\ \overset{H}{[6]}^3\ [4]^3\ [2]$
h H	$\overset{l}{[11]}\ \overset{h}{[9]}^3\ [7]^4\ [5]^4\ [3]^3\ [1]$
$\bar{h}$ H	$\overset{\bar{l}}{[11]}\ \overset{\bar{h}}{[9]}^3\ [7]^4\ [5]^4\ [3]^3\ [1]$
H H	$-\overset{L}{[10]}^3\ \overset{H}{[8]}^5\ [6]^6\ [4]^5\ [2]^3$

Table 7

state	s/m_1^2	c–series	u–series
A_L	1.41421	0.7596531	0.0705265
$A_l\ A_{\bar{l}}$	$\geq$ 2.00000	0.0844238	0.0229507
A_H	2.73205	0.0029236	0.001013
$A_L\ A_L$	$\geq$ 2.82843	0.0024419	0.0019380
$A_l\ A_{\bar{h}}$	$\geq$ 2.93185	0.0023884	0.0016745
$A_{\bar{l}}\ A_h$	$\geq$ 2.93185	0.0023884	0.0016745
$A_l\ A_l\ A_l$	$\geq$ 3.00000	0.0004215	0.0004925
$A_{\bar{l}}\ A_{\bar{l}}\ A_{\bar{l}}$	$\geq$ 3.00000	0.0004215	0.0004925
$A_l\ A_{\bar{l}}\ A_L$	$\geq$ 3.41421	0.00159	0.000251
$A_h\ A_{\bar{h}}$	$\geq$ 3.86370	0.0000504	0.0001476
$A_l\ A_l\ A_h$	$\geq$ 3.93185	0.000089	0.0002015
$A_{\bar{l}}\ A_{\bar{l}}\ A_{\bar{h}}$	$\geq$ 3.93185	0.000089	0.0002015
$A_l\ A_{\bar{l}}\ A_l\ A_{\bar{l}}$	$\geq$ 4.00000	0.0000959	0.000381
partial sum		0.8569765	0.1019449
exact value		0.8571429	0.1056624

Table 8

$F_L^{\ominus}$	1.261353947
$F_H^{\ominus}$	0.292037405

Table 9

$a^0_{l\bar{l}}$	6.283185307
$a^0_{L\,L}$	21.76559237
$a^1_{L\,L}$	9.199221756
$a^0_{l\bar{h}}$	25.22648264
$a^0_{h\,\bar{h}}$	414.1182423
$a^1_{h\,\bar{h}}$	565.6960386
$a^2_{h\,\bar{h}}$	175.0269632

Table 10

$$
\begin{aligned}
S_{11}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{2}{7}\right)} \\
S_{12}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)}\,\overset{\mathbf{3}}{\left(\tfrac{3}{7}\right)} \\
S_{13}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)}\,\overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)} \\
S_{22}(\theta) &= \overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)}\,\left(\tfrac{5}{7}\right)^2 \\
S_{23}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)}\,\overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)}\,\left(\tfrac{3}{7}\right)^2 \\
S_{33}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)}\,\left(\tfrac{3}{7}\right)^2\,\left(\tfrac{5}{7}\right)^2
\end{aligned}
$$

$$
m_a = \frac{\sin\frac{a\pi}{7}}{\sin\frac{\pi}{7}}\, m_1 \qquad a = 1,\, 2,\, 3
$$

Table 11: S–Matrix and mass ratios of the $[M(2/9)]_{(1,3)}$ model.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$0.8020765730\ i$	$1.445292044\ i$	$1.802249700\ i$
$F_2^{\mathcal{O}}$	-0.3139111350	-1.019263053	-1.584911372
$F_3^{\mathcal{O}}$	$-0.1373692458\ i$	$-0.5561967212\ i$	$-1.002231860\ i$

Table 12

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
H_{11}	3.127325930	10.15436351	15.78961011
H_{12}	1.430685375 i	8.370732785 i	16.23085987 i
a_{12}	1.603875472	0.3568958679	0.1588336037
H_{13}	−0.6754444788	−4.927973308	−11.07305551
a_{13}	2.493959207	0.6920214716	0.2469796037
H_{22}	2.936562411	30.95981497	74.85757540
a_{22}	5.740938811	1.246979604	0.5331878680

Table 13

	$\frac{\delta m_1}{\delta m_2}$		$\frac{\delta \mathcal{E}_{vac}}{m_1^{(0)} \delta m_1}$	
deformation	*numerical* ($\pm 3\%$)	*theoretical*	*numerical* ($\pm 3\%$)	*theoretical*
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,2}$	0.390	0.391396	−1.04	−1.03826
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,4}$	0.811	0.834681	−0.205	−0.205640

Table 14