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Diffusion Approach to Non-equilibrium Extremal Dynamics

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Contents

Nor Art and Science serve, alone;
Patience must in the work be shown.
A quiet spirit plods and plods at length;
Nothing but time can give the brew its strength.
And all, belonging thereunto,
It's rare and strange, howe'er you take it.

J.W.Goethe

Abstract

In this work we analyze tow extremal dynamics models with the analytic tools of the one-dimensional diffusion motion.

In the second chapter we find analytically the critical regime of the first return in origin distribution for a diffusive motion generated by a scale invariant equation on the semi-positive axis.

In the third chapter we study the avalanche width dynamics in the one dimensional evolution model. The chapter has tow parts: in the first one we obtain an analytic solution using a mean field approximation of the original dynamics; in the second part we present numerical evidence sustaining the idea that the universality class of the avalanche width dynamics is described by a diffusion equation.

In the last chapter we present an analytically solvable model for the earthquakes distribution. The approach starts from a linear growth equation connecting the released energy with the activity of the fault zone. The Gutenberg-Richter law and the Omori's law are obtained within the model.

Chapter 1

Introduction

1.1 General survey

In Nature, besides the the simple thermodynamic systems, there is a large class of systems which exhibits complex spatial and temporal structure. Examples can by found in interface growth, disorder systems, fluid dynamics, biology, chemistry, etc. Despite of totally different dynamics many of these complex systems present self-similar structures and asymptotic algebraic decay for the distribution probability characterizing them. In equilibrium statistical physics such a situation occurred in phase transition phenomena. Geometrically the phase transition point was characterized by the change of the free energy manifold from a simple convex shape to a generic double well shape [1, 2, 3]. This idea has been extended to complex systems; the evolution takes place in a landscape profile which shape is rugged due to competing interactions between the component parts of the system. Although a general theory connecting the landscape properties with the dynamical properties is missing; one can obtain new insights studying simple models [4].

In this work we present detailed analysis for models describing system in which large size, intermittent phenomena occur. The classical example are interface depining, earthquakes, floods in a river basin, biological extinction[5, 6]. All these systems evolve through events whose distribution exhibits scale invariance over several orders of magnitude despite the fact that the underlying dynamics differs from case to case. Some spatial structures of these systems are also scale invariant. The models describing them are able to produce large events from simple dynamical rules governing the interaction of the constitutive agents. This is a conceptually different point of view, until recently the large fluctuations were thought to have causes external of the system. Almost ten years ago Bak, Tang and Wiesenfeld [5] proposed the concept of Self Organized Criticality (SOC) to explain processes far from equilibrium in which the system is driven by extremal dynamics. The system's activity is initiated by the region in which some relevant parameter reached a critical value. After the burst of activity, for example earthquakes or biological extinction, the evolution is stationary until in another region the critical threshold is reached. This kind of behavior has been studied mainly through simple numerical models. Over the years the number of models increased but the analytical result are still behind. In the following we describe shortly the well known models without pretending to be exhaustive.

The sand pile

It is the first proposed model and it is thought to be a paradigm for the field. On a d dimensional lattice grains of sand are added at random to the lattice sites. If the height of the a column of sand is higher than a given threshold, the grains of sand are distributed to the neighboring sites. In this way avalanches of all scales are generated; the distribution of the cluster sizes covered by an avalanche decay algebraic with a critical exponent $\tau \approx 0.98$ [5]. There is an analytical solvable model [7] called the Abelian sand pile. The boundary properties of the avalanches scaling have been also investigated [8].

The evolution model

Conceptually it is the simplest model. To each site of a d dimensional lattice random number are assigned from a flat distribution on $(0,1)$. At each time step the site with the smallest number and its $2d$ neighbors are updated independently with new random numbers. In the stationary states the density of the random numbers is vanishingly small for $x < \lambda_{critical}$ and it is constant above $\lambda_{critical}$ [9]. An avalanche is the interval between two consecutive configurations with all the random number higher than $\lambda_{critical}$. The avalanche distribution and the activity first return distribution are scale invariant at large times. The spatial structure of an avalanche also exhibits scale invariance. Using numerical results and scaling arguments it has been shown that there are tow independent critical exponents characterizing the system [6].

Initially the model was proposed as an explanation for the large time dynamics of the ecosystem, the random numbers on the lattice site are thought to be a coarse description of the fitness for a given species. Nevertheless its agreement with the recorded data is poor [10]. Analytical solutions are available for the infinite range model [11, 12]. Using an improved mean field approximation we derived analytically a value of $\lambda_{critical}$ which is in a very good agreement with the numerical data. In chapter 2 we present in detail this result and a diffusion equation which describes the universality class of the model. Conceptually similar models have been proposed for interface growth [13] or flux creep [14].

Models for earthquakes and the Gutenberg Richter law

The first model which successfully described the universality observed in the earthquake energy distribution was proposed in the reference [15]. Its dynamic rules account for the interplay between the elastic and friction forces acting in the fault zone. The model idealizes the interaction between solid blocks connected by springs and situated between two moving plates. The dynamics is as follows:
(1) Each site of a bidimensional lattice is initialized random between 0 and F_{th}.
(2) If any $F_{i,j} > F_{th}$ then redistribute the force on $F_{i,j}$ to its neighbors according to the rule

$$\begin{aligned} F_{n,n} &\rightarrow F_{n,n} + \alpha F_{i,j} \\ Fi,j &\rightarrow 0 \end{aligned} \tag{1.1}$$

where $F_{n,n}$ are the strains of the four-nearest neighbors and α is constant depending on the elastic constants of the model.
(3) Repeat step 2 until the earthquake is fully evolved.
(4) Locate the block with the largest strain, F_{max}. Add $F_{th} - F_{max}$ to all sites and return to step 2.
The released energy decays algebraic; the critical exponent is close to the observed values. It depends on the constant α allowing for the variations observed in nature. A relatively similar model has been proposed in the reference [16].

The Gutenberg Richter law also was obtained from a model whose dynamics emerges from the fractal structure of the fault zone [17]. The critical exponent depends upon the fractal dimension of the curve modeling the fault.

In chapter 3 we present an analytically solvable model which also explains the variation of the critical exponents.

1.2 A local equation

All the above phenomena are interface processes: the individual composing a species are the interface between the background genotype and the external factors, the earthquakes appear predominantly at the contact zone of the tectonic plates [18], the avalanches on a sand pile do not affect the bulk of the pile. When an avalanche takes place we may consider it as an isolated system because its life time is much more smaller than the characteristic times of the stationary evolution. The part of the interface where the avalanche takes place undergoes a relaxation since it is a system whose constrains have been released. Therefore there is a quantity which is increasing during the avalanche accounting for the irreversible nature of the process .e.g.: in a sand pile the energy dissipated by friction of the falling grains, in an earthquake the released energy from the fault zone. If the interface is a thermodynamic system we can consider the Boltzmann entropy as monotonic parameter of time. A second parameter of interest is the the activity a which we assume to be a scalar equal to the number of agents involved in an avalanche. The avalanche stops when the activity reach zero value. Instead of time, we want to use as a parameter for describing the system's evolution the Boltzmann entropy or the released energy; generically we denote it by S. This way will allow us to propose a diffusion equation for the probabilities describing the avalanche, without involving the time coordinate. Through this procedure we try to find the most simple relation connecting the system's parameters. We think that this procedure can be useful in systems whose dynamics has no elementary time scale therefore the time evolution is not relevant.

We are interested in finding an equation for the probability $Q_S(a, a')$ of the avalanche to pass from the state of activity a to the state of activity a' releasing and amount of entropy(energy) S.

The basic postulates are: the interface has a characteristic released entropy(energy) for each jump $s_c(a)$ independent of history. The probability for a jump from $x = a$ in the set Γ of final states is $K(a, \Gamma)$. We impose the condition

$$K(a = 0, \Gamma) = 0$$

accounting for the fact that the avalanche stops when the activity vanishes.

In analytical term these postulates lead to an integral equation for the transition probabilities

$$Q_S(a, \Gamma) = \frac{1}{s_c(a)} \int_0^S e^{-\frac{S'}{s_c(a)}} dS' \int_\Omega K(a, dy) Q_{S-S'}(y; \Gamma)$$

For a set Γ containing a we must add the probability that no jump occurs before S has been released and thus we get

$$Q_S(a, \Gamma) = e^{-\frac{S}{s_c(a)}} + \frac{1}{s_c(a)} \int_0^S e^{-\frac{S'}{s_c(a)}} dS' \int_\Omega K(a, dy) Q_{S-S'}(y; \Gamma)$$

Differentiation with respect to S then reduces the tow equation to the same form

$$\frac{\partial Q_S(a, \Gamma)}{\partial S} = -\frac{1}{s_c(a)} Q_S(a, \Gamma) + \frac{1}{s_c(a)} \int_\Omega K(a, dy) Q_S(y; \Gamma)$$

This is the Kolmogorov's backwards equation written in the parameter S. The conceptually difficult point is to obtain an expression for the kernel $K(a,\Gamma)$. We can find a differential equation for it if we assume the hypothesis that for a small amount of released S the activity cannot change by a large value. This implies that the activation or the deactivation of the agents produce some irreversible effect. Mathematically this condition reads:

$$\frac{1}{S}\int_{|a-a'|>\delta} Q_S(a,da') \to 0 \quad S \to 0.$$

Under a broad range of condition [19] this implies that the density $q_S(a,a')$ satisfies a diffusion equation:

$$\frac{\partial q_S(a,a')}{\partial S} = d(a)\frac{\partial^2 q_S(a,a')}{\partial a^2} + v(a)\frac{\partial q_S(a,a')}{\partial a}$$

with $d(a)$ a positive function. The two functions $d(a), v(a)$ are the local variance and the local drift characterizing the diffusion process. The condition that the avalanche stops when the activity is zero requires the boundary condition

$$q_S(0,a) = 0.$$

The same kind of reasoning can be made if the activity is a vectorial field. In this case $d(a)$ is the covariance matrix and $v(a)$ is the drift vector. The probability of first return to the origin gives the avalanche distribution as function of the released quantity S. One can progress analytically if the functions $d(a)$ and $v(a)$ have simple forms. In the chapter III we use a linear growth hypothesis to formulate an analytically solvable model for earthquake distributions.

The thesis is structured as follow: In Chapter **2** we present the connection between the critical exponent of a scale invariant drift and the critical exponent of the first return time distribution for diffusive motion on the semi-positive axis. The mathematical results presented in this chapter will be used in the next two chapters.

In Chapter **3** we present a new approach to the evolution model based on the avalanche width dynamics. There are two main results. The mean field approach gives a $\lambda_{critical}$ in a very good agreement with the numerical results. The diffusion approach establish a connection between the critical exponent of the avalanche distribution and the critical exponent of the local variance at a given width. This section is a combination of numerical and analytic work.

Chapter 4 presents an analytically solvable model for the earthquake distribution. We implemented the diffusion equation using an linear growth hypothesis connecting the released energy and the activity of the fault during the earthquake. The Gutenberg-Richter law and Olami's law are retrieved together with an explanation for the variability of the critical exponents which is observed in nature.

The detailed mathematical computation are presented in the appendices A, B, C.

In Appendix D we reproduce, for completeness, the main mathematical results used in this thesis.

Chapter 2

Dynamical Phase Transition in 1D

2.1 Introduction

Random walk and continuous time random walk are invaluable tools in the analysis of the non-equilibrium statistical phenomena. The strength comes from the fact that these stochastic processes offer the simplest, yet not trivial, way to start the analysis of a physical system in which the interaction competes with a local noise. Their properties have been analyzed extensively on translation invariant lattices as well in disorder media [20, 21, 22, 23].

In the avalanche dynamics random walks are used as first approximation in deriving analytical results [11]. The first return time in origin probability distribution is connected with the avalanche distribution since an avalanche stops when the generic activity is zero. In one dimension, for a homogeneous random walk in the absence of any drift, the distribution of first return decay algebraic with the critical exponent 3/2. If a constant drift is turned on the probability distribution decay exponentially. A negative drift bounds the random walk in vicinity of the origin; a positive constant drift drive away the walker, the origin in not anymore a recurrent state, there is an positive probability that the random walk will not return in the origin.

In this chapter we show that if the drift added decay to zero fast enough the algebraic behavior is preserved in the certain region of the parameter space. Instead of the discreet case we shall use the diffusion equation which is the exact continuum limit [19] of a Markovian continuous time random walk. We start from a diffusion process described by the equation:

$$\frac{\partial p}{\partial t} = \frac{1}{x^{\alpha}}\frac{\partial^2 p}{\partial t^2} + \frac{v}{x^{\beta}}\frac{\partial p}{\partial x}, \quad x > 0. \tag{2.1}$$

where α, β, v are constants; the constant diffusion being set to 1. The functions $1/x^\alpha$ and v/x^β represent the local variance and the local drift obtained by a limiting procedure [19]. This is the backward equation and it has the same Green function as the forward (Fokker-Plank) equation. The time Laplace transform leads us to the ordinary differential equation:

$$\frac{1}{x^\alpha}p'' + \frac{v}{x^\beta}p' - \lambda p = p(x,0) \tag{2.2}$$

the Green function of this equation is constructed using tow positive, linear independent solution $\xi_\lambda, \eta_\lambda$ including the boundary conditions [19].

$$G_\lambda(x,y) = \begin{cases} \frac{\xi_\lambda(x)\eta_\lambda(y)}{W(y)} & if \quad y > x \\ \frac{\eta_\lambda(y)\xi_\lambda(x)}{W(y)} & if \quad y < x \end{cases} \tag{2.3}$$

where $W(y) = (\xi'_\lambda(y)\eta_\lambda(y) - \xi_\lambda(y)\eta'_\lambda(y))x^{-\alpha}$ is the Wronskian associated to the equation. The probability distribution of the first return in the origin can be obtained from the derivative in respect with time of the total mass

$$J(t) = \int_0^\infty G_t(x,y)dy$$

therefore we shall concentrate on the behavior for small λ of the Laplace transform of the total mass. The connection with the asymptotic behavior is made through the Tauberian theorem.

2.2 The case $\beta = \alpha + 1$

In this case the equation (2.1) is invariant to a scale transformation $t \to ct$, $x \to c^{\frac{1}{\alpha+2}}x$. Setting $\alpha = 0$, $v = 0$ we have the classical diffusion equation, $\alpha = -1$, $v = 0$ gives the linear growth equation. The solutions appearing in the equation 2.3 can be expressed in terms of Bessel function, see Appendix D.

$$\xi_\lambda(x) = x^k I_\nu(\delta x^\gamma) \tag{2.4}$$

$$\eta_\lambda(x) = x^k K_\nu(\delta x^\gamma) \tag{2.5}$$

where we performed a variable change $x \to \lambda^{-\frac{1}{\alpha+2}}x$, $k = (1-v)/2$, $\nu = \pm(1-v)/(\alpha+2) > 0$, $\gamma = 1 + \alpha/2$, $\delta = 1/\gamma$; I_ν and K_ν are the Bessel function of imaginary argument of first and third class [24]. The Wronskian can be easily computed

$$W(x) = \gamma x^{-\alpha-v}.$$

The Green function reads:

$$G_\lambda(x,y) = \begin{cases} \frac{1}{\gamma} x^k I_\nu(\delta x^\gamma) y^{1-k-\alpha} K_\nu(\delta y^\gamma) & if \quad y > x, \\ \gamma y^{1-k-\alpha} I_\nu(\delta y^\gamma) x^k K_\nu(\delta x^\gamma) & if \quad y < x \end{cases} \tag{2.6}$$

and the Laplace transform of the total mass is

$$\begin{aligned} J_\lambda(x) &= \frac{\lambda^{-1}}{\gamma} x^k K_\nu(\delta x^\gamma) \int_0^x dy\, y^{1-k+\alpha} I_\nu(\delta y^\gamma) \\ &+ \frac{\lambda^{-1}}{\gamma} x^k I_\nu(\delta x^\gamma) \int_x^\infty dy\, y^{1-k+\alpha} K_\nu(\delta y^\gamma) \end{aligned} \tag{2.7}$$

In computing the asymptotic behavior of the total mass we distinguish between the regimes: $v < 1$, when $\nu = (1-v)/(\alpha+2)$, and $v > 1$ when $\nu = -(1-v)/(\alpha+2)$.

A. The case $v < 1$.

Using the properties of the Bessel function one can write the total mass as:

$$\begin{aligned} J_\lambda(x) &= \frac{\lambda^{-1} x^\gamma}{\gamma} \frac{\pi/2}{\sin \nu\pi} \left[I_{\nu-1}(\delta x^\gamma) I_{-\nu}(\delta x^\gamma) - \frac{(\delta x^\gamma)^{\nu-1}}{2^{\nu-1}\Gamma(\nu)} I_{-\nu}(\delta x^\gamma) \right. \\ &\left. + \frac{(\delta x^\gamma)^{\nu-1}}{2^{\nu-1}\Gamma(\nu)} I_\nu(\delta x^\gamma) - I_\nu(\delta x^\gamma) I_{1-\nu}(\delta x^\gamma) \right] \\ &+ \frac{\lambda^{-1} x^k}{\gamma} I_\nu(\delta x^\gamma) C_\nu \end{aligned} \tag{2.8}$$

where C_ν is a positive constant

$$C_\nu = \frac{1}{\gamma} \int_1^\infty dy y^{1-\nu} K_\nu(\delta y) + \frac{\pi/2}{\sin \nu\pi} (I_{1-\nu}(\delta) - I_{\nu-1}(\delta))$$

Recovering the variable λ by $x \to \lambda^{\frac{1}{\alpha+2}} x$ we can analyze the asymptotic behavior.

In the case $\nu < 2$ the last term of the formula (2.8) will dominate when $\lambda \approx 0$ if $\nu < 1$ we have

$$J_\lambda(x) \approx \lambda^{-1+\nu}.$$

If $1 < \nu < 2$, under a proper normalization we have

$$J_\lambda(x) \approx 1 - A\lambda^{-1+\nu}$$

with A depending on x and ν. From the Tauberian theorem [19] we conclude that the asymptotic behavior of the total mass is

$$J_t(x) \approx t^{-\nu}$$

and the critical exponent of the first return time is

$$\tau = 1 + \nu = 1 + \frac{1-v}{\alpha+2}.$$

One can see that in the case of a null drift we recover $\tau = 3/2$ for simple diffusion ($\alpha = 0$) or the linear growth exponent $\tau = 2$ when $\lambda = -1$.
In the case in which $\nu > 2$ the linear term present in the first part of the equation 2.8 will prevail, therefore the asymptotic behavior will be an exponential decay. Some calculation shows that the characteristic time diverge as

$$tc \approx \frac{1}{\nu - 2} \text{ as } \nu \to 2^+.$$

B. The case $v > 1$
In this case the computations are easier, the total mass expression is:

$$J_\lambda(x) = \frac{\lambda^{-1}}{1+\gamma} x^k I_\nu(\delta x^\gamma) A_\nu$$

where

$$A_\nu = \frac{1}{\gamma}\int_0^\infty dy\, y^{1-\nu} K_\nu(\delta y) = \frac{2^{-\nu}}{\gamma}\delta^\nu \Gamma(1-\nu)$$

is a constant independent of x and λ. After the substitution $x \to \lambda^{\frac{1}{\alpha+2}} x$ one can see that as $\lambda \to 0$

$$J_\lambda(x) \approx \lambda^{-1}$$

This means that the diffusion process has no more adsorbing condition in the origin, as equation (2.6) shows. For a discreet model this situation corresponds to the case in which the origin is not a recurrent state, there is a positive probability for the random walk to escape to infinity. We summarize all the cases in the following table:

$\nu = \frac{1-v}{\alpha+2}$	the dominant term in $J_\lambda(x)$	the asymptotic behavior of total mass $J_t(x)$
$\nu < 0$	λ^{-1}	constant
$0 < \nu \leq 2$	$\lambda^{-1+\nu}$	algebraic decay $t^{-\nu}$
$2 < \nu$	$1 - t_c\lambda$	exponential decay, $t_c \approx (\nu - 2)^{-1}$

For $0 < \nu \leq 1$ the average of the first return time distribution diverges. For $1 < \nu \leq 2$ the variance diverges. Formally we may connect this situation with the phase transition classification scheme.

2.3 The general case

In the homogeneous equation

$$\frac{1}{x^\alpha}p'' + \frac{v}{x^\beta}p' - \lambda p = 0 \tag{2.9}$$

we make the substitution $x \to \lambda^{-\frac{1}{\alpha+2}}x$ and we get

$$\frac{1}{x^\alpha}p'' + \frac{v\lambda^{\frac{\beta-\alpha-1}{\alpha+2}}}{x^\beta}p' - p = 0$$

When $\beta > \alpha+1$ the drift term becomes vanishingly small in the limit $\lambda \to 0$. This scaling argument suggest that the critical exponent preserve its value

$$\tau = 1 + \frac{1}{\alpha+2}.$$

Using convexity properties of the total mass near the origin we proofed rigorously in the Appendix A that the above relation hold.

When $\beta < \alpha+1$ equation (2.9) shows that the drift became more important as the time increases. The random walk will be bounded to origin for a negative drift $v < 0$ and will escape to infinity for a positive one $v > 0$. We checked the these statements numerically simulating a random walk with $\alpha = 0, v = -0.5$. The observed asymptotic behavior of the first return time distribution fits with the analytic results presented in this chapter.

2.4 Final remark

The expression for the first return time in origin of the critical exponent are not connected with the divergence of the local variance and local drift in origin but with their scale invariance. We can choose $a > 0$ as an absorbing boundary condition, in this case the new Green function is

$$G_\lambda^\#(x,y) = G_\lambda(x,y) - \phi_\lambda(x,a)G_\lambda(a,y) \tag{2.10}$$

where

$$\phi_\lambda(x,y) = \left(\frac{x}{y}\right)^k \frac{K_\nu(\delta\lambda^{\frac{1}{2}}x^\gamma)}{K_\nu(\delta\lambda^{\frac{1}{2}}y^\gamma)}, \quad x > y \tag{2.11}$$

is the Laplace transform of the first arrival in a distribution (see Appendix D). Scaling $\lambda \to c\lambda$, $x \to c^{-\frac{1}{\alpha+2}}$, $c \to 0$ equation (2.10) becomes

$$G_\lambda^\#(x,y) = G_\lambda(x,y) - c^{-\nu}f(a,x,y,\lambda)$$

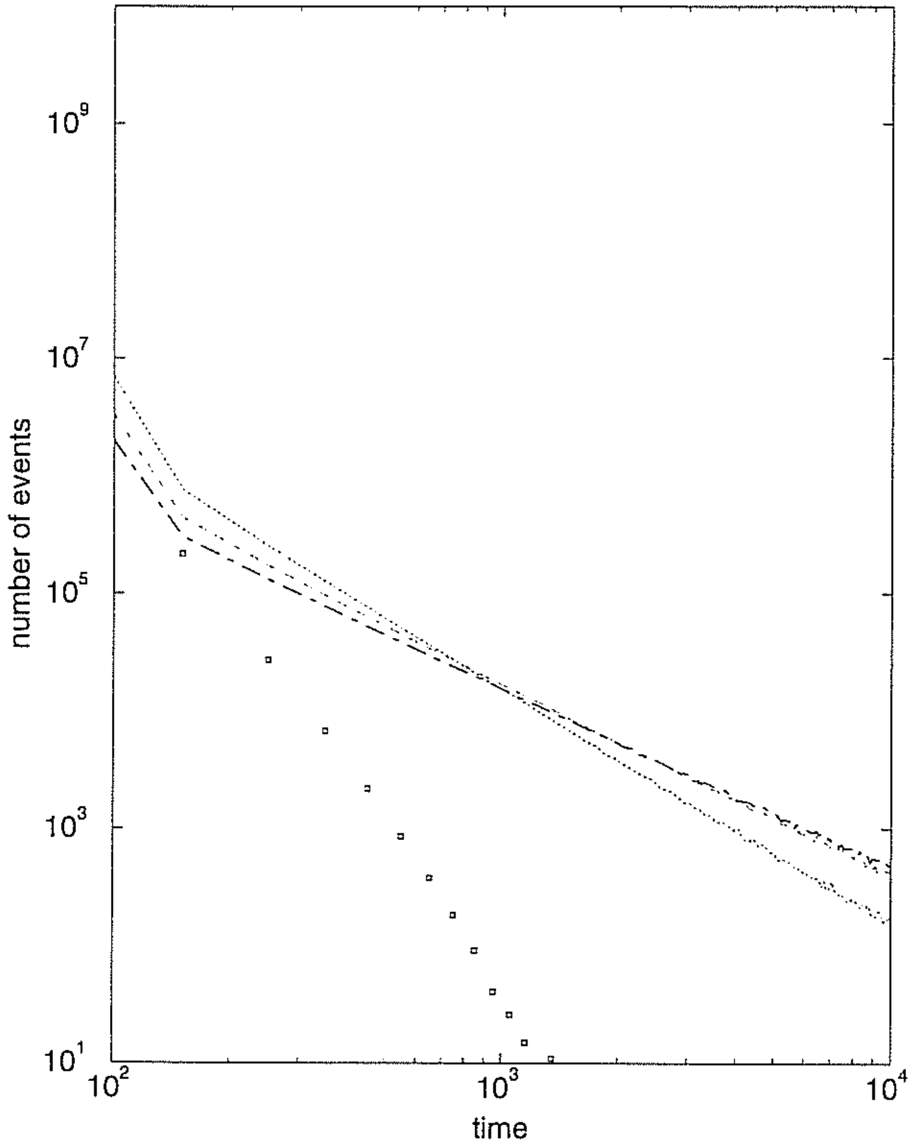

Figure 2.1: The asymptotic behavior of the first return time distribution for $\alpha = 0$; $\beta = 0.5(dots)$, $1(dotted - line)$, $1.5(dashedline)$, $2.5(dot - dashedline)$. One can see that the two last line approach each other asymptotically

with f a scale invariant function. The previous formula shows that in the case $\nu > 0$ the position of the point a is irrelevant. This feature is useful in perturbation computations since it avoids the divergence present in the origin. In this sense one can say that the equation (2.1) is renormalizable.

Chapter 3

One Dimensional diffusion process and the Bak & Sneppen evolution model

3.1 Introduction

Bak & Sneppen (BS)[9] model has been formulated as a coarse grained description of the biological evolution. It originates from models which tries to find an adaptive dynamics of the genotype in the space of mutations [25]. Using simple local dynamical rule the model is able to reach a stationary state in which the distribution of large size events is scale invariant. From this property one can argue that large events in biological system are characteristic to the evolution and not produced by external agents. Its mathematical simplicity attracted numerical studies [9], [26], [27] and mean-field analytic treatments [12], [11].

The model treats a number of N species interacting on an one dimensional chain (a simple pictures of the food chain). Each species has assigned a scalar parameter, called fitness, with values in $(0,1)$ interval. It is thought as a measure for the adaptability of species to the ecosystem in a coarse grain description. One step of the dynamics consists in choosing the site with the smallest fitness then new random independent values from $(0,1)$ interval are attributed to this site and to its two neighbors with a uniform distribution. For this system we define a λ-avalanche ($0 < \lambda < 1$) as the number of steps between two consecutive configuration with all the fitness values greater than λ. The spatial and temporal properties of the model presents scale invariance; through numerical simulation and scaling ansatz it has been shown that there are two independent exponents [6]. In the next sections we deal with three exponents: the critical exponent τ of the temporal distribution of the avalanches

$$p(t) \sim t^{-\tau}, \quad \tau \to \infty \tag{3.1}$$

the exponent γ characterizing the divergence of the temporal average as $\lambda \to \lambda_{critical}$

$$\bar{t} \sim |\lambda - \lambda_{critical}|^{-\gamma} \tag{3.2}$$

the avalanche mass dimension critical exponent D is describes the temporal evolution of the average avalanche spatial width

$$\bar{w}(t) \approx t^{\frac{1}{D}}. \tag{3.3}$$

We mention that the model shows also critical behavior in more than one dimension or with slightly modified dynamics [26], [27].

In this chapter we present two approaches based on random walk theory which focus on the dynamics of the avalanche spatial width. The difficult point is the presence of the memory effects in the width dynamics. In the section 3.3 we present a mean field approach but in the section 4.4 we are proposing a general diffusion equation which has the universality class of the model.

The chapter is organized as follow: Section 3.2 presents the general master equation for the fitness distribution and the derivation of the mean field equation found with probabilistic arguments in Ref. [9]. Section 3.3 introduces our analytical treatment based on a mapping in a random walk problem. In this approximation we compute exactly the value of $\lambda_{critical}$ and the critical exponents τ and γ. Section 4.4 presents a continuum diffusion equation which embeds the universality of BS model.

3.2 The Master Equation

The BS model is completely characterized by the probability $P(x_1, x_2, \ldots, x_N; t)$ to find the system in the state $(x_1, x_2, \ldots, x_N)$ at the time t given the initial distribution at $t = 0$. Because there are not memory effects, the evolution of the system is described by the following master equation

$$\begin{aligned} P(x_1, x_2, \ldots, x_N; t+1) &= \sum_i \int dx'_i dx'_{i-1} dx'_{i+1} P_{st}(i; x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1} \ldots, x_N) \\ &\quad \times P(x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1} \ldots, x_N; t), \end{aligned} \tag{3.4}$$

where periodic boundary conditions were assumed and $P_{st}(i; x_1, \ldots, x_N)$ is the probability to have activity at site i if the system is in the configuration

$(x_1, \ldots x_N)$. For the original one dimensional BS model

$$P_{st}(i; x_1, x_2, \ldots, x_N) = \prod_{j \neq i} \theta(x_j - x_i), \tag{3.5}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

At stationarity, integrating in (1) over $x_2, \ldots, x_N$ we get easily the following relation:

$$P_{ac}(1, x_1) + P_{ac}(2, x_1) + P_{ac}(N, x_1) - \frac{3}{N} = 0; \tag{3.6}$$

where

$$P_{ac}(i; x_j) = \int dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_N P_{st}(i; x_1, \ldots, x_N) \times P(x_1, \ldots, x_N)$$

is the the probability to have activity in site i when in site j the fitness has the value x_j. If in equation (3.6) we try a stationary self-consistent mean field solution of the form $p(x_1, \ldots, x_N) = p(x_1)p(x_2)\ldots p(x_N)$ after some algebra we get

$$\Big(1 - \frac{2}{N-1}\Big) Q^N(x) + \frac{2N}{N-1} Q(x) + 3x - 3 = 0 \tag{3.7}$$

with $Q(x) = \int_x^1 p(x')dx'$. Equation (3.7) was previously obtained in Ref.[12], in the $N \to \infty$ limit one finds $p(x) = 3/2 \quad x \in (\lambda_{critical}, 1)$ and $p(x) = 0$ when $x \in (0, \lambda_{critical})$, $\lambda_{critical} = 1/3$ whereas numerically $\lambda_{critical} \approx 2/3$ [9]. The statistical independence between the sites in the mean field solution allows the reduction of the problem at an one dimensional random walk on the positive semi-axis where the state n represents the state of the system with n fitness values greater than λ. The solution developed in Ref. [11] gives the same $\lambda_{critical}$ as predicted by equation.(3.7) and the critical exponents $\tau = 3/2, \ \gamma = 1$.

3.3 The random walk approach to the avalanche width dynamics

We remember here that the size of an λ avalanche is the number of steps between two consecutive events with no fitness below the the value λ, so, it is a quantity characterizing time intervals. For a system of size N, with free boundary conditions, we define the avalanche width at a given moment t as the number of sites between the most left species with the fitness less than λ and the most right species with the fitness less than λ. The species between these two sites can have any value of the fitness. This is a quantity which characterize the spatial structure of our system. The width dynamics has memory effects for the originally proposed dynamics, in the spirit of the mean-field we approximate the evolution of the avalanche width as following: at every step the species between right and left extrema are updated independently. Eventually one update also the right nearest neighbor of the right extremum species accounting for the fact that in the original dynamics an avalanche can increase its with only with one step. The complete randomness makes the movements of the two extrema completely equivalent and for this reason we have chosen to move only in one direction. In origin we also accept the double step.

With this change the avalanche width is a random variable without memory effects on a discreet set of states which now can be extended to the entire non-negative semi-axis with the state zero corresponding to the state with no species below λ and the state n to a realization of BS model with with n sites between the most left and the most right sites with the value of their corresponding fitness less than λ. The transition matrix of the model has the following form:

$$p = \begin{pmatrix} (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \dots \\ (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \dots \\ (1-\lambda)^3 & 3\lambda(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & \dots \\ (1-\lambda)^4 & 4\lambda(1-\lambda)^3 & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.8}$$

The formulas for the matrix elements are:

$$\begin{aligned} p_{00} &= (1-\lambda)^2,\ p_{01} = 2\lambda(1-\lambda),\ p_{02} = \lambda^2,\ p_{0j} = 0,\ j > 2; \\ p_{j0} &= (1-\lambda)^{j+1}, \quad j \geq 1; \\ p_{j1} &= (j+1)\lambda(1-\lambda)^j, \quad j \geq 1; \\ p_{j2} &= j\lambda^2(1-\lambda)^{j-1}, \quad j \geq 1; \\ p_{jl} &= \begin{cases} p_{j+1,i-1} & \text{if } l \geq 2 \text{ and } j \leq l-1; \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.9}$$

with the convention that p_{ij} is the transition probability from state the i to the state j and $i, j \in \{0, 1, 2, \dots\}$. The distribution probability of avalanches is the first return probability distribution for this random walk and it can be written as

$$p(n+2) = \sum_{i=1}^{\infty} p_{01}\tilde{p}_{1i}^{(n)} p_{i0} + \sum_{i=1}^{\infty} p_{02}\tilde{p}_{2i}^{(n)} p_{i0} \tag{3.10}$$

where $\tilde{p}_{ij}^{(n)}$ is the i, j element of the n-th power of the matrix $\tilde{p}$ obtained from the matrix p removing the row and the column zero and it is describing the evolution of the random walk outside of the origin. The first (second) term in the r.h.s. of equation.(3.10) represents the first return probability, after n steps, when the initial step is single, (double). For a site different from the origin the forward step can only be single, as matrix p shows.

As we are concerned only with asymptotic behavior of the model we may modify the first two columns of the transition matrix p such that to have the same elements on the diagonals of the $\tilde{p}$ matrix. Keeping the closure relation $\sum_j \tilde{p}_{ij} = 1$ we produce the following matrix:

$$p' = \begin{pmatrix} (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \dots \\ (1-\lambda)^2(1+2\lambda) & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & 0 & \dots \\ (1-\lambda)^3(1+3\lambda) & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & \dots \\ (1-\lambda)^4(1+4\lambda) & 4\lambda^2(1-\lambda)^3 & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.11}$$

The asymptotic behavior of the first return time distribution is the same for the both random walks described by the matrices p and p'. In fact in equation. (3.10) we can go a step further developing $\tilde{p}_{1i}^{(n)}$ with respect to site 1. The remaining matrix from the second column and second row is identical with the $\tilde{p}'$ matrix obtained from p' in the same way as $\tilde{p}$ from p.

$$p_{1i}^{(n)} = \sum_{\substack{n_1+\dots n_j \\ =n-n'}} \prod_{l=1}^{j} \Big(\sum_{k=2}^{\infty} \tilde{p}_{12} \tilde{p}_{2k}^{(n_l)} \tilde{p}_{k1} \Big) \Big((1-\delta_{1i}) \tilde{p}_{12} \tilde{p}_{2i}^{(n')} + \delta_{1i} \tilde{p}_{11}^{(n')} \Big) \tag{3.12}$$

with $n = n'+n_1+\dots n_j$ and δ_{ij} the Kronecker symbol. The terms in equation. (3.12) represent multiple returns in the site 1 before the last step to the origin. In the $n \to \infty$ limit the terms with n' and all the n_j bounded but one have the same asymptotic behavior as $\tilde{p}'^{(n)}_{1i}$ because they are generated by the same matrix; the other terms will decay exponentially, due to $\tilde{p}_{12}^{j}$ factor, or as a power of the leading term when there are two ore more unbounded exponents. If $i = 1$ n' has to be bounded to avoid the exponential decay.

In the Appendix B we present the computation for the generating function of the avalanches distribution probability (B.16). Using the generating function $R(z) = \sum_t z^t P(t)$ the average time for the avalanche distribution is

$$\bar{t} = \frac{dR(\xi)}{dz}\Big|_{z=1} = \frac{dR(\xi)}{d\xi}\frac{d\xi}{dz}\Big|_{z=1}. \tag{3.13}$$

From (B.16) we obtain that the mean time of an avalanche can be written as

$$\bar{t} \approx \Big|\lambda - \lambda_{critical}\Big|^{-\gamma} \tag{3.14}$$

when $|\lambda - \lambda_{critical}| \ll 1$, with the critical exponent $\gamma = 1$ and the critical value of λ, $\lambda_{critical} = 2/3$. We can also compute the asymptotic behavior of the avalanches probability distribution using equation (B.18)[19]. For $\lambda = 2/3$ we found

$$p(t) \approx t^{-\tau}, \quad t \to \infty, \tag{3.15}$$

with $\tau = 3/2$. For $\lambda \neq 2/3$ the decay is exponential. In the previous equations the critical exponent $\tau = 3/2$ and γ have values as obtained in the mean field solution [11], whereas $\lambda_{critical} = 2/3$ is in extremely good agreement with the critical value of λ found in numerical experiments [9], [6], [27].

In language of Markov chain one can say that $\lambda = 2/3$ is the transition point between persistent states ($\lambda \leq 2/3$) and transient states ($\lambda > 2/3$) [21]. Nevertheless λ is not a dynamical parameter for BS model, it introduces an "observational window" for a certain variable which we may choose from the set of statistical variables compatible with the dynamics of the BS model. SOC appears when there is at least one statistical variable with events at all scale lengths. In our approach λ-avalanches are bounded to origin for $\lambda < 2/3$ and they escape to ∞ for $\lambda > 2/3$. At $\lambda = 2/3$ we have the peculiar stationary state in which the average time of avalanches is diverging, therefore there are events on the all time scales. The approximation can be extended to arbitrary dimensions. The active sites can be included in a minimal convex volume V. We update independently all the sites in V and the sites next to V. The maximum diameter of this set will have the dynamics described by the transition matrix p, therefore the critical behavior will remain unchanged.

3.4 The continuous time random walk universality class

The main difference between the approximation proposed in the previous section and the BS model is that in our approximation the avalanche width may change at every step, meanwhile in the original dynamics the activity will fall inside of the avalanche and therefore the system will spend a characteristic time at a given size of the avalanche. A simple stochastic process with this kind of behavior is the continuous time random walk (CTRW).

The probability that the avalanche increasing or decreasing its width one or two steps is determinate by the dynamic rule of the system; on the other hand the probability for a step higher than tow sites depends upon the probability to have next to the extreme site an interval with all the fitness higher than λ. For a finite width the distribution of the "empty" sites will self average in time, therefore for large times there is a stationary probability of jumping from a given width to another one. The finite size of the width imply also that the time in which the activity will reach the boundary is exponential decaying for large time. These arguments make plausible the idea that a CTRW can describe the asymptotic behavior of the BS model. As we are interested in asymptotic behavior we take the continuum limit. In this frame the system is described by a diffusion equation of the type

$$\frac{\partial p}{\partial t} = a(x)\frac{\partial^2 p}{\partial x^2} + b(x)\frac{\partial p}{\partial x} \tag{3.16}$$

where $a(x)$ and $b(x)$ are the local variance and the local drift obtained by the limiting procedure from the discrete continuous time random walk. [19]. Avalanches of all sizes appear if the drift vanishes as the width increases. The functional form of this decay has influence on the critical exponents, as we have shown in the previous chapter. From the null drift condition we have for the BS model:

$$\lambda = \int^{\infty} p(i)i \tag{3.17}$$

where $p(i)$ is the transition probability of a backward jump. We note here that the null drift condition yields an lower bound for the $\lambda_{critical}$. Considering only the first two backward steps we have

$$\lambda = 3(1-\lambda)^2\lambda - (1-\lambda)\lambda^2,$$

with the positive solution $\lambda_{critical} = 0.5$. The backward jumps higher than two steps will increase $\lambda_{critical}$.

At criticality the distribution of the backward jumps has a algebraic tail as figure 3.1 shows, hence for a finite width avalanche i we have the drift of the transition probabilities proportional with

$$\int_i^{\infty} p(i)i \approx i^{-\beta+2} \tag{3.18}$$

This scale invariant structure will imply that the local variance of the transition probabilities will behave as

$$a(i) \approx c + d\, i^{-\beta+3} \quad i \gg 1 \tag{3.19}$$

with c, d, β constants. The asymptotic behavior for the coefficients of the equation (3.16) are obtained dividing the previous two expressions with the mean life time of an avalanche at each site. Numerically we found that the mean life-time behaves asymptotically as

$$\bar{t}_i \approx i^p \qquad p = 1.00 \pm 0.01$$

Hence the general equation describing the asymptotic behavior of the BS model is

$$\frac{\partial p}{\partial t} = \frac{K'}{x^p}\frac{\partial^2 p}{\partial x^2} + \frac{K}{x^\alpha}\frac{\partial^2 p}{\partial x^2} + \frac{v}{x^{\alpha+1}}\frac{\partial p}{\partial x} \tag{3.20}$$

where K, K', v are positive constants and $\alpha = \beta - 3 + p$. A change of scale $t \to ct$, $x \to c^{\frac{1}{\alpha+2}}$ $c < 1$ transforms the above equation in

$$\frac{\partial p}{\partial t} = c^{1-\frac{p+2}{\alpha+2}}\frac{K'}{x^p}\frac{\partial^2 p}{\partial x^2} + \frac{K}{x^\alpha}\frac{\partial^2 p}{\partial x^2} + \frac{v}{x^{\alpha+1}}\frac{\partial p}{\partial x}$$

This shows that the operator

$$\frac{K'}{x^p}\frac{\partial^2}{\partial x^2}$$

is a perturbation for the diffusion operator

$$\frac{K}{x^\alpha}\frac{\partial^2}{\partial x^2} + \frac{v}{x^{\alpha+1}}\frac{\partial}{\partial x}$$

at large times and if $\beta > 2$. We can use the Green function of the unperturbed operator to derive the critical exponents of the asymptotic behavior. The critical exponent of the avalanche temporal distribution, τ, can be obtained from the first return time distribution. Using the results of Chapter I we have

$$\tau = 1 + \frac{1 - v/K}{\alpha + 2}. \tag{3.21}$$

The avalanche mass dimension exponent D can be determinate from the asymptotic behavior in time of the average width. using equation (2.6) we get

$$w_{average} = \frac{\int_0^\infty y G_t(x,y)dy}{\int_0^\infty G_t(x,y)dy} \sim t^{\frac{1}{\alpha+2}}. \tag{3.22}$$

3.5 Numerical analysis

We measured by Monte Carlo simulation the local variance and the local drift in the interval $(10, 100)$. This corresponds to avalanche maximal temporal size of order 10^6 as the average maximum width scale $\bar{w}_{max} \approx t^{0.42}$, see figure 3.2. The average life time of an avalanche at a given width increases linearly as figure 3.3 shows. The critical exponent β, defined in (3.18), was measured from the local drift data and from the asymptotic behavior of the backward jumps distribution, see figures 3.1, 3.4. Their value agrees to $\beta = 2.40 \pm 0.05$. Using equation (3.22) we obtain for the mass dimension exponent $D^{-1} = 0.41 \pm 0.02$, value which is in agreement with the previous measurements [26, 6]. The precise measurement of the constants v and K is difficult due to the hight weight of the short backwards jumps in respect with the long ones, see figure 3.1. Hence the number of steps needed to average the distribution in every site is very large. For a more precise measurement of the ratio v/K, needed in equation (3.21), we filtered the data of the jump variance selecting them only if they are increasing but not more than rate allowed by the formula (3.19) were we set $\beta = -2.40$. This procedure proved not to be very sensitive at small variation of β and applied to the drift data yields the same value for β. By the linear fit of the selected data we obtained $K = 0.37 \pm 0.02$ and $v = 0.26 \pm 0.02$, see figures 3.5, 3.6. Using formula (3.21) we obtain for the first return time critical exponent $\tau = 1.12 \pm 0.05$ which agrees within the errors bar with the previous determinate values [26, 6].

There is also an indirect but more precise method of measuring the exponents α and $1 - v/K$ using formula (2.11). For small λ we have

$$\phi_\lambda(x,y) \sim \Big(\frac{x}{y}\Big)^{1-\frac{v}{K}}\Big(1 - \frac{1}{(\alpha+2)^2(\nu+1)}(y^{\alpha+2} - x^{\alpha+2})\lambda\Big), \qquad (3.23)$$

where $\alpha = 3 - \beta + p$ $y > x$. One can see that if $\lambda = 0$ the previous formula gives the probability for an avalanche to reach for the first time width y starting from the width x without passing trough the state with zero width. Numerical simulation gives a algebraic behavior, see figure 3.7, and we obtained $1 - v/K = 0.180 \pm 0.006$. The constant multiplying λ in the equation (3.23) is the characteristic time for an avalanche to touch for the first time width y starting from the width x. Numerically we found that it behaves algebraic with $\alpha + 2 = 2.42 \pm 0.02$, see figure 3.8. This two exponents allow us to compute the critical exponents of the avalanche size distribution $\tau = 1.074 \pm 0.004$ and $D^{-1} = 0.413 \pm 0.008$. These values are in good agreement with the previous determined ones in the reference [27], $\tau = 1.074 \pm 0.004$ $D^{-1} = 0.4114 \pm 0.0020$.

From the above numeric analyze we may conclude that the diffusion equation (3.20) describes the the universality class for the BS models. The basic facts which lead to this description are:

1. The underlying Markov chain reach stationarity at any finite width of the avalanche, therefore one can think in terms of a stationary probability transition from one width to another one.

2. The probability of the activity to remain inside of an avalanche decrease exponentially in time, this is the second ingredient needed to complete the CTRW picture.

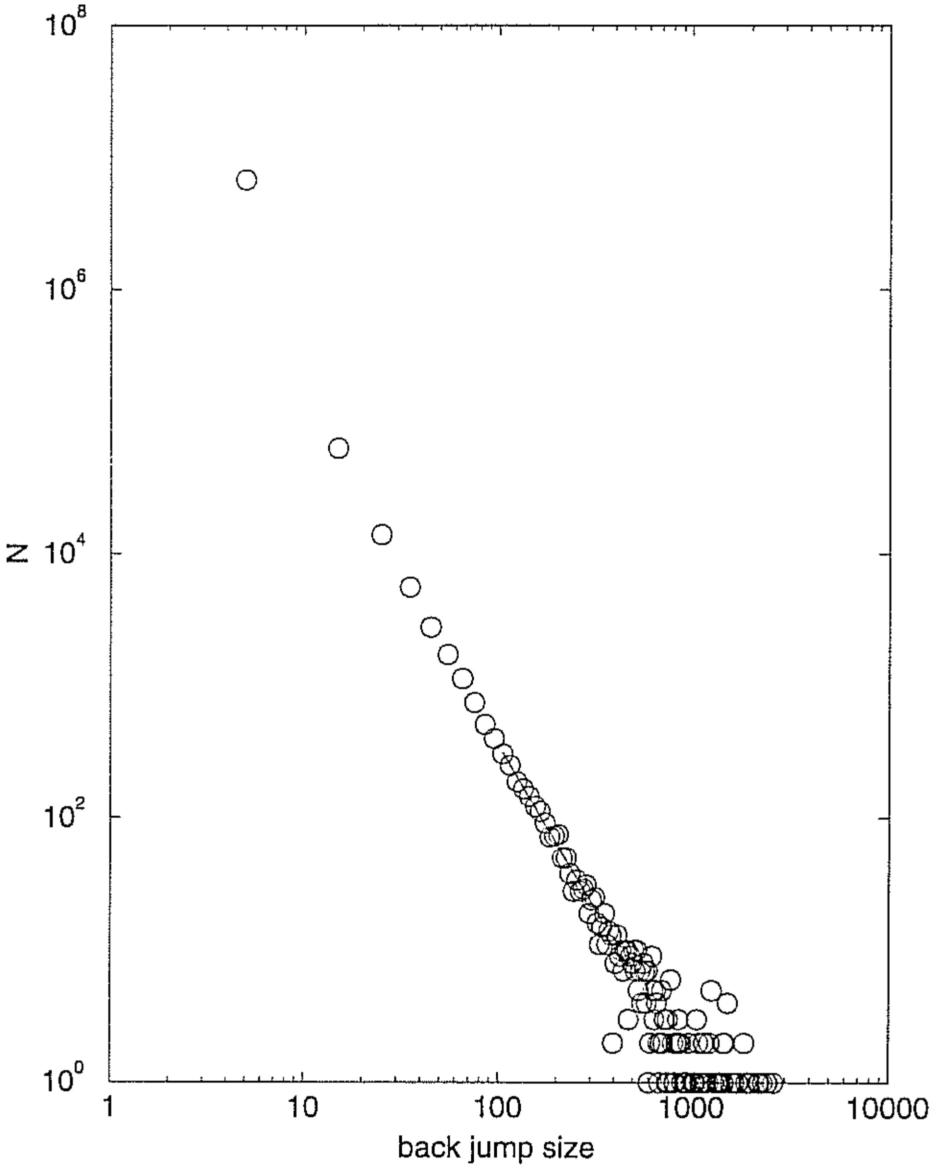

Figure 3.1: The distribution of the backwards jumps for avalanche width 2000. The short size jumps are dominating, this makes difficult producing accurate data. An temporal event averages over many sites and the statistics is better.

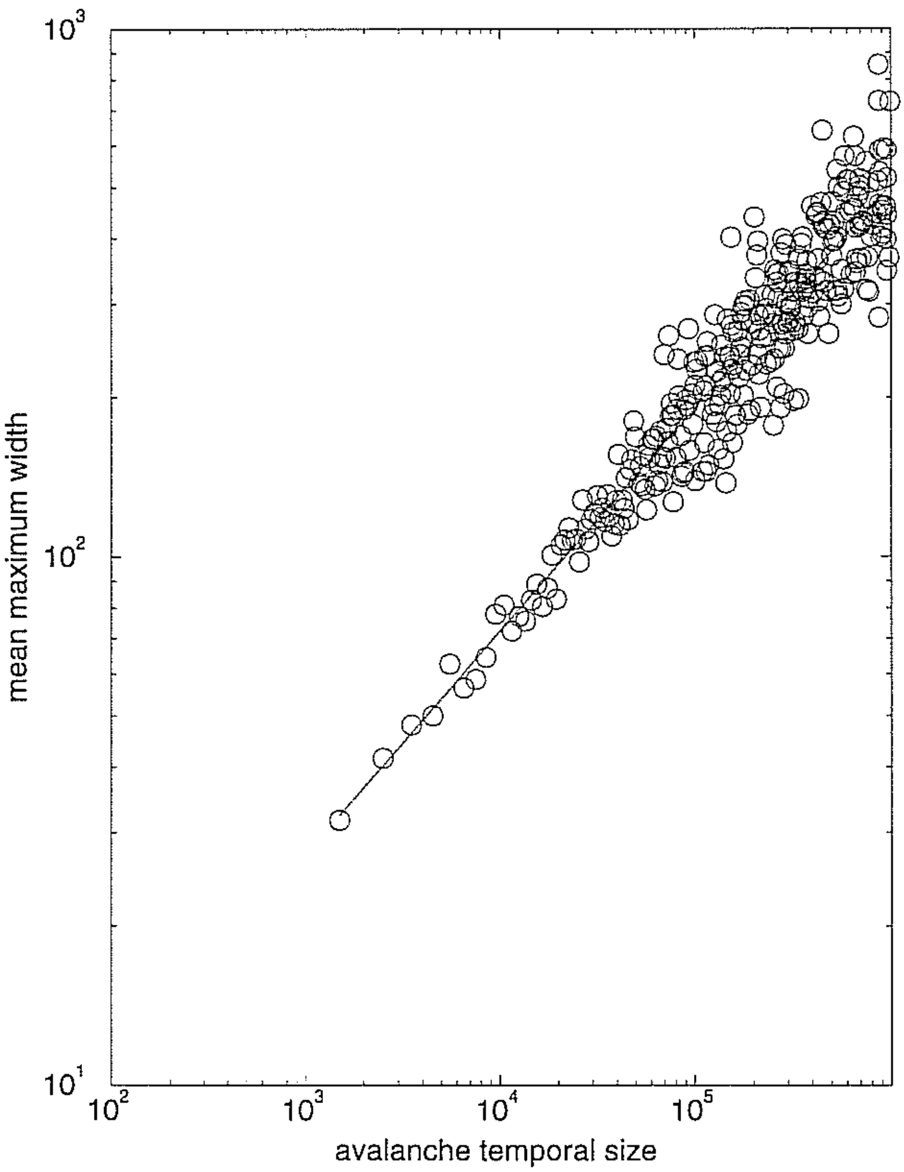

Figure 3.2: The mean maximum width function of the avalanche size. The critical exponent $\frac{1}{D} = 0.42 \pm 0.02$. This graphic shows that the temporal size of the avalanches is very large even their spatial width is less than 1000.

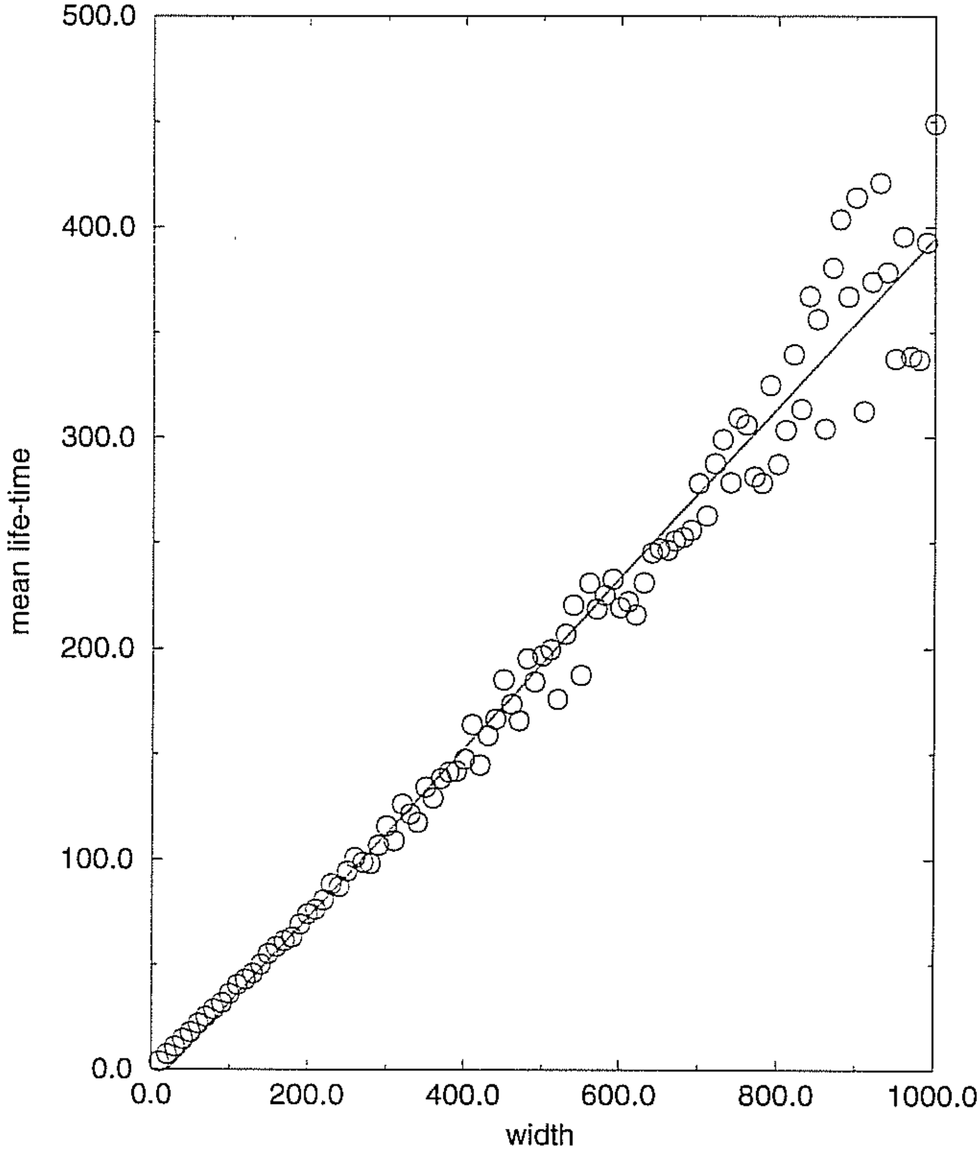

Figure 3.3: Average life-time versus width, the behavior is linear. This behavior is connected to the fact that the activity has a normal diffusion inside of the avalanche.

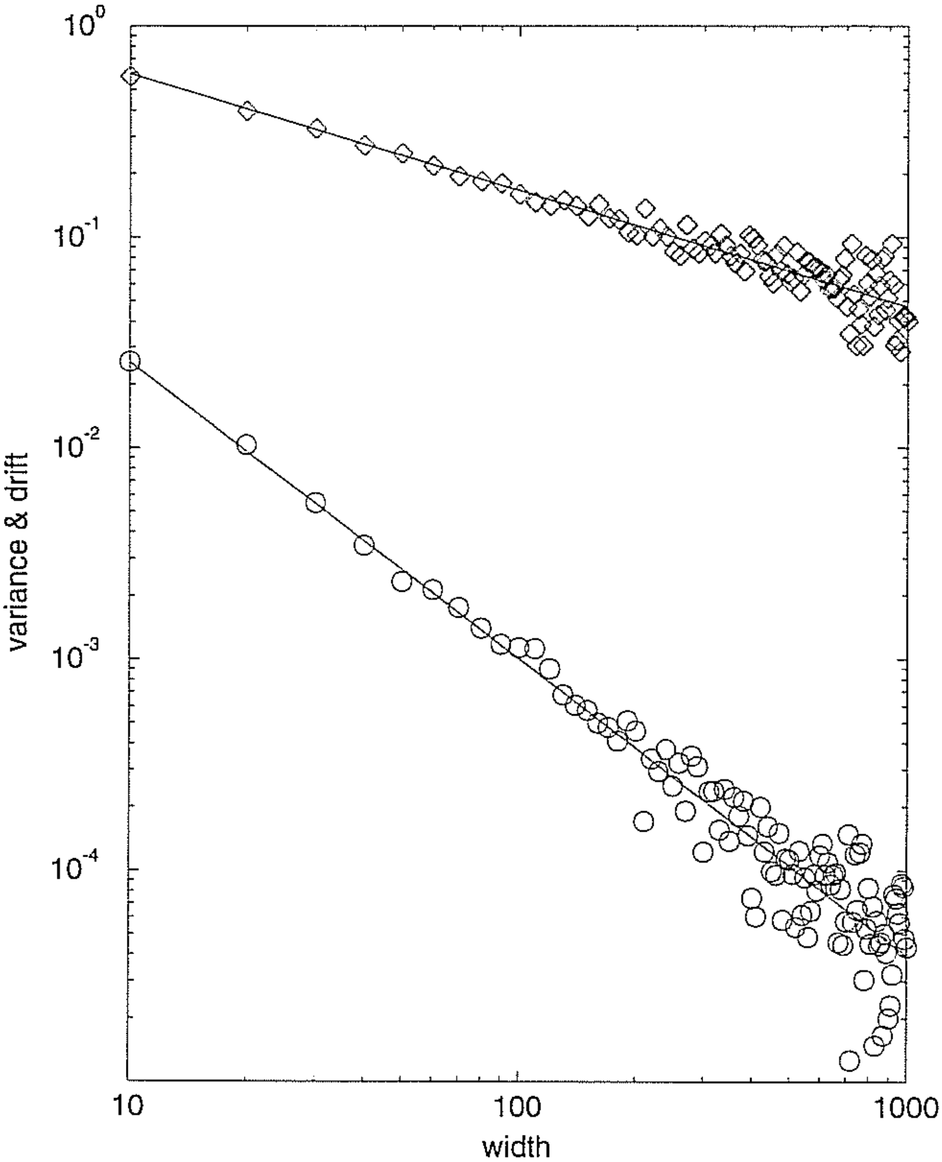

Figure 3.4: The local variance(◇) and the local drift(○). The exponent for the drift decay is -1.40 ± 0.05, the variance decay with the exponent -0.52 ± 0.02. Their difference is not exactly 1 due to the presence of the inhomogeneous term in equation (3.19).

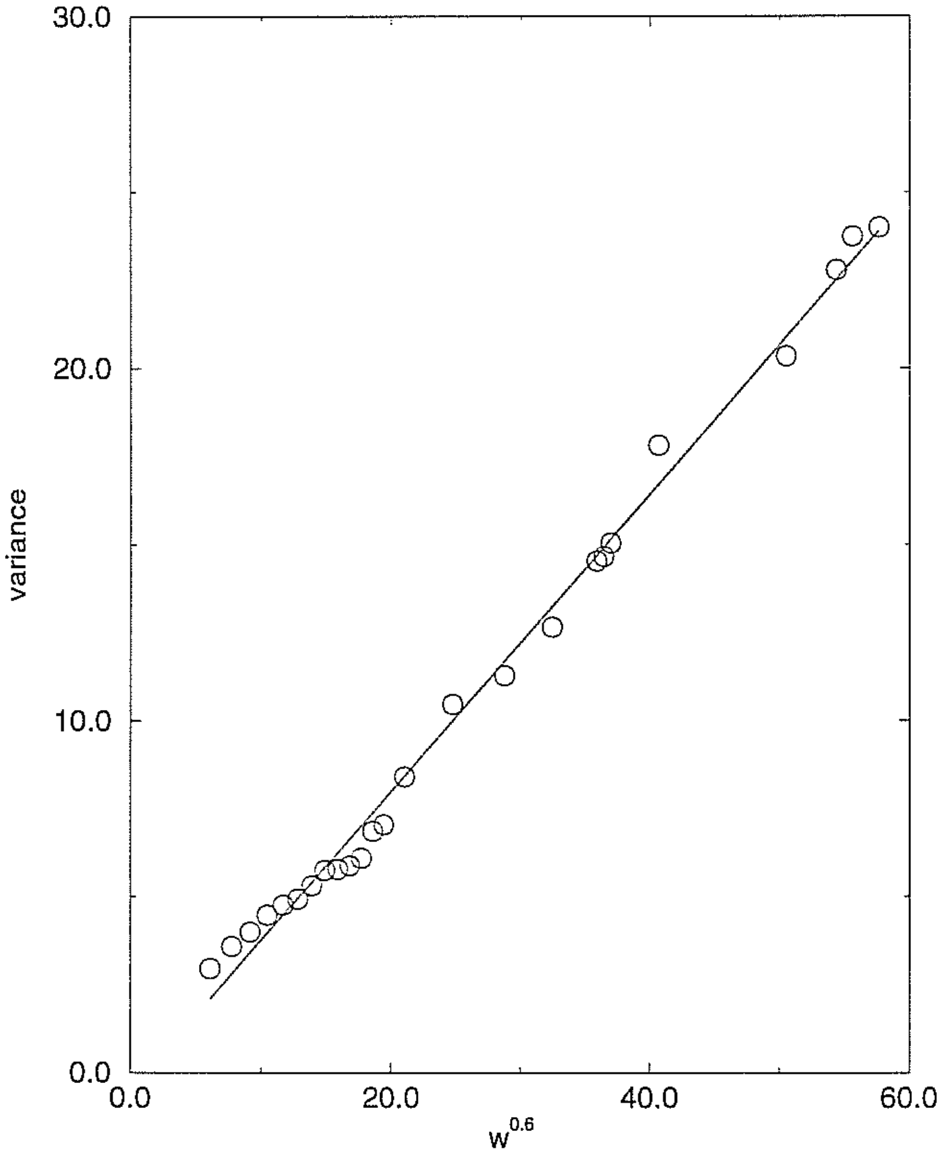

Figure 3.5: The local variance of the transition probability average width versus $w^{0.6}$, the coefficients of the linear fit $a_v + b_v x$ are: $a_v = -0.45 \pm 0.2$, $b_v = 0.42 \pm 0.01$. The monotonic data were selected together with the condition $(y_j - y_{j-1})/(w_i^{0.6} - w_{i-1}^{0.6}) < 1$.

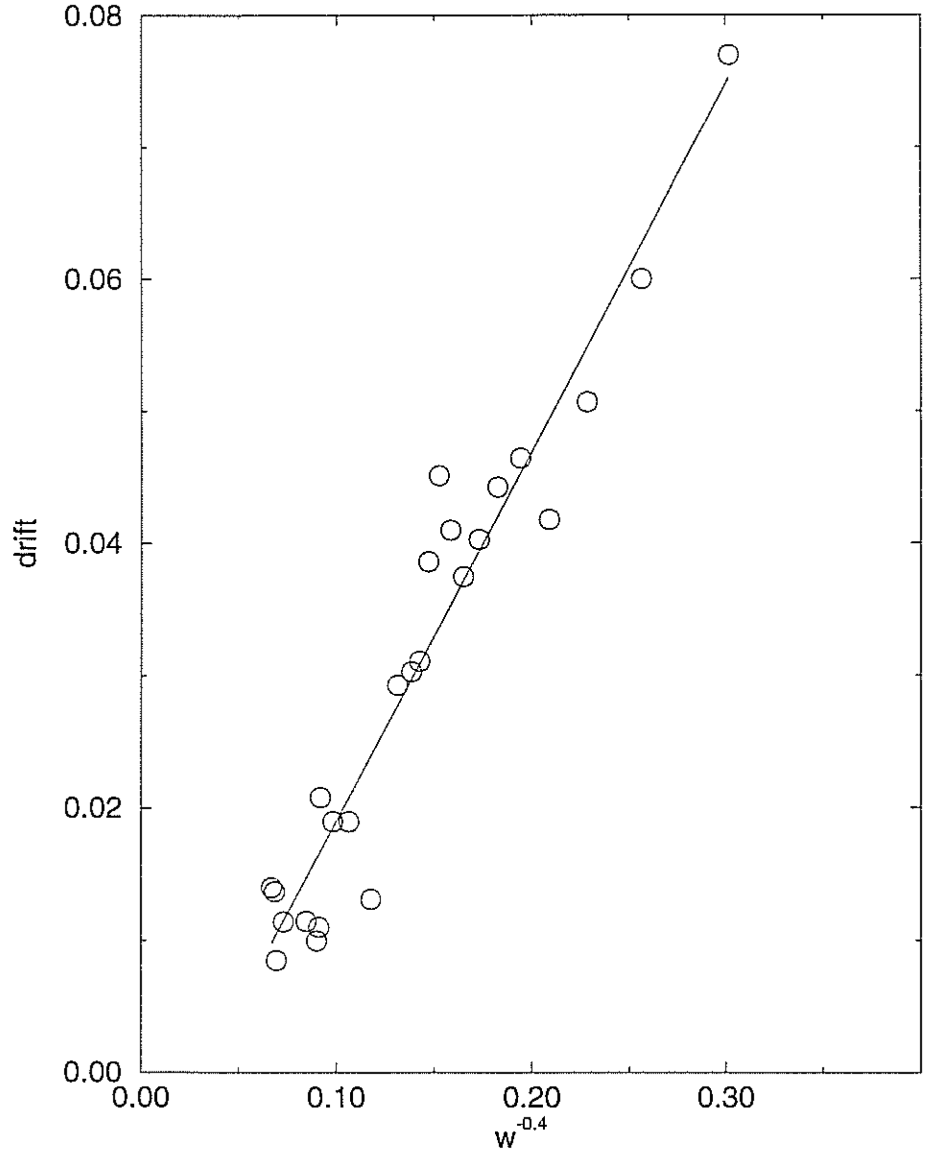

Figure 3.6: The mean of the local transition probability versus of $w^{-0.4}$. The coefficients of the linear fit $a_d + b_d x$ are: $a_d = 0.001 \pm 0.002$, $b_d = 0.27 \pm 0.02$.

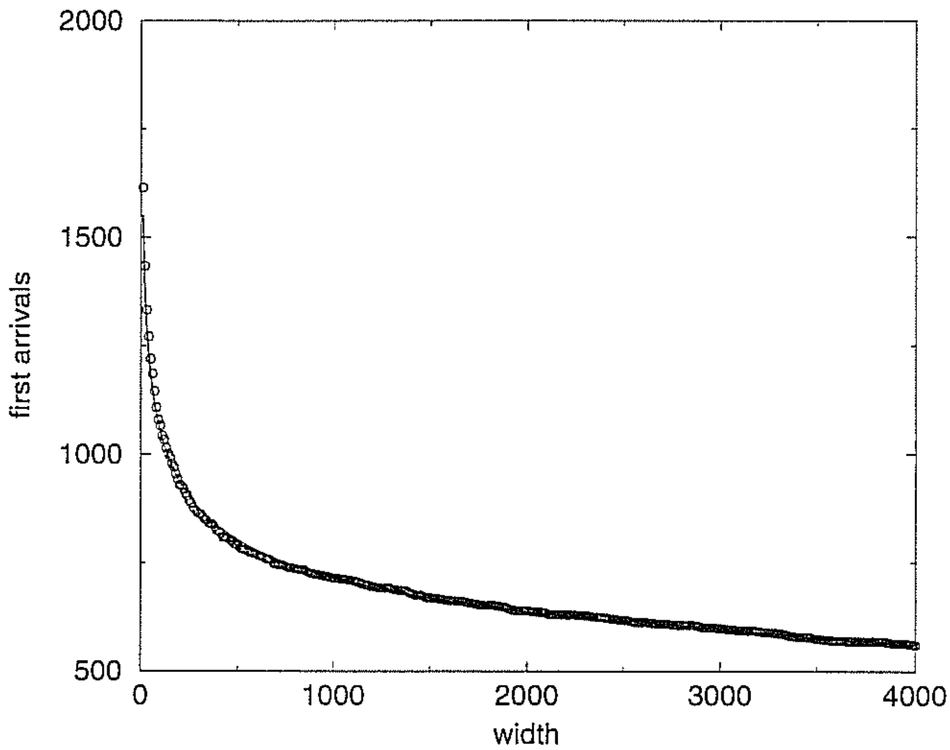

Figure 3.7: The probability of a first arrival to a width y starting from a width $x < y$ without touching the zero width state. The behavior is algebraic as the formula (3.23) predicts, the exponent has the value $1 - v/K \approx 0.180 \pm 0.006$.

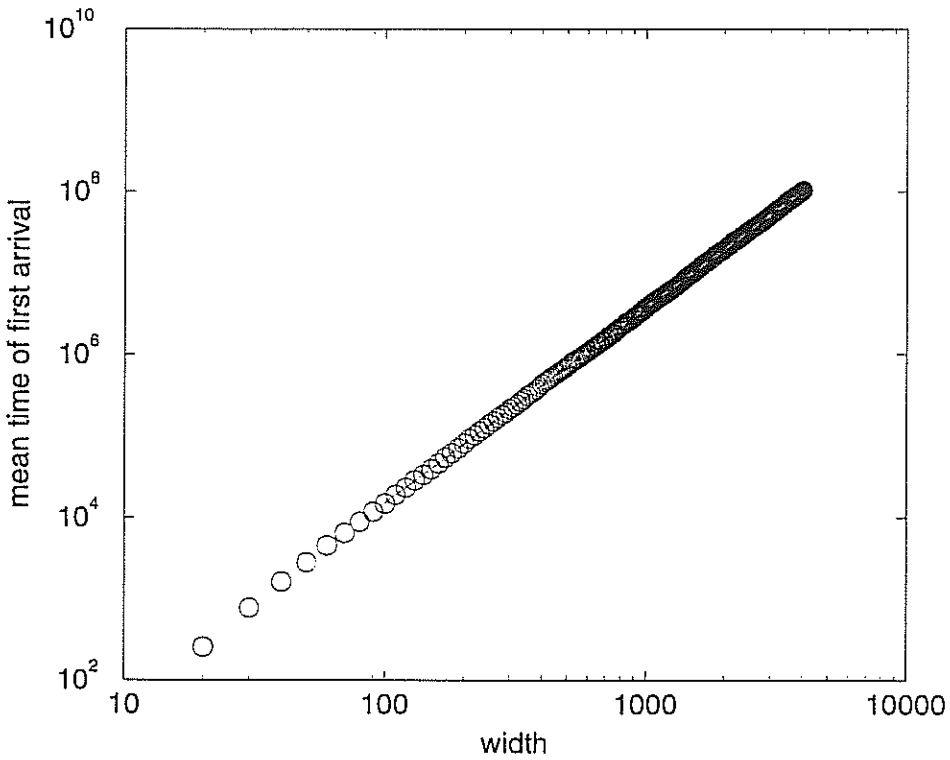

Figure 3.8: The behavior of the characteristic time for the first arrival in y starting from $x < y$ is algebraic for $x \ll y$ as formula (3.23) shows. The critical exponent has the value $\alpha + 2 = 2.42 \pm 0.02$ which gives $D^{-1} = 0.413 \pm 0.008$.

Chapter 4

Simple Equation for Earthquake Distributions

4.1 Introduction

An earthquake occurs when rocks being deformed suddenly break along a fault. The great majority of the earthquakes appear at the contact region of the tectonic plates which form the crust of the earth [18]. Due to their movement elastic energy accumulates in the fault zone, the earthquake release part of this energy in form of seismic waves. Recently it has been observed in the San Andreas fault a slow slip process without any seismic weave emission [28] . The complete understanding of the earthquake mechanism is a long term problem. The friction interaction of the rocks depends on the temperature and pressure in the fault, the finely fragmented rock accumulated between the sliding parts are also important. Beside of the particular properties which may vary from case to case there are tow simple and universal laws. The Gutenberg-Richter [29] law states that the the probability of the earthquakes decays algebraic function of the released energy

$$P(E) \approx E^{-b} \tag{4.1}$$

with $b \approx 2$ but it varies for different earth regions and also the value fluctuates in time for the same region [17, 30]. The Omori's law [31] state that the temporal distribution of the aftershocks following a main event has asymptotically an algebraic decay

$$N(t) \approx t^{-\alpha} \tag{4.2}$$

with α close to unity. In the recent years physicist came across in this field proposing simple statistical models as explanatory for the general feature of the earthquakes [17, 32, 33, 34, 16]. These models yields complex behavior from the cooperative dynamics of the elementary components although individually they obey simple dynamical rules. The predictions are obtained mainly through numerical simulation.

4.2 The equation

We propose an analytical solvable model which explains the Gutenberg-Richter and Omori laws in term of diffusive motion in the space of activity function of released energy. We characterize an earthquake by the number of energy sources in the fault zone and by the quantity of kinetic (vibration) energy E radiated outside of the volume surrounding the fault zone. Since we want to keep the model as simple as possible we consider the energy sources as primary identical agents of the process and their total number n will be called activity. The active agents contribute to the released kinetic energy, denoted by E, and in the same time, within a certain probability, an active agent can initiate an other agent or it can die. The model will be characterized by the distribution probability $P(n, E)$ to be in the state of activity n after it released an amount of energy E.

Our basic assumption is that the activity of the fault zone cannot change drastically if the released energy is small, therefore we propose to describe the probability $P(n, E)$ for a small amount of released energy dE by a one dimensional birth and death process [21]:

$$\begin{aligned} P(n, E + dE) &= (1 - (\lambda_n - \mu_n)dE)\, P(n, E) + \lambda_{n-1} dE\, P(n-1, E) \\ &\quad + \mu_{n+1} dE\, P(n+1, E) \end{aligned} \tag{4.3}$$

where $\lambda_n\, dE$ and $\mu_n\, dE$ are the probabilities to pass in a state with activity $n+1$, respective $n-1$, while releasing the amount of energy dE. The simplest expression for the rates λ_n and μ_n comes if we consider that each passive agent can be influenced with the same weight by any of the active agents therefore we consider the transition rates proportional with the number of the active agents $\lambda_n = n\lambda$ and $\mu_n = n\mu$ where λ and μ are individual rates and they depend upon the threshold which an agent has to surpass to become active and the energy which an active agent can transfer to a passive one. Since the activation of a passive agent requires a certain amount of energy we take $\lambda \le \mu$. If we multiply the equation:

$$\begin{aligned} \frac{dP(n, E)}{dE} &= -n(\lambda + \mu)P(n, E) + \lambda(n-1)P(n-1, E) \\ &\quad + \mu(n+1)P(n+1, E) \end{aligned} \tag{4.4}$$

with n and then we sum over all its value we will obtain the following relation for the mean activity $M(E)$:

$$M(E) = M(0)e^{(\lambda-\mu)E} \tag{4.5}$$

where $M(0)$ is the initial average activity activity. This relation shows that $1/|\lambda - \mu|$ can considered as a mean energy released by the earthquake. It diverges as $\lambda \rightarrow \mu$ and events of all size can appear in the system, therefore we set the study of the equation (4.4) to the case $\lambda = \mu$ later on we come back to general case.

The equation 4.4 can be treated easier in the continuum limit, operation which produce [19] the general diffusion equation

$$\frac{\partial p(n,E)}{\partial E} = a(n)\frac{\partial^2 p(n,E)}{\partial n^2} + b(n)\frac{\partial p(n,E)}{\partial n} \tag{4.6}$$

where $a(n)$ and $b(n)$ are the local variance and the local drift. Here we use the backward equation instead of the forward (Fokker-Planck) equation. Nevertheless they have the same Green function [19], $G_E(n_0, n)$, representing the probability to pass from the activity n_0 to the activity n releasing an amount of energy E. The active agents take energy from the background stress field. In reaction the background field can undergo a relaxation or perhaps the stress field increases, depending on the local fault configuration and structure. This effect has to be small comparatively with the the interaction between the agents as they transfer o each other large amounts of energy in short periods of time. We include this effect in our equation as a global drift term acting on the system activity.

From the assumption we have made before on the λ_n and μ_n it results that $a(n) = Dn$, $D = \lambda + \mu$, and $b(n) = v$ $|v| \ll D$. The equation (4.6) can be solved in the standard way [19],[35]. After the Laplace transform in the variable E we obtain an ordinary second order inhomogeneous equation

$$np'' - up' - \lambda p = p(n,0), \tag{4.7}$$

where $n \rightarrow Dn$, $\quad u = v/D$. Its Green function reads:

$$G_\lambda(n_0, n) = \begin{cases} 2n_0^{\frac{1-u}{2}} I_\nu(\sqrt{2\lambda}n_0^{\frac{1}{2}})n^{\frac{1+u}{2}} K_\nu(\sqrt{2\lambda}n^{\frac{1}{2}}) & if \quad n > n_0 \\ 2n^{\frac{1+u}{2}} I_\nu(\sqrt{2\lambda}n^{\frac{1}{2}})n_0^{\frac{1-u}{2}} K_\nu(\sqrt{2\lambda}n_0^{\frac{1}{2}}) & if \quad n \leq n_0 \end{cases} \tag{4.8}$$

where I_ν and K_ν are are the Bessel function of imaginary argument of first and second kind [24] and $\nu = 1-u$. Since $G_\lambda(0, n) = 0$ the equation (4.6) has adsorbing boundary condition in the origin. This property gives a natural way to obtain the earthquake distribution: an earthquake terminates after it release the energy E if the system activity is zero. The distribution of the first arrival in origin for the activity can be obtained from the derivative in respect with E of the "total mass"

$$J_\lambda(n_0) = \int_0^\infty dn\, G_\lambda(n_0, n) \tag{4.9}$$

Using the results found in Chapter 1 for the equation (4.9) one can see that the leading term in λ is

$$J_\lambda(n_0) \approx \lambda^{-u}, \quad \lambda \ll 1, \quad u < 1. \tag{4.10}$$

Therefore, using Tauberian Theorem [19], we conclude that the asymptotic behavior of the probability distribution for the earthquake distribution function of released energy is:

$$p(n_0, E) \approx E^{-b}, \quad b = 2 - u, \qquad u = v/D. \tag{4.11}$$

This asymptotic behavior remain unchanged upon the average over the initial activity distribution if we assume that the it has all the moments finite. From physical point of view it is natural to assume that the earthquake starts with a typical activity and the distribution of the initial activities decays exponentially with n_0.

In the case in which $\lambda - \mu = \epsilon$ is not zero, the Laplace transform of the diffusion equation reads:

$$nDp'' - (\epsilon n + v)p' - \lambda p = 0 \tag{4.12}$$

The following substitutions $x \to \frac{x}{\epsilon} D$ $u = v/D$ lead us to the equation satisfied by confluent (degenerate) hyper-geometric function

$$xp'' + (u - x)p' - \frac{\lambda}{\epsilon} p = 0 \tag{4.13}$$

Standard calculation C shows that a characteristic cut-off energy appear

$$E_c = -\frac{1}{\epsilon}(\psi(1) + \psi(1 - u))$$

where $\psi(z)$ is the logarithmic derivative of the Euler function [24]

$$\psi(z) = \frac{d}{dz}\Gamma(z).$$

The fact that the Gutenberg-Richter low is extended over six order of magnitudes practically sets $\epsilon = 0$ for this model.

The released energy E is a monotonic function of time but its variation is very irregular. Within this model one can explain the Omori's law making the following assumption: if the activity is very close to zero the released energy is a linear function of time, that means that the power emitted by the fault zone is constant.This happen if the characteristic relaxation time are very large. The energy released in these states of low activity is very small and it is dispersed in the vicinity of the fault zone until it will increase drastically and an aftershock will appear. Using time coordinate instead of energy and neglecting the time intervals of strong activity in respect with the intervals of weak activity the total mass behaves

$$J_t(n_0) \approx t^{-1+v}. \tag{4.14}$$

Since an aftershock can appear only if the activity is positive the equation (4.14) gives the critical exponent in the Omori's law $\alpha = 1 - v$ close to its observed value.

We have shown that the Gutenberg-Richter law can be obtained from simple hypothesis in the frame of the diffusive motion, an simple assumption on the weak active states give us an explanation of the Olami's law. Within the context of this model we predict

$$b - \alpha = 1$$

The parameter v allow an explanation for the non-universality observed in nature, the critical behavior is present on a finite interval of time in the case of a small difference between the death and birth rate.

Appendix A

The case $\beta > \alpha + 1$

The equation satisfied by the total mass is:

$$\frac{1}{x^\alpha} u_\lambda(v,x)'' + \frac{v}{x^\beta} u'_\lambda(v,x) - \lambda u_\lambda(v,x) = 1 \tag{A.1}$$

with $u_\lambda(v,0) = 0$ $\beta > \alpha + 1 > 0$. The variable substitution $x \to x^{-\frac{1}{\alpha+2}}$ yields the equation:

$$\frac{1}{x^\alpha} u_\lambda(v,x)'' + \frac{v\lambda^p}{x^\beta} u'_\lambda(v,x) - u_\lambda(v,x) = \lambda^{-1} \tag{A.2}$$

As we are interested in the limit of the ratio

$$(\lambda^p v, \lambda^\nu x)/u(0, \lambda^\nu x)$$

where $\nu = 1/(\alpha+2)$ $p = (\beta - \alpha - 1)/(\alpha + 2)$, we can set the inhomogeneous term to 1. It can be seen from equation (2.7) that

$$u_\lambda(0,x) \approx Dx \ , x \ll 1 \tag{A.3}$$

with D a constant. In every $x \neq 0$ $u_\lambda(v,x)$ is analytical in v [35] therefore

$$\exists C_x \text{ such that } |u_\lambda(0,x) - u_\lambda(v,x)| < C_x v < C_x \sqrt{v} \tag{A.4}$$

for v small enough. As $u_\lambda(v,0) = u_\lambda(0,0) = 0$ the converges in a vicinity of the origin is uniform i.e. $C = max\{C_x\}$ is finite. If $v < 0$ one can see that $u''_\lambda > 0$ therefore the function is convex and monotone in a vicinity of the origin. Using (A.3) we can write

$$u_\lambda(0,x) - u_\lambda(v,x) < Dx < D\sqrt{x} \tag{A.5}$$

for x enough small. Multiplying the equations (A.4) and (A.5) and dividing by $u_\lambda(0,x)$ we obtain

$$\left|1 - \frac{u_\lambda(v,x)}{u_\lambda(0,x)}\right| < Av\frac{x}{u_\lambda(0,x)} \tag{A.6}$$

Substituting $x \to x\lambda^\nu$ and $v \to v\lambda^p$ we obtain

$$\left|1 - \frac{u_\lambda(v\lambda^p, x\lambda^\nu)}{u_\lambda(0,x)}\right| < A\lambda^p \tag{A.7}$$

for λ enough small. We have used the fact that

$$\frac{x}{u_\lambda(0,x)}$$

is bounded as $x \to 0$. Equation (A.7) shows that the total mass has approach to 0 independent of the value of v.

In the case $v > 0$ $u_\lambda(v,x)$ is a concave function an we use the following inequality

$$u_\lambda(v,x) - u_\lambda(0,x) < u_\lambda(v,x)$$

for small enough x. The final result is the same.

Appendix B

The Generating Function

We present the detailed calculation for the generating function of the avalanches probability distribution for the random walk defined in section 3.3. For this propose we shall use the special form of the matrix $\tilde{p}'$ obtained from the matrix p' (3.11) removing the line and the row with index zero. This matrix has equal diagonal elements and we can write it as a linear combination of one-diagonal matrices I_i defined as follow:

$$(I_i)_{kl} = \begin{cases} \delta_{k+i,l} & i \geq 0, \\ \delta_{k,l+i} & i < 0 \end{cases} \tag{B.1}$$

I_0 being the identity matrix. From the definitions (B.1) we can compote the commutator $T^{(i)} = I_1 I_{-i} - I_{-i} I_1$; for $i > 0$ we have

$$(T^{(i)})_{kl} = \delta_{i+1,1} \tag{B.2}$$

and for $ij > 0$ we have the property

$$I_i I_j = I_{i+j}. \tag{B.3}$$

The matrix $\tilde{p}'$ can be expressed as:

$$\begin{aligned} \tilde{p}' &= \lambda^2 I_1 + 2\lambda^2(1-\lambda)I_0 + \sum_{i=1}^{\infty}(i+2)\lambda^2(1-\lambda)^{i+1} I_{-i} \\ &= \lambda^2 I_1 \sum_{i=0}^{\infty}(i+1)(1-\lambda)^i I_{-i} = \lambda^2 I_1 A \end{aligned} \tag{B.4}$$

where $A = \sum_{i=0}^{\infty}(i+1)(1-\lambda)^i I_{-i}$. Using eq. (B.3) it is easy to compute the n-th power of this matrix

$$A^n = \sum_{j=0}^{\infty} \binom{2n+j-1}{j} (1-\lambda)^j I_{-j}. \tag{B.5}$$

From eq. (B.2) we can compute the commutator $T_n = I_1 A_n - A_n I_1$ which has only the first column non zero

$$(T_n)_{kl} = \binom{2n+k+1}{k}(1-\lambda)^k \delta_{l1} \tag{B.6}$$

consequently,

$$(I_n T_n)_{j1} = \binom{3n+j-1}{j+n}(1-\lambda)^{j+n}. \tag{B.7}$$

All the previous mentioned properties lead us to the relation

$$(I_1 A)^n = I_n A^n - \sum_{i=0}^{n-2} I_i T_i (I_1 A)^{n-i-2}. \tag{B.8}$$

Eq. (B.8) imply the following equation for the generating matrix $G(z) = \sum_{i=0}^{\infty} (\lambda^2 I_1 A)^i z^i$

$$G(z) = F(z) - \sum_{i=1}^{\infty} I_i T_i \lambda^{2(i+1)} z^{i+1} G(z) \tag{B.9}$$

where $F(z) = \sum_{i=1}^{\infty} I_i A^i z^i$, z complex number. The sums which are appearing in eq.(B.9) can be done in the following way:

$$\begin{aligned} u_j(z) &= \sum_{k=1}^{\infty} (I_k T_k)_{j1} \lambda^{2(k+1)} z^{k+1} = \sum_{k=1}^{\infty} \binom{3k+j-1}{j+k} (1-\lambda)^{j+k} \lambda^{2(k+1)} z^{k+1} \\ &= (1-\lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{(k+j)!} (3k+j-1) \dots 2k \xi^{2k-1} \\ &= (1-\lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{(k+j)!} \frac{d^{k+j}}{d\xi^{k+j}} \xi^{3k+j-1} \qquad \text{(B.10)} \\ &= (1-\lambda)^{j-1} \xi^3 \sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_\Gamma d\eta \frac{\eta^{3k+j-1}}{(\eta-\xi)^{j+k+1}} \\ &= (1-\lambda)^{j-1} \xi^3 \frac{1}{2\pi i} \oint_\Gamma d\eta \frac{\eta^{j+2}}{(\eta-\xi)^{j+1}(-\eta^3+\eta-\xi)} \qquad \text{(B.11)} \end{aligned}$$

where $\xi^2 = (1-\lambda)\lambda^2 z$. We can perform the summation if $|\eta^3/(\eta-\xi)| < 1$, this set is not empty for $0 < \xi < 2/3\sqrt{1/3}$. There is a annulus with inner radius and external radius obtained from the the positive solutions of the equation $(r+\xi)^3 - r = 0$; more than that, one of the roots of the polynomial $-\eta^3+\eta-\xi$ is inside of the minimal integration contour Γ for $0 < \xi < 2/3\sqrt{1/3}$ and the other two are outside of the maximal integration contour. Using the above mentioned properties of the matrices $\{I_k\}$ (B.3) we can compute the elements of the matrix $I_n A^n$ which appear in the expression of the generating matrix $F(z)$:

$$\begin{aligned}(I_n A^n)_{1j} &= \binom{3n-j}{n-j+1}(1-\lambda)^{n-j+1}, \quad (I_n A^n)_{21} = \binom{3n}{n+1}(1-\lambda)^{n+1},\\ (I_n A^n)_{2j} &= (I_n A^n)_{1\,j-1} \quad j > 1.\end{aligned}$$

Eq.(3.10) shows that we need to compute only the first two rows in the generating matrices $G(z)$ and $F(z)$. The general formula for these matrix elements of $F(z)$ is

$$F_{1j}(z) = \delta_{j1} + \sum_{n=j-1}^{\infty} (1-\lambda)^{n-j+1} \binom{3n-j}{n-j+1} \lambda^{2n} z^n.$$

The previous series can be summed following the same computational path as in eq.(B.10). If $j = 1$ we get

$$F_{1,1}(\xi) = 1 + \frac{\xi}{2\pi i} \oint_\Gamma \frac{\eta^2}{(\eta-\xi)(-\eta^3+\eta-\xi)},$$

for $j > 1$ we have the following expression:

$$F_{1j}(\xi) = \frac{\xi}{(1-\lambda)^{j-1}} \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{2j}}{-\eta^3+\eta-\xi},$$

$F_{2j}(z) = F_{1j-1}(z)$, $j > 1$, because $I_n A^n$ has equal elements on diagonals, and by direct calculation

$$F_{21}(z) = \sum_{n=1}^{\infty} \binom{3n}{n+1} (1-\lambda)^{n+1} z^n = \frac{1-\lambda}{2\pi i} \xi \oint_\Gamma \frac{\eta^3}{(\eta-\xi)^2(-\eta^3+\eta-\xi)}.$$

In all the above formula the contour Γ is the same as the that one used in eq.(B.10) and $\xi^2 = (1-\lambda)\lambda^2 z$. Solving eq.(B.9) we obtain for the first two rows the solutions in terms of previously computed functions $u_1(z), u_2(z), F_{1j}(z), F_{2j}$:

$$\begin{aligned}G_{1,j}(z) &= \frac{F_{1j}(z)}{1+u_1(z)},\\ G_{2,j}(z) &= F_{2j}(z) - \frac{u_2(z)}{1+u_1(z)} F_{1j}(z).\end{aligned}$$

Residue theorem allow us to compute the generating functions in term of the third solutions of the polynomial $-\eta^3+\eta-\xi$, $\eta_3(\xi)$, that solution which lies inside of the integration contour Γ in the above integrals.

$$G_{1j}(\xi) = \frac{1}{(1-\lambda)^{j-1}}\frac{\eta_3(\xi)^{2j}}{\xi^2} \qquad i \geq 1, \tag{B.12}$$

$$G_{2j}(\xi) = -\frac{1}{(1-\lambda)^{j-2}}\Big(\frac{1}{\xi^2}-2\Big)\frac{\eta_3(\xi)^{2j}}{\xi^2} \qquad j \geq 1; \tag{B.13}$$

with $\xi^2 = \lambda^2(1-\lambda)z$ and

$$\eta_3(\xi) = -\frac{1-i\sqrt{3}}{2^{2/3}(-27\xi+\sqrt{729\xi^2-108})^{1/3}} - \frac{(1+i\sqrt{3})(-27\xi+\sqrt{729\xi^2-108})^{1/3}}{6\,2^{1/3}}. \tag{B.14}$$

From eq. (3.10) for the avalanches probability distribution one can write the generating function:

$$\begin{aligned} R(z) &= (1-\lambda)^2 z + z^2 p_{01}\sum_{i=1}^{\infty} G_{1i}(z)p_{i0} + z^2 p_{02}\sum_{i=1}^{\infty} G_{2i}(z)p_{i0} \\ &= (1-\lambda)^2 z + 2\lambda(1-\lambda)z^2\sum_{i=1}^{\infty}\frac{1}{1+u_1(z)}F_{1i}(z)(1-\lambda)^{i+1}(1+(i+1)\lambda) \\ &\quad + z^2\lambda^2\sum_{i=1}^{\infty}(F_{2i}(z) - \frac{u_2(z)}{1+u_1(z)}F_{1i}(z))(1-\lambda)^{i+1}(1+(i+1)\lambda). \end{aligned} \tag{B.15}$$

The series which are appearing above can be summed and the closed expression for generating function reads:

$$\begin{aligned} R(\xi(z)) &= -2\frac{1-\lambda}{\lambda}\xi^2 + \frac{2(1-\lambda)}{\lambda^3}\xi^2\eta_3(\xi)^2 \qquad \text{(B.16)} \\ &\quad \times\Big(1+2\lambda+\frac{\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\big(1+\lambda\frac{3-2\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\big)\Big) \\ &\quad + \frac{1-\lambda}{\lambda^2}(1-2\xi^2)\eta_3(\xi)^2 \qquad \text{(B.17)} \\ &\quad \times\Big(1+2\lambda+(1+3\lambda)\eta_3(\xi)^2+\frac{\eta_3(\xi)^4}{1-\eta_3(\xi)^2}\big(1+\lambda\frac{4-3\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\big)\Big). \end{aligned}$$

Making the substitution $z = \epsilon^{-s}$ we obtain

$$1 - R(s) \approx s^{\frac{1}{2}}, \quad \text{if} \quad \lambda = \frac{2}{3}, \quad s \to 0. \tag{B.18}$$

If $\lambda \neq 2/3$ $R(s)$ is an analytical function in origin, therefore the probability distribution function decays exponentially.

Appendix C

First Return Time for the Linear growth Equation

We present here the detailed solution of the equation (4.12) and the characteristic life time derivation. The starting equation is:

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + (v - \epsilon x)\frac{\partial u}{\partial x}. \tag{C.1}$$

The temporal Laplace transform leave us with ordinary inhomogeneous differential equation

$$xu'' + (v - \epsilon x)u' - \lambda u = -u(x,0) \tag{C.2}$$

The computation of the first return time requires absorbing boundary condition for equation C.2 in the origin. The Green function is constructed using to positive linear independent solution $\xi_\lambda(x), \eta_\lambda(x)$ from which the first one is chosen zero in the origin. The general form of the Green function is [19]

$$G_\lambda(x,y) = \begin{cases} \frac{\xi_\lambda(x)\eta_\lambda(y)}{W(y)} & if \quad x < y, \\ \frac{\xi_\lambda(y)\eta_\lambda(x)}{W(y)} & if \quad x > y \end{cases} \tag{C.3}$$

The total mass is defined as

$$J_\lambda = \int_0^\infty dy\, G_\lambda(x,y) \tag{C.4}$$

and it is the solution of the equation

$$xu'' + (v - \epsilon x)u' - \lambda u = 1$$

The following change of variable and parameters: $x \to \frac{\lambda}{\epsilon}$ map the homogeneous equation associated with C.2 in the equation satisfied by the confluent Hypergeometric function

$$xp'' + (u - x)p' - \frac{\lambda}{\epsilon}p = 0. \tag{C.5}$$

We choose

$$\xi_\lambda = x^{1-v}\Phi\Big(\frac{\lambda}{\epsilon} - v + 1, 2 - v; x\Big) \tag{C.6}$$

$$\eta_\lambda(x) = \Psi\Big(\frac{\lambda}{\epsilon}, v; x\Big) = \tag{C.7}$$

$$\frac{\Gamma(1-v)}{\Gamma(\frac{\lambda}{\epsilon} - v + 1)})\Phi\Big(\frac{\lambda}{\epsilon}, v; x\Big) + \frac{\Gamma(v-1)}{\Gamma(\frac{\lambda}{\epsilon})}x^{1-v}\Phi\Big(\frac{\lambda}{\epsilon} - v + 1, 2 - v; x\Big)$$

where $\Gamma(x)$ is the Euler's function [24] and

$$\Phi(\alpha, \gamma; x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \dots$$

is the confluent Hypergeometric function [24]. The Wronskian is

$$W(x) = (\xi_\lambda(x)'\eta_\lambda(x) - \xi_\lambda(x)\eta_\lambda(x)')x = \frac{\Gamma(2-v)}{\Gamma(\frac{\lambda}{\epsilon} - v + 1)}x^{1-v}e^x.$$

We have used the asymptotic properties of the function $\xi_\lambda(x)$ and $\eta_\lambda(x)$ to determinate the constant [35]. The total mass can be written

$$\begin{aligned} J_\lambda(x) &= \epsilon^{-1}\frac{\Gamma(\frac{\lambda}{\epsilon} - v + 1)}{\Gamma(2-v)}\Big[\Psi\Big(\frac{\lambda}{\epsilon} - v + 1; x\Big) \\ &\times \int_0^x dy\, e^{-y}\Phi\Big(\frac{\lambda}{\epsilon} - v + 1, 2 - v; x\Big) \\ &+ x^{1-v}\Psi\Big(\frac{\lambda}{\epsilon} - v + 1, 2 - v; x\Big)\int_x^\infty dy\, y^{v-1}e^{-y}\Psi\Big(\frac{\lambda}{\epsilon}, v; y\Big)\Big] \end{aligned} \tag{C.8}$$

If $v > 0$ the second integral can be taken from origin and its result is known [24]. The remaining term satisfy the inhomogeneous equation but the inhomogeneous term is $-1/\epsilon$, so we can write

$$J_\lambda(x) = \frac{1}{2}\epsilon^{-v}\frac{\Gamma(\frac{\lambda}{\epsilon} - v + 1)}{\Gamma(\frac{\lambda}{\epsilon} + 1)}\frac{\Gamma(v)}{\Gamma(2-v)}x^{1-v}\Phi\Big(\frac{\lambda}{\epsilon} - v + 1, 2 - v; \epsilon x\Big) \tag{C.9}$$

where we have recovered the initial x variable. If $\epsilon \ll 1$ only the terms containing $\frac{\lambda}{\epsilon}$ are relevant. In this order, using the properties of the Γ function we obtain:

$$J_\lambda(x) = \frac{1}{2}\epsilon^{-v}\frac{\Gamma(v)}{\Gamma(2-v)}x^{1-v}\Gamma(1-v)\left(1+\frac{\lambda}{\epsilon}(\psi(1)+\psi(1-v)\right)\text{(C.10)}$$

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) \quad \text{(C.11)}$$

therefrom the characteristic time comes

$$t_c = -\frac{1}{\epsilon}(\psi(1)+\psi(1-v)) \quad \text{(C.12)}$$

As the involved function are analytic in the complex plane and the change of sign for v does not imply any divergences in the initial formula (C.8) of the total mass one can cross in the region with $v < 0$ $|v| < 1$, and the relation founded above must hold.

Appendix D

Mathematical Addenda

D.1 Tauberian theorem

We reproduce here from the reference [19] the Tauberian theorem and two useful related theorems. The theorems are valid for generic measure on the semi-positive axis, $U(t)$ whose Laplace transform exists.

$$\omega(\tau) = \int_0^\infty e^{-\tau x} U\{dx\}$$

We recall that a positive function L defined on $(0, \infty)$ varies slowly at ∞ if for every fixed x

$$\frac{L(tx)}{L(t)} \longrightarrow 1, \qquad t \to \infty.$$

L varies slowly at 0 if this relation holds as $t \to 0$, that is, if $L(1/x)$ varies slowly at ∞. With the above notation we have:

Theorem 1 *If L is slowly varying at infinity and $0 \leq \rho < \infty$ then each of the relation*

$$\omega(\tau) \sim \tau^{-\rho} L\left(\frac{1}{\tau}\right), \qquad \tau \longrightarrow 0,$$

and

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^{\rho} L(t), \qquad t \longrightarrow \infty$$

implies each other.

If the U poses a density u which is ultimately monotone, that is, monotone in some interval (x_0, ∞), we have

Theorem 2 *Let $0 < \rho < \infty$. If u has an ultimately monotone derivative u then as $\lambda \to 0$ and $x \to \infty$, respectively,*

$$\omega(\lambda) \sim \frac{1}{\lambda^\rho} L(\frac{1}{\lambda}) \quad \textit{iff} \quad u(x) \sim \frac{1}{\Gamma(\rho)} x^{\rho-1} L(x)$$

If $f(x)$ is an integrable function on the interval $(0, \infty)$ with the Laplace transform

$$\phi(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx$$

and it is ultimately monotone from the above theorems we have the equivalence

$$\phi(0) - \phi(\lambda) \sim \lambda^{1-\rho} L(1/\lambda) \quad \text{and} \quad f(x) \sim \frac{1}{\Gamma(\rho - 1)} x^{\rho-2} L(x)$$

where $0 < \rho < 1$ and $\Gamma(z)$ is the Euler function [24].

D.2 The Imaginary argument Bessel function I_ν and K_ν

Bessel function $Z_\nu(z)$ are solution of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{dZ_\nu}{dz} + (1 - \frac{\nu^2}{z^2}) Z_\nu = 0.$$

The Bessel function of first kind are

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)}, \qquad |arg\, z| < \pi.$$

Two positive on the real axis and linearly independent solutions are the Bessel function of imaginary argument

$$\begin{array}{rcll} I_\nu(z) & = & e^{-\frac{\pi}{2}\nu i} J_\nu(e^{\frac{\pi}{2} i} z) & -\pi < arg\, z \le \frac{\pi}{2}, \\ I_\nu(z) & = & e^{\frac{3}{2}\pi\nu i} J_\nu(e^{-\frac{3}{2}\pi i} z) & \frac{\pi}{2} < arg\, z \le \pi \\ , K_\nu(z) & = & \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, & \nu \text{ not an integer} \end{array}$$

The following recursion relation are useful

$$\begin{array}{rcl} (\frac{d}{zdz})^m \{z^\nu I_\nu(z)\} & = & z^{\nu-m} I_{\nu-m}(z) \\ (\frac{d}{zdz})^m \{z^{-\nu} I_\nu(z)\} & = & z^{-\nu-m} I_{\nu+m}(z). \end{array}$$

The diffusion equation (4.6) is connected with the following equation

$$u'' + \frac{1-2\alpha}{z} u' + \left[(\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2\gamma^2}{z^2} \right] u = 0 \tag{D.1}$$

which has the general solution

$$u = z^\alpha Z_\nu(\beta z^\gamma)$$

with Z_ν a generic Bessel function.

D.3 The formal solution of the diffusion equation on the positive semi-axis

We start with the general equation

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x}$$

where $a > 0$ represents the local variance and b the local drift. The time Laplace transform yields an ordinary inhomogeneous second order differential equation

$$a\,u'' + b\,u' - \lambda\,u = -u(x,0). \tag{D.2}$$

Its solution can be expressed using the Green function

$$u_\lambda(x) = \int_0^\infty dy G_\lambda(x,y)u(x,0), \tag{D.3}$$

and we see that The Green function is the Laplace transform of the transition probabilities density $q_t(x,y)$ describing the diffusion process. We take as known that the path variable $X(t)$ depend continuously on t. Let $X(0) = x_0$ and denote by $F(t,x,y)$ the probability that the point y will be reached before t. We call F the distribution of the first passage time from x to y and denote its Laplace transform by $\phi_\lambda(x,y)$.

For $x < y < z$ the event $X(t) = z$ takes place iff a first passage through y occurs at some epoch $\tau < t$ and is followed by a transition from y to z within the time $t - \tau$. Thus $q(x,z)$ represents the convolution of $F(t,x,y)$ and $q_t(y,z)$, whence

$$G_\lambda(x,z) = \phi_\lambda(x,y)G_\lambda(y,z), \qquad x < y < z.$$

Fix a point y and choose $u(x,0)$ concentrated on (y,∞). Multiply the previous equation by $u(z,0)$ and integrate with respect to z. In view of (D.3) the result is

$$u_\lambda(x) = \phi_\lambda(x,y)u_\lambda(y), \qquad x < y, \tag{D.4}$$

while (D.2) requires that for y fixed $\phi_\lambda(x, y)$ satisfy the differential equation

$$a(x)\frac{\partial^2 \phi_\lambda}{\partial x^2} + b(x)\frac{\partial \phi_\lambda}{\partial x} - \lambda\phi_\lambda = 0.$$

Therefore it has to be a positive solution of the homogeneous equation bounded in origin. Assuming it exist we denote it with $\xi_\lambda(x)$. Since (D.4) shows that $\phi_\lambda(x, y) \to 1$ as $x \to y$ we have

$$\phi_\lambda(x, y) = \frac{\xi_\lambda(x)}{\xi_\lambda(y)}, \qquad x < y, \tag{D.5}$$

where $\xi_\lambda(x)$ is a positive and bounded in the origin solution of the homogeneous equation. A similar argument applies when $x > y$ but we have to choose a positive solution bounded at ∞. By direct computation one can see that the Green function has the form

$$G_\lambda(x, y) = \begin{cases} \frac{\xi_\lambda(x)\eta_\lambda(y)}{W(y)} & if \quad y > x \\ \frac{\eta_\lambda(y)\xi_\lambda(x)}{W(y)} & if \quad y < x \end{cases} \tag{D.6}$$

where $W(y) = (\xi'_\lambda(y)\eta_\lambda(y) - \xi_\lambda(y)\eta'_\lambda(y))a(x)$ is the Wronskian associated to the equation.

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