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FOR ADVANCED STUDIES

Scaling symmetries of evolutionary systems, time–dependent Hamiltonian structures, and monodromy–preserving deformations

Thesis submitted for the degree of
"Doctor Philosophiæ"

CANDIDATE
Monica Ugaglia

SUPERVISOR
Prof. Boris Dubrovin

November 11, 1998

SISSA - SCUOLA
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TRIESTE
Via Beirut 2-4

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To Giulia.

Не ужасное ль в уме,
Борис, Борис,
Не ужасное ль в уме
Вы замыслили, о, братья?

Forse orribili cose nella mente,
Borís, Borís!
Forse orribili cose nella mente
voi avete tramato, o fratelli?

VELIMIR CHLÉBNIKOV

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Chapter 0

Introduction

0.1 Outline

The six Painlevé equations are known since the beginning of the century [Fu, Schl, G]. They appear for the first time in the project of classifying ODEs on the basis of the singularities they admits. The singularities of the solutions of an ODE are *fixed*, if they do not depend on the initial data, or *movable*, otherwise. The singularities other than poles are called *critical points.* A linear ODE has only fixed critical points; indeed any singularity of a solution coincides with a singularity of the coefficients. A nonlinear ODE admits also movable singularities. A first order equation admits only poles (of any order) or algebraic branch points; Moreover, Fuchs [F] proved that the only equation without movable critical points of the form

$$u_t = F(u,t),$$

where F is rational in u and locally analytic in t, is the Riccati equation:

$$u_t = a(t) + b(t)u + c(t)u^2,$$

with $a(t), b(t), c(t)$ analytical.

Painlevé, Gambier and others (see [I] for a complete review) proved an analogue of the Fuchs result for the second order ODE of the form

$$u_{tt} = F(t,u,u_t),$$

where F is algebraic in u, rational in u_t and locally analytic in t. In this case the solutions admits poles, algebraic and logarithmic branch points and essential singularities.

The requirement that an ODE has no movable critical points is known as the *Painlevé property.* Painlevé and Gambier found fifty equations satisfying this property. Six among them, the so called *Painlevé equations*, turned out not to be integrable in terms of known functions. Their solutions were called *Painlevé transcendents.* (We list in Appendix 0.A the six Painlevé equations).

At the end of 70s these equations were included in the framework of the theory of integrable systems [ARS, FN1, JMU]. Indeed, there are several connections between Painlevé property and integrability. In this work we will analyze two of these connections: namely the relationship between ODEs with Painlevé property and the self–similar reductions of completely integrable PDEs, and the analogy between Inverse Scattering Transform (IST) method (involving completely integrable PDEs) and Monodromy Preserving Deformation method (involving ODEs with Painlevé property).

Starting from the work of Flaschka, Newell [FN1], and the Japanese school [JMU] on one side and of Ablowitz, Segur, Ramani [AS,ARS] on the other, this twofold connection between equations with Painlevé property and partial differential equations solvable via Inverse Scattering Transform (IST) has been developed in great detail.

Here we will recall some basic facts about these two topics.

0.1.1 Self–similar solutions

A completely integrable PDE is an infinite dimensional Hamiltonian system (see Sect 0.2 for details). Let us concentrate our attention on a particular kind of solutions of PDEs, namely the so called *self–similar* solutions: although the properties we enunciate hold for every soliton equation, we sketch here, as an example, the case of the modified Korteweg–de Vries equation (mKdV).

The mKdV equation

$$u_t = 6u^2 u_x - u_{xxx}$$

is completely integrable and admits the self–similar solution

$$u(x,t) = (3t)^{-\frac{1}{3}} w(z), \qquad z = x(3t)^{-\frac{1}{3}}.$$

This follows from the observation that, if $u(x,t)$ satisfies KdV, then so does $ku(kx, k^3 t)$, i.e. a self–similar solution is invariant under such a scaling. In this case, the function $w(z)$ satisfies the second Painlevé equation

$$w'' = 2w^3 + zw + a,$$

where a is an arbitrary integration constant.

This, and the analogous cases of KdV and Sine–Gordon equations are analyzed in detail in Chapter 4. Coming back to the general case, the fact that, in all known examples, the reductions obtained in this way possess the Painlevé property, led Ablowitz, Ramani and Segur [ARS] to formulate the so–called *Painlevé conjecture*, which formalizes this relationship, asserting that every ODE arising as a similarity reduction of a PDE solvable via the IST method (i.e. completely integrable), has the Painlevé property.

In this work we will concentrate our attention on the Hamiltonian aspect of the problem: the starting PDE is an (infinite dimensional) Hamiltonian system and its Hamiltonian structure is well known (see [FT]); on the other hand, also the reduced ODE of Painlevé type can be read as a Hamiltonian system, and for the simplest examples of these restrictions the Hamiltonian structure is known. For instance, the Hamiltonian description of the classical six Painlevé equations was found by Okamoto, [O]. Whereas the relationship between the starting PDE and the reduced ODE is clear and has been investigated quite a lot (see, for instance [AC,AS]), the relationship between the starting Hamiltonian structure and the reduced one has not been investigated. This work will give a contribution in understanding this relationship.

Our first result (see Chapter 2) is a universal construction of the reduced Hamiltonian structure starting from the Hamiltonian structure of the original hierarchy of PDEs.

This is all about the reduction from PDEs to ODEs of Painlevé type. Now we will briefly discuss the analogy between spectrum preserving deformations and monodromy preserving deformations.

0.1.2 IST and MPDE

As it is well known, completely integrable PDEs can be obtained as spectrum preserving deformations of differential linear operator and this is at the basis of the IST method. On the other hand, ODE of of Painlevé type can be obtained as Monodromy Preserving Deformation Equations (MPDE) of differential operator with rational coefficients. We recall here some basic facts about both these methods.

Completely integrable PDEs are characterized by the fact that the initial value problem can be exactly solved via the inverse scattering method. The latter was initially employed by Gardner, Greene, Kruskal and Miura [GGKM] to solve the initial value problem for the KdV equation. The generalization of this technique is applied to other infinite dimensional evolution equations and it is known as IST method.

The idea was to solve the KdV equation with initial condition

$$u(x,0) = f(x),$$

where $f(x)$ decays sufficiently rapidly as $|x| \to \infty$, by relating it to the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + u(x,t).$$

If one deforms the potential $u(x,t)$ in t, such a deformation preserves the spectrum of L iff $u(x,t)$ satisfies the KdV equation.

Indeed, imposing $\lambda_t = 0$, the KdV equation can be read as compatibility of

$$Lv = v_{xx} + u(x,t)v = \lambda v$$

with

$$v_t = (u_x + \gamma)v - (2u + 4\lambda)v_x.$$

The latter equation represents the time evolution of the eigenfunctions of L, where γ is an arbitrary constant and λ is the spectral parameter. The eigenvalues of L and the behavior of the eigenfunctions determine the *scattering data* of the problem.
The IST method consists in three steps: firstly (*direct problem*) one constructs a map from $u(x,0)$ to the set of the scattering data $S(\lambda, 0)$ of the problem at the time $t = 0$. As a second step one let the scattering data evolve, and finally (*inverse problem*) one construct the map from $S(\lambda, t)$ to $u(x,t)$.

The basic idea is to shift the evolution from the space of potentials $u(x,t)$ to the space of monodromy data, where the dynamic is quite simple.

The method has been put in a more general framework (see, for instance, [AC] for a complete reference): one may consider the two linear systems

$$\frac{d\phi}{dx} = X(x,t)\phi \tag{0.1a}$$

and

$$\frac{d\phi}{dt} = T(x,t)\phi, \tag{0.1b}$$

for a vector ϕ and $n \times n$ matrices X, T. In this case the compatibility gives

$$X_t - T_x = [T, X];$$

this integrability condition turn into a completely integrable PDE: for Instance, if $\phi = (\phi_1, \phi_2)^T$, and one chose as operators

$$X = \begin{pmatrix} -i\xi & u \\ u & i\xi \end{pmatrix}$$

and

$$T = \begin{pmatrix} -4i\xi^3 - 2iu^2\xi & 4z^2u + 2i\xi u_x - u_{xx} + 2u^3 \\ 4\xi^2 u - 2i\xi u_x - u_{xx} + 2u^3 & 4i\xi^3 + 2iu^2\xi \end{pmatrix},$$

the compatibility condition coincides with the mKdV equation, and this means that the scattering data of the problem are preserved if u, as function of t, satisfies the mKdV equation.

We will use this example to introduce the MPDE framework: we recall that, performing a self–similar reduction of mKdV one obtains the second Painlevé.

Indeed, following [FN1] one can consider self–similar solutions of the system (0.1) itself: if $\phi(x,t,\xi)$ satisfies (0.1), then so does $\phi(kx, k^3t, k^{-1}\xi)$. One defines, in the case of self–similar u, $\psi(x,\xi) \equiv \phi(x, \frac{1}{3}, \xi)$, so that

$$\phi(x,t,\xi) = \psi(z,\zeta),$$

where $\zeta = \xi(3t)^{\frac{1}{3}}$. ψ satisfies

$$\frac{d\psi}{dx} = \tilde{X}(x,t)y \tag{0.2a}$$

and

$$\frac{d\psi}{dt} = \tilde{T}(x,t)y, \tag{0.2b}$$

with

$$\tilde{X} = \begin{pmatrix} -i\zeta & v \\ v & i\zeta \end{pmatrix}$$

and

$$\tilde{T} = \begin{pmatrix} -i(4\zeta^2 - 2v^2 + z) & 4\zeta v + 2i\frac{dv}{dz} + \frac{a}{\zeta} \\ 4\zeta v - 2i\frac{dv}{dz} + \frac{a}{\zeta} & i(4\zeta^2 - 2v^2 + z) \end{pmatrix}.$$

The compatibility of systems (0.2) gives the second Painlevé

$$\frac{d^2 v}{dz^2} = 2v^3 + zv + a.$$

In the case of IST, the compatibility of (0.1a) and (0.1b) entails the invariance of the scattering data. Here the compatibility of (0.2a) and (0.2b) still entails some invariance, but the preserved objects are the *monodromy data* of the linear operator with rational coefficients $\tilde{T}$. What is a scattering data is well known, here we briefly recall the definition of monodromy data. Let us consider in the complex domain a differential equation with rational coefficients

$$\frac{dy}{dz} = A(z)y(z) \tag{0.3}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix}, \qquad A(z) = \begin{pmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & .. & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & .. & .. & .. & a_{nn} \end{pmatrix}$$

An arbitrary solution $y(z)$ of (0.3) is locally holomorphic but globally multivalued; the poles of $A(z)$ are singularities of the solution. Fixing a basis $y^{(1)}, \ldots, y^{(n)}$ in the n-dimensional space of solutions we construct the *fundamental* $n \times n$ matrix

$$Y(z) = (y^{(1)}, \ldots, y^{(n)})$$

satisfying the matrix version of (0.3)

$$\frac{dY(z)}{dz} = A(z)Y(z).$$

The fundamental matrix solution is in general a multivalued analytic function. The branch points of the solution are the singular points for the operator $A(z)$. We are interested, in particular, in the so called Fuchsian systems, i.e. systems of the form

$$\frac{d\Phi}{dz} = \sum_{i=1}^{n} \frac{A_i}{z - a_i}\Phi.$$

Near the point $z = a_j$ a fundamental matrix of solutions $Y_j(z)$ exists such that

$$\Phi_j(z) = W_j(z)(z - a_j)^{\hat{A}_j}$$

where $\hat{A}_j$ is the diagonalization of A_J, and $W_j(z)$ converges for small $|z - a_j|$. Such kind of singularities is called *Fuchsian.*

If one continues $\Phi_j(z)$ along a path encircling the point $z = a_j$, the columns of the resulting matrix are linear combinations of the columns of $\Phi_j(z)$; hence there exists a matrix M_j such that

$$\Phi_j(z) \mapsto \Phi_j(z)M_j.$$

The matrix M_j is called *monodromy* matrix around the point $z = a_j$.
Also the point ∞ is Fuchsian; the general solution can be expressed as

$$\Phi_\infty(\lambda) = W^{(\infty)}(\lambda)\left(\frac{1}{\lambda}\right)^{\hat{A}_\infty},$$

where $W^{(\infty)}$ converges at $|\lambda| \to \infty$ and $\hat{A}_\infty$ is the diagonalization of A_∞.

The matrices M_j are the monodromy data of the system. We will describe in more detail the problem in Chapter 6, where we will introduce also the so called *irregular* singularities, which give rise to the *Stokes phenomenon.* In this case we have additional monodromy data. This brief recall is enough to explain what a MPDE is, i.e. the equation describing the deformations of the operator $A(z)$, as a function of the singular points, which preserve the monodromy data.

In particular, the linear system (0.2b) has a regular singular point at zero and an irregular singular point at the infinity.

The monodromy data of the system, describing the behavior of the solutions near the singularities, remain unchanged under deformation of v satisfying PII.

A central point in the IST method is that the scattering map connecting the space of potentials $u(x)$, and the space of scattering data is a Poisson map, i.e. it preserves the Hamiltonian structure (in particular it is possible to choose, in the target space, a combination of the scattering data forming a system of action–angle variables).

The starting PDE is a Hamiltonian system on an infinite dimensional space, w.r.t. the linear Poisson bracket we will describe in detail in Section 0.2. The time evolution of the scattering data, on the other hand, is still an infinite dimensional Hamiltonian system and the associated Poisson bracket is quadratic [ZF]. The understanding of this structure was a fundamental step for the subsequent development of soliton theory, and of his quantum version: the Poisson structure on the space of the scattering data can be read as the semi-classical limit of quantum group [AC].

The analogy between the two processes of scattering and monodromy transform suggests that, also in the MPDE case, the complete understanding of the Hamiltonian structure should be important: in particular, the analogue of the scattering map, the monodromy transform, i.e. the map associating the monodromy data to the linear operator with rational coefficient, plays an essential role. In the case of Fuchsian systems it has been proved [R] that MPDE, i.e. the Schlesinger system (see 6.2.2), is the semi-classical limit of the Knizhnik–Zamolodchikov equation. However, the Hamiltonian nature of the monodromy transform has not been understood in full generality (cf. [FN2], where a very particular case was under investigation).

0.1.3 Main results

This work contains three main results: concerning the reduction of PDEs to ODEs of Painlevé type, we prove that the finite dimensional Hamiltonian structure of the ODEs is obtained from the Hamiltonian structure of the starting hierarchy of PDEs, via scaling reduction, i.e. a reduction on the set of stationary points of a time–dependent first integral. Particularly, we construct the time–dependent Hamiltonian functions of the reduced flows. In the time–independent case this procedure coincides with the well known stationary–flow reduction discovered by Bogoyavlenskii and Novikov [BN]. As an application we present the case of PI, PII, PIII, PVI and also certain higher order systems appeared recently in the theory of Frobenius manifolds [D1].

As a byproduct of these investigations we discover a very general Lagrangian formalism of the procedure of reduction of an evolutionary system

$$u(x)_t = F(x, t, u(x), u_x(x), \ldots, u^{(m)}(x))$$

on the manifold of stationary points

$$\frac{\delta I}{\delta u(x)} = 0$$

of a local integral

$$I = \int L(x, t, u(x), u_x(x), \ldots, u^{(n)}(x)) dx.$$

Namely, we prove that this restriction is again a Lagrangian system with the Lagrangian function Λ, such that

$$\frac{d\Lambda}{dx} = \frac{dL}{dt}.$$

As we have already explained, scaling reductions of evolutionary PDEs can be recasted into the form of MPDE for certain linear differential operators with rational coefficients. So, after completing the description of a natural Hamiltonian structure of these equations, our work goes in the direction of understanding the Hamiltonian nature of the monodromy transform. This question was formulated in [FN2] and solved in an example of a MPDE of a particular second order differential operator. However, the general algebraic properties of the arising class of Poisson brackets on the space of monodromy data remained unclear. The technique of [FN2] seems not to work in the general case. The authors of the papers [AM,FR,KS,Hi] consider the important case of MPDE of Fuchsian systems in the general setting of symplectic structures on the moduli space of flat connections (see, for instance, [Au]). They do not write, however, the Poisson bracket on the space of the monodromy data in a closed form. MPDE of non–Fuchsian operators and the Poisson structure on their monodromy data were not considered in these papers.

The next result is a solution of the problem of computing the Poisson structure in the monodromy data coordinates in the presence of irregular singularities. We derive explicit formulae for an important example of operators with one regular and one irregular singularity. The MPDE of which play a central role in the theory of Frobenius manifolds [D1]. In the particular case of MPDE for the operator of order 3 they coincide with PVI.

0.1.4 Plan of the work

The work is structured as follows: after recalling, in Section 0.2, some basic facts about the Hamiltonian structure of the completely integrable PDEs, in Chapter 1 we briefly summarize the method of reduction of evolutionary flows on the manifold of stationary points of their integral, introduced by Bogoyavlenskii and Novikov [BN]. In Section 1.2 we consider this reduction method to a non necessarily Hamiltonian evolution equation.

Chapter 2 contains the generalization of this procedure to scaling symmetries. The reduced flow is a time–dependent Hamiltonian system, and in Theorem 2.1 we give the relationship between the infinite–dimensional Hamiltonian structure and the reduced one.

Chapter 3 is devoted to a Lagrangian approach to the problem: after describing the general framework, in Theorem 3.1 we give the procedure of reduction and we construct the reduced Lagrangian function. In Section 3.2 we establish the relationship with the Hamiltonian approach. As an application we study the Lagrangian reduction of KdV on the manifold of the fixed points of the 7–th flow.

Chapter 4 contains the application of the theory to the scaling reductions from KdV, mKdV and Sine–Gordon equations respectively to Painlevé I, Painlevé II and III. These examples are studied both from the Hamiltonian and the Lagrangian point of view.

In Chapter 5 we study the n-waves equation and his scaling reduction to a system of commuting Hamiltonian flows on the Lie algebra $\mathfrak{so}(n)$. The reduced system is a non-autonomous Hamiltonian system w.r.t. the Poisson structure of $\mathfrak{so}(n)$. In particular, for $n = 3$, adding an additional symmetry condition, one arrives at Painlevé VI equation.

In Chapter 6 we derive the same non-autonomous Hamiltonian system on $\mathfrak{so}(n)$ as monodromy preserving deformation equation for a linear operator with rational coefficient and we construct the monodromy map from $\mathfrak{so}(n)$ to the space of the monodromy data of the operator. Here the MPDE is still Hamiltonian w.r.t. a quadratic Poisson bracket which we explicitly construct.

0.2 Infinite dimensional Hamiltonian structures

Let us consider the phase space $\mathfrak{M}$ of smooth maps of the circle into some smooth n-dimensional manifold. Actually we can forget about the boundary conditions when dealing with local functionals only. We denote by $\mathfrak{F}$ the space of smooth functionals on $\mathfrak{M}$ of the form

$$F(u) = \int f(x, u(x), u_x(x), \ldots, u^{(m)}(x))dx,$$

where the density f depends only on a finite number of derivatives of u. On the space $\mathfrak{F}$ the variational derivative $\frac{\delta F}{\delta u^i(x)}$ is defined by

$$\delta F = \int \frac{\delta F}{\delta u^i(x)} \delta u^i(x) dx.$$

Explicitly,

$$\frac{\delta F}{\delta u^i(x)} = \frac{\partial f}{\partial u^i(x)} + \sum (-1)^k \frac{d^k}{dx^k} \frac{\partial f}{\partial u^{i(k)}(x)}.$$

One can define on $\mathfrak{M}$ the (formal) Poisson brackets

$$\{u^i(x), u^j(y)\} = w^{ij}(x,y) = \sum_{k=0}^{m} A_k^{ij} \delta^{(k)}(x-y),$$

where A_k^{ij} depends on x and on a finite number of derivatives of u. This induces on $\mathfrak{F}$ the Poisson bracket

$$\{F, G\} = \int \frac{\delta F}{\delta u^i(x)} P^{ij} \frac{\delta G}{\delta u^j(x)} dx,$$

where

$$P^{ij} = \sum_{k=0}^{m} A_k^{ij} (\frac{d}{dx})^k.$$

A Hamiltonian system on $\mathfrak{M}$ has then the form

$$u_t^i(x) = \{u^i(x), H\} = P^{ij} \frac{\delta H}{\delta u^j(x)}.$$

In particular, we consider so called Gardner–Zakharov–Faddev bracket $P^{ij} = \delta^{ij} \frac{d}{dx}$. In this case a Hamiltonian system has the form

$$u_t(x) = \{u(x), H\} = \frac{d}{dx} \frac{\delta H}{\delta u(x)} \tag{0.3}$$

with Poisson bracket

$$\{F, G\} = \int \frac{\delta F}{\delta u(x)} \frac{d}{dx} \frac{\delta G}{\delta u(x)} dx.$$

Let us consider a first integral

$$I = \int L(x, u(x), u_x(x), \ldots, u^{(n)})dx.$$

The generalized Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = 0 \tag{0.3}$$

is an ODE of order $2n$ which fixes the $2n$–dimensional manifold $\mathfrak{S}$ of the stationary points of the first integral I. The functional L is the Lagrangian of the x–flow defined by (0.3). If L is non-degenerate, then it defines also on $\mathfrak{S}$ the natural system of canonical coordinates

$$q_i = u^{(i-1)}, \quad i = 1, 2, ..., n$$
$$p_i = \frac{\delta I}{\delta u^{(i)}},$$

and equation (0.3) can be put in the Hamiltonian form

$$\begin{cases} (p_i)_x = -\frac{\partial H}{\partial q_i} \\ (q_i)_x = \frac{\partial H}{\partial p_i}, \end{cases}$$

where H is the generalized Legendre transform of L:

$$-L + \sum_{i=1}^{n} \frac{\delta I}{\delta u^{(i)}} u^{(i)}$$

which, in terms of the canonical coordinates takes the form:

$$H = -L + \sum_{1}^{n} p_i \frac{dq_i}{dx}.$$

0.A Appendix

We tabulate here the six Painlevé equations:

$$\mathrm{PI}: \quad \frac{d^2w}{dz^2} = 6w^2 + z$$

$$\mathrm{PII}: \quad \frac{d^2w}{dz^2} = 2w^3 + zw + \alpha$$

$$\mathrm{PIII}: \quad \frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}$$

$$\mathrm{PIV}: \quad \frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3w^2}{2} + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\mathrm{PV}: \quad \frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$$

$$\mathrm{PVI}: \quad \frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right) + \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)$$

Chapter 1
On reduction of an evolutionary system on the manifold of stationary points of its integral

1.1 Reduction of Hamiltonian evolutionary flows

For what concerns the scaling reduction as source of nonlinear ODEs with Painlevé property, we will read it in the framework of the Hamiltonian reductions developed in particular by Bogoyavlenskii and Novikov [BN].

Let us consider a completely integrable PDE with one spatial variable and a symmetry, i.e. a first integral of the PDE. Due to commutativity, the submanifold of the stationary points of the symmetry is invariant w.r.t. the evolution equation. Bogoyavlenskii and Novikov discovered that the restriction of the initial PDE to the finite dimensional invariant submanifold is a Hamiltonian system of ODEs. Actually, they proved this result for the more general case of an integrable Hamiltonian hierarchy of PDEs with one spatial variable. Due to the commutativity of the flows of the hierarchy, the submanifold of stationary points of any flow, or of any of their linear combination, is invariant w.r.t. any of the flows of the hierarchy. In this case the restriction of any of the flows to the finite dimensional invariant submanifold is a Hamiltonian system of ODEs w.r.t. a natural Poisson bracket. Bogoyavlenskii and Novikov also found a universal scheme to construct the Hamiltonians of the reductions in terms of the Hamiltonians of the original hierarchy.

The fact that the reduced flow is Hamiltonian is well established and in literature it is possible to find various different approaches to the problem; in particular, in [GD], [R] Hamiltonian systems of Lax type are considered, in [Al] a different parameterization of the manifold $\mathfrak{S}$ is given, a bihamiltonian approach is developed in [T], [FMT].

We will concentrate our attention on the mentioned framework developed by Bogoyavlenskii and Novikov, in order to describe the Hamiltonian structure in terms of the infinite dimensional one. Here we summarize the scheme of procedure.

The starting point is the generic element of a hierarchy of commuting Hamiltonian flows:

$$\frac{du}{dt_k} = \frac{d}{dx}\frac{\delta I_k}{\delta u(x)}$$

with $I_k = \int L_k(u(x), u_x(x), \ldots, u^{(n_k)}(x))dx$.

We will describe the reduction procedure of the k-flow on the finite dimensional manifold of the stationary points of the j–th flow:

$$\frac{du}{dt_j} = \frac{d}{dx}\frac{\delta I_j}{\delta u(x)} = 0,$$

where $I_j = \int L_j(u(x), u_x(x), \ldots, u^{(n_j)}(x))dx$.

The manifold is invariant w.r.t. all the flows of the hierarchy, due to the commutativity

$$\frac{d}{dt_j}\frac{d}{dt_k} = \frac{d}{dt_k}\frac{d}{dt_j}.$$

The equation defining the stationary manifold can be rewritten as

$$\frac{\delta I_j}{\delta u(x)} = -a$$

for constant a, this implies that the functional

$$I_j^{(a)} = \int L_j^{(a)} dx = \int [L_j + au] dx$$

must satisfy

$$\frac{\delta I_j^{(a)}}{\delta u(x)} = 0.$$

This Euler–Lagrange equation plays a crucial role in the present approach: it is an ODE of order $2n_j$ and it defines a $(2n_j+1)$-dimensional submanifold $\mathfrak{S}_j$ foliated in a family of $2n_j$-dimensional symplectic manifolds $\mathfrak{S}_j^{(a)}$, parametrized by the values a.

But the equation gives also a system of canonical coordinates on $\mathfrak{S}_j^{(a)}$, the so–called Ostrograndskii coordinates. The reduced x–flow on $\mathfrak{S}_j^{(a)}$ is automatically given; it turns out to be Hamiltonian and the Hamiltonian function is obtained from L via Lagrange transform.

This touches upon a fundamental point, namely the privileged status of the x variable. Indeed one consider u as a function of x, while the times t_i play the role of parameters. We will attempt to put all them at the same level, firstly (Chapter 2) allowing the symmetries to depend on times, secondly (Chapter 3), facing the problem from a Lagrangian point of view.

Coming back to the starting problem, Bogoyavlenskii and Novikov proved that the reduced t_j–flows are all Hamiltonian and gave the algorithm to produce the Hamiltonian functions $(-Q_{k,j}^{(a)})$:

$$\frac{\delta I_j^{(a)}}{\delta u(x)} \frac{d}{dx} \frac{\delta I_k}{\delta u(x)} \equiv \frac{d}{dx} Q_{k,j}^{(a)}.$$

Notice that this definition makes sense because of the commutativity of the flows of the initial hierarchy: in fact

$$\{I_k, I_j\} = \int \frac{\delta I_j}{\delta u(x)} \frac{d}{dx} \frac{\delta I_k}{\delta u(x)} dx = 0$$

implies the existence of a function $Q_{k,j}$, which depends on the derivatives of u up to order $2n-1$ (where n is the maximum between n_k and n_j), such that

$$\frac{\delta I_j}{\delta u(x)} \frac{d}{dx} \frac{\delta I_k}{\delta u(x)} \equiv \frac{d}{dx} Q_{k,j},$$

and $Q_{k,j}^{(a)} = Q_{k,j} + a\frac{\delta I_j}{\delta u}$. We will analyze in more detail the result of [BN], summarized in

Theorem 1.B: *Let us consider the Hamiltonian flow* X_1:

$$\frac{du}{dt} = \frac{d}{dx} \frac{\delta I_1}{\delta u(x)}, \tag{1.1}$$

where $I_1 = \int L_1(u(x), u(x)_x, ..., u(x)^{(n_1)}) dx$ *and the Hamiltonian stationary flow* X:

$$\frac{du}{ds} = \frac{d}{dx} \frac{\delta I}{\delta u(x)} = 0, \tag{1.2}$$

where $I = \int L(u(x), u(x)_x, ..., u(x)^{(n)}) dx$ and $n \geq n_1$.

The flow X_1 reduces on the manifold of the stationary points of the X flow and the reduced flow is Hamiltonian, with Hamiltonian function $(-Q^{(a)})$ defined via the relation

$$\left(\frac{\delta I}{\delta u(x)}+a\right)\frac{d}{dx}\frac{\delta I_1}{\delta u(x)}=\frac{d}{dx}Q^{(a)}, \tag{1.3}$$

where a is a constant.

Proof: The proof can be found in [BN] (TH.1); here we only sketch the scheme of procedure: as a first step, one describes the manifold defined by the generalized Euler–Lagrange equation

$$\frac{\delta I^{(a)}}{\delta u(x)}=0 \tag{1.4}$$

which follows from (1.2), with $I^{(a)}=\int(L+au)dx$. On this manifold one introduces a system of canonical coordinates obtained via generalized Lagrange transform (here one supposes that the generalized Lagrangian L is non-degenerate). In these coordinates the Euler–Lagrange equation is a Hamiltonian system and it describes the reduced x–flow. Rewriting the identity (1.3) in canonical coordinates and using the fact that $n\geq n_1$, one immediately obtains that $Q^{(a)}$ and the Hamiltonian H of the reduced x–flow commute. Using (1.3) it is straightforward to write also the first step of the inductive proof by means of which one obtains

$$\frac{dp_i}{dt}=\frac{\partial Q^{(a)}}{\partial q_i}=-\{p_i,Q^{(a)}\},$$
$$\frac{dq_i}{dt}=-\frac{\partial Q^{(a)}}{\partial p_i}=-\{q_i,Q^{(a)}\},$$

for $\quad i=1,2,...,n.$

1.2 Reduction of more general evolutionary flows

The generalization of this result to the case of $n<n_1$ is developed in [Mo], where it is also proved that the starting flow must not necessarily be Hamiltonian. The proof presented in [Mo] contains some rather obscure points and a unjustified assumption. Hence before generalizing this theorem to the case we are interested in, i.e. the time dependent one, we propose here a slightly different proof.

Theorem 1.1: *If the generic k–th element of a hierarchy of commuting evolution equations:*

$$\frac{du}{dt_k}=F_k(u(x),u_x(x),.....,u^{(m_k)}(x)),$$

admits a first integral $I=\int L(u,u_x,...,u^{(n_s)})dx$, such that

$$\frac{\delta I}{\delta u(x)}=0,$$

then it is Hamiltonian on the manifold $\mathfrak{S}$ of the stationary points of the symmetry with Hamiltonian function $(-\tilde{Q}_k)$, which is the reduction on $\mathfrak{S}$ of the function $(-Q_k)$ such that

$$\left(\frac{\delta I}{\delta u(x)}\right)F_k\equiv\frac{d}{dx}Q_k \tag{1.5}$$

Proof: We prove the theorem in three steps: first we describe the submanifold $\mathfrak{S}$ of stationary points of the integral I, where we introduce the system of canonical coordinates, in accordance with [BN] and [Mo]; then we construct the functional Q_k, satisfying the identity (1.5) and its reduction $\tilde{Q}_k$ on the manifold $\mathfrak{S}$. Here the commutation relation between $(-\tilde{Q}_k)$ and the Hamiltonian function H of the reduced x–flow is obtained. Finally we prove that the restricted t_k–flow is Hamiltonian on $\mathfrak{S}$, with Hamiltonian function $(-\tilde{Q}_k)$.

1. The set $\mathfrak{S}$ is the $2n_s$–dimensional manifold of the solutions of the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = 0. \tag{1.6}$$

It is invariant under the k–th flow and it naturally carries a system of canonical coordinates:

$$q_i = u^{(i-1)}, \tag{1.7a}$$

$$p_i = \frac{\delta I}{\delta u^{(i)}}, \tag{1.7b}$$

for $i = 1, 2, ..., n_s$. Here it is necessary the non-degeneracy of the generalized Lagrangian L.

Inverting relations (1.7), one can express the derivatives $u, u_x, ..., u^{(2n_s-1)}$ in terms of the canonical coordinates p_i and q_i; explicitly:

$$\begin{cases} u^{(n_s)} = (q_{n_s})_x = g_1(q_1, \dots, q_{n_s}, p_{n_s}) \\ u^{(n_s+1)} = g_2(q_1, \dots, q_{n_s}, p_{n_s}, p_{n_s-1}) \\ \dots\dots \\ u^{(2n_s-1)} = g_n(q_1, \dots, q_{n_s}, p_{n_s}, \dots, p_1). \end{cases}$$

Notice that the following identities hold:

$$\begin{cases} (p_1)_x + \frac{\partial H}{\partial q_1} \equiv -\frac{\delta I}{\delta u} \\ (p_i)_x + \frac{\partial H}{\partial q_i} \equiv 0 \qquad i > 1 \\ (q_i)_x - \frac{\partial H}{\partial p_i} \equiv 0, \end{cases} \tag{1.8}$$

where H is the generalized Legendre transform of L:

$$-L + \sum_{i=1}^{n_s} \frac{\delta I}{\delta u^{(i)}} u^{(i)}$$

which, in terms of the canonical coordinates takes the form:

$$H = -L + \sum_{1}^{n_s} p_i \frac{dq_i}{dx}.$$

The first identity in (1.8) allows us to explicitly express the higher derivatives $u^{(m)}$ for $m \geq 2n_s$ in terms of the p_i, q_i and $p_1^{(l)}$ with $l = 1, \dots, m - 2n_s + 1$:

$$\begin{cases} u^{(2n_s)} = g_{n+1}(q_1, , \dots q_{n_s}, p_{n_s}, \dots, p_1, (p_1)_x) \\ \dots\dots \\ u^{(m)} = g_{m-n_s+1}(q_1, , \dots q_{n_s}, p_{n_s}, \dots, p_1, \dots, (p_1)^{(m-2n+1)}). \end{cases}$$

On $\mathfrak{S}$ $(p_1)_x + \frac{\partial H}{\partial q_1} \equiv 0$, and system (1.8) is a canonical Hamiltonian system, with Hamiltonian function H, giving the reduced x–flow.

2. Now we will show that also the t–flow, when reduced on $\mathfrak{S}$ is Hamiltonian with Hamiltonian function $(-\tilde{Q}_k)$. To this end we first observe that $Q_k^{(a)}$ is a function of u and of its x–derivatives up to order $(m_k + n_s)$, so that it can be rewritten in terms of (p_i, q_i) and $p_1^{(l)}$ up to order $l = m_k - n_s + 1$.

We denote by $\tilde{f}$ a function $f(u(x), \ldots, u(x)^{(j)})$ reduced on $\mathfrak{S}$; notice that, if $j \geq 2n_s$, then the reduction can be done using the relation

$$(p_1)_x = -\frac{\partial H}{\partial q_1}. \tag{1.9}$$

The function $\tilde{f}$ does depend explicitly only on the p_i and q_i, for $i = 1, \ldots, n_s$. In fact differentiating (1.9) one obtains the derivatives $p_1^{(l)}$ in terms of the canonical coordinates (p_i, q_i).

We rewrite explicitly the equation (1.5) for the function Q_k:

$$\begin{aligned}
-\left((p_1)_x + \frac{\partial H}{\partial q_1}\right)\frac{dq_1}{dt_k} &\equiv \\
&\equiv \frac{\partial Q_k}{\partial p_1}(p_1)_x + \sum_{i=1}^{m} \frac{\partial Q_k}{\partial (p_1)^{(i)}}(p_1)^{(i+1)} + \sum_{i=2}^{n_s} \frac{\partial Q_k}{\partial p_i}(p_i)_x + \sum_{i=1}^{n_s} \frac{\partial Q_k}{\partial q_i}(q_i)_x \\
&\equiv \frac{\partial Q_k}{\partial x} + \frac{\partial Q_k}{\partial p_1}(p_1)_x + \sum_{i=1}^{m} \frac{\partial Q_k}{\partial (p_1)^{(i)}}(p_1)^{(i+1)} - \sum_{i=2}^{n_s} \frac{\partial Q_k}{\partial p_i}\frac{\partial H}{\partial q_i} + \sum_{i=1}^{n_s} \frac{\partial Q_k^{(a)}}{\partial q_i}\frac{\partial H}{\partial p_i}.
\end{aligned} \tag{1.10}$$

At this point we need the

Lemma 1.1: *On the submanifold $\mathfrak{S}$ the following relation holds:*

$$\left(\widetilde{\frac{\partial Q_k}{\partial (p_1)^{(j)}}}\right) = 0 \qquad \forall j \geq 1. \tag{1.11}$$

Proof: See Appendix 1.A

From Lemma 1.1 it follows that, on the submanifold $\mathfrak{S}$ (where $(p_1)_x = -\frac{\partial H}{\partial q_1}$), eq. (1.10) reduces to:

$$\sum_{i=1}^{n_s} \frac{\partial \tilde{Q}_k}{\partial p_i}\frac{\partial H}{\partial q_i} - \sum_{i=1}^{n_s} \frac{\partial \tilde{Q}_k}{\partial q_i}\frac{\partial H}{\partial p_i} = \{H, \tilde{Q}_k\} = 0. \tag{1.12}$$

This completes the second step in the proof of the theorem.

3. Now we will construct the Hamiltonian system inductively; to this end we need a further lemma:

Lemma 1.2: *The fundamental relation*

$$\frac{dq_1}{dt_k} = -\frac{\partial \tilde{Q}_k}{\partial p_1}. \tag{1.13}$$

holds.

Proof: See Appendix 1.A

For simplicity, here and in the following we omit the *"tilde"* sign: Q_k will indicate the reduced function on $\mathfrak{S}$.

Now, we assume that $\frac{dq_i}{dt_k} = -\frac{\partial Q_k}{\partial p_i} = -\{q_i, Q_k\}$ and we prove inductively (the calculations are the same as in [BN]) that the same relation holds for q_{i+1}. Indeed,

$$\frac{dq_{i+1}}{dt_k} = (\frac{dq_i}{dt_k})_x = -\frac{d}{dx}\{q_i, Q_k\} = -\{\{q_i, Q_k\}, H\}.$$

Using the Jacobi identity and the commutativity we get

$$\frac{dq_{i+1}}{dt_k} = -\{q_{i+1}, Q_k\}, \qquad i = 1, 2, ..., n_s - 1.$$

Now we prove $\frac{dp_i}{dt_k} = -\{p_i, Q_k\}$ by induction, starting from p_{n_s}.

This comes from the commutativity of the flows

$$\frac{d}{dt_k}\frac{d}{dx}q_{n_s} = \frac{d}{dx}\frac{d}{dt_k}q_{n_s};$$

explicitly:

$$\begin{aligned}\frac{d}{dt_k}\left(\frac{d}{dx}q_{n_s}\right) =& \frac{d}{dt_k}\left(\frac{\partial H}{\partial p_{n_s}}\right) = \\ =& -\sum_{i=1}^{n_s} \frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{d}{dt_k}p_{n_s} = \\ =& -\sum_{i=1}^{n_s} \frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{d}{dt_k}p_{n_s}.\end{aligned}$$

On the other hand, using the Jacobi identity and the commutativity, one can write

$$\begin{aligned}\frac{d}{dx}\left(\frac{dq_{n_s}}{dt_k}\right) =& - \{\{q_{n_s}, Q_k\}, H\} = \\ =& - \{q_{n_s}, \{Q_k, H\}\} - \{\{q_{n_s}, H\}Q_k\} = \\ =& - \{\frac{\partial H}{\partial p_{n_s}}, Q_k\} = \\ =& -\sum_{i=1}^{n_s} \frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{\partial Q_k}{\partial q_{n_s}}.\end{aligned}$$

Comparing the two expressions and noticing that $\frac{\partial^2 H}{\partial p_{n_s}^2} \neq 0$ because of the non-degeneracy, we get

$$\frac{dp_{n_s}}{dt_k} = -\{p_{n_s}, Q_k\} = \frac{\partial Q_k}{\partial q_{n_s}}.$$

- Let us now suppose that $\frac{dp_i}{dt_k} = \{p_i, Q_k\}$ and deduce the same for p_{i-1}. Indeed,

$$\begin{aligned}\frac{d}{dx}\left(\frac{dp_i}{dt_k}\right) =& \{\frac{\partial Q_k}{\partial q_i}, H\} = -\{\{p_i, Q_k\}, H\} = -\{\{p_i, H\}, Q_k = \\ =& \sum_{j=1}^{n_s-1} \frac{\partial^2 H}{\partial q_i \partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{\partial Q_k}{\partial q_{n_s}} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{\partial Q_k}{\partial q_{i-1}}\end{aligned}$$

and

$$\frac{d}{dt_k}\left(\frac{dp_i}{dx}\right) = -\frac{d}{dt_k}\left(\frac{\partial H}{\partial q_i}\right) =$$

$$= -\sum_{j=1}^{n_s-1}\frac{\partial^2 H}{\partial q_i \partial q_j}\frac{dq_j}{dt_k} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{dp_{n_s}}{dt_k} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{dp_{i-1}}{dt_k} =$$

$$= \sum_{j=1}^{n_s-1}\frac{\partial^2 H}{\partial q_i \partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{\partial Q_k}{\partial q_{n_s}} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{dp_{i-1}}{dt_k},$$

where $\frac{\partial^2 H}{\partial q_i \partial p_{i-1}} = 1$. Comparing the two expressions we get $\frac{dp_{i-1}}{dt_k} = \frac{\partial Q_k}{\partial q_{i-1}}$; hence it follows that

$$\frac{dp_i}{dt_k} = \frac{\partial Q_k}{\partial q_i} = -\{p_i, Q_k\}, \qquad i = 1, 2, ..., n_s$$

Q.E.D.

1.A Appendix

Proof of Lemma 1.1: We observe that the recursive relation

$$\left(\frac{\partial Q_k}{\partial (p_1)^{(j-1)}}\right) = \frac{\partial}{\partial (p_1)^{(j)}}\left(\frac{d}{dx}Q_k\right) - \frac{d}{dx}\left(\frac{\partial Q_k}{\partial (p_1)^{(j)}}\right)$$

holds for $j > 1$. Indeed

$$\frac{\partial}{\partial (p_1)^{(j)}}\left(\frac{d}{dx}Q_k\right) = \sum_{i=1}^{n_s}\left(\frac{\partial^2 Q_k}{\partial q_i \partial (p_1)^{(j)}}\right)(q_i)_x + \sum_{i=1}^{n_s}\left(\frac{\partial^2 Q_k}{\partial p_i \partial (p_1)^{(j)}}\right)(p_i)_x +$$

$$+ \sum_{i=1}^{m_k-n_s+1}\left(\frac{\partial^2 Q_k}{\partial (p_1)^{(i)}\partial (p_1)^{(j)}}\right)(p_1)_x^{(i)} + \left(\frac{\partial Q_k}{\partial (p_1)^{(j-1)}}\right).$$

When we reduce on $\mathfrak{S}$:

$$\left(\widetilde{\frac{\partial Q_k}{\partial (p_1)^{(j-1)}}}\right) = -\frac{d}{dx}\left(\widetilde{\frac{\partial Q_k}{\partial (p_1)^{(j)}}}\right).$$

But Q_k depends on $(p_1)^{(j)}$ up to a finite order, then

$$\left(\widetilde{\frac{\partial Q_k}{\partial (p_1)^{(j)}}}\right) = 0 \qquad \forall j \geq 1.$$

Q.E.D.

Proof of Lemma 1.2: The expansion of $\frac{d}{dx}Q_k$ in powers of $(p_1)_x$ near the point $\frac{\partial H}{\partial q_1}$ reads

$$\left(\widetilde{\frac{d}{dx}Q_k}\right) + \left[\frac{d}{d(p_1)_x}\left(\widetilde{\frac{d}{dx}Q_k}\right)\right]\left((p_1)_x + \frac{\partial H}{\partial q_1}\right) + \Theta\left((p_1)_x + \frac{\partial H}{\partial q_1}\right)^2.$$

The zero order term is

$$\left(\widetilde{\frac{d}{dx}Q_k}\right) = 0,$$

and the first order coefficient is

$$\left[\frac{d}{d(p_1)_x}\widetilde{\left(\frac{d}{dx}Q_k\right)}\right] = \left[\frac{d}{dx}\widetilde{\left(\frac{\partial}{\partial(p_1)_x}Q_k\right)}\right] + \widetilde{\left(\frac{\partial Q_k}{\partial p_1}\right)}$$
$$= \left[\frac{d}{dx}\widetilde{\left(\frac{\partial}{\partial(p_1)_x}Q_k\right)}\right] + \frac{\partial \tilde{Q}_k}{\partial p_1} - \sum_{i=2}^{m_k-n_s+1}\left(\frac{\partial Q_k}{\partial(p_1)^{(i)}}\widetilde{\frac{\partial(p_1)^{(i)}}{\partial p_1}}\right), \tag{1.14}$$

where the only non zero term is $\frac{\partial \tilde{Q}_k}{\partial p_1}$, by virtue of Lemma 1.1. Then we obtain the power series expansion of $\frac{d}{dx}Q_k$ up to the first order:

$$\frac{\partial \tilde{Q}_k}{\partial p_1}\left((p_1)_x + \frac{\partial H}{\partial q_1}\right)$$

this, compared with the left–hand side of eq. (1.10), gives the requested relation.

Q.E.D.

Chapter 2
Scaling reductions of evolutionary systems: Hamiltonian formulation

2.1. Scaling symmetries and their Hamiltonians

In the Introduction we presented the reduction from a PDE to an ODE with the Painlevé property. It is based on the existence of the self–similar solutions. In other words this means that certain solutions are invariant under the so called scaling transformations. Hence we reformulate the problem in terms of symmetries: a self similar solution is a fixed point for a given symmetry. We will call such kind of symmetry *scaling symmetries*. Obviously, a scaling symmetry does depends upon both the space variable and the time (or the "times", in the case of a hierarchy of PDE).

We will develop in detail, in Chapter 4, some example; here we briefly sketch the idea for the mKdV equation

$$u_t = 6u^2 u_x - u_{xxx}.$$

It admits the self–similar solution

$$u(x,t) = (3t)^{-\frac{1}{3}} w(z), \qquad z = x(3t)^{-\frac{1}{3}}.$$

Such a solution satisfies the evolution equation

$$xu_x + 3tu_t + u = 0,$$

which can be read as a Hamiltonian stationary flow

$$u_s = xu_x + 3tu_t + u = \frac{d}{dx}\frac{\delta I}{\delta u(x)} = 0$$

for $I = \int(\frac{3}{2}t(u^4 + u_x^2) + \frac{u^2 x}{2})dx$. This means that the set of self–similar solutions coincides with the set $\mathfrak{S}$ of the fixed points for the scaling symmetry I.

One may ask what happens when reducing the initial PDE on $\mathfrak{S}$. The case is analogue to the stationary flow reduction, apart from the time–dependence. In fact, one may still reduce the initial t–flow on $\mathfrak{S}$, where it is represented by a non-autonomous Hamiltonian system. In the following we will give the proof of this fact, together with the prescriptions to construct the time–dependent Hamiltonian function.

Notice that, on $\mathfrak{S}$, it is possible to choose a particular system of coordinates, the scaling coordinates, in which the reduced flow is an ODE of Painlevé type; in particular, for the example of mKdV, it coincides with Painlevé II. We will discuss these and other examples in Chapter 4.

In this Chapter we extend the Bogoyavlenskii–Novikov scheme to finite dimensional invariant submanifolds specified by scaling symmetries of the hierarchy. Of course any of the flows is a symmetry but there are more general symmetries. They can be represented as linear combinations of the flows of the hierarchy, with coefficients depending explicitly on x and on the time variables. These more general finite dimensional manifolds are invariant w.r.t. only some part of the equations of the hierarchy. We will show that the restriction of these flows on the finite dimensional manifolds admits a natural description as Hamiltonian system with time dependent Hamiltonian.
First of all we recall some facts about the time dependent symmetries (see [GOS], [SM]).

Given an evolution equation

$$u_t = f(x, u_x, \ldots, u^{(j)})$$

a local symmetry is defined by

$$u_s = g(x, t, u_x, \ldots, u^{(k)}) = 0$$

under the commutativity condition

$$\frac{dg}{dt} = \frac{df}{ds}$$

If the evolution equation has the Hamiltonian form

$$\frac{du}{dt} = \frac{d}{dx}\frac{\delta I}{\delta u(x)},$$

for the functional $I_1 = \int L_1(x, u(x), u_x(x), \ldots, u^{(n)}(x))dx$ and also

$$u_s = g(x, t, u_x, \ldots, u^{(m)}) = \frac{d}{dx}\frac{\delta I_{(s)}}{\delta u(x)}$$

then the condition on g to be a symmetry is

$$\{I_1, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} = 0.$$

If one considers a part of a hierarchy, the local symmetries may depend on more than one time. Explicitly for the KdV hierarchy there exist three types of local symmetries: the shift along any of the times

$$u_s = u_{t_m},$$

the Galilean symmetry

$$u_s = 1 + 6t_1 u_x + 10t_2 u_{t_1} + \ldots + 2(2m+1)t_m u_{t_{m-1}}$$

and the scaling symmetry

$$u_s = 2u + xu_x + 3t_1 u_{t_1} + 5t_2 u_{t_2} + \ldots + (2m+1)t_m u_{t_m}.$$

They commute with the x–flow and with the first m time–flows of the hierarchy. In a more general way, we consider any linear combination of these symmetries, with constant coefficients. This corresponds to a linear combination of the flows of the hierarchy, with coefficients depending on x and on the first m times.

In terms of Hamiltonian functions this means that we are considering a symmetry defined by:

$$\frac{du}{ds} = \{u(x), I\} = 0,$$

where $I = \int L(u, u_x, \ldots, u^{(n_s)}, x, t_1, t_2, \ldots, t_s)dx$ and L is a linear combination of the densities L_i, with coefficients depending on x and on the times:

$$L = \sum_{i=0}^{s} c_i(x, t_1, t_2, \ldots, t_s) L_i(x, u(x), u_x(x), \ldots, u^{(n_i)}(x))$$

We call such a kind of symmetry a *scaling*. It defines a $2n_i$–dimensional submanifold $\mathfrak{S}$ in the space of the solutions of the system.

2.2 Self–similar solutions of evolutionary systems and time–dependent Hamiltonian formalism

In this Section we generalize Theorem 1.1 to the case of scaling symmetries. The procedure scheme is similar, indeed Lemma 1.1 and 1.2 hold also in the time dependent case. The main difference consists, obviously, in the fact that the Hamiltonian functions of the reduced flows do not commute, the commutativity being replaced by a zero–curvature relation, which takes into account the time dependence. Our results are formalized in the following Theorem 2.1. We postpone the comments to the end of the chapter.

Theorem 2.1: *If the generic k–th element of a hierarchy of commuting evolution equations:*

$$\frac{du}{dt_k} = F_k(x, u(x), u_x(x), \ldots, u^{(m_k)}(x)),$$

admits a nondegenerate scaling symmetry

$$I = \int L(u, u_x, \ldots, u^{(n_s)}, x, t_1, t_2, \ldots, t_s)dx,$$

then it is Hamiltonian on the manifold $\mathfrak{S}$ of the stationary points of the symmetry with time dependent Hamiltonian function $(-\tilde{Q}_k)$, which is the reduction on $\mathfrak{S}$ of

$$Q_k = \Lambda_k - \sum_{i=1}^{n_s} p_i \frac{dq_i}{dt_k}, \tag{2.1}$$

where p_i, q_i are the canonical coordinates on $\mathfrak{S}$, expressed in terms of $u, u_x, \ldots, u^{(2n_s-1)}$, and

$$\frac{dL}{dt_k} = \frac{d\Lambda_k}{dx}. \tag{2.2}$$

Proof: The proof is articulated in three steps: we first describe the submanifold $\mathfrak{S}$ of stationary points of the symmetry I, where we introduce a system of canonical coordinates; then we deduce, on $\mathfrak{S}$, a *zero–curvature* equation for $(-\tilde{Q}_k)$ and the Hamiltonian function $\tilde{H}$ of the reduced x–flow. Finally we prove that the restricted t_k–flow is Hamiltonian on $\mathfrak{S}$,with Hamiltonian function $(-\tilde{Q}_k)$.

1) The manifold $\mathfrak{S}$ is the $2n_s$–dimensional manifold of the solutions of the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = 0. \tag{2.3}$$

It is invariant under the k–th flow and it naturally carries a system of canonical coordinates:

$$q_i = u^{(i-1)}, \qquad i = 1, 2, \ldots, n_s \tag{2.4a}$$

$$p_i = \frac{\delta I}{\delta u^{(i)}}, \tag{2.4b}$$

obtained via generalized Lagrange transform (here we suppose that the generalized Lagrangian L is nondegenerate). Observe that the p_i's depend on x and on the times $t_1, t_2, .., t_s$. From now on, we will indicate simply them with t.

Inverting relations (2.4), one can express the derivatives $u, u_x, \ldots, u^{(2n_s-1)}$ in terms of the canonical coordinates p_i and q_i, x and t; explicitly:

$$\begin{cases} u^{(n_s)} = (q_{n_s})_x = g_1(x, t, q_1, \ldots, q_{n_s}, p_{n_s}) \\ u^{(n_s+1)} = g_2(x, t, q_1, \ldots, q_{n_s}, p_{n_s}, p_{n_s-1}) \\ \ldots\ldots \\ u^{(2n_s-1)} = g_n(x, t, q_1, \ldots, q_{n_s}, p_{n_s}, \ldots, p_1). \end{cases}$$

Observe that (2.4) gives the identities:

$$\begin{cases} (p_1)_x + \frac{\partial H}{\partial q_1} \equiv -\frac{\delta I}{\delta u} \\ (p_i)_x + \frac{\partial H}{\partial q_i} \equiv 0, \qquad i > 1 \\ (q_i)_x - \frac{\partial H}{\partial p_i} \equiv 0, \end{cases} \tag{2.5}$$

where H is the generalized Legendre transform of L:

$$-L + \sum_{i=1}^{n_s} \frac{\delta I}{\delta u^{(i)}} u^{(i)}$$

which, in terms of the canonical coordinates takes the form:

$$H = -L + \sum_{1}^{n_s} p_i \frac{dq_i}{dx}.$$

The first among identities (2.5) allows us to explicitly express the higher derivatives $u^{(m)}$ for $m \geq 2n_s$ in terms of x,t, p_i, q_i and $p_1^{(l)}$ with $l = 1, \dots, m - 2n_s + 1$:

$$\begin{cases} u^{(2n_s)} = g_{n+1}(x, t, q_1, \dots, q_{n_s}, p_{n_s}, \dots, p_1, (p_1)_x) \\ \dots\dots \\ u^{(m)} = g_{m-n_s+1}(x, t, q_1, \dots, q_{n_s}, p_{n_s}, \dots, p_1, \dots, (p_1)^{(m-2n+1)}). \end{cases}$$

On $\mathfrak{S}$ it reduces to $(p_1)_x + \frac{\partial H}{\partial q_1} \equiv 0$, and the system (2.5) is a canonical Hamiltonian system, with Hamiltonian function H, giving the reduced x–flow.

Now we will show that also the t_k–flow reduces on $\mathfrak{S}$ with Hamiltonian function $(-\tilde{Q}_k)$.

Indeed we first observe that $Q_k^{(a)}$ is a function of x,t, u and its x–derivatives up to the order $(m_k + n_s)$, then it can be rewritten in terms of x, t, (p_i, q_i) and $p_1^{(l)}$ up to the order $l = m_k - n_s + 1$.

We denote by $\tilde{f}$ a function $f(x, t, u(x), \dots, u(x)^{(j)})$ reduced on $\mathfrak{S}$; notice that, if $j \geq 2n_s$, then the reduction can be done using the relation

$$(p_1)_x = -\frac{\partial H^{(a)}}{\partial q_1}. \tag{2.6}$$

Then $\tilde{f}$ does explicitly depend only on the p_i and q_i, for $i = 1, \dots, n_s$ and on the time t^k. In fact, differentiating (2.6) one obtains the derivatives $p_1^{(l)}$ in terms of the canonical coordinates (p_i, q_i).

2) We consider the derivative

$$\begin{aligned} \frac{dL}{dt_k} &= \frac{\partial L}{\partial t_k} + \sum_{i=1}^{n_s} \frac{\partial L}{\partial q_i}\frac{dq_i}{dt_k} + \sum_{i=1}^{n_s} \frac{\partial L}{\partial p_i}\frac{dp_i}{dt_k} = \\ &= -\frac{\partial H}{\partial t_k} - \sum_{i=1}^{n_s} \frac{\partial H}{\partial q_i}\frac{dq_i}{dt_k} + \sum_{i=1}^{n_s} p_i \frac{d^2 q_i}{dx dt_k}, \end{aligned} \tag{2.7}$$

analogously, noticing that Λ_k depends on x,t,(p_i, q_i) and $p_1^{(l)}$ up to the order $l = m_k - n_s + 1$, we have:

$$\begin{aligned} \frac{d\Lambda_k}{dx} &= \frac{\partial Q_k}{\partial x} + \sum_{i=1}^{n_s} \frac{\partial Q_k}{\partial q_i}\frac{dq_i}{dx} + \sum_{i=1}^{n_s} \frac{\partial Q_k}{\partial p_i}\frac{dp_i}{dx} + \sum_{i=1}^{m_k-n_s+1} \frac{\partial Q_k}{\partial (p_1)^{(i)}}\frac{d}{dx}(p_1)^{(i)} + \\ &+ \sum_{i=1}^{n_s} \frac{dp_i}{dx}\frac{dq_i}{dt_k} + \sum_{i=1}^{n_s} p_i \frac{d^2 q_i}{dx dt_k} = \\ &= \frac{\partial Q_k}{\partial x} + \sum_{i=1}^{n_s} \frac{\partial Q_k}{\partial q_i}\frac{\partial H}{\partial p_i} - \sum_{i=2}^{n_s} \frac{\partial Q_k}{\partial p_i}\frac{\partial H}{\partial q_i} + \sum_{i=1}^{m_k-n_s+1} \frac{\partial Q_k}{\partial (p_1)^{(i)}}\frac{d}{dx}(p_1)^{(i)} + \\ &- \sum_{i=2}^{n_s} \frac{\partial H}{\partial q_i}\frac{dq_i}{dt_k} + \sum_{i=1}^{n_s} p_i \frac{d^2 q_i}{dx dt_k} + \frac{dq_1}{dt_k}(p_1)_x + \frac{\partial Q_k}{\partial p_1}(p_1)_x. \end{aligned} \tag{2.8}$$

Then equation (2.2) gives:

$$\frac{\partial H}{\partial t_k}+\frac{\partial Q_k}{\partial x}+\sum_{i=1}^{n_s}\frac{\partial Q_k}{\partial q_i}\frac{\partial H}{\partial p_i}-\sum_{i=2}^{n_s}\frac{\partial Q_k}{\partial p_i}\frac{\partial H}{\partial q_i}+\frac{\partial Q_k}{\partial p_1}(p_1)_x+$$
$$+\sum_{i=1}^{m_k-n_s+1}\frac{\partial Q_k}{\partial (p_1)^{(i)}}\frac{d}{dx}(p_1)^{(i)}+\frac{dq_1}{dt_k}(p_1)_x+\frac{\partial H}{\partial q_1}\frac{dq_1}{dt_k}=0,$$

which can be rewritten as

$$\frac{\partial H}{\partial t_k}+\frac{dq_1}{dt_k}\left((p_1)_x+\frac{\partial H}{\partial q_1}\right)=-\frac{d}{dx}Q_k \tag{2.9}$$

We observe that Lemma 1.1 holds also in the time–dependent case, i.e. the relation

$$\left(\widetilde{\frac{\partial}{\partial (p_1)^{(j)}}Q_k}\right)=0 \qquad \forall j>1. \tag{2.10}$$

holds. Hence, on the submanifold $\mathfrak{S}$, eq. (2.9) reduces to:

$$\frac{\partial H}{\partial t_k}+\frac{\partial \tilde{Q}_k}{\partial x}+\sum_{i=1}^{n_s}\frac{\partial \tilde{Q}_k}{\partial q_i}\frac{\partial H}{\partial p_i}-\sum_{i=1}^{n_s}\frac{\partial \tilde{Q}_k}{\partial p_i}\frac{\partial H}{\partial q_i}=0$$

This is a *zero–curvature* equation:

$$\{(-\tilde{Q}_k),H\}+\frac{\partial(-\tilde{Q}_k)}{\partial x}-\frac{\partial H}{\partial t}=0.$$

3) Lemma 1.2 can be generalized to the time–dependent case and gives

$$\frac{dq_1}{dt_k}=-\frac{\partial \tilde{Q}_k}{\partial p_1}. \tag{2.11}$$

Indeed, the expansion of $\frac{d}{dx}Q_k$ in powers of $(p_1)_x$ near the point $\frac{\partial \tilde{H}}{\partial q_1}$ has still the expression

$$\left(\widetilde{\frac{d}{dx}Q_k}\right)+\left[\frac{d}{d(p_1)_x}\widetilde{\left(\frac{d}{dx}Q_k\right)}\right]\left((p_1)_x+\frac{\partial H}{\partial q_1}\right)+\Theta\left((p_1)_x+\frac{\partial H}{\partial q_1}\right)^2.$$

In this case the zero order term is not zero, but

$$\left(\widetilde{\frac{d}{dx}Q_k}\right)=-\frac{\partial H}{\partial t_k}$$

by virtue of the zero–curvature equation. The first order coefficient is

$$\left[\frac{d}{d(p_1)_x}\widetilde{\left(\frac{d}{dx}Q_k\right)}\right]=\left[\frac{d}{dx}\left(\widetilde{\frac{\partial}{\partial (p_1)_x}Q_k}\right)\right]+\left(\widetilde{\frac{\partial Q_k}{\partial p_1}}\right)$$
$$=\left[\frac{d}{dx}\left(\widetilde{\frac{\partial}{\partial (p_1)_x}Q_k}\right)\right]+\frac{\partial \tilde{Q}_k}{\partial p_1}-\sum_{i=2}^{m_k-n_s+1}\left(\widetilde{\frac{\partial Q_k}{\partial (p_1)^{(i)}}\frac{\partial (p_1)^{(i)}}{\partial p_1}}\right), \tag{2.12}$$

where the only non zero term is $\frac{\partial \tilde{Q}_k}{\partial p_1}$, by virtue of Lemma 1.1. Then we obtain the power series expansion of $\frac{d}{dx}Q_k$ up to the first order:

$$-\frac{\partial H}{\partial t_k}+\frac{\partial \tilde{Q}_k}{\partial p_1}\left((p_1)_x+\frac{\partial H}{\partial q_1}\right).$$

This, compared with the left–hand side of eq. (2.9), gives the relation (2.11).

For simplicity, here and in the following we omit the " *hat*" sign: Q_k will indicate the reduced function on $\mathfrak{S}$. Now, we assume that $\frac{dq_i}{dt_k} = -\frac{\partial Q_k}{\partial p_i} = -\{q_i, Q_k\}$ and we prove inductively that the same relation holds for q_{i+1}. The scheme of the procedure is the same as in Theorem 1.1, the only differences being the contributions of the partial derivatives in t and x. Indeed,

$$\frac{dq_{i+1}}{dt_k} = (\frac{dq_i}{dt_k})_x = -\frac{d}{dx}\{q_i, Q_k\} = -\{\{q_i, Q_k\}, H\} - \{q_i, \frac{\partial Q_k}{\partial x}\}.$$

Using the Jacobi identity and the zero–curvature relation we get

$$\frac{dq_{i+1}}{dt_k} = -\{\frac{\partial H}{\partial t_k}, q_i\} - \{q_{i+1}, Q_k\}, \qquad i = 1, 2, ..., n_s - 1.$$

Here the term $\{\frac{\partial H}{\partial t_k}, q_i\}$ is zero for every $i \neq n_s$ since $\widetilde{(\frac{\partial L}{\partial t_k})} = -\frac{\partial H}{\partial t_k}$. Indeed L depends on u and on the derivatives of u up to the order n_s. This means that, restricted on $\mathfrak{S}$, it depends on $q_1, q_2, \ldots, q_{n_s+1}$. Then, there is no dependence on the p_i for $i \neq n_s$. Finally we get

$$\begin{aligned}\{\frac{\partial H}{\partial t_k}, q_{n_s}\} =& \{\widetilde{(\frac{\partial L}{\partial t_k})}, q_{n_s}\} \neq 0,\\ \{\frac{\partial H}{\partial t_k}, q_i\} =& 0 \qquad i < n_s.\end{aligned}$$

Hence we have proved that

$$\frac{dq_i}{dt_k} = -\{q_i, Q_k\}, \qquad i = 1, 2, ..., n_s.$$

Now we prove that $\frac{dp_i}{dt_k} = -\{p_i, Q_k\}$ by induction, starting from p_{n_s}.
This comes from the commutativity of the flows

$$\frac{d}{dt_k}\frac{d}{dx}q_{n_s} = \frac{d}{dx}\frac{d}{dt_k}q_{n_s};$$

explicitly:

$$\begin{aligned}\frac{d}{dt_k}\left(\frac{d}{dx}q_{n_s}\right) =& \frac{d}{dt_k}\left(\frac{\partial H}{\partial p_{n_s}}\right) =\\ =& \frac{\partial}{\partial t_k}\frac{\partial H}{\partial p_{n_s}} - \sum_{i=1}^{n_s}\frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{d}{dt_k}p_{n_s} =\\ =& \{q_{n_s}, \frac{\partial H}{\partial t_k}\} - \sum_{i=1}^{n_s}\frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{d}{dt_k}p_{n_s}.\end{aligned}$$

On the other hand, using the Jacobi identity and the *zero–curvature* equation, one can write

$$\begin{aligned}\frac{d}{dx}\left(\frac{dq_{n_s}}{dt_k}\right) =& -\frac{\partial}{\partial x}\frac{dQ_k}{dt_k} - \{\{q_{n_s}, Q_k\}, H\} =\\ =& -\{q_{n_s}, \{Q_k, H\}\} - \{\{q_{n_s}, H\}Q_k\} - \{q_{n_s}, \frac{\partial Q_k}{\partial x}\} =\\ =& \{q_{n_s}, \frac{\partial H}{\partial t_k}\} - \{\frac{\partial H}{\partial p_{n_s}}, Q_k\} =\\ =& \{q_{n_s}, \frac{\partial H}{\partial t_k}\} - \sum_{i=1}^{n_s}\frac{\partial H}{\partial p_{n_s}\partial q_i}\frac{\partial Q_k}{\partial p_i} + \frac{\partial^2 H}{\partial p_{n_s}^2}\frac{\partial Q_k}{\partial q_{n_s}}.\end{aligned}$$

Comparing the two expressions and noticing that $\frac{\partial^2 H}{\partial p_{n_s}^2} \neq 0$ because of nondegeneracy, we get

$$\frac{dp_{n_s}}{dt_k} = -\{p_{n_s}, Q_k\} = \frac{\partial Q_k}{\partial q_{n_s}}.$$

- Now we suppose that $\frac{dp_i}{dt_k} = \{p_i, Q_k\}$ and we deduce the same for p_{i-1}. Indeed,

$$\begin{aligned}\frac{d}{dx}\left(\frac{dp_i}{dt_k}\right) =& \{\frac{\partial Q_k}{\partial q_i}, H\} + \frac{\partial}{\partial x}\frac{\partial Q_k}{\partial q_i} = \\ =& -\{\{p_i, Q_k\}, H\} - \{p_i, \frac{\partial Q_k}{\partial x}\} = \\ =& -\{\{p_i, H\}, Q_k\} + \{p_i, \frac{\partial H}{\partial t_k}\} = \\ =& \{p_i, \frac{\partial H}{\partial t_k}\} + \sum_{j=1}^{n_s-1} \frac{\partial^2 H}{\partial q_i \partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{\partial Q_k}{\partial q_{n_s}} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{\partial Q_k}{\partial q_{i-1}}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt_k}\left(\frac{dp_i}{dx}\right) =& -\frac{d}{dt_k}\left(\frac{\partial H}{\partial q_i}\right) = \\ =& -\frac{\partial}{\partial t_k}\frac{\partial H}{\partial q_i} - \sum_{j=1}^{n_s-1} \frac{\partial^2 H}{\partial q_i \partial q_j}\frac{dq_j}{dt_k} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{dp_{n_s}}{dt_k} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{dp_{i-1}}{dt_k} = \\ =& \{p_i, \frac{\partial H}{\partial t_k}\} + \sum_{j=1}^{n_s-1} \frac{\partial^2 H}{\partial q_i \partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial^2 H}{\partial q_i \partial p_{n_s}}\frac{\partial Q_k}{\partial q_{n_s}} - \frac{\partial^2 H}{\partial q_i \partial p_{i-1}}\frac{dp_{i-1}}{dt_k},\end{aligned}$$

where $\frac{\partial^2 H}{\partial q_i \partial p_{i-1}} = 1$. Comparing the two expressions we get $\frac{dp_{i-1}}{dt_k} = \frac{\partial Q_k}{\partial q_{i-1}}$; hence it follows that

$$\frac{dp_i}{dt_k} = \frac{\partial Q_k}{\partial q_i} = -\{p_i, Q_k\}, \qquad i = 1, 2, ..., n_s$$

Q.E.D.

2.2 Concluding remarks

Summarizing, we start from a partial differential equation of order m on the functional space $\mathfrak{M}$, describing the evolution of the function $u(x)$ in the time t and a first integral $I = \int L dx$, where the density L is of order n in x and depends on t. The scaling symmetry I satisfies the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = 0. \tag{2.13}$$

This is the Lagrangian system, with Lagrangian L, giving the evolution evolution of $u(x)$ in x. It is an ordinary differential equation of order $2n$. If L is nondegenerate, the space of solutions is a $2n$ dimensional manifold $\mathfrak{S}$, which naturally carries a system of canonical coordinates. In these coordinates, system (2.13) is Hamiltonian, with Hamiltonian function H, obtained from L via Legendre transform:

$$H = -L + \sum_{i=1}^{n} p_i \frac{dq_i}{dx}.$$

Following the scheme of [BN], one may reduce on $\mathfrak{S}$ also the equation of the evolution in t; it results to be Hamiltonian and the Hamiltonian function can be constructed as in (1.5):

$$\left(\frac{\delta I}{\delta u(x)}\right)\frac{du(x)}{dt} \equiv \frac{d}{dx}Q.$$

Indeed, the definition of a first integral $\frac{dI}{dt} = 0$ implies that $\frac{\delta I}{\delta u(x)}\frac{du(x)}{dt}$ must be a total derivative.

But also, as Theorem 2.1 states, one may adopt a slightly different perspective: from the fact that I is a first integral, one deduces that $\frac{dL}{dt}$ must be the total derivative in x of a functional Λ. Obviously there exists a relation between Q and Λ, precisely

$$-Q = -\Lambda + \sum_{i=1}^{n} p_i \frac{dq_i}{dt}.$$

We will take it as definition of Q. This because it looks very similar to the definition of the Hamiltonian function H of the x–flow.
Here a symmetry in x and t seems to show up: one could be tempted to read the definition of Q as a Legendre transform and hence to read Λ as the Lagrangian of the t–flow. But this is not completely true: indeed the coordinates q_i and p_i are obtained from the Lagrangian L, so that they are not, a priori, good coordinates for Λ. In the next chapter we will perform a change of coordinates on $\mathfrak{S}$, in order to read Λ as Lagrangian function.

Chapter 3

Scaling reductions of evolutionary systems: Lagrangian formulation

3.1 General framework

The basic idea is to develop a reduction method dealing on the same footing with all the times of a hierarchy, included the "time" x. The starting point is an evolution PDE

$$u(x)_t = F(x, u(x), u_x(x), \ldots.., u^{(m)}(x)), \tag{3.1}$$

in the space $\mathfrak{M}$ described in Section 0.2. The first step of our construction is to read u as a function of x and t and to consider equation (3.1) as a definition of $u^{(m)}(x,t)$ in terms of $x, u(x,t), u_x(x,t), \ldots.., u^{(m-1)}(x,t)$ and $u_t(x,t)$.

This corresponds to consider as "coordinates" in $\mathfrak{M}$, instead of $u(x,t)$ and its derivatives in x:

$$u, u_x, u_{xx}, \ldots$$

(here and in the following u indicate the function $u(x,t)$), the system

$$u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt}, \ldots, u_t^{(m-1)}, u_{tt}, \ldots$$

By virtue of the reversibility of (3.1) in $u^{(m)}(x,t)$ it is possible to perform this "change of variables". If one introduce the vector

$$\bar{u} = (u, u_x, \ldots, u^{(m-1)}),$$

the new system of "coordinates" in $\mathfrak{M}$ is given by $\bar{u}(x,t)$ and its derivatives in t:

$$\bar{u}, \bar{u}_t, \bar{u}_{tt}, \ldots$$

At this point one takes a first integral of eq (3.1), i.e. a functional

$$I = \int L\Big(x, t, u(x), u_x(x), \ldots, u^{(n)}(x)\Big)\, dx \tag{3.2}$$

in the space $\mathfrak{M}$, such that

$$\frac{\delta I}{\delta u(x)} = 0. \tag{3.3}$$

This Euler–Lagrange equation defines a finite dimensional manifold $\mathfrak{S}$, i.e. the set of the fixed points of I. Indeed the Euler–Lagrange equation (3.3) is an ODE of order $2n$, so that the space of the solutions is a $2n$–dimensional manifold; $\mathfrak{S}$ is modeled on this space, having as coordinates certain combinations of the initial values, i.e. of the first $(2n-1)$ x–derivatives of $u(x)$ evaluated at x_0.

In Lemma 3.2 below, we rewrite the definition of the manifold $\mathfrak{S}$ in terms of $\bar{u}(t), \bar{u}_t(t), \ldots$ and of a functional

$$J = \int \Lambda\Big(x, t, \bar{u}(t), \bar{u}_t(t), \ldots, \bar{u}^{(\beta)}(t)\Big)\, dt,$$

where Λ can be calculated from L (see eq. (3.4)), and the order β of derivation in t depends on the ratio between m and n, as we will show in detail in Section 3.3.
In Theorem 3.1 we will prove that $\Lambda\Big(x, t, \bar{u}(t), \bar{u}_t(t), \dots, \bar{u}^{(\beta)}(t)\Big)$ is the generalized Lagrangian for the t–flow reduced on $\mathfrak{S}$. Indeed equation (3.1) can be rewritten in form of a Euler–Lagrange equation:

$$\frac{\delta J}{\delta \bar{u}(t)} = 0,$$

for the vector

$$\frac{\delta J}{\delta \bar{u}(t)} = \left(\frac{\delta J}{\delta u(t)}, \frac{\delta J}{\delta u_x(t)}, \dots, \frac{\delta J}{\delta u^{(m-1)}(t)}\right),$$

where

$$\frac{\delta J}{\delta u^{(i)}(t)} = \frac{\partial \Lambda}{\partial u^{(i,0)}} + \sum (-1)^{\alpha} \frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial \Lambda}{\partial u^{(i,\alpha)}}.$$

In the multiindex (i, α) the Latin character indicates the order in the x–derivative, the Greek indicate the order in the t–derivative.
Explicitly, equation (3.1) reads

$$\begin{aligned}
\frac{\partial \Lambda}{\partial u} + \sum (-1)^{\alpha} \frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial \Lambda}{\partial u^{(\alpha)}} &= 0 \\
\frac{\partial \Lambda}{\partial u_x} + \sum (-1)^{\alpha} \frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial \Lambda}{\partial u_x^{(\alpha)}} &= 0 \\
&\vdots \\
\frac{\partial \Lambda}{\partial u^{(m-1)}} + \sum (-1)^{\alpha} \frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial \Lambda}{\partial u^{(m-1,\alpha)}} &= 0.
\end{aligned}$$

We will formalize these facts in the following Theorem 3.1; here we give an idea of the proof, ignoring all the calculations, that we will concentrate in Lemma 3.1 and 3.2, the proof of which is postponed in Appendix 3.A.

The proof is performed in four steps: firstly we define the new system of "coordinates" in $\mathfrak{M}$, and we give some useful relations between the new and the old "coordinates". As a second step we rewrite the Lagrangian density $L\big(u(x)\big)$ in terms of the new "coordinates" and we construct, starting from $\hat{L}\big(\bar{u}(t)\big)$, the Lagrangian density $\hat{\Lambda}\big(\bar{u}(t)\big)$.

The third step consists in recovering the relation between performing the variation of L (in x) and of $\hat{\Lambda}$ (in t). The most relevant relation is that given in Lemma 3.1. This relation is necessary to rewrite the Euler–Lagrange equation (3.2), defining $\mathfrak{S}$, as a condition on $\hat{\Lambda}$. The explicit form of this condition is given in Lemma 3.2.

Finally we prove that, under this condition, i.e. after performing the reduction on $\mathfrak{S}$, the starting evolution equation (3.1), reads as an Euler–Lagrange equation for $\hat{\Lambda}$.

The method of Hamiltonian reduction described in Chapter 2 allows us to put a canonical system of coordinates $\{p_i, q_i\}$ on $\mathfrak{S}$ (see formula (2.4)). These coordinates are obtained from L via generalized Lagrange transform, so that they are, in a certain sense, adapted to the x–flow. This means that in these coordinates the reduced x–flow is a Hamiltonian system. Theorem 2.1 also gives the explicit form of the Hamiltonian function

$$H = -L + \sum_i p_i (q_i)_x.$$

The method of Lagrangian reduction which we describe in this Chapter, still allows us to define a system of canonical coordinates: we will call it $\{\tilde{p}_i, \tilde{q}_i\}$. These coordinates are obtained from $\hat{\Lambda}$, i.e. they are adapted to the t–flow; in fact we will prove (see Section 3.3) that, in these coordinates, the reduced t–flow is a Hamiltonian system, with Hamiltonian function

$$-\hat{Q} = -\hat{\Lambda} + \sum_i \tilde{p}_i(\tilde{q}_i)_t.$$

When rewritten in terms of $\{p_i, q_i\}$, the Hamiltonian $\hat{Q}$ coincides with the Hamiltonian function Q constructed by Bogoyavlenskii and Novikov.

In this sense the alternative definition of Q given by us in Theorem 2.1:

$$-Q = -\Lambda + \sum_i p_i(q_i)_t,$$

is a Legendre transformation, if one uses the right system of canonical coordinates (see below).

3.2 Lagrangian reduction

Theorem 3.1: *If the generic k-th element of a hierarchy of commuting evolution equations:*

$$\frac{du}{dt_k} = F_k(u(x), u_x(x), \ldots\ldots, u^{(m_k)}(x)), \tag{3.3}$$

admits a first integral

$$I = \int L(x, t_k, u(x), u_x(x), \ldots, u^{(n)}(x))dx,$$

then it reduces on the manifold $\mathfrak{S}$ of the stationary points of the first integral to a Lagrangian motion in t_k, for the time dependent Lagrangian function Λ_k, defined by:

$$\frac{dL}{dt_k} = \frac{d\Lambda_k}{dx}. \tag{3.4}$$

Proof: (in the following we will use the shortened form $t_k = t$, the same holds for every time of the hierarchy and we consider the case $m \leq n < 2m$. The same holds in the case $(\alpha - 1)m \leq n < \alpha m$, as we will show in Section 3.3). We prove the theorem in four steps:

1. *Change of "coordinates"*: Let us suppose that the evolutionary equation (3.3) depends on $u(x)$ and on its x–derivatives up to finite order m, and that this equation is invertible in $u^{(m)}$. In this case we can read (3.3) as a definition of $u^{(m)}$ in terms of $u, u_x, \ldots, u^{(m-1)}$ and u_t:

$$u^{(m)} = f_0\left(u, u_x, \ldots, u^{(m-1)}, u_t\right). \tag{3.5}$$

Differentiating eq. (3.5) in x one obtains all the x–derivatives of u of order greater then m in terms of $u, u_x, \ldots, u^{(m-1)}$ and their t–derivatives:

$$\begin{cases} u^{(m+1)} = f_1\big(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt} \ldots, u_{xt}\big) \\ \vdots \\ u^{(2m-1)} = f_{m-1}\big(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt} \ldots, u_t^{(m-1)}\big) \\ u^{(2m)} = f_m\big(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt} \ldots, u_t^{(m-1)}, u_{tt}\big) \\ \vdots \\ u^{(m+n)} = f_n\big(u, u_x, \ldots, u^{(m-1)}, u_t, u_{xt} \ldots, u_t^{(m-1)}, u_{tt}, u_{xtt}, \ldots, u_{tt}^{(n-m)}\big). \end{cases}$$

Explicitly, the first relation has the form

$$u^{(m+1)} = \frac{\partial u^{(m)}}{\partial x} + \frac{\partial u^{(m)}}{\partial u} u_x + \ldots + \frac{\partial u^{(m)}}{\partial u^{(m-1)}} u^{(m)} + \frac{\partial u^{(m)}}{\partial u_t} u_{xt},$$

and in general, using the multiindex notation introduced in Section 3.1,

$$u^{(m+j)} = \frac{\partial u^{(m+j-1)}}{\partial x} + \sum_{k=0}^{m-1} \sum_{\beta=0}^{\alpha} \frac{\partial u^{(m+j-1)}}{\partial u^{(k,\beta)}} u^{(k+1,\beta)},$$

where the higher order α in the t–derivative is fixed by $(\alpha - 1)m \leq j < \alpha m$.

This completes the construction of the map from

$$u, u_x, \ldots, u^{(m-1)}, u^{(m)}, \ldots, u^{(n)}, \ldots, u^{(2m-1)}, u^{(2m)}, \ldots\ldots$$

to the new system of "coordinates"

$$u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u_t^{(m-1)}, u_{tt}, u_{xtt}, \ldots, u_{tt}^{(n-m)}, \ldots\ldots$$

Here below we list some noteworthy relationships between the two system (they will be useful in the following):

$$u_t^{(m)} = \frac{\partial u^{(m)}}{\partial t} + \frac{\partial u^{(m)}}{\partial u} u_t + \ldots + \frac{\partial u^{(m)}}{\partial u^{(m-1)}} u_t^{(m-1)} + \frac{\partial u^{(m)}}{\partial u_t} u_{tt}, \tag{3.6a}$$

$$\frac{d}{dx}\left(\frac{\partial u^{(i)}}{\partial u_t}\right) = \frac{\partial u^{(i+1)}}{\partial u_t} - \frac{\partial u^{(i)}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u_t} \tag{3.6b}$$

$$\frac{d}{dx}\left(\frac{\partial u^{(i)}}{\partial u_t^{(k)}}\right) = \frac{\partial u^{(i+1)}}{\partial u_t^{(k)}} - \frac{\partial u^{(i)}}{\partial u_t^{(k-1)}} \tag{3.6c}$$

$$\frac{d}{dx}\left(\frac{\partial u^{(i)}}{\partial u^{(k)}}\right) = \frac{\partial u^{(i+1)}}{\partial u^{(k)}} - \frac{\partial u^{(i)}}{\partial u^{(k-1)}} - \frac{\partial u^{(i)}}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u^{(k)}} \tag{3.6d}$$

$$\frac{d}{dt}\left(\frac{\partial u^{(i)}}{\partial u_t}\right) = \frac{\partial u_t^{(i)}}{\partial u_t} - \frac{\partial u^{(i)}}{\partial u}. \tag{3.6e}$$

2. *Lagrangian densities*: The Lagrangian L defining the symmetry, depends on $u, u_x, \ldots, u^{(n)}$, so that its derivative $\frac{dL}{dt}$ depends on $u, u_x, \ldots, u^{(m+n)}$. In terms of the new "coordinates" one may rewrite L as

$$\hat{L}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u_t^{(m+n)}) \tag{3.7}$$

and

$$\begin{aligned}\widehat{\left(\frac{dL}{dt}\right)} = &\widehat{\left(\frac{\partial L}{\partial t}\right)} + \widehat{\left(\frac{\partial L}{\partial u}\right)} u_t + \ldots + \widehat{\left(\frac{\partial L}{\partial u^{(m-1)}}\right)} u_t^{(m-1)} + \\ &+ \widehat{\left(\frac{\partial L}{\partial u^{(m)}}\right)} u_t^{(m)}(u, \ldots, u_{tt}) + \widehat{\left(\frac{\partial L}{\partial u^{(m+1)}}\right)} u_t^{(m+1)}(u, \ldots, u_{tt}, u_{xtt}) + \ldots + \\ &+ \widehat{\left(\frac{\partial L}{\partial u^{(n)}}\right)} u_t^{(n)}(u, \ldots, u_{tt}, \ldots, u_{tt}^{(n-m)})\end{aligned} \tag{3.8a}$$

Of course, (3.8a) coincides with

$$\frac{d\hat{L}}{dt}=\frac{\partial\hat{L}}{\partial t}+\frac{\partial\hat{L}}{\partial u}u_t+\ldots+\frac{\partial\hat{L}}{\partial u^{(m-1)}}u_t^{(m-1)}+\frac{\partial\hat{L}}{\partial u_t}u_{tt}+\ldots+\frac{\partial\hat{L}}{\partial u_t^{(n-m)}}u_{tt}^{(n-m)}, \tag{3.8b}$$

where

$$\frac{\partial\hat{L}}{\partial t}=\left(\widehat{\frac{\partial L}{\partial t}}\right)+\sum_{k=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial t} \tag{3.9a}$$

$$\frac{\partial\hat{L}}{\partial u^{(i)}}=\left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right)+\sum_{k=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial u^{(i)}}\qquad i=0,\ldots,m-1 \tag{3.9b}$$

$$\frac{\partial\hat{L}}{\partial u_t^{(i)}}=\sum_{k=m+i}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial u_t^{(i)}}\qquad i=0,\ldots,m-1. \tag{3.9c}$$

From the fact that I is a first integral, it follows that there exists a functional

$$\hat{\Lambda}(x,t,u,u_x,\ldots,u^{(m-1)},u_t,\ldots,u_t^{(m-1)},u_{tt},\ldots,u_{tt}^{(n-m-1)}),$$

such that

$$\frac{d\hat{L}}{dt}=\frac{d\hat{\Lambda}}{dx},$$

where

$$\begin{aligned}\frac{d\hat{\Lambda}}{dx}&=\frac{\partial\hat{\Lambda}}{\partial x}+\frac{\partial\hat{\Lambda}}{\partial u}u_x+\ldots+\frac{\partial\hat{\Lambda}}{\partial u^{(m-2)}}u^{(m-1)}+\\&+\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}}u^{(m)}(u,u_x,\ldots,u^{(m-1)},u_{tt})+\frac{\partial\hat{\Lambda}}{\partial u_t}u_{xt}+\ldots+\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-2)}}u_t^{(m-1)}+\\&+\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}}u_t^{(m)}(u,\ldots,u_{tt})+\frac{\partial\hat{\Lambda}}{\partial u_{tt}}u_{xtt}+\ldots+\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(n-m-1)}}u_{tt}^{(n-m)}.\end{aligned} \tag{3.10}$$

3. *Variations*: Our aim is to reduce equation (3.3) on the space $\mathfrak{S}$ of the stationary points of $I=\int L dx$. This finite–dimensional manifold is defined by the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)}=0.$$

This is a variational equation in the old "coordinates" $u, u_x, \ldots$; how can we define the same manifold $\mathfrak{S}$ in terms of the new "coordinates"? We must express $\frac{\delta I}{\delta u(x)}$ in terms of Λ and its variations. To this end we recall that

$$\frac{\delta I}{\delta u(x)}=\frac{\partial L}{\partial u}+\sum(-1)^j\frac{d^j}{dt^j}\frac{\partial L}{\partial u^{(j)}}.$$

and we first express the terms

$$\frac{\delta L}{\delta u^{(j)}(x)}$$

for $j>0$ in terms of $\hat{\Lambda}$ and the the new "coordinates", namely:

Lemma 3.1: *The following recurrence relation holds:*

$$\frac{\partial\hat{\Lambda}}{\partial u_t^{(i)}}-\frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}\right)=\left(\widehat{\frac{\delta I}{\delta u^{(i+1)}(x)}}\right)+\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-i)}}{\partial u^{(i)}}\quad i=1,\ldots,m-1 \tag{3.11}$$

Proof: see Appendix 3.A

The proof of Lemma 3.1 is based on the comparison of (3.8) and (3.10) and their partial derivatives w.r.t. $u_t^{(j)}$ and $u_{tt}^{(j)}$. With a similar technique, and using equation (3.11), one can proves the fundamental

Lemma 3.2: *The (generalized) Euler–Lagrange equation*

$$\frac{\delta I}{\delta u(x)} = 0$$

is equivalent to the condition:

$$\frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} - \frac{d}{dt}\left(\frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}}\right) = 0. \tag{3.12}$$

Proof: see Appendix 3.A

Introducing the functional

$$J = \int \hat{\Lambda}\Big(x, t, u(t), u_x(t), \ldots, u^{(m-1)}(t), u_t(t), \ldots, u_{tt}^{(n-m-1)}(t)\Big)\,dt,$$

equation (3.12) reads

$$\frac{\delta J}{\delta u^{(m-1)}(t)} = 0.$$

Notice that the object in the left hand side is the last component of the vector

$$\frac{\delta J}{\delta \bar{u}(t)} = \left(\frac{\delta J}{\delta u(t)}, \frac{\delta J}{\delta u_x(t)}, \ldots, \frac{\delta J}{\delta u^{(m-1)}(t)}\right).$$

4. *Reduced evolutionary equation*: Here we prove that all the components of the vector $\frac{\delta J}{\delta \bar{u}(t)}$ are zero. This can be done recursively, by mean of

Lemma 3.3: *The following recurrence relation holds*:

$$\frac{\delta J}{\delta u^{(i-1)}(t)} = \frac{\delta J}{\delta u^{(m-1)}(t)}\frac{\partial u^{(m)}}{\partial u^{(i)}} - \frac{d}{dx}\left(\frac{\delta J}{\delta u^{(i)}(t)}\right) \qquad i = 1, \ldots, m-1. \tag{3.13a}$$

Proof: see Appendix 3.A

Indeed, Lemma 3.2 states that the $(m-1)$–th component $\frac{\delta J}{\delta u^{(m-1)}(t)}$ is zero when reduced on $\mathfrak{S}$, hence, by virtue of (3.13a), all the components of $\frac{\delta J}{\delta \bar{u}(t)}$ vanish on $\mathfrak{S}$.
This is the Euler–Lagrange equation for the Lagrangian

$$\hat{\Lambda}(x, t, u, u_x, \ldots, u^{(m-1)}, u_t, \ldots, u_t^{(m-1)}, u_{tt}, \ldots, u_{tt}^{(n-m)}).$$

Q.E.D.

Remark: Equation (3.13a) can be rewritten as

$$\left[\frac{\partial\hat\Lambda}{\partial u^{(i-1)}}-\frac{d}{dt}\left(\frac{\partial\hat\Lambda}{\partial u_t^{(i-1)}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial\hat\Lambda}{\partial u_{tt}^{(i-1)}}\right)\right]=$$

$$=\left[\frac{\partial\hat\Lambda}{\partial u^{(m-1)}}-\frac{d}{dt}\left(\frac{\partial\hat\Lambda}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u^{(i)}}-\frac{d}{dx}\left[\frac{\partial\hat\Lambda}{\partial u^{(i)}}-\frac{d}{dt}\left(\frac{\partial\hat\Lambda}{\partial u_t^{(i)}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial\hat\Lambda}{\partial u_{tt}^{(i)}}\right)\right]. \tag{3.13b}$$

for $i=1,\dots,m-1$. And The Euler–Lagrange equation reads

$$\begin{cases}\frac{\partial\Lambda}{\partial u^{(i)}}-\frac{d}{dt}\frac{\partial\Lambda}{\partial u_t^{(i)}}=0 & i=n-m,\dots,m-1\\ \frac{\partial\Lambda}{\partial u^{(i)}}-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(i)}}\right)+\frac{d^2}{dt^2}\frac{\partial\Lambda}{\partial u_{tt}^{(i)}}=0 & i=0,\dots,n-m-1.\end{cases} \tag{3.14}$$

3.3 Relation with the Hamiltonian reduction

Theorem 3.1 provides an alternative definition of the space $\mathfrak{S}$, and of the relative system of canonical coordinates:

$$\begin{cases}\tilde q_i=q_i=u^{(i-1)} & i=1,\dots,m\\ \tilde q_{m+i}=(q_i)_t=u_t^{(i-1)} & i=1,\dots,n-m\\ \tilde p_i=\frac{\delta J}{\delta u_t^{(i-1)}(t)} & i=1,\dots,m\\ \tilde p_{m+i}=\frac{\partial\hat\Lambda}{\partial u_{tt}^{(i-1)}} & i=1,\dots,n-m\end{cases} \tag{3.15}$$

We consider now the Hamiltonian $(-Q)$ of the reduced t–flow, defined in Theorem 2.1

$$Q=\Lambda-\sum_{i=1}^{n}p_i(q_i)_t=\Lambda-\sum_{i=0}^{n-1}\frac{\delta I}{\delta u^{(i+1)}(t)}u_t^{(i)}$$

At the end of the Chapter 2 we noticed how this expression looks very similar to a Legendre transform, but it is not; here we will show that actually the Legendre transform of the Lagrangian $\hat\Lambda$ gives the Hamiltonian $\hat Q$, where $\hat Q$ is written in the coordinate system relative to $\hat\Lambda$.

Firstly we rewrite Q in the coordinate system (3.15):

$$\begin{aligned}-\hat Q&=-\hat\Lambda+\sum_{i=0}^{n-1}\left(\widehat{\frac{\delta I}{\delta u^{(i+1)}(x)}}\right)\left(\widehat{u_t^{(i)}}\right)\\&=-\hat\Lambda+\sum_{i=0}^{m-1}\left[\left(\widehat{\frac{\delta I}{\delta u^{(i+1)}(x)}}\right)+\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(i)}}\right]u_t^{(i)}+\\&+\sum_{i=0}^{n-m-1}\sum_{j=m+i+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u_t^{(i)}}u_{tt}^{(i)}.\end{aligned}$$

Using Lemma 3.1 we get

$$-\hat Q=-\hat\Lambda+\sum_{i=0}^{m-1}\frac{\delta J}{\delta u_t^{(i)}}u_t^{(i)}+\sum_{i=0}^{n-m-1}\frac{\partial\hat\Lambda}{\partial u_{tt}^{(i)}}u_{tt}^{(i)}=-\hat\Lambda+\sum_{i=1}^{n}\tilde p_i(\tilde q_i)_t, \tag{3.16}$$

for $\Lambda\big(x,t,\tilde q_i,(\tilde q_1)_t,\dots,(\tilde q_{n-m})_t\big)$.

3.4 Concluding remarks

The case considered in Theorem 3.1 is the more general one. Indeed, for $(\alpha-1)m < n \leq \alpha m$, the Euler–Lagrange equation defining $\mathfrak{S}$:

$$\left(\widehat{\frac{\delta I}{\delta u(x)}}\right) = 0$$

into the new "coordinates", is a differential equation in $u, \dots, u^{(m-1)}, \dots, u^{(n-m,\alpha)}$.

The Lagrangian L transforms into

$$\hat{L}(x, t, u, u_x, \dots, u^{(m-1)}, u_t, \dots, u_t^{(m-1)}, u_{tt}, \dots, u^{(n-m,\alpha-1)})$$

and we can define the new Lagrangian

$$\hat{\Lambda}(x, t, u, u_x, \dots, u^{(m-1)}, u_t, \dots, u_t^{(m-1)}, u_{tt}, \dots, u^{(n-m-1,\alpha)}).$$

The proof of the theorem is the same, one has only to consider the identities

$$\frac{\partial}{\partial u^{(i,\beta)}}\left(\widehat{\frac{dL}{dt}}\right) = \frac{\partial}{\partial u^{(i,\beta)}}\left(\frac{d\hat{\Lambda}}{dx}\right)$$

for $i = 1, \dots, m-1$ and $\beta = 1, \dots, \alpha$.

In particular, the manifold $\mathfrak{S}$ is defined by

$$\frac{\delta J}{\delta u^{(m-1,\alpha-1)}(t)} = 0 \tag{3.17}$$

and it naturally carries the canonical system of coordinates

$$\begin{cases} \hat{q}_{\beta m+i} = u^{(i-1,\beta)} & i = 1, \dots, m; \quad \beta = 0, \dots, \alpha-2 \\ \hat{q}_{(\alpha-1)m+i} = u^{(i-1,\alpha-1)} & i = 1, \dots, n-(\alpha-1)m \\ \hat{p}_i = \frac{\delta J}{\delta(\hat{q}_i)_t} & i = 1, \dots, n \end{cases} \tag{3.18}$$

In the following we will consider in detail the case $\alpha = 1$, i.e. $n < m$, which occurs in the applications we are interested in (see next chapter). In this case L and $\hat{L}$ coincide and

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u}u_t + \dots + \frac{\partial L}{\partial u^{(n)}}u_t^{(n)}. \tag{3.19}$$

The new Lagrangian is

$$\hat{\Lambda}(x, t, u, u_x, \dots, u^{(m-1)}, (u)_t, \dots, u_t^{(n-1)})$$

with

$$\frac{dL}{dt} = \frac{d\Lambda}{dx} = \frac{\partial \Lambda}{\partial x} + \frac{\partial \Lambda}{\partial u}u_x + \dots + \frac{\partial \Lambda}{\partial u^{(m-1)}}u^{(m)} + \frac{\partial \Lambda}{\partial u_t}u_{xt} + \dots + \frac{\partial \Lambda}{\partial u_t^{(n-1)}}u_t^{(n)}. \tag{3.20}$$

In this case Lemma 3.1 reduces to the following recurrence relation:

$$\frac{\delta I}{\delta u^{(i)}(x)} = \frac{\partial \Lambda}{\partial u_t^{(i-1)}} \qquad i = 1, \dots, n \tag{3.21}$$

and the proof is based on the identity

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{dL}{dt}\right) = \frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\Lambda}{dx}\right) \qquad i = 0, \dots, n-1, \tag{3.22}$$

observing that, for $i \geq 1$,

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{dL}{dt}\right) = \frac{\partial L}{\partial u^{(i)}}.$$

In fact, L does not depend on $u_t^{(i)}$, and

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\Lambda}{dx}\right) = \frac{d}{dx}\left(\frac{\partial \Lambda}{\partial u_t^{(i)}}\right) + \frac{\partial \Lambda}{\partial u_t^{(i-1)}}.$$

In particular the first step, $i = n$, follow directly from the fact that the only dependence of $u_t^{(n)}$ in both (3.19) and (3.20) is the one explicitly shown, so that

$$\frac{\partial L}{\partial u^{(n)}} = \frac{\partial \Lambda}{\partial u_t^{(n-1)}}. \tag{3.23}$$

On the other hand, from (3.22) for the index $i = 0$, one obtains the fundamental relation

$$\frac{\delta I}{\delta u(x)} = \frac{\partial \Lambda}{\partial u^{(m-1)}} \frac{\partial u^{(m)}}{\partial u_t}, \tag{3.24}$$

and, since $\frac{\partial u^{(m)}}{\partial u_t}$ is always nonzero, the condition that defines the submanifold $\mathfrak{S}$ is

$$\frac{\partial \Lambda}{\partial u^{(m-1)}} = 0.$$

The relative system of canonical coordinates is given by:

$$\begin{cases} \hat{q}_i = u^{(i-1)}, \\ \hat{p}_i = \frac{\partial \Lambda}{\partial u_t^{(i-1)}} \end{cases}$$

For $i = 1, \dots, n$.

The reduced t–flow is Lagrangian, with Lagrangian Λ.
Indeed, from the identity

$$\frac{\partial}{\partial u^{(i)}}\left(\frac{dL}{dt}\right) = \frac{\partial}{\partial u^{(i)}}\left(\frac{d\Lambda}{dx}\right) \qquad i = 1, \dots, m-1,$$

and using the Lemma, one obtains on the subspace $\mathfrak{S}$:

$$\begin{cases} \frac{\partial \Lambda}{\partial u^{(i)}} = 0 & i = n, \dots, m-1 \\ \frac{\partial \Lambda}{\partial u^{(i)}} - \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_t^{(i)}}\right) = 0 & i = 0, \dots, n-1 \end{cases}$$

which is the Euler–Lagrange equation for the Lagrangian $\Lambda(x, t, u, u_x, \dots, u^{(m-1)}, u_t, \dots, u_t^{(n-1)})$.

3.5 Example: KdV with t_7 fixed

We will give below an example of how does Theorem 3.1 works for the first non trivial case, $n = m$. We study the Lagrangian reduction of the KdV equation

$$u_t = 6uu_x - u_{xxx} \tag{3.25}$$

on the stationary manifold of the t_7–flow.

The Lagrangian density of the t_7–flow, reduced to the normal form (here I mean that L does not contains total derivatives), depends on the x–derivatives of $u(x,t)$ up to order $n = 3$, and has the expression

$$L = 7u^5 + 35u^2u_x^2 + 7uu_{xx}^2 + \frac{1}{2}(u^{(3)})^2. \tag{3.26}$$

The submanifold $\mathfrak{S}$ of the stationary points defined by the Euler–Lagrange equation for L gives the $n+m = 6$ derivative in terms of the first five, explicitly

$$u^{(6)} = 14uu^{(4)} + 28u_xu_{xxx} - 70u^2u_{xx} + 21u_{xx}^2 - 70uu_x^2 + 35u^4. \tag{3.27}$$

From the relation

$$\frac{dL}{dt} = \frac{d\Lambda}{dx}$$

one construct the Lagrangian $\Lambda(u, u_x, \ldots, u^{(5)})$. By direct calculation

$$\begin{aligned}\Lambda = &- u^{(3)}u^{(5)} + \frac{1}{2}(u^{(4)})^2 - 14uu_{xx}u^{(4)} + 10u(u^{(3)})^2 + 14u_xu_{xx}u_{xxx} + \\ &-70u^2u_xu^{(3)} - (u^{(2)})^3 + 77u^2(u_{xx})^2 + 70uu_x^2u_{xx} - 35u^4u_{xx} + \\ &-\frac{35}{2}u_x^4 + 280u^3u_x^2 + 35u^6.\end{aligned} \tag{3.28}$$

The evolution equation (3.27) is the definition of u_{xxx} in terms of (u, u_x, u_{xx}, u_t), explicitly:

$$u_{xxx} = 6uu_x - u_t.$$

Differentiating this relation in x one obtains

$$\begin{cases} u^{(4)} = 6uu_{xx} + 6u_x^2 - u_{xt} \\ u^{(5)} = 18u_xu_{xx} + 36u^2 - 6uu_t - u_{xxt} \\ u^{(6)} = 18u_{xx}^2 + 180uu_x^2 + 36u^2u_{xx} - 30u_xu_t - 12uu_{xt} + u_{tt} \end{cases}$$

which is a map from the "coordinates"

$$u, u_x, u_{xx}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, \ldots$$

into

$$u, u_x, u_{xx}, u_t, u_{xt}, u_{xxt}, u_{tt}, \ldots$$

The Lagrangian L depends on u, u_x, u_{xx}, u_{xxx}; in the new "coordinates"

$$\hat{L}(u, u_x, u_{xx}, u_t) = 7u^5 + 53u^2u_x^2 + 7uu_{xx}^2 - 6uu_xu_t + \frac{1}{2}u_t^2$$

Its derivative $\frac{d\hat{L}}{dt}$ looks like

$$\frac{d\hat{L}}{dt} = \frac{\partial \hat{L}}{\partial t} + \frac{\partial \hat{L}}{\partial u}u_t + \frac{\partial \hat{L}}{\partial u_x}u_{xt} + \frac{\partial \hat{L}}{\partial u_{xx}}u_{xxt} + \frac{\partial \hat{L}}{\partial u_t}u_{tt}.$$

And there exist a functional $\hat{\Lambda}$ depending on $u, u_x, u_{xx}, u_t, u_{xt}, u_{xxt}$, explicitly

$$\begin{aligned}\hat{\Lambda} =& 35u^6 + 4u^3u_x^2 + \frac{1}{2}u_x^4 - 6u_x^2u_{xt} + \frac{1}{2}u_{xt}^2 + \\ & -35u^4u_{xx} - 2uu_x^2u_{xx} + 8uu_{xt}u_{xx} + 11u^2u_{xx}^2 + \\ & -u_{xx}^3 + 6uu_xu_{xxt} + 22u^2u_xu_t + 4u_xu_{xx}u_t - u_{xxt}u_t + 4uu_t^2\end{aligned}$$

such that

$$\begin{aligned}\frac{d\hat{L}}{dt} = \frac{d\hat{\Lambda}}{dx} = \frac{\partial\hat{\Lambda}}{\partial x} + \frac{\partial\hat{\Lambda}}{\partial u}u_x + + \frac{\partial\hat{\Lambda}}{\partial u_x}u_{xx} + \frac{\partial\hat{\Lambda}}{\partial u_{xx}}u^{(3)} + \\ + \frac{\partial\hat{\Lambda}}{\partial u_t}u_{xt} + \frac{\partial\hat{\Lambda}}{\partial u_{xt}}u_{xxt} + \frac{\partial\hat{\Lambda}}{\partial u_{xxt}}u_t^{(3)},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial L}{\partial t} &= 0\\ \frac{\partial L}{\partial u} &= 35u^4 + 106uu_x^2 + 7u_{xx}^2 - 6u_xu_t\\ \frac{\partial L}{\partial u_x} &= 106u^2u_x - 6uu_t\\ \frac{\partial L}{\partial u_{xx}} &= 14uu_{xx}\\ \frac{\partial L}{\partial u_t} &= -6uu_x + u_t\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\hat{\Lambda}}{\partial x} &= 0\\ \frac{\partial\hat{\Lambda}}{\partial u} &= 210u^5 + 12u^2u_x^2 - 140u^3u_{xx} - 2u_x^2u_{xx} + 8u_{xt}u_{xx} + 22uu_{xx}^2 + 6u_xu_{xxt} + 44uu_xu_t + 4u_t^2\\ \frac{\partial\hat{\Lambda}}{\partial u_x} &= 8u^3u_x + 2u_x^3 - 12u_xu_{xt} - 4uu_xu_{xx} + 6uu_{xxt} + 22u^2u_t + 4u_{xx}u_t\\ \frac{\partial\hat{\Lambda}}{\partial u_{xx}} &= -35u^4 - 2uu_x^2 + 8uu_{xt} + 22u^2u_{xx} - 3u_{xx}^2 + 4u_xu_t\\ \frac{\partial\hat{\Lambda}}{\partial u_t} &= 22u^2u_x + 4u_xu_{xx} - u_{xxt} + 8uu_t\\ \frac{\partial\hat{\Lambda}}{\partial u_{xt}} &= -6u_x^2 + u_{xt} + 8uu_{xx}\\ \frac{\partial\hat{\Lambda}}{\partial u_{xxt}} &= 6uu_x - u_t\end{aligned}$$

Lemma 3.2 states that the condition $\frac{\delta I}{\delta u} = 0$, which defines the submanifold $\mathfrak{S}$, is equivalent to the condition

$$\frac{\partial\hat{\Lambda}}{\partial u_{xx}} - \frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_{xxt}}\right) = 0,$$

explicitly:

$$u_{tt} = 35u^4 + 2uu_x^2 - 22u^2u_{xx} + 3u_{xx}^2 + 2u_xu_t - 2uu_{xt}.$$

The Euler–Lagrange equation for $\hat{\Lambda}$ reads

$$\begin{aligned}
\frac{\partial\hat{\Lambda}}{\partial u_{xx}} - \frac{d}{dt}(\frac{\partial\hat{\Lambda}}{\partial u_{xxt}}) &= u_{tt} - 35u^4 - 2uu_x^2 + 22u^2u_{xx} - 3u_{xx}^2 - 2u_xu_t + 2uu_{xt} = 0 \\
\frac{\partial\hat{\Lambda}}{\partial u_{x}} - \frac{d}{dt}(\frac{\partial\hat{\Lambda}}{\partial u_{xt}}) &= 8u^3u_x + 2u_x^3 - 4uu_xu_{xx} - 2uu_{xxt} + 22u^2u_t - 4u_{xx}u_t - u_{xtt} = 0 \\
\frac{\partial\hat{\Lambda}}{\partial u} - \frac{d}{dt}(\frac{\partial\hat{\Lambda}}{\partial u_{t}}) &= 210u^5 + 12u^2u_x^2 - 140u^3u_{xx} - 2u_x^2u_{xx} + 4u_{xt}u_{xx} + 22uu_{xx}^2 + \\
&+ 2u_xu_{xxt} - 4u_t^2 - 22u^2u_{xt} + u_{xxtt} - 8uu_{tt} = 0.
\end{aligned} \tag{3.29}$$

In this case L is nondegenerate, so that on $\mathfrak{S}$ we can define the system of canonical coordinates

$$\begin{cases} q_i = u^{(i-1)} & i = 1,2,3 \\ p_i = \frac{\partial\hat{\Lambda}}{\partial u_t^{(i-1)}} & i = 1,2,3 \end{cases}$$

which reads

$$\begin{aligned}
\tilde{q}_1 &= u \\
\tilde{q}_2 &= u_x \\
\tilde{q}_3 &= u_{xx} \\
\tilde{p}_1 &= 22u^2u_x + 4u_xu_{xx} - u_{xxt} + 8uu_t \\
\tilde{p}_2 &= -6u_x^2 + u_{xt} + 8uu_{xx} \\
\tilde{p}_3 &= 6uu_x - u_t
\end{aligned}$$

We will now solve the problem from the Hamiltonian point of view: starting from L and following Theorem 1.1 one construct the canonical coordinates $\{p_i, q_i\}$ on $\mathfrak{S}$:

$$\begin{aligned}
q_1 &= u \\
q_2 &= u_x \\
q_3 &= u_{xx} \\
p_1 &= \frac{\delta I}{\delta u_x} = 70u^2u_x - 14u_xu_{xx} - 14uu_{xxx} + u^{(5)} \\
p_2 &= \frac{\delta I}{\delta u_{xx}} = 14uu_{xx} - u^{(4)} \\
p_3 &= \frac{\delta I}{\delta u_{xxx}} = u_{xxx}
\end{aligned}$$

and the Hamiltonian function

$$Q = \Lambda - \sum_{i=1}^{3} p_1(q_i)_t.$$

By direct calculation one obtains

$$\begin{aligned}
Q = 35u^6 - 140u^3u_x^2 - \frac{35}{2}u_x^4 - 35u^4u_{xx} + 70uu_x^2u_{xx} - 7u^2u_{xx}^2 - u_{xx}^2 + 84u^2u_xu_{xxx} + \\
- 18u_xu_{xx}u_{xxx} - 10uu_{xxx}^2 + 6u_x^2u^{(4)} + 6uu_{xx}u^{(4)} - \frac{1}{2}(u^{(4)})^2 - 6uu_xu^{(5)} + u_{xxx}u^{(5)},
\end{aligned}$$

and in canonical coordinates

$$Q = 35q_1^6 + 280q_1^3q_2^2 - \frac{35}{2}q_2^4 - 35q_1^4q_3 + 70q_1q_2^2q_3 - 21q_1^2q_3^2 +$$
$$- q_3^3 - 6q_1q_2p_1 - 6q_2^2p_2 + 8q_1q_3p_2 - \frac{1}{2}p_2^2 - 70q_1^2q_2p_3 - 4q_2q_3p_3 + p_1p_3 + 4q_1p_3^3.$$

The corresponding Hamiltonian system reads

$$\begin{aligned}
\dot{q}_1 &= 6q_1q_2 - p_3 \\
\dot{q}_2 &= 6q_2^2 - 8q_1q_3 + p_2 \\
\dot{q}_3 &= 70q_1^2q_2 + 4q_2q_3 - p_1 - 8q_1p_3 \\
\dot{p}_1 &= 210q_1^5 + 840q_1^2q_2^2 - 140q_1^3q_3 + 70q_2^2q_3 - 42q_1q_3^2 - 6q_2p_1 + 8q_3p_2 - 140q_1q_2p_3 + 4p_3^2 \\
\dot{p}_2 &= 560q_1^3q_2 - 70q_2^3 + 140q_1q_2q_3 - 6q_1p_1 - 12q_2p_2 - 70q_1^2p_3 - 4q_3p_3 \\
\dot{p}_3 &= -35q_1^4 + 70q_1q_2^2 - 42q_1^2q^3 - 3q_3^2 + 8q_1p_2 - 4q_2p_3
\end{aligned}$$

Rewriting this system in coordinates $\{\tilde{p}_i, \tilde{q}_i\}$ one obtains exactly (3.29).

3.A Appendix

Proof of Lemma 3.1: we prove the Lemma in two parts:

- firstly we prove the relation

$$\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}} = \sum_{j=i+m+1}^{n} \left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right) \frac{\partial u^{(j-i)}}{\partial u_t^{(i)}} \qquad i = 0, \ldots, n-m-1. \tag{a.1}$$

For convenience we can explicitly rewrite eq. (3.8a), using (3.9):

$$\begin{aligned}
\left(\widehat{\frac{dL}{dt}}\right) =& \left[\left(\widehat{\frac{\partial L}{\partial t}}\right) + \sum_{j=m}^{n} \left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right) \frac{\partial u^{(j)}}{\partial t}\right] + \\
&+ \sum_{i=0}^{m-1} \left[\left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right) + \sum_{j=m}^{n} \left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right) \frac{\partial u^{(j)}}{\partial u^{(i)}}\right] u_t^{(i)} + \\
&+ \sum_{i=0}^{n-m} \left[\sum_{j=m+i}^{n} \left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right) \frac{\partial u^{(j)}}{\partial u_t^{(i)}}\right] u_{tt}^{(i)}.
\end{aligned} \tag{a.2}$$

Notice that the arguments of in the square brackets depend on u and its x–derivatives upon the order $m-1$ and on u_t and its x–derivatives upon the order $n-m$, so that the dependence on $u_t^{(i)}$ for $n-m+1 \leq i \leq m-1$ and on $u_{tt}^{(j)}$, for every j is only the explicit one. Analogously

$$\begin{aligned}
\frac{d\hat{\Lambda}}{dx} =& \left[\frac{\partial\hat{\Lambda}}{\partial x} + \frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}} \frac{\partial u^{(m)}}{\partial t}\right] + \sum_{i=1}^{m} \frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}} u^{(i)} + \\
&+ \left[\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}} \frac{\partial u^{(m)}}{\partial u}\right] u_t + \sum_{i=1}^{m-1} \left[\frac{\partial\hat{\Lambda}}{\partial u_t^{(i)}} + \frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}} \frac{\partial u^{(m)}}{\partial u^{(i)}}\right] u_t^{(i)} + \\
&+ \frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}} \frac{\partial u^{(m)}}{\partial u_t} u_{tt} + \sum_{i=1}^{n-m} \frac{\partial\hat{\Lambda}}{\partial u_t^{(i-1)}{}_t} u_{tt}^{(i)}.
\end{aligned} \tag{a.3}$$

The i–th step of $(a.1)$ is obtained from the obvious identity

$$\frac{\partial}{\partial u_{tt}^{(i)}}\left(\widehat{\frac{dL}{dt}}\right)=\frac{\partial}{\partial u_{tt}^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right)\qquad i=1,\ldots,n-m.$$

Indeed, from $(a.2)$ and $(a.3)$, it follows that

$$\frac{\partial}{\partial u_{tt}^{(i)}}\left(\widehat{\frac{dL}{dt}}\right)=\sum_{j=m+i}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u_{t}^{(i)}},\tag{a.4}$$

and

$$\frac{\partial}{\partial u_{tt}^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right)=\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}\right)+\frac{\partial\Lambda}{\partial u_{tt}^{(i-1)}}.\tag{a.5}$$

In particular, at the first step, $i=n-m$ one obtains the basic relation

$$\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(n-m-1)}}=\left(\widehat{\frac{\partial L}{\partial u^{(n)}}}\right)\frac{\partial u^{(n)}}{\partial u_{t}^{(n-m)}}\equiv\left(\widehat{\frac{\delta I}{\delta u^{(n)}(x)}}\right)\frac{\partial u^{(n)}}{\partial u_{t}^{(n-m)}}.\tag{a.6}$$

Substituting $(a.6)$ into the further step of the recurrence, one finds

$$\frac{\partial\Lambda}{\partial u_{tt}^{(n-m-2)}}=\left(\widehat{\frac{\delta I}{\delta u^{(n-1)}(x)}}\right)\frac{\partial u^{(n-2)}}{\partial u_{t}^{(n-m-2)}}+\left(\widehat{\frac{\delta I}{\delta u^{(n)}(x)}}\right)\frac{\partial u^{(n-1)}}{\partial u_{t}^{(n-m-2)}}$$

and so on. This gives relation $(a.2)$.

- The second step is the proof of the relation

$$\frac{\partial\hat{\Lambda}}{\partial u_{t}^{(i)}}=\left(\widehat{\frac{\delta I}{\delta u^{(i+1)}(x)}}\right)+\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-i)}}{\partial u^{(i)}}\qquad i=n-m,\ldots,m-1,\tag{a.7}$$

which is a part of $(a.1)$, indeed, for $i\geq n-m$, the partial derivative $\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}$ vanishes. Very much as in the previous case, equation $(a.7)$ follows from the identity

$$\frac{\partial}{\partial u_{tt}}\left(\widehat{\frac{dL}{dt}}\right)=\frac{\partial}{\partial u_{tt}}\left(\frac{d\hat{\Lambda}}{dx}\right)$$

Using $(a.2)$ one can rewrite the left hand side as

$$\frac{\partial}{\partial u_{tt}}\left(\widehat{\frac{dL}{dt}}\right)=\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u_{t}}.\tag{a.8}$$

On the other hand, from $(a.3)$, one has

$$\frac{\partial}{\partial u_{tt}}\left(\frac{d\hat{\Lambda}}{dx}\right)=\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}}\right)+\frac{\partial\hat{\Lambda}}{\partial u_{t}^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_{t}},\tag{a.9}$$

where

$$\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}}\right)=\frac{d}{dx}\left(\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u_{t}}\right).$$

We develop the right hand side, recalling (3.6b):

$$\frac{d}{dx}\left(\frac{\partial u^{(i)}}{\partial u_t}\right) = \frac{\partial u^{(i+1)}}{\partial u_t} - \frac{\partial u^{(i)}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t},$$

obtaining

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial \hat{\Lambda}}{\partial u_{tt}}\right) &= \sum_{j=m+1}^{n}\left[\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u_t} + \left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j)}}{\partial u_t} - \left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(m)}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t}\right] \\
&= \left[\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(m+1)}(x)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t} + \sum_{j=m+1}^{n}\left[\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right) + \frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(j+1)}(x)}}\right)\right]\frac{\partial u^{(j)}}{\partial u_t} + \\
&- \sum_{j=m+1}^{n}\left[\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(m)}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t}\right] = \\
&= \left[\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(m+1)}(x)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t} + \sum_{j=m+1}^{n}\left[\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u_t} - \left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(m)}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t}\right].
\end{aligned}$$

Inserting in $(a.9)$ and equating it to $(a.8)$ we obtain

$$\left[\left(\widehat{\frac{\partial L}{\partial u^{(m)}}}\right) - \frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(m+1)}(x)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t} = \frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t} - \left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(m)}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t}.$$

But the term $\frac{\partial u^{(m)}}{\partial u_t}$ is nonzero by definition, so that

$$\frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}} = \left(\widehat{\frac{\delta I}{\delta u^{(m)}(x)}}\right) + \sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}},$$

which is the first step of the recurrence $(a.7)$, and so on.

• Finally, since $(a.1)$ for $i > n-m$ coincides with $(a.7)$, it remains to prove it for $i \leq n-m$. These relations can be obtained from $(a.2)$ and $(a.7)$ together with the identity

$$\frac{\partial}{\partial u_t^{(i)}}\left(\widehat{\frac{dL}{dt}}\right) = \frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right), \qquad i = 1, \ldots, n-m.$$

Indeed, starting from the index $i = n-m$ and using (3.9), one may write

$$\begin{aligned}
\frac{\partial}{\partial u_t^{(n-m)}}\left(\widehat{\frac{dL}{dt}}\right) &= \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial u_t^{(n-m)}}\right) + \frac{\partial \hat{L}}{\partial u^{(n-m)}} = \\
&= \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial u^{(n)}}\frac{\partial u^{(n)}}{\partial u_t^{(n-m)}}\right) + \left(\widehat{\frac{\partial L}{\partial u^{(n-m)}}}\right) + \sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u^{(n-m)}}. \qquad (a.10)
\end{aligned}$$

On the other hand

$$\frac{\partial}{\partial u_t^{(n-m)}}\left(\frac{d\hat{\Lambda}}{dx}\right) = \frac{d}{dx}\left(\frac{\partial \hat{\Lambda}}{\partial u_t^{(n-m)}}\right) + \frac{\partial \Lambda}{\partial u_t^{(n-m-1)}} + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}.$$

Performing the same steps as in the previous case, one obtains

$$\frac{\partial \hat{\Lambda}}{\partial u_t^{(n-m-1)}} - \frac{d}{dt}\left(\frac{\partial \hat{\Lambda}}{\partial u_{tt}^{(n-m-1)}}\right) = \left(\widehat{\frac{\delta I}{\delta u^{(n-m)}(x)}}\right) + \sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(n-m-1)}}$$

which is the first recursive step of $(a.1)$.

Q.E.D.

Proof of Lemma 3.2: We prove the Lemma by mean of the equivalence

$$\left(\widehat{\frac{\delta I}{\delta u(x)}}\right) = \left[\frac{\partial \hat{\Lambda}}{\partial u^{(m-1)}} - \frac{d}{dt}\left(\frac{\partial \hat{\Lambda}}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t}$$

which follows from the identity

$$\frac{\partial}{\partial u_t}\left(\frac{d\hat{L}}{dt}\right) = \frac{\partial}{\partial u_t}\left(\frac{d\hat{\Lambda}}{dx}\right).$$

Indeed, expanding, one has

$$\begin{aligned}\frac{\partial}{\partial u_t}\left(\frac{d\hat{L}}{dt}\right) &= \frac{d}{dt}\frac{\partial \hat{L}}{\partial u_t} + \frac{\partial \hat{L}}{\partial u} = \\ &= \frac{d}{dt}\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u_t} + \left(\widehat{\frac{\partial L}{\partial u}}\right) + \sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u} = \\ &= \frac{d}{dt}\left[\frac{d}{dx}\left(\frac{\partial \Lambda}{\partial u_{tt}}\right) + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t}\right] + \left(\widehat{\frac{\partial L}{\partial u}}\right) + \sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u},\end{aligned}$$

where the last equality follows from the equivalence of $(a.8)$ and $(a.9)$. Expanding the right hand side

$$\begin{aligned}\frac{\partial}{\partial u_t}\left(\frac{d\hat{L}}{dt}\right) &= \frac{d}{dx}\left[\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_{tt}}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t} + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u_t}\right) + \\ &+ \left(\widehat{\frac{\partial L}{\partial u}}\right) + \sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u},\end{aligned} \qquad (a.11)$$

On the other hand

$$\begin{aligned}\frac{\partial}{\partial u_t}\left(\frac{d\Lambda}{dx}\right) &= \frac{d}{dx}\left(\frac{\partial \Lambda}{\partial u_t}\right) + \frac{\partial \Lambda}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t} + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u} + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u_t}\right) = \\ &= \frac{d}{dx}\left(\frac{\partial \Lambda}{\partial u_t}\right) + \frac{\partial \Lambda}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u_t} + \\ &+ \left[\left(\widehat{\frac{\delta I}{\delta u^{(m)}(x)}}\right) + \sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}}\right]\frac{\partial u^{(m)}}{\partial u} + \frac{\partial \Lambda}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u_t}\right).\end{aligned} \qquad (a.12)$$

Comparing $(a.11)$ and $(a.12)$ one obtains

$$\begin{aligned}&\frac{d}{dx}\left[\left(\frac{\partial \Lambda}{\partial u_t}\right) - \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_{tt}}\right)\right] + \left[\left(\frac{\partial \Lambda}{\partial u^{(m-1)}}\right) - \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t} = \\ &= \left(\widehat{\frac{\partial L}{\partial u}}\right) + \sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u} - \left[\left(\widehat{\frac{\delta I}{\delta u^{(m)}(x)}}\right) + \sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}}\right]\frac{\partial u^{(m)}}{\partial u}.\end{aligned} \qquad (a.13)$$

But Lemma 3.1 states that

$$\left[\left(\frac{\partial \Lambda}{\partial u_t}\right) - \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial u_{tt}}\right)\right] = \left(\widehat{\frac{\delta I}{\delta u_x(x)}}\right) - \sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u},$$

so that

$$\frac{d}{dx}\left[\left(\frac{\partial\Lambda}{\partial u_t}\right)-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_{tt}}\right)\right]=\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u_x(x)}}\right)-\frac{d}{dx}\left[\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\right]\frac{\partial u^{(j-1)}}{\partial u}+$$
$$+\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{d}{dx}\left(\frac{\partial u^{(j-1)}}{\partial u}\right)=$$
$$=\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u_x(x)}}\right)-\frac{d}{dx}\left[\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\right]\frac{\partial u^{(j-1)}}{\partial u}+$$
$$+\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\left(\frac{\partial u^{(j)}}{\partial u}-\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}}\frac{\partial u^{(m-1)}}{\partial u}\right).$$

Substituting in $(a.13)$ we get

$$\left[\left(\frac{\partial\Lambda}{\partial u^{(m-1)}}\right)-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t}=\left[\left(\widehat{\frac{\partial L}{\partial u}}\right)-\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u_x(x)}}\right)\right]+$$
$$+\sum_{j=m}^{n}\left[\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)-\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(j+1)}(x)}}\right)\right]\frac{\partial u^{(j)}}{\partial u}+$$
$$-\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\left(\frac{\partial u^{(j)}}{\partial u}-\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}}\frac{\partial u^{(m-1)}}{\partial u}\right)+$$
$$+\left[\left(\widehat{\frac{\delta I}{\delta u^{(m)}(x)}}\right)-\sum_{j=m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(j)}(x)}}\right)\frac{\partial u^{(j-1)}}{\partial u^{(m-1)}}\right]\frac{\partial u^{(m)}}{\partial u}.$$

All the terms cancels but

$$\left[\left(\frac{\partial\Lambda}{\partial u^{(m-1)}}\right)-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u_t}=\left[\left(\widehat{\frac{\partial L}{\partial u}}\right)-\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u_x(x)}}\right)\right]$$

Q.E.D.

Proof of Lemma 3.3: Relation (3.13) follows from the identities

$$\frac{\partial}{\partial u^{(i)}}\left(\frac{d\hat{L}}{dt}\right)=\frac{\partial}{\partial u^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right)\qquad i=1,\ldots,m-1, \tag{a.14}$$

and

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\hat{L}}{dt}\right)=\frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right)\qquad i=1,\ldots,m-1. \tag{a.15}$$

Starting from $(a.14)$, we can write

$$\frac{\partial}{\partial u^{(i)}}\left(\frac{d\hat{L}}{dt}\right)=\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial u^{(i)}}\right)=\frac{d}{dt}\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u^{(i)}}+\frac{d}{dt}\left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right),$$

and

$$\frac{\partial}{\partial u^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right)=\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u^{(i)}}\right)+\frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}}+\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}+\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u^{(i)}}\right).$$

which give

$$\frac{d}{dt}\left[\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u^{(i)}}+\left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right)\right] = \frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u^{(i)}}\right)+\frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}}+\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}+\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u^{(i)}}\right). \quad (a.16)$$

On the other hand, in $(a.15)$

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\hat{L}}{dt}\right)\frac{\partial\hat{L}}{\partial u^{(i)}}+\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial u_t^{(i)}}\right) = \left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right)+\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u^{(i)}}+\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial u_t^{(i)}}\right)$$

and

$$\frac{\partial}{\partial u_t^{(i)}}\left(\frac{d\hat{\Lambda}}{dx}\right) = \frac{d}{dx}\left(\frac{\partial\Lambda}{\partial u_t^{(i)}}\right)+\frac{\partial\Lambda}{\partial u_t^{(i-1)}}+\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}$$

which give

$$\left(\widehat{\frac{\partial L}{\partial u^{(i)}}}\right)+\sum_{j=m}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(j)}}}\right)\frac{\partial u^{(j)}}{\partial u^{(i)}} = -\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial u_t^{(i)}}\right)+\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(i)}}\right)+\frac{\partial\Lambda}{\partial u_t^{(i-1)}}+\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}.$$

Performing the derivative w.r.t. t and substituting into $(a.16)$ one gets

$$-\frac{d^2}{dt^2}\left(\frac{\partial\hat{L}}{\partial u_t^{(i)}}\right)+\frac{d}{dx}\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(i)}}\right)+\frac{d}{dt}\frac{\partial\Lambda}{\partial u_t^{(i-1)}}+\frac{d}{dt}\left[\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}\right] =$$

$$= \frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u^{(i)}}\right)+\frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}}+\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}}\frac{\partial u^{(m)}}{\partial u^{(i)}}+\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}}\frac{d}{dt}\left(\frac{\partial u^{(m)}}{\partial u^{(i)}}\right),$$

which gives

$$-\frac{d^2}{dt^2}\left(\frac{\partial\hat{L}}{\partial u_t^{(i)}}\right) = \frac{d}{dx}\left[\left(\frac{\partial\hat{\Lambda}}{\partial u^{(i)}}\right)-\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t^{(i)}}\right)\right]+\left[\frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}}-\frac{d}{dt}\frac{\partial\Lambda}{\partial u_t^{(i-1)}}\right]+$$

$$+\left[\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}}-\frac{d}{dt}\frac{\partial\Lambda}{\partial u_t^{(m-1)}}\right]\frac{\partial u^{(m)}}{\partial u^{(i)}}. \quad (a.17)$$

The left hand side of $(a.17)$ is zero if $i > n-m$.
If $i < n-m$,

$$\frac{\partial\hat{L}}{\partial u_t^{(i)}} = \sum_{k=m+i}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial u_t^{(i)}} = \left[\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i-1)}}+\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}\right)\right]. \quad (a.18)$$

Indeed, using $(a.1)$, and relation $(3.6b)$, which we rewrite here below,

$$\frac{d}{dx}\left(\frac{\partial u^{(k)}}{\partial u_t^{(i)}}\right) = \frac{\partial u^{(k+1)}}{\partial u_t^{(i)}}-\frac{\partial u^{(k)}}{\partial u^{(i-1)}}$$

one obtains

$$\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i-1)}}+\frac{d}{dx}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}\right) = \sum_{k=i+m}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(k)}(x)}}\right)\frac{\partial u^{(k-1)}}{\partial u_t^{(i-1)}}+\frac{d}{dx}\left[\sum_{k=i+m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(k)}(x)}}\right)\frac{\partial u^{(k-1)}}{\partial u_t^{(i)}}\right] =$$

$$= \sum_{k=i+m+1}^{n}\left[\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(k)}(x)}}\right)\right]\frac{\partial u^{(k-1)}}{\partial u_t^{(i)}}+\sum_{k=i+m+1}^{n}\left(\widehat{\frac{\delta I}{\delta u^{(k)}(x)}}\right)\frac{\partial u^{(k)}}{\partial u_t^{(i)}}+\left(\widehat{\frac{\delta I}{\delta u^{(i+m)}(x)}}\right)\frac{\partial u^{(i+m-1)}}{\partial u_t^{(i-1)}} =$$

$$= \sum_{k=i+m+1}^{n}\left(\widehat{\frac{\partial L}{u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial u_t^{(i)}}+\left[\frac{d}{dx}\left(\widehat{\frac{\delta I}{\delta u^{(m+i+1)}(x)}}\right)\right]\frac{\partial u^{(m+i)}}{\partial u_t^{(i)}} =$$

$$= \sum_{k=m+i}^{n}\left(\widehat{\frac{\partial L}{\partial u^{(k)}}}\right)\frac{\partial u^{(k)}}{\partial u_t^{(i)}}$$

where the last identity follows from the fact that

$$\frac{\partial u^{(k)}}{\partial u_t^{(l)}} = 0$$

if $k - l < m$. In (3.6), this implies

$$\frac{\partial u^{(k+m)}}{\partial u_t^{(k)}} = \frac{\partial u^{(k+m+j)}}{\partial u_t^{(k+j)}}.$$

Finally, substituting $(a.18)$ into $(a.17)$ gives (3.13)

$$\left[\frac{\partial\hat{\Lambda}}{\partial u^{(i-1)}} - \frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_t^{(i-1)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i-1)}}\right)\right] =$$

$$= \left[\frac{\partial\hat{\Lambda}}{\partial u^{(m-1)}} - \frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_t^{(m-1)}}\right)\right]\frac{\partial u^{(m)}}{\partial u^{(i)}} - \frac{d}{dx}\left[\frac{\partial\hat{\Lambda}}{\partial u^{(i)}} - \frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_t^{(i)}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial\hat{\Lambda}}{\partial u_{tt}^{(i)}}\right)\right].$$

Q.E.D.

Chapter 4

Applications to Painlevé equations

In this Chapter we study some applications of Theorem 2.1 and Theorem 3.1. to show how the finite dimensional Hamiltonian structure of Painlevé equations comes from an infinite dimensional structure via the above procedure.

4.1 PI as scaling reduction of KdV

At the beginning we study the problem following the scheme illustrated in chapter 2, this means that we focus our attention on the Hamiltonian structure of the problem, then we will apply the framework of Theorem 3.1.

We consider the KdV equation

$$u_t = 6uu_x - u_{xxx}. \tag{4.1}$$

i.e. the $t = t_1$–flow in the KdV hierarchy (1.2); it admits the nondegenerate *scaling* symmetry

$$I_{(s)} = \int (u^3 + \frac{u_x^2}{2} + 2ux + 6tu^2)dx, \tag{4.2}$$

which depends on x, u, u_x, t. We note that $L = [L_1 + 4xL_{-1} + 12tL_0]$,where $I_{-1} = \int L_{-1}dx = \int \frac{u(x)}{2}dx$ and $I_0 = \int L_0 dx = \int \frac{u^2(x)}{2}dx$ are the first Hamiltonians of the KdV hierarchy.

Theorem 2.1 states that the t–flow is Hamiltonian on the manifold $\mathfrak{S}$ of the stationary points of the symmetry, i.e. $\mathfrak{S}$ is the 2–dimensional manifold of the solutions of the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = u_{xx} - 3u^2 - 2x - 12tu = 0. \tag{4.3}$$

It is invariant under the t–th flow and it naturally carries the system of canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\delta I}{\delta u_x} = u_x \end{cases}$$

Notice that the identities

$$\begin{cases} p_x + \frac{\partial H}{\partial q} \equiv -\frac{\delta I}{\delta u} \\ q_x - \frac{\partial H}{\partial p} \equiv 0, \end{cases} \tag{4.3}$$

hold, where H is the generalized Legendre transform of L:

$$H = -L + \frac{\delta I}{\delta u_x}u_x.$$

The first of identities (4.4) allows us to express the higher derivatives $u^{(m)}$ for $m \geq 2$ in terms of x,t, p, q and $p^{(l)}$ with $l = 1, \ldots, m - 2 + 1$.

On $\mathfrak{S}$ $p_x + \frac{\partial H}{\partial q_1} \equiv 0$, and the system (4.4) reduces to the canonical Hamiltonian system

$$\begin{cases} p_x = 3q^2 + 2x + 12tq \\ q_x = p \equiv 0, \end{cases}$$

for the Hamiltonian function

$$H = -L + u_x^2 = \frac{p^2}{2} - q^3 - 2qx - 6tq^2 \tag{4.5}$$

giving the reduced x–flow. This system is equivalent to the second order ODE in the variable q :

$$q'' = -3q^2 - 2x - 12tq - a. \tag{4.6}$$

The space $\mathfrak{S}$ is the set of the stationary points of the scaling symmetry (4.2); this means that $\mathfrak{S}$ carries a "natural" system of canonical coordinates $\{w_i, \pi_i\}$, given by the self–similar function of u, i.e; combinations of u, x, t in the variable $z(x,t)$ invariant w.r.t. the scaling. We'll call them scaling coordinates. In this case

$$\begin{cases} w = \frac{q}{2} + t \\ \pi = 2p \end{cases}$$

with $z = x - 6t^2$.

In terms of the scaling coordinates the system reads

$$\begin{cases} \frac{d\pi}{dz} = -\frac{\partial \mathfrak{H}}{\partial w} \\ \frac{dw}{dz} = \frac{\partial \mathfrak{H}}{\partial \pi}. \end{cases}$$

for the Hamiltonian

$$\mathfrak{H} = \frac{\pi^2}{8} - 8w^3 - 4wz + 8t^3 + 4tz.$$

The system is equivalent to the ODE:

$$w'' = 6w^2 + z, \tag{4.7}$$

that is exactly Painlevé I. The Hamiltonian $\mathfrak{H}$ differs from the usual PI Hamiltonian for the terms in z, t that do not enter in the Hamiltonian system.

We now construct the time dependent Hamiltonian function $(-\tilde{Q})$, that is the reduction on $\mathfrak{S}$ of

$$-Q = -\Lambda + p\frac{dq}{dt},$$

where p, q are expressed in terms of u, u_x, and $\Lambda(x, t, u, u_x, u_{xx}, u_{xxx})$, calculated from

$$\frac{dL}{dt} = \frac{d\Lambda}{dx}.$$

has the form

$$\Lambda = 6t(4u^3 - 2uu_{xx} + u_x^2) + 2x(3u^2 - u_{xx}) + \frac{9}{2}u^4 + \frac{1}{2}u_{xx}^2 + 2u_x - 3u^2u_{xx} + 6uu_x^2 - u_xu_{xxx}.$$

By direct calculation one obtains

$$Q = 12t(2u^3 + \frac{u_x^2}{2} - uu_{xx}) + \frac{u_{xx}^2}{2} - 3u^2u_{xx} + \frac{9}{2}u^4 + 2u_x + 2x(3u^2 - u_{xx}), \tag{4.8}$$

This reduces on $\mathfrak{S}$ to

$$\tilde{Q} = 12t(\frac{p^2}{2} - q^3 - 6tq^2 - 2xq) + 2p - 2x^2 \tag{4.9}$$

Theorem 2.1 states that $(-\tilde{Q})$ is the Hamiltonian for the reduced t-flow,i.e., in terms of p and q

$$\begin{cases} \dot{q} = -2\,(6tp + 1) = -\frac{\partial \tilde{Q}}{\partial p} \\ \dot{p} = -12t\,(3q^2 + 2x + 12tq) = \frac{\partial \tilde{Q}}{\partial q}, \end{cases} \tag{4.10}$$

Notice that system (4.10), written in terms of the scaling coordinates w and z, gives the same Painlevé I.

Remark: In this case the evolution equation is Hamiltonian and it can be written in the form

$$u_t = \{u(x), I_1\} = \frac{d}{dx}\frac{\delta I_1}{\delta u},$$

where $I_1 = \int L_1 dx$ with density

$$L_1 = u^3 + \frac{u_x^2}{2}.$$

On the other hand the scaling symmetry defines the stationary flow

$$\frac{du}{ds} = 12tu_x + u_t + 2 = 0.$$

which is Hamiltonian:

$$\frac{du}{ds} = \{u(x), I\} = \frac{d}{dx}\frac{\delta I}{\delta u} = 0.$$

The s-flow and the t-flow commute, but the Hamiltonian generating the scaling depends explicitly on the time t, so that the relation

$$\frac{dI_{(s)}}{dt} = \{I_{(s)}, I_1\} + \frac{\partial I_{(s)}}{\partial t} = 0$$

holds. Hence we have an alternative way to define the reduced Hamiltonian Q, following Theorem 1.1:

$$\frac{d}{dx}Q = \frac{dL}{dt} - \frac{\partial I}{\partial u}\frac{d}{dx}\frac{\partial I_1}{\partial u}.$$

In this case the relation (4.2) follows as a consequence.

System (4.10) i.e. the reduction of the t–flow on $\mathfrak{S}$, can be obtained from the Lagrangian point of view; indeed one can consider the evolution equation (4.1) as the definition of u_{xxx} in terms of (u, u_x, u_{xx}, u_t), explicitly:

$$u_{xxx} = 6uu_x - u_t.$$

Differentiating this relation in x one obtains

$$\begin{cases} u^{(4)} = 6uu_{xx} + 6u_x^2 - u_{xt} \\ u^{(5)} = 18u_x u_{xx} + 36u^2 - 6uu_t - u_{xxt} \\ \vdots \end{cases}$$

which is a map from the "coordinates"

$$u, u_x, u_{xx}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, \ldots$$

into

$$u, u_x, u_{xx}, u_t, u_{xt}, u_{xxt}, u_{tt}, \ldots$$

The Lagrangian L depends on u, u_x and hence its derivative $\frac{dL}{dt}$ looks like

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u}u_t + \frac{\partial L}{\partial u_x}u_{xt}.$$

And there exist a functional $\hat{\Lambda}$ depending on u, u_x, u_{xx}, u_t such that

$$\frac{dL}{dt} = \frac{d\hat{\Lambda}}{dx} = \frac{\partial\hat{\Lambda}}{\partial x} + \frac{\partial\hat{\Lambda}}{\partial u}u_x + \frac{\partial\hat{\Lambda}}{\partial u_x}u_{xx} + \frac{\partial\hat{\Lambda}}{\partial u_{xx}}u^{(3)} + \frac{\partial\hat{\Lambda}}{\partial u_t}u_{xt}.$$

Here, in terms of the new coordinates

$$\hat{\Lambda} = 6t(4u^3 - 2uu_{xx} + u_x^2) + 2x(3u^2 - u_{xx}) + \frac{9}{2}u^4 + \frac{1}{2}u_{xx}^2 + 2u_x - 3u^2u_{xx} + u_xu_t,$$

$\hat{L} \equiv L$, and

$$\frac{\partial L}{\partial t} = 6u^2$$
$$\frac{\partial L}{\partial u} = 3u^2 + 2x + 12tu$$
$$\frac{\partial L}{\partial u_x} = u_x$$

$$\frac{\partial\hat{\Lambda}}{\partial x} = 6u^2 - 2u_{xx}$$
$$\frac{\partial\hat{\Lambda}}{\partial u} = 72tu^2 - 12tu_{xx} + 12xu + 18u^3 - 6uu_{xx}$$
$$\frac{\partial\hat{\Lambda}}{\partial u_x} = 12tu_x + u_t + 2$$
$$\frac{\partial\hat{\Lambda}}{\partial u_{xx}} = -12tu + u_{xx} - 2x - 3u^2$$
$$\frac{\partial\hat{\Lambda}}{\partial u_t} = u_x$$

The condition $\frac{\delta I}{\delta u} = 0$, that defines the submanifold $\mathfrak{S}$, is equivalent to the condition

$$\frac{\partial\hat{\Lambda}}{\partial u_{xx}} = 0$$

Hence we have an alternative definition of the space $\mathfrak{S}$, and an alternative way to defines the canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\partial\hat{\Lambda}}{\partial u_t} = u_x \end{cases}$$

Theorem 3.1 states that the reduced t–flow is Lagrangian ,with Lagrangian Λ, in this case it is easy to verify it, indeed, on $\mathfrak{S}$,

$$\begin{cases} \frac{\partial\Lambda}{\partial u_{xx}} = -12tu + u_{xx} - 2x - 3u^2 = 0 \\ \frac{\partial\Lambda}{\partial u_x} = 12tu_x + u_t + 2 = 0 \\ \frac{\partial\Lambda}{\partial u} - \frac{d}{dt}\left(\frac{\partial\Lambda}{\partial u_t}\right) = 72tu^2 - 12tu_{xx} + 12xu + 18u^3 - 6uu_{xx} - u_{xt} = 0, \end{cases}$$

where the first equation is the definition of the submanifold $\mathfrak{S}$ itself, the other two reproduces (4.9), indeed they can be rewritten as

$$\begin{cases} u_t = -12tu_x - 2 \\ u_{xt} = -12t(3u^2 + 2x + 12tu) \end{cases}$$

4.2 PII as scaling reduction of mKdV

One can repeat the same procedure as in section 4.1 starting from the mKdV equation

$$u_t = 6u^2 u_x - u_{xxx}. \tag{4.11}$$

It admits the nondegenerate *scaling* symmetry

$$I = \int (\frac{3}{2}t(u^4 + u_x^2) + \frac{u^2 x}{2})dx, \tag{4.12}$$

which depends on x, u, u_x, t. We notice that $L = 3tL_1 + \frac{u^2 x}{2}$.

Here $\mathfrak{S}$ is the 2–dimensional manifold of the solutions of the Euler–Lagrange equation

$$\frac{\delta I}{\delta u(x)} = u_{xx} - \frac{1}{3t}(6tu^3 + ux) = 0. \tag{4.13}$$

It naturally carries the system of canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\delta I}{\delta u_x} = 3tu_x. \end{cases}$$

As in the previous case we read the Euler–Lagrange equation as a reduced x–flow with Hamiltonian

$$H = \frac{p^2}{6t} - \frac{3}{2}tq^4 - \frac{1}{2}q^2 x \tag{4.14}$$

where H is the generalized Legendre transform of L:

$$H = -L + \frac{\delta I}{\delta u_x}u_x = -L + 3tu_x^2.$$

The system is equivalent to the second order ODE in the variable q :

$$q_{xx} = 2q^3 + \frac{1}{3t}qx.$$

The scaling coordinates are now

$$\begin{cases} w = (3t)^{\frac{1}{3}} q \\ \pi = \frac{p}{(3t)^{\frac{1}{3}}} \end{cases}$$

in the variable $z = \frac{x}{(3t)^{\frac{1}{3}}}$, and the system transforms into

$$\begin{cases} \frac{d\pi}{dz} = -\frac{\partial \mathfrak{H}}{\partial w} \\ \frac{dw}{dz} = \frac{\partial \mathfrak{H}}{\partial \pi}. \end{cases}$$

for the Hamiltonian

$$\mathfrak{H} = \frac{1}{2}(\pi^2 - w^4) - \frac{1}{2}zw^2.$$

The system is equivalent to the ODE:

$$w'' = 2w^3 + zw. \tag{4.15}$$

that is exactly Painlevé II.

We now construct the time dependent Hamiltonian function $(-\tilde{Q})$, that is the reduction on $\mathfrak{S}$ of

$$-Q=-\Lambda+p\frac{dq}{dt},$$

where

$$\Lambda=t\big(6u^6-6u^3u_{xx}+18u^2u_x^2+\frac{3}{2}u_{xx}^2-3u_xu_{xxx}\big)+x\big(\frac{3}{2}u^4+\frac{1}{2}u_x^2-uu_{xx}\big)+uu_x.$$

By direct calculation one obtains

$$Q=6t(u^6-u^3u_{xx}+\frac{1}{4}u_{xx}^2)+x(\frac{3}{2}u^4-uu_{xx}+\frac{1}{2}u_x^2)+uu_x, \tag{4.16}$$

which on $\mathfrak{S}$ reduces to

$$\tilde{Q}=\frac{x}{2}(\frac{p^2}{9t^2}-q^4)+\frac{1}{3t}pq-\frac{1}{6t}q^2x^2 \tag{4.17}$$

and is the Hamiltonian for the reduced t-flow . In fact

$$\begin{cases}\dot{q}=-\frac{q}{3t}-\frac{xp}{9t^2}=-\frac{\partial\tilde{Q}}{\partial p}\\ \dot{p}=\frac{p}{3t}-\frac{qx^2}{3t}-2q^3x=\frac{\partial\tilde{Q}}{\partial q},\end{cases} \tag{4.18}$$

Notice that also the system (4.18), written in w and z, gives Painlevé II.

Remark: The evolution equation is Hamiltonian and can be written in the form

$$u_t=\{u(x),I_1\}=\frac{d}{dx}\frac{\delta I_1}{\delta u},$$

where $I_1=\int L_1dx$ with density

$$L_1=\frac{1}{2}(u^4+u_x^2)$$

On the other hand the scaling symmetry defines the Hamiltonian stationary flow

$$\frac{du}{ds}=xu_x+3tu_t+u=\frac{d}{dx}\frac{\delta I_{(s)}}{\delta u}=0.$$

We now deduce system (4.18) from the Lagrangian point of view, reading the evolution equation (4.11) as the definition of u_{xxx} in terms of (u,u_x,u_{xx},u_t), explicitly:

$$u_{xxx}=6u^2u_x-u_t.$$

Differentiating this relation in x one obtains

$$\begin{cases}u^{(4)}=6u^2u_{xx}+12uu_x^2-u_{xt}\\ u^{(5)}=36uu_xu_{xx}+36u^4u_x+12u_x^3-6u^2u_t-u_{xxt}\end{cases}$$

which is a map from the "coordinates"

$$u,u_x,u_{xx},u^{(3)},u^{(4)},u^{(5)},u^{(6)},\ldots$$

into

$$u,u_x,u_{xx},u_t,u_{xt},u_{xxt},u_{tt},\ldots$$

Here

$$\hat{\Lambda} = t\big(6u^6 - 6u^3 u_{xx} + \frac{3}{2}u_{xx}^2 - 3u_x u_{xxx} + 3u_x u_t\big) + x\big(\frac{3}{2}u^4 + \frac{1}{2}u_x^2 - uu_{xx}\big) + uu_x.$$

and

$$\frac{\partial L}{\partial t} = \frac{3}{2}(u^4 + u_x^2)$$
$$\frac{\partial L}{\partial u} = 6tu^3 + ux$$
$$\frac{\partial L}{\partial u_x} = 3tu_x$$

$$\frac{\partial \Lambda}{\partial x} = \frac{3}{2}(u^4 + u_x^2) - uu_{xx}$$
$$\frac{\partial \Lambda}{\partial u} = 36tu^5 - 18tu^2 u_{xx} + 6xu^3 - xu_{xx} + u_x$$
$$\frac{\partial \Lambda}{\partial u_x} = xu_x + u + 3tu_t$$
$$\frac{\partial \Lambda}{\partial u_{xx}} = 3tu_{xx} - 6tu^3 - ux$$
$$\frac{\partial \Lambda}{\partial u_t} = 3tu_x$$

The condition $\frac{\delta I}{\delta u} = 0$, that defines the submanifold $\mathfrak{S}$, is equivalent to the condition

$$\frac{\partial \Lambda}{\partial u_{xx}} = 3tu_{xx} - 6tu^3 - ux = 0$$

Hence we have an alternative definition of the space $\mathfrak{S}$, and an alternative way to defines the canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\partial \Lambda}{\partial u_t} = 3tu_x \end{cases}$$

On $\mathfrak{S}$

$$\begin{cases} \frac{\partial \Lambda}{\partial u_{xx}} = 3tu_{xx} - 6tu^3 - ux = 0 \\ \frac{\partial \Lambda}{\partial u_x} = xu_x + 3tu_t + u = 0 \\ \frac{\partial \Lambda}{\partial u} - \frac{d}{dt}\big(\frac{\partial \Lambda}{\partial u_t}\big) = -18tu^2 u_{xx} + 36tu^5 + 6xu^3 - xu_{xx} - 2u_x - 3tu_{xt} = 0, \end{cases}$$

where the first defines the submanifold, the second one gives the motion of u and the third the motion of u_x, hence one can rewrite them as

$$\begin{cases} 3tu_t = -xu_x - u \\ 3tu_{xt} = u_x - 2u^3 x - \frac{x^2 u}{3t}, \end{cases}$$

which coincides with (4.18).

4.3 PIII as scaling reduction of Sine-Gordon

A particular case of Painlevé III equation can be obtained as reduction of the Sine–Gordon equation

$$\begin{cases} u_t = v = \frac{\delta I_1}{\delta v} \\ v_t = u_{xx} - \sin\, u = -\frac{\delta I_1}{\delta u}, \end{cases} \tag{4.19}$$

via the scaling

$$\begin{cases} \frac{du}{ds} = xv + tu_x = \frac{\delta I_{(s)}}{\delta v} = 0 \\ \frac{dv}{ds} = x(u_{xx} - \sin\ u) + tv_x + u_x = -\frac{\delta I_{(s)}}{\delta u} = 0, \end{cases} \tag{4.20}$$

where $I_1 = \int L_1 dx$, $I_{(s)} = \int L dx$, with the Hamiltonians

$$L_1 = \frac{1}{2}(v^2 + u_x^2) - \cos\ u$$

and

$$L = \frac{x}{2}(v^2 + u_x^2) - x \cos\ u + tvu_x = xL_1 + tvu_x$$

w.r.t. the Poisson bracket

$$\{F, G\} = \int (\frac{\delta f}{\delta u(x)} \frac{\delta g}{\delta v(x)} - \frac{\delta f}{\delta v(x)} \frac{\delta g}{\delta u(x)})\ dx.$$

The scaling reduction equation means

$$\frac{\delta I_{(s)}}{\delta u(x)} = \frac{\delta I_{(s)}}{\delta v(x)} = 0,$$

which defines the submanifold $\mathfrak{S}$:

$$u_x = -\frac{xv}{t}, \tag{4.21}$$

with the canonical coordinates

$$\begin{cases} p = xu_x + tv = \frac{t^2 - x^2}{t} v \\ q = u \end{cases}$$

in $\mathfrak{S}$.

The equation defining $\mathfrak{S}$ can be written as an Hamiltonian system in canonical form describing the reduced x-flow:

$$\begin{cases} (p)_x = -\frac{\partial H}{\partial q} \\ (q)_x = \frac{\partial H}{\partial p} \end{cases}$$

where

$$H = \frac{x}{2} \frac{p^2}{x^2 - t^2} + x\ \cos\ q. \tag{4.22}$$

In terms of the scaling coordinates

$$\begin{cases} w = q \\ \pi = \frac{-2z}{t} p \end{cases}$$

in the variable $z = \frac{x^2 - t^2}{2}$, the Hamiltonian system transform into

$$\begin{cases} \frac{d\pi}{dz} = -\frac{\partial \mathfrak{H}}{\partial w} \\ \frac{dw}{dz} = \frac{\partial \mathfrak{H}}{\partial \pi}. \end{cases}$$

for the Hamiltonian

$$\mathfrak{H} = -\frac{1}{4} \frac{\pi^2}{z} - \cos\ w.$$

The system is equivalent to PainlevéIII:

$$2zw'' + 2w' - \sin\ w = 0.$$

Let us now construct the time dependent Hamiltonian function $(-\tilde{Q})$, that is the reduction on $\mathfrak{S}$ of

$$-Q = -\Lambda + p\frac{dq}{dt},$$

where

$$\Lambda = xu_x v + t\left(\frac{1}{2}(v^2 + u_x^2) - \cos u\right).$$

By direct calculation one obtains

$$Q = t\left(\frac{1}{2}(v^2 - u_x^2) - \cos u\right)$$

which on $\mathfrak{S}$ reduces to

$$\tilde{Q} = -t\left(\frac{1}{2}\frac{p^2}{t^2 - x^2} - \cos\ q\right).$$

This is the Hamiltonian for the reduced t-flow . In fact

$$\begin{cases} \dot{q} = v = -\frac{t}{x}u_x = -t\frac{p}{t^2-x^2} = -\frac{\partial \tilde{Q}}{\partial p} \\ \dot{p} = v + xv_x + tv_t = t\ \sin\ q = \frac{\partial \tilde{Q}}{\partial q}. \end{cases} \tag{4.23}$$

Note that also the system (4.23), written in w and z, gives Painlevé III.
Remark : We now deduce system (4.23) from the Lagrangian point of view, reading the evolution equation (4.19) as the definition of v in terms of u_t, explicitly:

$$\begin{cases} v = u_t \\ u_{xx} = v_t + \sin u = u_{tt} + \sin u \end{cases}$$

Differentiating this relation in x one obtains

$$\begin{aligned} v_x &= u_{xt} \\ v_{xx} &= v_{tt} - v\cos u = u_{ttt} - u_t \cos u \\ &\vdots \\ u_{xxx} &= u_{ttt} + u_x cos u \\ &\vdots \end{aligned}$$

which is a map from the "coordinates"

$$u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots$$

into

$$u, u_x, u_t, u_{xt}, u_{tt}, u_{xtt}, \ldots$$

Here

$$\hat{\Lambda} = xu_x u_t + t\left(\frac{1}{2}(u_t^2 + u_x^2) - \cos u\right)$$

and

$$\hat{L} = tu_x u_t + x\left(\frac{1}{2}(u_t^2 + u_x^2) - \cos u\right)$$

which give

$$\begin{aligned} \frac{\partial \hat{L}}{\partial t} &= \frac{1}{2}(u_t^2 + u_x^2) - \cos u \\ \frac{\partial \hat{L}}{\partial u} &= -\sin u \\ \frac{\partial \hat{L}}{\partial u_x} &= tu_t + xu_x \\ \frac{\partial \hat{L}}{\partial u_t} &= tu_x + xu_t \end{aligned}$$

and

$$\frac{\partial\hat{\Lambda}}{\partial x} = \frac{1}{2}(u_t^2 + u_x^2) - \cos u$$
$$\frac{\partial\hat{\Lambda}}{\partial u} = -t\sin u$$
$$\frac{\partial\hat{\Lambda}}{\partial u_x} = xu_t + tu_x$$
$$\frac{\partial\hat{\Lambda}}{\partial u_t} = xu_x + tu_t$$

The condition $\frac{\delta I}{\delta u} = 0$, that defines the submanifold $\mathfrak{S}$, is equivalent to the condition

$$\frac{\partial\hat{\Lambda}}{\partial u_x} = xu_t + tu_x = 0$$

Hence we have an alternative definition of the space $\mathfrak{S}$, and an alternative way to defines the canonical coordinates:

$$\begin{cases} q = u \\ p = \frac{\partial\hat{\Lambda}}{\partial u_t} = xu_x + tu_t \end{cases}$$

On $\mathfrak{S}$

$$\begin{cases} \frac{\partial\hat{\Lambda}}{\partial u_x} = xu_t + tu_x = 0 \\ \frac{\partial\hat{\Lambda}}{\partial u} - \frac{d}{dt}\left(\frac{\partial\hat{\Lambda}}{\partial u_t}\right) = -t\ \sin u - xu_{xt} - u_t - tu_{tt} = 0, \end{cases}$$

where the first defines the submanifold, and the second

$$-t\sin u - xu_{xt} - u_t - tu_{tt} = 0$$

coincides with (4.23).

Chapter 5

Self–similar solutions of n–waves equation and Hamiltonian MPDEs

5.1 n–waves equations and their symmetries

Let us consider the equation

$$u_t - v_x - [u, v] = 0, \tag{5.1}$$

where

$$u = [\gamma, a] \quad v = [\gamma, b] \quad a = diag\ (a^1, ...a^n) \quad b = diag\ (b^1, ...b^n) \tag{5.2}$$

and γ is a function of x, t.

Following [DS] it is possible to rewrite (5.1) as an infinite dimensional Hamiltonian system on the space $\mathfrak{M}$ of functions of x with values in $Mat(n, C)$ with the inner product

$$(u, v) = \int \mathrm{Tr}\ (u(x)v(x))dx.$$

On the space $\mathfrak{F}$ of functionals

$$F = \int f(x, u, u_x, ...u^{(k)})\ dx$$

one can define $\nabla_u F \in \mathfrak{M}$ by

$$\frac{d}{d\epsilon} F(u + \epsilon w)|_{\epsilon=0} = (\nabla_u F, w)$$

and the Poisson structure P with the Poisson bracket

$$\{F, G\}(u) = (\nabla_u F, [\nabla_u G, \frac{d}{dx} + u]) \tag{5.3}$$

The n-waves equation (5.1) is a Hamiltonian system w.r.t. this Poisson structure:

$$u_t = PdI_1 = [\nabla_u I_1, \frac{d}{dx} + u] = [-v, \frac{d}{dx} + u], \tag{5.4}$$

where

$$I_1 = \int L_1 dx = -\frac{1}{2} \int Tr\ (uv)dx$$

that in components of γ gives

$$I_1 = \int \sum_i \sum_k (\frac{b_i}{a_k - a_i} u_{ik} u_{ki})dx = \int \sum_i \sum_k [(b_i - b_k)(a_i - a_k)\gamma_{ik}\gamma_{ki}]dx. \tag{5.5}$$

For ($n = 3$, $u^T = -u$) one can reduce to a particular case of P VI equation (see [D], where the Hamiltonian structure for this particular case of P VI is derived from the Hamiltonian structure of the n–waves equation.), imposing the scaling

$$\frac{du}{ds} = tu_t + xu_x + u = 0. \tag{5.6}$$

It admits the Hamiltonian form

$$\frac{du}{ds} = [\nabla_u I_{(s)}, \frac{d}{dx} + u] = 0 \tag{5.7}$$

where

$$I_{(s)} = \int L dx = -\frac{1}{2}\int Tr\ (tuv + xu^2)dx = \int \sum_i \sum_k (t\frac{b_i}{a_k - a_i} - \frac{x}{2})u_{ik}u_{ki}dx$$

and

$$\nabla_u I_{(s)} = -tv - xu, \tag{5.8}$$

or, in terms of γ:

$$I_{(s)} = \int \sum_i \sum_k [(b_i - b_k)(a_i - a_k)t + (a_i - a_k)^2 x]\gamma_{ik}\gamma_{ki}dx \tag{5.9}$$

We emphasize the fact that the t–flow and the s–flow commute, so that

$$\{I_1, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} = 0.$$

By substituting:

$$\int [Tr(\nabla_u I_1 [\nabla_u I_{(s)}, \frac{d}{dx} + u]) + \frac{1}{2}Tr(uv)]dx = 0.$$

Then there exists a function $Q_{(t)}(x, t, u, v)$ such that

$$Tr(-v[-tv - xu, \frac{d}{dx} + u] + \frac{uv}{2}) = -\frac{d}{dx}Q_{(t)}.$$

By direct calculation (see Appendix A) we obtain

$$Q_{(t)} = \frac{1}{2}Tr(xuv + tv^2) = \frac{1}{2}\sum_{i,j}[(a_j - a_i)(b_i - b_j)x + (b_j - b_i)^2 t]\gamma_{ij}\gamma_{ji}$$

As in the previous examples, $Q_{(t)}$ is the Hamiltonian for the reduced t–flow. We now describe this flow.

We start by rewriting the system

$$\begin{cases} u_t - v_x - [u, v] & = 0 \\ tu_t + xu_x + u & = 0 \end{cases} \tag{5.10}$$

in terms of γ, i.e. we solve

$$[\gamma_t, a] \ = \ [\gamma_x, b] + [[\gamma, a], [\gamma, b]]$$

under the condition

$$\gamma_x = -\frac{t}{x}\gamma_t - \frac{1}{x}\gamma.$$

This gives

$$[\gamma_t, ax + tb] + [\gamma, b] = [[\gamma, ax], [\gamma, b]]$$

but, because of the commutativity of b with itself,

$$\frac{d}{dt}[\gamma, ax + tb] = [[\gamma, ax + bt], [\gamma, b]]. \tag{5.11}$$

Then we identify $\mathfrak{S}_s$ with the space of matrices

$$q = [\gamma, ax + bt] = ux + vt,$$

so that $u = \mathrm{ad}_{(\mathrm{ax+bt})}\mathrm{ad}_{\mathrm{a}}^{-1}\mathrm{q}$, $v = \mathrm{ad}_{(\mathrm{ax+bt})}\mathrm{ad}_{\mathrm{b}}^{-1}\mathrm{q}$, or, in terms of the matrix elements:

$$q_{ij} = [(a_j - a_i)x + (b_j - b_i)t]\gamma_{ij}$$

On $\mathfrak{G}$ the equation (5.11) has the Lax form

$$q_t = [q, v] = [q, \mathrm{ad}_{(\mathrm{ax+bt})}\mathrm{ad}_{\mathrm{b}}^{-1}\mathrm{q}]$$

with the Hamiltonian function

$$H_{(t)} = \frac{1}{2}\mathrm{Tr}(qv) = \frac{1}{2}\mathrm{Tr}(xuv + tv^2). \tag{5.12}$$

This coincides with $Q_{(t)}$.

One may change the role of x and t. This means that one considers the system (5.1) on the space of functions $v(t)$:

$$v_x = [\nabla_v I_0, \frac{d}{dt} + v],$$

where $I_0 = \int H_0 dt = -\frac{1}{2}\int \mathrm{Tr}(uv)\ dt$ and one integrates in the variable t. The scaling (5.8) can be read as an Hamiltonian equation

$$\frac{dv}{ds} = [\nabla_v I_{(s)}, \frac{d}{dt} + v] = 0, \tag{5.13}$$

where

$$I_{(s)} = \int L dt = -\frac{1}{2}\int \mathrm{Tr}\ (xuv + tv^2)\ dt \tag{5.14}$$

and

$$\nabla_v I_{(s)} = -tv - xu \tag{5.15}$$

Commutativity of the flows is equivalent to

$$\{I_0, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial x} = 0;$$

in our case

$$\int [\mathrm{Tr}(\nabla_v I_0[\nabla_v I_{(s)}, \frac{d}{dt} + v]) + \frac{1}{2}Tr(uv)]dt = 0.$$

Then there exists a function $Q_{(x)}(x, t, u, v)$ such that

$$\mathrm{Tr}(-u[-tv - xu, \frac{d}{dt} + v] + \frac{uv}{2}) = -\frac{d}{dt}Q_{(x)}.$$

By direct calculation (see Appendix A) we obtain

$$Q_{(x)} = \frac{1}{2}Tr(tuv + xu^2), \tag{5.16}$$

in components:

$$Q_{(x)} = \frac{1}{2}\sum_{i,j}[(a_j - a_i)(b_i - b_j)t + (a_j - a_i)^2 x]\gamma_{ij}\gamma_{ji}$$

Now we study the x–flow on the reduced manifold defined by the scaling equation : the system (5.10) gives

$$\begin{cases} u_t - v_x - [u, v] &= 0 \\ tv_t + xv_x + v &= 0. \end{cases} \tag{5.17}$$

In terms of γ this becomes

$$\frac{d}{dx}[\gamma, ax+tb] = [[\gamma, ax+bt], [\gamma, a]],$$

that is a Lax equation on $\mathfrak{S}_s$:

$$q_x = [q, u] = [q, \mathrm{ad}_{(ax+bt)} \mathrm{ad}_a^{-1} q] \tag{5.18}$$

with Hamiltonian function

$$H_{(x)} = \frac{1}{2}\mathrm{Tr}(qu) = \frac{1}{2}\mathrm{Tr}(xu^2 + tuv). \tag{5.19}$$

This coincides with $Q_{(x)}$.

In fact one can rewrite the scaling as a zero–curvature equation in two ways:

$$\frac{du}{ds} = q_x + [u, q] = 0 \tag{5.20}$$

and

$$\frac{du}{ds} = q_t + [v, q] = 0. \tag{5.21}$$

Therefore one may rewrite them in terms of q as

$$q_x = [q, ad_{(ax+bt)}^{-1} ad_a q]$$

and

$$q_t = [q, ad_{(ax+bt)}^{-1} ad_b q].$$

5.2 Commuting time–dependent Hamiltonian flows on $\mathfrak{so}(n)$

We can do exactly the same using the coordinates

$$t_i = xa_i + tb_i,$$

and the corresponding derivatives $\frac{d}{dt_i}$, with

$$\frac{d}{dx} = \sum_i a_i \frac{d}{dt_i}$$

and

$$\frac{d}{dt} = \sum_i b_i \frac{d}{dt_i}.$$

The starting equation is now

$$\frac{d}{dt_k} u_i - \frac{d}{dt_i} u_k - [u_i, u_k] = 0 \tag{5.22}$$

where

$$u_i = [\gamma, E_i] \qquad (u_i)_{kl} = \gamma_{kl}\delta_{ik} - \gamma_{kl}\delta_{il}$$

and $(E_i)_{kl} = \delta_{ik}\delta_{kl}$. We impose the scaling

$$\frac{d}{ds} u_k = \sum_i t_i \frac{d}{dt_i} u_k \ + \ u_k = 0 \tag{5.23}$$

For every k one can define, on the space $\mathfrak{F}_k$ of functionals

$$F = \int f(t_k, u, \frac{du}{dt_k}, ..., \frac{d^m u}{dt_k^m})\ dt_k$$

with

$$\frac{d}{d\epsilon} F(u_k + \epsilon w)|_{\epsilon=0} = (\nabla_{u_k} F, w),$$

a Poisson structure $P^{(k)}$ with the Poisson bracket

$$\{F, G\}(u_k) = (\nabla_{u_k} F, [\nabla_{u_k} G, \frac{d}{dt_k} + u_k])$$

The n-waves equation (5.22) is Hamiltonian w.r.t. the Poisson structure $P^{(k)}$ in $\mathfrak{F}_k$:

$$\frac{d}{dt_i} u_k = [\nabla_{u_k} I_i, \frac{d}{dt_k} + u_k] = [-u_i, \frac{d}{dt_k} + u_k], \tag{5.24}$$

where

$$I_i = \int L_i dt_k = -\frac{1}{2} \int \mathrm{Tr}\ (u_i u_k)\ dt_k. \tag{5.25}$$

On $\mathfrak{F}_k$ we can reduce to P VI equation imposing the scaling (5.23), which admits the Hamiltonian form

$$\frac{d}{ds} u_k = [\nabla_{u_k} I_{(s)}, \frac{d}{dt_k} + u_k] = [-\sum_j t_j u_j, \frac{d}{dt_k} + u_k] = 0 \tag{5.26}$$

where

$$I_{(s)} = \int L dt_k = -\frac{1}{2} \int \mathrm{Tr} \sum_j t_j u_j u_k dt_k = -\frac{1}{2} \int \mathrm{Tr}(\sum_{j \neq k} t_j u_j u_k + t_k u_k^2) dt_k \tag{5.27}$$

The commutativity of the flows is equivalent to

$$\{I_i, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t_i} = 0;$$

in our case

$$\int [\mathrm{Tr}(\nabla_{u_k} I_i [\nabla_{u_k} I_{(s)}, \frac{d}{dt_k} + u_k]) + \frac{1}{2} Tr(u_i u_k)] dt_k = -\int \partial_k Q_{(i)} dt_k.$$

By direct calculation (following the scheme in Appendix A) we obtain

$$Q_{(i)} = \frac{1}{2} \mathrm{Tr} \sum_j t_j u_j u_i \ = \ \sum_j (t_i - t_j) \gamma_{ij} \gamma_{ji} \tag{5.28}$$

The scaling equation defines the submanifold $\mathfrak{S}_s$. One can consider on $\mathfrak{S}_s$ the system of coordinates given by the matrix elements of q:

$$q = [\gamma, \sum_j t_j E_j] = [\gamma, U],$$

where U is the diagonal matrix $\mathrm{diag}(t_1, \ldots, t_n)$; explicitly

$$q_{ij} = (t_j - t_i) \gamma_{ij}. \tag{5.29}$$

As in the previous cases, $Q_{(i)}$ is the Hamiltonian for the t_i-flow on the reduced manifold.

Starting now from $\mathfrak{F}_i$, $i \neq k$, we can reduce on the same submanifold $\mathfrak{S}$ and construct the Hamiltonian function $Q_{(k)}$.

Indeed, the scaling (5.23) for every k produces on $\mathfrak{S}_s$ the Lax equation

$$q_k = [q, u_k] = 0, \tag{5.30}$$

with Hamiltonian functions

$$H_k = \frac{1}{2}\mathrm{Tr}(qu_k) = \frac{1}{2}\mathrm{Tr}\sum_j t_j u_j u_k = \frac{1}{2}\sum_{j\neq k}\frac{q_{jk}q_{kj}}{t_k - t_j}. \tag{5.31}$$

These coincide with the $Q_{(k)}$ constructed above. Observing that $\gamma = \mathrm{ad}_{\mathrm{U}}^{-1}\mathrm{q}$ one can rewrite

$$u_k = \mathrm{ad}_{\mathrm{E_k}}\mathrm{ad}_{\mathrm{U}}^{-1}\mathrm{q}.$$

In the case $q^T = -q$ eqs. (5.30) are the Monodromy Preserving Deformation equations for the linear differential operator

$$\Lambda = \frac{d}{d\lambda} - U - \frac{q}{\lambda}$$

that give Painlevé VI, for $n = 3$, and the higher–order analogues, for $n > 3$.

5.A Appendix

Let us consider the following explicit expressions:

$$\begin{aligned}
I_t &= -\frac{1}{2}\int \mathrm{Tr}\ (uv)dx \\
\nabla I_t &= -v \\
I_{(s)} &= -\frac{1}{2}\int \mathrm{Tr}\ (xu^2 + tuv)dx \\
\nabla I_{(s)} &= -tv - xu \\
\{I_t, I_{(s)}\} &= (-v, tu_t + xu_x + u) = \\
&= -\int \mathrm{Tr}\ (tvv_x + tv[u,v] + xvu_x + uv)dx \\
\frac{\partial I_{(s)}}{\partial t} &= I_t = -\frac{1}{2}\int \mathrm{Tr}\ (uv). \\
\{I_t, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} &= -\int \mathrm{Tr}\ (tvv_x + tv[u,v] + xvu_x + \frac{1}{2}uv)dx
\end{aligned} \tag{A.1}$$

In (A.1) the relations

$$\begin{aligned}
\mathrm{Tr}\ v\ [u,v] &= 0 \\
\mathrm{Tr}\ (vv_x) &= \frac{1}{2}\frac{d}{dx}\mathrm{Tr}\ (v^2) \\
\mathrm{Tr}\ (xvu_x + \frac{1}{2}uv) &= \frac{1}{2}\frac{d}{dx}\mathrm{Tr}\ (xuv)
\end{aligned}$$

hold. In fact, in terms of γ_{ij} one can write

$$\begin{aligned}
\mathrm{Tr}\ (xvu_x) &= \sum_i\sum_k x(b_k - b_i)(a_i - a_k)\gamma_{ik}(\gamma_x)_{ki} = \\
&= \sum_i\sum_k x(b_i - b_k)(a_k - a_i)(\gamma_x)_{ki}\gamma_{ik} = \\
&= \mathrm{Tr}\ (xv_x u),
\end{aligned}$$

which implies

$$\frac{d}{dx}\mathrm{Tr}\ (xuv) = 2\mathrm{Tr}\ (xvu_x) + \mathrm{Tr}\ (uv).$$

Then:

$$\{I_t, I_{(s)}\} - \frac{\partial I_{(s)}}{\partial t} = -\frac{1}{2}\int \frac{d}{dx}\mathrm{Tr}\ (xuv + tv^2)\ dx.$$

Chapter 6

Poisson structure on the Stokes matrices

As we said in the Introduction, many authors were inspired by the parallelism between the technique of soliton theory based on the spectral transform and that of the MPDE theory based on the monodromy transform. in previous Chapters we discussed the fact that, in both cases, one deals with certain classes of Hamiltonian systems, namely, with infinite-dimensional Hamiltonian structures of evolutionary equations and of their finite-dimensional invariant submanifolds in soliton theory, and with remarkable finite-dimensional time-dependent Hamiltonian systems in the MPDE theory.

We also recalled that one of the first steps in soliton theory was understanding of the Hamiltonian nature of the spectral transform as the transformation of the Hamiltonian system to the action-angle variables [ZF]. Further development of these ideas was very important for development of the Hamiltonian approach to the theory of solitons [FT] and for the creation of a quantum version of this theory.

In the general theory of MPDE it remains essentially an open question to understand the Hamiltonian nature of the monodromy transform, i.e., of the map associating the monodromy data to the linear differential operator with rational coefficients. This question was formulated in [FN2] and solved in an example of a MPDE of a particular second order linear differential operator. However, the general algebraic properties of the arising class of Poisson brackets on the spaces of monodromy data remained unclear. The technique of [FN2] seems not to work for more general case. The authors of the papers [AM, FR, KS, Hi] consider the important case of MPDE of Fuchsian systems in a more general setting of symplectic structures on the moduli space of flat connections (see, e.g., [Au]) not writing, however, the Poisson bracket on the space of monodromy data in a closed form. MPDE of non-Fuchsian operators and Poisson structure on their monodromy data were not considered in these papers.

This Chapter is based on the paper [Ug], where we solve the problem of computing the Poisson structure of MPDE in the monodromy data coordinates for one particular example of the operators with one regular and one irregular singularity of Poincaré rank 1

$$\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$$

where U is a diagonal matrix with pairwise distinct entries and V is a skewsymmetric $n \times n$ matrix.

Recently MPDE of this operators proved to play a fundamental role in the theory of Frobenius manifolds [D, D1]. The Poisson structure of MPDE for the operator Λ coincides with the standard linear Poisson bracket on the Lie algebra $\mathfrak{so}(n) \ni V$. The most important part of the monodromy data is the Stokes matrix (see the definition below). This is an upper triangular matrix $S = S(V, U)$ with all diagonal entries being equal to 1. Generically S determines other parts of the monodromy data. It turns out that, although the monodromy map

$$V \mapsto S$$

is given by complicated transcendental functions, the Poisson bracket on the space of Stokes matrices is given by very simple degree two polynomials (see formula (6.22) below). The technique of [KS] was important in the derivation of this main result.

We hope that this interesting new class of polynomial Poisson bracket and their quantization (cf. [R, Ha2]) deserves a further investigation that we are going to continue in subsequent publications.

6.1 Systems with irregular singularity

6.1.1 Stokes phenomenon

In this Section we will concentrate our attention on the linear systems

$$\frac{dY}{dz} = (U + \frac{V}{z})Y, \qquad z \in \mathbb{C}, \tag{6.2}$$

where U is a diagonal $n \times n$ matrix with distinct entries $u_1, u_2, \ldots, u_n$ and $V = (v_{ij}) \in \mathfrak{so}(n,\mathbb{C})$, with nonresonant eigenvalues $(\mu_1, \mu_2, \ldots, \mu_n)$ (i.e. $\mu_i - \mu_j \notin Z \setminus 0$). The solutions of the system (6.2) have two singular points, 0 and ∞.

• Near the Fuchsian point $z = 0$ a fundamental matrix of solutions $Y_0(z)$ exists such that

$$Y_0(z) = W(z)z^\theta = [W_0 + W_1 z, \ldots]z^\theta, \tag{6.3}$$

where θ is the diagonalization of V, $\theta = W_0^{-1}VW_0 = diag(\mu_1, \mu_2, \ldots, \mu_n)$, and $W(z)$ converges for small $|z|$.

If one continues $Y_0(z)$ along a path encircling the point $z = 0$, the columns of the resulting matrix are linear combinations of the columns of $Y_0(z)$; hence there exists a matrix M_0 such that

$$Y_0(z) \mapsto Y_0(z)M_0.$$

The matrix M_0 is called *monodromy* matrix around zero. In our case $M_0 = exp(2\pi i\theta)$.

• At ∞ the solution has an *irregular* singularity of Poincaré rank 1. This means that it is possible to construct a formal series

$$\Gamma(z) = \mathbb{1} + \frac{\Gamma_1}{z} + \frac{\Gamma_2}{z^2} + \ldots$$

where $V = [\Gamma_1, U] +$ *diagonal*, i.e. $\Gamma_1 = (\gamma_{ij}) = (\frac{v_{ij}}{u_j - u_i})$ for $i \neq j$, and to define certain sectors $\mathfrak{S}_i$ in which a fundamental matrix of solutions Y_i exists with asymptotic behavior

$$Y_i \sim Y_\infty = \Gamma(z)e^{zU}, \tag{6.4}$$

for $|z| \to \infty$ in $\mathfrak{S}_i$. This means that $\Gamma(z)$ is the asymptotic expansion of $Y_i e^{-zU}$.

In different sectors one has different solutions, and this fact is known as *Stokes phenomenon.* The matrices connecting the solutions in different sectors are called *Stokes matrices.*

A complete and detailed description of the phenomenon can be found in [BJL1], [Si], [IN],[Ue]; here we will concentrate our attention on the particular operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ (see also [D]).

Following [D] we define an *admissible* line for the system (6.2) as a line l through the origin on the z-plane such that

$$\text{Re}\, z(u_i - u_j)|_{z\in l} \neq 0 \qquad \forall i \neq j.$$

We denote the half-lines

$$l_+ = z : \arg z = \psi \qquad l_- = z : \arg z = \psi - \pi,$$

where ψ is a fixed real value of the argument.

The line l lies in the intersection $\mathfrak{S}_+ \cup \mathfrak{S}_-$ of the two sectors

$$\mathfrak{S}_R : \ \psi - \pi - \epsilon < \arg z < \psi + \epsilon$$

and

$$\mathfrak{S}_L : \ \psi - \epsilon < \arg z < \psi + \pi + \epsilon.$$

Here ϵ is a sufficiently small positive number.

Theorem 6.1 : *There exists a unique solution $Y_L(z)$ analytic in the sector $\mathfrak{S}_L$ with the asymptotic behavior*

$$Y_L(z) \sim Y_\infty;$$

the same holds for $Y_R(z)$ in $\mathfrak{S}_L$.

Proof: See [BJL1].

S_+ and S_- are the Stokes matrices connecting the two solutions in $\mathfrak{S}_+$, resp. in $\mathfrak{S}_-$, i.e.

$$Y_L(z) = Y_R(z)S_+, \qquad z \in \mathfrak{S}_+$$

and

$$Y_L(z) = Y_R(z)S_-, \qquad z \in \mathfrak{S}_-.$$

From the skew-symmetry $V^T = -V$ it follows

$$S_- = S_+^T.$$

Moreover, one can prove that, given an admissible line, it is possible to order the elements u_i, i.e., to perform a conjugation

$$\Lambda(z) \mapsto P^{-1}\Lambda(z)P,$$

where P is the matrix of the permutation in such a way that the Stokes matrix $S \equiv S_+$ is upper triangular.

Remark: The full set of monodromy data for the operator Λ consists of the Stokes matrix S but also of the monodromy matrix at the point 0 and of the matrix C connecting the solution (6.3) near zero with a solution near the infinity:

$$Y_0(z) = Y_L(z)C.$$

The monodromy data $\{S, M_0, C\}$ satisfy certain constraints described in [D1]. Particularly,

$$C^{-1}S^{T^{-1}}S\,C = M_0.$$

So, in the generic case (i.e., the diagonalizable and nonresonant one) under consideration the diagonal entries of M_0 $e^{2\pi i\mu_1}, \ldots, e^{2\pi i\mu_n}$ are the eigenvalues of $S^{T^{-1}}S$ and C is the diagonalizing transformation for this matrix. The ambiguity in the choice of the diagonalizing transformation does not affect the operator Λ. So, the $\frac{n(n-1)}{2}$ entries of the Stokes matrix S can serve as local coordinates near a generic point of the space of monodromy data of the operator Λ (see details in [D], [D1]).

6.1.2 Monodromy Preserving Deformation Equations

MPDE describe how should the matrix V be deformed, as a function of the "coordinates" u_i, in order to preserve the monodromy data. MPDE are the analogue of the isospectral equations in soliton theory. The MPDE for the operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ are obtained (see [Ue], [D]) as compatibility equations of the system (6.2) with the system

$$\frac{\partial Y}{\partial u_i} = (zE_i + V_i)Y,$$

where $V_i = [E_i, \Gamma_1] = -ad_{E_i}ad_U^{-1}V$ and $(E_i)^a_b = \delta^a_i\delta^i_b$. These equations admit the Lax form

$$\frac{\partial V}{\partial u_i} = [V, V_i]. \tag{6.5}$$

One can write the MPDE as a Hamiltonian system on the space of the skewsymmetric matrices V with the standard linear Poisson bracket for $V = (v_{ab}) \in \mathfrak{so}(n)$:

$$\{v_{ab}, v_{cd}\} = v_{ad}\delta_{bc} + v_{bc}\delta_{ad} - v_{bd}\delta_{ac} - v_{ac}\delta_{bd}. \tag{6.6}$$

Indeed, the Lax equation (6.5) can be rewritten as

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V,u)\},$$

for the Hamiltonian function

$$H_i = \frac{1}{2}\sum_{j\neq i} \frac{v_{ij}^2}{u_i - u_j}. \tag{6.7}$$

In this case, the Poisson bracket is linear but the dynamic of the problem is very complicated; in the following we will show how, very much as in the case of isospectral equations, it is possible to find a different coordinate system (the entries of the Stokes matrix) in which the dynamic of the evolution is trivial, but the Poisson structure is quadratic. The technique developed here consists in building up the monodromy map $V \to S$ passing through an auxiliary Fuchsian system. The MPDE for the system (6.2) can be represented also as MPDE for an appropriate Fuchsian system

$$\frac{d\chi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\chi,$$

which we shall describe in the next section. The basic idea to construct the Poisson bracket on the space of Stokes matrices is to include the map from $V \in \mathfrak{so}(n)$ to $S \in \mathcal{S}$ into the following commutative diagram of Poisson maps

$$\begin{array}{ccc} \mathfrak{so}(n) & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{A}/\mathcal{G} & \longrightarrow & \mathfrak{M}/GL(n,\mathbb{C}) \end{array} \tag{6.8}$$

where $\mathcal{A}/\mathcal{G}$ is the space of residues $\{A_i\}$ of the connection $A = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} d\lambda$ modulo the action of the gauge group $\mathcal{G}$, as we will explain in section 6.2.2, and $\mathfrak{M}/GL(n,\mathbb{C})$ is the space of the monodromy data of the Fuchsian system (section 6.2.3), i.e. the space of n-dimensional representations of the free group with n generators.

6.2 Related Fuchsian system

6.2.1 Fuchsian system

One can relate the system (6.2), with one regular and one irregular singularity to a system with $n+1$ Fuchsian singularities:

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{B_i}{\lambda - u_i}\Phi, \tag{6.9}$$

where

$$B_i = -E_i\left(V + \frac{1}{2}\mathbb{1}\right), \quad i = 1,\ldots,n$$

and

$$B_\infty = V + \frac{1}{2}\mathbb{1}.$$

Such a relation is well known in the domain of differential equations, see, e.g. [BJL], [Sch].

Now we will briefly describe the monodromy data of the system (6.9).

In this case u_j is a Fuchsian singular points and, as in (6.3), the general solution near u_j can be expressed as

$$\Phi_j(\lambda) = W^{(j)}(\lambda)(\lambda - u_j)^{\hat{B}_j},$$

where $W^{(j)}(\lambda) = W_0^{(j)} + (\lambda - u_j)W_1^{(j)} + \dots$ converges for small $|\lambda - u_j|$ and $\hat{B}_j = -\frac{1}{2}E_j$ is the diagonalization of B_j.
We denote M_j the monodromy matrix along the path γ_j encircling the point u_j w.r.t. the basis Φ_∞ we define in (6.10) below. The matrix M_j is conjugated with the matrix $exp(2\pi i\hat{B}_j)$.
Also the point ∞ is Fuchsian; the general solution can be expressed as

$$\Phi_\infty(\lambda) = W^{(\infty)}(\lambda)\left(\frac{1}{\lambda}\right)^{\hat{B}_\infty}, \tag{6.10}$$

where $W^{(\infty)}(\lambda) = W_0^{(\infty)} + \frac{W_1^{(\infty)}}{\lambda} + \dots$ converges at $|\lambda| \to \infty$ and $\hat{B}_\infty = diag(\frac{1}{2} + \mu_1, \dots, \frac{1}{2} + \mu_n)$ is the diagonalization of B_∞. Indeed, the following relation holds in the space of the residues :

$$-\sum_{i=1}^{n} B_i = B_\infty = \frac{1}{2}\mathbb{1} + V.$$

In this basis the monodromy matrix $M_\infty = -e^{2\pi i\theta}$. We assume that the loops $\gamma_1, \dots, \gamma_n$ and γ_∞ are chosen in such a way that

$$M_1 M_2 \dots M_n M_\infty = 1. \tag{6.11}$$

6.2.2 Monodromy Preserving Deformation equations

We now want to deduce the MPDE for the system (6.9). This amounts to find how can the matrix B_j be deformed as function of $u_1, u_2, \dots, u_n$ in order to preserve the monodromy matrices $M_1, \dots, M_n, M_\infty$. The answer is given by
Theorem 6.1 (Schlesinger): *If the fundamental solution near infinity is normalized as in (6.10) and A_∞ is a constant diagonal matrix with nonresonant elements, then the dependence of the A_j on the position of the poles of the Fuchsian system*

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\Phi$$

is given, in order to preserve the monodromy, by

$$\begin{cases} \frac{\partial A_i}{\partial u_j} = \frac{1}{u_i - u_j}[A_i, A_j] & i \neq j \\ \frac{\partial A_j}{\partial u_j} = -\sum_{i\neq j} \frac{[A_i, A_j]}{u_i - u_j}, \end{cases}$$

Proof: it can be found in [Si].

Note that system (6.9) does not satisfy the hypotheses of the Schlesinger theorem, because $B_\infty = (V + \frac{1}{2}\mathbb{1})$ is not diagonal.

In order to apply the Schlesinger theorem it is sufficient to perform the gauge transformation

$$B_i \mapsto A_i = W_0^{-1} B_i W_0, \tag{6.12}$$

where W_0 is the matrix of eigenvectors of V normalized in such a way that

$$\frac{\partial W_0}{\partial u_i} = \mathrm{ad}_{\mathrm{E_i}} \mathrm{ad}_{\mathrm{U}}^{-1} \mathrm{V}. \tag{6.13}$$

Indeed, substituting $\Phi = W_0 \chi$, the system (6.9) transforms into

$$\frac{d\chi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \chi \tag{6.14}$$

and the Schlesinger system follows from the compatibility of (6.14) with

$$\frac{\partial \chi}{\partial u_i} = -\frac{A_i}{\lambda - u_i} \chi.$$

(See [D]).

The Schlesinger system can be rewritten in the Hamiltonian form

$$\frac{dA_i}{du_j} = \{A_i, \mathfrak{H}_j\}$$

with the Hamiltonians

$$\mathfrak{H}_j = -\sum_{k \neq j} \frac{\mathrm{Tr}(A_j A_k)}{u_j - u_k}$$

w.r.t. the linear Poisson bracket

$$\{(A_i)^a_b, (A_j)^c_d\} = \delta_{ij} \left(\delta^a_d (A_i)^c_b - \delta^c_b (A_j)^a_d \right). \tag{6.15}$$

This corresponds to taking, for every u_i, the residue $A_i \in \mathfrak{gl}(n,\mathbb{C})$ with the natural Poisson bracket on $\mathfrak{gl}(n,\mathbb{C})$. The residues relative to different singular points commute. In other words (see [KS],[FR],[A]) this corresponds to read the matrices A_i as residues of a flat connection (with values in the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$) on the Riemann surface with $n+1$ punctures:

$$A = \sum_{i=0}^{n} \frac{A_i}{\lambda - u_i} d\lambda$$

(in our case $u_0 = \infty$). On the space of flat connections modulo gauge transformations it is defined the Poisson bracket

$$\{A^a(\mu), A^b(\nu)\} = -f^{ab}_c \frac{A^c(\mu) - A^c(\nu)}{\mu - \nu},$$

where f^{ab}_c are the structure constants of $\mathfrak{g}$ w.r.t. the basis $\{e_a\}$ and

$$A^a(\mu) = \sum_{i=0}^{n} \frac{A^a_i}{\mu - u_i}, \qquad A(\mu) = A^a(\mu) e_a.$$

This Poisson bracket gives (6.14).

Now we can perform the first step in the construction of the map between V and S, that is we pass from $\mathfrak{so}(n)$ to the space $\mathcal{A}/\mathcal{G}$, where

$$\mathcal{A} = \{V, A_1, \dots, A_n | \sum_{i=0}^{n} A_i = 0\}$$

is the family of the residues of $A(\lambda)$ and $\mathcal{G}$ is the gauge group.

Lemma 6.1 : *The map $V \in \mathfrak{so}(n) \mapsto (V, A_1, \ldots, A_n) \in \mathcal{A}/\mathcal{G}$ is a Poisson map.* (Cf. [Ha1],[Hi])

Proof: We must compare the Poisson brackets on the two spaces. In $\mathfrak{so}(n)$ one has the natural coordinates $\{v_{ab}\}$, with the Poisson bracket (6.6). The natural coordinates in the quotient space $\mathcal{A}/\mathcal{G}$ are the traces of the products of the matrices A_i, so that we consider the brackets

$$\begin{aligned}\{\mathrm{Tr}(A_iA_k), \mathrm{Tr}(A_jA_l)\} &= \{(A_i)^a_b(A_k)^b_a, (A_j)^c_d(A_l)^d_c\} = \\ &= (A_i)^a_b(A_j)^c_d\{(A_k)^b_a, (A_l)^d_c\} + (A_i)^a_b(A_l)^d_c\{(A_k)^b_a, (A_j)^c_d\}+ \\ &+ (A_k)^b_a(A_j)^c_d\{(A_i)^a_b, (A_l)^d_c\} + (A_k)^b_a(A_l)^d_c\{(A_i)^a_b, (A_j)^c_d\}\end{aligned} \tag{6.16a}$$

and

$$\begin{aligned}\{\mathrm{Tr}(A_iV),\mathrm{Tr}(A_jV)\} &= \{(A_i)^a_bV^b_a, (A_j)^c_dV^d_c\} = \\ &= (A_i)^a_b(A_j)^c_d\{V^b_a, V^d_c\} + V^b_aV^d_c\{(A_i)^a_b, (A_j)^c_d\}.\end{aligned} \tag{6.16b}$$

On $\mathcal{A}/\mathcal{G}$ by direct calculation, using the bracket (6.15), one obtains

$$\begin{aligned}\{\mathrm{Tr}(A_iA_k), \mathrm{Tr}(A_jA_l)\} &= \delta_{kl}\mathrm{Tr}\big(A_iA_jA_k - A_kA_jA_i\big) + \delta_{kj}\mathrm{Tr}\big(A_iA_lA_k - A_kA_lA_i\big)+ \\ &+ \delta_{il}\mathrm{Tr}\big(A_kA_jA_i - A_iA_jA_k\big) + \delta_{ij}\mathrm{Tr}\big(A_kA_lA_i - A_iA_lA_k\big) = \\ &= 2\Big(\delta_{kl}\mathrm{Tr}\big(A_iA_jA_k\big) + \delta_{kj}\mathrm{Tr}\big(A_iA_lA_k\big) + \delta_{il}\mathrm{Tr}\big(A_kA_jA_i\big) + \delta_{ij}\mathrm{Tr}\big(A_kA_lA_i\big)\Big)\end{aligned} \tag{6.17}$$

Indeed, $A_i = -E_i(V + \frac{1}{2}\mathbb{1})$ implies

$$\mathrm{Tr}\big(A_iA_jA_k\big) = -\mathrm{Tr}\big(A_kA_jA_i\big) = v_{ij}v_{jk}v_{ki}. \tag{6.18}$$

On the other hand, $\mathrm{Tr}(A_iA_j) = -v_{ij}^2$, hence

$$\begin{aligned}\{\mathrm{Tr}(A_iA_k), \mathrm{Tr}(A_jA_l)\} &= 4v_{ik}v_{jl}\{v_{ik}, v_{jl}\} \\ &= 4\big(\delta_{kl}v_{ij}v_{jk}v_{ki} - \delta_{kj}v_{ik}v_{kl}v_{li} + \delta_{il}v_{ik}v_{kj}v_{ji} + \delta_{ij}v_{ik}v_{kl}v_{li}\big)\end{aligned} \tag{6.19}$$

where we have used the bracket (6.6). By means of (6.18) it is easy to check that it coincides with (6.17).

The same can be done for equation (6.16b). Indeed, using the bracket on the A_i matrices and observing that

$$\mathrm{Tr}\big(A_iA_jV\big) = -\mathrm{Tr}\big(VA_jA_i\big) = \sum_{k\neq i\neq j} v_{ij}v_{ki}v_{jk},$$

one finds

$$\{\mathrm{Tr}(A_iV), \mathrm{Tr}(A_jV)\} = 2\mathrm{Tr}(A_iA_jV).$$

On the other hand $\mathrm{Tr}(A_iV) = -\sum_{k\neq i} v_{ki}^2$, that gives

$$\{\mathrm{Tr}(A_iV), \mathrm{Tr}(A_jV)\} = 4\sum_{k\neq i}\sum_{l\neq j} v_{ki}v_{lj}\{v_{ki}, v_{lj}\} = -4\sum_{k\neq i} v_{ki}v_{kj}v_{ij}$$

which coincides with (6.16b). Q.E.D

Lemma 6.2 : *The MPDE for the system (6.2) and its related Fuchsian system coincide.*

Proof: It follows immediately from Lemma 6.1 by a straightforward calculation using (6.13), that MPDE for the Fuchsian system (6.14) after the gauge transformation (6.12) coincide with (6.5). Actually, one can see that the pull back of the Hamiltonian

$$\mathfrak{H}_j = -\sum_{k\neq j} \frac{\mathrm{Tr}(A_j A_k)}{u_j - u_k} = -\sum_{k\neq j} \frac{\mathrm{Tr}(B_j B_k)}{u_j - u_k}$$

is exactly equal to H_j, as defined in (6.7).

6.2.3 Poisson structure on monodromy data

In this section we will perform the second step of our construction, that is we will map the Poisson structure of $\mathcal{A}/\mathcal{G}$ into the space of monodromy data of the Fuchsian system; this is shown in the following well-known (see, e.g., [Hi])

Theorem 6.2: *The monodromy map*

$$\mathcal{A}/\mathcal{G} \to \mathfrak{M}/SL(n,\mathbb{C})$$

where $\mathfrak{M} = \{M_0, M_1, \ldots, M_n | M_1 M_2 \ldots M_n M_0 = \mathbb{1}\}$, *is a Poisson map.*

To actually compute the Poisson bracket on the space of monodromy data, i.e., on the space of n-dimensional representations of the free group with n generators we will use, following [KS] (Th. 4.2), the following technique. We construct the skewsymmetric bracket

$$\left\{(M_i)^a_b, (M_j)^c_d\right\} = i\pi\Big((M_j M_i)^c_b \delta^a_d + (M_i M_j)^a_d \delta^c_b - (M_i)^c_b (M_j)^a_d - (M_j)^c_b (M_i)^a_d\Big) \qquad i<j \qquad (6.20a)$$

$$\left\{(M_i)^a_b, (M_i)^c_d\right\} = i\pi\Big((M_i^2)^c_b \delta^a_d - (M_i^2)^a_d \delta^c_b\Big). \qquad (6.20b)$$

on the space $\mathfrak{M}$ of the monodromy matrices. As it was proved in [KS], when restricted to the space of representations $\mathfrak{M}/SL(n,\mathbb{C})$, this bracket defines a Poisson structure on the quotient induced by the monodromy map. Observe that the eigenvalues of the matrices M_i are the Casimirs of the Poisson bracket, i.e., the functions Poisson commuting with all others (see [KS]).

6.3. Poisson structure on the Stokes matrices

6.3.1 Connecting the monodromy data of the two systems

In the previous section we have seen that the space of monodromy data of a Fuchsian system carries a natural Poisson structure. In this section we will show that this structure induces a Poisson bracket on the space of Stokes matrices of the related system we studied in chapter 1. To this end we consider the relation between the monodromy matrices $M_1, M_2, \ldots, M_n$ of the Fuchsian system and the Stokes matrix S.

In section 6.2.1 we claimed that the two systems

$$\frac{dY}{dz} = (U + \frac{V}{z})Y$$

and

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\Phi$$

are related, in the sense that, (see Lemma 6.3), the MPDE for the operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ can be represented also as MPDE for the operator $A(\lambda) = \frac{d}{d\lambda} - \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}$.

For a detailed analysis of the transform connecting the two system see [D1]; here we will concentrate our attention on the relation between the monodromy data of the two systems.
Following Theorem 6.2, we are interested in the quotient of the space of the monodromy data of the Fuchsian system w.r.t. the $GL(n,\mathbb{C})$ conjugations. So, we can choose a particular basis of solution of the system and work with the corresponding monodromy matrices.

Theorem 6.3 : *Suppose that $\left(S + S^T\right)$ is nondegenerate; then there exists a unique basis of solutions (which depends on the particular choice of the branchcuts in the complex λ–plane) $\{\Phi^{(j)}(\lambda)\}$ of the Fuchsian system (6.9), such that*

- *Near u_i the solution has the behavior*

$$\Phi_a^{(i)} \sim \frac{1}{\sqrt{u_i - \lambda}} \delta_a^i.$$

- *the monodromy matrices are reflections, i.e., going around the singularity u_i the solutions transform as*

$$\begin{aligned} \Phi^{(i)} &\to -\Phi^{(i)} \\ \Phi^{(j)} &\to \Phi^{(j)} - 2g_{ij}\Phi^{(i)} \end{aligned}$$

where $G = (g_{ij}) = \frac{1}{2}\left(S + S^T\right)$ is the Gram matrix of the following invariant bilinear form w.r.t. the chosen basis

$$g_{ij} = \left(\Phi^{(i)}, \Phi^{(j)}\right) := \Phi^{(i)T}\left(U - \lambda\right)\Phi^{(j)}.$$

Invariance means that g_{ij} does not depend on λ neither on $u_1, \ldots, u_n$.

Proof: See [D1], Th.5.3.

Remark: Φ and Y_L are related by the Laplace transform

$$(Y_L)_a^{(j)}(z) = \frac{-\sqrt{z}}{2\sqrt{\pi}} \int_{\gamma_{(j)}} \Phi_a^{(j)}(\lambda) e^{\lambda z} d\lambda$$

where $\gamma_{(j)}$ is a fixed path in the λ–plane; analogously for Y_R.

In the $\{\Phi^{(j)}(\lambda)\}$ basis the i–th monodromy matrix M_i has the form

$$M_i = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ -2g_{1i} & -2g_{2i} & \ldots & -1 & \ldots & -2g_{ni} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{pmatrix},$$

$$2g_{ij} = 2g_{ji} = s_{ij} \quad \text{for } i < j.$$

This is a reflection w.r.t. the hyperplane normal to the vector $\Phi^{(i)}$.

The Coxeter identity ([B]) gives

$$M_1 M_2 \ldots M_n = -S^{-1} S^T.$$

Lemma 6.3: *The following relations hold (all the indices are pairwise distinct)*

$$\mathrm{Tr}(M_iM_j) = n - 4 + s_{ij}^2 \tag{6.21a}$$

$$\mathrm{Tr}(M_kM_iM_jM_i) = n - 4 + (s_{kj} - s_{ij}s_{ik})^2 \tag{6.21b}$$

$$\mathrm{Tr}(M_iM_jM_lM_k) = n - 8 + s_{ij}^2 + s_{ik}^2 + s_{il}^2 + s_{jk}^2 + s_{jl}^2 + s_{kl}^2 - s_{ij}s_{ik}s_{jk} + \\ - s_{ik}s_{il}s_{kl} - s_{jk}s_{jl}s_{kl} - s_{ij}s_{il}s_{jl} + s_{ij}s_{il}s_{jk}s_{kl}. \tag{6.21c}$$

Proof: The fact that the M_i are reflections and that $S + S^T = 2G$ geometrically reads into

$$-2\cos\alpha_{ij} = s_{ij}$$

where α_{ij} is the angle between the two hyperplanes normal to $\Phi^{(i)}$ and $\Phi^{(j)}$.

On the other hand, the products M_iM_j are rotations by the angle $2\alpha_{ij}$ and this provides the relation (6.21a), indeed

$$\mathrm{Tr}(M_iM_j) = n - 2 + 2\cos(2\alpha_{ij}) = n - 4 + s_{ij}^2.$$

To obtain relation (6.21b) we observe that the product $M_iM_jM_i$ is still a reflection, w.r.t. the mirror normal to the vector $M_i(\Phi^{(j)})$. This means that the product $M_kM_iM_jM_i$ is a rotation by the angle 2β, where

$$-2\cos\beta = \left(M_i(\Phi^{(j)}), \Phi^{(k)}\right) = (\Phi^{(j)} - s_{ij}\Phi^{(i)}, \Phi^{(k)}) = s_{kj} - s_{ij}s_{ik}$$

so that $\mathrm{Tr}(M_kM_iM_jM_i) = n - 2 + 2\cos(2\beta) = n - 4 + (s_{kj} - s_{ij}s_{ik})^2$.

Finally, (6.21c) can be obtained directly in the case of the 4×4 reflection matrices M_i. Indeed, for ordered indices i, j, k, l, the Coxeter identity gives

$$M_iM_jM_kM_l = -S_{ijkl}^{-1}S_{ijkl}^T,$$

where

$$S_{ijkl} = \begin{pmatrix} 1 & s_{ij} & s_{ik} & s_{il} \\ 0 & 1 & s_{jk} & s_{jl} \\ 0 & 0 & 1 & s_{kl} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

An easy calculation gives the result.
The same result holds also in dimension $n > 4$. Indeed, one can observe that, for every n, the product of four matrices acts nontrivially in the 4–dimensional subspace spanned by the vectors normal to the mirrors of the reflections M_i, M_k, M_l, M_j. It is equal to the identity in the orthogonal complement to the 4-dimensional subspace.

Q.E.D.

6.3.2 Poisson bracket

Combining all the above facts, we can conclude our construction proving the following main

Theorem 6.4: *1. The following formulae*

$$\{s_{ik}, s_{il}\} = \frac{i\pi}{2}(2s_{kl} - s_{ik}s_{il}) \qquad i < k < l \tag{6.22a}$$

$$\{s_{ik}, s_{jk}\} = \frac{i\pi}{2}(2s_{ij} - s_{ik}s_{jk}) \qquad i < j < k \tag{6.22b}$$

$$\{s_{ik}, s_{kl}\} = \frac{i\pi}{2}(s_{ik}s_{kl} - 2s_{il}) \qquad i < k < l \tag{6.22c}$$

$$\{s_{ik}, s_{jl}\} = 0 \qquad i < k < j < l \tag{6.22d}$$

$$\{s_{ik}, s_{jl}\} = 0 \qquad i < j < l < k \tag{6.22e}$$

$$\{s_{ik}, s_{jl}\} = i\pi(s_{ij}s_{kl} - s_{il}s_{kj}) \qquad i < j < k < l. \tag{6.22f}$$

define a Poisson bracket on the space $\mathcal{S}$ of Stokes matrices.

2. The monodromy map

$$\mathfrak{so}(n) \to \mathcal{S}$$

associating the Stokes matrix $S \in \mathcal{S}$ to the operator $\Lambda = \frac{d}{dz} - U - \frac{V}{z}$, $V \in \mathfrak{so}(n)$, is a Poisson map.

3. The eigenvalues of $S^{-1}S^T$ are the Casimir functions of the Poisson bracket.

4. The Poisson bracket (6.22) is invariant w.r.t. the action of the braid group B_n on the space of braid matrices.

Proof: *1.* As a first step we explicitly write the restriction of the bracket (6.20) to the space of representations. By direct calculation one obtains

$$\begin{aligned}\{\mathrm{Tr}(M_iM_k),&\mathrm{Tr}(M_jM_l)\} = \{(M_i)^a_b(M_k)^b_a, (M_j)^c_d(M_l)^d_c\} = \\ &= (M_i)^a_b(M_j)^c_d\{(M_k)^b_a, (M_l)^d_c\} + (M_i)^a_b(M_l)^d_c\{(M_k)^b_a, (M_j)^c_d\}+ \\ &+ (M_k)^b_a(M_j)^c_d\{(M_i)^a_b, (M_l)^d_c\} + (M_k)^b_a(M_l)^d_c\{(M_i)^a_b, (M_j)^c_d\}.\end{aligned} \tag{6.23}$$

where we mean summation over repeated indices; using (6.21a), one can rewrite the left hand sides of (6.23) as

$$\{\mathrm{Tr}(M_iM_k), \mathrm{Tr}(M_jM_l)\} = \{n-4+s_{ik}^2, n-4+s_{jl}^2\} = 4s_{ik}s_{ij}\{s_{ik}, s_{ij}\}. \tag{6.24}$$

Now one has to distinguish between three essentially different cases, in correspondence with the different order of the indices.

• $i < k < j < l$ or $i < j < l < k$:

For $i < k < j < l$ all the addenda in the right hand side of (6.23) involve a Poisson bracket of the form (6.20) with correctly ordered indices. Here we write explicitly only the first one:

$$i\pi\mathrm{Tr}\Big(M_iM_jM_lM_k + M_iM_kM_lM_j - M_iM_lM_jM_k - M_iM_kM_jM_l \Big).$$

The others have a similar form, and it is easy to see that they cancel pairwise (the first with the second and the third with the fourth).

The same happens when $i < j < l < k$, since the only difference is a change of sign in the two last elements. Hence it follows

$$\{\mathrm{Tr}(M_iM_k), \mathrm{Tr}(M_jM_l)\} = 0 \qquad i < k < j < l \tag{6.25a}$$

$$\{\mathrm{Tr}(M_iM_k), \mathrm{Tr}(M_jM_l)\} = 0 \qquad i < j < l < k \tag{6.25b}$$

Using (6.21b) one immediately obtains equations (6.22d/e)

• $i < j < k < l$

Here the different order of the indices induces a change of sign in the second addendum, which becomes equal to the first. Equation (6.23) gives

$$\begin{aligned}\{\mathrm{Tr}(M_iM_k), \mathrm{Tr}(M_jM_l)\} =& 2i\pi\mathrm{Tr}\Big(M_iM_jM_lM_k + M_iM_kM_lM_j - M_iM_lM_jM_k - M_iM_kM_jM_l \Big) \\ =& 4i\pi s_{ik}s_{jl}(s_{ij}s_{kl} - s_{il}s_{kj}),\end{aligned}$$

where the last equality follows from Lemma 3.1. Using eq.(6.24) we obtain immediately eq. (6.22f)

• $i = j < k < l$ or $i < j = k < l$ or $i < j < k = l$

If two indices coincide, for instance $i = j < k < l$, the other two cases are analogous, we find

$$\begin{aligned}\{\mathrm{Tr}(M_iM_k),\mathrm{Tr}(M_iM_l)\} &= \{(M_i)^a_b(M_k)^b_a, (M_i)^c_d(M_l)^d_c\} = \\ &= (M_i)^a_b(M_i)^c_d\{(M_k)^b_a, (M_l)^d_c\} + (M_i)^a_b(M_l)^d_c\{(M_k)^b_a, (M_i)^c_d\}+ \\ &+ (M_k)^b_a(M_i)^c_d\{(M_i)^a_b, (M_l)^d_c\} + (M_k)^b_a(M_l)^d_c\{(M_i)^a_b, (M_i)^c_d\}.\end{aligned}$$

The first and the third addendum cancel, the last is zero (because $M_i^2 = \mathbb{1}$), and it remains:

$$\begin{aligned}\{\mathrm{Tr}(M_iM_k), \mathrm{Tr}(M_iM_l)\} &= 2i\pi\Big((\mathrm{Tr}(M_i^2M_lM_k) - \mathrm{Tr}(M_iM_kM_iM_l)\Big) \\ &= 2i\pi[(n-4+s_{kl}^2) - (n-4+s_{kl}^2+s_{ik}^2s_{il}^2 - 2s_{kl}s_{ik}s_{il})] \\ &= 2i\pi s_{ik}s_{il}(2s_{kl} - s_{ik}s_{il}),\end{aligned}$$

where the second equality follows from (6.21a) and (6.21b). Using (6.24) this leads to (6.22a/b/c).

2. It follows from the commutativity of the diagram (6.8), where all the arrows are Poisson maps

3. As we have said above, the eigenvalues of the monodromy matrices are the Casimir functions for this Poisson structure. Particularly, applying to M_∞ we obtain, due to (6.11), the needed statement. Practically it is more convenient to use the coefficients of the characteristical polynomial $det(S^{-1}S^T - \mu\mathbb{1})$ as the basic Casimirs.

4. Recall [D], that the natural action of the braid group B_n with n strands on the space of Stokes matrices is generated by the following transformations corresponding to the standard generators $\sigma_1, \ldots, \sigma_{n-1}$

$$\sigma_i : \; S \mapsto K_iS\,K_i$$

where the matrix $K_i = K_i(S)$ has the form

$$\begin{aligned}&K_{jj} = 1, \quad j = 1, \ldots, n; \quad j \neq i, i+1 \\ &K_{ii} = -s_{ii+1}, \quad K_{ii+1} = K_{i+1i} = 1, \quad K_{i+1i+1} = 0.\end{aligned}$$

Other matrix entries of K_i vanish. According to [D] this action describes the structure of analytic continuation of the solutions of MPDE. Our Poisson bracket is obviously invariant w.r.t. analytic continuation.

Q.E.D.

Example 1. $n = 3$. In this case the space of Stokes matrices has dimension 3. Denoting $x = s_{12}$, $y = s_{13}$, $z = s_{23}$ we obtain,

$$\begin{aligned}\{x,y\} &= \frac{i\pi}{2}(2z - xy) \\ \{y,z\} &= \frac{i\pi}{2}(2x - yz) \\ \{z,x\} &= \frac{i\pi}{2}(2y - zx).\end{aligned}$$

Our Poisson bracket coincides, within the constant factor $-\frac{i\pi}{2}$, with that of [D].

Example 2. $n = 4$. For convenience of the reader we write here down, omitting the constant factor $\frac{i\pi}{2}$, the Poisson bracket on the six-dimensional space of the Stokes matrices of the form

$$S = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\{p,q\} = (2x - pq) \qquad \{x,y\} = (2z - xy)$$
$$\{p,r\} = (2y - pr) \qquad \{y,z\} = (2x - yz)$$
$$\{q,r\} = (2z - qr) \qquad \{z,x\} = (2y - zx)$$

$$\begin{array}{lll} \{x,p\} = (2q - xp) & \{q,x\} = (2p - qx) & \{r,x\} = 0 \\ \{y,p\} = (2r - yp) & \{q,y\} = 2(pz - rx) & \{r,y\} = (2p - ry) \\ \{p,z\} = 0 & \{z,q\} = (2r - zq) & \{r,z\} = (2q - rz) \end{array} \tag{6.26}$$

The Casimirs of this Poisson bracket are

$$C_1 = -4 + p^2 + q^2 + r^2 + x^2 + y^2 + z^2 - pqx - pry - qrz - xyz + prxz$$

and

$$C_2 = 6 - 2(+p^2 + q^2 + r^2 + x^2 + y^2 + z^2) + 2(-pqx - pry - qrz - xyz) - 2(pqyz + qrxy) + p^2r^2 + q^2y^2 + r^2x^2$$

On the 4-dimensional level surfaces of the Casimirs the Poisson bracket (6.26) induces a symplectic structure. These surfaces and the symplectic structures on them are invariant w.r.t. the following action of the braid group B_4:

$$\sigma_1 : (p,q,r,x,y,z) \mapsto (-p, x - pq, y - pr, q, r, z)$$
$$\sigma_2 : (p,q,r,x,y,z) \mapsto (q - px, p, r, -x, z - xy, y)$$
$$\sigma_3 : (p,q,r,x,y,z) \mapsto (p, r - qz, q, y - xz, x, -z)$$

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