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Obstacle problems for monotone operators with measure data

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Contents

Introduction

Given a bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$, and a strictly monotone elliptic operator A in divergence form, we study obstacle problems for the operator A in Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$, when the forcing term μ is a measure on Ω and the obstacle ψ is an arbitrary function on Ω.

Obstacle problems when the forcing term belongs to the dual of the energy space have been studied as part of the theory of Variational Inequalities (for which we refer to well known books such as [38] and [55]). In this frame the problem consists in finding a function $u \in W_0^{1,p}(\Omega)$ (the energy space), which is above the obstacle ψ, such that

$$\begin{cases} \langle A(u), v-u\rangle \geq \langle F, v-u\rangle, \\ \forall v \in W_0^{1,p}(\Omega), v \geq \psi \text{ in } \Omega. \end{cases} \tag{0.0.1}$$

For such problems a wide abstract theory has been developed, and we know that, if the datum F belongs to the dual $W^{-1,p'}(\Omega)$ of the energy space, and if there exists at least a function $z \in W_0^{1,p}(\Omega)$ above the obstacle, then there exists one and only one solution.

Moreover, we recall that the solution of (0.0.1) is also characterized as the smallest function $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} A(u) - F \geq 0 \text{ in } \mathcal{D}'(\Omega) \\ u \geq \psi \text{ in } \Omega, \end{cases} \tag{0.0.2}$$

or, equivalently, u is the smallest function in $W_0^{1,p}(\Omega)$, greater than or equal to ψ, such that

$$\begin{cases} A(u) - F = \lambda & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.3}$$

for some nonnegative measure λ of $W^{-1,p'}(\Omega)$.

Trying to extend this theory to problems where the forcing term is a measure various difficulties arise.

We recall that, already in the case of equations, the term $\langle \mu, u\rangle$ has not always a meaning when μ is a measure and $u \in W_0^{1,q}(\Omega)$, $q \leq N$. Moreover, simple examples show that the solution of equations with measure data cannot be expected to belong to the energy space $W_0^{1,p}(\Omega)$ determined by the growth assumptions on the operator. When $N \geq 2$, indeed, the solution of the Laplace equation in a ball with the Dirac mass at the center as datum, does not belong to $H_0^1(\Omega)$, but only to $W_0^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$. Hence the classical formulation (0.0.1) of the variational inequality fails.

Also the use of the characterization (0.0.2) to define the obstacle problem with measure data is not possible because another problem arises: an example by J. Serrin (see [52] and, for more details, [51]) shows that, when A is a particular linear elliptic operator with discontinuous coefficients, the homogeneous equation

$$\begin{cases} Au = 0 & \text{in } \mathcal{D}'(\Omega) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution v which does not belong to $H_0^1(\Omega)$. The function $u \equiv 0$ is obviously the unique solution in $H_0^1(\Omega)$.

So (0.0.2) in general does not determine the solution of the obstacle problem: indeed, with such A and v, if we choose $\psi \equiv -\infty$, and if u were the minimal supersolution, then we would have $u \leq u + tv$ a.e. in Ω, for any t in $\mathbb{R}$, which implies $v \equiv 0$, i.e. a contradiction.

To overcome these difficulties, when the forcing term is a measure, we introduce a formulation for obstacle problems, based on a suitable notion of solution to equations with measure data, which ensures existence and uniqueness results.

We briefly recall that, in the linear case, i.e. $A(u) = -\operatorname{div}(\mathcal{A}(x)\nabla u)$, where $\mathcal{A}$ is a uniformly elliptic matrix with $L^\infty(\Omega)$ coefficients, the problem of finding a solution of

$$\begin{cases} Au = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.4}$$

when μ is a bounded Radon measure on Ω, was investigated by G. Stampacchia, who introduced and studied in [53] a notion of solution using duality and regularity arguments. This allowed him to prove both existence and uniqueness results.

Stampacchia's framework, which heavily relies on a duality argument, cannot be extended to the general nonlinear case, except in the case $p = 2$, where Stampacchia's ideas continue to work if the operator is strongly monotone and Lipschitz continuous. In this setting, indeed, we can use the notion of solution, namely the reachable solution, considered by F. Murat in [48] to solve (uniquely) the Dirichlet problem (0.0.4), when μ is a bounded Radon measure.

In the general nonlinear monotone case, when μ is a bounded Radon measure vanishing on all sets of p-capacity zero (the capacity defined starting from $W_0^{1,p}(\Omega)$: see Section 1.1), we may use the notion of entropy solution introduced in [3] and [10], which ensures us that (0.0.4) has a unique entropy solution.

Using these types of solutions we give a definition for unilateral problems with measure data quite similar to the characterization given by (0.0.3) in the classical setting.

We say that a function u solves the Obstacle Problem when the forcing term μ is a bounded Radon measure and the obstacle ψ is an arbitrary function on Ω, if u is the smallest function with the following properties: $u \geq \psi$ in Ω and u is the unique solution, in an appropriate sense, of a problem of the form

$$\begin{cases} A(u) = \mu + \lambda & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.5}$$

for some bounded Radon measure $\lambda \geq 0$. More precisely, if A is linear, we assume that u is the solution of (0.0.5) in the sense of Stampacchia; if $p = 2$ and A is strongly monotone and Lipschitz continuous, we assume that u is the reachable solution of (0.0.5); finally, when A is a general nonlinear strictly monotone operator, and μ, λ are bounded Radon measure which vanish on sets of p-capacity zero, then u is the entropy solution.

The measure λ which corresponds to the solution u of the unilateral problem relative to A, μ, and ψ, is called the obstacle reaction associated with u.

From now on we will call Obstacle Problems the ones with measure data, according to the previous definition, and Variational Inequalities these with data in the dual of the energy space, solved in the variational sense.

The outline of the thesis is as follows.

After giving the definition and some preliminary results on p-capacity and on measures, we study in Chapter 1 the various notions of solutions to equations with measure data, making a slight reelaboration of known results.

In Chapter 2 (which contains the results of [23]) we prove existence and uniqueness of solutions of obstacle problems with measure data when A is a linear differential operator. The only restriction required on the choice of the obstacle is that there exists a nonnegative bounded Radon measure ρ such that the solution (of Stampacchia) of equation (0.0.4) with datum ρ is above the obstacle. This condition is similar to the one needed for the variational case, but it is not comparable to that. In Section 2.5 we will discuss these conditions in deeper details, after proving that the variational solution to the obstacle problem (problem (0.0.1)) coincides with the new one (Definition 2.1.1) when both make sense. In other words we will show that the new formulation of the Obstacle Problem is consistent with the classical one.

In Section 2.4 we will give some stability results. Section 2.6 provides a characterization of the solution in terms of approximating sequences of Variational Inequalities, and in Section 2.7 we study some properties of the solution of the Obstacle Problem for the class of bounded Radon measures, that do not charge the sets of capacity zero.

Chapter 3 (containing the results of [39]) is devoted to the study of the Obstacle Problem for a strictly monotone operator $A(u) = -\operatorname{div}(a(x, \nabla u))$ acting on $W_0^{1,p}(\Omega)$, $p > 1$, when the forcing term is a bounded Radon measure μ vanishing on all sets of p-capacity zero. We obtain existence and uniqueness results, and consistency with the classical theory of Variational Inequalities.

We study also some properties of the obstacle reactions associated with the solutions of the Obstacle Problems, obtaining the Lewy-Stampacchia inequality: the measure λ corresponding to the Obstacle Problem relative to A, μ, and ψ, satisfies

$$\lambda \leq (\mu - A(\psi))^{-},$$

when $A(\psi)$ is a bounded Radon measure vanishing on all sets of p-capacity zero.

Furthermore, as in the classical framework, in Section 3.4 we will show that the solution found can be characterized by the complementarity system. More precisely, Theorem 3.1.7 shows that the solution u of the Obstacle Problem relative to A, μ, and ψ is the only entropy solution of (0.0.5) such that $u = \psi$ λ-almost everywhere in Ω, and $u \geq \psi$ in Ω. We also find a more technical characterization of the solution of the Obstacle Problem, which in the case of ψ C_p-quasi upper bounded in Ω (see Section 1.1) turns out to be similar to (0.0.1). In this framework we recover the definitions given by L. Boccardo and T. Gallouët in [8] and by L. Boccardo and G.R. Cirmi in [5]-[6] when $\mu = f \in L^1(\Omega)$, and by P. Oppezzi and A. M. Rossi in [49]-[50] in a more general case.

At this point we would like to extend the theory of Chapter 3 to an arbitrary bounded Radon measure μ, dropping the assumption that μ vanishes on all sets of p-capacity zero.

The main difficulty is that, in our approach to obstacle problems when the data do not belong to the dual of the energy space, we need a suitable notion of solution to equations with measure data, which ensures existence and uniqueness results. This is still an open problem in the case $p \neq 2$, and it is solved by F. Murat in [48] when $p = 2$: he proved that, if A is strongly monotone and Lipschitz continuous, then there exists a unique reachable solution of (0.0.4) (see Theorem 1.5.3 and Theorem 1.5.13).

Thus, in Chapter 4 (which contains the results of [40]) we deal with the Obstacle Problem for a strongly monotone and Lipschitz continuous operator A, when $p = 2$ and the forcing term is a bounded Radon measure. We obtain existence and uniqueness results, and consistency with the theory of Chapter 3; hence, with the classical theory of Variational Inequalities.

Also in this case we obtain the Lewy-Stampacchia inequality.

Finally we investigate the interaction between obstacles and data, and in particular the complementarity conditions. If the negative part μ^- of the datum μ vanishes on sets of capacity zero, so does the obstacle reaction, provided that there exists a bounded Radon measure ρ absolutely continuous with respect to capacity, such that the solution of (0.0.4) relative to ρ is greater than or equal to ψ. By Theorem 3.1.7 we will deduce that in this case the obstacle reaction is concentrated on the contact set $\{u = \psi\}$, whenever the obstacle ψ is quasi upper semicontinuous (see Section 1.1). So we concentrate our attention on the case where $\mu_s^- \neq 0$. We recall (see Proposition 1.2.1) that every bounded Radon measure μ can be decomposed in a unique way as $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to capacity and μ_s is concentrated on a set of capacity zero, and $\mu_s = \mu_s^+ - \mu_s^-$. When the obstacle is controlled from above and from below in an appropriate way, we will prove (see Theorem 4.1.5) that the solution u of the Obstacle Problem relative to A, μ, and ψ does not depend on μ_s^-, while the obstacle reaction has the form $\lambda = \lambda_1 + \mu_s^-$, where λ_1 is a nonnegative measure, absolutely continuous with respect to capacity, which is concentrated on the contact set $\{u = \psi\}$.

Finally Chapter 5 (containing the results of [41]) develops the theme of continuous dependence with respect to data.

In the context of Chapter 3 we study the convergence properties of the solutions of the Obstacle Problem, under simultaneous perturbation of the operator, the forcing term, and the obstacle.

In [11], [26], and [12] the authors proved some results on the convergence of Variational Inequalities for strictly monotone operators of the form $A(u) = -\operatorname{div}(a(x, \nabla u))$, when both the operator and the obstacle are perturbed. For F_h, $F \in W^{-1,p'}(\Omega)$, and for ψ_h, $\psi : \Omega \to \overline{\mathbb{R}}$, they considered a sequence of Variational Inequalities

$$\begin{cases} u_h \in W_0^{1,p}(\Omega), u_h \geq \psi_h, \\ \langle -\operatorname{div}(a_h(x, \nabla u_h)), v - u_h \rangle \geq \langle F_h, v - u_h \rangle, \\ \forall v \in W_0^{1,p}(\Omega), v \geq \psi_h, \end{cases} \tag{0.0.6}$$

and the corresponding convex sets

$$K_{\psi_h} := \{z \in W_0^{1,p}(\Omega) : z \geq \psi_h \ C_p\text{-q.e. in } \Omega\},$$

$$K_{\psi} := \{z \in W_0^{1,p}(\Omega) : z \geq \psi \ C_p\text{-q.e. in } \Omega\},$$

assuming that (see Definition 5.2.1 and Definition 4.3.1)

$$F_h \text{ converges to } F \text{ strongly in } W_0^{1,p}(\Omega),$$

$$a_h \ G\text{-converges to } a,$$

$$K_{\psi_h} \text{ converges to } K_\psi \text{ in the sense of Mosco.}$$

Theorem 3.1 of [26] shows that

$$u_h \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),$$

$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \quad \text{weakly in } L^{p'}(\Omega)^N,$$

$$\int_\Omega a_h(x, \nabla u_h)\nabla u_h \, dx \to \int_\Omega a(x, \nabla u)\nabla u \, dx,$$

where u is the solution of the Variational Inequality relative to A, F, and ψ ($A(u) = -\operatorname{div}(a(x, \nabla u))$).

In this chapter we extend the stability result stated above to the case when the forcing term μ is a bounded Radon measure which vanishes on all sets of p-capacity zero.

We will adopt the notion of solution considered in Chapter 3 to solve uniquely the obstacle problem (denoted by $OP_0(A, \mu, \psi)$) relative to A, μ, and ψ.

We consider a sequence of obstacle problems $OP_0(A_h, \mu_h, \psi_h)$, when the measures μ_h vanish on sets of p-capacity zero, and we assume that

$$\mu_h(B) \to \mu(B), \text{ for every Borel set } B \subseteq \Omega,$$

$$a_h \ G\text{-converges to } a,$$

$$K_{\psi_h} \text{ converge to } K_\psi \text{ in the sense of Mosco.}$$

Denoting the solutions of $OP_0(A_h, \mu_h, \psi_h)$ and $OP_0(A, \mu, \psi)$ by u_h and u, respectively, we will prove in Theorem 5.4.1 that

$$T_j(u_h) \rightharpoonup T_j(u) \quad \text{weakly in } W_0^{1,p}(\Omega), \text{ for every } j > 0,$$

$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \text{ weakly in } L^q(\Omega)^N, \text{ for every } q < \tfrac{N}{N-1},$$

$$\int_\Omega a_h(x, \nabla u_h)\nabla T_j(u_h) \, dx \to \int_\Omega a(x, \nabla u)\nabla T_j(u) \, dx, \text{ for every } j > 0,$$

where $T_j(\cdot)$ is the truncation function at level j.

In the special case where $a_h = a$, for every h, we obtain also that $T_j(u_h)$ converges to $T_j(u)$ strongly in $W_0^{1,p}(\Omega)$, for every $j > 0$.

The content of the Chapters 2, 3, 4, and 5 corresponds to the papers [23] with P. Dall'Aglio, [39], [40], and [41].

Chapter 1
Equations with measure data

1.1. Capacity

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and K be a compact subset of Ω. For $p > 1$, the p-capacity of K with respect to Ω is defined as:

$$C_p(K) = \inf\left\{ \int_\Omega |\nabla\varphi|^p dx : \varphi \in C_c^\infty(\Omega), u \geq \chi_K \right\},$$

where χ_K is the characteristic function of K. This definition can be extended to any open subset A of Ω in the following way:

$$C_p(A) = \sup\{C_p(K) : K \text{ compact}, K \subseteq A\}.$$

Finally, it is possible to define the p-capacity of any set $E \subseteq \Omega$ as:

$$C_p(E) = \inf\{C_p(A) : A \text{ open}, E \subseteq A\}.$$

We say that a property $\mathcal{P}(x)$ holds C_p-quasi everywhere (C_p-q.e.) in a set $E \subseteq \Omega$, if it holds for all $x \in E$ except for a subset N of E with $C_p(N) = 0$.

A function $v : \Omega \to \overline{\mathbb{R}}$ is C_p-quasi Borel if there exists a Borel function $u : \Omega \to \overline{\mathbb{R}}$ such that $v = u$ C_p-q.e. in Ω. A function $v : \Omega \mapsto \overline{\mathbb{R}}$ is said to be C_p-quasi continuous (resp. C_p-quasi upper semicontinuous) if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$, with $C_p(E) < \varepsilon$, such that the restriction of v to $\Omega \setminus E$ is a continuous (resp. upper semicontinuous) function with values in $\overline{\mathbb{R}}$. Thus, every C_p-quasi continuous (resp. C_p-quasi upper semicontinuous) function v is C_p-quasi Borel.

We recall also that if u and v are C_p-quasi continuous functions and $u \leq v$ a.e. in Ω then also $u \leq v$ C_p-q.e. in Ω.

It is well known that every $u \in W_0^{1,p}(\Omega)$ has a C_p-quasi continuous representative, which is uniquely defined (and finite) up to a set of p-capacity zero. In the sequel we shall always identify u with its C_p-quasi continuous representative, so that the pointwise values of a function $u \in W_0^{1,p}(\Omega)$ are defined C_p-quasi everywhere.

With this convention for every subset B of Ω we have

$$C_p(B) = \inf\left\{\int_\Omega |\nabla u|^p dx\right\},$$

where the infimum is taken over all functions $u \in W_0^{1,p}(\Omega)$ such that $u = 1$ C_p-q.e. on B, and $u \geq 0$ C_p-q.e. on Ω (we use the convention $\inf\emptyset = +\infty$).

We also have that if $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ then there exists a subsequence which converges C_p-quasi everywhere.

A set $E \subseteq \Omega$ is said to be C_p-quasi open if for every $\varepsilon > 0$ there exists an open set A such that $E \subseteq A \subseteq \Omega$ and $C_p(A \setminus E) \leq \varepsilon$.

A set $E \subseteq \Omega$ is said to be C_p-quasi Borel if there exists a Borel set B, with $E \subseteq B \subseteq \Omega$, such that $C_p(B \setminus E) = 0$.

For every $j > 0$ we define the truncation function $T_j : \mathbb{R} \mapsto \mathbb{R}$ by

$$T_j(t) = \begin{cases} t & \text{if } |t| \leq j \\ j\,\text{sign}(t) & \text{if } |t| > j. \end{cases}$$

Let us consider the space $\mathcal{T}_0^{1,p}(\Omega)$ of all functions $u : \Omega \mapsto \overline{\mathbb{R}}$ which are almost everywhere finite and such that $T_j(u) \in W_0^{1,p}(\Omega)$ for every $j > 0$. It is easy to see that every function $u \in \mathcal{T}_0^{1,p}(\Omega)$ has a C_p-quasi continuous representative with values in $\overline{\mathbb{R}}$, that will always be identified with the function u. Moreover, for every $u \in \mathcal{T}_0^{1,p}(\Omega)$ there exists a measurable function $\Phi : \Omega \mapsto \mathbb{R}^N$ such that $\nabla T_j(u) = \Phi\chi_{\{|u|\leq j\}}$ a.e. in Ω (see Lemma 2.1 in [3]). This function Φ, which is unique up to almost everywhere equivalence, will be denoted by ∇u. It is possible to prove (see [33]) that ∇u is the approximate gradient of u in the sense of Geometric Measure Theory (see Definition 3.1.2 of [34]). Moreover ∇u coincides with the distributional gradient of u whenever $u \in \mathcal{T}_0^{1,p}(\Omega) \cap L^1_{\text{loc}}(\Omega)$ and $\nabla u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$.

In the special case where $p = 2$, we write capacity and cap$(\cdot)$ to denote, respectively, p-capacity and $C_p(\cdot)$, and, in general, we omit the prefix C_p to quasi everywhere, quasi Borel, etc..

1.2. Preliminaries about measures

Let $\mathcal{M}_b(\Omega)$ the space of Radon measures μ on Ω whose total variation $|\mu|$ is bounded on Ω, while $\mathcal{M}_{b,0}^p(\Omega)$ is the special subspace of $\mathcal{M}_b(\Omega)$ of all measures, which are absolutely continuous with respect to the p-capacity, that is, a measure $\mu \in \mathcal{M}_b(\Omega)$ belongs to $\mathcal{M}^p_{b,0}(\Omega)$ if and only if $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $C_p(B) = 0$. We denote the positive cones of $\mathcal{M}_b(\Omega)$ and $\mathcal{M}^p_{b,0}(\Omega)$ by $\mathcal{M}^+_b(\Omega)$ and $\mathcal{M}^{p,+}_{b,0}(\Omega)$, respectively. Moreover, for a measure $\mu \in \mathcal{M}_b(\Omega)$, μ^+ and μ^- will be the positive and negative part of μ, respectively.

It is well known that, if μ belongs to $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$, then μ is in $\mathcal{M}^p_{b,0}(\Omega)$, every u in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ is summable with respect to μ and

$$\langle \mu, u \rangle = \int_\Omega u \, d\mu,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W^{1,p}_0(\Omega)$, while in the right hand side u denotes the C_p-quasi continuous representative and, consequently, the pointwise values of u are defined μ-almost everywhere.

We recall that for a measure $\mu \in \mathcal{M}_b(\Omega)$, and a Borel set $B_0 \subseteq \Omega$, the measure $(\mu \llcorner B_0)(B) = \mu(B_0 \cap B)$ for any Borel set $B \subseteq \Omega$. If a measure $\mu \in \mathcal{M}_b(\Omega)$ is such that $\mu = \mu \llcorner B_0$ for a certain Borel set B_0, the measure μ is said to be concentrated on B_0.

The following result is the analogue of the Lebesgue decomposition theorem, and can be proved in the same way (see Lemma 2.1 of [35]).

Proposition 1.2.1. *For every measure $\mu \in \mathcal{M}_b(\Omega)$ there exists a unique pair of measures (μ_a, μ_s), with $\mu_a \in \mathcal{M}^p_{b,0}(\Omega)$ and μ_s concentrated on a set of p-capacity zero, and $\mu = \mu_a + \mu_s$. If μ is nonnegative, so are μ_a and μ_s.*

The measures μ_a and μ_s will be called the absolutely continuous and the singular part of μ with respect to the p-capacity.

Also a decomposition theorem for measures in $\mathcal{M}^p_{b,0}(\Omega)$ is known (see Theorem 2.2 of [24]):

Theorem 1.2.2. *For every measure $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ there exists a nonnegative measure $\gamma \in \mathcal{M}^+_b(\Omega) \cap W^{-1,p'}(\Omega)$ and a Borel measurable function $g \in L^1(\Omega, \gamma)$ such that $\mu(A) = (g\gamma)(A)$ for every C_p-quasi Borel subset A of Ω. If μ is nonnegative, so is g.*

Starting from this result, it can be proved a further decomposition result.

Theorem 1.2.3. *Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, then for every $\varepsilon > 0$ there exist $f_\varepsilon \in L^1(\Omega)$ and $F_\varepsilon \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ such that $\mu = f_\varepsilon + F_\varepsilon$ and*

$$\|f_\varepsilon\|_{L^1(\Omega)} \le \|\mu\|_{\mathcal{M}_b(\Omega)}, \qquad \|F_\varepsilon\|_{W^{-1,p'}(\Omega)} \le \varepsilon.$$

Proof. The proof of Theorem 2.1 in [10] shows that if μ is an element of $\mathcal{M}^p_{b,0}(\Omega)$, then μ can be written as $\mu = f + F$, where $f \in L^1(\Omega)$, $F \in W^{-1,p'}(\Omega)$, and

$$\|f\|_{L^1(\Omega)} \le \|\mu\|_{\mathcal{M}_b(\Omega)}, \qquad \|F\|_{W^{-1,p'}(\Omega)} \le 1.$$

If we write μ as $\varepsilon \dfrac{\mu}{\varepsilon}$ and we apply the same result to $\dfrac{\mu}{\varepsilon}$, we easily conclude. □

By Theorem 1.2.2, if a measure μ belongs to $\mathcal{M}^p_{b,0}(\Omega)$, then every $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ is summable with respect to μ.

As concerning the approximation of measures in $\mathcal{M}^p_{b,0}(\Omega)$, it is useful to state a lemma, which is quite simple, but is proved for the sake of completeness.

Lemma 1.2.4. *Let $\mu_i \in \mathcal{M}^p_{b,0}(\Omega)$ $(i = 1,2)$ be such that $\mu_1 \le \mu_2$; then there exists $\mu_i^n \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ $(i = 1,2)$ such that μ_i^n converges to μ_i strongly in $\mathcal{M}_b(\Omega)$ and $\mu_1^n \le \mu_2^n$.*

Proof. It suffices to consider the decomposition of $\mu_2 - \mu_1$ in the sense of Theorem 1.2.2:

$$0 \le \mu_2 - \mu_1 = g\gamma,$$

where $\gamma \in W^{-1,p'}(\Omega) \cap \mathcal{M}^+_b(\Omega)$ and $g \in L^1(\Omega,\gamma)$, $g \ge 0$. On the other hand, μ_1 decomposes as $\mu_1 = g_1\gamma_1$, with $\gamma_1 \in W^{-1,p'}(\Omega) \cap \mathcal{M}^+_b(\Omega)$ and $g_1 \in L^1(\Omega,\gamma_1)$. We take $\mu_1^n := T_n(g_1)\gamma_1$ and $\mu_2^n := T_n(g)\gamma + \mu_1^n$, and we can conclude. □

Remark 1.2.5. The proof of the previous lemma shows that, if $\mu \in \mathcal{M}^{p,+}_{b,0}(\Omega)$, then there exists a nondecreasing sequence $\mu_n \in W^{-1,p'}(\Omega) \cap \mathcal{M}^+_b(\Omega)$ such that μ_n converges to μ strongly in $\mathcal{M}_b(\Omega)$.

As usual, we identify $\mathcal{M}_b(\Omega)$ with the dual of the Banach space $C_0(\Omega)$ of continuous functions that are zero on the boundary; so that the duality is $\langle \mu, u\rangle = \int_\Omega u\,d\mu$, for every u in $C_0(\Omega)$ and the norm is $\|\mu\|_{\mathcal{M}_b(\Omega)} = |\mu|(\Omega)$.

Definition 1.2.6. If μ_h, $\mu \in \mathcal{M}_b(\Omega)$, we say that μ_k converges to μ $*$-weakly in $\mathcal{M}_b(\Omega)$ if

$$\lim_{h\to+\infty} \int_\Omega u \, d\mu_h = \int_\Omega u \, d\mu,$$

for every $u \in C_0(\Omega)$.

For nonnegative measures we have a characterization of the $*$-weak convergence in terms of convergence of sets.

Proposition 1.2.7. *Given μ_h, $\mu \in \mathcal{M}_b^+(\Omega)$, the following conditions are equivalent:*

1. *μ_h converges to μ $*$-weakly in $\mathcal{M}_b(\Omega)$;*
2. *$\mu(A) \le \liminf_{h\to+\infty} \mu_h(A)$, for every A open subset of Ω,*
   $\mu(K) \ge \limsup_{h\to+\infty} \mu_h(K)$, for every K compact subset of Ω.

Concerning the weak convergence in $\mathcal{M}_b(\Omega)$, the following result shows that it is stronger than the $*$-weak one.

Proposition 1.2.8. *Given $\mu_h, \mu \in \mathcal{M}_b(\Omega)$, the following conditions are equivalent:*

1. *μ_h converges to μ weakly in $\mathcal{M}_b(\Omega)$;*
2. *$\lim_{h\to+\infty} \mu_h(B) = \mu(B)$, for every Borel set B contained in Ω.*

The proof of this proposition (see, e.g., Theorem 6.6 in [2]) relies on the Vitali-Hahn-Sacks Theorem (see, e.g., Theorem 6.4 in [2]), which is similar to the Banach-Steinhaus uniform boundedness theorem and gives a useful condition for the equiintegrability of a sequence of summable functions.

Theorem 1.2.9. *Let ν be a measure in $\mathcal{M}_b(\Omega)$, let g_h be a sequence in $L^1(\Omega, \nu)$ and set $\mu_h = g_h \nu$. Assume that, for every Borel set $B \subseteq \Omega$, the $\lim_{h\to+\infty} \mu_h(B)$ exists and is finite; then g_h is equiintegrable.*

Remark 1.2.10. Notice that in general it is not possible to approximate any measure $\mu \in \mathcal{M}_b(\Omega)$ by means of measures in $\mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$ with respect to the weak topology of $\mathcal{M}_b(\Omega)$. Indeed the weak closure of $\mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$ is $\mathcal{M}_{b,0}^p(\Omega)$.

In the last part of this section we give a weak notion of convergence in capacity, similar to that one considered in [16], and some properties related to it.

Definition 1.2.11. Let u_j, $u : \Omega \to \mathbb{R}$ be C_p-quasi Borel functions. We say that u_j converges to u weakly in capacity if, for every measure $\mu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$, u_j converges to u in μ-measure, i.e.,

$$\lim_{j\to+\infty} \mu(\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\}) = 0, \tag{1.2.1}$$

for every $\varepsilon > 0$.

Remark 1.2.12. Actually, Definition 1.2.11 is not equivalent to Definition 3.1 of [16], where the measures μ are positive elements of $W^{-1,p'}(\Omega)$, hence positive Radon measures (not bounded), and the convergence in μ-measure is only local. However, it is easy to check that (1.2.1) turns out to be equivalent to the condition considered in Definition 3.1 of [16], when $\mu \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$.

The following proposition (see Proposition 3.5 in [16]) shows the relationship between weak convergence in $W_0^{1,p}(\Omega)$ and weak convergence in capacity.

Proposition 1.2.13. *Let u_j, $u \in W_0^{1,p}(\Omega)$ be such that u_j converges to u weakly in $W_0^{1,p}(\Omega)$. Then u_j converges to u weakly in capacity.*

Proof. First of all, we note that $|u_j - u|$ converges to 0 weakly in $W_0^{1,p}(\Omega)$. Thus, if $\mu \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$, we have

$$\lim_{j\to+\infty} \int_\Omega |u_j - u| \, d\mu = \lim_{j\to+\infty} \langle \mu, |u_j - u| \rangle = 0,$$

which implies that u_j converges to u in μ-measure.

Let now $\mu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$; by Theorem 1.2.2, we may decompose μ as $\mu = g\gamma$, where $\gamma \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$ and $g \in L^1(\Omega, \gamma)$, with $g \geq 0$.

Hence, u_j converges to u in γ-measure, and, for every $t \geq 0$,

$$\mu(\{|u_j - u| > t\}) = \int_\Omega g\chi_{\{|u_j-u|>t\}} \, d\gamma,$$

which tends to zero as j goes to infinity, by the Lebesgue dominated convergence theorem. □

Finally, the next result follows immediately from the dual definition of the capacity (see Proposition 2.2 in [16] and Theorem 2.6.12 in [56]).

Proposition 1.2.14. *For every C_p-quasi Borel set $E \subset \Omega$ with positive capacity, there exists a measure $\mu \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$ such that $\mu(E) = 1$ and $\mu(\Omega \setminus E) = 0$.*

Next lemma will be used several times in the following.

Lemma 1.2.15. *Let $\mu_h, \mu \in \mathcal{M}_{b,0}^p(\Omega)$ be such that μ_h converges to μ weakly in $\mathcal{M}_b(\Omega)$. Let $\Phi_h, \Phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be such that $\sup_h \|\Phi_h\|_{L^\infty(\Omega)}$ is bounded and Φ_h converges to Φ weakly in $W_0^{1,p}(\Omega)$. Then*

$$\lim_{h \to +\infty} \int_\Omega \Phi_h \, d\mu_h = \int_\Omega \Phi \, d\mu.$$

Proof. We define the measure $\nu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$ as $\nu := \sum_{h=1}^{\infty} \frac{1}{2^h} \frac{|\mu_h|}{|\mu_h|(\Omega)}$, so that $|\mu_h|$ is absolutely continuous with respect to ν. This implies that $\mu_h = g_h \nu$, with $g_h \in L^1(\Omega, \nu)$; on the other hand, thanks to Proposition 1.2.8, we have that $\mu_h(B)$ tends to $\mu(B)$, for every Borel set $B \subseteq \Omega$. Applying Theorem 1.2.9 we deduce that the sequence g_h is equiintegrable, and, in conclusion, it converges to a function g weakly in $L^1(\Omega, \nu)$, with $\mu = g\nu$.

Now, we can prove that $\int_\Omega \Phi_h \, d\mu_h$ tends to $\int_\Omega \Phi \, d\mu$, when Φ_h, Φ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $\sup_h \|\Phi_h\|_{L^\infty(\Omega)} < +\infty$, and Φ_h converges to Φ weakly in $W_0^{1,p}(\Omega)$. By Proposition 1.2.13, indeed, the convergence of Φ_h to Φ is, in particular, in ν-measure. At this point, it is easy to obtain that

$$\lim_{h \to +\infty} \int_\Omega \Phi_h \, d\mu_h = \lim_{h \to \infty} \int_\Omega \Phi_h g_h \, d\nu = \int_\Omega \Phi g \, d\nu = \int_\Omega \Phi \, d\mu.$$

□

Actually, we can extend Definition 1.2.11 to the case where u_j and u are arbitrariness functions with real extended values.

Definition 1.2.16. Let u_j, $u : \Omega \to \overline{\mathbb{R}}$ be arbitrary functions. We say that u_j converges to u weakly in capacity if, for every measure $\mu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$, u_j converges to u in μ^*-measure, where μ^* is the outer measure associated with μ. It means that, for every $\varepsilon > 0$,

$$\lim_{j \to +\infty} \mu^*(\{x \in \Omega : |\Phi(u_j(x)) - \Phi(u(x))| > \varepsilon\}) = 0, \tag{1.2.2}$$

where $\Phi : \overline{\mathbb{R}} \to [0, 1]$ is an increasing and continuous function.

Remark 1.2.17. Let us point out that the above definition does not depend on Φ.

Remark 1.2.18. It is easy to check that Definition 1.2.16 turns out to be equivalent to Definition 1.2.11, when u_j and u are C_p-quasi Borel functions with real values.

1.3. Assumptions on the operator

Let p and p' two real numbers, with $p > 1$, $p' > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $a : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ be a Carathéodory function such that for almost every x in Ω and for every ξ, η in $\mathbb{R}^N$ $(\xi \neq \eta)$:

$$|a(x,\xi)| \leq c_0[k_0(x) + |\xi|^{p-1}], \tag{1.3.1}$$

$$a(x,\xi)\xi \geq c_1|\xi|^p - k_1(x), \tag{1.3.2}$$

$$(a(x,\xi) - a(x,\eta))(\xi - \eta) > 0, \tag{1.3.3}$$

$$a(x,0) = 0, \tag{1.3.4}$$

where c_0 and c_1 are two positive real constants, k_0 is a nonnegative function in $L^{p'}(\Omega)$ and k_1 is a nonnegative function in $L^1(\Omega)$.

If $\xi, \eta \in \mathbb{R}^N$, we denote the scalar product in $\mathbb{R}^N$ between ξ and η by $\xi\eta$, while the tensorial product $\xi \otimes \eta$ is defined as $\xi \otimes \eta = (\xi_i \eta_j)_{i,j=1,..,N}$.

Thanks to hypotheses (1.3.1)$-$(1.3.4) the operator $A : u \mapsto -\text{div}(a(x,\nabla u))$ maps $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ and for every $F \in W^{-1,p'}(\Omega)$ there exists a unique function $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} A(u) = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.5}$$

in the weak sense, that means

$$\int_\Omega a(x,\nabla u)\nabla v \, dx = \langle F, v \rangle, \tag{1.3.6}$$

for every $v \in W_0^{1,p}(\Omega)$ (see, e.g., [45]).

In order to study the elliptic problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.7}$$

when μ is a bounded Radon measure, we can not use the variational formulation (1.3.6), since, in general, the term $\langle \mu, v \rangle$ has not always meaning when μ is a measure and $v \in W_0^{1,r}(\Omega)$, with $r \le N$. Moreover, simple examples show that the solution cannot be expected to belong to the energy space $W_0^{1,p}(\Omega)$ determined by the growth conditions on the operator. When $N \ge 2$, indeed, the solution of the Laplace equation in a ball, with μ the Dirac mass at the center, does not belong to $H_0^1(\Omega)$ but only to $W_0^{1,q}(\Omega)$, with $q < \frac{N}{N-1}$. Thus, it is necessary to change the functional setting in order to prove existence result.

Notice that, if $p > N$, then the Sobolev embedding theorem implies that $\mathcal{M}_b(\Omega)$ is contained in $W^{-1,p'}(\Omega)$, so that (1.3.7) is a particular case of (1.3.5). Therefore, we shall always assume that $1 < p \le N$.

1.4. The linear case

In the linear case, i.e., if $p = 2$ and $a(x, \nabla u) = \mathcal{A}(x)\nabla u$, where $\mathcal{A}$ is an $N \times N$ matrix such that

$$|\mathcal{A}(x)\xi| \le c_0|\xi| \text{ and } \mathcal{A}(x)\xi\xi \ge c_1|\xi|^2, \ \forall \xi \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega, \tag{1.4.1}$$

with c_0 and c_1 two positive real constants, problem (1.3.7) was studied by G. Stampacchia, who introduced and studied in [53] a notion of solution defined by duality.

Definition 1.4.1. A function $u_\mu \in L^1(\Omega)$ is a solution in the sense of Stampacchia (also called solution by duality) of the equation

$$\begin{cases} Au_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.2}$$

if

$$\int_\Omega u_\mu g \, dx = \int_\Omega u_g^* \, d\mu, \quad \forall g \in L^\infty(\Omega), \tag{1.4.3}$$

where u_g^* is the solution of

$$\begin{cases} A^* u_g^* = g & \text{in } \Omega \\ u_g^* \in H_0^1(\Omega) \end{cases}$$

and A^* is the adjoint of A.

For this theory we need to assume that the boundary $\partial\Omega$ has the following property, which is satisfied in particular when $\partial\Omega$ is Lipschitz.

Definition 1.4.2. We say that a bounded open subset of $\mathbb{R}^N$ is regular if there exist two positive constants $\delta \in (0,1)$ and R_0 such that, for every $x_0 \in \partial\Omega$ and for all $R < R_0$, we have

$$|B_R(x_0) \setminus \Omega| > \delta\, |B_R(x_0)|,$$

where $B_R(x_0)$ denotes the ball of center x_0 and radius R, and $|\cdot|$ the Lebesgue measure in $\mathbb{R}^N$.

Actually, the theory of Stampacchia works under slightly weaker but more complicated assumptions as said in [53] (see Definition 6.2).

The notion of solution by duality relies on the following regularity results due to G. Stampacchia (see [53]) and to E. De Giorgi (see [31]).

Before stating these theorems we recall that, for $\gamma \in (0,1)$,

$$C^{0,\gamma}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}) : \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} < +\infty \right\},$$

is the space of γ-Hölder continuous functions on $\overline{\Omega}$, with the norm

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

Theorem 1.4.3. *Let Ω be a regular (in the sense of Definition 1.4.2) subset of $\mathbb{R}^N$, $\mathcal{A}$ satisfy (1.4.1), $g \in L^m(\Omega)$, with $m > \frac{N}{2}$, and u_g be the (weak) solution in $H_0^1(\Omega)$ of the Dirichlet problem*

$$\begin{cases} Au = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \in C^{0,\gamma}(\overline{\Omega})$, and

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u_g(x)-u_g(y)|}{|x-y|^\gamma} \le C(\|g\|_{L^m(\Omega)}),$$

where the constants C and γ depend on Ω, N, c_0, c_1 and m.

Theorem 1.4.4. *Let Ω be a bounded open set of $\mathbb{R}^N$, $\mathcal{A}$ satisfy (1.4.1), $g \in L^m(\Omega)$, with $m > \frac{N}{2}$, and u_g be the (weak) solution in $H_0^1(\Omega)$ of the Dirichlet problem*

$$\begin{cases} Au = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|u_g\|_{L^\infty(\Omega)} \le C(\|g\|_{L^m(\Omega)}),$$

where the constant C depends on $|\Omega|$, N, c_0, c_1, and m.

Corollary 1.4.5. *Let Ω be a regular (in the sense of Definition 1.4.2) subset of $\mathbb{R}^N$, $\mathcal{A}$ satisfy (1.4.1), $g \in L^m(\Omega)$, with $m > \frac{N}{2}$, and u_g be the (weak) solution in $H_0^1(\Omega)$ of the Dirichlet problem*

$$\begin{cases} Au = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u_g \in C_0(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$ and

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C(\|g\|_{L^m(\Omega)}), \tag{1.4.4}$$

where the constants C and γ depend on Ω, N, c_0, c_1, and m.

Thanks to Corollary 1.4.5 we can, first of all, notice that Definition 1.4.1 makes sense. In the first term of (1.4.3) $g \in L^\infty(\Omega)$ and $u_\mu \in L^1(\Omega)$, in the second one $u_g^* \in C_0(\Omega)$.

The next theorem regards the existence and uniqueness of u_μ.

Theorem 1.4.6. *Let Ω be a regular subset of $\mathbb{R}^N$, and let $\mu \in \mathcal{M}_b(\Omega)$. Then, for every q, with $1 < q < \frac{N}{N-1}$, there exists a unique $u_\mu \in W_0^{1,q}(\Omega)$ solution of (1.4.2) in the sense of Definition 1.4.1.*

The solution u_μ satisfies in particular

$$\int_\Omega \mathcal{A}(x)\nabla u_\mu \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu, \quad \text{for every } \varphi \in C_c^\infty(\Omega), \tag{1.4.5}$$

i.e., u_μ is a solution of (1.4.2) which belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, and satisfies the equation in the distributional sense. Let us emphasize that (1.4.5) is not enough to characterize the (unique) solution in the sense of Definition 1.4.1 when the coefficients of the matrix $\mathcal{A}$ are discontinuous (see [52]).

It is worth to notice that the theory of Stampacchia is consistent with the variational one. More precisely, if the datum μ belongs to $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, then the solution obtained by duality coincides with the variational one (which satisfies (1.4.2) in the weak sense).

From now on we will use the following notation: u_μ denotes the solution of the equation

$$\begin{cases} Au_\mu = \mu & \text{in } \Omega \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ is a measure in $\mathcal{M}_b(\Omega)$ or an element of $H^{-1}(\Omega)$. In the former case we refer to Definition 1.4.1, in the latter to the usual variational one.

Another important fact is the continuous dependence with respect to data converging in the $*$-weak topology of $\mathcal{M}_b(\Omega)$.

Theorem 1.4.7. *Let Ω be a regular subset of $\mathbb{R}^N$. Let μ_n, $\mu \in \mathcal{M}_b(\Omega)$ be such that μ_n converges to μ $*$-weakly in $\mathcal{M}_b(\Omega)$, then u_{μ_n} tends to u_μ strongly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, and, for every $j > 0$, $T_j(u_{\mu_n})$ tends to $T_j(u_\mu)$ weakly in $H_0^1(\Omega)$.*

Remark 1.4.8. By the previous theorem we deduce that, when $\mu \in \mathcal{M}_b(\Omega)$, u_μ is the unique solution in the sense of distributions of the equation $Au = \mu$ (u_μ satisfies (1.4.5)) which can be obtained as limit of solutions u_n to the problem

$$\begin{cases} Au_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where f_n is a sequence of smooth functions (e.g., in $C_c^\infty(\Omega)$) converging to μ in the $*$-weak topology of measures.

1.5. Reachable Solutions

Following these ideas, in the nonlinear case, the first attempt to solve problem (1.3.7) was done by L. Boccardo and T. Gallouët, who proved in [7] and [9], under the hypothesis $p > 2 - \frac{1}{N}$, the existence of a solution of (1.3.7) which satisfies

$$\begin{cases} u \in W_0^{1,s}(\Omega), \ \forall s < \frac{N(p-1)}{N-1}, \\ \int_\Omega a(x, \nabla u)\nabla\varphi \, dx = \int_\Omega \varphi \, d\mu, \text{ for every } \varphi \in C_c^\infty(\Omega). \end{cases} \tag{1.5.1}$$

Note that this framework coincides with the framework given by (1.4.5) if $p = 2$. In (1.5.1) the exponent $\frac{N(p-1)}{N-1}$ is sharp. Hence, the restriction on p is motivated by the fact that, if $p \leq 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} \leq 1$: in order to obtain the existence of a solution for p close to 1, it is necessary to go out of the framework of classical Sobolev spaces.

Definition 1.5.1. Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in W^{-1,p'}(\Omega)$. We say that a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is a reachable solution of the problem

$$\begin{cases} A(u) = \mu + F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5.2}$$

if there exist two sequences μ_n and u_n such that

(i) $\mu_n \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$ and μ_n converges to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$;

(ii) $u_n \in W_0^{1,p}(\Omega)$, and u_n solves the Dirichlet problem

$$\begin{cases} A(u_n) = \mu_n + F & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5.3}$$

in the weak sense;

(iii) u_n converges to u a.e. in Ω.

Remark 1.5.2. The notion of reachable solution was introduced in [28] (see Definition 2.3) considering smooth approximations μ_n and F_n of μ and F such that
(i)' $\mu_n \in C_c^\infty(\Omega)$ and μ_n converges to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$,
(ii)' $F_n \in C_c^\infty(\Omega)$ and F_n converges to F strongly in $W^{-1,p'}(\Omega)$;
while u_n is the solution of

$$\begin{cases} A(u_n) = \mu_n + F_n & \text{in } \Omega \\ u_n \in W_0^{1,p}(\Omega). \end{cases}$$

Thanks to Proposition 3.4 of [28], Definition 1.5.1 turns out to be equivalent to Definition 2.3 of [28], when $F = 0$.

Theorem 1.5.3. *Let $\mu \in \mathcal{M}_b(\Omega)$ and $F \in W^{-1,p'}(\Omega)$; under hypotheses (1.3.1), (1.3.2), (1.3.3), and (1.3.4), there exist a reachable solution u of (1.5.2). Moreover, assuming (i)–(iii), $|\nabla u_n|^{p-1}$ is bounded in $L^q(\Omega)$ for every $q < \frac{N}{N-1}$, $T_j(u_n)$ converges to $T_j(u)$ weakly in $W_0^{1,p}(\Omega)$ for every $j > 0$, ∇u_n converges to ∇u a.e. in Ω, and $a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ strongly in $L^q(\Omega)^N$ for every $q < \frac{N}{N-1}$.*

Before proving Theorem 1.5.3, we state two results (see Lemma 4.1 and Lemma 4.2 of [3]), in which you control the measure of the level sets of a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ and of its gradient ∇u, when $T_j(u)$ is bounded in $W_0^{1,p}(\Omega)$ in an appropriate way.

Lemma 1.5.4. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be such that*

$$\int_{\{|u|<j\}} |\nabla u|^p \, dx \le (j+1)M, \tag{1.5.4}$$

for every $j > 0$. If $1 < p < N$, there exists $C = C(N,p) > 0$ such that

$$|\{|u| > h\}| \le C M^{\frac{N}{N-p}} \frac{1}{h^{p_1}}, \quad \forall h \ge 1, \tag{1.5.5}$$

where $p_1 = \dfrac{N(p-1)}{N-p}$. If $p = N$, for every $r > 1$, there exists $C = C(N,r) > 0$ such that

$$|\{|u| > h\}| \le C M^r \frac{1}{h^{r(N-1)}}, \quad \forall h \ge 1. \tag{1.5.6}$$

Lemma 1.5.5. *Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ satisfy (1.5.4) for every $j > 0$. If $1 < p < N$, then, for every $h \geq 1$,*

$$|\{|\nabla u| > h\}| \leq C(N,p) M^{\frac{N}{N-1}} \frac{1}{h^{p_3}}, \tag{1.5.7}$$

where $p_3 = \dfrac{N(p-1)}{N-1}$. If $p = N$, then, for every l, with $N-1 < l < N$, and for every $h \geq 1$,

$$|\{|\nabla u| > h\}| \leq C(N,l) M^{\frac{l}{N-1}} \frac{1}{h^{l}}. \tag{1.5.8}$$

Moreover in the proof of Theorem 1.5.3, we will need the following standard lemma of measure theory.

Lemma 1.5.6. *Let $(X, \mathcal{M}, m)$ a measurable space such that $m(X) < +\infty$. Let $\gamma : X \to [0, +\infty]$ a measurable function such that $m(\{x : \gamma(x) = 0\}) = 0$. Then for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that*

$$\int_E \gamma(x)\, dm \leq \delta_\varepsilon \Rightarrow m(E) \leq \varepsilon.$$

Proof of Theorem 1.5.3. Let μ_n be in $\mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$, converging to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, and let u_n be the solution of (1.5.3). If we take $v = T_j(u_n)$ in the variational formulation (1.3.6) satisfied by u_n, applying in the first term (1.3.2) and in the last one Young's inequality, we get:

$$c_1 \int_\Omega |\nabla T_j(u_n)|^p\, dx \leq j\|\mu_n\|_{\mathcal{M}_b(\Omega)} + \|k_1\|_{L^1(\Omega)} + c_2\|F\|^{p'}_{W^{-1,p'}(\Omega)} + \frac{c_1}{2} \int_\Omega |\nabla T_j(u_n)|^p\, dx,$$

and then

$$\int_\Omega |\nabla T_j(u_n)|^p\, dx \leq (j+1)M, \tag{1.5.9}$$

for every $j > 0$. Let us prove that $u_n \to u$ in measure. To begin with, we observe that for $t, \varepsilon > 0$ we have

$$\{|u_n - u_m| > t\} \subseteq \{|u_n| > h\} \cup \{|u_m| > h\} \cup \{|T_h(u_n) - T_h(u_m)| > t\}. \tag{1.5.10}$$

Thanks to Lemma 1.5.4, we choose h large enough such that

$$|\{|u_n| > h\}| < \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, since $T_j(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for all $j > 0$ we can assume that $T_j(u_n)$ is a Cauchy sequence in measure. By (1.5.10), this proves that $u_n \to u$ in measure.

We now prove that ∇u_n converges to some function v in measure, and, therefore, after passing to a suitable subsequence, we can always assume that the convergence is a.e. in Ω. To prove this we show that ∇u_n is a Cauchy sequence in measure. Let again t and $\varepsilon > 0$. Then

$$\{|\nabla u_n - \nabla u_m| > t\} \subseteq \{|\nabla u_n| > B\} \cup \{|\nabla u_m| > B\} \cup \{|u_n - u_m| > j\}$$
$$\cup \{|u_n - u_m| \le j, |\nabla u_n| \le B, |\nabla u_m| \le B, |\nabla u_n - \nabla u_m| > t\}.$$

We first choose B large enough in order to have

$$|\{|\nabla u_n| > B\}| < \varepsilon, \quad \text{for all } n \in \mathbb{N}$$

(this is possible by Lemma 1.5.5).

Since $a(x, \cdot)$ is continuous for almost every $x \in \Omega$, assumption (1.3.3) implies that there exists a real valued function $\gamma(x)$ such that $|\{x : \gamma(x) = 0\}| = 0$ and

$$(a(x,\xi) - a(x,\eta))(\xi - \eta) \ge \gamma(x), \quad \text{q.o. in } \Omega \text{ and } \forall \xi, \eta \in K(\xi,\eta),$$

where $K(\xi,\eta) = \{\xi, \eta \in \mathbb{R}^N : |\xi|, |\eta| \le B, |\xi - \eta| \ge t\}$ is a compact set. Hence

$$\int_{K(\nabla u_n, \nabla u_m) \cap \{|u_n - u_m| \le j\}} \gamma(x)$$
$$\le \int_{K(\nabla u_n, \nabla u_m) \cap \{|u_n - u_m| \le j\}} (a(x, \nabla u_n) - a(x, \nabla u_m))(\nabla u_n - \nabla u_m)\, dx$$
$$= \int_{K(\nabla u_n, \nabla u_m) \cap \{|u_n - u_m| \le j\}} (a(x, \nabla u_n) - a(x, \nabla u_m))\nabla T_j(u_n - u_m)\, dx.$$

If we use $T_j(u_n - u_m)$ in (1.3.6) we can say that the last integral is less than or equal to $2jM$, where $M \ge \|\mu_n\|_{\mathcal{M}_b(\Omega)}$, for every $n \in \mathbb{N}$.

Let now δ_ε given from Lemma 1.5.6. We choose j such that $j < \frac{\delta_\varepsilon}{2M}$, then $|K(\nabla u_n, \nabla u_m) \cap \{|u_n - u_m| \le j\}| < \varepsilon$.

Finally, thanks to the fact that u_n is a Cauchy sequence in measure, we obtain the desired compactness result.

In conclusion, since $T_j(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, we have established the following facts:

$u \in \mathcal{T}_0^{1,p}(\Omega)$,

u_n converges to u a.e. in Ω,

$T_j(u_n)$ converges to $T_j(u)$ weakly in $W_0^{1,p}(\Omega)$, for every $j > 0$,

∇u_n converges to ∇u a.e. in Ω.

To obtain the boundedness of $|\nabla u_n|^{p-1}$ in $L^q(\Omega)$, for every q, with $1 < q < \frac{N}{N-1}$, we observe that, by (1.5.7) of Lemma 1.5.5, we have

$$\begin{aligned}\int_\Omega |\nabla u_n|^{q(p-1)}\,dx &= |\Omega| + q(p-1)\int_1^{+\infty} t^{q(p-1)-1}|\{|\nabla u_n| > t\}|\,dt \\ &\le |\Omega| + q(p-1)\int_1^{+\infty} t^{q(p-1)-1}\frac{c}{t^{\frac{N(p-1)}{N-1}}}\,dt < +\infty,\end{aligned}$$

since $\frac{N(p-1)}{N-1} - q(p-1) + 1 > 1$ for every $q < \frac{N}{N-1}$. Thus

$$|\nabla u_n|^{p-1} \text{ is bounded in } L^q(\Omega), \text{ for every } q < \tfrac{N}{N-1}. \tag{1.5.11}$$

If $p = N$ we obtain again (1.5.11) using (1.5.8) instead of (1.5.7); to derive it we proceed as before: the only change consists in observing that, once $1 < q < \frac{N}{N-1}$ is fixed, it is then possible to choose $N-1 < l < N$ such that $l > q(N-1)$, so that

$$\int_1^{+\infty} t^{q(N-1)-1}\frac{c}{t^l}\,dt < +\infty.$$

Then also $a(x,\nabla u_n)$ is bounded in $L^q(\Omega)^N$, thanks to (1.3.1). On the other hand the almost everywhere convergence of ∇u_n to ∇u implies that $a(x,\nabla u_n)$ converges to $a(x,\nabla u)$ a.e. in Ω. At this point it is easy to prove that

$$a(x,\nabla u_n) \text{ converges to } a(x,\nabla u) \text{ strongly in } L^q(\Omega)^N, \text{ for every } q < \tfrac{N}{N-1},$$

which concludes the proof. □

Remark 1.5.7. Let us note that, thanks to (1.5.9), the reachable solutions of (1.5.2) satisfy (1.5.4), with M greater than or equal to $c_3(\|\mu_n\|_{\mathcal{M}_b(\Omega)} \vee (\|k_1\|_{L^1(\Omega)} + \|F\|^{p'}_{W^{-1,p'}(\Omega)}))$, where c_3 depends only on p and c_1. Moreover, by standard arguments of capacity theory, (1.5.4) implies

$$C_p(\{|u| > j\}) \le \frac{M(j+1)}{j^p}; \tag{1.5.12}$$

that is, if u is a reachable solution of (1.5.2), then (the C_p-quasi continuous representative of) u is finite up to a set of p-capacity zero.

Remark 1.5.8. Using the convergence of $a(x,\nabla u_n)$ to $a(x,\nabla u)$ we can prove for u the distributional formulation

$$\int_\Omega a(x,\nabla u)\nabla\varphi\,dx = \int_\Omega \varphi\,d\mu + \langle F,\varphi\rangle, \text{ for every } \varphi \in C_c^\infty(\Omega), \tag{1.5.13}$$

starting from the same equality satisfied by u_n and taking the limit as n tends to $+\infty$.

Remark 1.5.9. We note that if $p \geq 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} > 1$, so that, taking into account that $\frac{N(p-1)}{N-1} < \frac{N(p-1)}{N-p}$, by Lemma 1.5.4 and Lemma 1.5.5 we easily deduce that $u \in W_0^{1,s}(\Omega)$, for every $s < \frac{N(p-1)}{N-1}$. Moreover, u_n converges to u strongly in $W_0^{1,s}(\Omega)$.

Remark 1.5.10. Let $\mu \in \mathcal{M}_b(\Omega)$ and $F = 0$. If u is a reachable solution of (1.3.7) and we assume (i)-(iii) of Definition 1.5.1, we can not expect the strong convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_n)$ to $T_j(u)$ (see [13] for a counterexample). On the other hand, thanks to the next proposition (see [14]), for every reachable solution u of (1.3.7) there exist two sequences $\mu_n \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$ and u_n satisfying (i)-(iii) of Definition 1.5.1, such that $T_j(u_n)$ converges to $T_j(u)$ strongly in $W_0^{1,p}(\Omega)$, for every $j > 0$.

Proposition 1.5.11. *Let $\mu \in \mathcal{M}_b(\Omega)$ and u be a reachable solution of (1.3.7). Under hypotheses (1.3.1), (1.3.2), (1.3.3), (1.3.4), we have*

$$A(T_n(u)) \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega),$$

$$A(T_n(u)) \rightharpoonup \mu \ \ast\text{-weakly in } \mathcal{M}_b(\Omega).$$

Remark 1.5.12. By Remark 1.4.8 we have that, in the linear case, the reachable solution coincides with that obtained by duality by Stampacchia. Thus, in particular, it is unique.

On the other hand we just noted that (1.5.13) is not enough to ensure uniqueness. Indeed, in the linear case, Stampacchia's definition, which implies uniqueness requires stronger conditions on the solutions, namely that the equation is satisfied for a larger space of test functions.

As a matter of fact, for a general monotone operator A the uniqueness of the reachable solution is still an open problem, except in the case $p = 2$, where Stampacchia's ideas continue to work if A is strongly monotone and Lipschitz continuous. In this special case, indeed, F. Murat (in [48]) proved also the uniqueness of the reachable solution.

Let us consider this particular setting.

We assume that Ω is a regular (in the sense of Definition (1.4.2)) subset of $\mathbb{R}^N$, and we consider $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function such that for every ξ, η in $\mathbb{R}^N$ ($\xi \neq \eta$), and for almost every x in Ω,

$$|a(x,\xi) - a(x,\eta)| \leq c_0|\xi - \eta|, \tag{1.5.14}$$

$$(a(x,\xi)-a(x,\eta))(\xi-\eta)\geq c_1|\xi-\eta|^2, \tag{1.5.15}$$

$$a(x,0)=0, \tag{1.5.16}$$

where c_0 and c_1 are two positive real constants.

Theorem 1.5.13. *Assuming (1.5.14), (1.5.15), and (1.5.16), for every $\mu \in \mathcal{M}_b(\Omega)$ and for every $F \in H^{-1}(\Omega)$ there exists a unique reachable solution of problem (1.5.2).*

Proof. Let us consider two sequences μ_n, $\hat{\mu}_n \in \mathcal{M}_b(\Omega)\cap H^{-1}(\Omega)$, and the corresponding solutions u_n and $\hat{u}_n$ of (1.5.3), such that

$$\mu_n \rightharpoonup \mu \text{ and } \hat{\mu}_n \rightharpoonup \mu \text{ *-weakly in } \mathcal{M}_b(\Omega)$$
$$u_n \to u \text{ and } \hat{u}_n \to \hat{u} \text{ a.e. in } \Omega.$$

By Theorem 1.5.3 and Remark 1.5.9 we also know that, for every $q<\frac{N}{N-1}$ and for every $j>0$,

$$u_n \to u \text{ strongly in } W_0^{1,q}(\Omega) \text{ and } T_j(u_n) \rightharpoonup T_j(u) \text{ weakly in } H_0^1(\Omega),$$

as well as

$$\hat{u}_n \to \hat{u} \text{ strongly in } W_0^{1,q}(\Omega) \text{ and } T_j(\hat{u}_n) \rightharpoonup T_j(\hat{u}) \text{ weakly in } H_0^1(\Omega).$$

We consider the equations (1.3.6) satisfied by u_n and $\hat{u}_n$, respectively; they give:

$$\int_\Omega (a(x,\nabla u_n)-a(x,\nabla \hat{u}_n))\nabla\varphi\,dx = \langle \mu_n-\hat{\mu}_n,\varphi\rangle, \tag{1.5.17}$$

for every $\varphi \in H_0^1(\Omega)$.

We will use the following lemma.

Lemma 1.5.14. *Let a satisfy conditions (1.5.14), (1.5.15), and (1.5.16); then, for almost every $x\in\Omega$ and for every ξ_1, $\xi_2 \in \mathbb{R}^N$, there exists a $N\times N$ matrix $\mathcal{M}=\mathcal{M}(x,\xi_1,\xi_2)$ such that*

$$|\mathcal{M}\eta| \leq C_0|\eta| \quad \textit{and} \quad \mathcal{M}\eta\eta \geq C_1|\eta|^2, \tag{1.5.18}$$

$$a(x,\xi_1)-a(x,\xi_2)=\mathcal{M}(\xi_1-\xi_2), \tag{1.5.19}$$

where C_0 and C_1 are two positive constants that depend only on c_0 and c_1

Let us define the measurable function $\mathcal{M}_n(x) := \mathcal{M}(x, \nabla u_n(x), \nabla \hat{u}_n(x))$, with $\mathcal{M}$ as in the previous lemma. Thanks to (1.5.18) the operator $M_n u = -\mathrm{div}(\mathcal{M}_n(x)\nabla u)$ is a (linear) uniformly (with respect to n) bounded and elliptic operator from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$.

Thus, we can rewrite (1.5.17) as

$$\int_\Omega \mathcal{M}_n(x)\nabla(u_n - \hat{u}_n)\nabla\varphi\, dx = \langle \mu_n - \hat{\mu}_n, \varphi\rangle, \tag{1.5.20}$$

for every $\varphi \in H_0^1(\Omega)$.

Now we choose as φ the solution w_n of the Dirichlet problem

$$\begin{cases} M_n^* w_n = g & \text{in } \Omega \\ w_n \in H_0^1(\Omega), \end{cases}$$

where g is a function in $L^\infty(\Omega)$.

We can apply Corollary 1.4.5 to w_n and deduce by the Ascoli-Arzelà Theorem that w_n converges (up to a subsequence, still denoted by w_n) to a function $w \in C_0(\Omega)$ uniformly in $\overline{\Omega}$. Rewriting the equation (1.5.20) with $\varphi = w_n$, we obtain

$$\int_\Omega g(u_n - \hat{u}_n)\, dx = \int_\Omega w_n\, d(\mu_n - \hat{\mu}_n),$$

and, passing to the limit

$$\int_\Omega g(u - \hat{u})\, dx = 0.$$

For the arbitrarity of the function $g \in L^\infty(\Omega)$, we conclude that $u = \hat{u}$. □

We will prove now the important Lemma 1.5.14.

Proof of Lemma 1.5.14. Let us denote by λ and ν the vectors $\lambda := \xi_1 - \xi_2$ and $\nu := a(x, \xi_1) - a(x, \xi_2)$. Thanks to (1.5.14) and (1.5.15) we have

$$\nu\lambda \geq c_1|\lambda|^2 \quad \text{and} \quad |\nu| \leq c_0|\lambda|.$$

Define the matrix $\mathcal{M} := C_0\mathcal{I} + (s\lambda + t\nu)\otimes(s\lambda + t\nu)$, with $\mathcal{I}$ the identity $N \times N$ matrix, and C_0, s, t to be determined, and check that $\mathcal{M}$ satisfies (1.5.19), i.e.,

$$\nu = C_0\lambda + (s\lambda + t\nu)\otimes(s\lambda + t\nu)\lambda = C_0\lambda + (s\lambda + t\nu)(s|\lambda|^2 + t\nu\lambda),$$

which holds if

$$1 - t(s|\lambda|^2 + t\nu\lambda) = 0 \quad \text{and} \quad C_0 + s(s|\lambda|^2 + t\nu\lambda) = 0.$$

If we assume that $\nu\lambda - C_0|\lambda|^2 > 0$, which is true if $C_0 < c_1$, the choice of t and s will be

$$t = \pm\frac{1}{\sqrt{\nu\lambda - C_0|\lambda|^2}}, \quad s = \mp\frac{C_0}{\sqrt{\nu\lambda - C_0|\lambda|^2}}.$$

Thus $\mathcal{M}$ will be

$$\mathcal{M} = C_0\mathcal{I} + \frac{1}{\nu\lambda - C_0|\lambda|^2}(C_0\lambda - \nu) \otimes (C_0\lambda - \nu).$$

It can be easily proved that $\mathcal{M}$ satisfies the ellipticity condition with $0 < C_0 (< c_1)$, and the boundedness estimate with $C_1 = C_0 + \dfrac{C_0^2 + c_0^2}{c_1 - C_0}$. □

Remark 1.5.15. If we assume that the function a is of class C^1 with respect to the variable ξ, we can find $\mathcal{M}$ in a more direct way.

Indeed, we can write the difference $a(x,\xi_1) - a(x,\xi_2)$ as

$$\begin{aligned} a(x,\xi_1) - a(x,\xi_2) &= \int_0^1 \frac{d}{dt} a(x, \xi_2 + t(\xi_1 - \xi_2))\, dt \\ &= \left(\int_0^1 a_\xi(x, \xi_2 + t(\xi_1 - \xi_2))\, dt\right)(\xi_1 - \xi_2), \end{aligned}$$

from which we can define $\mathcal{M} = \mathcal{M}(x,\xi_1,\xi_2) := \int_0^1 a_\xi(x,\xi_2 + t(\xi_1 - \xi_2))\, dt$.

Moreover, $\mathcal{M}$ satisfies the ellipticity condition with the constant c_1 and the boundedness estimate with c_0.

The same tools used to prove the uniqueness of the reachable solution can be easily applied to obtain a stability result for problem (1.3.7).

Theorem 1.5.16. *Assume (1.5.14), (1.5.15), and (1.5.16). Let $F \in H^{-1}(\Omega)$ and μ_k, $\mu \in \mathcal{M}_b(\Omega)$ be such that*

$$\mu_k \rightharpoonup \mu \quad *\text{-weakly in } \mathcal{M}_b(\Omega),$$

and let u_k and u be the reachable solutions with data $\mu_k + F$ and $\mu + F$. Then

$$u_k \to u \quad \text{strongly in } W_0^{1,q}(\Omega), \text{ for every } q < \tfrac{N}{N-1},$$

$$T_j(u_k) \rightharpoonup T_j(u) \quad \textit{weakly in } H_0^1(\Omega), \textit{ for every } j > 0.$$

Proof. Let us consider two sequences μ_k^n, $\mu^n \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, and the corresponding solutions u_k^n and u^n of (1.5.3), such that

$$\mu_k^n \rightharpoonup \mu_k \ \text{ and } \ \mu^n \rightharpoonup \mu \ *\text{-weakly in } \mathcal{M}_b(\Omega)$$
$$u_k^n \to u_k \ \text{ and } \ u^n \to u \ \text{ a.e. in } \Omega.$$

By Theorem 1.5.3 and Remark 1.5.9, we also know that, for every $q < \frac{N}{N-1}$ and for every $j > 0$,

$$u_k^n \to u_k \ \text{ strongly in } W_0^{1,q}(\Omega) \ \text{ and } \ T_j(u_k^n) \rightharpoonup T_j(u_k) \ \text{ weakly in } H_0^1(\Omega),$$

as well as

$$u^n \to u \ \text{ strongly in } W_0^{1,q}(\Omega) \ \text{ and } \ T_j(u^n) \rightharpoonup T_j(u) \ \text{ weakly in } H_0^1(\Omega).$$

We consider the equations (1.3.6) satisfied by u_k^n and u^n, respectively; they give:

$$\int_\Omega (a(x, \nabla u_k^n) - a(x, \nabla u^n)) \nabla \varphi \, dx = \int_\Omega \varphi \, d(\mu_k^n - \mu^n),$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. The previous equation can be written as

$$\int_\Omega \mathcal{M}_k^n(x) \nabla (u_k^n - u^n) \nabla \varphi \, dx = \int_\Omega \varphi \, d(\mu_k^n - \mu^n), \tag{1.5.21}$$

where $\mathcal{M}_k^n(x) = \mathcal{M}(x, \nabla u_k^n(x), \nabla u^n(x))$ is given by Lemma 1.5.14. Now we choose as φ the solution of the Dirichlet problem

$$\begin{cases} M_k^{n*} w_k^n = g & \text{in } \Omega \\ w_k^n \in H_0^1(\Omega), \end{cases}$$

where g is a function in $L^\infty(\Omega)$.

We can apply Corollary 1.4.5 to w_k^n to deduce by the Ascoli-Arzelà theorem that w_k^n converges (up to subsequences) to a function $w_k \in C_0(\Omega)$ uniformly in $\overline{\Omega}$. On the other hand, thanks to this convergence, also w_k satisfies (1.4.4), and, by the same arguments, w_k converges (up to subsequences) to $w \in C_0(\Omega)$ uniformly in $\overline{\Omega}$. Thus, rewriting the equation (1.5.21) with $\varphi = w_k^n$, we obtain

$$\int_\Omega g(u_k^n - u^n) \, dx = \int_\Omega w_k^n \, d(\mu_k^n - \mu^n),$$

and, passing to the limit, first as $n \to +\infty$, then as $k \to +\infty$, we have

$$\lim_{k\to+\infty} \int_\Omega g(u_k - u)\, dx = 0.$$

On the other hand, for every $j > 0$, $T_j(u_k)$ satisfies (1.5.4) (see Remark 1.5.7). Hence, by Lemma 1.5.4 and Lemma 1.5.5 (see also Remark 1.5.9), we get

$$u_k \to u \;\text{ strongly in } W_0^{1,q}(\Omega), \text{ for every } q < \tfrac{N}{N-1},$$

$$T_j(u_k) \rightharpoonup T_j(u) \;\text{ weakly in } H_0^1(\Omega), \text{ for every } j > 0$$

(the result is true for the whole sequence u_k by the uniqueness of the limit function u). □

Finally, we state a comparison principle concerning the reachable solutions.

Theorem 1.5.17. *Assume (1.5.14), (1.5.15), and (1.5.16). For $i = 1, 2$, let u_i be the reachable solution of (1.3.7) relative to $\mu_i \in \mathcal{M}_b(\Omega)$. Suppose that $\mu_1 \le \mu_2$, then $u_1 \le u_2$ almost everywhere in Ω.*

Proof. The result is well known when $\mu_i \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$.

Approximate, in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, $\mu_2 - \mu_1$ and μ_1 with $\mu_n \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ and $\mu_n^1 \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, respectively. Moreover, define $\mu_n^2 = \mu_n + \mu_n^1$, which is greater than or equal to μ_n^1. Thanks to Theorem 1.5.16 we can conclude. □

1.6. Entropy Solutions

In the nonlinear case, when μ is a bounded Radon measure vanishing on all sets of p-capacity zero, other types of solution of (1.3.7) have been proposed. The notion of entropy solution, of *SOLA*, and of renormalized solution were introduced respectively in [3], [20], and [46].

These three frameworks, which are actually equivalent, are successful since they allow one to prove existence, uniqueness and continuity of the solutions with respect to the datum μ.

Definition 1.6.1. Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$. We say that a function $u \in \mathcal{T}^{1,p}_0(\Omega)$, satisfying (1.5.4) for every $j > 0$, is an entropy solution of (1.3.7) if

$$\int_\Omega a(x, \nabla u)\nabla(T(u-\varphi)\omega + \Phi)\,dx = \int_\Omega (T(u-\varphi)\omega + \Phi)\,d\mu, \tag{1.6.1}$$

where $\Phi \in W^{1,q'}_0(\Omega)$, for every $q < \frac{N}{N-1}$, $\omega \in C^1(\overline{\Omega})$, $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, and $T : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function such that

$$T(0) = 0 \text{ and } T'(t) = 0 \text{ if } |t| \geq k \text{ for some } k > 0.$$

Remark 1.6.2. Notice that both integrals in (1.6.1) are well defined. First of all, we remark that, if $q < \frac{N}{N-1}$, then $q' > N$, so that, by Sobolev embedding theorems, $W^{1,q'}_0(\Omega) \subseteq C(\overline{\Omega})$. Moreover $\nabla T(u-\varphi) = 0$ a.e. where $|u| > k + \|\varphi\|_{L^\infty(\Omega)}$. Hence, remembering that $u \in \mathcal{T}^{1,p}_0(\Omega)$, the second member offers no difficulty since $T(u-\varphi)\omega + \Phi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathcal{M}^p_{b,0}(\Omega)$. As to the first member we observe that

$$a(x, \nabla u)\nabla T(u-\varphi)\,\omega = a(x, \nabla T_{k+\|\varphi\|_{L^\infty(\Omega)}}(u))\nabla T(u-\varphi)\,\omega \in L^1(\Omega)^N,$$

thanks to (1.3.1). On the other hand, by Lemma 1.5.5, for every $q < \frac{N}{N-1}$, $|\nabla u|^{p-1} \in L^q(\Omega)$, so that, by (1.3.1), $a(x, \nabla u) \in L^q(\Omega)^N$, and

$$a(x, \nabla u)\nabla\omega T(u-\varphi) + a(x, \nabla u)\nabla\Phi \in L^1(\Omega)^N.$$

Remark 1.6.3. Starting from equation (1.6.1) we have, for every $j > 0$ and for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u)\nabla T_j(u-\varphi)\,dx = \int_\Omega T_j(u-\varphi)\,d\mu, \tag{1.6.2}$$

Moreover, it holds:

$$\int_\Omega a(x, \nabla u)\nabla\Phi\,dx = \int_\Omega \Phi\,d\mu, \tag{1.6.3}$$

for every $q < \frac{N}{N-1}$ and for every $\Phi \in W^{1,q'}_0(\Omega)$.

Remark 1.6.4. If $F \in W^{-1,p'}(\Omega)$ ($F = \mathrm{div} f$, with $f \in L^{p'}(\Omega)^N$) we can consider as data also $\mu + F$, the definition of entropy solution being

$$\int_\Omega a(x, \nabla u)\nabla(T(u-\varphi)\omega + \Phi)\,dx = \int_\Omega (T(u-\varphi)\omega + \Phi)\,d\mu + \langle F, T(u-\varphi)\omega + \Phi\rangle. \tag{1.6.4}$$

Theorem 1.6.5. *Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ and $F \in W^{-1,p'}(\Omega)$; under assumptions (1.3.1), (1.3.2), (1.3.3), and (1.3.4), there exists at least an entropy solution u of (1.3.7) relative to $\mu + F$.*

Proof. Following the "classical" procedure, our first step consists in approximating the measure μ with a sequence $\mu_n \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$, converging to μ in the weak topology of $\mathcal{M}_b(\Omega)$ (which is stronger than the $*$ weak convergence). Then, it is well known that there exists a weak solution $u_n \in W^{1,p}_0(\Omega)$ of

$$-\text{div}(a(x, \nabla u_n)) = \mu_n + F. \tag{1.6.5}$$

By Theorem 1.5.3, we know that a subsequence of u_n (still denoted by u_n) satisfy:

u_n converges to u a.e. in Ω,
$T_j(u_n)$ converges to $T_j(u)$ weakly in $W^{1,p}_0(\Omega)$,
∇u_n converges to ∇u a.e. in Ω,
$a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ strongly in $L^q(\Omega)^N$, for every $q < \frac{N}{N-1}$.

Moreover, by Remark 1.5.7, u satisfies (1.5.12), and so (the C_p-quasi continuous representative of) u is finite up to a set of p-capacity zero.

Now we prove that $T_j(u_n)$ strongly converges to $T_j(u)$ in $W^{1,p}_0(\Omega)$, for every $j > 0$. Let $h > j$ and let us take $w_n = T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u))$ as test function in (1.6.5). Then, if we set $l = 4j + h$, it is easy to see that $\nabla w_n = 0$ where $|u_n| > l$; therefore, thanks to hypothesis (1.3.4), we can write

$$\int_\Omega a(x, \nabla T_l(u_n))\nabla w_n \, dx = \int_\Omega w_n \, d\mu_n + \langle F, w_n \rangle.$$

Splitting the integral in the left hand side on the sets where $|u_n| \leq j$ and where $|u_n| > j$ we get (remember that $a(x,\xi)\xi \geq 0$):

$$\begin{aligned}&\int_\Omega a(x, \nabla T_l(u_n))\nabla T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u)) \, dx \\ &\geq \int_\Omega a(x, \nabla T_j(u_n))\nabla(T_j(u_n) - T_j(u)) \, dx - \int_{\{|u_n|>j\}} |a(x, \nabla T_l(u_n))||\nabla T_j(u)| \, dx,\end{aligned}$$

and then from the equation it follows:

$$\int_\Omega (a(x,\nabla T_j(u_n)) - a(x,\nabla T_j(u))\nabla(T_j(u_n)-T_j(u))\,dx$$
$$\leq \int_{\{|u_n|>j\}} |a(x,\nabla T_l(u_n))||\nabla T_j(u)|\,dx + \int_\Omega T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u))\,d\mu_n$$
$$+\langle F, T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u))\rangle - \int_\Omega a(x,\nabla T_j(u))\nabla(T_j(u_n)-T_j(u))\,dx. \tag{1.6.6}$$

Now, if we take $T_{2j}(u_n - T_h(u_n))$ in (1.6.5) we can proceed as in the beginning of the proof of Theorem 1.5.3 in order to have

$$\int_\Omega |\nabla T_{2j}(u_n - T_h(u_n))|^p\,dx \leq (2j+1)M_1,$$

where M_1 is a positive constant that does not depend on h and n. Since

$$T_{2j}(u_n - T_h(u_n)) \rightharpoonup T_{2j}(u - T_h(u)) \text{ weakly in } W_0^{1,p}(\Omega),$$

we get

$$\int_\Omega |\nabla T_{2j}(u - T_h(u))|^p\,dx \leq (2j+1)M_1,$$

from which we can deduce:

$$\langle F, T_{2j}(u - T_h(u))\rangle \leq M_2 \int_{\{|u|>h\}} |f|^{p'}\,dx,$$

where M_2 depends on j but not on h. Therefore by the absolute continuity of integral we get:

$$\lim_{h\to+\infty} \langle F, T_{2j}(u - T_h(u))\rangle = 0.$$

Since (1.5.12) implies also

$$\lim_{h\to+\infty} \int_\Omega T_{2j}(u - T_h(u))\,d\mu = 0,$$

we can fix a positive real number h_ε sufficiently large to have

$$\int_\Omega T_{2j}(u - T_{h_\varepsilon}(u))\,d\mu + \langle F, T_{2j}(u - T_{h_\varepsilon}(u))\rangle \leq \varepsilon.$$

Now we take $h = h_\varepsilon$ in formula (1.6.6) (then $l = l_\varepsilon$), and observe that $|a(x, \nabla T_l(u_n))|$ is bounded in $L^{p'}(\Omega)$ while $\chi_{\{|u_n|>j\}}|\nabla T_j(u)|$ converges strongly to zero in $L^p(\Omega)$ as n tends to infinity; this allows us to write

$$\lim_{n\to+\infty} \int_{\{|u_n|>j\}} |a(x, \nabla T_l(u_n))||\nabla T_j(u)|\, dx = 0. \tag{1.6.7}$$

Furthermore, it is easy to see that, as n tends to infinity,

$$T_{2j}(u_n - T_h(u_n) + T_j(u_n) - T_j(u)) \rightharpoonup T_{2j}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p}(\Omega),$$

so that, passing to the limit in (1.6.6), by means of (1.6.7) and Lemma 1.2.15, we deduce

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega (a(x, \nabla T_j(u_n)) - a(x, \nabla T_j(u))\nabla(T_j(u_n) - T_j(u))\, dx \\ \le \int_\Omega T_{2j}(u - T_{h_\varepsilon}(u))\, d\mu + \langle F, T_{2j}(u - T_{h_\varepsilon}(u))\rangle \le \varepsilon,\end{aligned}$$

that is to say, since ε is arbitrary small,

$$\lim_{n\to+\infty} \int_\Omega (a(x, \nabla T_j(u_n)) - a(x, \nabla T_j(u))\nabla(T_j(u_n) - T_j(u))\, dx = 0.$$

In conclusion (see [44]), we have that, for every $j > 0$, $T_j(u_n)$ converges to $T_j(u)$ strongly in $W_0^{1,p}(\Omega)$. We point out that this convergence implies that, for every $j > 0$, $a(x, \nabla T_j(u_n))$ converges to $a(x, \nabla T_j(u))$ strongly in $L^{p'}(\Omega)^N$.

Now we show that u is an entropy solution. In order to prove equality (1.6.4) we take a bounded Lipschitz function $T : \mathbb{R} \to \mathbb{R}$ such that $T(0) = 0$ and $T'(t) = 0$ when $|t| > k$; we also choose a smooth function $\omega \in C^1(\overline{\Omega})$, $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\Phi \in W_0^{1,q'}(\Omega)$, for every $q < \frac{N}{N-1}$, and apply the test function $T(u_n - \varphi)\omega + \Phi$ to equation (1.6.5) to get

$$\int_\Omega a(x, \nabla u_n)\nabla(T(u_n - \varphi)\omega + \Phi)\, dx = \int_\Omega (T(u_n - \varphi)\omega + \Phi)\, d\mu_n + \langle F, T(u_n - \varphi)\omega + \Phi\rangle,$$

which can be rewritten as

$$\begin{aligned}\int_\Omega a(x, \nabla T_L(u_n))\nabla T(u_n - \varphi)\,\omega\, dx + \int_\Omega a(x, \nabla u_n)\nabla\omega T(u_n - \varphi)\, dx \\ + \int_\Omega a(x, \nabla u_n)\nabla\Phi\, dx = \int_\Omega (T(u_n - \varphi)\omega + \Phi)\, d\mu_n + \langle F, T(u_n - \varphi)\omega + \Phi\rangle.\end{aligned}$$

where $L = k + \|\varphi\|_{L^\infty(\Omega)}$.

Taking into account the convergence results previously obtained and Lemma 1.2.15, we can easily pass to the limit as n tends to infinity, proving that u is an entropy solution. □

Now we settle the question of uniqueness. The main tool of the uniqueness proof is an estimate on the decay of the energy of the entropy solution on the sets where it is large.

Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, $F \in W^{-1,p'}(\Omega)$ ($F = \mathrm{div} f$, with $f \in L^{p'}(\Omega)^N$), and u be an entropy solution of (1.3.7) relative to $\mu + F$. As we noticed in Remark 1.6.3, for every $j > 0$ and for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, u satisfies

$$\int_\Omega a(x, \nabla u)\nabla T_j(u - \varphi)\,dx = \int_\Omega T_j(u - \varphi)\,d\mu + \langle F, T_j(u - \varphi)\rangle. \tag{1.6.8}$$

Thus, choosing $\varphi = T_i(u)$, for $i > 0$, and observing that $\nabla T_j(u - T_i(u)) = \nabla u$ where $i < |u| \le i + j$, and is zero elsewhere, we can write

$$\int_{\{i<|u|\le i+j\}} a(x, \nabla u)\nabla u\,dx \le j|\mu|(\{|u| > i\}) + \int_{\{i<|u|\le i+j\}} f\nabla u\,dx.$$

Using (1.3.2) (in the left hand side), and Young's inequality (in the right hand side), we get, setting $B_{i,j} = \{i < |u| \le i + j\}$ and $A_i = \{|u| > i\}$,

$$c_1 \int_{B_{i,j}} |\nabla u|^p\,dx \le j|\mu|(A_i) + \int_{B_{i,j}} |k_1|\,dx + c\int_{B_{i,j}} |f|^{p'}\,dx + \frac{c_1}{2}\int_{B_{i,j}} |\nabla u|^p\,dx.$$

Hence, in particular, we have

$$\lim_{i\to+\infty} \int_{B_{i,j}} |\nabla u|^p\,dx = 0, \tag{1.6.9}$$

since μ vanishes on all sets of p-capacity zero, and we can apply (1.5.12). As a matter of fact, (1.6.9) is true only if the datum μ belongs to $\mathcal{M}^p_{b,0}(\Omega)$, and not for a general reachable solution relative to $\mu \in \mathcal{M}_b(\Omega)$.

Theorem 1.6.6. *Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, $F \in W^{-1,p'}(\Omega)$, u and v two entropy solutions of the equation*

$$-div(a(x, \nabla u)) = \mu + F,$$

under assumptions (1.3.1), (1.3.2), (1.3.3), and (1.3.4). Then $u = v$.

Proof. Choose $T_j(u - T_i(v))$ as test function in (1.6.8) (written for u) and $T_j(v - T_i(u))$ as test function in (1.6.8) (written for v). Add the equations, so that

$$\int_\Omega (a(x, \nabla u) - f)\nabla T_j(u - T_i(v))\,dx + \int_\Omega (a(x, \nabla v) - f)\nabla T_j(v - T_i(u))\,dx$$

$$= \int_\Omega (T_j(u - T_i(v)) + T_j(v - T_i(u)))\, d\mu.$$

The right hand side of the preceding relation tends to zero as i tends to infinity, since $T_j(\cdot)$ is odd. For the left hand side, let us define (we have omitted the dependence on $x \in \Omega$ for the sake of brevity)

$$E_0 = \{|u - v| \le j, |u| \le i, |v| \le i\},$$

$$E_1 = \{|u - T_i(v)| \le j, |v| > i\}, \quad E_2 = \{|u - T_i(v)| \le j, |v| \le i, |u| > i\}$$

$$E_1' = \{|v - T_i(u)| \le j, |u| > i\}, \quad E_2' = \{|v - T_i(u)| \le j, |u| \le i, |v| > i\},$$

so that $\Omega \cup \{|u - T_i(v)| > j\} \cup \{|v - T_i(u)| > j\} = E_0 \cup E_1 \cup E_2 = E_0 \cup E_1' \cup E_2'$. First of all, note that $\nabla T_j(u - T_i(v)) = 0$ almost everywhere on $\{|u - T_i(v)| > j\}$, as well as $\nabla T_j(v - T_i(u)) = 0$ on $\{|v - T_i(u)| > j\}$. On E_0 the left hand side is equal to

$$\int_{E_0} (a(x, \nabla u) - a(x, \nabla v))\nabla(u - v)\, dx.$$

On E_1 (and on E_1' with u exchanged with v), recalling that $a(x, \xi)\xi \ge 0$, we have

$$\int_{E_1} (a(x, \nabla u) - f)\nabla u\, dx \ge -\int_{E_1} f\nabla u\, dx.$$

By Hölder inequality and thanks to the inclusion $E_1 \subseteq B_{i-j,2j}$, we have

$$\int_{E_1} f\nabla u\, dx \le \|f\|_{L^{p'}(E_1)^N} \left(\int_{E_1} |\nabla u|^{p'}\, dx\right)^{\frac{1}{p}} \le \|f\|_{L^{p'}(E_1)^N} \left(\int_{B_{i-j,2j}} |\nabla u|^{p'}\, dx\right)^{\frac{1}{p}},$$

which tends to zero as i goes to infinity, thanks to (1.6.9), that is what we wanted to prove; hence

$$\limsup_{i\to+\infty} \left(\int_{E_1} (a(x, \nabla u) - f)\nabla u\, dx + \int_{E_1'} (a(x, \nabla v) - f)\nabla v\, dx\right) \ge 0.$$

Analogously, on E_2 (and the same estimate can be done on E_2'), we have,

$$\int_{E_2} (a(x, \nabla u) - f)\nabla(u - v)\, dx \ge -\int_{E_2} a(x, \nabla u)\nabla v\, dx - \int_{E_2} f\nabla(u - v)\, dx.$$

Reasoning as before, the second term tends to zero as i tends to infinity since $E_2 \subseteq B_{i,j}$, as well as $E_2 \subseteq \{i - j < |v| \le i\}$. For the first term we use (1.3.1), obtaining

$$\int_{E_2} a(x, \nabla u)\nabla v\, dx \le c_0 \left(\int_{B_{i,j}} [|\nabla u|^{p-1} + k_0]^{p'}\, dx\right)^{\frac{1}{p'}} \left(\int_{\{i-j<|v|\le i\}} |\nabla v|^p\, dx\right)^{\frac{1}{p}},$$

and the right hand side tends to zero thanks to (1.6.9). Summing up the results obtained for E_0, E_1, E_1', E_2, E_2', we have

$$\lim_{i\to+\infty}\int_{E_0}(a(x,\nabla u)-a(x,\nabla v))\nabla(u-v)\,dx=0,$$

that is

$$\int_{\{|u-v|\le j\}}(a(x,\nabla u)-a(x,\nabla v))\nabla(u-v)\,dx=0,$$

for every $j>0$. Thus, by (1.3.3), $\nabla u=\nabla v$ a.e. in Ω. Now we consider, for every $i>0$, the function $T_j(T_i(u)-T_i(v))$, which belongs to $W_0^{1,p}(\Omega)$. Since $\nabla u=\nabla v$ a.e. in Ω, it is easy to prove that, for $i>j$,

$$\int_\Omega|\nabla T_j(T_i(u)-T_i(v))|^p\,dx\le\int_{\{i-j<|u|<i\}}|\nabla u|^p\,dx+\int_{\{i-j<|v|<i\}}|\nabla v|^p\,dx.$$

By (1.6.9) the right hand side of the previous inequality tends to zero as i goes to infinity; in particular, it is bounded uniformly with respect to i. Therefore the function $T_j(T_i(u)-T_i(v))$ is bounded in $W_0^{1,p}(\Omega)$ uniformly with respect to i. Since this function converges to $T_j(u-v)$ almost everywhere in Ω as i tends to infinity, we conclude that $T_j(u-v)$ belongs to $W_0^{1,p}(\Omega)$ and that

$$\int_\Omega|\nabla T_j(u-v)|^p\,dx=0.$$

Hence, $T_j(u-v)=0$ a.e. in Ω, for every $j>0$, and, in conclusion, $u=v$. □

Remark 1.6.7. As a matter of fact, the proof of Theorem 1.6.6 shows that formulation (1.6.2) relative to $\mu+F$ characterizes uniquely the function u.

Regarding the stability of the entropy solutions with respect to the datum $\mu+F$, we have the following result, which slightly improves Theorem 1.2 of [42]. We remark that in this framework the "natural" continuous dependence concerns not the solutions themselves, but their truncations.

Theorem 1.6.8. *Let $\mu_k\in\mathcal{M}^p_{b,0}(\Omega)$ and $F_k\in W^{-1,p'}(\Omega)$ be such that*

$$\begin{aligned}&\mu_k\rightharpoonup\mu \text{ weakly in } \mathcal{M}_b(\Omega),\\ &F_k\to F \text{ strongly in } W^{-1,p'}(\Omega);\end{aligned}\tag{1.6.10}$$

let u_k be the entropy solutions of (1.3.7) relative to $\mu_k + F_k$, and let u be the entropy solution of (1.3.7) relative to $\mu + F$. Then

$$\lim_{k\to\infty} T_j(u_k) = T_j(u) \text{ strongly in } W_0^{1,p}(\Omega),$$

for every $j > 0$.

Proof. By Remark 1.6.3 we have, for every $j > 0$ and every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, \nabla u_k)\nabla T_j(u_k - \varphi)\, dx = \int_\Omega T_j(u_k - \varphi)\, d\mu_k + \langle F_k, T_j(u_k - \varphi)\rangle. \tag{1.6.11}$$

Taking $\varphi = 0$ we can reason as in Theorem 1.5.3 and get

$$\int_\Omega |\nabla T_j(u_k)|^p\, dx \le (j+1)M. \tag{1.6.12}$$

As we have already seen in the proof of Theorem 1.5.3, it implies that there exist a subsequence, still denoted by u_k, and a function u^* such that

$$\begin{aligned} &u^* \in \mathcal{T}_0^{1,p}(\Omega),\\ &u_k \text{ converges to } u^* \text{ a.e. in } \Omega,\\ &T_j(u_k) \text{ converges to } T_j(u^*) \text{ weakly in } W_0^{1,p}(\Omega), \text{ for every } j > 0,\\ &\nabla u_k \text{ converges to } \nabla u^* \text{ a.e. in } \Omega, \end{aligned} \tag{1.6.13}$$

with u^* satisfying (1.6.12).

Let us now take $\varphi_k = T_h(u_k) - T_n(u_k) + T_n(u^*)$, $h > n$, and $j = 2n$ in (1.6.11) in order to have:

$$\begin{aligned} &\int_\Omega (a(x, \nabla u_k) - f_k)\nabla T_{2n}(u_k - T_h(u_k) + T_n(u_k) - T_n(u^*))\, dx\\ &= \int_\Omega T_{2n}(u_k - T_h(u_k) + T_n(u_k) - T_n(u^*))\, d\mu_k. \end{aligned}$$

Henceforth, thanks to (1.6.13), we can repeat the proof of Theorem 1.6.5 with u_k instead of u_n; thus we find, in the same way, that, for every $j > 0$,

$$T_j(u_k) \to T_j(u^*) \text{ strongly in } W_0^{1,p}(\Omega).$$

Moreover, this convergence allows us to deduce that

$$a(x, \nabla T_j(u_k)) \to a(x, \nabla T_j(u)) \text{ strongly in } L^{p'}(\Omega)^N;$$

therefore, if we observe that

$$\int_\Omega a(x,\nabla u_k)\nabla T_j(u_k-\varphi)\,dx = \int_\Omega a(x,\nabla T_{j+\|\varphi\|_{L^\infty(\Omega)}}(u_k))\nabla T_j(u_k-\varphi)\,dx,$$

we can pass to the limit in (1.6.11) as k tends to infinity and get:

$$\int_\Omega a(x,\nabla u^*)\nabla T_j(u^*-\varphi)\,dx = \int_\Omega T_j(u^*-\varphi)\,d\mu + \langle F, T_j(u^*-\varphi)\rangle,$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and every $j>0$; this means that $u^* = u$, thanks to Remark 1.6.7; this implies that the whole sequence $T_j(u_k)$ (and not only a subsequence) converge to $T_j(u)$. □

Remark 1.6.9. By the previous theorem we deduce that the entropy solution u of (1.3.7) is the unique solution of the equation (1.6.3) which can be obtained as limit of solutions u_n to the problem

$$\begin{cases} Au_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where f_n is a sequence of smooth functions (e.g., in $C_c^\infty(\Omega)$) converging to μ in the weak topology of $\mathcal{M}_b(\Omega)$. Hence, when $p=2$ and the operator is Lipschitz continuous and strongly monotone, u coincides with the unique reachable solution.

Furthermore, we have the following comparison principle about the entropy solutions.

Theorem 1.6.10. *Assume (1.3.1), (1.3.2), (1.3.3), and (1.3.4). For $i=1,2$, let u_i be the entropy solution of (1.3.7) relative to $\mu_i \in \mathcal{M}_{b,0}^p(\Omega)$. Suppose that $\mu_1 \le \mu_2$, then $u_1 \le u_2$ almost everywhere in Ω.*

Proof. The result is well known when $\mu_i \in W^{-1,p'}(\Omega)\cap \mathcal{M}_b(\Omega)$. Using Lemma 1.2.4 and Theorem 1.6.8 we can conclude. □

Finally, we have to say that the notion of renormalized solution (see [46]) was extended in [29] to the case of a general measure $\mu \in \mathcal{M}_b(\Omega)$. In that paper the authors proved the existence of such a solution, and introduced other equivalent definitions, which show that all the renormalized solutions are constructed by approximating, in an appropriate way, the measure μ (with respect to the $*$-weak convergence of measures), so that they are reachable solutions. In particular, when $p=2$ and the operator is Lipschitz continuous and strongly monotone, this implies that the renormalized solution is unique and coincides with the unique reachable solution.

Definition 1.6.11. Assume that a satisfies (1.3.1), (1.3.2), (1.3.3), (1.3.4), and let μ be a measure in $\mathcal{M}_b(\Omega)$. We say that a function $u \in \mathcal{T}_0^{1,p}(\Omega)$, satisfying (1.5.4) for every $j > 0$, is a renormalized solution of problem (1.3.7) if

$$\int_\Omega a(x, \nabla u)\nabla(h(u)\varphi)\,dx = \int_\Omega h(u)\varphi\,d\mu_a + h(+\infty)\int_\Omega \varphi\,d\mu_s^+ - h(-\infty)\int_\Omega \varphi\,d\mu_s^-, \quad (1.6.14)$$

where $\varphi \in W_0^{1,q'}(\Omega$, for every $q < \frac{N}{N-1}$, $h \in W^{1,\infty}(\mathbb{R})$ such that h' has compact support in $\mathbb{R}$. Here $h(+\infty)$ and $h(-\infty)$ are the limits of $h(s)$ at $+\infty$ and $-\infty$ respectively (note that h is constant for $|s|$ large).

Remark 1.6.12. As in (1.6.1), every term in (1.6.14) is well defined.

Chapter 2
Linear obstacle problems with measure data

2.1. Definition of the problem

In this chapter we consider the obstacle problem with measure data for a linear differential operator A, for which we prove existence and uniqueness of solutions together with some stability results.

Consider first the objects that won't change throughout the chapter.

Let Ω be a regular subset of $\mathbb{R}^N$ (for the notion of regularity see Definition 1.4.2).

Let $Au = -\text{div}(\mathcal{A}(x)\nabla u)$ be a linear elliptic operator with coefficients in $L^\infty(\Omega)$, that is $\mathcal{A}(x)$ is an $N \times N$ matrix satisfying (1.4.1).

Consider a function $\psi : \Omega \to \overline{\mathbb{R}}$, and define the sets

$$J_\psi := \{z \in \mathcal{T}_0^{1,2}(\Omega) : \ z \geq \psi \ \text{ q.e. in } \Omega\}, \quad K_\psi := \{z \in H_0^1(\Omega) : \ z \geq \psi \ \text{ q.e. in } \Omega\}.$$

We recall that, for any datum $F \in H^{-1}(\Omega)$ the Variational Inequality with obstacle ψ

$$\begin{cases} u \in K_\psi, \\ \langle Au, v-u \rangle \geq \langle F, v-u \rangle, \\ \forall v \in K_\psi \end{cases} \tag{2.1.1}$$

is denoted by $VI(F,\psi)$, and makes sense whenever the set K_ψ is nonempty.

We want to arrive to a suitable definition of obstacle problems with measure data. As we have seen, we can not use the variational formulation (2.1.1), because the term $\langle \mu, v-u \rangle$ may not be defined. Also the use of the characterization (0.0.2) is not possible because this, in general, does not determine the solution of the obstacle problem.

To avoid these problems we give the following definition, in which, roughly speaking, we choose the minimum element among those functions v, above the obstacle, such that $Av - \mu$ is not only nonnegative in the sense of distributions but is actually a nonnegative bounded Radon measure, and the equation is solved in the sense of Stampacchia.

We shall use the notations of Section 1.4. Hence, in the sequel we shall shortly say that u_μ is a solution of the equation $Au = \mu$ or that u_μ is a solution of the problem

$$\begin{cases} Au = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ is a measure in $\mathcal{M}_b(\Omega)$ or an element of $H^{-1}(\Omega)$.

Definition 2.1.1. We say that $u \in J_\psi$ is a solution of the Obstacle Problem with datum μ and obstacle ψ if

1. there exists a nonnegative bounded Radon measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that
$$u = u_\mu + u_\lambda;$$
2. for any $\nu \in \mathcal{M}_b^+(\Omega)$, such that $v = u_\mu + u_\nu$ belongs to J_ψ, we have
$$u \leq v \quad \text{q.e. in } \Omega.$$

The positive measure λ, which is uniquely defined, will be called the obstacle reaction relative to u. This problem will be shortly indicated by $OP(\mu, \psi)$.

To show that for any datum μ there exists one and only one solution, we introduce the set
$$\mathcal{F}_\psi(\mu) := \left\{ v \in J_\psi : \ \exists \nu \in \mathcal{M}_b^+(\Omega) \text{ s.t. } v = u_\mu + u_\nu \right\}.$$
We will prove that $\mathcal{F}_\psi(\mu)$ has a minimum element, that is a function $u \in \mathcal{F}_\psi(\mu)$ such that $u \leq v$ q.e. in Ω for any other function $v \in \mathcal{F}_\psi(\mu)$. This is clearly the solution of the Obstacle Problem according to the Definition 2.1.1. If this solution exists it is obviously unique.

To ensure that $\mathcal{F}_\psi(\mu)$ is nonempty, we require that
$$\exists \rho \in \mathcal{M}_b(\Omega) \ : \ u_\rho \geq \psi \text{ q.e. in } \Omega; \tag{2.1.2}$$
thus, for every $\mu \in \mathcal{M}_b(\Omega)$, the set $\mathcal{F}_\psi(\mu)$ contains the function $u_{\mu^+} + u_\rho$.

The proof of existence will be first worked out for the case of a nonpositive obstacle (Section 2.2): this is based on an approximation technique. The obstacle reactions associated with the solutions for regular data are shown to satisfy an estimate on the masses, which allows to pass to the limit and obtain the solution in the general case. Then the proof is easily extended to the case of general obstacle (Section 2.3)

2.2. Nonpositive obstacles

Throughout this chapter we assume the obstacle to be nonpositive. In this frame both J_ψ and K_ψ are nonempty.

We begin with a preparatory result which will be proved in two steps.

Lemma 2.2.1. *Let $\psi \leq 0$ an let $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ such that μ^+ and μ^- belong to $H^{-1}(\Omega)$. Let u be the solution of $VI(\mu, \psi)$ and λ the obstacle reaction associated with u. Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}.$$

Proof. Observe that the function u_{μ^+} is nonnegative and hence greater than or equal to ψ, belongs to $H_0^1(\Omega)$, and

$$Au_{\mu^+} - \mu \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

By (0.0.2) we have

$$u = u_\mu + u_\lambda \leq u_{\mu^+} \quad \text{q.e. in } \Omega,$$

and, by linearity,

$$u_\lambda \leq u_{\mu^-} \quad \text{q.e. in } \Omega. \tag{2.2.1}$$

We will prove that this implies

$$\lambda(\Omega) \leq \mu^-(\Omega) \tag{2.2.2}$$

which is equivalent to the thesis.

To prove (2.2.2) we note that, thanks to (2.2.1)

$$\int_\Omega w\, d\mu^- = \langle A^* w, u_{\mu^-} \rangle \geq \langle A^* w, u_\lambda \rangle = \int_\Omega w\, d\lambda, \tag{2.2.3}$$

for every $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$, such that $A^* w \geq 0$ in $\mathcal{D}'(\Omega)$.

It is now easy to find a sequence w_n in $H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $w_n \nearrow 1$ q.e. in Ω, and $A^* w_n \geq 0$ in $\mathcal{D}'(\Omega)$. For instance, one can choose as w_n the A^*-capacitary potential (see [36], chapter 9) of J_n, where J_n is an invading family of compact subsets of Ω.

Passing to the limit in (2.2.3), as $n \to \infty$, we obtain (2.2.2). □

Theorem 2.2.2. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$. Let u be the solution of $VI(\mu, \psi)$ and let λ be the obstacle reaction relative to u. Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}. \tag{2.2.4}$$

Proof. Thanks to Lemma 3.3 in [28] there exists a sequence of smooth functions f_n such that

$$||f_n - \mu||_{H^{-1}(\Omega)} \leq \frac{1}{n} \quad \text{and} \quad ||f_n||_{L^1(\Omega)} \leq ||\mu||_{\mathcal{M}_b(\Omega)}.$$

Thanks to the next Lemma, the sequence f_n satisfies

$$f_n^{\pm} \rightharpoonup \mu^{\pm} \ *\text{-weakly in } \mathcal{M}_b(\Omega) \text{ and } ||f_n^{\pm}||_{L^1(\Omega)} \to ||\mu^{\pm}||_{\mathcal{M}_b(\Omega)}.$$

Let u_n and u be the solutions of $VI(f_n, \psi)$ and $VI(\mu, \psi)$, respectively. We know from the general theory (see, for instance, [38]) that $u_n \to u$ in $H_0^1(\Omega)$. So the measures λ_n and λ associated with u_n and u, respectively, satisfy

$$\lambda_n \to \lambda \text{ in } H^{-1}(\Omega),$$

$$||\lambda_n||_{\mathcal{M}_b(\Omega)} \leq ||f_n^-||_{L^1(\Omega)}.$$

So $\lambda_n \rightharpoonup \lambda$ $*$-weakly in $\mathcal{M}_b(\Omega)$, and we get the inequality (2.2.4). □

The following lemma is quite simple, but is proved here for the sake of completeness.

Lemma 2.2.3. *Let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \rightharpoonup \mu \ *\text{-weakly in } \mathcal{M}_b(\Omega) \text{ and } ||\mu_n||_{\mathcal{M}_b(\Omega)} \to ||\mu||_{\mathcal{M}_b(\Omega)}$$

then

$$\mu_n^+ \rightharpoonup \mu^+ \text{ and } \mu_n^- \rightharpoonup \mu^- \ *\text{-weakly in } \mathcal{M}_b(\Omega),$$

and

$$||\mu_n^+||_{\mathcal{M}_b(\Omega)} \to ||\mu^+||_{\mathcal{M}_b(\Omega)} \text{ and } ||\mu_n^-||_{\mathcal{M}_b(\Omega)} \to ||\mu^-||_{\mathcal{M}_b(\Omega)}. \tag{2.2.5}$$

Proof. Observe that

$$||\mu_n^{\pm}||_{\mathcal{M}_b(\Omega)} \leq ||\mu_n||_{\mathcal{M}_b(\Omega)}.$$

so, up to a subsequence,

$$\mu_n^+ \rightharpoonup \alpha \text{ and } \mu_n^- \rightharpoonup \beta \ *\text{-weakly in } \mathcal{M}_b(\Omega);$$

where $\alpha - \beta = \mu$. Hence, we can compute

$$\begin{aligned} ||\alpha||_{\mathcal{M}_b(\Omega)} + ||\beta||_{\mathcal{M}_b(\Omega)} &\leq \liminf ||\mu_n^+||_{\mathcal{M}_b(\Omega)} + \liminf ||\mu_n^-||_{\mathcal{M}_b(\Omega)} \\ &\leq \liminf ||\mu_n||_{\mathcal{M}_b(\Omega)} = ||\mu||_{\mathcal{M}_b(\Omega)}; \end{aligned}$$

from which we easily deduce that $\alpha = \mu^+$, $\beta = \mu^-$. Therefore the whole sequences μ_n^+ and μ_n^- converge to μ^+ and μ^- respectively. Moreover, as

$$\limsup_{n\to+\infty} \|\mu_n^+\|_{\mathcal{M}_b(\Omega)} + \liminf_{n\to+\infty} \|\mu_n^-\|_{\mathcal{M}_b(\Omega)}$$
$$\leq \lim_{n\to+\infty} \|\mu_n\|_{\mathcal{M}_b(\Omega)} = \|\mu\|_{\mathcal{M}_b(\Omega)} = \|\mu^+\|_{\mathcal{M}_b(\Omega)} + \|\mu^-\|_{\mathcal{M}_b(\Omega)}$$

we obtain easily the first relation in (2.2.5). The second one is obtained in a similar way. □

In order to proceed we need to prove that when both the classical formulation for the obstacle problem and the new one, given in Definition 2.1.1, make sense then the solutions, when they exist, are the same. At present we prove it for a nonpositive obstacle, and we will prove it in the general case in Section 1.3.

Lemma 2.2.4. *Let μ be an element of $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and ψ a nonpositive function; then the solution of $VI(\mu, \psi)$ coincides with the solution of $OP(\mu, \psi)$.*

Proof. Let u be the solution of $VI(\mu, \psi)$ and λ be the corresponding obstacle reaction. Thanks to Theorem 2.2.2 it is an element of $\mathcal{M}_b(\Omega)$; so $u \in \mathcal{F}_\psi(\mu)$. Take v an element in $\mathcal{F}_\psi(\mu)$, then $v = u_\mu + u_\nu$, with $\nu \in \mathcal{M}_b^+(\Omega)$, and $v \geq \psi$ q.e. in Ω.

Consider the approximation of ν, given by $AT_k(u_\nu) =: \nu_k$. This is such that $\nu_k \rightharpoonup \nu$ $*$-weakly in $\mathcal{M}_b(\Omega)$ and $\nu_k \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ (see Proposition 1.5.11). Set $v_k = u_\mu + u_{\nu_k} = u_\mu + T_k(u_\nu)$. Since trivially $T_k(u_\nu) \nearrow u_\nu$ q.e. in Ω, we have

$$v_k \nearrow v \quad \text{q.e. in } \Omega.$$

Denote now the solutions of $VI(\mu, \psi_k)$ by u_k, where ψ_k are the functions defined by

$$\psi_k := \psi \wedge v_k.$$

From $\psi_k \leq \psi_{k+1}$ q.e. in Ω it easily follows that $u_k \leq u_{k+1}$ q.e. in Ω. Then there exists a function u^* such that $u_k \nearrow u^*$ q.e. in Ω.

So, passing to the limit in $u_k \geq \psi_k$ q.e. in Ω we obtain $u^* \geq \psi$ q.e. in Ω.

Moreover it is easy to see that $\|u_k\|_{H_0^1(\Omega)} \leq C$. So, thanks to Lemma 1.2 in [27] we get that u^* is a quasi continuous function of $H_0^1(\Omega)$ such that

$$u_k \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega).$$

Moreover it can be easily proved that u^* satisfies the variational formulation 2.1.1; thus, $u^* = u$ q.e. in Ω.

Naturally, from the minimality of u_k, we deduce

$$u_k \leq v_k \quad \text{q.e. in } \Omega.$$

so, passing to the limit as $k \to +\infty$ we conclude that $u \leq v$ q.e. in Ω. Since this is true for every $v \in \mathcal{F}_\psi(\mu)$, the function u is the minimum in $\mathcal{F}_\psi(\mu)$, i.e. the solution of $OP(\mu, \psi)$. □

We are now in position to prove that, for every $\mu \in \mathcal{M}_b(\Omega)$ and for every $\psi \leq 0$, there exists a solution to the Obstacle Problem according to Definition 2.1.1.

Theorem 2.2.5. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a solution of $OP(\mu, \psi)$.*

Proof. Consider the function u_μ and define

$$AT_k(u_\mu) =: \mu_k.$$

We know from Proposition 1.5.11 that

$$\mu_k \rightharpoonup \mu \ \ast\text{-weakly in } \mathcal{M}_b(\Omega)$$

and $\mu_k \in H^{-1}(\Omega)$.

Let u_k be the solution of $VI(\mu_k, \psi)$ and denote

$$Au_k - \mu_k =: \lambda_k,$$

which we know from Theorem 2.2.2 to be a measure in $\mathcal{M}_b^+(\Omega)$ such that

$$\|\lambda_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu_k^-\|_{\mathcal{M}_b(\Omega)}. \tag{2.2.6}$$

Up to a subsequence $\lambda_k \rightharpoonup \lambda$ $\ast$-weakly in $\mathcal{M}_b(\Omega)$, and, thanks to Theorem 1.4.7, $u_k \to u$ strongly in $W_0^{1,q}(\Omega)$, with $u = u_\mu + u_\lambda$, and also $T_h(u_k) \rightharpoonup T_h(u)$ weakly in $H_0^1(\Omega)$, for all $h > 0$.

Now the set

$$K_{T_h(\psi)} = \{v \in H_0^1(\Omega) \ : \ v \geq T_h(\psi) \ \text{ q.e. in } \Omega\}$$

is closed and convex in $H_0^1(\Omega)$, so it is also weakly closed. Since, clearly, $T_h(u_k) \geq T_h(\psi)$ q.e. in Ω, passing to the limit as $k \to +\infty$ we get that also $T_h(u) \in K_{T_h(\psi)}$, hence $T_h(u) \geq T_h(\psi)$ q.e. in Ω for all $h > 0$. Passing to the limit as $h \to +\infty$ we get $u \geq \psi$ q.e. in Ω. In conclusion we deduce $u \in \mathcal{F}_\psi(\mu)$.

To show that u is minimal, take $v \in \mathcal{F}_\psi(\mu)$ so that $v \geq \psi$ and $v = u_\mu + u_\nu$.

Let $v_k = u_{\mu_k} + u_\nu$ so that $v_k = T_k(u_\mu) + u_\nu$ and $v_k \to v$ strongly in $W_0^{1,q}(\Omega)$.

Since $\psi \leq 0$, we have that $v_k \geq \psi$ q.e. in Ω. As u_k is the minimum of $\mathcal{F}_\psi(\mu_k)$, by Lemma 2.2.4, we obtain $u_k \leq v_k$ a.e. in Ω and in the limit $u \leq v$ a.e. in Ω. Hence u solves $OP(\mu, \psi)$. □

From formula (2.2.6) we see that to extend (2.2.4) to the case of $\mu \in \mathcal{M}_b(\Omega)$ we just need to show that

$$||\mu_k^-||_{\mathcal{M}_b(\Omega)} \to ||\mu^-||_{\mathcal{M}_b(\Omega)};$$

this is proved in the following Proposition.

Proposition 2.2.6. *Let $\psi \leq 0$ and $\mu \in \mathcal{M}_b(\Omega)$. Let u be the solution of $OP(\mu, \psi)$ and λ the corresponding obstacle reaction. Then*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||\mu^-||_{\mathcal{M}_b(\Omega)}.$$

Proof. What we need is implicit in [14]; we recall the main steps of that proof, having a closer look to the constants involved.

Let f_n be a smooth approximation of μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, such that $||f_n||_{L^1(\Omega)} \leq ||\mu||_{\mathcal{M}_b(\Omega)}$, and let u_n be the solutions of

$$\begin{cases} Au_n = f_n & \text{in } H^{-1}(\Omega) \\ u_n \in H_0^1(\Omega). \end{cases}$$

Consider, for $\delta > 0$, the Lipschitz continuous functions h_δ defined by

$$\begin{cases} h_\delta(s) = 1 & \text{if } |s| \leq k \\ h_\delta(s) = 0 & \text{if } |s| \geq k + \delta \\ |h'_\delta(s)| = \dfrac{1}{\delta} & \text{if } k \leq |s| \leq k + \delta, \end{cases}$$

and S_δ defined by

$$\begin{cases} S_\delta(s) = 0 & \text{if } |s| \le k \\ S_\delta(s) = \operatorname{sign}(s) & \text{if } |s| \ge k+\delta \\ S'_\delta(s) = \dfrac{1}{\delta} & \text{if } k \le |s| \le k+\delta. \end{cases}$$

Using the equation, we can see that $-\operatorname{div}(h_\delta(u_n)\mathcal{A}(x)\nabla u_n)$ belongs to $L^1(\Omega)$ and that

$$\begin{aligned} &\int_\Omega |-\operatorname{div}(h_\delta(u_n)\mathcal{A}(x)\nabla u_n)|\,dx \\ &\le \int_\Omega |f_n| \left(h_\delta(u_n) + S_\delta^+(u_n) + S_\delta^-(u_n)\right)\,dx \\ &= \int_\Omega |f_n|\,dx \le ||\mu||_{\mathcal{M}_b(\Omega)}. \end{aligned}$$

This implies

$$||\mu_k||_{\mathcal{M}_b(\Omega)} \le ||\mu||_{\mathcal{M}_b(\Omega)},$$

(recall that $\mu_k = AT_k(u_\mu)$) and we conclude thanks to Lemma 2.2.3. □

2.3. The general existence theorem

We come now to prove the existence and uniqueness of the solution to the Obstacle Problem, without the technical assumption that the obstacle be nonpositive. From now on the only hypothesis will be (2.1.2).

Theorem 2.3.1. *Let ψ satisfy (2.1.2) and let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a (unique) solution of $OP(\mu,\psi)$. Moreover, the corresponding obstacle reaction λ satisfies*

$$||\lambda||_{\mathcal{M}_b(\Omega)} \le ||(\mu-\rho)^-||_{\mathcal{M}_b(\Omega)}. \tag{2.3.1}$$

Proof. It is enough to show that we can refer to the case $\psi \le 0$. Indeed define

$$\varphi := \psi - u_\rho,$$

which is, obviously, nonpositive.

By Theorem 2.2.5 there exists v minimum in $\mathcal{F}_\varphi(\mu-\rho)$, and we prove that the function $u := v + u_\rho$ is the minimum of $\mathcal{F}_\psi(\mu)$.

Trivially $u \geq \psi$ and, denoted the nonnegative obstacle reaction associated to v by λ, we have $u = v + u_\rho = u_\mu + u_\lambda$, which shows that u is an element of $\mathcal{F}_\psi(\mu)$. Let us observe also that λ satisfies (2.2.1), which means

$$||\lambda||_{\mathcal{M}_b(\Omega)} \leq ||(\mu-\rho)^-||_{\mathcal{M}_b(\Omega)}.$$

Consider now a function $w \in \mathcal{F}_\psi(\mu)$. By similar computations we deduce that $w - u_\rho$ belongs to $\mathcal{F}_\varphi(\mu-\rho)$ and, by the minimality of v, $v \leq w - u_\rho$, so that we conclude $u \leq w$ q.e. in Ω, and λ is the obstacle reaction associated to u. □

2.4. Some stability results

In this section we want to show some results of continuous dependence of the solutions on the data.

The following proposition concerns the problem of stability with respect to the obstacle, which, however, is not true in general (see Remark 2.7.2).

Proposition 2.4.1. *Let $\psi_n : \Omega \to \overline{\mathbb{R}}$ be obstacles such that*

$$\psi_n \leq \psi \quad \text{and} \quad \psi_n \to \psi \quad \text{q.e. in } \Omega,$$

ψ satisfies (2.1.2), and let u_n and u be the solutions of $OP(\mu,\psi_n)$ and $OP(\mu,\psi)$, respectively. Then

$$u_n \to u \quad \text{strongly in } W_0^{1,q}(\Omega).$$

We also obtain that $u_n \to u$ q.e. in Ω and that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $H_0^1(\Omega)$, for all $k > 0$.

Proof. Since u is trivially in $\mathcal{F}_{\psi_n}(\mu)$ for any n we have

$$u_n \leq u \qquad \text{q.e. in } \Omega. \tag{2.4.1}$$

To every minimum u_n there corresponds a nonnegative obstacle reaction λ_n, satisfying inequality (2.3.1), so we obtain that, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \quad *\text{-weakly in } \mathcal{M}_b(\Omega)$$

$$u_n \to \hat{u} \quad \text{strongly in } W_0^{1,q}(\Omega)$$

and

$$\hat{u} = u_\mu + u_{\hat{\lambda}}.$$

Hence, from (2.4.1), $\hat{u} \le u$ a.e. in Ω, and also q.e. in Ω. On the other side, we have to prove that $\hat{u} \ge \psi$ q.e. in Ω, in order to obtain $\hat{u} \in \mathcal{F}_\psi(\mu)$, and so $u \le \hat{u}$ q.e. in Ω.

First consider the case when $\psi_n \le \psi_{n+1}$ q.e. in Ω.

From this fact it follows that $u_n \le u_{n+1}$ q.e. in Ω, and then $T_k(u_n) \le T_k(u_{n+1})$ q.e. in Ω, for all $k > 0$. Hence this sequence has a quasi everywhere limit. On the other hand, the fact that $\mu + \lambda_n \rightharpoonup \mu + \hat{\lambda}$ $*$-weakly in $\mathcal{M}_b(\Omega)$ implies that $T_k(u_n) \rightharpoonup T_k(\hat{u})$ weakly in $H^1_0(\Omega)$ and then, by Lemma 1.2 of [27], $T_k(u_n) \to T_k(\hat{u})$ q.e. in Ω. Since this holds for all $k > 0$ we get also

$$u_n \to \hat{u} \quad \text{q.e. in } \Omega,$$

since u_n and $\hat{u}$ are finite up to sets of capacity zero (see Chapter 1). Then, passing to the limit in $u_n \ge \psi_n$ q.e. in Ω we get $\hat{u} \ge \psi$ q.e. in Ω.

If the sequence ψ_n is not increasing, consider

$$\varphi_n := \inf_{k \ge n} \psi_k, \tag{2.4.2}$$

so that $\varphi_n \nearrow \psi$ q.e. in Ω and $\varphi_n \le \psi_n$ q.e. in Ω. If $\overline{u}_n$ is the solution of $OP(\mu, \varphi_n)$ it is easy to see, using Definition 2.1.1, that $\overline{u}_n \le u_n \le u$ q.e. in Ω. Applying the first case to $\overline{u}_n$ and passing to the limit we get $u_n \to u$ q.e. in Ω.

By the uniqueness of the function u, the whole sequence u_n converges to u. ☐

As for stability with respect to the right-hand side, we will show later that in general it is not true that if

$$\mu_n \rightharpoonup \mu \quad *\text{-weakly in } \mathcal{M}_b(\Omega)$$

then

$$u_n \to u \quad \text{strongly in } W^{1,q}_0(\Omega),$$

where u_n and u are the solutions relative to μ_n and μ with the fixed obstacle ψ.

However we can give now the following stability result.

Proposition 2.4.2. *Let μ_n and μ be measures in $\mathcal{M}_b(\Omega)$ such that*

$$\mu_n \to \mu \text{ strongly in } \mathcal{M}_b(\Omega),$$

then

$$u_n \to u \text{ strongly in } W_0^{1,q}(\Omega)$$

where u_n and u are the solutions of $OP(\mu_n, \psi)$ and of $OP(\mu, \psi)$, respectively.

Proof. Let λ_n be the obstacle reactions associated to u_n, then

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu_n - \rho)^-\|_{\mathcal{M}_b(\Omega)},$$

so, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \text{ *-weakly in } \mathcal{M}_b(\Omega)$$

and

$$u_n \to \hat{u} \text{ strongly in } W_0^{1,q}(\Omega)$$

$$T_k(u_n) \rightharpoonup T_k(\hat{u}) \text{ weakly in } H_0^1(\Omega), \ \forall k > 0$$

where $\hat{u} = u_\mu + u_{\hat{\lambda}}$.

As $T_k(u_n) \geq T_k(\psi)$ q.e. in Ω for every $k \geq 0$, and for every n, we have $T_k(\hat{u}) \geq T_k(\psi)$ q.e. in Ω for every $k > 0$.

Passing to the limit as $k \to +\infty$ we obtain that $\hat{u}$ belongs to $\mathcal{F}_\psi(\mu)$.

Let $v \in \mathcal{F}_\psi(\mu)$, with ν the associated measure. Consider now v_n the Stampacchia solution relative to $\zeta_n := \mu_n + (\mu_n - \mu)^- + \nu$. Since $\zeta_n \to \mu + \nu$ strongly in $\mathcal{M}_b(\Omega)$, the sequence v_n converges strongly in $W_0^{1,q}(\Omega)$ to v.

Moreover $v_n \geq v \geq \psi$ q.e. in Ω; hence $v_n \in \mathcal{F}_\psi(\mu_n)$, then $u_n \leq v_n$ q.e. in Ω, and, in the limit,

$$\hat{u} \leq v \text{ a.e. in } \Omega,$$

and hence also q.e. in Ω.

By the uniqueness of the function limit, the whole sequence u_n converges to u. □

Remark 2.4.3. Thanks to this last result we can say that the solutions obtained in this paper coincide with those given by L. Boccardo and G.R. Cirmi in [5]-[6] when the data are $L^1(\Omega)$ functions.

As said above we give now the counterexample showing that in general there is not stability with respect to *-weakly convergent data.

Example 2.4.4 Let $\Omega = (0,1)^N$ with $N \geq 3$, $A = -\Delta$ and $\psi \equiv 0$.

The construction follows the one made by D. Cioranescu and F. Murat in [18].

For each $n \in \mathbb{N}$, divide the whole of Ω into small cubes of side $\frac{1}{n}$. In the centre of each of them take two balls: $B_{\frac{1}{2n}}$, inscribed in the cube, and B_{r_n} of ray $r_n = \left(\frac{1}{2n}\right)^{\frac{N}{N-2}}$.

In each cube define w_n to be the capacitary potential of B_{r_n} with respect to $B_{\frac{1}{2n}}$ extended by zero in the rest of the cube.

Hence

$$\Delta w_n = \mu_n,$$

with

$$\mu_n \rightharpoonup 0 \text{ both weakly in } H^{-1}(\Omega) \text{ and } *\text{-weakly in } \mathcal{M}_b(\Omega).$$

(see [18]). Thus $w_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

Let u_n be the solution of $VI(\mu_n, 0)$. Using w_n as test function in the Variational Inequality we get $||u_n||_{H_0^1(\Omega)} \leq C$. By contradiction assume that its $H_0^1(\Omega)$-weak limit is zero.

Consider the function $z_n := u_n + w_n$ which must then converge to zero weakly in $H_0^1(\Omega)$. Obviously $z_n \geq w_n$ q.e. in Ω and then $z_n \geq 1$ on $\bigcup B_{r_n}$. Hence if we define the obstacles

$$\psi_n := \begin{cases} 1 & \text{in } \bigcup B_{r_n} \\ 0 & \text{elsewhere,} \end{cases}$$

we have $z_n \geq \psi_n$. Call v_n the function realizing

$$\min_{\substack{v \geq \psi_n \\ v \in H_0^1(\Omega)}} \int_\Omega |\nabla v|^2 \, dx$$

(v_n solves $VI(0, \psi_n)$). A simple computation yields

$$-\Delta z_n = -\Delta u_n - \Delta w_n \geq 0.$$

Then $z_n \geq v_n \geq 0$, so that

$$v_n \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega).$$

But this is not possible because a Γ-convergence result contained in [15] says that there exists a constant $c > 0$ such that v_n tends to the minimum point of

$$\min_{\substack{v \geq 0 \\ v \in H_0^1(\Omega)}} \int_\Omega |\nabla v|^2 \, dx + c \int_\Omega |(v-1)^-|^2 \, dx$$

which is not zero.

2.5. Comparison with the classical solutions

As announced, in this section, we want to show that the new formulation of Obstacle Problem is consistent with the classical one.

To talk about the equivalence of the two formulations it is necessary that both make sense. So we will work under the hypothesis that $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and that the obstacle ψ satisfies

$$\exists z \in H_0^1(\Omega) \text{ s.t. } z \geq \psi \quad \text{q.e. in } \Omega, \tag{2.5.1}$$

$$\exists \rho \in \mathcal{M}_b^+(\Omega) \text{ s.t. } u_\rho \geq \psi \quad \text{q.e. in } \Omega, \tag{2.5.2}$$

which ensure that K_ψ and J_ψ are nonempty.

Later on we will discuss these conditions in deeper details. Actually, in the next lemma we will consider a stronger hypothesis on the obstacle, but we will see in Remark 4.4.1 that it turns out to be equivalent to (2.5.1) and (2.5.2).

Lemma 2.5.1. *If there exists a measure $\sigma \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ such that $u_\sigma \geq \psi$ q.e. in Ω, then the solutions of $VI(\mu, \psi)$ and of $OP(\mu, \psi)$ coincide.*

Proof. Let u be the solution of $VI(\mu, \psi)$. Subtracting u_σ to it, and with the same technique as in the proof of Theorem 2.3.1, we return to the case of negative obstacle and we can use Lemma 2.2.4. □

Theorem 2.5.2. *Under the hypotheses (2.5.1) and (2.5.2), the solutions of $VI(\mu, \psi)$ and of $OP(\mu, \psi)$ coincide.*

Proof. As a first step consider the case of an obstacle bounded from above by a constant M. The measure $\rho_M := AT_M(u_\rho)$ is in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and $T_M(u_\rho) \geq \psi$ so that we are in the hypotheses of the previous lemma.

If, instead, ψ is not bounded, we consider $\psi \wedge k$, and, with respect to this new obstacle, conditions (2.5.1) and (2.5.2) are satisfied by the function $T_k(u_\rho)$.

Hence we can apply the first step and say that u_k, solution of $VI(\mu, T_k(\psi))$, is also the solution of $OP(\mu, T_k(\psi))$.

On the other hand, from the classical theory we know that the sequence u_k tends in $H_0^1(\Omega)$ to the solution of $VI(\mu, \psi)$, while from Proposition 2.4.1 u_k converges in $W_0^{1,q}(\Omega)$ to the solution of $OP(\mu, \psi)$. □

Remark 2.5.3. Let us assume that ψ satisfies (2.5.1) and (2.5.2). Then we can consider the solution w of $VI(0,\psi)$, and the corresponding obstacle reaction σ, which is a nonnegative element of $H^{-1}(\Omega)$, hence a nonnegative Radon measure. Thanks to Theorem 2.5.2, w is also the solution of $OP(0,\psi)$, and by (2.3.1), $\sigma \in \mathcal{M}_b(\Omega)$. Thus, condition (2.5.2) is satisfied by $\rho = \sigma \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$.

A little attention is required in treating conditions (2.5.1) and (2.5.2). Each one is necessary for the corresponding problem to be nonempty, but together they can be somewhat weakened.

First of all we underline that no one of the two conditions is implied by the other. This is seen with the following examples.

Example 2.5.4. Let $\Omega = (-1,1) \subset \mathbb{R}$ and let $A = -\Delta = -u''$. Take $\psi \in \mathrm{H}_0^1(-1,1)$ such that $-\psi''$ is an unbounded positive Radon measure. For instance we may take $\psi = (1-|x|)(1-\log(1-|x|))$.

Now (2.5.1) is trivially true, and the solution of $VI(0,\psi)$ is ψ itself. If also (2.5.2) were true, then ψ would be also the solution of $OP(0,\psi)$. But this is not possible, because, being $-\psi''$ an unbounded measure, we can not write it as u_λ for some $\lambda \in \mathcal{M}_b^+(\Omega)$.

Example 2.5.5. Let $N \geq 3$, $A = -\Delta$ and $\rho = \delta_{x_0}$, the Dirac delta in a fixed point $x_0 \in \Omega$.

Take $\psi = u_{\delta_{x_0}}$, the Green function with pole at x_0. Then (2.5.2) holds, but if also (2.5.1) held we would have $\psi \in \mathrm{L}^{2^*}(\Omega)$ which is not true.

On the other side we already saw in the proof of Theorem 2.5.2 that if we add to condition (2.5.2) the assumption that the obstacle be bounded, this is enough for (2.5.1) too to hold.

Moreover, if (2.5.1) is satisfied and we assume that the obstacle is "controlled near the boundary" also condition (2.5.2) is true:

Assume that (2.5.1) holds and there exists a compact $J \subset \Omega$, such that $\psi < 0$ in $\Omega \setminus J$. Then also (2.5.2) holds. Indeed just take as ρ the obstacle reaction corresponding to u, the solution of $VI(0,\psi)$. Then

$$\mathrm{supp}\rho \subset J,$$

and $\rho \in \mathcal{M}_b^+(\Omega)$.

A finer condition expressing the "control near the boundary" is

(2') $\exists J$ compact $\subset \Omega$ and $\exists \tau \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$: $u_\tau \geq \psi$ in $\Omega \setminus J$.

In conclusion we want to remark that, in general, in classical Variational Inequalities, the obstacle reaction associated to the solution is indeed a Radon measure, but it is not always bounded, as Example 2.5.4 shows.

On the other side, in the new setting, the minimum of $\mathcal{F}_\psi(\mu)$ is not, in general, an element of $H_0^1(\Omega)$.

Hence the two formulations do not overlap completely and no one is included in the other.

2.6. Approximation properties

As we have seen so far, if we have a sequence μ_n $*$-weakly convergent to μ, we can not deduce convergence of solutions, but, from (2.3.1) we have

$$||\lambda_n||_{\mathcal{M}_b(\Omega)} \leq ||(\mu_n - \rho)^-||_{\mathcal{M}_b(\Omega)},$$

where the λ_n are the obstacle reactions relative to the solutions u_n. So, up to a subsequence,

$$\lambda_n \rightharpoonup \hat{\lambda} \ *\text{-weakly in } \mathcal{M}_b(\Omega)$$

and

$$u_n \rightharpoonup \hat{u} = u_\mu + u_{\hat{\lambda}} \ \text{ strongly in } W_0^{1,q}(\Omega).$$

With the same argument used in the proof of Theorem 2.2.5 we can show that $\hat{u} \geq \psi$ q.e. in Ω. Hence $\hat{u} \geq u$, the minimum of $\mathcal{F}_\psi(\mu)$.

On the other hand, in Theorem 2.3.1 we have obtained the solution of $OP(\mu, \psi)$ as a limit of the solutions to $OP(\mathcal{A}T_n(u_{\mu-\rho}) + \rho, \psi)$. We remark that if ρ belongs to $\mathcal{M}_b(\Omega) \cap H_0^1(\Omega)$ then the approximating problems are actually Variational Inequalities.

Thanks to these two facts, when $\rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, we can characterize the solution u of $OP(\mu, \psi)$ by approximation with solutions of Variational Inequalities with data in V:

1. For every sequence μ_n in $\mathcal{M}_b(\Omega)$, with $\mu_n \rightharpoonup \mu$ $*$-weakly in $\mathcal{M}_b(\Omega)$, we have

$$s\text{-}W_0^{1,q}(\Omega) \text{ - } \lim_{n\to\infty} u_n \geq u.$$

2. There exists a sequence $\mu_n \in V$, with $\mu_n \rightharpoonup \mu$ $*$-weakly in $\mathcal{M}_b(\Omega)$ such that

$$s\text{-}W_0^{1,q}(\Omega)\ \text{-} \lim_{n\to\infty} u_n = u$$

In other words:

$$u = \min\left\{ s\text{-}W_0^{1,q}(\Omega)\ \text{-} \lim_{n\to+\infty} u_n \right\}$$

where the minimum is taken over all u_n, solutions of $VI(\mu_n, \psi)$, with $\mu_n \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, μ_n converging to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$.

2.7. Measures vanishing on sets of capacity zero

We show now an example (suggested by L. Orsina and A. Prignet) in which the solution of the Obstacle Problem with right-hand side measure does not touch the obstacle, though it is not the solution of the equation.

Example 2.7.1. Let $N \geq 2$, Ω be the ball $B_1(0)$, and $A = -\Delta$. Take the datum μ a negative measure concentrated on a set of capacity zero and the obstacle ψ negative and bounded below by a constant $-h$. Let u be the solution of $OP(\mu, \psi)$, then $u = u_\mu + u_\lambda$. We want to show that $\lambda = -\mu$.

First observe that, for minimality, $u \leq 0$; on the other hand $u \geq -h$, so that $u = T_h(u)$ and hence $u \in H_0^1(\Omega)$. This implies that the measure $\mu+\lambda$ is in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, which is contained in $\mathcal{M}_{b,0}(\Omega)$, the measures which are zero on the sets of capacity zero. In other words $\lambda = -\mu + \hat{\lambda}$, with $\hat{\lambda}$ a measure in $\mathcal{M}_{b,0}(\Omega)$, and so positive, since λ is positive. Then $u \geq 0$, and finally $u = 0$. Thus the solution can be far above the obstacle, but the obstacle reaction is nonzero, and is exactly $-\mu$.

Remark 2.7.2. This example shows also that in general there is no continuous dependence on the obstacles. Indeed, if $h \to +\infty$, then the solution of $OP(\mu, -h)$ is identically zero for each h, while the solution of $OP(\mu, -\infty)$ is u_μ.

In the next chapter we will see that, if $\mu \in \mathcal{M}_{b,0}(\Omega)$, then the above phenomenon is avoided.

Consider, as datum, a measure in $\mathcal{M}_{b,0}(\Omega)$. In this case we can use Theorem 1.2.3: for any such measure μ there exists a function f in $L^1(\Omega)$ and a functional F in

$\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, such that $\mu = f + F$. If, in addition $\mu \geq 0$, then also f can be taken to be nonnegative.

We want to show that also the obstacle reaction λ belongs to $\mathcal{M}_{b,0}(\Omega)$ and that in this particular case we can write our Obstacle Problem in a variational way, that is with and "entropy formulation".

We begin by considering the case of a nonpositive obstacle.

Lemma 2.7.3. *Let $\psi \leq 0$ and let $\mu_1, \mu_2 \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $VI(\mu_1, \psi)$ and $VI(\mu_2, \psi)$, respectively.*

If $\mu_1 \leq \mu_2$ then $\lambda_1 \geq \lambda_2$.

Proof. This proof is inspired by Lemma 2.5 in [30]. We easily have that $u_1 \leq u_2$.

Take now a function $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, and set

$$\varphi_\varepsilon := \varepsilon\varphi \wedge (u_2 - u_1) \in H_0^1(\Omega).$$

Now, using the hypothesis that $\mu_1 \leq \mu_2$ and monotonicity of A, compute

$$\begin{aligned}\langle \lambda_1, \varepsilon\varphi - \varphi_\varepsilon \rangle &\geq \langle Au_1, \varepsilon\varphi - \varphi_\varepsilon \rangle - \langle \mu_2, \varepsilon\varphi - \varphi_\varepsilon \rangle \\ &= \langle Au_1 - Au_2, \varepsilon\varphi - \varphi_\varepsilon \rangle + \langle \lambda_2, \varepsilon\varphi - \varphi_\varepsilon \rangle \\ &\geq \varepsilon \int_{\{u_2 - u_1 \leq \varepsilon\varphi\}} \mathcal{A}(x)\nabla(u_1 - u_2)\nabla\varphi + \varepsilon\langle \lambda_2, \varphi \rangle - \langle \lambda_2, \varphi_\varepsilon \rangle.\end{aligned}$$

Now, using u_1 as a test function in $VI(\mu_2, \psi)$ and the fact that $u_2 - u_1 \geq \varphi_\varepsilon \geq 0$ we easily get $\langle \lambda_2, \varphi_\varepsilon \rangle = 0$.

Since, also, $-\langle \lambda_1, \varphi_\varepsilon \rangle \leq 0$ we obtain

$$\langle \lambda_1, \varphi \rangle \geq \int_{\{u_2 - u_1 \leq \varepsilon\varphi\}} \mathcal{A}(x)\nabla(u_1 - u_2)\nabla\varphi + \langle \lambda_2, \varphi \rangle.$$

Passing to the limit as $\varepsilon \to 0$ and observing that

$$\int_{\{u_2 - u_1 \leq \varepsilon\varphi\}} \mathcal{A}(x)\nabla(u_1 - u_2)\nabla\varphi \longrightarrow \int_{\{u_2 = u_1\}} \mathcal{A}(x)\nabla(u_1 - u_2)\nabla\varphi = 0,$$

we get the thesis. □

Let us see now what can we say more if $\mu \in \mathcal{M}_{b,0}(\Omega)$, still in the case of nonpositive obstacle.

Lemma 2.7.4. *Let $\psi \leq 0$ and let $\mu \in \mathcal{M}_{b,0}(\Omega)$ then the obstacle reaction relative to the solution of $OP(\mu, \psi)$ is also in $\mathcal{M}_{b,0}(\Omega)$.*

Proof. It is not restrictive to assume μ to be negative. Indeed, if $\mu = \mu^+ - \mu^-$, then also μ^+ and μ^- are in $\mathcal{M}_{b,0}(\Omega)$. Hence the minimum of $\mathcal{F}_\psi(\mu)$ can be written as $u_{\mu^+} + v$ with v minimum in $\mathcal{F}_{\psi - u_{\mu^+}}(-\mu^-)$, and the same obstacle reaction λ; and so we are in the case of a negative measure.

Consider now the decomposition $\mu = f + F$ with $f \leq 0$. And let $\mu_k := T_k(f) + F$ so that $\mu_k \to \mu$ strongly in $\mathcal{M}_b(\Omega)$.

Let u_k be the solution of $OP(\mu_k, \psi)$. It is also the solution of $VI(\mu_k, \psi)$ so that $\lambda_k \in \mathcal{M}_{b,0}(\Omega)$.

Thanks to Proposition 2.4.2 we have that $u_k \to u = u_\mu + u_\lambda$ strongly in $W_0^{1,q}(\Omega)$ and that $\lambda_k \rightharpoonup \lambda$ $*$-weakly in $\mathcal{M}_b(\Omega)$.

From the fact that $\mu_k \geq \mu_{k+1}$ and from Lemma 2.7.3 we obtain that $\lambda_k \leq \lambda_{k+1}$. Hence if we define

$$\hat{\lambda}(B) := \lim_{k \to \infty} \lambda_k(B) \qquad \forall B \text{ Borel set in } \Omega,$$

we know from classical measure theory that it is a bounded Radon measure, it is in $\mathcal{M}_{b,0}(\Omega)$, since all λ_k are, and necessarily coincides with λ. So $\lambda \in \mathcal{M}_{b,0}(\Omega)$. □

In order to pass to a signed obstacle observe first that the minimal hypothesis (2.1.2) becomes necessarily

$$\exists \sigma \in \mathcal{M}_{b,0}(\Omega) \, : \, u_\sigma \geq \psi. \tag{2.7.1}$$

Once we have noticed this, it is easy to use the result for a negative obstacle, as we did in the proof of Theorem 2.3.1 and obtain the following result.

Theorem 2.7.5. *Let ψ satisfy hypothesis (2.7.1), and let μ be in $\mathcal{M}_{b,0}(\Omega)$. Then the obstacle reaction relative to the solution of $OP(\mu, \psi)$ belongs to $\mathcal{M}_{b,0}(\Omega)$ as well.*

Remark 2.7.6. In the previous chapter we saw that, if $\mu \in \mathcal{M}_{b,0}(\Omega)$, then u_μ coincides with the unique entropy solution of $Au = \mu$. Hence, by Theorem 2.7.5, we can characterize the solution u of $OP(A,\mu)$ as the minimum element of the set

$$\{v \in J_\psi : v \text{ entropy solution of } Au = \mu + \nu, \text{ with } \nu \in \mathcal{M}^+_{b,0}(\Omega)\}.$$

Thanks to this fact, the theory developed in the next chapter is consistent with this one just treated.

Chapter 3
Nonlinear obstacle problems with special measure data

3.1. Assumptions and main results

In this chapter we consider the obstacle problem with measure data associated with a nonlinear elliptic differential operator A of monotone type, mapping $W_0^{1,p}(\Omega)$, $p > 1$, into its dual $W^{-1,p'}(\Omega)$.

As we just observed, the first difficulty in order to study obstacle problems when the forcing term is a measure consists in giving a notion of solution to the obstacle problem, which must be based on a suitable notion of solution to equations with measure data.

In Chapter 1 (see Section 1.6) we studied the notion of entropy solution for the problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

when A is a nonlinear elliptic operator of monotone type, and μ is a measure vanishing on all sets of p-capacity zero.

In this chapter Ω will be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, p and p' two real numbers, with $1 < p \leq N$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, and $a : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ a Carathéodory function satisfying (1.3.1), (1.3.2), (1.3.3) and (1.3.4).

Finally, we consider a function $\psi : \Omega \mapsto \overline{\mathbb{R}}$ such that:

$$\psi \leq u_\rho \; C_p\text{-q.e. in } \Omega, \tag{3.1.2}$$

where ρ is an element of $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ and u_ρ is the variational solution of

$$\begin{cases} A(u) = \rho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, we define the convex set

$$K_\psi := \{z \in W_0^{1,p}(\Omega) : z \geq \psi \; C_p\text{-q.e. in } \Omega\}.$$

Without loss of generality we may suppose that ψ is C_p-quasi upper semicontinuous thanks to the following Proposition (see Proposition 1.5 in [25]).

Proposition 3.1.1. *Let $\psi : \Omega \mapsto \overline{\mathbb{R}}$ satisfying*

$$\exists z \in W_0^{1,p}(\Omega) \text{ such that } z \geq \psi \ C_p\text{-q.e. in } \Omega. \tag{3.1.3}$$

Then there exists a C_p-quasi upper semicontinuous function $\hat{\psi} : \Omega \mapsto \overline{\mathbb{R}}$ such that:

1. *$\hat{\psi} \geq \psi$ C_p-q.e. in Ω;*
2. *if $\varphi : \Omega \mapsto \overline{\mathbb{R}}$ is C_p-quasi upper semicontinuous and $\varphi \geq \psi$ C_p-q.e. in Ω, then $\varphi \geq \hat{\psi}$ C_p-q.e. in Ω.*

Thus, in particular, $K_\psi = K_{\hat{\psi}}$.

We recall that, for any datum $F \in W^{-1,p'}(\Omega)$ the unilateral problem relative to A, F, and the obstacle ψ (denoted by $VI(A, \mu, \psi)$) is the problem of finding a function u such that

$$\begin{cases} u \in K_\psi, \\ \langle A(u), v - u\rangle \geq \langle F, v - u\rangle, \\ \forall v \in K_\psi. \end{cases} \tag{3.1.4}$$

This problem has a unique solution whenever K_ψ is nonempty.

Characterization 1. The solution u can be characterized (see, e.g., chapters II and III in [38]) as the smallest function in $W_0^{1,p}(\Omega)$, greater than or equal to ψ, such that

$$\begin{cases} A(u) - F = \lambda & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.5}$$

for some nonnegative element λ of $W^{-1,p'}(\Omega)$.

Characterization 2. Finally, when the obstacle ψ is C_p-quasi upper semicontinuous u is also characterized (see, e.g., Theorem 3.2 in [1]) by the complementarity system

$$\begin{cases} u \in K_\psi, \\ A(u) = F + \lambda, \\ \lambda \in W^{-1,p'}(\Omega), \ \lambda \geq 0, \\ \lambda(\{u - \psi > 0\}) = 0, \end{cases} \tag{3.1.6}$$

where the pointwise values of u are defined C_p-quasi everywhere. Since λ is a nonnegative element of $W^{-1,p'}(\Omega)$, by the Riesz Representation Theorem, it is a nonnegative Radon measure; this explains the meaning of the last line of (3.1.6), which can be written also as $u = \psi$ λ-almost everywhere in Ω.

Using the notion of entropy solution we introduce a definition for unilateral problems with measure data quite similar to *Characterization 1.*

Definition 3.1.2. We say that u is the solution of the Obstacle Problem with datum $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ and obstacle ψ (denoted by $OP_0(A,\mu,\psi)$) if

1. there exists a measure $\lambda \in \mathcal{M}^{p,+}_{b,0}(\Omega)$ such that u is the entropy solution of (3.1.1) relative to $\mu+\lambda$, and $u \geq \psi$ C_p-quasi everywhere in Ω.

2. for any $\nu \in \mathcal{M}^{p,+}_{b,0}(\Omega)$ such that the entropy solution v of (3.1.1) relative to $\mu+\nu$ satisfies $v \geq \psi$ C_p-quasi everywhere in Ω, we have $u \leq v$ C_p- q.e. in Ω.

By definition, it is clear that, if such a solution exists, it is unique.

The nonnegative measure λ, which is uniquely defined, will be called the obstacle reaction relative to u, or the measure associated with it.

Observe that assumption (3.1.2) is satisfied if (3.1.3) holds, and there exists a compact $J \subset \Omega$ such that $\psi < 0$ in $\Omega \setminus J$. Indeed, we take as ρ the obstacle reaction corresponding to the solution of $VI(A,0,\psi)$. Then, by (3.1.6) supp$\rho \subset J$, and $\rho \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$. On the other hand, Example 2.5.4 shows that, in general, (3.1.3) does not imply (3.1.2).

In Definition 3.1.2 we have specified the formulation of obstacle problems we will adopt in this chapter. Let us note, however, that in this definition, since u and v are C_p-quasi continuous, it is enough to prove $u \leq v$ a.e. in Ω to obtain also the inequality C_p-q.e. in Ω.

In Section 3.3, by an approximation technique, we will prove the following existence theorem.

Theorem 3.1.3. *Let ψ satisfy (3.1.2) and let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$. Then there exists a unique solution of $OP_0(A,\mu,\psi)$. Moreover the corresponding obstacle reaction λ satisfies*

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu-\rho)^-\|_{\mathcal{M}_b(\Omega)}. \tag{3.1.7}$$

In Section 3.4 we will prove also the following theorem about continuous dependence on data.

Theorem 3.1.4. *Let ψ satisfy hypothesis (3.1.2), let μ_n, $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ and λ_n and λ the measures associated to the solution u_n and u of $OP_0(A,\mu_n,\psi)$ and $OP_0(A,\mu,\psi)$, respectively. If $\mu_n \to \mu$ strongly in $\mathcal{M}_b(\Omega)$, then $\lambda_n \to \lambda$ strongly in $\mathcal{M}_b(\Omega)$. Moreover, for every $k>0$, $T_k(u_n)$ converges to $T_k(u)$ strongly in $W^{1,p}_0(\Omega)$.*

Remark 3.1.5. Under slightly stronger hypotheses on the obstacle, when the data are $L^1(\Omega)$ functions, the solutions considered in this chapter coincide, for uniqueness reasons, with those given by L. Boccardo and T. Gallouët in [8] and by L. Boccardo and G.R. Cirmi in [6]. Indeed, these solutions are obtained as limit of solutions of variational obstacle problems, whose data are smooth and converge strongly in $\mathcal{M}_b(\Omega)$. By Theorem 3.1.4 these solutions satisfy Definition 3.1.2.

Our setting allows us to prove the Lewy-Stampacchia inequality: first proved in [43] it has been extended by various authors to different cases. It has become a powerful tool for proving existence and regularity results.

Theorem 3.1.6. *Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ and u be the solution of $OP_0(A,\mu,u_\rho)$ (u_ρ defined in (3.1.2)). If we denote by λ the obstacle reaction associated with u, it holds*

$$\lambda \le (\mu-\rho)^-. \tag{3.1.8}$$

Finally we will show that the solution found can be characterized by the complementarity system.

Theorem 3.1.7. *Let μ be in $\mathcal{M}^p_{b,0}(\Omega)$ and ψ satisfy (3.1.2); then the following statements are equivalent:*

(1) u is the solution of $OP_0(A,\mu,\psi)$ and λ is the associated obstacle reaction;

(2) $u \ge \psi$ C_p-q.e. in Ω, $\lambda \in \mathcal{M}^{p,+}_{b,0}(\Omega)$, u is the entropy solution of (3.1.1) relative to $\mu+\lambda$, and

$$\begin{cases} \int_\Omega T_k(u-\varphi)\,d\lambda \le \int_\Omega T_k(v-\varphi)\,d\lambda \\ \forall \varphi \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega),\ \forall v \in \mathcal{T}^{1,p}_0(\Omega), v \ge \psi; \end{cases} \tag{3.1.9}$$

(3) $u \ge \psi$ C_p-q.e. in Ω, $\lambda \in \mathcal{M}^{p,+}_{b,0}(\Omega)$, u is the entropy solution of (3.1.1) relative to $\mu+\lambda$, and

$$u = \psi \ \lambda\text{- a.e. in } \Omega. \tag{3.1.10}$$

Remark 3.1.8. Observe that if ψ is C_p-q.e. upper bounded, we can consider in (3.1.9) $\varphi \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$, $\varphi \ge \psi$ C_p-q.e. in Ω and $v=\varphi$, so that, taking into account that u is the entropy solution of (3.1.1) relative to $\mu+\lambda$, for every $k>0$, u satisfies

$$\int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi)\,dx \le \int_\Omega T_k(u-\varphi)\,d\mu, \tag{3.1.11}$$

which is quite similar to the usual variational formulation (3.1.4). Formula (3.1.11) was just obtained in [6] when the datum μ is a function in $L^1(\Omega)$. This is an alternative proof of the fact that a solution of $OP_0(A,\mu,\psi)$ coincides with that given by L. Boccardo and T. Gallouët in [8], and by L. Boccardo and M. R. Cirmi in [6].

At the end of Section 3.4 we will show that a solution of $OP_0(A,\mu,\psi)$ is also a renormalized solution of the obstacle problem, according to the definition of [49]-[50].

3.2. Preparatory results

We give a result concerning the solutions of obstacle problems in the variational framework.

Theorem 3.2.1. *Let ψ satisfy hypothesis (3.1.2) and let μ in $W^{-1,p'}(\Omega)\cap\mathcal{M}_b(\Omega)$. Let u be the solution of $VI(A,\mu,\psi)$ and λ be the obstacle reaction relative to u. Then λ satisfies (3.1.7).*

Proof. We observe that we can consider, without loss of generality, the case of a nonpositive obstacle. Suppose that we have proved inequality (3.1.7) in this case (with $\rho=0$). Now, if ψ is a general obstacle satisfying hypothesis (3.1.2), we consider the new obstacle $\psi-u_\rho$, which is nonpositive, and the operator $B(v)=-\mathrm{div}(a(x,\nabla v+\nabla u_\rho)-a(x,\nabla u_\rho))$, which is of the same type of A. The measure λ associated with the solution v of $VI(B,\mu-\rho,\psi-u_\rho)$ satisfies the inequality (3.1.7). Now it is easy to check that the function $u=v+u_\rho$ is the solution of $VI(A,\mu,\psi)$ and λ is the measure associated with u.

Moreover, by an approximation argument we may suppose that μ^+ and μ^- belong to $W^{-1,p'}(\Omega)$ (as in Theorem 2.2.2).

The proof of (3.1.7) for a nonpositive obstacle ($\rho=0$) consists of two steps.

Step 1. Suppose that there exists a positive number δ such that $\psi\le-\delta$ C_p-q.e. in Ω.

Observe that the function u_{μ^+}, solution of

$$\begin{cases} A(u_{\mu^+})=\mu^+ & \text{in } \Omega \\ u_{\mu^+}=0 & \text{on } \partial\Omega, \end{cases}$$

is nonnegative and hence greater than or equal to ψ, belongs to $W_0^{1,p}(\Omega)$, and

$$A(u_{\mu^+})-\mu=\mu^- \text{ in } \Omega.$$

So $u \le u_{\mu^+}$ (by *Characterization 1*). For $\varepsilon > 0$ let us consider the truncated function $T_\varepsilon(u_{\mu^+} - u) = (u_{\mu^+} - u) \wedge \varepsilon$; it is nonnegative and less than or equal to ε.
Let us compute $\langle \lambda, T_\varepsilon(u_{\mu^+} - u)\rangle$, where $\langle \cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$:

$$\begin{aligned}
&\langle \lambda, T_\varepsilon(u_{\mu^+} - u)\rangle \\
&= \langle A(u), T_\varepsilon(u_{\mu^+} - u)\rangle - \langle \mu, T_\varepsilon(u_{\mu^+} - u)\rangle \\
&= \langle A(u) - A(u_{\mu^+}), T_\varepsilon(u_{\mu^+} - u)\rangle + \langle \mu^-, T_\varepsilon(u_{\mu^+} - u)\rangle \\
&\le \langle \mu^-, T_\varepsilon(u_{\mu^+} - u)\rangle,
\end{aligned}$$

where in the last inequality we used the monotonicity of A. Then, using the C_p-quasi-continuous representatives of functions in $W_0^{1,p}(\Omega)$, we can write

$$\int_\Omega T_\varepsilon(u_{\mu^+} - u)\, d\lambda \le \int_\Omega T_\varepsilon(u_{\mu^+} - u)\, d\mu^-.$$

By the positivity of the measures λ and μ^- and the properties of the truncated function, we deduce that

$$\lambda(\{u_{\mu^+} - u > \varepsilon\}) \le \mu^-(\Omega).$$

When ε tends to 0, the previous formula becomes

$$\lambda(\{u_{\mu^+} - u > 0\}) \le \mu^-(\Omega).$$

Now, if we prove that $\lambda(\{u_{\mu^+} = u\}) = 0$ we get the result. It suffices to observe that $\lambda(\{u_{\mu^+} = u\}) \le \lambda(\{u \ge 0\}) \le \lambda(\{u > \psi\})$, which is zero thanks to (3.1.6).
Step 2. Suppose only that $\psi \le 0$ C_p-q.e. in Ω.

Let us define, for all $\delta > 0$, the sequence of functions $\psi_\delta := \psi - \delta$, and consider the solutions u_δ of $VI(A, \mu, \psi_\delta)$. If we call λ_δ the measures associated with u_δ, by the previous step we get $\lambda_\delta(\Omega) \le \mu^-(\Omega)$.

As δ tends to zero the solution u_δ tends to u strongly in $W_0^{1,p}(\Omega)$, hence λ_δ tends to λ strongly in $W^{-1,p'}(\Omega)$, and this implies $\lambda(\Omega) \le \mu^-(\Omega)$. □

We compare now the two problems $VI(A, \mu, \psi)$ and $OP_0(A, \mu, \psi)$ when the forcing term μ is an element of $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$.

Proposition 3.2.2. *Let μ be an element of $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ and let ψ satisfy (3.1.2); then the solution of $VI(A,\mu,\psi)$ is the solution of $OP_0(A,\mu,\psi)$.*

Proof. Let u be the solution of $VI(A,\mu,\psi)$ and λ be the corresponding obstacle reaction, which is a nonnegative element of $W^{-1,p'}(\Omega)$. Thanks to Theorem 3.2.1 it belongs to $\mathcal{M}_b(\Omega)$, hence to $\mathcal{M}^{p,+}_{b,0}(\Omega)$. Thus u is the entropy solution of (3.1.1) relative to $\mu+\lambda$. Take $v \geq \psi$ C_p-q.e. in Ω, v the entropy solution of (3.1.1) corresponding to $\mu+\nu$, where ν belongs to $\mathcal{M}^{p,+}_{b,0}(\Omega)$. We want to prove that $u \leq v$ almost everywhere in Ω. Consider ν_k an approximation of ν (see Lemma 1.2.4 and Remark 1.2.5):

$$\nu_k \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega),\ \nu_k \geq 0,\ \nu_k \nearrow \nu \text{ strongly in } \mathcal{M}_b(\Omega);$$

and v_k the solution of (3.1.1) corresponding to $\mu+\nu_k$. Thanks to Theorem 1.6.10 the sequence v_k is nondecreasing and tends to v in the sense of Theorem 1.5.16. Thus, in particular, v_k tends to v C_p-q.e. in Ω. Let $\psi_k = \psi \wedge v_k$ and denote the solutions of $VI(A,\mu,\psi_k)$ by u_k. Naturally, from the minimality of u_k (see *Characterization 1*), we deduce

$$u_k \leq v_k \text{ a.e. in } \Omega.$$

Since $u_k \leq u_{k+1}$ C_p-q.e. in Ω, u_k converges to a function u^* C_p-q.e. in Ω. Thus, $u^* \geq \psi$ C_p-q.e. in Ω. It is easy to check that u_k is bounded in $W^{1,p}_0(\Omega)$; thanks to Lemma 1.2 in [27], u^* is (the C_p-quasi continuous representative of) a function of $W^{1,p}_0(\Omega)$ and u_k converges to u^* weakly in $W^{1,p}_0(\Omega)$. Moreover, it can be easily proved that u^* satisfies the variational formulation (3.1.4) and, consequently, u^* coincides with u. Hence, passing to the limit, we conclude that $u \leq v$ a.e. in Ω. □

Remaining in the variational framework we state a result that generalizes Lemma 2.7.3 to the nonlinear operator A. We omit the proof, because we may use the same tools as in the linear case. In the next section we will extend it to general measures in $\mathcal{M}^p_{b,0}(\Omega)$.

Lemma 3.2.3. *Let ψ satisfy (3.1.2) and let μ_1, $\mu_2 \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $VI(A,\mu_1,\psi)$ and $VI(A,\mu_2,\psi)$, respectively. If $\mu_1 \leq \mu_2$, then $\lambda_1 \geq \lambda_2$.*

3.3. Proof of the existence theorem

In this section we will prove Theorem 3.1.3; to simplify the exposition, it is convenient to divide the proof into various lemmas.

Lemma 3.3.1. *Let ψ satisfy (3.1.2) and let $\mu_n \in \mathcal{M}^p_{b,0}(\Omega)$ be a nondecreasing sequence of measures converging to μ strongly in $\mathcal{M}_b(\Omega)$; suppose that, for every n, problem $OP_0(A, \mu_n, \psi)$ has a solution u_n and let λ_n be the measure associated with it. If λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$, then there exists a solution u of $OP_0(A, \mu, \psi)$, and λ is the obstacle reaction relative to u.*

Proof. First we observe that μ and λ belong to $\mathcal{M}^p_{b,0}(\Omega)$, by the strong convergence of μ_n and λ_n in $\mathcal{M}_b(\Omega)$. Now we can use Theorem 1.5.16 about continuous dependence of entropy solutions: for every $k > 0$, the sequence $T_k(u_n)$ converges to $T_k(u)$ in the strong topology of $W^{1,p}_0(\Omega)$, u being the entropy solution of (3.1.1) relative to $\mu + \lambda$. The strong convergence in $W^{1,p}_0(\Omega)$ implies that, up to a subsequence, still denoted by u_n, $T_k(u_n)$ converges to $T_k(u)$ C_p-q.e. in Ω; thus $T_k(u) \geq T_k(\psi)$ C_p-q.e. in Ω, and, letting k tend to infinity, $u \geq \psi$ C_p-q.e. in Ω. Let us take now the entropy solution v of (3.1.1) relative to $\mu + \nu$, with ν in $\mathcal{M}^{p,+}_{b,0}(\Omega)$, and assume that $v \geq \psi$ C_p-q.e. in Ω. Since $\mu \geq \mu_n$ we can write $\mu + \nu = \mu_n + \nu_n$, with ν_n in $\mathcal{M}^{p,+}_{b,0}(\Omega)$. As u_n is the solution of $OP_0(A, \mu_n, \psi)$, we obtain that $u_n \leq v$ a.e. in Ω, and, passing to the limit, $u \leq v$ a.e. in Ω. In conclusion, u is the solution of $OP_0(A, \mu, \psi)$, according to Definition 3.1.2. □

Lemma 3.3.2. *Let ψ satisfy (3.1.2) and let $\mu_n \in \mathcal{M}^p_{b,0}(\Omega)$ be a nonincreasing sequence of measures converging to μ strongly in $\mathcal{M}_b(\Omega)$; suppose that, for every n, problem $OP_0(A, \mu_n, \psi)$ has a solution u_n and let λ_n be the measure associated with it. If λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$, then there exists a solution u of $OP_0(A, \mu, \psi)$, and λ is the obstacle reaction relative to u.*

Proof. As in the previous lemma, for every $k > 0$, $T_k(u_n)$ converges to $T_k(u)$ in the strong topology of $W^{1,p}_0(\Omega)$, u being the entropy solution of (3.1.1) relative to $\mu + \lambda$; moreover $u \geq \psi$ C_p-q.e. in Ω. Let us take the entropy solution v of (3.1.1) relative to $\mu + \nu$, with ν in $\mathcal{M}^{p,+}_{b,0}(\Omega)$, and assume that $v \geq \psi$ C_p-q.e. in Ω. Define the sequence v_n as the entropy solution of (3.1.1) relative to $\mu_n + \nu$. Since $\mu_n \geq \mu$, by the comparison principle of entropy solutions (see Theorem 1.6.10), we obtain $v_n \geq v \geq \psi$, and, thanks to Theorem 1.5.16, v_n converges to v a.e in Ω. By the definition of Obstacle Problems (Definition 3.1.2) we have that $u_n \le v_n$ a.e. in Ω, and, passing to the limit, $u \le v$ a.e. in Ω. Thus u is the solution of $OP_0(A, \mu, \psi)$. □

Lemma 3.3.3. *Let ψ satisfy (3.1.2) and let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ with $\mu^- \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$. Then there exists a solution of $OP_0(A, \mu, \psi)$, and the corresponding obstacle reaction satisfies (3.1.7).*

Proof. Since μ^+ is a nonnegative measure in $\mathcal{M}^p_{b,0}(\Omega)$, thanks to Remark 1.2.5, there exists a nondecreasing sequence $\mu_n^+ \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b^+(\Omega)$, converging to μ^+ in the strong topology of $\mathcal{M}_b(\Omega)$. Let us define $\mu_n := \mu_n^+ - \mu^-$, so that $\mu_n \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ and μ_n is a nondecreasing sequence converging to μ strongly in $\mathcal{M}_b(\Omega)$. By Proposition 3.2.2 the solution u_n of $VI(A, \mu_n, \psi)$ is a solution of $OP_0(A, \mu_n, \psi)$. Thanks to Theorem 3.2.1 the corresponding obstacle reaction λ_n satisfies

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \le \|(\mu_n - \rho)^-\|_{\mathcal{M}_b(\Omega)}. \tag{3.3.1}$$

As $\mu_n \le \mu_{n+1}$, by Lemma 3.2.3, we have that $\lambda_n \ge \lambda_{n+1}$. Hence, if we define

$$\lambda(B) := \lim_{n\to\infty} \lambda_n(B) \qquad \text{for every } B \text{ Borel set in } \Omega,$$

we know from measure theory that λ is a nonnegative Borel measure, it is bounded because λ_n is nonincreasing, and it is in $\mathcal{M}^p_{b,0}(\Omega)$, since all λ_n are. Besides, λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$. By Lemma 3.3.1, there exists u solution of $OP_0(A, \mu, \psi)$, and λ is the measure associated with it. Moreover, passing to the limit in (3.3.1), we obtain (3.1.7). □

Lemma 3.3.4. *Let ψ satisfy (3.1.2) and let μ_1, $\mu_2 \in \mathcal{M}^p_{b,0}(\Omega)$, with μ_1^-, $\mu_2^- \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacles corresponding to the solutions u_1 and u_2 of $OP_0(A, \mu_1, \psi)$ and $OP_0(A, \mu_2, \psi)$, respectively. If $\mu_1 \le \mu_2$, then $\lambda_1 \ge \lambda_2$.*

Proof. Since μ_1^+ and $\mu_2 - \mu_1$ are nonnegative measures in $\mathcal{M}^p_{b,0}(\Omega)$, thanks to Remark 1.2.5 there exist two nondecreasing sequences $\mu_{1,n}^+$, $\mu_n \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b^+(\Omega)$, converging strongly in $\mathcal{M}_b(\Omega)$ to μ_1^+ and $\mu_2 - \mu_1$, respectively. Let us define $\mu_{1,n} := \mu_{1,n}^+ - \mu_1^-$ and $\mu_{2,n} := \mu_n + \mu_{1,n}$; thus, for $i = 1, 2$, $\mu_{i,n}$ belongs to $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$, and $\mu_{i,n}$ is a nondecreasing sequence converging to μ_i strongly in $\mathcal{M}_b(\Omega)$. For $i = 1, 2$, consider the solution $u_{i,n}$ of $VI(A, \mu_{i,n}, \psi)$ and the corresponding obstacle reaction $\lambda_{i,n}$. Since $\mu_{1,n} \leq \mu_{2,n}$, by Lemma 3.2.3, we have that $\lambda_{1,n} \geq \lambda_{2,n}$. Using the same arguments of the proof of Lemma 3.3.3, we obtain that, for $i = 1, 2$, $\lambda_{i,n}$ converges to λ_i strongly in $\mathcal{M}_b(\Omega)$. Thus we get the result. □

Proof of Theorem 3.1.3. Consider μ^- and approximate it in the strong topology of $\mathcal{M}_b(\Omega)$ with a nondecreasing sequence $\mu_n^- \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b^+(\Omega)$ (see Remark 1.2.5). Defining $\mu_n := \mu^+ - \mu_n^-$, by Lemma 3.3.3 we can consider the solutions u_n of $OP_0(A, \mu_n, \psi)$ and the measures λ_n associated with them. By Lemma 3.3.3 we know that λ_n satisfies

$$\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu_n - \rho)^-\|_{\mathcal{M}_b(\Omega)}. \tag{3.3.2}$$

Since μ_n is nonincreasing, by Lemma 3.3.4, λ_n is nondecreasing. Thus, defining

$$\lambda(B) := \lim_{n \to \infty} \lambda_n(B) \qquad \text{for every } B \text{ Borel set in } \Omega,$$

we know from measure theory that λ is a nonnegative Borel measure, it is bounded because λ_n satisfies (3.3.2), and it is in $\mathcal{M}_{b,0}^p(\Omega)$, since all λ_n are. Moreover, λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$. By Lemma 3.3.2, there exists u solution of $OP_0(A, \mu, \psi)$, and λ is the corresponding obstacle reaction. Moreover, passing to the limit in (3.3.2) we obtain (3.1.7). □

Corollary 3.3.5. *Let ψ satisfy (3.1.2) and let μ_1, $\mu_2 \in \mathcal{M}_{b,0}^p(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $OP_0(A, \mu_1, \psi)$ and $OP_0(A, \mu_2, \psi)$, respectively. If $\mu_1 \leq \mu_2$ then $\lambda_1 \geq \lambda_2$.*

Proof. It suffices to consider μ_1^- and approximate it in the strong topology of $\mathcal{M}_b(\Omega)$ with a nondecreasing sequence $\mu_{1,n}^- \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b^+(\Omega)$ (see Remark 1.2.5). Defining $\mu_{1,n} := \mu_1^+ - \mu_{1,n}^-$ and $\mu_{2,n} := \mu_{1,n} + \mu_2 - \mu_1$, for $i = 1, 2$, we have that $\mu_{i,n}$ is a nonincreasing sequence converging to μ_i strongly in $\mathcal{M}_b(\Omega)$. Moreover, $\mu_{1,n} \leq \mu_{2,n}$. We consider the solutions $u_{i,n}$ of $OP_0(A, \mu_{i,n}, \psi)$ and the corresponding obstacle reactions $\lambda_{i,n}$, which are nondecreasing in n and satisfy $\lambda_{1,n} \geq \lambda_{2,n}$ (by Lemma 3.3.4). Using the same tools of the proof of Theorem 3.1.3, we obtain that $\lambda_{i,n}$ converges to λ_i strongly in $\mathcal{M}_b(\Omega)$. In conclusion $\lambda_1 \geq \lambda_2$. □

At this point we are able to check the validity of the Lewy-Stampacchia inequality.

Proof of Theorem 3.1.6. Consider the solution v of $OP_0(A, \rho - (\mu - \rho)^-, u_\rho)$ and its obstacle reaction λ'. It is easy to check that $v = u_\rho$, so that $\lambda' = (\mu - \rho)^-$. On the other hand we have that $\mu \geq \rho - (\mu - \rho)^-$, and, by the previous corollary, $\lambda \leq \lambda'$, so that the Lewy-Stampacchia inequality (3.1.8) is proved. □

3.4. Proof of the stability result and of the complementarity conditions

In this section we will prove Theorem 3.1.4 and Theorem 3.1.7.

Proof of Theorem 3.1.4. Since μ_n converges to μ strongly in $\mathcal{M}_b(\Omega)$, there exists a subsequence μ_{n_j} of μ_n such that

$$\sum_{j=1}^{+\infty} \|\mu_{n_j} - \mu\|_{\mathcal{M}_b(\Omega)} < +\infty.$$

Let us define $\mu'_j := \mu - \sum_{i=j}^{+\infty}(\mu_{n_i} - \mu)^-$ and $\mu''_j := \mu + \sum_{i=j}^{+\infty}(\mu_{n_i} - \mu)^+$; it is easy to check that

$$\mu'_j \nearrow \mu, \quad \mu''_j \searrow \mu \text{ strongly in } \mathcal{M}_b(\Omega)$$

and

$$\mu'_j \leq \mu_{n_j} \leq \mu''_j.$$

For every $j \geq 1$, let u'_j and u''_j be the solutions of $OP_0(A, \mu'_j, \psi)$ and $OP_0(A, \mu''_j, \psi)$, respectively, and let λ'_j and λ''_j be the corresponding associated measures. Reasoning as in the proof of Theorem 3.1.3, we obtain that u'_j and u''_j converge to u (in the sense of Theorem 1.5.16), while λ'_j and λ''_j converge to λ in the strong topology of $\mathcal{M}_b(\Omega)$. On the other hand, thanks to Corollary 3.3.5, we have $\lambda''_j \leq \lambda_{n_j} \leq \lambda'_j$, so that λ_{n_j} converges to λ strongly in $\mathcal{M}_b(\Omega)$. Finally, since the result does not depend on the subsequence all λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$.

We can apply Theorem 1.5.16 to obtain the convergence of $T_k(u_n)$ to $T_k(u)$ in the strong topology of $W_0^{1,p}(\Omega)$, for every $k > 0$. □

Proof of Theorem 3.1.7. We will divide the proof in three steps.
Step 1. (1) $\Rightarrow$ (2).

It suffices to prove (3.1.9). We proceed by an approximation argument. Let $\mu_n \in W^{-1,p'}(\Omega)\cap\mathcal{M}_b(\Omega)$ be such that μ_n converges to μ in the strong topology of $\mathcal{M}_b(\Omega)$. If we consider the solution u_n of $VI(A,\mu_n,\psi)$ and the measure λ_n associated to it, thanks to (3.1.4) we have

$$\begin{cases} \langle \lambda_n, w-u_n\rangle \geq 0 \\ \forall w \in K_\psi. \end{cases} \tag{3.4.1}$$

We choose as test function in (3.4.1) $w = u_n - T_k(u_n-\varphi)+T_k(v-\varphi)$, where $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $v \in \mathcal{T}_0^{1,p}(\Omega)$, with $v \geq \psi$ C_p-q.e. in Ω, and we obtain

$$\langle \lambda_n, T_k(u_n-\varphi)\rangle \leq \langle \lambda_n, T_k(v-\varphi)\rangle.$$

By Theorem 3.1.4 we know that λ_n converges to λ strongly in $\mathcal{M}_b(\Omega)$; hence, thanks to Lemma 1.2.15, we can pass to the limit and we get the result.
Step 2. (2) $\Rightarrow$ (3).

Let t be a positive real number. Observe that the set $\{u-\psi>t\}$ is C_p-quasi open, because u is C_p-quasi continuous and ψ is C_p-quasi upper semicontinuous. Thanks to Lemma 1.5 in [24] there exists an increasing sequence v_n of nonnegative functions in $W_0^{1,p}(\Omega)$ which converges to $\chi_{\{u-\psi>t\}}$ C_p-q.e. in Ω. The function $u-tv_n \in \mathcal{T}_0^{1,p}(\Omega)$ is greater than or equal to ψ, thus we can apply (3.1.9) with $v = u - tv_n$; observing that $u-tv_n \leq u$ and λ is a nonnegative measure, we get

$$\int_\Omega T_k(u-\varphi)\,d\lambda = \int_\Omega T_k(u-tv_n-\varphi)\,d\lambda,$$

for every φ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Passing to the limit as n goes to infinity, we get

$$\int_{\{u-\psi>t\}} T_k(u-\varphi)\,d\lambda = \int_{\{u-\psi>t\}} T_k(u-t-\varphi)\,d\lambda. \tag{3.4.2}$$

Now we choose $\varphi = T_h(u)$ in (3.4.2). Let us estimate the left hand side of (3.4.2):

$$\Big|\int_{\{u-\psi>t\}} T_k(u-T_h(u))\,d\lambda\Big| \leq k\lambda(\{|u|>h\}),$$

which tends to zero if h tends to infinity (recall that u is finite up to a set of p-capacity zero and λ vanishes on the sets of p-capacity zero).

In the same way we can split the integral in the right hand side into two parts:

$$\int_{\{u-\psi>t\}\cap\{|u|\le h\}} T_k(-t)\,d\lambda + \int_{\{u-\psi>t\}\cap\{|u|>h\}} T_k(u-t-T_h(u))\,d\lambda;$$

as before, the second integral tends to zero if h goes to infinity.

In conclusion, we obtain $T_k(t)\,\lambda(\{u-\psi>t\})=0$, from which $\lambda(\{u-\psi>t\})=0$, for every $t>0$. Letting t tend to zero, we get the result.

Step 3. (3) $\Rightarrow$ (1).

We have to prove that, for any ν in $\mathcal{M}^{p,+}_{b,0}(\Omega)$ such that the entropy solution v relative to $\mu+\nu$ is greater than or equal to ψ, we have $u\le v$ almost everywhere in Ω.

By Definition 1.6.1, for every $k>0$, u satisfies (in particular)

$$\int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi)^+dx - \int_\Omega T_k(u-\varphi)^+d\mu = \int_\Omega T_k(u-\varphi)^+d\lambda, \tag{3.4.3}$$

for every φ in $W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$.

Similarly, for every $k>0$, v satisfies

$$-\int_\Omega a(x,\nabla v)\nabla T_k(v-\varphi)^-dx + \int_\Omega T_k(v-\varphi)^-d\mu = -\int_\Omega T_k(v-\varphi)^-d\nu. \tag{3.4.4}$$

We choose $\varphi=T_h(v)$ in (3.4.3) and $\varphi=T_h(u)$ in (3.4.4), and we add the two equations.

For the left hand side we can use the same tools of the proof of uniqueness of entropy solutions (see Theorem 1.6.6); thus we obtain

$$\begin{aligned}&\int_{\{0<u-v\le k,\ |u|\le h,\ |v|\le h\}} (a(x,\nabla u)-a(x,\nabla v))\nabla(u-v)dx\\ &\le \omega_k(h) + \int_\Omega T_k(u-T_h(v))^+d\lambda - \int_\Omega T_k(v-T_h(u))^-d\nu,\end{aligned} \tag{3.4.5}$$

where $\omega_k(h)$ tends to zero if h goes to infinity.

Since ν is a nonnegative measure we can rewrite (3.4.5) as

$$\int_{\{0<u-v\le k,\ |u|\le h,\ |v|\le h\}} (a(x,\nabla u)-a(x,\nabla v))\nabla(u-v)dx \le \omega_k(h) + \int_\Omega T_k(u-T_h(v))^+d\lambda;$$

now we let h tend to infinity, so that

$$\int_{\{0<u-v\le k\}} (a(x,\nabla u)-a(x,\nabla v))\nabla(u-v)dx \le \int_\Omega T_k(u-v)^+d\lambda.$$

Observing that $\{u-v>0\} \subseteq \{u-\psi>0\}$, since $v \geq \psi$, the term in the right hand side is zero, by hypothesis (3.1.10). In conclusion

$$\int_{\{0<u-v\leq k\}} (a(x,\nabla u) - a(x,\nabla v))\nabla(u-v)dx \leq 0,$$

for every $k>0$; by (1.3.3) this implies $\nabla(u-v)^+ = 0$ almost everywhere in Ω. Now we consider, for every $i>0$, the function $T_j(T_i(u)-T_i(v))^+$, which belongs to $W_0^{1,p}(\Omega)$. Since $\nabla(u-v)^+=0$ a.e. in Ω, it is easy to prove that, for $i>j$,

$$\int_\Omega |\nabla T_j(T_i(u)-T_i(v))^+|^p\, dx \leq \int_{\{i-j<|u|<i\}} |\nabla u|^p\, dx + \int_{\{i-j<|v|<i\}} |\nabla v|^p\, dx.$$

By (1.6.9) the right hand side of the previous inequality tends to zero as i goes to infinity; in particular, it is bounded uniformly with respect to i. Therefore the function $T_j(T_i(u)-T_i(v))^+$ is bounded in $W_0^{1,p}(\Omega)$ uniformly with respect to i. Since this function converges to $T_j(u-v)^+$ almost everywhere in Ω as i tends to infinity, we conclude that $T_j(u-v)^+$ belongs to $W_0^{1,p}(\Omega)$ and that

$$\int_\Omega |\nabla T_j(u-v)^+|^p\, dx = 0.$$

Hence, $T_j(u-v)^+ = 0$ a.e. in Ω, for every $j>0$, and, in conclusion, $u \leq v$. □

Remark 3.4.1. If ψ is C_p-q.e. upper bounded, we point out that the solution u of $OP_0(A,\mu,\psi)$ satisfies

$$\int_\Omega a(x,\nabla u)\nabla(h(u)(u-\varphi))\, dx \leq \int_\Omega h(u)(u-\varphi)\, d\mu, \tag{3.4.6}$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \geq \psi$ C_p-q.e. in Ω, and for every $h \in C_c^1(\mathbb{R})$, $h \geq 0$ in $\mathbb{R}$. Indeed, working as in the proof of the first step of Theorem 3.1.7, we choose in (3.4.1) $w = u_n - \alpha h(u_n)(u_n-\varphi)$, with h and φ as before and α a positive constant such that $\alpha\|h\|_\infty \leq 1$. Passing to the limit we obtain (3.4.6). Thus u is also a renormalized solution of the obstacle problem according to the definition of [49]-[50].

Chapter 4

On a class of nonlinear obstacle problems with measure data

4.1. Assumptions and main results

We have observed that, if we want to study, in a nonlinear framework, unilateral problems when the forcing term is a general bounded Radon measure, we fall into the context considered by F. Murat in [48] to solve uniquely the Dirichlet problem

$$\begin{cases} A(u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

when μ is a bounded Radon measure.

Actually, in Section 1.5 we have seen that, if A is strongly monotone and Lipschitz continuous, there exists a unique reachable solution of (4.1.1) (see Theorem 1.5.3 and Theorem 1.5.13).

We recall that, in the general nonlinear context of Chapter 3, the uniqueness of a reachable solution still remains an open problem.

Let us make our assumptions more precise. We will consider a regular (in the sense of Definition 1.4.2) subset of $\mathbb{R}^N$, $N \geq 2$, and an operator A of the form

$$A(u) = -\mathrm{div}(a(x, \nabla u)),$$

with $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function satisfying (1.5.14), (1.5.15), and (1.5.16).

Moreover, $\psi : \Omega \to \overline{\mathbb{R}}$ is an arbitrary function such that

$$\psi \leq u_\rho \text{ q.e. in } \Omega, \tag{4.1.2}$$

where $\rho \in \mathcal{M}_b(\Omega)$ and u_ρ is the reachable solution of

$$\begin{cases} A(u) = \rho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 4.1.1. We say that u is the solution of the Obstacle Problem with datum $\mu \in \mathcal{M}_b(\Omega)$ and obstacle ψ (denoted by $OP(A, \mu, \psi)$) if

1. there exists a measure $\lambda \in \mathcal{M}_b^+(\Omega)$ such that u is the reachable solution of (4.1.1) relative to $\mu + \lambda$, and $u \geq \psi$ quasi everywhere in Ω.
2. for any $\nu \in \mathcal{M}_b^+(\Omega)$ such that the reachable solution v of (4.1.1) relative to $\mu + \nu$ satisfies $v \geq \psi$ q.e. in Ω, we have $u \leq v$ a.e. in Ω.

By definition, it is clear that, if such a solution exists, it is unique.

Since u and v are quasi continuous, the inequality $u \leq v$ a.e. in Ω is equivalent to $u \leq v$ q.e. in Ω.

The nonnegative measure λ, which is uniquely defined, will be called the obstacle reaction relative to u, or the measure associated with it.

To show that for any datum μ there exists one and only one solution, we introduce the set

$$\mathcal{F}_\psi^A(\mu) = \{v \geq \psi \text{ q.e.}, v \text{ reachable sol. of (4.1.1) relative to } \mu + \nu, \nu \in \mathcal{M}_b^+(\Omega)\}.$$

We will prove that $\mathcal{F}_\psi^A(\mu)$ has a minimum element, i.e., a function $u \in \mathcal{F}_\psi^A(\mu)$ such that $u \leq v$ a.e. in Ω for any other function $v \in \mathcal{F}_\psi^A(\mu)$. This is clearly the solution of the Obstacle Problem according to Definition 4.1.1.

Observe that, thanks to (4.1.2), $\mathcal{F}_\psi^A(\mu)$ is nonempty, because it contains the reachable solution of (4.1.1) relative to $\mu^+ + \rho$ (see Theorem 1.5.17).

In Section 4.3 we will prove the following theorem.

Theorem 4.1.2. *Let ψ satisfy (4.1.2) and let $\mu \in \mathcal{M}_b(\Omega)$. Then there exists a unique solution of $OP(A, \mu, \psi)$.*

We emphasize that, when A is a linear operator, Definition 3.1.2 turns out to be equivalent to Definition 2.1.1. As a matter of fact, proving Theorem 4.1.2 we give also an alternative proof to Theorem 2.3.1.

Without loss of generality we may suppose that ψ is quasi upper semicontinuous thanks to the following Proposition (it is a consequence of Proposition 3.1.1).

Proposition 4.1.3. *Let $\psi : \Omega \mapsto \overline{\mathbb{R}}$. Then there exists a quasi upper semicontinuous function $\hat{\psi} : \Omega \mapsto \overline{\mathbb{R}}$ such that:*

1. *$\hat{\psi} \geq \psi$ q.e. in Ω;*
2. *if $\varphi : \Omega \mapsto \overline{\mathbb{R}}$ is quasi upper semicontinuous and $\varphi \geq \psi$ q.e. in Ω, then $\varphi \geq \hat{\psi}$ q.e. in Ω.*

Our setting allows us to prove also the Lewy-Stampacchia inequality.

Theorem 4.1.4. *Let $\mu \in \mathcal{M}_b(\Omega)$ and u the solution of $OP(A, \mu, u_\rho)$ (u_ρ defined in (4.1.2)). If we denote by λ the obstacle reaction associated with u, it holds*

$$\lambda \leq (\mu - \rho)^-. \tag{4.1.3}$$

In Section 4.4 we will show that the solution to the Obstacle Problem considered in the previous chapter (see Definition 3.1.2) coincides with the new one (Definition 4.1.1) when both make sense.

Finally, Section 4.5 is devoted to the study of the interaction between obstacles and data. The aim is to obtain the complementarity conditions, but we will see that this is not possible in general. An important role in this problem is played by the space $\mathcal{M}_{b,0}(\Omega)$ of all bounded Radon measures on Ω which vanish on the sets of capacity zero. If the negative part μ^- of the datum μ belongs to $\mathcal{M}_{b,0}(\Omega)$, so does the obstacle reaction, provided that there exists a measure $\rho \in \mathcal{M}_{b,0}(\Omega)$ such that the solution of (4.1.1) relative to ρ is greater than or equal to ψ. In this case the obstacle reaction is concentrated on the contact set $\{u = \psi\}$, whenever the obstacle ψ is quasi upper semicontinuous (Theorem 4.5.1). So we concentrate our attention on the case $\mu^- \notin \mathcal{M}_{b,0}(\Omega)$. In this case μ can be decomposed in a unique way as $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathcal{M}_{b,0}(\Omega)$ and μ_s is concentrated on a set of capacity zero, and $\mu_s = \mu_s^+ - \mu_s^-$, with $\mu_s^- \neq 0$. We prove the following theorem, dealing with the case where the obstacle is controlled from above and from below.

Theorem 4.1.5. *Let $\mu \in \mathcal{M}_b(\Omega)$ and let $\psi : \Omega \to \overline{\mathbb{R}}$ be quasi upper semicontinuous satisfying*

$$u^A_{-\rho-\tau} \leq \psi \leq u^A_\rho, \tag{4.1.4}$$

where $\rho \in \mathcal{M}_{b,0}(\Omega)$ and $\tau \in \mathcal{M}_b(\Omega)$, with $\tau \perp \mu_s^-$ (here $u^A_{-\rho-\tau}$ and u^A_ρ are the reachable solutions of (4.1.1) relative to $-\rho-\tau$ and ρ, respectively). Let u and u_1 be the solutions *of $OP(A,\mu,\psi)$ and $OP(A,\mu_a+\mu_s^+,\psi)$, respectively, and λ and λ_1 be the corresponding obstacle reactions. Then $u=u_1$ a.e. in Ω, $\lambda=\lambda_1+\mu_s^-$, and $u=\psi$ λ_1-a.e. in Ω.*

This shows that, under these assumptions, the solution u of $OP(A,\mu,\psi)$ does not depend on μ_s^-, while the obstacle reaction has the form $\lambda_1+\mu_s^-$.

In [22] this theorem was proved in the linear case, investigating the behaviour of the potential of two mutually singular measures near their singular points (Lemma 3.3 and Lemma 3.4). Actually we extend this result to our (nonlinear) context giving alternative proofs.

4.2. Preliminary results

We need some results about the reachable solutions.

If $a(x,\xi)$ satisfies hypotheses (1.5.14), (1.5.15), (1.5.16), and U is a measurable function from Ω into $\mathbb{R}^N$, define the function $b:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ as

$$b(x,\xi)=a(x,\xi+U(x))-a(x,U(x)),$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$. It is easy to check that $b(x,\xi)$ satisfies (1.5.14), (1.5.15), (1.5.16) too. Hence the operator B defined as $B(u)=-\mathrm{div}(b(x,\nabla u))$ is of the same kind of A.

We can prove now the following result.

Proposition 4.2.1. *Let μ and ν be two measures in $\mathcal{M}_b(\Omega)$, u and v the corresponding reachable solutions. Consider the function $b(x,\xi)=a(x,\xi+\nabla v(x))-a(x,\nabla v(x))$ and the associated operator B. Then $w=u-v$ is the reachable solution of*

$$\begin{cases} B(w)=\mu-\nu & \text{in } \Omega \\ w=0 & \text{on } \partial\Omega. \end{cases} \tag{4.2.1}$$

Proof. For every $j>0$, define

$$\mu_j:=A(T_j(u)) \text{ and } \nu_j:=A(T_j(v)).$$

We know from Proposition 1.5.11 that

$$\mu_j \rightharpoonup \mu \text{ and } \nu_j \rightharpoonup \nu \ *\text{-weakly in } \mathcal{M}_b(\Omega);$$

moreover μ_j, $\nu_j \in H^{-1}(\Omega)$. For j, $n > 0$ let us consider w_n^j the variational solution of

$$\begin{cases} B(w_n^j) = \mu_j - \nu_n & \text{in } \Omega \\ w_n^j \in H_0^1(\Omega). \end{cases} \tag{4.2.2}$$

It means that

$$\int_\Omega (a(x, \nabla w_n^j + \nabla v) - a(x, \nabla v))\nabla\varphi \, dx$$
$$= \int_\Omega a(x, \nabla T_j(u))\nabla\varphi \, dx - \int_\Omega a(x, \nabla T_n(v))\nabla\varphi \, dx,$$

for every $\varphi \in H_0^1(\Omega)$. Splitting Ω into the sets where $|v| \leq n$ and where $|v| > n$ we get (remember that $a(x, 0) = 0$):

$$\begin{aligned} &\int_{\{|v|\leq n\}} (a(x, \nabla w_n^j + \nabla v) - a(x, \nabla T_j(u)))\nabla\varphi \, dx \\ &+ \int_{\{|v|>n\}} (a(x, \nabla w_n^j + \nabla v) - a(x, \nabla v) - a(x, \nabla T_j(u)))\nabla\varphi \, dx = 0. \end{aligned} \tag{4.2.3}$$

For $i > 0$, let us use $\varphi = T_i(w_n^j + T_n(v) - T_j(u))$ as function test in (4.2.3). Noting that $\nabla T_i(w_n^j + T_n(v) - T_j(u)) = 0$ where $|w_n^j + T_n(v) - T_j(u)| > i$, and using (1.5.15), we have:

$$\begin{aligned} &c_1 \int_{\{|v|\leq n\}} |\nabla T_i(w_n^j + T_n(v) - T_j(u))|^2 \, dx \\ &+ \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} (a(x, \nabla w_n^j + \nabla v) - a(x, \nabla v))\nabla(w_n^j - T_j(u)) \, dx \\ &- \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} a(x, \nabla T_j(u)))\nabla(w_n^j - T_j(u)) \, dx \leq 0; \end{aligned}$$

and then, thanks again to the hypothesis of strong monotonicity (1.5.15), it follows:

$$\begin{aligned} &c_1 \int_{\{|v|\leq n\}} |\nabla T_i(w_n^j + T_n(v) - T_j(u))|^2 \, dx \\ &+ c_1 \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} |\nabla w_n^j|^2 \, dx \\ &+ c_1 \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} |\nabla T_j(u)|^2 \, dx \\ &- \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} (a(x, \nabla w_n^j + \nabla v) - a(x, \nabla v))\nabla T_j(u) \, dx \\ &- \int_{\{|v|>n\}\cap\{|w_n^j + T_n(v) - T_j(u)|\leq i\}} a(x, \nabla T_j(u)))\nabla w_n^j \, dx \leq 0. \end{aligned}$$

Now, if we apply the hypothesis (1.5.14) and then the Young inequality on the last two integrals (denoted by $I_1 + I_2$), we deduce

$$I_1 + I_2 \geq - c_0\varepsilon \int_{\{|v|>n\}\cap\{|w_n^j+T_n(v)-T_j(u)|\leq i\}} |\nabla w_n^j|^2 \, dx - \frac{c_0}{\varepsilon} \int_{\{|v|>n\}\cap\{|w_n^j+T_n(v)-T_j(u)|\leq i\}} |\nabla T_j(u)|^2 \, dx,$$

with $\varepsilon > 0$ such that $c_1 - c_0\varepsilon \geq 0$. In conclusion we have

$$c_1 \int_{\{|v|\leq n\}} |\nabla T_i(w_n^j + T_n(v) - T_j(u))|^2 \, dx \leq \left(\frac{c_0}{\varepsilon} - c_1\right) \int_{\{|v|>n\}\cap\{|w_n^j+T_n(v)-T_j(u)|\leq i\}} |\nabla T_j(u)|^2 \, dx,$$

being $\frac{c_0}{\varepsilon} - c_1 > 0$. Passing to the limit as n tends to infinity we have:

$$\liminf_{n\to\infty} \int_{\{|v|\leq n\}} |\nabla T_i(w_n^j + T_n(v) - T_j(u))|^2 \, dx \leq 0.$$

Thanks to Theorem 1.5.16, for every $q < \frac{N}{N-1}$, w_n^j converges to w^j strongly in $W_0^{1,q}(\Omega)$, w^j being the reachable solution of (4.2.1) relative to $\mu_j - \nu$, as well as w^j tends to w; thus, by Fatou Lemma,

$$\int_{\{|w^j+v-T_j(u)|<i\}} |\nabla(w^j + v - T_j(u))|^2 \, dx = 0,$$

for every $i > 0$. We conclude that, for every $j > 0$, $w^j + v - T_j(u) = 0$ a.e. in Ω, and, finally, letting j tend to infinity, $w + v - u = 0$ a.e. in Ω. □

In the next lemma we shall use the notation u_μ^A to indicate the reachable solution relative to an operator A of

$$\begin{cases} A(u_\mu^A) = \mu & \text{in } \Omega \\ u_\mu^A = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in \mathcal{M}_b(\Omega)$.

Lemma 4.2.2. *Let μ, ν be in $\mathcal{M}_b(\Omega)$ and u_μ^A, u_ν^A be the corresponding reachable solutions. Assume also that $u_\mu^A \geq u_\nu^A$; then, for every $\sigma \in \mathcal{M}_b(\Omega)$ there exists a function $b(x,\xi)$ satisfying (1.5.14), (1.5.15), (1.5.16), such that $u_{\mu+\sigma}^B \geq u_{\nu+\sigma}^B$, $u_{\mu+\sigma}^B$ and $u_{\nu+\sigma}^B$ being the reachable solutions relative to the operator B (associated to $b(x,\xi)$) and the measures $\mu+\sigma$ and $\nu+\sigma$, respectively.*

Proof. Let us define the function $a' : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ as $a'(x,\xi) = a(x,\xi + \nabla u_\nu^A(x)) - a'(x, \nabla u_\nu^A(x))$, and consider the associated operator A'. By the previous proposition we have that $u_\mu^A - u_\nu^A = u_{\mu-\nu}^{A'}$. By the same proposition we deduce also that, if we consider the reachable solution $u_{-\nu-\sigma}^{A'}$ and the function $b(x,\xi) = a'(x,\xi+\nabla u_{-\nu-\sigma}^{A'}(x)) - a'(x, \nabla u_{-\nu-\sigma}^{A'}(x))$, the operator B associated with it is such that $u_{\nu+\sigma}^B = u_0^{A'} - u_{-\nu-\sigma}^{A'} = u_{-\nu-\sigma}^{A'}$ and $u_{\mu+\sigma}^B = u_{\mu-\nu}^{A'} - u_{-\nu-\sigma}^{A'} = u_\mu^A - u_\nu^A - u_{-\nu-\sigma}^{A'}$, from which we obtain $u_{\mu+\sigma}^B \geq u_{\nu+\sigma}^B$. □

In the sequel we shall use often a simple fact, that is worth stating on its own.

Lemma 4.2.3. *Let μ, $\nu \in \mathcal{M}_b(\Omega)$. Then u is the solution of $OP(A,\mu,\psi)$ if and only if $u - u_\nu^A$ is the solution of $OP(B, \mu-\nu, \psi - u_\nu^A)$, where B is the operator associated with the function $b(x,\xi) := a(x,\xi + \nabla u_\nu^A(x)) - a(x, \nabla u_\nu^A(x))$. Moreover, the obstacle reaction is the same.*

Proof. Let u be the solution of $OP(A,\mu,\psi)$ and λ be the obstacle reaction associated with it. Of course, the function $u - u_\nu^A$ is greater than or equal to $\psi - u_\nu^A$, and, by Proposition 4.2.1, we have that $u_{\mu-\nu+\lambda}^B = u - u_\nu^A$. So, $u - u_\nu^A$ belongs to $\mathcal{F}_{\psi - u_\nu^A}^B(\mu-\nu)$. If we take $v \in \mathcal{F}_{\psi-u_\nu^A}^B(\mu-\nu)$, it is easy to check (using again Proposition 4.2.1) that $v + u_\nu^A \in \mathcal{F}_\psi^A(\mu)$, and, by the minimality of u, we get $u \leq v + u_\nu^A$, i.e., $u - u_\nu^A \leq v$; so that $u - u_\nu^A$ solves $OP(B, \mu-\nu, \psi - u_\nu^A)$. For the same reasons, the converse is also true. □

Moreover, we have the following lemma.

Lemma 4.2.4. *Let v_1 and v_2 the reachable solutions of (4.1.1) relative to $\mu+\nu_1$ and $\mu+\nu_2$, respectively, with $\mu \in \mathcal{M}_b(\Omega)$ and ν_1, $\nu_2 \in \mathcal{M}_b^+(\Omega)$. Then there exists $\nu \in \mathcal{M}_b^+(\Omega)$ such that $v_1 \wedge v_2$ is the reachable solution of (4.1.1) relative to $\mu+\nu$. Moreover, for $i = 1,2$,*

$$\|\nu\|_{\mathcal{M}_b(\Omega)} \leq \|\nu_i\|_{\mathcal{M}_b(\Omega)}. \tag{4.2.4}$$

Proof. We approximate the measures μ and ν_i $(i = 1, 2)$ by some sequences $\mu_n \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, $\nu_i^n \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ such that μ_n, ν_i^n converge to μ, ν_i, respectively, in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, and $\|\nu_i^n\|_{\mathcal{M}_b(\Omega)}$ converges to $\|\nu_i\|_{\mathcal{M}_b(\Omega)}$. Given such approximations, for $i = 1, 2$, let v_i^n the unique solution of the following problem

$$\begin{cases} A(v_i^n) = \mu_n + \nu_i^n & \text{in } \Omega \\ v_i^n \in H_0^1(\Omega). \end{cases}$$

From the general theory, we know that $v_1^n \wedge v_2^n$ is the solution of the variational inequality $VI(A, \mu_n, v_1^n \wedge v_2^n)$, so that there exists a nonnegative Radon measure ν_n (in $H^{-1}(\Omega)$) such that $A(v_1^n \wedge v_2^n) = \mu_n + \nu_n$ (see *Characterization 1* in Section 3.1). Moreover, thanks to Theorem 3.2.1, we obtain $(i = 1, 2)$:

$$\|\nu_n\|_{\mathcal{M}_b(\Omega)} \leq \|\nu_i^n\|_{\mathcal{M}_b(\Omega)}, \tag{4.2.5}$$

from which ν_n converges, up to a subsequence (still denoted by ν_n), to a measure $\nu \in \mathcal{M}_b^+(\Omega)$, and, by Theorem 1.5.16, $v_1^n \wedge v_2^n$ converges to v, the reachable solution of (4.1.1) relative to $\mu + \nu$, so that $v_1 \wedge v_2 = v$. Passing to the limit in (4.2.5) we get (4.2.4), using the $*$-weak lower semicontinuity of $\|\cdot\|_{\mathcal{M}_b(\Omega)}$ for the left-hand side. □

4.3. Existence and uniqueness for the Obstacle Problem

In this section we will prove Theorem 4.1.2 and Theorem 4.1.4.

Proof of Theorem 4.1.2. As announced we will prove that the set $\mathcal{F}_\psi^A(\mu)$ (defined in Section 4.1) has a minimum element. We already observed (in Section 4.1) that, thanks to hypothesis (4.1.2), this set is nonempty. Noting that $\mathcal{F}_\psi^A(\mu) \subset W_0^{1,q}(\Omega)$, for any q, $1 < q < \frac{N}{N-1}$, we take a sequence $v_n \in \mathcal{F}_\psi^A(\mu)$ dense with respect to the topology of $W_0^{1,q}(\Omega)$. Let us define $u_n := v_1 \wedge .... \wedge v_n$; by Lemma 4.2.4 there exists $\lambda_n \in \mathcal{M}_b^+(\Omega)$ such that u_n is the reachable solution of (4.1.1) relative to $\mu + \lambda_n$ and $\lambda_n(\Omega)$ is equibounded (with respect to n), thanks to (4.2.4). Hence λ_n converges, up to a subsequence still denoted by λ_n, to some $\lambda \in \mathcal{M}_b^+(\Omega)$ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$ and, by Theorem 1.5.16, u_n converges to u, the reachable solution of (4.1.1) relative to $\mu + \lambda$, in the sense that, for every $j > 0$, $T_j(u_n)$ tends to $T_j(u)$ weakly in $H_0^1(\Omega)$. On the other hand, $u_n \geq \psi$ q.e. in Ω. Now the set

$$K_{T_j(\psi)} = \{v \in H_0^1(\Omega) \ : \ v \geq T_j(\psi) \ \text{ q.e. in } \Omega\}$$

is convex and closed in $H_0^1(\Omega)$, so it is also weakly closed. Since, clearly, $T_j(u_n) \geq T_j(\psi)$ q.e. in Ω, passing to the limit as $n \to +\infty$ we get that also $T_j(u) \in K_{T_j(\psi)}$, hence $T_j(u) \geq T_j(\psi)$ q.e. in Ω for all $j > 0$. Passing to the limit as $j \to +\infty$ we get $u \geq \psi$ q.e. in Ω. Hence u is in $\mathcal{F}_\psi^A(\mu)$. By construction, for every $i \leq n$, $u_n \leq v_i$ a.e. in Ω, and as n goes to infinity, $u \leq v_i$, for every $i \geq 1$. Now, if $v \in \mathcal{F}_\psi^A(\mu)$, by density, there exists a sequence i_k such that v_{i_k} converges to v strongly in $W_0^{1,q}(\Omega)$; then we deduce easily that $u \leq v$ a.e. in Ω. □

Now we want to prove that the obstacle reaction λ associated to the solution u of $OP(A, \mu, \psi)$ satisfies inequality

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu - \rho)^-\|_{\mathcal{M}_b(\Omega)}. \tag{4.3.1}$$

To do this we need a proposition, which has an intrinsic interest, concerning the approximation properties of the Obstacle Problem.

But first we have to recall the definition given by U. Mosco in [47] about a convergence of sets.

Definition 4.3.1. Let K_n be a sequence of subsets of a Banach space X. The strong lower limit

$$s\text{-}\liminf_{n\to+\infty} K_n$$

of the sequence K_n is the set of all $v \in X$ such that there exists a sequence v_n converging to v strongly in X, with $v_n \in K_n$, for n large.

The weak upper limit

$$w\text{-}\limsup_{n\to+\infty} K_n$$

of the sequence K_n is the set of all $v \in X$ such that there exists a sequence v_k converging to v weakly in X and a sequence of integers n_k converging to $+\infty$, such that $v_k \in K_{n_k}$.

The sequence K_n converges to the set K in the sense of Mosco, shortly $K_n \overset{M}{\to} K$, if

$$s\text{-}\liminf_{n\to+\infty} K_n = w\text{-}\limsup_{n\to+\infty} K_n = K.$$

For every $\psi : \Omega \to \overline{\mathbb{R}}$, let us define the convex set

$$K_\psi = \{z \in H_0^1(\Omega) : z \geq \psi \ \text{ q.e. in } \Omega\}.$$

Proposition 4.3.2. *Let $\psi \leq 0$ and let u be the solution of $OP(A, \mu, \psi)$, with $\mu \in \mathcal{M}_b(\Omega)$. For every sequence $\mu_k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ converging to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, there exists a sequence $\psi_k \leq \psi$, with $K_{\psi_k} \xrightarrow{M} K_\psi$, such that the solution u_k of $VI(A, \mu_k, \psi_k)$ satisfies*

$$u_k \to u \quad \text{strongly in } W_0^{1,q}(\Omega), \text{ for every } q < \frac{N}{N-1},$$

$$T_j(u_k) \rightharpoonup T_j(u) \quad \text{weakly in } H_0^1(\Omega), \text{ for every } j > 0.$$

Proof. First of all we observe that ψ satisfies (4.1.2) with $\rho = 0$. Let λ be the obstacle reaction associated with u, and approximate it, in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, by a sequence $\hat{\lambda}_h \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$. Let w_k be the variational solution of (4.1.1) relative to $\mu_k + \hat{\lambda}_h$, so that w_k converges to u in the sense of Theorem 1.5.16. Defining $\psi_k = \psi \wedge w_k$, the measure λ_k associated to the solution u_k of $VI(A, \mu_k, \psi_k)$ satisfies the inequality

$$\|\lambda_k\|_{\mathcal{M}_b(\Omega)} \leq \|\mu_k^-\|_{\mathcal{M}_b(\Omega)}, \tag{4.3.2}$$

thanks to Theorem 3.2.1. Then λ_k converges, up to a subsequence (still denoted by λ_k) to a measure $\lambda' \in \mathcal{M}_b^+(\Omega)$ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$; by Theorem 1.5.16 u_k tends to the reachable solution u' of (4.1.1) relative to $\mu + \lambda'$ in the sense that $T_j(u_k)$ converges to $T_j(u')$ weakly in $H_0^1(\Omega)$, for every $j > 0$. Thanks to the next two lemmas we have that $K_{\psi_k} \xrightarrow{M} K_\psi$, and by results of [25], $K_{T_j(\psi_k)} \xrightarrow{M} K_{T_j(\psi)}$ too. Then $T_j(u') \geq T_j(\psi)$ q.e. in Ω, for every $j > 0$, and finally $u' \geq \psi$ q.e. in Ω. By definition of $OP(A, \mu, \psi)$, $u' \geq u$ a.e in Ω. On the other hand, by *Characterization 1* in Section 3.1, $u_k \leq w_k$ and, passing to the limit, $u' \leq u$ so that $u = u'$ a.e. in Ω. Let us observe that by the uniqueness of u, the whole sequence u_k tends to u. □

The following two lemmas are quite simple, but are proved here for the sake of completeness.

Lemma 4.3.3. *Let $\psi : \Omega \to \overline{\mathbb{R}}$ an arbitrary function; let $\nu \in \mathcal{M}_b(\Omega)$ and v be the reachable solution of (4.1.1) relative to ν. Assume also that $v \geq \psi$ q.e. in Ω. For every $\nu_k \in \mathcal{M}_b(\Omega)$ converging to ν in the $*$-weak topology of $\mathcal{M}_b(\Omega)$, let v_k be the reachable solution of (4.1.1) relative to ν_k, and $\psi_k := \psi \wedge v_k$. Then, ψ_k converges to ψ weakly in capacity, in the sense of Definition 1.2.16.*

Proof. We know, by Theorem 1.5.16, that, for every $j > 0$, $T_j(v_k)$ converges to $T_j(v)$ weakly in $H_0^1(\Omega)$, and by Proposition 1.2.13 weakly in capacity. Since (the quasi continuous representative of) v_k and v are quasi Borel functions, thanks to (1.5.12) it is easy to check that, for every $\mu \in \mathcal{M}^+_{b,0}(\Omega)$, v_k converges to v in μ-measure; hence ψ_k converges to ψ in μ^*-measure, i.e., weakly in capacity, in the sense of Definition 1.2.16.

□

Lemma 4.3.4. *Let ψ satisfy*

$$\exists \zeta \in H^1_0(\Omega) \text{ such that } \zeta \geq \psi \text{ q.e. in } \Omega; \tag{4.3.3}$$

and $\psi_k : \Omega \to \overline{\mathbb{R}}$ such that $\psi_k \leq \psi$ q.e. in Ω, and ψ_k converges to ψ weakly in capacity, in the sense of Definition 1.2.16. Then $K_{\psi_k} \stackrel{M}{\to} K_\psi$.

Proof. First of all we observe that

$$K_\psi \subseteq s\text{-}\liminf_{k\to\infty} K_{\psi_k} \subseteq w\text{-}\limsup_{k\to\infty} K_{\psi_k},$$

since $\psi_k \leq \psi$. It remains to prove that $w\text{-}\limsup_{k\to\infty} K_{\psi_k} \subseteq K_\psi$. Consider a function $z \in H^1_0(\Omega)$ such that there exists a sequence z_h converging to z weakly in $H^1_0(\Omega)$ and a sequence of integers k_h converging to $+\infty$ such that $z_h \in K_{\psi_{k_h}}$. Then $z_h \geq \psi_{k_h}$ q.e. in Ω, which implies that $z_h \geq \psi_{k_h}$ μ-a.e. in Ω, for every $\mu \in \mathcal{M}^+_{b,0}(\Omega)$. By Proposition 1.2.13 we know that z_h converge weakly in capacity to z, as well as, by hypothesis, ψ_{k_h} converge to ψ. Let us point out that, if a sequence converges weakly in capacity, then a subsequence converges μ^*-a.e. in Ω. So we obtain that $z \geq \psi$ μ-a.e. in Ω, which, by Proposition 1.2.14, implies $z \geq \psi$ q.e. in Ω, i.e. $z \in K_\psi$. □

Remark 4.3.5. If, in Proposition 4.3.2, we choose μ_k such that $\|\mu_k^-\|_{\mathcal{M}_b(\Omega)}$ converges to $\|\mu^-\|_{\mathcal{M}_b(\Omega)}$ (such an approximation exists), passing to the limit in (4.3.2) we obtain

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \|\mu^-\|_{\mathcal{M}_b(\Omega)}, \tag{4.3.4}$$

using the $*$-weak lower semicontinuity of $\|\cdot\|_{\mathcal{M}_b(\Omega)}$ for the left-hand side.

We shall prove now the inequality (3.1.7), without the technical assumption that the obstacle be nonpositive.

Proposition 4.3.6. *Let ψ satisfy (4.1.2) and let $\mu \in \mathcal{M}_b(\Omega)$. Then the obstacle reaction λ associated with the solution u of $OP(A,\mu,\psi)$ satisfies (4.3.1).*

Proof. We can refer to the case $\psi \le 0$. Indeed define $\varphi := \psi - u_\rho$, which is obviously negative, and consider the operator B associated with the function $b(x,\xi) := a(x,\xi + \nabla u_\rho(x)) - a(x, \nabla u_\rho(x))$. By Theorem 4.1.2 we can consider the solution v of $OP(B,\mu - \rho, \psi - u_\rho)$ and, by Lemma 4.2.3 we have that $v = u - u_\rho$, and λ is the measure associated with it. On the other hand, thanks to (4.3.4), λ satisfies

$$\|\lambda\|_{\mathcal{M}_b(\Omega)} \le \|(\mu - \rho)^-\|_{\mathcal{M}_b(\Omega)}.$$

□

Now we state a simple consequence of Proposition 4.3.2: we shall use it to prove an important result concerning the solutions to Obstacle Problems with measure data, that is the Lewy-Stampacchia inequality.

Corollary 4.3.7. *Let $\psi \le 0$ and let $\mu \in \mathcal{M}_b(\Omega)$, and $\mu_k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, converging to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$. Consider the sequence ψ_k of Proposition 4.3.2 and any $\varphi_k \le \psi_k$, with $K_{\varphi_k} \overset{M}{\to} K_\psi$, and let w_k and u the solutions of $VI(A,\mu_k,\varphi_k)$ and $OP(A,\mu,\psi)$, respectively, with λ_k and λ the corresponding obstacle reactions. Then*

$$w_k \to u \quad \textit{strongly in } W_0^{1,q}(\Omega), \textit{ for every } q < \frac{N}{N-1},$$

$$T_j(u_k) \rightharpoonup T_j(u) \quad \textit{weakly in } H_0^1(\Omega), \textit{ for every } j > 0,$$

and λ_k converges to λ in the $*$-weak topology of measures.

Proof. As in Proposition 4.3.2 denote by u_k the solution of $VI(A,\mu_k,\psi_k)$. Since $\varphi_k \le \psi_k$, we easily get (see *Characterization 1* in Section 3.1) that $w_k \le u_k$ q.e. in Ω. On the other hand, thanks to (3.1.7) the obstacle reaction λ_k associated with w_k is equibounded (with respect to k). Up to a subsequence (still denoted by λ_k), λ_k converges to a measure λ' $*$-weakly in $\mathcal{M}_b(\Omega)$, and w_k tends to the reachable solution w of (4.1.1) relative to $\mu + \lambda'$, in the sense of Theorem 1.5.16. As in the proof of Proposition 4.3.2 we deduce that $w \ge \psi$ q.e. in Ω, so that we can compare w with u getting $w \ge u$. Finally, passing to the limit in $w_k \le u_k$, we obtain the result. Hence we have that $\lambda' = \lambda$, and, by the uniqueness of u, the whole sequence λ_k converges to λ $*$-weakly in $\mathcal{M}_b(\Omega)$. □

We prove now a result that extends the variational setting of Lemma 3.2.3 to the general case of measures in $\mathcal{M}_b(\Omega)$.

Proposition 4.3.8. *Let ψ satisfy (4.1.2) and let μ_1, $\mu_2 \in \mathcal{M}_b(\Omega)$. Let λ_1 and λ_2 be the reactions of the obstacle corresponding to the solutions u_1 and u_2 of $OP(A, \mu_1, \psi)$ and $OP(A, \mu_2, \psi)$, respectively. If $\mu_1 \leq \mu_2$, then $\lambda_1 \geq \lambda_2$.*

Proof. First of all we observe that, by Lemma 4.2.3 it is enough to consider the case $\psi \leq 0$ q.e. in Ω.

Since $\mu_2 - \mu_1 \geq 0$, we consider a sequence $\eta_k \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ and a sequence $\mu_1^k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ approximating $\mu_2 - \mu_1$ and μ_1, respectively, in the $*$-weak topology of measures. Defining $\mu_2^k := \eta_k + \mu_1^k$, we have $\mu_2^k \geq \mu_1^k$. Let us consider the sequences ψ_1^k and ψ_2^k of Proposition 4.3.2 and the solutions u_1^k and u_2^k of $VI(A, \mu_1^k, \psi_1^k \wedge \psi_2^k)$ and $VI(A, \mu_2^k, \psi_1^k \wedge \psi_2^k)$, respectively. Concerning the reactions λ_1^k and λ_2^k associated with u_1^k and u_2^k, respectively, we have that

$$\lambda_1^k \geq \lambda_2^k, \tag{4.3.5}$$

thanks to Lemma 3.2.3. By Lemma 4.3.3 we have that $\psi_1^k \wedge \psi_2^k$ converges to ψ weakly in capacity (in the sense of Definition 1.2.16), and by Lemma 4.3.4 we deduce that $K_{\psi_1^k \wedge \psi_2^k} \xrightarrow{M} K_\psi$. Hence the previous corollary shows that u_1^k and u_2^k converge to u_1 and u_2, as well as λ_1^k and λ_2^k converge to λ_1 and λ_2 $*$-weakly in $\mathcal{M}_b(\Omega)$. Passing to the limit in (4.3.5) we get the result. □

At this point we are able to check the validity of the Lewy-Stampacchia inequality.

Proof of Theorem 4.1.4. Consider the solution v of $OP(A, \rho - (\mu - \rho)^-, u_\rho)$ and its obstacle reaction λ'. It is easy to check that $u_\rho \in \mathcal{F}_{u_\rho}^A(\rho - (\mu - \rho)^-)$, so that $v \leq u_\rho$. Hence $v = u_\rho$ and $\lambda' = (\mu - \rho)^-$. On the other hand we have that $\mu \geq \rho - (\mu - \rho)^-$, and, by the previous proposition, $\lambda \leq \lambda'$, so that the Lewy-Stampacchia inequality (4.1.3) is proved. □

4.4. Comparison with the classical solutions

In this section we want to show that the new formulation of Obstacle Problem is consistent with that one given in Chapter 3.

To speak about the equivalence of the two formulations it is necessary that both make sense. So we will work under the hypotheses that $\mu \in \mathcal{M}_{b,0}(\Omega)$ and that the obstacle ψ satisfies (4.1.2), with $\rho \in \mathcal{M}_{b,0}(\Omega)$.

Actually in the previous chapter the hypothesis on the obstacle is slightly more restrictive, in the sense that the measure ρ in (4.1.2) is assumed to be in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$. Since, now, the operator satisfies the stronger hypotheses (1.5.14), (1.5.15), (1.5.16), we can extend the results of Chapter 3 to the case $\rho \in \mathcal{M}_{b,0}(\Omega)$, arguing as in Lemma 4.2.3.

In Theorem 4.1.2 we proved that the problem $OP_0(A, \mu, \psi)$ has a unique solution u. We will show that u solves $OP(A, \mu, \psi)$, so that the obstacle reaction λ associated with the solution of $OP(A, \mu, \psi)$ is in $\mathcal{M}^+_{b,0}(\Omega)$. Moreover u can be characterized by the complementarity system (Theorem 3.1.7).

Theorem 4.4.1. *Let μ be an element of $\mathcal{M}_{b,0}(\Omega)$ and let ψ satisfy (4.1.2), with $\rho \in \mathcal{M}_{b,0}(\Omega)$; then the solution u of $OP_0(A, \mu, \psi)$ solves $OP(A, \mu, \psi)$.*

Proof. We denote the operator associated with $b(x, \xi) = a(x, \xi + \nabla u_\rho(x)) - a(x, \nabla u_\rho)$ by B. Arguing as in Lemma 4.2.3, we easily prove that $u - u_\rho$ is the solution of $OP_0(B, \mu - \rho, \psi - u_\rho)$. We will show that $u - u_\rho$ solves $OP(B, \mu - \rho, \psi - u_\rho)$, and, finally, by Lemma 4.2.3 u solves $OP(A, \mu, \psi)$.

By Definition 3.1.2 we know that $u - u_\rho$ is the entropy solution of

$$\begin{cases} B(v) = \mu - \rho + \lambda & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4.1}$$

λ being a measure in $\mathcal{M}^+_{b,0}(\Omega)$. Since every entropy solution is a reachable solution, we have that $u - u_\rho \in \mathcal{F}^B_{\psi - u_\rho}(\mu - \rho)$. Take v an element in $\mathcal{F}^B_{\psi - u_\rho}(\mu - \rho)$, then v is the reachable solution of (4.4.1) relative to $\mu - \rho + \nu$, $\nu \in \mathcal{M}^+_b(\Omega)$, and $v \geq \psi - u_\rho$ q.e. in Ω. Consider $\nu_k \in \mathcal{M}^+_b(\Omega) \cap H^{-1}(\Omega)$ an approximation of ν in the $*$-weak topology of measures, and v_k the entropy solution of (4.4.1) corresponding to $\mu - \rho + \nu_k$; so that v_k tends to v in the sense of Theorem 1.5.16. Let $\psi_k = (\psi - u_\rho) \wedge v_k$ and denote by w_k the solution of $OP_0(B, \mu - \rho, \psi_k)$. Naturally, by the minimality of w_k we deduce

$$w_k \leq v_k \quad \text{a.e. in } \Omega. \tag{4.4.2}$$

In Lemma 4.3.4 (see also Lemma 4.3.3) we proved that $K_{\psi_k} \overset{M}{\to} K_{\psi - u_\rho}$; on the other hand, in Corollary 5.4.6 it is proved that, under this hypothesis, w_k converges to $u - u_\rho$, in particular in the sense of Theorem 1.5.16. So, passing to the limit in (4.4.2) we obtain $u - u_\rho \leq v$ a.e. in Ω. Since this is true for every $v \in \mathcal{F}^B_{\psi - u_\rho}(\mu - \rho)$, the function $u - u_\rho$ is the minimum in $\mathcal{F}^B_{\psi - u_\rho}(\mu - \rho)$, i.e. the solution of $OP(B, \mu - \rho, \psi - u_\rho)$. □

Remark 4.4.2. By Theorem 4.4.1 and Theorem 3.2.2 we deduce also that, when $\mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ and ψ satisfies (4.1.2) with $\rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, then the solution of $VI(A, \mu, \psi)$ is the solution of $OP(A, \mu, \psi)$.

4.5. Interactions between obstacles and data

Thanks to Theorem 4.4.1, we can restate Theorem 3.1.7:

Theorem 4.5.1. *Let $\mu \in \mathcal{M}_{b,0}(\Omega)$ and ψ be a quasi upper semicontinuous function which satisfies (4.1.2) with $\rho \in \mathcal{M}_{b,0}(\Omega)$. Then the following facts are equivalent:*

1. *u is the solution of $OP(A, \mu, \psi)$ and λ is the corresponding reaction;*
2. *$\lambda \in \mathcal{M}^+_{b,0}(\Omega)$, u is the reachable solution of (4.1.1) relative to $\mu + \lambda$, $u \geq \psi$ q.e. in Ω, and $u = \psi$ λ-q.e. in Ω.*

Nevertheless, this fact is no longer true when we pass to consider general data in $\mathcal{M}_b(\Omega)$: in Example 2.7.2 the solution of the obstacle problem with right-hand side measure does not touch the obstacle, though it is not the solution of the equation.

Remark 4.5.2. Actually, thanks to Lemma 4.2.3, Theorem 4.5.1 is true even if just the negative part μ^- of μ belongs to $\mathcal{M}_{b,0}(\Omega)$.

Therefore we have to concentrate our attention on the case $\mu^- \notin \mathcal{M}_{b,0}(\Omega)$. Then μ can be decomposed in a unique way as $\mu = \mu_a + \mu_s$, with $\mu_a \in \mathcal{M}_{b,0}(\Omega)$ and μ_s concentrated on a set of capacity zero (see Lemma 1.2.1), and $\mu_s = \mu_s^+ - \mu_s^-$, with $\mu_s^- \neq 0$.

In all this section we shall use the notation of Lemma 4.2.2.

The proof of Theorem 4.1.5 relies on the following lemma, which has an intrinsic interest.

Lemma 4.5.3. *Let $\mu \in \mathcal{M}_b(\Omega)$ and $\rho \in \mathcal{M}_{b,0}(\Omega)$ such that $u^A_\mu \le u^A_\rho$ a.e. in Ω; then $\mu^+ \in \mathcal{M}_{b,0}(\Omega)$.*

Proof. We recall that $\mu = \mu_a^+ + \mu_s^+ - \mu_a^- - \mu_s^-$. We consider the solution v of $OP(A, \mu_a, u^A_\mu)$ and its obstacle reaction λ_1: by Theorem 4.5.1 we know that $\lambda_1 \in \mathcal{M}_{b,0}(\Omega)$. On the other hand by Lewy-Stampacchia inequality (4.1.3) we have that $\lambda_1 \le (\mu_a - \mu_a - \mu_s^+ + \mu_s^-)^- = \mu_s^+$. These two facts imply that $\lambda_1 = 0$, so that $v = u^A_{\mu_a} \ge u^A_\mu$. Lemma 4.2.2 allows us to find an operator B such that $0 = u^B_0 \ge u^B_{\mu_s}$. Let us prove that this implies $\mu_s^+ = 0$. We note indeed (see Chapter 1) that the function $u^B_{\mu_s}$ is a renormalized solution of the equation

$$\begin{cases} B(u) = \mu_s & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5.1}$$

that means (see Definition 4.5.1)

$$\begin{aligned} &\int_\Omega b(x, \nabla u^B_{\mu_s}) \nabla (h(u^B_{\mu_s})\varphi)\, dx \\ &= h(+\infty) \int_\Omega \varphi \, d\mu_s^+ - h(-\infty) \int_\Omega \varphi\, d\mu_s^-, \end{aligned} \tag{4.5.2}$$

for every $\varphi \in C^\infty_c(\Omega)$ and for every $h \in W^{1,\infty}(\mathbb{R})$ such that h' has compact support in $\mathbb{R}$ (here $h(+\infty)$ and $h(-\infty)$ are the limits of $h(t)$ at $+\infty$ and $-\infty$ respectively). At this point, choosing in (4.5.2) two admissible functions h_1 and h_2, with $h_1(t) = h_2(t)$ for every $t \le 0$, we do not change the left-hand side of (4.5.2), since $u^B_{\mu_s} \le 0$, and, consequently

$$h_1(+\infty) \int_\Omega \varphi\, d\mu_s^+ = h_2(+\infty) \int_\Omega \varphi\, d\mu_s^+.$$

Since this equality holds for every $\varphi \in C^\infty_c(\Omega)$ and for every such pair h_1, h_2, we obtain that $\mu_s^+ = 0$. □

Corollary 4.5.4. *Let μ, $\tau \in \mathcal{M}_b(\Omega)$, $\tau \perp \mu^+$, and $\rho \in \mathcal{M}_{b,0}(\Omega)$ such that $u^A_\mu \le u^A_{\rho+\tau}$ a.e. in Ω; then $\mu^+ \in \mathcal{M}_{b,0}(\Omega)$.*

Proof. First of all we observe that the measure τ can be assumed to be positive, replacing it with its positive part. By Lemma 4.2.2 there exists an operator B such that $u^B_{\mu-\tau} \le u^B_\rho$ a.e. in Ω, so that $(\mu - \tau)^+ \in \mathcal{M}_{b,0}(\Omega)$, thanks to the previous result. On the other hand $(\mu - \tau)^+ = \mu^+$, since $\tau \perp \mu^+$. □

Proof of Theorem 4.1.5. We can apply Theorem 4.5.1 to u_1 (see Remark 4.5.2), so that $\lambda_1 \in \mathcal{M}_{b,0}(\Omega)$ and $u_1 = \psi$ λ_1-a.e. in Ω.

Since $\mu \le \mu_a + \mu_s^+$, by Proposition 4.3.8, we get $\lambda \ge \lambda_1$, that is $\lambda = \lambda_1 + \lambda_2$, with $\lambda_2 \in \mathcal{M}_b^+(\Omega)$. On the other hand, by the minimality of u it is easy to check that $u \le u_1$ a.e. in Ω, that means $u^A_{\mu_a+\mu_s^+-\mu_s^-+\lambda_1+\lambda_2} \le u^A_{\mu_a+\mu_s^++\lambda_1}$. Lemma 4.2.2 implies the existence of an operator B such that $u^B_{\lambda_2-\mu_s^-} \le 0$ a.e. in Ω. Then, by Lemma 4.5.3, $(\lambda_2 - \mu_s^-)^+ \in \mathcal{M}_{b,0}(\Omega)$.

Thanks to hypothesis (4.1.4) we have $u \ge \psi \ge u^A_{-\rho-\tau}$, which implies, by Lemma 4.2.2, $u^C_{\mu_a+\mu_s^++\lambda_1+\rho+\tau} \ge u^C_{\mu_s^- -\lambda_2}$ a.e. in Ω, for some operator C of the same kind of A. Now $(\mu_s^+ + \tau) \perp (\mu_s^- - \lambda_2)^+$, since $\mu_s^+ \perp \mu_s^-$, $\tau \perp \mu_s^-$ and $\lambda_2 \ge 0$. So $(\mu_s^- - \lambda_2)^+ \in \mathcal{M}_{b,0}(\Omega)$, by Corollary 4.5.4.

As $(\mu_s^- - \lambda_2)^- = (\lambda_2 - \mu_s^-)^+ \in \mathcal{M}_{b,0}(\Omega)$, we conclude that $\lambda_2 - \mu_s^- \in \mathcal{M}_{b,0}(\Omega)$. Therefore $\lambda_2 = \sigma + \mu_s^-$, with $\sigma \in \mathcal{M}_{b,0}^+(\Omega)$, and $u = u^A_{\mu_a+\mu_s^++\lambda_1+\sigma}$. Thus $u \ge u_1$ a.e. in Ω, that implies $u = u_1$, and what we had to prove. $\square$

Chapter 5
Stability with respect to data

5.1. Assumptions and notations

The problem we deal with in this chapter regards the behaviour of the Obstacle Problem in the sense of Definition 3.1.2 under perturbation of the operator, of the forcing term, and of the obstacle.

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$. Let p be a real constant, $1 < p \leq N$, and let p' its dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$.

Given two constants c_0, $c_1 > 0$ and two constants α and β, with $0 \leq \alpha \leq 1 \wedge (p-1)$ and $p \vee 2 \leq \beta < +\infty$, we consider the family $\mathcal{L}(c_0, c_1, \alpha, \beta)$ of Carathéodory functions $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that:

$$|a(x,\xi) - a(x,\eta)| \leq c_0(1 + |\xi| + |\eta|)^{p-1-\alpha}|\xi - \eta|^{\alpha}, \tag{5.1.1}$$

$$(a(x,\xi) - a(x,\eta))(\xi - \eta) \geq c_1(1 + |\xi| + |\eta|)^{p-\beta}|\xi - \eta|^{\beta}, \tag{5.1.2}$$

$$a(x,0) = 0, \tag{5.1.3}$$

for almost every $x \in \Omega$, for every $\xi, \eta \in \mathbb{R}^N$.

We note that, if $a \in \mathcal{L}(c_0, c_1, \alpha, \beta)$, conditions (1.3.1), (1.3.2), (1.3.3), and (1.3.4) are satisfied, with k_0 and k_1 replaced by positive real constants depending on c_0, c_1, α, and β.

Remark 5.1.1. For a particular choice of the constants α and β, i.e. if $1 < p \leq 2$, $\alpha = p - 1$, and $\beta = 2$, the inequalities (5.1.1) and (5.1.2) become

$$|a(x,\xi) - a(x,\eta)| \leq c_0|\xi - \eta|^{p-1},$$

$$(a(x,\xi) - a(x,\eta))(\xi - \eta) \geq c_1(1 + |\xi| + |\eta|)^{p-2}|\xi - \eta|^2.$$

Moreover, if $2 \leq p < +\infty$, $\alpha = 1$, and $\beta = p$, the continuity and monotonicity assumptions (5.1.1) and (5.1.2) for the function a take the form

$$|a(x,\xi) - a(x,\eta)| \leq c_0(1 + |\xi| + |\eta|)^{p-2}|\xi - \eta|,$$

$$(a(x,\xi) - a(x,\eta))(\xi - \eta) \geq c_1|\xi - \eta|^p.$$

5.2. G-convergence and Mosco-convergence

The study of the properties of the solutions to the obstacle problems under perturbations of the operator a is based on a notion of convergence in $\mathcal{L}(c_0, c_1, \alpha, \beta)$, called G-convergence.

Definition 5.2.1. We say that a sequence of functions $a_h(\cdot,\cdot)$ belonging to $\mathcal{L}(c_0, c_1, \alpha, \beta)$ G-converges to a function $a(\cdot,\cdot)$ satisfying the same hypotheses (possibly with different constants $\tilde{c}_0, \tilde{c}_1, \tilde{\alpha}, \tilde{\beta}$) if for any $F \in W^{-1,p'}(\Omega)$, the solution u_h of

$$\begin{cases} A_h(u_h) = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2.1}$$

satisfies

$$u_h \rightharpoonup u \ \text{ weakly in } W_0^{1,p}(\Omega) \tag{5.2.2}$$

and

$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \text{ weakly in } L^{p'}(\Omega)^N, \tag{5.2.3}$$

where u is the unique solution of

$$\begin{cases} A(u) = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2.4}$$

The following theorem justifies the definition of G-convergence.

Theorem 5.2.2. *Any sequence $a_h(x,\xi)$ of functions belonging to $\mathcal{L}(c_0, c_1, \alpha, \beta)$ admits a subsequence which G-converges to a function $a(x,\xi) \in \mathcal{L}(\tilde{c}_0, \tilde{c}_1, \frac{\alpha}{\beta-\alpha}, \beta)$, where $\tilde{c}_0, \tilde{c}_1$ depend only on $N, p, \alpha, \beta, c_0, c_1$*

This compactness theorem was obtained by L. Tartar (see [54] and Theorem 1.1 of [32]) in the case of nonlinear monotone operators defined from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, when $p = 2$ and the functions $a_h \in \mathcal{L}(c_0, c_1, 1, 2)$, and then extended in the version of Theorem 5.2.2 in [17] (see Theorem 4.1).

The investigation of the properties of obstacle problems when the obstacle varies relies on a notion of convergence for sequences of convex sets introduced by U. Mosco in [47] (see Definition 4.3.1 in the previous chapter).

Mosco proved that this type of convergence is the right one for the stability of Variational Inequalities with respect to obstacles. This is the main theorem of his theory.

Theorem 5.2.3. *Let K_{ψ_h} and K_ψ be nonempty. Then*

$$K_{\psi_h} \xrightarrow{M} K_\psi$$

if and only if, for any $F \in W^{-1,p'}(\Omega)$,

$$u_h \to u \text{ strongly in } W_0^{1,p}(\Omega),$$

where u_h and u are the solutions of $VI(A,F,\psi_h)$ and $VI(A,F,\psi)$, respectively.

Several stability results can be proved as corollaries of this theorem by Mosco. In particular, the strong convergence

$$\psi_h \to \psi \text{ strongly in } W_0^{1,p}(\Omega)$$

easily implies the convergence of K_{ψ_h} to K_ψ in the sense of Mosco, but the weak convergence

$$\psi_h \rightharpoonup \psi \text{ weakly in } W^{1,r}(\Omega),\ r > p,$$

also implies the same result (see [11], [1]). Moreover, if

$$\psi_h \leq \psi \ C_p\text{-q.e. in } \Omega, \quad \psi_h \to \psi \ C_p\text{-q.e. in } \Omega,$$

then K_{ψ_h} converges to K_ψ in the sense of Mosco.

A necessary and sufficient condition for the convergence of K_{ψ_h}, expressed in terms of the convergence of the p-capacity of the level sets $\{x \in \Omega : \psi_h(x) > t\}$ has been given in [25].

Remark 5.2.4. It has been proved in [25] that if K_{ψ_h} converges to K_ψ in the sense of Mosco, then also $K_{T_i(\psi_h)}$ converges to $K_{T_i(\psi)}$ in the sense of Mosco, for every $i > 0$.

5.3. Preliminary results

Under assumptions (1.3.1), (1.3.2), (1.3.3), and (1.3.4), we recall (see Chapter 1) that, if u is the entropy solution of the equation

$$\begin{cases} A(u) = \mu + F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3.1}$$

when $\mu \in \mathcal{M}^p_{b,0}(\Omega)$ and $F \in W^{-1,p'}(\Omega)$, then, for every $j > 0$, we have the following estimate:

$$\int_\Omega |\nabla T_j(u)|^p \, dx \le M(j+1), \tag{5.3.2}$$

where the constant M depends on $\|\mu\|_{\mathcal{M}_b(\Omega)}$, $\|F\|_{W^{-1,p'}(\Omega)}$, p, c_0, c_1, $\|k_0\|_{L^{p'}(\Omega)}$, and $\|k_1\|_{L^1(\Omega)}$. Moreover, it holds:

$$\lim_{i\to+\infty} \int_{\{i<|u|\le i+j\}} |\nabla u|^p \, dx = 0, \tag{5.3.3}$$

for every $j > 0$.

Finally, for every $j > 0$, (5.3.2) implies

$$C_p(\{|u| > j\}) \le \frac{M(j+1)}{j^p}. \tag{5.3.4}$$

Proposition 5.3.1. *Assume (1.3.1), (1.3.2), (1.3.3), (1.3.4). Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, $F \in W^{-1,p'}(\Omega)$, and let u be the entropy solution of (5.3.1). Then, for every $z \in W^{1,p}_0(\Omega)$ the function $u - z$ belongs to $\mathcal{T}^{1,p}_0(\Omega)$; more precisely, for every $j > 0$, we have:*

$$\|T_j(u-z)\|^p_{W^{1,p}_0(\Omega)} \le M(j+1), \tag{5.3.5}$$

where the constant M depends only on $\|\mu\|_{\mathcal{M}_b(\Omega)}$, $\|F\|_{W^{-1,p'}(\Omega)}$, $\|z\|_{W^{1,p}_0(\Omega)}$, p, c_0, c_1, $\|k_0\|_{L^{p'}(\Omega)}$, and $\|k_1\|_{L^1(\Omega)}$.

Proof. Let us consider a sequence $\mu_n \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ such that μ_n converges to μ strongly in $\mathcal{M}_b(\Omega)$. Denoting the variational solution of the problem (5.3.1) relative to $\mu_n + F$ by u_n, we know that u_n tends to u in the sense of Theorem 1.6.8. If $z \in W^{1,p}_0(\Omega)$, define the operator $B(v) = -\mathrm{div}(a(x, \nabla v + \nabla z) - a(x, \nabla z))$, which is of the same type of A, satisfying (1.3.2), (1.3.1), (1.3.3), and (1.3.4) with different coercitivity and growth parameters depending by c_0, c_1, k_0, k_1, p, and z. Let v_n be the solution of

$$\begin{cases} B(v) = \mu_n + F - A(z) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega; \end{cases} \tag{5.3.6}$$

that is

$$\langle B(v_n), w \rangle = \langle \mu_n + F - A(z), w \rangle,$$

or, equivalently

$$\int_\Omega a(x, \nabla v_n + \nabla z) \nabla w \, dx = \langle \mu_n + F, w \rangle,$$

for every $w \in W_0^{1,p}(\Omega)$. By the uniqueness of the solution of (5.3.6), it follows that $u_n = v_n + z$, and, since v_n tends to the entropy solution v of the problem (5.3.6) relative to $\mu + F - A(z)$ (see Theorem 1.6.8) we obtain that $u = v + z$.

At this point, the result follows by (5.3.2). □

Remark 5.3.2. By the previous proposition we deduce also that if z_n, $z \in W_0^{1,p}(\Omega)$, with z_n converging to z weakly in $W_0^{1,p}(\Omega)$, then, for every $j > 0$, $T_j(u - z_n)$ converges to $T_j(u - z)$ weakly in $W_0^{1,p}(\Omega)$, where u is the entropy solution of (5.3.1) relative to $\mu \in \mathcal{M}_{b,0}^p(\Omega)$.

5.4. Convergence results

We consider a sequence a_h of functions in $\mathcal{L}(c_0, c_1, \alpha, \beta)$, a sequence of measures $\rho_h \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$, and the variational solution $u_{\rho_h}^{A_h}$ of

$$\begin{cases} A_h(u_{\rho_h}^{A_h}) = \rho_h & \text{in } \Omega \\ u_{\rho_h}^{A_h} \in W_0^{1,p}(\Omega). \end{cases}$$

We assume that

$$\sup_h \|\rho_h\|_{\mathcal{M}_b(\Omega)} < +\infty \tag{5.4.1}$$

and that the function ψ_h satisfies:

$$\psi_h \le u_{\rho_h}^{A_h} \ C_p\text{-q.e. in } \Omega. \tag{5.4.2}$$

Moreover we suppose that

$$\psi \le 0 \ C_p\text{-q.e. in } \Omega. \tag{5.4.3}$$

We can now state the main result of this section.

Theorem 5.4.1. *Let a_h be a sequence in $\mathcal{L}(c_0, c_1, \alpha, \beta)$, which G-converges to a function a, and let A_h and A be the operators associated to a_h and a, respectively. Let us assume (5.4.1), (5.4.2), and (5.4.3), with K_{ψ_h} converging to K_ψ in the sense of Mosco. Finally, consider μ_h, $\mu \in \mathcal{M}_{b,0}^p(\Omega)$, with μ_h converging to μ weakly in $\mathcal{M}_b(\Omega)$. Then the solutions u_h and u of the obstacle problems $OP_0(A_h, \mu_h, \psi_h)$ and $OP_0(A, \mu, \psi)$, respectively, satisfy*

$$T_j(u_h) \rightharpoonup T_j(u) \quad \textit{weakly in } W_0^{1,p}(\Omega), \textit{ for every } j > 0, \tag{5.4.4}$$

$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \quad \textit{weakly in } L^q(\Omega)^N, \textit{ for every } q < \tfrac{N}{N-1}, \tag{5.4.5}$$

$$\int_\Omega a_h(x, \nabla u_h) \nabla T_j(u_h)\, dx \to \int_\Omega a(x, \nabla u) \nabla T_j(u)\, dx, \textit{ for every } j > 0. \tag{5.4.6}$$

Remark 5.4.2. By formal modifications we can prove Theorem 5.4.1 replacing (5.4.3) with (3.1.2) and

$$\psi \leq M \ \ C_p\text{-q.e. in } \Omega,$$

where M is a positive constant.

Proof of Theorem 5.4.1. To simplify the exposition, it is convenient to divide the proof into various steps.

Step 1. We will prove (5.4.4).

Proof of Step 1. Let us recall that the solution u_h of the obstacle problem $OP_0(A_h, \mu_h, \psi_h)$ is the entropy solution of the equation (5.2.1) relative to $\mu + \lambda_h$, i.e., in particular, for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for every $j > 0$, u_h satisfies:

$$\int_\Omega a_h(x, \nabla u_h) \nabla T_j(u_h - \varphi)\, dx = \int_\Omega T_j(u_h - \varphi)\, d\mu_h + \int_\Omega T_j(u_h - \varphi)\, d\lambda_h, \tag{5.4.7}$$

where the obstacle reaction $\lambda_h \in \mathcal{M}_{b,0}^{p,+}(\Omega)$ satisfies (3.1.7), i.e.

$$\|\lambda_h\|_{\mathcal{M}_b(\Omega)} \leq \|(\mu_h - \rho_h)^-\|_{\mathcal{M}_b(\Omega)}.$$

Combining the previous estimate with (5.4.1) and (5.3.2), we obtain that, for every $j > 0$,

$$\int_\Omega |\nabla T_j(u_h)|^p\, dx \leq Mj, \tag{5.4.8}$$

where the constant M does not depend on j and h. Working as in the proof of Theorem 1.5.3 we have that there exists a subsequence of u_h (still denoted by u_h) and a function $u^* \in \mathcal{T}_0^{1,p}(\Omega)$ such that u_h converges to u^* a.e. in Ω and, for every $j > 0$, $T_j(u_h)$ converges to $T_j(u^*)$ weakly in $W_0^{1,p}(\Omega)$. Since also $K_{T_j(\psi_h)}$ converges to $K_{T_j(\psi)}$ in the sense of Mosco (see Remark 5.2.4), by the weakly convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h)$ to $T_j(u^*)$ we deduce that $T_j(u^*) \geq T_j(\psi)$ C_p-q.e. in Ω, for every $j > 0$, so that $u^* \geq \psi$ C_p-q.e. in Ω. Furthermore, by the weak convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h)$ to $T_j(u^*)$ we obtain that also $T_j(u^*)$ satisfies (5.4.8), which implies (5.3.4) for u^*.

Let us consider a function $\Phi \in W_0^{1,p}(\Omega)$, with $\Phi \geq \psi$, and the solution w_h of the variational inequality $VI(A_h, A(\Phi), \psi_h)$. By Theorem 3.1 of [26], w_h satisfies:

$$w_h \rightharpoonup w \ \text{ weakly in } W_0^{1,p}(\Omega),$$

$$a_h(x,\nabla w_h) \rightharpoonup a(x,\nabla w) \text{ weakly in } L^{p'}(\Omega)^N,$$

$$\langle a_h(x,\nabla w_h), w_h\rangle \to \langle a(x,\nabla w), w\rangle,$$

where w is the solution of $VI(A, A(\Phi), \psi)$; so that $w = \Phi$ (see *Characterization 1* of Section 3.1).

Moreover, w_h satisfies

$$\begin{aligned}\int_\Omega a_h(x,\nabla w_h)\nabla(w_h - v)\,dx &\le \langle A(\Phi), w_h - v\rangle \\ &= \int_\Omega a(x,\nabla\Phi)\nabla(w_h - v),\end{aligned} \tag{5.4.9}$$

for every $v \in W_0^{1,p}(\Omega)$, with $v \ge \psi_h$. Now, using the monotonicity of the operator A_h, we can rewrite (5.4.7) as

$$\int_\Omega a_h(x,\nabla\varphi)\nabla T_j(u_h - \varphi)\,dx \le \int_\Omega T_j(u_h - \varphi)\,d\mu_h + \int_\Omega T_j(u_h - \varphi)\,d\lambda_h, \tag{5.4.10}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We would like to use w_h in (5.4.10), but, a priori, we do not know that w_h is a bounded function. Let us note, nevertheless, that if a function φ is in $W_0^{1,p}(\Omega)$, for every $i > 0$, we can use $T_i(\varphi)$ as function test in (5.4.10). Observe now that, letting i tend to infinity, $T_i(\varphi)$ converges to φ strongly in $W_0^{1,p}(\Omega)$, so that, on one hand $a_h(x,\nabla T_i(\varphi))$ tends to $a_h(x,\nabla\varphi)$ strongly in $L^{p'}(\Omega)^N$, on the other $T_j(u_h - T_i(\varphi))$ converges to $T_j(u_h - \varphi)$ weakly in $W_0^{1,p}(\Omega)$, as observed in Remark 5.3.2. Now we can rewrite (5.4.10) for every $\varphi \in W_0^{1,p}(\Omega)$, and, in particular, choosing w_h as function test we obtain

$$\begin{aligned}\int_\Omega a_h(x,\nabla w_h)\nabla T_j(u_h - w_h)\,dx &\le \int_\Omega T_j(u_h - w_h)\,d\mu_h + \int_\Omega T_j(u_h - w_h)\,d\lambda_h \\ &\le \int_\Omega T_j(u_h - w_h)\,d\mu_h,\end{aligned} \tag{5.4.11}$$

where the last inequality follows by the complementarity system (3.1.10) and by the fact that $w_h \ge \psi_h$ C_p-q.e. in Ω.

The choice of the function $v = v_h := w_h - T_j(w_h - u_h)$ as test in (5.4.9) is admissible and gives:

$$\int_\Omega a_h(x,\nabla w_h)\nabla T_j(w_h - u_h)\,dx \le \int_\Omega a(x,\nabla\Phi)\nabla T_j(w_h - u_h)\,dx, \tag{5.4.12}$$

which, with (5.4.11), implies

$$\int_\Omega a(x,\nabla\Phi)\nabla T_j(u_h - w_h)\,dx \le \int_\Omega T_j(u_h - w_h)\,d\mu_h. \tag{5.4.13}$$

By the estimate (5.3.5), it is easy to prove that $T_j(u_h - w_h)$ converges to $T_j(u-\Phi)$ weakly in $W_0^{1,p}(\Omega)$, and, thanks to Lemma 1.2.15 we easily pass to the limit in (5.4.13). In conclusion, we obtain

$$\int_\Omega a(x,\nabla\Phi)\nabla T_j(u^* - \Phi)\,dx \le \int_\Omega T_j(u^* - \Phi)\,d\mu, \tag{5.4.14}$$

for every $\Phi \in W_0^{1,p}(\Omega)$, $\Phi \ge \psi$ C_p-q.e. in Ω.

Thanks to the next lemma, we have the following fact:

$$\int_\Omega a(x,\nabla u^*)\nabla T_j(u^* - \varphi)\,dx \le \int_\Omega T_j(u^* - \varphi)\,d\mu,$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \ge \psi$ C_p-q.e. in Ω. As observed in Remark 3.1.8, the previous formulation characterizes uniquely the function u^*. Thus, having denoted the solution of $OP_0(A,\mu,\psi)$ by u, we have $u^* = u$; this implies that the whole sequence $T_j(u_h)$ (and not only a subsequence) converge to $T_j(u)$.

Hence, to conclude, we have to prove the following lemma, which is inspired by Lemma 1.2 of [4]. We give here the proof for the sake of completeness.

Lemma 5.4.3. *Assume μ be in $\mathcal{M}^p_{b,0}(\Omega)$ and ψ satisfy (5.4.3). Under hypotheses (1.3.1), (1.3.2), (1.3.3), and (1.3.4), a solution of*

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega), u \ge \psi, \\ \int_\Omega a(x,\nabla\Phi)\nabla T_j(u-\Phi)\,dx \le \int_\Omega T_j(u-\Phi)\,d\mu, \\ \forall j>0, \forall \Phi \in W_0^{1,p}(\Omega), \Phi \ge \psi, \end{cases} \tag{5.4.15}$$

satisfying (1.5.4), is also a solution of

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega), u \ge \psi, \\ \int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi)\,dx \le \int_\Omega T_k(u-\varphi)\,d\mu, \\ \forall k>0, \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \varphi \ge \psi. \end{cases} \tag{5.4.16}$$

The converse is also true.

Proof. Let u be a solution of (5.4.15) and $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \ge \psi$. The choice of $\Phi = tT_i(u) + (1-t)\varphi$, with $i>0$ and $t\in(0,1)$, in (5.4.15) is admissible and gives

$$\begin{cases} I_i \le J_i \\ I_i = \int_\Omega a(x, t\nabla T_i(u) + (1-t)\nabla\varphi)\nabla T_j(u - tT_i(u) - (1-t)\varphi)\,dx \\ J_i = \int_\Omega T_j(u - tT_i(u) - (1-t)\varphi)\,d\mu. \end{cases} \tag{5.4.17}$$

Now,

$$I_i = \int_{\{|u|\le i\}} a(x, t\nabla u + (1-t)\nabla\varphi)\nabla T_j((1-t)(u-\varphi))\,dx$$
$$+ \int_{\{|u|>i\}\cap\{|u-tT_i(u)-(1-t)\varphi|\le j\}} a(x,(1-t)\nabla\varphi)\nabla(u-(1-t)\varphi)\,dx,$$

since $\nabla T_j(u - tT_i(u) - (1-t)\varphi) = 0$ where $|u - tT_i(u) - (1-t)\varphi| > j$. The set $\{|u| > i\} \cap \{|u - tT_i(u) - (1-t)\varphi| \le j\}$ is empty if we choose $i > \|\varphi\|_{L^\infty(\Omega)}$ and $0 < j \le (1-t)(i - \|\varphi\|_{L^\infty(\Omega)})$; hence

$$I_i = \int_{\{|u|\le i\}} a(x, t\nabla u + (1-t)\nabla\varphi)\nabla T_j((1-t)(u-\varphi))\,dx.$$

Let us consider J_i:

$$J_i = \int_{\{|u|\le i\}} T_j((1-t)(u-\varphi))\,d\mu + \int_{\{|u|>i\}} T_j(u - tT_i(u) - (1-t)\varphi)\,d\mu$$
$$\le \int_{\{|u|\le i\}} T_j((1-t)(u-\varphi))\,d\mu + j|\mu|(\{|u|>i\}).$$

Now we pass to the limit as i tends to $+\infty$ in (5.4.17); taking into account (5.3.4), we obtain, by the previous remarks about I_i and J_i that

$$I := \lim_{i\to+\infty} I_i = \int_\Omega a(x, t\nabla u + (1-t)\nabla\varphi)\nabla T_j((1-t)(u-\varphi))\,dx$$
$$\le \lim_{i\to\infty} J_i = \int_\Omega T_j((1-t)(u-\varphi))\,d\mu =: J,$$

for every $j > 0$. Let us write I as

$$I = (1-t)\int_{\{(1-t)|u-\varphi|\le j\}} a(x, t\nabla u + (1-t)\nabla\varphi)\nabla(u-\varphi)\,dx,$$

while

$$J = (1-t)\int_{\{(1-t)|u-\varphi|\le j\}} (u-\varphi)\,d\mu + \int_{\{(1-t)(u-\varphi)>j\}} j\,d\mu + \int_{\{(1-t)(u-\varphi)<-j\}} (-j)\,d\mu.$$

Let $k > 0$ and j such that $j = k(1-t)$, so that $I \le J$ implies

$$(1-t)\int_\Omega a(x, t\nabla u + (1-t)\nabla\varphi)\nabla T_k(u-\varphi)\,dx \le (1-t)\int_\Omega T_k(u-\varphi)\,d\mu$$

Dividing by $(1-t)$ and passing to the limit with respect to $t \to 1^-$, we obtain (5.4.16).

The converse is just the monotonicity of the operator A. Let us note that, if u solves (5.4.16), then u satisfies (1.5.4), since we can choose $\varphi = 0$ as test in (5.4.16) and use (1.3.2). □

Step 2. Denoting the obstacle reactions of u_h and u by λ_h and λ, respectively, we will prove that

$$\lim_{h\to+\infty}\int_\Omega \Phi_h\,d\lambda_h = \int_\Omega \Phi\,d\lambda, \tag{5.4.18}$$

for every $\Phi \in W_0^{1,q'}(\Omega)$, with $q < \frac{N}{N-1}$, and for every $\Phi_h \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, with $\sup_h \|\Phi_h\|_{L^\infty(\Omega)} < +\infty$, converging to Φ strongly in $W_0^{1,p}(\Omega)$.

Proof of Step 2. For every $i > 0$ and for every $t \in \mathbb{R}$, $t \neq 0$, we consider the solution v_h of the variational inequality $VI(A_h, A(T_i(u)+t\Phi_h), \psi_h + t\Phi_h)$ and the obstacle reaction η_h associated with it. Observing that $A(T_i(u)+t\Phi_h)$ converges to $A(T_i(u)+t\Phi)$ strongly in $W^{-1,p'}(\Omega)$, and that $K_{\psi_h+t\Phi_h}$ converges to $K_{\psi+t\Phi}$ in the sense of Mosco, we can apply Theorem 3.1 of [26] to deduce:

$$v_h \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega),$$

$$\eta_h \rightharpoonup \eta \quad \text{weakly in } W^{-1,p'}(\Omega),$$

where v is the solution of $VI(A, A(T_i(u)+t\Phi), \psi + t\Phi)$ and η is the obstacle reaction associated with it. On the other hand, thanks to (5.4.3), for every $i > 0$, we have that $T_i(u) \geq \psi$ C_p-q.e. in Ω, so that $v = T_i(u) + t\Phi$ and $\eta = 0$ (see *Characterization 1* of Section 3.1).

Consider now, for every $l > 0$ and for every $j > 0$, the inequality

$$\int_\Omega (a_h(x,\nabla u_h) - a_h(x,\nabla T_l(v_h)))\nabla T_j(u_h - T_l(v_h)\,dx \geq 0,$$

which follows by the monotonicity of a_h. If we use the entropy formulation (1.6.2) of u_h in the previous inequality we obtain

$$\int_\Omega T_j(u_h - T_l(v_h))\,d\lambda_h + \int_\Omega T_j(u_h - T_l(v_h))\,d\mu_h \geq \int_\Omega a_h(x,\nabla T_l(v_h))\nabla T_j(u_h - T_l(v_h))\,dx; \tag{5.4.19}$$

passing to the limit as l tends to $+\infty$ thanks to Proposition 5.3.1 (see also Remark 5.3.2), and using the variational formulation (3.1.5) satisfied by v_h, we rewrite (5.4.19) as

$$\begin{cases} I_h + II_h \geq III_h + IV_h \\ I_h = \int_\Omega T_j(u_h - v_h)\,d\lambda_h \\ II_h = \int_\Omega T_j(u_h - v_h)\,d\mu_h \\ III_h = \int_\Omega a(x,\nabla(T_i(u)+t\Phi_h))\nabla T_j(u_h - v_h)\,dx \\ IV_h = \langle \eta_h, T_j(u_h - v_h)\rangle. \end{cases} \tag{5.4.20}$$

By the complementarity system (3.1.6), we have that

$$IV_h = \int_\Omega T_j(u_h - \psi_h - t\Phi_h)\, d\eta_h \geq \int_\Omega T_j(-t\Phi_h)\, d\eta_h = \langle \eta_h, T_j(-t\Phi_h)\rangle,$$

which tends to 0 as h goes to $+\infty$, i.e.

$$\liminf_{h\to+\infty} IV_h \geq 0. \tag{5.4.21}$$

Moreover, by (5.3.5), it is easy to check that $T_j(u_h - v_h)$ converges to $T_j(u - T_i(u) - t\Phi)$ weakly in $W_0^{1,p}(\Omega)$, so that we can apply Lemma 1.2.15 to deduce that

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j(u - T_i(u) - t\Phi)\, d\mu. \tag{5.4.22}$$

Since, thanks to (5.1.1), $a(x, \nabla(T_i(u) + t\Phi_h))$ converges to $a(x, \nabla(T_i(u) + t\Phi))$ strongly in $L^{p'}(\Omega)^N$, we pass to the limit also in III_h, obtaining

$$\lim_{h\to+\infty} III_h = \int_\Omega a(x, \nabla(T_i(u) + t\Phi))\nabla T_j(u - T_i(u) - t\Phi)\, dx. \tag{5.4.23}$$

Combining (5.4.21), (5.4.22) and (5.4.23) we have

$$\begin{aligned} &\liminf_{h\to+\infty} \int_\Omega T_j(u_h - v_h)\, d\lambda_h + \int_\Omega T_j(u - T_i(u) - t\Phi)\, d\mu \\ &\geq \int_\Omega a(x, \nabla(T_i(u) + t\Phi))\nabla T_j(u - T_i(u) - t\Phi)\, dx, \end{aligned} \tag{5.4.24}$$

which can be written also as

$$\begin{cases} \liminf\limits_{h\to+\infty} I_h - I^i \geq II^i \\ I^i = \int_\Omega T_j(u - T_i(u) - t\Phi)\, d\lambda \\ II^i = \int_\Omega (a(x, \nabla(T_i(u) + t\Phi)) - a(x, \nabla u))\nabla T_j(u - T_i(u) - t\Phi)\, dx, \end{cases} \tag{5.4.25}$$

using the entropy formulation satisfied by u. By the complementarity system (3.1.10), we have that

$$\liminf_{h\to+\infty} I_h = \liminf_{h\to+\infty} \int_\Omega T_j(\psi_h - v_h)\, d\lambda_h;$$

on the other hand, since $v_h \geq \psi_h + t\Phi_h$ we obtain by the previous equality that

$$\liminf_{h\to+\infty} I_h \leq \liminf_{h\to+\infty} \int_\Omega T_j(-t\Phi_h)\, d\lambda_h. \tag{5.4.26}$$

On the other hand, thanks to (5.3.4) it is easy to check that

$$\lim_{i\to+\infty} I^i = \int_\Omega T_j(-t\Phi)\,d\lambda. \tag{5.4.27}$$

Finally, in II^i we split the integral into the sets where $|u|\le i$ and where $|u|>i$, getting

$$\begin{aligned} II^i = &\int_{\{|u|\le i\}} (a(x,\nabla(u+t\Phi)) - a(x,\nabla u))\nabla T_j(-t\Phi)\,dx \\ &+ \int_{\{|u|>i\}\cap\{|u-T_i(u)-t\Phi|\le j\}} (a(x,\nabla(t\Phi)) - a(x,\nabla u))\nabla(u-t\Phi)\,dx, \end{aligned}$$

since $\nabla T_j(u-T_i(u)-t\Phi)=0$ where $|u-T_i(u)-t\Phi|>j$. Let us observe that $\{|u-T_i(u)-t\Phi|\le j\}\subseteq\{|u|\le i+j+|t|\|\Phi\|_{L^\infty(\Omega)}\}$, so that, by the growth conditions assumed on a and by (5.3.3), it is easy to prove that

$$\lim_{i\to+\infty}\int_{\{|u|>i\}\cap\{|u-T_i(u)-t\Phi|\le j\}} (a(x,\nabla(t\Phi)) - a(x,\nabla u))\nabla(u-t\Phi)\,dx = 0, \tag{5.4.28}$$

as well as

$$\begin{aligned} &\lim_{i\to+\infty}\int_{\{|u|\le i\}} (a(x,\nabla(u+t\Phi)) - a(x,\nabla u))\nabla T_j(-t\Phi)\,dx \\ &= \int_\Omega (a(x,\nabla(u+t\Phi)) - a(x,\nabla u))\nabla T_j(-t\Phi)\,dx, \end{aligned} \tag{5.4.29}$$

since $a(x,\nabla(u+t\Phi))-a(x,\nabla u)\in L^q(\Omega)^N$ and, by hypothesis, $\Phi\in W_0^{1,q'}(\Omega)$. Combining (5.4.26), (5.4.27), (5.4.28) and (5.4.29) we have

$$\begin{aligned} &\liminf_{h\to+\infty}\int_\Omega T_j(-t\Phi_h)\,d\lambda_h - \int_\Omega T_j(-t\Phi)\,d\lambda \\ &\ge \int_\Omega (a(x,\nabla(u+t\Phi)) - a(x,\nabla u))\nabla T_j(-t\Phi)\,dx, \end{aligned}$$

and, for $j\ge |t|(\|\Phi_h\|_{L^\infty(\Omega)}\vee\|\Phi\|_{L^\infty(\Omega)})$,

$$\liminf_{h\to+\infty}\int_\Omega -t\Phi_h\,d\lambda_h + t\int_\Omega \Phi\,d\lambda \ge -t\int_\Omega (a(x,\nabla(u+t\Phi)) - a(x,\nabla u))\nabla\Phi\,dx.$$

At this point, dividing by $|t|$ and passing to the limit with respect to $t\to 0$, we obtain (5.4.18).

Step 3. We will prove (5.4.5).

Proof of Step 3. We recall that u_h satisfies (1.6.3), i.e.,

$$\int_\Omega a_h(x, \nabla u_h)\nabla\Phi\, dx = \int_\Omega \Phi\, d\mu_h + \int_\Omega \Phi\, d\lambda_h, \tag{5.4.30}$$

for every $\Phi \in W_0^{1,q'}(\Omega)$, with $1 < q < \frac{N}{N-1}$. We just observed that $W_0^{1,q'}(\Omega) \subseteq C(\overline{\Omega})$, so that, thanks to Lemma 1.2.15 and (5.4.18), we can pass to the limit as h goes to $+\infty$ in the last two terms of (5.4.30), obtaining

$$\begin{aligned}\lim_{h\to+\infty} \int_\Omega a_h(x, \nabla u_h)\nabla\Phi\, dx &= \int_\Omega \Phi\, d\mu + \int_\Omega \Phi\, d\lambda \\ &= \int_\Omega a(x, \nabla u)\nabla\Phi\, dx,\end{aligned}$$

where the last equality follows by the equation (1.6.3) satisfied by u. In other words, we proved that

$$-\mathrm{div}(a_h(x, \nabla u_h)) \rightharpoonup -\mathrm{div}(a(x, \nabla u)) \;\text{ weakly in } W^{-1,q}(\Omega), \text{ for every } q < \frac{N}{N-1}.$$

On the other hand, $a_h(x, \nabla u_h)$ is equibounded (with respect to h) in the L^q-norm, thanks to Lemma 1.5.5. By this fact we easily deduce that

$$a_h(x, \nabla u_h) \rightharpoonup \sigma \text{ weakly in } L^q(\Omega)^N, \tag{5.4.31}$$

where $\mathrm{div}(a(x, \nabla u) - \sigma) = 0$. As we will see later, to prove (5.4.5), it is enough to show, by Minty's trick, that

$$\lim_{h\to+\infty} \int_\Omega a_h(x, \nabla u_h)\nabla\Phi\omega\, dx = \int_\Omega a(x, \nabla u)\nabla\Phi\omega\, dx, \tag{5.4.32}$$

for every $\Phi \in W_0^{1,q'}(\Omega)$ and for every $\omega \in C^1(\overline{\Omega})$.

With minor changes with respect to the proof of *Step 2*, we will prove (5.4.32). Let $\Phi \in W_0^{1,q'}(\Omega)$ and $t \in \mathbb{R}$, with $t \neq 0$; then the solution v_h of $VI(A_h, A(T_i(u)+t\Phi), \psi_h + t\Phi)$ and the obstacle reaction η_h associated with it are such that

$$v_h \rightharpoonup T_i(u) + t\Phi \;\text{ weakly in } W_0^{1,p}(\Omega),$$

$$a_h(x, \nabla v_h) \rightharpoonup a(x, \nabla(T_i(u) + t\Phi)) \;\text{ weakly in } L^{p'}(\Omega)^N,$$

$$\eta_h \rightharpoonup 0 \;\text{ weakly in } W^{-1,p'}(\Omega),$$

since $T_i(u) + t\Phi$ is the solution of $VI(A, A(T_i(u) + t\Phi), \psi + t\Phi)$. By the monotonicity assumption on $a_h(x, \cdot)$ we have, for every $l, j > 0$

$$\int_\Omega (a_h(x, \nabla u_h) - a_h(x, \nabla T_l(v_h))\nabla T_j(u_h - T_l(v_h))\, \omega\, dx \geq 0,$$

where $\omega \in C^1(\overline{\Omega})$, with $\omega \geq 0$. For convenience we write the previous inequality in the form

$$\int_\Omega (a_h(x, \nabla u_h) - a_h(x, \nabla T_l(v_h))\nabla (T_j(u_h - T_l(v_h))\omega)\, dx$$
$$\geq \int_\Omega (a_h(x, \nabla u_h) - a_h(x, \nabla T_l(v_h))\nabla \omega\, T_j(u_h - T_l(v_h))\, dx,$$

which gives, using the entropy formulation (1.6.1) of u_h and letting l tend to $+\infty$, as in the proof of *Step 2*,

$$\begin{cases} I_h + II_h + III_h \geq IV_h + V_h \\ I_h = \int_\Omega T_j(u_h - v_h)\omega\, d\lambda_h \\ II_h = \int_\Omega T_j(u_h - v_h)\omega\, d\mu_h \\ III_h = -\int_\Omega a_h(x, \nabla u_h)\nabla\omega\, T_j(u_h - v_h)\, dx \\ IV_h = \int_\Omega a_h(x, \nabla v_h)\nabla(T_j(u_h - v_h)\omega)\, dx \\ V_h = -\int_\Omega a_h(x, \nabla v_h)\nabla\omega\, T_j(u_h - v_h)\, dx. \end{cases}$$

The same tools used to deduce (5.4.26) give:

$$I_h \leq \int_\Omega T_j(-t\Phi)\omega\, d\lambda_h;$$

choosing $j \geq |t|\|\Phi\|_{L^\infty(\Omega)}$ and using the formulation (1.6.3) satisfied by u_h, we have:

$$I_h \leq -t\int_\Omega a_h(x, \nabla u_h)\nabla(\Phi\omega)\, dx + t\int_\Omega \Phi\omega\, d\mu_h. \tag{5.4.33}$$

Thanks to the variational formulation satisfied by v_h we write IV_h as

$$IV_h = \int_\Omega a(x, \nabla(T_i(u) + t\Phi))\nabla(T_j(u_h - v_h)\omega)\, dx + \langle \eta_h, T_j(u_h - v_h)\omega\rangle$$

and we obtain that

$$\liminf_{h\to+\infty} IV_h \geq \int_\Omega a(x, \nabla(T_i(u) + t\Phi))\nabla(T_j(u - T_i(u) - t\Phi)\omega)\, dx, \tag{5.4.34}$$

since we can work as in the proof of (5.4.21) and (5.4.23). Analogously, as we prove (5.4.22), we have also that

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j(u - T_i(u) - t\Phi)\, \omega\, d\mu. \tag{5.4.35}$$

On the other hand, it is easy to check that

$$\lim_{h\to+\infty} III_h = t \lim_{h\to+\infty} \int_\Omega a_h(x,\nabla u_h)\nabla\omega\,\Phi\,dx + \int_\Omega \sigma\nabla\omega(-t\Phi - T_j(u - T_i(u) - t\Phi))\,dx, \tag{5.4.36}$$

as well as

$$\lim_{h\to+\infty} V_h = -\int_\Omega a(x,\nabla(T_i(u)+t\Phi))\nabla\omega\,T_j(u - T_i(u) - t\Phi)\,dx. \tag{5.4.37}$$

Combining (5.4.33), (5.4.34), (5.4.35), (5.4.36), and (5.4.37) we obtain

$$\begin{aligned}
&\liminf_{h\to+\infty} -t\int_\Omega a_h(x,\nabla u_h)\nabla\Phi\omega\,dx + t\int_\Omega \Phi\omega\,d\mu \\
&+ \int_\Omega T_j(u - T_i(u) - t\Phi)\,\omega\,d\mu + \int_\Omega \sigma\nabla\omega(-t\Phi - T_j(u - T_i(u) - t\Phi))\,dx \\
&\geq \int_\Omega a(x,\nabla(T_i(u)+t\Phi))\nabla T_j(u - T_i(u) - t\Phi)\,\omega\,dx,
\end{aligned}$$

which gives, letting $i\to+\infty$

$$\liminf_{h\to+\infty} -t\int_\Omega a_h(x,\nabla u_h)\nabla\Phi\omega\,dx \geq -t\int_\Omega a(x,\nabla(u+t\Phi))\nabla\Phi\,\omega\,dx.$$

Finally, dividing by $|t|$ and passing to the limit with respect to $t\to 0$, we obtain (5.4.32).

Combining (5.4.32) and (5.4.31), we have

$$\int_\Omega (\sigma - a(x,\nabla u))\nabla\Phi\,\omega\,dx = 0, \tag{5.4.38}$$

for every $\Phi\in W_0^{1,q'}(\Omega)$, with $q<\frac{N}{N-1}$, and for every $\omega\in C^1(\overline{\Omega})$.

Let $\xi\in\mathbb{R}^N$, with $\xi\neq 0$, and let $\zeta\in C_c^\infty(\Omega)$; then the choice of $\Phi(x)=\xi x\,\zeta(x)$ in (5.4.38) is admissible, and gives

$$\int_\Omega (\sigma - a(x,\nabla u))\xi\,\zeta\omega\,dx = 0,$$

since $\xi x\,\omega(x)\in C^1(\overline{\Omega})$. Now we let ζ tend to 1, obtaining

$$\int_\Omega (\sigma - a(x,\nabla u))\xi\,\omega\,dx = 0,$$

for every $\omega\in C^1(\overline{\Omega})$, and, finally, $(\sigma(x) - a(x,\nabla u(x)))\xi = 0$, for every $\xi\in\mathbb{R}^N$ and for almost every $x\in\Omega$, so that (5.4.5) is proved.

Step 4. We will prove the lower semicontinuity of the "energy", that is

$$\int_\Omega a(x,\nabla u)\nabla T_j(u)\,dx \le \liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h)\,dx, \tag{5.4.39}$$

for every $j>0$.

Proof of Step 4. To prove (5.4.39) we need an approximation result for the G-convergence (see Lemma 2.3 of [26]).

Lemma 5.4.4. *Let a_h be a sequence in $\mathcal{L}(c_0,c_1,\alpha,\beta)$ G-converging to a function a, and let A_h and A be the operators associated to a_h and a, respectively. Let $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and v_h the solution of (5.2.1) relative to $A(v)$. Then there exist a decreasing sequence ε_h converging to 0 and a sequence $w_h\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that*

$$w_h \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{5.4.40}$$

$$(a_h(x,\nabla w_h)-a_h(x,\nabla v_h))\to 0 \quad \text{strongly in } L^{p'}(\Omega)^N, \tag{5.4.41}$$

$$|w_h(x)-v(x)|\le\varepsilon_h \quad C_p\text{-q.e. in } \Omega. \tag{5.4.42}$$

Let v, $w_h\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as in Lemma 5.4.4. By the monotonicity assumption on $a_h(x,\cdot)$ we have, for every $j>0$:

$$\int_\Omega (a_h(x,\nabla u_h)-a_h(x,\nabla w_h))\nabla T_j(u_h-w_h)\,dx\ge 0.$$

We use the entropy formulation of u_h to obtain

$$\int_\Omega T_j(u_h-w_h)\,d\lambda_h+\int_\Omega T_j(u_h-w_h)\,d\mu_h-\int_\Omega a_h(x,\nabla w_h)\nabla T_j(u_h-w_h)\,dx\ge 0.$$

Let us rewrite the previous inequality as

$$\begin{cases} I_h+II_h+III_h\ge 0\\ I_h=\int_\Omega T_j(u_h-w_h)\,d\lambda_h\\ II_h=\int_\Omega T_j(u_h-w_h)\,d\mu_h\\ III_h=-\int_\Omega a_h(x,\nabla w_h)\nabla T_j(u_h-w_h)\,dx,\end{cases}$$

and the term III_h as

$$\begin{aligned} III_h=&\int_\Omega (a_h(x,\nabla v_h)-a_h(x,\nabla w_h))\nabla T_j(u_h-w_h)\,dx\\ &-\int_\Omega a_h(x,\nabla v_h)\nabla T_j(u_h-w_h)\,dx.\end{aligned}$$

By (5.3.5), $T_j(u_h - w_h)$ is uniformly bounded (with respect to h) in $W_0^{1,p}(\Omega)$, so that $T_j(u_h - w_h)$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_j(u-v)$. Thanks to this fact and to (5.4.41), it is easy to pass to the limit in the first term of III_h. For the second one it is sufficient to use the definition of v_h and, again, the weak convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h - w_h)$, so that

$$\lim_{h\to+\infty} III_h = -\int_\Omega a(x,\nabla v)\nabla T_j(u-v)\,dx. \tag{5.4.43}$$

Analogously we have

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j(u-v)\,d\mu, \tag{5.4.44}$$

since we can apply Lemma 1.2.15. Finally, thanks to (5.4.42) and by the lipschitzianity of the truncated function, we have:

$$\liminf_{h\to+\infty} I_h = \liminf_{h\to+\infty} \int_\Omega T_j(u_h - v)\,d\lambda_h. \tag{5.4.45}$$

Combining (5.4.43), (5.4.44), and (5.4.45) we obtain

$$\liminf_{h\to+\infty} \int_\Omega T_j(u_h - v)\,d\lambda_h + \int_\Omega T_j(u-v)\,d\mu \ge \int_\Omega a(x,\nabla v)\nabla T_j(u-v)\,dx, \tag{5.4.46}$$

for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Let $t \in (0,1)$; for $i > 0$, we use $v = tT_i(u)$ as function test in (5.4.46). Since $tT_i(u) \ge tT_i(\psi)$ C_p-q.e. in Ω and since $K_{tT_i(\psi_h)}$ converges to $K_{tT_i(\psi)}$ in the sense of Mosco (see Remark 5.2.4), there exist $k \in \mathbb{N}$ and a sequence z_h converging to $tT_i(u)$ strongly in $W_0^{1,p}(\Omega)$ such that $z_h \in K_{tT_i(\psi_h)}$, for every $h \ge k$. We consider the function $\Phi_h = T_i(z_h) - tT_i(u)$, which belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and tends to 0 strongly in $W_0^{1,p}(\Omega)$; so we can use (5.4.18) and the lipschitzianity of the truncated function to deduce that

$$\liminf_{h\to+\infty} \int_\Omega T_j(u_h - tT_i(u))\,d\lambda_h = \liminf_{h\to+\infty} \int_\Omega T_j(u_h - T_i(z_h))\,d\lambda_h. \tag{5.4.47}$$

Moreover, since, for every $h \ge k$, $T_i(z_h) \ge tT_i(\psi_h)$ C_p-q.e. in Ω, we estimate the right hand side of (5.4.47) as

$$\begin{aligned}\liminf_{h\to+\infty} \int_\Omega T_j(u_h - T_i(z_h))\,d\lambda_h &\le \liminf_{h\to+\infty} \int_\Omega T_j(u_h - tT_i(\psi_h))\,d\lambda_h \\ &= \liminf_{h\to+\infty} \int_\Omega T_j(u_h - tT_i(u_h))\,d\lambda_h,\end{aligned}$$

where the last equality follows by the complementarity system (3.1.10). Finally, using the entropy formulation (1.6.2) of u_h, we get

$$\begin{aligned}
&\liminf_{h\to+\infty}\int_\Omega T_j(u_h - T_i(z_h))\,d\lambda_h \\
\leq&\liminf_{h\to+\infty}\left(\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h - tT_i(u_h))\,dx - \int_\Omega T_j(u_h - tT_i(u_h))\,d\mu_h\right) \\
=&\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h - tT_i(u_h))\,dx - \int_\Omega T_j(u - tT_i(u))\,d\mu,
\end{aligned} \tag{5.4.48}$$

since $T_j(u_h - tT_i(u_h))$ converges to $T_j(u - tT_i(u))$ weakly in $W_0^{1,p}(\Omega)$ and we can apply Lemma 1.2.15.

Hence, using in (5.4.46) $v = tT_i(u)$ and combining (5.4.47) and (5.4.48), we obtain

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h - tT_i(u_h))\,dx \geq \int_\Omega a(x,t\nabla T_i(u))\nabla T_j(u - tT_i(u))\,dx. \tag{5.4.49}$$

We denote $\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h - tT_i(u_h))\,dx$ by J_h, and we split Ω into the sets where $|u_h| \leq i$ and where $|u_h| > i$, so that

$$\begin{aligned}
J_h = &\int_{\{|u_h|\leq i\}} a_h(x,\nabla u_h)\nabla T_j((1-t)(u_h))\,dx \\
&+ \int_{\{|u_h|>i\}\cap\{|u_h - tT_i(u_h)|\leq j\}} a_h(x,\nabla u_h)\nabla u_h\,dx.
\end{aligned}$$

Observing that $\{|u_h - tT_i(u_h)| \leq j\} \subseteq \{|u_h| \leq j + ti\}$, if we choose $j < (1-t)i$, we have that $\{|u_h| > i\} \cap \{|u_h - tT_i(u_h)| \leq j\}$ is empty, and

$$J_h = \int_{\{|u_h|\leq i\}} a_h(x,\nabla u_h)\nabla T_j((1-t)(u_h))\,dx \leq \int_\Omega a_h(x,\nabla u_h)\nabla T_j((1-t)(u_h))\,dx,$$

since the integrand is nonnegative. Analogously

$$\int_\Omega a(x,t\nabla T_i(u))\nabla T_j(u - tT_i(u))\,dx = \int_{\{|u|\leq i\}} a(x,t\nabla T_i(u))\nabla T_j((1-t)u)\,dx,$$

so that (5.4.49) becomes

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j((1-t)(u_h))\,dx \geq \int_{\{|u|\leq i\}} a(x,t\nabla T_i(u))\nabla T_j((1-t)u)\,dx.$$

Letting i tend to $+\infty$, we rewrite the previous inequality as

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j((1-t)(u_h))\,dx \ge \int_\Omega a(x,t\nabla u)\nabla T_j((1-t)u)\,dx$$

or, equivalently,

$$(1-t)\liminf_{h\to+\infty}\int_{\{(1-t)|u_h|\le j\}} a_h(x,\nabla u_h)\nabla u_h\,dx \ge (1-t)\int_{\{(1-t)|u|\le j\}} a(x,t\nabla u)\nabla u\,dx \tag{5.4.50}$$

for every $j>0$. Let $n>0$ and $j=(1-t)n$; then we can rewrite (5.4.50) as

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_n(u_h)\,dx \ge \int_\Omega a(x,t\nabla u)\nabla T_n(u)\,dx.$$

Finally, letting t tend to 1^-, we obtain (5.4.39).

Step 5. We will prove (5.4.6).

Proof of Step 5. The proof is quite similar to that of *Step 2*, so we will often refer to it.

Let $t>0$; then, for every $k>0$, we have that $tT_k(u)\ge tT_k(\psi)$ C_p-q.e. in Ω. Since $K_{tT_k(\psi_h)}$ converges to $K_{tT_k(\psi)}$ in the sense of Mosco (see Remark 5.2.4), there exist $n\in\mathbb{N}$ and a sequence Φ_h converging to $tT_k(u)$ strongly in $W_0^{1,p}(\Omega)$ such that $\Phi_h\in K_{tT_k(\psi_h)}$ for every $h\ge n$. For $i>0$ we consider the solution v_h of $VI(A_h, A(T_i(u)+tT_k(u)), \psi_h+\Phi_h)$ and the obstacle reaction η_h associated with it; as in the proof of *Step 2*, we deduce by Theorem 3.1 of [26] that

$$v_h \rightharpoonup T_i(u)+tT_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega),$$

$$\eta_h \rightharpoonup 0 \quad \text{weakly in } W^{-1,p'}(\Omega),$$

since $T_i(u)+tT_k(u)$ is the solution of $VI(A, A(T_i(u)+tT_k(u)), \psi+tT_k(u))$. We have also, by (5.4.24), that

$$\begin{aligned}&\liminf_{h\to+\infty}\int_\Omega T_j(u_h-v_h)\,d\lambda_h+\int_\Omega T_j(u-T_i(u)-tT_k(u))\,d\mu\\&\ge\int_\Omega a(x,\nabla(T_i(u)+tT_k(u))\nabla T_j(u-T_i(u)-tT_k(u))\,dx.\end{aligned} \tag{5.4.51}$$

On the other hand, by (5.4.26), we have that

$$\begin{aligned}\liminf_{h\to+\infty}\int_\Omega T_j(u_h-v_h)\,d\lambda_h &\le \liminf_{h\to+\infty}\int_\Omega T_j(-\Phi_h)\,d\lambda_h\\&\le \liminf_{h\to+\infty}\int_\Omega T_j(-tT_k(\psi_h))\,d\lambda_h\\&= \liminf_{h\to+\infty}\int_\Omega T_j(-tT_k(u_h))\,d\lambda_h,\end{aligned} \tag{5.4.52}$$

where the last inequalities follow, on one hand, by the fact that $\Phi_h \in K_{tT_k(\psi_h)}$, for h large enough, on the other, by the complementarity system (3.1.10). Thanks to (5.4.52) we rewrite (5.4.51) as

$$\begin{aligned}&\liminf_{h\to+\infty}\int_\Omega T_j(-tT_k(u_h))\,d\lambda_h + \int_\Omega T_j(u-T_i(u)-tT_k(u))\,d\mu \\ &\geq \int_\Omega a(x,\nabla(T_i(u)+tT_k(u))\nabla T_j(u-T_i(u)-tT_k(u))\,dx.\end{aligned} \tag{5.4.53}$$

Let us choose $j \geq tk$ and $i > k$; if we split the integral in the right hand side of (5.4.53) into the sets where $|u| \leq i$ and where $|u| > i$ we obtain, by (5.1.3):

$$-t\int_{\{|u|\leq i\}} a(x,\nabla(u+tT_k(u))\nabla T_k(u)\,dx = -t\int_\Omega a(x,\nabla(T_k(u)(1+t)))\nabla T_k(u)\,dx. \tag{5.4.54}$$

As in the proof of *Step 2*, we let i tend to $+\infty$ in (5.4.53), so that, using (5.4.54), we easily get

$$-t\limsup_{h\to+\infty}\int_\Omega T_k(u_h)\,d\lambda_h - t\int_\Omega T_k(u)\,d\mu \geq -t\int_\Omega a(x,\nabla(T_k(u)(1+t)))\nabla T_k(u)\,dx,$$

or, equivalently, using the entropy formulation of u_h

$$\begin{aligned}&-t\limsup_{h\to+\infty}\left(\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,dx - \int_\Omega T_k(u_h)\,d\mu_h\right) - t\int_\Omega T_k(u)\,d\mu \\ &\geq -t\int_\Omega a(x,\nabla(T_k(u)(1+t)))\nabla T_k(u)\,dx,\end{aligned} \tag{5.4.55}$$

for every $k > 0$. On the other hand, Lemma 1.2.15 implies that

$$\lim_{h\to+\infty}\int_\Omega T_k(u_h)\,d\mu_h = \int_\Omega T_k(u)\,d\mu,$$

so that (5.4.55) becomes

$$-t\limsup_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,dx \geq -t\int_\Omega a(x,\nabla(T_k(u)(1+t)))\nabla T_k(u)\,dx.$$

Finally, dividing by t and passing to the limit as $t \to 0^+$, we have, for every $k > 0$

$$\limsup_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,dx \leq \int_\Omega a(x,\nabla u)\nabla T_k(u)\,dx,$$

which, combined with (5.4.39) gives (5.4.6). □

Remark 5.4.5. If we choose in Theorem 5.4.1 as obstacles $\psi_h = \psi = -\infty$, we recover Theorem 3.2 of [4], which concerns the continuous dependence of the entropy solutions under perturbations of the operator A.

Corollary 5.4.6. *Let a be a sequence in $\mathcal{L}(c_0, c_1, \alpha, \beta)$ and A be the operator associated with it. Let us assume (5.4.1), (5.4.2) (with $A_h = A$, for every $h > 0$), and (5.4.3), with K_{ψ_h} converging to K_ψ in the sense of Mosco. Finally, consider μ_h, $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, with μ_h converging to μ weakly in $\mathcal{M}_b(\Omega)$. Then the solutions u_h and u of the obstacle problems $OP_0(A, \mu_h, \psi_h)$ and $OP_0(A, \mu, \psi)$, respectively, satisfy*

$$T_j(u_h) \to T_j(u) \text{ strongly in } W^{1,p}_0(\Omega), \text{ for every } j > 0. \tag{5.4.56}$$

Proof. By Theorem 5.4.1 (with $a_h = a$, for every $h > 0$) we have that $T_j(u_h)$ converges to $T_j(u)$ weakly in $W^{1,p}_0(\Omega)$, and

$$\int_\Omega a(x, \nabla u_h) \nabla T_j(u_h)\, dx \to \int_\Omega a(x, \nabla u) \nabla T_j(u)\, dx, \text{ for every } j > 0. \tag{5.4.57}$$

On the other hand, if the function a is fixed, working as in the proof of Theorem 1.5.3, it can be proved that ∇u_h converges to ∇u almost everywhere in Ω. Since $a(x, \cdot)$ is a Carathéodory function, also $a(x, \nabla T_j(u_h))$ tends to $a(x, \nabla T_j(u))$ almost everywhere in Ω.

Moreover, thanks to (5.1.1), $a(x, \nabla T_j(u_h))$ is uniformly (with respect to h) bounded in $L^{p'}(\Omega)^N$; so we deduce that

$$a(x, \nabla T_j(u_h)) \rightharpoonup a(x, \nabla T_j(u)) \text{ weakly in } L^{p'}(\Omega)^N. \tag{5.4.58}$$

Combining (5.4.57) and (5.4.58), we have:

$$\lim_{h \to +\infty} \int_\Omega (a(x, \nabla T_j(u_h)) - a(x, \nabla T_j(u))) \nabla (T_j(u_h) - T_j(u))\, dx = 0,$$

which implies that $T_j(u_h)$ converges to $T_j(u)$ strongly in $W^{1,p}_0(\Omega)$. □

Remark 5.4.7. Other results in the case $a_h = a$, under different hypotheses on μ_h and ψ_h, can be found in [19].

Remark 5.4.8. We just noted (see Remark 2.7.2) that, if the datum μ is a general measure of $\mathcal{M}_b(\Omega)$, there is no continuous dependence with respect to the convergence of the obstacles in the sense of Mosco.

Actually, in [21], it is showed, in the linear framework of Chapter 2, how the continuous dependence on the obstacles is influenced by the behaviour of solutions due to the singular components of the forcing term μ, that we have studied in Chapter 4. In particular (see Theorem 3.14 of [21]), for $\mu \in \mathcal{M}_b(\Omega)$, calling u_h and u the solutions of $OP(\mu, \psi_h)$ and $OP(\mu, \psi)$, respectively (see Definition 2.1.1), the Mosco convergence of K_{ψ_h} to K_ψ ($K_{\psi_h} \stackrel{M}{\to} K_\psi$) implies:

$$\text{if } \mu^- \in \mathcal{M}_{b,0}(\Omega), \text{ then, for every } q < \tfrac{N}{N-1},\ u_h \to u \text{ strongly in } W_0^{1,q}(\Omega).$$

Other results in this sense can be found in [21].

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