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A New Description of the Jacobian Super KP Hierarchy

Thesis submitted for the degree of
"*Doctor Philosophiae*"

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SUPERVISOR
Cesare Reina

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Contents

1 Introduction.

As for any other theory, a wide comprehension of the theory of soliton equations has been achieved only after the development of different techniques, which allowed the investigation of its various aspects. The efforts of the people involved in such investigations have furthermore produced a deeper understanding of many related branches of mathematics. Among the several concepts that have been used, either introduced from scratch or already available, we can for example cite, to name only a few, the theory of algebraic curves; the universal (and the Sato) Grassmannian; the Krichever map; pseudo-differential operators; Lax, Zakharov–Shabat equations and iso-spectral deformations; Bäcklund–Darboux–Lie transformations; bi-Hamiltonian structures; Faà di Bruno polynomials, recurrence relations and Riccati equations; and so on.

On the other hand, stimulated both by physical research and mathematical interest, several attempts have been made to extend the theory to the super-symmetric domain. The first "surprise" in this direction has been the realization that such an extension is by no means unique. Indeed, two non-isomorphic super KP theories have been defined up to now: the MRSKP of Manin and Radul [28] and the JSKP of Mulase and Rabin [33, 37].

While constructing a generalisation of a theory, one has to choose the concepts and features that he considers characterising the theory, and then he has to develop the new theory trying to keep consistence with these. The starting point of Manin and Radul, in this context, was the pseudo-differential operator and Lax equation approach. Studying their theory, Mulase and (independently) Rabin discovered that on the algebraic geometric side that SKP theory is not good. The reason leading to this conclusion is the fact that the flows of MRSKP not only move a point in the super Jacobian of a super curve, but deform the curve itself. In the context of integrable systems this is surely a discouraging property. In fact, a fundamental requirement for such a system, that can not be put apart, is to evolve along straight lines on suitable tori (in this case the super Jacobian).

Mulase and Rabin were mainly interested in the algebraic geometry of SKP, e.g. in extending the successful (even if not constructive) characterisation of Jacobians among Abelian varieties to the super-commutative domain, and so on. It is in studying these issues that they finally came up with their JSKP, where now only the super line bundle moves along the flows of the theory.

Having at our disposal a certain number of different (but equivalent) approaches to the KP hierarchy, it is thus tempting to follow other ways in defining a super KP theory and look at the corresponding outcomes. The aim of the present investigation is exactly to provide, in the context of super KP hierarchies, some of the techniques that have proved to be successful for the investigation of the KP theory. Therefore, starting from the bi-Hamiltonian formulation of KP [10, 5], formalised by the Faà di Bruno polynomials and recurrence relations, we give a new definition of the JSKP hierarchy and study it by systematically adapting the corresponding methods. This will hopefully shed new light both on JSKP and on KP.

We have divided the exposition of the subsequent material in two Parts. In Part I we aim at providing a general but nevertheless accurate view of the theory, so we give the definitions and present different descriptions of the Jacobian super KP hierarchy. Section 2 is motivational in character: we briefly review the KP theory emphasising the role of Faà di Bruno polynomials and the associated conservation equations, which are equivalent to KP. This allows us to introduce the main characters and to offer our definition of the JSKP hierarchy in Section 3. Then in Section 4 we provide the link to the theory of Mulase and Rabin, showing that our JSKP is isomorphic to their hierarchy. Finally, in Section 5 we give a Lax representation by means of two super pseudo-differential operators, while in Section 6 we present a hyper-cohomological interpretation of the theory along the same lines of [15].

Part II is devoted to the detailed study of JSKP, a tool of central importance being the theory of Darboux transformations (Section 7). In Section 8 we explain the connection with the DKP hierarchy introduced in [25] to give a geometric description of the Darboux transformation, while in Section 9 we use that technique to linearise the flows on the super universal Grassmannian. Finally, in Section 10 we move our attention to the problem of reductions, considering two simple examples. For the convenience of the reader we also include an Appendix where useful formulae have been collected.

We wish to thank G. Falqui for having proposed to study this interesting subject and for many useful and enlightening discussions, C. Reina for the enjoyable and fruitful work we did together during the staying at SISSA for the Ph.D., and all the people whose help has contributed to the development of the present work.

Part I
A panoramic view of JSKP.

2 The KP hierarchy: a brief review.

In this Section we trace the origins of our definition of JSKP, which are rooted in the bi-Hamiltonian description of KP. As we have said in the Introduction, there are many different routes to the KP hierarchy [23, 11, 38, 9, 10, 5, 40, 32, 31]. Let us start from a monic ordinary differential operator of order n

$$P := \partial^n + \sum_{j=0}^{n-2} p_j(x)\partial^j,$$

where x is a formal parameter which we can regard, for example, as a coordinate on $\mathbb{R}$ and $\partial = \frac{d}{dx}$. Our aim now is to study the spectrum of the operator P:

$$P\psi = \lambda\psi.$$

Of course, to define a spectrum we need to specify on which class of functions P acts, e.g. by considering suitable boundary conditions at infinity. For instance, if we specify the boundary conditions in such a way that P becomes a bounded operator then its spectrum, if not empty, has necessarily to be discrete, so it carries no geometric properties. Working instead only on a formal neighbourhood of the origin of $\mathbb{C}$ and acting on the space of power series in x with complex coefficients, the spectrum of P turns out to be the entire complex line $\mathbb{C}$. Needless to say, any single eigenspace can be degenerate, so to completely characterise the spectrum we simply consider the subalgebra of all ordinary differential operators Q commuting with P. A lemma of Schur [39] then shows that this is a commutative algebra $\mathcal{A}_P$, moreover there exists a pseudo-differential operator $S = 1 + \sum_{j>0} s_j\partial^{-j}$ such that $\mathcal{A} := S \cdot \mathcal{A}_P \cdot S^{-1}$ is a ring of pseudo-differential operators with constant coefficients. If $\mathcal{A}$ is bigger than $\mathbb{C}$ then it is the ring of meromorphic functions on a complete irreducible algebraic curve $\mathcal{C}$ whose only poles are situated at a precise fixed smooth point $p_\infty \in \mathcal{C}$ (i.e. it is the coordinate ring of the affine curve $\mathcal{C}\setminus\{p_\infty\}$): the *spectral curve* of the operator P (or, more precisely, of the commutative algebra $\mathcal{A}_P$). The name is explained as follows. Any common eigenvector ψ of the differential operators belonging to $\mathcal{A}_P$ can be interpreted as a point of the curve: indeed, it defines a homomorphism $\varphi : \mathcal{A}_P \to \mathbb{C}$ by sending the operator Q to its eigenvalue $\lambda_Q(\psi)$ associated with ψ. Of course, the curve $\mathcal{C}$ covers $\mathbb{P}^1$ by means of the map which associates to a point ψ the eigenvalue $\lambda_P(\psi)$. Then $\operatorname{Spec} \mathcal{A}_P \simeq \operatorname{Spec} \mathcal{A}$ is the pre-image of $\mathbb{P}^1 \setminus \{\infty\}$, as can be easily seen: let us consider a differential operator $Q \in \mathcal{A}_P$ whose order is prime with respect to that of P (here we are simplifying things a little bit, see below the definition of rank of $\mathcal{A}_P$). Then P and Q must necessarily satisfy a (minimal) polynomial equation $p(P, Q) = 0$, which corresponds to a planar model $\tilde{\mathcal{C}}$ of our curve $\mathcal{C}$. In fact, $\mathcal{C}$ is a normalisation of $\tilde{\mathcal{C}}$. The points of $\tilde{\mathcal{C}}$ are in one to one correspondence with the pairs $(\lambda_P(\psi), \lambda_Q(\psi))$. Now, the composition of the normalisation map $\mathcal{C} \to \tilde{\mathcal{C}}$ and the projection map $\tilde{\mathcal{C}} \to \mathbb{P}^1$ given by $(\lambda_P(\psi), \lambda_Q(\psi)) \mapsto \lambda_P(\psi)$ is the covering map we were searching for.

Let us now ask the following simple question: which is the most general deformation of P which preserves its spectrum? More formally, we are looking for a differential operator Q (obviously not in $\mathcal{A}_P$) such that

$$\begin{cases} \frac{\partial \psi}{\partial t} = Q\psi \\ \\ \frac{\partial P}{\partial t} = [Q, P] \end{cases}.$$

Equivalently, we require that the eigenvalues of P do not change with time, $\lambda_{P(t)}(\psi(t)) = \lambda_{P(t_0)}(\psi(t_0))$, which is the concept of *iso-spectral deformation.*

It turns out that the generic form of such an operator is the linear combination of the basic operators $(P^{\frac{r}{n}})_+$, where the subscript $+$ means that we have to take the differential part of the pseudo-differential operator $P^{\frac{r}{n}}$. This formulation of the the solution of the iso-spectral deformation problem admits a simple generalisation which leads directly to the KP hierarchy: let L be a monic pseudo-differential operator of the form

$$L := \partial + \sum_{j>0} u_j \partial^{-j}$$

and define the following hierarchy of evolution equations

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad k > 0.$$

It is an easy exercise to proof the compatibility of these equations, which form the so called KP hierarchy. Other than being a generalisation, the hierarchy also allows for the treatment of the iso-spectral deformation problem in a way independent of the order n of P. Indeed, it can be shown that the condition $P := L^n = (L^n)_+$ for a power of L to be a purely differential operator is compatible with the KP flows moreover, in such a case, the evolution equations for P are exactly that of iso-spectral deformation which, therefore, can be seen as reductions of the KP hierarchy (the n-GD hierarchies of Gel'fand and Dickey). On the other hand, the concept of spectral curve (and others which we introduce presently) extends to KP.

The information contained in the couple $(P, \mathcal{A}_P)$ (and more generally in L) is far richer than the mere concept of spectral curve. To explain this, we have to introduce the *universal Grassmannian.* In the following we shall be interested mainly in the formal algebraic aspects of the theory and not in the analytical ones, so we shall define only the algebraic version of the Grassmannian, leaving the aspects of convergence of the involved series to actual functions to the existing literature (see e.g. [36, 40]).

Let $V = \mathbb{C}((z^{-1}))$ be the field of formal Laurent series in the variable z^{-1} and let $V_- := \mathbb{C}[[z^{-1}]] \cdot z^{-1}$ be its subspace of formal power series without constant term. V has a natural filtration

$$\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots,$$

where $V_j = \mathbb{C}[[z^{-1}]] \cdot z^j$, which defines a natural complete topology on it. We can regard therefore both V and V_- as topological vector spaces and define the universal Grassmannian $Gr(V, V_-)$ as the set of close infinite dimensional vector subspaces W of V which are compatible with V_- in the following precise sense: $W \in Gr(V, V_-)$ if and only if the restriction π_W of the canonical projection $\pi : V \to V/V_-$ to W is a Fredholm operator, i.e. it has finite dimensional kernel and cokernel. Denoting by i_W the index $\dim(\ker \pi_W) - \dim(\operatorname{coker} \pi_W)$ of π_W, we see that $Gr(V, V_-)$ is the disjoint union of the denumerable set of its components Gr_j, $j \in \mathbb{Z}$, indexed by i_W. Moreover, each Gr_j has a scheme structure by means of projective limits.

Now, to any monic pseudo-differential operator L of order one as above we can associate a point of the universal Grassmannian by means of the *Sato construction* [38, 31], which runs as follows. First of all we notice that there exists a (not unique) invertible pseudo-differential operator

$$S = 1 + \sum_{j>0} s_j \partial^{-j}$$

such that $L = S \partial S^{-1}$, hence the information encoded in L is completely characterised by S. Let $\mathcal{E}$ be the ring of pseudo-differential operators with coefficients in $\mathbb{C}[[x]]$, $\mathcal{D}$ its subring of differential operators and $\mathcal{E}_-$ the subspace of the purely pseudo-differential ones. Then, clearly, $\mathcal{E} = \mathcal{D} \oplus \mathcal{E}_-$. Sending ∂ to z we get an isomorphism ρ between $E/E \cdot x$ and V, which transforms the above decomposition of $\mathcal{E}$ into the decomposition $\mathbb{C}[z] \oplus V_-$ of V and maps $S^{-1}\mathcal{D}$ to an element $W_L \in Gr_0$.

Conversely, to any point W of the big cell of Gr_0 one can associate a monic pseudo-differential operator L_W of order one by means of the Baker-Akhiezer function [23, 40] φ_W of W. The most important property of the universal Grassmannian is, it can be shown (see [38, 40]), that the flows of the KP hierarchy are linear there.

Denote by $\mathcal{A}_L$ the subring of V which consists of those functions f for which $f \cdot W_L \subset W_L$. Like before, if $\mathcal{A}_L$ is bigger than $\mathbb{C}$, then it corresponds to an irreducible complete algebraic curve $\mathcal{C}$, of which $\operatorname{Spec} \mathcal{A}_L$ is an open affine subset missing exactly one well identified smooth point p_∞. The extra information we alluded to is the subspace $W_L \subset V$ which, when $\mathcal{C}$ exists, is a finitely generated $\mathcal{A}_L$-module. As explained for example in [32, 40], this is the module of holomorphic sections of a torsion free sheaf $\mathcal{F} \to \mathcal{C}$ over $\mathcal{C} \setminus \{p_\infty\}$. It is possible to associate a *rank* r to the ring $\mathcal{A}_L$: it is, by definition [32], the greatest common divisor of the order of pole at p_∞ of the functions $f \in \mathcal{A}_L$, and turns out to be exactly the rank of $\mathcal{F}$ as an $\mathcal{O}_\mathcal{C}$- module.

Summarising, the geometric datum we have attached to L consists of an irreducible complete algebraic curve $\mathcal{C}$, a smooth point p_∞ on it, a local (formal) coordinate z^{-1} at p_∞, a torsion free sheaf $\mathcal{F}$ of rank r over $\mathcal{C}$ and a local (formal) trivialisation η of $\mathcal{F}$ on the domain where the coordinate is defined. Conversely, the *Krichever map* [23, 40, 31] produces a point of Gr out of any given quintet $(\mathcal{C}, p_\infty, z^{-1}, \mathcal{F}, \eta)$ as above by defining $W = \Gamma(\mathcal{C} \setminus \{p_\infty\}, \mathcal{F})$. When discussing algebraic geometric properties of the hierarchies under study, later on, we shall assume for simplicity that $\mathcal{C}$ is smooth, p_∞ is not Weierstrass for $\mathcal{C}$ and $\mathcal{F}$ is an invertible sheaf (equivalently, a line bundle). We shall not dwell anymore on the discussion of KP along these lines, preferring instead to introduce the Faà di Bruno polynomials and the associated formalism.

The Faà di Bruno polynomials naturally arise in the study of the bi-Hamiltonian properties of KP [10, 5]. In the sequel we closely follow [13]. The technique that plays a prominent role here is the method of Poisson pencils to construct integrable Hamiltonian systems, where one considers a manifold M endowed with two Poisson structures P_0 and P_1 which are compatible in the sense that $P_\lambda := P_0 + \lambda P_1$ is a *linear pencil of Poisson structures*, i.e. P_λ is a Poisson structure for any λ. The simplest example is given by a Poisson manifold $(M, \{\cdot,\cdot\})$ together with a vector field X on it whose task is to deform the Poisson bracket to another one. We denote by

$$\{f,g\}' = \{X(f),g\} + \{f,X(g)\} - X(\{f,g\})$$

and

$$\{f,g\}'' = \{X(f),g\}' + \{f,X(g)\}' - X(\{f,g\}')$$

the first two Lie derivatives of the bracket along X and we require that the second derivative identically vanishes on M. Then the pull-back $\{f,g\}_\lambda := \{f \circ \phi_{-\lambda}, g \circ \phi_{-\lambda}\} \circ \phi_\lambda$ of the Poisson bracket of M with respect to the flow $\phi_\lambda : M \to M$ defined by X is linear in λ and defines the above mentioned pencil. In the situation just described M is called an *exact bi-Hamiltonian manifold* and X its *Liouville vector field*, underlining the analogy with the case of exact symplectic manifolds.

The basic idea behind the method is to construct integrable Hamiltonian systems by means of the *Casimir functions* of the pencil. In the case of an odd–dimensional manifold endowed with a Poisson pencil of maximal rank, $\{\cdot,\cdot\}_\lambda$ has a unique Casimir function H_λ which, according to Gel'fand and Zakharevich [16], is a polynomial in λ of degree n, where $2n+1 = \dim M$,

$$H_\lambda = H_0\lambda^n + H_1\lambda^{n-1} + \cdots + H_n,$$

whose leading coefficient H_0 is the Casimir function of $\{\cdot,\cdot\}'$, while the "constant term" H_n is the Casimir of $\{\cdot,\cdot\}$. The coefficients H_j satisfy the recurrence relations

$$\{\cdot, H_{j+1}\}' = \{\cdot, H_j\}$$

and therefore are in involution with respect to all the brackets of the pencil:

$$\{H_j, H_k\}_\lambda = 0.$$

When M is compact, the level surfaces of the H_j's are n-dimensional tori defining a Lagrangian foliation of M.

We get a completely integrable (in the sense of Liouville) bi-Hamiltonian system by introducing the vector fields

$$X_\lambda(f) := \{f, H_\lambda\}' = \{f, H'_\lambda\}_\lambda,$$

where the second Hamiltonian function is $H'_\lambda := X(H_\lambda)$. The integrability comes by observing that $X_\lambda(H_j) = 0$, and the polynomial family of vector fields

$$X_\lambda = X_0\lambda^n + X_1\lambda^{n-1} + \cdots + X_n$$

is called *the canonical hierarchy* defined on the exact bi-Hamiltonian manifold M.

A concrete example of a canonical hierarchy is the KdV theory [13]. Here, the manifold M is the space of scalar–valued C^∞ functions on the circle S^1, the Liouville vector field is defined by

$$\dot{u} := X(u) = 1$$

and the Poisson pencil, given as a one-parameter family of skew-symmetric maps from the cotangent to the tangent bundle, reads

$$\dot{u} = (P_\lambda)_u v = -\frac{1}{2}v_{xxx} + 2(u+\lambda)v_x + u_x v,$$

where x is a coordinate on S^1, u represents a point of M, v is a covector at u and the value of v on the tangent vector $\dot{u}$ at u is given by

$$\langle v, \dot{u}\rangle = \int_{S^1} v(x)\dot{u}(x)dx.$$

The Casimir function H_λ and its derivative H'_λ along X can be written as integrals

$$H = 2z\int_{S^1} hdx$$

and

$$H' = \int_{S^1} h'dx$$

of the local densities

$$h(z) = z + \sum_{j>0} h_j z^{-j}$$

and

$$h'(z) = 1 + \sum_{j>0} h'_j z^{-j},$$

which are Laurent series in $z^{-1} = \sqrt{\lambda^{-1}}$. It turns out [13] that if h and h' are the unique solutions of the Riccati system

$$h_x + h^2 = u + z^2$$

$$-\frac{1}{2}h'_x + hh' = z$$

admitting the above asymptotic expansion, then their integrals H_λ and H'_λ are respectively the Casimir function of the Poisson pencil of KdV and its second Hamiltonian function. The canonical hierarchy of M then admits several representations. The one we are interested in is a consequence of the complete integrability condition $X_\lambda(H_j) = 0$ and can be expressed by saying that the local Hamiltonian density $h(z)$ must obey local conservation laws of the form

$$\frac{\partial h}{\partial t_j} = \partial_x H^{(j)},$$

where the "current densities" $H^{(j)}$ are given by

$$H^{(2j)} = \lambda^j$$

and

$$H^{(2j+1)} = -\frac{1}{2}(\lambda^j v)_{+,x} + h(\lambda^j v)_+,$$

the subscript + meaning to take the positive part of the expansion in powers of z. The last equation can also be written as [13]

$$H^{(2j+1)} = z^{2j}\left(-\frac{1}{2}v_x + hv\right) + \frac{1}{2}(z^{2j}v)_{-,x} - h(z^{2j}v)_-$$

or

$$H^{2j+1} = \sum_{l=1}^{j}\left[-\frac{1}{2}v_{j-l,x}(z^{2l}\cdot 1) + v_{j-l}(z^{2l}\cdot h)\right].$$

The first of these two expressions shows that $H^{(j)} = z^j + O(z^{-1})$, since by the second Riccati equation above we have $z^{2j}\left(-\frac{1}{2}v_x + hv\right) = z^{2j+1}$, while the interpretation of the second one needs some more work. We consider the Faà di Bruno iterates of $h^{(0)} = 1$ at h defined by

$$h^{(j+1)} = (\partial_x + h)h^{(j)}$$

and denote by W the linear space spanned by them over the ring of C^∞ functions on S^1. Then, the first Riccati equation above translates into

$$z^2 = h^{(2)} - uh^{(0)},$$

showing that $z^2 \in W$. Moreover, applying the operators $(\partial_x + h)^j$ to the above equation we see that $z^2 W \subset W$, in particular $z^{2j} \cdot 1 \in W$ and $z^{2j} \cdot h \in W$, hence $H^{(j)} \in W$ too, i.e.

$$H^{(j)} = \sum_{k=0}^{j} c_k^j h^{(k)}.$$

The importance of this result is that the Hamiltonian origin of the currents $H^{(j)}$ is completely encoded into their expansion as a sum of the Faà di Bruno polynomials $h^{(k)}$. Furthermore, this property does not require h to be a solution of the Riccati equation, allowing therefore for generalisations of the KdV hierarchy which now we describe. Since here we are interested only in algebraic properties, we substitute the circle S^1 with the formal scheme $X = \operatorname{Spec} \mathbb{C}[[x]]$.

Definition 2.1 Let h be a monic (formal) Laurent series in z^{-1}

$$h := z + \sum_{j>0} h_j z^{-j},$$

whose coefficients h_j belong to $\mathbb{C}[[x]]$, and define as above the Faà di Bruno polynomials by

$$\begin{cases} h^{(k+1)} := \partial_x h^{(k)} + h h^{(k)} \\ h^{(0)} := 1 \end{cases}.$$

It turns out that these Laurent series can be computed for any $k \in \mathbb{Z}$ and, by definition of $Gr(V, V_-)$, it follows that the (formal) family of subspaces $W_x \subset V$ spanned by the positive iterates $h^{(k)}$, $k \geq 0$, constitutes a (formal) curve in the universal Grassmannian. Denote by W_X such a family, thought of as a subspace of $V_X = V \otimes_{\mathbb{C}} \mathbb{C}[[x]]$, and introduce the "current densities" $H^{(k)}$ by requiring them to be the unique elements of W_X of the form

$$H^{(k)} = z^k + \sum_{j>0} H_j^k z^{-j}.$$

The *KP hierarchy* is defined to be the set of "conservation laws"

$$\partial_k h = \partial_x H^{(k)},$$

where $\partial_k := \frac{\partial}{\partial t_k}$. Observe that $H^{(1)} = h$, so we can identify the first time t_1 with x.

□

The link between the present formalism and the previous one, which used pseudo-differential operators, is given by an analogy with the dynamics of the rigid body, where one can envisage two equivalent descriptions: that in the *space* coordinate system and that in the *body* coordinate system. Like there, the Faà di Bruno formalism constitutes a kind of *space representation*, while the pseudo-differential operator picture is a *body representation* [5]. To obtain the second from the first, one defines [5] the map $\Phi : V_X \to \Psi DO$, from our space of formal Laurent series in z^{-1}, with formal power series in x as coefficients, to the algebra of formal pseudo-differential operators in x, by

$$\Phi(h^{(k)}) = \partial^k$$

on the Faà di Bruno basis and extending it to the whole of V_X by linearity. The Lax operator of the KP hierarchy is then

$$L := \Phi(z) = \Phi\left(h^{(1)} + \sum_{j>0} u_j h^{(-j)}\right) = \partial + \sum_{j>0} u_j \partial^{-j}.$$

One can show that such an L evolves along the usual KP flows

$$\partial_k L = [(L^k)_+, L].$$

The way the hierarchy is constructed (or, alternatively, the compatibility of its equations) implies that $\partial_j H^{(k)} = \partial_k H^{(j)}$, whence the current densities can be obtained as derivatives of a suitable function, say $H^{(k)} = \partial_k \log \psi$; in particular, $h = \partial_x \log \psi$. The function $\psi = (1 + \sum_{j>0} \psi_j z^{-j}) \exp \sum_{k>0} t_k z^k$, defined up to multiplication by a constant monic Laurent series of order zero, is the *Baker-Akhiezer function* of the hierarchy. Along with that, one can also define the τ-function, as explained in [5], by taking a "dual" picture in which dual Faà di Bruno polynomials and dual current densities are introduced.

Another important feature of this formalism is that the operator $\partial_k + H^{(k)}$ "preserves" W_X, in the sense that $(\partial_k + H^{(k)}) \cdot W_X \subset W_X$. This is the starting point for the study of reductions of KP. For instance, the n-GD hierarchy is given by the compatible constraint $H^{(n)} = z^n$. Moreover, one can easily rewrite the conservation laws as equations involving only the current densities $H^{(k)}$, obtaining in this way the so called *Central System* [6, 5], a dynamical system generalising KP. Observe that the above flows of KP on the universal Grassmannian are not the linear ones of [38, 40], however they can be linearised as in [13].

In order to not repeat ourselves, we shall explain these and other properties directly in the super case. In the next Section, motivated by the above discussion, we shall give a new definition of the Jacobian super KP hierarchy introducing the super-relatives of h and $H^{(k)}$.

3 Super Faà di Bruno polynomials and JSKP.

We come now to the central object of our investigations: the JSKP hierarchy [33, 37, 2, 41]. To define it we need to reintroduce some of the concepts we have briefly discussed in the previous Section. The first is the *super universal Grassmannian.* Fix once and for all a Grassmann algebra Λ over $\mathbb{C}$: it is needed for several reasons, e.g. at the algebraic geometric level to be able to consider non-split $1|1$ super curves, but most importantly to study functorial properties of the hierarchy, which we shall not do here. The next thing to do is to supplement our bosonic coordinates z^{-1} and x with their fermionic "super-partners", which we call θ and φ respectively. We denote by $\bar{f}$ the parity of a homogeneous element f, e.g. $\bar{x} = 0$, $\bar{\varphi} = 1$.

Let $V := \Lambda((z^{-1})) \oplus \Lambda((z^{-1})) \cdot \theta$ be the quotient ring of the ring of formal power series in z^{-1} and θ over Λ and let $V_- := \Lambda[[z^{-1}, \theta]] \cdot z^{-1}$. As before, we have a natural filtration on V which makes it and its Λ-submodule V_- complete topological spaces. Then, the super Grassmannian $SGr_\Lambda := SGr_\Lambda(V, V_-)$ is the set of closed free Λ-submodules W of V which are compatible with V_- in the sense that the restriction π_W of the natural projection $\pi : V \to V/V_-$ to W is a Fredholm operator, meaning that its kernel (respectively cokernel) is a Λ-submodule (respectively a Λ-quotient module) of a finite rank free Λ-module (compare with [2]). As usual, SGr_Λ acquires a super-scheme structure by means of a projective limit of finite dimensional super Grassmannians.

The role the circle S^1 (or its replacement we have utilised above) played in KP is now performed by the super-space $X := \operatorname{Spec} B$, where $B := \Lambda[[x, \varphi]]$. Later on, when introducing the multi-times t_k, $k > 0$, X will in its turn be substituted by $T := \operatorname{Spec} B_T$ where

$$B_T := B[[t_k]]_{k>0} := \varprojlim_n B[[t_1, \cdots, t_n]].$$

Finally, the odd derivation operator $\delta := \partial_\varphi + \varphi\partial_x$ takes the place of ∂. The main reason for considering this type of derivation, apart from supersymmetry, is that its square is ∂_x, providing a direct connection with KP.

In the following, however, we shall define the super Central System where there are neither special super-space coordinates as x, φ, nor special super-derivation operators such as δ.

Now it is time to introduce the main character, the principal actor, of the theory: the super Faà di Bruno polynomial $\hat{h}$. It is an odd formal Laurent series belonging to $V_X := V \otimes_\Lambda B$ and of the form

$$\hat{h}(z,\theta;x,\varphi) := \nu(z;x) + \theta a(z;x) + \varphi h(z;x) + (\theta\varphi)\psi(z;x),$$

where Latin letters have been used for even quantities while Greek letters for odd ones. We specify exactly the content of the components a, h, ν and ψ by requiring two things: the first is the existence of suitable "super current densities" $\hat{H}^{(k)}$, while the second is the possibility of identifying the second time t_2 with x (it will be not possible, however, to identify neither t_1 nor any other odd time with φ). It turns out that the two requirements can be satisfied if and only if a is holomorphic with invertible zeroth order coefficient (which we assume, for simplicity, to be equal to 1)

$$a(z;x) := 1 + \sum_{j>0} a_j(x) z^{-j},$$

h has the usual form (its zeroth order coefficient as been set to zero so as to give the usual KP hierarchy when reducing to $a = 1$, $\nu = \psi = 0$)

$$h(z;x) := z + \sum_{j>0} h_j(x) z^{-j},$$

ν has no poles and its zeroth order coefficient is constant (zero, for simplicity)

$$\nu(z;x) := \sum_{j>0} \nu_j(x) z^{-j}$$

and ψ is holomorphic too (again we assume that its zeroth order coefficient vanishes)

$$\psi(z;x) := \sum_{j>0} \psi_j(x) z^{-j}.$$

Observe that the restriction we have put on some coefficients, just to simplify the discussion, will be compatible with the equations of the hierarchy (which will state that the only evolving coefficients are those of negative order). One can understand our choice simply as a reduction of a more general hierarchy. Indeed, accepting to discard the second requirement $t_2 \sim x$, we could also work with ν having a simple pole and ψ having its zeroth order coefficient restored. Of course, our results will extend to this case, however all the formulae will correspondingly become extremely complicated.

To the super Faà di Bruno polynomial $\hat{h}$ we associate (for $k \in \mathbb{N}$) its iterates

$$\begin{cases} \hat{h}^{(k+1)} := (\delta + \hat{h}) \cdot \hat{h}^{(k)} \\ \hat{h}^{(0)} := 1 \end{cases}.$$

Lemma 3.1 *Let $\hat{\mu} := \delta(\hat{h}) = h - \theta\psi + \varphi\partial_x\nu - (\theta\varphi)\partial_x a$. Then, for any $k \in \mathbb{N}$*

$$\begin{cases} \hat{h}^{(2k+2)} = (\partial_x + \hat{\mu}) \cdot \hat{h}^{(2k)} = (\partial_x + \hat{\mu})^{k+1} \cdot 1 \\ \hat{h}^{(2k+3)} = (\partial_x + \hat{\mu}) \cdot \hat{h}^{(2k+1)} = (\partial_x + \hat{\mu})^{k+1} \cdot \hat{h} \end{cases}.$$

Proof. We have

$$(\delta + \hat{h})^2 = \delta^2 + \delta(\hat{h}) - \hat{h}\delta + \hat{h}\delta + \hat{h}^2 = \delta^2 + \delta(\hat{h}) = \partial_x + \hat{\mu},$$

so we get

$$\hat{h}^{(2k+2)} = (\delta + \hat{h})^2 \cdot \hat{h}^{(2k)} = (\partial_x + \hat{\mu}) \cdot \hat{h}^{(2k)} = (\delta + \hat{h})^{2k+2} \cdot 1 = (\partial_x + \hat{\mu})^{k+1} \cdot 1$$

and

$$\hat{h}^{(2k+3)} = (\delta + \hat{h})^2 \cdot \hat{h}^{(2k+1)} = (\partial_x + \hat{\mu}) \cdot \hat{h}^{(2k+1)} = (\delta + \hat{h})^{2k+2} \cdot \hat{h} = (\partial_x + \hat{\mu})^{k+1} \cdot \hat{h}.$$

□

To facilitate computations it is convenient to introduce the following notation:

$$\begin{cases} \hat{h}^{(2k)} = h^{(k)} - \theta\psi^{(k)} + \varphi\omega^{(k)} - (\theta\varphi)b^{(k)} \\ \hat{h}^{(2k-1)} = \nu^{(k)} + \theta a^{(k)} + \varphi\eta^{(k)} + (\theta\varphi)\chi^{(k)} \end{cases},$$

where the components are Laurent series of the form

$$\nu^{(k)} = \sum_{j>0} \nu_j^{(k)} z^{k-j-1}$$

$$h^{(k)} = z^k + \sum_{j>0} h_j^{(k)} z^{k-j-1}$$

$$a^{(k)} = z^{k-1} + \sum_{j>0} a_j^{(k)} z^{k-j-1}$$

$$\psi^{(k)} = \sum_{j>0} \psi_j^{(k)} z^{k-j-1}$$

$$\eta^{(k)} = z^k + \sum_{j>0} \eta_j^{(k)} z^{k-j-1}$$

$$\omega^{(k)} = \sum_{j>0} \omega_j^{(k)} z^{k-j-1}$$

$$\chi^{(k)} = \sum_{j>0} \chi_j^{(k)} z^{k-j-1}$$

$$b^{(k)} = \sum_{j>0} b_j^{(k)} z^{k-j-1}.$$

The super Faà di Bruno recurrence relation above then translates into the following ones:

$$\begin{cases} \nu^{(k+1)} = (\partial_x + h)\nu^{(k)} \\ \nu^{(1)} = \nu \end{cases},$$

$$\begin{cases} h^{(k+1)} = (\partial_x + h)h^{(k)} \\ h^{(0)} = 1 \end{cases},$$

$$\begin{cases} a^{(k+1)} = (\partial_x + h)a^{(k)} - \psi\nu^{(k)} \\ a^{(1)} = a \end{cases},$$

$$\begin{cases} \psi^{(k+1)} = (\partial_x + h)\psi^{(k)} + \psi h^{(k)} \\ \psi^{(0)} = 0 \end{cases},$$

$$\begin{cases} \eta^{(k+1)} = (\partial_x + h)\eta^{(k)} + (\partial_x \nu)\nu^{(k)} \\ \eta^{(1)} = h \end{cases},$$

$$\begin{cases} \omega^{(k+1)} = (\partial_x + h)\omega^{(k)} + (\partial_x \nu)h^{(k)} \\ \omega^{(0)} = 0 \end{cases},$$

$$\begin{cases} \chi^{(k+1)} = (\partial_x + h)\chi^{(k)} + \psi\eta^{(k)} + (\partial_x \nu)a^{(k)} - (\partial_x a)\nu^{(k)} \\ \chi^{(1)} = \psi \end{cases},$$

$$\begin{cases} b^{(k+1)} = (\partial_x + h)b^{(k)} - \psi\omega^{(k)} - (\partial_x \nu)\psi^{(k)} + (\partial_x a)h^{(k)} \\ b^{(0)} = 0 \end{cases}.$$

In turn, these equations allow us to compute the coefficients of the various components of $\hat{h}^{(k)}$:

$$\begin{cases} \nu_1^{(k+1)} = \nu_1^{(k)} \\ \nu_2^{(k+1)} = \nu_2^{(k)} + \partial_x \nu_1^{(k)} \\ \nu_j^{(k+1)} = \nu_j^{(k)} + \partial_x \nu_{j-1}^{(k)} + \sum_{l=1}^{j-2} h_{j-l-1}\nu_l^{(k)} \\ \qquad j > 2 \end{cases},$$

$$\begin{cases} h_1^{(k+1)} = h_1^{(k)} + h_1 \\ h_2^{(k+1)} = h_2^{(k)} + h_2 + \partial_x h_1^{(k)} \\ h_j^{(k+1)} = h_j^{(k)} + h_j + \partial_x h_{j-1}^{(k)} + \sum_{l=1}^{j-2} h_{j-l-1}h_l^{(k)} \\ \qquad j > 2 \end{cases},$$

$$\begin{cases} a_1^{(k+1)} = a_1^{(k)} \\ a_2^{(k+1)} = a_2^{(k)} + \partial_x a_1^{(k)} + h_1 \\ a_j^{(k+1)} = a_j^{(k)} + \partial_x a_{j-1}^{(k)} + h_{j-1} + \sum_{l=1}^{j-2} \left(h_{j-l-1} a_l^{(k)} - \psi_{j-l-1} \nu_l^{(k)} \right) \\ \qquad\quad j > 2 \end{cases},$$

$$\begin{cases} \psi_1^{(k+1)} = \psi_1^{(k)} + \psi_1 \\ \psi_2^{(k+1)} = \psi_2^{(k)} + \psi_2 + \partial_x \psi_1^{(k)} \\ \psi_j^{(k+1)} = \psi_j^{(k)} + \psi_j + \partial_x \psi_{j-1}^{(k)} + \sum_{l=1}^{j-2} \left(h_{j-l-1} \psi_l^{(k)} + \psi_{j-l-1} h_l^{(k)} \right) \\ \qquad\quad j > 2 \end{cases},$$

$$\begin{cases} \eta_1^{(k+1)} = \eta_1^{(k)} + h_1 \\ \eta_2^{(k+1)} = \eta_2^{(k)} + h_2 + \partial_x \eta_1^{(k)} \\ \eta_j^{(k+1)} = \eta_j^{(k)} + h_j + \partial_x \eta_{j-1}^{(k)} + \sum_{l=1}^{j-2} \left(h_{j-l-1} \eta_l^{(k)} + (\partial_x \nu_{j-l-1}) \nu_l^{(k)} \right) \\ \qquad\quad j > 2 \end{cases},$$

$$\begin{cases} \omega_1^{(k+1)} = \omega_1^{(k)} + \partial_x \nu_1 \\ \omega_2^{(k+1)} = \omega_2^{(k)} + \partial_x \nu_2 + \partial_x \omega_1^{(k)} \\ \omega_j^{(k+1)} = \omega_j^{(k)} + \partial_x \nu_j + \partial_x \omega_{j-1}^{(k)} + \sum_{l=1}^{j-2} \left(h_{j-l-1} \omega_l^{(k)} + (\partial_x \nu_{j-l-1}) h_l^{(k)} \right) \\ \qquad\quad j > 2 \end{cases},$$

$$\begin{cases} \chi_1^{(k+1)} = \chi_1^{(k)} + \psi_1 \\ \chi_2^{(k+1)} = \chi_2^{(k)} + \psi_2 + \partial_x \chi_1^{(k)} + \partial_x \nu_1 \\ \chi_j^{(k+1)} = \chi_j^{(k)} + \psi_j + \partial_x \chi_{j-1}^{(k)} + \partial_x \nu_{j-1} \\ \quad + \sum_{l=1}^{j-2} \left(h_{j-l-1} \chi_l^{(k)} + \psi_{j-l-1} \eta_l^{(k)} + (\partial_x \nu_{j-l-1}) a_l^{(k)} - (\partial_x a_{j-l-1}) \nu_l^{(k)} \right) \\ \qquad\quad j > 2 \end{cases},$$

$$\begin{cases} b_1^{(k+1)} = b_1^{(k)} + \partial_x a_1 \\ b_2^{(k+1)} = b_2^{(k)} + \partial_x a_2 + \partial_x b_1^{(k)} \\ b_j^{(k+1)} = b_j^{(k)} + \partial_x a_j + \partial_x b_{j-1}^{(k)} \\ \quad + \sum_{l=1}^{j-2} \left(h_{j-l-1} b_l^{(k)} - \psi_{j-l-1}\omega_l^{(k)} - (\partial_x \nu_{j-l-1})\psi_l^{(k)} + (\partial_x a_{j-l-1}) h_l^{(k)} \right) \\ \qquad j > 2 \end{cases}.$$

As we see, the Faà di Bruno iterates can be computed by backward recurrence also for negative k and therefore form a basis of V_X. Let $\mathfrak{m}$ be the ideal $\langle x, \varphi\rangle \subset B$, $p = \operatorname{Spec} B/\mathfrak{m} = \operatorname{Spec}\Lambda \hookrightarrow X$ be the (unique) closed Λ-point of X and define

$$\begin{cases} W_X := \operatorname{span}_B \left\{ \hat{h}^{(k)},\ k \geq 0 \right\} \\ W_p := W_X \times_X p = \operatorname{span}_\Lambda \left\{ \tilde{h}^{(k)},\ k \geq 0 \right\} \end{cases},$$

where the $\tilde{h}^{(k)}$'s are the reductions mod $\mathfrak{m}$ of the $\hat{h}^{(k)}$'s:

$$\tilde{h}^{(k)} = \hat{h}^{(k)} \bmod \mathfrak{m} \in V = V_X \times_X p = V_X/\mathfrak{m}V_X.$$

We can think of W_X as a family of subspaces of V. In fact, we can prove the following

Proposition 3.1 *Let $\hat{h}$, W_X and W_p be defined as above. Then W_p belongs to the big cell of the index* 0 *component of the super universal Grassmannian SGr_Λ.*

Proof. Clearly, it is enough to show that for any $k \geq 0$ there exists an element $\hat{H}^{(k)} \in W_X$ of the form

$$\begin{cases} \hat{H}^{(2k)} = z^k \bmod V_{X-} \\ \hat{H}^{(2k+1)} = \theta z^k \bmod V_{X-} \end{cases},$$

where $V_{X-} = V_- \otimes_\Lambda B$. By construction

$$\begin{cases} \hat{H}^{(0)} = 1 \\ \hat{H}^{(1)} = \hat{h}^{(1)} - \varphi \hat{h}^{(2)} \\ \hat{H}^{(2)} = \hat{h}^{(2)} \end{cases}.$$

The others can be computed recursively: suppose we have defined $\hat{H}^{(j)}$ for $0 \leq j < k$; if $k = 2n$ is even then

$$\hat{H}^{(k)} = \hat{h}^{(k)} - \sum_{j=1}^{n-1} \left(h_j^{(n)} + \varphi\omega_j^{(n)}\right) \hat{H}^{(k-2j-2)} - \sum_{j=1}^{n-1} \left(\psi_j^{(n)} + \varphi b_j^{(n)}\right) \hat{H}^{(k-2j-1)},$$

while if $k = 2n - 1$ is odd then

$$\begin{aligned} \hat{H}^{(k)} = \hat{h}^{(k)} - \varphi\hat{h}^{(k+1)} &- \textstyle\sum_{j=1}^{n-1} \left(\nu_j^{(n)} + \varphi(\eta_j^{(n)} - h_j^{(n)})\right) \hat{H}^{(k-2j-1)} \\ &- \textstyle\sum_{j=1}^{n-1} \left(a_j^{(n)} + \varphi(\chi_j^{(n)} - \psi_j^{(n)})\right) \hat{H}^{(k-2j)} \end{aligned}.$$

□

Therefore, we can interpret W_X as a (formal) curve inside SGr_Λ. We have prepared thus all the ingredients needed for the following

Definition 3.1 (JSKP) Let $\hat{h} \in V_T = V \otimes_\Lambda B_T$ be defined as above, where now its components depend also on the times t_k, $k > 0$, and compute its Faà di Bruno iterates $\hat{h}^{(k)}$ and the basis $\{\hat{H}^{(k)},\ k \geq 0\}$ of W_T as explained. The *Jacobian super KP hierarchy* is the set of "super conservation laws"

$$\partial_k \hat{h} = (-1)^k \delta \hat{H}^{(k)}, \qquad k > 0.$$

□

Let us work out some simple consequences of the definition. The evolution equations are simply the super-commutativity conditions

$$[\delta + \hat{h}, \partial_k + \hat{H}^{(k)}] = 0$$

and imply that

$$\left(\partial_k + \hat{H}^{(k)}\right) \cdot W_T \subset W_T.$$

Indeed,

$$\begin{aligned} \left(\partial_k + \hat{H}^{(k)}\right) \cdot \hat{h}^{(l)} &= \left(\partial_k + \hat{H}^{(k)}\right) \cdot \left(\delta + \hat{h}\right)^l \cdot 1 \\ &= (-1)^{kl} \left(\delta + \hat{h}\right)^l \cdot \left(\partial_k + \hat{H}^{(k)}\right) \cdot 1 \\ &= (-1)^{kl} \left(\delta + \hat{h}\right)^l \cdot \hat{H}^{(k)} \end{aligned}$$

and by definition $\hat{H}^{(k)} \in W_T$, $(\delta + \hat{h}) \cdot W_T \subset W_T$. In turn, this implies that $\partial_j \hat{H}^{(k)} = (-1)^{jk} \partial_k \hat{H}^{(j)}$, as the following simple argument shows: let $V_{T-} := V_- \otimes_\Lambda B_T$, so $V_T = W_T \oplus V_{T-}$. Then, by the above property, $\partial_j \hat{H}^{(k)}$ is the V_{T-}-component of $-\hat{H}^{(j)}\hat{H}^{(k)}$, while $\partial_k \hat{H}^{(j)}$ is the V_{T-}-component of $-\hat{H}^{(k)}\hat{H}^{(j)} = -(-1)^{jk}\hat{H}^{(j)}\hat{H}^{(k)}$. Finally we obtain the compatibility of the evolution equations

$$\partial_j \partial_k \hat{h} = (-1)^{j+k} \delta \partial_j \hat{H}^{(k)} = (-1)^{jk+j+k} \delta \partial_k \hat{H}^{(j)} = (-1)^{jk} \partial_k \partial_j \hat{h}$$

and

$$[\partial_j + \hat{H}^{(j)}, \partial_k + \hat{H}^{(k)}] = 0.$$

From the above discussion, we see that it is possible to describe the theory in terms of the $\hat{H}^{(k)}$'s only, avoiding the need to introduce the super-space variables x and φ and the super-derivative δ which up to now played a special role. We have

$$\begin{aligned} \partial_{2k}\hat{H}^{(2j)} + \hat{H}^{(2k)}\hat{H}^{(2j)} \;=\;& \hat{H}^{(2j+2k)} + \sum_{l=1}^{j}\left(\hat{H}^{2k}_{0,l}\hat{H}^{(2j-2l)} + \hat{H}^{2k}_{1,l}\hat{H}^{(2j-2l+1)}\right) \\ & + \sum_{l=1}^{k}\left(\hat{H}^{2j}_{0,l}\hat{H}^{(2k-2l)} + \hat{H}^{2j}_{1,l}\hat{H}^{(2k-2l+1)}\right), \end{aligned}$$

$$\begin{aligned} \partial_{2k}\hat{H}^{(2j+1)} + \hat{H}^{(2k)}\hat{H}^{(2j+1)} \;=\;& \hat{H}^{(2j+2k+1)} + \sum_{l=1}^{j}\hat{H}^{2k}_{0,l}\hat{H}^{(2j-2l+1)} \\ & + \sum_{l=1}^{k}\left(\hat{H}^{2j+1}_{0,l}\hat{H}^{(2k-2l)} + \hat{H}^{2j+1}_{1,l}\hat{H}^{(2k-2l+1)}\right), \end{aligned}$$

$$\begin{aligned} \partial_{2k+1}\hat{H}^{(2j)} + \hat{H}^{(2k+1)}\hat{H}^{(2j)} \;=\;& \hat{H}^{(2j+2k+1)} + \sum_{l=1}^{k}\hat{H}^{2j}_{0,l}\hat{H}^{(2k-2l+1)} \\ & + \sum_{l=1}^{j}\left(\hat{H}^{2k+1}_{0,l}\hat{H}^{(2j-2l)} + \hat{H}^{2k+1}_{1,l}\hat{H}^{(2j-2l+1)}\right), \end{aligned}$$

$$\partial_{2k+1}\hat{H}^{(2j+1)} + \hat{H}^{(2k+1)}\hat{H}^{(2j+1)} = \sum_{l=1}^{j}\hat{H}^{2k+1}_{0,l}\hat{H}^{(2j-2l+1)} - \sum_{l=1}^{k}\hat{H}^{2j+1}_{0,l}\hat{H}^{(2k-2l+1)},$$

where we used the following decomposition of the $\hat{H}^{(k)}$'s

$$\begin{cases} \hat{H}^{(2k)} = z^k + \sum_{j>0}\left(\hat{H}^{2k}_{0,j} z^{-j} + \hat{H}^{2k}_{1,j}\theta z^{-j}\right) \\ \\ \hat{H}^{(2k+1)} = \theta z^k + \sum_{j>0}\left(\hat{H}^{2k+1}_{0,j} z^{-j} + \hat{H}^{2k+1}_{1,j}\theta z^{-j}\right) \end{cases}.$$

The dynamical system defined by these equations is indeed more general than JSKP and, borrowing the terminology of [6, 13], we call it the *super Central System* (SCS). The properties we have deduced in the previous computations hold also for SCS and will be of key importance in the study of the two theories.

4 Identification with the usual hierarchy.

In this Section we show that our hierarchy is indeed isomorphic to that defined by Mulase and Rabin. Before doing that, however, let us prove the iso-spectrality property within our formalism.

Definition 4.1 Let W be a point of SGr_Λ and $W_X \subset V_X$, $W_T \subset V_T$ be defined as in the previous Section. Then

$$\mathcal{A}_W := \Lambda\langle f \in V \mid f \text{ homogeneous, } f \cdot W \subset W\rangle,$$

$$\mathcal{A}_{X/T} := \Lambda\langle f \in V \mid f \text{ homogeneous, } f \cdot W_{X/T} \subset W_{X/T}\rangle,$$

where homogeneous refers to the $\mathbb{Z}_2$-grading of V and $\Lambda\langle\cdots\rangle$ is the ring generated over Λ by the elements inside the angle brackets.
□

The above Λ-algebras are the would-be coordinate rings of the affine part $\mathcal{C} \setminus \{p_\infty\}$ of the "super spectral curves". While, for special W, $\mathcal{A}_W$ can obviously be bigger than Λ, and therefore correspond to an effective super curve, the same could not be said for $W_{X/T}$.

First of all we give a criterion for computing $\mathcal{A}_X$ (an analogous criterion works for $\mathcal{A}_T$):

Lemma 4.1 *Let $\hat{h}$ and W_X be defined as above.*

i. Let f be a homogeneous element of V_X, then $f \cdot W_X \subset W_X$ if and only if, for all $k \geq 0$, $f^{(k)} := \delta^k(f) \in W_X$.

ii. $\mathcal{A}_X$ *is the maximal homogeneous* Λ*-algebra contained in* $W_X \cap V$*, where the intersection is taken in* V_X *(of course* $V \subset V_X$*).*

Proof.

i. This follows by induction using the easily verifiable commutation relations

$$f \cdot (\delta + \hat{h})^k = \sum_{j \geq 0} (-1)^{\frac{j(j+1)}{2} + k\bar{f}} \begin{bmatrix} k \\ j \end{bmatrix} (\delta + \hat{h})^{k-j} f^{(j)},$$

where $\begin{bmatrix} k \\ j \end{bmatrix}$ is the super binomial coefficient (see Appendix A).

ii. By definition $1 \in W_X$, so if $f \in \mathcal{A}_X$ then $f \cdot 1 = f \in W_X$. Conversely, suppose that $f \in W_X$ does not depend on x and φ. Without loss of generality we can assume that it is homogeneous. Then $f^{(k)} = 0$ for $k > 0$, hence by part i. $f \cdot W_X \subset W_X$, i.e. $f \in \mathcal{A}_X$.

□

Denote by $\mathfrak{m}_T$ the ideal of B_T generated by φ and all the times t_k, $k > 0$ (remember that we set $x = t_2$), let $p_T = \operatorname{Spec} B_T/\mathfrak{m}_T = \operatorname{Spec} \Lambda \hookrightarrow T$ be the closed Λ-point of T and

$$W_{p_T} = W_T \times_T p_T = W_T \bmod \mathfrak{m}_T W_T \subset V.$$

We have

Theorem 4.1 (Iso-spectrality of JSKP) *Let* $\hat{h}$*,* W_T *and* $W_{p_T} \in SGr_\Lambda$ *be defined as above. Then*

$$\mathcal{A}_T = \mathcal{A}_{W_{p_T}}.$$

Proof. Obviously $\mathcal{A}_T \subset \mathcal{A}_{W_{p_T}}$, so let $\tilde{f} \in \mathcal{A}_{W_{p_T}}$ be homogeneous. Let $\tilde{H}^{(k)} = \hat{H}^{(k)} \bmod \mathfrak{m}_T$. We can find constants $c_0, \cdots, c_k \in \Lambda$ such that

$$\tilde{f} = \sum_{j=0}^{k} c_j \tilde{H}^{(j)}.$$

Put $f = \sum_{j=0}^{k} c_j \hat{H}^{(j)}$, so $\tilde{f} = f \bmod \mathfrak{m}_T$. If we show that $f = \tilde{f}$ then the proof follows by using part ii. of Lemma 4.1.

Let us consider first the dependence of f on x and φ. We prove by induction on k that $f^{(k)} = 0 \bmod \mathfrak{m}_T$ for any $k > 0$. Having showed that, we get $\partial_x f = \partial_\varphi f = 0$. Now,

$$\tilde{f}\tilde{h} = f\hat{h} \bmod \mathfrak{m}_T = \left((-1)^{\bar{f}+1} f^{(1)} + (-1)^{\bar{f}}(\delta + \hat{h})f\right) \bmod \mathfrak{m}_T \in W_{p_T},$$

but $(\delta + \hat{h})f \in W_T$ whence $f^{(1)} = 0 \bmod \mathfrak{m}_T$ (indeed $f^{(1)} \in W_-$). Therefore, let us proceed by induction assuming that $f^{(j)} = 0 \bmod \mathfrak{m}_T$ for $0 < j < k$. Then

$$\begin{aligned}
\tilde{f}\tilde{h}^{(k)} &= f\hat{h}^{(k)} \bmod \mathfrak{m}_T \\
&= (-1)^{k\bar{f}} \left((\delta + \hat{h})^k f + \sum_{j=1}^{k-1} (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k \\ j \end{bmatrix} (\delta + \hat{h})^{k-j} f^{(j)} \right. \\
&\qquad \left. + (-1)^{\frac{k(k+1)}{2}} f^{(k)} \right) \bmod \mathfrak{m}_T \\
&= (-1)^{k\bar{f}} \left((\delta + \hat{h})^k f + \sum_{j=1}^{k-1} g_{k,j} f^{(j)} + (-1)^{\frac{k(k+1)}{2}} f^{(k)} \right) \bmod \mathfrak{m}_T \\
&= (-1)^{k\bar{f}} \left((\delta + \hat{h})^k f + (-1)^{\frac{k(k+1)}{2}} f^{(k)} \right) \bmod \mathfrak{m}_T \in W_{p_T}
\end{aligned}$$

and since $(\delta + \hat{h})^k f \in W_T$ we get $f^{(k)} = 0 \bmod \mathfrak{m}_T \in W_{p_T}$. To finish the proof, we apply the same argument with

$$\hat{H}^{(k,l)} := (\partial_k + \hat{H}^{(k)})^l \cdot 1 \in W_T$$

and its reduction mod $\mathfrak{m}_T$, $\tilde{H}^{(k,l)}$, in place of $\hat{h}^{(k)}$ and $\tilde{h}^{(k)}$ respectively, obtaining $\partial_k^l f = 0 \bmod \mathfrak{m}_T$, i.e. f does not depend on the times t_k too.
□

Now we construct the isomorphism with the theory of Mulase and Rabin by means of the super Baker-Akhiezer function of our hierarchy.

Remember that we have verified that $\partial_j \hat{H}^{(k)} = (-1)^{jk} \partial_k \hat{H}^{(j)}$, which implies the existence of a (not unique) even function Ψ of the form

$$\Psi(z, \theta; \varphi, \mathbf{t}) = \left(1 + \sum_{j>0} (\alpha_j(\varphi, \mathbf{t}) z^{-j} + \beta_j(\varphi, \mathbf{t}) \theta z^{-j}) \right) e(z, \theta; \varphi, \mathbf{t}),$$

where

$$e(z, \theta; \varphi, \mathbf{t}) = \exp \left(\theta\varphi + \sum_{j>0} (t_{2j} z^j + t_{2j-1} \theta z^{j-1}) \right),$$

such that

$$\hat{H}^{(k)} = \partial_k \log \Psi.$$

Any such function is called a Baker-Akhiezer function of JSKP. Let now S be the even, invertible, super pseudo-differential operator

$$S = 1 + \sum_{j>0} s_j(\varphi, \mathbf{t})\delta^{-j}$$

such that

$$\Psi(\varphi, \mathbf{t}) = S \cdot e(z, \theta; \varphi, \mathbf{t}).$$

Notice that all the information present in the JSKP hierarchy is encoded in its Baker-Akhiezer function Ψ and hence in S.

Exploiting the commutation relations we have used in the proof of part i. of Lemma 4.1 we find that

$$\partial_k \Psi = H^{(k)}\Psi = \psi \sum_{j=0}^{k} \gamma_j \hat{h}^{(j)} = \Psi \sum_{j=0}^{k} \gamma_j (\delta + \hat{h})^j \cdot 1 = B_k \cdot \Psi,$$

where B_k is a super differential operator of order k and of degree k mod 2.

On the other hand, however, we have

$$\begin{aligned} \partial_{2k}\Psi &= \partial_{2k}(S \cdot e) = (\partial_{2k} S) \cdot e + S \cdot z^k e \\ &= (\partial_{2k} S) \cdot e + S\delta^{2k} \cdot e \\ &= \left((\partial_{2k} S)S^{-1} + S\delta^{2k} S^{-1}\right) S \cdot e \\ &= \left((\partial_{2k} S)S^{-1} + S\delta^{2k} S^{-1}\right) \cdot \Psi = B_{2k} \cdot \Psi \end{aligned}$$

and

$$\begin{aligned} \partial_{2k-1}\Psi &= \partial_{2k-1}(S \cdot e) = (\partial_{2k-1} S) \cdot e + S \cdot \theta z^{k-1} e \\ &= (\partial_{2k-1} S) \cdot e + S(\delta^{2k-1} - \varphi\delta^{2k}) \cdot e \\ &= \left((\partial_{2k-1} S)S^{-1} + S(\delta^{2k-1} - \varphi\delta^{2k})S^{-1}\right) \cdot \Psi \\ &= B_{2k-1} \cdot \Psi. \end{aligned}$$

Since $(\partial_k S)S^{-1} = ((\partial_k S)S^{-1})_-$ is a purely pseudo-differential operator (i.e. it has no differential part) we get

$$\partial_{2k} S = -(S\delta^{2k} S^{-1})_- S = -(S\partial_x^k S^{-1})_- S$$

and

$$\partial_{2k-1} S = -(S(\delta^{2k-1} - \varphi\delta^{2k})S^{-1})_- S = -(S\partial_\varphi \partial_x^{k-1} S^{-1})_- S,$$

which are the equations that Mulase and Rabin defined for JSKP.

5 Lax description of JSKP.

An important issue to be investigated is the possible existence of a Lax representation of JSKP. We know how to recover the Lax operator and the Lax equations for KP out of the Faà di Bruno polynomials and current densities, so we apply the same machinery to our situation. We shall see that for JSKP two Lax operators are needed. Let

$$\theta = \hat{h}^{(1)} - \varphi \hat{h}^{(2)} + \sum_{j \geq 0} u_j \hat{h}^{(-j)} = \sum_{j \geq 0} \alpha_j \hat{h}^{(2-j)}$$

and

$$z = \hat{h}^{(2)} + \sum_{j \geq 0} v_j \hat{h}^{(-j)} = \sum_{j \geq 0} \beta_j \hat{h}^{(2-j)}$$

be the expressions of θ and z in terms of the Faà di Bruno basis of V_X. We define a map $\hat{\Phi} : V_X \to S\Psi DO_X$ to the algebra of formal super pseudo-differential operators on X by letting

$$\hat{\Phi}(\hat{h}^{(k)}) = \delta^k$$

and extending it to the whole of V_X by B-linearity. Then, to θ and z we associate the two operators

$$\begin{cases} \hat{L}_1 = \hat{\Phi}(\theta) = \delta - \varphi \delta^2 + \sum_{j \geq 0} u_j \delta^{-j} = \sum_{j \geq 0} \alpha_j \delta^{2-j} \\ \\ \hat{L}_2 = \hat{\Phi}(z) = \delta^2 + \sum_{j \geq 0} v_j \delta^{-j} = \sum_{j \geq 0} \beta_j \delta^{2-j} \end{cases} .$$

Let us examine some useful properties of the map $\hat{\Phi}$. First of all, by definition we have

$$\hat{\Phi} \circ \hat{\pi}_+ = \hat{\Pi}_+ \circ \hat{\Phi},$$

where $\hat{\pi}_+$ is the projection $V_X \to W_X$ and $\hat{\Pi}$ is the projection $S\Psi DO_X \to SDO_X$ to the subring of super differential operators. Secondly,

Lemma 5.1 *With the above notations, we have*

$$\begin{cases} \hat{\Phi}(z^k \hat{h}^{(j)}) = \delta^j \cdot \hat{L}_2^k \\ \\ \hat{\Phi}(\theta z^k \hat{h}^{(j)}) = (-1)^j \delta^j \cdot \hat{L}_1 \cdot \hat{L}_2^k \end{cases} .$$

Moreover, $[\hat{L}_1, \hat{L}_1] = 0$ *and* $[\hat{L}_1, \hat{L}_2] = 0$.

Proof. The clue comes from the observation that $\hat{\Phi}$ sends the operator $\delta+\hat{h}$ to left multiplication by δ. We recall the useful formula

$$\delta^k f = \sum_{l\geq 0}(-1)^{\bar{f}(k-l)}\begin{bmatrix}k\\ l\end{bmatrix} f^{(l)}\delta^{k-l},$$

where f is homogeneous and $f^{(l)}=\delta^l(f)$, as in the proof of Lemma 4.1. Let us compute $\hat{\Phi}(\theta\hat{h}^{(k)})$. We proceed by induction. First of all we find

$$\begin{aligned}
\hat{\Phi}(\theta\hat{h}^{(1)}) &= -\hat{\Phi}\left((\delta+\hat{h})\theta\right) = -\hat{\Phi}\left((\delta+\hat{h})\sum_{j\geq 0}\alpha_j\hat{h}^{(2-j)}\right)\\
&= -\hat{\Phi}\left(\sum_{j\geq 0}(-1)^{\bar{\alpha}_j}\alpha_j\hat{h}^{(3-j)} + \sum_{j\geq 0}(\delta\alpha_j)\hat{h}^{(2-j)}\right)\\
&= -\left(\sum_{j\geq 0}(-1)^{\bar{\alpha}_j}\alpha_j\delta^{3-j} + \sum_{j\geq 0}(\delta\alpha_j)\delta^{2-j}\right) = \delta\cdot\hat{L}_1.
\end{aligned}$$

Then, we use the inductive assumption that $\hat{\Phi}(\theta\hat{h}^{(j)}) = (-1)^j\delta^j\cdot\hat{L}_1$ for $0\leq j<k$. Thus

$$(\delta+\hat{h})^{k-1}\theta = \sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l-1)}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l)}\hat{h}^{(k-j-l+1)},$$

showing that

$$\begin{aligned}
\hat{\Phi}(\theta\hat{h}^{(k)}) &= (-1)^k\hat{\Phi}\left((\delta+\hat{h})^k\theta\right)\\
&= (-1)^k\hat{\Phi}\left((\delta+\hat{h})\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l-1)}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l)}\hat{h}^{(k-j-l+1)}\right)\\
&= (-1)^k\hat{\Phi}\left(\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l)+l}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l)}\hat{h}^{(k-j-l+2)}\right.\\
&\qquad\left.+\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l-1)}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l+1)}\hat{h}^{(k-j-l+1)}\right)\\
&= (-1)^k\left(\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l)+l}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l)}\delta^{k-j-l+2}\right. \times
\end{aligned}$$

$$+\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l-1)}\begin{bmatrix}k-1\\ l\end{bmatrix}\alpha_j^{(l+1)}\delta^{k-j-l+1}\Bigg)$$

$$\begin{aligned}
&= (-1)^k\Bigg(\sum_{j\geq 0}(-1)^{k\bar{\alpha}_j}\alpha_j\delta^{k-j+2}\\
&\quad+\sum_{j\geq 0}\sum_{l\geq 1}(-1)^{\bar{\alpha}_j(k-l)}\left((-1)^l\begin{bmatrix}k-1\\ l\end{bmatrix}+\begin{bmatrix}k-1\\ l-1\end{bmatrix}\right)\alpha_j^{(l)}\delta^{k-j-l+2}\Bigg)\\
&= (-1)^k\sum_{j\geq 0}\sum_{l\geq 0}(-1)^{\bar{\alpha}_j(k-l)}\begin{bmatrix}k\\ l\end{bmatrix}\alpha_j^{(l)}\delta^{k-j-l+2}\\
&= (-1)^k\delta^k\cdot\hat{L}_1.
\end{aligned}$$

The other relations can be proved by using the same technique. In particular, the computation of $\hat{\Phi}(\theta z^k)$ can be performed in two ways. The first is by expressing z^k as a series in the Faà di Bruno iterates $\hat{h}^{(j)}$ (to do this it is better to exploit the formula $\hat{\Phi}(z^k)=\hat{L}_2^k$) and using the formula for $\hat{\Phi}(\theta\hat{h}^{(j)})$ we have just proved to show that

$$\hat{\Phi}(\theta z^k)=\hat{L}_2^k\cdot\hat{L}_1,$$

while the second is by expressing θ as a series in the $\hat{h}^{(j)}$'s and using the formula for $\hat{\Phi}(z^k\hat{h}^{(j)})$, which yields

$$\hat{\Phi}(\theta z^k)=\hat{L}_1\cdot\hat{L}_2^k.$$

Since $\hat{\Phi}(\theta z^k)$ does not depend on the method used to compute it, it follows that $\hat{L}_1$ and $\hat{L}_2$ commute. Finally,

$$0=-\hat{\Phi}(\theta^2)=-\hat{\Phi}\left(\theta(\hat{h}^{(1)}-\varphi\hat{h}^{(2)}+\sum_{j\geq 0}u_j\hat{h}^{(-j)})\right)=\hat{L}_1^2=\frac{1}{2}[\hat{L}_1,\hat{L}_1].$$

□

The next step is to compute the image under $\hat{\Phi}$ of the derivative, with respect to the time t_k, of $\hat{h}^{(j)}$. The easiest way to perform this calculation is to use the super-commutativity of the operators $\partial_k+\hat{H}^{(k)}$ and $\delta+\hat{h}$. We have

$$\begin{aligned}
\hat{\Phi}(\partial_k\hat{h}^{(j)}) &= \hat{\Phi}\left((\partial_k+\hat{H}^{(k)})\hat{h}^{(j)}-\hat{H}^k\hat{h}^{(j)}\right)\\
&= \hat{\Phi}\left((\partial_k+\hat{H}^{(k)})(\delta+\hat{h})^j1-\hat{H}^k\hat{h}^{(j)}\right)\\
&= \hat{\Phi}\left((-1)^{jk}(\delta+\hat{h})^j\hat{H}^{(k)}-\hat{H}^k\hat{h}^{(j)}\right)
\end{aligned}$$

hence, using the expression of $\hat{H}^{(k)}$ as a series in θ and z (see the end of Section 3) and the previous lemma, we find

$$\hat{\Phi}(\partial_k \hat{h}^{(j)}) = (-1)^{jk} \sum_{l>0} \left([\delta^j, \hat{H}^k_{0,l}] \cdot \hat{L}_2^{-l} + [\delta^j, \hat{H}^k_{1,l}] \cdot \hat{L}_1 \cdot \hat{L}_2^{-l} \right).$$

We are now ready to compute the time derivative of $\hat{L}_1$ and $\hat{L}_2$. We do it only for the first operator, since the other calculation is identical: the ∂_k derivative of the expression of θ in terms of the $\hat{h}^{(j)}$'s yields

$$\sum_{j\geq 0} (\partial_k \alpha_j) \hat{h}^{(2-j)} = - \sum_{j\geq 0} (-1)^{k(j+1)} \alpha_j \partial_k \hat{h}^{(2-j)}$$

and applying $\hat{\Phi}$ we get

$$\begin{aligned}
\partial_k \hat{L}_1 &= -(-1)^k \sum_{j\geq 0} \sum_{l>0} \left([\alpha_j \delta^{2-j}, \hat{H}^k_{0,l}] \hat{L}_2^{-l} + [\alpha_j \delta^{2-j}, \hat{H}^k_{1,l}] \hat{L}_1 \hat{L}_2^{-l} \right) \\
&= -(-1)^k \sum_{l>0} \left([\hat{L}_1, \hat{H}^k_{0,l}] \hat{L}_2^{-l} + [\hat{L}_1, \hat{H}^k_{1,l}] \hat{L}_1 \hat{L}_2^{-l} \right) \\
&= -(-1)^k \left[\hat{L}_1, \sum_l (\hat{H}^k_{0,l} \hat{L}_2^{-l} + \hat{H}^k_{1,l} \hat{L}_1 \hat{L}_2^{-l}) \right],
\end{aligned}$$

where the last equality comes from the super-commutativity of $\hat{L}_1$ and $\hat{L}_2$. The last step to conclude the computation is to observe that

$$\begin{aligned}
\hat{\Phi}(\hat{H}^{(2k)}) &= \hat{\Phi}(\hat{\pi}_+(z^k)) = \hat{\Pi}_+(\hat{\Phi}(z^k)) = (\hat{L}_2^k)_+ \\
&= \hat{L}_2^k + \sum_l \left(\hat{H}^{2k}_{0,l} \hat{L}_2^{-l} + \hat{H}^{2k}_{1,l} \hat{L}_1 \hat{L}_2^{-l} \right)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\Phi}(\hat{H}^{(2k+1)}) &= \hat{\Phi}(\hat{\pi}_+(\theta z^k)) = \hat{\Pi}_+(\hat{\Phi}(\theta z^k)) = (\hat{L}_1 \hat{L}_2^k)_+ \\
&= \hat{L}_1 \hat{L}_2^k + \sum_l \left(\hat{H}^{2k+1}_{0,l} \hat{L}_2^{-l} + \hat{H}^{2k+1}_{1,l} \hat{L}_1 \hat{L}_2^{-l} \right).
\end{aligned}$$

Thus, we find

$$\begin{cases}
\partial_{2k} \hat{L}_j = [(\hat{L}_2^k)_+, \hat{L}_j] = -[(\hat{L}_2^k)_-, \hat{L}_j] \\
\partial_{2k+1} \hat{L}_j = [(\hat{L}_1 \hat{L}_2^k)_+, \hat{L}_j] = -[(\hat{L}_1 \hat{L}_2^k)_-, \hat{L}_j]
\end{cases},$$

where $j = 1, 2$. Notice that it is not possible to write these equations using only one operator: another natural operator we could introduce is

$$\hat{L} := \hat{L}_1 + \varphi \hat{L}_2 = \hat{\Phi}(\theta + \varphi z),$$

but there are no relations expressing $\hat{L}_1$ and $\hat{L}_2$ in terms of $\hat{L}$ only.

6 A hyper-cohomological interpretation of the Jacobian super KP hierarchy.

As for the case of KP [15], the JSKP hierarchy admits, in the formalism of super Faà di Bruno polynomials, a hyper-cohomological interpretation (see also [32]). The aim of this Section is to explain that issue, which is based on deformation theory. The discussion will be somewhat technical and will include a more detailed discussion of super-spaces and sheaves on them. We shall start from a few central definitions and then exploit them to achieve the above mentioned result. The interested reader can find more information in [3, 27, 42, 43].

Definition 6.1 A *super-space* is a locally ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of locally super-commutative rings on a topological space X. A *morphism of super-spaces* is a morphism of locally ringed spaces preserving the grading of the structure sheaves. A super-space whose structure sheaf is even, i.e. $\mathcal{O}_{X,1} = 0$ (here $\mathcal{O}_{X,j}$ is the component of degree $j = 0, 1$), is called *purely even.*
□

To a super-space there are always associated two sheaves of ideals: $\mathcal{J}_X := \mathcal{O}_{X,1} + \mathcal{O}_{X,1}^2$ and $\mathcal{N}$, which is the subsheaf of nilpotent elements. These ideals allow us to supplement X with two spaces, its associated graded super-space $\mathrm{Gr}\, X := (X, \mathrm{Gr}\, \mathcal{O}_X = \oplus_{j \geq 0} \mathcal{J}_X^j / \mathcal{J}_X^{j+1})$ and the underlying (reduced) space $X_{red} := (X, \mathcal{O}_X / \mathcal{N})$, which is purely even. A super-space X is said to be *split* if it is isomorphic to $\mathrm{Gr}\, X$. It is well known that in the category of smooth super-manifolds, all super-spaces are split [27]. Also, in the category of (smooth, holomorphic or algebraic) super-manifolds $X_{red} = X_{rd}$, where the structure sheaf of the last is $\mathcal{O}_X / \mathcal{J}_X$.

To work in the algebraic geometric category, one introduces the concept of affine super-space, namely the spectrum of a super-commutative ring $\mathcal{A}$, whose odd component $\mathcal{A}_1$ is a finite $\mathcal{A}_0$-module, together with the obvious structure sheaf. Then, a super-space X is said to be a *super-scheme* if it has a covering by (open) affine super-spaces. Its quotient space $(X, \mathcal{O}_{X,0})$ is then a usual scheme and $\mathcal{O}_{X,1}$ is a coherent sheaf of $\mathcal{O}_{X,0}$-modules.

A morphism $X \to Y$ between two super-schemes can be treated as a family of super-schemes parametrised by Y. The study of the behaviour of the fibres of such a morphism near a fixed point of the base brings the attention to the concept of *germs* of families. More rigorously, one introduces the concept of germs of morphism as classes of morphisms under an obvious equivalence relation. Since in this work we are not concerned with the issue of actual convergence, we define a germ of a morphism as a morphism to a one-point formal super-scheme $(p, \tilde{\mathcal{O}}_p)$ and, in this Section, all the rings are super-commutative algebras over the formal local ring $\tilde{\mathcal{O}}_p = \Lambda[[y, \eta]]/I$, where y (respectively η) is a set of even (respectively odd) variables and I is a suitable homogeneous ideal.

Another tool needed in the discussion of deformation theory is the concept of tangent space of a (super-) space at a point. In the algebraic geometric setting, this is defined in the following way: let $\mathcal{O}_{X,x}$ be the local ring of a point $x \in X$ and $\mathfrak{m}_x$ its maximal ideal. Then, the (Zariski) tangent space of X at x is the super-space $T_xX = \mathrm{Hom}\,(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$ and the dimension of X at x is defined to be the dimension of T_xX (X is *regular* if its dimension at x equals its dimension at every other point $y \in X$). The origin of the definition is obvious: a tangent vector v at x is a $\mathbb{C}$-valued derivation of the germs of functions at x, i.e. of the elements of $\mathcal{O}_{X,x}$. Any such function can be written as the sum $f_0 + f$, where $f_0 \in \mathcal{O}_{X,x}/\mathfrak{m}_x \simeq \mathbb{C} \cdot 1 \hookrightarrow \mathcal{O}_{X,x}$ and $f \in \mathfrak{m}_x$. Since v is a derivation it must give 0 on f_0, so it restricts to a map $\mathfrak{m}_x \to \mathbb{C}$. However, the super Leibniz rule $v(fg) = v(f)g_0 + (-1)^{\bar{v}\bar{f}} f_0 v(g)$ implies that it vanishes on the products fg, i.e. on $\mathfrak{m}_x^2$.

The best way to exploit the concept of super tangent vectors is to introduce the one-point super-scheme $\mathcal{D} := (\{*\}, \mathcal{O}_{\mathcal{D}} = \mathbb{C}[\varepsilon, \zeta]/(\varepsilon^2, \varepsilon\zeta))$, where ε is an even variable and ζ an odd one. It is then a standard fact, easy to show, that the tangent space of X at x is the set of morphisms $\mathcal{D} \to X$ whose image is exactly x. The super-scheme $\mathcal{D}$ will be used later in connection with infinitesimal deformations.

We come now to the definition of coherent sheaves on a super-scheme.

Definition 6.2 A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a super-scheme X is said to be *coherent* if for each point $x \in X$ there exists an affine neighbourhood U of

x such that $\mathcal{F}|_U$ is isomorphic to the sheaf associated to a finitely generated $\Gamma(U, \mathcal{O}_X)$-module M_U.
□

As for ordinary algebraic geometry [17], this concept is well behaved only in the category of Noetherian super-schemes, to which belong all the super-schemes we shall consider. This sheaves are very important due to their features, such as the finite dimensionality of their cohomology groups on compact super-schemes, good functorial properties under a suitable class of morphisms and so on. The characteristic we need is the existence of Stein super-spaces and hence of Stein coverings: a super-scheme X is Stein if X_{rd} is a Stein scheme. Thus [42]

Proposition 6.1 *Let $(X, \mathcal{O}_X)$ be a Stein super-scheme and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Then $H^j(X, \mathcal{F}) = 0$ for $j > 0$.* □

Let us now consider the problem of deformation of sheaves on a super-scheme and of their sections. The general theory can be found in [42], here we are interested only in invertible sheaves, i.e. super line bundles. Accordingly, we follow [43]. Recall that an invertible sheaf $\mathcal{L}$ is a locally free evenly generated $\mathcal{O}_X$-module of rank $1|0$. It is the sheaf of sections of a super line bundle that, abusing notations, we still call $\mathcal{L}$. Any affine open sub-super-scheme U of X is Stein and can be consequently used to trivialise $\mathcal{L}$. Therefore, let $\{U_j\}_{j\in J}$ be a covering of X by open affine sub-super-schemes and let g_{jk} be the transition functions of $\mathcal{L}$ on the intersections $U_{jk} = U_j \cap U_k$, $j, k \in J$: they are even invertible functions $g_{jk} \in \Gamma(U_{jk}, \mathcal{O}^{\times}_{X,ev})$ satisfying the cocycle conditions $g_{jk}g_{kj} = 1$ and $g_{jk}g_{kl}g_{lj} = 1$. Invertible sheaves can also be described in terms of super Cartier divisors, which are collections $D = \{(U_j, f_j)\}_{j\in J}$ of even non-zero rational functions f_j defined, up to even invertible regular functions, on the open affine subsets of the given covering and agreeing in the intersections U_{ij} up to an element of $\Gamma(U_{jk}, \mathcal{O}^{\times}_{X,ev})$. Thus, a super Cartier divisor D on X is a section of the sheaf $\mathcal{M}^{\times}_{X,ev}/\mathcal{O}^{\times}_{X,ev}$. To the divisor D we associate the invertible sheaf $\mathcal{L}$ whose local sections on U_j span the module $f_j^{-1}\Gamma(U_j, \mathcal{O}_X)$. It is an obvious consequence of the definition that these local sheaves glue together giving an invertible sheaf on X. The transition functions of $\mathcal{L}$ are therefore $g_{jk} = f_j/f_k$. Clearly, the set of isomorphism classes of invertible sheaves forms an Abelian group under tensor multiplication. This group is called the *Picard group* of X and is denoted by Pic(X). The degree of $\mathcal{L}$ is the degree of $\mathcal{L}_{red}$.

Definition 6.3 Let X be a super-scheme, $\mathcal{L}$ an invertible sheaf on X, $s = \{(U_j, f_j)\}_{j\in J}$ a global section of $\mathcal{L}$ and $(Y, \mathcal{O}_Y, y)$ a pointed super-scheme. A *Y-family of invertible sheaves on X* is an invertible sheaf $\mathcal{L}_Y$ over $X \times Y$. A *deformation of* $(\mathcal{L}, s)$ over the pointed super-scheme (Y, y) is a triple $(\mathcal{L}_Y, \sigma, \rho)$ where

i. $\mathcal{L}_Y$ is a Y-family of invertible sheaves on X,

ii. σ is a global section of $\mathcal{L}_Y$ and

iii. ρ is an isomorphism $\rho : \mathcal{L} \to \iota^*\mathcal{L}_Y$, where $\iota : X \hookrightarrow X \times Y$ is the embedding identifying X with $X \times \{y\}$, such that $\rho^*\iota^*\sigma = s$.

Two deformations $(\mathcal{L}_Y, \sigma, \rho)$ and $(\mathcal{N}_Y, \tau, \xi)$ of $(\mathcal{L}, s)$ over (Y, y) are isomorphic if and only if there exists an isomorphism of sheaves $\eta : \mathcal{L}_Y \to \mathcal{N}_Y$ compatible with ρ and ξ ($\xi = \iota^*(\eta) \circ \rho$) and such that $\sigma = \eta^*\tau$. The line bundle $\mathcal{L}_Y|_{X\times\{y\}} \simeq \mathcal{L}$ is sometimes called the *central fibre* of the deformation. Finally, an *infinitesimal deformation of* $(\mathcal{L}, s)$ is a deformation over the one-point super-scheme $\mathcal{D}$ we introduced before.
□

Of course, one can define the notion of morphism, making the set of deformations a category. However, here we want only to classify the infinitesimal deformations of the couple $(\mathcal{L}, s)$.

As before, let $\{U_j\}_{j\in J}$ be a covering by open affine sub-super-schemes of X and denote by $U_{j_1,\cdots,j_k}$ the intersection $\bigcap_{l=1}^k U_{j_l}$, by $\mathcal{O}_{j_1,\cdots,j_k}$ the super-commutative ring of sections of $\mathcal{O}_X$ over $U_{j_1,\cdots,j_k}$ and by $\mathcal{L}_{j_1,\cdots,j_k}$ the $\mathcal{O}_{j_1,\cdots,j_k}$-module of sections of $\mathcal{L}$ over $U_{j_1,\cdots,j_k}$. Finally, define

$$\begin{cases} \mathcal{O}_{j_1,\cdots,j_k}[\varepsilon,\zeta] := \mathcal{O}_{j_1,\cdots,j_k} \otimes \mathcal{O}_\mathcal{D} \\ \mathcal{L}_{j_1,\cdots,j_k}[\varepsilon,\zeta] := \mathcal{L}_{j_1,\cdots,j_k} \otimes \mathcal{O}_\mathcal{D} \\ U_{j_1,\cdots,j_k}[\varepsilon,\zeta] := \operatorname{Spec} \mathcal{O}_{j_1,\cdots,j_k}[\varepsilon,\zeta] = U_{j_1,\cdots,j_k} \times \mathcal{D} \end{cases} .$$

Then, $\{U_j[\varepsilon,\zeta]\}_{j\in J}$ is an open affine covering of $X \times \mathcal{D}$ and the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_j & \longrightarrow & \mathcal{O}_j[\varepsilon,\zeta]^\times_{ev} & \longrightarrow & \mathcal{O}^\times_{j,ev} & \longrightarrow & 1 \\ & & f & \mapsto & 1 + \varepsilon f_0 + \zeta f_1 & & & & \end{array} ,$$

where f_0 and f_1 are the even and odd components of f, yields an isomorphism $\mathrm{Pic}\,(U_j[\varepsilon,\zeta]) \simeq \mathrm{Pic}\,(U_j)$. Thus, if $\mathcal{L}_\mathcal{D}$ is an infinitesimal deformation of $\mathcal{L}$ then

$$\mathcal{L}_\mathcal{D}|_{U_j[\varepsilon,\zeta]} \simeq \left(\mathcal{L}|_{U_j}\right)[\varepsilon,\zeta],$$

so it is described as the gluing of the last modules by means of a suitable isomorphism

$$G_{jk} : \mathcal{L}_{jk}[\varepsilon,\zeta] \xrightarrow{\simeq} \mathcal{L}_{jk}[\varepsilon,\zeta],$$

which in turn is given by the transition matrix

$$G_{jk} = g_{jk} \begin{pmatrix} 1 & 0 & 0 \\ \alpha_{jk} & 1 & 0 \\ \beta_{jk} & 0 & 1 \end{pmatrix},$$

where we express an element $\sigma_j \in \mathcal{L}_j[\varepsilon,\zeta]$, $\sigma_j = \tilde{s}_j + \varepsilon\tau_j + \zeta\xi_j$ as a column vector $(\tilde{s}_j,\tau_j,\xi_j)^t$, $\alpha_{jk} \in \mathcal{O}_{jk,ev}$, $\beta_{jk} \in \mathcal{O}_{jk,odd}$ and g_{jk} is the transition function of $\mathcal{L}$. The cocycle condition for G_{jk} implies that $\{\alpha_{jk}+\beta_{jk}\}_{jk}$ is a 1-cocycle c_1 on X with values in $\mathcal{O}_X$. Clearly, if we change g_{jk} and c_1 by coboundaries we get an isomorphic infinitesimal deformation of the invertible sheaf $\mathcal{L}$. Hence, the set of isomorphism classes of infinitesimal deformations of $\mathcal{L}$ is isomorphic to $H^1(X,\mathcal{O}_X)$. If we have a deformation $\mathcal{L}_Y$ of $\mathcal{L}$ over (Y,y) and $v : \mathcal{D} \to Y$ is a tangent vector to Y at y, then the pull-back of $\mathcal{L}_Y$ under $id_X \times v$ is an infinitesimal deformation of $\mathcal{L}$ and corresponds by the above argument to a class $[c_1] \in H^1(X,\mathcal{O}_X)$. This defines a map $KS : T_yY \to H^1(X,\mathcal{O}_X)$ which is known as the Kodaira–Spencer map of the deformation.

Now we consider the deformation $\sigma \in H^0(X \times \mathcal{D}, \mathcal{L}_\mathcal{D})$ of $s \in H^0(X,\mathcal{L})$. Let us write the local expression of σ as above: $\sigma_j = s_j + \varepsilon\tau_j + \zeta\xi_j$ (we have substituted $\tilde{s}_j$ with s_j because we know that σ is a deformation of s). Then, the cocycle condition for σ to be a global section reads

$$\begin{cases} \tau_j - g_{jk}\tau_k = \alpha_{jk}s_k \\ \\ \xi_j - g_{jk}\xi_k = \beta_{jk}s_k \end{cases}.$$

The meaning of the two equations is the following (see e.g. [43] and the Appendix of [15]): the triple $(\{U_j\}_j, \{g_{jk}^{-1}(\alpha_{jk}+\beta_{jk})\}_{jk}, \{\tau_j+\xi_j\}_j)$ gives rise to a class $\gamma_1 \in \mathbb{H}^1_s(X,\mathcal{C}^\cdot)$ of the hyper-cohomology of the complex

$$\mathcal{C}^\cdot : \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{s\cdot} \mathcal{L} \longrightarrow 0$$

of sheaves on X. The set of isomorphism classes of infinitesimal deformations of $(\mathcal{L}, s)$ is isomorphic to $\mathbb{H}^1_s(X, \mathcal{C}^\cdot)$, and a corresponding Kodaira–Spencer map can be defined for any deformation.

In the present context we are mostly interested in $1|1$ super curves $\mathcal{C}$ over Λ [2], in invertible sheaves over them and in the associated super Krichever map. However, not all such super curves are good, because the definition of the super universal Grassmannian requires a special property for the space of sections of super line bundles on $\mathcal{C}_{red} \setminus \{p_\infty\}$, where p_∞ is the reduced point associated to an irreducible effective divisor on $\mathcal{C}$: it has to be a free Λ-module. A sufficient condition for $\mathcal{C}$ to satisfy this constraint is to be a generic SKP curve:

Definition 6.4 (SKP curve) A Λ-super curve $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ is called an *SKP curve* [2] if its split structure sheaf $\mathcal{O}^{sp}_\mathcal{C} := \mathcal{O}_\mathcal{C} \otimes_\Lambda \Lambda/\mathfrak{m}$ is of the form $\mathcal{O}^{red}_\mathcal{C}|\mathcal{S}$, where $\mathfrak{m}$ is the maximal ideal of nilpotent elements of Λ, $\mathcal{S}$ is an invertible $\mathcal{O}^{red}_\mathcal{C}$-module (a "reduced" line bundle) of degree zero and, following the notations of [2], $\cdot|\cdot$ denotes a direct sum of free Λ-modules, with on the left an evenly generated summand and on the right an odd one. If $\mathcal{S} \neq \mathcal{O}^{red}_\mathcal{C}$ then $\mathcal{C}$ is called a *generic* SKP curve.
□

The super Krichever map [34, 37] associates a point W of SGr_Λ to the geometric datum $(\mathcal{C}, D, \{z^{-1}, \theta\}, \mathcal{L}, \eta)$ of

i. a generic SKP curve $\mathcal{C}$,

ii. an irreducible divisor D on $\mathcal{C}$ whose reduced support is a smooth point p_∞,

iii. formal coordinates z^{-1} and θ at p_∞,

iv. an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ and

v. a formal trivialisation η of $\mathcal{L}$ at p_∞.

Let $\mathcal{L}(\infty D) = \lim_{n\to\infty} \mathcal{L}(nD)$ be the sheaf of sections of $\mathcal{L}$ with at most an arbitrary pole at D, then the point W is $\eta\left(H^0(\mathcal{C}, \mathcal{L}(\infty D))\right)$.

Theorem 6.1 (Bergvelt,Rabin[2]) *The Λ-module $H^0(\mathcal{C}, \mathcal{L}(\infty D))$ is free, hence $W \in SGr_\Lambda$.* □

It is not clear to us if the condition on $\mathcal{C}$ to be a generic SKP curve is also necessary for W to be a point of SGr_Λ. Anyway, in this Section we are interested only in the inverse of the super Krichever map, when defined.

In Section 4 we have associated a super-commutative ring $\mathcal{A}_T$ to the family of points $W_T \subset SGr_\Lambda$ defined by JSKP. Here we assume that $\mathcal{A}_T$ is bigger than Λ, allowing us to define a super curve over Λ and a T-family of sheaves $\mathcal{L}_T$ over it as in [34]. We let $\mathcal{L}$ be the pull-back of this sheaf on $\mathcal{C}$ by the embedding $\iota : \mathcal{C} \simeq \mathcal{C} \times \{p_T\} \hookrightarrow \mathcal{C} \times T$. Furthermore, we suppose that $\mathcal{C}$ is smooth and that $\mathcal{L}_T$ is an invertible sheaf on $\mathcal{C} \times T$, in particular $\mathcal{L}$ is an invertible sheaf on $\mathcal{C}$, of degree equal to the genus of $\mathcal{C}$, whose space of global sections is a free Λ-module of rank $1|1$. We let σ be any non-zero even global section of $\mathcal{L}_T$ which does not vanish at p_∞ and call s its pull-back under ι. Clearly enough, we can interpret $(\mathcal{L}_T, \sigma, id_{\mathcal{L}})$ as a (formal) deformation of the couple $(\mathcal{L}, s)$ and apply the machinery we have discussed above.

We can use σ (respectively s) to trivialise $\mathcal{L}_T$ (respectively $\mathcal{L}$) on the formal neighbourhood $U_0 \times T$ of $p_\infty \times T$ (respectively U_0). Next, let σ_1 be a section of $\mathcal{L}_T$ trivialising it on the open affine subset $U_1 \times T$ whose support is $(\mathcal{C} \setminus \{p_\infty\}) \times T$ and $s_1 = \iota^* \sigma_1$, which trivialises $\mathcal{L}$ on U_1. Since here we do not prove it, we also assume that, given the transition function g_{10} of $\mathcal{L}$ associated to the above trivialisation, the trivialisation of $\mathcal{L}_T$ can be constructed in such a way to produce the transition function $G_{10} = g_{10}\Psi$, where Ψ is the Baker–Akhiezer function of JSKP we computed in Section 4. As it has been proved in [15], this is possible for the KP hierarchy.

To give the promised hyper-cohomological interpretation of JSKP we manipulate the cocycle equations for σ we have written above. First of all we change notation substituting ε and ζ with x and φ respectively and writing $\tau = \partial_x \sigma$, $\xi = \partial_\varphi \sigma$, $\alpha_{10} = G_{10}^{-1} \partial_x G_{10}$ and $\beta_{10} = G_{10}^{-1} \partial_\varphi G_{10}$. Then the cocycle conditions become

$$\begin{cases} G_{10}^{-1} \partial_x \sigma_1 = \partial_x \sigma_0 + (G_{10}^{-1} \partial_x G_{10}) \sigma_0 \\ G_{10}^{-1} \partial_\varphi \sigma_1 = \partial_\varphi \sigma_0 + (G_{10}^{-1} \partial_\varphi G_{10}) \sigma_0 \end{cases},$$

whence

$$G_{10}^{-1} \delta \sigma_1 = (\delta + G_{10}^{-1} \delta G_{10}) \cdot \sigma_0.$$

First of all, we notice that $(\delta\sigma_1, (\delta + G_{10}^{-1}\delta G_{10}) \cdot \sigma_0)$ is a (meromorphic) section of $\mathcal{L}_T$ whose localisation at p_∞ belongs to W_T. Secondly, the above choice of trivialisation implies that $G_{10}^{-1} \delta G_{10} = \hat{h}$, $\hat{h}^{(0)} := \sigma_0 = 1$ and the above equation is exactly the Faà di Bruno recurrence relation. More generally, the cocycle conditions on σ to be a deformation of s along the times t_j are precisely the characterising properties of the vector fields ∂_j of SCS: $(\partial_j + \hat{H}^{(j)}) \cdot W_T \subset W_T$.

Summarising, the JSKP hierarchy can be interpreted as the set of cocycle conditions for the deformation of the spectral super line bundle and of its meromorphic sections which, by means of the super Krichever map, define a point W of the super universal Grassmannian. One can also understand the above statement as another proof of the iso-spectrality of the Jacobian super KP hierarchy.

Part II

A detailed study.

7 Darboux transformations.

One of the most important techniques, which has proved to be very effective in the construction of large classes of explicit solutions of soliton equations, is the method of Darboux transformations [8, 18]. As it is well known, the Darboux transformation is a way to connect two systems of differential equations enabling to produce a solution of the second once a solution of the first has been supplied. An example of such transformation is provided by the Miura map in the KdV theory and the associated modified KdV hierarchy (mKdV) [1, 4, 10, 19, 22, 30, 35].

Suppose that a differential operator L factors as

$$L = PQ$$

and consider the operator

$$\tilde{L} := QP.$$

By the above definition we see that Q intertwines L and $\tilde{L}$, that is

$$\tilde{L}Q = QL.$$

Consequently, if ψ is an eigenfunction of L then $\tilde{\psi} := Q\psi$ turns out to be an eigenfunction of $\tilde{L}$. For KdV, which is also the case studied by Darboux, L

is given by

$$L := -\partial_x^2 + u.$$

Assume that $L\varphi = 0$, then

$$L = (-\partial_x - v)(\partial_x - v),$$

where $v = \partial_x \log \varphi$. Indeed, by its definition v satisfies the Riccati equation

$$v_x + v^2 = u,$$

which is the necessary and sufficient condition for factorising L as above. The last equation corresponds to the definition of the Miura map, which relates the KdV equation to the modified KdV one. The new potential is given by $\tilde{u} := u - 2v_x$ and

$$\tilde{\psi} = (\partial_x - v)\psi$$

is an eigenfunction of $\tilde{L}$ corresponding to the eigenvalue λ whenever ψ satisfies $L\psi = \lambda\psi$. The ideas involved in the Darboux method admit several generalisations, see for instance [29, 20, 21, 24], here we are mostly interested in the concept of Darboux coverings introduced by Magri, Pedroni and Zubelli [25, 26].

Consider three vector fields X, Y and Z on three manifolds M, N and P, respectively.

Definition 7.1 [25] The vector field Y *intertwines* X and Z if there exists a pair of maps $(\mu : N \to M, \sigma : N \to P)$ such that $X = \mu_* Y$ and $Z = \sigma_* Y$. Moreover, if $X = Z$ and N is a fibre bundle on $M = P$, then Y is said to be a *Darboux covering* of X:

$$\begin{array}{c} N, Y \\ \mu \Big\downarrow \Big\downarrow \sigma \\ M, X \end{array}$$

□

This concept is useful for constructing both solutions and invariant submanifolds of the vector field X: if U is a chart on M with coordinate x and

$V \subset \mu^{-1}(U)$ a chart on N adapted to the projection μ and with fibred coordinates (x, a), then the local expression of the above vector fields is

$$\dot{x} = X(x)$$

$$\dot{a} = Y(x, a),$$

where the first equation is that of X on U. Then, any integral curve $x(t)$ of X can be lifted to an integral curve $(x(t), a(t))$ of Y by solving the second equation, which is controlled by $x(t)$. Therefore, we get a new integral curve of X by setting

$$\tilde{x}(t) = \sigma(x(t), a(t)).$$

The last equation can also be interpreted as a "symmetry transformation" of the dynamical system described by X, depending on a solution of the auxiliary system, controlled by X itself, for a.

The application of the formalism we have just described to KP leads directly to the DKP hierarchy, as explained by Magri, Pedroni and Zubelli.

Definition 7.2 Let M be the affine space of (formal) monic Laurent series in z^{-1} with coefficients in $\mathbb{C}[[x]]$ and of the form

$$h(z, x) = z + \sum_{j>0} h_j(x) z^{-j}$$

and let N be the affine space of couples (h, a) where h is as above and a is a monic Laurent series of the form

$$a(z, x) = z + \sum_{j \geq 0} a_j(x) z^{-j}.$$

Define two maps $\mu, \sigma : N \to M$ by

$$\mu(h, a) = h$$

and

$$\sigma(h, a) = h + \frac{\partial_x a}{a}.$$

Finally, let $H^{(k)}$ and $\tilde{H}^{(k)}$ be the current densities associated to h and $\tilde{h}$, respectively. The *DKP hierarchy* is the hierarchy of evolution equations on N defined by

$$\begin{cases} \partial_k h = \partial_x H^{(k)} \\ \partial_k a = a(\tilde{H}^{(k)} - H^{(k)}) \end{cases},$$

where as usual $\partial_k = \frac{\partial}{\partial t_k}$.
□

DKP is a Darboux covering, in the sense of Definition 7.1, of the KP hierarchy

$$\partial_k h = \partial_x H^{(k)}.$$

Indeed, it is clear that μ_* maps the vector fields ∂_j of DKP to those of KP. As for σ_*, we have

$$\begin{aligned}
\partial_k\left(\frac{\partial_x a}{a}\right) &= \frac{a\partial_x\partial_k a - (\partial_x a)(\partial_k a)}{a^2} \\
&= \frac{a\partial_x(a\tilde{H}^{(k)} - aH^{(k)}) - a(\partial_x a)(\tilde{H}^{(k)} - H^{(k)})}{a^2} \\
&= \partial_x\tilde{H}^{(k)} - \partial_x H^{(k)}
\end{aligned}$$

and finally

$$\partial_k\tilde{h} = \partial_k h + \partial_k\left(\frac{\partial_x a}{a}\right) = \partial_x\tilde{H}^{(k)}.$$

A result of Magri, Pedroni and Zubelli, the fact that the modified KP hierarchy is the reduction of DKP defined by $a = h + a_0$, shows that this hierarchy naturally arises in the framework of soliton equations. Instead of continuing the discussion of the properties of DKP, for which we refer the reader to [26, 25], in the following Sections we shall apply the method of Darboux coverings to the study of both the linearisation of JSKP (in the formalism of Faà di Bruno polynomials) and its reductions.

8 JSKP and Darboux transformations.

Before proceeding with the program we outlined at the end of the last Section, our first concern is to give a remarkable connection between the Jacobian super KP hierarchy and DKP, which points out the relevance of the JSKP theory. First of all, we observe that the role a has in DKP does indeed not depend on the order of its pole, since it appears in a homogeneous way in all the equations. Hence, we see that the bosonic degrees of freedom of JSKP are exactly the degrees of freedom of DKP: our a is z^{-1} times the a of DKP. It is thus tempting to conjecture a relation between the two hierarchies. In fact, we can prove the following

Proposition 8.1 *Let $\hat{h}$ and $\hat{H}^{(k)}$ be defined as in Section 3.*

i. *The constraint $\nu = \psi = 0$ is compatible with the even flows of the JSKP hierarchy.*

ii. *The reduction $JSKP_{bos}$ of the even flows of JSKP given by setting $\nu = \psi = 0$ is isomorphic to DKP, i.e. if $\hat{h}$ is a solution of $JSKP_{bos}$, then $(h, z^{-1}a)$ is a solution of DKP and vice versa.*

Proof.

i. Looking at the recurrence relations we introduced in Section 3, we see that

$$\begin{cases} \hat{h}^{(2k-1)} = \theta a^{(k)} + \varphi h^{(k)} \\ \hat{h}^{(2k)} = h^{(k)} - (\theta\varphi) b^{(k)} \end{cases},$$

which implies that

$$\hat{H}^{(2k)} = H^{(k)} - (\theta\varphi) K^{(k)},$$

where $H^{(k)}$ is the k-th current density of KP and $K^{(k)}$ is some power series in z^{-1} and x of the form

$$K^{(k)}(z, x) = \sum_{j>0} K_j^k(x) z^{-j}.$$

The evolution equations for the even flows of JSKP are then

$$\begin{cases} \partial_{2k}\nu = 0 \\ \partial_{2k} a = K^{(k)} \\ \partial_{2k} h = \partial_x H^{(k)} \\ \partial_{2k}\psi = 0 \end{cases},$$

showing that the constraint $\nu = \psi = 0$ is compatible with them.

ii. In the proof of i. we have established also that h evolves according to KP. We need only to understand better the evolution of a. We have to show that $K^{(k)}/a + H^{(k)}$ is the k-th current density of KP associated to

$$\tilde{h} = h + \frac{\partial_x a}{a}.$$

To achieve this result we perform the following "gauge transformation"

$$\hat{h}^{(k)} \mapsto \check{h}^{(k)} := \left(-\frac{1}{a} + \theta\varphi\right) \hat{h}^{(k)}.$$

For instance, we get

$$\begin{cases} \check{h}^{(0)} = -\frac{1}{a} + (\theta\varphi) \cdot 1 \\ \check{h}^{(1)} = -\theta - \varphi\frac{h}{a} \\ \check{h}^{(2)} = -\frac{h}{a} + (\theta\varphi) \cdot \left(h + \frac{\partial_x a}{a}\right) = -\frac{h}{a} + (\theta\varphi) \cdot \tilde{h} \end{cases},$$

moreover

$$\left(-\frac{1}{a} + \theta\varphi\right) (\delta + \hat{h})^2 \left(-\frac{1}{a} + \theta\varphi\right)^{-1} = \partial_x + h + \frac{\partial_x a}{a} = \partial_x + \tilde{h},$$

which shows that the $\theta\varphi$-component of $\check{h}^{(2k)}$ is exactly $\tilde{h}^{(k)}$, and

$$\left(-\frac{1}{a} + \theta\varphi\right) (\partial_{2k} + \hat{H}^{(2k)}) \left(-\frac{1}{a} + \theta\varphi\right)^{-1} = \partial_{2k} + H^{(k)} + \frac{K^{(k)}}{a},$$

which therefore implies, together with the previous result, that

$$H^{(k)} + \frac{K^{(k)}}{a} = \tilde{H}^{(k)}.$$

□

As explained in the Introduction, the importance of having produced JSKP out of a formalism based on the Faà di Bruno polynomials and current densities is that we can apply all the associated machinery to it. As a second example of this paradigm, after the Lax representation we described in Section 5, we can introduce Darboux transformations (and a D-JSKP hierarchy) for the Jacobian super KP theory. We observe that the equation

$$\delta \cdot \sum_{j>0} (\alpha_j + \theta\beta_j + \varphi\gamma_j + (\theta\varphi)\delta_j)\, z^{-j} = \sum_{j>0} (a_j + \theta b_j + \varphi c_j + (\theta\varphi) d_j)\, z^{-j}$$

can be solved by

$$\begin{cases} \partial_x \alpha_j = c_j \\ \partial_j \beta = -d_j \\ \gamma_j = a_j \\ \delta_j = -b_j \end{cases}$$

so, given two Laurent series $\hat{h}$ and $\hat{k}$ of the usual form, we can always find a monic even power series

$$\hat{p} = 1 + p + \theta\zeta + \varphi\xi + (\theta\varphi)q$$

with $\bar{p} = \bar{q} = 0$, $\bar{\zeta} = \bar{\xi} = 1$ and

$$\begin{cases} p = \sum_{j>0} p_j z^{-j} \\ q = \sum_{j>0} q_j z^{-j} \\ \zeta = \sum_{j>0} \zeta_j z^{-j} \\ \xi = \sum_{j>0} \xi_j z^{-j} \end{cases},$$

such that

$$\hat{k} = \hat{h} + \frac{\delta\hat{p}}{\hat{p}}.$$

Indeed, first of all we solve as explained above the equation $\delta\hat{q} = \hat{k} - \hat{h}$ and then we put $\hat{p} = \exp\hat{q}$. Thus, we can safely introduce the

Definition 8.1 (D-JSKP) Let $\hat{N}$ be the affine space of couples of monic formal Laurent series $(\hat{h}, \hat{p})$, let

$$\hat{k} = \hat{h} + \frac{\delta\hat{p}}{\hat{p}}$$

and let $\hat{K}^{(k)}$ be the k-th super current density associated to $\hat{k}$. The *Darboux–Jacobian super KP hierarchy* is the set of compatible evolution equations

$$\begin{cases} \partial_k \hat{h} = (-1)^k \delta \hat{H}^{(k)} \\ \partial_k \hat{p} = \hat{p}(\hat{K}^{(k)} - \hat{H}^{(k)}) \end{cases}.$$

□

If we let $\hat{M}$ be the affine space of the monic formal Laurent series $\hat{h}$ and we define two maps $\hat{\mu}, \hat{\sigma} : \hat{N} \to \hat{M}$ by

$$\hat{\mu}(\hat{h}, \hat{p}) = \hat{h}$$

and

$$\hat{\sigma}(\hat{h}, \hat{p}) = \hat{h} + \frac{\delta \hat{p}}{\hat{p}},$$

then the same computations we did for DKP show that D-JSKP is a Darboux covering of JSKP. The use of D-JSKP, of course, is to provide Darboux transformations in the super-symmetric case, producing new solutions of JSKP from already computed ones. We shall study more accurately this hierarchy in Section 10 in connection with reductions of JSKP.

9 Linearisation.

As we have remarked, the evolution equations of the Jacobian super KP hierarchy we have studied are not linear. To obtain the linearised version we can exploit Darboux covering techniques as it has been done in [13] for KP. The idea is to find a Darboux covering which intertwines the super Central System SCS defined in Section 3 with a new hierarchy whose linearisation is easier.

To this end, let $\hat{\mathcal{M}}$ be the space of sequences of Laurent series $\{\hat{W}^{(k)}\}_{k\geq 0}$ of the form

$$\begin{cases} \hat{W}^{(2k)} = z^k + \sum_{j>0} \left(\hat{W}^{2k}_{0,j} z^{-j} + \hat{W}^{2k}_{1,j} \theta z^{-j} \right) \\ \hat{W}^{(2k+1)} = \theta z^k + \sum_{j>0} \left(\hat{W}^{2k+1}_{0,j} z^{-j} + \hat{W}^{2k+1}_{1,j} \theta z^{-j} \right) \end{cases},$$

where $\bar{\hat{W}}^{(k)} = k \bmod 2$. The third manifold $\hat{\mathcal{P}}$ of Definition 7.1 is just a copy of $\hat{\mathcal{M}}$ formed by the sequences $\{\hat{H}^{(k)}\}_{k\geq 0}$. Finally, the manifold $\hat{\mathcal{N}}$ is the cartesian product $\hat{\mathcal{M}} \times \hat{\mathcal{G}}$ of $\hat{\mathcal{M}}$ by the group of even invertible formal power series $\hat{w}$ of the form

$$\hat{w} = 1 + \sum_{j>0} \left(\hat{w}_{0,j} z^{-j} + \hat{w}_{1,j} \theta z^{-j} \right).$$

The next step is to define suitable vector fields $\hat{\mathcal{X}}$, $\hat{\mathcal{Y}}$ and $\hat{\mathcal{Z}}$ on $\hat{\mathcal{M}}$, $\hat{\mathcal{N}}$ and $\hat{\mathcal{P}}$, respectively. The vector field $\hat{\mathcal{Z}}$ is any vector field of SCS, which is completely characterised by

$$(\partial_k + \hat{H}^{(k)}) \cdot W_T \subset W_T.$$

The flow can be identified by using an index, so we call this vector field $\hat{\mathcal{Z}}_k$. To define $\hat{\mathcal{X}}$ we introduce the subspace $W_T^{(\hat{W})}$ of V_T spanned over B_T by the $\hat{W}^{(j)}$'s. Then, if $k = 2n$ is even we let $\hat{\mathcal{X}}_k$ be the vector field characterised by the property

$$(\partial_k + z^n) \cdot W_T^{(\hat{W})} \subset W_T^{(\hat{W})},$$

while if $k = 2n + 1$ we let $\hat{\mathcal{X}}_k$ be the vector field characterised by

$$(\partial_k + \theta z^n) \cdot W_T^{(\hat{W})} \subset W_T^{(\hat{W})}.$$

As for SCS, we can write down the equations defining $\hat{\mathcal{X}}_k$ by comparing coefficients: if $k = 2n$

$$\partial_k \hat{W}^{(j)} + z^n \hat{W}^{(j)} = \hat{W}^{(j+2n)} + \sum_{l=1}^{n} \left(\hat{W}^j_{0,l} \hat{W}^{(2n-2l)} + \hat{W}^j_{1,l} \hat{W}^{(2n-2l+1)} \right),$$

while if $k = 2n + 1$

$$\begin{cases} \partial_k \hat{W}^{(2j)} + \theta z^n \hat{W}^{(2j)} = \hat{W}^{(2j+2n+1)} + \sum_{l=1}^n \hat{W}^{2j}_{0,l} \hat{W}^{(2n-2l+1)} \\ \partial_k \hat{W}^{(2j+1)} + \theta z^n \hat{W}^{(2j+1)} = -\sum_{l=1}^n \hat{W}^{2j+1}_{0,l} \hat{W}^{(2n-2l+1)} \end{cases}.$$

The definition of $\hat{\mathcal{Y}}_k$ is obtained imposing the further condition

$$(\partial_k + z^n) \cdot \hat{w} \in W_T^{(\hat{W})}$$

if $k = 2n$ and

$$(\partial_k + \theta z^n) \cdot \hat{w} \in W_T^{(\hat{W})}$$

if $k = 2n + 1$.

Definition 9.1 Following [13], we call *super Sato System* (SS) the family of vector fields $\{\hat{\mathcal{X}}_k\}_{k>0}$ on $\hat{\mathcal{M}}$ and *super Darboux-Sato System* (SDS) the family of vector fields $\{\hat{\mathcal{Y}}_k\}_{k>0}$ on $\hat{\mathcal{N}}$.
□

The next step is to define the maps $\hat{\mu} : \hat{\mathcal{N}} \to \hat{\mathcal{M}}$ and $\hat{\sigma} : \hat{\mathcal{N}} \to \hat{\mathcal{P}}$. The first is as usual the projection

$$(\{\hat{W}^{(k)}\}_{k\geq 0}, \hat{w}) \mapsto \{\hat{W}^{(k)}\}_{k\geq 0},$$

while the second is defined by imposing the intertwining condition

$$\hat{w} \cdot W_T = W_T^{(\hat{W})},$$

which holds if and only if

$$\begin{cases} \hat{w}\hat{H}^{(2j)} = \hat{W}^{(2j)} + \sum_{l=1}^{j} \left(\hat{w}_{0,l}\hat{W}^{(2j-2l)} + \hat{w}_{1,l}\hat{W}^{(2j-2l+1)}\right) \\ \hat{w}\hat{H}^{(2j+1)} = \hat{W}^{(2j+1)} + \sum_{l=1}^{j} \hat{w}_{0,l}\hat{W}^{(2j-2l+1)} \end{cases}.$$

Definition 9.2 We say that the sequence $\{\hat{H}^{(k)}\}_{k\geq 0}$ is related to the sequence $\{\hat{W}^{(k)}\}_{k\geq 0}$ by the Darboux transformation generated by $\hat{w}$ if $\hat{w} \cdot W_T = W_T^{(\hat{W})}$.
□

Then [13]:

Lemma 9.1 *The SDS system is a Darboux covering of SS intertwining it with SCS.*

Proof. Clearly, we need only to prove that $\hat{\sigma}_*(SDS) = SCS$. This follows by observing that the definitions of SDS and $\hat{\sigma}$ imply

$$\begin{cases} \partial_{2k}\hat{w} + z^k\hat{w} = \hat{w}\hat{H}^{2k} \\ \partial_{2k+1}\hat{w} + \theta z^k\hat{w} = \hat{H}^{2k+1} \end{cases},$$

so

$$\begin{cases} \hat{w} \cdot (\partial_{2k} + \hat{H}^{(2k)}) = (\partial_{2k} + z^k) \cdot \hat{w} \\ \hat{w} \cdot (\partial_{2k+1} + \hat{H}^{(2k+1)}) = (\partial_{2k} + \theta z^k) \cdot \hat{w} \end{cases}.$$

Hence, we get

$$\begin{aligned} \hat{w} \cdot (\partial_{2k} + \hat{H}^{2k}) \cdot W_T &= (\partial_{2k} + z^k) \cdot \hat{w}W_T \\ &= (\partial_{2k} + z^k) \cdot W_T^{(\hat{W})} \subset W_T^{(\hat{W})} \end{aligned}$$

and

$$\begin{aligned}\hat{w}\cdot(\partial_{2k+1}+\hat{H}^{2k+1})\cdot W_T &= (\partial_{2k+1}+\theta z^k)\cdot\hat{w}W_T\\ &= (\partial_{2k+1}+\theta z^k)\cdot W_T^{(\hat{W})}\subset W_T^{(\hat{W})},\end{aligned}$$

showing that $(\partial_k+\hat{H}^{(k)})\cdot W_T\subset W_T$, i.e. the SCS.
□
We consider now the map $\hat{\rho}:\hat{\mathcal{M}}\to\hat{\mathcal{N}}$ defined by

$$\{\hat{W}^{(k)}\}_{k\geq 0}\mapsto(\{\hat{W}^{(k)}\}_{k\geq 0},\hat{W}^{(0)})$$

and the corresponding map $\hat{\sigma}\circ\hat{\rho}:\hat{\mathcal{M}}\to\hat{\mathcal{P}}$.

Lemma 9.2 *The submanifold $\hat{\rho}(\hat{\mathcal{M}})$ of $\hat{\mathcal{N}}$ is a section of $\hat{\mu}$ invariant under SDS.*

Proof. The previous definitions imply

$$\left\{\begin{array}{l}\partial_{2j}(\hat{w}-\hat{W}^{(0)})=-z^j(\hat{w}-\hat{W}^{(0)})\\ \qquad+\sum_{l=1}^{j}\left((\hat{w}_{0,l}-\hat{W}^0_{0,l})\hat{W}^{(2j-2l)}+(\hat{w}_{1,l}-\hat{W}^0_{1,l})\hat{W}^{(2j-2l+1)}\right)\\ \\ \partial_{2j+1}(\hat{w}-\hat{W}^{(0)})=-\theta z^j(\hat{w}-\hat{W}^{(0)})+\sum_{l=1}^{j}(\hat{w}_{0,l}-\hat{W}^0_{0,l}))\hat{W}^{(2j-2l+1)}\end{array}\right.,$$

proving the lemma.
□
The above lemma motivates the following

Definition 9.3 The map $\hat{\sigma}\circ\hat{\rho}:\hat{\mathcal{M}}\to\hat{\mathcal{P}}$ is the *super Miura* map relating SS to SCS.
□

We have now to linearise the super Sato system. To achieve the result it is better to introduce the infinite even matrix $\mathcal{W}$ defined by

$$\mathcal{W}_{jk}:=\left\{\begin{array}{ll}\hat{W}^j_{0,\frac{k+2}{2}} & \text{for } k \text{ even}\\ \\ \hat{W}^j_{1,\frac{k+1}{2}} & \text{for } k \text{ odd}\end{array}\right.,\quad j,k\geq 0,$$

and the associated matrix $\widetilde{\mathcal{W}}$ whose entries are

$$\widetilde{\mathcal{W}}_{jk} := (-1)^{\bar{\mathcal{W}}_{jk}}\mathcal{W}_{jk} = (-1)^{j+k}\mathcal{W}_{jk}.$$

Then, an easy computation shows that the flows of the SS hierarchy translate into the following Riccati type evolution equations:

$$\begin{cases} \partial_{2n}\mathcal{W} + \mathcal{W}\Lambda_2^{t\,n} - \Lambda_2^n\mathcal{W} = \mathcal{W}\Gamma_{2n}\mathcal{W} \\ \partial_{2n+1}\mathcal{W} + \widetilde{\mathcal{W}}\Lambda_1\Lambda_2^{t\,n} - \Lambda_1\Lambda_2^n\mathcal{W} = \widetilde{\mathcal{W}}\Gamma_{2n+1}\mathcal{W} \end{cases},$$

where ^{t} means ordinary transposition (not super transposition), Λ_1 is the odd shift matrix with entries

$$(\Lambda_1)_{jk} := \frac{1-(-1)^k}{2}\delta_{k,j+1},$$

Λ_2 is the even shift matrix with entries

$$(\Lambda_2)_{jk} := \delta_{k,j+2},$$

Γ_{2n} is the even convolution matrix defined by

$$(\Gamma_{2n})_{jk} := \frac{1-(-1)^k}{2}\delta_{k,2n-j} + \frac{1-(-1)^{k+1}}{2}\delta_{k,2n-j-2}$$

and finally Γ_{2n+1} is the odd convolution matrix given by

$$(\Gamma_{2n+1})_{jk} := \frac{1-(-1)^k}{2}\delta_{k,2n-j-1}.$$

Observe that these matrices satisfy the relations

$$[\Lambda_1, \Lambda_1] = [\Lambda_1, \Lambda_2] = [\Lambda_1, \Lambda_2^t] = 0,$$

$$\Lambda_2^t\Gamma_n = \Gamma_n\Lambda_2,$$

$$\Lambda_1\Gamma_{2n} = \Gamma_{2n}\Lambda_1$$

and

$$\Lambda_1\Gamma_{2n+1} = \Gamma_{2n+1}\Lambda_1 = 0,$$

which imply the compatibility of the above system of matrix Riccati equations.

Proposition 9.1 *The infinite even matrix $\mathcal{W}$ is a solution of the above matrix Riccati equations if and only if it has the form $\mathcal{W} = \mathcal{V}\cdot\mathcal{U}^{-1}$, where $\mathcal{U}$ and $\mathcal{V}$ are infinite even matrices satisfying the constant coefficients linear system*

$$\begin{cases} \partial_{2n}\mathcal{U} = {\Lambda_2^t}^n\mathcal{U} - \Gamma_{2n}\mathcal{V} \\ \partial_{2n+1}\mathcal{U} = \Lambda_1{\Lambda_2^t}^n\mathcal{U} - \Gamma_{2n+1}\mathcal{V} \\ \partial_{2n}\mathcal{V} = \Lambda_2^n\mathcal{V} \\ \partial_{2n+1}\mathcal{V} = \Lambda_1\Lambda_2^n\mathcal{V} \end{cases}$$

with, of course, $\mathcal{U}$ invertible.

Proof. The proof is exactly the same of the commutative case, once we have observed that for two matrices $\mathcal{U}$ and $\mathcal{V}$ the following relations hold:

$$\begin{cases} \partial_{2k}(\mathcal{U}\mathcal{V}) = (\partial_{2k}\mathcal{U})\mathcal{V} + \mathcal{U}(\partial_{2k}\mathcal{V}) \\ \partial_{2k+1}(\mathcal{U}\mathcal{V}) = (\partial_{2k}\mathcal{U})\mathcal{V} + \widetilde{\mathcal{U}}(\partial_{2k}\mathcal{V}) \quad . \\ \widetilde{\mathcal{U}\mathcal{V}} = \widetilde{\mathcal{U}}\widetilde{\mathcal{V}} \Rightarrow \widetilde{\mathcal{U}^{-1}} = \widetilde{\mathcal{U}}^{-1} \end{cases}$$

Thus, if $\mathcal{U}$ and $\mathcal{V}$ solve the system of linear equations of the statement and if we let $\mathcal{W} = \mathcal{V}\mathcal{U}^{-1}$, then

$$\begin{aligned} \partial_{2n}\mathcal{W} &= (\partial_{2n}\mathcal{V})\mathcal{U}^{-1} - \mathcal{V}\mathcal{U}^{-1}(\partial_{2n}\mathcal{U})\mathcal{U}^{-1} \\ &= \Lambda_2^n\mathcal{V}\mathcal{U}^{-1} - \mathcal{V}\mathcal{U}^{-1}{\Lambda_2^t}^n + \mathcal{V}\mathcal{U}^{-1}\Gamma_{2n}\mathcal{V}\mathcal{U}^{-1} \\ &= -\mathcal{W}{\Lambda_2^t}^n + \Lambda_2^n\mathcal{W} + \mathcal{W}\Gamma_{2n}\mathcal{W} \end{aligned}$$

and

$$\begin{aligned} \partial_{2n+1}\mathcal{W} &= (\partial_{2n+1}\mathcal{V})\mathcal{U}^{-1} - \widetilde{\mathcal{V}}\widetilde{\mathcal{U}}^{-1}(\partial_{2n+1}\mathcal{U})\mathcal{U}^{-1} \\ &= \Lambda_1\Lambda_2^n\mathcal{V}\mathcal{U}^{-1} - \widetilde{\mathcal{V}}\widetilde{\mathcal{U}}^{-1}\Lambda_1{\Lambda_2^t}^n + \widetilde{\mathcal{V}}\widetilde{\mathcal{U}}^{-1}\Gamma_{2n+1}\mathcal{V}\mathcal{U}^{-1} \\ &= -\widetilde{\mathcal{W}}\Lambda_1{\Lambda_2^t}^n + \Lambda_1\Lambda_2^n\mathcal{W} + \widetilde{\mathcal{W}}\Gamma_{2n}\mathcal{W}. \end{aligned}$$

Therefore, if we look for a solution $\mathcal{W}$ of the Riccati matrix equations of SS with initial condition $\mathcal{W}(0) = \mathcal{W}_0$, we have simply to solve the linear system above imposing the initial conditions $\mathcal{V}(0) = \mathcal{W}_0$ and $\mathcal{U}(0) = \mathbb{I}$. As we already noticed, the necessary condition

$$\Gamma_{2n+1}\Lambda_1 = 0$$

for the integrability of the linear system holds.
□

Of course, the computations given in the proposition are only formal: to make sense of them one should also introduce a suitable notion of convergence for the intervening series in infinite variables. However, notice that the constraint "$\mathcal{W}_{jk} = 0$ when either $j \geq J$ or $k \geq K$" is compatible with the evolution equations for $\mathcal{W}$, allowing us to consider reductions where only the finite submatrix $\mathcal{W}_{JK}$ of $\mathcal{W}$ consisting of its first J rows and K columns does not vanish. Obviously, $\mathcal{W}_{JK}$ evolves according to the reduced Riccati equations

$$\begin{cases} \partial_{2n}\mathcal{W}_{JK} + \mathcal{W}_{JK}{\Lambda^t_{2,KK}}^n - \Lambda^n_{2,JJ}\mathcal{W}_{JK} = \mathcal{W}_{JK}\Gamma_{2n,KJ}\mathcal{W}_{JK} \\ \partial_{2n+1}\mathcal{W}_{JK} + \widetilde{\mathcal{W}}_{JK}(\Lambda_1{\Lambda_2^t}^n)_{KK} - \Lambda_{1,JJ}\Lambda^n_{2,JJ}\mathcal{W}_{JK} = \widetilde{\mathcal{W}}_{JK}\Gamma_{2n+1,KJ}\mathcal{W}_{JK} \end{cases}$$

yielding "finite type" solutions (i.e. depending only on finitely many times) of SS and hence of SCS and JSKP. Observe that the compatibility of the reduced system requires K to be even, in which case $(\Lambda_1{\Lambda_2^t}^n)_{KK} = \Lambda_{1,KK}{\Lambda^t_{2,KK}}^n$.

Example 9.1 Just to show an example, we compute the solution of SS associated to $J = 3$ and $K = 4$. To simplify notations let us call $W := \mathcal{W}_{34}$,

$$A_1 := \Lambda_{1,33} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 := \Lambda_{2,33} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_1 := \Lambda_{1,44} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 := \Lambda^t_{2,44} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $C_k := \Gamma_{k,43}$. The relevant (i.e. different from zero) convolution matrices are

$$C_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_4 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_5 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that $A_2^2 = 0$ and $B_2^2 = 0$, so the solution of SS (or SCS) will depend only on the first six times. We solve the Riccati system for W by introducing the 4×4 matrix U and the 3×4 matrix V which are solutions of the following linear Cauchy problems:

$$\begin{cases} \partial_{2k} V = A_2^k V \\ \partial_{2k+1} V = A_1 A_2^k V \\ V(0) = W(0) \end{cases} \qquad \begin{cases} \partial_{2k} U = B_2^k U - C_{2k} V \\ \partial_{2k+1} U = B_1 B_2^k U - C_{2k+1} V \\ U(0) = \mathbb{I} \end{cases}$$

and then putting $W := VU^{-1}$. First of all we find that

$$V = \exp \sum_{j>0} (t_{2j} A_2^j + t_{2j-1} A_1 A_2^{j-1}) V(0) = \begin{pmatrix} 1 & t_1 & t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W(0).$$

Then we solve the system for U by introducing the matrix U_0 defined by

$$U = \exp \sum_{j>0} (t_{2j} B_2^j + t_{2j-1} B_1 B_2^{j-1}) U_0 = (\mathbb{I} + t_1 B_1 + t_2 B_2 + (t_3 + t_1 t_2) B_1 B_2) U_0$$

and evolving as

$$\begin{cases} \partial_{2k} U_0 = -(\mathbb{I} - t_1 B_1 - t_2 B_2 - (t_3 - t_1 t_2) B_1 B_2) C_{2k} V \\ \partial_{2k+1} U_0 = -(\mathbb{I} + t_1 B_1 - t_2 B_2 + (t_3 - t_1 t_2) B_1 B_2) C_{2k+1} V \end{cases}.$$

The equations for U_0 are easily solvable and we get

$$U_0 = \mathbb{I} - \begin{pmatrix} t_2 & t_3 & t_4 + \frac{1}{2} t_2^2 \\ 0 & t_2 & 0 \\ t_4 - \frac{1}{2} t_2^2 & t_5 - t_2 t_3 & t_6 - \frac{1}{3} t_2^3 \\ 0 & t_4 - \frac{1}{2} t_2^2 & 0 \end{pmatrix} W(0).$$

In order to write down an effective solution, we choose simple initial conditions, e.g.

$$W(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$V = \begin{pmatrix} 0 & 0 & 0 & t_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & t_1 & 0 & -t_3 - t_1 t_2 \\ 0 & 1 & 0 & -t_2 \\ t_2 & t_3 + t_1 t_2 & 1 & t_1(1 - t_4 - \frac{1}{2}t_2^2) - t_5 - t_2 t_3 \\ 0 & t_2 & 0 & 1 - t_4 - \frac{1}{2}t_2^2 \end{pmatrix}.$$

Finally, we find

$$\hat{W}^{(0)} = 1 - \frac{2t_1 t_2 \theta z^{-1}}{\tau} + \frac{2t_1 \theta z^{-2}}{\tau},$$

$$\hat{W}^{(1)} = \theta - \frac{2t_2 \theta z^{-1}}{\tau} + \frac{2\theta z^{-2}}{\tau},$$

$$\hat{W}^{(2)} = z,$$

$$\hat{W}^{(2k)} = z^k \qquad \text{for } k > 1$$

$$\hat{W}^{(2k+1)} = \theta z^k \qquad \text{for } k > 0,$$

where $\tau = 2 + t_2^2 - 2t_4$. We can thus compute the first super currents of SCS

$$\hat{H}^{(1)} = \theta - \frac{2t_2 \theta z^{-1}}{\tau} + \frac{2\theta z^{-2}}{\tau},$$

$$\hat{H}^{(2)} = z - \frac{2t_1(2 - t_2^2 - 2t_4)\theta z^{-1}}{\tau^2} - \frac{4t_1 t_2 \theta z^{-2}}{\tau^2},$$

$$\hat{H}^{(3)} = \theta z$$

and

$$\hat{H}^{(4)} = z^2 - \frac{4t_1 t_2 \theta z^{-1}}{\tau^2} + \frac{4t_1 \theta z^{-2}}{\tau^2}.$$

As explained in the next section, we obtain a solution of JSKP after substituting t_2 and t_1 with x and $\varphi + t_1$ respectively and putting $\hat{h} = \hat{H}^{(1)} + \varphi\hat{H}^{(2)}$:

$$\begin{cases} \nu = 0 \\ a = 1 - \frac{2xz^{-1} - 2z^{-2}}{2+x^2-2t_4} \\ h = z \\ \psi = -2t_1 \frac{(2-x^2-2t_4)z^{-1}+2xz^{-2}}{(2+x^2-2t_4)^2} \end{cases}.$$

From the above form of the $W^{(k)}$'s it easily follows that the only non-trivial super currents are $H^{(1)}$, $H^{(2)}$ and $H^{(4)}$, in accordance with the fact that the solution of JSKP we have just computed depends on the times t_1 and t_4 only. □

10 Reductions.

An important issue in the theory of integrable systems is the study of their reductions. We have already met one such reduction in Section 7 when discussing the connections between JSKP and the Darboux transformations in KP theory through DKP. Another obvious reduction is obtained from D–JSKP (or, equivalently, DKP) by letting $a = 1$: it is just KP. In this Section we consider more genuine reductions of JSKP, involving all the supercommuting components of $\hat{h}$. Our discussion is based on the reductions of the SCS hierarchy and parallels those of [6, 14], thus it is important to understand how to get the Jacobian super KP hierarchy from the super Central System. Once these reductions are considered in the realm of the JSKP theory, however, we expect some problems to arise, as shown in Example 10.2. Since the application to reductions, even if very important, is not the central theme of the present work, we leave the study of the above mentioned problems to future investigations.

As before, let $\hat{H}^{(k)}$, $k \geq 0$, be the homogeneous formal Laurent series of parity k mod 2 with constant coefficients in Λ and of the form

$$\begin{cases} \hat{H}^{(2k)} = z^k + \sum_{j>0}(\hat{H}^{2k}_{0,j} z^{-j} + \hat{H}^{2k}_{1,j}\theta z^{-j}) \\ \hat{H}^{(2k+1)} = \theta z^k + \sum_{j>0}(\hat{H}^{2k+1}_{0,j} z^{-j} + \hat{H}^{2k+1}_{1,j}\theta z^{-j}) \end{cases}.$$

Define $W := \mathrm{span}_\Lambda\{\hat{H}^{(k)} | k \in \mathbb{N}\} \in SGr_\Lambda$. Then $V = W \oplus V_-$ and, calling π_- the projection onto V_- parallel to W, the super Central System can be equivalently defined by

$$\partial_j \hat{H}^{(k)} = -\pi_-(\hat{H}^{(j)}\hat{H}^{(k)}),$$

which implies that $\partial_j \hat{H}^{(k)} = (-1)^{jk}\partial_k \hat{H}^{(j)}$. Notice in particular that $\partial_1 \hat{H}^{(1)} = 0$. Now, for $k > 1$ we have

$$(-1)^k(\partial_1 + t_1\partial_2)\hat{H}^{(k)} = \partial_k(\hat{H}^{(1)} + t_1\hat{H}^{(2)}),$$

suggesting that in order to get JSKP we should put $t_2 = x$ and $t_1 = \varphi$. However, for $k = 1$ we get

$$-(\partial_1 + t_1\partial_2)\hat{H}^{(1)} = -t_1\partial_1\hat{H}^{(2)} \neq \partial_1(\hat{H}^{(1)} + t_1\hat{H}^{(2)}),$$

showing that we can not simply identify t_1 with φ. Instead, the right way to proceed is to substitute x to t_2, $\varphi + t_1$ to t_1, and to define $\hat{h} := \hat{H}^{(1)} + \varphi\hat{H}^{(2)}$. Then, $\partial_1\hat{H}^{(k)} = \partial_\varphi\hat{H}^{(k)}$ for any k proving, together with the above computations, that $\partial_k\hat{h} = (-1)^k\delta\hat{H}^{(k)}$, i.e. the JSKP hierarchy (observe also that $(\partial_j + \hat{H}^{(j)})W_T \subset W_T$ implies $(\delta + \hat{h})W_T \subset W_T$).

The basic tools we need to understand the problem at hand are the equations we have just recalled and the super-commutativity of the vector fields ∂_k, $k > 0$. The first kind of reduction we consider is the restriction to the invariant submanifolds of the vector fields of SCS. We observe that the invariant submanifold $\hat{\mathcal{Z}}_l$ of ∂_l is defined by

$$\hat{H}^{(l)} = \text{const.}$$

Indeed, the compatibility of the evolution equations of the hierarchy implies

$$\partial_l\hat{H}^{(k)} = (-1)^{kl}\partial_k\hat{H}^{(l)} = 0.$$

At this point, the discussion depends on the parity of l.

Proposition 10.1 *The manifold $\hat{\mathcal{Z}}_l$ is characterised by the condition*

$$\hat{H}^{(l)} \cdot W \subset W.$$

Moreover, if $l = 2n$ is even then $\hat{\mathcal{Z}}_l$ can be identified with the space of the $(l-1)$-tuples $\{\hat{H}^{(j)}\}_{j=1}^{l-1}$.

Proof. Since SCS is defined by $(\partial_l + \hat{H}^{(l)})W_T \subset W_T$ and $\partial_l w = 0$, for any $w \in W$, we get $\hat{H}^{(l)} \cdot W \subset W$. Conversely, suppose that $\hat{H}^{(l)} \cdot W \subset W$. Then $\partial_l \hat{H}^{(k)} = -\pi_-(\hat{H}^{(l)}\hat{H}^{(k)}) = 0$.

Now, if $l = 2n$ is even we can use the equations of SCS to write (recursively) $\hat{H}^{(k)}$ with $k \geq l$ in terms of those with $0 \leq k \leq l-1$:

$$\begin{aligned} \hat{H}^{(2k+l)} &= \hat{H}^{(l)}\hat{H}^{(2k)} - \sum_{j=1}^{n}\left(\hat{H}^{2k}_{0,j}\hat{H}^{(2n-2j)} - \hat{H}^{2k}_{1,j}\hat{H}^{(2n-2j+1)}\right) \\ &\quad - \sum_{j=1}^{k}\left(\hat{H}^{2n}_{0,j}\hat{H}^{(2k-2j)} - \hat{H}^{2n}_{1,j}\hat{H}^{(2k-2j+1)}\right), \end{aligned}$$

$$\begin{aligned} \hat{H}^{(2k+l+1)} &= \hat{H}^{(l)}\hat{H}^{(2k+1)} - \sum_{j=1}^{k}\hat{H}^{2n}_{0,j}\hat{H}^{(2k-2j+1)} \\ &\quad - \sum_{j=1}^{n}\left(\hat{H}^{2k+1}_{0,j}\hat{H}^{(2n-2j)} - \hat{H}^{2k+1}_{1,j}\hat{H}^{(2n-2j+1)}\right), \end{aligned}$$

concluding the proof.
□

In case $l = 2n+1$ is odd we can only write $\hat{H}^{(2k+1)}$, with $k > n$, in terms of the even super current densities and the odd ones with $k < n$. The equations given by the membership of $\hat{H}^{(l)}\hat{H}^{(2k+1)}$ in W yield odd constraints. For JSKP, the condition of the proposition is equivalent to $\delta\hat{H}^{(l)} = 0$ (see Lemma 4.1) and, once we have solved this equation, the above odd constraints are *automatically* satisfied.

Continuing with the case l even, we can express the operator $\partial_k + \hat{H}^{(k)}$ as a matrix $\mathcal{H}_k$, functionally depending on z and θ, acting on the right on the vectors of the basis $\{\hat{H}^{(0)} = 1, \cdots, \hat{H}^{(l-1)}\}$. As usual we order the basis by putting first the even currents and then the odd ones. Thus

$$(\partial_k + \hat{H}^{(k)}) \cdot (v_0, v_1) = (v_0, v_1)\begin{pmatrix} \mathcal{H}_{k,00} & \mathcal{H}_{k,01} \\ \mathcal{H}_{k,10} & \mathcal{H}_{k,11} \end{pmatrix},$$

where we have collected in v_0 the even basis vectors, in v_1 the odd ones and we have written the matrix in block form. We obtain the following super zero curvature (or super Zakharov–Shabat) representation for the restriction of SCS to $\hat{\mathcal{Z}}_l$:

$$\partial_j \mathcal{H}_k^{\langle j \rangle} - (-1)^{jk} \partial_k \mathcal{H}_j^{\langle k \rangle} + [\mathcal{H}_j, \mathcal{H}_k] = 0,$$

where

$$\mathcal{H}_k^{\langle j \rangle} = \begin{pmatrix} \mathcal{H}_{k,00} & \mathcal{H}_{k,01} \\ (-1)^j \mathcal{H}_{k,10} & (-1)^j \mathcal{H}_{k,11} \end{pmatrix}.$$

If we further restrict to $\hat{\mathcal{Z}}_l \cap \hat{\mathcal{Z}}_k$ for any given k (either even or odd) we get the Lax representation

$$\partial_j \mathcal{H}_k^{\langle j \rangle} + [\mathcal{H}_j, \mathcal{H}_k] = 0.$$

From the point of view of the Lax operators we introduced in Section 5 for JSKP, these reductions can be called super n-Gel'fand–Dickey hierarchies, because they correspond to $\hat{L}_2^n$ (respectively $\hat{L}_1 \hat{L}_2^n$) being a purely differential operator. While for an even monic super differential operator $\hat{P}$ of order $2n$ in δ of the form

$$\hat{P} = \delta^{2n} + \sum_{j=0}^{2n-2} p_j \delta^{2n-j-2}$$

both existence and uniqueness of a $2n$-th root $\hat{L}$ need not be true, see [28], in Appendix A we prove that such an operator has a uniquely defined n-th root $\hat{L}_2$. Hence, the super $2n$-GD hierarchy is defined by

$$\begin{cases} \partial_{2k} \hat{P} = [(\hat{P}^{\frac{k}{n}})_+, \hat{P}] \\ \partial_{2k} \hat{L}_1 = [(\hat{P}^{\frac{k}{n}})_+, \hat{L}_1] \\ \partial_{2k+1} \hat{P} = [(\hat{L}_1 \hat{P}^{\frac{k}{n}})_+, \hat{P}] \\ \partial_{2k} \hat{L}_1 = [(\hat{L}_1 \hat{P}^{\frac{k}{n}})_+, \hat{L}_1] \end{cases},$$

where $\hat{L}_1$ is, as in Section 5, an odd super pseudo-differential operator of order 2 of the form

$$\hat{L}_1 = \delta - \varphi \delta^2 + \sum_{j \geq 0} u_j \delta^{-j}$$

such that $[\hat{L}_1, \hat{L}_1] = 0$ and super-commuting with $\hat{P}$.

Similarly, we consider a non-monic odd super differential operator of order $2n+2$

$$\hat{Q} = \delta^{2n+1} - \varphi\delta^{2n+2} + \sum_{j=0}^{2n} q_j \delta^{2n-j},$$

such that $[\hat{Q}, \hat{Q}] = 0$, and a monic even super pseudo-differential operator

$$\hat{L}_2 = \delta^2 + \sum_{j\geq 0} v_j \delta^{-j}$$

super-commuting with $\hat{Q}$. Then, the super $(2n+1)$-GD hierarchy is defined by

$$\begin{cases} \partial_{2k}\hat{Q} = [(\hat{L}_2^{\frac{k}{2}})_+, \hat{Q}] \\ \partial_{2k}\hat{L}_2 = [(\hat{L}_2^{\frac{k}{2}})_+, \hat{L}_2] \\ \partial_{2k+1}\hat{Q} = [(\hat{Q}\hat{L}_2^{\frac{k-n}{2}})_+, \hat{Q}] \\ \partial_{2k}\hat{L}_2 = [(\hat{Q}\hat{L}_2^{\frac{k-n}{2}})_+, \hat{L}_2] \end{cases}.$$

We give now two simple examples of the above reductions for JSKP. Observe that the condition $\delta\hat{H}^{(l)} = 0$ does not uniquely define the current, so $\hat{\mathcal{Z}}_l$ indeed foliates according to its choice. As for KP, see [5, 6, 14], we consider currents of the form

$$\hat{H}^{(2n)} = z^n$$

or

$$\hat{H}^{(2n+1)} = \theta z^n.$$

The first example is a restriction to the invariant submanifold of an odd flow.

Example 10.1 We consider the case $l = 1$, i.e. $\theta W_T \subset W_T$. Since $\hat{H}^{(1)} = \nu + \theta a$, we see that for JSKP the submanifold $\hat{\mathcal{Z}}_1$ is defined by

$$\begin{cases} \nu = 0 \\ a = 1 \end{cases}.$$

Accordingly, we get

$$\begin{cases} \hat{h}^{(2k)} = h^{(k)} - \theta\psi^{(k)} \\ \\ \hat{h}^{(2k-1)} = \theta h^{(k-1)} + \varphi h^{(k)} + (\theta\varphi)\psi^{(k)} \end{cases},$$

so the super current densities become

$$\begin{cases} \hat{H}^{(2k)} = H^{(k)} - \theta\Psi^{(k)} \\ \\ \hat{H}^{(2k-1)} = \theta H^{(k-1)} \end{cases},$$

where $H^{(k)}$ is the k-th KP current density and

$$\Psi^{(k)} = \psi^{(k)} - \sum_{j=1}^{k-1} \left(\psi_j^{(k)} H^{(k-j-1)} + h_j^{(k)} \Psi^{(k-j-1)} \right).$$

These computations confirm that $H^{(2k+1)} = \theta H^{(2k)}$, as expected. The further constraints we mentioned above in this case do not appear since we have taken $l = 1$, the lowest possible value, and therefore there are no odd currents of order less than l, while the conditions for those with $2k+1 > l$ simply reduce to the trivial identity $0 = 0$. The evolution equations for the reduced hierarchy thus become

$$\begin{cases} \partial_{2k} h = \partial_x H^{(k)} \\ \\ \partial_{2k}\psi = \partial_x \Psi^{(k)} \end{cases}$$

and

$$\begin{cases} \partial_{2k+1} h = 0 \\ \\ \partial_{2k+1}\psi = \partial_x H^{(k)} \end{cases}.$$

We see that h evolves along the flows of KP.
□

The following example shows which kind of problems can arise in connection with the reductions we are studying.

Example 10.2 We take $l = 4$. A simple computation shows that

$$\begin{aligned}\hat{H}^{(4)} &= (h' + h^2 - 2h_1 - 2\psi_1\nu) \\ &\quad -\theta(\psi' + 2h\psi - 2\psi_1 a) \\ &\quad +\varphi(\nu'' + 2h\nu' - 2\nu_1' - 2a_1'\nu) \\ &\quad -(\theta\varphi)(a'' + 2ha' - 2a_1'a),\end{aligned}$$

where $'$ indicates the derivative with respect to x. Imposing $\hat{H}^{(4)} = z^2$ we get the following recurrence relations

$$\begin{cases} h_2 = -\frac{1}{2}h_1' + \psi_1\nu_1 & \\ h_k = -\frac{1}{2}h_{k-1}' - \frac{1}{2}\sum_{j=1}^{k-2} h_{k-j-1}h_j + \psi_1\nu_{k-1} & \text{for } k > 2 \end{cases},$$

$$\begin{cases} \psi_2 = -\frac{1}{2}\psi_1' + a_1\psi_1 & \\ \psi_k = -\frac{1}{2}\psi_{k-1}' - \sum_{j=1}^{k-2} h_{k-j-1}\psi_j + a_{k-1}\psi_1 & \text{for } k > 2 \end{cases},$$

$$\begin{cases} \nu_2' = -\frac{1}{2}\nu_1'' + a_1'\nu_1 & \\ \nu_k' = -\frac{1}{2}\nu_{k-1}'' - \sum_{j=1}^{k-2} h_{k-j-1}\nu_j' + a_1'\nu_{k-1} & \text{for } k > 2 \end{cases},$$

and

$$\begin{cases} a_2' = -\frac{1}{2}a_1'' + a_1a_1' & \\ a_k' = -\frac{1}{2}a_{k-1}'' - \sum_{j=1}^{k-2} h_{k-j-1}a_j' + a_{k-1}a_1' & \text{for } k > 2 \end{cases}.$$

These equations allow us to compute recursively the coefficients h_j, ψ_j, ν_j and a_j for $j > 1$ in terms of h_1, ψ_1, ν_1 and a_1 by means of quadratures. For instance, we get

$$\begin{cases} h_3 = \frac{1}{4}h_1'' - \frac{1}{2}\psi_1'\nu_1 - \frac{1}{2}h_1^2 - \psi_1\nu_1' + \psi_1\int a_1'\nu_1 \\ \psi_3 = \frac{1}{4}\psi_1'' - \frac{1}{2}a_1\psi_1' - h_1\psi_1 - a_1'\psi_1 + \psi_1\int a_1a_1' \\ \nu_3 = \frac{1}{4}\nu_1'' - \frac{1}{2}a_1'\nu_1 - \frac{1}{2}\int a_1'\nu_1' - \int h_1\nu_1' + \int(a_1'\int a_1'\nu_1) \\ a_3 = \frac{1}{4}a_1'' - \frac{1}{2}a_1a_1' - \frac{1}{2}\int a_1'^2 - \int h_1a_1' + \int(a_1'\int a_1a_1') \end{cases},$$

where integrals such as $\int a_1 a_1'$ have not been simplified to signal the need of an integration constant. Thus, the degrees of freedom in this reduction of JSKP are h_1, a_1, ν_1, ψ_1 and the above mentioned integration constants.

In order to compute the equations for the evolving degrees of freedom we have only to calculate the coefficients of order 1 of the super current densities. More precisely, we write

$$\hat{H}^{(k)} = (z^{\lfloor k/2 \rfloor} \text{ or } \theta z^{\lfloor k/2 \rfloor}) + \sum_{j>0} \left(\hat{H}_j^{k,1} + \theta \hat{H}_j^{k,\theta} + \varphi \hat{H}_j^{k,\varphi} + (\theta\varphi) \hat{H}_j^{k,\theta\varphi} \right) z^{-j},$$

where $\lfloor y \rfloor$ is the largest integer not greater than y, whence

$$\begin{cases} \partial_{2k}\nu_1 = \hat{H}_1^{2k,\varphi} \\ \partial_{2k}a_1 = -\hat{H}_1^{2k,\theta\varphi} \\ \partial_{2k}h_1 = \partial_x \hat{H}_1^{2k,1} \\ \partial_{2k}\psi_1 = -\partial_x \hat{H}_1^{2k,\theta} \end{cases}$$

and

$$\begin{cases} \partial_{2k+1}\nu_1 = -\hat{H}_1^{2k+1,\varphi} \\ \partial_{2k+1}a_1 = -\hat{H}_1^{2k+1,\theta\varphi} \\ \partial_{2k+1}h_1 = \partial_x \hat{H}_1^{2k+1,1} \\ \partial_{2k+1}\psi_1 = \partial_x \hat{H}_1^{2k+1,\theta} \end{cases}$$

In turn, the coefficients $\hat{H}_1^{k,\cdot}$ can be computed using the recurrence relations we wrote in Section 3, Proposition 3.1, that define $\hat{H}^{(k)}$. After some cumbersome calculations (hoping to have not mistaken some signs) we get the first evolution equations:

$$\begin{cases} \partial_3\nu_1 = 0 \\ \partial_3 a_1 = -\nu_1' \\ \partial_3 h_1 = \frac{1}{2}\nu_1'' - a_1\nu_1' \\ \partial_3\psi_1 = \frac{1}{2}a_1'' - a_1 a_1' + h_1' \end{cases},$$

$$\begin{cases} \partial_5 \nu_1 = -\nu_1 \nu_1' \\ \partial_5 a_1 = -a_1' \nu_1 - a_1 \nu_1' \\ \partial_5 h_1 = \frac{1}{4}\nu_1''' - \frac{1}{2} a_1 \nu_1'' - 2h_1 \nu_1' - h_1' \nu_1 - a_1' \nu_1' \\ \qquad + a_1^2 \nu_1' - \nu_1' \int a_1 a_1' \\ \partial_5 \psi_1 = \frac{1}{4} a_1''' - \frac{1}{2} a_1 a_1'' - \frac{1}{2} a_1'^2 - 2h_1 a_1' - h_1' a_1 \\ \qquad + a_1^3 - 2a_1^2 a_1' + \psi_1' \nu_1 + \psi_1 \nu_1' - a_1' \int a_1 a_1' \end{cases}$$

and

$$\begin{cases} \partial_6 \nu_1 = \frac{1}{4}\nu_1''' - \frac{3}{2} a_1' \nu_1' + 3h_1 \nu_1' \\ \partial_6 a_1 = \frac{1}{4} a_1''' - \frac{3}{2} a_1'^2 - 3h_1 a_1' + 6\psi_1 \nu_1' \\ \partial_6 h_1 = \frac{1}{4} h_1''' - 3h_1 h_1' - \frac{3}{2}\psi_1 \nu_1'' - \frac{3}{2}\psi_1' \nu_1' \\ \partial_6 \psi_1 = \frac{1}{4}\psi_1''' - \frac{3}{2} a_1'' \psi_1 - \frac{3}{2} a_1' \psi_1' - 3h_1 \psi_1' - 3h_1' \psi_1 \end{cases}$$

We see that the evolution equations for the time t_6 are a super-symmetric extension of the KdV equation, which can be retrieved by setting $a_1 = \nu_1 = \psi_1 = 0$. Moreover, as could have been anticipated from the form of the equation $\hat{H}^{(4)} = z^2$, this reduction of JSKP involves integral differential equations for its evolution flows. By contrast, let us compute the reduction $l = 4$ of SCS. Proposition 10.1 says that the degrees of freedom are the coefficients of the first three super currents $\hat{H}^{(1)}$, $\hat{H}^{(2)}$ and $\hat{H}^{(3)}$. Adopting the usual notation, we find for instance that

$$\hat{H}^{(5)} = \hat{H}^{(4)}\hat{H}^{(1)} - \hat{H}^1_{1,1}\hat{H}^{(3)} - \hat{H}^1_{0,1}\hat{H}^{(2)} - \hat{H}^1_{1,2}\hat{H}^{(1)} - \hat{H}^1_{0,2}$$

and

$$\hat{H}^{(6)} = \hat{H}^{(4)}\hat{H}^{(2)} - \hat{H}^2_{1,1}\hat{H}^{(3)} - \hat{H}^2_{0,1}\hat{H}^{(2)} - \hat{H}^2_{1,2}\hat{H}^{(1)} - \hat{H}^2_{0,2}.$$

To link with the computations for JSKP we observe that, as in the beginning of the Section, $\hat{h} = \hat{H}^{(1)} + \varphi \hat{H}^{(2)}$ so we abuse somehow notations by renaming the coefficients of the first two super currents as follows:

$$\hat{H}^1_{0,j} = \nu_j, \quad \hat{H}^1_{1,j} = a_j,$$

$$\hat{H}^2_{0,j} = h_j, \quad \hat{H}^2_{1,j} = -\psi_j.$$

Then, making use of the equations $\partial_j \hat{H}^{(k)} = (-1)^{jk} \partial_k \hat{H}^{(j)}$ of SCS, we get for instance

$$\begin{cases} \partial_5 \nu_1 = -\partial_1 \left(\nu_3 - h_1 \nu_1 - a_1 \hat{H}^3_{0,1} - a_2 \nu_1 \right) \\ \partial_5 a_1 = -\partial_1 \left(a_3 + \nu_1 \psi_1 - a_1 \hat{H}^3_{1,1} - a_1 a_2 \right) \\ \partial_5 h_1 = \partial_2 \left(\nu_3 - h_1 \nu_1 - a_1 \hat{H}^3_{0,1} - a_2 \nu_1 \right) \\ \partial_5 \psi_1 = -\partial_2 \left(a_3 + \nu_1 \psi_1 - a_1 \hat{H}^3_{1,1} - a_1 a_2 \right) \end{cases}$$

and

$$\begin{cases} \partial_6 \nu_1 = \partial_1 \left(h_3 - h_1^2 + \psi_1 \hat{H}^3_{0,1} + \psi_2 \nu_1 \right) \\ \partial_6 a_1 = -\partial_1 \left(\psi_3 - h_1 \psi_1 - \psi_1 \hat{H}^3 1, 1 - a_1 \psi_2 \right) \\ \partial_6 h_1 = \partial_2 \left(h_3 - h_1^2 + \psi_1 \hat{H}^3_{0,1} + \psi_2 \nu_1 \right) \\ \partial_6 \psi_1 = \partial_1 \left(\psi_3 - h_1 \psi_1 - \psi_1 \hat{H}^3 1, 1 - a_1 \psi_2 \right) \end{cases} .$$

Now, an effective comparison with the equations we obtained for JSKP would require more computations, which we do not perform since this is not the point we want to make. We observe only that the result we have obtained for SCS is reasonably in accordance with the JSKP case. What we want to stress, however, is the fact that *the evolution equations for the present reduction of SCS are all of differential type*: in fact, the flows of SCS and the expressions of the high order super currents in terms of the others are all given by equations of that type. Accordingly, the mechanism responsible for the appearance of non local equations is the "spatialisation" of the dynamical SCS system. Such a phenomenon has already been observed for reductions of the KP hierarchy [12].
□

Another kind of reductions we can perform is the projection onto the space of orbits of the vector field ∂_l. We call $\hat{Q}_l$ the space of the solutions of the l-th flow of SCS. The compatibility of the evolution equations of SCS then implies that the vector fields ∂_k induce super-commuting flows on $\hat{\mathcal{Q}}_l$. We have

Proposition 10.2 *If $l = 2n$ is even then the quotient space $\hat{\mathcal{Q}}_l$ can be identified with the space of the l-tuples $\{\hat{H}^{(j)}\}_{j=1}^{l}$.*

Proof. Using the definition of SCS given at the end of Section 3 we can write a recurrence relation for the $\hat{H}^{(j)}$'s with $j > l$:

$$\begin{aligned}\hat{H}^{(2j+l)} &= \partial_l \hat{H}^{(2j)} + \hat{H}^{(l)}\hat{H}^{(2j)} - \sum_{k=1}^{j}\left(\hat{H}^{l}_{0,k}\hat{H}^{(2j-2k)} - \hat{H}^{l}_{1,k}\hat{H}^{(2j-2k+1)}\right)\\ &\quad - \sum_{k=1}^{n}\left(\hat{H}^{2j}_{0,k}\hat{H}^{(l-2k)} + \hat{H}^{2j}_{1,k}\hat{H}^{(l-2k+1)}\right),\end{aligned}$$

$$\begin{aligned}\hat{H}^{(2j+l+1)} &= \partial_l \hat{H}^{(2j+1)} + \hat{H}^{(l)}\hat{H}^{(2j+1)} - \sum_{k=1}^{j}\hat{H}^{l}_{0,k}\hat{H}^{(2j-2k+1)}\\ &\quad - \sum_{k=1}^{n}\left(\hat{H}^{2j+1}_{0,k}\hat{H}^{(l-2k)} + \hat{H}^{2j+1}_{1,k}\hat{H}^{(l-2k+1)}\right).\end{aligned}$$

□

As for the process of reduction by restriction we discussed above, if $l = 2n+1$ is odd then we can only express the odd super current densities of order greater than l by means of the others, obtaining at the same time a set of odd constraints for these last.

Of course, we can obtain new reductions simply by combining those we have discussed up to now. Another kind of reduction of JSKP, the last we consider here, is instead related to the D-JSKP hierarchy we introduced in Section 8 (see [7, 25] for an analogous discussion in the context of KP and DKP). Recall that to the couple $(\hat{h}, \hat{p})$ we associated a new Faà di Bruno polynomial

$$\hat{k} := \hat{h} + \frac{\delta\hat{p}}{\hat{p}}$$

and the corresponding super currents $\hat{K}^{(j)}$. Then

Proposition 10.3 *The submanifold $\mathcal{S}_l$ of $\hat{N}$ (see Definition 8.1) characterised by*

$$z^{l/2}\hat{p} \in W_X$$

for l even, or by

$$\theta z^{(l-1)/2}\hat{p} \in W_X$$

for l odd, is invariant under the flows of the D-JSKP hierarchy.

Consequently, the submanifold

$$\mathcal{T}_l := \hat{\mu}(\mathcal{S}_l)$$

of $\hat{M}$ is invariant under JSKP.

Proof. We give the proof only for $l = 2n$ even, the other proof is the same up to some obvious change of signs. The condition $(\hat{h}, \hat{p}) \in \mathcal{S}_l$ implies the existence of some coefficients $\alpha_j(x, \varphi)$, $j = 0, \cdots, l$ such that

$$z^n \hat{p} = \sum_{j=0}^{l} \alpha_j \hat{H}^{(j)}.$$

Let $W_T^{\hat{k}} := \operatorname{span}_{B_T}\{\hat{K}^{(j)} |\, j \geq 0\}$. Then, by definition we have

$$\hat{p}(\delta + \hat{k}) = (\delta + \hat{h})\hat{p},$$

hence

$$\hat{p}(\delta + \hat{k})^j = (\delta + \hat{h})^j \hat{p},$$

implying that

$$z^l \hat{p} W_T^{\hat{k}} \subset W_T.$$

Therefore,

$$(\partial_k + \hat{H}^{(k)}) z^l \hat{p} = z^l \hat{p} \hat{K}^{(k)} \in W_T,$$

i.e.

$$z^l \partial_k \hat{p} + \sum_{j=0}^{l} (-1)^{jk} \alpha_j \hat{H}^{(k)} \hat{H}^{(j)} \in W_T.$$

Using now the property $\partial_k \hat{H}^{(j)} + \hat{H}^{(k)} \hat{H}^{(j)} \in W_T$ of JSKP and comparing the coefficients of z^j and θz^j for $j = 0, \cdots, l$, we get

$$z^l \partial_k \hat{p} - \sum_{j=0}^{l} (-1)^{jk} \alpha_j \partial_k \hat{H}^{(j)} = \sum_{j=0}^{l} (\partial_k \alpha_j) \hat{H}^{(j)},$$

i.e.

$$\partial_k \left(z^l \hat{p} - \sum_{j=0}^{l} \alpha_j \hat{H}^{(j)} \right) = 0$$

that is what we had to prove.

□

Obviously, any intersection $\mathcal{S}_{j_1,\cdots,j_n} := \cap_{k=1}^{n} \mathcal{S}_{j_k}$, is then invariant under D-JSKP and, consequently, $\mathcal{T}_{j_1,\cdots,j_n} := \hat{\mu}(\mathcal{S}_{j_1,\cdots,j_n})$ is invariant under the Jacobian super KP hierarchy. As for KP, see [7], we can readily understand this type of reductions in the Lax formalism of Section 5. Let

$$\hat{L}_{(2k)} := \hat{\Phi}(z^k \hat{p})$$

and

$$\hat{L}_{(2k+1)} := \hat{\Phi}(\theta z^k \hat{p}).$$

Of course, $z^k \hat{p} \in W_X$ implies that $\hat{L}_{(2k)}$ is purely differential, while $\theta z^k \hat{p} \in W_X$ implies that $\hat{L}_{(2k+1)}$ is purely differential. Then, using the properties of the map $\hat{\Phi}$, we deduce for instance that $\hat{\mathcal{T}}_{k,k+2n}$ implies

$$\hat{L}_{(k+2n)} = \hat{L}_{(k)} \cdot \hat{L}_2^n$$

while $\hat{\mathcal{T}}_{k,k+2n+1}$ implies

$$\hat{L}_{(k+2n)} = \hat{L}_{(k)} \cdot \hat{L}_1 \hat{L}_2^n,$$

in particular $\hat{\mathcal{T}}_{0,n}$ is the super n-GD hierarchy. Borrowing the terminology from KP theory, we say that $\hat{\mathcal{T}}_{k,k+n}$ is a *rational reduction* of JSKP.

11 Conclusions.

As we said in the introductory Section, any mathematical theory can not be given the status of a well developed one unless we have a deep understanding of all its many facets. In this respect, we can say without any worries that the super KP theory still presents us a long way to be walked before to achieve a level of comprehension similar to that of its parent, KP. In this investigation, armed with faith in the adage which says that "anything that can be done can also be *super-done*", we have exploited and adapted the techniques associated with the Faà di Bruno formalism to study the Jacobian super KP hierarchy.

The reason why our approach is isomorphic to that of Mulase and Rabin has its roots in the form of the evolution equations that, as for KP, in the realm of algebraic geometry admit an interpretation as cocycle conditions for the deformation of a super line bundle over a super curve and of its sections holomorphic except at a point. Indeed, this feature, which has been explained in Section 6, was already used by Rabin in [37] to define JSKP.

In our opinion, the importance of this work is to have extended the formalism to the super case, allowing to exploit its power in the study of the diverse properties of the theory. We have been able in Section 4 to give a simple proof of the iso-spectrality property but also to give a Lax description of the hierarchy (Section 5), which was lacking up to now. In Section 8 we have studied more closely the relations with KP, proving that the even flows of JSKP are really those of the DKP hierarchy introduced in connection with the Darboux transformations: this highlights the naturalness of the Jacobian super KP hierarchy. Moreover, we have also defined super Darboux transformations and the corresponding super hierarchy (D-JSKP) which, as in the ordinary case, brings with itself reductions of JSKP. These and other reductions, such as the super n-GD hierarchies, have been considered in Section 10, where we have also studied two examples, super 1-GD and super 4-GD, which pointed out some difficulties. Finally, in Section 9 we have shown how to linearise JSKP and to obtain finite type solutions in an easy way.

We think that the present work really opens the way to more profound investigations of JSKP, having implemented some of the methods that proved to be very helpful in the study of the KP hierarchy.

A Some useful formulae.

In this Appendix we recall the definition and some properties of the super binomial coefficients, allowing us to define the ring of formal super pseudo-differential operators and to prove a result about the roots of suitable pseudo-differential operators. First of all, the ordinary binomial coefficients are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

and more generally for any n but $k \geq 0$ by

$$\binom{n}{k} := \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

Following [28, 33] we define for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ the super binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 0 & \text{if } 0 \leq n < k \text{ or } (n,k) \equiv (0,1) \bmod 2 \\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{otherwise} \end{cases},$$

where $\lfloor x \rfloor$ is the largest integer not greater than x. The super binomial coefficients satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix} + (-1)^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

and

$$\sum_{j=0}^{k} (-1)^{\frac{j(j+1)}{2} + j(n-k)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k-j \end{bmatrix} = 0.$$

Let now x and φ be an even and an odd variable respectively and define $B := \Lambda[[x, \varphi]]$, the ring of formal power series in x and φ with coefficients in a Grassmann algebra Λ over $\mathbb{C}$. The ring $S\Psi DO_X$ of formal super pseudo-differential operators over $X = \operatorname{Spec} B$ is the space of formal series

$$L_n := \sum_{j \geq 0} u_j \delta^{n-j}, \qquad u_j \in B,$$

where $\delta = \partial_\varphi + \varphi \partial_x$. We say that the operator L_n as above has order n. The product of two such operators is defined by means of the super Leibniz rule

$$\delta^k \cdot f = \sum_{j \geq 0} (-1)^{\bar{f}(k-j)} \begin{bmatrix} k \\ j \end{bmatrix} f^{(j)} \delta^{k-j},$$

where $f \in B$ is a homogeneous element of degree $\bar{f}$ and $f^{(j)} = \delta^j(f)$. The properties of the super binomial coefficients can be used to verify the following adjoint super Leibniz rule

$$f \delta^k = \sum_{j \geq 0} (-1)^{\frac{j(j+1)}{2} + k\bar{f}} \begin{bmatrix} k \\ j \end{bmatrix} \delta^{k-j} f^{(j)}.$$

We can give a structure of complete topological space to $S\Psi DO_X$ by means of the natural filtration

$$\cdots S\Psi DO_{X,k-1} \subset S\Psi DO_{X,k} \subset S\Psi DO_{X,k+1} \subset \cdots,$$

where $S\Psi DO_{X,k}$ is the subspace of operators of order not greater than k.

Proposition A.1 *Let*

$$\hat{P} = \delta^{2n} + \sum_{j\geq 0} p_j \delta^{2n-j-2}$$

be an even monic super pseudo-differential operator of order $2n > 0$. *Then there exists a unique even monic operator*

$$\hat{L} = \delta^2 + \sum_{j\geq 0} u_j \delta^{-j}$$

such that $\hat{L}^n = \hat{P}$.

Proof. We proceed by induction: suppose that we have proved that there exists an operator $\hat{L}_k$, uniquely defined up to $O(\delta^{-k-1})$, such that $\hat{L}_k^n = \hat{P} + O(\delta^{2n-k-3})$. For $k = -1$ we obviously have $\hat{L}_{-1} = \delta^2$. Then we let $\hat{L}_{k+1} = \hat{L}_k + u_{k+1}\delta^{-k-1}$ so

$$\hat{L}_{k+1}^n = \hat{L}_k^n + \sum_{j=0}^{n-1} \hat{L}_k^j u_{k+1}\delta^{-k-1}\hat{L}_k^{n-j-1} + R_k,$$

where R_k has order at most $\max_{j\leq n-2}\{2j - (n-1)(k+1)\} = 2n - 2k - 6 \leq 2n - k - 4$, while

$$\sum_{j=0}^{n-1} \hat{L}_k^j u_{k+1}\delta^{-k-1}\hat{L}_k^{n-j-1} = \sum_{j=0}^{n-1} u_{k+1}\delta^{-k-1}\hat{L}_k^{n-1} + O(\delta^{2n-k-4}),$$

so

$$\hat{L}_{k+1}^n = \hat{L}_k^n + nu_{k+1}\delta^{2n-k-3} + O(\delta^{2n-k-4})$$

and we can compute u_{k+1} in such a way that $\hat{L}_{k+1}^n = \hat{P} + O(\delta^{2n-k-4})$.
□

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