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Approximation of Free-Discontinuity Problems

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CONTENTS

INTRODUCTION

In many problems of the Calculus of Variations involving competing bulk and surface energies, we encounter functionals of the form

$$\mathcal{E}(u,K) = \int_{\Omega\setminus K} f(x,u,\nabla u)\,dx + \int_K \vartheta(x,u^+,u^-,\nu)\,d\mathcal{H}^{n-1}, \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ is a fixed bounded open subset of $\mathbf{R}^n$, K is a (sufficiently smooth) unknown closed hypersurface with normal ν, $\mathcal{H}^{n-1}$ indicating the $(n-1)$-dimensional Hausdorff measure, and the unknown function u belongs to a class of (sufficiently smooth) functions defined in $\Omega \setminus K$ with traces $u^{\pm}$ on both sides of K. As K can be interpreted as the set of discontinuity points of u problems of this kind are usually labelled as "free-discontinuity problems", according to a terminology introduced by De Giorgi. Examples of energies of the form (1) arise from problems in Computer Vision, Fracture Mechanics, the theory of Liquid Crystals, Minimal Surfaces (see for example [39], [11], the Introduction of [23]). Among them, we recall the functional introduced by Mumford and Shah in [57] to study the problem of image reconstruction in computer vision,

$$E(u,K) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + c_1\mathcal{H}^1(K) + c_2\int_{\Omega\setminus K} |u-g|^2\,dx\,. \tag{2}$$

In this case $\Omega \subset \mathbf{R}^2$ parameterizes the input picture taken from a camera, g is interpreted as the grey level of the input picture and c_1 and c_2 are contrast parameters. If (u,K) minimizes E, then u provides a piecewise-smooth approximation of g, while K is expected to detect the relevant contour of the objects in the picture.

Another important application comes from the study of fracture mechanics in hyperelastic and brittle media. In this case $\Omega \subset \mathbf{R}^3$ is the reference configuration of a body, K is the crack surface, and u represents the deformation in the unfractured part of the body. In the framework of Griffith's theory of brittle fracture, the energy necessary to produce the fracture is proportional to the crack surface and, in the general non-isotropic case, may depend on the orientation of the crack. If the deformation outside the fracture can be modeled by an elastic energy density independent of the crack, then we are led to study energies of the form

$$\int_\Omega W(\nabla u)\,dx + \lambda\mathcal{H}^2(K) \tag{3}$$

if the body is isotropic, or more in general of the form

$$\int_\Omega W(\nabla u)\,dx + \int_K \phi(\nu)\,d\mathcal{H}^2,$$

W being an elastic bulk energy density.

The treatment of minimum problems involving functionals as in (1) following the direct methods of the calculus of variations present many difficulties, as no topology on closed sets is available that provides compactness for sequence of pairs (u_j, K_j) under the condition $\sup_j \mathcal{E}(u_j, K_j) < +\infty$. It is therefore necessary to formulate a weak version of this kind of problems. This purpose has led De Giorgi and Ambrosio to introduce the space of *special functions of bounded variation* (see [39], [14]). A function $u \in L^1(\Omega)$ belongs to $SBV(\Omega)$ if and only if its distributional derivative Du is a bounded measure that can be split into a bulk and a surface term. This definition can be further specified: if $u \in SBV(\Omega)$ and S_u stands for the complement of the set of the Lebesgue points for u then a measure-theoretical normal ν_u to S_u can be defined $\mathcal{H}^{n-1}$-a.e. on S_u, together with the traces $u^\pm$ on both sides of S_u; moreover, the approximate gradient ∇u exists a.e. on Ω, and we have

$$Du = \nabla u\,\mathcal{L}^n + (u^+ - u^-)\nu_u\,\mathcal{H}^{n-1} \llcorner S_u\,.$$

In $SBV(\Omega)$ we can thus consider a weak version of the energies (1), formally substituting K by S_u, so that $\mathcal{E}$ becomes

$$\int_\Omega f(x, u, \nabla u)\,dx + \int_{S_u} \vartheta(x, u^+, u^-, \nu_u)\,d\mathcal{H}^{n-1}\,. \tag{4}$$

An important and extensively studied model case is given by the Mumford-Shah functional

$$E(u) = \alpha \int_\Omega |\nabla u|^2\,dx + \beta\,\mathcal{H}^{n-1}(S_u)\,, \tag{5}$$

which corresponds to the "leading term" in (2). Under some suitable growth and convexity assumptions on f and ϑ in (4), lower semicontinuous and compactness results by Ambrosio assure the existence of solutions to problems involving this kind of energies (see [7]–[10]). Various regularity results show that for a wide class of problems the weak solution u in $SBV(\Omega)$ provides a solution to the original free-discontinuity problem, taking $K = \overline{S(u)}$ (see [40], [33], [13], [19], [42]).

If the existence theory developed in SBV-spaces allows to solve free-discontinuity problems from a theoretical point of view, new difficulties arise when we look for explicit solutions to these problems or try to provide numerical algorithms to approximate such solutions. Indeed, the detection of the unknown discontinuity hypersurface by applying the usual numerical methods has revealed to be a non-trivial task. Moreover, in the study of evolution problems related to free discontinuity energies, one clashes with the impossibility of flowing such energies, since they are not differentiable in any reasonable norm. To bypass these difficulties, an object of recent research has been to provide variational approximations of functionals as in (4), and in particular of the Mumford-Shah functional, by differentiable energies defined on spaces of smooth functions, easier to be handled numerically. The natural notion of convergence for these types of problems has turned out to be that of De Giorgi's Γ-*convergence* (see [41], [37], [28]). Indeed, this notion guarantees that, under suitable equicoerciveness assumptions, minima and minimizers of the approximating functionals converge to the corresponding minima and minimizers of the limit functional.

Purpose of this thesis is to present some recent results on this subject in an organic way and to link them with the existing literature. The proof of most of these results relies on a general scheme, illustrated in Chapter 2, which allows to reduce the n-dimensional problems to a 1-dimensional analysis, by slicing and density techniques. This procedure will be followed in the n-dimensional proofs of the approximation results of Chapter 3 and 4. To treat more general vector problems, it is necessary to develop different arguments, an example of which will be described in Chapter 5.

The first approximation we mention is that suggested by Modica and Mortola [55] to approximate the perimeter functional via elliptic functionals . The approximating functionals have the form

$$F_\varepsilon(u) = \int_\Omega \left(\frac{W(u)}{\varepsilon} + \varepsilon|\nabla u|^2\right) dx,$$

for $u \in H^1(\Omega)$, where $W : \mathbf{R} \to [0,+\infty)$ is a continuous function such that $W(z) = 0$ if and only if $z \in \{0,1\}$. Under this assumption, F_ε Γ-converges as $\varepsilon \to 0+$ with respect to the $L^1(\Omega)$-topology to the functional

$$F(u) = \begin{cases} c\mathcal{H}^{n-1}(S_u) = c\,\mathrm{Per}_\Omega(\{u=1\}) & \text{if } u \in SBV(\Omega),\ u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where c is a suitable constant depending on W. This model has been extensively studied, mostly in connection with phase transition problems. This result has been generalized in many ways; in particular we mention a recent paper by Fonseca and Mantegazza, who show that, if in $F_\varepsilon(u)$ the penalization term $\varepsilon|\nabla u|^2$ is replaced by $\varepsilon^3|\nabla^2 u|^2$ for $u \in H^2(\Omega)$, the resulting limit functional is of the same form as $F(u)$ above (see [47]). High-order singular perturbation are used also in the approximation of more general free-discontinuity energies (see below).

Following the idea developed by Modica and Mortola, Ambrosio and Tortorelli provided in [15] and [16] a first approximation via Γ-convergence of the Mumford-Shah functional, by introducing an additional function variable which in the limit approaches $1-\chi_{S_u}$. In [16] they proved that the family of functionals

$$G_\varepsilon(u,v) = \int_\Omega v^2|\nabla u|^2\,dx + \frac{1}{2}\int_\Omega \left(\varepsilon|\nabla v|^2 + \frac{1}{\varepsilon}(1-v)^2\right) dx\,, \tag{6}$$

defined for $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$, $0 \le v \le 1$, Γ-converges as $\varepsilon \to 0+$ with respect to the $(L^1(\Omega))^2$-topology to the functional

$$G(u,v) = \begin{cases} E(u) & \text{if } v = 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

which is equivalent to E, as far as minimum problems are concerned. This construction has been recently extended by Focardi to the vectorial case (see [46]), obtaining in the limit functionals of the more general form

$$\int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u} \varphi(\nu_u)\,d\mathcal{H}^{n-1}\,,$$

where f is quasiconvex in the gradient variable and satisfies superlinear growth condition and φ is a norm. Neverthless, the adaptation of the Ambrosio and Tortorelli approximation to obtain as limits more complex surface energies, depending also on the traces u^+, u^-, does not seem to follow easily from their approach. In Chapter 4, we study a variant of the Ambrosio Tortorelli construction by considering functionals of the form

$$G_\varepsilon(u,v) = \int_\Omega v^2|\nabla u|\,dx + \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v)^2\Big)\,dx. \tag{7}$$

Even though the form of the functionals in (7) is quite similar to that in (6), the domain of the limiting functional will be different. Indeed, as we have $G_\varepsilon(u,1) \le \int_\Omega |\nabla u|$, it is clear that the limit of these functionals will be finite if $u \in BV(\Omega)$. In fact we prove (Theorem 4.1 and Example 4.6) that G_ε converge to the functional

$$G(u,v) = \begin{cases} |Du|(\Omega \setminus S_u) + \displaystyle\int_{S_u} \frac{|u^+ - u^-|}{1+|u^+ - u^-|}\,d\mathcal{H}^{n-1} & \text{if } v = 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where $|Du|(A)$ denotes the total variation on A of Du. This approach can be pushed further to construct a variational approximation for a wide class of nonconvex functionals defined on spaces of generalized functions of bounded variation. In particular, we extend this procedure to limiting functionals of the more general form

$$G(u) = \int_\Omega f(|\nabla u|)\,dx + \lambda|D^s u|(\Omega \setminus S_u) + \int_{S_u} \vartheta(|u^+ - u^-|)\,d\mathcal{H}^{n-1},$$

with f of linear growth, where $|D^s u|$ denotes the singular part of $|Du|$ with respect to the Lebesgue measure. These functionals provide a weak formulation for problems in fracture mechanics involving crack initiation energies of Barenblatt type, i.e. depending on the size of the crack opening (see [11], [17], [54]), and are used to explain softening and fracture phenomena (see [27], where they are also derived from an atomic model).

An alternative approach to approximation, suggested by heuristic arguments, is the use of functionals of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx, \tag{8}$$

with f satisfying the conditions

$$f'(0)=\alpha \text{ and } \lim_{t\to+\infty} f(t)=\beta, \tag{9}$$

with $0<\alpha,\beta<+\infty$. Unfortunately, the use of local integral functionals of the form (8) as approximate energies is forbidden in the framework of Γ-convergence, as easy convexity arguments show. Nevertheless, non-convex integrands of this type can be exploited, provided we slightly modify the functionals in (8). In Chapter 3, we remove the convexity constraint in ∇u by a singular perturbation approach, by considering energies of the form

$$F_\varepsilon(u)=\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx+\varepsilon^\gamma\int_\Omega |Hu|^2\,dx, \tag{10}$$

where Hu denotes the Hessian matrix of u. Sections 3.1.1 and 3.1.2 deal with the 1-dimensional case. In Section 3.1.1 we show that if f satisfies conditions (9) then the Γ-limit of F_ε as $\varepsilon\to 0^+$ can be expressed on $SBV(\Omega)$ as

$$F(u)=\alpha\int_\Omega |u'|^2\,dx+c\sum_{S_u}\sqrt{|u^+-u^-|}$$

when $\gamma=3$ (the other cases giving trivial limits), with c explicitly computable from β. In Section 3.1.2, we recover in the limit the Mumford-Shah functional, by replacing f by more complex f_ε in (10). This approach has been generalized by Bouchitté, Dubs and Seppecher (see [22]). In Section 3.2 we carry on this study to the higher-dimensional case, when the Γ-limit takes the form

$$F(u)=\alpha\int_\Omega |\nabla u|^2\,dx+c\int_{S_u}\sqrt{|u^+-u^-|}\,d\mathcal{H}^{n-1}.$$

The extension of the 1-dimensional case is not obvious, and requires some non-trivial improvements of previous techniques. While a lower bound for the Γ-limit can be obtained by reducing to the 1-dimensional case through standard slicing procedures, the proof of the upper bound requires some extra care to show that the presence of second derivatives of u does not introduce a dependence on the curvature of S_u in the surface part of the limit energy. To this end, we use a density theorem of functions with polyhedral jump set S_u in $SBV(\Omega)$ by Cortesani and Toader ([36]). For functions of this class the possible effect of Hu is restricted to a neighbourhood of a set of Hausdorff dimension $n-2$, which can be then more easily taken care of.

A different approach can be followed by considering non-local functionals. In [26] Braides and Dal Maso provided an approximation of the Mumford-Shah functional by functional of the form

$$E_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\Big(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|^2\, dy\Big)\, dx\,,$$

defined for $u \in H^1(\Omega)$, where f is a suitable non-decreasing continuous function satisfying the limit conditions (9). These functionals are non-local in the sense that their energy density at a point $x \in \Omega$ depends on the behaviour of u in the whole set $B_\varepsilon(x) \cap \Omega$. This method is also suitable for the approximation of more general surface energies (see [29] and [35]). Even though we do not include functionals of this type in this presentation, we point out that some techniques for [26] are very close in the spirit to those used in Chapter 5.

Another type of non-local approximation of the Mumford-Shah functional is that, proposed by de Giorgi and studied by Gobbino in [50], by "finite difference" functionals of the form

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{1}{\varepsilon} f(\varepsilon|D^\xi_\varepsilon u(x)|^2)\rho(\xi)\, dx\, d\xi,$$

where f satisfies conditions (9), ρ is a simmetric convolution kernel and $D^\xi_\varepsilon u(x)$ denotes the difference quotient $\frac{1}{\varepsilon}\,(u(x+\varepsilon\xi) - u(x))$. The introduction of the convolution kernel ρ has been proposed by De Giorgi to overcome the anisotropy that obviously results if we take difference quotients only in the coordinate directions (see e.g. [34], [30]). As in the previous case, this procedure can be adapted to approximate more general functionals (see [51]).

In Chapter 5, we push further this approach to provide an approximation of functionals as in (3), when $W(\nabla u)$ is a linear elasticity density of type $\mu|\mathcal{E}u|^2 + \frac{\lambda}{2}|\operatorname{div} u|^2$, where $\mathcal{E}u = \frac{1}{2}(\nabla u + \nabla^t u)$ denotes the symmetric part of the gradient of u, for $u \in \mathcal{C}^1(\Omega\backslash K; \mathbf{R}^n)$, and μ, λ represent in the applications the Lamé constant of the material under consideration. In this case W is degenerate as a quadratic form with respect to ∇u, and the framework of SBV functions is not suitable. A weak formulation of free-discontinuity problems related to this kind of bulk energy density is obtained by replacing $SBV(\Omega)$ by the space of *special functions of bounded deformation* (see [12], [18], [60]). A function $u \in L^1(\Omega; \mathbf{R}^n)$ belongs to $SBD(\Omega)$ if its symmetrized distributional derivative Eu is a bounded measure whose singular part with respect to the Lebesgue measure is concentrated on an $(n-1)$-dimensional set J_u. As in the SBV case, for $u \in SBD(\Omega)$ we can define ν_u, u^+, u^- $\mathcal{H}^{n-1}$-a.e. on J_u; moreover the approximate gradient ∇u exists a.e. on Ω and we have

$$Eu = \frac{1}{2}(\nabla u + \nabla^t u)\mathcal{L}^n + (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where $\odot$ denotes the symmetric tensor product. For such functions, $\mathcal{E}u$ and $\operatorname{div} u$ are defined in an approximate sense.

Then, the weak formulation of functionals as in (3) in the n-dimensional case for bulk energy density as described above is provided by functionals of the type

$$\mu \int_\Omega |\mathcal{E}u|^2\,dx + \frac{\lambda}{2}\int_\Omega |\operatorname{div} u(x)|^2\,dx + \gamma\mathcal{H}^{n-1}(J_u) \tag{11}$$

for $u \in SBD(\Omega)$.

In order to approximate (11) one may try to "symmetrize" the effect of the difference quotient by considering the family of functionals

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{1}{\varepsilon} f\left(\varepsilon |\langle D_\varepsilon^\xi u(x), \xi\rangle|^2\right) \rho(\xi)\,dx\,d\xi.$$

In this case, roughly speaking, $\langle D_\varepsilon^\xi u(x), \xi\rangle$ approaches $\langle \mathcal{E}u(x)\xi, \xi\rangle$ as $\varepsilon \to 0$ in the region where u is smooth. Analogous computations as those in [50] show that we obtain as Γ-limit energies of type (11). The main drawback of this approximation is that the two coefficients μ, λ of the limit functional are related by a fixed ratio, depending only on n.

In Chapter 5, dealing with the 2-dimensional case, we introduce in the model a suitable discretization of the divergence, call it $\operatorname{div}_\varepsilon^\xi u$, and consider functionals of the form

$$F_\varepsilon(u) := \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{1}{\varepsilon} f\left(\varepsilon\left(|\langle D_\varepsilon^\xi u(x), \xi\rangle|^2 + \theta|\operatorname{div}_\varepsilon^\xi u(x)|^2\right)\right) \rho(\xi)\,dx\,d\xi,$$

with θ a strictly positive parameter (for more precise definitions see Section 5.1). We prove that with suitable choices of f, ρ and θ we can approximate all the functionals of type (11).

The main technical issue of Chapter 5 is that in the proof we cannot reduce to the 1-dimensional case by an integral-geometric approach as in [50], due to the presence of the divergence term. Instead we directly use a discretization argument, that leads us to study the limiting behaviour of families of discrete functionals of the form

$$\mathcal{F}_\varepsilon^\xi(u) = \sum_{\alpha\in\varepsilon\mathbf{Z}^2} \varepsilon f\left(\varepsilon\left(|\langle D_\varepsilon^\xi u(\alpha), \xi\rangle|^2 + \theta|\operatorname{div}_\varepsilon^\xi u(\alpha)|^2\right)\right).$$

We then recover the limiting behaviour of F_ε by an integration argument which leads to (11).

The content of Chapters 3, 4 and 5 is published in the papers [3], [6], [4], [5] and is the result of a research activity carried on by the Author during his graduate studies at the International School for Advanced Studies in Trieste, in collaboration with A. Braides, M.S. Gelli, M. Focardi and J.Shah.

1

PRELIMINARIES

This chapter is devoted to the main results about Γ-convergence, BV and SBV functions, BD and SBD functions, which provides the necessary background to deal with the approximation problems treated in the following chapters.

1.1 Γ-convergence and relaxation

In this section we introduce the notions of Γ-convergence and relaxation and state their main properties. In what follows $X = (X, d)$ is a metric space. For a comprehensive introduction to Γ-convergence we refer to Dal Maso [37] (see also Part II of [28]).

Definition 1.1 *We say that a sequence $F_j : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if for all $u \in X$ we have*

(i) (lower limit inequality) *for every sequence (u_j) converging to u*

$$F(u) \le \liminf_j F_j(u_j);$$

(ii) (existence of a recovery sequence) *for any $\eta > 0$ there exists a sequence (u_j) converging to u such that*

$$F(u) + \eta \ge \limsup_j F_j(u_j).$$

The function F is called the Γ-limit of (F_j) (with respect to d), and we write $F = \Gamma\text{-}\lim_j F_j$. If (F_ε) is a family of functionals indexed by $\varepsilon > 0$, then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_j F_{\varepsilon_j}$ for all (ε_j) converging to 0.

The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 1.2 *Let $F = \Gamma\text{-}\lim_j F_j$, and let a compact set $K \subset X$ exist such that $\inf_X F_j = \inf_K F_j$ for all j. Then*

$$\exists \min_X F = \lim_j \inf_X F_j.$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ then its limit is a minimum point for F.

The definition of Γ-convergence can be given pointwise on X. It is convenient to introduce also the notion of Γ-lower and upper limit, as follows.

Definition 1.3 *Let* $F_\varepsilon : X \to [-\infty, +\infty]$ *and* $u \in X$. *We define*

$$F'(u) := \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \; u_\varepsilon \to u\};$$

$$F''(u) := \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \; u_\varepsilon \to u\}.$$

If $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u)$ *then the common value is called the* Γ-limit of (F_ε) at u, *and is denoted by* $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$. *Note that this definition is in accord with the previous one, and that* F_ε *Γ-converges to* F *if and only if* $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ *at all points* $u \in X$.

Remark 1.4 The following properties of Γ-convergence are easily derived from its definition:

(a) if $F = \Gamma\text{-}\lim_j F_j$ and G is a continuous function then

$$F + G = \Gamma\text{-}\lim_j (F_j + G);$$

(b) the Γ-lower and upper limits define lower semicontinuous functions.

(c) if (X, d) is a topological vector space and $F = \Gamma\text{-}\lim_j F_j$, with F_j convex for all j, then F is convex.

We recall also the notion of *relaxed functional.*

Definition 1.5 *Let* $F : X \to [-\infty, +\infty]$. *Then the* relaxed functional $\overline{F}$ *of* F, *or* relaxation *of* F, *is the greatest lower semicontinuous functional less than or equal to* F, *i.e.,*

$$\overline{F}(u) = \sup\{G(u) : \; G \textit{ is lower semicontinuous and } G \le F\}.$$

Remark 1.6 (i) If $F_j = F$ for all $j \in \mathbb{N}$, then we get that

$$\Gamma\text{-}\lim_j F_j = \overline{F}. \tag{1.1}$$

This in particular implies that $\overline{F}$ can be described as follows

$$\overline{F}(u) = \inf\{\liminf_j F_j(u_j) : \; u_j \to u\};$$

(ii) the Γ-lower and upper limits enjoy the following property.

$$\Gamma\text{-}\liminf_j F_j(u) = \Gamma\text{-}\liminf_j \overline{F}_j(u),$$

$$\Gamma\text{-}\limsup_j F_j(u) = \Gamma\text{-}\limsup_j \overline{F}_j(u).$$

Finally, from (1.1) and Theorem 1.2 one immediately deduced the following well-known Weiestrass Theorem.

Theorem 1.7 *Let $K \subset X$ be a compact set such that* $\inf_X F = \inf_K F$ *. Then*

$$\exists \min_X \overline{F} = \inf_X F.$$

Moreover, the minimum points for $\overline{F}$ are exactly all the limits of converging sequences (u_j) such that $\lim_j F(u_j) = \inf_X F$.

1.2 Measure theory. Basic notation

Let Ω be an open subset of $\mathbf{R}^n$. We denote by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ the families of open and Borel subsets of Ω, respectively. The set of all measures $\mu : \mathcal{B}(\Omega) \to \mathbf{R}^N$ is denoted by $\mathcal{M}(\Omega; \mathbf{R}^N)$. If $N = 1$ we simply write $\mathcal{M}(\Omega)$ instead of $\mathcal{M}(\Omega; \mathbf{R})$ and we denote by $\mathcal{M}_+(\Omega)$ the set of all measures taking values in $[0, +\infty)$. If $B \in \mathcal{B}(\Omega)$, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. For any $\mu \in \mathcal{M}(\Omega; \mathbf{R}^N)$ we will indicate by $|\mu|$ the total variation of μ, that is the measure in $\mathcal{M}_+(\Omega)$ defined by

$$|\mu|(B) = \sup\Big\{\sum_{i \in \mathbf{N}} |\mu(B_i)| : \ B_i \text{ disjoint, } B = \bigcup_i B_i\Big\}.$$

for any $B \in \mathcal{B}(\Omega)$. $\mathcal{M}(\Omega; \mathbf{R}^N)$ is a Banach space when equipped with the norm $\|\mu\| := |\mu|(\Omega)$ and it is the dual of $C_0(\Omega; \mathbf{R}^N)$, closure with respect to the uniform convergence of the space $C_c(\Omega; \mathbf{R}^N)$ of continuous functions with compact support in Ω. By virtue of the duality above, a notion of weak* convergence on $\mathcal{M}(\Omega; \mathbf{R}^N)$ can be introduced: we say that a sequence $(\mu_j) \subset \mathcal{M}(\Omega; \mathbf{R}^N)$ converges weakly to μ (and we write $\mu_j \rightharpoonup \mu$) if for any $\phi \in C_0(\Omega; \mathbf{R}^N)$

$$\lim_{j \to +\infty} \int_\Omega \phi \, d\mu_j = \int_\Omega \phi \, d\mu.$$

Note that by the lower semicontinuity of the dual norm with respect to weak* convergence we have that $\mu \mapsto |\mu|(\Omega)$ is weakly lower semicontinuous; i.e., $|\mu|(\Omega) \leq \liminf_j |\mu_j|(\Omega)$ if $\mu_j \rightharpoonup \mu$.

Finally we include the following proposition on the supremum of a family of measures which will be useful in the sequel and that can be easily deduced from the regularity properties of positive measures.

Proposition 1.8 *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be an open-set function super-additive on open sets with disjoint compact closures (i.e., $\mu(A \cup B) \geq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}(\Omega)$ with $\overline{A} \cap \overline{B} = \emptyset$, $\overline{A} \cup \overline{B} \subset\subset \Omega$), let $\lambda \in \mathcal{M}_+(\Omega)$, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi \, d\lambda$ for all $A \in \mathcal{A}(\Omega)$.*

1.3 Discontinuity points, jump points and approximate differentiability

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbf{R}^n$; $B_\rho(x)$ is the open ball with center x and radius $\rho > 0$. The boundary of the unit ball $B_1(0)$ is denoted by S^{n-1}. The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbf{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively. If $B \subseteq \mathbf{R}^n$ is a Borel set, we will also use the notation $|B|$ for $\mathcal{L}^n(B)$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified. We use standard notation for Lebesgue and Sobolev spaces.

Let $u \in L^1_{loc}(\Omega; \mathbf{R}^m)$. The complement of the Lebesgue set of u will be called the *discontinuity set* of u and denoted by S_u, i.e., $x \notin S_u$ if and only if

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z| \, dy = 0$$

for some $z \in \mathbf{R}$. If z exists then it is unique and we denote it by $\tilde{u}(x)$. By the Lebesgue Differentiation Theorem, the set S_u is Lebesgue-negligible and $\tilde{u}$ is a Borel function equal to u a.e.

Moreover, we say that $x \in \Omega$ is a *jump point* of u, and we denote by J_u the set of all such points for u, if there exist $a, b \in \mathbf{R}$ and $\nu \in S^{n-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0} \rho^{-n} \int_{B^+_\rho(x,\nu)} |u(y) - a| \, dy = 0, \quad \lim_{\rho \to 0} \rho^{-n} \int_{B^-_\rho(x,\nu)} |u(y) - b| \, dy = 0, \tag{1.2}$$

where $B^\pm_\rho(x,\nu) := \{y \in B_\rho(x) : \pm \langle y - x, \nu(x) \rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (1.2) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. Notice that J_u is a Borel subset of S_u.

We say that u is *approximately differentiable* at a Lebesgue point x if there exists a $m \times n$ matrix L such that

$$\lim_{\rho \to 0} \rho^{-n-1} \int_{B_\rho(x)} |u(y) - \tilde{u}(x) - L(y - x)| \, dx = 0. \tag{1.3}$$

If u is approximately differentiable at x, then L, uniquely determined by (1.3), will be called the *approximate gradient* of u at x and denoted by $\nabla u(x)$.

1.4 BV functions

For the general theory of functions of bounded variation we refer to [14] [44],[49]. In this section and in the following one we just recall some definitions and results we will use in the sequel.

From now on we will suppose that Ω is a bounded open set of $\mathbf{R}^n$.

Definition 1.9 *Let $u \in L^1(\Omega)$. We say that u is a* function of bounded variation *on Ω if its distributional derivative is a measure; i.e., there exists $\mu \in \mathcal{M}(\Omega; \mathbf{R}^n)$ such that*

$$\int_\Omega uD\phi\, dx = -\int_\Omega \phi\, d\mu$$

for all $\phi \in C_c^1(\Omega)$. The measure μ will be denoted by Du. The space of all functions of bounded variation on Ω will be denoted by $BV(\Omega)$.

If $u \in BV(\Omega)$, then u is approximately differentiable a.e. in Ω and S_u turns out to be *countably $\mathcal{H}^{n-1}$-rectifiable*, or, briefly, *rectifiable*, i.e.,

$$S_u = N \cup \bigcup_{i \in \mathbf{N}} K_i, \tag{1.4}$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 $(n-1)$ dimensional manifold Γ_i. Moreover $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and the vector ν_u is normal to S_u in the sense that, if S_u is represented as in (1.4), then $\nu_u(x)$ is normal to Γ_i for $\mathcal{H}^{n-1}$-a.e. $x \in K_i$.

If we define

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega),$$

then $BV(\Omega)$ turns out to be a Banach space. The next theorem shows that its embedding in $L^1(\Omega)$ is compact.

Theorem 1.10 *If $(u_j) \subset BV(\Omega)$ and $\sup_j \left(\|u_j\|_{L^1(\Omega)} + |Du_j|(\Omega)\right) < +\infty$ then there exists a subsequence converging in $L^1(\Omega)$ to some $u \in BV(\Omega)$.*

We say that a set $E \subset \mathbf{R}^n$ is a *set of finite perimeter in Ω* if $\chi_E \in BV(\Omega)$. The quantity $|D\chi_E|(\Omega)$ is called the *perimeter of E in Ω*. We will set $\partial^* E = S(\chi_E)$, the *reduced boundary* of u in Ω, and $\nu_E = \nu_{\chi_E}$.

We recall that if $u \in BV(\Omega)$, then for a.e. $t \in \mathbf{R}$ the set $\{u > t\}$ is of finite perimeter in Ω, and the *co-area formula* holds:

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\, dt. \tag{1.5}$$

We recall also the *Fleming and Rishel co-area formula*: let u be a Lipschitz function; then for every $v \in BV(\Omega)$ we have that

$$\int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u>t\}} \tilde{v}\, d\mathcal{H}^{n-1}. \tag{1.6}$$

Now we describe the structure of the distributional derivative of BV functions. By using the Radon-Nykodym Theorem, we may write

$$Du = D^a u + D^s u,$$

where $D^a u$ is the absolutely continuous part with respect to $\mathcal{L}^n$ and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides a.e. with the approximate gradient ∇u of u, i.e.

$$D^a u = \nabla u \mathcal{L}^n.$$

We may further decompose the singular part $D^s u$ into two mutually singular measures as

$$D^s u = D^i u + D^c u,$$

where we have set

$$D^j u := D^s u \llcorner S_u \; ; \; D^c u := D^s u \llcorner (\Omega \setminus S_u);$$

$D^j u$ and $D^c u$ are called respectively the *jump part* and the *Cantor part* of Du. Moreover we can characterize $D^j u$ as

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u \tag{1.7}$$

and the following property for $D^c u$ holds

$$|D^c u|(B) = 0 \text{ for any } B \in \mathcal{B}(\Omega) \text{ with } \mathcal{H}^{n-1}(B) < +\infty.$$

Let $g : \Omega \to S^{n-1}$ such that $Du = g|Du|$. Since all parts of derivative of u are mutually singular, we have

$$D^a u = g|D^a u|, \quad D^j u = g|D^j u|, \quad D^c u = g|D^c u|.$$

In particular, by (1.7), we have, up to a change of sign,

$$g(x) = \nu_u(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_u.$$

For this reason, with a slight abuse of notation, we will set $g = \nu_u$.

Let us see how BV functions can be characterized through their one dimensional sections. We first introduce some notation. Let $\xi \in S^{n-1}$ and let $\Pi_\xi := \{y \in \mathbf{R}^n : \langle y, \xi \rangle = 0\}$ be the linear hyperplane orthogonal to ξ. If $y \in \Pi_\xi$ and $E \subset \mathbf{R}^n$ we set

$$E_{\xi y} := \{t \in \mathbf{R} : y + t\xi \in E\}. \tag{1.8}$$

Moreover, if $u : E \to \mathbf{R}$ we define the function $u_{\xi,y} : E_{\xi y} \to \mathbf{R}$ by

$$u_{\xi,y}(t) = u(y + t\xi). \tag{1.9}$$

Theorem 1.11 (a) *Let $u \in BV(\Omega)$. Then, for all $\xi \in S^{n-1}$ the function $u_{\xi,y}$ belongs to $BV(\Omega_{\xi,y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$ and*

$$\int_{\Pi_\xi} |Du_{\xi,y}|(\Omega_{\xi y})\, d\mathcal{H}^{n-1}(y) = |\langle Du, \xi \rangle|(\Omega) < +\infty.$$

Moreover

$$\langle D^k u(B), \xi \rangle = \int_{\Pi_\xi} D^k u_{\xi,y}(B_{\xi y})\, d\mathcal{H}^{n-1}(y), \tag{1.10}$$

for any $B \in \mathcal{B}(\Omega)$ *and* $k = a, j$ *or* c; *for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi_\xi$

$$\langle \nabla u(y + t\xi), \xi \rangle = u'_{\xi,y}(t) \text{ for a.e } t \in \Omega_{\xi,y},$$

$$S_{u_{\xi,y}} = (S_u)_{\xi y} = \{t \in \mathbf{R} : y + t\xi \in S_u\}$$

$$u^{\pm}_{\xi,y}(t) = u^{\pm}(y + t\xi) \ \forall t \in S_{u_{\xi,y}}$$

for an appropriate choice of $\nu_{u_{\xi,y}}$; *for all Borel functions* g

$$\int_{\Pi_\xi} \sum_{t \in S_{u_{\xi,y}}} g(t)\, d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x) |\langle \nu_u, \xi \rangle| d\mathcal{H}^{n-1}.$$

(b) *Conversely, if* $u \in L^1(\Omega)$ *and for all* $\xi \in \{e_1, \ldots, e_n\}$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \Pi_\xi$ $u_{\xi,y} \in BV(\Omega_{\xi,y})$ *and*

$$\int_{\Pi_\xi} |Du_{\xi,y}|(\Omega_{\xi,y})\, d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in BV(\Omega)$.

Remark 1.12 From (1.10) one can deduce also that

$$|\langle D^k u, \xi \rangle|(B) = \int_{\Pi_\xi} |D^k u_{\xi,y}|(B_{\xi y})\, d\mathcal{H}^{n-1}(y).$$

for any $B \in \mathcal{B}(\Omega)$ and $k = a, j$ or c.

We will consider also the larger space of generalized functions of bounded variation defined as follows.

Definition 1.13 *A function* $u \in L^1(\Omega)$ *is a* generalized function of bounded variation *if for each* $T > 0$ *the* truncated function $u_T = (-T) \vee (T \wedge u)$ *belongs to* $BV(\Omega)$. *The space of these functions will be denoted by* $GBV(\Omega)$.

The generalized functions of bounded variation inherit most of the main features of BV functions. Namely, if $u \in GBV(\Omega)$, then u is approximately differentiable a.e. in Ω and S_u turns out to be countably $\mathcal{H}^{n-1}$-rectifiable. Note that $\nabla u_T = \nabla u$ a.e. on $\{u = u_T\}$ and $\nabla u_T = 0$ a.e. on $\{u \neq u_T\} = \{|u| > T\}$. Moreover, $S(u) = \bigcup S(u_T)$ and $u^{\pm}$ coincide with the limit of the corresponding quantities for u_T as $T \to \infty$.

Finally we define the cantor part of the derivative of a function $u \in GBV(\Omega)$ as the measure $|D^c u| : \mathcal{B}(\Omega) \to [0, +\infty]$ defined by

$$|D^c u|(B) = \sup_{T>0} |D^c u_T|(B) = \lim_{T \to +\infty} |D^c u_T|(B).$$

If $u \in BV(\Omega)$, then $|D^c u|$ coincides with the usual notion of total variation of $D^c u$.

1.5 SBV functions

Definition 1.14 *We say that $u \in BV(\Omega)$ is a* special function of bounded variation *if its distributional derivative has no Cantor part, i.e., $D^c u = 0$. The space of special functions of bounded variation on Ω is denoted by $SBV(\Omega)$.*

It turns out that for any $u \in SBV(\Omega)$ we have

$$Du = D^a u + D^j u = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u,$$

and, in particular, the total variation of u is given by

$$|Du| = |\nabla u|\mathcal{L}^n + |u^+ - u^-|\mathcal{H}^{n-1} \llcorner S_u$$

The main property of SBV functions is the following compactness theorem due to Ambrosio (see [7],[9]).

Theorem 1.15 *Let $\phi : [0,+\infty) \to [0,+\infty]$, $\theta : (0,+\infty) \to (0,+\infty]$ be lower semicontinuous increasing functions and assume that*

$$\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty, \quad \lim_{t \to 0} \frac{\theta(t)}{t} = +\infty. \tag{1.11}$$

Let $(u_k) \subset SBV(\Omega)$ such that

$$\sup_k \|u_k\|_{BV(\Omega)} < +\infty \tag{1.12}$$

and

$$\sup_k \left\{ \int_\Omega \phi(|\nabla u_k|)\, dx + \int_{S_{u_k}} \theta(|u_k^+ - u_k^-|)\, d\mathcal{H}^{n-1} \right\} < +\infty, \tag{1.13}$$

Then we may extract a subsequence (not relabelled) (u_k) which converges in $L^1(\Omega)$ to some $u \in SBV(\Omega)$. Moreover ∇u_k weakly converges to ∇u in $L^1(\Omega; \mathbf{R}^n)$ and

$$\liminf_{k \to +\infty} \int_\Omega \phi(|\nabla u_k|)\, dx \geq \int_\Omega \phi(|\nabla u|)\, dx$$

if ϕ is convex;

$$\liminf_{k \to +\infty} \int_{S_{u_k}} \theta(|u_k^+ - u_k^-|)\, d\mathcal{H}^{n-1} \geq \int_{S_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{n-1}$$

if θ is concave.

Remark 1.16 In the theorem above condition (1.12) can be replaced by the condition: $\sup_k \|u_k\|_\infty < +\infty$. Indeed, one can easily check that if this last assumption is satisfied and (1.11) and (1.13) hold, then $\|u_k\|_{BV(\Omega)}$ is uniformly bounded.

For any $p \geq 1$ we will consider also the auxiliary space

$$SBV^p(\Omega) := \{u \in SBV(\Omega) : \nabla u \in L^p(\Omega; \mathbf{R}^n),\ \mathcal{H}^{n-1}(S_u) < +\infty\},$$

which replace in the framework of special functions of bounded variation the Sobolev space $W^{1,p}(\Omega)$. The subspace of $u \in SBV^\infty(\Omega)$ such that $\nabla u = 0$ a.e. in Ω is denoted by $SBV_0(\Omega)$.

In analogy with the strong density results of smooth functions in $W^{1,p}(\Omega)$, functions in $SBV^p(\Omega)$ can be approximated in a "strong sense" by functions which have a "regular" jump set and are smooth outside. This can be formally expressed as follows.

Definition 1.17 *We call $\mathcal{W}(\Omega)$ the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:*

(i) $\mathcal{H}^{n-1}\left(\overline{S_w} \setminus S_w\right) = 0$;

(ii) $\overline{S_w}$ *is the intersection of Ω with the union of a finite number of $(n-1)$-dimensional simplexes;*

(iii) $w \in W^{k,\infty}\left(\Omega \setminus \overline{S_w}\right)$ *for every $k \in \mathbf{N}$.*

The following density result of $\mathcal{W}(\Omega)$ in $SBV^p(\Omega)$ is due to Cortesani and Toader [36] (see also [25]).

Theorem 1.18 *Assume that $\partial\Omega$ is Lipschitz and let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence (w_h) in $\mathcal{W}(\Omega)$ such that*

$$w_h \to u \quad \text{strongly in } L^1(\Omega), \tag{1.14}$$

$$\nabla w_h \to \nabla u \quad \text{strongly in } L^p(\Omega), \tag{1.15}$$

$$\limsup_{h\to+\infty} \|w_h\|_\infty \leq \|u\|_\infty \tag{1.16}$$

and

$$\limsup_{h\to+\infty} \int_{S_{w_h}} \phi(w_h^+, w_h^-, \nu_{w_h})\, d\mathcal{H}^{n-1} \leq \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1} \tag{1.17}$$

for every upper semicontinuous function $\phi : \mathbf{R} \times \mathbf{R} \times S^{n-1} \to [0, +\infty)$ such that

$$\phi(a, b, \nu) = \phi(b, a, -\nu),$$

for every $a, b \in \mathbf{R}$ and $\nu \in S^{n-1}$.

Remark 1.19 Under the additional assumption that $1 < p \leq 2$ the structure of the jump set of the functions w_h given by Theorem 1.18 can be further improved by using a capacitary argument. In particular, it is possible to obtain that $\overline{S_{w_h}}$ is the intersection of Ω with the union of a finite number of pairwise disjoint simplexes.

We recall also a relaxation result in $BV(\Omega)$ of isotropic functionals with domain in $SBV^2(\Omega)$ and that is a particular case of a theorem by Bouchitté, Braides and Buttazzo [20].

Theorem 1.20 *Let $g : \mathbf{R} \to [0, +\infty]$ be a lower semicontinuous function with*

$$g(0) = 0, \qquad \lim_{t \to 0^+} \frac{g(t)}{t} = 1,$$

and such that the map $t \to g(|t|)$ is subadditive and locally bounded. Let $F : BV(\Omega) \to [0, +\infty]$ be defined by

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u| \, dx + \int_{S_u} g(|u^+ - u^-|) \, d\mathcal{H}^{n-1} & \text{if} \quad u \in SBV^2(\Omega) \cap L^\infty(\Omega) \\ +\infty & \text{otherwise in } BV(\Omega). \end{cases}$$

Then the relaxation of F with respect to the $L^1(\Omega)$-topology is given on $BV(\Omega)$ by the functional

$$\overline{F}(u) = \int_\Omega |\nabla u| \, dx + \int_{S_u} g(|u^+ - u^-|) \, d\mathcal{H}^{n-1} + |D^c u|(\Omega).$$

As in the BV setting, we generalize the definition of SBV functions as follows.

Definition 1.21 *A function $u \in L^1(\Omega)$ is a* generalized special function of bounded variation *if for each $T > 0$ the truncated function $u_T = (-T) \vee (T \wedge u)$ belongs to $SBV(\Omega)$, i.e., if $u \in GBV(\Omega)$ and $|D^c u|(\Omega) = 0$. The space of these functions will be denoted by $GSBV(\Omega)$.*

The following closure and lower semicontinuous theorem in $GSBV(\Omega)$ can be deduced from Theorem 1.15, by applying a truncation argument and taking into account Remark 1.16.

Theorem 1.22 *Let $\phi : [0, +\infty) \to [0, +\infty]$, $\theta : (0, +\infty) \to (0, +\infty]$ be lower semicontinuous increasing functions verifying* (1.11). *Let $(u_k) \subset GSBV(\Omega)$ be such that*

$$\sup_k \left\{ \int_\Omega (\phi(|\nabla u_k|) + g(|u_k|)) \, dx + \int_{S_{u_k}} \theta(|u_k^+ - u_k^-|) \, d\mathcal{H}^{n-1} \right\} < +\infty.$$

If $u_k \to u$ in $L^1(\Omega)$, then $u \in GSBV(\Omega)$ and ∇u_k weakly converges to ∇u in $L^1(\Omega; \mathbf{R}^n)$. Moreover

$$\liminf_{k\to+\infty}\int_\Omega \phi(|\nabla u_k|)\,dx \geq \int_\Omega \phi(|\nabla u|)\,dx$$

if ϕ is convex;

$$\liminf_{k\to+\infty}\int_{S_{u_k}} \theta(|u_k^+ - u_k^-|)\,d\mathcal{H}^{n-1} \geq \int_{S_u} \theta(|u^+ - u^-|)\,d\mathcal{H}^{n-1}$$

if θ is concave.

The following theorem describes the properties of 1-dimensional sections of $GSBV(\Omega)$-functions and can be easily deduced from Theorem 1.11.

Theorem 1.23 (a) *Let $u \in GSBV(\Omega)$. Then, for all $\xi \in S^{n-1}$ the function $u_{\xi,y}$ belongs to $GSBV(\Omega_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$. Moreover for such y we have*

$$u'_{\xi,y}(t) = \langle \nabla u(y+t\xi), \xi\rangle \quad \textit{for a.a. } t \in \Omega_{\xi y},$$

$$S_{u_{\xi,y}} = \{t \in \mathbf{R} : y + t\xi \in S_u\},$$

$$u^{\pm}_{\xi,y}(t) = u^{\pm}(y+t\xi)$$

for an appropriate choice of $\nu_{u_{\xi,y}}$ and for all Borel functions g

$$\int_{\Pi_\xi} \sum_{t\in S_{u_{\xi,y}}} g(t)\,d\mathcal{H}^{n-1}(y) = \int_{S_u} g(x)|\langle \nu_u, \xi\rangle|\,d\mathcal{H}^{n-1}.$$

(b) *Conversely, if $u \in L^1(\Omega)$ and for all $\xi \in \{e_1, ..., e_n\}$ and for a.a. $y \in \Pi_\xi$ $u_{\xi y} \in SBV(\Omega_{\xi y})$ and*

$$\int_{\Pi_\xi} |Du_{\xi y}|\,d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in SBV(\Omega)$.

We underline that Theorems 1.11 and 1.23 play a key role when we want to reduce some variational problems in n-dimensional domains to one-dimensional problems. This fact will be highlighted in the following chapters.

Finally we introduce the space $GSBV^p(\Omega)$ by setting

$$GSBV^p(\Omega) := \{u \in GSBV(\Omega) :\ \nabla u \in L^p(\Omega;\mathbf{R}^n),\ \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

1.6 BD and SBD functions

We recall some definitions and basic results on functions with bounded deformation. For a general treatment of this subject we refer to [12] (see also [18],[43],[60]).

Definition 1.24 *Let $u \in L^1(\Omega;\mathbf{R}^n)$. We say that u is a* function of bounded deformation *on Ω if its symmetric distributional derivative $Eu := \frac{1}{2}(Du + D^t u)$, is a $n \times n$ matrix-valued measure on Ω. The space of all functions of bounded deformation will be denoted by $BD(\Omega)$.*

For every $\xi \in \mathbf{R}^n$, let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi v = \langle Dv, \xi\rangle$. For every function $u : \Omega \to \mathbf{R}^n$ let us define the function $u^\xi : \Omega \to \mathbf{R}$ by $u^\xi(x) := \langle u(x), \xi\rangle$.

Theorem 1.25 *If $u \in BD(\Omega)$, then $D_\xi u^\xi \in \mathcal{M}(\Omega)$ and*

$$D_\xi u^\xi = \langle Eu\xi, \xi\rangle.$$

Conversely, let $\xi_1, \ldots, \xi_n$ be a basis of $\mathbf{R}^n$ and let $u \in L^1(\Omega;\mathbf{R}^n)$; then $u \in BD(\Omega)$ if $D_\xi u^\xi \in \mathcal{M}(\Omega)$ for every ξ of the form $\xi_i + \xi_j$, $i, j = 1, \ldots, n$.

If $u \in BD(\Omega)$, then u is approximately differentiable a.e. in Ω and J_u turns out to be countably $\mathcal{H}^{n-1}$-rectifiable, but it is not known whether $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ or not.

As in the BV case, we can decompose Eu as

$$Eu = E^a u + E^j u + E^c u,$$

where $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^n$ with density

$$\mathcal{E}u := \frac{1}{2}\left(\nabla u + \nabla^t u\right);$$

$E^j u$ and $E^c u$ are respectively the *jump part* and the *cantor part* of Eu and are defined by

$$E^j u := E^s u \llcorner J_u, \quad E^c u := E^s u \llcorner (\Omega \setminus J_u),$$

where $E^s u$ is the singular part of Eu with respect to $\mathcal{L}^n$. Moreover, we can characterize $E^j u$ as

$$E^j u = \left(u^+ - u^-\right) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where $\odot$ is the symmetric tensor product defined by $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$, $a \otimes b$ denoting the matrix whose entries are $a_i b_j$.

Definition 1.26 *We say that $u \in BD(\Omega)$ is a* special function of bounded deformation *in Ω, and we write $u \in SBD(\Omega)$, if $E^c u = 0$.*

In analogy with the BV functions, we may characterize the spaces $BD(\Omega)$ and $SBD(\Omega)$ by means of suitable one-dimensional sections, for which we introduce an appropriate notation. Given $y, \xi \in \mathbf{R}^n$, $\xi \neq 0$, $E \subset \mathbf{R}^n$, let $\Pi_\xi, E_{\xi y}$ be defined as before Theorem 1.11. If $u : E \to \mathbf{R}^n$, we define the function $u^{\xi,y} : E_{\xi y} \to \mathbf{R}$ by

$$u^{\xi,y}(t) := \langle u\,(y + t\xi)\,, \xi\rangle.$$

Moreover, if $u \in BD(\Omega)$ we set

$$J_u^\xi := \left\{x \in J_u : \langle u^+(x) - u^-(x), \xi\rangle \neq 0\right\}.$$

Note that, since $\mathcal{H}^{n-1}\left(\{\xi \in S^{n-1} : \langle u^+(x) - u^-(x), \xi\rangle = 0\}\right) = 0$ for every $x \in J_u$, by Fubini's Theorem we have

$$\mathcal{H}^{n-1}\left(J_u \setminus J_u^\xi\right) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in S^{n-1}.$$

Theorem 1.27 (a) *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{R}^n$, $\xi \neq 0$. Then $u^{\xi,y} \in BV(\Omega_{\xi y})$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$,*

$$\int_{\Pi_\xi} |Du^{\xi,y}|(B_{\xi y})\, d\mathcal{H}^{n-1}(y) = |D_\xi u^\xi|(B) < +\infty \tag{1.18}$$

for every $B \in \mathcal{B}(\Omega)$, and

$$\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u\,(y + t\xi)\,\xi, \xi\rangle$$

$$J_{u^{\xi,y}} = \left(J_u^\xi\right)_{\xi y}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$ and for a.e. $t \in \Omega_{\xi y}$.

(b) *Conversely, let $u \in L^1(\Omega; \mathbf{R}^n)$ and let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathbf{R}^n$. If for every ξ of the form $\xi_i + \xi_j$,*

$$u^{\xi,y} \in BV(\Omega_{\xi y}) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi_\xi,$$
$$\int_{\Pi_\xi} \left|Du^{\xi,y}\right|(\Omega_{\xi y})\, d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in BD(\Omega)$.

Moreover, if $u \in BD(\Omega)$, then $u \in SBD(\Omega)$ if and only if $u^{\xi,y} \in SBV(\Omega_{\xi y})$ for every ξ of the form $\xi_i + \xi_j$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$.

The following compactness result in $SBD(\Omega)$ is due to Bellettini, Coscia and Dal Maso (see [18]) and its proof is based on slicing techniques and on the characterization of $SBD(\Omega)$ provided by Theorem 1.27.

Theorem 1.28 *Let $(u_k) \subset SBD(\Omega)$ be such that*

$$\sup_k \left(\int_\Omega |\mathcal{E}u_k|^2\, dx + \mathcal{H}^{n-1}(J_{u_k}) + \|u_k\|_\infty\right) < +\infty.$$

Then there exist a subsequence (not relabelled) (u_k) converging in $L^1_{loc}(\Omega;\mathbf{R}^n)$ to a function $u \in SBD(\Omega)$. Moreover $\mathcal{E}u_k$ weakly converges to $\mathcal{E}u$ in $L^2(\Omega;\mathbf{R}^{n^2})$ and

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_{u_k}) \geq \mathcal{H}^{n-1}(J_u).$$

We state now a lower semicontinuity result in SBD that can be proved by following the same ideas and strategy of the proof of Theorem 1.28.

Theorem 1.29 *Let $u_j, u \in SBD(\Omega)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbf{R}^n)$ and*

$$\sup_j \int_\Omega |\langle \mathcal{E}u_j(x)\xi,\xi\rangle|^2\,dx + \int_{J^\xi_{u_j}} |\langle \nu_{u_j},\xi\rangle|\,d\mathcal{H}^{n-1} < +\infty \tag{1.19}$$

for $\xi \in \mathbf{R}^n \setminus \{0\}$. Then $\langle \mathcal{E}u_j(x)\xi,\xi\rangle \to \langle \mathcal{E}u(x)\xi,\xi\rangle$ weakly in $L^2(\Omega)$ and

$$\int_{J^\xi_u} |\langle \nu_u,\xi\rangle|\,d\mathcal{H}^{n-1} \leq \liminf_j \int_{J^\xi_{u_j}} |\langle \nu_{u_j},\xi\rangle|\,d\mathcal{H}^{n-1}.$$

In particular, if (1.19) holds for every $\xi \in \{\xi_1,\ldots,\xi_n\}$ orthogonal basis in $\mathbf{R}^n$, then $\operatorname{div} u_j \to \operatorname{div} u$ weakly in $L^2(\Omega)$.

Finally, we introduce the following subspace of $SBD(\Omega)$.

$$SBD^2(\Omega) := \left\{ u \in SBD(\Omega) : \int_\Omega |\mathcal{E}u(x)|^2\,dx + \mathcal{H}^{n-1}(J_u) < +\infty \right\}.$$

This space is the domain of the free-discontinuity energies we will consider in Chapter 5.

2

A GENERAL APPROACH TO APPROXIMATION

2.1 The method

In this section we describe a general method which has been successfully used to prove the approximations via Γ-convergence of functionals in GBV and GSBV we will present. This method allows to reduce the n-dimensional problems to a 1-dimensional analysis.

As a model case, we study the Γ-convergence of a family of functionals $F_\varepsilon : L^1(\Omega) \to [0, +\infty]$. Then, according to the definition given in Section 1.1, we deal with lower and upper inequality for F' and F'', respectively.

2.1.1 *A lower inequality by slicing*

The procedure we follow to give an estimate from below of the Γ-liminf of F_ε is based on slicing techniques and can be summarized as follows.

1. We "localize" the functionals F_ε highlighting their dependence on the set of integration, defining functionals $F_\varepsilon(\cdot, A)$ for all open subsets $A \subset \Omega$;

2. For all $\xi \in S^{n-1}$ and for all $y \in \Pi_\xi$, we find functionals $F_\varepsilon^{\xi,y}(v, I)$, defined for $I \subset \mathbf{R}$ and $v \in L^1(I)$, such that, using the notation introduced before Theorem 1.11 and setting

$$F_\varepsilon^\xi(u, A) = \int_{\Pi_\xi} F_\varepsilon^{\xi,y}(u_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y),$$

we have $F_\varepsilon(u, A) \geq F_\varepsilon^\xi(u, A)$.

3. We compute the $\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^{\xi,y}(v, I) = F^{\xi,y}(v, I)$ and define

$$F^\xi(u, A) = \int_{\Pi_\xi} F^{\xi,y}(u_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y)\,.$$

4. From Fatou's Lemma we have, if $u_\varepsilon \to u$,

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) &\geq \liminf_{\varepsilon\to 0^+} F_\varepsilon^\xi(u_\varepsilon, A)\\
&= \liminf_{\varepsilon\to 0^+} \int_{\Pi_\xi} F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi} \liminf_{\varepsilon\to 0^+} F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y)
\end{aligned}$$

$$\geq \int_{\Pi_\xi} F^{\xi,y}(u_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y)$$
$$= F^\xi(u, A).$$

Hence, we deduce that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A) \geq F^\xi(u, A),$$

for all $\xi \in S^{n-1}$;

5. From estimates from below on $F^{\xi,y}$, and Theorem 1.23(b) or Theorem 1.11(b), we deduce that if $u \in L^\infty(\Omega)$ then $\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A)$ is finite only if $u \in SBV(A)$ or $u \in BV(A)$, according to the cases. By an approximation argument we see that if $u \in L^1(\Omega)$ only this holds if $u \in GSBV(A)$ ($u \in GBV(A)$);

6. We prove the existence of Borel functions f^ξ, g^ξ, h^ξ such that setting

$$f^{\xi,y}(t, s, z) = f^\xi(y + t\xi, s, z),$$
$$g^{\xi,y}(t, v, w) = g^\xi(y + t\xi, v, w),$$
$$h^{\xi,y}(t, s, z) = h^\xi(y + t\xi, s, z)$$

we have

$$F^{\xi,y}(v, I) \geq \int_I f^{\xi,y}(t, v, v')\, dt + \sum_{S_v} g^{\xi,y}(t, v^+, v^-) + \int_I h^{\xi,y}\left(t, \tilde{v}, \frac{dD^c v}{d|D^c v|}\right) d|D^c v|,$$

and by Theorem 1.23 or Theorem 1.11 we deduce that

$$F^\xi(u, A) \geq \int_A f^\xi(x, u, \langle\nabla u, \xi\rangle)\, dx + \int_{S_u\cap A} g^\xi(x, u^+, u^-)|\langle\nu_u, \xi\rangle| d\mathcal{H}^{n-1} + \int_A h^\xi\left(x, \tilde{u}, \frac{d\langle D^c u, \xi\rangle}{d|\langle D^c u, \xi\rangle|}\right) d|\langle D^c u, \xi\rangle|$$

if $u \in GSBV(\Omega)$ ($u \in GBV(\Omega)$);

7. We check that if $u \in GSBV(\Omega)$ ($u \in GBV(\Omega)$) then the set function

$$\mu(A) = \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A)$$

is superadditive on open sets with disjoint compact closures. Take u such that $\mu(\Omega) < +\infty$. Using Proposition 1.8 we conclude that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A) \geq \int_A \tilde{f}(x, u, \nabla u)\, dx + \int_{S_u\cap A} \tilde{g}(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1}$$

$$+\int_A \tilde{h}\left(x, u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u|$$

if $u \in GSBV(\Omega)(u \in GBV(\Omega))$, where

$$\tilde{f}(x, s, z) = \sup_i f^{\xi_i}(x, s, \langle z, \xi_i \rangle),$$

$$\tilde{g}(x, v, w, \nu) = \sup_i g^{\xi_i}(x, v, w)|\langle \nu, \xi_i \rangle|,$$

$$\tilde{h}(x, u, \nu) = \sup_i h^{\xi_i}\left(x, u, \frac{\langle \nu, \xi_i \rangle}{|\langle \nu, \xi_i \rangle|}\right)|\langle \nu, \xi_i \rangle|,$$

and (ξ_i) is a fixed sequence in S^{n-1}. By varying (ξ_i) we can optimize the estimate. If the domain of the limit functional is $GSBV(\Omega)$, we have $h^\xi \equiv 0$.

2.1.2 *An upper inequality by density*

While it is usually difficult to prove directly a meaningful upper inequality for $F''(u)$ on the whole $L^1(\Omega)$, a recovery sequence can often be easily constructed if the target function u has some special structure. In order to give an upper estimate of F'' by some functional $F : L^1(\Omega) \to [0, +\infty]$ it is therefore useful to proceed as follows.

Step 1. Define a subset $\mathcal{D}$ of $L^1(\Omega)$, dense in $\{F < +\infty\}$ (the domain of F), such that for each $u \in L^1(\Omega)$ such that $F(u) < +\infty$ we can find a sequence $(u_j) \subset \mathcal{D}$ such that $u_j \to u$ in $L^1(\Omega)$, and $F(u) = \lim_j F(u_j)$;

Step 2. Prove that we have $F''(u) \leq F(u)$ for each $u \in \mathcal{D}$.

By the lower semicontinuity of F'' we then conclude that

$$F''(u) \leq \liminf_j F''(u_j) \leq \lim_j F(u_j) = F(u)$$

if $F(u) < +\infty$, so that $F'' \leq F$ on $L^1(\Omega)$.

Finally we underline that the construction of recovery sequences in the 1-dimensional case suggests the corresponding construction in higher dimension for any $u \in \mathcal{D}$, as we will see in the sequel.

2.2 Examples of approximation

In this section we will illustrate some examples of approximation of free discontinuity problems whose proof can be recovered, up to slight modifications, by following the procedure described in the previous section. We will not enter into the details of the proofs, but in each example we will show how this procedure can be applied.

2.2.1 *Approximation of the perimeter by elliptic functionals*

The first approximation we address is that of the perimeter. Since sets of finite perimeter in Ω may be identified with their characteristic functions as a subset of $SBV(\Omega)$, we can define the perimeter functional as follows:

$$P(u) = \begin{cases} |Du|(\Omega) = \mathcal{H}^{n-1}(S_u) & \text{if } u \in SBV(\Omega) \text{ and } u \in \{0,1\} \text{ a.e.} \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

The functional $P : L^1(\Omega) \to [0,+\infty]$ is lower semicontinuous with respect to the $L^1(\Omega)$-convergence.

The following result is due to Modica and Mortola (see [55] and [23]).

Theorem 2.1 *Let Ω be a bounded open set with Lipschitz boundary. Let $W : \mathbf{R} \to [0,+\infty)$ be a continuous function such that $W(z) = 0$ if and only if $z \in \{0,1\}$, and let $F_\varepsilon : L^1(\Omega) \to [0,+\infty]$ be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)\,dx + \varepsilon \int_\Omega |\nabla u|^2\,dx & \text{if } u \in W^{1,2}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then F_ε Γ-converges to $c_W P$ with respect to the $L^1(\Omega)$-convergence, where $c_W = 2\int_0^1 \sqrt{W(s)}\,ds$.

For the proof, following the steps outlined in Section 2.1.1, one can choose

$$F_\varepsilon^{\xi,y}(v,I) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I W(v)\,dt + \varepsilon \int_I |v'|^2\,dt & \text{if } u \in W^{1,2}(I) \\ +\infty & \text{otherwise} \end{cases}$$

(independent of y). We then have, by Fubini's Theorem,

$$F_\varepsilon^{\xi}(u,A) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_A W(u)\,dx + \varepsilon \int_I |\langle Du,\xi\rangle|^2\,dx & \text{if } \langle \xi, Du\rangle << \mathcal{L}_n \\ +\infty & \text{otherwise.} \end{cases}$$

Note that, since $|\langle \nu,\xi\rangle| \le |\nu|$ for all $\nu \in \mathbf{R}^n, \xi \in \mathcal{S}^{n-1}$, we have that $F_\varepsilon^\xi \le F_\varepsilon$.

Then one computes $F^{\xi,y}(v,I) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon^{\xi,y}(v,I)$ and obtains

$$F^{\xi,y}(v,I) = \begin{cases} c_W \#(S_v) & \text{if } v \in \{0,1\} \text{ a.e. on } I, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, if $u \in SBV(A)$ and $u \in \{0,1\}$ a.e., from Theorem 1.23 we have

$$F^\xi(u,A) = c_W \int_{A\cap S_u} |\langle \nu_u, \xi\rangle| d\mathcal{H}^{n-1}(y)\,.$$

For what concern the upper inequality, one can prove, under the hypothesis $\partial\Omega$ Lipschitz, that the set $\mathcal{D}$ of all characteristic functions of subsets of Ω which are restrictions of sets of class C^∞ in a neighbourhood of Ω is dense in the domain of P in the sense specified in Section 2.1.2. Then, if $u \in \mathcal{D}$, one can construct a recovery sequence $v_\varepsilon(x)$ letting it depend on dist (x, S_u).

More precisely, one proves in the 1-dimensional case that, if $u = \chi_{[0,+\infty)}$, a recovery sequence is given by

$$u_\varepsilon(t) = \begin{cases} v(\frac{t}{\varepsilon}) & \text{if } |t| \leq T\varepsilon \\ u(t) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $v \in W^{1,2}(-T,T)$ and $T > 0$ are such that

$$\int_{-T}^{T} \left(W(v) + |v'|^2\right) dt \leq 2c_w + \eta$$

and $v(-T) = 0$, $v(T) = 1$. The existence of v and T satisfying the condition above for any $\eta > 0$ is deduced, once proved that

$$2c_W = \min\left\{\int_{-\infty}^{+\infty} \left(W(u) + |u'|^2\right) dt,\ u \in W^{1,2}_{loc}(\mathbf{R}),\ u(-\infty) = 0,\ u(+\infty) = 1\right\}.$$

Then, in higher dimension, if $u = \chi_E \in \mathcal{D}$, a recovery sequence is given by

$$v_\varepsilon(x) = u_\varepsilon(d(x)),$$

where u_ε is defined by (2.1) and $d(x) = \text{dist}\,(x, \Omega \setminus E) - \text{dist}\,(x, E)$ is the *signed distance function* to ∂E.

2.2.2 *Approximation of the Mumford-Shah functional by elliptic functionals*

Pushing further the idea of the previous approximation and introducing an auxiliary variable to take care of the surface part, one obtains the following approximation of the Mumford-Shah functional (see [16] and [23]).

Theorem 2.2 *Let Ω be a bounded open set with Lipschitz boundary. Let $V : \mathbf{R} \to [0,+\infty)$ be a continuous function such that $V(z) = 0$ if and only if $z = 1$, let $\psi : [0,1] \to [0,+\infty)$ be a lower semicontinuous and increasing function, with $\psi(z) = 0$ if and only if $z = 0$ and $\psi(1) = 1$. Let $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ be defined by*

$$G_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \left(\psi(v)|\nabla u|^2 + \frac{1}{\varepsilon}V(v) + \varepsilon|\nabla v|^2\right) dx & \textit{if } u,v \in W^{1,2}(\Omega) \\ & \textit{and } 0 \leq v \leq 1 \\ +\infty & \textit{otherwise,} \end{cases} \tag{2.2}$$

and let $G : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ be defined by

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + 4c_V \mathcal{H}^{n-1}(S_u) & \textit{if } u \in GSBV(\Omega) \textit{ and } v = 1 \textit{ a.e.} \\ +\infty & \textit{otherwise,} \end{cases}$$

where $c_V = \int_0^1 \sqrt{V(s)}\, ds$. Then we have $\Gamma\text{-}\lim_{\varepsilon\to0^+} G_\varepsilon(u,v) = G(u,v)$.

In this case, following again the strategy described in Section 2.1.1, one simply chooses

$$G_\varepsilon^{\xi,y}(u,v,I)=\begin{cases}\displaystyle\int_I\Big(\psi(v)|u'|^2+\frac{1}{\varepsilon}V(v)+\varepsilon|v'|^2\Big)\,dt & \text{if } u,v\in W^{1,2}(I)\\ & \text{and } 0\le v\le 1\\ +\infty & \text{otherwise}\end{cases}$$

(independent of y). We then have, by Fubini's Theorem,

$$G_\varepsilon^{\xi}(u,v,A)=\begin{cases}\displaystyle\int_A\Big(\psi(v)|\langle\nabla u,\xi\rangle|^2+\frac{1}{\varepsilon}V(v)+\varepsilon|\langle\nabla u,\xi\rangle|^2\Big)\,dx\\ \qquad\qquad \text{if } \langle\xi,Du\rangle<<\mathcal{L}_n,\ \langle\xi,Dv\rangle<<\mathcal{L}_n,\ 0\le v\le 1\\ +\infty \qquad\qquad\qquad \text{otherwise.}\end{cases}$$

Then one computes $G^{\xi,y}(u,v,I)=\Gamma\text{-}\lim_{\varepsilon\to 0}G_\varepsilon^{\xi,y}(u,v,I)$ and obtain

$$G^{\xi,y}(u,v,I)=\begin{cases}\displaystyle\int_I|u'|^2\,dt+4c_V\#(S_v\cap I) & \text{if } u\in SBV(I)\\ & \text{and } v=1 \text{ a.e. on } I,\\ +\infty & \text{otherwise.}\end{cases}$$

Thus, if $u\in GSBV(A)$ and $v=1$ a.e., from Theorem 1.23(a) we have

$$G^{\xi}(u,v,A)=\int_A|\langle\nabla u,\xi\rangle|^2\,dx+4c_V\int_{A\cap S_u}|\langle\xi,\nu_u\rangle|d\mathcal{H}^{n-1}.$$

By applying a truncation argument it suffices to prove the upper inequality for $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$ and $v=1$ a.e. Then , by virtue of Theorem 1.18, a dense family in $SBV^2(\Omega)\cap L^\infty(\Omega)$ in the sense specified in Section 2.1.2 is given by the set $\mathcal{D}=\mathcal{W}(\Omega)$.

Let us show how to construct a recovery sequence in the 1-dimensional case. By simplicity, take $u\in SBV(\Omega)$, $u'\in L^2(\Omega)$ and $S_u=\{0\}$. Then, fixed $\xi_\varepsilon=o(\varepsilon)$, a recovery sequence is given by the pair $(u_\varepsilon,v_\varepsilon)$, with $u_\varepsilon\in W^{1,2}(\Omega)$, $u_\varepsilon(t)=u(t)$ if $|t|>\xi_\varepsilon$, and v_ε defined by

$$v_\varepsilon(t)=\begin{cases}0 & \text{if } |t|\le\xi_\varepsilon\\ v\Big(\dfrac{|t|-\xi_\varepsilon}{\varepsilon}\Big) & \text{if } \xi_\varepsilon<|t|<\xi_\varepsilon+\varepsilon T\\ 1 & \text{if } |t|\ge\xi_\varepsilon+\varepsilon T\,,\end{cases}$$

where $v\in W^{1,2}(0,T)$ and $T>0$ are such that

$$\int_0^T (V(v) + |v'|^2)\, dt \le 2c_V + \eta,$$

and $v(0) = 0$, $v(T) = 1$.

In higher dimension, for $u \in \mathcal{D}$, a recovery sequence can be constructed by proceeding as above, letting v_ε depend on dist (x, S_u), analogously to the previous example.

2.2.3 *Finite difference approximation of the Mumford-Shah functional*

In this example we illustrate the approximation of the Mumford-Shah functional by a family of non-local functionals depending on measurable functions through some difference quotient (see [50] and [23]).

Let $\rho : \mathbf{R}^n \to [0, +\infty)$ be a symmetric mollifier (i.e., $\rho(x) = \psi(|x|)$), and let $f : [0, +\infty) \to [0, +\infty)$ be a Borel function with $f(0) = 0$, such that for all $c > 0$ $\inf\{f(t) : t \ge c\} > 0$, $a, b > 0$ exist with

$$\lim_{t\to 0^+} \frac{f(t)}{t} = a, \qquad \lim_{t\to+\infty} f(t) = b,$$

and $f(t) \le \min\{at, b\}$.

We define the functionals $F_\varepsilon : L^1_{\rm loc}(\mathbf{R}^n) \to [0, +\infty)$ as

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f\Big(\frac{(u(x+\varepsilon\xi) - u(x))^2}{\varepsilon}\Big)\rho(\xi)\, dx\, d\xi\,.$$

Theorem 2.3 *The functionals F_ε Γ-converge as $\varepsilon \to 0^+$ with respect to the $L^1_{\rm loc}(\mathbf{R}^n)$-convergence to the Mumford-Shah functional F defined by*

$$F(u) = \begin{cases} A\displaystyle\int_{\mathbf{R}^n} |\nabla u|^2\, dx + B\mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV_{\rm loc}(\mathbf{R}^n) \\ +\infty & \text{otherwise,} \end{cases}$$

where A, B are defined by

$$A = a\omega_n \int_0^{+\infty} t^{n+1}\psi(t)\, dt, \qquad B = 2b\omega_{n-1}\int_0^{+\infty} t^n\psi(t)\, dt$$

(where $\rho(\xi) = \psi(|\xi|)$).

In this case one does not follow exactly the steps outlined in Section 2.1.1, but applies the same argument based on slicing techniques. Indeed one has

$$F_\varepsilon(u) = \int_{\mathbf{R}^n}\int_{\Pi_\xi} |\xi| F^1_\varepsilon(u_{\xi,y})\, d\mathcal{H}^{n-1}(y)\rho(\xi)\, d\xi\,, \tag{2.3}$$

where

$$F^1_\varepsilon(v) = \frac{1}{\varepsilon}\int_{\mathbf{R}} f\Big(\frac{(v(t+\varepsilon)-v(t))^2}{\varepsilon}\Big)\,dt,$$

(independent of ξ and y).

Then one proves that (F^1_ε) Γ-converges to the Mumford-Shah functional F^1 whose value on $SBV(\mathbf{R})$ is

$$F^1(v) = a\int_{\mathbf{R}} |v'|^2\,dt + b\#(S_v)\,.$$

By applying Fatou's Lemma and Theorem 1.23, one obtains that $\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u)$ is finite only if $u \in GSBV_{loc}(\mathbf{R}^n)$ and

$$\begin{aligned}\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u) &\geq \int_{\mathbf{R}^n}\int_{\Pi_\xi} |\xi| F^1(u_{\xi,y})\,d\mathcal{H}^{n-1}(y)\rho(\xi)\,d\xi \\ &= \int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n} a|\langle\nabla u,\xi\rangle|^2\,dx + b\int_{S_u}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^{n-1}\Big)\rho(\xi)\,d\xi\,.\end{aligned}$$

Easy computations show that the last term in the inequality above is equal to $F(u)$.

For what concern the upper inequality, we underline that in this case any density result is needed, since one proves that

$$F_\varepsilon(u) \leq F(u)$$

for any $\varepsilon > 0$ and $u \in L^1_{loc}(\mathbf{R}^n)$. To this end one can reduce once more to the 1-dimensional case. In fact, by (2.3) it suffices to verify that

$$F^1_\varepsilon(v) \leq F^1(v)$$

for any $v \in SBV(\mathbf{R})$.

3

APPROXIMATIONS BY HIGH-ORDER PERTURBATIONS

In this chapter we provide an approximation of functionals in GSBV, and the Mumford-Shah functional in particular in the 1-dimensional case, by local functionals with a high-order singular perturbation. It is clear that approximation with local integral functionals depending only on first derivatives is not possible. In fact, from standard convex analysis arguments, the relaxation with respect to the $L^1(\Omega)$-convergence of functionals of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(\nabla u)\, dx \qquad u \in W^{1,1}(\Omega)$$

is given by

$$\overline{F}_\varepsilon(u) = \int_\Omega f_\varepsilon^{**}(\nabla u)\, dx \qquad u \in W^{1,1}(\Omega).$$

where f_ε^{**} denotes the lower semicontinuous and convex envelope of f_ε. Hence, by Remark 1.6(ii) and Remark 1.4(c) the Γ-limit of functionals as above must be convex, and it can be easily checked that functionals on $GSBV(\Omega)$ cannot be convex.

The convexity in the gradient variable can be overcome considering higher-order derivatives. Since high derivatives do not appear in the limit, they may be introduced as a singular perturbation.

The next section is devoted to the 1-dimensional results. An extension to higher dimension will be treated in Section 3.2.

The results of this chapter are contained in [3] and [6].

3.1 The 1-dimensional case

3.1.1 *Surface energies generated by high-order singular perturbation*

Theorem 3.1 *Let $p > 1$, and let $f : [0, +\infty) \to [0, +\infty)$ be a lower semicontinuous increasing function, such that $\alpha, \beta \in \mathbf{R}$ exist with*

$$\alpha = \lim_{t \to 0^+} \frac{f(t)}{t}, \qquad \beta = \lim_{t \to +\infty} f(t).$$

Let I be a bounded open subset of $\mathbf{R}$, and let $F_\varepsilon : L^1(I) \to [0, +\infty]$ be defined by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_I f(\varepsilon |u'|^2)\, dt + \varepsilon^{2p-1} \int_I |u''|^p\, dt & \text{if } u \in W^{2,p}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the Γ*-limit* $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u) = F(u)$ *with respect to the* $L^1(I)$*-convergence, where*

$$F(u) = \begin{cases} \alpha \int_I |u'|^2\,dx + m(\beta)\sum_{S_u}\sqrt{|u^+ - u^-|} & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$m(s) = \min_{T>0} \min\Big\{2sT + \int_{-T}^{T} |\varphi''|^p\,dt : \ \varphi(\pm T) = \pm 1/2, \varphi'(\pm T) = 0\Big\}$$

for all $s > 0$.

Remark 3.2 The choice of the power ε^{2p-1} follows from the scaling argument leading to the definition of $m(b)$, which will be clear in the proofs (see also Corollary 3.11).

Remark 3.3 For all $z \in \mathbf{R}$ and $s > 0$ let

$$m(s,z) = \min_{\eta>0} \min\Big\{2s\eta + \int_{-\eta}^{\eta} |v''|^p\,dt : \ v \in W^{2,p}(-\eta,\eta), \\ v(\pm\eta) = \pm\frac{z}{2}, \ v'(\pm\eta) = 0\Big\}.$$

If we set

$$c_p = \min\Big\{\int_{-1}^{1} |\varphi''|^p\,dt : \varphi \in W^{2,p}(-1,1), \ \varphi(\pm 1) = \pm\frac{1}{2}, \ \varphi'(\pm 1) = 0\Big\},$$

then the substitution $v(t) = z\,\varphi(t/\eta)$ gives

$$\begin{aligned} m(s,z) &= \min_{\eta>0}\Big\{2\eta s + |z|^p \eta^{1-2p} c_p\Big\} \\ &= s^{(2p-1)/2p}\sqrt{|z|}\Big(2^{2p-1}c_p(2p-1)\Big)^{1/2p}\Big(1 + \frac{1}{c_p(2p-1)}\Big) \\ &= s^{(2p-1)/2p} m(1)\sqrt{|z|} = m(s)\sqrt{|z|}. \end{aligned}$$

If $p = 2$ then c_2 is easily computed, noticing that the solution φ is the third order polynomial satisfying the given boundary conditions. In this case, $m(s) = s^{\frac{3}{4}}\Big(2\sqrt{3/2} + \sqrt{2/3}\Big)$.

The proof of Theorem 3.1 will be obtained as a consequence of some simpler propositions which deal with lower and upper Γ-limits separately. Before stating and proving them, we define a "localized version" of our functionals, which highlights their behaviour as set functions, by setting

$$F_\varepsilon(u,I)=\begin{cases}\frac{1}{\varepsilon}\int_I f(\varepsilon|u'|^2)\,dt+\varepsilon^{2p-1}\int_I|u''|^p\,dt & \text{if } u\in W^{2,p}(I)\\ +\infty & \text{otherwise,}\end{cases} \tag{3.1}$$

and

$$F(u,I)=\begin{cases}\alpha\int_I|u'|^2\,dx+m(\beta)\sum_{I\cap S_u}\sqrt{|u^+-u^-|} & \text{if } u\in SBV(I)\\ +\infty & \text{otherwise,}\end{cases} \tag{3.2}$$

for all $u\in L^1_{\rm loc}(\mathbf{R})$ and $I\subseteq\mathbf{R}$ bounded open set.

Proposition 3.4 *Let $f(t)=\min\{at,b\}$ and let $u_\varepsilon\in W^{2,p}(I)$. Then there exists $v_\varepsilon\in SBV(I)$ such that*

$$a\int_I|v_\varepsilon'|^2\,dt=\frac{1}{\varepsilon}\int_I f(\varepsilon|u_\varepsilon'|^2)\,dt\,, \tag{3.3}$$

$$\int_{S_{v_\varepsilon}}\sqrt{|v_\varepsilon^+-v_\varepsilon^-|}\,d\#\le\frac{F_\varepsilon(u_\varepsilon)}{m(b)}\,, \tag{3.4}$$

and

$$\|u_\varepsilon-v_\varepsilon\|_{L^1(I)}\le\varepsilon\, c\,(F_\varepsilon(u_\varepsilon))^3.$$

Proof Set

$$D_\varepsilon=\{t\in I:\ \varepsilon|u_\varepsilon'|^2>b/a\}.$$

Since D_ε is open, we can write $D_\varepsilon=\bigcup_{k\in\mathbf{N}}I_\varepsilon^k$ as the union of disjoint open intervals $I_\varepsilon^k=(a_\varepsilon^k,b_\varepsilon^k)$. It is not restrictive to suppose that $[a_\varepsilon^k,b_\varepsilon^k]\subset I$ for all k. Note that

$$\frac{1}{\varepsilon}|D_\varepsilon|b\le C:=F_\varepsilon(u_\varepsilon).$$

Consider v_ε defined by

$$v_\varepsilon(t)=\begin{cases}u_\varepsilon(t) & \text{if } t\in I\setminus D_\varepsilon\\ u_\varepsilon(a_\varepsilon^k)+u_\varepsilon'(a_\varepsilon^k)(t-a_\varepsilon^k) & \text{if } t\in I_\varepsilon^k.\end{cases} \tag{3.5}$$

As $|v_\varepsilon'|^2=b/\varepsilon a$ on D_ε, we have

$$a\int_I|v_\varepsilon'|^2\,dt=a\int_{I\setminus D_\varepsilon}|u_\varepsilon'|^2\,dt+\frac{b}{\varepsilon}|D_\varepsilon|=\frac{1}{\varepsilon}\int_I f(\varepsilon|u_\varepsilon'|^2)\,dt\le C\,;$$

i.e., (3.3).

If $t\in I_\varepsilon^k$ we get, using Hölder's inequality,

$$|v_\varepsilon(t)-u_\varepsilon(t)|=\int_{a_\varepsilon^k}^t|v_\varepsilon'-u_\varepsilon'|\,ds\le\int_{a_\varepsilon^k}^t\int_{a_\varepsilon^k}^s|u_\varepsilon''(\tau)|\,d\tau\,ds$$

$$\leq \Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^p\Big)^{1/p} \int_{a_\varepsilon^k}^t (s-a_\varepsilon^k)^{(p-1)/p}\,ds$$
$$= \frac{p}{2p-1}\Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^p\Big)^{1/p} (t-a_\varepsilon^k)^{(2p-1)/p};$$

hence, integrating on I_ε^k,

$$\int_{I_\varepsilon^k} |v_\varepsilon - u_\varepsilon|\,dt \leq c\Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^p\Big)^{1/p} |I_\varepsilon^k|^{(3p-1)/p}.$$

We eventually obtain

$$\begin{aligned}\|u_\varepsilon - v_\varepsilon\|_{L^1(I)} = \int_{D_\varepsilon} |v_\varepsilon - u_\varepsilon|\,dt &\leq c\sum_{k\in\mathbf{N}}\Big(\int_{I_\varepsilon^k} |u_\varepsilon''|^p\Big)^{1/p} |I_\varepsilon^k|^{(3p-1)/p}\\ &\leq c\Big(\int_{D_\varepsilon} |u_\varepsilon''|^p\Big)^{1/p}\Big(\sum_{k\in\mathbf{N}} |I_\varepsilon^k|^{(3p-1)/(p-1)}\Big)^{p-1/p}\\ &\leq c\frac{C^{1/p}}{\varepsilon^{(2p-1)/p}}|D_\varepsilon|^{(3p-1)/p} \leq c\,C^3\,\varepsilon\end{aligned}$$

as required.

The function v_ε is discontinuous only at the points b_ε^k. Set

$$z_\varepsilon^k = |v_\varepsilon^+(b_\varepsilon^k) - v_\varepsilon^-(b_\varepsilon^k)|, \qquad w_\varepsilon = u_\varepsilon - v_\varepsilon.$$

Since $u_\varepsilon'' = w_\varepsilon''$ on I_ε^k, we have

$$\frac{1}{\varepsilon}\int_{I_\varepsilon^k} f(\varepsilon|u_\varepsilon'|^2)\,dt + \varepsilon^{2p-1}\int_{I_\varepsilon^k} |u_\varepsilon''|^p\,dt = \frac{b}{\varepsilon}|I_\varepsilon^k| + \varepsilon^{2p-1}\int_{I_\varepsilon^k} |w_\varepsilon''|^p\,dt$$
$$\geq \min\Big\{\frac{b}{\varepsilon}|I_\varepsilon^k| + \varepsilon^{2p-1}\int_{I_\varepsilon^k} |\varphi''|^p\,dt:\ \varphi\in W^{2,p}(I_\varepsilon^k),\ \varphi(a_\varepsilon^k)=0,\ \varphi(b_\varepsilon^k)=z_\varepsilon^k,$$
$$\varphi'(a_\varepsilon^k) = \varphi'(b_\varepsilon^k) = 0\Big\}$$
$$\geq \min_{\eta>0}\min\Big\{2\eta b + \int_{-\eta}^{\eta} |\psi''|^p\,dt:\ \psi\in W^{2,p}(-\eta,\eta),\ \psi(\pm\eta) = \pm\frac{z_\varepsilon^k}{2}, \psi'(\pm\eta)=0\Big\}$$
$$= m(b)\sqrt{z_\varepsilon^k};$$

the last equality being shown in Remark 3.3. Hence,

$$\sum_{k\in\mathbf{N}}\sqrt{|v_\varepsilon^+(b_\varepsilon^k) - v_\varepsilon^-(b_\varepsilon^k)|} \leq \frac{C}{m(b)}. \tag{3.6}$$

By (3.3) and (3.6) we get that $v_\varepsilon\in SBV(I)$, $S_{v_\varepsilon} = (b_\varepsilon^k)_k$, and (3.4) holds, so that the proof is complete. □

Proposition 3.5 *Let $f(t) = \min\{at, b\}$. Let (u_ε) be a bounded family in $L^1(I)$ satisfying $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$ and let v_ε be defined as in (3.5). Then for every sequence of positive numbers (ε_j) converging to 0 there exists a subsequence (not relabeled) and a function $u \in SBV(I)$ such that $v_{\varepsilon_j} \to u$ in $L^1(I)$,*

$$v'_{\varepsilon_j} \rightharpoonup u' \quad \textit{weakly in } L^2(I)\,, \tag{3.7}$$

$$\int_{I\cap S_u} \sqrt{|u^+ - u^-|}\, d\# \leq \liminf_j \int_{I\cap S_{v_{\varepsilon_j}}} \sqrt{|v^+_{\varepsilon_j} - v^-_{\varepsilon_j}|}\, d\#\,. \tag{3.8}$$

Proof By (3.3) and (3.4) we get that $\sup_j \|v_{\varepsilon_j}\|_{BV(I)} < +\infty$ and

$$\sup_j \Big(\int_I |v'_{\varepsilon_j}|^2\, dt + \sum_{I\cap S_{v_{\varepsilon_j}}} \sqrt{|v^+_{\varepsilon_j} - v^-_{\varepsilon_j}|} \Big) < +\infty\,.$$

Then the thesis follows from Theorem 1.15, applied with $\phi(t) = t^2$ and $\theta(t) = \sqrt{|t|}$. □

Proposition 3.6 *If $f(t) = \min\{at, b\}$ then for all $u \in L^1_{\rm loc}(\mathbf{R})$ we have*

$$F(u, I) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u, I)\,,$$

for all bounded open sets $I \subset \mathbf{R}$, where F_ε is defined in (3.1) and F in (3.2), with a, b in place of α, β.

Proof It is not restrictive to suppose that $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u, I) < +\infty$. Let (ε_j) be a sequence of positive numbers converging to 0 and let $u_{\varepsilon_j} \to u$ be a sequence in $L^1_{\rm loc}(\mathbf{R})$ such that the limit $\lim_j F_{\varepsilon_j}(u_{\varepsilon_j}, I)$ exists and equals the $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u, I)$.

Let v_{ε_j} be defined by (3.5). Note that $v_{\varepsilon_j} \to u$ in $L^1_{\rm loc}(\mathbf{R})$. With fixed $\delta \in (0, 1)$, by (3.3) and (3.4) (applied with δb in place of b) we obtain

$$\begin{aligned} F_{\varepsilon_j}(u_{\varepsilon_j}, I) &\geq (1-\delta)\frac{1}{\varepsilon_j}\int_I f(\varepsilon_j |u'_{\varepsilon_j}|^2)\, dt \\ &\quad + \delta\frac{1}{\varepsilon_j}\int_I f(\varepsilon_j |u'_{\varepsilon_j}|^2)\, dt + {\varepsilon_j}^{2p-1}\int_I |u''_{\varepsilon_j}|^p\, dt \\ &\geq (1-\delta) a \int_I |v'_{\varepsilon_j}|^2\, dt + m(\delta b) \sum_{I\cap S_{v_{\varepsilon_j}}} \sqrt{|v^+_{\varepsilon_j} - v^-_{\varepsilon_j}|}\,. \end{aligned}$$

By (3.7) and (3.8) we deduce that

$$\liminf_{\varepsilon_j\to 0^+} F_{\varepsilon_j}(u_{\varepsilon_j}, I) \geq (1-\delta) a \int_I |u'|^2\, dt + m(\delta b) \sum_{I\cap S_u} \sqrt{|u^+ - u^-|}\,.$$

Then, we can apply Proposition 1.8 with

$$\mu(I) = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,I),$$

$$\lambda = \mathcal{L}_1 + \sum_{t\in S_u} \sqrt{|u^+ - u^-|}\delta_t,$$

and, if $(\delta_i) = \mathbf{Q}\cap(0,1)$,

$$\psi_i(x) = \begin{cases} (1-\delta_i)a|u'(x)|^2 & \text{a.e. on } I\setminus S_u \\ m(\delta_i b) = \delta_i^{(2p-1)/2p} m(b) & \text{on } S_u, \end{cases}$$

obtaining the thesis. □

Proposition 3.7 *Under the hypotheses of Theorem* 3.1 *we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,I) \geq F(u,I)$$

for all $u \in L^1_{\text{loc}}(\mathbf{R})$, *and for all bounded open sets* $I \subset \mathbf{R}$.

Proof Let (a_i) (b_i) be sequences of positive numbers such that $\sup_i a_i = \alpha$, $\sup_i b_i = \beta$, and

$$f_i(t) := \max\{a_i t, b_i\} \leq f(t) \qquad \text{for all } t \geq 0\,.$$

From Proposition 3.6 we have that $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,I)$ is finite only if $F(u,I)$ is finite, and that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,I) \geq a_i \int_I |u'|^2\,dt + m(b_i) \sum_{I\cap S_u} \sqrt{|u^+ - u^-|}\,.$$

The thesis follows as in the proof of Proposition 3.6 taking now

$$\psi_i(x) = \begin{cases} a_i|u'(x)|^2 & \text{a.e. on } I\setminus S_u \\ m(b_i) & \text{on } S_u \end{cases}$$

in Proposition 1.8. □

In the sequel $u(t\pm)$ and $u'(t\pm)$ denote the right-hand side and left-hand side limits of u and u' at t, respectively.

Proposition 3.8 *Let* $u \in SBV(I)$ *satisfy* $\#(S_u) < +\infty$, $u \in W^{2,p}(I\setminus S_u)$, *and* $u'(t\pm) = 0$ *on* S_u. *Then there exists a family* (u_ε) *converging to* u *in* $L^1(I)$ *such that* $\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \leq F(u)$.

Proof Since the construction of u_ε will modify u only in a small neighbourhood of S_u, we can suppose that $I=(-1,1)$ and $S_u=\{0\}$. Moreover, by a translation argument we can suppose also that $u^+(0)+u^-(0)=0$. Let $z=u^+(0)-u^-(0)$, and let η and v be the minimizing pair in the definition of $m(\beta,z)$. If we set $v_\varepsilon(x)=v(x/\varepsilon)$ then we have $v_\varepsilon(\pm\varepsilon\eta)=\pm z/2$, $v'_\varepsilon(\pm\varepsilon\eta)=0$, and

$$F_\varepsilon(v_\varepsilon,(-\varepsilon\eta,\varepsilon\eta))\le 2\eta\beta+\varepsilon^{2p-1}\int_{-\eta}^{\eta}|v''_\varepsilon|^p\,dt=m(\beta,z).$$

We then define

$$u_\varepsilon(x)=\begin{cases} v_\varepsilon(x) & \text{if } x\in(-\varepsilon\eta,\varepsilon\eta)\\ u(x+\varepsilon\eta) & \text{if } x\le-\varepsilon\eta\\ u(x-\varepsilon\eta) & \text{if } x\ge\varepsilon\eta,\end{cases}$$

so that $u_\varepsilon\in W^{2,p}(I)$, $u_\varepsilon\to u$ in $L^1(I)$, and

$$F_\varepsilon(u_\varepsilon)\le\frac{1}{\varepsilon}\int_I f(\varepsilon|u'|^2)\,dt+\varepsilon^{2p-1}\int_I|u''|^p\,dt+m(\beta,z).$$

Note that $f(\varepsilon|u'|^2)/\varepsilon\le K|u'|^2$ for some constant K, and that $f(\varepsilon|u'|^2)/\varepsilon\to\alpha|u'|^2$ a.e. on I; hence, after applying Lebesgue's Dominated Convergence Theorem, we obtain

$$\limsup_{\varepsilon\to0+}F_\varepsilon(u_\varepsilon)\le\alpha\int_I|u'|^2\,dt+m(\beta,z)=F(u),$$

and the thesis. □

Proposition 3.9 *Let $u\in SBV(I)$ satisfy $\#(S_u)<+\infty$ and $u'\in L^2(I)$. Then there exists a sequence (u_j) in $SBV(I)$ such that $S_{u_j}\subseteq S_u$, $u_j\in W^{2,p}(I\setminus S_u)$, $u'_j(t\pm)=0$ on S_u, $u_j\to u$ in $L^\infty(I)$, $u'_j\to u'$ in $L^2(I)$ and $u_j(t\pm)\to u(t\pm)$ on S_u.*

Proof It is not restrictive to suppose $I=(a,b)$. Let $S_u=\{x_1,\dots,x_N\}$, with $x_i<x_{i+1}$, and set $x_0=a$, $x_{N+1}=b$. Let (v_j) be a sequence of functions in $C^\infty(I\setminus S_u)$ converging strongly to u in $H^1(x_i,x_{i+1})$ for all $i\in\{0,1,\dots,N\}$. For all $j\in\mathbf{N}$, and $i\in\{0,1,\dots,N\}$, let u^i_j be the solution to the minimum problem

$$\min\Big\{\int_{x_i}^{x_{i+1}}|v'|^2\,dt+j\int_{x_i}^{x_{i+1}}|v_j-v|^2\,dt:\ v\in H^1(x_i,x_{i+1})\Big\}.$$

Note that u^i_j is also a classical solution of the Euler equation $v''=j(v-v_j)$ with the Neumann conditions $v'(x_i)=v'(x_{i+1})=0$. The function u_j defined by $u_j=u^i_j$ on (x_i,x_{i+1}) satisfies the required conditions. Note that $u_j\to u$ in $W^{1,2}(I\setminus S_u)$, and then also in $L^\infty(I)$. In particular $u_j(t\pm)\to u(t\pm)$ on S_u. □

The following proposition concludes the proof of Theorem 3.1.

Proposition 3.10 *We have* $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon) \le F(u)$ *for all* $u \in SBV(I)$.

Proof We use the notation $F'' = \Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon$, and we suppose without loss of generality that $I = (a,b)$.

If $u \in SBV(I)$ with $\#(S_u) < +\infty$ and $u' \in L^2(I)$, let (u_j) be given by Proposition 3.9. By Proposition 3.8 we have that $F''(u_j) \le F(u_j)$ for all j. Moreover, by Proposition 3.9 $\lim_j F(u_j) = F(u)$. By the lower semicontinuity of F'' we then obtain

$$F''(u) \le \liminf_j F''(u_j) \le \liminf_j F(u_j) = F(u). \tag{3.9}$$

Let now $u \in SBV(I)$ satisfy $F(u) < +\infty$ and $\#(S_u) = +\infty$. If $S_u = \{x_1, x_2, \ldots\}$, set $z_i = u^+(x_i) - u^-(x_i)$ and

$$u_k = u - \sum_{j=k+1}^{\infty} z_i \chi_{(x_i,b)}.$$

We have $u'_k = u'$, $S_{u_k} = \{x_1, \ldots, x_k\}$, $u_k^+(x_i) - u_k^-(x_i) = z_i$ on S_{u_k}, and $\lim_k F(u_k) = F(u)$. By (3.9) we have $F''(u_k) \le F(u_k)$. Using the lower semicontinuity of F'' again, we obtain the required inequality. □

We show now that the Γ-limit in Theorem 3.1 is trivial if we replace the exponent $2p-1$ by a different one.

Corollary 3.11 *Let f satisfy the hypotheses of Theorem 3.1 and let $\gamma > 0$, $p > 1$. Let I be a bounded open subset of* $\mathbf{R}$, *and let $F_\varepsilon^\gamma : L^1(I) \to [0,+\infty]$ be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|^2)\,dt + \varepsilon^\gamma \int_I |u''|^p\,dt & \text{if } u \in W^{2,p}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) = F^\gamma(u)$ with respect to the $L^1(I)$-convergence, where $F^\gamma(u) = 0$ for all $u \in L^1(I)$ if $\gamma > 2p-1$, $F^\gamma = F$ as in Theorem 3.1 if $\gamma = 2p-1$, and

$$F^\gamma(u) = \begin{cases} \alpha \displaystyle\int_I |u'|^2\,dx & \text{if } u \in H^1(I) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.10}$$

if $\gamma < 2p-1$.

Proof The case $\gamma = 2p-1$ is dealt with in Theorem 3.1. Fixed $\gamma > 2p-1$, take $\varepsilon_0 > 0$; for all positive $\varepsilon < \varepsilon_0$ we have

$$F_\varepsilon^\gamma(u) \le \varepsilon_0^{\gamma-2p+1}\Big(\frac{1}{\varepsilon}\int_I \varepsilon_0^{2p-1-\gamma} f(\varepsilon|u'|^2)\,dt + \varepsilon^{2p-1}\int_I |u''|^p\,dt\Big).$$

We can apply Theorem 3.1 with $\varepsilon_0^{2p-1-\gamma} f$ in the place of f to obtain for all $u \in SBV(I)$

$$\begin{aligned}
&\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma(u)\\
&\le \varepsilon_0^{\gamma-2p+1}\Big(\varepsilon_0^{2p-1-\gamma}\alpha\int_I |u'|^2\,dt + \varepsilon_0^{(2p-1-\gamma)\frac{2p-1}{2p}} m(\beta)\int_{S_u}\sqrt{|u^+-u^-|}\,d\#\Big)\\
&= \alpha\int_I |u'|^2\,dt + \varepsilon_0^{(\gamma-2p+1)/2p} m(\beta)\int_{S_u}\sqrt{|u^+-u^-|}\,.
\end{aligned}$$

Letting $\varepsilon_0 \to 0+$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) \le \alpha\int_I |u'|^2\,dt$$

for all $u \in SBV(I)$; in particular for every piecewise constant function v we have $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma(v) = 0$. By a density argument and by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma$ we deduce that $F^\gamma \equiv 0$.

Let now $\gamma < 2p-1$. Arguing as above we deduce that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) = +\infty \text{ if } u \in L^1(I)\setminus SBV(I)$$

and for all $\varepsilon_0 > 0$

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) \ge \alpha\int_I |u'|^2\,dt + \varepsilon_0^{(\gamma-2p+1)/2p} m(\beta)\int_{S_u}\sqrt{|u^+-u^-|}$$

if $u \in SBV(I)$. Letting $\varepsilon_0 \to 0+$ we have then that

$$\begin{aligned}
&\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) = +\infty \quad \text{if } u \in L^1(I)\setminus H^1(I)\\
&\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) \ge \alpha\int_I |u'|^2\,dt \quad \text{if } u \in H^1(I).
\end{aligned}$$

On the other hand, if $u \in W^{2,p}(I)$ we have

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) \le \limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma(u) = \alpha\int_I |u'|^2\,dt.$$

By a density argument and by the lower semicontinuity of $\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\gamma$ we deduce (3.10). □

Finally an application to the convergence of minimum problems is derived.

Corollary 3.12 *Let $g \in L^2(I)$ and $\lambda > 0$ be fixed. Then for all $\varepsilon > 0$ there exists a minimum point u_ε for the problem*

$$\min\Big\{\frac{1}{\varepsilon}\int_I f(\varepsilon|v'|^2)\,dt+\varepsilon^{2p-1}\int_I |v''|^p\,dt+\lambda\int_I |v-g|^2\,dt:\ v\in W^{2,p}(I)\Big\},\quad(3.11)$$

and for every sequence (ε_j) *of positive numbers converging to* 0 *there exists a subsequence (not relabeled) such that* u_{ε_j} *converges in* $L^1(I)$ *to a function* $u\in SBV(I)$*, which minimizes*

$$\min\Big\{\alpha\int_I |v'|^2+m(\beta)\int_{S_v}\sqrt{|v^+-v^-|}\,d\#+\lambda\int_I |v-g|^2\,dt:\ v\in SBV(I)\Big\}.\quad(3.12)$$

Moreover, the minimum values (3.11) *converge to* (3.12).

Proof The existence of minimum points for 3.11 is assured by an application of the direct methods of the calculus of variations. In fact, the functional in (3.11) is coercive in $W^{2,p}(I)$ and lower semicontinuous (see e.g. [53]). Note that the value in (3.11) is less than or equal to $\lambda\int_I |g|^2\,dt$. Hence, we have $\sup_{\varepsilon>0}F_\varepsilon(u_\varepsilon)<+\infty$ and $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^2(I)}<+\infty$. Hence by Proposition 3.4 from every sequence (ε_j) of positive numbers converging to 0+ we can extract a further subsequence (not relabeled) such that u_{ε_j} converges in $L^1(I)$ to some $u\in SBV(I)$. The minimality of u and the convergence of minimum values follow from Theorem 1.2. □

3.1.2 *Approximation of the Mumford-Shah functional by high-order perturbations*

In this section we show that it is possible to approximate the Mumford-Shah functional using the singular perturbation method introduced in the previous section.

For all $\varepsilon>0$ let $a_\varepsilon\ge 1$ with

$$\lim_{\varepsilon\to 0+}a_\varepsilon=+\infty\qquad\text{and}\qquad\lim_{\varepsilon\to 0+}\varepsilon a_\varepsilon=0.$$

With fixed $K>0$ set

$$C_\varepsilon=\frac{K}{4\sqrt{\varepsilon\,a_\varepsilon}},$$

and

$$f_\varepsilon(z)=\begin{cases} z & \text{if } z\le 1\\ a_\varepsilon & \text{if } 1<z<(1+C_\varepsilon)^2\\ 0 & \text{if } z\ge(1+C_\varepsilon)^2\,.\end{cases}$$

Theorem 3.13 *Let* I *be a bounded open subset of* $\mathbf{R}$. *The functionals* $F_\varepsilon: L^1(I)\to[0,+\infty]$ *given by*

$$F_\varepsilon(u)=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I f_\varepsilon(\varepsilon|u'|^2)\,dt+\varepsilon^3\int_I |u''|^2\,dt & \text{if } u\in H^2(I)\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converge in the $L^1(I)$-topology as $\varepsilon \to 0^+$ to the functional $F : L^1(I) \to [0,+\infty]$ given by

$$F(u) = \begin{cases} \displaystyle\int_I |u'|^2\,dt + K\#(S_u) & \text{if } u \in SBV(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Proof We first prove the lower semicontinuity inequality. Let $u_\varepsilon \to u$ in $L^1(I)$ with $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon, I) < +\infty$, and let

$$D_\varepsilon = \{\varepsilon|u_\varepsilon'|^2 > 1\} = \bigcup_{k\in\mathbf{N}} I_\varepsilon^k$$

in the notation of Proposition 3.6. Again, it is not restrictive to suppose that $\overline{I}_\varepsilon^k \subset I$ for all k. Note that for all I_ε^k we have the estimate

$$\frac{1}{4\sqrt{\varepsilon}}(I_\varepsilon^k)^2 \le \int_{I_\varepsilon^k} |u_\varepsilon|\,dx \le c,$$

so that $(I_\varepsilon^k)^2 \le c\sqrt{\varepsilon}$.

We divide D_ε into two families:

$$D_\varepsilon^1 = \bigcup\{I_\varepsilon^k \subset D_\varepsilon : \varepsilon|u_\varepsilon'|^2 < (1+C_\varepsilon)^2 \text{ on } I_\varepsilon^k\}, \qquad D_\varepsilon^2 = D_\varepsilon \setminus D_\varepsilon^1.$$

If $I_\varepsilon^k \subset D_\varepsilon^2$ we have

$$\frac{1}{\varepsilon}\int_{I_\varepsilon^k} f(\varepsilon|u_\varepsilon'|^2)\,dt + \varepsilon^3\int_{I_\varepsilon^k} |u_\varepsilon''|^2\,dt \tag{3.13}$$

$$\ge 2\min_{\eta>0}\min\Big\{\frac{1}{\varepsilon}\eta a_\varepsilon + \varepsilon^3\int_0^\eta |u''|^2\,dt : u'(0)=0,\ u'(\eta) = C_\varepsilon/\sqrt{\varepsilon}\Big\}.$$

Such a minimum can be easily computed, after remarking that it is equivalent to

$$\begin{aligned} &\min_{\eta>0}\min\Big\{\eta a_\varepsilon + \int_0^\eta |u''|^2\,dt : u'(0)=0,\ u'(\eta) = C_\varepsilon\sqrt{\varepsilon}\Big\} \\ &= \min_{\eta>0}\min\Big\{\eta a_\varepsilon + \int_0^\eta |w'|^2\,dt : w(0)=0,\ w(\eta) = C_\varepsilon\sqrt{\varepsilon}\Big\}, \end{aligned}$$

and see, by our choice of C_ε, that the minimizing pair in (3.13) is

$$\eta = \frac{\varepsilon K}{4a_\varepsilon}, \qquad u(x) = \frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}x^2,$$

(up to an additive constant for u) which gives

$$\frac{1}{\varepsilon}\int_{I^k_\varepsilon} f(\varepsilon|u'_\varepsilon|^2)\,dt + \varepsilon^3\int_{I^k_\varepsilon}|u''_\varepsilon|^2\,dt \geq K.$$

This shows that $\#(\{k : I^k_\varepsilon \subset D^2_\varepsilon\})$ is equibounded. We can suppose that $\#(\{k : I^k_\varepsilon \subset D^2_\varepsilon\}) = N$, independent of ε. Since $|I^k_\varepsilon| \to 0$ as $\varepsilon \to 0^+$, we can suppose also that D^2_ε shrinks to a finite set H. Of course, $\#(H) \leq N$.

For all $t > 0$ let $I_t = \{x \in I : \operatorname{dist}(x, H) > t\}$. For $j \in \mathbb{N}$, we define

$$f^j(s) = \begin{cases} s & \text{if } s \leq 1 \\ j & \text{if } s > 1 \end{cases}$$

and F^j_ε as in (3.1) with f^j in place of f; then, for fixed t and j, we have, for ε small enough $F_\varepsilon(u_\varepsilon, I_t) \geq F^j_\varepsilon(u_\varepsilon, I_t)$. By applying Proposition 3.7 we get that $u \in SBV(I_t)$ and

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I_t) \geq \int_{I_t}|u'|^2 + m(j,1)\sum_{I_t\cap S_u}\sqrt{|u^+ - u^-|}\,.$$

By the arbitrariness of j we then have that $u \in H^1(I_t)$ for all $t > 0$, and

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I_t) \geq \int_{I_t}|u'|^2. \tag{3.14}$$

By the arbitrariness of t we obtain that $u \in SBV(I)$ and $S_u \subset H$. On the other hand, we clearly have, for fixed t,

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I\setminus \overline{I}_t) \geq \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, D^2_\varepsilon) \geq KN \geq K\#(S_u). \tag{3.15}$$

By (3.14), (3.15), and the arbitrariness of t we get

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2\,dt + K\#(S_u).$$

Now, it suffices to construct a recovery sequence in the case $u \in SBV(-1,1)$ with $S_u = \{0\}$, $u \in H^2((-1,1)\setminus\{0\})$ and $u'(0\pm) = 0$. The general case follows from Propositions 3.9 and 3.10.

We suppose without loss of generality that $u^+(0) > u^-(0)$. Let $z = u^+(0) - u^-(0)$, and let

$$\eta_\varepsilon = \frac{z\sqrt{\varepsilon}}{2(1+C_\varepsilon)} + \frac{(1+C_\varepsilon)\,\varepsilon\sqrt{\varepsilon}}{2\sqrt{a_\varepsilon}}.$$

For ε small enough the following function is well-defined and belongs to the space $H^2(-1,1)$:

$$u_\varepsilon(x) = \begin{cases} u(x+\eta_\varepsilon) & \text{if } x \le -\eta_\varepsilon \\ u^-(0) + \frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}(x+\eta_\varepsilon)^2 & \text{if } -\eta_\varepsilon < x < -\eta_\varepsilon + \frac{(1+C_\varepsilon)\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}} \\ u^-(0) + \frac{(1+C_\varepsilon)^2\varepsilon}{2\sqrt{a_\varepsilon}} & \\ \quad + \frac{1+C_\varepsilon}{\sqrt{\varepsilon}}(x+\eta_\varepsilon - \frac{(1+C_\varepsilon)\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}}) & \text{if } |x| \le \eta_\varepsilon - \frac{(1+C_\varepsilon)\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}} \\ u^+(0) - \frac{\sqrt{a_\varepsilon}}{2\varepsilon^2}(x-\eta_\varepsilon)^2 & \text{if } \eta_\varepsilon - \frac{(1+C_\varepsilon)\varepsilon\sqrt{\varepsilon}}{\sqrt{a_\varepsilon}} < x < \eta_\varepsilon \\ u(x-\eta_\varepsilon) & \text{if } x \ge \eta_\varepsilon. \end{cases}$$

The function u_ε is obtained "filling the gap of u" with two minimizers for the minimum problem in (3.13) joined by a steep affine function, with slope $(1+C_\varepsilon)/\sqrt{\varepsilon}$ so that it gives no contribution to the integrals. A direct computation gives

$$\limsup_{\varepsilon\to 0+} F_\varepsilon(u_\varepsilon) = \int_I |u'|^2\,dt + K = F(u),$$

and the proof is concluded. □

Corollary 3.14 *Let $g \in L^2(I)$, $\lambda > 0$, and let (ε_j) be a sequence of positive numbers converging to 0. Prove that, for every sequence (u_j) of minimizers of the problems*

$$\min\Big\{\frac{1}{\varepsilon_j}\int_I f_{\varepsilon_j}(\varepsilon_j|u'|^2)\,dt + \varepsilon_j^3\int_I |u''|^2\,dt + \lambda\int_I |u-g|^2\,dt:\ u \in H^2(I)\Big\}$$

there exists a subsequence converging in $L^1(I)$ to a minimizer of the problem

$$\min\Big\{\int_I |u'|^2\,dt + \#(S_u) + \lambda\int_I |u-g|^2\,dt:\ u \in SBV(I)\Big\},$$

and we have also convergence of the minimum values.

Proof Since we have $\sup\|u_j - g\|_{L^2(I)} < +\infty$ the sequence (u_j) is bounded in $L^1(I)$. Repeating the reasoning of the first part of the proof of Theorem 3.13 we get that up to subsequences u_j converges in $L^1(I)$ to some $u \in SBV(I)$. The minimality of u and the convergence of minimum values follow from Theorem 1.2. □

3.2 The n-dimensional case

In this section we provide an extension to higher dimension of the result stated in Theorem 3.1. To this end, in the singular perturbation term we replace $|u''|$ by $\|Hu\|$, where Hu denotes the Hessian matrix of u and $\|\cdot\|$ is the norm on the space $M^{n\times n}$ of $n\times n$ real matrices defined by $\|M\| = \sup_{|\xi|=1}|\langle M\xi,\xi\rangle|$. The use of a different norm will be discussed in Section 3.2.2.

3.2.1 *The main result*

Theorem 3.15 *Let f satisfy the hypotheses of Theorem 3.1 for some α and β in $\mathbf{R}$. Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and let $F_\varepsilon : L^2(\Omega) \to [0, +\infty]$ be defined by*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f(\varepsilon|\nabla u|^2)\, dx + \varepsilon^3 \int_\Omega \|Hu\|^2\, dx & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then for any $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0+} F_\varepsilon(u) = F(u)$ with respect to the $L^2(\Omega)$-topology, where

$$F(u) = \begin{cases} \alpha \displaystyle\int_I |\nabla u|^2\, dx + m(\beta) \int_{S_u} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1} & \\ \qquad\qquad\qquad \text{if } u \in GSBV(\Omega) \cap L^2(\Omega) & \\ +\infty \qquad\qquad \text{otherwise} & \end{cases}$$

and $m(\beta) = \beta^{\frac{3}{4}} \left(2\sqrt{3/2} + \sqrt{2/3}\right)$.

Remark 3.16 Note that in Theorem 3.1 $F_\varepsilon, F \equiv +\infty$ on $L^1(I) \setminus L^2(I)$ and (F_ε) Γ-converges to F also with respect to the $L^2(I)$-topology.

The proof of Theorem 3.15 will be a consequence of the propositions in the rest of the chapter, which deal with lower and upper Γ-limits separately.

Remark 3.17 Note that we do not recover the Γ-limit of F_ε on the whole $L^2(\Omega)$. The result would be complete if we showed that for all $u \in GSBV(\Omega) \cap L^2(\Omega)$ there exists a sequence $u_j \in GSBV^2(\Omega) \cap L^2(\Omega)$ such that $u_j \to u$ and $F(u_j) \to F(u)$. The corresponding result for the Mumford-Shah functional relies on regularity results which are not extensible to our case. However the *lower semicontinuity inequality* holds for any $u \in L^2(\Omega)$, as stated in Proposition 3.18.

We define also a "localized version" of our functionals by setting

$$F_\varepsilon(u, A) = \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_A f(\varepsilon|\nabla u|^2)\, dt + \varepsilon^3 \int_A \|Hu\|^2\, dt & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.16}$$

and

$$F(u, A) = \begin{cases} \alpha \displaystyle\int_A |\nabla u|^2\, dx + m(\beta) \int_{A\cap S_u} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1} & \\ \qquad\qquad\qquad \text{if } u \in GSBV(A) \cap L^2(A) & \\ +\infty \qquad\qquad \text{otherwise,} & \end{cases}$$

(3.17)

for all $u \in L^2(\Omega)$ and $A \subseteq \Omega$ open set.

Proposition 3.18 *We have*

$$F(u, A) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A)\,,$$

for all $u \in L^2(\Omega)$ and for all $A \in \mathcal{A}(\Omega)$.

Proof We recover now the thesis from the 1-dimensional case, by using the method described in Section 2.1.1.

Fix $\xi \in S^{n-1}$, $A \in \mathcal{A}(\Omega)$, $v \in H^2(\Omega)$ and let $A_{\xi y}$, $v_{\xi,y}$ be defined as in (1.8) and (1.9). Since

$$v'_{\xi,y}(t) = \langle \nabla v(y+t\xi), \xi\rangle \quad \text{and} \quad v''_{\xi,y}(t) = \langle (Hv(y+t\xi))\,\xi, \xi\rangle,$$

we have, by Fubini's Theorem,

$$\begin{aligned}
&F_\varepsilon(v, A)\\
&= \int_{\Pi_\xi}\int_{A_{\xi y}} \Big(\frac{1}{\varepsilon} f(\varepsilon|\nabla v(y+t\xi)|^2) + \varepsilon^3\|Hv(y+t\xi)\|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi}\int_{A_{\xi y}} \Big(\frac{1}{\varepsilon} f(\varepsilon|\langle\nabla v(y+t\xi),\xi\rangle|^2) + \varepsilon^3|\langle (Hv(y+t\xi))\,\xi,\xi\rangle|^2\Big)\,dt\,d\mathcal{H}^{n-1}(y)\,.\\
&= \int_{\Pi_\xi} F_\varepsilon^{\xi,y}(v_{\xi,y}, A_{\xi y})\,d\mathcal{H}^{n-1}(y),
\end{aligned}$$

where $F_\varepsilon^{\xi,y}(g, I)$ (independent of ξ and y) is defined as in (3.1) in the case $p = 2$ for all $g \in L^2_{\rm loc}(\mathbf{R})$ and $I \subseteq \mathbf{R}$ bounded open set. Notice that, according to the notation of Section 2.1.1, we have in this case

$$F_\varepsilon^\xi(v, A) = \frac{1}{\varepsilon}\int_A f(\varepsilon|\langle\nabla v, \xi\rangle|^2) + \varepsilon^3\int_A |\langle (Hv)\,\xi,\xi\rangle|^2\,dx.$$

Let $u_\varepsilon \to u$ in $L^2(\Omega)$ be such that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) < +\infty$. Then, by Fubini's Theorem and Fatou's Lemma, $(u_\varepsilon)_{\xi,y} \to u_{\xi,y}$ in $L^2(A_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$. Hence, by Proposition 3.7 and again by Fatou's Lemma, we have that $u_{\xi,y} \in SBV(A_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$ and

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\ & F_\varepsilon(u_\varepsilon, A)\\
&\geq \int_{\Pi_\xi} \liminf_{\varepsilon\to 0} F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y}, A_{\xi y})\,d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi}\Big(\alpha\int_{A_{\xi y}} |u'_{\xi,y}|^2\,dt + m(\beta)\sum_{S_{u_{\xi,y}}\cap A_{\xi y}} \sqrt{|u^+_{\xi,y} - u^-_{\xi,y}|}\,\Big)\,d\mathcal{H}^{n-1}(y).
\end{aligned}$$

(3.18)

Let $T > 0$ and set $u_T = (-T) \vee (u \wedge T)$. By (3.18) we deduce that

$$\int_{\Pi_\xi} |D(u_T)_{\xi,y}|(A_{\xi y})\, d\mathcal{H}^{n-1}(y) < +\infty$$

for every $\xi \in S^{n-1}$ and for a.e. $y \in \Pi_\xi$. Then by Theorem 1.23(b) $u_T \in SBV(A)$, i.e. $u \in GSBV(A)$.

Moreover, by (3.18) and Theorem 1.23(a), we get

$$\begin{aligned}\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A)\\ \geq \alpha \int_A |\langle \nabla u, \xi\rangle|^2\, dx + m(\beta) \int_{S_u\cap A} \sqrt{|u^+ - u^-|}\,|\langle \nu_u, \xi\rangle|\, d\mathcal{H}^{n-1}(y).\end{aligned} \tag{3.19}$$

For a fixed $u \in GSBV(\Omega)$ consider the superadditive increasing function μ defined on $\mathcal{A}(\Omega)$ by

$$\mu(A) := \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A)$$

and the measure

$$\lambda := \mathcal{L}_n \llcorner \Omega + \sqrt{|u^+ - u^-|}\mathcal{H}^{n-1} \llcorner S_u.$$

Fixed a sequence $(\xi_i)_{i\in\mathbb{N}}$, dense in S^{n-1}, we have, by (3.19),

$$\gamma(A) \geq \int_A \psi_i(x)\, d\lambda \quad \forall i \in \mathbb{N},$$

where

$$\psi_i = \begin{cases} |\langle \nabla u, \xi_i\rangle|^2 & \text{if } x \in A \setminus S_u \\ |\langle \nu_u, \xi_i\rangle| & \text{if } x \in S_u. \end{cases}$$

Hence, by applying Proposition 1.8, we get

$$\begin{aligned}F(u, A) &\geq \Gamma\text{-}\liminf_{\varepsilon\to 0+} F_\varepsilon(u, A) \geq \int_A \sup_i \psi_i(x)\, d\lambda\\ &= \alpha \int_A |\nabla u|^2\, dx + m(\beta) \int_{S_u\cap A} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}(y),\end{aligned}$$

as desired. □

The rest of the section is devoted to the proof of the upper inequality for the Γ-limit.

Proposition 3.19 *Let Ω' be a bounded open set such that $\Omega \subset\subset \Omega'$ and let E be such that $E = E' \cap \Omega$, where E' is a set of finite perimeter in Ω' such that $\partial E' \cap \Omega'$ is a smooth $(n-1)$-manifold. Then*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(z\chi_E, A) \le m(\beta)\sqrt{|z|}\mathcal{H}^{n-1}(\partial E \cap A)$$

for all $z \in \mathbf{R}$, for all $A \in \mathcal{A}(\Omega)$.

Proof Let

$$d(x) := d(x, \mathbf{R}^n \setminus E') - d(x, E'), \qquad x \in \mathbf{R}^n.$$

Under our hypotheses there exists $\delta > 0$ such that $d \in C^\infty(\overline{B_\delta \cap \Omega})$, where

$$B_\delta := \{x \in \mathbf{R}^n : |d(x)| < \delta\}.$$

Moreover $|\nabla d| = 1$ on B_δ.

Consider the recovery sequence $v_\varepsilon : \mathbf{R} \to \mathbf{R}$ defined by

$$v_\varepsilon(t) = \begin{cases} z & \text{if } t \ge \varepsilon\eta \\ v\left(\frac{t}{\varepsilon}\right) + \frac{z}{2} & \text{if } t \in (-\varepsilon\eta, \varepsilon\eta) \\ 0 & \text{if } t \le -\varepsilon\eta, \end{cases}$$

where (η, v) is the minimizing pair in the definition of $m(\beta, z)$ for $p = 2$ (see Remark 3.3). Note that $\lim_{\varepsilon\to 0+} \mathcal{F}_\varepsilon(v_\varepsilon, (-\delta, \delta)) = m(\beta)\sqrt{|z|}$ (see Proposition 3.8).

Define

$$\widetilde{v}_\varepsilon(x) = v_\varepsilon(d(x)), \qquad x \in \Omega.$$

We have $\widetilde{v}_\varepsilon \in H^2(\Omega)$ and $\widetilde{v}_\varepsilon \to z\chi_E$ in $L^2(\Omega)$. Indeed

$$\int_\Omega |\widetilde{v}_\varepsilon(x) - z\chi_E|^2 \, dx \le 2|z|^2 \mathcal{L}^n(B_{\eta\varepsilon}) \to 0.$$

Using the co-area formula (1.6), we get

$$\begin{aligned} F_\varepsilon(\widetilde{v}_\varepsilon, A) = & \frac{1}{\varepsilon}\int_{A\cap B_{\eta\varepsilon}} f(\varepsilon|v'_\varepsilon(d)\nabla d|^2)\, dx \\ & + \varepsilon^3 \int_{A\cap B_{\eta\varepsilon}} ||v''_\varepsilon(d)(\nabla d \otimes \nabla d) + v'_\varepsilon(d) Hd||^2 \, dx \\ \le & \frac{1}{\varepsilon}\int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A: d(x) = t\}} f(\varepsilon|v'_\varepsilon(t)|^2)\, d\mathcal{H}^{n-1} \\ & + \varepsilon^3 \int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A: d(x)=t\}} (|v''_\varepsilon(t)|^2 + |v'_\varepsilon(t)|^2 ||Hd||^2_\infty)\, d\mathcal{H}^{n-1} \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\eta\varepsilon}^{\eta\varepsilon} \Big(\frac{1}{\varepsilon} f(\varepsilon|v_\varepsilon'(t)|^2) + \varepsilon^3 |v_\varepsilon''(t)|^2\Big) \mathcal{H}^{n-1}(\{x \in A : d(x) = t\})\, dt \\
&\quad + \varepsilon \int_{-\eta\varepsilon}^{\eta\varepsilon} |v'(\frac{t}{\varepsilon})|^2 \|Hd\|_\infty^2\, \mathcal{H}^{n-1}(\{x \in A : d(x) = t\}) dt \\
&\leq \sup_{t\in(-\eta\varepsilon,\eta\varepsilon)} \mathcal{H}^{n-1}(\{x \in A : d(x) = t\}) \mathcal{F}_\varepsilon(v_\varepsilon, (-\eta\varepsilon, \eta\varepsilon)) + o(\varepsilon);
\end{aligned} \tag{3.20}
$$

hence, since

$$
\lim_{t\to 0+} \mathcal{H}^{n-1}(\{x \in A : d(x) = t\}) = \mathcal{H}^{n-1}(\partial E \cap A),
$$

we have

$$
\limsup_{\varepsilon\to 0} F_\varepsilon(\widetilde{v}_\varepsilon, A) \leq m(\beta)\sqrt{|z|}\, \mathcal{H}^{n-1}(\partial E \cap A)
$$

for all $A \in \mathcal{A}(\Omega)$ and for all $z \in \mathbf{R}$. □

Proposition 3.20 *Let $u = \sum_{i=1}^k z_i \chi_{E_i}$ with E_i closed polyhedra and $\overset{o}{E}_i \cap \overset{o}{E}_j = \emptyset$ if $i \neq j$. Then*

$$
\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon(u, A) \leq F(u, A)
$$

for all $A \in \mathcal{A}(\Omega)$.

Proof Up to further subdivide the family (E_i), it is not restrictive to suppose that each E_i is convex. Set

$$
E_{k+1} := \Omega \setminus \bigcup_{i=1}^k E_i,
$$

$$
\mathcal{I} := \{(i,j) : i, j \in \{1, ..., k+1\}, i < j, \mathcal{H}^{n-1}(E_i \cap E_j) \neq 0\}.
$$

For $r > 0$, let

$$
B^r := \{x \in \Omega : d(x, \bigcup_{i=1}^k \partial E_i) < r\}
$$

and, for $(i,j) \in \mathcal{I}$, let

$$
B_{ij}^r := \{x \in \Omega : d(x, E_i \cap E_j) < r\}.
$$

Then, for a fixed $\delta > 0$, we can find $g_{ij} \in C_0^\infty(\mathbf{R}^n)$ such that

$$
g_{ij} = 1 \quad \text{on } B_{ij}^\delta, \quad g_{ij} = 0 \quad \text{on } \Omega \setminus B_{ij}^{2\delta},
$$

$$
\|\nabla g_{ij}\|_\infty \leq \frac{c}{\delta}, \quad \|Hg_{ij}\|_\infty \leq \frac{c}{\delta^2},
$$

for any $(i,j) \in \mathcal{I}$, with c independent of δ, i and j. Define

$$h_{ij} := g_{ij}\Big(\sum_{(l,m)\in\mathcal{I}} g_{lm} + \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big)^{-1},$$

$$h_0 := \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big(\sum_{(l,m)\in\mathcal{I}} g_{lm} + \prod_{(l,m)\in\mathcal{I}} (1-g_{lm})\Big)^{-1}.$$

Note that

$$\sum_{(i,j)\in\mathcal{I}} h_{ij} + h_0 = 1,$$

$$h_{ij} = 1 \quad \text{on } B_{ij}^{\delta} \setminus \bigcup_{(l,m)\neq(i,j)} B_{lm}^{2\delta}, \quad h_0 = 1 \quad \text{on } \Omega \setminus B^{2\delta}.$$

Moreover, an easy computation shows that

$$\|\nabla h_{ij}\|_\infty \le \frac{c}{\delta}, \quad \|\nabla h_0\|_\infty \le \frac{c}{\delta},$$

$$\|Hh_{ij}\|_\infty \le \frac{c}{\delta^2}, \quad \|Hh_0\|_\infty \le \frac{c}{\delta^2}.$$

Let $v_{ij}^\varepsilon \in H^2(\Omega)$ be the recovery sequence, constructed as in Proposition 3.19, related to

$$u_{ij} = \begin{cases} z_i - z_j & \text{on } S_{ij}^+ \\ 0 & \text{on } S_{ij}^-, \end{cases}$$

where S_{ij} is the hyperplane containing $E_i \cap E_j$, that is uniquely determined since E_i, E_j are convex and belong to $\mathcal{I}$, and

$$S_{ij}^{\pm} := \{x \in \Omega : x = y \pm t\nu_{ij},\ y \in S_{ij},\ t \in \mathbf{R}^+\},$$

where ν_{ij} is the internal normal to E_i in $E_i \cap E_j$. We can now choose $\delta = \eta\varepsilon$ with η as in Proposition 3.19 and set

$$v_\varepsilon := \sum_{(i,j)\in\mathcal{I}} h_{ij}(v_{ij}^\varepsilon + z_j) + h_0 u.$$

We have that $v_\varepsilon \in H^2(\Omega)$ and $v_\varepsilon \to u$ in $L^2(\Omega)$. Indeed

$$\int_\Omega |v_\varepsilon - u|^2\, dx \le \|u\|_\infty^2 \mathcal{L}^n(B^{2\eta\varepsilon}) = O(\varepsilon).$$

Moreover,

$$F_\varepsilon(v_\varepsilon, A) \le \sum_{(i,j)\in\mathcal{I}} F_\varepsilon\Big(g_{ij}v_{ij}^\varepsilon + (1-g_{ij})u, \Big(B_{ij}^{2\eta\varepsilon} \setminus \bigcup_{(l,m)\neq(i,j)} B_{lm}^{2\eta\varepsilon}\Big) \cap A\Big)$$

$$+c\sum_{(i,j)\neq(l,m)}\int_{B_{ij}^{2\eta\varepsilon}\cap B_{lm}^{2\eta\varepsilon}\cap A}\Big(\frac{1}{\varepsilon}+\varepsilon^3\sup_{(i,j)\in\mathcal{I}}\{\|Hh_{ij}\|_\infty^2+\|Hv_{ij}^\varepsilon\|_\infty^2$$

$$+\|Hh_0\|_\infty^2+\|\nabla h_{ij}\|_\infty^2\|\nabla v_{ij}^\varepsilon\|_\infty^2\}\Big)\,dx$$

$$\leq\sum_{(i,j)\in\mathcal{I}}F_\varepsilon(v_{ij}^\varepsilon,B_{ij}^{\eta\varepsilon}\cap A)$$

$$+c\sum_{(l,m)\neq(i,j)}\left(\frac{1}{\varepsilon}+\varepsilon^3\left(\frac{1}{\eta^4\varepsilon^4}+\frac{1}{\varepsilon^2}\frac{1}{\eta^2\varepsilon^2}+\frac{1}{\varepsilon^4}\right)\right)\mathcal{L}^n(B_{ij}^{2\eta\varepsilon}\cap B_{lm}^{2\eta\varepsilon})$$

$$=O(\varepsilon)+\sum_{(i,j)\in\mathcal{I}}m(\beta)\sqrt{|z_i-z_j|}\,\mathcal{H}^{n-1}(E_i\cap E_j\cap A).$$

Note that $\mathcal{L}^n(B_{ij}^{2\eta\varepsilon}\cap B_{lm}^{2\eta\varepsilon})=O(\varepsilon^2)$ if $(i,j)\neq(l,m)$, due to the fact that $\mathcal{H}^{n-2}(E_i\cap E_j\cap E_l\cap E_m)<+\infty$. □

Remark 3.21 By using a density argument and the previous proposition, it can be easily seen that if $u\in L^\infty(\Omega)$ and $u=\sum_{i=1}^{+\infty}z_i\chi_{E_i}$, with E_i closed polyhedra, $\mathring{E}_i\cap\mathring{E}_j=\emptyset$ and $\mathcal{H}^{n-1}(S_u)<+\infty$, then

$$F''(u,A)\leq\sum_{i<j}m(\beta)\sqrt{|z_i-z_j|}\mathcal{H}^{n-1}(E_i\cap E_j\cap A)$$

for all $A\in\mathcal{A}(\Omega)$.

From now on, if $A',A\in\mathcal{A}(\Omega)$ are such that $A'\subset A$ and $\operatorname{dist}(A',\Omega\setminus A)>0$, by a *cut-off function between A' and A* we mean a function $\phi\in C^\infty(\overline{\Omega})$ with $0\leq\phi\leq1$ and such that $\phi=1$ on A' and $\phi=0$ on $\Omega\setminus A$.

Proposition 3.22 *Let $A',A,B\in\mathcal{A}(\Omega)$ be such that $A'\subset A$ and $\operatorname{dist}(A',\Omega\setminus A)>0$ and let ϕ be a cut-off function between A' and A. Then for all $u,v\in L^2(\Omega)$ and for all u_ε, $v_\varepsilon\in H^2(\Omega)$ such that $u_\varepsilon\to u$, $v_\varepsilon\to v$ in $L^2(\Omega)$ and*

$$F''(u,A)=\lim_{\varepsilon\to0^+}F_\varepsilon(u_\varepsilon,A),\ \ F''(v,B)=\lim_{\varepsilon\to0^+}F_\varepsilon(v_\varepsilon,B),$$

the following estimate holds

$$F''(\phi u+(1-\phi)v,A'\cup B)$$

$$\leq F''(u,A)+F''(v,B)+c\|\nabla\phi\|_\infty^2\|u-v\|_{L^2((A\setminus\overline{A'})\cap B)}^2$$

$$+c\limsup_{\varepsilon\to0}\left(F_\varepsilon(u_\varepsilon,(A\setminus\overline{A'})\cap B)+F_\varepsilon(v_\varepsilon,(A\setminus\overline{A'})\cap B)\right).$$

Proof. Set

$$S = (A \setminus \overline{A'}) \cap B.$$

Then, using the convexity of $|\cdot|^2$, $\|\cdot\|^2$, and the monotonicity of f, we have

$$\begin{aligned}
&F_\varepsilon(\phi u_\varepsilon + (1-\phi)v_\varepsilon, A' \cup B) \\
&\quad = F_\varepsilon(u_\varepsilon, A') + F_\varepsilon(v_\varepsilon, B \setminus A) \\
&\quad + \frac{1}{\varepsilon}\int_S f(\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi + \phi\nabla u_\varepsilon + (1-\phi)\nabla v_\varepsilon|^2)\,dx \\
&\quad + \varepsilon^3 \int_S \|2\nabla\phi \otimes \nabla(u_\varepsilon - v_\varepsilon) + (u_\varepsilon - v_\varepsilon)(H\phi) + \phi(Hu_\varepsilon - Hv_\varepsilon)\|^2\,dx \\
&\quad \le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
&\quad + \frac{1}{\varepsilon}\int_S f(2\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi|^2 + 2\varepsilon\phi|\nabla u_\varepsilon|^2 + 2\varepsilon(1-\phi)|\nabla v_\varepsilon|^2)\,dx \\
&\quad + c\varepsilon^3 \int_S (|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2 + \|Hu_\varepsilon - Hv_\varepsilon\|^2)\,dx \\
&\quad \le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
&\quad + \frac{1}{\varepsilon}\int_S \left(f(4\varepsilon|(u_\varepsilon - v_\varepsilon)\nabla\phi|^2) + f(4\varepsilon|\nabla u_\varepsilon|^2) + f(4\varepsilon|\nabla v_\varepsilon|^2)\right)\,dx \\
&\quad + c\varepsilon^3 \int_S (|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2 + \|Hu_\varepsilon\|^2 + \|Hv_\varepsilon\|^2)\,dx.
\end{aligned}$$

By our hypotheses on f, it can be easily proved that for all $\alpha > 0$

$$f(\alpha t) \le c_\alpha f(t) \quad \forall t > 0.$$

Thus we have

$$\begin{aligned}
&F_\varepsilon(\phi u_\varepsilon + (1-\phi)v_\varepsilon, A' \cup B) \\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
&\quad + c\Big(F_\varepsilon(u_\varepsilon, S) + F_\varepsilon(v_\varepsilon, S)\Big) + c\frac{1}{\varepsilon}\int_S f(\varepsilon|\nabla\phi|^2|u_\varepsilon - v_\varepsilon|^2)\,dx \\
&\quad + c\varepsilon^3 \int_S \Big(|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2\Big)\,dx \\
&\le F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\
&\quad + c\Big(F_\varepsilon(u_\varepsilon, S) + F_\varepsilon(v_\varepsilon, S)\Big) + c\int_S |\nabla\phi|^2|u_\varepsilon - v_\varepsilon|^2\,dx \\
&\quad + c\varepsilon^3 \int_S \Big(|\nabla\phi|^2|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\|H\phi\|^2\Big)\,dx. \qquad (3.21)
\end{aligned}$$

Note that, by Gagliardo-Nirenberg inequality (see [32]), the equiboundness of the family

$$\Big(\|u_\varepsilon - v_\varepsilon\|_{L^2(S)} + \varepsilon^3\|Hu_\varepsilon - Hv_\varepsilon\|^2_{L^2(S)}\Big)_\varepsilon$$

implies that

$$\lim_{\varepsilon\to 0^+} \varepsilon^3 \int_S |\nabla u_\varepsilon - \nabla v_\varepsilon|^2\, dx = 0$$

The thesis follows passing to the limit as $\varepsilon \to 0^+$ in (3.21). □

Corollary 3.23 *Let A', A, $B \in \mathcal{A}(\Omega)$ be as in Proposition 3.22. Then for any u, $v \in L^2(\Omega)$, $N \in \mathbf{N}$ there exists a cut-off function ϕ_N between A' and A, such that*

$$F''(\phi_N u + (1-\phi_N)v, A'\cup B) \\ \leq (1+\frac{c}{N})\,(F''(u,A)+F''(v,B)) + c\left|\frac{N}{\mathrm{dist}\,(A',\Omega\setminus A)}\right|^2 \|u-v\|^2_{L^2((A\setminus\overline{A'})\cap B)}. \tag{3.22}$$

Proof We will use a standard argument by De Giorgi. Let $N \in \mathbf{N}$ be fixed and set $d = \mathrm{dist}\,(A', \Omega\setminus A)$. Then for $k \in \{0, ..., N\}$ consider the set

$$A_k^N = \left\{x \in A : \mathrm{dist}\,(x, A') < k\frac{d}{N}\right\},$$

For any $k \in \{0, ..., N-1\}$ we can find a cut-off function ϕ_k^N between A_k^N and A_{k+1}^N such that

$$\|\nabla\phi_k^N\|_\infty \leq 2\frac{N}{d}. \tag{3.23}$$

Set

$$S_k^N = \left(A_{k+1}^N \setminus \overline{A}_k^N\right) \cap B.$$

Then, by (3.21) and taking into account (3.23), we get

$$\begin{aligned} F_\varepsilon\ &(\phi_k^N u_\varepsilon + (1-\phi_k^N)v_\varepsilon, A'\cup B) \\ &\leq F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B) \\ &\quad + c\Big(F_\varepsilon(u_\varepsilon, S_k^N) + F_\varepsilon(v_\varepsilon, S_k^N)\Big) + c\frac{N^2}{d^2}\int_{S_k^N} |u_\varepsilon - v_\varepsilon|^2\, dx \\ &\quad + c\varepsilon^3 \int_{S_k^N} \Big(|\nabla\phi_k^N|^2 |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2 \|H\phi_k^N\|^2\Big)\, dx. \end{aligned}$$

Thus, there exists $k_N \in \{0, \dots, N-1\}$ such that

$$\begin{aligned} &F_\varepsilon(\phi_{k_N}^N u_\varepsilon + (1-\phi_{k_N}^N)v_\varepsilon, A'\cup B) \\ &\leq \frac{1}{N}\sum_{k=0}^{N-1} F_\varepsilon(\phi_k^N u_\varepsilon + (1-\phi_k^N)v_\varepsilon, A'\cup B) \end{aligned}$$

$$\leq \left(1 + \frac{c}{N}\right) (F_\varepsilon(u_\varepsilon, A) + F_\varepsilon(v_\varepsilon, B))$$
$$+c\frac{N^2}{d^2} \int_S |u_\varepsilon - v_\varepsilon|^2 \, dx + c(N)\varepsilon^3 \int_S \left(|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 + |u_\varepsilon - v_\varepsilon|^2\right) \, dx.$$

The thesis follows taking $\phi_N = \phi^N_{k_N}$ and letting ε tend to 0^+. □

Remark 3.24 Note that for any $A \in \mathcal{A}(\Omega)$ and $u \in L^2(\Omega)$ such that $u|_A \in H^1(A)$ we have

$$F''(u, A) \leq \alpha \int_A |\nabla u|^2 \, dx. \tag{3.24}$$

Indeed, if $u|_A \in H^2(A)$ it suffices to choose $u_\varepsilon = u$ as the recovery sequence, while in the general case (3.24) follows by the density of $H^2(A)$ in $H^1(A)$ and the lower semicontinuity of F''.

Proposition 3.25 *Let $u \in SBV^2(\Omega)$. Then*

$$F''(u, \Omega) \leq \alpha \int_\Omega |\nabla u|^2 \, dx + m(\beta) \int_\Omega \sqrt{|u^+ - u^-|} \, d\mathcal{H}^{n-1}.$$

Proof *Step 1.* Suppose that $u \in \mathcal{W}(\Omega)$. In particular, for any $h \in \mathbf{N}$ the sets

$$B_h = \left\{ x \in \Omega \ : \ d(x, S_u) < \frac{1}{h} \right\}$$

satisfy

$$\mathcal{L}^n(B_h) \leq \frac{c}{h}.$$

Then, fixed a sequence (ρ_h) decreasing to 0, we can find $(v_h) \subset SBV_0(\Omega)$ satisfying the hypotheses of Proposition 3.20 such that

$$\|u - v_h\|_\infty \leq \rho_h, \quad \mathcal{H}^{n-1}((S_{v_h} \cap B_h) \setminus S_u) \leq \frac{1}{\rho_h} \int_{B_h} |\nabla u| \, dx + O(1). \tag{3.25}$$

Indeed by the co-area formula (1.5) we have

$$|Du|(B_h \setminus S_u) = \int_{B_h} |\nabla u| \, dx = \sum_{j \in I_h} \int_{j\rho_h}^{(j+1)\rho_h} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap (B_h \setminus S_u)) \, dt,$$

where $I_h = \{j \in \mathbf{Z} : |j| \leq \frac{\|u\|_\infty}{\rho_h} + 2\}$. Hence, by the Mean Value Theorem, for every $j \in \mathbf{Z}$ we can find $t_h^j \in (j\rho_h, (j+1)\rho_h)$ such that

$$\rho_h \mathcal{H}^{n-1}(\partial^* \{u > t_h^j\} \cap (B_h \setminus S_u)) \leq \int_{B_h} |\nabla u| \, dx. \tag{3.26}$$

Let P_h^j be a polyhedron such that

$$B_h \cap \{u > (j+1)\rho_h\} \subseteq P_h^j \subseteq B_h \cap \{u > j\rho_h\} \tag{3.27}$$

and

$$\mathcal{H}^{n-1}(\partial P_h^j \cap B_h) \leq \mathcal{H}^{n-1}(\partial^* \{u > t_h^j\} \cap B_h) + \rho_h 2^{-|j|}. \tag{3.28}$$

Then we can define $v_h \in SBV_0(B_h)$ by setting

$$v_h(x) = (j+1)\rho_h \quad \text{on } P_h^j \setminus P_h^{(j+1)}.$$

Taking into account (3.26)–(3.28), it is not difficult to verify that v_h satisfies (3.25).

By Corollary 3.23 we can find for any $N \in \mathbf{N}$ a cut-off function ϕ_N^h between B_{2h} and B_h, such that (3.22) holds with $A = B_h$, $A' = B_{2h}$, $B = \Omega \setminus \overline{B}_{3h}$, i.e.

$$F''(\phi_N^h v_h + (1-\phi_N^h)u, \Omega) \leq \Big(1 + \frac{1}{N}\Big)\Big(F''(u, \Omega \setminus \overline{S_u}) + F''(v_h, B_h)\Big) + cN^2 h^2 \rho_h^2 \mathcal{L}^n(B_h \setminus B_{2h}). \tag{3.29}$$

By Proposition 3.20,

$$\begin{aligned} F''(v_h, B_h) &\leq m(\beta) \int_{B_h \cap S_{v_h}} \sqrt{|v_h^+ - v_h^-|}\, d\mathcal{H}^{n-1} \\ &\leq 2\|u\|_\infty m(\beta) \mathcal{H}^{n-1}(S_{v_h} \cap B_h) \setminus S_u) + O(1) \\ &\quad + m(\beta) \int_{S_u} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.30}$$

Choose

$$\rho_h = h^{-\frac{1}{2}} \Big(\int_{B_h} |\nabla u|^2\, dx\Big)^{\frac{1}{4}}.$$

Then, by (3.25) and (3.30), $\limsup_h F''(v_h, B_h) \leq m(\beta) \int_{S_u} \sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}$. Letting $h \to +\infty$ in (3.29), by the lower semicontinuity of F'', we get the thesis by the arbitrariness of N.

Step 2. Suppose now that $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. By Theorem 1.18 there exists $(w_h) \subseteq \mathcal{W}(\Omega)$ such that (1.14)–(1.17) hold with $\phi(a, b, \nu) = \sqrt{|a-b|}$. Then we get

$$\begin{aligned} F''(u, \Omega) &\leq \liminf_{h \to +\infty} F''(w_h, \Omega) \\ &\leq \lim_{h \to +\infty} \left(\alpha \int_\Omega |\nabla w_h|^2\, dx + m(\beta) \int_{S_{v_h}} \sqrt{|w_h^+ - w_h^-|}\, d\mathcal{H}^{n-1} \right) \end{aligned}$$

$$= F(u, \Omega).$$

Step 3. We recover the general case by a truncation argument. Let $u \in GSBV^2(\Omega)$ and let $u_h = (-h) \vee u \wedge h$. By the Monotone Convergence Theorem we have

$$\lim_{h \to +\infty} F(u_h, \Omega) = F(u, \Omega).$$

Since $u_h \to u$ in $L^2(\Omega)$, by Step 2 we get the thesis. □

3.2.2 *Some generalizations*

The choice of the norm $\|\cdot\|$ that occurs in the definition of F_ε in the singular perturbation term is justified by the technical fact that such norm seems the most natural to recover the lower semicontinuity inequality in the n-dimensional case from the 1-dimensional one, by applying a slicing technique. From a numerical point of view, it is easier to deal with other norms, like the euclidean one. In this section we replace $\|\cdot\|$ by a generic norm and give a sufficient condition in order to have a generalization of the previous results.

Let $\varphi : M^{n\times n} \to \mathbf{R}^+$ be a norm and let $F_\varepsilon^\varphi, F^\varphi : L^2(\Omega) \to \mathbf{R}^+$ be the functionals defined by

$$F_\varepsilon^\varphi(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega f(\varepsilon|\nabla u|^2)\,dx + \varepsilon^3 \int_\Omega \varphi^2(Hu)\,dx & \text{if } u \in H^2(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

$$F^\varphi(u) = \begin{cases} \alpha \displaystyle\int_\Omega |\nabla u|^2\,dx + m(\beta)\int_{S_u} \sqrt{\varphi(\nu_u \otimes \nu_u)}\ \sqrt{|u^+ - u^-|}\,d\mathcal{H}^{n-1} & \\ \qquad\qquad \text{if } u \in GSBV(\Omega) \cap L^2(\Omega) & \\ +\infty & \text{otherwise,} \end{cases}$$

where f and $m(\beta)$ are as in Theorem 3.15. A localized version of the functionals F_ε^φ, F^φ is obtained by extending in a natural way the definitions (3.16) and (3.17).

Proposition 3.26 *Let the norm φ satisfy the following property: for any $M \in M^{n\times n}$*

$$\varphi(M) \geq |\langle M\xi, \xi\rangle| \varphi(\xi \otimes \xi) \qquad \textit{for all} \quad \xi \in \mathbf{R}^n\,,\ |\xi| = 1. \tag{3.31}$$

Then, for any $u \in GSBV^2(\Omega) \cap L^2(\Omega)$ there exists the Γ-limit $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon^\varphi(u) = F^\varphi(u)$ with respect to the $L^2(\Omega)$-topology.

Remark 3.27 It can be easily verified that condition (3.31) is satisfied, in particular, by the euclidean norm. In this case the limit functional F^φ coincides with the functional F defined in the previous section.

Proposition 3.28 *Under the hypotheses of Proposition* 3.26 *we have*

$$F^{\varphi}(u,A) \leq \Gamma\text{-}\liminf_{\varepsilon\to 0+} F^{\varphi}_{\varepsilon}(u,A)$$

for all $u \in L^2(\Omega)$ *and for all* $A \in \mathcal{A}(\Omega)$.

Proof Proceeding as in the proof of Proposition 3.18, taking into account (3.31), we get

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F^{\varphi}_{\varepsilon}(u,A) \geq \alpha \int_A |\langle \nabla u, \xi\rangle|^2\,dx$$
$$+m(\beta)\int_{S_u\cap A} \sqrt{\varphi(\xi\otimes\xi)}\sqrt{|u^+ - u^-|}|\langle \nu_u,\xi\rangle|\,d\mathcal{H}^{n-1}(y).$$

Since, again by (3.31), $\sup_{|\xi|=1}\sqrt{\varphi(\xi\otimes\xi)}|\langle\nu_u,\xi\rangle| = \sqrt{\varphi(\nu_u\otimes\nu_u)}$, we get the thesis using Proposition 1.8. □

We state now the analogue of Proposition 3.19.

Proposition 3.29 *Let* $E \subseteq \Omega$ *satisfy the hypotheses of Proposition* 3.19. *Then*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} F_{\varepsilon}(z\chi_E, A) \leq m(\beta)\sqrt{|z|}\int_{A\cap\partial E} \sqrt{\varphi(\nu_E\otimes\nu_E)}\,d\mathcal{H}^{n-1}$$

for all $A \in \mathcal{A}(\Omega)$.

Proof Let $\widetilde{v_\varepsilon}$ be as in the proof of Proposition 3.19. Arguing as in (3.20), we get

$$\begin{aligned}
F_\varepsilon(\widetilde{v}_\varepsilon, A) &= \frac{1}{\varepsilon}\int_{A\cap B_{\eta\varepsilon}} f(\varepsilon|v'_\varepsilon(d)\nabla d|^2\,dx \\
&\quad +\varepsilon^3\int_{A\cap B_{\eta\varepsilon}} \varphi^2(v''_\varepsilon(d)(\nabla d\otimes\nabla d) + v'_\varepsilon(d)Hd)\,dx \\
&\leq \frac{1}{\varepsilon}\int_{-\eta\varepsilon}^{\eta\varepsilon} dt \int_{\{x\in A:d(x)=t\}} f(\varepsilon|v'_\varepsilon(t)|^2)\,d\mathcal{H}^{n-1} \\
&\quad +\varepsilon^3\int_{-\eta\varepsilon}^{\eta\varepsilon} dt\int_{\{x\in A:d(x)=t\}} |v''_\varepsilon(t)|^2\varphi^2(\nabla d\otimes\nabla d)\,d\mathcal{H}^{n-1} + o(\varepsilon)\,.
\end{aligned} \tag{3.32}$$

By the uniform continuity of the functions $f_1(x) = \varphi^2(\nabla d(x)\otimes\nabla d(x))$, $f_2(x) = \sqrt{\varphi(\nabla d(x)\otimes\nabla d(x))}$, for any fixed $\gamma > 0$, we can find $\delta > 0$ such that

$$|x_1 - x_2| < \delta \;\Rightarrow\; |f_i(x_1) - f_i(x_2)| \leq \gamma \quad i = 1,2\,.$$

Let $(M_k)_{k=1,\ldots,N}$ be a finite family of measurable, pairwise disjoint sets such that $\Omega \subseteq \bigcup_{k=1}^N M_k$ and diam $(M_k) < \delta$, and let $x_k \in M_k$ be fixed. Then, by (3.32), we obtain

$$F_\varepsilon(\widetilde{v}_\varepsilon, A) \le \sum_{k=1}^{N} \int_{-\eta\varepsilon}^{\eta\varepsilon} \Big(\frac{1}{\varepsilon} f(\varepsilon|v_\varepsilon'(t)|^2) + \varepsilon^3 |v_\varepsilon''(t)|^2 \varphi^2(\nu_E(Px_k)\otimes\nu_E(Px_k))\Big)\, dt$$
$$\times \sup_{t\in(-\eta\varepsilon,\eta\varepsilon)} \mathcal{H}^{n-1}(\{x\in M_k\cap A : d(x)=t\}) + \gamma c + o(\varepsilon),$$

where by Px_k we denote the projection of x_k on ∂E. Hence, letting $\varepsilon \to +\infty$ we get

$$\limsup_{\varepsilon\to 0+} F_\varepsilon(\widetilde{v}_\varepsilon, A)$$
$$\le m(\beta) \sum_{k=1}^{N} \sqrt{|z|} \int_{M_k\cap\partial E\cap A} \sqrt{\varphi(\nu_E(Px_k)\otimes(Px_k))}\, d\mathcal{H}^{n-1} + c\gamma$$
$$\le c\gamma + m(\beta)\sqrt{|z|}\int_{A\cap\partial E} \sqrt{\varphi(\nu_E\otimes\nu_E)}\, d\mathcal{H}^{n-1}.$$

The thesis follows by the arbitrariness of γ. □

Proposition 3.30 *Under the hypotheses of Proposition* 3.26 *we have*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon^\varphi(u,\Omega) \le \alpha \int_\Omega |\nabla u|^2\, dx + m(\beta)\int_\Omega \sqrt{\varphi(\nu_u\otimes\nu_u)}\sqrt{|u^+ - u^-|}\, d\mathcal{H}^{n-1}.$$

for $u \in GSBV^2(\Omega)\cap L^2(\Omega)$.

Proof First we note that the analogues of Propositions 3.20, 3.22 and Corollary 3.23 hold, since the argument used in the proofs is not affected by the particular choice of the norm in the definition of F_ε^φ. Hence the thesis follows proceeding as in the proof of Proposition 3.25, noticing that, in this case, we apply Theorem 1.18 taking in (1.17) $\phi(a,b,\nu) = \sqrt{\varphi(\nu\otimes\nu)}\sqrt{|a-b|}$. □

4

APPROXIMATION BY FUNCTIONALS INVOLVING THE L^1-NORM OF THE GRADIENT

In this chapter we provide an approximation of free-discontinuity energies by using a variant of the Ambrosio-Tortorelli construction we described in Section 2.2.2. In the next section the approximating functionals are obtained just by replacing $|\nabla u|^2$ by $|\nabla u|$ in (2.2). In Section 4.2 we push this approach further to construct an approximation for a wide class of functionals on GBV and GSBV by a double limit procedure.

The results of this chapter are contained in [4].

4.1 The main result

Theorem 4.1 *Let $W : [0,1] \to [0,+\infty)$ be a continuous function such that $W(x) = 0$ if and only if $x = 1$, and let $\psi : [0,1] \to [0,1]$ be an increasing lower semicontinuous function with $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(t) > 0$ if $t \neq 0$. Let $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty)$ be defined by*

$$G_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi(v)|\nabla u| + \frac{1}{\varepsilon}W(v) + \varepsilon|\nabla v|^2\Big)\,dx & \text{if } u,v \in H^1(\Omega) \\ & \text{and } 0 \leq v \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the $\Gamma\text{-}\lim_{\varepsilon\to 0+} G_\varepsilon(u,v) = G(u,v)$ with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence, where

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega |\nabla u|\,dx + \int_{S_u} g(|u^+ - u^-|)\,d\mathcal{H}^{n-1} + |D^c u|(\Omega) & \text{if } u \in GBV(\Omega) \\ & \text{and } v = 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$g(z) := \min\{\psi(t)z + 2c_W(t) : 0 \leq t \leq 1\}, \tag{4.1}$$

with $c_W(t) := 2\int_t^1 \sqrt{W(s)}\,ds$.

The proof of the theorem above will be a consequence of the propositions in the rest of the section. Before entering into the details of the proof, we define also a 'localized version' of our functionals as follows:

$$G_\varepsilon(u,v,A)=\begin{cases}\displaystyle\int_A\Big(\psi(v)|\nabla u|+\frac{1}{\varepsilon}W(v)+\varepsilon|\nabla v|^2\Big)\,dx & \text{if } u,v\in H^1(\Omega)\\ & \text{and } 0\le v\le 1 \text{ a.e.}\\ +\infty & \text{otherwise.}\end{cases}$$

and

$$G(u,v,A)=\begin{cases}\displaystyle\int_A|\nabla u|\,dx+\int_{S_u\cap A}g(|u^+-u^-|)\,d\mathcal{H}^{n-1}+|D^c u|(A) & \\ & \text{if } u\in GBV(\Omega) \text{ and } v=1 \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}$$

for any $A\subseteq\Omega$ bounded open set.

Remark 4.2 By the assumptions on ψ and W, it can be easily proved that g satisfies the following properties
(i) g is increasing, $g(0)=0$ and

$$\lim_{z\to+\infty}g(z)=2c_W(0)=4\int_0^1\sqrt{W(s)}\,ds;$$

(ii) g is subadditive, i.e.

$$g(z_1+z_2)\le g(z_1)+g(z_2)\qquad\forall z_1,\ z_2\in\mathbf{R}^+;$$

(iii) g is Lipschitz-continuous with Lipschitz constant 1;
(iv) $g(z)\le z$ for all $z\in\mathbf{R}^+$ and

$$\lim_{z\to0^+}\frac{g(z)}{z}=1;$$

(v) for any $T>0$ there exists a constant $c_T>0$ such that $z\le c_T\,g(z)$ for all $z\in[0,T]$.

Proposition 4.3 *Let $n=1$. Then $G(u,v)\le\Gamma\text{-}\liminf_{\varepsilon\to0^+}G_\varepsilon(u,v)$ for all $u,v\in L^1(\Omega)$.*

Proof. It suffices to consider the case in which the right-hand side is finite. Let $\varepsilon_j\to0^+$, $u_j\to u$ and $v_j\to v$ in $L^1(\Omega)$ be such that $\lim_j G_{\varepsilon_j}(u_j,v_j)=\Gamma\text{-}\liminf_{\varepsilon\to0^+}G_\varepsilon(u,v)$. Up to passing to subsequences we may suppose

$$u_j\to u,\ \text{and } v_j\to v \text{ a.e.}\tag{4.2}$$

We have

$$\int_\Omega W(v_j)\,dx<c\varepsilon_j;$$

hence, by the continuity of W, for any $\eta > 0$ $\mathcal{L}^1(\{x \in \Omega : W(v(x)) > \eta\}) = \lim_j \mathcal{L}^1(\{x \in \Omega : W(v_j(x)) > \eta\}) = 0$. We conclude that $W(v) = 0$ a.e., i.e. $v = 1$ a.e.

We now use a discretization argument. By simplicity, we suppose that $\Omega = (a, b)$ (otherwise we split Ω into its connected components). Let $N \in \mathbf{N}$ and consider the intervals

$$I_N^k = \Big(a + \frac{(k-1)}{N}(b-a), a + \frac{k}{N}(b-a)\Big), \qquad k \in \{1, .., N\}.$$

Up to passing to subsequences we may suppose that

$$\lim_j \inf_{I_N^k} v_j$$

exists for all $N \in \mathbf{N}$ and $k \in \{1, .., N\}$. Let $z \in (0, 1)$ be fixed and consider the set

$$J_N^z = \Big\{k \in \{1, .., N\} : \lim_j \inf_{I_N^k} v_j \le z\Big\}.$$

Note that for any (α, β) interval in $\mathbf{R}$ and for any $w \in H^1(\alpha, \beta)$ we have, by Young's inequality,

$$\int_\alpha^\beta \Big(\frac{1}{\varepsilon} W(w) + \varepsilon |w'|^2\Big)\, dx \ge 2 \int_\alpha^\beta \sqrt{W(w)}|w'|\, dx \ge 2 \Big| \int_{w(\alpha)}^{w(\beta)} \sqrt{W(s)}\, ds \Big|.$$

From this inequality we deduce, following an argument as in [16], that

$$\Big(2 \int_z^1 \sqrt{W(s)}\, ds\Big) \# J_N^z \le \lim_j G_{\varepsilon_j}(u_j, v_j) < +\infty.$$

Then

$$\# J_N^z \le C$$

with C independent of N. Hence, up to a subsequence, we may suppose

$$J_N^z = \{k_1^N, .., k_L^N\}$$

with L independent of N, and up to a further subsequence that there exist $S = \{t_1, .., t_L\} \subset [a, b]$ such that

$$\lim_{N \to +\infty} \frac{k_i^N}{N} = t_i$$

for any $i \in \{1, .., L\}$. For every $\eta > 0$ we have

$$I_N^k \subset S_\eta := S + [-\eta, \eta]$$

for all $k \in J_N^z$ and for N large enough. Then

$$\liminf_j G_{\varepsilon_j}(u_j, v_j) \ge \liminf_j G_{\varepsilon_j}(u_j, v_j, \Omega \setminus S_\eta)$$

$$
\begin{aligned}
&+\liminf_j \sum_{i=1}^{L} G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta)) \\
&\geq \liminf_j \psi(z) \int_{\Omega\setminus S_\eta} |u_j'|\, dt \\
&\quad + \sum_{i=1}^{L} \liminf_j G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta)). \qquad (4.3)
\end{aligned}
$$

With fixed $i \in \{1,\dots,L\}$, we focus our attention on the term $G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta))$. By definition and by (4.2), we have that for any $\delta > 0$ there exist $x_1,\ x_2 \in (t_i-\eta, t_i+\eta)$ such that

$$
\begin{aligned}
&\lim_j u_j(x_1) = u(x_1) < \operatorname*{ess\text{-}inf}_{(t_i-\eta,t_i+\eta)} u + \delta, \\
&\lim_j u_j(x_2) = u(x_2) > \operatorname*{ess\text{-}sup}_{(t_i-\eta,t_i+\eta)} u - \delta, \\
&\lim_j v_j(x_1) = \lim_j v_j(x_2) = 1. \qquad (4.4)
\end{aligned}
$$

Let $x_j^i \in [x_1, x_2]$ be such that $v_j(x_j^i) = \inf_{[x_1,x_2]} v_j$. Then we obtain the following estimate:

$$
\begin{aligned}
G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta)) &\geq G_{\varepsilon_j}(u_j, v_j, (x_1, x_2)) \\
&\geq \psi(v_j(x_j^i)) \Big| \int_{x_1}^{x_2} u_j'\, dx \Big| + 2\int_{x_1}^{x_2} \sqrt{W(v_j)} |v_j'|\, dx \\
&\geq \psi(v_j(x_j^i)) |u_j(x_2) - u_j(x_1)| \\
&\quad + 2\int_{v_j(x_j^i)}^{v_j(x_1)} \sqrt{W(s)}\, ds + 2\int_{v_j(x_j^i)}^{v_j(x_2)} \sqrt{W(s)}\, ds \\
&\geq \inf_{t\in[0,1]} \Big\{ \psi(t) |u_j(x_2) - u_j(x_1)| \\
&\quad + 2\Big(\int_t^{v_j(x_1)} \sqrt{W(s)}\, ds + \int_t^{v_j(x_2)} \sqrt{W(s)}\, ds \Big) \Big\}. \qquad (4.5)
\end{aligned}
$$

Letting $j \to +\infty$ and taking into account (4.4), we get

$$
\begin{aligned}
&\liminf_j G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta)) \\
&\geq \inf_{t\in[0,1]} \Big\{ \psi(t) \Big| \operatorname*{ess\text{-}sup}_{(t_i-\eta,t_i+\eta)} u - \operatorname*{ess\text{-}inf}_{(t_i-\eta,t_i+\eta)} u - 2\delta \Big| + 4\int_t^1 \sqrt{W(s)}\, ds \Big\}.
\end{aligned}
$$

Thus, by the arbitrariness of $\delta > 0$,

$$
\liminf_j G_{\varepsilon_j}(u_j, v_j, (t_i-\eta, t_i+\eta)) \geq g\Big(\operatorname*{ess\text{-}sup}_{(t_i-\eta,t_i+\eta)} u - \operatorname*{ess\text{-}inf}_{(t_i-\eta,t_i+\eta)} u \Big) \qquad (4.6)
$$

Now we turn back to the estimate (4.3). Since $\sup_j G_{\varepsilon_j}(u_j, v_j) < +\infty$, by (4.3) we get the equiboundness of $\int_{\Omega\setminus S_\eta} |u_j'|\,dt$. Hence $u \in BV(\Omega \setminus S_\eta)$ and, by (4.3) and (4.6),

$$\liminf_j G_{\varepsilon_j}(u_j, v_j) \geq \psi(z)|Du|(\Omega \setminus S_\eta) + \sum_{i=1}^{L} g\Big(\operatorname*{ess\text{-}sup}_{(t_i-\eta,t_i+\eta)} u - \operatorname*{ess\text{-}inf}_{(t_i-\eta,t_i+\eta)} u\Big). \quad (4.7)$$

By the arbitrariness of η, we deduce that $u \in BV(\Omega \setminus S)$, i.e., since S is finite, $u \in BV(\Omega)$. Then, letting $\eta \to 0$ in (4.7), we get

$$\liminf_j G_{\varepsilon_j}(u_j, v_j) \geq \psi(z)|Du|(\Omega \setminus S) + \sum_{i=1}^{L} g(|u^+ - u^-|(t_i))$$

$$\geq \psi(z)|Du|(\Omega \setminus S_u) + \sum_{t\in S_u} \Big(g(|u^+ - u^-|(t)) \wedge \psi(z)|u^+ - u^-|(t)\Big). \quad (4.8)$$

Finally, letting $z \to 1$ in (4.8) we obtain the required inequality, since $g(t) \leq t$.
□

We recover, now, the n-dimensional analogue of the previous inequality, by using the method described in Section 2.1.1.

Proposition 4.4 *Let $n \in \mathbb{N}$. Then $G(u,v) \leq \Gamma\text{-}\liminf_{\varepsilon\to0^+} G_\varepsilon(u,v)$ for all $u, v \in L^1(\Omega)$.*

Proof In the following we will use the notation introduced before Theorem 1.11, and we set as usual $G' = \Gamma\text{-}\liminf_{\varepsilon\to0^+} G_\varepsilon$.

Let $\xi \in S^{n-1}$ be fixed. For any $u, v \in H^1(\Omega)$, $0 \leq v \leq 1$, we have, by Fubini's Theorem,

$$\begin{aligned}
&G_\varepsilon(u, v, A) \\
&= \int_{\Pi_\xi} \int_{A_{\xi y}} \Big(\psi(v(y + t\xi))|\nabla u(y + t\xi)| \\
&\qquad + \frac{1}{\varepsilon} W(v(y + t\xi)) + \varepsilon|\nabla v(y + t\xi)|^2\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\
&\geq \int_{\Pi_\xi} \int_{A_{\xi y}} \Big(\psi(v_{\xi,y}(t))|u_{\xi,y}'| + \frac{1}{\varepsilon} W(v_{\xi,y}(t)) + \varepsilon|v_{\xi,y}'(t)|^2\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\
&= \int_{\Pi_\xi} G_\varepsilon^{\xi,y}(u_{\xi,y}, v_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y), \qquad (4.9)
\end{aligned}$$

where $G_\varepsilon^{\xi,y}$ (independent of ξ and y) is defined by

$$G_\varepsilon^{\xi,y}(u, v, I) = \begin{cases} \displaystyle\int_I \Big(\psi(v)|u'| + \frac{1}{\varepsilon} W(v) + \varepsilon|v''|^2\Big)\, dt & \text{if } u, v \in H^1(I) \text{ and } 0 \leq v \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

for any $u, v \in L^1(I)$ and $I \subset \mathbf{R}$ open and bounded. Notice that, according to the notation of Section 2.1.1, we have in this case

$$G_\varepsilon^\xi(u,v,A) = \int_A \Big(\psi(v)|\langle\nabla u,\xi\rangle| + \frac{1}{\varepsilon}W(v) + \varepsilon|\langle\nabla v,\xi\rangle|^2\Big)\,dx.$$

Let $u_\varepsilon \to u$, $v_\varepsilon \to v$ in $L^1(\Omega)$ be such that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty. \tag{4.10}$$

Then, as in the proof of Proposition 4.3, $v = 1$ a.e. Moreover, by Fubini's Theorem, $(u_\varepsilon)_{\xi,y} \to u_{\xi,y}, (v_\varepsilon)_{\xi,y} \to 1$ in $L^1(\Omega_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$.

Thus by Proposition 4.3 and by Fatou's Lemma we get that $u_{\xi,y} \in BV(A_{\xi y})$ for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$ and

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon, A)\\
&\geq \int_{\Pi_\xi} \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon((u_\varepsilon)_{\xi y}, (v_\varepsilon)_{\xi,y}, A_{\xi y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi}\Big(\int_{A_{\xi y}} |u'_{\xi,y}|\,dt + \int_{S_{u_{\xi,y}}\cap A_{\xi y}} g(|u^+_{\xi,y} - u^-_{\xi,y}|)\,d\#\Big)\,d\mathcal{H}^{n-1}(y)\\
&\quad + \int_{\Pi_\xi}\Big(|D^c u_{\xi,y}|(A_{\xi y})\Big)\,d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{4.11}$$

Let $T > 0$ and set

$$u_T = (-T) \vee (u \wedge T).$$

Since g is increasing, it is clear that we decrease the last term in (4.11) if we substitute u by u_T. Moreover, since $u_T \in L^\infty(\Omega)$, with $\|u_T\|_\infty \leq T$, by Remark 4.2(v), we have

$$|u_T^+ - u_T^-| \leq c_T\, g(|u_T^+ - u_T^-|)$$

for a suitable constant c_T depending only on T. Then, by (4.10) and (4.11), we have

$$\int_{\Pi_\xi} |Du_T|(A_{\xi y})\, d\mathcal{H}^{n-1}(y) < +\infty.$$

Thus, applying Theorem 1.11, we get that $u_T \in BV(\Omega)$ and, by the arbitrariness of (u_ε) and(v_ε),

$$\begin{aligned}
&G'(u,1,A)\\
&\geq \int_A |\langle\nabla u_T,\xi\rangle|\,dx + \int_{S_u\cap A} g(|u_T^+ - u_T^-|)|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^{n-1} + |\langle D^c u_T,\xi\rangle|(A)
\end{aligned} \tag{4.12}$$

for all $A \in \mathcal{A}(\Omega)$ and $\xi \in S^{n-1}$.

Consider the superadditive increasing function defined on $\mathcal{A}(\Omega)$ by

$$\mu(A) := G'(u, 1, A)$$

and the measure

$$\lambda := \mathcal{L}^n \llcorner \Omega + g(|u_T^+ - u_T^-|)\mathcal{H}^{n-1} \llcorner S_{u_T} + |D^c u_T|.$$

Fixed a sequence $(\xi_i)_{i \in \mathbf{N}}$, dense in S^{n-1}, we have, by (4.12),

$$\mu(A) \geq \int_A \psi_i \, d\lambda$$

for all $i \in \mathbf{N}$, where

$$\psi_i(x) = \begin{cases} |\langle \nabla u_T(x), \xi_i \rangle| & \mathcal{L}^n \text{ a.e. on } \Omega \\ |\langle \nu_u(x), \xi_i \rangle| & |D^c u_T| \text{ a.e. on } \Omega \setminus S_{u_T} \\ |\langle \nu_u(x), \xi_i \rangle| & \mathcal{H}^{n-1} \text{ a.e. on } S_{u_T}. \end{cases}$$

Hence, applying Proposition 1.8, we get

$$G'(u, 1, A) \geq \int_A |\nabla u_T| \, dx + \int_{S_{u_T} \cap A} g(|u_T^+ - u_T^-|) \, d\mathcal{H}^{n-1} + |D^c u_T|(A) \quad (4.13)$$

for all $A \in \mathcal{A}(\Omega)$. In particular

$$G'(u, 1, \Omega) \geq \int_\Omega |\nabla u_T| \, dx + \int_{S_{u_T}} g(|u_T^+ - u_T^-|) \, d\mathcal{H}^{n-1} + |D^c u_T|(\Omega). \quad (4.14)$$

Finally, by the arbitrariness of $T > 0$, $u \in GBV(\Omega)$ and the thesis follows letting $T \to +\infty$ in (4.14). □

Proposition 4.5 *We have*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0^+} G_\varepsilon(u, v) \leq G(u, v)$$

for all $u, v \in L^1(\Omega)$.

Proof It suffices to prove the inequality for $v = 1$ a.e. Since we will use density and relaxation arguments, we divide the proof into five steps, passing from a particular choice of u to the general one. In the following we will use the notation $G'' = \Gamma\text{-}\limsup_{\varepsilon \to 0^+} G_\varepsilon$.

Step 1. Suppose that $u \in \mathcal{W}(\Omega)$ and

$$\overline{S}_u = \Omega \cap K$$

with K a $(n-1)$-dimensional simplex. Up to a translation and rotation argument, we can suppose that K is contained in the hyperplane $\Pi := \{x_n = 0\}$. Set

$$h(y) := u^+(y) - u^-(y), \quad y \in \overline{S}_u.$$

By our hypotheses on u, h is regular on $\overline{S}_u$; hence, fixed $\delta > 0$, we can find a triangulation $\{T_i\}_{i=1}^N$ of $\overline{S}_u$ such that

$$|h(y_1) - h(y_2)| < \delta \quad \text{if } y_1, y_2 \in T_i.$$

Let $h_\delta : \overline{S}_u \to \mathbf{R}$ be defined as

$$h_\delta(y) := z_i \quad y \in T_i,$$

where $z_i := \min\{h(y) : \ y \in \overline{T}_i\}$. Since $\|h - h_\delta\|_\infty < \delta$, by Remark 4.2 (iii), we have that

$$\int_{S_u} g(h_\delta(y))\, d\mathcal{H}^{n-1} \leq \int_{S_u} g(h(y))\, d\mathcal{H}^{n-1} + \delta \mathcal{H}^{n-1}(\overline{S}_u).$$

Let x_{z_i} realize the minimum in (4.1) for $z = z_i$. Fixed $\eta > 0$, there exists $T(\eta) > 0$ such that

$$\min\left\{\int_0^T \Big(|v'|^2 + W(v)\Big)\, dt : v \in H^1(0,T),\ v(0) = x_{z_i},\ v(T) = 1\right\} \leq c_W(x_{z_i}) + \eta \tag{4.15}$$

for all $T \geq T(\eta)$ and for any $i = 1,..,N$. Let $v(z_i, \cdot)$ realize the minimum in (4.15).

For $r > 0, \varepsilon > 0$ and $i \in \{1,..,N\}$, set

$$B_r := \Big\{(y,t) \in \Omega : \ y \in \overline{S}_u,\ |t| < r\Big\} \quad \text{and} \quad T_i^\varepsilon := \Big\{y \in T_i : \ d(y, \partial T_i) > \varepsilon\Big\},$$

and let $\phi_\varepsilon^i : \mathbf{R}^{(n-1)} \to \mathbf{R}$ be a cut-off function between T_i^ε and T_i such that $\|\nabla \phi_\varepsilon^i\|_\infty < C\varepsilon^{-1}$. Fix a sequence (ξ_ε) such that $\lim_{\varepsilon \to 0+} \frac{\xi_\varepsilon}{\varepsilon} = 0$, set $T_\varepsilon := T(\eta)\varepsilon + \xi_\varepsilon$, and define

$$v_\varepsilon(y,t) := \begin{cases} 1 & \text{if } \ (y,t) \in \Omega \setminus B_{T_\varepsilon} \\ \phi_\varepsilon^i(y) v_\varepsilon^i(t) + (1 - \phi_\varepsilon^i(y)) & \text{if } \ y \in T_i,\ |t| < T_\varepsilon, \end{cases}$$

where

$$v_\varepsilon^i(t) := \begin{cases} x_{z_i} & \text{if } \ |t| < \xi_\varepsilon \\ v\Big(z_i, \frac{|t| - \xi_\varepsilon}{\varepsilon}\Big) & \text{if } \ \xi_\varepsilon < |t| < T_\varepsilon. \end{cases}$$

We have that $(v_\varepsilon) \in H^1(\Omega)$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$ as $\varepsilon \to 0+$. Hence, we get

$$\begin{aligned}
&\int_\Omega \left(\varepsilon|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon)\right) dx \qquad (4.16)\\
&= \sum_{i=1}^N \int_{T_i^\varepsilon} 2\int_{\xi_\varepsilon}^{T_\varepsilon} \frac{1}{\varepsilon}\left(\left|v'\left(z_i, \frac{|t|-\xi_\varepsilon}{\varepsilon}\right)\right|^2 + W\left(v\left(z_i, \frac{|t|-\xi_\varepsilon}{\varepsilon}\right)\right)\right) dt\, d\mathcal{H}^{n-1}(y)\\
&\quad + \sum_{i=1}^N \int_{T_i}\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \left(\varepsilon|\nabla\phi_\varepsilon^i(y)|^2|x_{z_i}-1|^2 + \frac{1}{\varepsilon}W(v_\varepsilon(y,t))\right) dt\, d\mathcal{H}^{n-1}(y)\\
&\quad + \sum_{i=1}^N \int_{T_i\setminus T_i^\varepsilon}\int_{\xi_\varepsilon}^{T_\varepsilon} \left(\varepsilon|\nabla\phi_\varepsilon^i(y)|^2\left|v\left(z_i, \frac{|t|-\xi_\varepsilon}{\varepsilon}\right)-1\right|^2\right.\\
&\qquad \left. + \frac{1}{\varepsilon}|\phi_\varepsilon^i(y)|^2\left|v'\left(z_i, \frac{|t|-\xi_\varepsilon}{\varepsilon}\right)\right|^2\right) dt\, d\mathcal{H}^{n-1}(y)\\
&\quad + \sum_{i=1}^N \int_{T_i\setminus T_i^\varepsilon}\int_{\xi_\varepsilon}^{T_\varepsilon} \frac{1}{\varepsilon}W(v_\varepsilon(y,t))\, dt\, d\mathcal{H}^{n-1}(y)\\
&\le \sum_{i=1}^N \int_{T_i^\varepsilon} 2\int_0^T \left(|v'(z_i,t)|^2 + W(v(z_i,t))\right) dt\, d\mathcal{H}^{n-1}(y)\\
&\quad + c\frac{\xi_\varepsilon}{\varepsilon}\mathcal{H}^{n-1}(S_u) + c(\eta)\sum_{i=1}^N \mathcal{H}^{n-1}(T_i\setminus T_i^\varepsilon)\\
&\le \sum_{i=1}^N 2\int_{T_i} c_W(x_{z_i})\, d\mathcal{H}^{n-1}(y) + 2\eta\mathcal{H}^{n-1}(S_u) + O(\varepsilon).
\end{aligned}$$

We now construct a recovery sequence u_ε. Let

$$\tilde{u}_\varepsilon(z_1,z_2,t) = \begin{cases} z_1 & -T_\varepsilon < t < -\xi_\varepsilon \\ \frac{z_2-z_1}{2\xi_\varepsilon}(t+\xi_\varepsilon)+z_1 & |t|<\xi_\varepsilon \\ z_2 & \xi_\varepsilon < t < T_\varepsilon \end{cases}$$

and set

$$u_\varepsilon(y,t) = \begin{cases} u(y,t) & |t| > T_\varepsilon \\ \tilde{u}_\varepsilon\big(u(y,-T_\varepsilon), u(y,T_\varepsilon), t\big) & |t| < T_\varepsilon\ . \end{cases}$$

It can be easily verified that $u_\varepsilon \in H^1(\Omega)$ and $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0^+$. Moreover, we have

$$\int_\Omega \psi(v_\varepsilon)|\nabla u_\varepsilon|\, dx \le \sum_{i=1}^N \int_{T_i^\varepsilon}\int_{-\xi_\varepsilon}^{\xi_\varepsilon} \frac{1}{2\xi_\varepsilon}\psi(x_{z_i})|u(y,T_\varepsilon) - u(y,-T_\varepsilon)|\, dt\, d\mathcal{H}^{n-1}(y)$$

$$+\int_{\Omega\setminus B_{t_\varepsilon}} |\nabla u|\,dx + c\mathcal{H}^{n-1}(T_i\setminus T_i^\varepsilon) + O(\varepsilon)$$
$$= \int_\Omega |\nabla u|\,dx + \sum_{i=1}^N \int_{T_i} \psi(x_{z_i})|u^+ - u^-|(y)\,d\mathcal{H}^{n-1}(y) + O(\varepsilon). \tag{4.17}$$

Letting, now, ε tend to 0^+, we obtain, by (4.16) and (4.17),

$$\begin{aligned}
G''(u,1) &\le \limsup_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon)\\
&\le \int_\Omega |\nabla u|\,dx + \sum_{i=1}^N \int_{T_i} (|u^+ - u^-|(y)\psi(x_{z_i}) + 2c_W(x_{z_i}))\,d\mathcal{H}^{n-1}(y) + c\eta\\
&\le \int_\Omega |\nabla u|\,dx + \sum_{i=1}^N \int_{T_i} (z_i\psi(x_{z_i}) + 2c_W(x_{z_i}))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta)\\
&= \int_\Omega |\nabla u|\,dx + \int_{S_u} g(h_\delta(y))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta)\\
&\le \int_\Omega |\nabla u|\,dx + \int_{S_u} g(|u^+ - u^-|(y))\,d\mathcal{H}^{n-1}(y) + c(\eta+\delta).
\end{aligned}$$

Letting η and δ tend to 0^+, we obtain the required inequality.

In order to use the same construction as above in the case $\overline{S}_u = \Omega \cap \left(\bigcup_{i=1}^M K_i\right)$, with $M > 1$, we now show that we can replace (u_ε) by a new sequence $(\hat{u}_\varepsilon)$ such that $\hat{u}_\varepsilon \neq u$ only in a small neighbourhood of K. To this end we again use a cut-off argument. Set

$$K_\varepsilon := \{y \in \Pi : d(y,K) < \varepsilon\}$$

and let $\phi_\varepsilon : \mathbf{R}^{n-1} \to \mathbf{R}$ be a cut-off function between K and K_ε with $|\nabla\phi_\varepsilon|_\infty \le c\varepsilon^{-1}$. Define

$$\hat{u}_\varepsilon(y,t) := \phi_\varepsilon(y)u_\varepsilon(y,t) + (1-\phi_\varepsilon(y))u(y,t) \qquad (y,t)\in\Omega.$$

We have

$$\begin{aligned}
\hat{u}_\varepsilon(y,t) &= u_\varepsilon(y,t) \ \text{ if } (y,t)\in B_{T_\varepsilon},\\
\hat{u}_\varepsilon(y,t) &= u(y,t) \ \text{ if } (y,t)\in \Omega\setminus K_\varepsilon\times(-T_\varepsilon, T_\varepsilon).
\end{aligned} \tag{4.18}$$

Then

$$\int_{\Omega\setminus B_{T_\varepsilon}} |\nabla\hat{u}_\varepsilon|\,dx \le \int_{\Omega\setminus K_\varepsilon\times(-T_\varepsilon,T_\varepsilon)} |\nabla u|\,dx$$
$$+ \int_{\Omega\cap(K_\varepsilon\setminus K)} \int_{-T_\varepsilon}^{T_\varepsilon} \Big(|\nabla\phi_\varepsilon(y)||u_\varepsilon(y,t) - u(y,t)|\Big)\,dt\,d\mathcal{H}^{n-1}(y)$$

$$+\int_{\Omega\cap(K_\varepsilon\setminus K)}\int_{-T_\varepsilon}^{T_\varepsilon}\Big(\phi_\varepsilon(y)|\nabla u_\varepsilon(y,t)|$$
$$+(1-\phi_\varepsilon(y))|\nabla u(y,t)|\Big)\,dt\,d\mathcal{H}^{n-1}(y)$$
$$\leq\int_\Omega|\nabla u|\,dx+c\frac{T_\varepsilon}{\varepsilon}\mathcal{H}^{n-1}(K_\varepsilon\setminus K)+O(\varepsilon).$$

Thus

$$\limsup_{\varepsilon\to0^+}\int_{\Omega\setminus B_{T_\varepsilon}}|\nabla\hat{u}_\varepsilon|\,dx=\int_\Omega|\nabla u|\,dx,$$

and, by (4.18), we still have

$$\limsup_{\varepsilon\to0^+}G_\varepsilon(\hat{u}_\varepsilon,v_\varepsilon)\leq G(u,1)+c(\eta+\delta).$$

Step 2. If $u\in\mathcal{W}(\Omega)$ with $\overline{S}_u=\Omega\cap\left(\bigcup_{i=1}^M K_i\right)$ and $K_i\cap K_j=$ if $i\neq j$, we can generalize in a very natural way the construction of the recovery sequences $\hat{u}_\varepsilon$ and v_ε in Step 1, since this construction modifies u and v only in a small neighbourhood of each sets K_i.

Step 3. Let $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$. Then, applying Theorem 1.18 with $\phi(a,b,\nu)=g(|a-b|)$ and taking into account Remark 1.19, there exists a sequence (w_j) as in Step 2 such that

$$w_j\to u \text{ in } L^1(\Omega), \text{ and } \limsup_j G(w_j,1)\leq G(u,1).$$

Then, by the previous steps and by the lower semicontinuity of G''

$$G''(u,1)\leq\liminf_j G''(w_j,1)\leq\liminf_j G(w_j,1)\leq G(u,1).$$

Step 4. Since g satisfies the hypotheses of Theorem 1.20, the relaxation with respect to $L^1(\Omega)$-topology of the functional

$$F(u):=\begin{cases}G(u,1) & \text{if } u\in SBV^2(\Omega)\cap L^\infty(\Omega)\\ +\infty & \text{otherwise in } BV(\Omega)\end{cases}$$

is given by

$$\overline{F}(u)=G(u,1)$$

for all $u\in BV(\Omega)$. Then by the previous steps and by the lower semicontinuity of G'' we get

$$G''(u,1)\leq\overline{F}(u)=G(u,1)$$

for any $u\in BV(\Omega)$.

Step 5. We recover the general case by a truncation argument. Let $u \in GBV(\Omega)$ and let $u_j = (-j) \vee (u \wedge j)$. Then

$$\lim_j G(u_j, 1) = G(u, 1).$$

Since $u_j \to u$ in $L^1(\Omega)$ we get the thesis by the lower semicontinuity of G''. □

Example 4.6 We illustrate with a few simple examples the behaviour of the function g, given by (4.1), with different choices of ψ.

Let $W(v) = (1-v)^2/4$, so that $c_W(t) = (1-t)^2/2$. We then have

(a) if $\psi(v) = v^2$ then $g(z) = |z|/(1+|z|)$;

(b) if $\psi(v) = v$ then $g(z) = \begin{cases} |z| - (z^2/4) & \text{if } |z| \le 2 \\ 1 & \text{if } |z| > 2; \end{cases}$

(c) if $\psi(v) = \begin{cases} 0 & \text{if } v = 0 \\ 1 & \text{otherwise,} \end{cases}$ then $g(z) = \min\{|z|, 1\}$.

We see that the 'bulk term' and of the 'surface term' (i.e. the first and the second term in (4.1)) play different roles in these examples. Note that in (a) we always have interaction between these two terms i.e. both terms contribute to the value $g(z)$ contrary to what happens in the Ambrosio Tortorelli case. The interaction also occurs in (b) for $|z| < 2$. Note moreover that in the third case the minimal t in the definition of $g(z)$ does not vary with continuity at $z = 1$.

4.2 Approximation of general functionals

In this section we show how Theorem 4.1 can be used to obtain an approximation of general (isotropic) energies defined on GSBV and GBV by a double limit. The set Ω will be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary.

Proposition 4.7 *Let W and ψ be defined as in Theorem 4.1, let $f : [0, +\infty) \to [0, +\infty)$ be a convex and increasing function satisfying*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = 1, \tag{4.19}$$

and let $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty)$ be defined by

$$G_\varepsilon(u, v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi(v) f(|\nabla u|) + \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2\Big)\, dx & \text{if } u, v \in H^1(\Omega) \\ & \text{and } 0 \le v \le 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the Γ-$\lim_{\varepsilon \to 0+} G_\varepsilon(u, v) = G(u, v)$ with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence, where

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)dx + \int_{S_u} g(|u^+ - u^-|)d\mathcal{H}^{n-1} + |D^c u|(\Omega) & \\ \qquad\qquad \textit{if } u \in GBV(\Omega) \textit{ and } v = 1 \textit{ a.e.} \\ +\infty & \textit{otherwise,} \end{cases}$$

and g is defined in (4.1).

Proof The estimate for the Γ-lim inf can be performed as in Proposition 4.3, noting that in (4.5) we obtain, by Jensen's inequality,

$$G_{\varepsilon_j}(u_j, v_j, (t_i - \eta, t_i + \eta)) \geq \psi(v_j(x_j^i))|x_2 - x_1| f\Big(\frac{u(x_2) - u(x_1)}{x_2 - x_1}\Big) + 2\int_{x_1}^{x_2} \sqrt{W(v_j)}|v_j'|\, dx,$$

from which the lower bound can be easily obtained taking into account (4.19). The rest of the proof can be obtained following Propositions 4.4 and 4.5. □

Remark 4.8 Let $K > 0$ and $N \geq 2$, let

$$0 = a_0 < a_1 < \cdots < a_N = 1, \qquad 0 = b_N < b_{N-1} \cdots < b_0 = K,$$

and let f and W be as in the previous proposition. Then there exists ψ satisfying the hypotheses in Theorem 4.1 such that, if $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty)$ is defined by

$$G_\varepsilon(u,v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi(v) f(|\nabla u|) + \frac{K}{\varepsilon} W(v) + \varepsilon K |\nabla v|^2\Big)\, dx & \text{if } u, v \in H^1(\Omega) \\ & \text{and } 0 \leq v \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

then the thesis of the previous proposition holds with $g : [0, +\infty) \to [0, +\infty)$ given by

$$g(z) = \min\{a_i z + b_i\}.$$

In fact, in this case the formula for g can be easily inverted, obtaining ψ as the piecewise constant function given by $\psi(0) = 0$ and

$$\psi(\xi) = a_i \qquad \text{if } c_W^{-1}(b_{i-1}/2) < \xi \leq c_W^{-1} l(b_i/2),$$

where c_W is defined in Theorem 4.1.

Proposition 4.9 *Let W be as in Theorem* 4.1. *Let $\varphi, \vartheta : [0, +\infty) \to [0, +\infty)$ be functions satisfying*

(i) *φ is convex and increasing,* $\lim_{t \to +\infty} \varphi(t)/t = +\infty$*;*

(ii) ϑ *is concave,* $\lim_{t\to 0^+}\vartheta(t)/t=+\infty$.

Then there exist two increasing sequences of functions (φ_j) *and* (ψ_j), *and a sequences of positive real numbers* (k_j), *converging to* $\sup\vartheta$, *such that if we define*

$$G^j_\varepsilon(u,v)=\begin{cases}\displaystyle\int_\Omega\Big(\psi_j(v)\varphi_j(|\nabla u|)+\frac{k_j}{\varepsilon}W(v)+k_j\varepsilon|\nabla v|^2\Big)\,dx & \text{if } u,v\in H^1(\Omega)\\ & \text{and } 0\le v\le 1 \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}\tag{4.20}$$

then for every $j\in\mathbb{N}$ *there exist the limits*

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+}G^j_\varepsilon(u,v)=:G^j(u,v)$$
$$\Gamma\text{-}\lim_{j\to+\infty}G^j(u,v)=\lim_{j\to+\infty}G^j(u,v)=G(u,v)$$

with respect to the $L^1(\Omega)\times L^1(\Omega)$*-convergence, where*

$$G(u,v)=\begin{cases}\displaystyle\int_\Omega\varphi(|\nabla u|)\,dx+\int_{S_u}\vartheta(|u^+-u^-|)d\mathcal{H}^{n-1} & \text{if } u\in GSBV(\Omega)\\ & \text{and } v=1 \text{ a.e.}\\ +\infty & \text{otherwise.}\end{cases}$$

Proof Let $\vartheta_j:[0,+\infty)\to[0,+\infty)$ be functions of the form

$$\vartheta_j(z)=\min\{A^j_i z+B^j_i\},$$

with $0=A^j_0<\cdots<A^j_j=j$ converging increasingly to ϑ, and let $\varphi_j:[0,+\infty)\to[0,+\infty)$ be convex increasing functions with

$$\lim_{t\to+\infty}\frac{\varphi_j(t)}{t}=j,$$

converging increasingly to φ. Let $k_j=\max\vartheta_j$.

Set $g_j=\vartheta_j/j$, $K_j=k_j/j$ and $f_j=\varphi_j/j$. By the previous remark, applied with $g=g_j$, $f=f_j$ and $K=K_j$, we can find $\psi=:\psi_j$ such that if we let $G^j_\varepsilon:L^1(\Omega)\times L^1(\Omega)\to[0,+\infty)$ be defined by (4.20) then there exists the $\Gamma\text{-}\lim_{\varepsilon\to0^+}G^j_\varepsilon(u,v)=G^j(u,v)$ with respect to the $L^1(\Omega)\times L^1(\Omega)$-convergence, where

$$G^j(u,v)=\begin{cases}\displaystyle\int_\Omega\varphi_j(|\nabla u|)dx+\int_{S_u}\vartheta_j(|u^+-u^-|)d\mathcal{H}^{n-1}+j|D^cu|(\Omega) & \\ & \text{if } u\in GBV(\Omega) \text{ and } v{=}1 \text{ a.e.}\\ +\infty & \text{otherwise.}\end{cases}$$

Since the functionals G^j converge increasingly to G, they also Γ-converge to G as $j \to +\infty$. □

Remark 4.10 If φ is convex and even, ϑ is concave and even, and

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = \lim_{t \to o^+} \frac{\vartheta(t)}{t} = M,$$

then there exist (φ_j), (ψ_j) and (k_j) such that the functionals G^j defined above Γ-converge with respect to the $L^1(\Omega) \times L^1(\Omega)$-convergence to

$$G(u,v) = \begin{cases} \displaystyle\int_\Omega \varphi(|\nabla u|)dx + \int_{S_u} \vartheta(|u^+ - u^-|)d\mathcal{H}^{n-1} + M|D^c u|(\Omega) & \\ \qquad\qquad\qquad\qquad \text{if } u \in GBV(\Omega) \text{ and } v = 1 \text{ a.e.} & \\ +\infty \qquad\qquad \text{otherwise.} & \end{cases}$$

The proof can be obtained directly from Remark 4.8, using the approximation argument of Proposition 4.9.

5

FINITE DIFFERENCE APPROXIMATIONS IN *SBD*

In this chapter we provide a variational approximation in dimension 2 of functionals of the type

$$\mu \int_\Omega |\mathcal{E}u(x)|^2\,dx + \frac{\lambda}{2}\int_\Omega |\operatorname{div} u(x)|^2\,dx + \gamma\mathcal{H}^2(J_u) \tag{5.1}$$

defined on the space $SBD(\Omega)$, where $\Omega \subset \mathbf{R}^2$ is a bounded open set.

To this end, we push further the approach outlined in Section 2.2.3, by "symmetrizing" the effect of the difference quotient and introducing in the model a suitable discretization of the divergence. In the proof of the Γ-liminf inequality, we cannot apply the method described in Chapter 2, due to the presence of the divergence term. Instead we use a discretization argument, that leads us to study the limiting behaviour of families of discrete functionals. The proof of the Γ-limsup inequality will be consequence of a pointwise convergence result.

The results of this chapter are contained in [5].

5.1 The main result

We introduce first a discretization of the divergence. If $\xi = (\xi^1, \xi^2) \in \mathbf{R}^2$, we denote by $\xi^\perp$ the vector in $\mathbf{R}^2$ orthogonal to ξ defined by $\xi^\perp := (-\xi^2, \xi^1)$. Fix $\xi, \zeta \in \mathbf{R}^2 \setminus \{0\}$; for $\varepsilon > 0$ and for any $u : \mathbf{R}^2 \to \mathbf{R}^2$ define

$$\begin{aligned} &D_\varepsilon^\xi u(x) := \langle u(x+\varepsilon\xi) - u(x), \xi\rangle, \\ &\operatorname{div}_\varepsilon^{\xi,\zeta} u(x) := D_\varepsilon^\xi u(x) + D_\varepsilon^\zeta u(x), \\ &|D_{\varepsilon,\xi} u(x)|^2 := |D_\varepsilon^\xi u(x)|^2 + |D_\varepsilon^{-\xi} u(x)|^2, \\ &|\operatorname{Div}_{\varepsilon,\xi} u(x)|^2 := |\operatorname{div}_\varepsilon^{\xi,\xi^\perp} u(x)|^2 + |\operatorname{div}_\varepsilon^{\xi,-\xi^\perp} u(x)|^2 \\ &\qquad\qquad + |\operatorname{div}_\varepsilon^{-\xi,\xi^\perp} u(x)|^2 + |\operatorname{div}_\varepsilon^{-\xi,-\xi^\perp} u(x)|^2. \end{aligned} \tag{5.2}$$

We underline that this is only one possible definition of discretized divergence that seems to agree with mechanical models of neighbouring atomic interactions.

We can give also the following alternative definition

$$\begin{aligned} &D_\varepsilon^\xi u(x) := \langle u(x+\varepsilon\xi) - u(x-\varepsilon\xi), \xi\rangle, \\ &|D_{\varepsilon,\xi} u(x)|^2 := \frac{1}{2}|D_\varepsilon^\xi u(x)|^2 \\ &|\operatorname{Div}_{\varepsilon,\xi} u(x)|^2 := |D_\varepsilon^\xi u(x) + D_\varepsilon^{\xi^\perp} u(x)|^2. \end{aligned} \tag{5.3}$$

This second definition can be motivated by the fact that from a numerical point of view it gives better approximations of the divergence as $\varepsilon \to 0$.

In the definition of the approximating functionals in the sequel we will implicitly mean that one among definitions (5.2) and (5.3) is used. We remark that the choice of one or the other definition does not affect the convergence results.

Let Ω be a bounded open set of $\mathbf{R}^2$ and let $f : [0,+\infty) \to [0,+\infty)$ be an increasing function, such that $a, b > 0$ exist with

$$a := \lim_{t\to 0+} \frac{f(t)}{t}, \quad b := \lim_{t\to+\infty} f(t) \tag{5.4}$$

and $f(t) \leq (at) \wedge b$ for any $t \geq 0$. Moreover, for any $\xi \in \mathbf{R}^2$ let $\rho(\xi) = \psi(|\xi|)$, where $\psi : [0,+\infty) \to [0,+\infty)$ is such that for some $M > 0$ $\operatorname{ess-inf}_{|t|\leq M} \psi(t) > 0$ and $\int_0^{+\infty} t^5 \psi(t)\, dt < +\infty$.

For $\varepsilon > 0$, define $F_\varepsilon : L^1(\Omega; \mathbf{R}^2) \to [0, +\infty]$ as

$$F_\varepsilon(u) := \int_{\mathbf{R}^2} \rho(\xi) \mathcal{F}_\varepsilon^\xi(u)\, d\xi,$$

where

$$\mathcal{F}_\varepsilon^\xi(u) := \frac{1}{\varepsilon} \int_{\Omega_\varepsilon^\xi} f\left(\frac{1}{\varepsilon} \left(|D_{\varepsilon,\xi} u(x)|^2 + \theta |\mathrm{Div}_{\varepsilon,\xi} u(x)|^2 \right) \right) dx \tag{5.5}$$

with $\theta > 0$ and

$$\Omega_\varepsilon^\xi := \{ x \in \mathbf{R}^2 : [x - \varepsilon\xi, x + \varepsilon\xi] \cup [x - \varepsilon\xi^\perp, x + \varepsilon\xi^\perp] \subset \Omega \},$$

where we denote by $[x, y]$ the segment between x and y.

Theorem 5.1 *F_ε Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ with respect to the $L^1(\Omega;\mathbf{R}^2)$-convergence to the functional $F : L^\infty(\Omega;\mathbf{R}^2) \to [0,+\infty]$ given by*

$$F(u) := \begin{cases} \mu \int_\Omega |\mathcal{E}u(x)|^2\, dx + \lambda \int_\Omega |\operatorname{div} u(x)|^2\, dx + \gamma \mathcal{H}^1(J_u) & \\ & \text{if } u \in SBD(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mu := a \int_{\mathbf{R}^2} \rho(y) \left(|y|^4 - 4y_1^2 y_2^2 \right) dy,$$
$$\lambda := a \int_{\mathbf{R}^2} \rho(y) \left(4\theta |y|^4 + 2y_1^2 y_2^2 \right) dy,$$
$$\gamma := 2b \int_{\mathbf{R}^2} \rho(y) \left(|y_1| \vee |y_2| \right) dy.$$

Moreover, F_ε converges to F pointwise on $L^\infty(\Omega;\mathbf{R}^2)$.

The proof of the theorem above will be consequence of propositions in Sections 5.3 and 5.4.

Remark 5.2 Notice that $\mu = a\int_{\mathbf{R}^2}\rho(y)\left(y_1^2-y_2^2\right)^2\,dy$, so that μ, λ and γ are all positive. Moreover, the summability assumption on ψ easily yields the finiteness of such constants.

Remark 5.3 Notice that the domain of F is $L^\infty(\Omega;\mathbf{R}^2)\cap SBD^2(\Omega)$.

Remark 5.4 We underline that for any positive coefficients μ, λ and γ, we can choose f,ρ and θ such that the limit functional has the form

$$\mu\int_\Omega|\mathcal{E}u(x)|^2\,dx+\lambda\int_\Omega|\operatorname{div}u(x)|^2\,dx+\gamma\mathcal{H}^1(J_u).$$

By dropping the divergence term in (5.5)(i.e. $\theta=0$), one can consider the sequence of functionals $G_\varepsilon : L^1(\Omega;\mathbf{R}^2)\to[0,+\infty]$ defined by

$$G_\varepsilon(u):=\int_{\mathbf{R}^2}\rho(\xi)\frac{1}{\varepsilon}\int_{\widetilde{\Omega}_\varepsilon^\xi}f\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(x)|^2\right)\,dx\,d\xi$$

with $\widetilde{\Omega}_\varepsilon^\xi := \{x\in\mathbf{R}^2 : [x-\varepsilon\xi,x+\varepsilon\xi]\subset\Omega\}$ and ρ as above. By following the same procedure summarized in Section 2.2.3 to prove Theorem 2.3, it can be proved the following result.

Theorem 5.5 *G_ε Γ-converges on $L^\infty(\Omega;\mathbf{R}^2)$ with respect to the $L^1(\Omega;\mathbf{R}^2)$-convergence to the functional $G : L^\infty(\Omega;\mathbf{R}^2)\to[0,+\infty]$ given by*

$$G(u):=\begin{cases}\mu'\int_\Omega|\mathcal{E}u(x)|^2\,dx+\lambda'\int_\Omega|\operatorname{div}u(x)|^2\,dx\;+\;\gamma'\mathcal{H}^1(J_u) & \\ & \text{if } u\in SBD(\Omega)\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$\mu' := a\int_{\mathbf{R}^2}\rho(y)\left(|y|^4-4y_1^2y_2^2\right)\,dy,$$
$$\lambda' := 2a\int_{\mathbf{R}^2}\rho(y)y_1^2y_2^2\,dy,$$
$$\gamma' := 2b\int_{\mathbf{R}^2}\rho(y)|y_1|\,dy.$$

Remark 5.6 Notice that, although the definition of G_ε corresponds in some sense to taking $\theta=0$ in (5.5), its Γ-limit G is not equal to F for $\theta=0$.

Remark 5.7 The restriction to $L^\infty(\Omega;\mathbf{R}^2)$ in Theorems 5.1 and 5.5 is technical in order to characterize the Γ-limit. For a function u in $L^1(\Omega;\mathbf{R}^2)\setminus L^\infty(\Omega;\mathbf{R}^2)$, by following the procedure of the proof of Proposition 5.10 below, one can deduce from the finiteness of the Γ-limits that the one dimensional sections of u belong to $SBV(\Omega_{\xi y})$. Anyway, since condition (1.18) is not in general satisfied, one cannot conclude that $u\in SBD(\Omega)$. On the other hand this condition is satisfied if $u\in BD(\Omega)$, so that Theorems 5.1 and 5.5 still hold if we replace $L^\infty(\Omega;\mathbf{R}^2)$ by $BD(\Omega)$.

5.2 Preliminary Lemmas

In this section we state and prove some preliminary results, that will be used in the sequel.

Let $\mathcal{B}:=\{\xi_1,\dots,\xi_n\}$ an orthogonal basis of $\mathbf{R}^n$. Then for any measurable function $u:\mathbf{R}^n\to\mathbf{R}^n$ and $y\in\mathbf{R}^n$ define

$$T_y^{\varepsilon,\mathcal{B}}u(x):=u\left(\varepsilon y+\varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right)$$

where $[z]_{\mathcal{B}}:=\sum\limits_{i=1}^{n}\left[\frac{\langle z,\xi_i\rangle}{|\xi_i|^2}\right]\xi_i$.

Notice that $T_y^{\varepsilon,\mathcal{B}}u$ is constant on each cell $\alpha+\varepsilon Q_{\mathcal{B}}$, $\alpha\in\varepsilon\bigoplus\limits_{i=1}^{n}\xi_i\mathbf{Z}$, where $Q_{\mathcal{B}}:=\{x\in\mathbf{R}^n:0<\langle x,\xi_i\rangle\le|\xi_i|^2\}$. The following result generalizes Lemma 3.36 in [23].

Lemma 5.8 *Let $u_\varepsilon\to u$ in $L^1_{loc}(\mathbf{R}^n;\mathbf{R}^n)$, then $T_y^{\varepsilon,\mathcal{B}}u_\varepsilon\to u$ in $L^1_{loc}(\mathbf{R}^n;\mathbf{R}^n)$ for a.e. $y\in Q_{\mathcal{B}}$.*

Proof For the sake of simplicity we assume $\mathcal{B}=\{e_1,\dots,e_n\}$. It suffices to prove that for any compact set K of $\mathbf{R}^n$

$$\lim_{\varepsilon\to0}\int_{(0,1)^n}\int_K\left|u_\varepsilon\left(\varepsilon y+\varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right)-u(x)\right|\,dx\,dy=0. \tag{5.6}$$

Then fix K and call I_ε the double integral in (5.6). By Fubini's Theorem and the change of variable $\varepsilon y+\varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\to y$ we get

$$\begin{aligned}
I_\varepsilon&=\int_K\int_{(0,1)^n}\left|u_\varepsilon\left(\varepsilon y+\varepsilon\left[\frac{x}{\varepsilon}\right]_{\mathcal{B}}\right)-u(x)\right|\,dy\,dx\\
&\le\int_K\frac{1}{\varepsilon^n}\int_{x+\varepsilon(0,1)^n}|u_\varepsilon(y)-u(x)|\,dy\,dx\\
&\le\int_K\frac{1}{\varepsilon^n}\int_{x+\varepsilon(0,1)^n}\left(|u_\varepsilon(y)-u_\varepsilon(x)|+|u_\varepsilon(x)-u(x)|\right)\,dy\,dx.
\end{aligned}$$

The further change of variable $y\to x+\varepsilon y$ and Fubini's Theorem yield

$$I_\varepsilon \le \int_{(0,1)^n} \int_K |u_\varepsilon(x+\varepsilon y) - u_\varepsilon(x)|\, dx\, dy + \int_K |u_\varepsilon(x) - u(x)|\, dx,$$

thus the conclusion follows by the uniform continuity of the translation operator for strongly converging sequences in $L^1_{loc}(\mathbf{R}^n;\mathbf{R}^n)$. □

For $\xi \in \mathbf{R}^2 \setminus \{0\}$ and $\mathcal{B} = \{\xi, \xi^\perp\}$, we will denote the operators $T_y^{\varepsilon,\mathcal{B}}$ and $[\cdot]_\mathcal{B}$ by $T_y^{\varepsilon,\xi}$ and $[\cdot]_\xi$, respectively.

Lemma 5.9 *Let J be a countably $\mathcal{H}^{n-1}$-rectifiable set and define*

$$J_\varepsilon^\xi := \{x \in \mathbf{R}^n : x = y + t\xi \text{ with } t \in (-\varepsilon, \varepsilon) \text{ and } y \in J\} \tag{5.7}$$

for $\xi \in \mathbf{R}^n$ and

$$J_\varepsilon^{\xi_1,\ldots,\xi_r} := \bigcup_{i=1}^r J_\varepsilon^{\xi_i} \tag{5.8}$$

for $\xi_1, \ldots, \xi_r \in \mathbf{R}^n$, r being a positive integer. Then, if $\mathcal{H}^{n-1}(J) < +\infty$

$$\limsup_{\varepsilon \to 0} \frac{\mathcal{L}^n\left(J_\varepsilon^{\xi_1,\ldots,\xi_r}\right)}{\varepsilon} \le 2 \int_J \sup_i |\langle \nu, \xi_i \rangle|\, d\mathcal{H}^{n-1}, \tag{5.9}$$

where $\nu(x)$ is the unitary normal vector to J at x.

Proof First note that by Fubini's Theorem and the Generalized Coarea Formula (see [14])

$$\mathcal{L}^n\left(J_\varepsilon^\xi\right) \le 2\varepsilon \int_{\Pi_\xi} \#\left(J_\varepsilon^\xi\right)_{\xi y} d\mathcal{H}^{n-1}(y) = 2\varepsilon \int_J |\langle \nu, \xi \rangle|\, d\mathcal{H}^{n-1},$$

hence

$$\mathcal{L}^n\left(J_\varepsilon^{\xi_1,\ldots,\xi_r}\right) \le 2\varepsilon \int_J \sum_{i=1}^r |\langle \nu, \xi_i \rangle|\, d\mathcal{H}^{n-1} \le 2r\varepsilon \sup_i |\xi_i| \mathcal{H}^{n-1}(J). \tag{5.10}$$

By the very definition of rectifiability there exist countably many compact subsets K_i of C^1 graphs such that

$$\mathcal{H}^{n-1}\left(J \setminus \bigcup_{i\ge 1} K_i\right) = 0,$$

and $\mathcal{H}^{n-1}(K_i \cap K_j) = 0$ for $i \ne j$. Thus, by (5.10) for any $M \in \mathbf{N}$ we have

$$\frac{\mathcal{L}^n\left(J_\varepsilon^{\xi_1,\ldots,\xi_r}\right)}{\varepsilon} \le \sum_{1\le i\le M} \frac{\mathcal{L}^n\left((K_i)_\varepsilon^{\xi_1,\ldots,\xi_r}\right)}{\varepsilon} + 2r \sup_i |\xi_i| \mathcal{H}^{n-1}\left(J \setminus \bigcup_{1\le i\le M} K_i\right),$$

hence, first letting $\varepsilon \to 0$ and then $M \to +\infty$ it follows

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{L}^n\left(J_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon} \leq \sum_{i\geq 1} \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^n\left((K_i)_\varepsilon^{\xi_1,..,\xi_r}\right)}{\varepsilon}.$$

Thus, it suffices to prove (5.9) for J compact subset of a C^1 graph. Up to an outer approximation with open sets we may assume J open. Furtherly, splitting J into its connected components, we can reduce ourselves to prove the inequality for J connected. For such a J (5.9) follows by an easy computation. □

5.3 Estimate from below

We "localize" the functionals $\mathcal{F}_\varepsilon^\xi$ as

$$\mathcal{F}_\varepsilon^\xi(u,A) := \frac{1}{\varepsilon}\int_{A_\varepsilon^\xi} f\left(\frac{1}{\varepsilon}\left(|D_{\varepsilon,\xi}u(x)|^2 + \theta|\mathrm{Div}_{\varepsilon,\xi}u(x)|^2\right)\right)\,dx,$$

for any $u \in L^1(\Omega;\mathbf{R}^2)$, $A \in \mathcal{A}(\Omega)$, with

$$A_\varepsilon^\xi := \{x \in \mathbf{R}^2 : [x-\varepsilon\xi, x+\varepsilon\xi] \cup [x-\varepsilon\xi^\perp, x+\varepsilon\xi^\perp] \subset A\}.$$

Proposition 5.10 *For any $u \in L^\infty(\Omega;\mathbf{R}^2)$,*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \geq F(u).$$

Proof We assume for the proof that definitions (5.2) hold. A simple adaptation of the argument we are going to use can be applied to recover the proof in the case that definitions (5.3) hold.

Step 1 Let us first assume $f(t) = (at) \wedge b$. Let $\varepsilon_j \to 0$, $u_j \in L^1(\Omega;\mathbf{R}^2)$, $u \in L^\infty(\Omega;\mathbf{R}^2)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbf{R}^2)$ and $\liminf_j F_{\varepsilon_j}(u_j) = \lim_j F_{\varepsilon_j}(u_j) < +\infty$. In particular for a.e. $\xi \in \mathbf{R}^2$ such that $\rho(\xi) \neq 0$ we have that $\liminf_j \mathcal{F}_{\varepsilon_j}^\xi(u_j) < +\infty$. Fix such a $\xi \in \mathbf{R}^2$ and $A \in \mathcal{A}(\Omega)$. Up to passing to a subsequence we may assume that $\liminf_j \mathcal{F}_{\varepsilon_j}^\xi(u_j,A) = \lim_j \mathcal{F}_{\varepsilon_j}^\xi(u_j,A) < +\infty$. We now adapt to our case a "discretization" argument used in the proof of Proposition 3.38 of [23]. If we define

$$g_j(x) := \begin{cases} f\left(\frac{1}{\varepsilon_j}\left(|D_{\varepsilon_j,\xi}u(x)|^2 + \theta|\mathrm{Div}_{\varepsilon_j,\xi}u(x)|^2\right)\right) & \text{if } x \in A_{\varepsilon_j}^\xi \\ 0 & \text{otherwise in } \mathbf{R}^2, \end{cases}$$

we can write

$$\mathcal{F}_{\varepsilon_j}^\xi(u_j,A) = \frac{1}{\varepsilon_j}\int_{\mathbf{R}^2} g_j(x)\,dx = \frac{1}{\varepsilon_j}\sum_{\alpha\in\varepsilon_j(\mathbf{Z}\xi\oplus\mathbf{Z}\xi^\perp)}\int_{\alpha+\varepsilon_j\widetilde{Q}_\xi} g_j(x)\,dx$$

$$= \frac{1}{\varepsilon_j} \sum_{\alpha \in \varepsilon_j(\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp)} \int_{\varepsilon_j \widetilde{Q}_\xi} g_j(x+\alpha)\,dx = \int_{\widetilde{Q}_\xi} \phi_j(x)\,dx,$$

where

$$\phi_j(x) := \sum_{\alpha \in \varepsilon_j(\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp)} \varepsilon_j g_j(\varepsilon_j x + \alpha),$$

$$\widetilde{Q}_\xi := \{x \in \mathbf{R}^2 : 0 \le \langle x, \xi\rangle < |\xi|^2, 0 \le \langle x, \xi^\perp\rangle < |\xi^\perp|^2\}.$$

Fix $\eta > 0$ and set

$$C^j_{\eta,\xi} := \left\{ x \in \widetilde{Q}_\xi : |\xi|^2 \phi_j(x) \le \mathcal{F}^\xi_{\varepsilon_j}(u_j, A) + \eta \right\}.$$

Then

$$|\widetilde{Q}_\xi \setminus C^j_{\eta,\xi}| \le |\xi|^2 \frac{\mathcal{F}^\xi_{\varepsilon_j}(u_j, A)}{\mathcal{F}^\xi_{\varepsilon_j}(u_j, A) + \eta} \le c < |\xi|^2.$$

Consider now u_j and u extended to 0 outside Ω and apply Lemma 5.8. By Egoroff's Theorem, there exists a measurable set B in $\widetilde{Q}_\xi$ with $|B| < \frac{|\xi|^2 - c}{2}$ such that $T^{\varepsilon_j,\xi}_x u_j \to u$ in $L^1(\Omega; \mathbf{R}^2)$ uniformly with respect to $x \in \widetilde{Q}_\xi \setminus B$. Thus for any $j \in \mathbf{N}$ we can choose $x_j \in C^j_{\eta,\xi} \setminus B$ such that

$$T^{\varepsilon_j,\xi}_{x_j} u_j \to u \text{ in } L^1(\Omega; \mathbf{R}^2) \tag{5.11}$$

and

$$\begin{aligned} &\mathcal{F}^\xi_{\varepsilon_j}(u_j, A) + \eta \ge |\xi|^2 \phi_j(x_j) \\ &\ge |\xi|^2 \varepsilon_j \sum_{\substack{\alpha \in \varepsilon_j(\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp) \\ \alpha \in A^\xi_{\varepsilon_j} - \varepsilon_j x_j}} f\left(\frac{1}{\varepsilon_j} \left(|D_{\varepsilon_j,\xi} u(\alpha + \varepsilon_j x_j)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u(\alpha + \varepsilon_j x_j)|^2 \right) \right). \end{aligned} \tag{5.12}$$

We can suppose for the sake of notation that $x_j = 0$ for all j. Notice, now, that we can split the lattice $\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp$ into an union of disjoint sub-lattices as

$$\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp = Z^\xi \cup (Z^\xi + \xi) \cup (Z^\xi + \xi^\perp) \cup (Z^\xi + (\xi + \xi^\perp))$$

where $Z^\xi := 2\mathbf{Z}\xi \oplus 2\mathbf{Z}\xi^\perp$. We confine, then, our attention to the sequence

$$\mathcal{F}_j(A) := \sum_{\alpha \in Z_j(A)} \varepsilon_j f\left(\frac{1}{\varepsilon_j} \left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2 \right) \right)$$

where $Z_j(A) := A^\xi_{\varepsilon_j} \cap \varepsilon_j Z^\xi$. Set

$$I_j := \left\{ \alpha \in \varepsilon_j \left(\mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp \right) \, : |D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2 > \frac{b}{a} \varepsilon_j \right\}$$

and let (v_j) be the sequence in $SBV(\Omega; \mathbf{R}^2)$, whose components are piecewise affine, uniquely determined by

$$\langle v_j(x), \xi \rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j \xi), \xi \rangle & x \in (\alpha + \varepsilon_j Q_\xi) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \cap I_j \\ \langle u_j(\alpha), \xi \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^\xi_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi \rangle & x \in (\alpha + \varepsilon_j Q_{\xi,+}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \\ \langle u_j(\alpha), \xi \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{-\xi}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi \rangle & x \in (\alpha + \varepsilon_j Q_{\xi,-}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \end{cases}$$

$$\langle v_j(x), \xi^\perp \rangle := \begin{cases} \langle u_j(\alpha - \varepsilon_j \xi^\perp), \xi^\perp \rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \cap I_j \\ \langle u_j(\alpha), \xi^\perp \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{\xi^\perp}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi^\perp \rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,+}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j \\ \langle u_j(\alpha), \xi^\perp \rangle + \frac{1}{\varepsilon_j |\xi|^2} D^{-\xi^\perp}_{\varepsilon_j} u_j(\alpha) \langle x - \alpha, \xi^\perp \rangle & \\ & x \in (\alpha + \varepsilon_j Q_{\xi^\perp,-}) \cap \Omega \\ & \alpha \in \varepsilon_j Z^\xi \setminus I_j, \end{cases}$$

where

$$Q_\xi := \{ x \in \mathbf{R}^2 \, : \, |\langle x, \xi \rangle| \le |\xi|^2, \, |\langle x, \xi^\perp \rangle| \le |\xi|^2 \}$$
$$Q_{\xi,\pm} := \{ x \in Q_\xi \, : \, \pm \langle x, \xi \rangle \ge 0 \}.$$

In order to clarify this construction, we note that, in the case $\xi = e_1$, $v_j = (v_j^1, v_j^2)$ is the sequence whose component v_j^i is piecewise affine along the direction e_i and piecewise constant along the orthogonal direction, for $i = 1, 2$.
It is easy to check that, by (5.11), v_j still converges to u in $L^1(\Omega; \mathbf{R}^2)$. Let us fix $r > 0$ and consider $A_r := \{ x \in A \, : \, \mathrm{dist}\,(x, \mathbf{R}^2 \setminus A) > r \}$. Note that, by construction, for j large we have

$$\sum_{\alpha \in Z_j(A) \setminus I_j} \left(|D_{\varepsilon_j,\xi} u_j(\alpha)|^2 + \theta |\mathrm{Div}_{\varepsilon_j,\xi} u_j(\alpha)|^2 \right)$$
$$\ge \frac{1}{2|\xi|^2} \int_{A_r} |\langle \mathcal{E} v_j(x) \xi, \xi \rangle|^2 \, dx + \theta |\xi|^2 \int_{A_r} |\mathrm{div}\, v_j(x)|^2 \, dx$$

and

$$\varepsilon_j \#\{Z_j(A)\cap I_j\}$$
$$\geq \frac{1}{2|\xi|^2}\max\left\{\int_{J^{\xi}_{v_j}\cap A_r}|\langle\nu_{v_j}(y),\xi\rangle|\,d\mathcal{H}^1(y),\int_{J^{\xi^\perp}_{v_j}\cap A_r}|\langle\nu_{v_j}(y),\xi^\perp\rangle|\,d\mathcal{H}^1(y)\right\}$$

Then, for j large and for any fixed $\delta\in[0,1]$,

$$\begin{aligned}\mathcal{F}_j(A)\geq&\sum_{\alpha\in Z_j(A)\setminus I_j} a\left(|D_{\varepsilon_j,\xi}u_j(\alpha)|^2+\theta|\mathrm{Div}_{\varepsilon_j,\xi}u_j(\alpha)|^2\right)\\&+b\varepsilon_j\#\{Z_j(A)\cap I_j\}\\\geq&\frac{a}{2|\xi|^2}\int_{A_r}|\langle\mathcal{E}v_j(x)\xi,\xi\rangle|^2\,dx+a\theta|\xi|^2\int_{A_r}|\mathrm{div}\,v_j(x)|^2\,dx\\&+\frac{b}{2|\xi|^2}\delta\int_{J^{\xi}_{v_j}\cap A_r}|\langle\nu_{v_j}(y),\xi\rangle|\,d\mathcal{H}^1(y)\\&+\frac{b}{2|\xi|^2}(1-\delta)\int_{J^{\xi^\perp}_{v_j}\cap A_r}|\langle\nu_{v_j}(y),\xi^\perp\rangle|\,d\mathcal{H}^1(y).\end{aligned}\tag{5.13}$$

In particular by applying a slicing argument and taking into account the notation used in Theorem 1.27, by Fatou's Lemma, we get

$$\begin{aligned}+\infty>&\liminf_j\mathcal{F}_j(A)\\\geq&\frac{1}{2|\xi|^2}\int_{\Pi_\xi}\liminf_j\left(a\int_{(A_r)_{\xi y}}|\dot{v}^{\xi,y}_j|^2\,dt+b\#\left(J\left(v^{\xi,y}_j\right)\right)\right)\,d\mathcal{H}^1(y).\end{aligned}\tag{5.14}$$

Note that, taking into account also the divergence term and the second surface term in (5.13), we can obtain an analog of the inequality (5.14) for $\xi^\perp$. By Theorem 1.22 and since $u\in L^\infty(\Omega;\mathbf{R}^2)$, we deduce that $u^{\zeta,y}\in SBV\big((A_r)_{\zeta_y}\big)$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi_\xi$ and

$$c\geq\int_{\Pi_\xi}|Du^{\zeta,y}|\big((A_r)_{\zeta y}\big)\,d\mathcal{H}^1(y)\tag{5.15}$$

for $\zeta=\xi,\xi^\perp$. Moreover, by the assumption on ρ we can choose ξ such that that $\rho(\xi)\neq 0,\rho(\xi+\xi^\perp)\neq 0$, thus we can repeat all the argument by replacing ξ by $\xi+\xi^\perp$ and obtain that (5.15) holds in particular for $\zeta=\xi,\xi^\perp,\xi+\xi^\perp$. Then by Theorem 1.27, we get that $u\in SBD(A_r)$ for any $r>0$. Moreover, since the estimate in (5.15) is uniform with respect to r, we conclude that $u\in SBD(A)$.

Going back to (5.13), by applying Theorem 1.29 and then letting $r\to 0$, we get

$$\liminf_j \mathcal{F}_j(A) \geq \frac{a}{2|\xi|^2}\int_A |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + a\theta|\xi|^2\int_A |\mathrm{div}\,u(x)|^2\,dx$$
$$+\frac{b}{2|\xi|^2}\Big(\delta\int_{J_u^\xi\cap A}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+(1-\delta)\int_{J_u^{\xi^\perp}\cap A}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big),$$

for any $\delta\in[0,1]$.

Note that, using the inequality above with $A=\Omega$ and taking into account that an analogous inequality holds by replacing ξ by $\xi+\xi^\perp$, it can be easily checked that $\mathcal{E}u\in L^2(\Omega;\mathbf{R}^{2\times 2})$ and $\mathcal{H}^1(J_u)<+\infty$. Then, by Proposition 1.8 applied with

$$\mu(A)=\liminf_j\mathcal{F}_j(A),$$
$$\lambda:=\frac{a}{2|\xi|^2}\mathcal{L}^2\llcorner\Omega+\frac{b}{2|\xi|^2}\mathcal{H}^1\llcorner J_u,$$
$$\psi_h(x)=\begin{cases}\left(|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2+\theta|\xi|^2|\mathrm{div}\,u(x)|^2\right) & \text{on }\Omega\setminus J_u\\[1ex] \delta_h|\langle\nu_u,\xi\rangle| & \text{on }J_u^\xi\setminus J_u^{\xi^\perp}\\[1ex] (1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on }J_u^{\xi^\perp}\setminus J_u^\xi\\[1ex] \delta_h|\langle\nu_u,\xi\rangle|+(1-\delta_h)|\langle\nu_u,\xi^\perp\rangle| & \text{on }J_u^\xi\cap J_u^{\xi^\perp},\end{cases}$$

with $\delta_h\in\mathbf{Q}\cap[0,1]$, we get

$$\liminf_j\mathcal{F}_j(\Omega)\geq\frac{a}{2|\xi|^2}\int_\Omega|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx+a\theta|\xi|^2\int_\Omega|\mathrm{div}\,u(x)|^2\,dx$$
$$+\frac{b}{2|\xi|^2}\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+\int_{J_u^{\xi^\perp}\setminus J_u^\xi}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1$$
$$+\int_{J_u^\xi\cap J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).$$

Finally, since the argument above is not affected by the choice of the sub-lattices in which $\mathbf{Z}\xi\oplus\mathbf{Z}\xi^\perp$ has been split and taking into account (5.12), we obtain

$$\liminf_j\mathcal{F}^\xi_{\varepsilon_j}(u_j)\geq 2a\int_\Omega|\langle\mathcal{E}u(x)\xi,\xi\rangle|^2\,dx+4a\theta|\xi|^4\int_\Omega|\mathrm{div}\,u(x)|^2\,dx$$
$$+2b\Big(\int_{J_u^\xi\setminus J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\,d\mathcal{H}^1+\int_{J_u^{\xi^\perp}\setminus J_u^\xi}|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1$$
$$+\int_{J_u^\xi\cap J_u^{\xi^\perp}}|\langle\nu_u,\xi\rangle|\vee|\langle\nu_u,\xi^\perp\rangle|\,d\mathcal{H}^1\Big).$$

Recalling that for a.e. $\xi\in\mathbf{R}^2$ $\mathcal{H}^1(J_u\setminus J_u^\xi)=0$, by integrating with respect to ξ and by Fatou's Lemma we get

$$\liminf_j F_{\varepsilon_j}(u_j)\geq\int_{\mathbf{R}^2}\rho(\xi)\liminf_j\mathcal{F}^\xi_{\varepsilon_j}(u_j)\,d\xi$$

$$\geq \int_{\mathbf{R}^2} \rho(\xi) \left(\int_\Omega 2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4 |\mathrm{div}\, u(x)|^2 \, dx \right) d\xi$$
$$+ \int_{\mathbf{R}^2} 2b\rho(\xi) \left(\int_{J_u} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1 \right) d\xi$$
$$= \int_\Omega \left(\int_{\mathbf{R}^2} \rho(\xi) \left(2a|\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 + 4a\theta|\xi|^4 |\mathrm{div}\, u(x)|^2\right) d\xi \right) dx$$
$$+ \int_{J_u} \left(\int_{\mathbf{R}^2} 2b\rho(\xi) |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\xi \right) d\mathcal{H}^1.$$

The expressions for μ, λ, γ follow after a simple computation.

Step 2 If f is any increasing positive function satisfying (5.4), we can find two sequences of positive numbers (a_i) and (b_i) such that $\sup_i a_i = a$, $\sup_i b_i = b$ and $f(t) \geq (a_i t) \wedge b_i$ for any $t \geq 0$. By Step 1, $\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u)$ is finite only if $F(u)$ is finite and

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \geq \mu_i \int_\Omega |\mathcal{E}u(x)|^2 \, dx + \lambda_i \int_\Omega |\mathrm{div}\, u(x)|^2 \, dx + \gamma_i \mathcal{H}^1(J_u)$$

with $\sup_i \mu_i = \mu$, $\sup_i \lambda_i = \lambda$ and $\sup_i \gamma_i = \gamma$. The thesis follows using once more Proposition 1.8. □

5.4 Estimate from above

Set, for $u \in SBD^2(\Omega) \cap L^\infty(\Omega, \mathbf{R}^2)$,

$$\mathcal{F}^\xi(u) := 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi\rangle|^2 \, dx + 4a\theta|\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2 \, dx$$
$$+2b\Big(\int_{J_u^\xi \setminus J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \, d\mathcal{H}^1 + \int_{J_u^{\xi^\perp} \setminus J_u^\xi} |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1$$
$$+ \int_{J_u^\xi \cap J_u^{\xi^\perp}} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle| \, d\mathcal{H}^1\Big).$$

The following proposition will be crucial for the proof of the Γ-limsup inequality.

Proposition 5.11 *Let $u \in SBD^2(\Omega) \cap L^\infty(\Omega, \mathbf{R}^2)$, then*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^\xi(u) \leq \mathcal{F}^\xi(u).$$

Proof As in the proof of Proposition 5.10 we will assume that definitions 5.2 hold, but the argument we use can be easily adapted to the other case.

Using the notation of Lemma 5.9, set

$$J_u^\varepsilon := \left(J_u^\xi \setminus J_u^{\xi^\perp}\right)_\varepsilon^\xi \cup \left(J_u^{\xi^\perp} \setminus J_u^\xi\right)_\varepsilon^{\xi^\perp} \cup \left(J_u^\xi \cap J_u^{\xi^\perp}\right)_\varepsilon^{\xi,\xi^\perp}.$$

Since $f(t) \leq b$, by Lemma 5.9 there follows

$$\begin{aligned} \limsup_{\varepsilon\to 0} \mathcal{F}^\xi_\varepsilon(u) &\leq \limsup_{\varepsilon\to 0} \mathcal{F}^\xi_\varepsilon\left(u, \Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right) + b \limsup_{\varepsilon\to 0} \frac{\mathcal{L}^2\left(J^\varepsilon_u\right)}{\varepsilon} \\ &\leq \limsup_{\varepsilon\to 0} \mathcal{F}^\xi_\varepsilon\left(u, \Omega^\xi_\varepsilon \setminus J^\varepsilon_u\right) \\ &\quad +2b\left(\int_{J^\xi_u \setminus J^{\xi^\perp}_u} |\langle \nu_u, \xi\rangle|\, d\mathcal{H}^1 + \int_{J^{\xi^\perp}_u \setminus J^\xi_u} |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1 \right. \\ &\quad \left. + \int_{J^\xi_u \cap J^{\xi^\perp}_u} |\langle \nu_u, \xi\rangle| \vee |\langle \nu_u, \xi^\perp\rangle|\, d\mathcal{H}^1\right). \end{aligned}$$

Let us prove that for a.e. $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and for $\zeta \in \{\pm\xi, \pm\xi^\perp\}$

$$D^\zeta_\varepsilon u(x) = \langle u(x+\varepsilon\zeta) - u(x), \zeta\rangle = \int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta, \zeta\rangle\, ds. \tag{5.16}$$

Let, for instance, $\zeta = \xi$, then using the notation of Theorem 1.27 if $x \in \Omega^\xi_\varepsilon \setminus J^\varepsilon_u$ and $x = y + t\xi$, with $y \in \Pi_\xi$, we get

$$\langle u(x+\varepsilon\xi) - u(x), \xi\rangle = u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t).$$

Since $u \in SBD(\Omega)$, for $\mathcal{H}^1$-a.e. $y \in \Pi_\xi$ we have that $u^{\xi,y} \in SBV\left(\left(\Omega^\xi_\varepsilon\right)_{\xi y}\right)$, $\dot{u}^{\xi,y}(t) = \langle \mathcal{E}u(y+t\xi)\xi, \xi\rangle$ for $\mathcal{L}^1$ a.e. $t \in \left(\Omega^\xi_\varepsilon\right)_{\xi y}$ and $J_{u^{\xi,y}} = \left(J^\xi_u\right)_{\xi y}$. Thus

$$\begin{aligned} &u^{\xi,y}(t+\varepsilon) - u^{\xi,y}(t) \\ &= \int_t^{t+\varepsilon} \langle \mathcal{E}u(y+s\xi)\xi, \xi\rangle\, ds + \sum_{s \in \left(J^\xi_u\right)_{\xi y}} \left(\left(u^{\xi,y}\right)^+(s) - \left(u^{\xi,y}\right)^-(s)\right) \end{aligned}$$

and, since $\left(J^\xi_u\right)_{\xi y} \cap [t, t+\varepsilon] = \emptyset$, (5.16) follows.

Moreover, Jensen's inequality, Fubini's Theorem and (5.16) yield

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \left|D^\zeta_\varepsilon u(x)\right|^2 dx &= \int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} \frac{1}{\varepsilon^2}\left|\int_0^\varepsilon \langle \mathcal{E}u(x+s\zeta)\zeta, \zeta\rangle\, ds\right|^2 dx \\ &\leq \int_\Omega |\langle \mathcal{E}u(x)\zeta, \zeta\rangle|^2\, dx, \end{aligned} \tag{5.17}$$

for $\zeta = \pm\xi$.

Let us also prove that

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega^\xi_\varepsilon \setminus J^\varepsilon_u} |\mathrm{div}^{\xi,\xi^\perp}_\varepsilon u|^2\, dx \leq |\xi|^4 \int_\Omega |\mathrm{div}\, u(x)|^2\, dx. \tag{5.18}$$

Setting

$$g(x) := |\xi|^2 \operatorname{div} u(x)$$

and

$$g_\varepsilon(x) := \frac{1}{\varepsilon} \operatorname{div}_\varepsilon^{\xi,\xi^\perp} u(x) \mathcal{X}_{\Omega_\varepsilon^\xi \setminus J_u^\varepsilon}(x),$$

(5.18) follows if we prove prove that

$$\|g - g_\varepsilon\|_{L^2(\Omega)} \to 0. \tag{5.19}$$

Note that

$$g(x) = \langle \mathcal{E}u(x)\xi, \xi \rangle + \langle \mathcal{E}u(x)\xi^\perp, \xi^\perp \rangle,$$

and that by (5.16) on $\Omega_\varepsilon^\xi \setminus J_u^\varepsilon$ we have

$$\operatorname{div}_\varepsilon^{\xi,\xi^\perp} u(x) = \int_0^\varepsilon \langle \mathcal{E}u(x + s\xi)\xi, \xi \rangle + \langle \mathcal{E}u(x + s\xi^\perp)\xi^\perp, \xi^\perp \rangle \, ds.$$

Thus, by absolute continuity and Jensen's inequality we get

$$\begin{aligned}\|g - g_\varepsilon\|_{L^2(\Omega)}^2 \leq o(1) + 2|\xi|^4 \int_{\Omega_\varepsilon^\xi} \frac{1}{\varepsilon} \int_0^\varepsilon |\mathcal{E}u(x + s\xi) - \mathcal{E}u(x)|^2 \, ds \, dx \\ +2|\xi|^4 \int_{\Omega_\varepsilon^\xi} \frac{1}{\varepsilon} \int_0^\varepsilon |\mathcal{E}u(x + s\xi^\perp) - \mathcal{E}u(x)|^2 \, ds \, dx.\end{aligned}$$

Applying Fubini's Theorem and then extending $\mathcal{E}u$ to 0 outside Ω yield

$$\begin{aligned}\|g - g_\varepsilon\|_{L^2(\Omega)}^2 \leq o(1) + 2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x + s\xi) - \mathcal{E}u(x)|^2 \, dx \, ds \\ +2|\xi|^4 \frac{1}{\varepsilon} \int_0^\varepsilon \int_\Omega |\mathcal{E}u(x + s\xi^\perp) - \mathcal{E}u(x)|^2 \, dx \, ds,\end{aligned}$$

and so (5.19) follows by the continuity of the translation operator in $L^2(\Omega; \mathbf{R}^{2\times 2})$.

Of course, using the same argument, we can claim that the analogous inequalities of (5.18), obtained by replacing $(\xi, \xi^\perp)$ by one among the pairs $(\xi, -\xi^\perp)$, $(-\xi, \xi^\perp)$, $(-\xi, -\xi^\perp)$, hold true.

Eventually, since $f(t) \leq at$, by (5.17) and (5.18) we get

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^\xi \left(u, \Omega_\varepsilon^\xi \setminus J_u^\varepsilon\right) \leq 2a \int_\Omega |\langle \mathcal{E}u(x)\xi, \xi \rangle|^2 \, dx + 4a\theta |\xi|^4 \int_\Omega |\operatorname{div} u(x)|^2 \, dx$$

and the conclusion follows. □

Remark 5.12 Arguing as in the proof of Proposition 5.11 we infer that the functionals defined by

$$\mathcal{G}_\varepsilon^\xi(u) := \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^\xi} g\left(\frac{1}{\varepsilon}|D_{\varepsilon,\xi}u(x)|^2\right)\,dx,$$

where $g(t) := (at)\wedge b$, satisfy the estimate

$$\mathcal{G}_\varepsilon^\xi(u) \le 2a\int_\Omega |\langle \mathcal{E}u(x)\xi,\xi\rangle|^2\,dx + 2b\int_{J_u^\xi} |\langle \nu_u,\xi\rangle|\,d\mathcal{H}^1,$$

for any $u \in SBD(\Omega)$.

Moreover, by the subadditivity of g and since $f(t)\le g(t)$ by hypothesis, there holds

$$\mathcal{F}_\varepsilon^\xi(u) \le c\left(\mathcal{G}_\varepsilon^\xi(u) + \mathcal{G}_\varepsilon^{\xi^\perp}(u)\right) \le c\mathcal{F}^\xi(u).$$

Let us now prove the Γ-limsup inequality which easily follows by Proposition 5.11 and Remark 5.12.

Proposition 5.13 *For any* $u \in L^\infty(\Omega;\mathbf{R}^2)$,

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u) \le F(u).$$

Proof We can reduce ourselves to prove the inequality for $u \in SBD^2(\Omega)$. For such a u the recovery sequence is provided by the function itself. Indeed, by Proposition 5.11, Remark 5.12 and Fatou's Lemma, we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(u) &\le \int_{\mathbf{R}^2} \rho(\xi)\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon^\xi(u)\,d\xi\\ &\le \int_{\mathbf{R}^2}\rho(\xi)\mathcal{F}^\xi(u)\,d\xi = F(u).\end{aligned}$$

□

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