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SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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FOR ADVANCED STUDIES

ANNA CAPIETTO

CONTINUATION THEOREMS FOR PERIODIC BOUNDARY VALUE PROBLEMS

Thesis submitted for the degree of Doctor Philosophiae

Academic Year 1989/90

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Il presente lavoro costituisce la tesi presentata dalla Dott.ssa Anna Capietto, sotto la direzione del Prof. Fabio Zanolin, al fine di ottenere l'attestato di ricerca postuniversitaria "Doctor Philosophiae" presso la S.I.S.S.A., classe di Matematica, settore di Analisi Funzionale e Applicazioni. Ai sensi del Decreto del Ministero della Pubblica Istruzione 24.4.1987, n. 419, tale diploma è equipollente al titolo di "Dottore di ricerca in Matematica".

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To my family

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Introduction

The object of this thesis is the study of the following boundary value problem

$$\dot{x} = F(t,x) \tag{0.1}$$
$$x(0) = x(T), \tag{0.2}$$

where $F : [0,T]\times C\to\mathbf{R}^m$ is a continuous function, $C\subseteq\mathbf{R}^m$ and $T>0$.
In particular, the existence of at least one solution $x(\cdot)$ to (0.1)-(0.2) such that $x(t)\in C$, for all $t\in[0,T]$, is investigated.
It is important to recall, once for all, that if $F : [0,T]\times\mathbf{R}^m\to\mathbf{R}^m$ is T-periodic in the first variable, then any solution of (0.1)-(0.2) is the restriction of a classical C^1, T-periodic solution of (0.1) defined on the whole real line.
The results of Chapters 2 to 6 have been developed by the author during her permanence at the International School for Advanced Studies; in particular, Chapter 2 and Chapters 3, 4, 5 are joint works with Fabio Zanolin and with Jean Mawhin and Fabio Zanolin, respectively.

The periodic boundary value problem is a classical topic in the qualitative theory of ODEs, which was initiated by H. Poincaré. Models arising from several branches of the physical sciences (e.g., mechanics, biology, mathematical economy) have been motivating (and stimulating) for the last decades the work of many mathematicians, in several directions and with different methods.

In this thesis, existence theorems for (0.1)-(0.2) are performed by means of "topological degree" methods. In the last 25 years (even if Poincaré himself used topological degree arguments), many results have been obtained in this general framework for the solvability of various boundary value problems associated to (0.1) in the particular case of $C=\mathbf{R}^m$. A short and definitely not exhaustive account of some of the classical techniques for the periodic BVP is given below.

Generally speaking, the existence of solutions to (0.1)-(0.2) is obtained by proving the existence of zeros (or fixed points) of some operator defined in a suitable space. The most important concept is the "degree"; more precisely, both the "finite-dimensional" Brouwer degree and its "infinite-dimensional" analogue, which was developed by J. Leray and J. Schauder [90], are used. A classical procedure consists of the following three steps:

(i) write $F(t,x) = f(t,x;1)$, where $f = f(t,x;\lambda) : [0,T]\times\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$, as to imbed (0.1) in a family of parametrized equations

$$\dot{x} = f(t,x;\lambda), \tag{0.3}$$

where f is an auxiliary function related to F, and $\lambda \in]0,1[$;

(ii) perform "transversality conditions" for the solutions of (0.3) in order to show that the "continuation" from $\lambda = 1$ to $\lambda \to 0^+$ is admissible;

(iii) show that the Brouwer degree of a suitable autonomous map (related to F) is nonzero.

In this way, by means of (i) it is possible to carry problem (0.1)-(0.2) to a "simpler" one; more precisely, by (ii) one can prove that the "degree" of the operator whose zeros are the solutions of (0.1)-(0.2) is equal to the finite-dimensional Brouwer degree of a corresponding autonomous map with values in $\mathbf{R}^m$. Indeed, the application of (ii) and (iii) corresponds to the use of the homotopy and existence properties of the degree.

It is the procedure described above which motivates the word "continuation" for the theorems like those performed in this thesis.

A brief description of earlier continuation theorems for the case $C=\mathbf{R}^m$ can be useful before explaining the core of the results of this thesis. In Stoppelli's pioneering work [139], the "simpler" problem obtained after the continuation is an autonomous equation whose T-periodic solutions consist of an odd number of nondegenerate equilibria; another possibility is to obtain a linear equation having only the trivial T-periodic solution (see e.g. [98], references for Th. IV.5). However, such an approach will only succeed in problems having an odd degree.

In a more general situation, two main approaches in order to perform step (ii) have been developed. By clG, frG we denote, respectively, the closure and the boundary of a set $G \subset \mathbf{R}^m$.

On the one hand, by the Liapunov-Schmidt reduction, problem (0.1)-(0.2) is transformed into an equivalent coincidence equation in the space of T-periodic functions:

$$Lx = Nx, \tag{0.4}$$

with L a linear (not necessarily invertible) operator and N a (nonlinear) Nemitzky operator. Under rather general hypotheses, (0.4) is replaced by an equivalent fixed-point problem

$$x = Mx.$$

In this framework, J. Mawhin has proved the following:

THEOREM A [98]. *Let* $f:[0,T]\times\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ *be continuous and let* $G\subset\mathbf{R}^m$ *be an open bounded set. Assume*

(a1) *for any* $x(\cdot)$ *solution of* $\dot{x} = \lambda f(t,x;\lambda)$, $\lambda\in\,]0,1[$, *with* $x(0) = x(T)$ *and* $x(t)\in \mathrm{cl}G$ *for all* t, *it follows that* $x(t)\in G$ *for all* t;

(a3) $d_B(\bar{f}_0,G,0) \neq 0$, *with* $\bar{f}_0 := \frac{1}{T}\int_0^T f(s,x(s),0)ds$.

Then, there is a solution $x(\cdot)$ *of* $x^{\cdot} = f(t,x;1)$, *satisfying* (0.2) *and such that* $x(t)\in \mathrm{cl}G$ *for all* t.

Assumptions (a1)-(a2) correspond, respectively, to (ii) and (iii); the "continuation" is performed via the homotopy

$$(\lambda,x) \longmapsto (1-\lambda)\,\frac{1}{T}\int_0^T f(s,x(s),\lambda)ds + \lambda f(t,x;\lambda) \;=\; \mathcal{f}(t,x;\lambda), \tag{0.5}$$

leading for $\lambda = 0$ to the integro-differential system

$$\dot{x} - \frac{1}{T}\int_0^T f(s,x(s),0)ds = 0, \tag{0.6}$$

whose T-periodic solutions are constant and given by the zeros of the function $\bar{f}_0 : z\mapsto \frac{1}{T}\int_0^T f(s,x(s),0)ds$. In this case, the "coincidence degree" (in a suitable space of T-periodic functions) of the operator associated to the left-hand member of (0.6) can be computed in terms of the Brouwer degree in $\mathbf{R}^m$ of $\bar{f}_0$.

A second point of view has been developed by M.A. Krasnosel'skii, and is originated by the study of the fixed points of the translation operator (Poincaré-Andronov map) $\pi_T : x(0)\longmapsto\pi(T,x(0))$. Recall that if π is a dynamical system in $\mathbf{R}^m$ induced by a Cauchy problem, then $\pi(t,x)$ denotes the value at time t of the unique solution of the Cauchy problem with initial value x(0). In this setting, it is worth recalling the celebrated Krasnosel'skii theorem:

THEOREM B [79]. *Let* $F:[0,T]\times\mathbf{R}^m\to\mathbf{R}^m$ *be continuous and such that uniqueness and global existence for the solutions of the associated Cauchy problems is guaranteed.*
Assume:

(b1) *there is no solution* $x(\cdot)$ of $x^{\cdot} = F(t,x)$ *such that* $x(0) = x(k)\in \mathrm{fr}G$ *for some* $0 < k < T$;

(b2) $F(0,z) \neq 0$ *for* $z\in \mathrm{fr}G$;

(b3) $d_B(F(0,\cdot),G,0) \neq 0$.

Then, there is a solution $x(\cdot)$ *of* $x^{\cdot} = F(t,x)$ *satisfying* (1.2) *and such that* $x(0) \in \mathrm{cl}G$.

Assumptions (b1)-(b3) correspond to (ii), (iii) and the needed homotopy carrying problem (0.1)-(0.2) to a simpler one is the following:

$$(\lambda,z) \longmapsto f(t,z;\lambda) := \begin{cases} \dfrac{z - \pi(\lambda T,z)}{\lambda} h(\lambda) & \text{for } \lambda \neq 0 \\ -F(0,z) & \text{for } \lambda = 0, \end{cases}$$

where $h(\cdot) : [0,1] \to \mathbf{R}^+$ is a continuous function such that $h(0) = 1/T$, $h(1) = 1$.
Further developments along this direction have been achieved by R. Srzednicki [138] on the line of Ważewski's method.

Both Mawhin's and Krasnosel'skii's theorem have found useful applications in the literature (see for instance the references in [7,55,81,98,116,131,132]. For other different but related results see [87,129,138].

In recent years, some results have been obtained in the case when the underlying space has not a linear structure. For instance, the situations in which the set C is a regular manifold [11,52,53], a convex set or a conical shell [18,33,44,56] have been investigated. More precisely, in [11,33,44,52,53] the properties of the Poincaré map are used, while in [18,56] coincidence degree arguments in function spaces are employed. In any case, the positive (negative) invariance of the set C is a key assumption.

Generalizations of the above quoted results are given in Chapters 2 to 6, where extensive comparisons and comments are performed.

Before describing the main results of this thesis, we have to mention that problem (0.1)-(0.2) has been treated by a different functional-analytic approach also by L. Cesari [21], and that isolated results based on various different techniques (e.g., implicit function theorem, shooting method, Poincaré-Birkhoff theorem) can be found in the literature. On the other hand, it must be noticed that problem (0.1)-(0.2) has been successfully tackled by many authors in the completely different framework of "variational methods". If the problem has a variational structure, then it is possible to define a functional whose critical points correspond to the T-periodic solutions (cf. [104]). These different approaches, however, will not be studied in this work; mention of them will be made in several remarks.

In what follows, a description of the contributions obtained, by means of continuation methods, to the periodic boundary value problem is given.

The first goal has been to study the case in which $C \subsetneq \mathbf{R}^m$. Assume that C is a flow-invariant ENR (Euclidean Neighbourhood Retract). Recall that a metric space U is an ANR (Absolute Neighbourhood Retract) if and only if it is homeomorphic to a neighbourhood retract of a Banach space V. If V is finite dimensional, we say that U is an ENR (Euclidean Neighbourhood Retract). In this situation, step (i) is developed using the homotopy (0.5), and an assumption analogous to (a1) in Theorem A is made for the system

$$\dot{x} = \lambda\, f(t,x;\lambda).$$

The difficulty arises because the set $\Gamma = \{y(\cdot): [0,T] \to C\}$ is not a linear space. Hence, the usual Leray-Schauder degree is not available. The approach proposed consists, roughly speaking, of embedding (0.1)-(0.2) in a functional-analytic framework (which is different from Mawhin's one) and to use the fixed point index introduced by A. Granas [59] for arbitrary ANRs . This is possible because (cf. [75]) the space Γ is an ANR if and only if C is an ENR. The operator defined on Γ whose fixed points are the solutions of (0.1)-(0.2) is constructed by means of the family of processes induced by the Cauchy problem associated to (0.5). In this way, the approaches described by Theorems A and B are unified and previous results (valid for some particular choice of the ENR) such as [18,44,53,56] are extended. In the applications, the set C is, in a first example, a domain with holes and, in another one, an (m−1)-dimensional symplex; in the former, the Euler-Poincaré characteristic is a useful concept for the computation of the "index of rest points" of the dynamical system induced (for $\lambda=0$) by (0.5). These results are motivated by possible applications to hydrodynamics and biology, respectively.

The definition and the main properties of the fixed point index for arbitrary ANRs are recalled in Chapter 1 (on the lines of [13, Chap. 1]), where a description is also given of the "index of rest points"; the latter is the "finite dimensional analogue" of the fixed point index, i.e. it has the the role played by the Brouwer degree with respect to the Leray-Schauder degree in classical results such as Theorems A and B.

The case when the set C is a closed convex set with nonempty interior has been treated in [18]; the results for ENRs are contained in Chapter 2. The operator defined on the space Γ which is introduced in Section 2 of Chapter 2 is used also in Sections 5 of Chapter 3 and 2 of Chapter 6, where results improving Theorems A and B also in the case $C = \mathbf{R}^m$ are generalized to this more general situation. We also point out that, since the proof of these results is based on processes, it is necessary to require the uniqueness for the solutions of the Cauchy problems which are taken into account. Hence, for simplicity, the considered vector fields are assumed to be lipschitzian in the x-variable. A "continuous" version of our results (cf. [18,53,98]) can be performed by means of a standard pertrubation argument based on Weierstrass-Stone and Ascoli-Arzelà theorems.

Another aspect which has been studied consists of working on step (ii) (at first, for $C=\mathbf{R}^m$) in order to get continuation theorems suitable for some special cases of problem (0.1)-(0.2).

First (Chapter 3), perturbed systems of the form

$$\dot{x} = g(x) + e(t)$$

have been considered.
In this situation, one is naturally led to consider the homotopy

$$(\lambda,x) \longmapsto f(t,x;\lambda) = g(x) + \lambda e(t),$$

which connects $F(t,x) = f(t,x;1)$ to the function $g(x)$ (for $\lambda=0$). More generally, one may assume that the homotopized field $f(t,x;\lambda)$ is such that there exists an autonomous function f_0 satisfying $f(t,x;0) = f_0(x)$. In the above situation, however, the computation of the coincidence degree of the left-hand member of the autonomous differential equation

$$\dot{x} - f_0(x) = 0$$

is made more difficult by the presence of possible non constant closed orbits with period less than T. Indeed, it is shown that a connection still holds between the coincidence degree and the Brouwer degree of f_0, so that step (iii) can be accomplished. In the proof of this result, the crucial point is the use of an approximation procedure for the map f_0 based on the Kupka-Smale's theorem [22,119], which ensures the existence of a sequence (φ_k) of C^1 functions, $(\varphi_k) \to f_0$, such that for each $\sigma>0$, for every compact subset K of $\mathbf{R}^m$ and for all $k \in \mathbf{N}$, system $\dot{x} = \varphi_k(x)$ has *finitely many* rest points or closed orbits with period less or equal than σ which are contained in K.
A generalization in the framework of Chapter 2 to the case when the phase-space is a flow-invariant ENR is possible, provided that an additional "Kupka-Smale approximation property" holds; this is the case, for example, when the set C is a regular manifold or a convex set.
The applications given deal with the case when $F(t,x) = g(x) + e(t,x)$, and the homotopized field used is

$$f(t,x;\lambda) = g(x) + \lambda e(t,x).$$

On the one hand, it is possible to obtain results of perturbational type, assuming that $|e|_\infty$ is sufficiently small. In this case, earlier contributions of Amel'kin-Gaishun-Ladis [1], Berstein-Halanay [9], Cronin [27,28,29], Gomory [57], Halanay [62,64], Hale-Somolinos [68], Lando [84,85], Reissig [129], Srzednicki [137,138], Ward [145] are generalized in various ways.
On the other hand, results of global type are obtained assuming that the function g is positively homogeneous of some order; in this way, it is possible to improve theorems due to Dancer [31], Fonda-Habets [47], Fonda-Zanolin [48], Fučik [49], Krasnosel'skii-Zabreiko [81], Lazer-McKenna [88], Lasota [86], Muhamadiev [110,111].

Chapter 4 is concerned with the following boundary value problem for a retarded functional differential equation (RFDE)

$$\dot{x}(t) = F(t,x_t), \quad t\in \mathbf{R}, \tag{0.7}$$

$$x(0) = x(T), \tag{0.8}$$

where $F : \mathbf{R}\times\mathcal{C}_r \to \mathbf{R}^m$ is continuous, T-periodic in the first variable, takes bounded sets into bounded sets and $\mathcal{C}_r := \mathcal{C}([-r,0],\mathbf{R}^m)$, $x_t(\theta) := x(t+\theta)$, $\theta\in[-r,0]$.
As in Chapter 3, problem (0.7)-(0.8) is imbedded in a family of parametrized equations

$$\dot{x} = f(t,x_t;\lambda),$$

where the function $f: \mathbf{R}\times\mathcal{C}_r\times[0,1]\to\mathbf{R}^m$ is such that $f(t,\varphi;1) = F(t,\varphi)$, for all $t\in\mathbf{R}$, $\varphi\in\mathcal{C}_r$; moreover, the existence of a map $f_0 : \mathcal{C}_r\to\mathbf{R}^m$ such that $f_0(\varphi) = f(t,\varphi;0)$, for all $t\in\mathbf{R}$, $\varphi\in\mathcal{C}_r$ is assumed.
A result on the computation of the coincidence degree of the left-hand member of the autonomous retarded functional differential equation

$$\dot{x}(t) - f_0(x_t) = 0 \tag{0.9}$$

in terms of the Brouwer degree of $f_{0|\mathbf{R}^m}$ is performed, on the lines of Chapter 3.
A basic ingredient in the proof is an extension due to Mallet-Paret [93] of the Kupka-Smale's theorem in the case of a RFDE. It is worth noticing that the different nature of (0.9) requires at various steps nontrivial variants of the arguments developed in Chapter 3, and even completely different ones due in particular to the fact that time-scaling involves modifications of the delay in a RFDE. More precisely, it is necessary to use the fact that the "finitely many" singular orbits are, in the RFDE case, hyperbolic. Using the result on the computation of the degree, a continuation theorem for the T-periodic solutions of (0.7) is proved.

Another aspect of the research developed in this thesis is the object of Chapter 5, which is devoted to the so-called "superlinear" periodic boundary value problem. This class of nonlinearities arises when dealing with a second-order equation of the form

$$\ddot{x} + g(x) = p(t), \tag{0.10}$$

where p is T-periodic. First, conditions for the existence of T-periodic solutions of (0.9) must exclude the well-known resonant situation

$$\ddot{x} + (n\omega)^2 x = \cos n\omega t, \tag{0.11}$$

where $\omega = \dfrac{2\pi}{T}$ and $n\in \mathbf{Z}_+$.

This can be obtained by assuming, for example, that $g(x)\cdot x < 0$ for $|x|$ large (in a variational setting, this corresponds to the coercivity of the associated action functional). Another general way to exclude counterexample (0.11) consists in not allowing g to be linear. If one wants to avoid boundedness restrictions on g, one is led to consider the class of superlinear nonlinearities, i.e. functions g such that

$$\frac{g(x)}{x} \to +\infty \quad \text{as } |x| \to +\infty.$$

Earlier results on this topic can be found in the works of Morris [106,107,108,109], Ehrmann [39,40], Harvey [71], Cesari [21], Micheletti [105], Fučik-Lovicar [50], Struwe [140], W. Y. Ding [36], Bahri-Berestycki [3,4], Rabinowitz [126,127]. These authors have used various methods, which are briefly described in Chapter 5.

In the above situation, the main difficulty in developing the procedure explained at the beginning of this Introduction is in step (ii), since, roughly speaking, in the superlinear case no a priori bounds are available.

This problem is overcome (in the framework of coincidence degree for L-compact pertrubations of linear Fredholm mappings of index zero) by a continuation result (Lemma 1), where the new ingredient is a functional φ which is proper on the set Σ of possible solutions of the homotopic family of equations and which avoids two values during this homotopy. For the applications to periodic BVPs, the significant special case is that of a functional taking only positive integer values on large norm solutions and whose positive integer level sets have a bounded intersection with Σ. In particular, the functional used is closely related to the mapping counting the number of rotations around the origin of the solutions of a planar differential system. Applications are provided for planar systems with linear growth, for superlinear planar hamiltonian systems and for "weakly coupled" systems of the form

$$\ddot{x}_k + g_k(x_k) = p_k(t,x_k), \qquad k = 1,\dots,m.$$

Note that when a priori bounds are available for the (possible) solutions of an operator equation $x = M(x,\lambda)$, where $M : U\times[0,1]\to U$ and Ω is a bounded open subset of a normed space U containing Σ, then the choice

$$\varphi(x;\lambda) := \begin{cases} -\mathrm{dist}(x, \mathrm{fr}\Omega) & \text{for } x\in \Omega, \\ \mathrm{dist}(x,\mathrm{fr}\Omega) & \text{for } x\notin \Omega \end{cases} \tag{0.12}$$

is suitable for the applications of Lemma 1; therefore, solutions of the classical continuation theorems can be reobtained.

It is also worth mentioning that for the proof of Lemma 1 it is crucial to investigate attentively the structure of the solution set Σ. It is interesting to compare this approach with earlier work in [46,51,53]; in [51], the existence of a connected branch of solutions to the parametrized system

$$\dot{x} = \lambda\, f(t,x;\lambda)$$

(for a function f as in Theorem A) is proved using the concept of "regular map". The results of Chapter 5, however, are independent of [51].

In Chapter 6 the method initiated in Chapter 2 is used in order to generalize to ANRs the abstract continuation theorem of Chapter 5. On the other hand, setting $\Sigma := \{(x,\lambda)\in\Omega\times[0,1] : x = M(x,\lambda)\}$, the existence of a closed connected branch $C\subset\Sigma$ of solutions to the operator equation

$$x = M(x,\lambda)$$

(with $M : \Omega\times[0,1]\to U$ a completely continuous operator defined on a bounded open subset of U, a metric ANR) such that $C\cap(\Sigma_0\times\{0\}) \neq \emptyset$, and, for $C^+ := \{x\in\Sigma : \lambda > 0\}$, $C^- := \{x\in\Sigma : \lambda<0\}$, the following alternative holds : either $C^{\pm}$ is unbounded or $C^{\pm} \cap \partial\Omega \neq \emptyset$. With this result, which generalizes [51] to ENRs by the use of the fixed point index, it is possible to give a different proof of the abstract continuation theorem; on the other hand, choosing φ like in (0.12), it is possible to reobtain the usual existence results for differential systems in flow-invariant ENRs when a priori bounds are available. Indeed, if this is the case the second alternative in the result described above is excluded so that, for $\lambda = 1$, existence for the operator equation $x = M(x;1)$ is proved.

Finally, we point out that the use of the functional φ originated by the time-map may provide information for other boundary value problems (e.g. for the two-point BVP). Moreover, an investigation has started on a (possible) deeper use of the functional φ used in Chapters 5 and 6 in relation with the fundamental homotopy group of some suitable space. In this direction, the use of the degree for equivariant maps might be fruitful (cf. [6]).

In this thesis, the number of formulas, sections, theorems, corollaries, lemmas, propositions refers, if not otherwise specified, to the chapter in which it appears.

Chapter 1

Preliminaries and notations

1. Introduction

In this Chapter we describe the most important concepts used in this thesis and we recall some well-known facts which are employed in what follows. In Section 2 we give the axioms and the main properties of the fixed point index; in Section 3 we describe the Euler-Poincaré characteristic; in Section 4 we introduce the "index of rest points". In Section 5, we give a list of notations.

2. The fixed point index

In this Section, we introduce in an axiomatic way the fixed point index for a rather general class of spaces and maps. To this aim, we follow [59,113].
First, we introduce the class of spaces for which we define the fixed point index.

DEFINITION 1. *A metric space* U *is an ABSOLUTE NEIGHBOURHOOD RETRACT (ANR) if and only if for every metric space* V, *for every closed subset* M *of V and for every continuous map* $f : M \to U$ *there exists a continuous extension* $\tilde{f}$ *of* f *which is defined in an open set containing* M.

REMARK 1. We point out that, equivalently, U is an ANR if and only if U is homeomorphic to a subset V of a Banach space B and V is a neighbourhood retract of B. This is a consequence of the classical Arens-Eells embedding theorem [59, p. 221].

DEFINITION 2. *If* U *is an ANR and the Banach space* B *in the above remark is* R^m, *then* U *is called an EUCLIDEAN NEIGHBOURHOOD RETRACT (ENR)* .

Now we recall some elementary properties and examples of ANRs (for a general treatment of ANRs, see [75]).

(a) (Dugundji) [83, § 53]. If U is a closed convex subset of a normed linear space, then U is an ANR (indeed, U is a retract of the space).

(b) If U is a closed subset of a normed linear space V and if there exists a family $\{C_j\}_{j\in J}$ of closed convex subsets of V such that $U = \bigcup_{j\in J} C_j$ and $\{C_j\}_{j\in J}$ is a locally finite covering of U, then U is an ANR.

(c) A retract of an ANR is an ANR.

(d) Every open subset of an ANR is an ANR.

This latter fact implies that any ANR is locally ANR. The converse of proposition (d) is also true; namely:

(e) (Hanner) [69]. If U is a metric space and every $x\in U$ is contained in an open neighbourhood N_x which is an ANR, then U is an ANR.

In particular, a metrizable Banach manifold is an ANR (this fact will be extensively used in the sequel). Finally, we recall:

(f) (West) [147]. Every compact ANR is homotopically equivalent to a compact polyhedron (see also Section 3).

(g) ([75]). Let U be a compact metrizable space and V a metrizable space. Let d be a distance which defines the topology of V. Consider the function space $\Omega = \{f : U \to V, \text{continuous}\}$, endowed with the distance d^*, where $d^*(f,g) := \sup_{x\in U} d(f(x), g(x))$. Then, Ω with the d^*-topology is an ANR if and only if V is an ANR.

(h) Let U_1, U_2 and $U_1\cap U_2$ be ANRs; then, $U_1\cup U_2$ is an ANR.

Let U be an ANR and let W be an open subset of U. Let $f : W \to U$ be a continuous function. Let us give the following

DEFINITION 3. *The triple* (U,W,f) *is called ADMISSIBLE if the set* $S = \{x\in W: f(x) = x\}$ *is compact (possibly empty) and there exists an open neighbourhood* V *of* S *such that* $\mathrm{cl}V\subset W$ *and* $f_{|\mathrm{cl}V}$ *is compact.*

REMARK 2. Frequently, the axioms of the fixed point index are given in a less abstract framework, i.e. one assumes that W is a (bounded) open subset of U and $f: \mathrm{cl}W \to U$ is a compact map such that $f(x) \neq x$ for all $x \in \mathrm{fr}W$. Indeed, if this is true, then the triple (U,W,f) is admissible.

THE AXIOMS

To any given admissible triple (U,W,f) we associate an integer

$$i_U(f,W)$$

called the fixed point index of f on W (relatively to U) satisfying the following properties:

I. EXCISION

Let W' be an open subset of W with $S \subset W'$ and let $f' = f_{|W'} : W' \to U$. Then,

$$i_U(f,W) = i_U(f',W').$$

(Note that the triple (U,f',W') is admissible.)

II. ADDITIVITY

Assume that $W = \bigcup_{i=1}^{n} W_i$ and let $f_i := f_{|W_i}$, $S_i := S \cap W_i$. If $S_i \cap S_j = \emptyset$, $i \neq j$, then

$$i_U(f,W) = \sum_{i=1}^{n} i_U(f_i,W_i)\ .$$

III. FIXED POINT PROPERTY

If $i_U(f,W) \neq 0$, then $S \neq \emptyset$, i.e. the map f has a fixed point.

IV. HOMOTOPY

Let $H : W \times [0,1] \to U$ be a continuous homotopy, and let $H_t : W \to U$ be defined by $H_t(x) := H(t,x)$. Assume that $S := \bigcup_{t \in [0,1]} \{x \in W : H_t(x) = x\}$ is compact and there is an open neighbourhood V of S such that $\mathrm{cl}V \subset W$ and $H_{|\mathrm{cl}V \times [0,1]}$ is a compact mapping. Then,

$$i_U(H_0,W) = i_U(H_1,W).$$

V. MULTIPLICATIVITY

If the triples (W_1,U_1,f_1), (W_2,U_2,f_2) are admissible, then

$$i_{U_1 \times U_2}(f_1 \times f_2, W_1 \times W_2) = i_{U_1}(f_1,W_1) \cdot i_{U_2}(f_2,W_2)\ .$$

VI. COMMUTATIVITY

Let W_1, W_2 be open subsets of U_1, U_2, respectively; assume that $f_1 : W_1 \to U_2$, $f_2 : W_2 \to U_1$ are continuous maps and that the map f_1 is compact in a neighbourhood of $\{x \in W_1 : f_2 f_1(x) = x\}$ (or the map f_2 is compact in a neighbourhood of $\{x \in W_2 : f_1 f_2(x) = x\}$). Consider the composite maps:

$$f_2 f_1 : f_1^{-1}(W_2) \to U_1$$

$$f_1 f_2 : f_2^{-1}(W_1) \to U_2.$$

If one of the triples

$$(f_1^{-1}(W_2), U_1, f_2 f_1) \ , \ (f_2^{-1}(W_1), U_2, f_1 f_2)$$

is admissible, then so is the other and, in this case,

$$i_{U_1}(f_2 f_1, f_1^{-1}(W_2)) = i_{U_2}(f_1 f_2, f_2^{-1}(W_1)).$$

VII. NORMALIZATION

If $W = U$ and the map f is compact, then the Lefschetz number of f is defined and

$$i_U(f,W) = \Lambda(f).$$

For the definition and properties of the Lefschetz number, we refer to [59,89].

REMARK 3. We point out that the axioms given above are not independent. For instance, axiom III (fixed point property) is an easy consequence of the additivity and excision axioms.

Now, we recall a useful property of the fixed point index which follows from the commutativity axiom:

PROPOSITION 1 (Contraction property of the fixed point index). *Let* W *be an open subset of* U *and* $f: W \to U$ *a continuous map for which the index* $i_U(f,W)$ *is defined. If a metric ANR* V *is a subset of* U *such that the inclusion* $j : V \subset U$ *is continuous and* $f(W) \subset V$, *then*

$$i_U(f,W) = i_V(f_{|W \cap V}, W \cap V).$$

REMARK 4. First of all, we point out that if we assume that U is a compact ANR and we denote by Id_U the identity map, then $i_U(Id_U,U) = \Lambda(Id_U)$; this number, which depends only on the set U, has many important topological properties, which we recall in Section 3.
Furthermore, we remark that the normalization axiom is, essentially, the Lefschetz fixed point theorem.
It seems interesting to recall also the "weak" form of the normalization axiom (and to compare it with the analogous property of the Brouwer degree):

VII BIS. "WEAK" NORMALIZATION

If the triple (W,U,f) is admissible and $f(x) = p$ for all x, then

$$i_U(f,W) = \begin{cases} 1 & \text{if } p \in W \\ 0 & \text{if } p \notin W \end{cases} .$$

The proof of the existence of the fixed point index for ANRs is omitted for brevity. See [59, Th. 7.1, Th. 10.1] for details.

3. The Euler-Poincaré characteristic

In this Section we give the definition and some remarks about the Euler-Poincaré characteristic of a set, another important tool in this thesis. Although it is defined in algebraic topology, in recent years it has turned out to be very useful from the point of view of analysis too. Accordingly, after the abstract definition, we briefly present an intuitive explanation of this important concept; as a consequence, we outline the way in which it can be viewed, and used, by analysts.
Let $C \subset R^m$ be a compact ENR (indeed, it is sufficient to consider a set which has the homotopy type of a polyhedron).

DEFINITION 4. *The Lefschetz number of the identity map* Id_C *is called the EULER-POINCARE' CHARACTERISTIC of the set* C, *and it is denoted by* $\chi(C)$.

The following useful formulas can be proved by the axioms and properties of singular homology:

$$\chi(\{P\}) = 1, \quad \chi(B[0,1]) = 1, \quad \chi(S(0,1)) = 1 + (-1)^m, \quad \chi(S_h) = 2 - 2h,$$

where S_h denotes an orientable surface of genus h (see [37, p.106]).
Furthermore, we recall that if M is a compact manifold and if the dimension of M is odd, then $\chi(M) = 0$ (see [142]).

We now outline the way in which the Euler-Poincaré characteristic, a purely algebraic topology object, can be used in order to apply topological methods in the search of periodic solutions to differential systems. Roughly speaking, in many cases the Euler-Poincaré characteristic of certain subsets of R^m turns out to be equal to the "index of rest points"(cf. Section 4). In other words, in this thesis (expecially in the applications) the Euler-Poincaré characteristic "plays the role" of the topological degree and/or the fixed point index, i.e. the hypothesis $\chi(C) \neq 0$ in some cases implies (the existence of fixed points of suitable operators and consequently) the existence of solutions to some boundary value problem.
In some sense, we establish a link between the topological nature of a manifold (or, in general, of a compact ENR) and the possible kinds of singularities of a vector field on such a set.

Actually, this link is in the classical Poincaré-Hopf theorem; indeed, if we denote by i_x the "index of isolated singularities" x of a given vector field on a (sufficiently regular) manifold M, then

$$\sum_x i_x = \chi(M).$$

Let T(M) denote the tangent bundle of M. The famous Poincaré-Hopf theorem states that if $\chi(M) \neq 0$ then any smooth vector field $v : M \rightarrow T(M)$ must vanish somewhere.
In the case $m = 2$, for example, since $\chi(S(0,1)) = 2$, a vector field defined on $S(0,1)$ has at least one singular point.
The results we present in this thesis include the Poincaré-Hopf theorem as a particular case.

Before ending this section we observe that, among others, H. Groemer, V.A. Efremovic and Yu.B. Rudjak [38,60] have given a characterization of the Euler-Poincaré characteristic for compact polyhedra by means of the following axioms:

1. <u>ADDITIVITY</u>

$$\chi(C_1 \cup C_2) = \chi(C_1) + \chi(C_2) - \chi(C_1 \cap C_2).$$

2. <u>NORMALIZATION</u>

$$\chi(\emptyset) = 0, \quad \chi(\{P\}) = 1.$$

Indeed, we point out that since compact ANRs are homotopically equivalent to compact polyhedra (see [147]) then we have a characterization of the Euler-Poincaré characteristic for compact ANRs by adding to the additivity and normalization axioms the following property:

3. <u>HOMOTOPY EQUIVALENCE</u>

If C_1 and C_2 have the same homotopy type, then

$$\chi(C_1) = \chi(C_2).$$

In this way, the Euler-Poincaré characteristic is determined by a number of axioms, just as in the case of the Brouwer degree and the fixed point index, independently of its construction through (the Lefschetz number and) singular homology.

4. The index of rest points

In this Section we introduce another important concept for this thesis, the index of rest points, by means of the fixed point index (cf. Section 2). In some of our main results (Theorem 1 in Chapter 2, Theorem 5 in Chapter 3, Theorem 2 and Corollary 2 in Chapter 6) the "index of rest points" plays the role of the Brouwer degree in classical theorems for the periodic BVP (see [79,81,98]).
Let U be an ENR, and let π be a dynamical system in U. Let Ω be an open subset of U. Assume that there are no rest points of π in $\mathrm{fr}\Omega$ and Ω is relatively compact. Then, we know (see Section 2) that the fixed point index $i_U(\pi_t,U)$ (where $\pi_t : x \longmapsto \pi(t,x)$) has a constant value for $0 < t < \varepsilon$, provided that ε is sufficiently small. In this situation, we can give the following:

DEFINITION 5 [137]. *The INDEX OF REST POINTS of the dynamical system π in the set Ω is given by the formula:*

$$I(\pi,\Omega) := \lim_{\varepsilon\to 0^+} i_U(\pi_t,\Omega).$$

In what follows, we often deal with a dynamical system π which is induced by an (autonomous) differential system of the type

$$\dot{x} = f_0(x), \tag{1.1}$$

where $f_0 : \mathbf{R}^m \to \mathbf{R}^m$ is locally lipschitzian.

A concept analogous to the index of rest points has been introduced by Furi and Pera in [53] for flows on manifolds satisfying suitable assumptions. More precisely, for a vector field $f_0(\cdot)$ as in (1.1) they define $\chi(f)$, the "Euler characteristic of the vector field f"; the properties of this characteristic are analogous to those of the fixed point index (i.e. the solution, excision, additivity, homotopy, normalization properties hold). Indeed, if π is the dynamical system induced by (1.1), then we have:

$$I(\pi,\Omega) = \chi(-f).$$

Now, we recall (following [137]) some properties of the index of rest points.

PROPOSITION 2 [137, Prop.4.3, Th.5.1, Th.6.1].

(i) *Assume that* W_1 *and* W_2 *are open,* $W_1 \subset W_2$, $\mathrm{cl}W_2$ *is compact and there are no rest points in* $(\mathrm{cl}W_2)\backslash W_1$ *; then:*

$$I(\pi,W_1) = I(\pi,W_2).$$

(ii) *Assume that* $W_1,W_2,...,W_r$ *are open subsets of* U*such that* $W_i \cap W_j = \emptyset$ *for* $i \neq j$ *and there are no rest points in* $U \setminus (\bigcup_{i=1}^{r} W_i)$*; then,*

$$I(\pi,U) = \sum_{i=1}^{r} I(\pi,W_i).$$

(iii) *Assume that* U *is compact; then,*

$$I(\pi,U) = \chi(U).$$

(iv) *Assume that there are no rest points in the set* $\mathrm{cl}\Omega$*; then,*

$$I(\pi,\Omega) = 0.$$

(v) *Assume that* π *is generated by the equation* $\dot{x} = Ax$, *where* A *is a real nonsingular matrix; let* k *denote the number of its eigenvalues having positive real parts; then, for any open set* Ω, $0 \in \Omega$,

$$I(\pi,\Omega) = (-1)^k.$$

(vi) *Assume that* π *is a dynamical system in* R^m *generated by a function* $f_0 : R^m \to R^m$. *Let* $\Omega \subset R^m$ *be an open bounded set and suppose that* $f_0(x) \neq 0$ for $x \in \mathrm{fr}\Omega$*; then,*

$$I(\pi,\Omega) = (-1)^m \, d_B(f_0,\Omega,0).$$

(vii) *Assume that there is a compact subset* K *of* U *such that, for every* $x \in U$, $\pi([0,\infty[,x) \cap K \neq \emptyset$*; then,*

- U *is of finite type;*
- K *has a rest point provided that* $\chi(U) \neq 0$*;*
- *If there are no rest points in* frK, *then*

$$I(\pi,\mathrm{int}K) = \chi(U).$$

(viii) [137, Theorem 4.4]. *Let* B *be a "block" and* b^- *the set of "egress points". Assume that* B, b^- *are* ENRs. *Then,*

$$I(\pi,B) = \chi(B) - \chi(b^-).$$

(If a set B is a "block", then, roughly speaking, each point of frB is, in Ważewski's terminology, a "strict ingress point" or a "strict egress point", i.e. "sliding points" are not allowed. We refer to [24] for an extensive treatment of the concept of "block").

A homotopy property for the index of rest points has been proved in the particular case of tangent vector fields by Furi and Pera in [53]; for the general case, we refer to Lemma 2 in Chapter 2.

We end this Section with an important remark concerning the computation of the index of rest points.

REMARK 5. First, we point out that in this thesis we use the index of rest points in a situation which is slightly more general than the one of Definition 5. Roughly speaking, we consider a "flow invariant" ENR $C \subset \mathbf{R}^m$ and a bounded set $G \subset C$, open relatively to C.
Let us now recall some facts about the computation of the index of rest points.

If $G = C$ (C compact), then $I(\pi^0, G) = \chi(C)$, where χ denotes the Euler-Poincaré characteristic (see [53,137]).

If $\mathrm{cl}G \subset \mathrm{int}C$, then $I(\pi^0, G) = (-1)^m d_B(f_0, G, 0)$ (see [81,137]).

If C is a closed convex set with non-empty interior, then $I(\pi^0, G) = i_C(r(I + f_0), G)$, $r : \mathbf{R}^m \to C$ being the canonical projection (see [18]).

If C is a manifold (satisfying suitable assumptions) and f_0 is a vector field tangent to C, then $I(\pi^0,G) = \chi(-f_0)$, where χ is the "characteristic of the vector field" f_0 introduced in [53].

If π^0 is dissipative, i.e. there is a compact set $\mathcal{K} \subset C$ such that for each $x \in C$ there is $t_x \geq 0$ with $\pi^0(x,t) \in \mathcal{K}$ for all $t \geq t_x$, then C is of finite type and $I(\pi^0, G) = \chi(C)$ for every $G \supseteq \mathcal{K}$ (see [137, Th. 6.1]).

Finally, if clG is a "block", then the index of rest points may be computed by means of Proposition 2, (viii).

5. Notations

We denote by **N**, **Z**, **R**, the sets of natural numbers, integers and reals, respectively; we also consider $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$, $\mathbf{R}_+ = [0,+\infty[$ and $\mathbf{R}^+ =]0,+\infty[$.

The m-dimensional real euclidean space $\mathbf{R}^m$ is endowed with the usual scalar product $(\cdot\mid\cdot)^{1/2}$, norm $|\cdot|$ and distance $d(\cdot,\cdot)$. The vectors of the canonical orthonormal basis in $\mathbf{R}^m$ are denoted by e_j, $(j=1,\ldots,m)$.
Given two subsets C_1, C_2 of $\mathbf{R}^m$, we denote by

$$\mathrm{dist}(C_1,C_2) := \inf\{|a-b|: a\in C_1,\ b\in C_2\},$$

the distance between the sets C_1 and C_2; we also set $\mathrm{diam}(C_1) := \sup\{|a-b|: a,\ b\in C_1\}$.
For a closed convex set $K\subset\mathbf{R}^m$ we denote by $N(z,K)$ the set of (nonzero) outer normals to K at $z\in \mathrm{fr}K$.

Given any metric space V, for $A\subset B\subset V$, by $\mathrm{int}_B A$, $\mathrm{fr}_B A$, $\mathrm{cl}_B A$ we mean, respectively, the interior, boundary and closure of the set A relatively to B; $\mathrm{card}A$ is the cardinality of the set A. We omit the subscript whenever no confusion occurs. The open and closed ball of center x_0 and radius $R>0$ are denoted by $B(x_0,R)$ and $B[0,R]$; we also set $S(x,R) := \mathrm{fr}B(x,R)$. Let Ω be a subset of the product space $V\times[0,1]$. Then, we denote by Ω_λ, $\lambda\in[0,1]$, the section of Ω at λ, that is $\Omega_\lambda := \{x\in V : (x,\lambda)\in\Omega\}$; moreover, we set: $(\partial\Omega)_\lambda := \{x\in\Omega : (x,\lambda)\in \mathrm{fr}_{V\times[0,1]}\Omega\}$. Observe that, in general, $(\partial\Omega)_\lambda \not\subset \mathrm{fr}(\Omega_\lambda)$.

If U is a normed space, $|\cdot|_U$ denotes its norm and I_U the identity operator in U; as a usual convention, the subscript is omitted for $U = \mathbf{R}^m$.

The norm of a linear bounded operator L between normed spaces is denoted by $\|L\|$.

We shall deal with the following Banach spaces:

$$Y := \mathcal{C}([0,T],\mathbf{R}^m), \qquad X := \{x\in Y: [0,T]\to\mathbf{R}^m,\ x(0) = x(T)\},$$
$$\mathcal{C}_r := \mathcal{C}([-r,0],\mathbf{R}^m), \qquad \mathcal{C}_T := \{x(\cdot) : \mathbf{R}\to \mathbf{R}^m,\ x(0) = x(T)\},$$

with the sup-norm $|\cdot|_\infty$ and distance d^*, $d^*(x_1,x_2) = |x_1-x_2|_\infty$.
Moreover, for any $1\le q\le+\infty$, we denote by $|\cdot|_q$ the L^q-norm of a function $u(\cdot)$ belonging to the Lebesgue space $L^q([0,T],\mathbf{R}^m)$; we set $Z := L^1([0,T],\mathbf{R}^m)$.

For a map $u\in L^1([0,T],\mathbf{R}^m)$, we define

$$\bar{u} := \frac{1}{T}\int_0^T u(s)ds,$$

the mean value of $u(\cdot)$ on $[0,T]$, and for a function $y = y(t;\lambda) : [0,T]\times[0,1]\to\mathbf{R}^m$, we write $\bar{y}_\lambda := \frac{1}{T}\int_0^T y(s,\lambda)ds$.

By d_B and deg we mean, respectively, the usual Brouwer degree in $\mathbf{R}^m$ and the Leray-Schauder degree in a normed vector space.
A final preliminary to our results is needed.

Let $C \subset \mathbf{R}^m$ be a closed set. We denote by

$$T(z;C) := \{v \in \mathbf{R}^m : \liminf_{h\to 0^+} d(z + hv, C) / h = 0\}$$

the (Bouligand) tangent cone to C at z.

Recall that, according to a classical theorem of M. Nagumo [112], given a continuous function $F : J\times C \to \mathbf{R}^m$, where $J \subset \mathbf{R}$ is a nondegenerate interval with interior I, for each $(t_0,x_0) \in I\times C$ the Cauchy problem

$$\begin{cases} \dot{x} = F(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a solution $x(\cdot) : \mathrm{dom}\,x(\cdot) \to C$ defined on a right maximal neighbourhood of t_0 if and only if

$$F(t,z) \in T(z;C) \quad \text{for all } t \in I,\ z \in \mathrm{fr}C. \tag{1.2}$$

Equivalently, if $F^* : J\times\mathbf{R}^m \to \mathbf{R}^m$ is any continuous extension of F, then (1.2) ensures that the set C is (weakly) positively invariant with respect to the equation $\dot{x} = F^*(t,x)$, i.e. for each $(t_0,x_0) \in I\times C$ there is at least a solution $x(\cdot)$ of $\dot{x} = F^*(t,x)$ with $x(t_0)=x_0$ such that $x(t) \in C$ in its right maximal interval of existence. Accordingly, since we are interested in solutions lying in the set C, there will be no loss of generality if we assume $F(t,\cdot)$ defined on the whole space $\mathbf{R}^m$ whenever (1.2) is assumed.

Chapter 2

An existence theorem for the periodic boundary value problem in flow-invariant ENRs with applications

1. Introduction

Let $C \subset \mathbf{R}^m$ be a closed ENR (Euclidean Neighbourhood Retract) and let $F : [0,T] \times C \to \mathbf{R}^m$ be a continuous function. The aim of this Chapter (which is based on [19]) is to prove the existence of a solution $x(\cdot)$ to the periodic BVP

$$\dot{x} = F(t,x) \tag{1.1}$$

$$x(0) = x(T) \tag{1.2}$$

such that, for all $t \in [0,T]$, $x(t)$ belongs to a certain subset of C.
In the Introduction, we have recalled the celebrated results for the case $C = \mathbf{R}^m$ by J. Mawhin and M.A. Krasnosel'skii (Theorems A and B) and some contributions obtained in the case when the underlying space has not a linear structure.
In this Chapter, an unifying result is provided, both regarding the methods and with respect to the properties required for the set C. Indeed, problem (1.1)-(1.2) is embedded in a functional-analytic framework and, at the same time, the properties of the translation operator are used. On the other hand, since regular manifolds, closed convex sets and conical shells are examples of ENRs, the result given below takes into account all the situations mentioned above.

In Section 2 the main result of this Chapter is proved (Theorem 1). Instead of the Brouwer degree, we use the index of rest points of an associated flow. As for the proof, the crucial point is based on the study of nonlinear operators in the space of continuous functions depending on some processes which are defined from (1.1). Corollaries 1 and 4 extend Theorems A and B (respectively) to flow-invariant ENRs.

In Section 3 Krasnosel'skii's method of guiding functions is adapted to the case of ENRs, and a computation of an analogue of the "index of non-degeneracy" for a potential function is performed (see [79, p.84]).

In Section 4 more concrete examples which illustrate the range of applicability of the above results are given; the first example (suggested by possible applications to hydrodynamics [76,128]) deals with a set which is a domain with holes; then, as a simple corollary, a result related to the Floquet problem is obtained. In the last example a periodicity theorem for the generalized hypercycle equation (see [41,73,134]) is performed. In this case, the set C is an $(m-1)$-dimensional symplex,and therefore all the theorems quoted in the introduction are not applicable (cf. Remark 10).

Finally, we point out that, without loss of generality, we can assume (if it is convenient) $F : [0,T]\times A \to \mathbf{R}^m$, with A any open set such that $C \subset A \subset \mathbf{R}^m$ (cf. Remark 2).

2. The main result

Throughout this Section, we suppose that $C \subset \mathbf{R}^m$ is an ENR.
As usual, we deal with the periodic boundary value problem

$$\dot{x} = F(t,x) \tag{2.1}$$

$$x(0) = x(T)\ , \tag{2.2}$$

where

$$F(t,x) := f(t,x;1)$$

and $f = f(t,x;\lambda) := [0,T]\times\mathbf{R}^m\times[0,1] \to \mathbf{R}^m$ is a continuous function which is locally lipschitzian in x. As it was mentioned in the Introduction, such assumption is not strictly necessary in our proofs, but it avoids the requirement of the uniqueness of the solutions to all the Cauchy problems which will be considered henceforth.
In what follows, we denote by Γ the complete metric space of the continuous functions $x(\cdot):[0,T]\to C$ endowed with the distance d^*, $d^*(x_1,x_2):=|x_1-x_2|_\infty$. From [75, p.186], we know that (Γ,d^*) is a (metric) ANR. We want to prove the existence of solutions to (2.1)-(2.2) belonging to certain subsets of Γ.
To this end, we produce a continuation theorem (on the line of [94,98]) involving the averaged system

$$\dot{x} = \bar{f}_0(x) \tag{2.3}$$

Observe that the map $\bar{f}_0$ is locally lipschitzian; accordingly, (2.3) induces a local dynamical system $\bar{\pi}$ with phase space $\mathbf{R}^m$. We also note that if the set C is positively invariant for $\dot{x} = f(t,x;0)$, then the same property is true for $\bar{\pi}$ (see Lemma 2 in Section 3).
We further remark that if $G \subset C$ is a bounded set, open relatively to C, such that

$$\bar{f}_0(x) \neq 0 \ \text{ for all } x \in \mathrm{fr}_C G \tag{2.4}$$

holds, then there is $\varepsilon_0 > 0$ such that the map $\bar{\pi}_\varepsilon : x \mapsto \bar{\pi}(\varepsilon,x)$ is fixed point free on $\mathrm{fr}_C G$, for all $0 < \varepsilon \leq \varepsilon_0$. Therefore, whenever C is positively invariant for $\bar{\pi}$ and (2.4) holds, the fixed point index $i_C(\bar{\pi}_\varepsilon,G)$ is defined and it is constant with respect to ε, for all $0 < \varepsilon \leq \varepsilon_0$. In this situation, according to Chapter 2, the index of rest points

$$I(\bar{\pi},G) := \lim_{\varepsilon\to 0^+} i_C(\bar{\pi}_\varepsilon , G) \tag{2.5}$$

is well defined.
For the computation of the index of rest points, we refer to Remark 5 in Section 4 of Chapter 1.

The main result of this Chapter is the following. Note that points of C are identified with constant functions.

THEOREM 1. *Assume*

(g1) C *is positively invariant for* $\dot{x} = f(t,x;\lambda)$, $\lambda\in [0,1]$.

Let $\Omega\subset\Gamma$ *be an open bounded set such that the following conditions are satisfied:*

(g2) *there is no* $x(\cdot)\in \mathrm{fr}\Gamma\Omega$, *with* $x(0) = x(T)$, *such that*

$$\dot{x} = \lambda f(t,x;\lambda), \quad \lambda\in]0,1[; \tag{2.1_λ}$$

(g3) $\bar{f}_0(z) \neq 0$ *for all* $z\in C\cap \mathrm{fr}_\Gamma\Omega$;

(g4) $I(\bar{\pi},\Omega\cap C) \neq 0$.

Then, (2.1)-(2.2) *has at least one solution* $x(\cdot)\in \mathrm{cl}_\Gamma\Omega$.

REMARK 1. Observe that in the particular case $C = \mathbf{R}^m$ assumption (g1) is trivially verified, while condition (g4) is equivalent to $d_B(\bar{f}_0,\Omega\cap\mathbf{R}^m,0) \neq 0$, so that we obtain [94, Th. 2]. Actually, in [94] the local lipschitzianity of f is not supposed; however, in the special case $C = \mathbf{R}^m$ we can relax such regularity assumption on f using a standard perturbation argument.

The following result is crucial for the proof.

LEMMA 1. *Assume* (g1). *Then, for each* $\alpha,\beta \geq 0$ *and* $0 \leq \lambda_i \leq 1$, $i=1,2$, C *is flow-invariant for* $\dot{x}= \alpha f(t,x;\lambda_1) + \beta\, \bar{f}_{\lambda_2}(x)$.

Proof. At first, we observe that the function $\alpha f(t,x;\lambda_1) + \beta \bar{f}_{\lambda_2}(x)$ is locally lipschitzian in x, so that the uniqueness for the solutions of the associated Cauchy problems is guaranteed. Recall that, by the characterization of flow-invariant sets in terms of tangent cones (see [2,25]), (g1) implies that $f(t,z;\lambda) \in T(z;C)$, for all $t \in [0,T]$, $z \in \mathrm{fr}C$ and $\lambda \in [0,1]$, where $T(z;C)$ is a suitable tangent cone to C at z. Without loss of generality (see [121, Th.3.9]), we can assume that $T(z;C)$ is closed and convex (for instance, the Clarke tangent cone can be chosen). Then, by the mean value theorem [2, p.21], $\bar{f}_\lambda(z) \in T(z;C)$ for all $z \in \mathrm{fr}C$ and $\lambda \in [0,1]$. Finally, the convexity and the cone property of $T(z;C)$ imply that

$\alpha f(t,z;\lambda_1) + \beta \bar{f}_{\lambda_2}(z) \in T(z;C)$.

The proof is complete. ♦

Proof of Theorem 2. At first, we prove our result under the supplementary assumption that there is a constant $A > 0$ such that

$$|f(t,x;\lambda)| \leq A \tag{2.6}$$

for all $t \in [0,T]$, $x \in \mathbf{R}^m$, $\lambda \in [0,1]$. The general situation will be examined at the end of the proof.
Without loss of generality, we also suppose that (g2) holds with $\lambda \in]0,1]$ in (2.1_λ) (otherwise, the result is already proved for $x \in \mathrm{fr}_\Gamma \Omega$).
We begin with some technical preliminaries.
Let $\varepsilon \in]0,T[$ be arbitrarily small but fixed. We define the following functions:

$$\theta(\lambda) := (\lambda T - \varepsilon) / (T - \varepsilon) \ , \qquad \varepsilon/T \leq \lambda \leq 1;$$

$$\phi(\theta,t) := [\varepsilon + \theta(T - \varepsilon)]\,(t / T) \ , \qquad 0 \leq \theta \leq 1, \ \ 0 \leq t \leq \theta;$$

$$g(s,y;\theta) := \begin{cases} f\big(sT/\phi(\theta,T),y;\lambda(\theta)\big) \ , & 0 \leq \theta \leq 1 \ , 0 \leq s \leq \phi(\theta,T) \ , \ y \in \mathbf{R}^m \ , \\ f(T,y;\lambda(\theta)) \ , & 0 \leq \theta \leq 1 \ , \ s > \phi(\theta,T) \ , \ y \in \mathbf{R}^m \ . \end{cases}$$

Observe that $g: \mathbf{R}_+ \times \mathbf{R}^m \times [0,1] \rightarrow \mathbf{R}^m$ is continuous and such that uniqueness and global existence for the associated Cauchy problems are guaranteed. Accordingly, if we denote by $u(\sigma,z,\cdot;\theta)$ the solution of

$$\begin{aligned} \dot{y} &= g(s,y;\theta) \\ y(\sigma) &= z \end{aligned} \tag{2.7}$$

then a one-parameter family of processes is defined. Using (g1), it can be easily checked that, for each $\theta \in [0,1]$, the set C is positively invariant for the corresponding process u. We further note that, since

$$\lambda = \lambda(\theta) = \phi(\theta,T)/T,$$

then the function y(s) is a solution of (2.7) for $s\in[0,\phi(\theta,T)]$ if and only if the function

$$x(t) := y(\phi(\theta,T)t/T) = y(\phi(\theta,t)) \tag{2.8}$$

is a solution of (2.1_λ) with $t\in[0,T]$.
The existence of solutions to (2.1)-(2.2) will be achieved producing a fixed point for a suitable operator defined on Γ. We will carry out this programme using the properties of the fixed point index for compact operators in metric ANRs (see [59]); more precisely, some admissible homotopies will be constructed.
As a first step, we introduce a nonlinear operator M defined on $\Gamma\times[0,1]$ as follows:

$$M(x,\theta) := u(0,x(T),\phi(\theta,\cdot);\theta) \ , \ \theta\in[0,1].$$

By the flow-invariance of C, $M:\Gamma\times[0,1]\to\Gamma$; moreover, by the Ascoli-Arzelà theorem, M is compact on $cl_\Gamma\Omega\times[0,1]$. Using the definition of u and (2.8), it is immediately seen that x is a fixed point of $M(\cdot,\theta)$ for some $\theta\in[0,1]$ if and only if x is a solution of (2.1_λ) with $\lambda\in[\varepsilon/T,1]$ and $x(0) = x(T)$. In particular, (2.1)-(2.2) is solvable if and only if $M(\cdot,1)$ has a fixed point. Hence, this claim and assumption (g2) imply that $M(x,\theta)\neq x$ for $x\in fr_\Gamma\Omega$ and $\theta\in[0,1]$. Therefore, M is an admissible homotopy and so

$$i_\Gamma(M(\cdot,1),\Omega) = i_\Gamma(M(\cdot,0),\Omega). \tag{2.9}$$

Secondly, we denote by $v(\sigma,z,\cdot;\mu)$ the solution of

$$\dot{y} = (1-\mu)\,\bar{f}_0(y) + \mu g(s,y;0)$$

$$y(\sigma) = z,$$

with $\mu\in[0,1]$.
As before, a one-parameter family of processes is defined. By Lemma 1 we have that, for each $\mu\in[0,1]$, the set C is flow-invariant for the corresponding process v as well. Now, we consider another nonlinear operator $N(x,\mu)$, defined on $\Gamma\times[0,1]$ as follows:

$$N(x,\mu) := v(0,x(T),\phi(0,\cdot);\mu).$$

Arguing as before, $N:\Gamma\times[0,1]\to\Gamma$ and it is compact on $cl_\Gamma\Omega\times[0,1]$. Moreover,

$$N(x,1) = M(x,0).$$

We want to prove that N is an admissible homotopy. To this end, we observe that x is a fixed point of $N(\cdot,\mu)$ if and only if $x(\cdot)$ is a solution of

$$\dot{x} = (\varepsilon/T)[(1-\mu)\,\bar{f}_0(x) + \mu f(t,x;\varepsilon/T)] \tag{2.10}$$

with $x(0) = x(T)$.

We claim that there is $\varepsilon_0 > 0$ (small enough) such that $N(x,\mu) \neq x$ for all $x \in fr_\Gamma\Omega$ and $\mu \in [0,1]$, provided that $\varepsilon \in]0,\varepsilon_0]$. (Recall that the function g and, consequently, the operator N depend on the constant ε chosen at the beginning of the proof). In fact, assume the contrary, i.e. that for each $n \in \mathbf{N}$ there are $\varepsilon_n \in [0,T]$ with $\lim \varepsilon_n = 0$, $\mu_n \in [0,1]$ and $x_n \in fr_\Gamma\Omega$ such that $N(x_n,\mu_n) = x_n$. Then, from (2.10) and (2.6) we have:

$$|\dot{x}_n|_\infty \leq (\varepsilon_n/T)A\,. \tag{2.11}$$

Moreover, as Ω is bounded there is a constant $R > 0$, independent of n, such that $|x_n|_\infty \leq R$. By the Ascoli-Arzelà theorem, we get that there is $x^* \in fr_\Gamma\Omega$ such that (up to a subsequence) $x_n(\cdot) \to x^*(\cdot)$ in the d^*-metric.

Clearly, $x^*(t) = \text{constant} = x^* \in C \cap fr_\Gamma\Omega$ (use (2.11)). We can also assume (passing, possibly, to a further subsequence) that $\lim \mu_n = \mu^* \in [0,1]$. Taking the mean value of (2.10) and dividing by (ε_n/T), we obtain, for each n,

$$0 = \Big[(1-\mu_n)\frac{1}{T}\int_0^T \bar{f}_0(x_n(t))dt \;+\; \mu_n\frac{1}{T}\int_0^T f(t,x_n(t);\varepsilon_n/T)dt\Big]\,.$$

Passing to the limit as $n \to +\infty$, we get

$$0 = (1-\mu^*)\;\bar{f}_0(x^*) + \mu^*\frac{1}{T}\int_0^T f(t,x^*;0)dt = \;\bar{f}_0(x^*)\;,\ \text{with}\;\; x^* \in fr_\Gamma\Omega\,.$$

Thus, a contradiction with (g3) is reached. Hence, the claim is proved and we can write:

$$i_\Gamma(M(\cdot,0),\Omega) = i_\Gamma(N(\cdot,1),\Omega) = i_\Gamma(N(\cdot,0),\Omega). \tag{2.12}$$

Finally, we define a third homotopy. Let $\bar{\pi}$: $\mathbf{R}\times\mathbf{R}^m \to \mathbf{R}^m$ be the dynamical system induced by $\dot{x} = \bar{f}_0(x)$ and observe that, with the notation introduced along the proof,

$$\bar{\pi}(t,z) = v(0,z,t;0).$$

By Lemma 1, C is positively invariant with respect to $\bar{\pi}$. A nonlinear operator H is defined on $\Gamma\times[0,1]$ as follows:

$$H(x,\beta) := \bar{\pi}((1-\beta)\varepsilon + \beta\phi(0,\cdot),x(T)).$$

As before, $H : \Gamma\times[0,1]\to\Gamma$ and it is compact on $cl_\Gamma\Omega\times[0,1]$. Moreover,

$$N(x,0) = H(x,1).$$

In this case, $x\in\Gamma$ is a fixed point of $H(\cdot,\beta)$ if and only if $x(t) \equiv y((1-\beta)\varepsilon + \beta\varepsilon(t/T))$, with $y:[0,\varepsilon]\to C$ an ε-periodic solution of

$$\dot{y} = \bar{f}_0(y)$$
$$y(0) = x(T).$$

We claim that there is $\varepsilon_1\in\,]0,\varepsilon_0]$ such that $H(x,\beta) \neq x$ for all $x\in fr_\Gamma\Omega$ and $\beta\in[0,1]$, provided that $\varepsilon\in\,]0,\varepsilon_1]$. Assume the contrary; then for each $n\in N$ there are $\varepsilon_n\in[0,T]$ with $\lim \varepsilon_n = 0$, $\beta_n\in[0,1]$ and $x_n\in fr_\Gamma\Omega$ such that $H(x_n,\beta_n) = x_n$. We consider the auxiliary functions $z_n\in\Gamma$ defined by $z_n(t):=y_n(\varepsilon_n t/T)$, where $\dot{y}_n = \bar{f}_0(y_n)$ and $y_n(0) = y_n(\varepsilon_n) = x_n(T)$. For such z_n we have:

$$\begin{aligned}\dot{z}_n &= (\varepsilon_n/T)\bar{f}_0(z_n),\\ z_n(0) &= z_n(T) = x_n(T)\end{aligned} \tag{2.13}$$

and

$$x_n(t) = z_n((1-\beta_n)T + \beta_n t). \tag{2.14}$$

Arguing as in the preceding claim, we easily get $|\dot{z}_n|_\infty \leq (\varepsilon_n/T)A$ and $|z_n(0)| \leq R$ (with $R > 0$ a suitable constant independent of n). Again, the Ascoli-Arzelà theorem implies that (passing, possibly, to subsequences) $z_n(\cdot)\to z^*(\cdot)$ in the d^*-metric, with $z^*(\cdot) \equiv z^* =$ constant and $\lim\beta_n = \beta^*\in[0,1]$. Furthermore by (2.14) $x_n(\cdot)\to z^*$ in the d^*-metric, with $z^*\in C\cap fr_\Gamma\Omega$. Taking the mean value of (2.13) and dividing by (ε_n/T) we get

$$\frac{1}{T}\int_0^T \bar{f}_0(z_n(t))dt = 0\,;$$

hence, passing to the limit as $n\to+\infty$, we have $\bar{f}_0(z^*) = 0$, with $z^*\in C\cap fr_\Gamma\Omega$ and a contradiction with (g3) is reached.

Therefore, the claim is proved and we can write:

$$i_\Gamma(N(\cdot,0),\Omega) = i_\Gamma(H(\cdot,1),\Omega) = i_\Gamma(H(\cdot,0),\Omega). \tag{2.15}$$

By definition of H, we have:

$$H(x,0) = \bar{\pi}(\varepsilon, x(T)) = \bar{\pi}_\varepsilon(x(T)),$$

where $\bar{\pi}_\varepsilon$ is the ε-Poincaré map ($0 < \varepsilon \leq \varepsilon_1$). Hence, since $H(\cdot,0):\Gamma \to C$, by the contraction property of the fixed point index (see Proposition 1 in Section 2 of Chapter 1) we have:

$$i_\Gamma(H(\cdot,0),\Omega) = i_C(H(\cdot,0),\Omega \cap C) = i_C(\bar{\pi}_\varepsilon,\Omega \cap C). \tag{2.16}$$

(Observe that, as a consequence of the last claim, $\bar{\pi}_\varepsilon$ is fixed point free on $fr_C(\Omega \cap C) \subset C \cap fr_\Gamma \Omega$).

In conclusion, we have proved that, via (2.9), (2.12), (2.15) and (2.16), the integer $i_C(M(\cdot,1),\Omega) = i_C(\bar{\pi}_\varepsilon,\Omega \cap C)$ is constant with respect to ε, for $\varepsilon > 0$ small enough.

Then,

$$i_\Gamma(M(\cdot,1),\Omega) = \lim_{\varepsilon \to 0^+} i_C(\bar{\pi}_\varepsilon,\Omega \cap C) = I(\bar{\pi},\Omega \cap C) .$$

Assumption (g4) provides (see Chapter 1) the existence of a fixed point $x \in \Omega$ of $M(\cdot,1)$. Therefore, the conclusion is established.

In the case when (2.6) is not satisfied, the proof can be repeated for the equation

$$\dot{x} = f(t,x;1)\cdot\rho(|x|), \tag{2.17}$$

where $\rho:\mathbf{R}_+ \to [0,1]$ is lipschitzian and such that $\rho(x) = 1$ for $|x| \leq R$, $\rho(x) = 0$ for $|x| \geq 2R$ and $cl_\Gamma \Omega \subset B(0,R)$.

Of course, the local flow $\bar{\pi}$ induced by (2.3) coincides with the flow induced by $\dot{x} = \bar{f}_0(x)\rho(|x|)$ in a neighbourhood of $\Omega \cap C$ and, moreover, any solution of (2.17)-(2.2) such that $x \in cl_\Gamma \Omega$ is also a solution of (2.1)-(2.2).

The proof is complete. ♦

REMARK 2. As we mentioned in Section 5 of Chapter 1, the flow-invariance condition (g1) may be stated in an equivalent geometrical manner using tangent cones. Indeed, (g1) holds if and only if

(h1) $f(t,z;\lambda) \in T(z;C)$, *for all* $t \in [0,T]$, $z \in frC$, $\lambda \in [0,1]$

is satisfied (see [2,25]). Hence, if $f(t,x;\lambda)$ is defined only for $x \in C$, then (h1) ensures that all the processes considered in the proof of Theorem 1 are defined.

A standard situation in which the function $f(t,\cdot;\lambda)$ is defined just on the set C occurs, for example, when C is a regular manifold; in this case, $f(t,z;\lambda)\in A\cap -A$, where $A = T(z;C)$, whenever f is a tangent vector field and so (h1) holds. Accordingly, our result is general enough to be applied to the setting considered in [11,52,53].
If C is a convex set like [18], then (h1) reduces to $(f(t,z;\lambda)|\eta) \leq 0$ for each $\eta\in N(z,C)$.

We end this Section with the following

REMARK 3. It is possible to obtain a variant of Theorem 1 assuming, besides (g2) and (g3), the following conditions which replace (g1) and (g4):

(g_1^-) C *is negatively invariant for* $\dot{x}=f(t,x;\lambda)$, $\lambda\in[0,1]$;

(g_4^-) $\lim_{\varepsilon\to 0^-} i_C(\bar{\pi}_\varepsilon,\Omega\cap C) \neq 0$.

This can be accomplished by the standard change of variables $t\longmapsto T-t$ which transforms equation (2.1) into $\dot{x}=-f(s,x;1)$, where $s=T-t$.
Observe that if, furthermore, the critical set

$$\mathcal{Z}= \{z\in C: \bar{f}_0(z) = 0\}$$

is compact and $\Omega\cap C\supset\mathcal{Z}$, then $\lim_{\varepsilon\to 0^-} i_C(\bar{\pi}_\varepsilon,\Omega\cap C)$ is exactly $\chi(\bar{f}_0)$, the "characteristic of the vector field" $\bar{f}_0$ defined in [53]. It is also clear that (g_1^-) is equivalent to

(h_1^-) $f(t,z;\lambda)\in -T(z;C)$, *for all* $t\in[0,T]$, $z\in \mathrm{fr}C$, $\lambda\in[0,1]$.

so that the situation considered in [53] fits our hypotheses (see also Remark 5 below).

Now, we present, as immediate corollaries of Theorem 1, an extension to ENRs of two classical results of existence of solutions for the periodic problem (2.1)-(2.2). Namely, only a (suitable) different choice of the set $\Omega\subset\Gamma$ is needed.
In what follows, $G\subset C$ denotes a bounded set which is open relatively to C. Observe that $\mathrm{cl}_C G = \mathrm{cl}G$.
As a first application, in the lines of theorem A we can prove the following:

COROLLARY 1. *Assume* (g1) *and suppose that the following conditions are satisfied*:

(h2) *for any* $x(\cdot)$, *solution of* (2.1_λ)-(2.2) *such that* $x(t)\in \mathrm{cl}G$ *for all* $t\in[0,T]$, *it follows that* $x(t)\in G$ *for all* $t\in[0,T]$;

(h3) $\bar{f}_0(z) \neq 0$ *for all* $z\in \mathrm{fr}_C G$;

(h4) $I(\bar{\pi},G) \neq 0$.

Then, (2.1)-(2.2) *has at least one solution* $x(\cdot)$ *such that* $x(t)\in \mathrm{cl}G$, *for all* $t\in[0,T]$.

Proof. In the setting of Theorem 1 we define:

$$\Omega = \{x\in\Gamma: x(t)\in G,\ \forall t\in[0,T]\}.$$

It can be checked that Ω is bounded and open relatively to Γ.
Furthermore, the following facts hold true:

$$\Omega\cap G = G;$$

$$\mathrm{cl}_\Gamma\Omega \subset \{x\in\Gamma: x(t)\in \mathrm{cl}G,\ \forall t\in[0,T]\}; \qquad \mathrm{fr}_\Gamma\Omega \subset \{x\in\Gamma: x(t)\in \mathrm{cl}G,\ \forall t \text{ and } \exists t_0 \text{ with } x(t_0)\in \mathrm{fr}_C G\}.$$

Hence, (h2) and (h4) imply (g2) and (g4), respectively. Finally, (g3) follows from (h3) since

$$\mathrm{fr}_C G \subset C\cap \mathrm{fr}_\Gamma\Omega.$$

Therefore, Theorem 1 applies and the proof is complete. ♦

REMARK 4. Hypothesis (h2) is a transversality condition at boundary points as considered in theorem 3.1. However, in (h2) not all the boundary is concerned, but only points of $\mathrm{fr}_C G$ are taken into account. This advantage is balanced by a weak boundary condition which is implicitly required in (g1). As we already observed in the previous Section, the flow-invariance assumption (g1) is equivalent to the cone condition (h1). However, since in Corollary 4.3 we study solutions lying in $\mathrm{cl}_C G$, we realize that it is possible to obtain a slight improvement of Corollary 1 by relaxing (h1). Namely, we have:

COROLLARY 1'. *Besides* (h2), (h3), (h4), *assume*

(h_1') $f(t,z;\lambda)\in T(z;C)$

for all $t \in [0,T]$, $z \in \mathrm{fr}C \cap \mathrm{cl}G$, $\lambda \in [0,1]$.
Then, the same conclusion of Corollary 1 *holds.*

The proof of this result can be achieved via a standard perturbation argument based on the Ascoli-Arzelà theorem (see [44] for an analogous situation).
In the particular case when C is a closed convex set with nonempty interior, Corollary 1' can be seen as a consequence of [18, Th. 1].

A simple application of Corollary 1 is based on the fact that assumption (h2) is fulfilled whenever a priori bounds for the solutions of (2.1_λ)-(2.2) can be produced. Accordingly, we have (recall Remark 3):

COROLLARY 2. *Assume that, for all* $t \in [0,T]$, $z \in \mathrm{fr}C$ *and* $\lambda \in [0,1]$,

$$f(t,z;\lambda) \in T(z;C) \quad (\textit{respectively}, \ f(t,z;\lambda) \in -T(z;C)).$$

Suppose that there is a compact set $K \subset C$ *containing all the solutions of* (2.1_λ)-(2.2) *and such that* $\{z \in C: \bar{f}_0(z) = 0\} \subset K$. *Let* $G \subset C$ *be a bounded set, open relatively to* C, *such that* $K \subset G$ *and suppose that*

$$\lim_{\varepsilon \to 0^+} i_C(\bar{\pi}_\varepsilon, G) \neq 0 \quad (\textit{respectively}, \ \lim_{\varepsilon \to 0^-} i_C(\bar{\pi}_\varepsilon, G) \neq 0) .$$

Then, (2.1)-(2.2) *has at least one solution with values in* K.

Observe that, by the excision property of the fixed point index, the limits $\lim_{\varepsilon \to 0^\pm} i_C(\bar{\pi}_\varepsilon, G) \neq 0$ are independent of the choice of $G \supset K$. The above proposition clearly contains [53, Th. 2.4]; in fact, according to the notations introduced in [53], $\lim_{\varepsilon \to 0^-} i_C(\bar{\pi}_\varepsilon, G) = \chi(\bar{f}_0)$.
We note that there is no loss of generality, in our setting, if we take $K = B[0,R_0] \cap C$ and $G = B(0,R) \cap C$ for any $R > R_0$. In this way, we obtain a generalization to arbitrary ENRs of an useful principle due to Mawhin [94, Th.4]. Finally, we remark that Corollary 2 is suitable for C non-compact. Indeed, if C is compact then we can choose $K = G = C$ and $f(t,x;\lambda) \equiv F(t,x)$. Accordingly, Corollary 2 recovers a classical result on the existence of periodic orbits in compact positively (negatively) invariant ENRs with non-zero Euler characteristic (cf. Poincaré-Hopf theorem).

As a second application, by means of another choice of the set Ω in Theorem 1 we prove two corollaries of our main result which are in the lines of the well-known Krasnosel'skii theorem [79, Th.6.1].

COROLLARY 3. *Besides* (g1), (h3) *and* (h4), *assume further*

(h_2') *there is no solution of* (2.1_λ)-(2.2) *with* $x(0)\in fr_C G$;

(h5) $|f(t,x;\lambda)| \leq A|x| + B$, *for all* $t\in [0,T]$, $x\in C$, $\lambda\in [0,1]$.

Then, (2.1)-(2.2) *has at least one solution* $x(\cdot)$ *such that* $x(0)\in cl_C G$.

Proof. First of all, we note that (h5) ensures the global existence for all the Cauchy problems associated to (2.1_λ) with initial values in C.
Then, there is a constant $R > 0$, independent of λ, such that $|x|_\infty < R$ for every $x(\cdot)$ solution of (2.1_λ) with $x(0)\in clG$.
In this situation, the appropriate definition of the set Ω (in order to apply Theorem 1) is the following:

$$\Omega := \{x\in\Gamma\colon x(0)\in G,\ |x|_\infty < R\}.$$

Obviously, Ω is bounded and open relatively to Γ. Observe that $\Omega\cap C = G$ and $fr_\Gamma\Omega \subset \{x\in\Gamma\colon x(0)\in fr_C G,\ |x|_\infty \leq R\}\cup\{x\in\Gamma\colon x(0)\in clG,\ |x|_\infty = R\}$. Then, by the choice of R, it is immediately seen that (h_2') implies (g2). Since $fr_C G \subset C\cap fr_\Gamma\Omega$, arguing like in the proof of the previous corollary, from (h3) and (h4) we obtain (g3) and (g4), respectively. Then we can apply Theorem 1 and the proof is complete. ♦

As a consequence of Corollary 3 we immediately get an extension of Krasnosel'skii's theorem to arbitrary ENRs. Precisely, we consider the equation:

$$\dot{x} = g(t,x) \tag{2.18}$$

with $g : [0,T]\times\mathbf{R}^m\to\mathbf{R}^m$ continuous, locally lipschitzian in x and such that the (forward) global existence for the solutions of the associated Cauchy problems with initial values in C is guaranteed. Then, we have:

COROLLARY 4. *Suppose that the following conditions are satisfied:*

(k1) C *is positively invariant for equation* (2.18);

(k2) *there is no solution* $x(\cdot)$ *of* (2.18) *such that* $x(0) = x(k)\in fr_C G$,
for some $0 < k < T$;

(k3) $g(0,z) \neq 0$ *for* $z\in fr_C G$.

Let π^0 *be the (local) flow induced by* $\dot{x} = g(0,x)$ *and assume:*

(k4) $I(\pi^0,G) \neq 0$.

Then, (2.18)-(2.2) *has at least one solution* $x(\cdot)$ *with* $x(0)\in clG$.

According to Krasnosel'skii's terminology, assumption (k2) means that the points of $fr_C G$ are points of "T-irreversibility".

Proof. By the global existence, there is a constant $R > 0$ such that $|x|_\infty < R$ for every $x(\cdot)$ solution of (2.18) with $x(0)\in clG$. Let $\rho:R^m\to[0,1]$ be a locally lipschitzian function such that $\rho(x) = 1$ for $|x| \leq R$ and $\rho(x) = 0$ for $|x| \geq 2R$.
Now we define, for $\lambda\in[0,1]$, $f(t,x;\lambda):=\rho(x)g(\lambda t,x)$ and observe that (by the choice of R, $\rho(\cdot)$) $x\in\Gamma$ is a solution of (2.1_λ)-(2.2) with $x(0)\in fr_C G$ if and only if $y(t):=x(t/\lambda)$ is a solution of $\dot{y} = g(t,y)$ with $y(0) = y(\lambda T)\in fr_C G$. Then, (h_2') follows from (k2). We also remark that (k1) implies (g1) and (k3), (k4) are nothing but (h3), (h4) respectively. Finally, (h5) is fulfilled with $A = 0$ and $B = \sup\{|g(t,x)|,\ t\in[0,T],\ x\in C,\ |x|\leq R\}$. Then, Corollary 3 applies and the result is achieved. ♦

An analogous result was obtained by R. Srzednicki in [137, Th. 1], whenever $g:R\times R^m\to R^m$ is T-periodic in the t-variable. More precisely, according to the notations of [137], let us set P:= clG and let p denote the process induced by (2.18). Then, the same conclusion of Corollary 4 holds provided that, instead of (k2), we assume the existence of a closed subset P^- of P such that

$$P^-\times R = \{(z,\sigma)\in P\times R : \exists\ (\varepsilon_n)\downarrow 0,\ p(\sigma,z,\varepsilon_n)\notin P\}.$$

However, we point out that Corollary 4 is not contained in [137], as easy examples show (see [13,129]). Indeed, it is sufficient to find a set P which is a " for Ważewski set" for π^0 but not for p.

REMARK 5. Straightforward variants of Corollaries 2 and 3 may be easily obtained following Remarks 2 and 3. In particular, the case of C negatively invariant may be treated as well.
The above result clearly generalizes [79, Th.6.1]; in the special case in which C is a manifold and g is a tangent vector field Corollary 4 reduces to an existence result following from Furi and Pera bifurcation theorem [53, Th. 2.2].

3. On the "index of nondegeneracy"

Let $C\subset R^m$ be a closed ENR. We consider the equation

$$\dot{x} = g(t,x), \tag{3.1}$$

with $g : [0,T]\times\mathbf{R}^m \to \mathbf{R}^m$ continuous, locally lipschitzian in x and such that the global existence for the solutions of the associated Cauchy problems, with initial values in C, is guaranteed. Our aim is to obtain a consequence of Corollary 4 in which the transversality condition (k2) follows by means of some explicit geometrical hypotheses on the vector field g. More precisely, we examine an extension to ENRs of the concept of guiding function (see[79,81]).
Now, we introduce the concept of guiding function relatively to the set C.

DEFINITION 1. *Let* $\Phi:\mathbf{R}^m \to \mathbf{R}^m$ *be a continuously differentiable function with* $\nabla\Phi$ *locally lipschitzian on* C. *We say that* Φ *is a guiding function for the equation* (3.1) *relatively to* C *if there is* $R_0 > 0$ *such that* $B(0,R_0)\cap C \neq \emptyset$ *(to avoid trivialities) and*

$$(\nabla\Phi(x) \mid g(t,x)) > 0 \tag{3.2}$$

for all $t\in[0,T]$, $x\in C$ *and* $|x| \geq R_0$.

In particular, it follows that $\{x\in C:\nabla\Phi(x) = 0\} \subset B(0,R_0)\cap C$.
We confine ourselves to guiding functions satisfying the additional condition:

(Φ1) C *is positively invariant for*

$$\dot{x} = \nabla\Phi(x). \tag{3.3}$$

Then, if we denote by π^Φ the (local) flow induced by (3.3), we have that, by (3.2) and (ϕ1), the index of rest points $I(\pi^\Phi,B(0,R_0)\cap C)$ is defined for any $R \geq R_0$ (see Section 4 of Chapter 1) and it is constant with respect to $R \geq R_0$ by the excision property. Hence, the integer

$$J_C(\Phi,\infty) := \lim_{R\to+\infty} I(\pi^\phi,B(0,R)\cap C) \tag{3.4}$$

is well defined.

REMARK 6. Up to now, we have just followed, verbatim, the corresponding definition of guiding function in $\mathbf{R}^m$ given by Krasnosel'skii ([79, § 6.3]), modulo the natural modifications due to the more general setting. Now, we explain the meaning of (3.4) in some particular cases. If $C = \mathbf{R}^m$, then (Φ1) is vacuously satisfied and

$$J_C(\Phi,\infty) = (-1)^m\,\gamma(\Phi,\infty),$$

where γ is the "index of non-degeneracy" of Φ, according to [79, p.84]).
If C is a (regular) manifold, it turns out that

$$J_C(\Phi,\infty) = \chi(-\nabla\Phi),$$

where $\chi(-\nabla\Phi)$ is the characteristic of the (tangent) vector field $-\nabla\Phi$, according to [53, p.325].
If C is compact, then

$$J_C(\Phi,\infty) = I(\pi^\Phi,C) = \chi(C), \tag{3.5}$$

where $\chi(C)$ is the Euler-Poincaré characteristic of C.

For the proof of the next theorem, we need a preliminary result relating homotopic fields with the indexes of the corresponding flows.
Let $h = h(x;\lambda):\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ be continuous and such that, for each $\lambda\in[0,1]$, the solutions for the Cauchy problems

$$\begin{cases} \dot{x} = F(t,x) & (3.6) \\ x(t_0) = x_0 & (3.7) \end{cases}$$

are unique. We denote by π^λ the local flow induced by (3.6).
Then we have, for G as in Section 2:

LEMMA 2. *Let* G *be a bounded subset of* $\mathbf{R}^m$, *open relatively to* C. *Assume that, for each* $\lambda\in[0,1]$, C *is positively invariant with respect to equation* (3.6). *If*

(L1) $h(x;\lambda) \neq 0$

holds for all $x\in \mathrm{fr}_C G$ *and* $\lambda\in[0,1]$, *then*

$$I(\pi^0,G) = I(\pi^1,G). \tag{3.8}$$

Particular cases of this result have been already examined in [53]. For the reader's convenience, we give the complete proof in the general situation.

Proof. We set

$$\eta := \inf\{|h(z;\lambda)|: z \in fr_C G, \lambda \in [0,1]\};$$

by (L1), $\eta > 0$. We define $x(t,z;\lambda)$ to be the solution of (3.6)-(3.7) and observe that, according to the notations previously introduced:

$$x(\varepsilon,z;\lambda) = \pi_\varepsilon^\lambda(z).$$

First of all, we note that there is $K > 0$ such that $x(\cdot)$ is defined on $[0,K]$, for each $z \in fr_C G$ and $\lambda \in [0,1]$. Then, the set

$$\mathcal{B} = \{x(t,z;\lambda): t \in [0,K], z \in fr_C G, \lambda \in [0,1]\}$$

is a compact subset of C.

Finally, let $M > 0$ be such that

$$|h(w;\lambda)| \leq M$$

for each $w \in \mathcal{B}$ and $\lambda \in [0,1]$.

Fix ε_0 such that $0 < \varepsilon_0 < K$. Then, for any $\varepsilon \in]0,\varepsilon_0]$, we have:

$$x(\varepsilon,z;\lambda) - z = \varepsilon \int_0^1 h(x(\theta\varepsilon,z;\lambda);\lambda)\, d\theta$$

$$= \varepsilon \int_0^1 \left[h(x(\theta\varepsilon,z;\lambda);\lambda) - h(z;\lambda)\right] d\theta + \varepsilon h(z;\lambda)\,.$$

Since

$$|x(\theta\varepsilon,z;\lambda) - z| \leq \varepsilon_0 M$$

for every $\theta \in [0,1]$, $\varepsilon \in]0,\varepsilon_0]$, $z \in fr_C G$ and $\lambda \in [0,1]$, by the uniform continuity of h on $\mathcal{B}\times[0,1]$ we have:

$$|h(x(\theta\varepsilon,z;\lambda);\lambda) - h(z;\lambda)| < \eta/2$$

for ε_0 small enough. Hence, we obtain:

$$(1/\varepsilon)|x(\varepsilon,z;\lambda) - z| \geq |h(z;\lambda)| - \eta/2 \geq \eta/2$$

for all $z \in fr_C G$, $\lambda \in [0,1]$, $\varepsilon \in]0,\varepsilon_0]$.

Then, we have proved that

$$i_C(x(\varepsilon,\cdot;\lambda),G) = \text{constant}$$

for all $\lambda \in [0,1]$, $\varepsilon \in]0,\varepsilon_0]$.
Therefore, (3.8) follows immediately. ♦

Now we are in position to state the main result of this Section. As before, we denote by Γ the complete metric space of the continuous functions $x(\cdot) : [0,T] \to C$ endowed with the distance d^*, $d^*(x_1,x_2) := |x_1-x_2|_\infty$.

THEOREM 2. *Let* Φ *be a guiding function for equation* (3.1) *relatively to* C *and suppose that* C *is positively invariant for* (3.1) *and* (3.3).
Then, there is a solution $x(\cdot) \in \Gamma$ *to* (3.1)-(1.2) *(i.e. a* T*–periodic solution), provided that*

(Φ2) $\quad J_C(\Phi,\infty) \neq 0.$

Proof. We apply Corollary 4 with respect to the set $G = B(0,R) \cap C$, where $R > R^*$, and $R^* := \sup\{x(t):\ t,t_0 \in [0,T],\ \dot{x} = g(t,x),\ x(t_0) \leq R_0\}$. Then, (k2) and (k3) follow from the definition of guiding function, arguing like in [81, p.48]. Finally, we observe that (3.2) implies that the function $h(x;\lambda) := (1-\lambda)g(0,x) + \lambda\nabla\Phi(x)$ satisfies (L1) of Lemma 2, so that

$$I(\bar{\pi},G) = I(\pi^0,G) = I(\pi^1,G) = J_C(\Phi,\infty)$$

Then, (Φ2) implies (k4) and the proof is complete. ♦

Clearly, Theorem 2 is an extension of Krasnosel'skii's result [79, Th.6.5], [81, Th.13.1] to the case of a flow-invariant ENR. In [79,81], various criteria are proposed in order to evaluate $\gamma(\Phi,\infty)$ for $C = R^m$. In particular, it is proved that

$$\gamma(\Phi,\infty) = (-1)^m, \text{ for } \lim_{|x| \to +\infty} \Phi(x) = -\infty$$

and

$$\gamma(\Phi,\infty) = 1, \text{ for } \lim_{|x| \to +\infty} \Phi(x) = +\infty.$$

On the same line and combining arguments from [79,137], we can prove an analogous result for $J_C(\Phi,\infty)$.

LEMMA 3. *Let* Φ *be a guiding function relatively to* C, *verifying* (Φ1) *and*

$$(\Phi 3) \qquad \lim_{\substack{|x| \to +\infty \\ x \in C}} \Phi(x) = -\infty .$$

Then, C *is of finite type and* $J_C(\Phi,\infty) = \chi(C)$.

Essentially, this result follows from (vii) in Proposition 2. However, according to our hypotheses, equation (3.3) does not induce a dynamical system as required in [137] but just a (local) semi-flow on C. Thus, we give the details of the proof for the reader's convenience.

Proof. Obviously, if C is bounded then (Φ3) is vacuously satisfied and Remark 6 immediately gives the result. Hence, we consider the case of C unbounded.
First, we observe that

$$\nabla\Phi(x) \neq 0 \tag{3.9}$$

for all $x \in C$, $|x| \geq R_0$. Now, we fix $c^* \in \mathbf{R}$ such that

$$c^* \leq \inf\{\Phi(x): x \in C, |x| \leq R_0\}.$$

Then, for any $c \leq c^*$, we consider the sets:

$$K_c := \{x \in C: \Phi(x) \geq c\},$$
$$L_c := \{x \in C: \Phi(x) = c\},$$
$$M_c := \{x \in C: \Phi(x) \leq c\}.$$

By (Φ1), (Φ3) and (3.9), it follows that for every $c \leq c^*$, we have: K_c is compact and flow-invariant for (3.3), $L_c = fr_C K_c = fr_C M_c$ and each $x \in L_c$ is a strict egress point for M_c (according to Ważewski [146]). Let $x \in M_{c^*}$; we want to show that there is $t_x \geq 0$ such that $\pi^\Phi(t_x,x) \in L_{c^*}$. Indeed, let us assume $\Phi(x) = c < c^*$; then, there is $\eta > 0$ such that $|\nabla\Phi(y)| \geq \eta$ for every $y \in K_c \setminus K_{c^*}$. Following [79, Lemma 6.5], the function $\phi(t) := \Phi(\pi^\Phi(t,x))$ is such that $\phi(t) \geq c$ for all $t \geq 0$ and $\dot{\phi}(t) \geq \eta^2$ for all $t \geq 0$ such that $\pi^\Phi(t,x) \in K_c \setminus K_{c^*}$. Then, arguing by contradiction, it can be seen that the solution of (3.3) with initial value x meets L_c at a time t_x, with $t_x \leq (c^* - c)/\eta^2$. Note that such t_x is unique. If $x \in L_{c^*}$, the claim follows with $t_x = 0$.
By Ważewski's Lemma, we know that the map $x \mapsto t_x$, $x \in M_{c^*}$, is continuous (see [24,146]) and so K_{c^*} is a strong deformation retract of C via the homotopy

$$(x,\lambda) \mapsto \pi^\Phi(\lambda t_x,x), \quad x \in M_{c^*}$$
$$(x,\lambda) \mapsto x, \qquad\qquad x \in K_{c^*}.$$

Then, K_{c^*} is a compact ENR and C and K_{c^*} have the same homotopy type. Accordingly, C is of finite type and

$$\chi(C) = \chi(K_{c^*}) = \chi(K_c)$$

for every $c \leq c^*$.

Now, it is clear that $\pi^\Phi(t,x) \neq x$ for every $x \in M_{c^*}$ and $t > 0$ (see also the proof of Theorem 2); hence, for any $\varepsilon > 0$, $i_C(\pi^\Phi_\varepsilon, B(0,R) \cap C)$ is defined whenever $B(0,R) \supset K_{c^*}$. Fix such an R and let $c \leq c^*$ be such that $K_c \supset B[0,R] \cap C$. Then, using the excision and contraction properties of the fixed point index, we can write:

$$i_C(\pi^\Phi_\varepsilon, B(0,R) \cap C) = i_C(\pi^\Phi_\varepsilon, \mathrm{int}_C K_c) = i_{K_C}(\pi^\Phi_\varepsilon, K_c).$$

On the other hand, π^Φ_ε is homotopic to the identity Id on K_c (moving the points along the semi-orbits); consequently,

$$i_{K_C}(\pi^\Phi_\varepsilon, K_c) = i_{K_C}(\mathrm{Id}, K_c) = \chi(K_c)\ .$$

From the above inequalities, letting $\varepsilon \to 0^+$, $R \to +\infty$ and recalling the definition of $J_C(\Phi,\infty)$, we have the conclusion. ♦

REMARK 7. A simple application of Lemma 3 can be performed when C is convex and flow-invariant with respect to (3.3). Indeed, in such a case $\chi(C) = 1$ and so (Φ2) holds. We notice that, even in this simple situation, the validity of (Φ2) is not ensured if

$$\lim_{\substack{|x| \to +\infty \\ x \in C}} \Phi(x) = +\infty \tag{3.10}$$

is assumed instead of (Φ3). For instance, it is easy to prove that when $C \setminus B(0,R)$ (R large enough) is contractible, then (Φ1) and (3.10) imply $J_C(\Phi,\infty) = 0$.

We finally notice that in the proof of Lemma 3 we have shown that the flow π^Φ is dissipative; indeed, one can prove (suitably modifying the arguments in [74,137]) that $I(\Pi, B(0,R) \cap C) = \chi(C)$, for $R > 0$ large, whenever Π is a dissipative semi-flow on C.

If, furthermore, $g: R_+ \times R^m \to R^m$ is T-periodic in the t-variable, then, with a few changes in the proof of Lemma 3, one can also show that the process induced by (3.1) is dissipative on C, provided that (Φ1) and (Φ3) are satisfied. Hence, in such a particular case the existence of an T-periodic solution may be obtained using some extensions to ENRs of the known theorems for periodic dissipative processes (see [65, Ch.4]).

4. Applications

In this Section we present some applications of the previous results to the periodic BVP

$$\dot{x} = F(t,x) \tag{4.1}$$

$$x(0) = x(T) \tag{4.2}$$

with $F : \mathbf{R}_+ \times D \to \mathbf{R}^m$ continuous, T-periodic in the first variable and D a subset of $\mathbf{R}^m$. Obviously, in order to make problem (4.1)-(4.2) meaningful we shall further require that $x(t) \in D$, for all $t \in [0,T]$. We recall that the solutions of (4.1)-(4.2) may be extended to $\mathbf{R}_+$ as classical T-periodic solutions.
We examine two cases in which Corollary 1 may be applied. In both examples, the choice of the set $C \subset D$ will be suggested by the nature of the model represented by equation (4.1) and by the interest of finding solutions satisfying some particular properties.
In order to apply Corollary 1, we suppose that the function F is locally lipschitzian in x. However, we stress the fact that all the results contained in this Section are still true even if F is just continuous (use standard perturbation arguments).

EXAMPLE 1. We examine the case in which it is natural to choose C as a domain with holes. Such situation occurs, for instance, in hydrodynamics applications (see [76,128]); for instance, F may denote the velocity field of the flow and $x = x(\mathbf{a},t)$ the position vector at various times t of the "element" of fluid identified by the label **a**.
We are interested in the case in which

$$D = \mathcal{M} \setminus P,$$

where $\mathcal{M} \subset \mathbf{R}^m$ is a regular manifold and $P \subset \mathcal{M}$ is a compact set. Along the lines of the Poincaré-Hopf Theorem, we want to prove the existence of solutions to (4.1)-(4.2) using topological properties of the set D, combined with suitable assumptions on the tangent vector field F. For simplicity, and in view of the next application (Corollary 5), we confine ourselves to the simple case

$$\begin{aligned} P &= \{x_1,\ldots,x_n\} \\ \mathcal{M} &= S(0,R). \end{aligned} \tag{4.3}$$

In this situation, we assume

$$(F(t,x) \mid x) = 0, \quad \textit{for} \quad t \in [0,T],\ |x| = R,\ x \notin P, \tag{4.4}$$

so that F is a vector field tangent to the set D. It is clear that (4.4) ensures (via Nagumo's Theorem) the local existence for the solutions of (4.1) with initial values in $\mathcal{M} \setminus P$.
This case has some independent interest also in view of the study of homogeneous vector fields with singularities (see Corollary 5 below).
For P like in (4.3), we define:

$$\delta := \min\{|x_i - x_j| : i,j = 1,\dots,n.\ i \neq j\}.$$

Then, we have:

PROPOSITION 1. *Let* $A \cup B = \{1,\dots,n\}$, $A \cap B = \emptyset$, *and* $0 < \varepsilon < \min\{\delta/2, R/2\}$ *be such that*

$$(F(t,x) \mid x - x_i) \geq 0 \quad \textit{for} \ i \in A \tag{4.5}$$

$$(F(t,x) \mid x - x_i) \leq 0 \quad \textit{for} \ i \in B \tag{4.6}$$

for all $t \in [0,T]$, $x \in \mathcal{M}$ *and* $|x - x_i| = \varepsilon$.
Then, (4.1)-(4.2) *has at least one solution* $x(\cdot)$ *with* $x(t) \in D$ *for all* t, *provided that one of the following conditions holds:*

$$m \ \textit{even}, \quad \text{card}B \neq \text{card}A \tag{4.7}$$

$$m \ \textit{odd}, \quad n \neq 2. \tag{4.8}$$

Roughly speaking, our hypotheses mean that the flow enters in the holes surrounding x_i, $i \in B$, and escapes from the holes around x_i, $i \in A$. In such a situation, we only need conditions on the number of such holes.
Our example is a generalization of a similar one considered in [52, p.169], where $\mathcal{M} = \mathbf{R}^3$ and $n = \text{card}B = 1$.

Proof. We apply Corollary 1. First of all, we note that there exist two lipschitzian functions

$$k : \mathcal{M} \to \mathbf{R}^m \quad , \quad \rho : \mathcal{M} \to [0,1]$$

such that

$$k(x) = \begin{cases} -x_i + R^{-2}(x|x_i)x, & \text{for } i\in A,\ |x - x_i| = \varepsilon \\ x_i - R^{-2}(x|x_i)x, & \text{for } i\in B,\ |x - x_i| = \varepsilon \end{cases}$$

$$\rho(x) = \begin{cases} 0, & \text{if } \exists i\in B : |x - x_i| \le \varepsilon/2, \\ 1, & \text{if } \forall i\in B\ |x - x_i| \ge \varepsilon. \end{cases}$$

Then, we define:

$$C := \mathcal{M}\setminus\Big(\bigcup_{i\in A} B(x_i,\varepsilon) \cup \bigcup_{i\in B} B(x_i,\varepsilon/2)\Big)$$

$$G := C\setminus\big(\bigcup_{i\in B} B[x_i,\varepsilon]\big)$$

and

$$f(t,x;\lambda) := \rho(x)(\lambda F(t,x) + (1-\lambda)k(x)).$$

We observe that, using (4.4), (4.5) and the definitions of k and ρ:

$$(f(t,x;\lambda) \mid x) = 0, \qquad \text{for } x\in \mathcal{M}\setminus P \tag{4.9}$$

$$(f(t,x;\lambda) \mid x - x_i) \ge 0, \quad \text{for } i\in A,\ |x - x_i| = \varepsilon,\ x\in \mathcal{M} \tag{4.10}$$

$$(f(t,x;\lambda) \mid x - x_i) = 0, \quad \text{for } i\in B,\ |x - x_i| = \varepsilon/2,\ x\in \mathcal{M} \tag{4.11}$$

hold for all $t\in [0,T]$, $\lambda\in [0,1]$.
Then, (4.9),(4.10) and (4.11) imply that the set C is positively invariant and (j1) is satisfied (see also Remark 2).
We note that

$$fr_C G = \mathcal{M}\cap \big(\bigcup_{i\in B} S(x_i,\varepsilon)\big)$$

and, by (4.6) and the choice of k and ρ:

$$(f(t,x;\lambda) \mid x - x_i) < 0 \qquad \text{for } i\in B,\ |x - x_i| = \varepsilon,\ x\in \mathcal{M}$$

holds for all $t\in [0,T]$ and $\lambda\in\ ]0,1[$. Hence, the homotopized field λf is transversal at $fr_C G$ and so, by standard arguments, (h2) is satisfied.
Moreover, $\overline{f}_0(z) = \rho(z)k(z) = k(z)$ for all $z\in fr_C G$ so that (h3) holds.

Finally, (h4) may be computed by Srzednicki's formula as clG is a block, with $fr_C G$ its set of "egress points". Therefore

$$I(\bar{\pi}, G) = \chi(clG) - \chi(fr_C G).$$

As clG is the sphere S(0,R) with n pairwise non-intersecting holes, we have:

$$\chi(clG) = \begin{cases} n , & m \text{ even} \\ 2 - n , & m \text{ odd} . \end{cases}$$

On the other hand,

$$\chi(fr_C G) = \begin{cases} 2\mathrm{card}B , & m \text{ even} \\ 0 , & m \text{ odd} ; \end{cases}$$

hence,

$$I(\bar{\pi}, G) = \begin{cases} n - 2\mathrm{card}B , & m \text{ even} \\ 2 - n , & m \text{ odd} \end{cases}$$

and, using (4.7)-(4.8), (h4) is proved.

Thus, we can apply Corollary 1 and we obtain the existence of a solution $x(\cdot)$ of $\dot{x} = f(t,x;1)$ satisfying (4.2) and such that $x(t) \in clG$ for all $t \in [0,T]$. At last, we observe that $\rho \equiv 1$ on clG, so that $f(t,x;1) = F(t,x)$ and the proof is complete. ♦

REMARK 8. We point out that Proposition 1 is not contained in [18,79,98], since $\mathrm{int}C = \varnothing$. Moreover, neither the results in [52,53] can be directly applied since our assumptions do not imply the compactness of the set of T–periodic solutions for the equation $\dot{x} = \lambda f(t,x;\lambda)$.

We also remark that it is not difficult to adapt the proof of Proposition 1 to other choices of the manifold $\mathcal{M}$. However, the case $\mathcal{M} = S(0,R)$ is suitable for studying systems where the field F is homogeneous in x, as we show below.

EXAMPLE 2. We consider the following problem:

$$\dot{x} = g(t,x) \tag{4.12}$$

$$x(T) = \mu x(0) \tag{4.13}$$

where $\mu \in \mathbf{R}$ is a real parameter.

We are interested in proving the existence of a Floquet solution, i.e. a solution $x(\cdot)$ to (4.12)-(4.13), corresponding to some $\mu \in \mathbf{R} \setminus \{0\}$, with $x(t) \neq 0$ for all t.

Such a problem, or its slight variants, has been examined in [72] and, for g positively homogeneous, in [77].

It is easy to find examples in which (4.12)-(4.13) does not possess non-trivial solutions: for instance, the system $\dot{x}_1 = x_2, \dot{x}_2 = - ax_1$ $(a > 0)$ has a Floquet solution if and only if $T = k\pi/(a)^{1/2}$ for some $k \in \mathbf{N}$.

We show how Proposition 1 can be used in order to treat (4.12)-(4.13) in the particular case of (positively) homogeneous nonlinearities.

On the other hand, our result can be applied to the case of vector fields with singularities as well.

Accordingly, we consider the following situation.

Let $g : \mathbf{R}_+ \times \mathcal{W} \to \mathbf{R}^m$ be continuous, T-periodic in the first variable and locally lipschitzian in x. We suppose

$$\mathcal{W} := \mathbf{R}^m \setminus \left(\bigcup_{i=1}^{n} \sigma_i\right), \tag{4.14}$$

where each σ_i is a half line passing through the origin; furthermore, we assume

$$g(t,kx) = kg(t,x), \quad \textit{for all } t \in [0,T],\ x \in \mathcal{W} \textit{ and } k \in \mathbf{R}_+. \tag{4.15}$$

As remarked in [77], any function $x(\cdot)$ satisfying (4.12)-(4.13) may be extended to $\mathbf{R}_+$ as a solution such that $x(t + T) = \mu x(t)$ for all t.

In view of (4.14)-(4.15), we also observe that it is sufficient to know the behaviour of g on the set

$$D := S(0,1) \cap \mathcal{W} = S(0,1) \setminus \{w_1, \ldots, w_n\},$$

where, for each $i \in \{1,\ldots,n\}$, $\{w_i\} := \sigma_i \cap S(0,1)$. Then, the following result holds.

COROLLARY 5. *Suppose there is* $\eta > 0$ *such that, for each* $i = 1,\ldots,n$,

$$\textit{either } \liminf_{\substack{w \to w_i \\ w \in D}} (g(t,w) \mid w - w_i) \geq \eta, \textit{ or } \limsup_{\substack{w \to w_i \\ w \in D}} (g(t,w) \mid w - w_i) \leq -\eta, \tag{4.16}$$

holds uniformly in $t \in [0,T]$ *and assume*

$$|g(t,w)| \leq c_1 |w - w_i|^{-1} + c_2, \quad \textit{for all } t \in [0,T],\ w \in D, \tag{4.17}$$

with $c_1, c_2 \in \mathbf{R}_+$. *Let* ν *be the number of indices for which the liminf in* (4.16) *is positive. Then, there is* $\mu > 0$ *such that* (4.12)-(4.13) *has a solution* $x(\cdot)$ *with* $x(t) \in \mathcal{W}$ *for all* $t \in [0,T]$, *provided that one of the following conditions holds:*

$$m \text{ even}, \qquad \nu \neq n/2 \tag{4.18}$$

$$m \text{ odd}, \qquad n \neq 2. \tag{4.19}$$

It is clear, by the homogeneity of $g(t,\cdot)$, that if $x(\cdot)$ is any such solution, then $Kx(\cdot)$ is a (Floquet) solution too, for every $K > 0$.

Proof. First of all we observe that, since g is homogeneous, $x(\cdot)$ is a solution of (4.12) such that $x(t) \in \mathcal{W}$ for all t if and only if

$$x(t) = |x(t)|(x(t)/|x(t)|) = k(t)\, w(t),$$

where the functions $k: \mathbf{R}_+ \to \mathbf{R}$ and $w: \mathbf{R}_+ \to \mathbf{R}^m$ are solutions of

$$\dot{w}(t) = F(t,w) := g(t,w) - (w|g(t,w))w, \tag{4.20}$$

$$\dot{k}(t) = k(t)\,(w|g(t,w)), \tag{4.21}$$

respectively, such that $w(t) \in D$, $k(t) > 0$, for all t. Note that $F : \mathbf{R}_+ \times D \to \mathbf{R}^m$.
We apply Proposition 1 to equation (4.20) (in this situation, $\mathrm{card}A = \nu$). By the form of F, it is immediately seen that (4.4) is satisfied. In order to verify (4.5)-(4.6), we note that

$$(F(t,w) \mid w - w_i) = \big(g(t,w) - (w|g(t,w))w \mid w - w_i\big) =$$
$$= -(g(t,w) \mid w_i) + (w|g(t,w))\,(w \mid w_i).$$

Adding and subtracting $(g(t,w) \mid w)$, we get:

$$(F(t,w) \mid w - w_i) = (g(t,w) \mid w - w_i) - (1 - (w \mid w_i))\,(g(t,w) \mid w).$$

Using the equality $|w - w_i|^2 = 2(1 - (w \mid w_i))$, then it is easily seen that, taking the limit as $w \to w_i$, $w \in D$, (4.16) and (4.17) imply (4.5)-(4.6), with the obvious choice of the sets A and B. Conditions (4.18)-(4.19) are equivalent to (4.7)-(4.8).
Thus, we can apply Proposition 1 and obtain an T-periodic solution $w(\cdot)$ to $\dot{w} = F(t,w)$ such that $w(t) \in D$ for all t. Then, we insert it in (4.21) and obtain $k(\cdot)$ by direct computation solving (4.21) with the initial condition $k(0) = k_0 > 0$. Finally, since $w(0) = w(T)$, (4.13) follows immediately, with $\mu = k(T)/k(0) = |x(T)|/|x(0)|$. The proof is complete. ♦

REMARK 9. The proof of Corollary 5 can be repeated if, instead of (4.16), we assume that $\{1,...,n\} = A\cup B$, $A\cap B = \varnothing$, $\mathrm{card}A = \nu$, and

$$\liminf_{\substack{w\to w_i \\ w\in D}} \left(g(t,w) \mid w - w_i\right) / |w - w_i|^{\gamma} \geq \eta, \qquad i\in A$$

$$\limsup_{\substack{w\to w_i \\ w\in D}} \left(g(t,w) \mid w - w_i\right) / |w - w_i|^{\gamma} \leq -\eta, \quad i\in B$$

uniformly in $t\in[0,T]$, with $0\leq\gamma<2$. In this case (4.17) has to be replaced by

$$|g(t,w)| \leq c_1|w - w_i|^{-\beta} + c_2, \qquad i=1,...,n, \quad \beta < 2-\gamma,$$

for all $t\in[0,T]$, $w\in D$.
Theorem 1 in [77] is a particular case of our Corollary 5 for m odd, n = 0; on the other hand, [77, Th.2,3] can be straightforwardly obtained from Corollary 1 as well.

EXAMPLE 3. Our third example deals with system:

$$\dot{x}_i = x_i\,(q_i(t) + x_{i-1}A_i(t,x) - \Phi(t,x)), \quad i = 1,...,m, \tag{4.22}$$

where $q_i : \mathbf{R}_+\to \mathbf{R}$, $A_i : \mathbf{R}_+ \times \mathbf{R}^m \to\mathbf{R}$ are continuous and T-periodic in t, A_i are locally lipschitzian in x and

$$\Phi(t,x) := \sum_{i=1}^{m} x_i(q_i(t) + x_{i-1}A_i(t,x)). \tag{4.23}$$

In (4.22) and (4.23) the indices are counted mod m $(m\geq 2)$.
System (4.22) represent a generalization to the time-dependent case of the inhomogeneous generalized hypercycle. This equation was introduced by M. Eigen and P. Schuster [41] and successively studied by many authors for its significance in describing some models arising from the theory of self-organization of biological macromolecules, population genetics and animal behaviour (see [73,74,134]).
The peculiarity of system (4.22) lies in the fact that the $(m-1)$-dimensional simplex

$$S_m = \{x\in \mathbf{R}^m : x_i \geq 0\ \forall i, \sum_{i=1}^{m} x_i = 1\}$$

is invariant. Moreover, since x_i represents the (relative) concentration of the i-th species, it is interesting to study the behaviour of the solutions in

$$\overset{o}{S}_m = \{x \in S_m : x_i > 0 \ \forall i \}.$$

J. Hofbauer [73] examined (4.22) in two special cases: q_i = constant, $A_i(t,x) = k_i$ = constant, with $k_i>0$ (the inhomogeneous hypercycle [73, § 2.1]), and $q_i = 0$, $A_i(t,x) = A_i(x) \geq k_i > 0$ (the generalized hypercycle [73, § 2.2]). In [73], necessary and sufficient conditions for permanence are obtained; more precisely, it is proved that the inhomogeneous hypercycle admits a compact attractor in $\overset{o}{S}_m$ if and only if there exists an equilibrium point in $\overset{o}{S}_m$. Along these lines, we assume

$$A_i(t,x) \geq k_i > 0 \tag{4.24}$$

for all $t \in [0,T]$, $x \in S_m$, $i = 1,...,m$, and we consider the auxiliary system

$$\dot{x}_i = x_i \, (\, \bar{q}_i + x_{i-1}k_i - \Phi^*(x)), \quad i = 1,...,m, \tag{4.25}$$

with

$$\Phi^*(x) := \sum_{i=1}^{m} x_i(\bar{q}_i + x_{i-1}k_i).$$

If we assume, as in [73, th. 2], that (4.25) has an equilibrium point $p \in \overset{o}{S}_m$, then it is easy to prove the existence of at least one T-periodic solution of (4.22) with values in $\overset{o}{S}_m$, provided that, for all i,

$$|q_i - \bar{q}_i|_\infty < \delta \ , \quad \sup_{x \in S_m} |A_i(\cdot,x) - k_i| < \delta \ , \tag{4.26}$$

with δ a sufficiently small constant. This follows by standard arguments from degree theory or by considering system (4.22) as a small perturbation of the dissipative system (4.25). However, if (4.26) is not satisfied then small perturbation techniques cannot be applied. Our result, which is "global" in nature, provides an answer to such a problem. Indeed, we have:

PROPOSITION 2. *Assume that* (4.25) *has an equilibrium point* $p \in \overset{o}{S}_m$. *Then, there exists at least one* T*–periodic solution* $x(\cdot)$ *of* (4.22) *such that* $x(t) \in \overset{o}{S}_m$, *for all* $t \in [0,T]$.

Proof. We apply Corollary 1, with $C = S_m$, $f_i(t,x;\lambda) = x_i\big(q_i(t) + x_{i-1}[\lambda A_i(t,x) + (1-\lambda)k_i] - \phi(t,x;\lambda)\big)$, $i=1,...,m$, $\lambda \in [0,1]$, where

$$\phi(t,x;\lambda) := \sum_{i=1}^{m} x_i\big(q_i(t) + x_{i-1}[\lambda A_i(t,x) + (1-\lambda)k_i]\big)$$

and

$$G := \{x \in S_m : x_i > \rho,\ i = 1,\dots,m\},$$

with $\rho > 0$ a suitable constant which will be chosen along the proof.
Observe that system

$$\dot{x}_i = \lambda f_i(t,x;\lambda), \quad i = 1,\dots,m \tag{4.22$_\lambda$}$$

reduces to (4.22) for $\lambda = 1$ and that the averaged field $\bar{f}_0(x) = \langle f(\cdot,x;0)\rangle$ is the right-hand side of (4.25). By the form of (4.22$_\lambda$), it is immediately seen that assumption (j1) is satisfied.
Now, we show that there exists a constant $\rho > 0$ such that the set G defined above is a "bound set" for system (4.22$_\lambda$).
First of all, we observe that, since $p \in \overset{o}{S}_m$, we have:

$$\Phi^*(p) = \bar{q}_i + k_i p_{i-1}, \tag{4.27}$$

so that

$$\eta := (\Phi^*(p) - \max_{1\le i\le m} \bar{q}_i) > 0. \tag{4.28}$$

Using (4.27) and adding and substracting $\lambda x_i k_i x_{i-1}$ to the i-th equation of (4.22$_\lambda$) we see that (4.22$_\lambda$) may be rewritten as:

$$\dot{x}_i = \lambda x_i\big\{[q_i(t) - \bar{q}_i] + k_i[x_{i-1} - p_{i-1}] + \lambda x_{i-1}[A_i(t,x) - k_i] + \Phi^*(p) - \phi(t,x;\lambda)\big\}. \tag{4.29}$$

Let $x(\cdot)$ be an T-periodic solution of (4.22$_\lambda$) such that $x(t) \in \overset{o}{S}_m$ for each $t \in [0,T]$ and consider (as in [73, p. 237]) the function

$$P(t) := \prod_{i=1}^{m} (x_i(t))^{1/k_i}.$$

Since $P(x(t)) > 0$ for all t, we obtain:

$$\dot{P}(t)/P(t) = \Big(\sum_{i=1}^{m} k_i^{-1}(x_i(t))^{(1/k_i)-1}\cdot \dot{x}_i(t)\cdot \prod_{j\neq i}^{m}(x_j(t))^{1/k_j}\Big) \Big/ \prod_{i=1}^{m}(x_i(t))^{1/k_i} =$$

$$= \sum_{i=1}^{m} \dot{x}_i(t)/k_i x_i(t) = \lambda \sum_{i=1}^{m} k_i^{-1}[q_i(t) - \langle q_i\rangle] + \lambda \sum_{i=1}^{m} x_{i-1}(t) - \lambda \sum_{i=1}^{m} p_{i-1} +$$

$$+ \lambda^2 \sum_{i=1}^{m} k_i^{-1}\, x_{i-1}(t)(A_i(t,x) - k_i) + \lambda \sum_{i=1}^{m} k_i^{-1}\,[\Phi^*(p) - \phi(t,x;\lambda)].$$

By (4.24) and (4.28) we have:

$$\dot{P}(t)/P(t) \geq \lambda \sum_{i=1}^{m} k_i^{-1}\,[q_i(t) - \bar{q}_i] + \lambda \sum_{i=1}^{m} k_i^{-1}\,[\max_{1\leq i\leq m} \bar{q}_i + \eta - \phi(t,x;\lambda)]. \tag{4.30}$$

We set, for the sake of semplicity,

$$\alpha := \sum_{i=1}^{m} k_i^{-1}, \qquad Q := \max_{1\leq i\leq m} \bar{q}_i, \qquad M := \max_{1\leq i\leq m} \{A_i(t,x)\colon t\in[0,T],\ x\in S_m\}.$$

Then, taking the mean value on [0,T] in (4.27) and dividing by $\lambda>0$, we have:

$$\frac{1}{T}\int_0^T \phi(s,x(s);\lambda)ds \geq Q + \eta. \tag{4.31}$$

We claim that for every index $j = 1,\ldots,m$ there is some t_j such that

$$x_j(t_j) \geq \eta / M. \tag{4.32}$$

Indeed, assume, by contradiction, that there is j such that $x_j(t) < \eta / M$ for all t. Let $j = i-1$ and take the mean value in [0,T] of the equation

$$\dot{x}_i / x_i = \lambda\big(q_i(t) + x_{i-1}[\lambda A_i + (1-\lambda)k_i] - \phi(t,x;\lambda)\big). \tag{4.33}$$

Then we get (after a division by $\lambda > 0$):

$$\frac{1}{T}\int_0^T \phi(s,x;\lambda)ds \;\leq\; \bar{q}_i + M\langle x_{i-1}\rangle < Q + \eta$$

and a contradiction with (4.31) is achieved.

Therefore, the claim is proved.
From (4.32) we also have:

$$| \dot{x}_i(t) / x_i(t) | \leq 2(Q^* + M), \tag{4.34}$$

where

$$Q^* := \max_{1 \leq i \leq m} |q_i|_\infty .$$

Then, using (4.32) and (4.34), an easy computation shows that

$$x_i(t) \geq \rho_1 := (\eta/M) \exp(-2(Q^* + M)T), \tag{4.35}$$

for all $t \in [0,T]$.
By (4.35), we see that the set G previously defined verifies condition (h2) for any ρ such that $0 < \rho < \rho_1$.
In order to verify (h3)-(h4), it is sufficient to recall that the assumption $p \in \overset{o}{S}_m$ implies (by Hofbauer's Theorem) the existence of a compact set $K \subset \overset{o}{S}_m$ which is an attractor for the "averaged" system (4.25):

$$\dot{x}_i = (\bar{f}_0)_i(x) = x_i \{\bar{q}_i + x_{i-1}k_i - \Phi^*(x)\} , \quad i = 1,\dots,m .$$

Then, if we take the constant ρ in the definition of G small enough (i.e. $0 < \rho < \rho_2 := \min_{1 \leq i \leq m} \{x_i : x \in K\}$) and we denote by $\bar{\pi}$ the dynamical system induced by (4.25), we have:

$$I(\bar{\pi},G) = \chi(S_m) = 1$$

(see [137, Th.6.1]).
Finally, for $0 < \rho < \min\{\rho_1,\rho_2\}$, all the assumptions of Corollary 1 are fulfilled and the proof is complete. ♦

REMARK 10. We point out that the result proved in Proposition 2 can be obtained neither by means of our Theorem in [18], since $\text{int}C = \emptyset$, nor by Theorems A or B, which require G to be an open subset of $\mathbf{R}^m$.
Moreover, also the results in [138] cannot be used since the set G in our proof is *not* a block for system (4.22); we also remark that the results in [53] cannot be applied as well, since the phase space $C = S_m$ is not a differentiable manifold.

Further examples can be produced for differential systems of Lotka-Volterra type. In particular, a natural setting for the existence of positive periodic solutions is $C := \mathbb{R}^m_+$ (see [18]). Other examples can be found in [13].

Chapter 3

Continuation theorems for periodic perturbations of autonomous systems

1. Introduction

In this Chapter (which is based on [15]) we are concerned with the periodic boundary value problem

$$\dot{x} = F(t,x) \tag{1.1}$$

$$x(0) = x(T) \tag{1.2}$$

where $F : [0,T]\times\mathbf{R}^m \to \mathbf{R}^m$ is a Caratheodory function $(T > 0)$.
More precisely, the important situation (occuring in several applications) which corresponds to the case when the nonautonomous field $F(t,x)$ splits as

$$F(t,x) := g(x) + e(t,x) \tag{1.3}$$

is studied. In such a situation, it is natural to choose the homotopy field $f(t,x;\lambda) := g(x)+\lambda e(t,x)$, $\lambda\in[0,1]$. In general, this cannot be done with a standard continuation theorem like Mawhin's one.
The aim of this Chapter is to provide new continuation results for (1.1)-(1.2) which are particularly suitable for dealing with nonlinearities like (1.3).
To do this, it is assumed that

$$F(t,x) := f(t,x;1),$$

where $f = f(t,x;\lambda) : [0,T]\times\mathbf{R}^m\times[0,1] \to \mathbf{R}^m$ is a Caratheodory function such that for $\lambda = 0$ the map f is autonomous, i.e. there is a continuous function $f_0 : \mathbf{R}^m \to \mathbf{R}^m$ such that

$$f_0(x) = f(t,x;0),$$

for almost all $t\in[0,T]$ and all $x\in\mathbf{R}^m$.

In Section 2 it is shown (Theorem 1) that, whenever it exists, the coincidence degree of the left-hand member of an autonomous differential equation $\dot{x} = g(x)$, in the space of T-periodic functions, can be computed in terms of the Brouwer degree of g. The proof is performed by means of an "approximation" procedure for the map g based on the Kupka-Smale's theorem [22,119].

In Section 3 the above result is used to provide efficient continuation theorems specially for T-periodic perturbations of autonomous systems (Theorems 2,3).

In Section 4, applications (in two different directions) of the continuation theorem in Section 3 are given for the important case when F splits as

$$F(t,x) = g(x) + e(t,x)$$

and the performed homotopy is

$$f(t,x;\lambda) = g(x) + \lambda e(t,x).$$

In Section 5, an extension of Theorem 2 to flow-invariant Euclidean Neighbourhood Retracts is performed (Theorem 4). In such a general framework, continuous vector fields and the fixed point index of compact operators defined on the space of continuous functions which take values in the given ENR are used. Moreover, Theorem 4 enables to deal with some cases when the phase space is not the whole $\mathbf{R}^m$ but e.g. a regular manifold, a closed convex set or a conical shell.

2. The main result: an estimate for the degree

We deal with the periodic boundary value problem:

$$\dot{x} = F(t,x) \tag{2.1}$$

$$x(0) = x(T), \tag{2.2}$$

where

$$F(t,x) := f(t,x;1) \tag{2.3}$$

and $f = f(t,x;\lambda) : [0,T]\times\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ satisfies the Caratheodory conditions, i.e. $f(\cdot,x;\lambda)$ is (Lebesgue) measurable for each (x,λ), $f(t,\cdot;\cdot)$ is continuous for a.e. t and, for each $r>0$, there exists $\beta_r\in L^1([0,T],\mathbf{R})$ such that $|f(t,x;\lambda)|\leq\beta_r(t)$ holds for a.e. $t\in[0,T]$ and all $|x|\leq r$, $\lambda\in[0,1]$. Accordingly, solutions for $\dot{x} = f(t,x;\lambda)$ are intended in the generalized (i.e. Caratheodory) sense. With small abuse in the terminology, we call T-periodic any solution satisfying the boundary condition (2.2).

As we mentioned in the introduction, we assume that for $\lambda = 0$ the map f is autonomous, i.e. there exists a continuous function $f_0 : \mathbf{R}^m \to\mathbf{R}^m$ such that

$$f_0(x) := f(t,x;0) \tag{2.4}$$

for almost every $t\in[0,T]$ and each $x\in\mathbf{R}^m$. A particular but significant case in which such a situation occurs is when f splits as

$$f(t,x;\lambda) = f_0(x) + \lambda e(t,x;\lambda);$$

this is examined in detail in Section 4.

The proof of continuation results for problem (2.1)-(2.2) is based, essentially, on the homotopy invariance of the topological degree and on estimates for the degree of some operators associated to system

$$\dot{x} = f_0(x). \tag{2.5}$$

This second goal is achieved showing that, under certain circumstances, it is sufficient to evaluate the (finite dimensional) Brouwer degree of the vector field f_0. In this Section, we prove some results in which the above programme is developed for various operators related to (2.5).

In what follows, the (real) Banach spaces $Z := L^1([0,T],\mathbf{R}^m)$, $Y := \mathcal{C}([0,T],\mathbf{R}^m)$ and $X := \{x\in Y: x(0) = x(T)\}$, with their usual norms, are considered. Notice that points of $\mathbf{R}^m$ are identified with constant functions.

First, we recall some basic facts from coincidence degree theory, borrowing notation and terminology from [98]. We define $L : \mathrm{dom}L\subset X\to Z$, $Lx = \dot{x}$, a Fredholm mapping of index zero, with $\mathrm{dom}L = \{x\in X: x(\cdot) \text{ is absolutely continuous}\}$.

Let M_0 be the Nemitzky operator from X to Z induced by the map f_0, i.e. $M_0 : x(\cdot)\longmapsto f_0(x(\cdot))$. In this situation, problem (2.5)-(2.2) can be transformed into the equivalent coincidence equation:

$$Lx = M_0x, \quad x\in \mathrm{dom}L. \tag{2.6}$$

If we introduce the linear projectors $Q : Z\to \mathrm{coker}L \subset Z$, $Qz := \bar{z} = (1/T) \int_0^T z(s)ds$ and $P := Q_{|X} : X\to\ker L \subset X$ and we denote by $K_{P,Q} : Z\to\ker P\cap\mathrm{dom}L$ the generalized inverse of L, then equation (2.6) is equivalent to:

$$x = R_0(x) := Px + K_{P,Q}M_0x + JQM_0x = x - (JQ + K_{P,Q})(L - M_0)x \ ,$$

where $J : \mathrm{Im}Q = \mathbf{R}^m \to \ker L = \mathbf{R}^m$ is a linear isomorphism, so that $I - R_0 = T(L - M_0)$, for some linear isomorphism T (see [98,99,120,143]).
In the sequel, for simplicity, we take $J := I$ (the identity in $\mathbf{R}^m$).

Let $\Omega\subset X$ be bounded and open (relatively to X).
It is a standard fact to check that $R_0 : \mathrm{cl}_X\Omega\to X$ is compact. Therefore, the coincidence degree of L and M_0 in Ω is defined by:

$$D_L(L - M_0, \Omega) := \deg(I_X - R_0, \Omega, 0),$$

provided that

$$Lx \neq M_0 x \quad \text{for all } x \in fr_X\Omega \cap domL.$$

From [98, p.19] we know that the definition of the coincidence degree is independent of the projectors P and Q.

We note that a similar framework may be introduced by a different choice of the function spaces. In particular, the use of $Z := L^1([0,T],\mathbf{R}^m)$ is not necessary at this point. However, such a choice is convenient as we deal later with nonautonomous nonlinearities satisfying only the Caratheodory conditions (see [98, Ch.VI]).

The following theorem, which is crucial for the proof of Theorem 2 in next section, may be considered of some independent interest as a contribution to the coincidence degree theory.

THEOREM 1. *Assume that there is no* $x(\cdot) \in fr_X\Omega$ *such that* $\dot{x} = f_0(x)$. *Then,*

$$D_L(L - M_0, \Omega) = (-1)^m d_B(f_0\, , \Omega \cap \mathbf{R}^m, 0). \tag{2.7}$$

Proof. First of all, we observe that, as Ω is bounded, there is a constant $R > 0$ such that $|x|_\infty < R$, for every $x \in \Omega$. Furthermore, we point out that the assumption is equivalent to

$$Lx \neq M_0 x\,, \tag{2.8}$$

for all $x \in domL \cap fr_X\Omega$; therefore, the coincidence degree $D_L(L - M_0, \Omega)$ is well defined.

The proof is performed by means of a corollary of the Kupka-Smale's theorem [22, p.68]; this result ensures the existence of a sequence of $\mathcal{C}^1$-functions (φ_k), $\varphi_k : \mathbf{R}^m \to \mathbf{R}^m$, such that:

(a) $(\varphi_k) \to f_0$ uniformly on compact sets;

(b) for every compact subset K of $\mathbf{R}^m$ and for all $k \in \mathbf{N}$, system

$$\dot{x} = \varphi_k(x)$$

has finitely many singular orbits (i.e., rest points and closed orbits) with minimal period in [0,T+1] which are contained in K.

Let $N^{k,\mu}$ be the Nemitzky operator induced by the functions $x \mapsto \mu f_0(x) + (1 - \mu)\varphi_k(x)$, $\mu \in [0,1]$. We claim that there is $k_0 > 0$ such that, for all $k \geq k_0$ and for all $\mu \in [0,1]$

$$Lx \neq N^{k,\mu}x \quad \text{for all } x \in domL \cap fr_X\Omega. \tag{2.9}$$

This fact will imply, in particular, that

$$\varphi_k(z) \neq 0 \quad \text{for all } z \in \mathrm{fr}_X \Omega \cap \mathbf{R}^m, \ k \geq k_0. \tag{2.10}$$

Then, a classical compactness argument ensures that, for any $k \geq k_0$, there is $\delta_1 = \delta_1(k)$ such that

$$\varphi_k(y) \neq 0 \quad \text{for all } y \in B(\mathrm{fr}_X \Omega \cap \mathbf{R}^m, \delta_1). \tag{2.11}$$

To obtain (2.9), it is sufficient to observe that the sequence of operators $N^{k,\mu}$ converges, as $k \to +\infty$, to M_0 in Z uniformly on $\mathrm{cl}_X \Omega \times [0,1]$ and that, by (2.8),

$$\inf\{ |(L - M_0)x|_Z : x \in \mathrm{dom} L \cap \mathrm{fr}_X \Omega \} > 0.$$

Hence, the claim is proved and, using the homotopy property of the coincidence degree (see [55, Th. III.2]), we can write

$$D_L(L - M_0, \Omega) = D_L(L - N^{k,1}, \Omega) = D_L(L - N^{k,0}, \Omega), \tag{2.12}$$

for every $k \geq k_0$, and, in particular,

$$d_B(f_0, \Omega \cap \mathbf{R}^m, 0) = d_B(\varphi_K, \Omega \cap \mathbf{R}^m, 0). \tag{2.13}$$

Let us fix $k^* \geq k_0$. For brevity, we set

$$\varphi := \varphi_{k^*}, \ N_\varphi := N^{k^*,0}, \ \delta_1 := \delta_1(k^*).$$

Consider the singular orbits (i.e. rest points and closed orbits) with minimal period in [0,T+1] of the system

$$\dot{x} = \varphi(x). \tag{2.14}$$

By the Kupka-Smale's theorem, there exist finitely many such orbits which are contained in B(0,R). We denote these orbits by $S_1, \ldots, S_n$. They are mutually disjoint. Pick, for each $i=1,\ldots,n$ a point $z_i \in S_i$. Then, z_i is a periodic point (possibly a rest point). We can assume that z_i is a rest point for $1 \leq i \leq p$ ($p \geq 0$ an integer) and a periodic point for $p+1 \leq i \leq n$. We denote its minimal period by T_i ($p+1 \leq i \leq n$). We can also assume that $T_i \leq T$ for $p+1 \leq i \leq q$ and $T < T_i \leq T+1$ for $q+1 \leq i \leq n$. We denote by k_i the largest integer such that $k_i T_i \leq T$ ($p+1 \leq i \leq q$), so that $(k_i+1)T_i > T$ ($p+1 \leq i \leq q$). We denote by $x_i(\cdot)$ the solution of (2.14) with $x_i(0)=z_i$ ($p+1 \leq i \leq n$). We claim that for each T' such that

$$T < T' < \min\{(k_{p+1}+1)T_{p+1}, \ldots, (k_q+1)T_q, T_{q+1}, \ldots, T_n, T+1\} := \tau,$$

the problem

$$\dot{x} = \varphi(x) \quad , \quad x(0) = x(T') \tag{2.15}$$

has no solution $x(\cdot)$, with $x(t) \in B(0,R)$ for all t, other than the equilibria $z_1,\ldots,z_p$.
Indeed, if $x(\cdot)$ satisfies (2.15) and is contained in $B(0,R)$, then $S = \{x(t): 0 \le t \le T'\}$ is a singular orbit of (2.14) contained in $B(0,R)$. If it is not a rest point, then $S = S_i$ for some $p+1 \le i \le n$ and hence there exists $\alpha_i \in \mathbf{R}$ such that

$$x(t) = x_i(t + \alpha_i), \qquad 0 \le t \le T'.$$

In particular,

$$x_i(T' + \alpha_i) = x_i(\alpha_i).$$

This is impossible for $q+1 \le i \le n$ as then $T' < T_i$ and T_i is the smallest period. This is impossible for $p+1 \le i \le q$ as in this case $k_i T_i < T' < (k_i+1) T_i$.
Therefore the claim is proved.
Now, the solutions of (2.15) correspond, by the transformation

$$y(t) = x\left(\frac{T'}{T} t\right), \quad t \in [0,T],$$

to the solutions of the problem

$$\dot{y}(t) = \frac{T'}{T}\, \varphi(y(t)) \quad , \quad y(0) = y(T). \tag{2.16}$$

Thus, problem (2.16) has, by construction, no nontrivial (i.e. non-equilibrium) solution on $\mathrm{cl}_X \Omega$ and, by assumption, no rest point on $\mathrm{fr}_X \Omega$ (as its rest points are the same as those of (2.14), and all its possible solutions in $B(0,R)$ are rest points). Now, as (2.14) has no solution on $\mathrm{fr}_X \Omega$, the homotopy invariance of coincidence degree implies that

$$D_L(L - N_\varphi, \Omega) = D_L(L - \frac{T'}{T} N_\varphi, \Omega), \tag{2.17}$$

for all $T \le T' < \tau$. Fix some $T' \in (T, \tau)$.
Now, by excision,

$$D_L(L - \frac{T'}{T} N_\varphi, \Omega) = \sum_{\substack{1 \le j \le p \\ z_j \in \Omega}} D_L(L - \frac{T'}{T} N_\varphi, B(z_j, \delta)), \tag{2.18}$$

where

$$\delta = \min\{\delta_1, \eta/2\}, \;\; \eta = \min\{\mathrm{dist}(S_i,S_j) \colon 1 \le i \ne j \le n\}.$$

Now, the problems

$$\dot{x}(t) = \lambda \frac{T'}{T}\varphi(x(t)), \;\; \lambda \in (0,1]$$
$$x(0) = x(T)$$

have no solution on $\mathrm{fr}_X B(z_j,\delta)$.
Indeed, if there exists $\lambda^* \in]0,1]$ and $\dot{x}^*(\cdot) \in \mathrm{fr}_X B(z_j,\delta)$ such that

$$\dot{x}^* = \lambda^* \frac{T'}{T}\varphi(x^*(t)), \quad x^*(0) = x^*(T)$$

then

$$y(t) := x(\frac{T\,t}{\lambda^* T'})$$

will satisfy

$$\dot{y}(t) = \varphi(y(t)), \quad y(0) = y(\lambda^* T')$$

and hence $\{y(t) \colon t \in [0,\lambda^* T']\} = S_{i^*}$, for some $1 \le i^* \le n$. Moreover,

$$|y(t) - z_j| \le \delta \;\text{ for all } t \in [0,\lambda^* T']$$

so that, by the choice of δ, $i^*=j$ and $y(\cdot)$ is constant and equal to z_j for all $t \in [0,\lambda^* T']$, a contradiction. Thus we can argue as in [55, pp.28-29] and obtain

$$D_L(L - \frac{T'}{T}N_\varphi, B(z_j,\delta)) = d_B(-JQN_\varphi, B(z_j,\delta) \cap \mathbf{R}^m, 0) = (-1)^m\, d_B(\varphi, B(z_j,\delta) \cap \mathbf{R}^m, 0) \qquad (2.19)$$

for $1 \le j \le p$, $z_j \in \Omega$. Consequently, form (2.18) we have:

$$D_L(L - \frac{T'}{T}N_\varphi, B(z_j,\delta)) = (-1)^m \sum_{\substack{1 \le j \le p \\ z_j \in \Omega}} d_B(\varphi, B(z_j,\delta) \cap \mathbf{R}^m, 0) = (-1)^m\, d_B(\varphi, \Omega \cap \mathbf{R}^m, 0). \qquad (2.20)$$

The result follows by (2.17), (2.20), and (2.13). The proof is complete. ♦

Theorem 1 is a generalization of Lemma VI.1 in [98], where the case $f_0 = -\nabla V$, with $V \in \mathcal{C}^1(\mathbf{R}^m,\mathbf{R})$ and $\Omega \cap \mathbf{R}^m = B(0,r)$, $r>0$, is treated.

We remark that (2.19) holds for any linear orientation preserving isomorphism $J : R^m \to R^m$ (see [98]), and so (2.7) is independent of the choice of P, Q, J, whenever $\det J > 0$. In the more general case in which $J : ImQ=R^m \to kerL=R^m$ is an arbitrary linear isomorphism, we can write, instead of (2.7),

$$|D_L(L - M_0, \Omega)| = |d_B(f_0, \Omega \cap R^m, 0)|.$$

We remark that a different proof of Theorem 1, obtained in the framework of degree theory for equivariant maps, has been recently obtained by T. Bartsch and J. Mawhin [6].

From Theorem 1, using the duality theorems developed in [98, Ch.III] and [81, Ch. III], we can find other relations between the degree of some fixed point operators related to (2.5)-(2.2) and the Brouwer degree of f_0. To this end, the following maps $\Phi_i : Y \to Y$, i = 1,2,3, are defined:

$$\Phi_1(x)(t) := x(T) + \int_0^t f_0(x(s))ds,$$

$$\Phi_2(x)(t) := x(0) + \int_0^T f_0(x(s))ds + \int_0^t f_0(x(s))ds ,$$

$$\Phi_3(x)(t) := x(0) + (T-t) \int_0^T f_0(x(s))ds + \int_0^t f_0(x(s))ds.$$

All the Φ_i, i=1,2,3, are completely continuous and their corresponding fixed points are exactly the solutions of (2.5)-(2.2). Moreover, $\Phi_3|_X : X \to X$. Let $\Omega \subset Y$ be bounded and open (relatively to Y). In [98], the following equalities are proved, provided that there is no $x \in fr_Y \Omega$, solution of (2.5)-(2.2):

$$\deg(I_Y - \Phi_1, \Omega, 0) = \deg(I_Y - \Phi_2, \Omega, 0) = \deg(I_Y - \Phi_3, \Omega, 0) = \deg(I_X - \Phi_3|_X, \Omega \cap X, 0).$$

Indeed, it is sufficient to apply, respectively, Theorem III.1, Theorem III.4 and Proposition III.5 in [44, Ch.III]. Related results can be found in [81,68].
Now, we have:

COROLLARY 1. *Assume that there is no* $x \in X \cap fr_Y \Omega$ *such that* $\dot{x} = f_0(x)$. *Then, for* i=1,2,3,

$$\deg(I_Y - \Phi_i, \Omega, 0) = (-1)^m d_B(f_0, \Omega \cap R^m, 0). \tag{2.21}$$

Proof. It is sufficient to recall that, by [98, Th.III.7],

$$\deg(I_Y - \Phi_3, \Omega, 0) = D_L(L - M_0, \Omega \cap X)$$

and then Theorem 1 can be applied. ♦

In [111], the author stated the equality $\deg(I_Y - \Phi_1, \Omega, 0) = d_B(-f_0, \Omega \cap \mathbf{R}^m, 0)$ for the case when Ω is a ball and f_0 is positively homogeneous of order 1, assuming that equation (2.5) does not possess non trivial periodic solutions of *any* period. Hence, Corollary 1 improves Muhamadiev's theorem in [111, Th. 5. m=1] (see the next Section for a more detailed discussion).

Finally, we give an analogous result for the Poincaré map. Suppose that equation (2.5) defines a flow in $\mathbf{R}^m$, i.e. assume uniqueness and global existence for the solutions of the Cauchy problems associated to (2.5). For each $z \in \mathbf{R}^m$, we denote by $x(\cdot,z)$ the solution of (2.5) with $x(0,z) = z$. Thus, the Poincaré-Andronov operator on $[0,T]$ is defined by

$$U_0 z := x(T,z).$$

Let $G \subset \mathbf{R}^m$ be an open bounded set. Then, the following result holds.

COROLLARY 2. *Assume that* $U_0 z \neq z$ *for all* $z \in \mathrm{fr}G$. *Then,*

$$d_B(I - U_0, G, 0) = (-1)^m d_B(f_0, G, 0). \tag{2.22}$$

Proof. We fix $R > 0$ such that

$$R > \sup\{|x(t,z)| : 0 \le t \le T,\ z \in \mathrm{cl}G\}.$$

Then, for $\Omega := \{x \in Y : x(0) \in G,\ |x|_\infty < R\}$, we have:

$$\deg(I_Y - \Phi_1, \Omega, 0) = d_B(I - U_0, G, 0). \tag{2.23}$$

Indeed, (2.25) can be obtained either from [81, Th. 28.5], observing that $G \subset \mathbf{R}^m$ and $\Omega \subset Y$ have a "common core" with respect to the T−periodic boundary value problem (2.5)-(2.2), or from [98, Th. III.11, Cor. III.12]. Hence, Corollary 1 can be applied and the thesis follows. ♦

Recall that in [9] and [79] the equality $d_B(I - U, G, 0) = d_B(-f_0, G, 0)$ is proved under the stronger condition that $x(t,z) \neq z$ for all $t \in]0,T]$ and $z \in \mathrm{fr}G$ (that is, assuming that all the points of frG are of T-irreversibility [79]).

3. Existence theorems

In this Section, we use the notations introduced in Section 2. Recall that

$$f(t,x;0) = f_0(x), \qquad f(t,x;1) = F(t,x).$$

First, we give our main result for the solvability of

$$\dot{x} = F(t,x) \tag{3.1}$$

$$x(0) = x(T). \tag{3.2}$$

THEOREM 2. *Let* $\Omega \subset X$ *be an open bounded set such that the following conditions are satisfied:*

(p1) *there is no* $x(\cdot) \in fr_X\Omega$ *such that*

$$\dot{x} = f(t,x;\lambda), \qquad \lambda \in [0,1[; \tag{3.1$_\lambda$}$$

(p2)

$$d_B(f_0, \Omega \cap R^m, 0) \neq 0.$$

Then, (3.1)-(3.2) *has at least one solution* $x(\cdot) \in cl_X\Omega$.

Proof . We use the framework of coincidence degree theory as in Theorem 1. The classical Leray-Schauder continuation Theorem [90] could be used instead, by the equivalence at the beginning of Section 2. Beside the spaces and the operators considered there, we further define $M := M(x;\lambda) : X \times [0,1] \to Z$:

$$M(x;\lambda)(t) := f(t,x(t);\lambda).$$

Observe that $M(\cdot;0) = M_0$.

According to [98, Ch. VI], M is L-compact on $cl_X\Omega \times [0,1]$. We remark that $x(\cdot)$ is a solution of $\dot{x} = f(t,x;\lambda)$, $\lambda \in [0,1]$, with $x(0) = x(T)$, if and only if $x \in domL$ is a solution of the coincidence equation $Lx = M(x;\lambda)$, $\lambda \in [0,1]$. In particular, (3.1)-(3.2) is equivalent to $Lx = M(x;1)$ (according to (2.3) in Section 2).

Without loss of generality, we suppose that (p1) holds for $\lambda \in [0,1]$ in (3.1$_\lambda$). Otherwise, the result is proved for $x \in fr_X\Omega$. Accordingly, by the definition of $M(\cdot;\lambda)$ and using (p1) we have:

$$Lx \neq M(x;\lambda) \qquad \lambda \in [0,1],$$

for all $x \in domL \cap fr_X\Omega$. Thus, we can apply the homotopy property of the coincidence degree and obtain:

$$D_L(L - M_0, \Omega) = D_L(L - M(\cdot;0), \Omega) = D_L(L - M(\cdot;1), \Omega). \tag{3.3}$$

Assumption (p1) (for $\lambda = 0$) ensures that Theorem 1 can be applied, so that, using this result, (3.3) and (p2) imply :

$$|D_L(L - M(\cdot;1), \Omega)| = |d_B(f_0, \Omega \cap \mathbf{R}^m, 0)| \neq 0.$$

Hence, by the existence property of the coincidence degree, there is $\tilde{x} \in \mathrm{dom}L \cap \Omega$ such that $L\tilde{x} = M(\tilde{x};1)$; thus $\tilde{x}(\cdot)$ is a solution to (3.1)-(3.2), with $x(\cdot) \in \mathrm{dom}L \cap \Omega$. The proof is complete. ♦

An immediate consequence of Theorem 2 which is just based on a (suitable) choice of the set $\Omega \subset X$ is the following:

COROLLARY 3. *Let* G *be a bounded open subset of* $\mathbf{R}^m$. *Suppose that the following conditions are satisfied:*

(j1) *("bound set" condition)*
for any $x(\cdot)$, *solution of* (3.1_λ)-(3.2) *such that* $x(t) \in \mathrm{cl}G$ *for all* $t \in [0,T]$,
it follows that $x(t) \in G$ *for all* $t \in [0,T]$;

(j2)
$$d_B(f_0, G, 0) \neq 0.$$

Then, (3.1)-(3.2) *has at least one solution* $x(\cdot)$ *such that* $x(t) \in \mathrm{cl}G$, *for all* $t \in [0,T]$.

Proof. It is sufficient to define (in the setting of Theorem 2):

$$\Omega := \{x \in X : x(t) \in G \ \forall t \in [0,T]\}$$

and to check that (p1)-(p2) are fulfilled. For brevity, we omit the details. ♦

REMARK 1. Corollary 3 is a continuation theorem analogous to [94]. Namely, in [94] the bound set condition is required for equation

$$\dot{x} = \lambda h(t,x;\lambda), \ \lambda \in]0,1[, \tag{3.4}$$

with $h(t,x;1) = F(t,x)$, and, in place of (j2), the Brouwer degree of the averaged vector field $\bar{h}_0(z) := (1/T) \int_0^T h(s,z;0)ds$ is considered.

A comparison between the continuation theorem for (3.4) and Corollary 3 can be made by means of the following examples.

EXAMPLE 1. Let us consider the plane system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\mu x_1^+ + \nu x_1^- + p(t),$$

with $p \in L^1([0,T],\mathbf{R})$, $\mu>0$, $\nu>0$, $x^+ := \max\{x,0\}$, $x^- := \max\{-x,0\}$, which comes from the study of the equivalent second order scalar equation $\ddot{x} + \mu x^+ - \nu x^- = p(t)$.
It is easy to prove that Corollary 3 can be applied with $f(t,x;\lambda) := (x_2, -\mu x_1^+ + \nu x_1^- + \lambda p(t))$, $\lambda \in [0,1]$ and $G = B(0,R)$, for $R>0$ sufficiently large, provided that

$$n(\mu^{-1/2} + \nu^{-1/2}) \neq T/\pi, \quad \text{for every } n \in \mathbf{N}. \tag{3.5}$$

Indeed, in this case a priori bounds for the T–periodic solutions are available (see [31,49]). On the other hand, if we consider the system

$$\dot{x}_1 = \lambda x_2, \quad \dot{x}_2 = \lambda(-\mu x_1^+ + \nu x_1^- + p(t)), \; \lambda \in]0,1],$$

the a priori bounds for the T–periodic solutions can be found only if

$$\lambda^{-1} n(\mu^{-1/2} + \nu^{-1/2}) \neq T/\pi, \quad \text{for every } n \in \mathbf{N} \text{ and } \lambda \in]0,1]. \tag{3.6}$$

Note that (3.6) holds if and only if $\mu^{-1/2} + \nu^{-1/2} > T/\pi$.
Hence, it is easy to choose μ and ν such that (3.5) holds, while (3.6) does not. This elementary example shows that there are situations in which Theorem 2 may be more directly used. In Section 4 we provide some more substantial applications.

Example 1 deals with a periodically perturbed autonomous system. In the case of a general non-autonomous equation

$$\dot{x} = F(t,x),$$

a natural choice for the homotopy in applying Theorem 3 is to take

$$f(t,x;\lambda) = (1 - \lambda)\, \overline{F}(x) + \lambda F(t,x),$$

where $\bar{F}$ is the averaged vector field defined by

$$\bar{F}(x) = \frac{1}{T}\int_0^T F(s,x)ds.$$

EXAMPLE 2. We consider the problem

$$\dot{x} = h(t,x) + p(t) \tag{3.7}$$

$$x(0) = x(T), \tag{3.2}$$

where $h : [0,T]\times\mathbf{R}^m \to \mathbf{R}^m$ is a Caratheodory function positively homogeneous of order $\alpha \neq 1$ in x and $p \in L^1([0,T],\mathbf{R}^m)$. As usual, $\bar{h} : \mathbf{R}^m \to \mathbf{R}^m$ is defined by

$$\bar{h}(x) = \frac{1}{T}\int_0^T h(s,x)ds,$$

and we assume that $\bar{h}(z) \neq 0$ for $|z| = 1$ so that $d_B(\bar{h}, B(0,r),0)$ is defined and constant for each $r > 0$. Let us define $H : X\times[0,1] \to Z$ by

$$H(x;\lambda)(t) := (1-\lambda)\,\bar{h}(x(t)) + \lambda\, h(t,x(t)) + \lambda\delta(\alpha)p(t)\ ,$$

where $\delta(\alpha) = \max\{0,\frac{1-\alpha}{|1-\alpha|}\}$. We first show that there is some $r_0>0$ such that, for each $\lambda\in[0,1]$, the equation

$$Lx = H(x;\lambda)$$

has no solution x with $|x|_\infty = r$, for all $0 < r \leq r_0$ if $\alpha > 1$ and $r \geq r_0$ if $\alpha < 1$.
If this is not the case, there are sequences (x_k) in X and (λ_k) in [0,1] such that $|(x_k)|_\infty = r_k$, $r_k \leq 1/k$ if $\alpha > 1$, $r_k \geq k$ if $\alpha < 1$, and

$$\dot{x}_k = (1-\lambda_k)\,\bar{h}(x_k) + \lambda_k\, h(t,x_k) + \lambda_k\delta(\alpha)p(t)$$

$(k\in\mathbf{N})$. Letting $u_k = x_k / |x_k|_\infty = x_k/r_k$, we get

$$\dot{u}_k = r_k^{1-\alpha}\,[(1-\lambda_k)\,\bar{h}(u_k) + \lambda_k\, h(t,u_k)] + \lambda_k r_k^{-1}\delta(\alpha)p(t) \tag{3.8}$$

so that, a.e. on [0,T],

$$|\dot{u}_k(t)| \leq r_k^{1-\alpha}\beta(t) + \gamma(t) ,$$

for some $\beta \in L^1([0,T],\mathbf{R})$. Consequently, there are subsequences (λ_{j_k}), (u_{j_k}) and $\lambda^* \in [0,1]$, $v \in \mathcal{C}([0,T],\mathbf{R}^m)$, $|v|_\infty = 1$ such that $(u_{j_k}) \to v$ uniformly on [0,T] and $(\lambda_{j_k}) \to \lambda^*$. From

$$u_k(t) - u_k(0) = k^{1-\alpha} \int_0^t [(1-\lambda_k)\ \bar{h}(u_k(s)) + \lambda_k h(s,u_k(s))]ds,$$

we get

$$v(t) - v(0) = 0, \quad t \in [0,T],$$

so that v is constant and $|v|_\infty = 1$. From (3.8) we also get

$$0 = \int_0^T [(1 - \lambda_k)\ \bar{h}(u_k(s)) + \lambda_k\, h(s,u_k(s)) + \lambda_k r_k^{-\alpha}\delta(\alpha)p(s)]ds$$

and hence, letting $j_k \to +\infty$,

$$0 = T\,\bar{h}(v),$$

a contradiction.
Hence,

$$D_L(L - H(\cdot;1), B(0,r)) = D_L(L - H(\cdot;0), B(0,r))$$

and by Theorem 1 and our assumption,

$$D_L(L - H(\cdot;0), B(0,r)) = (-1)^m d_B(\bar{h},B(0,r),0) \neq 0.$$

Then, if $d_B(\bar{h},B(0,1),0) \neq 0$, (3.7)-(3.2) will have at least one solution for each $p \in L^1([0,T],\mathbf{R}^m)$ if $\alpha < 1$ and, when $\alpha > 1$, there will be some $\varepsilon_0 > 0$ such that, for $|p|_1 \leq \varepsilon_0$, one has

$$D_L(L - H(\cdot;1) - p, B(0,r_0)) = D_L(L - H(\cdot;1), B(0,r_0)) \neq 0$$

and (3.7)-(3.2) has at least one solution. This last situation is related to earlier work of Halanay [63] and Mawhin [97].

A simple consequence of Corollary 1 is based on the fact that, whenever a priori bounds for the solutions of (3.1_λ) can be performed, then the "bound set" condition (j1) is satisfied. More precisely, we have:

COROLLARY 4. *Assume that there is a compact set* $K \subset \mathbf{R}^m$ *containing all the solutions of* (3.1_λ)-(3.2) *and such that* $\{z \in \mathbf{R}^m : f_0(z) = 0\} \subset K$. *Let* $G \subset \mathbf{R}^m$ *be an open bounded set such that* $K \subset G$ *and suppose that*

(j2) $$d_B(f_0, G, 0) \neq 0.$$

Then, (3.1)-(3.2) *has at least one solution with values in* K.

A result analogous to Corollary 3 can be performed in the case when the phase space in not $\mathbf{R}^m$ but a closed convex subset C of $\mathbf{R}^m$ with non-empty interior, provided that a flow-invariance condition for the set C is satisfied. More precisely, we have:

THEOREM 3. *Let* $G \subset C$ *be a bounded set which is open relatively to* C, *where* $C \subset \mathbf{R}^m$ *is a closed convex set with* $\mathrm{int}C \neq \emptyset$. *Assume that the following conditions are satisfied:*

(c1) *for each* $u \in \mathrm{fr}C \cap G$ *there is* $\eta \in N(u)$ *such that*

$$(f(t,u;\lambda) \mid \eta) \leq 0 \quad \textit{for a.e. } t \in [0,T] \textit{ and } \lambda \in [0,1];$$

(c2) *for any* $x(\cdot)$, T*–periodic solution of*

$$\dot{x} = f(t,x;\lambda), \quad \lambda \in [0,1[,$$

such that $x(t) \in \mathrm{cl}_C G$ *for all* $t \in [0,T]$, *it follows that* $x(t) \in G$ *for all* $t \in [0,T]$;

(c3) *the fixed point index* $i_C(r(I + f_0),G)$ *is defined and*

$$i_C(r(I + f_0),G) \neq 0,$$

where $r : \mathbf{R}^m \to C$ *is the canonical projection.*

Then, (3.1)-(3.2) *has at least one solution* $x(\cdot)$ *such that* $x(t) \in \mathrm{cl}_C G$, *for all* $t \in [0,T]$.

For the proof of Theorem 3, it is sufficient to use the claims in Theorem 1 and to argue as in [18], where a continuation theorem for the existence of solutions to (3.1)-(3.2) which remain in a convex set is performed. Obviously, Corollary 4 can be modified accordingly.
In Section 5 we prove a continuation theorem which is a generalization of Theorem 2 to the case when the phase space is an Euclidean Neighbourhood Retract (ENR); however, the proof of this result is obtained by embedding (3.1)-(3.2) in a functional-analytic framework which is different from [98].

Another consequence of Corollary 3 can be deduced in the case of planar systems (m=2) for which equation (3.1_λ) takes the form

$$\dot{x}_1 = x_2 - \lambda g_1(x_1) + \lambda P(t), \qquad \dot{x}_2 = -g_2(t,x_1;\lambda), \tag{3.9}$$

where $g_1 : \mathbf{R}\to\mathbf{R}$ and $P : [0,T]\to\mathbf{R}$ are continuous functions and $g_2 : [0,T]\times\mathbf{R}\times[0,1]\to\mathbf{R}$ satisfies the Caratheodory conditions.
Systems like (3.9) come in a natural way from the study of the parametrized Liénard equation in the scalar case ($x\in\mathbf{R}$)

$$\ddot{x} + \lambda\psi_1(x)\dot{x} + \psi_2(t,x;\lambda) = \lambda p(t),$$

imposing $g_1(x_1) := \int_0^{x_1}\psi_1(s)ds$, $g_2 := \psi_2$, $P(t) := \int_0^t p(s)ds$ (usually, $\int_0^T p(s)ds = 0$ is also assumed in order to get $P(0) = P(T)$). In this particular situation, the following one-sided continuation theorem can be proved.

COROLLARY 5. *Suppose that* $g_2(t,z;0) := g_2(z)$ *and assume that there are constants* $R\geq d>0$ *such that*

$$g_2(t,z;\lambda)\cdot z > 0, \textit{ for a.e. } t\in[0,T] \textit{ and all } \lambda\in[0,1[, \quad |z|\geq d$$

and

$$\max\{x_1(t) : t\in[0,T]\} \neq R, \textit{ for any solution } (x_1(t),x_2(t)) \textit{ of } (3.9)\text{-}(3.2), \textit{with } \lambda\in[0,1].$$

Then, system (3.9) *has at least one* T*–periodic solution for* $\lambda=1$.

The proof of Corollary 5 can be performed through the construction of an open rectangle $G=]-M,R[\times]-M,M[\subset\mathbf{R}^2$ such that condition (j1) of Corollary 3 is satisfied with respect to the solutions of (3.9)-(3.2). The choice of the constant $M\geq R$ follows by the estimates developed in [103] and [115]. We omit the rest of the proof referring to these papers for the needed computations. We note that Corollary 5 (or some slight variants of it) is the basic tool for the proof of some recent results concerning the periodic

BVP for some Liénard and Duffing equations under one-sided growth restrictions on the restoring term ψ_2 (see [35,45]). Our result also improves [35, Lemma 1].

REMARK 2. We point out that the results of this section may be extended to the periodic BVP for n-th order differential systems :

$$x^{(n)} + F(t, x, \dot{x}, \ldots, x^{(n-1)}) = 0, \tag{3.10}$$

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0,1,\ldots,n-1, \tag{3.11}$$

with $F : [0,T]\times\mathbf{R}^{nm} \to \mathbf{R}^m$, by means of the standard reduction of (3.10)-(3.11) to the periodic BVP for a system of n first order equations in $\mathbf{R}^m$.
More precisely, we assume that there are $f : [0,T]\times\mathbf{R}^{nm}\times[0,1] \to \mathbf{R}^m$ which fulfils the Caratheodory assumptions and $f_0 : \mathbf{R}^{nm} \to \mathbf{R}^m$ such that

$$F(t, x, \dot{x}, \ldots, x^{(n-1)}) = -f(t, x, \dot{x}, \ldots, x^{(n-1)}; 1),$$

$$f_0(x, \dot{x}, \ldots, x^{(n-1)}) = -f(t, x, \dot{x}, \ldots, x^{(n-1)}; 0).$$

We also define $q_0 : \mathbf{R}^m \to \mathbf{R}^m$ by

$$q_0(z) := f_0(z, 0,\ldots,0), \quad z\in \mathbf{R}^m.$$

Then, we have

COROLLARY 6. *Assume that there is* R>0 *such that*

$$\max \{ |x_i|_\infty : i = 1,\ldots,n-1 \} < R,$$

for all possible solutions $x(\cdot)$ *of*

$$x^{(n)} = f(t, x, \dot{x}, \ldots, x^{(n-1)}; \lambda), \quad \lambda\in [0,1[,$$

satisfying the boundary condition (3.11). *Suppose that , for* $r \geq R$,

$$d_B(q_0, B(0,r), 0) \neq 0.$$

Then, (3.10)-(3.11) *has at least one solution.*

The proof follows straightforwardly from Corollary 4, arguing like in [95], and therefore it is omitted. We recall that in [32, p. 677] a similar result has been obtained for a second order scalar equation using a different approach based upon some equivariant degree theory.

As a final result, we give a continuation theorem based on the study of the Poincaré map. For each $z \in \mathbf{R}^m$, $\lambda \in [0,1]$, we denote by $x(\cdot,z;\lambda)$ the solution of $\dot{x} = f(t,x;\lambda)$ such that $x(0,z;\lambda) = z$. As usual, to do this, we assume uniqueness and global existence for the solutions of the Cauchy problems associated to (3.1_λ). The Poincaré-Andronov operator $U_\lambda = U_\lambda(z) : \mathbf{R}^m \to \mathbf{R}^m$ is defined as follows:

$$U_\lambda(z) = x(T,z;\lambda).$$

Then, we have:

THEOREM 4. *Let* $G \subset \mathbf{R}^m$ *be open and bounded. Assume that the following conditions are satisfied:*

(m1) $U_\lambda(z) \neq z$ *for all* $z \in \mathrm{fr}G$, $\lambda \in [0,1[$;

(m2) $d_B(f_0, G, 0) \neq 0$.

Then, (3.1)-(3.2) *has at least one solution* .

Proof. Without restrction, we can suppose that (m1) holds with $\lambda \in [0,1]$. Then, it is sufficient to observe that assumption (m1) ensures that the map $(I - U_\lambda)$ is an admissible homotopy, so that, by the homotopy invariance of the Brouwer degree,

$$d_B(I - U_1, G, 0) = d_B(I - U_0, G, 0).$$

Furthermore, Corollary 2 is applicable, so that

$$d_B(I - U_1, G, 0) = (-1)^m\, d_B(f_0, G, 0).$$

Hence, there is $z \in \mathbf{R}^m$ such that $U_1(z) = z$. The proof is complete. ♦

Extensions to differential-delay equations may be performed as well, combining Theorem 1 with the arguments developed in [96] (see Chapter 4).

4. Applications

In this section we deal with the problem of the existence of solutions $x(\cdot)$ to

$$\dot{x} = F(t,x) \tag{4.1}$$
$$x(0) = x(T) \tag{4.2}$$

such that $x(t) \in \mathrm{cl}G$ for all $t \in [0,T]$, where G is an open bounded subset of $\mathbf{R}^m$.
We state some consequences of Theorem 2 and of its corollaries which illustrate the range of applicability of our main result.

Throughout this section, we assume that the nonlinear field F splits as

$$F(t,x) = g(x) + e(t,x), \tag{4.3}$$

where the function $g : \mathbf{R}^m \to \mathbf{R}^m$ is continuous and $e : [0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ satisfies the Caratheodory assumptions. First, we consider the case of "small perturbations"; then, we study large perturbations of positively homogeneous vector fields.

4.a. Small perturbations

COROLLARY 7. *Assume that the following conditions are satisfied:*

(k1) *for any* T*–periodic solution* $x(\cdot)$ *of*

$$\dot{x} = g(x) \tag{4.4}$$

such that $x(t) \in \mathrm{cl}G$ *for all* $t \in [0,T]$, *it follows that* $x(t) \in G$ *for all* $t \in [0,T]$;

(k2) $d_B(g,G,0) \neq 0.$

Then, there is $\varepsilon_0 > 0$ *such that, for any forcing term* $e(\cdot,\cdot)$ *with* $|e(\cdot, z)|_\infty \le \varepsilon_0$ *for all* $z \in \mathrm{cl}G$, *system* (4.1) *has at least one* T*–periodic solution* $x(\cdot)$ *such that* $x(t) \in \mathrm{cl}G$ *for all* $t \in [0,T]$.

Proof. We apply Corollary 3 with $f_0 = g$. We imbed (4.1) in the family of parametrized equations

$$\dot{x} = f(t,x;\lambda) := g(x) + \lambda e(t, x), \ \ \lambda \in [0,1], \tag{4.1_λ}$$

and we claim that there is $\varepsilon_0 > 0$ such that for every function $e(\cdot, z)$ with $|e(\cdot, z)|_\infty \le \varepsilon_0$ for all z, the set G is a "bound set" for (4.1_λ).
Indeed, assume by contradiction that, for each $n \in \mathbf{N}$, there is a function e_n such that $|e_n(\cdot,z)|_\infty \le 1/n$ for all z and there is an T–periodic function $x_n(\cdot)$ satisfying

$$\dot{x}_n(t) = g(x_n(t)) + \lambda_n e_n(t,x_n(t)), \quad \lambda_n \in [0,1], \tag{4.5}$$

such that $x_n(t) \in clG$ for all t and $x_n(t_n) \in frG$ for some $t_n \in [0,T]$. By Ascoli-Arzelà's theorem we have that there is a T–periodic solution $x^*(\cdot)$ of (4.4), with $x^*(t) \in clG$ $\forall t$, such that (up to subsequences) $x_n \to x^*$ uniformly on [0,T]. Moreover, for $t_n \to t^*$, we have $x^*(t^*) \in frG$. Thus, passing to the limit in (4.5), a contradiction with (k1) is reached and the claim is proved, so that (j1) is satisfied for $e(\cdot)$ sufficiently small. Thus we can apply Corollary 3 and the proof is complete. ♦

With elementary changes in the proof it can be seen that the result is still true when $e(t,x) = e(t)$ and bounds for $|e|_1$ are considered.

Corollary 7 enables us to recover a number of previous results, thanks especially to the rather weak condition (k1).
For example, (k1) is satisfied whenever the flow π^0 induced by (4.4) is dissipative (i.e. there is a compact set $K \subset \mathbf{R}^m$ such that for each $x \in \mathbf{R}^m$ there is $t_x \geq 0$ with $\pi^0(t,x) \in K$ for all $t \geq t_x$); indeed, if this is the case, then $d_B(g,G,0) = (-1)^m \chi(\mathbf{R}^m) = (-1)^m$, for every $G \supseteq K$, where χ is the Euler-Poincaré characteristic (see [81],[138, Th.6.1]). Hence, Corollary 7 guarantees the existence of periodic solutions for small periodic perturbations of autonomous dissipative systems. In this manner, we recover some classical results contained in [29,68,123].
Now, we discuss other results for the existence of solutions to (4.1)-(4.2) where some conditions less general than (k1) are required.
In the two-dimensional case, J. Cronin [27,28,29] and A. C. Lando [84,85] deal with periodic perturbations of autonomous systems of the form:

$$\begin{aligned} \dot{x} &= X(x,y) + \varepsilon E_1(t) \\ \dot{y} &= Y(x,y) + \varepsilon E_2(t). \end{aligned} \tag{4.6}$$

Following Gomory's approach [57], the authors are led to construct a simple closed curve $\mathcal{J}$ (containing the origin in his interior) such that the unperturbed system

$$\begin{aligned} \dot{x} &= X(x,y) \\ \dot{y} &= Y(x,y) \end{aligned} \tag{4.7}$$

has no closed orbits intersecting $\mathcal{J}$.
Clearly, in this situation (k1) is satisfied and condition (k2) either is explicitly required (see [84,85]), or it is an implicit consequence of other hypotheses. For instance, in [27,28,29] it is assumed that "the point at infinity is strongly stable relative to (4.7)". However, in this case it can be proved that $d_B((X,Y), B(0,R), 0) = 1$, for R sufficiently large.

From the above discussion, it follows that Corollary 7 contains all the results proved in [27, Th.2], [28, Th. 6], [29, Th. 2], [84, Th. 3], [85, Th. 3].
On the other hand, we observe that none of the above quoted theorems is suitable for dealing with systems like

$$\begin{aligned} \dot{x} &= -y^3 + \varepsilon E_1(t) \\ \dot{y} &= x^3 + \varepsilon E_2(t). \end{aligned} \tag{4.8}$$

(see [28, p. 159]), while Corollary 7 applies.
We also note that many regularity hypotheses which are required in [27,28,29,84,85] are avoided using our approach.
Another condition (stronger than (k1)-(k2)) leading to the existence of T–periodic solutions of (4.6) was given in [1, Th. 2], where it is assumed that the origin is an isolated critical point with non-zero index and it is not an isochronous center of period T/k ($k \in \mathbb{N}$). In fact, in this case it is sufficient to take $G = B(0,\delta)$, with $\delta > 0$ sufficiently small. On the same line, see [9, Th. 2]. Finally, we mention that, by means of Corollary 7, we can give an easy proof of the

NEMITZKII'S CONJECTURE (first settled by A. Halanay [62]): *If the autonomous system* (4.7) *has a limit cycle, then there is least one* T*–periodic solution of* (4.6)*, for* ε *sufficiently small.*

Again Corollary 7 may be applied, choosing G such that frG is "sufficiently close" to the limit cycle.

In the higher dimensional case, Corollary 7 is an improvement of [1, Th. 1], [9, Th. 1], [64, Th. 3.13], where, besides (k2), various specific conditions are required, such as, e.g. [64], "The origin is the only critical point of (4.7) in a neighbourhood G of itself, and (4.7) has no periodic solutions of period $\tilde{T}$, $0<\tilde{T}\leq T$, passing through points of frG".
Corollary 7 is also a generalization of [138, Th. 4], [145, Th. 4.1, (c1)], where, instead of (k1), the existence of a compact isolating neighbourhood K for the flow induced by (4.4) is required. Indeed, if this is the case then G = intK is suitable for the validity of Corollary 7. Again, equation (4.8) provides an example of applicability of our result while [138,145] cannot be used (see also Example 3 below).

We end this subsection with an example of a system which is non-dissipative and such that, furthermore, the corresponding autonomous system has the origin as a global center. For related results see [50,118].

EXAMPLE 3. We deal with the forced nonlinear second order scalar equation:

$$\ddot{x} + \psi(x) = p(t), \tag{4.9}$$

where $\psi : \mathbf{R}\to\mathbf{R}$ is continuous and $p : \mathbf{R}_+\to\mathbf{R}$ is continuous, T–periodic and such that $\bar{p} := \frac{1}{T}\int_0^T p(s)ds = 0$. As is well-known, equation (4.9) is equivalent to the phase-plane system:

$$\dot{x} = y + P(t), \qquad \dot{y} = -\psi(x),$$

where

$$P(t) := \int_0^t p(s)ds.$$

We assume that the function ψ is continuously differentiable, odd and satisfies:

$$\psi(x)\cdot x > 0 \quad \text{for } |x| \neq 0, \tag{4.10}$$

$$\lim_{|x|\to\infty} \Psi(x) = +\infty, \quad \text{with } \Psi(x) := \int_0^x \psi(\xi)d\xi. \tag{4.11}$$

From (4.10) and (4.11), it follows that the origin in $\mathbf{R}^2$ is a global center for the autonomous system

$$(\dot{x},\dot{y}) = g(x,y) := (y,-\psi(x)), \tag{4.12}$$

so that there is no compact isolating neighbourhood clG of the origin (with G open).
Moreover, for any open bounded set $G\subset\mathbf{R}^2$,

$$d_B(g,G,0) = 1 \ \text{ for } 0\in G, \qquad d_B(g,G,0) = 0 \text{ for } 0\notin G.$$

Hence, Theorem 4.1 in [145] cannot be applied.
On the other hand, in order to use Corollary 7 it is sufficient to find a bound set G for (4.12), i.e. an open bounded set with $0\in G$ such that there is no T–periodic solution of (4.12) "tangent" to frG. To this end, we consider, for any c>0, the level set $\Psi_c := \{(x,y)\in\mathbf{R}^2: (1/2)y^2 + \Psi(x) < c\}$. Then, $\mathrm{fr}\Psi_c$ is a periodic orbit with minimum period:

$$T_c = \sqrt{2}\int_d^c \frac{1}{\sqrt{\Psi(c) - \Psi(\xi)}}d\xi, \qquad \text{with } d < 0 < c, \ \psi(d) = \psi(c).$$

Hence, it is sufficient to find $c > 0$ such that $T_c \neq T/n$ for all $n \in \mathbf{N}$. Such a choice of c is always possible if the continuous map $\tau :]0,+\infty[\to]0,+\infty[$, $c \longmapsto T_c$ is not constant.
Then, the following result follows from Corollary 7.

PROPOSITION 1. *For any continuous map* $\psi : \mathbf{R}\to\mathbf{R}$ *satisfying* (4.10)-(4.11) *and having a non-constant associated time-map* τ, *there is* $\varepsilon>0$ *such that equation* (4.9) *has a* T*–periodic solution for every* T*–periodic forcing term* $p(\cdot)$ *with* $|p|_1 \leq \varepsilon$.

Recall that if ψ is continuously differentiable and odd, then, by a classical theorem of M. Urabe [62, §13.3, Cor. 4], $\tau(\cdot)$ is constant if and only if $\psi : \mathbf{R}\to\mathbf{R}$ is linear.

4.b. Asymptotically homogeneous systems

In this subsection, we deal with perturbations of autonomous systems with positively homogeneous nonlinearity. More precisely, we consider equations of the form

$$\dot{x} = g(x) + e(t,x), \tag{4.13}$$

with $g : \mathbf{R}^m\to\mathbf{R}^m$ continuous and such that, for some $\alpha>0$,

(L1) $\quad g(kx) = k^{\alpha}\, g(x)$ *for all* $k > 0$, $x\in \mathbf{R}^m$

and $e : [0,T]\times\mathbf{R}^m\to\mathbf{R}^m$ satisfying the Caratheodory conditions and such that

(L2) $\quad \lim\limits_{|x|\to+\infty} (|e(t,x)| / |x|^{\alpha}) = 0$ *uniformly a.e. in* $t\in [0,T]$;

Systems of the form (4.13) have been widely studied; see, for instance, [81,86,88,101,110,111]. In [81, § 41], [110] the more general situation in which the function g may depend on t is considered too. However, as we show below, there are situations that can be settled in the framework of Corollary 4 but do not fit in [81,111].

In the first result of this subsection we consider the case of g homogeneous of degree one.

COROLLARY 8. *Assume* (L1)-(L2) *with* $\alpha = 1$. *Suppose that the following conditions are satisfied:*

(L3) $\quad x = 0$ *is the only* T*–periodic solution of*

$$\dot{x} = g(x); \tag{4.14}$$

(L4) $\quad d_B(g, B(0, R_0), 0) \neq 0,$ *for some* $R_0 > 0.$

Then, there is at least one T*–periodic solution to* (4.13).

Note that, from (L3), the origin is the unique singular point of g, and (L1)-(L4) imply that $d_B(g, B(0,R), 0) \neq 0$, for every R>0.

Proof. We apply Corollary 4, with $K = B[0,R]$ for $R > 0$ sufficiently large, $G = B(0,R_1)$, $R_1 > R$, $f_0 = g$, $f(t,x;\lambda) = g(x) + \lambda e(t,x)$. In order to find R and, as a consequence, to prove the existence of a priori bounds for the solutions of (4.1_λ), assume by contradiction that there is a sequence (x_n) of T–periodic functions, with $|x_n|_\infty \to +\infty$ and such that, for every $n \in N$,

$$\dot{x}_n = g(x_n) + \lambda_n\, e(t,x_n), \quad \lambda_n \in [0,1]. \tag{4.15}$$

Now we set, for all $n \in N$,

$$y_n(\cdot) := x_n(\cdot) \,/\, |x_n|_\infty\,,$$

so that, dividing (4.15) by $|x_n|_\infty$ and using (L1) we get

$$\dot{y}_n = g(y_n) + \lambda_n\, e(t,x_n) \,/\, |x_n|_\infty\,. \tag{4.16}$$

We observe that we can apply Ascoli-Arzelà's theorem; therefore, there exists y^*, with $|y^*|_\infty = 1$, such that (up to subsequences) $y_n \to y^*$ uniformly on $[0,T]$. Thus, taking the limit as $n \to +\infty$ in (4.16) and using (L2) we obtain:

$$\dot{y}^* = g(y^*).$$

Therefore, by (L3), $y^* \equiv 0$, which is a contradiction.
Thus, we have proved that there is $R > 0$ such that, for every T–periodic solution $x(\cdot)$ of (4.14) $x(t) \in B[0,R]$ for all $t \in [0,T]$. Therefore, using (L4) we see that Corollary 4 is applicable, and we get the existence of an T–periodic solution of (4.13) such that $x(t) \in B[0,R]$ for all $t \in [0,T]$. The proof is complete. ♦

REMARK 3. Corollary 8 is a generalization of the results (in the case $\alpha = 1$) in [111], where, besides (L1)-(L2)-(L4), the fact that there are no cycles or nontrivial equilibrium states for the autonomous system $\dot{x} = g(x)$ is assumed. In other words, (L3) must hold for periodic solutions of *any* period. Hence, the range of applicability of our corollary is wider than Muhamadiev's one. For instance, Muhamadiev's theorem does not apply to Example 1 in Section 3; indeed, in that situation, for μ, $\nu > 0$ the origin is a global center for the autonomous system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\mu x_1^+ + \nu x_1^-,$$

while our result applies provided that (μ,ν) does not belong to the Dancer-Fučik spectrum [49].
We also point out that, apparently, Corollary 8 (or, more precisely, the evaluation of the Leray-Schauder degree of the Nemitzkii operator induced by (4.14) in terms of the Brouwer degree of g) cannot be obtained by means of the techniques developed in [81]. To this regard, see the problem raised in [81, p. 253].

Now, we state the analogue of Corollary 8 for n-th order systems of the form

$$x^{(n)} + F(x, \dot{x}, \ldots, x^{(n-1)}) = e(t,x,\dot{x},\ldots,x^{(n-1)}). \tag{4.17}$$

COROLLARY 9. *Assume that the following conditions are satisfied:*

(f1) $F(kx, k\dot{x}, \ldots, kx^{(n-1)}) = kF(x, \dot{x}, \ldots, x^{(n-1)})$,

for all $k > 0$ *and* $(x, \dot{x}, \ldots, x^{(n-1)}) \in \mathbf{R}^{nm}$;

(f2) $$\lim_{|x|+|\dot{x}|+\ldots+|x^{(n-1)}| \to +\infty} |e(t,x,\dot{x},\ldots,x^{(n-1)})| \,/\, (|x|+|\dot{x}|+\ldots+|x^{(n-1)}|) = 0$$

uniformly a.e. in $t \in [0,T]$;

(f3) $x(t) = 0$, *for all* $t \in [0,T]$, *is the only* T*–periodic solution of*

$$x^{(n)} + F(x, \dot{x}, \ldots, x^{(n-1)}) = 0.$$

(f4) $d_B(q, B(0, r), 0) \neq 0$, *for* $r > 0$,

where $q(x) := F(x,0,\ldots,0)$.

Then, (4.17) *has at least one* T*–periodic solution.*

As for the proof, it is sufficient to repeat the argument in the proof of Corollary 9 and to apply Corollary 6.

REMARK 4. The particular case when $q(x) = F(x,0...,0) = \nabla V(x)$, with $V : R^m \rightarrow R$ a positively homogeneous potential of degree 2 has been considered by many authors (see, e.g., [81, §12.4], [88]). Corollary 9 improves the result in [88], where system

$$\ddot{x} + \nabla V(x) = p(t)$$

is studied. Indeed, besides assumptions analogous to (f1)-(f2)-(f3), it is assumed in [88] that $V(x)>0$ for $x\neq 0$ so that (f4) holds as well (see [81, Th. 12.6]). This remark shows that Corollary 9 contains the classical theorems in [31,49] on jumping nonlinearities, where asymptotically homogeneous autonomous equations are considered, and some of the results in [47,48]. An easier proof of the theorem in [88] has been recently obtained in [101].

As a second consequence of Corollary 4, we perform a result for asymptotically positively homogeneous systems of order α, with $\alpha\neq 1$.

COROLLARY 10. *Assume* (L1)-(L2)-(L4) *and suppose, respectively, either*

(L'3) $x = 0$ *is the only bounded solution of* $\dot{x} = g(x)$ (if $\alpha>1$);

or

(L"3) $g(x) \neq 0$ *for* $x\neq 0$ (if $\alpha<1$).

Then, (4.13) *has at least one* T*–periodic solution.*

The proof can be obtained by repeating essentially the proof of Corollary 8, or following Muhamadiev's argument [111].

By means of [111] and Corollary 4 it is easy to extend the result to systems of the form

$$\dot{x} = g(t,x) + e(t,x), \tag{4.18}$$

using an homotopy between (4.18) and either the "freezed" system $\dot{x} = g(0,x)$ for $\alpha>1$, or the "averaged" system $\dot{x} = \bar{g}(x)$ for $\alpha<1$.

We do not give any new contribution for (4.18), except for the abstract theorem we use; hence, we do not state such results in detail.

Finally, we mention that by means of the (continuation) Theorem 1 it is possible to obtain a result on the so-called "regular guiding functions" (see [10],[81, § 14]). In this way, we can easily recover [26,30,57], where a perturbation of a polynomial in $\mathbf{R}^2$ is studied.

5. An extension to flow-invariant ENRs

In Chapter 2, a variant of Mawhin's continuation theorem has been obtained for differential systems inducing a flow on some closed ENRs. In what follows we give a similar extension of Theorem 2, provided that a Kupka-Smale approximation property holds.

Let $C \subset \mathbf{R}^m$ be a closed ENR. In this section, our goal is to prove the existence of a solution $x(\cdot)$ to

$$\dot{x} = F(t,x) \tag{5.1}$$

$$x(0) = x(T) \tag{5.2}$$

such that, for all $t \in [0,T]$, $x(t)$ belongs to a certain subset of C.
The particular case when $C = \mathbf{R}^m$ has been treated in the previous sections.
As before, we assume that

$$F(t,x) := f(t,x;1), \tag{5.3}$$

where

$$f = f(t,x;\lambda) : [0,T] \times C \times [0,1] \to \mathbf{R}^m$$

is a continuous function which is locally lipschitzian in x, uniformly in t, λ. Once for all, we point out that such assumption is not strictly necessary in our proofs, but it provides the uniqueness of the solutions to all the Cauchy problems which we will consider henceforth.
Moreover, we assume that for $\lambda = 0$ the map f is autonomous, i.e. there exists a function $f_0 : C \to \mathbf{R}^m$ such that

$$f_0(x) = f(t,x;0) \tag{5.4}$$

for all $t \in [0,T]$, $x \in C$.
In this more general situation, we need a "flow-invariance" hypothesis ensuring that system

$$\dot{x} = f(t,x;\lambda) \tag{5.5}$$

induces a local process in C, for all $\lambda \in [0,1]$. More precisely, we want that, for each $(t_0,x_0) \in [0,T[\times C$ and for all $\lambda \in [0,1]$, the Cauchy problem

$$\dot{x} = f(t,x;\lambda), \qquad x(t_0) = x_0$$

has a solution $x(\cdot) : \mathrm{dom}\, x(\cdot) \to C$ defined on a right maximal neighbourhood of t_0. Nagumo's theorem (see [112,148]) ensures that this fact holds true if and only if

(i1) $$f(t,z;\lambda) \in T(z;C) \quad \textit{for all } t \in [0,T],\ z \in \mathrm{fr}C,\ \lambda \in [0,1],$$

where $T(z;C)$ is the (Bouligand) tangent cone to C at z. In other words, condition (i1) means that the function f is "subtangential" to C at $z \in \mathrm{fr}C$.

We refer to Section 5 of Chapter 1 for remarks concerning the validity of (i1) in some particular cases.

Now, we introduce the crucial "approximation"

PROPERTY (A). *If*

$$f_0(z) \in T(z;C) \qquad \textit{for all } z \in \mathrm{fr}C,$$

then there exists a sequence of locally lipschitzian functions (φ_k), $\varphi_k : C \to \mathbf{R}^m$ *such that:*

(a') $\varphi_k(z) \in T(z;C)$ *for all* $z \in \mathrm{fr}C$, $k \in \mathbf{N}$;

(b') $\varphi_k \to f_0$ *uniformly on compact sets ;*

(c') *for every compact subset* K *of* C *and for every* $k \in \mathbf{N}$ *system*

$$\dot{x} = \varphi_k(x)$$

has finitely many singular orbits (i.e., rest points and closed orbits) with minimal period in $[0,T+1]$ *which are contained in* K.

We stress the fact that if the set C is a manifold (with or without boundary) and f_0 is a tangent vector field to C, then property (A) is satisfied. Indeed, this is a consequence of the Kupka-Smale's theorem. In particular, property (A) is satisfied in the case when $C = \mathbf{R}^m$ (as in the above Sections). If the set C is a closed convex set with nonempty interior (as in [18]), then it is easy to prove that property (A) is satisfied. Indeed, if (5.5) holds one can show, by a standard perturbation argument, that there are sequences (ψ_k) and $(\delta_k)\downarrow 0$ such that $\psi_k \to f_0$ uniformly on compact sets and $(\psi_k(z) \mid \eta) \leq (-\delta_k \mid \eta) < 0$ for all

$z \in frC$, $\eta \in N(z,C)$. Now, by the Kupka-Smale's theorem, we have that, for each $k \in \mathbf{N}$, there is a sequence $(\varphi_{k,n})_{n=0}^{\infty}$ satisfying (a'), (b') and (c'). Finally, a diagonal argument leads to the conclusion.

In what follows, we denote, as usual, by Γ the complete metric space of the continuous functions $x(\cdot) : [0,T] \to C$ endowed with the distance d^*, $d^*(x_1, x_2) := |x_1 - x_2|_\infty$. We recall the following crucial result (see [75]): the space (Γ, d^*) is a metric ANR if and only if the set C is an ANR. This theorem will enable us to work with the fixed point index of compact operators defined in the function space Γ. For the definition and properties of the index of rest points, we refer to Section 4 of Chapter 1.
Notice that in what follows points of C will be identified with constant functions.
Now, we are in position to state the main result of this Chapter.

THEOREM 5. *Assume* (i1) *and* (A). *Let* $\Omega \subset \Gamma$ *be an open bounded set such that the following conditions are satisfied:*

(i2) *there is no* $x(\cdot) \in fr_\Gamma \Omega$ *such that*

$$\dot{x} = f(t,x;\lambda) \ , \quad \lambda \in [0,1[\ ; \tag{5.1$_\lambda$}$$

(i3) $I(\pi^0, \Omega \cap C) \neq 0$.

Then, (5.1)-(5.2) *has at least one solution* $x(\cdot) \in cl_\Gamma \Omega$.

Proof. We begin by observing that, as Ω is bounded, there is a constant $R > 0$ such that $|x|_\infty < R$, for every $x \in cl_\Gamma \Omega$.
Now, consider a sequence of locally lipschitzian functions (φ_k), $\varphi_k : C \to \mathbf{R}^m$, with $\varphi_k \to f_0$, uniformly on $C \cap B[0, R]$, and satisfying (a') and (c'), according to Property (A).
As a first step, we claim that, without loss of generality, we can suppose that

(d') for each $k \in \mathbf{N}$, the problem

$$\dot{x} = \varphi_k(x) \ , \ x(0) = x(T)$$

has no nontrivial solution in $C \cap B(0,R)$.

Indeed, consider the singular orbits (i.e. rest points and closed orbits) with minimal period in $[0,T+1]$ of the system $\dot{x} = \varphi_k(x)$. By (c'), there exist finitely many such orbits $S_1, \dots, S_{n_k}$, which are contained in $C \cap B(0,R)$. Let $z_i \ (1 \le i \le p_k)$ be the rest points among the S_i. Arguing as in the proof of Theorem 1 (from step (2.14) to step (2.17)), we can find, for each $k \in N$, a constant $\tau_k > T$ such that, for each $T < T' < \tau_k$, the problem

$$\dot{x} = \varphi_k(x) \ , \ x(0) = x(T')$$

has no solution in $C \cap B(0,R)$ other than the equilibria $z_1, \ldots, z_{p_k}$.
Hence, if we choose, for each $k \in N$,

$$T < T'_k < \min \{\tau_k , T + \frac{1}{K}\} \ ,$$

and define $\tilde{\varphi}_k := \frac{T'_k}{T} \varphi_k(z)$, for each $z \in C$, we get a sequence of locally lipschitzian functions $(\tilde{\varphi}_k)$, $\tilde{\varphi}_k : C \to \mathbf{R}^m$, with $\tilde{\varphi}_k \to f_0$, uniformly on $C \cap B[0,R]$, which satisfies (a') (by the cone property of $T(z; C)$) and such that the problem

$$\dot{x} = \tilde{\varphi}_k(x) \ , \ x(0) = x(T)$$

has no nontrivial solution in $C \cap B(0,R)$. The claim is proved.
Then, in the sequel, we can assume (d').
As a next step, we proceed along the lines of the proof of Theorem 1 in Chapter 2.
Without loss of generality, we suppose that (i2) holds with $\lambda \in [0,1]$ in (5.1_λ) (otherwise, the result is proved for $x \in fr_\Gamma \Omega$).
Let us consider the Cauchy problem

$$\begin{cases} y' = f(t,y;\lambda) & (5.6) \\ y(\sigma) = z \ . & (5.7) \end{cases}$$

We can assume f bounded, possibly replacing it by a modified function like $f(t,x;\lambda)\cdot\rho(|x|)$, as in the proof of Theorem 1 in Chapter 2. Since, in this situation, uniqueness and global existence for (5.6)-(5.7) are guaranteed, then if we denote by $u(\sigma,z,\cdot;\lambda)$ the solution of (5.6)-(5.7) a one-parameter family of processes is defined.
Besides, we introduce a one-parameter family of compact operators defined on $\Gamma \times [0,1]$ as follows:

$$M(x;\lambda) := u(0,x(T),\cdot\, ;\lambda) \ , \ \lambda \in [0,1].$$

By (i1), $M : \Gamma \times [0,1] \to \Gamma$; furthermore, Ascoli-Arzelà's theorem ensures that M is compact on $cl_\Gamma \Omega \times [0,1]$. By the definition of M, it is easily seen that x is a fixed point of $M(\cdot;\lambda)$ if and only if $x(\cdot)$

is a solution of (5.6) such that $x(0) = x(T)$. Accordingly, our aim is to prove the existence of a fixed point of the operator $M(\cdot\,; 1)$. This fact, together with (5.3), implies the thesis.
By assumption (i2), we have that $M(x; \lambda) \neq x$ for all $x \in \mathrm{fr}_\Gamma \Omega$ and $\lambda \in [0,1]$, so that M is an admissible homotopy and

$$i_\Gamma(M(\cdot\,; 1), \Omega) = i_\Gamma(M(\cdot\,; 0), \Omega).$$

Observe that the existence of a fixed point of the operator $M(\cdot\,; 0)$ is equivalent to the existence of a T–periodic solution of the autonomous system (5.8).
Now, by property (A) and the preceding claim, there is a sequence (φ_k) satisfying (a'), (b'), (c') and (d'), relatively to $C \cap B[0,R]$. Let us denote, for every $k \in \mathbf{N}$, by $\pi^{k,\mu} : \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ the dynamical system induced by

$$\dot{y} = \mu f_0(y) + (1 - \mu)\, \varphi_k(y)\,,$$

with $\mu \in [0,1]$.
Thus, for each $k \in \mathbf{N}$, a one-parameter family of dynamical systems is defined. Assumptions (i1) and (A) imply that the set C is flow-invariant for the dynamical systems $\pi^{k,\mu}$ as well. Indeed, this follows from the convexity and the cone property of $T(z;C)$ (cfr. Lemma 1 in Chapter 2 for the detailed proof of an analogous result).
On the other hand, assumption (i2) implies that $f_0(z) \neq 0$ for all $z \in \mathrm{fr}_\Gamma \Omega \cap C$; hence, by the compactness of $\mathrm{fr}_\Gamma \Omega \cap X$, we obtain that there is $\varepsilon_0 \in]0,T[$ such that

$$\pi^{k,0}(\varepsilon,z) = \pi^{0}(\varepsilon,z) = \pi^{0}_{\varepsilon}(z) \neq z, \qquad (5.8)$$

for all $z \in \mathrm{fr}_\Gamma \Omega \cap C$, $\varepsilon \in]0,\varepsilon_0]$.
Now, as before, we introduce, for every $k \in \mathbf{N}$, an operator N^k defined on $\Gamma \times [0,1]$ by

$$N^k(x;\mu) := \pi^{k,\mu}(\cdot,x(T));$$

observe that $N^k : \Gamma \times [0,1] \to \Gamma$ and that it is compact on $\mathrm{cl}_\Gamma \Omega \times [0,1]$.
Moreover,

$$N^k(\cdot;1) = M(\cdot;0),$$

for all $k \in \mathbf{N}$.
We claim that there is k_0 such that, for all $k \geq k_0$ and for all $\mu \in [0,1]$, $N_k(\cdot;\mu)$ is an admissible homotopy. This fact will imply, in particular, that if we denote by $\pi^k := \pi^{1,k}$ the flow induced by $\dot{x} = \varphi_k(x)$, then

$$\pi^k(\varepsilon_0,z) \neq z, \quad \text{for all } z\in C\cap \mathrm{fr}_\Gamma\Omega, \quad k\geq k_0. \tag{5.9}$$

Moreover, a classical compactness argument implies that, for any $k\geq k_0$, there is $\delta_1 = \delta_1(k) > 0$ such that

$$\pi^k(\varepsilon_0,y) \neq y, \quad \text{for all } y\in B(\mathrm{fr}_\Gamma\Omega\cap C,\delta_1). \tag{5.10}$$

Indeed, it is sufficient to observe that the sequence of operators (N^k) converges to $M(\cdot;0)$ uniformly on $\mathrm{cl}_\Gamma\Omega\times[0,1]$ and that

$$\inf\{d^*(x, M(x;0)) : x\in \mathrm{fr}_\Gamma\Omega\} > 0$$

(recall that $M(\cdot;0)$ is compact on $\mathrm{fr}_\Gamma\Omega$ and $\mathrm{fr}_\Gamma\Omega$ is closed). Hence, the claim is proved and we can write:

$$i_\Gamma(N_k(\cdot\ ;1), \Omega) = i_\Gamma(N_k(\cdot\ ;0), \Omega), \tag{5.11}$$

for every $k\geq k_0$ and, in particular,

$$i_C(\pi^k_{\varepsilon_0},\Omega\cap C) = i_C(\pi^0_{\varepsilon_0},\Omega\cap C). \tag{5.12}$$

Let us fix $k^* \geq k_0$. For brevity, we set $\varphi := \varphi_{k^*}$, $N := N^{k^*}$, $\pi^\varphi := \pi^{k^*}$, $\delta_1 := \delta_1(k^*)$.
Let $S_1,\dots, S_n \subset C\cap B(0,R)$ be the singular orbits (i.e. rest points and closed orbits) with minimal period in $[0, T+1]$ of the dynamical system π^φ induced by

$$\dot{x} = \varphi(x) \tag{5.13}$$

which are contained in $C\cap B(0,R)$.
Arrange the indexes so that z_i, for $1\leq i\leq p$, are the rest points of π^φ in $C\cap B(0,R)$, that is $z_i = S_i$, for $i = 1,\dots, p$.
By (d'), we know that $x(\cdot)\in \mathrm{cl}_\Gamma\Omega$ is a fixed point of the operator $N(\cdot;0)$ if and only if $x(t) = z_i$, for all $t\in [0, T]$, with $z_i\in \Omega\cap C$. Hence, by excision,

$$i_\Gamma(N(\cdot;0), \Omega) = \sum_{\substack{j=1\\ z_j\in\Omega}}^{p} i_\Gamma(N(\cdot;0), B(z_j, \delta)), \tag{5.14}$$

where

$$\delta = \min\{\delta_1, \eta/2\}, \qquad \eta = \min\{\mathrm{dist}(S_i, S_j), 1\leq i\neq j\leq n\}.$$

Now, we introduce a third homotopy by whom, roughly speaking, we "move along the orbits" of the dynamical system π^{φ}.

We define an operator H on $\Gamma\times[0,1]$ as follows:

$$H(x;\beta) := \pi^{\varphi}\,(x(T),\,(1-\beta)\varepsilon_0 + \beta\cdot).$$

As before, $H : \Gamma\times[0,1]\to\Gamma$ and it is compact on $cl_{\Gamma}\Omega\times[0,1]$. Moreover,

$$N(\cdot;0) = H(\cdot;1). \tag{5.15}$$

We observe that x is a fixed point of $H(\cdot;\beta)$ if and only if $x(t) \equiv y((1 - \beta)\varepsilon + \beta t)$, with $y(\cdot)$ a γ_0-periodic solution of

$$\dot{y} = \varphi(y), \qquad y(0) = x(T)$$

with $\gamma_0 := (1-\beta)\varepsilon + \beta T$.

By the same argument used in the proof of Theorem 1 (from step (2.18) to step (2.19)), we have that $H(\cdot;\beta)$ has no fixed points on $fr_{\Gamma}B(z_j, \delta)$, for each $j = 1,\ldots, p$ $(z_j\in\Omega)$ and each $\beta\in[0,1]$, so that

$$i_{\Gamma}(H(\cdot;\,1),\,B(z_j,\,\delta)) \;=\; i_{\Gamma}(H(\cdot;\,0),\,B(z_j,\,\delta))\,. \tag{5.16}$$

Since the only fixed points of $H(\cdot;\,0)$ in $cl_{\Gamma}\Omega$ are z_j, $j = 1,\ldots, p$, $(z_j\in\Omega)$, by excision it follows that

$$i_{\Gamma}(H(\cdot;\,0),\Omega) \;=\; \sum_{\substack{j=1\\ z_j\in\Omega}}^{p} i_{\Gamma}(H(\cdot;\,0),\,B(z_j,\,\delta)). \tag{5.17}$$

Therefore, from (5.14), (5.15), (5.16) and (5.17) we obtain that

$$i_{\Gamma}(N(\cdot;\,0),\,\Omega) = i_{\Gamma}(H(\cdot;\,0),\,\Omega).$$

Since $H(\cdot\,;\,0) : \Gamma \to C$, then by the contraction property of the fixed point index (see [113]) we have:

$$i_{\Gamma}(H(\cdot\,;\,0),\,\Omega) = i_{C}(H(\cdot\,;\,0),\,\Omega\cap C).$$

Furthermore, using the fact that

$$H(x;\,0) = \pi^{\varphi}(\varepsilon_0,\,x(T)) = \pi^{\varphi}_{\varepsilon_0}(x(T)),$$

we get

$$i_C(H(\cdot\,;0),\Omega\cap C) = i_C(\pi^{\varphi}_{\varepsilon_0},\Omega\cap C).$$

Finally, by the choice of k* , we can use (5.12), so that

$$i_C(\pi^{\varphi}_{\varepsilon_0},\Omega\cap C) = i_C(\pi^{0}_{\varepsilon_0},\Omega\cap C) = i_C(\pi^{0}_{\varepsilon},\Omega\cap C),$$

for $\varepsilon\in\,]0,\varepsilon_0]$.
In conclusion, we have proved that $i_\Gamma(M(\cdot\,;1),\Omega) = i_C(\pi^0_\varepsilon,\Omega\cap C)$ is constant with respect to ε, for $\varepsilon > 0$ small enough.
Then,

$$i_\Gamma(M(\cdot\,;1),\Omega) = \lim_{\varepsilon\to 0^+} i_C(\pi^0_\varepsilon,\Omega\cap C) = I(\pi^0,\Omega\cap C).$$

Assumption (i3) provides the existence of a fixed point $x\in\Omega$ of $M(\cdot\,;1)$. The proof is complete. ♦

REMARK 7. We point out that a generalization of Corollaries 3 and 4 to the case of flow-invariant ENRs can be performed arguing as in Section 3. Besides an analogous "bound set" or "a priori bounds" condition (respectively), it is sufficient to assume, according to Remark 7, that

$$I(\pi^0,G) \neq 0.$$

Chapter 4

The coincidence degree of some functional differential operators in spaces of periodic functions and related continuation theorems

1. Introduction

In Chapter 3, we have shown that if Ω is an open bounded set of the space X of continuous and T-periodic functions with values in $\mathbf{R}^m$ such that the autonomous equation

$$\dot{x}(t) - f(x(t)) = 0, \tag{1.1}$$

with $f : \mathbf{R}^m \to \mathbf{R}^m$ continuous, has no T-periodic solution on $\mathrm{fr}_X\Omega$, then the coincidence degree of the operator in X associated to the left-hand member of (1.1) is equal to $(-1)^m$ times the Brouwer degree of f, with respect to $\Omega\cap\mathbf{R}^m$. Of course, we identify here $\mathbf{R}^m$ with the space of constant mappings from $\mathbf{R}$ to $\mathbf{R}^m$.
The proof of the mentioned result depends upon the Kupka-Smale approximation theorem [82,136] for the closed orbits of autonomous systems and on some delicate degree computations.

In this Chapter (which is based on [17]) we state and prove the corresponding result for the autonomous retarded functional differential equation (RFDE)

$$\dot{x}(t) - f(x_t) = 0, \tag{1.2}$$

where $f : \mathcal{C}_r \to \mathbf{R}^m$ is continuous and takes bounded sets into bounded sets, $\mathcal{C}_r = \mathcal{C}([-r,0], \mathbf{R}^m)$ and, for each t, x_t is the element of $\mathcal{C}_r$ defined by $x_t(\theta) = x(t + \theta)$, $\theta\in[-r,0]$ (see [65] for these notations and a thorough treatment of RFDE).

In Section 2, we prove (Theorem 1) that the coincidence degree of the mapping in X defined by the left-hand member of (1.2) (which, if Ω is like above, is defined [96,100]) is again equal, up to a factor $(-1)^m$, to the Brouwer degree of the restriction of f to $\Omega\cap\mathbf{R}^m$. A basic ingredient in the proof is an extension to RFDE of the Kupka-Smale theorem due to Mallet-Paret [93] and following earlier generic results for fixed points of a RFDE defined on a compact manifold due to Oliva [114]. See also interesting remarks in [22] and [67].
Although our proof follows the same main lines as the one given in Theorem 1 in Chapter 3 for the ordinary differential equation case, the different nature of (1.2) requires at various stages nontrivial

variants of the arguments and even completely different ones due in particular to the fact that time-scaling involve modifications of the delay in a RFDE. We then relate the coincidence degree associate to a RFDE of the form

$$\dot{x}(t) - h(t,x_t) = 0,\ t\in \mathbf{R},$$

with $h(\cdot)$ T-periodic in t and positively homogeneous of degree $\alpha \neq 1$ in its second variable, to that associated to the autonomous RFDE

$$\dot{x}(t) - \bar{h}(x_t) = 0\ ,\quad t\in \mathbf{R},\quad \text{with}\quad \bar{h}(\varphi) = \frac{1}{T}\int_0^T h(s,\varphi(s))ds,$$

and we give a related existence theorem.

In Section 3 we state and prove a continuation thorem for the T-periodic solutions of non-autonomous RFDE

$$\dot{x}(t) = F(t,x_t)\ ,\ t\in \mathbf{R}$$

based upon the previous degree calculations.

2. The main result

Let $\mathcal{C}_r = \mathcal{C}([-r,0],\mathbf{R}^m)$, $r \geq 0$. and let $f : \mathcal{C}_r \to \mathbf{R}^m$ be continuous and such that it takes bounded sets into bounded sets.
In the first part of this Section, we consider the T-periodic solutions ($T > 0$) of the corresponding RFDE

$$\dot{x}(t) = f(x_t), \tag{2.1}$$

where $f : \mathcal{C}_r \to \mathbf{R}^m$ is continuous and takes bounded sets into bounded sets.
By a T-periodic solution of (2.1), we mean a function $x : \mathbf{R} \to \mathbf{R}^m$ of class $\mathcal{C}^1$ such that

$$x(t + T) = x(t)\ ,\ \ t\in \mathbf{R}$$

which satisfies (2.1) on $\mathbf{R}$. We denote by $\mathcal{C}_T$ the Banach space of continuous T-periodic functions $x : \mathbf{R} \to \mathbf{R}^m$ with the uniform norm $|x|_\infty = \max_{t\in \mathbf{R}} |x(t)|$.
If we define $L : \mathrm{dom}L \subset \mathcal{C}_T \to \mathcal{C}_T$ by $\mathrm{dom}L = \{x\in \mathcal{C}_T : x \text{ is of class } \mathcal{C}^1\}$ and $Lx = \dot{x}$, and $F : \mathcal{C}_T \to \mathcal{C}_T$ by $F(x)(t) = f(x_t)$, $t\in \mathbf{R}$ (Nemitzky operator), then it is well known [96,98] that L is a

Fredholm operator of index zero, F is L-completely continuous on $\mathcal{C}_T$ and the existence of T–periodic solutions of (2.1) is equivalent to the abstract equation

$$Lx = Fx\,, \quad x \in \operatorname{dom} L\,.$$

Moreover, if $\Omega \subset \mathcal{C}_T$ is an open bounded set such that

$$Lx \neq Fx, \quad x \in \operatorname{dom} L \cap \operatorname{fr}_{\mathcal{C}_T}\Omega,$$

then the coincidence degree $D_L(L - F, \Omega)$ is defined as the Leray-Schauder degree of an associated fixed point problem.

THEOREM 1. *Assume that* $\Omega \subset \mathcal{C}_T$ *is an open bounded set such that there is no* $x \in \operatorname{fr}_{\mathcal{C}_T}\Omega$ *such that* $\dot{x}(t) = f(x_t)$, $t \in \mathbf{R}$. *Then*

$$D_L(L - F, \Omega) = (-1)^m d_B(f|_{\mathbf{R}^m}, \Omega \cap \mathbf{R}^m, 0)\,.$$

Proof. First of all, we observe that, as Ω is bounded, there is a constant $R > 0$ such that $|x| < R$ for every $x \in \operatorname{cl}_{\mathcal{C}_T}\Omega$. Furthermore, we point out that the assumption is equivalent to

$$Lx \neq Fx\,, \tag{2.2}$$

for all $x \in \operatorname{dom} L \cap \operatorname{fr}_{\mathcal{C}_T}\Omega$; therefore, the coincidence degree $D_L(L - F, \Omega)$ is well defined, and as we also have

$$f(c) \neq 0$$

for all $c \in \mathbf{R}^m \cap \operatorname{fr}_{\mathcal{C}_T}\Omega$, the Brouwer degree $d_B(f|_{\mathbf{R}^m}, \Omega \cap \mathbf{R}^m, 0)$ is defined as well.

The proof is performed by means of Mallet-Paret's extension of the Kupka-Smale's Theorem [93]; this result ensures the existence of a sequence of C^1-functions (φ_k) , $\varphi_k : \mathcal{C}_T \to \mathbf{R}^m$, taking bounded sets into bounded sets, and such that

(a) $(\varphi_k) \to f$ uniformly on closed bounded sets

(b) for every closed bounded subset B of $\mathcal{C}_T$ and for all $k \in \mathbf{N}$, the equation

$$\dot{x} = \varphi_k(x_t)$$

has finitely many singular orbits in $\mathcal{C}_T$ (i.e., rest points and closed orbits) with minimal period in $[0,T+1]$ which are contained in B and they are all hyperbolic.

Let $N^{k,\mu}$ be the Nemitzky operator induced by the functions $x \to \mu f(x.) + (1-\mu)\varphi_k(x.)$, $\mu \in [0,1]$. We claim that there is $k_0 > 0$ such that, for all $k \geq k_0$ and for all $\mu \in [0,1]$,

$$Lx \neq N^{k,\mu}x \quad \text{for all } x \in \mathrm{dom}L \cap \mathrm{fr}_{\mathcal{C}_T}\Omega. \tag{2.3}$$

This fact will imply, in particular, that

$$\varphi_k(z) \neq 0 \quad \text{for all } z \in \mathrm{fr}_{\mathcal{C}_T}\Omega \cap \mathbf{R}^m \,,\ k \geq k_0 \,.$$

Then, a classical compactness argument ensures that, for any $k \geq k_0$, there is $\delta_1 = \delta_1(k)$ such that

$$\varphi_k(y) \neq 0 \quad \text{for all } y \in B(\mathrm{fr}_{\mathcal{C}_T}\Omega \cap \mathbf{R}^m, \delta_1)\,.$$

To obtain (2.3), it is sufficient to observe that the sequence of operators $N^{k,\mu}$ converges, as $k \to +\infty$, to F in $\mathcal{C}_T$ uniformly on $\mathrm{fr}_{\mathcal{C}_T}\Omega \times [0, 1]$ and that, by (2.2),

$$\inf\{\, |(L-F)x| : x \in \mathrm{dom}L \cap \mathrm{fr}\Omega \,\} > 0.$$

Hence, the claim is proved and, using the homotopy property of the coincidence degree (see [98, p. 15]), we can write

$$D_L(L-F, \Omega) = D_L(L - N^{k,1}, \Omega) = D_L(L - N^{k,0}, \Omega),$$

for every $k \geq k_0$ and, in particular,

$$d_B(f|_{\mathbf{R}^m}, \Omega \cap \mathbf{R}^m, 0) = d_B(\varphi_k|_{\mathbf{R}^m}, \Omega \cap \mathbf{R}^m, 0),$$

for every $k \geq k_0$.

Let us fix $k^* \geq k_0$. For brevity, we set

$$\varphi := \varphi_{k^*}\,, \quad N_\varphi := N := N^{k^*,0}\,, \quad \delta_1 := \delta_1(k^*).$$

Consider the singular orbits (i.e. rest points and closed orbits) with minimal period in $[0,T+1]$ of the equation

$$\dot{x} = \varphi(x_t). \tag{2.4}$$

By the Kupka-Smale's property, there exist finitely many such orbits which are contained in $B(0,R) \subset \mathcal{C}_T$ and they are hyperbolic. Recall that for a rest point, this means that the spectrum of the infinitesimal generator of its linearized equation contatins no purely imaginary values and, for a nonconstant periodic

solution, this means that the characteristic multiplier $\mu = 1$ of the linearized equation is simple and no other characteristic multiplier satisfies $|\mu| = 1$ (see [65]). We denote these orbits by $S_1,...,S_n$. They are mutually disjoint (two orbits of (2.4) may cross in $\mathcal{C}_r$ because uniqueness of the Cauchy problem only holds in the future, but this may not happen to closed orbits). Pick, for each $i = 1,...,n$ a point (in $\mathcal{C}_r$) $\phi_i \in S_i$. Then, ϕ_i is a periodic point (possibly a rest point). We can assume that ϕ_i is a rest point ζ_i for $1 \le i \le p$ ($p \ge 0$ an integer) and a periodic point for $p+1 \le i \le n$. We denote its minimal period by T_i ($p+1 \le i \le n$). We can also assume that $T_i \le T$ for $p+1 \le i \le q$ and $T < T_i \le T+1$ for $q+1 \le i \le n$. We denote by k_i the largest integer such that $k_i T_i \le T$ ($p+1 \le i \le q$), so that $(k_i+1)T_i > T$ ($p+1 \le i \le q$). We denote by $x^i(\cdot)$ the solution of (2.4) with $x_0^i = \phi_i$ ($p+1 \le i \le n$).
We claim that for each T' such that

$$T < T' < \min\{(k_{p+1}+1)T_{p+1}, \ldots, (k_q+1)T_q, T_{q+1}, \ldots, T_n, T+1\} := \tau ,$$

the problem

$$\dot{x} = \varphi(x_t) \ , \ x(t+T') = x(T') \tag{2.5}$$

has no solution $x(\cdot)$, with $x_t \in B(0, R)$, and hence $|x(t)| < R$, for all t, other than the equilibria $\zeta_1, \ldots, \zeta_p$.
Indeed, if x satisfies (2.5) and $x_t \in B(0, R)$, for all t, then $S = \{x_t : t \in \mathbf{R}\}$ is a singular orbit of (2.4) contained in $B(0,R)$. If it is not a rest point, then $S = S_i$ for some $p+1 \le i \le n$ and hence there exists $t_i \in \mathbf{R}$ such that

$$x_t = x^i_{t+t_i} \ , \ t \in \mathbf{R}$$

(indeed t_i is such that $x_{-t_i} = \phi_i = x_0^i$) .

In particular,

$$x^i(T' + t_i) = x^i(t_i).$$

This is impossible for $q+1 \le i \le n$ as then $T' < T_i$ and T_i is the smallest period. This is impossible for $p+1 \le i \le q$ as in this case $k_i T_i < T' < (k_i+1)T_i$.
Therefore the claim is proved.
Now, the solutions of (2.5) correspond, by the transformation

$$y(t) = x\left(\frac{T'}{T} t\right), \quad t \in \mathbf{R} ,$$

to the solutions of the problem

$$\dot{y}(t) = \frac{T'}{T}\varphi\big(y_t(\tfrac{T}{T'}(\cdot))\big) \ , \ y(t+T) = y(t) \, . \qquad (2.6)$$

Thus, problem (2.6) has, by construction, no nontrivial (i.e. non-equilibrium) solution on $\mathrm{cl}_{C_T}\Omega$ and, by assumption, no rest point on $\mathrm{fr}\Omega$ (as its rest points are the same as those of (2.8) and all its possible solutions in $B(0,R)$ are rest points). Defining

$$\Phi : C_T \times [T, \tau[\to C_T \, ,$$

by

$$\Phi(y, T')(t) = \frac{T'}{T}\varphi\big(y_t(\tfrac{T}{T'}(\cdot))\big) \ , \ t \in \mathbf{R},$$

we shall show that Φ is continuous so that we can apply a homotopy argument.
If $\varepsilon > 0$, $y \in C_T$, $T' \in [T, \tau[$ are given, then as $\{ y_t(\frac{T}{T'}(\cdot)) : t \in [0, T] \}$ is compact, there exists $\delta > 0$ such that

$$|\varphi(\phi) - \varphi\big(y_t(\tfrac{T}{T'}(\cdot))\big)| \le \varepsilon$$

whenever $t \in [0,T]$ (and hence whenever $t \in \mathbf{R}$) and $|\phi - y_t(\frac{T}{T'}(\cdot))|_{C_r} \le \delta$.
Now, y being uniformly continuous, there exists $\delta' > 0$, such that

$$|y(t') - y(t'')| \le \frac{\delta}{2}$$

when $|t' - t''| \le \delta'$ and hence if $T < T'' < \tau$ and $T'' \ge T'/2$, we shall have

$$|y\big(t + \tfrac{T}{T''}(\theta)\big) - y\big(t + \tfrac{T}{T'}(\theta)\big)| \le \frac{\delta}{2}$$

when

$$\left|\frac{T}{T''} - \frac{T}{T'}\right| |\theta| \le \delta'$$

which will be the case if

$$|T'' - T'| \le \frac{\delta'(T')^2}{2Tr} \, .$$

Now, if $z \in C_T$ is such that $|z - y|_\infty \le \delta/2$, we have

$$|z + (t + \frac{T}{T''}(\theta)) - y(t + \frac{T}{T'}(\theta))| \le |z(t + \frac{T}{T''}(\theta)) - y(t + \frac{T}{T''}(\theta))| +$$
$$+ |y(t + \frac{T}{T''}(\theta)) - y(t + \frac{T}{T'}(\theta))| \le |z - y|_\infty + \frac{\delta}{2} \le \delta$$

for all $t \in \mathbf{R}$ and $\theta \in [-r,0]$ and hence

$$|z_t(\frac{T}{T''}(\cdot)) - y_t(\frac{T}{T'}(\cdot))|_{\mathcal{C}_T} \le \delta.$$

Summarizing, if $z \in \mathcal{C}_T$ with $|z - y|_\infty \le \delta/2$, and if $T'' \in [T,\tau[$ with

$$|T'' - T'| \le \min\{\frac{T'}{2}, \frac{\delta'(T')^2}{2Tr}\},$$

we shall have

$$|\varphi(z_t(\frac{T}{T''}(\cdot))) - \varphi(y_t(\frac{T}{T'}(\cdot)))| \le \varepsilon$$

for all $t \in \mathbf{R}$ and this easily implies the continuity of Φ on $\mathcal{C}_T \times [T,\tau[$. Now, as (2.8) has no solution on $\mathrm{fr}_{\mathcal{C}_T}\Omega$, the homotopy invariance of the coincidence degree implies that

$$D_L(L - N_\varphi, \Omega) = D_L(L - \Phi(\cdot,T'),\Omega) \tag{2.7}$$

for all $T \le T' < \tau$.
Thus, by excision,

$$D_L(L - \phi(\cdot,T'),\Omega) = \sum_{\substack{1 \le j \le p \\ \zeta_j \in \Omega}} D_L(L - \Phi(\cdot,T'), B(\zeta_j,\delta)), \tag{2.8}$$

where

$$\delta = \min\{\delta_1, \eta/2\}, \quad \eta = \min\{\mathrm{dist}(S_i,S_j) : 1 \le i \ne j \le n\}.$$

But, by the choice of δ, $L - N_\varphi$ has no solution on $\mathrm{fr}B(\zeta_j,\delta)$ and hence, by homotopy invariance again we have, for all $T \le T' < \tau$,

$$D_L(L - \Phi(\cdot,T'), B(\zeta_j,\delta)) = D_L(L - N_\varphi, B(\zeta_j,\delta)), \quad 1 \le j \le p, \quad \zeta_j \in \Omega. \tag{2.9}$$

As φ is of class $\mathcal{C}^1$, the same is true for N_φ and hence, by the linearization property of the degree (see e.g. [98, Prop. VIII.3]),

$$D_L(L - N_\varphi, B(\zeta_j,\delta)) = D_L(L - N'_\varphi(\zeta_i), B(0,1)) , \quad 1 \le j \le p , \quad \zeta_j \in \Omega . \tag{2.10}$$

where $N'_\varphi(\zeta_i)$ has the form

$$N'_\varphi(\zeta_i)\phi = \dot{\varphi}(\zeta_i)\phi = \int_{-r}^{0} \phi(\theta)\, d\eta^j(\theta)$$

for some function η^j whose elements are of bounded variation ([65]).Moreover, the hyperbolicity of ζ_j implies that the corresponding characteristic equation

$$\det\Delta_j(\mu) = 0,$$

where

$$\det \Delta_j(\mu) = \mu I - \int_{-r}^{0} e^{\mu\theta} d\eta^j(\theta) ,$$

has all its roots with nonzero real part (see e.g. [65]). Consequently, the same is true for the characteristic equation of the equations in the family

$$\dot{x}(t) = \lambda\dot{\varphi}(\zeta_j)x_t , \qquad \lambda \in]0,1]$$

which therefore only admit the trivial T-periodic solution. A standard argument (see e.g. [98, Th. IV.12]) then shows that the same is true for the family

$$\dot{x}(t) = (1 - \lambda)\dot{\varphi}(\zeta_j)\overline{x} + \lambda\dot{\varphi}(\zeta_j)x_t , \qquad \lambda \in [0,1]$$

where

$$\overline{x} = \frac{1}{T}\int_0^T x(s)ds = Px.$$

Consequently, by the homotopy invariance, and a classical property of coincidence degree (see e.g. [98, Prop. II.12]),

$$\begin{aligned} D_L(L - N'_\varphi(\zeta_i) , B(0, 1)) &= D_L(L - N'_\varphi(\zeta_i)\, P, B(0, 1)) \\ &= (-1)^m d_B(\dot{\varphi}(\zeta_i)|_{R^m}, B(0, 1)\cap \mathbf{R}^m, 0) \qquad (2.11) \\ &= (-1)^m d_B(\varphi|_{R^m}, B(\zeta_i, \delta)\cap \mathbf{R}^m, 0) , \quad 1 \le j \le p , \zeta_j \in \Omega. \end{aligned}$$

Therefore, combining (2.7), (2.8), (2.9), (2.10) and (2.11), we obtain

$$D_L(L - N_\varphi,\Omega) = \sum_{\substack{1\le j\le p \\ \zeta_j\in\Omega}} (-1)^m d_B(\varphi|_{\mathbf{R}^m}, B(\zeta_j,\delta)\cap\mathbf{R}^m, 0)$$

$$= (-1)^m d_B(\varphi|_{\mathbf{R}^m}, \Omega\cap\mathbf{R}^m, 0)\ ,$$

and the proof is complete. ♦

An alternative proof of Theorem 1 has been obtained, in the framework of degree theory for equivariant maps, in [6] on the lines of an analogous result in the case of ordinary differential equations (see the remark at the end of the proof of Theorem 1 of Chapter 3).

Secondly, we consider the RFDE

$$\dot{x}(t) = h(t, x_t) + \delta(\alpha)p(t) \tag{2.12}$$

where $h : \mathbf{R}\times\mathcal{C}_T \to \mathbf{R}^m$, $(t, \varphi) \to h(t, \varphi)$ is a continuous mapping, taking bounded set into bounded sets, T-periodic in t and positively homogeneous of order $\alpha \neq 1$ in φ, $\delta(\alpha) = \max(0, (1-\alpha)/|1-\alpha|)$ and $p\in\mathcal{C}_T$. We define the averaged vector field $\bar{h} : \mathcal{C}_T \to \mathbf{R}^m$ by

$$\bar{h}(\varphi) = \frac{1}{T}\int_0^T h(s,\varphi)ds$$

and the corresponding Caratheodory mappings $H : \mathcal{C}_T \to \mathcal{C}_T$ and $\bar{H} : \mathcal{C}_T \to \mathcal{C}_T$ by

$$H(x)(t) = h(t, x_t)\ , \qquad \bar{H}(x)(t) = \bar{h}(x_t),$$

for all $t \in \mathbf{R}$.

THEOREM 2. *Assume that* $\bar{h}(z) \neq 0$ *for* $|z| = 1$ in $\mathbf{R}^m$. *Then, there exists* $r_0 > 0$ *such that, if* $\alpha < 1$ *and* $r \ge r_0$ *or* $\alpha > 1$ *and* $0 < r \le r_0$, (2.12) *has no* T*-periodic solutions* x *with* $|x| = r$ *and*

$$D_L(L - H - \delta(\alpha)p, B(0,r)) = D_L(L - \bar{H}, B(0,r)).$$

Proof. Let us define $\mathcal{H}: \mathcal{C}_T\times[0,1] \to \mathcal{C}_T$ by

$$\mathcal{H}(x, \lambda) = (1-\lambda)\,\bar{H}(x) + \lambda(H(x) + \delta(\alpha)p)\,.$$

We have only to show that there is some $r_0 > 0$ such that for each $r \geq r_0$ if $\alpha < 1$ or each $0 < r \leq r_0$ if $\alpha > 1$, and for each $\lambda \in [0,1]$, the equation

$$Lx = \mathcal{H}(x; \lambda)$$

has no solution x with $|x| = r$.

If this is not the case, there are sequences (r_k) in $\mathbf{R}_+$, (x_k) in $\mathcal{C}_T$ and (λ_k) in $[0,1]$ such that $|x_k| = r_k$, $r_k \leq 1/k$ if $\alpha > 1$, $r_k \geq k$ if $\alpha < 1$ and

$$\dot{x}_k(t) = (1-\lambda_k)\,\bar{h}((x_k)_t) + \lambda_k[h(t, (x_k)_t) + \delta(\alpha)p(t)]$$

$(k \in \mathbf{N})$. Letting $u_k = x_k / |x_k| = r_k^{-1}\, x_k$, we get

$$\dot{u}_k(t) = r_k^{\alpha-1}[(1-\lambda_k)\bar{h}((u_k)_t) + \lambda_k h(t, (u_k)_t)] + r_k^{-1}\lambda_k\,\delta(\alpha)p(t), \tag{2.13}$$

so that

$$|\dot{u}_k(t)| \leq r_k^{\alpha-1}\,\beta + \gamma$$

for some $\beta, \gamma > 0$ and all $t \in \mathbf{R}$. Consequently there are subsequences (λ_{j_k}), (u_{j_k}) and $\lambda^* \in [0,1]$, $v \in \mathcal{C}_T$, $|v| = 1$ such that $(u_{j_k}) \to v$ uniformly on $\mathbf{R}$ and $(\lambda_{j_k}) \to \lambda^*$.

From

$$u_k(t) - u_k 0) = r_k^{\alpha-1} \int_0^t [(1-\lambda_k)\,\bar{h}((u_k)_s) + \lambda_k\, h(s, (u_k)_s)]ds + r_k^{-1} \int_0^1 \delta(\alpha)p(s)\, ds,$$

we get

$$v(t) - v(0) = 0,\ t \in \mathbf{R},$$

so that v is constant and $|v| = 1$. From (2.13) we also get

$$0 = \int_0^T (1-\lambda_k)\,\bar{h}((u_k)_s) + \lambda_k\, h(s, (u_k)_s) + r_K^{-\alpha}\delta(\alpha)\lambda_k p(s)]ds = 0$$

and hence, letting $j_k \to \infty$,

$$0 = T\,\bar{h}(v),$$

a contradiction.
Hence, for $0 < r \le r_0$ if $\alpha > 1$ and $r \ge r_0$ if $\alpha < 1$, we have

$$D_L(L - H - \delta(\alpha)p, B(0, r)) = D_L(L - \mathcal{H}(\cdot,1), B(0,r)) =$$

$$= D_L(L - \mathcal{H}(\cdot, 0), B(0, r)) = D_L(L - \bar{H}, B(0, r)),$$

and the proof is complete. ◆

By using Theorem 1 and 2, the existence property of degree and its invariance for sufficiently small perturbations of the nonlinear term we immediately deduce the following existence result.

COROLLARY 1. *Assume that* h *satisfies the assumptions of* Theorem 2 *and that*

$$d_B(\bar{h}|_{\mathbf{R}^m}, B(0,1)\cap\mathbf{R}^m, 0) \neq 0.$$

Then, if $\alpha < 1$*, the RFDE*

$$\dot{x}(t) = h(t, x_t) + p(t) \tag{2.14}$$

has at least one T*-periodic solution for each* $p \in \mathcal{C}_T$ *and, if* $\alpha > 1$*, there exists* $\varepsilon_0 > 0$ *such that* (2.14) *has at least one* T*-periodic solution for each* $p \in \mathcal{C}_T$ *with* $|p|_\infty \le \varepsilon_0$.

3. An existence result

Let $F : \mathbf{R} \times \mathcal{C}_T \to \mathbf{R}^m$, $(t,\varphi) \to F(t,\varphi)$ be a continuous mapping, taking bounded sets into bounded sets and such that

$$F(t + T,\varphi) = F(t,\varphi)$$

for some $T > 0$ and all $t \in \mathbf{R}$ and $\varphi \in \mathcal{C}_T$. We consider the existence of T-periodic solutions of the RFDE

$$\dot{x}(t) = F(t, x_t), \quad t \in \mathbf{R}, \tag{3.1}$$

i.e. of solutions $x(\cdot)$ such that

$$x(t) = x(t+T), \quad t \in \mathbf{R}. \tag{3.2}$$

As it is the case in any continuation Theorem, we introduce a mapping $f : \mathbf{R}\times\mathcal{C}_T\times[0,1] \to \mathbf{R}^m$ which is continuous, takes bounded sets into bounded sets and is such that

$$f(t+T,\varphi,\lambda) = f(t,\varphi,\lambda)$$

for all $t \in \mathbf{R}$, $\varphi\in \mathcal{C}_T$, $\lambda\in [0,1]$,

$$f(t,\varphi\, 0) = f_0(\varphi)$$

for all $t \in \mathbf{R}$, $\varphi\in \mathcal{C}_T$ (i.e. $f(\cdot, \cdot; 0)$ is autonomous), and

$$f(t,\varphi,1) = F(t,\varphi)$$

for all $t\in \mathbf{R}$, $\varphi\in \mathcal{C}_T$.

THEOREM 3. *Let* $\Omega\subset\mathcal{C}_T$ *be an open bounded set such that the following conditions are satisfied:*

(p1) *there is no* $x\in \mathrm{fr}_{\mathcal{C}_T}\Omega$ *such that*

$$\dot{x}(t) = f(t,x_t,\lambda)\ ,\ t\in \mathbf{R},\ \lambda\in [0,1[; \tag{3.3}$$

(p2) $$d_B(f_{0|\mathbf{R}^m},\Omega\cap\mathbf{R}^m,0) \neq 0.$$

Then, (3.1)-(3.2) *has at least one solution* $x\in \mathrm{cl}_{\mathcal{C}_T}\Omega$.

Proof. We use the framework of coincidence degree as in Theorem 1. The classical Leray-Schauder continuation theorem [90] could be used instead by the equivalence stated at the beginning of Section 2. Besides the spaces and operators considered there, we further define $M := M(x;\lambda) : \mathcal{C}_T\times[0,1] \to \mathcal{C}_T$:

$$M(x;\lambda)(t) = f(t,x_t;\lambda).$$

Observe that $M(\cdot; 0) = M_0$, where

$$M_0 : \mathcal{C}_T \to \mathcal{C}_T,\quad x\to f_0(x.).$$

According to [98 , Chapter I], M is L -compact on $cl_{\mathcal{C}_T}\Omega\times[0,1]$. We remark that x is a T- periodic solution of $\dot{x}(t) = f(t, x_t;\lambda)$, $\lambda\in[0,1]$, if and only if $x\in \mathrm{dom}L$ is a solution of the coincidence equation $Lx = M(x;\lambda)$, $\lambda\in[0,1]$. In particular, (3.1)-(3.2) is equivalent to $Lx = M(x;1)$. Without loss of generality, we suppose that (p1) holds for $\lambda\in[0,1]$ in (3.3). Otherwise, the result is proved for $x\in fr_{\mathcal{C}_T}\Omega$. Accordingly, by the definition of $M(\cdot;\lambda)$ and using (p1) we have:

$$Lx \neq M(x;\lambda), \lambda\in[0,1]$$

for all $x\in \mathrm{dom}L\cap fr_{\mathcal{C}_T}\Omega$. Thus we can apply the homotopy property of the coincidence degree and obtain:

$$D_L(L - M_0,\Omega) = D_L(L - M(\cdot;\, 0),\Omega) = D_L(L - M(\cdot;\, 1),\Omega). \tag{3.4}$$

Assumption (p1) (for $\lambda = 0$) ensures that Theorem 1 can be applied, so that (3.4) and (p2) imply:

$$|D_L(L - M_0,\, \Omega)| = |d_B(f_{0|\mathbf{R}^m},\, \Omega\cap\mathbf{R}^m,\, 0)| \neq 0.$$

Hence, by the existence property of the coincidence degree, there is $\tilde{x}\in \mathrm{dom}L\cap\Omega$ such that $L\tilde{x} = M(\tilde{x};1)$; thus $\tilde{x}(\cdot)$ is a solution to (3.1)-(3.2), with $\tilde{x}\in \mathrm{dom}L\cap\Omega$. The proof is complete. ♦

Theorem 3 is particularly suitable for the study of T-periodic solutions of perturbed autonomous RFDE of the form

$$\dot{x}(t) = f(x_t + e(t)), \quad t\in\mathbf{R},$$

with $e\in\mathcal{C}_T$, through the homotopy

$$\dot{x}(t) = f(x_t) + \lambda e(t)\ ,\ t\in\mathbf{R}\ ,\ \lambda\in[0,1].$$

Chapter 5

A continuation approach to superlinear periodic boundary value problems

1. Introduction

This Chapter (which is based on [16]) deals with the problem of the existence of T-periodic solutions to the first order differential system

$$\dot{x} = F(t,x), \tag{1.1}$$

where $F : [0,T]\times\mathbf{R}^m \rightarrow \mathbf{R}^m$ is a Caratheodory function. In what follows, we prove some results for the solvability of the periodic BVP in the case when the dimension of the space is even. Such a limitation is motivated by our interest in applications to the second order equation

$$\ddot{u} + g(t,u,\dot{u}) = 0, \tag{1.2}$$

which takes the form of (1.1) when it is written as the equivalent system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -g(t,u,v). \end{aligned} \tag{1.3}$$

Clearly, in this case the new variable $x = (u,v)$ belongs to $\mathbf{R}^{2d}$ for $u\in\mathbf{R}^d$.

In order to introduce the main results of this Chapter, we examine some contributions to the solvability of the periodic problem for equation (1.2), beginning with the scalar case.
Conditions for the existence of T-periodic solutions of a second order equation of the form

$$\ddot{u} + g(u) = p(t) \tag{1.4}$$

where p is a given T-periodic function, must of course exclude the well known resonant situation

$$\ddot{u} + (n\omega)^2 u = \cos n\omega t, \tag{1.5}$$

(with $\omega = 2\pi/T$ and $n\in\mathbf{Z}_+$) whose solution set, given by

$$u(t) = A \cos n\omega t + B \sin n\omega t + (t/2\omega n) \sin n\omega t,$$

does not contain any T-periodic function. A first way consists in excluding functions g which have the same sign than u, by assuming for example that $g(u)u < 0$ for $|u|$ large. In a variational setting, this corresponds to the coercivity of the associated action functional (see e.g. [104, Chapter 1]). The case where $g(u)u \geq 0$ for $|u|$ large is more delicate. When g has a linear growth, known existence results impose more or less sharp conditions on g and p which exclude in particular the case where $g(u) = (n\omega)^2u$ and $p(t) = \cos n\omega t$ (see e.g. [34,99,117] and their references). Another general way to exclude the counterexample (1.5) consists in not allowing g to be linear. This is done in [144] by assuming essentially that g is bounded below or bounded above. If any boundedness restriction is avoided, the linear situation will be excluded by considering the class of superlinear nonlinearities, i.e. of functions g such that

$$g(u)/u \to +\infty \text{ as } |u| \to \infty . \tag{1.6}$$

This case appears to be the most delicate to treat, as shown by the relative scarcity and the technical difficulty of the papers devoted to this situation. In the first two ones in a series of four papers, Morris [106,107] considered in 1955 and 1958 the existence of infinitely many mT-periodic solutions (with $m \geq 1$ an integer) for the equation

$$\ddot{u} + 2u^3 = p(t) \tag{1.7}$$

when p is even, T-periodic, has mean value zero and is sufficiently smooth. Because of the symmetry properties of the restoring force and the forcing term, mT-periodic solutions of (1.7) are obtained as solutions verifying the Neumann boundary conditions

$$\dot{u}(0) = \dot{u}(mT/2) = 0. \tag{1.8}$$

Solutions of this Neumann problem near large amplitude solutions of the corresponding autonomous equation are then obtained through an implicit function argument on the associated shooting mapping. In 1957, and independently of Morris, Ehrmann used existence results proved in [39] by a shooting method for Sturm-Liouville problems for equations of the form

$$\ddot{u} + f(t, u, \dot{u}) = 0 \tag{1.9}$$

with

$$|f(t, u, v) - g(u)| \leq K_1|u| + K_2|v| + K_3$$

and g verifying (1.6), to obtain in [40] the existence of infinitely many T-periodic solutions of (1.9) when f satisfies suitable symmetry conditions.

In 1963, Harvey [71] extended Morris approach and results to equation (1.4) with p like in Morris paper and g satisfying (1.6) and some further conditions. More recent results on the existence of T-periodic solutions of equations of type (1.4) with symmetries are due to Cesari [21] (who deals with $g(u) = u^3$, $p(t) = \sin t$ and uses alternative method, some numerical estimates and Brouwer degree to prove the existence of at least one periodic solution), Micheletti [105] (who deals with $g(u) = 2u^3$, p(t) even, smooth and with mean value zero, and obtains infinitely many periodic solutions by reducing the problem to the use of the contraction mapping Theorem in the space of even periodic functions on neighborhoods of large amplitude solutions of the unperturbed equation), Castro and Lazer [20] (who consider systems, called "weakly-coupled", of the form

$$\ddot{u}_k + g_k(u_k) = p_k(t, u), \qquad (1 \le k \le m) \tag{1.10}$$

with each g_k odd and satisfying (1.6), p_k bounded and odd, and use a shooting argument and Miranda's Theorem to obtain infinitely many periodic solutions), and Schmitt and Mazzanti [133] (who obtain infinitely many periodic solutions for equation (1.4) with $g(u) = au + u^3$ by using shooting methods and nonstandard analysis).

It was Morris [108,109] who first proved in 1965 that equation (1.7) has, for each integer $m \ge 1$, infinitely many mT-periodic solutions when p is smooth but not necessarily symmetric. He uses Poincaré's operator together with an elementary fixed point theorem for area-preserving transformations in the plane. For the more general equation (1.4), the existence of one T-periodic solution was proved in 1975 by Fučik and Lovicar [50] for each Lebesgue integrable p by using the Poincaré's operator and a fixed point Theorem in $\mathbf{R}^2$, a result extended to (1.9) in 1980 by Struwe [140] with a similar approach. We also mention W. Y. Ding, who obtained in [36] the existence of infinitely many T-periodic solutions for (1.4), with g locally lipschitzian, $\lim_{|u| \to +\infty} g(u)/u = +\infty$ and $p \in L^\infty([0,T],\mathbf{R})$. The proof makes use of an extended version of the Poincaré-Birkhoff "twist" Theorem, which can be applied since the Poincaré's operator associated to the first order system

$$\dot{u} = v$$
$$\dot{v} = -g(v) + p(t)$$

is an area-preserving homeomorphism of the plane. However, such a property of the Poincaré's map is no more satisfied when an explicit dependence in $\dot{u}$, like in (1.9), is assumed.

As for variational methods, in 1984 Bahri and Berestycki [3] initiated the study of T-periodic solutions of Hamiltonian systems of the form

$$\dot{z} = \mathbf{J}\cdot\nabla H(z) + f(t)$$

with $z \in \mathbf{R}^m$, m even, and of the form

$$\ddot{u} + \nabla V(u) = f(t)$$

with $u \in \mathbf{R}^m$, when H or V are superquadratic at infinity and satisfy further restrictions. They prove the existence of infinitely many T-periodic solutions corresponding to arbitrary large critical values by using a minimax principle analogous to Kranosel'skii's one for perturbations of even functions [78] but where the $\mathbf{Z}/2\mathbf{Z}$ invariance is replaced by the S^1-action on periodic functions induced by time translations. Further results in this direction were obtained, by a related approach, by Rabinowitz [127], Pisani-Tucci [122] and Long [91].

The present Chapter proposes an approach for periodic solutions of differential equations which may be sublinear or superlinear by a Leray-Schauder's type continuation method. For this sake, we first state and prove in Section 2 (in the convenient frame of coincidence degree for L-compact perturbations of linear Fredholm mappings of index zero) a continuation theorem (Lemma 1) where the new ingredient is the use of a functional φ which is proper on the set Σ of possible solutions of the homotopic family of equations and which avoids two values during this homotopy. The significant special case for the application to periodic problems is that of a functional taking only positive integer values on large norm solutions and whose positive integer level sets have a bounded intersection with Σ. In differential equation problems, the functional we use is closely related to the mapping counting the number of rotations around the origin of the solution of a planar differential system. This is developed in Section 3, which contains a continuation theorem (Theorem 1) for the existence of T-periodic solutions of differential equations in $\mathbf{R}^m$ with m even. In the special case where $m = 2$ (Theorem 2), a basic assumption, besides the one on the "angular function" φ, is the existence of an a priori bound for the uniform norms of the solutions when the minimum on $[0,T]$ of their Euclidean norm is a priori bounded. Such an "elastic property" of the solutions can be checked by the use of some Liapunov-like functions, as shown in Proposition 3.

Section 4 gives applications of Theorem 2 to planar differential systems. A first application of Theorem 2 to systems with linear growth provides a nonresonance-type existence Theorem (Theorem 3) which is related to recent ones of Fabry [43], Fonda-Habets [47] and Habets-Metzen [61]. The idea here consists in establishing estimates on the functional φ which prevent it to take integer values. The second application to Theorem 2 deals with (not necessary Hamiltonian) perturbations of planar Hamiltonian systems

$$\dot{x} = \mathbf{J}[\nabla H(x) + p(t, x)]$$

with H superquadratic at infinity. One obtains (Theorem 4) the existence of at least one T-periodic solution under conditions which, specialized to (1.4) or (1.9), are more general than those of Fučik-Lovicar [50] and Struwe [140] mentioned above, and which, in the general case with p independent of x, do not seem to be covered by the variational techniques.

In Section 5, Theorem 1 is used to obtain an existence result (Theorem 5) for the "weakly-coupled" system (1.10), assuming that the function g has superlinear growth at infinity. In this case, an extension to system (1.10) of Fučik and Lovicar's theorem is achieved, which is modelled on the result of Castro and Lazer [20] for systems having symmetries.

Finally, we define, for a continuous function $Q : \mathrm{dom}Q \supseteq S^1 \to \mathbf{R}^+$, with $S^1 = \{(x_1,x_2) \in \mathbf{R}^2,\ x_1^2 + x_2^2 = 1\}$, the unit circle in $\mathbf{R}^2$, the value

$$\langle Q \rangle := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{Q(\cos\theta, \sin\theta)} ,$$

that is the integral average of $1/Q$ on S^1.

We also recall that a continuous map $Q : \mathbf{R}^2 \to \mathbf{R}$ is *positively homogeneous of degree* $k > 0$ and *positive definite* if $Q(tx) = t^k Q(x) > 0,\ \forall\ t \in \mathbf{R}^+,\ \forall\ x \in \mathbf{R}^2 \setminus \{0\}$. (Such functions are used in Section 4).

2. An abstract continuation theorem

In this Section we give an existence result (Theorem 1) for a coincidence equation in function spaces, whose corollaries are crucial for the proof of Theorems 1 and 2 on the solvability of the periodic BVP associated to (1.1). Accordingly, the following abstract setting is introduced.

Let $L : \mathrm{dom}L \subset \mathcal{X} \to \mathcal{Z}$ be a linear Fredholm mapping of index zero, where $\mathcal{X}$ and $\mathcal{Z}$ are real Banach spaces with norms $|\cdot|_{\mathcal{X}}$ and $|\cdot|_{\mathcal{Z}}$, respectively. Borrowing notation and terminology from [98], we consider continuous linear projectors $P : \mathcal{X} \to \mathcal{X}$, $Q : \mathcal{Z} \to \mathcal{Z}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\ker Q = \mathrm{Im}L$, $\mathcal{X} = \ker L \oplus \ker P$, $\mathcal{Z} = \mathrm{Im}L \oplus \mathrm{Im}Q$ and denote by $K_{P,Q} : \mathcal{Z} \to \mathrm{dom}L \cap \ker P$ the generalized inverse of L (see [98, pp.6-7]). We also fix a linear isomorphism $J : \mathrm{Im}Q \to \ker L$.

Let $N : \mathcal{X} \times [0,1] \to \mathcal{Z}$ be an L-*completely continuous* operator ([98, p.12]).

In such a framework, the equation

$$Lx = N(x,\lambda),\quad x \in \mathrm{dom}L, \tag{2.1}$$

is equivalent to

$$x = T(x,\lambda) := Px + JQN(x,\lambda) + K_{P,Q}N(x,\lambda). \tag{2.2}$$

We denote by

$$\Sigma := \{(x,\lambda) \in \mathrm{dom}L \times [0,1] : Lx = N(x,\lambda)\}$$

the (possibly empty) set of solutions (x,λ) of equation (2.1) and by Σ_λ $(0\leq\lambda\leq 1)$, the section of Σ at λ, that is

$$\Sigma_\lambda := \{x\in \mathrm{dom}L: (x,\lambda)\in\Sigma\}.$$

We observe that the sets Σ and Σ_λ $(0\leq\lambda\leq 1)$ are closed and, by the L-complete continuity of N (which, in turns, implies the complete continuity of T), they are locally compact, in the sense that the intersection of Σ (Σ_λ, respectively) with any bounded closed subset of $\mathcal{X}\times[0,1]$ ($\mathcal{X}$, respectively) is compact. In particular, Σ (or Σ_λ) is compact if it is bounded.
In what follows, we suppose:

(i1) Σ_0 *is bounded in* $\mathcal{X}$

(i.e. Σ_0 is compact) and define

$$\chi_0 := |D_L(L - N(\cdot,0),\mathcal{X})| = |D_L(L - N(\cdot,0),\Omega)|,$$

where $\Omega\supset\Sigma_0$ is any open bounded subset of $\mathcal{X}$ and the coincidence degree "D_L" is defined from the Leray-Schauder degree "deg" by the formula

$$|D_L(L - N(\cdot,0),\Omega)| = |\deg(I_{\mathcal{X}} - T(\cdot,0),\Omega,0)|.$$

From [98, pp.15-19], it follows that χ_0 is well defined; in particular, it is independent of the choice of P, Q, J and Ω. Moreover, $\Sigma_0\neq\emptyset$ if $\chi_0\neq 0$.
Accordingly, we further assume

(i2) $\chi_0 \neq 0.$

Finally, we introduce a functional $\varphi : \mathcal{X}\times[0,1]\to\mathbf{R}$ and suppose that

(i3) φ *is continuous on* $\mathcal{X}\times[0,1]$ *and proper on* Σ.

Recall that $\varphi_{|\Sigma}$ "proper" means that $\varphi^{-1}(K)\cap\Sigma$ is compact for each compact set $K\subset\mathbf{R}$ or, equivalently, that for any sequence $(x_n,\lambda_n)\in\Sigma$ such that $\varphi(x_n,\lambda_n)$ converges in $\mathbf{R}$, there is a subsequence (x_{n_k},λ_{n_k}) converging in Σ. We also note that if $\varphi : \mathcal{X}\times[0,1]\to\mathbf{R}$ is continuous, then the properness of $\varphi_{|\Sigma}$ is always guaranteed when Σ is bounded (and so compact), while for Σ unbounded (i3) holds provided that

$$\lim_{\substack{|x|_{\mathcal{X}}\to+\infty \\ (x,\lambda)\in\Sigma}} \Big(\inf_{\lambda\in[0,1]} |\varphi(x,\lambda)|\Big) = +\infty. \tag{2.3}$$

is fulfilled.
Under the assumptions (i1), (i2), (i3) listed above, we have that $\varphi(\cdot,0)$ is bounded on the compact nonempty set Σ_0 and hence the real constants

$$\varphi_- := \inf\{\varphi(x,0): x\in\Sigma_0\},$$
$$\varphi_+ := \sup\{\varphi(x,0): x\in\Sigma_0\}$$

are defined; moreover, φ_- and φ_+ are actually achieved by $\varphi(\cdot,0)$ on Σ_0.

We can state now a continuation lemma for the solvability of equation

$$Lx = N(x,1), \quad x\in \mathrm{domL}. \tag{2.4}$$

LEMMA 1. *Assume* (i1), (i2), (i3). *Suppose there are constants* c_-, c_+, *with*

$$c_- < \varphi_-, \; c_+ > \varphi_+, \tag{2.5}$$

such that

$$\varphi(x,\lambda) \notin \{c_-, c_+\} \tag{2.6}$$

for each $(x,\lambda)\in \mathrm{domL}\times]0,1[$, *satisfying equation* (2.1).
Then, equation (2.4) *has at least one solution.*

Proof. Assume, by contradiction, that (2.4) has no solution. From (2.5), we also have $\varphi(x,0) \notin \{c_-, c_+\}$, for each $x\in\Sigma_0$. Then, (2.6) yields

$$\varphi(\Sigma)\cap\{c_-, c_+\} = \emptyset. \tag{2.7}$$

We propose now two different ways to get the conclusion. As a first possibility, we can use a corollary of [51, Th.1.1] (see also [46]) which, in our case, ensures the existence of a (closed) unbounded connected set $\mathcal{C}$, with $\mathcal{C}\subset\Sigma$, such that $\mathcal{C}\cap\Sigma_0\times\{0\} \neq \emptyset$. Then, for the set $\varphi(\mathcal{C})\subset\mathbf{R}$, we have that

$\varphi(\mathcal{C})$ is unbounded (as $\varphi_{|\Sigma}$ is proper and $\mathcal{C}$ is unbounded),

$\varphi(\mathcal{C})$ is connected (as φ is continuous and $\mathcal{C}$ connected),

$\varphi(\mathcal{C}) \cap [\varphi_-,\varphi_+] \neq \emptyset$ (as $\mathcal{C}\cap\Sigma_0\times\{0\} \neq \emptyset$).

Hence, at least one of the unbounded intervals $]-\infty, \varphi_-]$, $[\varphi_+, +\infty[$ is contained in $\varphi(\mathcal{C})\subset\varphi(\Sigma)$.
By (2.5), we then get $\varphi(\Sigma)\cap\{c_-,c_+\} \neq \emptyset$ and a contradiction with (2.7) is produced.

Secondly, we give another proof based only on elementary properties of the Leray-Schauder degree. Assume, by contradiction, that (2.4) has no solution. For c_- and c_+ given in (2.5), we consider the sets (with $I = [0,1]$):

$$\mathcal{A} := \varphi^{-1}(]c_-,c_+[) \subset \mathcal{X}\times I$$
$$\Sigma^* := \varphi^{-1}([c_-,c_+])\cap\Sigma \subset \mathcal{X}\times I.$$

From (2.7) and (i3), it follows that $\Sigma^*\subset\mathcal{A}$, with Σ^* compact and $\mathcal{A}$ open in $\mathcal{X}\times I$. Then, by a standard covering of Σ^* with balls of small radius in $\mathcal{X}\times I$ contained in $\mathcal{A}$, we can find a set $\mathcal{B}$, bounded and open in $\mathcal{X}\times I$, such that

$$\Sigma^* \subset \mathcal{B} \subset \mathrm{cl}_{\mathcal{X}\times I}\mathcal{B} \subset \mathcal{A}. \tag{2.8}$$

We claim now that

$$Lx \neq N(x,\lambda),$$

whenever $x\in(\partial\mathcal{B})_\lambda$ and $\lambda\in[0,1]$, with

$$\mathcal{B}_\lambda := \{x\in\mathcal{X}: (x,\lambda)\in\mathcal{B}\}, \quad \text{for each } \lambda\in[0,1], \text{ and}$$

$$(\partial\mathcal{B})_\lambda := \{x\in\mathcal{X}: (x,\lambda)\in \mathrm{fr}_{\mathcal{X}\times I}\mathcal{B}\}.$$

Indeed, if $Lx = N(x,\lambda)$, for some $\lambda\in[0,1]$ and $x\in(\partial\mathcal{B})_\lambda$, then $(x,\lambda)\in\Sigma\cap\mathrm{cl}_{\mathcal{X}\times I}\mathcal{B}\subset\Sigma\cap\mathcal{A}=\Sigma^*$ and hence $x\in\mathcal{B}_\lambda$, a contradiction.

Using the homotopy invariance of coincidence degree in the slightly more general form
" $D_L(L - N(\cdot,\lambda),\Omega_\lambda)$ is independent of λ if $\Omega\subset\mathcal{X}\times I$ is an open bounded set such that $Lx\neq N(x,\lambda)$ for each $x\in(\partial\Omega)_\lambda$ and each $\lambda\in[0,1]$, where $\Omega_\lambda := \{x\in\mathcal{X}: (x,\lambda)\in\Omega\}$ ",
which easily follows from the definition of coincidence degree and the corresponding property of the Leray-Schauder degree (see e.g. [90, p. 60]), then we have

$$D_L(L - N(\cdot,0),\mathcal{B}_0) = D_L(L - N(\cdot,1),\mathcal{B}_1) = 0,$$

since equation (2.4) has no solution. Thus, we have

$$D_L(L - N(\cdot,0),\mathcal{B}_0) = 0. \tag{2.9}$$

On the other hand, by (2.5)

$$\Sigma_0 \times \{0\} \subset \Sigma^* \subset \mathcal{B}$$

and hence $\Sigma_0 \subset \mathcal{B}_0$, so that

$$|D_L(L - N(\cdot,0),\mathcal{B}_0)| = \chi_0 \neq 0,$$

a contradiction with (2.9) which completes the proof. ♦

EXAMPLE 1. Assume (i1) , (i2) and let $\Omega \subset \mathcal{X}$ be an open bounded set with $\Sigma_0 \subset \Omega$ and such that

$$Lx \neq N(x,\lambda)\ , \text{ for each } x \in \mathrm{dom}L \cap \mathrm{fr}_{\mathcal{X}}\Omega \text{ and } \lambda \in]0, 1[\ . \tag{2.10}$$

In this case, the solvability of (2.4) is ensured by standard properties of the coincidence degree (see [98]). However, we note that this situation easily fits into the framework of Lemma 1. Indeed, it is sufficient to define

$$\varphi(x;\lambda) := \begin{cases} -\operatorname{dist}(x,\mathrm{fr}\Omega) & \text{for } x \in \Omega, \\ \operatorname{dist}(x,\mathrm{fr}\Omega) & \text{for } x \notin \Omega \ ; \end{cases}$$

and observe that (i3) is satisfied and

$$-\operatorname{diam}(\mathrm{cl}_{\mathcal{X}}\Omega) \leq \varphi_-\ , \quad \varphi_+ < 0.$$

Then (2.10) implies the validity of (2.5)-(2.6) for $c_+ = 0$ and any $c_- < -\operatorname{diam}(\mathrm{cl}_{\mathcal{X}}\Omega)$. ♦

In the next Section, a particular choice of φ is considered. Accordingly, we state now some consequences of Lemma 1 which are more directly applicable to the subsequent examples.

COROLLARY 1. *Assume* (i1), (i2), (i3). *Suppose that* φ *is bounded below on* Σ *and there is a sequence* $(c_n)_n$ *of real numbers, with* $\lim_{n\to+\infty} c_n = +\infty$, *such that* $c_n \notin \varphi(\Sigma)$, *for all* $n \in \mathbb{N}$.
Then, equation (2.4) *has at least one solution.*

Proof. We immediately get the result from Lemma 1 with the choices

$$c_- < k\ , \text{ with } k \text{ a lower bound for } \varphi(\Sigma),$$

$$c_+ = c_n\ , \text{ for } n \text{ sufficiently large.}$$ ♦

In the next applications to the periodic problem to first order ODEs in the plane (Section 4), we deal with a functional

$$\varphi : \mathcal{X}\times[0, 1] \to \mathbf{R}_+ \qquad (2.11)$$

which is *continuous* and satisfies the following property:

(i4) $\exists\, R > 0$ *such that* $\varphi(\Sigma \setminus (B(0, R)\times[0, 1]) \subset \mathbf{Z}_+$.

We remark that in condition (i4) we do not assume the existence of points belonging to $\Sigma \setminus (B(0,R)\times[0,1])$. What we just require is that

if (x,λ) *is any solution of* (2.1), *with* $|x|_{\mathcal{X}}\geq R$ *and* $\lambda\in[0,1]$, *then* $\varphi(x,\lambda)\in \mathbf{Z}_+$.

Thus, assumption (i4) has to be considered as vacuously satisfied when Σ is bounded.
If (i4) holds, then, as a consequence, we get

(i5) $\exists\, M > 0$ *such that* $\varphi(\Sigma) \subset [0,M]\cup\mathbf{N}$.

In fact, it is sufficient to observe that $\Sigma\cap(B[0,R]\times[0,1])$ is compact and define $M := \sup\{\varphi(x,\lambda): (x,\lambda)\in \Sigma\cap(B[0,R]\times[0,1])\}$. Hence, (i4) implies (i5).
Then we have

COROLLARY 2. *Assume* (i1), (i2), (i4) *and suppose that*

(i6) $\varphi^{-1}(n)\cap\Sigma$ *is bounded, for each* $n\in \mathbf{Z}_+$

(with φ *continuous, as in* (2.11)*) holds.*
Then, equation (2.4) *has at least one solution.*

Proof. It can be easily checked that (i4) and (i6), together with the local compactness of Σ, imply that $\varphi_{|\Sigma}$ is proper, and so (i3) holds as φ is supposed to be continuous on $\mathcal{X}\times[0, 1]$.
On the other hand, (i5) (which, in turns, is implied by (i4)) ensures that $\varphi(\Sigma)$ is bounded below (see also (2.11)) and that $c_n := k + ((2n + 1)/2)\notin \varphi(\Sigma)$, for any fixed $k\in \mathbf{N}$, with $k \geq M$.
Then Corollary 1 applies and the result is achieved. ◆

We also remark that under (i4) the properness of $\varphi_{|\Sigma}$ is indeed equivalent to (i6) (in other words, if it is convenient, we can check (i6) through (2.3)).

Finally, we give a further consequence of Corollary 1 which extends Corollary 2 and is motivated by the study of the weakly coupled systems considered in Section 5. Once again, we stress the fact that the assumptions considered here are not the most general ones but they are chosen in order to make straightforward the application of the abstract existence results to the differential equations examined in the next Sections. To this purpose, the following situation is considered.

Let $\mathcal{X}_j$ ($j = 1, \ldots, \ell$) be closed linear subspaces of $\mathcal{X}$ such that $\mathcal{X} = \bigoplus_{j=1}^{\ell} \mathcal{X}_j$ and suppose that linear continuous projectors $\Pi_j : \mathcal{X} \to \mathcal{X}_j$, $\Pi_j(x) = x_j$, with $\|\Pi_j\| = 1$ ($j = 1, \ldots, \ell$) are selected, so that every $x \in \mathcal{X}$ can uniquely be expressed as $x = \sum_{j=1}^{\ell} x_j$. By the above assumptions, we have $|x_j|_{\mathcal{X}} \leq |x|_{\mathcal{X}} \leq \sum_{j=1}^{\ell} |x_j|_{\mathcal{X}}$, for each $x \in \mathcal{X}$. We also define $\tilde{\Pi}_j : \mathcal{X} \times \mathbf{R} \to \mathcal{X}_j$ by $\tilde{\Pi}_j(x,\lambda) = \Pi_j(x)$.
Then we suppose that, for each $j=1,\ldots,\ell$, there is a *continuous* functional $\varphi_j : \mathcal{X} \times [0,1] \to \mathbf{R}_+$ such that

(i*4) $\quad \forall j = 1, \ldots, \ell$, $\exists R_j > 0$ *such that* $\varphi_j(x,\lambda) \in \mathbf{Z}_+$, *for any* $(x,\lambda) \in \Sigma$ *with* $|x_j|_{\mathcal{X}} \geq R_j$,

(i*5) $\quad \forall j = 1, \ldots, \ell$, $\exists M_j > 0$, *such that* $\varphi_j(\Sigma) \subset [0,M_j] \cup \mathbf{N}$

and

(i*6) $\quad \forall j = 1, \ldots, \ell$, $\forall n \in Z_+$, $\tilde{\Pi}_j(\varphi_j^{-1}(n) \cap \Sigma)$ *is bounded,*

hold.
Essentially, (i*4), (i*5), (i*6) constitute an extension of the analogous conditions (i4), (i5), (i6). All these assumptions are motivated by some basic properties of the functional counting the number of revolutions of a periodic solution (see (3.13) and (3.32) in the next Section). Now, we can state the following

COROLLARY 3. *Assume* (i1), (i2), (i*4), (i*5), (i*6).
Then, equation (2.4) *has at least one solution.*

Proof. Let us set $M := \max\{M_j : j=1,\ldots,\ell\}$ and define the functional $\varphi : \mathcal{X} \times [0, 1] \to \mathbf{R}$ by

$$\varphi(x,\lambda) := \max\{\varphi_j(x,\lambda) : \ j=1,\ldots,\ell\}.$$

The map φ is continuous and satisfies (i5) (by (i*5)). Then, arguing as in the proof of Corollary 2, we can find a sequence $c_n \to +\infty$ with $c_n \notin \varphi(\Sigma)$, for all $n \in \mathbf{N}$. Moreover φ is bounded below (by definition of φ). Hence, in order to apply Corollary 1, we need to prove the properness of $\varphi_{|\Sigma}$. Accordingly, let $\mathcal{K} \subset \mathbf{R}_+$ be a compact set and take $n^* \in \mathbf{N}$ such that $\mathcal{K} \subset [0,n^*]$. We show that

$\varphi^{-1}([0,n^*])\cap\Sigma$ is compact.
Indeed, consider the sets

$$\mathcal{D}_j := \{(x,\lambda)\in\Sigma : \varphi_j(x,\lambda)\in[0, n^*]\}\ , j=1,\dots,\ell\ ,$$

$$\mathcal{R}_j := \{(x,\lambda)\in\Sigma : \varphi_j(x,\lambda)\in[0, n^*]\setminus \mathbf{Z}_+\}\ , j=1,\dots,\ell,$$

$$\mathcal{S}_j^{(k)} := \{(x,\lambda)\in\Sigma : \varphi_j(x,\lambda) = k\}\ , k = 0, 1, \dots, n^*\ ,\ j=1,\dots,\ell\ .$$

Clearly, $\mathcal{D}_j = \mathcal{R}_j \cup (\bigcup_{k=0}^{n^*} \mathcal{S}_j^{(k)})$, for each $j=1,\dots,\ell$.

By (i*4), we have $|x_j|_{\mathcal{X}} = |\Pi_j x|_{\mathcal{X}} < R_j$, for all $(x, \lambda)\in\mathcal{R}_j$.
On the other hand, (i*6) implies the boundedness of $\tilde{\Pi}_j(\mathcal{S}_j^{(k)})$, for every $k = 0,1, \dots,n^*$. As a consequence, we obtain that $\tilde{\Pi}_j(\mathcal{D}_j)$ is bounded, for each $j=1,\dots,\ell$. Hence, there is $R > 0$ such that

$$|\Pi_j x|_{\mathcal{X}} < R, \text{ for all } (x,\lambda)\in\mathcal{D}_j\ ,\quad j=1,\dots,\ell.$$

Finally, let $(x,\lambda)\in\varphi^{-1}([0, n^*])\cap\Sigma$, that is $(x,\lambda)\in\Sigma$ and $\varphi(x,\lambda)\le n^*$. By definition of φ, we have that $\varphi_j(x,\lambda)\le n^*$, for each $j=1,\dots,\ell$ and so $(x,\lambda)\in\mathcal{D}_j$, for $j=1,\dots,\ell$. By the above estimates, we get

$$|x_j|_{\mathcal{X}} < R,\ \text{ for } j=1,\dots,\ell\ ,$$

and therefore

$$(x,\lambda)\in B(0,\ \ell R)\times[0,1].$$

Then, we have proved that $\varphi^{-1}([0, n^*])\cap\Sigma$ is a bounded (and closed) subset of Σ, that is a compact set (by the local compactness of Σ).
As $\varphi_{|\Sigma}$ is proper, we have that (i3) is fulfilled and so Corollary 1 gives the thesis.
The proof is complete. ♦

3. Existence results

In this Section, we give a general continuation theorem for the solvability of the periodic boundary value problem on the interval $[0,T]$ $(T > 0)$:

$$\dot{x} = F(t, x) , \tag{3.1}$$

$$x(0) = x(T) . \tag{3.2}$$

More specific applications are considered in the next Sections 4 and 5.

After some (standard) preliminaries, we introduce a class of functionals whose properties (contained in Proposition 1, 2 and 3) are used for the proof of the existence results of this Chapter (Theorems 1, 2 of this Section and Theorems 3, 4 and 5 in Sections 4 and 5).

In what follows, we suppose that system (3.1) is imbedded into a one-parameter family of differential equations

$$\dot{x} = f(t,x;\lambda) , \tag{3.3}$$

where $\lambda \in [0,1]$ and

$$f(t,x;1) = F(t,x) . \tag{3.4}$$

The function $f : [0,T]\times\mathbf{R}^m\times[0,1] \to \mathbf{R}^m$ satisfies the Caratheodory conditions, i.e. $f(\cdot, x; \lambda)$ is (Lebesgue) measurable for each (x,λ), $f(t, \cdot; \cdot)$ is continuous for a.e. t and, for each $r > 0$, there exists $\beta_r \in L^1([0,T], \mathbf{R}_+)$ such that $|f(t,x;\lambda)| \leq \beta_r(t)$ holds for a.e. $t\in[0,T]$ and all $|x| \leq r$, $\lambda\in[0,1]$.
We also assume that system (3.3) is autonomous for $\lambda = 0$, that is

$$f(t,x;0) = f_0(x), \tag{3.5}$$

where $f_0 : \mathbf{R}^m \to \mathbf{R}^m$ is a continuous function. Solutions of (3.1) and (3.3) are intended in the generalized (Caratheodory) sense (see [66, p. 28]) and are called T-periodic provided that they are defined on $[0,T]$ and satisfy the boundary condition (3.2). It is well known that if $F : \mathbf{R}\times\mathbf{R}^m \to \mathbf{R}^m$ is continuous and T-periodic in the first variable, then any solution of (3.1)-(3.2) is the restriction on $[0,T]$ of a T-periodic solution of the class C^1 defined on the whole real line.
We consider now the spaces

$$X := \{x\in \mathcal{C}([0,T], \mathbf{R}^m) : x(0) = x(T)\} \ , \ |\cdot|_X := |\cdot|_\infty ,$$

$$Z := L^1([0,T], \mathbf{R}^m) \ , \ |\cdot|_Z = |\cdot|_1$$

and the operators

$$L : \mathrm{dom}L\subset X \to Z \ , \ (Lx)(t) := \dot{x}(t) ,$$

with $\mathrm{dom}L = \{x\in X : x \text{ is absolutely continuous}\}$,

$$N : X\times[0, 1] \to Z , \ N(x,\lambda)(t) := f(t, x(t); \lambda).$$

By the above assumptions on f, it follows that N is L-completely continuous, with L a Fredholm mapping of index zero (see [98, Chapters I, VI]).
Hence, with the notations of Section 2, we have that

$$\Sigma = \{(x,\lambda)\in \mathrm{dom}L\times[0,1]:\ \dot{x} = f(t,x;\lambda),\ x(0) = x(T)\}$$

and

$$\Sigma_0 = \{x\in \mathrm{dom}L : \dot{x} = f_0(x),\ x(0) = x(T)\}.$$

Observe that every $x(\cdot)\in\Sigma_0$ is actually of class C^1 and satisfies $\dot{x}(0) = \dot{x}(T)$ as well.
Then, condition (i1) is fulfilled if and only if

(h1) $\exists\, r_0 > 0$ *such that* $|x|_\infty < r_0$, *for every* T-*periodic solution* $x(\cdot)$ *of*

$$\dot{x} = f_0(x). \tag{3.6}$$

In this case, $\Sigma_0\subset B(0,r_0)$.
Since every $z\in\mathbf{R}^m$ such that $f_0(z) = 0$ is a T-periodic (constant) solution of (3.6), from (h1) we have that the Brouwer degree $d_B(f_0, B(0,r), 0)$ is defined and is constant with respect to $r \geq r_0$.
At this point, we can use Theorem 1 in Chapter 3 (with $\mathcal{X} = X$ and $\mathcal{Z} = Z$) according to which, for each $r \geq r_0$,

$$|D_L(L - N(\cdot, 0), B(0, r))| = |d_B(f_0, B(0, r), 0)| \tag{3.7}$$

holds, provided that (h1) is assumed (see Theorem 1 in Chapter 3). Finally, by (3.7) we have

$$\chi_0 = |d_B(f_0, B(0, r), 0)|\,,\ \text{for any}\ r \geq r_0 \tag{3.8}$$

and so (i2) is equivalent to

(h2) $d_B(f_0, B(0, r), 0) \neq 0$, *for any* $r \geq r_0$.

REMARK 1. We notice that, in some of the subsequent applications, the validity of (h1) will be guaranteed by means of

(h*1) $\exists\, r_0 > 0$ *such that every periodic solution (of any period)* $x(\cdot)$ *of* (3.6) *satisfies*

$$x(t)\in B(0,r_0)\subset\mathbf{R}^m\,,\ \forall t\in\mathbf{R}\,.$$

In this case, (3.7) can be obtained without invoking Theorem 1 in Chapter 3, but just using standards properties of the coincidence degree (see, for instance, [55, pp. 28-29] or [145, p. 208]).

We introduce now a suitable class of functionals $\varphi : X\times[0,1]\to\mathbf{R}$ in order to apply Lemma 1 and its corollaries.

Let $m\geq 2$ and let $\eta, \nu\in\mathbf{R}^m\setminus\{0\}$ be two (fixed) orthonormal vectors.
We set, for any $x\in\mathbf{R}^m$,

$$x_\eta := (x|\eta), \quad x_\nu := (x|\nu) \tag{3.9}$$

$$f_\eta(t,x;\lambda) := (f(t,x;\lambda) \mid \eta), \quad f_\nu(t,x;\lambda) := (f(t,x;\lambda) \mid \nu). \tag{3.10}$$

Observe that if $x(\cdot)$ is a solution of (3.3) then

$$\begin{aligned} \dot{x}_\eta(t) &= f_\eta(t,x(t);\lambda) \\ \dot{x}_\nu(t) &= f_\nu(t,x(t);\lambda), \end{aligned} \tag{3.11}$$

holds (with $x_\eta(t) := (x(t)|\eta)$ and $x_\nu(t) := (x(t)|\nu)$ defined from (3.9)).
Then, we define, for $(x,\lambda)\in X\times[0,1]$,

$$\varphi_{(\eta,\nu)}(x,\lambda) := \frac{1}{2\pi}\left|\int_0^T [x_\eta(t)\, f_\nu(t,x(t);\lambda) - x_\nu(t)\, f_\eta(t,x(t);\lambda)]\cdot \delta(x_\eta(t),x_\nu(t))dt\right|,$$

where $\delta : \mathbf{R}^2\to\mathbf{R}^+$ is defined as follows:

$$\delta(\alpha,\beta) := \begin{cases} 1 & \text{for } a^2+b^2<1, \\ -1 & \text{for } a^2+b^2\geq 1 \quad ; \end{cases} \tag{3.12}$$

Clearly, with the above positions, we have:

$$\varphi_{(\eta,\nu)}(x,\lambda) := \frac{1}{2\pi}\left|\int_0^T [x_\eta(t)\, f_\nu(t,x(t);\lambda) - x_\nu(t)\, f_\eta(t,x(t);\lambda)] \,/\, (x_\eta(t)^2+x_\nu(t)^2)dt\right|, \tag{3.13}$$

for any $x(\cdot)\in X$ such that

$$x_\eta(t)^2+x_\nu(t)^2\geq 1, \quad \forall\ t\in[0,T], \tag{3.14}$$

holds.
It can be easily checked that

$$\varphi_{(\eta,\nu)} : X\times[0,1]\to\mathbf{R}_+ \text{ is continuous.}$$

Moreover, $\varphi_{(\eta,\nu)} = \varphi_{(\nu,\eta)}$, for any pair (η,ν) of orthonormal vectors.

REMARK 2. The functional $\varphi_{(\eta,\nu)}$ defined above is a modification of the classical map which counts the number of rotations around the origin of the solutions of (3.11) (see [80, § 3.14-3.15], [8, § 2.5-2.10]). Actually, some fundamental properties of the standard "angular function" are preserved, since we have the following

PROPOSITION 1. *Let* $(x,\lambda)\in\Sigma$ *be such that* $x_\eta(t)^2+x_\nu(t)^2\geq 1$, *for all* $t\in[0,T]$. *Then,* $\varphi_{(\eta,\nu)}\in\mathbf{Z}_+$.

Proof. We introduce polar coordinates (θ,ρ) in the (η,ν)-plane such that

$$x_\eta(t) = \rho(t)\cos\theta(t), \quad x_\nu(t) = \rho(t)\operatorname{sen}\theta(t), \tag{3.15}$$

with

$$\rho(t) = (x_\eta(t)^2+x_\nu(t)^2)^{1/2} \tag{3.16}$$

and

$$\theta(t) = \mathbf{arg}(x_\eta(t)/\rho(t) + \mathbf{i}\; x_\nu(t)/\rho(t)), \tag{3.17}$$

provided that

$$(x_\eta(t),x_\nu(t)) \neq 0, \text{ for all } t\in[0,T]. \tag{3.18}$$

We note that we can suppose that ρ, $\theta : [0,T]\to\mathbf{R}$ are absolutely continuous functions such that

$$\rho(t) > 0 \quad \forall\; t \text{ and } \rho(0) = \rho(T), \qquad [\theta(T) - \theta(0)] / 2\pi \in \mathbf{Z} \tag{3.19}$$

hold, whenever x_η, $x_\nu : [0,T]\to\mathbf{R}$ are absolutely continuous, with $x_\eta(T) - x_\eta(0) = x_\nu(T) - x_\nu(0) = 0$ and (3.18) is fulfilled. Hence, in this case, (3.15) is well established and, by standard computations, we easily get

$$\dot\theta(t) = [\dot x_\nu(t)x_\eta(t) - \dot x_\eta(t)x_\nu(t)] / [x_\eta(t)^2+x_\nu(t)^2], \tag{3.20}$$

for a.e. $t\in[0,T]$.

Suppose now that $(x,\lambda)\in\Sigma$ and (3.14) is satisfied (as assumed in Proposition 1). Then, from the definition of $\delta(\cdot,\cdot)$ in (3.12), we have that $\varphi_{(\eta,\nu)}(x,\lambda)$ can be computed by (3.13) and so, using (3.11) and (3.20), we obtain

$$\varphi_{(\eta,\nu)}(x,\lambda) := \frac{1}{2\pi}\left| \int_0^T \dot{\theta}(t)dt \right| = |\theta(T) - \theta(0)| / 2\pi \in \mathbf{Z}_+ \tag{3.21}$$

(recall (3.19)).
Therefore, Proposition 1 is proved. ♦

We remark that all the discussion about polar coordinates performed in the above proof (from step (3.15) to step (3.20)) is completely independent of the assumptions in Proposition 1 (actually, such arguments lie on the validity of (3.18)). Accordingly, in what follows we can employ (3.15)-(3.20) without making explicit reference to the hypotheses of Proposition 1.

Now, we give some conditions which lead to the evaluation of upper and lower bounds for $\varphi_{(\eta,\nu)}(x,\lambda)$ when $(x,\lambda)\in\Sigma$ and $(x_\eta(t)^2+x_\nu(t)^2)$ is large. These estimates, which are a consequence of the form of the nonlinearity f, will be used in Sections 4 and 5. Namely, we have

PROPOSITION 2. *Suppose that there are a constant* $K_0 > 0$ *, a continuous function* $\Theta : S^1 \to R^+$ *and a measurable function* $\gamma\in L^1([0,T],R_+)$ *such that one of the following inequalities holds for a.e.* $t\in[0,T]$ *, all* $\lambda\in[0,1]$ *and each* $x\in R^m$ *such that* $|x_{\eta,\nu}|:=(x_\eta^2+x_\nu^2)^{1/2} \geq K_0$:

(w1) $$-\gamma(t) \leq f_\nu(t,x;\lambda)\cdot\frac{x_\eta}{|x_{\eta,\nu}|} - f_\eta(t,x;\lambda)\cdot\frac{x_\nu}{|x_{\eta,\nu}|} \leq \alpha_\lambda(t)\,(|x_{\eta,\nu}|\,\Theta\left(\frac{x_\eta}{|x_{\eta,\nu}|}\,,\,\frac{x_\nu}{|x_{\eta,\nu}|}\right) + \gamma(t)$$

(w2) $$\gamma(t) \geq f_\nu(t,x;\lambda)\cdot\frac{x_\eta}{|x_{\eta,\nu}|} - f_\eta(t,x;\lambda)\cdot\frac{x_\nu}{|x_{\eta,\nu}|} \geq -\alpha_\lambda(t)\,|x_{\eta,\nu}|\,\Theta\left(\frac{x_\eta}{|x_{\eta,\nu}|}\,,\,\frac{x_\nu}{|x_{\eta,\nu}|}\right) - \gamma(t)$$

(w3) $$f_\nu(t,x;\lambda)\cdot\frac{x_\eta}{|x_{\eta,\nu}|} - f_\eta(t,x;\lambda)\cdot\frac{x_\nu}{|x_{\eta,\nu}|} \geq \alpha_\lambda(t)\,|x_{\eta,\nu}|\,\Theta\left(\frac{x_\eta}{|x_{\eta,\nu}|}\,,\,\frac{x_\nu}{|x_{\eta,\nu}|}\right) - \gamma(t)$$

(w4) $$f_\nu(t,x;\lambda)\cdot\frac{x_\eta}{|x_{\eta,\nu}|} - f_\eta(t,x;\lambda)\cdot\frac{x_\nu}{|x_{\eta,\nu}|} \leq -\alpha_\lambda(t)\,|x_{\eta,\nu}|\,\Theta\left(\frac{x_\eta}{|x_{\eta,\nu}|}\,,\,\frac{x_\nu}{|x_{\eta,\nu}|}\right) + \gamma(t)$$

where, for each $\lambda\in[0,1]$, $\alpha_\lambda\in L^1([0,T],R)$ *and satisfies*

$$\int_0^T \alpha_\lambda(s)\,ds > 0.$$

Then, for every $\varepsilon > 0$, *there exists a constant* $R_\varepsilon \geq 1$ *(independent of* x *and* λ*) such that, for each* $(x,\lambda) \in \Sigma$ *satisfying*

$$x_\eta(t)^2 + x_v(t)^2 \geq R_\varepsilon^2 \text{ , for all } t \in [0,T] \text{ ,} \tag{3.22}$$

it follows that

$$\varphi_{(\eta,v)}(x, \lambda) \leq \int_0^T (\alpha_\lambda(s) \,/\, 2\pi \,\langle\Theta\rangle)\, ds + \varepsilon \tag{3.23}$$

in case (w1) *or* (w2), *and*

$$\varphi_{(\eta,v)}(x, \lambda) \geq \int_0^T (\alpha_\lambda(s) \,/\, 2\pi \,\langle\Theta\rangle)\, ds - \varepsilon \tag{3.24}$$

in case (w3) *or* (w4).

Proof. We prove only (3.23), assuming the validity of (w1). The investigation of the other cases is omitted, since it can be performed by obvious modifications of the argument developed below.

We consider the positive constants

$$A \geq \max\,\{1, K_0\}$$

and

$$\sigma := \min\{\,\Theta(a,b)\colon (a,b) \in S^1\}$$

and suppose that $(x,\lambda) \in \Sigma$ satisfies

$$x_\eta(t)^2 + x_v(t)^2 \geq A^2, \text{ for all } t \in [0,T]. \tag{3.25}$$

Using polar coordinates for $(x_\eta(t), x_v(t))$ as in (3.15), we can compute $\dot\theta(t)$ by (3.20), with $\dot x_\eta(t)$ and $\dot x_v(t)$ given by (3.11). Thus we obtain, for a.e. $t \in [0,T]$,

$$\dot\theta(t) = [f_v(t, x(t); \lambda) \cos\theta(t) - f_\eta(t, x(t); \lambda) \sin\theta(t)] \,/\, \rho(t).$$

Hence, from (w1) we get

$$-\frac{\gamma(t)}{\rho(t)} \leq \dot{\theta}(t) \leq \alpha_\lambda(t)\Theta(\cos\theta(t)\,,\sin\theta(t)) + \frac{\gamma(t)}{\rho(t)}.$$

Then, dividing by $\Theta(\cos\theta(t),\sin\theta(t))$ and recalling (3.25) and the definition of σ, we have

$$-\frac{\gamma(t)}{\sigma A} \leq \frac{\dot{\theta}(t)}{\Theta(\cos\theta(t)\,,\sin\theta(t))} \leq \alpha_\lambda(t) + \frac{\gamma(t)}{\sigma A}. \tag{3.26}$$

We integrate now (3.26) over $[0,T]$ and obtain

$$-|\gamma|_1/\sigma A \leq \int_0^T \frac{\dot{\theta}(t)}{\Theta(\cos\theta(t),\sin\theta(t))}dt =$$

$$= \int_{\theta(0)}^{\theta(T)} \frac{d\theta}{\Theta(\cos\theta,\sin\theta)} \leq \left(\int_0^T \alpha_\lambda(s)ds\right) + (|\gamma|_1/\sigma A).$$

For the rest of the proof, we set

$$L := \int_0^{2\pi} \frac{d\theta}{\Theta(\cos\theta,\sin\theta)} = 2\pi\,\langle\Theta\rangle, \quad \tau_\lambda := \int_0^T \alpha_\lambda(s)ds.$$

From (3.19) in Proposition 1, we have that

$$\theta(T) - \theta(0) = 2k\pi \ , \ \text{for some } k\in \mathbf{Z},$$

where (by (3.21))

$$|k| = \varphi_{(\eta,v)}(x,\lambda).$$

Hence, the above inequality can be written as

$$-|\gamma|_1 / \sigma A \leq kL \leq \tau_\lambda + (|\gamma|_1/\sigma A)$$

(using, in this step, the fact that $\Theta(\cos\theta,\sin\theta)$ is 2π-periodic) and, dividing by L, we get

$$-|\gamma|_1 / \sigma LA \leq k \leq (\tau_\lambda/L) + (|\gamma|_1 / \sigma LA).$$

Now, let $\varepsilon > 0$ be given. Without loss of generality, we can also suppose that $\varepsilon<1$. In this case, choosing $A > |\gamma|_1/\sigma L\varepsilon$, we obtain

$$-1 < -\varepsilon < k \leq (\tau_\lambda/L) + \varepsilon$$

and so, recalling that $k \in \mathbf{Z}$ we have

$$k = |k| = \varphi_{(\eta,\nu)}(x,\lambda) \leq (\tau_\lambda/L) + \varepsilon.$$

In the same manner, it is possible to prove that

$$-k = |k| = \varphi_{(\eta,\nu)}(x,\lambda) \leq (\tau_\lambda/L) + \varepsilon$$

holds in case (w2), while

$$k = |k| = \varphi_{(\eta,\nu)}(x,\lambda) \geq (\tau_\lambda/L) - \varepsilon$$

and

$$-k = |k| = \varphi_{(\eta,\nu)}(x,\lambda) \geq (\tau_\lambda/L) - \varepsilon$$

follow from (w3) and (w4), respectively (assuming $0<\varepsilon<1$, too).
The proof is complete. ♦

From the above argument it is clear that, in condition (w1) (respectively, (w2)), the existence of a lower bound (upper bound, respectively) for

$$(f_\nu x_\eta - f_\eta x_\nu) / (x_\eta^2 + x_\nu^2)^{1/2}$$

is employed only in order to prove that $k = |k|$ (respectively, $-k = |k|$), where $k = \theta(T) - \theta(0)$ is the number of counter-clockwise revolutions of $(x_\eta(t), x_\nu(t))$ around the origin in the (η,ν)-plane. Thus (3.23) can be achieved without assuming the validity of the left-hand side inequality in (w1) or (w2), provided that we are able to detect, in another way, the sign of k. This purpose can be accomplished if, for instance, the lower bound in (w1) (upper bound in (w2), respectively) is replaced by a condition like (w3) (or (w4), respectively). This remark will be used for the proof of Theorem 4 in the next Section.

At last, we observe that in the particular case when

$$\alpha_\lambda(t) = \alpha(t) = \text{constant w.r.to } \lambda \in [0,1],$$

it is possible to obtain more precise conclusions from (w1), (w2), (w3), (w4). Namely, in such a situation we have that

$$\varphi_{(\eta,v)}(x,\lambda) \leq \int_0^T \alpha(t)dt \; / \; 2\pi \; \langle\Theta\rangle \tag{3.27}$$

(for (w1) or (w2)) and

$$\varphi_{(\eta,v)}(x,\lambda) \geq \int_0^T \alpha(t)dt / 2\pi \; \langle\Theta\rangle \tag{3.28}$$

(for (w3) or (w4)) hold provided that $(x,\lambda)\in\Sigma$ satisfies

$$x_\eta(t)^2 + x_v(t)^2 \geq R_0{}^2 \,, \; \forall t\in [0,T],$$

with $R_0 \geq 1$ a suitable constant (independent of x and λ).
This result easily follows from (3.23), (3.24) (respectively) for ε sufficiently small, taking into account that $\varphi(x,\lambda)$ is an integer (for $R_0 \geq 1$, by Proposition 1).

After this preliminary list of the basic properties the functional $\varphi_{(\eta,v)}$, we are in position to state the main results of this Section. To this end, we confine ourselves to differential systems in spaces of even dimension. Accordingly, from now on, let us suppose

$$m = 2\ell \tag{3.29}$$

and set, for any $j = 1,\ldots,\ell$,

$$\varphi_j := \varphi_{(e_{2j-1},e_j)} \tag{3.30}$$

where $\{e_1,\ldots, e_m\}$ is the canonical orthonormal basis in $\mathbb{R}^m$. At the same time, we observe that

$$x_{e_{2j-1}} = x_{2j-1} = (x|e_{2j-1}) \;\;,\;\; x_{e_{2j}} = x_{2j} = (x|e_{2j}),$$

$$f_{e_{2j-1}} = f_{2j-1} = (f|e_{2j-1}) \;\;,\;\; f_{e_{2j}} = f_{2j} = (f|e_{2j}),$$

so that, for the pair $(e_{2j-1} \,, e_{2j})$, system (3.11) reduces to

$$\begin{cases} \dot{x}_{2j-1}(t) = f_{2j-1}(t,x(t);\lambda) \\ \dot{x}_{2j}(t) = f_{2j}(t,x(t);\lambda). \end{cases} \tag{3.31}$$

If $m>2$, we have further to require the following assumption (h*3), which essentially appears as a Caratheodory condition on (f_{2j-1},f_{2j}), uniform with respect to the x_k-components, for $k \neq 2j-1, 2j$:

(h*3) $\forall j = 1,\ldots,\ell$, $\forall r \geq 0$, $\exists \beta_r \in L^1([0,T],\mathbf{R}_+)$, *such that*

$$|(f_{2j-1}(t,x;\lambda) , f_{2j}(t,x;\lambda))| \leq \beta_r(t)$$

holds for a.e. $t\in[0,T]$, *all* $\lambda\in[0,1]$ *and each* $x\in\mathbf{R}^m$ *such that* $|(x_{2j-1},x_{2j})|\leq r$.

Then we have

THEOREM 1. *Assume* (h1), (h2), (h*3) *and suppose that for every* $j=1,\ldots,\ell$ *the following properties are satisfied :*

(h*4) $\forall r_1 \geq 0$ $\exists r_2 \geq r_1$ *such that, for each* $(x,\lambda)\in\Sigma$,

$$\min_{[0,T]} |(x_{2j-1}(t), x_{2j}(t))| \leq r_1 \Rightarrow \max_{[0,T]} |(x_{2j-1}(t), x_{2j}(t))| \leq r_2 ;$$

(h*5) $\forall n \in \mathbf{Z}_+$ $\exists K_n \geq 0$ *such that, for each* $(x,\lambda)\in\Sigma$,

$$\varphi_j(x,\lambda) = n \quad \Rightarrow \quad \min_{[0,T]} |(x_{2j-1}(t), x_{2j}(t))| \leq K_n .$$

Then, (3.1)-(3.2) *has at least one solution.*

As for the meaning (and fulfilment) of the above hypotheses, see Remark 3 below.

Proof. We apply Corollary 3 of Section 2.
Accordingly, we consider the subspaces $X_j \subset X$ $(j = 1,\ldots,\ell)$ defined by

$$X_j : = \{x(\cdot) = \mathrm{col}(x_k(\cdot))\in X: x_k = 0 \text{ for } k\notin \{2j-1, 2j\}\} ,$$

with projections $\Pi_j : X\to X_j$, $(j = 1,\ldots,\ell)$,

$$(\Pi_j x)(t) : = x_{2j-1}(t)\, e_{2j-1} + x_{2j}(t) e_{2j}$$

and see that all the structural assumptions are satisfied. The functionals $\varphi_j : X\times[0, 1]\to R_+$, $(j=1,\dots,\ell)$, are the same as in (3.30).
We have already observed that (i1), (i2) are equivalent to (h1), (h2) (respectively). Moreover, (h*4) and (h*5) clearly imply the validity of

(h*6) $\quad \forall\, n\in Z_+ \;\; \exists\, C_n \geq 0$ *such that* $\max_{[0,T]} |(x_{2j-1}(t)\,,\, x_{2j}(t))| \leq C_n,$

for each $(x,\lambda)\in\Sigma$ *such that* $\varphi_j(x,\lambda) = n,$

where the constant C_n is obtained as r_2 in (h*4) when K_n (given by (h*5)) plays the role of r_1. Since (h*6) is just a translation of the assumption (i*6) in Corollary 3, it is sufficient to verify the validity of (i*4) and (i*5).
Using condition (h*4) with $r_1=1$, we can find a suitable $r_2 = \tilde{r}$ such that if $(x,\lambda)\in\Sigma$ satisfies

$$|\Pi_j x|_X = \max_{[0,T]} |(x_{2j-1}(t),\, x_{2j}(t))| > \tilde{r},$$

then

$$x_{2j-1}(t)^2 + x_{2j}(t)^2 \geq 1, \text{ for all } t\in[0,T]$$

and hence Proposition 1 yields (i*4) with $R_j > \tilde{r}$ $(j=1,\dots,\ell)$.
If we define now, for every $j = 1,\dots,\ell$,

$$M_j := \sup\{\varphi_j(x,\lambda):\; x\in X,\, \lambda\in[0,1], \text{ with } |\Pi_j x|_X \leq \tilde{r}\},$$

from the definition of φ_j $(j=1,\dots,\ell)$ we get $M_j < +\infty$, as

$$M_j \leq \tilde{r}\, |\beta_{\tilde{r}}|_1 / 2\pi,$$

using (h*3). From this choice of the M_j, we immediately get (i*5), using again Proposition 1. Then Corollary 3 can be applied and the result follows. ◆

We note that in the two-dimensional case (m=2) assumption (h3) is contained in the Caratheodory conditions which are requested at the beginning of the Section. In this case, we can state a simplified version of Theorem 1, using Corollary 2. Moreover, for m=2, we consider a functional

$$\varphi = \varphi_{(e_1,e_2)},$$

which is defined as follows

$$\varphi(x,\lambda) := \frac{1}{2\pi}\left|\int_0^T (f(t,x(t);\lambda) \mid Jx(t))\ \delta(x(t))dt\right| \quad , \tag{3.32}$$

with δ like in (3.12) and $\mathbf{J} : \mathbf{R}^2 \to \mathbf{R}^2$ represented by the symplectic matrix

$$\mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Then we have the following existence result which ensures the solvability of

$$\dot{x} = F(t, x) \tag{3.1}$$

$$x(0) = x(T) \tag{3.2}$$

via the "continuation" through

$$\dot{x} = f(t,x;\lambda),\ \ \lambda \in [0,1] \tag{3.3}$$

with f Caratheodory and such that

$$f(t\ x;\ 0) = f_0(x), \qquad f(t,\ x;\ 1) = F(t,\ x).$$

THEOREM 2. *Let* $m = 2$ *and assume*

(h1) $\exists\, r_0 > 0$ *such that* $|x|_\infty < r_0$ *for every* T*-periodic solution of* $\dot{x} = f_0(x)$;

(h2) $d_B(f_0, B(0,r), 0) \neq 0$ *, for any* $r \geq r_0$;

(h4) $\forall\, r_1 \geq 0\ \exists\, r_2 \geq r_1$ *such that, for each* $x(\cdot)$ *solution of* (3.3)-(3.2) *(with* $\lambda \in [0,1]$*)*

$$\min_{[0,T]}\ (x_1(t)^2 + x_2(t)^2)^{1/2} \leq r_1 \ \Rightarrow\ |x|_\infty \leq r_2\ ;$$

(h5) $\forall n \in \mathbf{Z}_+\ \exists\, K_n \geq 0$ *such that, for each* $x(\cdot)$ *solution of* (3.3)-(3.2) *(with* $\lambda \in [0,1]$*)*

$$\varphi(x,\lambda) = n \ \Rightarrow\ \min_{[0,T]}\ (x_1(t)^2 + x_2(t)^2)^{1/2} \leq K_n$$

(where φ *is like in* (3.32)*)* .

Then, (3.1)-(3.2) *has at least one solution.*

The proof is omitted since it is essentially the same as that of Theorem 1 .

We end this Section with some remarks concerning the fulfilment of the above conditions.

REMARK 3. Assumptions (h*1), (h*2) or, respectively, (h1), (h2), concern the autonomous system $\dot{x} = f_0(x)$ and will be checked by a direct analysis of the vector field f_0. The structural hypothesis (h*3) depends on the particular form of system (3.3) and is satisfied in the case of weakly coupled systems. Conditions (h*5) and (h5) will be obtained with the aid of Proposition 2. Finally, we examine (h*4) and (h4).
These assumptions are somehow related to the global forward (or backward) continuability on $[0,T]$, uniformly in λ, of the solutions of the Cauchy problems associated to (3.3). Roughly speaking, (h*4)-(h4) mean that it is possible to find an (uniform) upper bound for the solutions of (3.3)-(3.2) provided that we are able to bound the solutions at some point. This situation already occurred in [102] where the result is achieved by the use of a suitable Liapunov-like function (see also [98, Chapter VI], [58] for a more extensive development of such technique related to the concept of guiding function [79,81]). In the light of the above quoted papers we give an auxiliary result (Proposition 3 below) for the validity of (h*4), (h4). The proof is performed in the case of equation (3.11) in order to ensure the applicability of our result to planar and higher order systems. Accordingly, we use the notation x_η , x_ν , f_η, f_ν given in (3.9), (3.10), where (η,ν) is a pair of orthonormal vectors in $\mathbf{R}^m$ $(m \geq 2)$. The set of solutions of (3.3)-(3.2) is denoted by Σ, as usual. Then we have the following

PROPOSITION 3. *Let* $V : \mathbf{R}^2 \to \mathbf{R}$ *be a continuously differentiable function, with* $\nabla V = (\dot{V}_1 , \dot{V}_2)$, *satisfying*

(v1)
$$\lim_{|z|\to+\infty} |V(z)| = +\infty$$

(for $z\in \mathbf{R}^2$*) and assume that there are a constant* $K \geq 0$ *and a measurable function* $\sigma\in L^1([0,T], \mathbf{R}_+)$ *such that the inequality*

(v2)
$$\dot{V}_1(x_\eta, x_\nu)f_\eta(t,x;\lambda) + \dot{V}_2(x_\eta, x_\nu) f_\nu(t,x;\lambda) \leq \sigma(t) |V(x_\eta, x_\nu)|$$

holds for a.e. $t\in [0,T]$, *all* $\lambda\in [0,1]$ *and each* $x\in \mathbf{R}^m$ *such that* $x_\eta^2 + x_\nu^2 \geq K^2$.
Then, $\forall\, r_1 \geq 0 \;\; \exists\, r_2 \geq r_1$ *such that, for each* $(x,\lambda)\in \Sigma$

$$\min_{[0,T]}(x_\eta(t)^2 + x_v(t)^2)^{1/2} \le r_1 \Rightarrow \max_{[0,T]}(x_\eta(t)^2 + x_v(t)^2)^{1/2} \le r_2 .$$

Proof. using (v1) we can find a constant $K_1 > K$ such that $|V(z)| > 0$, for all $|z| \ge K_1$.
Next we define

$$W(z) = \log |V(z)|$$

and observe that

$$\lim_{|z| \to +\infty} W(z) = +\infty \tag{3.33}$$

and

$$\nabla W(z) = V(z)^{-1} \cdot \nabla V(z), \tag{3.34}$$

with $W : \mathbf{R}^2 \setminus B[0,K_1] \to \mathbf{R}$ of class $\mathcal{C}^1$.
Let $(x,\lambda) \in \Sigma$ be such that

$$\min_{[0,T]}(x_\eta(t)^2 + x_v(t)^2)^{1/2} \le r_1 \tag{3.35}$$

and fix a constant c_1 with

$$c_1 > \max\{r_1, K_1\}. \tag{3.36}$$

Finally, we choose $t_0 = t_0(x,\lambda)$ such that

$$(x_\eta(t_0)^2 + x_v(t_0)^2)^{1/2} = \max_{[0,T]}(x_\eta(t)^2 + x_v(t)^2)^{1/2} := M(x).$$

If $M(x) > c_1$, using (3.35) and (3.36) we can find $t_1 \in [0,T]$ such that

$$(x_\eta(t_1)^2 + x_v(t_1)^2)^{1/2} = c_1.$$

Moreover, we note that by (3.2) we can always choose t_0 and t_1 such that the sign of $t_0 - t_1$ is the same of that of $V(z)$ for $|z| \ge K_1$ and

$$(x_\eta(t)^2 + x_v(t)^2)^{1/2} > c_1 \text{ for } t \in [t_1,t_0] \text{ (or } t \in [t_0,t_1]).$$

Now we consider the absolutely continuous function

$$w(t) := W(x_\eta(t), x_v(t))$$

for $t\in[t_1,t_0]$ (or $t\in[t_0,t_1]$, respectively) and, using (3.34) and (v2), from (3.11) we obtain

$$w(t_0) = w(t_1) + \int_{t_1}^{t_0} \dot{w}(s)ds \le w(t_1) + \left| \int_{t_1}^{t_0} \sigma(s)ds \right| \le$$

$$\le \max\{W(z): |z| = c_1\} + |\sigma|_1 \; := c_2$$

(see [98, p. 65] for a similar computation).
From (3.33) we have that there exists a constant $K_2 \ge K_1$ such that $W(z) > c_2$ for $|z| > K_2$ and hence we get

$$(x_\eta(t_0)^2 + x_v(t_0)^2)^{1/2} \le K_2 \, .$$

Thus the result is proved for

$$r_2 := \max\{c_1, K_2\}.$$

♦

Extensions of Proposition 3 could be easily obtained by suitably adapting the arguments developed in [98, Chapter VI]; however, our result, as stated above, is sufficient for all the applications contained in the next Sections. In particular, Proposition 3 applies whenever the field (f_η,f_v) is quasibounded in (x_η,x_v), uniformly with respect to the rest of the variables, that is when

$$(f_\eta(t,x;\lambda)^2 + f_v(t,x;\lambda)^2)^{1/2} \le a(t)(x_\eta^2 + x_v^2)^{1/2} + b(t) \tag{3.37}$$

holds for a.e. $t\in[0,T]$ and all $x\in\mathbf{R}^m$, $\lambda\in[0,1]$, with $a(\cdot), b(\cdot)\in L^1([0,T],\mathbf{R}_+)$.
Indeed, from (3.37) we get (v1) and (v2) by the choices $V(z) = |z|^2$, $K=1$, $\sigma(t) = 2(a(t) + b(t))$.

4. Applications to planar systems

In this section we give some corollaries of Theorem 2 of the periodic BVP

$$\dot{x} = -Jh(t,x), \tag{4.1}$$

$$x(0) = x(T), \tag{4.2}$$

where $h : [0,T]\times\mathbf{R}^2 \to \mathbf{R}^2$ satisfies the Caratheodory conditions and $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the previously considered symplectic matrix. We also set $\omega := 2\pi/T$.
Systems of the form (4.1) naturally arise when dealing with the second-order scalar equation

$$\ddot{u} + g(t,u,\dot{u}) = 0 \tag{4.3}$$

in the phase-plane: $x_1 = u$, $x_2 = \dot{u}$. In this case we set, for $x = (x_1,x_2)$,

$$h(t,x) := (g(t,x_1,x_2),x_2)$$

and condition (4.2) corresponds to the T-periodic boundary condition, for equation (4.3),

$$u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0. \tag{4.4}$$

Other possibilities to settle (4.3) into (4.1) are explored in the next examples 2 and 5. In any case, however, the equivalence between (4.4) and (4.2) will be guaranteed.
In what follows we treat two different cases, according to the fact that the nonlinear map $h(t,x)$ has linear or superlinear growth at infinity. In order to make more transparent the use of the continuation theorem, we do not consider the most general hypotheses for the solvability of (4.1)-(4.2). Actually, we prefer to focus our discussion on some assumptions which are meaningful for the examples, avoiding, at the same time, cumbersome technicalities.
We recall that $\bar{u}$, $\langle Q \rangle$ indicate, respectively, the mean value of $u \in L^1([0,T],\mathbf{R})$ and the integral average of $1/Q$ on the unit circle S^1, for $Q : \mathrm{dom}Q \supset S^1 \to \mathbf{R}^+$.

4.a. Equations with linear growth

Through this subsection, we consider the growth restriction

(j0) $\qquad \limsup_{|x|\to+\infty} |h(t,x)| / |x| \leq \ell(t)$, *uniformly a.e. in* $t \in [0,T]$,

with $\ell \in L^1([0,T],\mathbf{R}_+)$.
Our main result is the following.

THEOREM 3. *Assume* (j0). *Let* $S_1, S_2 : \mathbf{R}^2 \to \mathbf{R}$ *be positively homogeneous functions of degree two which are positive definite and satisfy*

$$S_1(x) \leq S_2(x), \quad \textit{for all } x \in \mathbf{R}^2. \tag{4.5}$$

Let $\alpha_1, \alpha_2 \in L^1([0,T],\mathbb{R})$ *be measurable functions with*

$$\bar{\alpha}_1 \le \bar{\alpha}_2, \tag{4.6}$$

such that

(j1) $$\liminf_{|x|\to+\infty} (h(t,x)|x) / S_1(x) \ \ge\ \alpha_1(t)$$

and

(j2) $$\limsup_{|x|\to+\infty} (h(t,x)|x) / S_2(x) \ \le\ \alpha_2(t)$$

hold, uniformly a.e. in $t \in [0,T]$.
Then, (4.1)-(4.2) *has at least one solution, provided that*

(j3) $$[\bar{\alpha}_1 / \langle S_1\rangle , \bar{\alpha}_2 / \langle S_2\rangle] \cap \omega\mathbb{Z} = \varnothing.$$

Assumption (j3) can be viewed as a generalization of the classical hypothesis of non-resonance with respect to the spectrum of the differential operator $x \to J\dot{x}$ with the T-periodic boundary conditions. This aspect will be clarified through some examples.

Proof. From (4.5) and (j3) it follows that $\bar{\alpha}_1 \cdot \bar{\alpha}_2 > 0$. Thus, we confine ourselves to the case in which

$$0 < \bar{\alpha}_1 \le \bar{\alpha}_2. \tag{4.7}$$

If $\bar{\alpha}_i < 0$ for i=1,2, the proof is completely similar and so it will be omitted.
In order to apply Theorem 2 we have to introduce an appropriate homotopic vector field $f(t,x;\lambda)$. To this end, we define

$$S(x) := (S_1(x) + S_2(x)) / 2,$$

$$\alpha(t) := (\alpha_1(t) + \alpha_2(t)) / 2$$

and set

$$f_0(x) := \begin{cases} -\ \bar{\alpha}\ S(x/|x|) \cdot Jx & , \text{ for } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$$

and, for $\lambda \in [0,1]$,

$$f(t,x;\lambda) := (1-\lambda)f_0(x) - \lambda J h(t,x).$$

As usual, we denote by Σ the set of the pairs (x,λ), where $x(\cdot)$ is a T-periodic solution of

$$\dot{x} = f(t,x;\lambda). \tag{4.8}$$

Observe that $f(t,x;0) = f_0(x)$ and that (4.8) becomes (4.1) for $\lambda = 1$.

At first, we check condition (h4) in Theorem 2. Indeed, by (j0) and the definition of $f_0(x)$, and using also the Caratheodory hypotheses for $h(t,x)$, we can find a, b $\in L^1([0,T],\mathbf{R}_+)$ such that

$$|f(t,x;\lambda)| \le a(t)|x| + b(t),$$

for all $x \in \mathbf{R}^2$, $\lambda \in [0,1]$ and a.e. $t \in [0,T]$.
Hence, we can apply Proposition 3 via (3.37) and (h4) is fulfilled.
Secondly, we prove (h5).
Using the homogeneity of S(x) and recalling the definition of the functional φ given in (3.32), we can write

$$\varphi(x,\lambda) := (1/2\pi)\left|\int_0^T [(1-\lambda)\bar{\alpha}S(x(s)) + \lambda(h(s,x(s)) \mid x(s))]\cdot \delta(x(s))ds\right|,$$

with δ as in (3.12). The needed estimate for φ is now obtained by means of Proposition 2. Having this in mind, and using (j3), we fix $\varepsilon>0$ such that

$$\left[\frac{\bar{\alpha}_1 - \varepsilon}{\omega\langle S_1\rangle} - \varepsilon, \frac{\bar{\alpha}_2 + \varepsilon}{\omega\langle S_2\rangle} + \varepsilon\right] \cap \mathbf{Z} = \varnothing. \tag{4.9}$$

Observe that $(\bar{\alpha}_1 - \varepsilon)/\omega\langle S_1\rangle > \varepsilon$ by (4.7) and (4.9).
From (j1) and the Caratheodory conditions on $h(t,x)$, it follows that there is $\beta_1 \in L^1([0,T],\mathbf{R}_+)$ such that

$$(h(t,x)|x) \ge (\alpha_1(t) - \varepsilon)\, S_1(x) - \beta_1(t)\, |x|,$$

for all $x \in \mathbf{R}^2$ and a.e. $t \in [0,T]$.
Hence we have, for $x \neq 0$, $\lambda \in [0,1]$ and a.e. $t \in [0,T]$,

$$\begin{aligned}[f_2(t,x;\lambda)x_1 - f_1(t,x;\lambda)x_2] / |x| &= -[(1-\lambda)\,\bar{\alpha}\, S(x) + \lambda(h(t,x)|x)] / |x| \\ &\le -[(1-\lambda)\,\bar{\alpha}S_1(x) + \lambda(\alpha_1(t) - \varepsilon)S_1(x) - \beta_1(t)\,|x|] / |x| \\ &= -[(1-\lambda)\,\bar{\alpha} + \lambda(\alpha_1(t) - \varepsilon)]\,|x|\, S_1(x/|x|) + \beta_1(t)\end{aligned}$$

(recalling also that $S(x) \geq S_1(x)$ from (4.5)).

Thus, we are under assumption (w4) of Proposition 2 and we can conclude that there is a constant $R_1 \geq 1$ such that, for each $(x,\lambda) \in \Sigma$ satisfying

$$|x(t)| \geq R_1 \quad \text{for all } t \in [0,T],$$

it follows that

$$\varphi(x,\lambda) \geq \left([(1-\lambda)\,\bar{\alpha} + \lambda(\bar{\alpha}_1 - \varepsilon)] \,/\omega\,\langle S_1\rangle\right) - \varepsilon \geq \frac{\bar{\alpha}_1 - \varepsilon}{\omega\langle S_2\rangle} - \varepsilon.$$

On the other hand, from (j2) and the Caratheodory conditions on $h(t,x)$, it follows that there is $\beta_2 \in L^1([0,T],\mathbf{R}_+)$ such that

$$(h(t,x)|x) \leq (\alpha_2(t) + \varepsilon)\, S_2(x) - \beta_2(t)\, |x|,$$

for all $x \in \mathbf{R}^2$ and a.e. $t \in [0,T]$.

Hence, arguing as above, it is possible to find a constant $R_2 \geq 1$ such that

$$\varphi(x,\lambda) \leq \left([(1-\lambda)\,\bar{\alpha} + \lambda(\bar{\alpha}_2 + \varepsilon)] \,/\omega\,\langle S_2\rangle\right) + \varepsilon \leq \frac{\bar{\alpha}_2 + \varepsilon}{\omega\langle S_2\rangle} + \varepsilon,$$

for each $(x,\lambda) \in \Sigma$ satisfying

$$|x(t)| \geq R_2 \quad \text{for all } t \in [0,T].$$

(In this case, we use (w2) and the remark at the end of Proposition 2).

Now we set

$$R := \max\{R_1, R_2\}$$

and claim that (h5) is fulfilled by taking, for any $n \in \mathbf{Z}_+$, $K_n = R$.

Indeed, let $\varphi(x,\lambda) = n \in \mathbf{Z}_+$ with $x(\cdot)$ solution of (4.8)-(4.2) and $\lambda \in [0,1]$. If, by contradiction, $\min_{[0,T]} |x(t)| > R$, then, from the above inequalities, we get $\frac{\bar{\alpha}_1 - \varepsilon}{\omega\langle S_2\rangle} - \varepsilon \leq \varphi(x,\lambda) \leq \frac{\bar{\alpha}_2 + \varepsilon}{\omega\langle S_2\rangle} + \varepsilon$ and so, by (4.9), we have $\varphi(x,\lambda) \notin Z$, a contradiction.

In this manner, (h5) is proved. We note that, by means of the above argument, we have also proved that $\min_{[0,T]} |x(t)| \leq R$ for every $(x,\lambda) \in \Sigma$. This fact, together with (h4) implies the existence of a priori bounds for

the T-periodic solutions of (4.8), that is there is $r_0>0$ such that $|x|_\infty<r_0$, for every $(x,\lambda)\in\Sigma$ (use (h4) with $r_1=R$ and choose $r_0 = r_2+1$). Thus, in particular, for $\lambda = 0$ we get (h1).
Finally, we prove (h2).
Indeed, it is sufficient to observe that, from the definition of $f_0(x)$, we have

$$-(1-\mu)Jz + \mu f_0(z) \neq 0,$$

for all $\mu\in[0,1]$ and $|z| = r \geq r_0$ (with r_0 the constant considered above). Hence, by the homotopy invariance of the Brouwer degree we obtain that

$$\chi_0 = |d_B(f_0, B(0,R), 0)| = |d_B(-\mathbf{J}, B(0,R), 0)| = 1,$$

for any $r \geq r_0$.
Therefore, all the assumptions of Theorem 2 are fulfilled and the proof is complete. ♦

REMARK 4. Theorem 3 is closely related to some recent results by C. Fabry [43] and A. Fonda and P. Habets [47], where the existence of a priori bounds for the T-periodic solutions of (4.1) is achieved by a phase-plane analysis in which the use of the "time-map" is crucial. Another analogy with [47] comes from the fact that a comparison with positively homogeneous functions is performed (see [47, § 4]). However, with respect to [47, Th.1], some componentwise conditions which are required for $-\mathbf{J}h$ are replaced here by similar assumptions on the scalar product $(h(t,x)|x)$.

In the next example, we set, for $v\in L^\infty([0,T],\mathbf{R})$,

$$v_* := \operatorname*{ess\,inf}_{[0,T]} v(t)\ ,\quad v^* := \operatorname*{ess\,sup}_{[0,T]} v(t).$$

EXAMPLE 2. Consider the second-order scalar equation

$$\frac{d}{dt}(w(t)\dot{u}) + g(t,u) = 0 \tag{4.10}$$

where $w(t)$ is a positive function (sufficiently smooth) with $w(0) = w(T)$ and $g : [0,T]\times\mathbf{R} \to \mathbf{R}$ is a Caratheodory function satisfying

$$\Gamma_1(t) \leq \liminf_{|u|\to+\infty} w(t)\, g(t,u)/u \leq \limsup_{|u|\to+\infty} w(t)\, g(t,u)/u \leq \Gamma_2(t)\ ,$$

uniformly a.e. in $t\in[0,T]$. We assume that, for $i = 1, 2$, $\Gamma_i\in L^\infty([0,T],\mathbf{R})$ and are such that

$$0 < (\Gamma_1)^* \leq (\Gamma_2)_* \text{ and } \overline{(\Gamma_1/w)} \leq \overline{(\Gamma_2/w)}\,. \tag{4.11}$$

Then, (4.10) has at least one T-periodic solution provided that

$$[\overline{(\Gamma_1/w)} / ((\Gamma_1)^*)^{1/2}\ ,\ \overline{(\Gamma_2/w)} / ((\Gamma_2)^*)^{1/2}\,] \cap \omega\mathbf{Z}_+ = \varnothing\,. \tag{4.12}$$

This result is essentially contained in [43, Th. 2], where $w(t) = 1$ and (4.11) is replaced by the (more general) condition $(\Gamma_1)^* > 0$, $(\Gamma_2)_* > 0$. By standard computations, it can be easily checked that the result fits into Theorem 3 when the following positions are made: $x = (x_1, x_2) := (u, w\dot{u})$, $h(t,x) := (g(t,x_1), x_2/w(t))$, $S_1(x) := (\Gamma_1)^* x_1^2 + x_2^2$, $S_2(x) := (\Gamma_2)_* x_1{}^2 + x_2{}^2$, $\alpha_1(t) := \Gamma_1(t) / (\Gamma_1)^* w(t)$, $\alpha_2(t) := \Gamma_2(t) / (\Gamma_2)_* w(t)$. In this case, (4.12) ensures the validity of (j3). For the computation of $\langle S_i \rangle$ $(i=1,2)$, it is sufficient to recall that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a \cos^2\theta + c\, \mathrm{sen}^2\theta} = \frac{1}{\sqrt{ac}}$$

(see [5]). ♦

EXAMPLE 3. Consider the second-order scalar equation

$$\ddot{u} + b(t,u,\dot{u})\,\dot{u} + g(t,u) = p(t,u,\dot{u}), \tag{4.13}$$

where b, g and p are Caratheodory functions, with b, p bounded and g satisfying (uniformly a.e. in t)

$$a_- \leq \liminf_{u\to-\infty} g(t,u)/u \leq \limsup_{u\to-\infty} g(t,u)/u \leq a^-$$

$$a_+ \leq \liminf_{u\to+\infty} g(t,u)/u \leq \limsup_{u\to+\infty} g(t,u)/u \leq a^+\,,$$

with $a_\pm$ and $a^\pm$ positive constants. Let $B > 0$ be such that

$$|b(t, u, v)| \leq 2B,$$

for all $u, v \in \mathbf{R}$ and a.e. $t \in [0,T]$.
If we also assume that $a_\pm > B^2$, then (4.13) has at least one T-periodic solution provided that there is $k \in \mathbf{Z}_+$ such that

$$\frac{T}{2(k+1)} < \frac{1}{\sqrt{a^- - B^2}} \cos^{-1}(B/\sqrt{a^-}) + \frac{1}{\sqrt{a^+ - B^2}} \cos^{-1}(B/\sqrt{a^+}) \leq$$

$$\leq \frac{1}{\sqrt{a_- - B^2}} \cos^{-1}(-B/\sqrt{a_-}) + \frac{1}{\sqrt{a_+ - B^2}} \cos^{-1}(-B/\sqrt{a_+}) < \frac{T}{2k} . \tag{4.14}$$

This result is a slight variation of [47, Proposition 2]. It can be easily deduced from Theorem 3 by the following positions: $x = (x_1, x_2) := (u, \dot{u})$, $h(t, x) := (b(t, x_1, x_2)x_2 + g(t, x_1) - p(t, x_1, x_2), x_2)$, $\alpha_1 = \alpha_2 = 1$,

$$S_1(x) := \begin{cases} a_- x_1^2 - 2B|x_1| \, |x_2| + x_2^2 , & \text{for } x_1 \leq 0 , \\ a_+ x_1^2 - 2B|x_1| \, |x_2| + x_2^2 , & \text{for } x_1 \geq 0 , \end{cases}$$

$$S_2(x) := \begin{cases} a^- x_1^2 + 2B|x_1| \, |x_2| + x_2^2 , & \text{for } x_1 \leq 0 , \\ a^+ x_1^2 + 2B|x_1| \, |x_2| + x_2^2 , & \text{for } x_1 \geq 0 . \end{cases}$$

In this case, (4.14) ensures the validity of (j3). The computation of $\langle S_i \rangle$ (i = 1, 2) is reduced to the estimate of integrals of the form

$$\int_0^{\pi/2} \frac{d\theta}{A\cos^2\theta \pm 2B\cos\theta\sin\theta + \sin^2\theta} = \frac{1}{\sqrt{A-B^2}} \cos^{-1}(\pm B/\sqrt{A}) ,$$

with $A = a_\pm , a^\pm$ (see [5,47]). ♦

For related results, see [47,61].

4.b. Superlinear equations

In this subsection we deal with equation (4.1) in the case when the growth restirction (j0) is no more satisfied.

For simplicity, we confine ourselves to the investigation of perturbed Hamiltonian systems with superlinear growth. More precisely, we suppose that

(H0) $$h(t,x) = \nabla H(x) + p(t,x) ,$$

with $H : \mathbf{R}^2 \to \mathbf{R}$ of class C^1 and $p : [0,T]\times\mathbf{R}^2 \to \mathbf{R}^2$ a Caratheodory function. We also assume

(H1)
$$\lim_{|x|\to +\infty} |H(x)| = +\infty ,$$

(H2)
$$\nabla H(x) \neq 0 \ \text{, for } |x| \geq r_0 > 0 ,$$

(H3)
$$\limsup_{|x|\to +\infty} \frac{(-Jp(t,x) \mid \nabla H(x))}{|H(x)|} \leq \ell(t) \ \text{, uniformly a.e. in } t\in[0, T] ,$$

with $\ell \in L^1([0, T], \mathbf{R}_+)$.
Besides the notations previously introduced, we also denote by

$$u^+(s) := \max\{u(s) , 0\} \ , \ u^-(s) := \max\{- u(s) , 0\},$$

the positive and the negative part of a function $u(\cdot) : \mathrm{dom}\, u(\cdot) \to \mathbf{R}$.

Our main result is the following

THEOREM 4. *Assume* (H0), (H1), (H2) *and* (H3). *Suppose that there is a sequence* $\{S_n\}$ *of positively homogeneous functions of degree two which are positive definite and there is a sequence* $\{\alpha_n\}$ *of Lebesgue integrable functions such that, for each* $n\in\mathbf{N}$,

(H4)
$$\liminf_{|x|\to+\infty} \frac{(\nabla H(x) \mid x) - (p(t,x) \mid x)^-}{S_n(x)} \geq \alpha_n(t)$$

holds, uniformly a.e. in $t\in[0,T]$.
Then, (4.1)-(4.2) *has at least one solution, provided that*

(H5)
$$\lim_{n\to+\infty} \bar{\alpha}_n / \langle S_n\rangle = +\infty .$$

Proof. We apply Theorem 2 and so we introduce an appropriate homotopic vector field $f(t,x;\lambda)$. To this end, we set, for $\lambda\in[0,1]$,

$$f(t,x;\lambda) := - J(\nabla H(x) + \lambda p(t, x)) + (1 - \lambda)\, q_0(x) , \tag{4.15}$$

where

$$q_0(x) := (E(\partial H(x) / \partial x_1) \ , \ E(\partial H(x) / \partial x_2))$$

and $E : \mathbf{R} \to \mathbf{R}$ is a continuous function, bounded in $\mathbf{R}$ and such that $E(s) \cdot s < 0$ for $s \neq 0$ (we could take, for instance, $E(s) = -\,tg^{-1}(s)$). By these assumptions, we get

$$|q_0(x)| \leq E_0 \ , \text{ for all } x \in \mathbf{R}^2 \ , \tag{4.16}$$

with $E_0 > 0$ a suitable constant and, from (H2),

$$(q_0(x) \mid \nabla H(x)) \ < \ 0 \ , \text{ for } |x| \ \geq \ r_0 \ . \tag{4.17}$$

From the above positions, we have that

$$f(t,x;1) = -\,\mathbf{J}\cdot h(t,x)$$

and

$$f(t,x;0) := f_0(x) = q_0(x) - \mathbf{J}\cdot\nabla H(x) \ .$$

We consider now the autonomous system

$$\dot{x} = f_0(x) \ . \tag{4.18}$$

By (H1) and (4.17), we are in the situation met in the guiding function method for (4.18), according to Theorem 4.3 and Corollary 4.4. of [131, p.188]. The proof of these theorems shows that the possible T-periodic solutions are a priori bounded by r_0 and the corresponding coincidence degree is equal in absolute value to the Brouwer degree of ∇H, which itself is equal to 1. Therefore, (h1) is satisfied. Moreover, using also the fact that $(f_0(x) \mid \nabla H(x)) < 0$ for $x \geq r_0$, we get

$$\chi_0 = |d_B(f_0, B(0, r) \ , 0)| = |d_B(\nabla H, B(0, r) \ , 0)| = 1 \ , \text{ for every } r \geq r_0 \ .$$

In this manner, (h2) is proved.
In order to prove (h4), we use Proposition 3 with the obvious choice $\eta = e_1$, $\nu = e_2$, $x_\eta = x_1$, $x_\nu = x_2$ and $V(x) := H(x)$. Clearly, in this case (v1) is exactly (H1). On the other hand, from (H1) and (H3), we get that there is $R > r_0$ such that

$$(-\,\mathbf{J}p(t, x) \mid \nabla H(x)) \leq (\ell(t) + 1) \ |H(x)| \ ,$$

for all $|x| \geq R$ and a.e. $t \in [0,T]$. Hence, recalling the definition of $f(t,x;\lambda)$ given in (4.15) and using also (4.17) we obtain that

$$(f(t, x; \lambda) \mid \nabla H(x)) \leq (\ell(t) + 1)\, |H(x)|,$$

holds for a.e. $t \in [0,T]$, all $\lambda \in [0,1]$ and each $x \in \mathbf{R}^2$ with $|x| \geq R$. Thus, (v2) is satisfied with $\sigma(t) := \ell(t) + 1$ and $K := R$ and therefore (h4) of Theorem 2 is fulfilled.
Finally, we prove (h5). As usual, we denote by Σ the set of T-periodic solutions of $\dot{x} = f(t,x;\lambda)$, $\lambda \in [0,1]$. Recalling (4.15) and the definition of the functional φ given in (3.32), we can write

$$\varphi(x, \lambda) = \frac{1}{2\pi} \left| \int_0^T [(\nabla H(x(s)) + \lambda p(s,x(s)) \mid x(s)) - (1 - \lambda)(q_0(x(s)) \mid Jx(s))]\, \delta(x(s))ds \right|$$

with δ as in (3.12). Now we produce estimates from below for φ using Proposition 2.
At first we observe that, from (H5), we can suppose (passing possibly to a subsequence), that, for each $n \in \mathbf{N}$,

$$\bar{\alpha}_n / \omega \langle S_n \rangle \geq n + 1 .$$

Then, for each $n \in \mathbf{N}$, we can choose $\varepsilon_n > 0$ such that

$$\frac{\bar{\alpha}_n - \varepsilon_n}{\omega \langle S_n \rangle} \geq n .$$

From (H4) and the Caratheodory conditions on $p(t,x)$, it follows that for each $n \in \mathbf{N}$ there is $\beta_n \in L^1([0,T], \mathbf{R}_+)$, such that

$$(\nabla H(x) \mid x) - (p(t, x) \mid x)^- \geq (\alpha_n(t) - \varepsilon_n)\, S_n(x) - \beta_n(t)\, |x| ,$$

for all $x \in \mathbf{R}^2$ and a.e. $t \in [0,T]$.
Hence we have, for $x \neq 0$, $\lambda \in [0,1]$ and a.e. $t \in [0,T]$,

$$\begin{aligned}
[f_2(t, x; \lambda)x_1 - f_1(t, x; \lambda)x_2] / |x| = \\
&= -[((\nabla H(x) \mid x) + \lambda(p(t, x) \mid x) - (1 - \lambda)(q_0(x) \mid Jx)] / |x| \\
&\leq -[(\alpha_n(t) - \varepsilon_n)S_n(x) - \beta_n(t)\, |x| - E_0\, |x|] / |x| \\
&\leq -(\alpha_n(t) - \varepsilon_n)\, |x|\, S_n(x / |x|) + (E_0 + \beta_n(t))
\end{aligned}$$

(recalling also (4.16)).
Thus we are under assumption (w4) of Proposition 2 with $\alpha_\lambda(t) := \alpha_n(t) - \varepsilon_n$ (constant with respect to $\lambda \in [0,1]$) and $\gamma(t) := E_0 + \beta_n(t)$. Accordingly, we can conclude that there is a constant $R_n \geq 1$ such that, for each $(x, \lambda) \in \Sigma$ satisfying

$$|x(t)| \geq R_n \quad \text{for all} \quad t \in [0,T]\ , \tag{4.19}$$

it follows that

$$\varphi(x,\lambda) \geq \frac{\bar{\alpha}_n - \varepsilon_n}{\omega \langle S_n \rangle} \geq n \tag{4.20}$$

(observe that we have used the remark at the end of Proposition 2 and that (4.20) corresponds to (3.28)). Now we claim that (h5) of Theorem 2 is fulfilled by taking, for any $n \in \mathbf{Z}_+$, $K_n = R_{n+1}$. Indeed, let $\varphi(x,\lambda) = n \in \mathbf{Z}_+$, with $x(\cdot)$ T-periodic solution of $\dot{x} = f(t, x; \lambda)$ for some $\lambda \in [0,1]$. If, by contradiction, $\min_{[0,T]} |x(t)| > R_{n+1}$, then, from (4.20), we get $\varphi(x, \lambda) \geq n + 1 > n$, a contradiction. Therefore, (h5) is proved. Thus, all the assumptions of Theorem 2 are satisfied and the proof is complete. ♦

REMARK 5. It can be easily checked that our result is still true if the hypothesis (H4) is replaced by

(H'4) $$\limsup_{|x| \to +\infty} \frac{(\nabla H(x) \mid x) + (p(t,x) \mid x)^+}{S_n(x)} \leq -\alpha_n(t) \quad \text{,\textit{uniformly a.e. in}} \ t \in [0, T]\ .$$

In this case, the only change in the proof consists in the use of (w3) of Proposition 2 instead of (w4).

We observe that the superlinear growth condition is contained in assumption (H5). This aspect will be clarified by the following examples.

EXAMPLE 4. Consider the second-order scalar equation

$$\ddot{u} + g(u) = q(t, u, \dot{u}) \tag{4.21}$$

where q is a Caratheodory function and $g : \mathbf{R} \to \mathbf{R}$ is continuous. We suppose that

$$\lim_{|x| \to +\infty} g(u) / u = +\infty \tag{4.22}$$

and

$$|q(t, u, v)| \leq K(|u| + |v|) + k(t) \tag{4.23}$$

for all $u, v \in \mathbf{R}$ and a.e. $t \in [0,T]$, with $K \geq 0$ and $k \in L^1([0, T], \mathbf{R}_+)$. Under these assumptions, M. Struwe in [140] proved the existence of T-periodic solutions for (4.21), using a fixed point theorem for

the Poincaré's operator due to S. Fučik and V. Lovicar [50] (actually, $k \in L^2([0, T], \mathbf{R}_+)$ is required in [140]). Now we indicate how to get such result from Theorem 4.
We make the following positions: $x = (x_1, x_2) := (u, \dot{u})$, $h(t,x) := \big(g(x_1) - q(t, x_1, x_2), x_2\big)$ and define

$$G(u) := \int_0^u g(s)ds .$$

Then $h(t,x)$ can be decomposed as in (H0), with

$$H(x) := G(x_1) + \frac{1}{2} x_2^2$$

and

$$p(t,x) := \big(- q(t, x_1, x_2), 0\big).$$

Hence, (H1) and (H2) easily follow from (4.22), while from (4.23) we get

$$\limsup_{|x| \to +\infty} \frac{(-Jp(t,x) \mid \nabla H(x))}{|H(x)|} =$$

$$= \limsup_{|x| \to +\infty} \frac{q(t,x_1,x_2) \cdot x_2}{H(x)} \leq 2K + k(t) ,$$

uniformly a.e. in $t \in [0,T]$, and thus (H3) is achieved (for the estimate of the "limsup" we also used the fact that $G(u) / u^2 \to +\infty$, as $|u| \to +\infty$).
In order to prove (H4), we first observe that, using (4.22), for each $n \in \mathbf{N}$ we can find a constant $c_n > 0$ such that

$$g(u) \cdot u \geq (2n^2 + 2K^2 + K) u^2 - c_n$$

holds for all $u \in \mathbf{R}$. Then, with the above positions, we have (for each $n \in \mathbf{N}$),

$$\begin{aligned}(\nabla H(x) \mid x) - (p(t, x) \mid x)^- &= g(x_1) x_1 + x_2^2 + (q(t, x_1, x_2) \cdot x_1)^- \\ &\geq g(x_1) x_1 + x_2^2 - Kx_1^2 - K|x_1| |x_2| - k(t) |x_1| \\ &\geq 2n^2 x_1^2 + 2K^2 x_1^2 - K|x_1| |x_2| + x_2^2 - k(t) |x_1| - c_n \\ &\geq 2n^2 x_1^2 + \frac{1}{2} x_2^2 - k(t) |x_1| - c_n .\end{aligned}$$

In this manner, (H4) is satisfied with

$$S_n(x) : = 2n^2x_1^2 + \frac{1}{2}x_2^2 \ , \ \alpha_n(t) = 1 - (k(t) / 2n^2) \, .$$

Finally, we compute $\bar{\alpha}_n / \langle S_n \rangle = n - (\bar{k} / 2n)$ and so (H5) is fulfilled. ♦

We stress the fact that our result is more general then the above quoted theorem by Struwe, even as far as the applicability to equations of the form (4.21) is concerned. This is shown by the next example.

EXAMPLE 5. Consider the second-order scalar (Liénard) equation

$$\ddot{u} + f(u)\,\dot{u} + \{u\}^k = q(t, u, \dot{u}), \tag{4.25}$$

where q is a Caratheodory function, $f : \mathbf{R} \to \mathbf{R}$ is continuous and (following the notation in [140]), $\{u\}^k : = u\,|u|^{k-1}$, with $k > 1$. We suppose that q is bounded and, for

$$F(u) : = \int_0^u f(s)ds \ ,$$

we assume

$$\liminf_{|u| \to +\infty} \ F(u) / u \ \geq -M > -\infty \tag{4.26}$$

$$\limsup_{|u| \to +\infty} \ F^2(u) / |u|^{k+1} < 4 \, . \tag{4.27}$$

Then, (4.25) has at least one T-periodic solution.

In order to prove this result in the framework of Theorem 4, we make the following positions:

$$x = (x_1, x_2) : = (u, \dot{u} + F(u)) \ , \ h(t,x) : = (\{x_1\}^k - q(t, x_1, x_2) \, , x_2 - F(x_1)).$$

For the decomposition of $h(t,x)$ according to (H0), we take

$$H(x) : = \frac{1}{k+1} \, |x_1|^{k+1} + \frac{1}{2}x_2^2$$

and

$$p(t,x) := (-q(t, x_1, x_2), -F(x_1)).$$

The hypotheses (H1) and (H2) are trivially satisfied, while (H3) follows from (4.26) as

$$\limsup_{|x|\to+\infty} \frac{(-Jp(t,x) \mid \nabla H(x))}{|H(x)|} =$$

$$= \limsup_{|x|\to+\infty} \frac{-F(x_1)\{x_1\}^k + q(t,x_1,x_2)x_2}{H(x)} \le (k+1)M,$$

uniformly a.e. in $t \in [0,T]$.

In order to prove (H4), we first observe that, using (4.27), there is a constant δ, with $0<\delta<1$ such that, for each $n \in \mathbf{N}$, we have

$$|u|^{k+1} - \frac{|F(u)|^2}{4(1-\delta)} \ge (n^2/\delta)\,|u|^2 - c_n,$$

for all $u \in \mathbf{R}$ (with $c_n > 0$, a suitable constant).

Let $B > 0$ be a bound for $|q(t, x_1, x_2)|$. Then, with the above positions, we get (for each $n \in \mathbf{N}$)

$$\begin{aligned}
(\nabla H(x) \mid x) - (p(t,x) \mid x)^- &= |x_1|^{k+1} + x_2^2 + (F(x_1)x_2 + q(t,x_1,x_2)x_1)^- \\
&\ge |x_1|^{k+1} + x_2^2 - |F(x_1)|\,|x_2| - B|x_1| \\
&= |x_1|^{k+1} + \delta x_2^2 + (1-\delta)\left[x_2^2 - \frac{|F(x_1)|}{(1-\delta)}|x_2| + \frac{F(x_1)^2}{4(1-\delta)^2}\right] - \frac{F(x_1)^2}{4(1-\delta)} - B|x_1| \\
&\ge |x_1|^{k+1} + \delta x_2^2 - \frac{F(x_1)^2}{4(1-\delta)} - B|x_1| \\
&\ge (n^2/\delta)\,x_1^2 + \delta x_2^2 - B|x_1| - c_n.
\end{aligned}$$

In this manner, (H4) is satisfied with

$$S_n(x) := (n^2/\delta)\,x_1^2 + \delta x_2^2, \quad \alpha_n(t) = 1.$$

Finally, we compute $\bar{\alpha}_n / \langle S_n \rangle = n$ and so (H5) is fulfilled. ♦

We note that if we want to apply [140, Theorem 2] to equation (4.25) we have to require the boundedness of $f(u)$ on $\mathbf{R}$. Such condition in turns implies

$$\limsup_{|u|\to+\infty} |F(u)|/|u| \le M < +\infty. \tag{4.28}$$

Now, it is easy to find examples of nonlinearities satisfying (for $k > 1$) (4.26) and (4.27) but not (4.28). An example in this direction is given, for instance, by

$$F(u) := \{u\}^{\alpha}\sin^2(g_0(u)) + u\cdot g_1(u), \tag{4.29}$$

where $1<\alpha<(k+1)/2$ and $g_0, g_1 : \mathbf{R}\to\mathbf{R}$ are continuously differentiable functions, with g_1 bounded on $\mathbf{R}$. We point out that even the theorem in [135] (which improves [140, Th.2]) is not applicable to equation (4.25) when $f(u) = \dot{F}(u)$, with $F(u)$ as in (4.29).

We also remark that the result stated in Example 5 is still true if the map q satisfies (4.23). Further generalizations can also be produced for equations of the form

$$\ddot{u} + f(u)\,\dot{u} + g(u) = q(t, u, \dot{u})$$

with g having superlinear growth as in (4.22) and $f(u)$ or $F(u)$ subject to suitable growth restrictions.

EXAMPLE 6. Consider the (planar) perturbed Hamiltonian system

$$\dot{x} = \mathbf{J}\cdot\nabla W(x) + q(t,x) \tag{4.30}$$

with $W : \mathbf{R}^2\to\mathbf{R}$ of class C^1 and $q : [0,T]\times\mathbf{R}^2\to\mathbf{R}^2$ a Caratheodory function. We suppose that

$$|q(t,x)| \le Q(t), \quad \text{for a.e. } t\in[0,T] \text{ and all } x\in\mathbf{R}^2, \tag{4.31}$$

with $Q\in L^1([0,T],\mathbf{R}_+)$ and there are $A, B\in\mathbf{R}_+$ such that

$$|\nabla W(x)| \le A\,|W(x)| + B, \quad \text{for all } x\in\mathbf{R}^2. \tag{4.32}$$

Assume also the superlinear growth condition

$$\lim_{|x|\to+\infty} |(\nabla W(x) \mid x)| \,/\, |x|^2 = +\infty. \tag{4.33}$$

Then, (4.30) has at least one T-periodic solution.

Indeed, we can apply Theorem 4 observing that the decomposition (H0) follows by setting

$$H(x) := -W(x),$$
$$p(t,x) := \mathbf{J}\cdot q(t,x).$$

Then, from (4.33), we get $\lim_{|x|\to+\infty} |\nabla H(x)| / |x| = +\infty$ (use the Cauchy-Schwarz inequality) and thus (H2) and (H1) (via (4.32)) are fulfilled. Moreover, (H3) follows from (4.31) and (4.32), with $\varrho(t) \leq (A+B)Q(t)$, as $|(-Jp(t,x) \mid \nabla H(x))| \leq Q(t)\, A\, |H(x)| + Q(t)\, B$, for a.e. $t\in[0,T]$ and all $x\in\mathbf{R}^2$. Finally, we note that (4.33) implies either $\lim_{|x|\to+\infty} ((\nabla H(x) \mid x)| / |x|^2 = +\infty$, or $\lim_{|x|\to+\infty} (\nabla H(x) \mid x)| / |x|^2 = -\infty$. Assume the former condition (the treatment of the latter being completely similar) and take $S_n(x) := n\, |x|^2$. Then we have (using (4.31)):

$$\frac{(\nabla H(x) \mid x) - (p(t,x) \mid x)^-}{S_n(x)} \geq \frac{(\nabla H(x) \mid x)}{n|x|^2} - \frac{Q(t)}{n|x|} \geq 1 - (Q(t) / n),$$

for a.e. $t\in[0,T]$ and $|x|$ large. Therefore, (H4) is reached with $\alpha_n(t) := 1 - (Q(t) / n)$. Hence, $\bar{\alpha}_n/\langle S_n\rangle = [1-(\bar{Q}/n)]\, n$ and (H5) follows. In this manner, all the assumptions of Theorem 4 are satisfied and the result is proved. ♦

Through the above example, we can make a comparison between Theorem 4 and some classical results concerning superlinear hamiltonian systems (see, e.g., [3,42,126]).
First of all, we note that, in general, system (4.30) has not a variational structure (for q depending on x) and so the theorems in [3,126] cannot be applied. On the other hand, we observe that we cannot guarantee, like in [3], the existence of more than one T-periodic solution. Examples in this direction can be easily obtained observing that if $x(\cdot)$ is any T-periodic solution of (4.30) then, necessarily,

$$\int_0^T (q(t,x(t)) \mid \nabla W(x(t)))\, dt = 0.$$

Hence, if we take q and W such that $q(t,0) \equiv 0$, $\nabla W(0) = 0$ and $(q(t,x) \mid \nabla W(x)) > 0$ for a.e. $t\in[0,T]$ and each $x\neq 0$, it is clear that the only T-periodic solution of (4.30) is the trivial one.
Secondly, we consider the case in which q is independent of x, i.e. $q(t,x) = q(t)$, with $q\in L^1([0,T],\mathbf{R}^2)$. In this situation, the theorems of P.H. Rabinowitz [126] and A. Bahri and H. Berestycki [3] can be applied, under further regularity assumptions on W and q and growth restrictions on W, provided that the superlinearity condition

$$0 < W(x) \leq k\, (\nabla W(x) \mid x), \text{ for every } x\in\mathbf{R}^2, |x|\geq R, \tag{4.34}$$

with $0 < k < 1/2$, is satisfied.
Since (4.34) implies that $W(x) \geq a\, |x|^{1/k} - b$, with $a, b \in \mathbf{R}_+$, we conclude that (4.34) implies (4.33), and so we realize that our superlinear condition is more general that (4.34) (it is easy to find examples where (4.33) holds but (4.34) is not satisfied).

Finally, with respect to the results by I. Ekeland in [42], we just observe that in our Example 6 no restriction on the L^1-norm of q has to be assumed, while in [42] the existence of solutions is obtained for $|q|_1$ sufficiently small.
For other theorems concerning the periodic BVP for Hamiltonian systems, see [104].
A major advantage of the above quoted results, based on critical point theory, lies on the fact that they can be applied to even-dimensional systems of the form (4.30) (with q(t,x) = q(t)), without restriction on the dimension of the space. On the other hand, all the examples presented here concern planar systems and so a condition on the dimension has been imposed from the beginning.
In the next section, we show how this problem can be overcomed, at least in the case of weakly-coupled systems.

5. An application to weakly coupled systems

In this Section we consider the problem of the existence of T-periodic solutions for the second-order vector equation

$$\ddot{u} + g(u) = p(t,u,\dot{u}), \tag{5.1}$$

where $g : \mathbf{R}^d \to \mathbf{R}^d$ is a continuous function and $p : [0,T]\times\mathbf{R}^d\times\mathbf{R}^d \to \mathbf{R}^d$ satisfies the Caratheodory assumptions. As usual, the T-periodic boundary condition for equation (5.1) is expressed by

$$u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0. \tag{5.2}$$

Our aim is to obtain an extension of the Theorem by Fučik and Lovicar [50] (recalled in Example 4) to systems; therefore, we examine only the case in which the map g grows faster than a linear function as $|u| \to +\infty$. We recall that in [50] the existence of a solution to (5.1)-(5.2) is proved for a scalar equation (d=1), with $p = p(t) \in L^1([0,T],\mathbf{R})$ and

$$\lim_{|u| \to +\infty} g(u) / u = +\infty. \tag{5.3}$$

Clearly, a trivial extension of this theorem to higher dimension works if system (5.1) is completely uncoupled, i.e. if $g(u) = \mathrm{col}(g_k(u))_{k=1,\dots,d}$ and $p = \mathrm{col}(p_k(t))_{k=1,\dots,d}$. However, as far as we know, no true generalization of this result has been obtained yet.
A major difficulty which arises in this direction lies on the fact that many useful tools which are linked to the possibility of developing a phase-plane analysis for the equivalent system

$$\dot{u} = y$$
$$\dot{y} = -\ g(u) + p(t,u,y) \qquad (5.4)$$

cannot be employed for d>1. A further problem is the absence of a priori bounds when g is superlinear. As a consequence, very few theorems have been produced for the solvability of (5.1)-(5.2) in the vector case and, in fact, we restrict our remarks to the papers of A. Castro and A.C. Lazer [20] and A. Bahri and H. Berestycki [4]. Actually, we exclude here from our discussion the results whose applicability is conditioned to the fulfilment of special bounds for the term p (like in [42]) or to the validity of symmetry conditions which imply that $u \equiv 0$ is a solution of (5.1) (like in [23]); in both such cases, the corresponding Theorems cannot be applied to equation

$$\ddot{u} + g(u) = p(t), \qquad (5.5)$$

with $p(\cdot)$ an arbitrary integrable function.
In [4], Bahri and Berestycki obtained the existence of infinitely many solutions for system (5.5) in the conservative case, i.e. for $g(u) = \nabla G(u)$, under the assumption

$$0 < G(u) \leq k(g(u) \mid u), \quad \text{for } |u| \geq R > 0, \qquad (5.6)$$

with $0 < k < 1/2$. From (5.6), by an integration it follows that $\lim\limits_{|u|\to+\infty} G(u) / |u|^2 = \lim\limits_{|u|\to+\infty} |g(u)| / |u| = +\infty$.
However, even in the scalar case, it can be easily seen that condition (5.6) does not cover all the possible superlinear terms. For instance, if $g(u)$ behaves like $u \cdot \log|u|$ in a neighbourhood of ∞, assumption (5.6) fails, while (5.3) is satisfied. On the other hand, in [20] Castro and Lazer obtained the existence of infinitely many T-periodic solutions for system (5.1) with $p = p(t,u)$ and $g(u) = \mathrm{col}(g_k(u_k))_{k=1,\dots,d}$, assuming the oddness of all the considered functions. Hence, in the scalar case, such a theorem can be applied to equation (5.5) only when g and p are odd.
Our main result, which generalizes Fučik and Lovicar's theorem to systems, deals (like in [20]), with a weakly-coupled system of the form

$$\ddot{u}_k + g_k(u_k) = p_k(t,u,\dot{u}), \qquad k = 1,\dots,d, \qquad (5.7)$$

with $g(u) = \mathrm{col}(g_k(u_k))_{k=1,\dots,d}$, $p_k(t,u,\dot{u}) = \mathrm{col}(p_k(t,u,\dot{u}))_{k=1,\dots,d}$ and ensures the existence of at least one T-periodic solution. We note that, in general, we cannot hope to get the existence of more than one solution for (5.1)-(5.2) or (5.7)-(5.2), because of the dependence in $\dot{u}$ of p. This fact is well-known even in the scalar case (see, for instance, [70, p. 39], [140, p. 288]), and does not require further explanations.

Now we state our main result.

THEOREM 5. *Suppose there is* $P \in L^1([0,T],\mathbf{R}_+)$ *such that*

(*I*1) $$|p_k(t,u,y)| \leq P(t), \quad \textit{for a.e. } t \in [0,T], \quad \textit{all } u, y \in \mathbf{R}^d, \ k = 1,\dots,d.$$

Assume

(*I*2) $$\lim_{|s| \to +\infty} g_k(s) / s = +\infty, \quad \textit{for each } k = 1,\dots,d.$$

Then, (5.7)-(5.2) *has at least one solution.*

Proof. In order to apply Theorem 1 in Section 3, we introduce an appropriate setting for the verification of the needed assumptions.
At first, we write equation (5.7) as an equivalent first order system in $\mathbf{R}^m$, with m=2d. To this end, we set

$$x = \mathrm{col}(x_i)_{i=1,\dots,2d} = (u_1, y_1, \dots, u_d, y_d),$$

so that (5.7) or, more precisely, system

$$\dot{u}_k = y_k$$

$$\dot{y}_k = -\, g_k(u_k) + p_k(t,u,y),$$

(k=1,...,d), takes the form

$$\dot{x} = F(t,x), \tag{5.8}$$

with $F = \mathrm{col}(F_i)_{i=1,\dots,2d} : [0,T] \times \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ a Caratheodory function such that, for each k=1,...d,

$$F_{2k-1}(t,x) = y_k$$
$$F_{2k}(t,x) = -\, g_k(u_k) + p_k(t,u,y).$$

The boundary condition (5.2) is clearly translated to $x(0) = x(T)$.
As a second step, we embed system (5.8) into a one-parameter family of differential equations. To this respect, we fix a scalar function $E : \mathbf{R} \to \mathbf{R}$ which is continuous and satisfies the conditions

$$E(s) \cdot s < 0 \quad \text{for } s \neq 0 \qquad \text{and} \quad |E(s)| \leq 1 \quad \text{for all } s \in \mathbf{R}.$$

Then we consider the system

$$\dot{x} = f(t,x;\lambda) \tag{5.9}$$

for $\lambda \in [0,1]$, where $f(t,x;\lambda)$ is the second term of

$$\dot{u}_k = y_k$$

$$\dot{y}_k = (1-\lambda)\, E(y_k) - g_k(u_k) + \lambda p_k(t,u,y),$$

(k=1,...,d).

As usual, we denote by Σ the set of pairs (x,λ), where $x(\cdot)$ is a T-periodic solution of (5.9) with $\lambda \in [0,1]$. In this case, Σ_0 is the set of T-periodic solutions of the autonomous equation

$$\dot{x} = f_0(x) = f(t,x;0),$$

which, by the above positions, is exactly equation

$$\dot{u}_k = y_k$$

$$\dot{y}_k = E(y_k) - g_k(u_k),$$

(k=1,...,d).

Clearly, system (5.10) is equivalent to the uncoupled second-order equation

$$\ddot{u}_k - E(\dot{u}_k) + g_k(u_k) = 0, \qquad k=1,...,d, \tag{5.10}$$

for which the only periodic solutions (of any period) are the constant ones. (To see this, suppose that $u(\cdot) = \mathrm{col}(u_k(\cdot))_{k=1,...,d}$ is a periodic solution of (5.10) with period $T_0>0$. Then, taking the scalar product of (5.10) with $\dot{u}(t)$ and integrating over $[0,T_0]$, we get

$$\sum_{k=1}^{d} \int_0^{T_0} E(\dot{u}_k(t))\, \dot{u}_k(t)dt = 0$$

and therefore, by assumption on $E(\cdot)$, we have $\dot{u}(t) = 0$, $\forall t \in [0,T]$).

From ($I2$), we can find a constant $R_0>0$ such that

$$g_k(s)\cdot s > 0 \quad \text{for } |s| \geq R_0, \qquad k=1,...,d.$$

Since we already know that all the periodic solutions of $\dot{x} = f_0(x)$ (of any period) are constants $\hat{x} \in \mathbf{R}^{2d}$, with $\hat{x} = (\hat{u}_1, 0, \hat{u}_2, 0, \ldots, \hat{u}_d, 0)$ and $g_k(\hat{u}_k) = 0$, for each $k=1,\ldots,d$, then we also get $|\hat{u}_k| < R_0$, for each $k=1,\ldots,d$.

In this manner, (h1) is satisfied via (h*1), for any r_0 with

$$r_0 \geq R_0\sqrt{2d}.$$

Moreover, setting, for each $k=1,\ldots,d$, $f_0^{(k)} : \mathbf{R}^2 \to \mathbf{R}^2$, $f_0^{(k)} : (a,b) \longmapsto (b, E(b) - g_k(a))$, by standard computations we have, for any $r \geq r_0$:

$$d_B(f_0, B(0,r), 0) = d_B(f_0, B(0,r_0), 0) =$$

$$d_B(f_0,]-R_0,R_0[^{2d}, 0) = \prod_{k=1}^{d} (d_B(f_0^{(k)},]-R_0,R_0[^2, 0))$$

$$= \prod_{k=1}^{d} (d_B(g_k,]-R_0,R_0[, 0)) = 1.$$

Thus, hypothesis (h2) is satisfied with $\chi_0 = 1$.

For the rest of the proof, we note that, with the above notation, we have, for any $j=1,\ldots,d$, $x \in \mathbf{R}^{2d}$ and $\lambda \in [0,1]$, that

$$x_{2j-1} = u_j , \qquad x_{2j} = y_j,$$

$$f_{2j-1}(t,x;\lambda) = y_j , \quad f_{2j}(t,x;\lambda) = (1-\lambda)\, E(y_j) - g_j(u_j) + \lambda\, p_j(t,u,y),$$

where $u = \mathrm{col}(u_j)_{j=1,\ldots,d}$, $\quad y = \mathrm{col}(y_j)_{j=1,\ldots,d}$.

Then, it is easy to check that assumption (h*3) follows from (I1), since

$$|(f_{2j-1}(t,x;\lambda)\, ,\, f_{2j}(t,x;\lambda))| \leq (y_j^2 + g_j(u_j)^2)^{1/2} + \sqrt{2d}\,(P(t) + 1)$$

holds for all $(x,\lambda) \in \mathbf{R}^{2d} \times [0,1]$ and a.e. $t \in [0,T]$.

As a next step, we check condition (h*4).

To this end, we employ Proposition 3 with $(\eta,\nu) = (e_{2j-1}, e_{2j})$ and define, for each $j=1,\ldots,d$, the functional

$$V_j : \mathbf{R}^2 \to \mathbf{R}$$

given by

$$V_j(a,b) := \frac{1}{2}b^2 + \int_0^a g_j(s)ds,$$

with $\nabla V_j = (\dot{V}_{j,1}, \dot{V}_{j,2})$.
From (I2), we immediately get that $\lim_{|z| \to +\infty} V_j(z) = +\infty$, for j=1,...,d, and hence (v1) of Proposition 3 holds. Moreover, by the definition of V_j and assumption (I2), we can find a constant K>0 such that, for each j=1,...,d,

$$|b| \le V_j(a,b), \quad \text{for } (a^2 + b^2)^{1/2} \ge K.$$

Hence, we get, for each j=1,...,d,

$$\dot{V}_{j,1}(x_{2j-1}, x_{2j})\, f_{2j-1}(t,x;\lambda) + \dot{V}_{j,2}(x_{2j-1}, x_{2j})\, f_{2j}(t,x;\lambda) =$$

$$= g_j(u_j)y_j + y_j((1-\lambda)\, E(y_j) - g_j(u_j) + \lambda p_j(t,u,y)) \le |y_j|\, |p_j(t,u,y)| \le P(t)\, V_j(u_k,y_k) =$$

$$= P(t)\, V_j(x_{2j-1}, x_{2j}),$$

for all $(x,\lambda) \in \mathbf{R}^{2d} \times [0,1]$, with $|(x_{2j-1}, x_{2j})| \ge K$ and a.e. $t \in [0,T]$.
Therefore, condition (v2) is also proved and the "elastic property" (h*4) follows from Proposition 3, for each j=1,...,d.

At last, we have to prove condition (h*5).
Accordingly, we recall that, for $(x,\lambda) \in \Sigma$ with

$$\min_{[0,T]} (x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2} = \min_{[0,T]} (u_j(t)^2 + y_j(t)^2)^{1/2} \ge 1,$$

we can write

$$\varphi_j(x,\lambda) = \frac{1}{2\pi} \left| \int_0^T \frac{x_{2j-1}(t)\, f_{2j}(t,x(t);\lambda) - x_{2j}(t)\, f_{2j-1}(t,x(t);\lambda)}{x_{2j-1}(t)^2 + x_{2j}(t)^2} dt \right|$$

$$= \frac{1}{2\pi} \left| \int_0^T - \frac{g_j(u_j(t))\, u_j(t) + y_j(t)^2 - (1-\lambda)\, E(y_j(t))\, u_j(t) - \lambda\, p_j(t,u(t),y(t))\, u_j(t)}{u_j(t)^2 + y_j(t)^2} dt \right|.$$

Recalling again hypothesis (I2), we observe that, for each $K \in \mathbf{R}^+$, we can find a constant $D_K \ge 0$ such that, for each j=1,...,d,

$$g_j(s)\cdot s \geq K^2 s^2 - D_K, \quad \text{for all } s\in \mathbf{R}$$

holds.

Hence, for every pair $(a,b)\in \mathbf{R}^2$, with $a^2 + b^2 \geq 1$ and a.e. $t\in [0,T]$, we can write (for $j=1,...,d$):

$$g_j(a)a + b^2 - (1-\lambda)\, E(b)a - \lambda\, p_j(t,u,y)a \geq K^2a^2 + b^2 - |a| - P(t)\,|a| - D_K \geq$$

$$\geq K^2 a^2 + b^2 - (P(t) + 1 + D_K)\,(a^2 + b^2)^{1/2}.$$

Therefore, we obtain that, for every $j=1,...,d$,

$$f_{2j}(t,x;\lambda)\cdot\frac{x_{2j-1}}{(x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2}} - f_{2j-1}(t,x;\lambda)\cdot\frac{x_{2j}}{(x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2}} \leq$$

$$-(x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2}\,\Theta_K\Big(\frac{x_{2j-1}}{(x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2}}, \frac{x_{2j}}{(x_{2j-1}(t)^2 + x_{2j}(t)^2)^{1/2}}\Big) + \gamma_K(t)$$

holds, for a.e. $t\in [0,T]$, all $\lambda\in [0,1]$ and each $x\in \mathbf{R}^m$ such that $(x_{2j-1}(t)^2 + x_{2j}(t)^2) \geq 1$, where

$$\Theta_K(a,b) := K^2a^2 + b^2$$

$$\gamma_K(t) := (P(t) + 1 + D_K).$$

Hence, we are under assumption (w4) of Proposition 2 and we can conclude (using also the remark at the end of Proposition 2), that for each $K\in \mathbf{R}^+$, there is a constant $C_K\geq 1$ such that for any $(x,\lambda)\in \Sigma$ satisfying

$$x_{2j-1}(t)^2 + x_{2j}(t)^2 \geq C_K^2 \quad \text{for all } t\in [0,T],$$

it follows that

$$\varphi(x,\lambda) \geq T / 2\pi\, \langle\Theta_K\rangle = KT / 2\pi.$$

(see (3.28) and observe that $\displaystyle\int_0^{2\pi} \frac{d\theta}{K^2\cos^2\theta + \sin^2\theta} = \frac{2\pi}{K}$).

Now, the proof of (h*5) can be straightforwardly obtained arguing by contradiction.

Thus, all the assumptions of Theorem 1 are fulfilled and the proof is complete. ♦

Chapter 6

Continuation results for operator equations in metric ANRs

1. Introduction

In this Chapter (which is based on [14]) we are concerned with the following boundary value problem

$$\dot{x} = F(t,x) \tag{1.1}$$
$$x(0) = x(T), \tag{1.2}$$

where $F : [0,T]\times C\to\mathbf{R}^m$ is continuous and $C\subset\mathbf{R}^m$ is a flow-invariant ENR (Euclidean Neighbourhood Retract). As it was shown in Section 5 of Chapter 3 (generalizing Theorem 1 of Chapter 2), problem (1.1)-(1.2) can be imbedded in a family of parametrized equations of the form

$$\dot{x} = f(t,x;\lambda), \tag{1.3}$$

where $f : [0,T]\times C\times[0,1]\to\mathbf{R}^m$ is continuous and such that $f(t,x;1) = F(t,x)$, for all $t\in[0,T]$, $x\in C$. Furthermore, we assume that f is locally lipschitzian and

$$f(t,x;0) = f_0(x), \tag{1.4}$$

for some function $f_0 : C\to\mathbf{R}^m$, and for all $t\in[0,T]$, $x\in C$. We denote by π_0 the local semi-dynamical system induced by

$$\dot{x} = f_0(x). \tag{1.5}$$

Then, (1.3) is studied by means of a corresponding system of parameter-depending operator equations

$$x = M(x;\lambda), \quad \lambda\in[0,1], \tag{1.6}$$

where $M(\cdot;\lambda) : \Gamma\to\Gamma$ is completely continuous, for every $\lambda\in[0,1]$, and the space $\Gamma := \{x : [0,T]\to C\}$ is endowed with the sup-norm. In this situation, the solutions of (1.1)-(1.2) are the fixed points of the operator $M(\cdot;1)$.

Our aim is to extend, in the situation described above, the results in Sections 2 and 3 of Chapter 5 (obtained in the framework of coincidence degree theory for L-compact perturbations of linear Fredholm mappings of index zero) to the general case of metric ANRs. We also give a general bifurcation result (Theorem 3) for operator equations of the form (1.6), in the case when the only assumption is the non-vanishing of the fixed point index of some (suitable) map.

In Section 2 we prove two abstract continuation results (Theorem 1 and Corollary 1) in the framework of the fixed point index theory for the operator equation

$$x = Gx,$$

where $G : \mathcal{X} \to \mathcal{X}$ is completely continuous and $\mathcal{X}$ is a metric ANR, using, as in Chapter 5, a functional $\varphi : \mathcal{X} \times [0,1] \to R$ which is proper on the set of possible solutions of the homotopic family of equations

$$x = \mathcal{G}(x;\lambda) \tag{1.7}$$

($\mathcal{G}(x;1) = Gx$, for all $x \in \mathcal{X}$), and which avoids two values during the homotopy. The use of the functional φ is crucial in cases when no a priori bounds for the solutions of (1.3) are available (cf. Example 1 of Chapter 5).

In Section 3 we apply Corollary 1 in order to prove (Theorem 2) the existence of at least one solution $x(\cdot) \in \Gamma$ to (1.1)-(1.2) such that $x(t) \in C$, for all $t \in [0,T]$. To prove this result, it is assumed that there exists a continuous functional $\psi : A \to Z_+$, $A \subset \Gamma_T := \{x \in \Gamma : x(0)=x(T)\}$, satisfying suitable assumptions. We point out that in Chapter 5 the functional used is a modification of the classical "time map" counting (for m=2) the number of rotations around the origin of the solutions of (1.3); in the more general situation considered in this Chapter, it can be of interest to provide a general definition for the functional ψ. This aspect is developed in the forthcoming paper [14].

In Section 4 we perform a result (Theorem 3) on the structure of the solution set of a parameter-dependent operator equation defined on a (metric) ANR; more precisely, given an open set $\Omega \subset \mathcal{X} \times \mathbf{R}$ such that $\Omega_0 := \{x \in \mathcal{X} : (x,0) \in \Omega\} \supseteq \Sigma_0 := \{x \in \mathcal{X} : x = \mathcal{G}_0(x)\}$ (Σ_0 bounded), we prove, under the only assumption of the non-vanishing of the fixed point index $i_{\mathcal{X}}(\mathcal{G}_0, \Omega_0)$, the existence of a closed connected subset $\mathcal{C}$ of solutions to (1.7) such that for the sets $\mathcal{C}^+ := \{(x,\lambda) \in \mathcal{C} : \lambda > 0\}$ and $\mathcal{C}^- := \{(x,\lambda) \in \mathcal{C} : \lambda < 0\}$ the following alternative holds: either $\mathcal{C}^+$ ($\mathcal{C}^-$) is unbounded or $\mathcal{C}^+ \cap \mathrm{fr}\Omega \neq \emptyset$ ($\mathcal{C}^- \cap \mathrm{fr}\Omega \neq \emptyset$).

The proof is based on the fundamental properties of the fixed point index and on a lemma from general topology on connected subsets of compact metric spaces; the use of this classical result was a crucial step also in the proof of similar results such as [46,51]. As an immediate consequence, taking $\mathcal{X} = \Gamma$, $G_\lambda = M_\lambda$, and using the index of rest points of the flow π_0 induced by (1.5) instead of the fixed point index, we obtain (Corollary 2) a result on the solutions to problem (1.1)-(1.2) analoguous to Theorem 3. Corollary 2 contains Theorem 2.1 in [53], Corollary 2 in [51], and several existence results such as Theorem 5 in Chapter 3, Theorem 1 and Corollaries 1 and 4 in Chapter 2.

2. A continuation theorem

In this Section, we perform a continuation theorem for an operator equation defined on a metric ANR (Absolute Neighbourhood Retract) by means of the fixed point index defined by Granas in [59]. We refer to [113, Chap. 1] for a complete treatment of this concept.

Let $\mathcal{X}$ be a metric ANR and let $G : \mathcal{X} \to \mathcal{X}$ be a completely continuous map.
We are concerned with the existence of at least one solution of the operator equation

$$Gx = x, \tag{2.1}$$

i.e of a fixed point of G.
We begin the study of this problem by a standard procedure, i.e. we consider a completely continuous operator $\mathcal{G} : \mathcal{X}\times[0,1]\to\mathcal{X}$ such that

$$\mathcal{G}(x;1) = Gx, \quad \text{for every } x\in\mathcal{X}.$$

Some estimates on the (possible) solutions of the parametrized equation

$$\mathcal{G}(x;\lambda) = x \tag{2.2}$$

are needed; to this end, we first define the following sets:

$$\Sigma = \{(x,\lambda)\in\mathcal{X}\times[0,1]: \ \mathcal{G}(x;\lambda) = x\},$$

$$\Sigma_\lambda = \{x\in\mathcal{X}: \ (x,\lambda)\in\Sigma\}.$$

Now, let

$$\varphi : \mathcal{X}\times[0,1]\to\mathbf{R}$$

be a continuous functional; the admissibility of the homotopies obtained by (2.2) will be proved by means of the properties of $\varphi_{|\Sigma}$.
As we mentioned in the introduction, in the applications of the main result of this Section (Theorem 1) to first order differential systems in $\mathbf{R}^m$ (m even) (developed in Section 3) the functional φ is a modification of the so-called "time map".

REMARK 1. We observe that the (continuous) functional $\varphi(\cdot;\lambda)$ is bounded whenever the set Σ_λ is bounded. Indeed, by the properties of the operator $\mathcal{G}$, if the set Σ_λ (Σ, respectively) is bounded, then it is compact. In this case, there exist $\min\{\varphi(\cdot;\lambda), x\in\Sigma_\lambda\}$ and $\max\{\varphi(\cdot;\lambda), x\in\Sigma_\lambda\}$ and they are finite.

We denote by

$$\varphi_- := \inf\{\varphi(\cdot;0),\ x\in\Sigma_0\},\qquad \varphi_+ := \sup\{\varphi(\cdot;0),\ x\in\Sigma_0\}.$$

The following lemma is a slightly more general form of the homotopy invariance property of the fixed point index (cf. [113, p. 26]) and is crucial for the proof of Lemma 1. Its analogue for the Leray-Schauder degree can be found in [90, p. 60], [12], [125, Lemma 1.8].
For an open bounded set $U\subset\mathcal{X}\times[0,1]$, we define

$$U_\lambda := \{x\in\mathcal{X} : (x,\lambda)\in U\}, \tag{2.3}$$

$$(\partial U)_\lambda := \{x\in\mathcal{X} : (x,\lambda)\in \mathrm{fr}_{\mathcal{X}\times[0,1]}U\}. \tag{2.4}$$

LEMMA 1. *Let* $U\subset\mathcal{X}\times[0,1]$ *be open and bounded. Assume that*

$$\mathcal{G}(x,\lambda) \neq x \quad \textit{for every}\quad x\in(\partial U)_\lambda,\quad \textit{for every}\ \lambda\in[0,1];$$

Then,

$$i_{\mathcal{X}}(\mathcal{G}(\cdot;\lambda), U_\lambda)\quad \textit{is independent of}\quad \lambda,\quad \lambda\in[0,1].$$

REMARK 2. We point out that a proof of Lemma 1 can be performed by means of the usual homotopy and multiplication properties. We also note that an analogue of Lemma 1 is valid for $U\subset\mathcal{X}\times\mathbf{R}$, as it was remarked by Leray and Schauder in [90, p. 59].

Now, we can state the main result of this Section.

THEOREM 1. *Assume*

(i1) $\varphi : \mathcal{X}\times[0,1] \to \mathbf{R}$ *is proper on* Σ;

(i2) Σ_0 *is bounded*;

(i3) *there are constants* $c_+, c_-\in\mathbf{R}$,

$$c_- < \varphi_-,\ c_+ > \varphi_+ \tag{2.5}$$

such that

$$\varphi(x;\lambda) \notin \{c_-, c_+\}.$$

Let $\Omega \supseteq \Sigma_0$ *be an open bounded subset of* $\mathcal{X}$ *and suppose that*

(i4) $$i_{\mathcal{X}}(\mathcal{G}(\cdot;0), \Omega) \neq 0.$$

Then, equation (2.1) *has at least one solution.*

We remark that, since $\Omega \supseteq \Sigma_0$ and the operators $\mathcal{G}(\cdot;\lambda)$ are completely continuous, then the fixed point index in (i4) is well defined.
As for the validity of (i1), we observe that, by the continuity of φ and recalling Remark 1, the properness of $\varphi_{|\Sigma}$ is guaranteed if Σ is bounded. However, as it was already observed in Chapter 5, if Σ is unbounded then assumption (i1) holds provided that

$$\lim_{\substack{d(x,x_0)\to+\infty \\ (x,\lambda)\in\Sigma}} \Big(\inf_{\lambda\in[0,1]} |\varphi(x,\lambda)| \Big) = +\infty.$$

for some $x_0 \in \mathcal{X}$.

Proof of Theorem 1. Assume, by contradiction, that (2.1) has no solution, i.e.

$$\mathcal{G}(x;1) \neq x, \quad \text{for all } x \in \mathcal{X}. \tag{2.6}$$

We argue, using the fixed point index for ANRs, along the lines of the proof of Lemma 1 in Chapter 5. First, let us consider the sets

$$S := \varphi^{-1}(]c_-,c_+[) \subset \mathcal{X}\times[0,1],$$

$$\Sigma^* := \varphi^{-1}([c_-,c_+]) \cap \Sigma \subset \mathcal{X}\times[0,1].$$

Observe that S is open in $\mathcal{X}\times[0,1]$ and, by (i1), Σ^* is compact.
By (2.6) and (i3),

$$\varphi(\Sigma) \cap \{c_-,c_+\} = \emptyset.$$

Hence, $\Sigma^* \subset S$. By a standard procedure, we can find a set $B \subset \mathcal{X}\times[0,1]$, bounded and open, such that

$$\Sigma^* \subset B \subset cl_{X\times[0,1]} B \subset S.$$

By (2.6), $i_X(\mathcal{G}(\cdot;1), B_1) = 0$. Now, we want to apply Lemma 1, with U = B. To this end, it is sufficient to prove that

$$\mathcal{G}(x;\lambda) \neq x$$

for every $x \in (\partial B)_\lambda$, $\lambda \in [0,1]$. Suppose not; then, $\mathcal{G}(x^*,\lambda^*) = x^*$ for some $\lambda^* \in [0,1]$, $x^* \in (\partial B)_{\lambda^*}$. Hence, $(x^*, \lambda^*) \in \Sigma \cap cl_{X\times[0,1]} B \subset \Sigma \cap S = \Sigma^*$, so that $(x^*, \lambda^*) \in B$, a contradiction. Thus, Lemma 1 gives

$$0 = i_X(\mathcal{G}(\cdot;1), B_1) = i_X(\mathcal{G}(\cdot;0), B_0) .$$

On the other hand, by (2.3),

$$\Sigma_0 \times \{0\} \subset \Sigma^* \subset B,$$

hence $\Sigma_0 \subset B_0$ and so, by the excision property of the fixed point index and (i4)

$$i_X(\mathcal{G}(\cdot;0), B_0) = i_X(\mathcal{G}(\cdot;0), \Omega) \neq 0.$$

Thus, a contradiction is achieved and the proof is complete. ♦

REMARK 3. Theorem 1 generalizes to metric ANRs Lemma 1 in Chapter 5, where an analogue continuation lemma is performed for an operator equation in normed spaces.
As it was pointed out in Example 1 in Chapter 5, with a suitable choice of the functional φ it can be easily seen that by Theorem 1 we can treat the classical situation when a priori bounds for the (possible) solutions of (2.2) are available, i.e. when (for $\Omega \supseteq \Sigma_0$ open and bounded) the condition

$$\mathcal{G}(x;\lambda) \neq x \quad \text{for every } x \in fr_X\Omega,\ \lambda \in]0,1[$$

is satisfied.
Indeed, it is sufficient to define

$$\varphi(x;\lambda) := \begin{cases} -\,dist(x, fr\Omega) & \text{for } x \in \Omega, \\ dist(x, fr\Omega) & \text{for } x \notin \Omega \ ; \end{cases}$$

then, since $-diam(cl_X\Omega) \leq \varphi_-$, $\varphi_+ < 0$, Theorem 1 can be applied, with $c_+ = 0$ and $c_- < diam(cl_X\Omega)$.

By Theorem 1 we can therefore reobtain some results on the existence of periodic solutions to differential systems in flow-invariant ENRs such as, for example, Theorem 1 in Chapter 2, Theorem 5 in Chapter 3, [53, Th. 2.1].
Finally, we point out that an alternative proof of Theorem 1 can be performed by means of Theorem 3 in Section 4; it is based on a careful study of the set of solutions of (2.2), on the lines of [46,51].

We state now a consequence of Theorem 1 which will be used to prove a result for the existence of periodic solutions to some differential system (Theorem 2). The meaning of (j1)-(j2) will be made clear in the applications developed in Section 3.

COROLLARY 1. *Let* A *be a subset of* $\mathcal{X}$. *Assume* (i2), (i4) *and let* $\psi : A \to Z_+$ *be a continuous functional. Assume*

(j1) $\exists\, R>0$ *such that,* $\forall x \in \Sigma_\lambda \setminus A \Rightarrow |x|_\infty \le R, \ \forall \lambda \in [0,1]$;

(j2) $\forall\, n \in Z_+ \ \exists\, K_n \ge 0$ *such that,* $\forall\, x \in A \cap \Sigma_\lambda,\ \psi(x) = n \Rightarrow |x|_\infty \le K_n,\ \forall \lambda \in [0,1]$.

Then, equation (2.1) *has at least one solution.*

Proof. In order to apply Theorem 1, we first need to construct a functional defined on the space $\mathcal{X}\times[0,1]$, and, subsequently, we must check (for the new functional) the validity of (i1) and (i3). By (j1),

$$\Sigma_\lambda \setminus A \subset B[x_0,R], \tag{2.7}$$

for some $x_0 \in \mathcal{X}$ and for every $\lambda \in [0,1]$. Define $\Sigma^* := \bigcup_{\lambda\in[0,1]} \Sigma_\lambda$; consider $R^* > R$ and the restriction of the functional ψ to the set $D := \Sigma^* \cap \big(A \cap (B[x_0,R^*] \setminus B(x_0,R))\big)$. We claim that D is a closed (bounded) subset of $\Sigma^* \cap B[x_0,R^*]$. Indeed, it is sufficient to prove that D is equal to the closed set $\Sigma^* \cap (B[x_0,R^*] \setminus B(x_0,R))$. By the definition, $D \subset \Sigma^* \cap (B[x_0,R^*] \setminus B(x_0,R))$; on the other hand, if there is $x^* \in \Sigma^* \cap (B[x_0,R^*] \setminus B(x_0,R))$ such that $x^* \notin A$, then, by (2.7), $x^* \in B[x_0,R]$, a contradiction. Hence, the claim is proved and, by Tietze's theorem, there exists a real valued functional ψ_1 defined on $\Sigma^* \cap B[x_0,R^*]$, a continuous extension of $\psi_{|D}$ such that

$$0 < \psi_1(x) \le M, \tag{2.8}$$

where $M := \sup\{\psi(x),\ x \in D\} < +\infty$. We note that, since $D \subset \Sigma^*$, by the properties of Σ^* it follows that D is compact. Now, we introduce a functional ψ_2 defined on Σ^* as follows:

$$\psi_2(x) = \begin{cases} \psi(x), & x \in \Sigma^* \cap (A \setminus B[x_0,R^*]) \\ \psi_1(x), & x \in \Sigma^* \cap B[x_0,R^*]. \end{cases}$$

We observe that $\Sigma^* \cap (A \setminus B[x_0,R^*]) = \Sigma^* \setminus B[x_0,R^*]$. One of the inclusions is obvious. On the other hand, it is sufficient to note that if, by contradiction, there exists $x^* \in \Sigma^* \setminus B[x_0,R^*]$, with $x^* \notin A$, then, by (2.7), $x^* \in B[x_0,R] \subset B[x_0,R^*]$, a contradiction. By the defintion of ψ_2, it follows that $\mathrm{Im}\psi_2 \subset \mathrm{Im}\psi \cup [0,M]$.
Furthermore, whenever $x \in \Sigma^*$, with $x \in \mathrm{fr}B[x_0,R^*]$, then, by construction of ψ_1, we have $\psi_1(x) = \psi(x)$. This argument, together with the continuity of ψ and ψ_1, proves the continuity of ψ_2.
Finally, again by Tietze's theorem, we can prove the existence of a real valued functional ψ_3 defined on $\mathcal{X} \times [0,1]$, a continuous extension of ψ_2.
Finally, we define $\varphi^* : \mathcal{X} \times [0,1] \to \mathbf{R}$ as follows: $\varphi^*(x,\lambda) := \psi_3(P(x,\lambda))$, where $P : \mathcal{X} \times [0,1] \to \mathcal{X}$ is such that $P(x,\lambda) = x$.
Now, in order to apply Theorem 1 to the functional φ^*, it is necessary to prove that φ^* is suitable for the validity of (i1) and (i3). As for (i1), it follows from (j1), (j2) and the compactness of the operators $\mathcal{G}(\cdot;\lambda)$. In order to get (i3), we observe that, by the definition of φ and the construction of φ^*, there is $M > 0$ such that $\varphi^*(\Sigma) \subset [0,M] \cup \mathbf{N}$; then, we can take $c_- = 0$; on the other hand, if we consider the sequence $c_n = M + (2n+1/2)$, it is easily seen that the choice $c_+ := c_{\bar{n}}$, for $\bar{n}$ sufficiently large, is suitable for the validity of (i3). Hence, Theorem 1 can be applied and the proof is complete. ♦

3. Applications to periodic boundary value problems

In this Section, we apply Corollary 1 to a differential system inducing a flow on a closed subset of the space $\mathbf{R}^m$. More precisely, let $C \subset \mathbf{R}^m$ be a closed ENR. Our aim is to prove the existence of at least one solution $x(\cdot)$ to the following boundary value problem:

$$\dot{x} = F(t,x) \tag{3.1}$$
$$x(0) = x(T), \tag{3.2}$$

such that $x(t)$ belongs to some subset of C for all $t \in [0,T]$. We assume that the function $F : [0,T] \times C \to \mathbf{R}^m$ is continuous and locally lipschitzian in x.
In order to apply the continuation results of Section 2, we introduce, by a standard procedure, a function $f = f(t,x;\lambda) : [0,T] \times C \times [0,1] \to \mathbf{R}^m$, continuous and locally lipschitzian in x, uniformly in t, λ, such that

$$f(t,x;1) = F(t,x), \tag{3.3}$$

for each $t \in [0,T]$, $x \in C$. Furthermore, we assume that

$$f(t,x;0) = f_0(x), \tag{3.4}$$

for some function $f_0 : C \to \mathbf{R}^m$, and for all $t \in [0,T]$, $x \in C$. We denote by π_0 the local semi-dynamical system induced by (3.4). Suitable homotopies on the (possible) T-periodic solutions of the parametrized system

$$\dot{x} = f(t,x;\lambda), \tag{3.5}$$

together with (3.4), will transform (3.1)-(3.2) in a simpler problem, as in Chapter 3.
Before giving a precise description of the functional frame in which we work (in the general case of ENRs), the following assumptions must be made. (See Remark 4 for comments on their verification).

(h1) *The set* C *is invariant for the flow induced by*

$$\dot{x} = f(t,x;\lambda), \quad \lambda \in [0,1].$$

The so-called "cone conditions" provide efficient equivalent formulations of assumption (h1) (cf. Chap. 1). These have been frequently used by many authors, and we omit for brevity more comments.
In the case of arbitrary ENRs, for the general homotopized system (3.5) we assume

(h2) **PROPERTY (A).** *If*

$$f_0(z) \in T(z;C) \tag{3.6}$$

for all $z \in \mathrm{fr}C$, *there exists a sequence of locally lipschitzian functions* (φ_k), $\varphi_k : C \to \mathbf{R}^m$ *such that:*

(a') $\varphi_k(z) \in T(z;\, C)$ *for all* $z \in \mathrm{fr}C$, $k \in \mathbf{N}$;

(b') $\varphi_k \to f_0$ *uniformly on compact sets*

and

(c') *for every compact subset* K *of* C *and for every* $k \in \mathbf{N}$ *system*

$$\dot{x} = \varphi_k(x)$$

has finitely many singular orbits (i.e., rest points and closed orbits) with minimal period in $[0,T+1]$ *which are contained in* K.

In order to apply Corollary 1, we introduce the following space

$$\Gamma := \{x(\cdot) : [0,T] \to C, \text{ continuous}\},$$

endowed with the distance d*, $d^*(x_1,x_2) := |x_1-x_2|_\infty$. The following result [75] is crucial: "the space Γ is an ANR if and only if the set C is an ENR". This theorem enables us to work with the fixed point index for ANRs.

Now, we introduce the operator, defined on Γ, whose fixed points $x(\cdot)$ (which we will obtain by means of Corollary 1), are solutions of (3.1)-(3.2) such that $x(t)\in C$, for all $t\in[0,T]$.

We use the idea introduced in Chapter 2 and developed in Chapter 3.

Let us consider the Cauchy problem

$$\dot{y} = f(t,x;\lambda) \tag{3.7}$$

$$y(\sigma) = z. \tag{3.8}$$

By the regularity assumptions on f, uniqueness and global existence for (3.7)-(3.8) are guaranteed. Hence, we can define the family of operators $M : \Gamma\times[0,1]\to\Gamma$ as follows:

$$M(x;\lambda) := u(0,x(T),\cdot;\lambda), \qquad \lambda\in[0,1], \tag{3.9}$$

where $u(\sigma,z,\cdot;\lambda)$ is a one-parameter family of processes induced by (3.7)-(3.8).

By (3.9), it is easily seen that $x\in\Gamma$ is a fixed point of $M(\cdot;1)$ if and only if x is a T-periodic solution of (3.1)-(3.2).

REMARK 4. Suppose that $C = \mathbf{R}^m$; then, assumptions (h1) and (h2) are obviously satisfied; observe also that, in this case, the index of rest points of the flow π_0 with respect to some $G\subset\mathbf{R}^m$, open and bounded, is such that $I(\pi_0, G) = (-1)^m d_B(f_0, G, 0)$, so that the classical situation when the phase space is $\mathbf{R}^m$ is recovered.

If C is a manifold (satisfying suitable assumptions), and $f(t,\cdot;\lambda)$ is a vector field tangent to C, then, since $f(t,z;\lambda) \in T(z;C)\cap -T(z;C)$, assumption (h1) is satisfied; by the Kupka-Smale theorem [22], (h2) holds as well. Moreover, $I(\pi_0, G) = \chi(-f_0)$, where χ is the "characteristic of the vector field" f_0 introduced in [53].

Now, we can give the main result of this Section.

In what follows, Γ_T denotes the set of functions $x\in\Gamma$ such that $x(0) = x(T)$.

We set:

$$S = \{(x,\lambda)\in\Gamma_T\times[0,1] : \dot{x} = f(t,x;\lambda)\}, \qquad S_\lambda := \{x\in\Gamma_T\colon (x,\lambda)\in S\}. \tag{3.10}$$

THEOREM 2. *Assume* (h1), (h2) *and* (h3) *there is* $r_0 > 0$ *such that, for any* $x(\cdot) \in \Gamma$, T-*periodic solution of* $\dot{x} = f_0(x)$, $|x|_\infty < r_0$;

(h4) $$I(\pi_0, B(0,r) \cap C) \neq 0 \quad \textit{for all} \quad r \geq r_0 .$$

Let A *be a subset of* Γ_T, *and suppose that there exists a continuous functional* $\psi : A \to \mathbf{Z}_+$ *such that* :

(h5) *there exists* $r_1 > 0$ *such that, for any* $x(\cdot) \notin A$, T-*periodic solution of* (3.5), $|x|_\infty < r_1$, *for every* $\lambda \in [0,1]$;

(h6) $\forall\, n \in \mathbf{Z}_+ \;\; \exists\, K_n \geq 0$ *such that*, $\forall\, x(\cdot)$,T-*periodic solution of* (3.5) , $\psi(x) = n \;\; \Rightarrow \;\; |x|_\infty \leq K_n$, $\forall \lambda \in [0,1]$.

Then, (3.1)-(3.2) *has at least one solution* $x(\cdot)$ *such that* $x(t) \in C$, *for all* $t \in [0,T]$.

It is worth noticing that Theorem 2 is specially suitable in the case when no a priori bounds are available; in this case, the use of the new ingredient of the functional ψ is crucial. See also Remark 3 in Section 2.

Proof. We apply Corollary 1 of Section 2, with $\mathcal{X} = \Gamma_T$ and $\mathcal{G}(x;\lambda) = M(x;\lambda)$. By the definition and Ascoli-Arzelà theorem it follows that the operator $M(\cdot;\lambda)$ is completely continuous. Recalling (3.4), it is clear that (h3) is the same as (i2). Moreover, (h5)-(h6) imply (j1)-(j2), respectively. In order to conclude the proof, it is sufficient to show that

$$I(\pi_0, B(0,r) \cap C) = i_{\mathcal{X}}(M(\cdot;0), B_{r_o}), \tag{3.11}$$

where $B_{r_o} = \{x \in \mathcal{X} : x(t) \in B(0,r), \text{ for all } t \in [0,T]\}$.
The above equality can be obtained by performing a series of homotopies which are shown to be admissible by the use of (h1), (k2), (h3). This procedure has been developed in Section 5 of Chapter 3] and we do not repeat the details here.
Hence, Corollary 1 is applicable and the proof is complete. ♦

Theorem 2 generalizes to closed ENRs Theorem 1 in Chapter 5, where an analogue continuation theorem is performed through the standard Liapunov-Schmidt reduction (on normed spaces) and the coincidence degree.

We end this Section with an illustration of Theorem 2 by means of two particular cases in which the phase space C of (3.1)-(3.2) is, respectively, $\mathbf{R}^m$ and the two-dimensonal sphere S^2, and a priori bounds for the solutions of (3.5) are not available.

The former situation has been treated in Chapter 5, the latter by M. Furi and M. P. Pera in [54]. In both cases, one is led to develop some estimates on the (possible) T-periodic solutions of (3.5) in order to construct some map $\xi : S^1 \to S^1$ and, therefore, to use the "degree" (or "winding number") of such a map. In this way, some admissible homotopies can be performed and a continuation theorem analogue to Corollary 1 is applied.

In Chapter 5, we introduced a functional ζ which is a modification of the classical "time-map" which counts (for m=2) the number of rotations around the origin of the solutions of (3.5). More precisely, in Section 3 of Chapter 5 we defined :

$$\zeta(x,\lambda) := \frac{1}{2\pi}\left|\int_0^T (f(t,x(t);\lambda) \mid Jx(t))\ \delta(x(t))dt\right| \quad ,$$

where $J : \mathbf{R}^2 \to \mathbf{R}^2$ is the usual sympletic matrix and $\delta(a,b) = 1$ for $a^2 + b^2 < 1$, $\delta(a,b) = \frac{1}{a^2+b^2}$, for $a^2+b^2 \geq 1$. In that situation, some estimates on the (possible) solutions of (3.5) were performed ensuring that, roughly speaking, the functional ζ takes only positive integer values on large norm solutions and whose corresponding level sets have a bounded intersection with Σ. In the framework proposed in this paper (which is slightly different from the one in Chapter 5), when (for simplicity) it is $C = \mathbf{R}^2$, we may take in Theorem 2

$$A = \{x \in \Gamma_T : x(t) \neq 0,\ \forall t \in [0,T]\};$$

then, for $x=(x_1,x_2) \in A$, we can define $\psi(x)$ to be the degree of the map $\xi : S^1 \to S^1$, $\xi(t) = (x_1(t)/\rho(t), x_2(t)/\rho(t))$, where $\rho(t) = ((x_1(t)^2 + x_2(t)^2)^{1/2}$. Note that (h5)-(h6) in Theorem 2 are analogue to (h4)-(h5) in Theorem 2 in Chapter 5. For more comments on the verification of (h5)-(h6) we refer to Propositions 2, 3 and Remark 3 in Chapter 5. By means of Theorem 2, which, in the particular case of $C = \mathbf{R}^m$ is analogue to Theorems 1, 2 in Chapter 5, all the applications to linear and superlinear periodic boundary value problems developed in Chapter 5 can be reobtained.

Secondly, we briefly describe the contribution given by M. Furi and M. P. Pera in [54] to the problem of the existence of forced oscillations for the spherical pendulum. In [54], any given orbit $x(\cdot)$ of (3.5) on the sphere is prevented to intersect some axis through the origin (which depends on the given orbit), and the classical "winding number" for some defined on S^1 with values in S^1 is used. In this situation, some estimates analogue to (h5)-(h6) are obtained from the geometrical properties of the problem. Then, the existence of forced oscillations is proved using a result [54, Th. 2] on the existence of an unbounded connected branch of nontrivial solutions to (3.5). For a similar result in the framework of arbitrary ANRs, see Corollary 3 in Section 4.

An investigation on a (possible) general definition for the functional ψ in Theorem 2 is in process; further developments on this subject will appear in a forthcoming paper [14].

REMARK 5. A result analogue to Theorem 2 can be performed for the non-autonomous retarded functional differential equation

$$\dot{x} = F(t,x_t).$$

It consists of generalizing to flow-invariant ENRs (in the framework of the fixed-point index theory) the formula for the computation of the coincidence degree obtained in Theorem 1 in Chapter 4. In this way, we can write the analogue of (3.11) and proceed as in the proof of Theorem 2.

4. A further result

In this Section, we give a theorem on the structure of the solution set of a parameter-dependent operator equation defined on a (metric) ANR, in the case when the only assumption is the non-vanishing of the fixed point index of some (suitable) map.
The proof is performed by means of the basic properties of the fixed point index.
Before stating this result, we recall a result from general topology which is crucial for the proof of Theorem 3.

LEMMA 2. [83, Ch.9]. *Let* K *be a compact metric space and let* A, B *be nonempty disjoint subsets of* K. *Then, either there exists a pair* (F_A,F_B) *of closed disjoint subsets of* K *such that* $F_A \supseteq A$, $F_B \supseteq B$, $F_A \cup F_B = K$, *or there exists a connected subset* D *of* $K \setminus (A \cup B)$ *whose closure meets both* A *and* B.

When the second alternative holds, we say that the set D "connects" A and B.

Let $\mathcal{X}$ be a metric ANR. Let Ω be an open subset of $\mathcal{X}\times\mathbf{R}$ and consider a completely continuous operator $\mathcal{G} : \mathrm{cl}\Omega \to \mathcal{X}$. Set $\mathcal{G}_\lambda(\cdot) := \mathcal{G}(\cdot;\lambda)$, $\lambda \in \mathbf{R}$. We deal with equation

$$x = \mathcal{G}_\lambda(x). \tag{4.1}$$

We use the following notations: $\Sigma := \{(x,\lambda) \in \mathrm{cl}\Omega : x = \mathcal{G}_\lambda(x)\}$, $\Sigma_+ := \{(x,\lambda) \in \Sigma : \lambda > 0\}$, $\Sigma_- := \{(x,\lambda) \in \Sigma : \lambda < 0\}$, $\Sigma_0 = \{(x,0) \in \mathrm{cl}\Omega : x = \mathcal{G}_0(x)\}$; as usual, $\Omega_0 = \{x \in \mathcal{X} : (x,0) \in \Omega\}$.

THEOREM 3. *Assume that* Σ_0 *is bounded, and*

(m1) $$i_{\mathcal{X}}(\mathcal{G}_0,\Omega_0) \neq 0.$$

Then, there exists a closed connected set $C \subset \Sigma$, $C \cap \Sigma_0 \neq \emptyset$, *such that for the sets* $C_+ := \{(x,\lambda) \in C: \lambda > 0\}$ *and* $C_- := \{(x,\lambda) \in C: \lambda < 0\}$ *the following alternative holds: either* (i) C_+ (C_-) *is unbounded or* (ii) $C_+ \cap \mathrm{fr}\Omega \neq \emptyset$ $(C_- \cap \mathrm{fr}\Omega \neq \emptyset)$.

Proof. First, we note that $\Sigma_+ \neq \emptyset$, $\Sigma_- \neq \emptyset$. It is a consequence of (m1) and of the homotopy property of the fixed point index.
Consider the sets $\Sigma^* := \Sigma \cup (\{+\infty\} \cup \{-\infty\})$, $\Sigma_+^* := \Sigma^+ \cup (\{+\infty\} \cup \{-\infty\})$, $\Sigma_-^* := \Sigma_- \cup (\{+\infty\} \cup \{-\infty\})$, $\Omega^* := \mathrm{cl}\Omega \cup (\{+\infty\} \cup \{-\infty\})$. We say that a subset O of Ω^* is an open neighbourhood of $\{+\infty\}$ ($\{-\infty\}$) if $\{+\infty\} \in \Omega$ ($\{-\infty\} \in \Omega$) and the set $O^* := \mathrm{cl}\Omega \setminus O$ is a closed subset of $\mathrm{cl}\Omega$ such that $O^* \cap \{(x,\lambda) \in \mathrm{cl}\Omega: \lambda > 0\}$ ($O^* \cap \{(x,\lambda) \in \mathrm{cl}\Omega : \lambda < 0\}$, resp.) is bounded. In this way, $\{+\infty\}$ and $\{-\infty\}$ are closed in the (Hausdorff) topology of Ω^*; if Ω is bounded, then $\{+\infty\}$ and $\{-\infty\}$ are isolated points.
Now, we apply Lemma 2 with $\Sigma^* = K$, $\Sigma_+^* = A$, $\Sigma_-^* = B$. If we can prove that there exists a connected subset $\hat{C}$ of Σ^* which "connects" Σ_+^* and Σ_-^*, then it will be sufficient for the completion of the proof to define $C := \mathrm{cl}_{\mathrm{cl}\Omega^*}\hat{C} \cap \mathrm{cl}\Omega$. Suppose not. Then, by Lemma 1, we can find two closed disjoint sets F_+^*, F_-^* such that $F_+^* \supset \Sigma_+^*$, $F_-^* \supset \Sigma_-^*$; moreover, $F_+^* \cup F_-^* = \Sigma^*$. Since Ω^* is Hausdorff, there exist two disjoint open subsets Ω_+^*, Ω_-^* such that $\Omega_+^* \supset F_+^*$, $\Omega_-^* \supset F_-^*$. We also set $\Omega_+ := \Omega_+^* \cap \Omega$, $\Omega_- := \Omega_-^* \cap \Omega$.
Note that the sets $\Omega_- \cap \{(x,\lambda) \in \mathrm{cl}\Omega : \lambda > 0\}$ and $\Omega_+ \cap \{(x,\lambda) \in \mathrm{cl}\Omega : \lambda < 0\}$ are bounded.
By the above construction, which is analogous to the one developed in the proof of Theorem 1.1 in [51], it follows that

$$\Sigma_0 \subset (\Omega_+^* \cap \Omega_-^*) \cap \Omega = \Omega_+ \cap \Omega_- . \tag{4.2}$$

Now, we apply the excision property of the fixed point index. To this end, consider the set $\Omega_0^- = \{x \in \mathcal{X} : (x,0) \in \Omega_-\}$; by (4.2), we obtain that $\mathcal{G}_0(x) \neq x$ for all $x \in \Omega_0 \setminus \Omega_0^-$. Hence

$$i_{\mathcal{X}}(\mathcal{G}_0, \Omega_0^-) \neq 0 . \tag{4.3}$$

Now, we claim that there is $\bar{x} \in \mathrm{fr}\Omega_\lambda^-$ such that $\bar{x}$ is a solution of (4.1) with $\lambda > 0$.
Suppose not. This means that $\mathcal{G}_\lambda(x) \neq x$ for every $(x,\lambda) \in (\mathrm{fr}\Omega_-)_\lambda$, $\lambda > 0$. Hence, we can apply Lemma 1 in Section 2 (cf. Remark 2) and obtain

$$i_{\mathcal{X}}(\mathcal{G}_\lambda,\Omega_\lambda^-) \;=\; \text{constant w.r.t. } \lambda > 0\,;$$

observe that $\Omega_\lambda^- = \{x\colon (x,\lambda)\in\Omega^-\}$ is empty for λ large; thus, $i_{\mathcal{X}}(\mathcal{G}_{\bar\lambda},\Omega_{\bar\lambda}) = 0$, for some $\bar\lambda > 0$. Hence, a contradiction with (4.3) is achieved and the claim is proved.
Now, observe that, by the construction performed in the first part of the proof, we have

$$\begin{aligned}\Sigma^*\cap \mathrm{fr}\Omega^- &= (F_+^*\cup F_-^*)\cap \mathrm{fr}\Omega\_ \\ &= (F_+^*\cap \mathrm{fr}\Omega\_)\cup(F_-^*\cup \mathrm{fr}\Omega\_) = \varnothing\,.\end{aligned}$$

This contradicts the above claim. Hence, there exists $\hat{C}\subset\Sigma^*$ which connects Σ_+^* and Σ_-^* and the set $\mathcal{C} := \mathrm{cl}_{\mathrm{cl}\Omega^*}\hat{C} \cap \mathrm{cl}\Omega$ satisfies either (i) or (ii). The proof is complete. ♦

Theorem 3 generalizes Theorem 1.1 in [51] to metric ANRs; we point out that in [51], instead of the fixed-point index, the concept of "regular map" is used.

REMARK 6. It is easy to see that if, under the hypotheses of Theorem 3, $\Omega = U\times\mathbf{R}$, $U\subset\mathcal{X}$, open and bounded and

$$\mathcal{G}(x,\lambda) \neq x, \qquad \text{for all } x\in \mathrm{fr}_{\mathcal{X}}U, \qquad \lambda\in[0,1],$$

holds, then (ii) is impossible; thus, the subset $\mathcal{C}$ obtained in Theorem 3 is unbounded in the variable λ and, taking $\lambda = 1$, the existence of at least one solution of equation $\mathcal{G}(x;1) = x$ is proved. See [51, Cor. 2.1] for a similar result in the case when $\mathcal{G}(x,\lambda) = \lambda\mathcal{G}^*(x)$, with $M^* : \mathrm{cl}U\to\mathcal{X}$ compact.
Furthermore, we remark that, by the choice $\Omega = \mathcal{X}\times\mathbf{R}$, it is possible to prove the analogue of Rabinowitz non-bifurcation result [124, Th. 3.2] for ANRs.

REMARK 7. By means of Theorem 3 we can provide a different proof of Theorem 1; it consists of the use the continuity of the real-valued functional φ, together with the connectedness of the unbounded set $\mathcal{C}\subset\Sigma$ (in the case considered in Theorem 1, the alternative (ii) is impossible). More precisely, the fact that solutions having a sufficiently large norm take values in the *discrete* set $\mathbf{Z}\subset\mathbf{R}$ is crucial for the proof. We omit the details, which are similar those developed in the proof of Lemma 1 in Chapter 5.

Finally, we examine the case when the operator $\mathcal{G}$ in Theorem 3 arises from the study of the boundary value problem

$$\dot{x} = F(t,x) = f(t,x;1) \tag{3.1}$$
$$x(0) = x(T), \tag{3.2}$$

(with $f : [0,T]\times C\times \mathbf{R}\to\mathbf{R}^m$) whose phase space is a closed ENR $C\subset\mathbf{R}^m$. This aspect has been developed in Section 3. According to (3.10) (with a slight abuse of notation), we recall that $S = \{(x,\lambda)\in\Gamma_T\times\mathbf{R}: \dot{x} = f(t,x;\lambda)\}$, $S_0 = \{x\in\Gamma_T: \dot{x} = f_0(x)\}$, where $\Gamma_T := \{x(\cdot) : [0,T]\to C$, continuous and such that $x(0)=x(T)\}$.

COROLLARY 2. *Assume* (h1), (h2) *and*

(n1) S_0 *is bounded ;*

Let $\Omega\subset\Gamma\times\mathbf{R}$, $\Omega_0\supseteq S_0$ *be an open set, and assume that*

(n1) $I(\pi_0,\Omega_0\cap C) \neq 0$.

Then, there exists a closed connected set $D\subset S$, $D\cap(S_0\times\{0\}) \neq \emptyset$, *such that for the sets* $D^+ := \{(x,\lambda)\in D: \lambda > 0\}$ *and* $D^- := \{(x,\lambda)\in D: \lambda < 0\}$ *the following alternative holds:*
either (i) D^+ (D^-) *is unbounded* or (ii) $D^+\cap\mathrm{fr}\Omega \neq\emptyset$ $(D^-\cap\mathrm{fr}\Omega\neq\emptyset)$.

We remark that, according to the definition and properties of the index of rest points (cf. Chapter 1), we have $I(\pi_0,\Omega_0\cap C) := I(\pi_0,\tilde{\Omega}_0\cap C)$, for any open bounded set $\tilde{\Omega}_0\subset\Gamma$ such that $S_0\subset\tilde{\Omega}_0\subset\Omega_0$.

Proof. In the framework of Section 3, it is sufficient to prove that, under the given conditions,

$$I(\pi_0,\Omega_0\cap C) = i_\Gamma(M_0,\Omega_0), \tag{4.4}$$

(the operator $M(\cdot;\lambda)$) is defined in (3.9)) and to apply Theorem 3 with $\mathcal{G}(\cdot;\lambda) = M(\cdot;\lambda)$. The argument needed for the proof of (4.4) is the same as the one used to get (3.10), again on the lines of Section 5 in Chapter 3. ♦

REMARK 8. Corollary 2 shows that, in the general framework of the fixed point index theory for arbitrary ANRs it is possible to improve many results (obtained with different techniques) concerning various periodic boundary value problems. On the one hand, Corollary 2 is a global bifurcation result; it contains Theorem 2.1 in [53], where the phase space is an m-dimensional manifold satisfying suitable assumptions and the "characteristic of the vector field" $\chi(f_0)$ is assumed to be non-zero. Corollary 2 also generalizes Theorem 2.1 in [51], where, in order to perform a proof based on Taylor's formula, more regularity for the operators arising from the given system of ODEs is required. On the other hand, from

Corollary 2 several existence results can be obtained too; more precisely, if a priori bounds for the solutions of the parametrized system (3.5) are available, then, according to Remark 6, we reobtain, among others, Theorem 5 in Chapter 3, Theorem 1 in Chapter 2 and Corollaries 1 and 4 in Chapter 2; in these last ones, we considered the following different choices for the set Ω:

$$\Omega = \{x\in\Gamma : x(t)\in G, \forall t\in[0,T]\}, \quad \Omega = \{x\in\Gamma: x(0)\in G, |x|_\infty<R\},$$

for $G \subset C$ bounded and open relatively to C.
Finally, since, according to Remark 7, Theorem 1 (and, as a consequence, Theorem 2) can be proved by means of Theorem 3, it turns out that the detailed study of the solution set of a parameter-dependent equation developed in this Section, together with the idea of using an integer-valued functional ψ as in Theorem 2 has a crucial importance also when there are no a priori bounds, like, e.g., in the "superlinear case" studied in Chapter 5 for $C = \mathbf{R}^m$.

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Author's address: Dipartimento di Matematica, Università di Torino
Via Carlo Alberto, 10 - 10123 TORINO - ITALY