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Facets of

Holographic Duality

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Contents

Introduction

The ultimate goal of theoretical physics must be the description of our universe in the simplest and most unified fashion. The great progress made in the last century has left us a rich heritage: Perhaps the most beautiful instance is General Relativity. Assuming only the equivalence principle, Einstein was able to formulate a classical theory of Gravity in a completely geometrical language, and his predictions have been tested with very high accuracy. The other big pillar on which theoretical physics stands nowadays, is Quantum Mechanics. Names like Planck, Heisenberg, Scrhödinger, Fermi, Dirac, Feynman have taken us from the initial quantum mechanical hypothesis to the formulation of Quantum Electrodynamics (QED) — the first example where a well established classical field theory has been given a completely new description in terms of quantized fields.

Following these ideas, it was finally possible to unify, in the so-called Standard Model, QED with the theory of weak interactions, previously described by the four-fermion effective theory. The underlying physical idea, the gauge principle, has also proved to be fruitful in describing strong interactions, with the formulation of Quantum Chromodynamics (QCD).

Standard Model predictions are tested every day in big accelerators, and so far they have been confirmed with incredible precision. Although much progress has been made in understanding more deeply QCD, a full satisfactory explanation of its most relevant feature, namely confinement, does not exist yet. Thus, it is a task for the new generations of physicists to complete this program.

Moreover, the final aim is to find, if it exists at all, the Theory of Everything — a resounding name for something that should be able at least to unify the classical theory of Gravitation with the realm of quantum world.

In an attempt to find such kind of theory, a striking mechanism was proposed by Kaluza and Klein [1, 2]: The idea that our universe possesses more than four dimensions. Even if the extra dimensions are not experienced by the every-day person, these could provide a framework for mixing together General Relativity and Quantum Field Theory. As we know, unfortunately, these first attempts failed.

Happily, in the last twenty five years, new theoretical tools appeared on the market. The two main revolutionary ideas are probably supersymmetry and string theory. They were born [3, 4, 5, 6] and have grown together.

Supersymmetry is a symmetry that relates bosons and fermions. Imposing supersymmetry to hold reduces drastically the number of possible theories. For instance, at the classical level, there is only one super Yang–Mills theory in ten dimensions, and no consistent supersymmetric gauge theory in more than ten dimensions [7]. Moreover, gauging supersymmetry immediately leads to gravity, as one is actually making Poincaré symmetry local [8, 9]. This, in combination with the old Kaluza–Klein theories, has led to the formulation of various supergravities. Again the structure of different supergravities is much constrained in various dimensions, the maximal dimension now being 11, and the more supersymmetry, the more restricted are these theories. Although important, so far supergravity has not managed to be turned in a consistent quantum theory by itself.

String theory improves the scenario by abandoning the concept of point-like particle, and using instead a one-dimensional quantized object: The string. The original motivation for introducing strings was explaining strong interactions. Some time had to pass before the scientific community accepted the idea that string theory could be used to describe gravity coupled to other fields [10]. Requiring non-anomalous reparameterization invariance of the string world-sheet and, again, supersymmetry, results in a low number of ten-dimensional different superstring theories. Furthermore, cancellation of anomalies in target space [11], restricts even further their number, so that to this day we know of five consistent superstring theories: Type I, IIA, IIB, and the two Heterotic strings. Remarkably, each of them corresponds, in its low energy limit, to a known ten-dimensional supergravity.

A major break-through in understanding string theory was achieved with the discovery of dualities relating these five theories [12, 13, 14, 15, 16]. Actually, these duality relations are conjectures, as they are non-perturbative in nature, and proving them would require a knowledge of these theories beyond perturbation theory. However, they can be successfully tested for some of their typical features, such as their BPS spectra, low energy actions, and supersymmetries. Thus, we now think of the five superstrings as a unique theory, that we can study perturbatively in different regimes.

However, a puzzle remains. Namely, the dualities require the existence of an eleven dimensional quantum theory as the strong coupling limit of Type IIA string theory. On the other hand its low energy limit, eleven-dimensional supergravity, has no known parent string theory describing it. Its quantum generalization is still searched for, and for some reason, it has been called M-Theory. Sometimes one refers to M-Theory as the yearned Theory of Everything, thus including all superstrings, $d = 11$ supergravity, and everything else possible.

A second formidable impetus for studying strings came after the discovery of D(irichelet)-branes: These are extended objects where open strings can end. The remarkable conjecture by Polchinski [17] is that D-branes are the microscopic description of some previously known supergravity solution. These states are also

required by the dualities, and are useful in testing them, as they are BPS, thus preserving some fraction of supersymmetry, and can be easily followed from one regime to another. They provide an unreplaceable tool for computing microscopically black hole entropies, as one can use the known particle content of the world-volume theory, *i.e.* the low-energy limit of the corresponding open string theory.

An important property of D-branes is that, opposite to the string itself, they couple to RR potentials of the closed string sector. It is satisfactory to realize that these fields also have their own sources.

Based on this dual description of D-branes, Maldacena [18] has made the very interesting proposal that closed string theory, or its supergravity approximation, should be dual, in some appropriate limit, to the D-brane world-volume theory, typically a supersymmetric Yang–Mills theory. Let us point out the motivations that led to the conjecture, focusing on the original setup, relying on the dual description of the D3-brane of Type IIB string theory. First, one notices that the near-horizon limit of the 3-brane solution of Type IIB supergravity, *i.e.* asymptotically near the location of the D-brane, has $AdS_5 \times S^5$ geometry, whose isometry group is $SO(4,2) \times SO(6)$. Note also that this background is an exact solution of Type IIB supergravity that is maximally supersymmetric, having $\mathcal{N} = 8$. It turns out that the isometry group can be extended to include the unbroken supercharges, to the $SU(2,2|4)$ supergroup.

On the other hand, the world-volume theory arising on a stack of N D3-branes is a super Yang–Mills theory, maximally supersymmetric in $d = 4$, namely with $\mathcal{N} = 4$. Its supercharges transform in a representation of the $SU(4) \simeq SO(6)$ R-symmetry group. Moreover, it can be proven that this theory is exactly conformal, so that the $SO(3,1)$ Lorentz invariance gets enhanced to the four dimensional conformal group $SO(4,2)$, and the complete symmetry of the theory is the same of the isometries of the near-horizon brane, although the algebraic realization is different.

In order to let the dimension-full radial coordinate going to zero, one has to simultaneously let $\alpha' \to 0$. Moreover, in order to trust the supergravity limit of string theory, curvatures should be small: This implies $g_{st}N \gg 1$, thus finally the number of branes N must be large. These are the limits in which the duality can be tested.

Here the meaning of *duality*, or *correspondence*, is that the two theories are actually two descriptions of the same physics, in the spirit of strong–weak dualities. This duality, named anti de Sitter – conformal field theory correspondence (AdS/CFT), has been given a more precise form in two subsequent papers [19, 20], where the QFT generating function has been identified with the on-shell supergravity action.

Not surprisingly, the AdS/CFT correspondence is better established when supersymmetry is present. In fact Maldacena's initial proposal concerned the maximally supersymmetric compactification of Type IIB supergravity, and its dual partner, the four-dimensional, maximally extended super Yang–Mills. After that work, many

beautiful checks of the conjecture have been performed, and also a myriad of generalizations to different situations have been proposed. For a review, see [21].

Among these developments, let us first focus on a non-standard application proposed by Polyakov [22]. It concerns the first of the two outstanding problems of theoretical physics mentioned above, namely a suitable approach to gauge theories, with particular attention to the confinement problem. The proposal combines the idea of the "confining string" [23, 24], with the increasing understanding of D-branes and the AdS/CFT correspondence itself.

The idea is to consider a non-supersymmetric string theory of a very particular kind. This theory has the same world-sheet description of the superstring, but an appropriate GSO projection removes all fermions from its spectrum. In addition, a new sector must be included in order to preserve modular invariance, containing a tachyonic scalar field at its lowest level, and no massless fields. The same physical requirement forces to double the RR sectors as well, thus inducing new potentially interesting characteristics, following from the doubling of D-branes types. As this theory, known as Type 0 [1][25, 26], contains a tachyon, it was somehow overlooked.

However, one should notice that these theories, as Polyakov pointed out, are more manageable than the usual bosonic string theory. First, their D-branes carry RR charge, and the coupling to the tachyon may provide a novel mechanism for its stabilization. Second, among different D-brane settings that one can construct, there are some which do not contain an open string tachyon, contrary to bosonic string. It turns out that the world-volume theory living on these branes is a type of non-supersymmetric Yang–Mills theory, with at most some adjoint scalar fields, or fermions in the bifundamental.

These observations, together with Maldacena's idea, have led Polyakov to conjecture that ordinary Yang–Mills theory should possess some conformal phase, dictated by the corresponding AdS solutions of Type 0 gravity[2]. His proposal is actually even more radical, and embraces the possibility of extending Type 0 theory and their pertinent duality in less then ten dimensions. Despite the difficulties in dealing with non-critical string theory, there is at least some evidence that Type 0 theory can be extended this way. Besides the reasons named before, let us mention that they have a "diagonal" partition function, that seems naively to be modular invariant regardless the number of dimensions. Also, whatever mechanism for tachyon condensation it would lead in any case to a non-zero effective central charge.

Polyakov's idea is perhaps a step towards a satisfactory explanation of Yang–Mills theory in terms of strings. Even if conformal QCD is not exactly what one would expect, one should still appreciate the fact that this approach leads to physically relevant non-supersymmetric theories. Moreover, there are examples of solutions that are not conformal [28, 29]. Let us finally notice that the use of the

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[1]The "0" refers to the number of supersymmetries, as usual.

[2]See [27] for a recent discussion of this idea.

AdS/CFT duality may also shed some light on non-supersymmetric and non-critical strings, definitely two irksome subjects.

As far as the other goal of string theory is concerned, namely unification of gravity and quantum field theory, the AdS/CFT correspondence has proven useful as well. In this context, we want to stress perhaps one of the most appealing, at least for us, extensions of AdS/CFT that has been proposed. Namely, if one relaxes the requirement of conformal invariance, dual couplings cease to be constant and become space-time dependent, lending themselves to be interpreted as quantum field theory running couplings. Thus the subject of renormalization group (RG) comes into the game and opens up a new direction in theoretical physics, that has been named Holographic RG.

The name refers to some earlier ideas in quantum gravity, based on counting degrees of freedom in black hole physics [30, 31]. Namely, it was proposed that, in a gravitational theory, the number of microstates enclosed in a given volume is proportional to the surface at its boundary.

In a diffeomorphism invariant theory, the concept of a point has no invariant meaning, so that it is impossible to define local observables. The Holographic Principle states that a theory containing gravity, e.g. string theory, should be describable in terms of a dual local quantum field theory on the boundary of space-time. From a reversed point of view, this means that some quantum theory may enjoy an alternative language in terms of gravity. The AdS/CFT correspondence in its simplest instance, is a very sharp example of this feature: A (super)gravity theory in AdS background is holographically dual to a (super) conformal field theory. The study of a holographic version of the Renormalization Group is the next problem one should naturally address. In our opinion, the consequences of a better understanding of this fascinating phenomenon can lead to interesting progress, both in gravity and in quantum field theory. In particular, many attractive features are already evident in considering the gravity approximation of string theory, even without using supersymmetry.

There are many evidences that the radial coordinate of AdS, when extended to deformed solutions, has the interpretation of an energy scale, namely the cutoff scale of a bare theory. One can define suitable beta functions and, moreover, it seems that powerful results known in two dimensions can in some cases be extended to higher dimensional theories. It is possible for instance to prove a holographic C–theorem, holding under some mild assumptions. In this framework one can also study flows connecting different fixed points, generally a difficult problem.

Among several questions that holographic duality poses us, there is the issue of the role of supersymmetry. So far, both the duality conjectures and the AdS/CFT itself, in its simplest form, had relied heavily on supersymmetry arguments. On the other hand, Holography seems really to be a general feature of some theories, regardless their amount of supersymmetry. The renormalization group flows and the

holographic Weyl anomaly [32] are highly non-trivial tests of this correspondence, that do not seem to rely on supersymmetry, but make use just of diffeomorphism invariance and the idea of Wilsonian renormalization group.

We feel that an urgent question is trying to understand to what extent this correspondence holds, and whether supersymmetry plays any role. For addressing these issues one should try to understand a general setup, including different kinds of fields, in addition to the metric and the scalar fields.

As holography involves a distinguished coordinate parameterizing the energy scale, and as flows are first order equations, a promising framework to attack these problems can be the Hamiltonian formulation of (super)gravity. In particular, it has been suggested [33] to use the Hamilton–Jacobi approach, as it naturally deals with the on-shell action, which is central interest in studying properties of the dual QFT. Using this framework, we have started investigating the role of fermions and form fields in the holographic correspondence [34].

Plan of the thesis

We now present in more detail the content of this thesis. We refrained from giving an extensive introduction for non experts, either to string theories, or to the AdS/CFT correspondence[3]. Rather, we have collected the material resulted from the work done in the last two years in connection to Type 0 string theory, AdS/CFT correspondence, and Holography, expanding it and discussing it in more detail. Moreover we have included some unpublished results that were achieved in the course of research, and which we feel can be a useful complement to the main material.

The dissertation is organized as follows:

Chapter 1. Very basic facts about strings and branes are recalled. The aim of this chapter is to set the notation and prepare the ground for discussing Type 0 string theories.

Chapter 2. Contains a detailed overview of Type 0 string theories. In Section 2.1, we discuss the construction of closed string spectra, stressing their relationships with Type II theories, from the point of view of orbifold constructions. We illustrate the Type 0 D-branes, and their mapping under orbifold operations. Then we discuss some world-sheet aspects, and their use in computation of effective actions. Finally, in Section 2.4 the possibility of extending Type 0 theories in dimensions different from ten is analyzed, and a concrete proposal for their effective theories is given.

Chapter 3. In the first section, by means of simple examples we introduce the idea of applying the AdS/CFT correspondence to Type 0 theories. In Section 3.2 explicit

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[3]The interested reader can find a comprehensive review of the subject in [21].

solutions of the Type 0 gravities are provided, for a general range of dimensions. After studying some of their properties, namely stability, entropy, and dual Wilson loops, we comment on their dual field theory meaning. Finally, we introduce the subject of holographic flows, to be discussed in much grater details later, by deforming the above Type 0 solutions.

Chapter 4. The first section contains an extensive introduction to holographic flows, with attention to some subtleties that can arise in implementing them. In the next two sections we give two applications of these ideas, illustrating also why the Hamiltonian formalism can be useful in deriving and interpreting the solutions. The first concerns the case of a single scalar field arising from $d = 7$ $\mathcal{N} = 1$ gauged supergravity. The analysis is complemented with some numerical calculations. The second example touches an independent issue — that of SCFT's dual to compactifications on "non-spherical" manifolds, and holographic flows among them. The section contains also the computation of a two-scalar effective action arising from compactification of M-Theory on the manifold $N(1,1)$. In Section 4.4 we motivate the use of Hamilton–Jacobi theory for studying flows, and Holography in general. Finally, we comment on holographic anomalies and present a novel way of deriving the holographic Weyl anomaly.

Chapter 5. Here we focus on some features of Holography including spin-$\frac{1}{2}$ fermions and form fields, in the framework of Hamilton–Jacobi theory. First we motivate our study, also giving an example concerning spinors in the AdS/CFT correspondence. In Section 5.3 we derive, in any dimension and signature, the ADM Hamiltonian for a generic theory of gravity coupled to spin-$\frac{1}{2}$ fermions and antisymmetric tensor fields. We then discuss under which conditions the system gives rise to a so-called Callan–Symanzik equation following from the zero-energy constraint. In Section 5.5 we complete the discussion considering the full set of Hamiltonian constraints, regarding them as Ward identities in the dual holographic theory. Some amusing conditions follow from the diffeomorphism constraint. We also present an expansion up to second non trivial order of the on-shell action. Finally the conclusions are given.

Appendices. Contain formulae useful in doing Hamiltonian reduction and the complete analysis of fermionic phase-spaces. Both for the complex and the real cases.

Chapter 1

Strings & branes

Before specializing to a more detailed review of Type 0 string theories in later chapters, we begin by recalling some basic facts about strings and branes. This will set the notation and prepare ground for discussing properties of Type 0 theories, such as their D-brane content, spectra, and low energy actions.

1.1 Strings

We will use the Ramond-Neveu-Schwarz (RNS) covariant formulation, so that different sectors of Type II and Type 0 theories can be described in essentially the same language in terms of left and right moving string excitations. We will not discuss the Heterotic string and consider only $\mathcal{N} = (1,1)$ world-sheet supersymmetry. The RNS action is derived from a two-dimensional action enjoying reparameterization, local Lorentz, and Weyl invariances, and supersymmetry [35]. After fixing the conformal gauge, $\eta_{\alpha\beta} = (-1,1)$ and setting to zero the world-sheet gravitino, this reads

$$S = -\frac{1}{2\pi} \int d^2\sigma \left(\partial^\alpha X_\mu \partial_\alpha X^\mu - i\bar{\psi}^\mu \slashed{\partial} \psi_\mu \right) . \tag{1.1}$$

We have set $\alpha' = \frac{1}{2}$. The target space metric is ten-dimensional flat $\eta_{\mu\nu} = (-1, 1 \ldots 1)$. The coordinates X^μ are target space vectors and world-sheet scalars, whereas ψ^μ are world-sheet Majorana spinors and target space vectors.

The action (1.1) must be stationary under variations with respect to fields. This implies the Laplace and Dirac equations of motion

$$\Box X^\mu = 0 \tag{1.2}$$

$$\slashed{\partial} \psi^\mu = 0 , \tag{1.3}$$

and the vanishing of the following contribution from the variations at the boundary

$$\delta S = \frac{1}{\pi} \oint d\sigma_\alpha \epsilon^{\alpha\beta} (\delta X^\mu \partial_\beta X_\mu + \frac{i}{2} \delta\bar{\psi}^\mu \rho_\beta \psi_\mu) = 0 . \tag{1.4}$$

Accordingly, the solutions of (1.2), (1.3) have to be supplemented with suitable boundary conditions. The closed string solves automatically (1.4), because of (anti)-periodicity. The open string conditions leave instead more possibilities. Namely, at the boundary, bosons must satisfy

$$N: \quad \partial_n X^\mu = 0 \qquad (1.5)$$

$$D: \quad \partial_t X^\mu = 0 , \qquad (1.6)$$

where the vanishing of the normal derivative is the Neumann boundary condition, while the vanishing of the tangential derivative is the Dirichlet one. For fermions one has to equate the two Weyl components up to a sign

$$+: \quad \psi_+^\mu = \psi_-^\mu \qquad (1.7)$$

$$-: \quad \psi_+^\mu = -\psi_-^\mu . \qquad (1.8)$$

At tree level, *i.e.* taking the world-sheet to be the strip, the solutions of equations (1.2), (1.3) with the boundary conditions above are easily obtained and can be expressed as sums of left and right moving modes. For the open string, one can choose different boundary conditions for each of the two disconnected components of the boundary, thus there are four possibilities: NN, ND, DN, and DD.

For the closed string the solutions read[1]

$$X^\mu = x^\mu + ip^\mu \sigma_1 + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-nz} + \tilde{\alpha}_n^\mu e^{-n\bar{z}} \right) \qquad (1.9)$$

$$\psi^\mu = \frac{1}{\sqrt{2}} \sum_n \psi_n^\mu e^{-nz} \qquad (1.10)$$

$$\bar{\psi}^\mu = \frac{1}{\sqrt{2}} \sum_n \tilde{\psi}_n^\mu e^{-n\bar{z}} . \qquad (1.11)$$

The sum in (1.9) runs over integers ($\mathbb{Z}$), while those in (1.10) and (1.11) extend on integer or half-integer ($\mathbb{Z} + \frac{1}{2}$) values depending on whether one is considering (1.8) or (1.7) b.c. respectively. Different fermion moddings are denoted in general as Ramond (R) and Neveu-Schwarz (NS) sectors. We will use the following definitions for R and NS sectors:

R = same boson and fermion modding
NS = opposite boson and fermion modding.

Therefore, in the usual situation where bosons have integer modding, the R sector consists of integer fermion modding, whereas the NS one consists of half-integer fermion modding.

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[1] We have defined $z = \sigma_1 + i\sigma_2$.

As far as the open string is concerned, the boundary conditions reduce by half the independent oscillators, in particular the left and right moving modes are actually the same: $\alpha_n^\mu = \tilde{\alpha}_n^\mu$ and $\psi_n^\mu = \tilde{\psi}_n^\mu$. Moreover, for DD directions, in (1.9) $p^\mu \sigma_1$ is replaced by $\frac{d^\mu}{\pi}\sigma_2$, while for ND and DN ones, these terms are both absent and bosons have half-integer modding.

Physical quantities such as scattering amplitudes and the partition function are given by summing over the full spectrum. In principle one should include all possible boundary conditions in the sum — however consistency requirements such as modular invariance or supersymmetry impose additional constraints on how different sectors are combined. This is taken into account by the so-called GSO projection [6]. The GSO projection operator is defined by

$$P_\pm = \frac{1}{2}(1 \pm (-1)^F) \,, \tag{1.12}$$

where F is the world-sheet fermion number. The operator $(-1)^F$ in the fermionic sectors contains a Clifford action, namely, on the zero modes

$$(-1)^F = \Gamma_\chi \,, \tag{1.13}$$

so that each fermionic sector is of definite chirality. The standard projection operator is P_+. This removes the tachyon from the open string spectrum. For the closed string, there is a projector for both left moving and right moving sectors.

In canonical quantization the oscillator modes become creation and annihilation operators obeying (anti)commutation rules. The spectra are constructed acting with them on the Fock vacuum. The Hamiltonian is the sum of the zero mode contribution, which depends on boundary conditions, and a contribution given by oscillators

$$H_{\text{open}} = H_0 + N - a \,. \tag{1.14}$$

Generically, $H_0 = p^2 + \frac{d^2}{\pi^2}$, where the momentum term comes from NN directions, while the d^2 shift comes from DD ones. $N = N_B + N_F$ includes bosonic and fermionic number operators respectively and a is the zero-point energy coming from normal ordering: Its value, together with space-time dimensionality $d = 10$, is fixed requiring closure of Lorentz operator algebra and cancellation of the conformal anomaly. The contributions to a from a single physical field is summarized here, according to its modding

	$\mathbb{Z}$	$\mathbb{Z} + \frac{1}{2}$
bosons	$-\frac{1}{24}$	$+\frac{1}{48}$
fermions	$+\frac{1}{24}$	$-\frac{1}{48}$

For instance a fermion in the R sector, with NN or DD boundary conditions, has $a = +\frac{1}{24}$. The open string spectrum is obtained by imposing the mass-shell condition $H_{\text{open}} = 0$ on physical states

$$m^2 \;=\; \frac{d^2}{\pi^2} + N - a \;.\tag{1.15}$$

The above boundary conditions can be imposed independently for each coordinate. A coordinate obeying (1.6) is fixed, so that choosing $9 - p$ D b.c. one defines a $(p+1)$-dimensional hyper-plane, which is called a Dp-brane. In this language the full Lorentz invariant NN boundary conditions, which define standard open strings, can be regarded as a space-filling D9-brane.

The open string lowest state, with no oscillators excited, is tachyonic. The GSO$_+$ projection removes it from the spectrum, and also has the virtue of rendering the full spectrum supersymmetric. The resulting massless spectrum[2] is that of Table 1.1.

	NS+	R+
I	A_μ	ψ

Table 1.1: Type I sectors and low laying fields.

One therefore gets a massless vector multiplet. The low energy theory for open strings is then $d = 10$ SYM, and its effective action is completely fixed by supersymmetry. It is moreover the Yang–Mills sector of Type I supergravity.

Let us now go to the closed string. The left and right moving degrees of freedom are independent and they behave effectively as two open strings. The physical spectra can then be constructed easily as direct products of open string states. The mass-shell condition

$$H_{\text{closed}} = H_0 + N + \tilde{N} - a - \tilde{a} \;=\; 0\tag{1.16}$$

must be supplemented with the level matching condition

$$N - a \;=\; \tilde{N} - \tilde{a} \;.\tag{1.17}$$

Where tilded quantities refer to right moving degrees of freedom.

The left and right R ground states are space-time spinors, whereas the NS ones are scalars, so that RR and NSNS states are space-time bosons, while RNS and NSR states are space-time fermions. The lowest NS state being tachyonic, implies that the

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[2]For notation see below.

NSNS closed string sector contains a tachyon as well[3] — a suitable GSO projection will remove it from the spectrum. In particular, one should project both left and right NS sectors with the operators P_+ and $\tilde{P}_+$. On the other hand the R states can still be projected arbitrarily. Fixing, say, the right sector to be projected with $\tilde{P}_+$[4], then the left R sector can be projected either with P_+ or P_-. This is reflected in the chirality properties of space-time fields and is the origin of the two distinct Type IIA and Type IIB theories, which are respectively non-chiral and chiral.

We will use the following notation for closed string sectors:

$$(NS\pm, NS\pm), \quad (NS\pm, R\pm), \quad (R\pm, R\pm) . \tag{1.18}$$

The two entries refer to left and right sectors respectively. The signs denote which GSO projection has been performed, *i.e.* whether P_+ or P_- have been used.

To summarize, with the above GSO projection, the massless spectra are those in Table 1.2.

	$(NS+, NS+)$	$(NS+, R+)$	$(R\pm, NS+)$	$(R\pm, R+)$
IIA	$\phi, B_{\mu\nu}, g_{\mu\nu}$	λ^1, ψ^1_μ	$\hat{\lambda}^2, \hat{\psi}^2_\mu$	$A_\mu, A_{\mu\nu\rho}$
IIB	$\phi, B_{\mu\nu}, g_{\mu\nu}$	$\lambda^1, \hat{\psi}^1_\mu$	$\lambda^2, \hat{\psi}^2_\mu$	$A, A_{\mu\nu}, A^+_{\mu\nu\rho\sigma}$

Table 1.2: Type II sectors and low laying fields.

The NS+NS+ sector is in common for both IIA and IIB theories, as well as any consistent string theory[5]. It includes the graviton, the dilaton, and the antisymmetric tensor field. The RR sectors are peculiar for each of the two theories. They also characterize the relevant D-brane spectrum. The NSR sectors represent fermionic degrees of freedom and include two spin-$\frac{1}{2}$ dilatini and two gravitini. Hatted fields have opposite chiralities with respect to the others.

This field content corresponds to the degrees of freedom of Type IIA and Type IIB supergravities[6]. The pertinent low energy effective actions can be extracted from on-shell computation of scattering amplitudes involving correlation functions at tree-level in string perturbation theory. Alternatively, at least for the NSNS sector, they can be deduced by requiring conformal invariance of the world-sheet sigma model,

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[3]The level matching condition (1.17) is such that the NS− sector can not be combined with NS+, R+, or R− ones, so that there are no tachyons in the fermionic RNS and NSR sectors. See below.

[4]An overall sign in the RR sector is unobservable in Type II theories.

[5]In addition to the bosonic string, we will see that this sector is also present in Type 0 theories.

[6]Supergravity theories are extensively treated in [36, 37, 38] and references therein.

i.e. vanishing of the beta-functions of strings propagating on a general curved space-time background. In Type II theories however, supersymmetry determine the form of the effective action completely. For the bosonic part, the general form reads

$$S \; = \; \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{g}\mathrm{e}^{-2\phi} \left(\mathrm{R} + 4|\mathrm{d}\phi|^2 - \frac{1}{2}|\mathrm{d}B|^2 - \frac{1}{2}\sum_p |\tilde{F}_{p+2}|^2 \right) \; , \quad (1.19)$$

as far as the NSNS and RR kinetic terms are concerned. The sum extends over all different RR fields-strengths, whose ranks change depending on whether one considers Type IIA or IIB theories, and the tilde denotes appropriate NSNS-shifts. There is a subtlety for the self-dual RR 4-form of Type IIB: The self-duality constraint makes the contribution to the action to vanish. One way of circumvent this problem is to use the naive action and impose self-duality as a constraint supplementing the equations of motion. For both theories there are Chern–Simons terms as well, namely

$$S_{\mathrm{CS}}^{\mathrm{IIA}} \; = \; -\frac{1}{4\kappa^2} \int B \wedge \mathrm{d}A_3 \wedge \mathrm{d}A_3 \qquad\qquad (1.20)$$

$$S_{\mathrm{CS}}^{\mathrm{IIB}} \; = \; -\frac{1}{4\kappa^2} \int A_4 \wedge \mathrm{d}B \wedge \mathrm{d}A_2 \; . \qquad\qquad (1.21)$$

Let us also recall that there exist further guiding principles fixing the form of the actions above. In the IIA case, the action can be derived by dimensional reduction of eleven-dimensional supergravity, which is unique. Type IIB supergravity instead should posses an $SL(2,\mathbb{R})$ invariance group mixing the two 2-form potentials and the two scalars, that is the classical version of the full quantum symmetry group $SL(2,\mathbb{Z})$ of Type IIB string theory.

For completeness we also notice that there exist a generalization of Type IIA supergravity whose origin is not eleven-dimensional. As in the ordinary IIA theory there are 2-form and 4-form RR field-strengths and, by Poincaré duality, a 6-form and an 8-form as well, it is natural to include a 10-form $F_{10} = \mathrm{d}A_9$. Its equation of motion reads

$$\mathrm{d}\,^*F_{10} \; = \; 0 \; , \qquad\qquad (1.22)$$

and since $^*F_{10}$ is a scalar this implies that it is constant. Therefore it does not contain propagating degrees of freedom. Nevertheless it has energy density and it should be considered as a physical field. In fact it carries the charge of a domain-wall 8-brane. This extension goes under the name of massive Type IIA supergravity [39]. We will discuss slightly more extensively this field in Section 2.4 in the context of Type 0 effective theories. Its role will be relevant in finding AdS solutions in Chapter 3.

1.2 Branes

Imposing Dirichlet boundary conditions on, say, the last $9 - p$ directions introduces the open strings living on a Dp-brane. Ten-dimensional Lorentz invariance is broken to $SO(p, 1) \times SO(9 - p)$, and the effective open string theory is confined on the $(p + 1)$-dimensional world-volume of the brane, with $SO(p, 1)$ Lorentz symmetry and global $SO(9 - p)$ R-symmetry. The low energy field content obtained in these cases corresponds to the dimensional reduction of $d = 10$ SYM to $p + 1$ dimensions. The effective actions are also determined by supersymmetry.

High dimensional branes are somehow excluded from the considerations we are making here. They comprise the 7-brane, the domain-wall 8-brane, and the space-time filling 9-brane. These should be discussed separately, case by case, because they alter the space-time at infinity.

A major breakthrough occurred after the work of Polchinski [17], who noticed that D-branes are a microscopic description of some supergravity solutions, previously known as p-branes. Let us briefly summarize their most relevant features. The explicit form of an extremal p-brane, that is a solution of the equations following from (1.19), (1.20 or 1.21) can be found for instance in [40], and reads[7]

$$
\begin{aligned}
\mathrm{d}s^2 &= H_p(r)^{-\frac{1}{2}} \left(-\mathrm{d}t^2 + \mathrm{d}x^2 \right) + H_p(r)^{+\frac{1}{2}} \mathrm{d}y^2 \,, \\
\mathrm{e}^\phi &= H_p(r)^{\frac{3-p}{4}} \,, \\
A_{01\cdots p} &= \kappa^{-1} \left[1 - H_p(r)^{-1} \right] \,.
\end{aligned}
\tag{1.23}
$$

Here $(t, x^1, \ldots, x^p)$ are the coordinates along the brane, while $(y^{p+1}, \ldots, y^9)$ are the transversal one, $r^2 = y^2$ and the harmonic function reads[8]

$$
H_p = 1 + \kappa \left(\frac{c_p}{r} \right)^{7-p} \,.
\tag{1.24}
$$

Their tension goes like $T \sim g_{\mathrm{st}}^{-1}$ and they also carry a nonzero charge associated to the RR $(p + 1)$-form potential A

$$
q_p = \frac{1}{\sqrt{2}} \int_{\mathrm{S}^{8-p}} {}^* F_{p+2} \sim g_{\mathrm{st}}^{-1} \,.
\tag{1.25}
$$

Therefore, in appropriate units, they have mass equal to charge, suggesting that they preserve some fraction of supersymmetry. Indeed, this can be verified by explicitly computing the supersymmetry variations of fermion fields in the background (1.23), (1.24): It turns out that there is a chiral Killing spinor, thus half supersymmetry is preserved.

---

[7]These are solutions in the string frame, cf. Eq. (1.19).

[8]c_p is a numerical coefficient.

These solutions can be thought of as a special limit of a wider family of non supersymmetric solutions [40], parameterized by two constants r_- and r_+, that modify the form of the function H_p. In general, they indicate the position, in transverse direction, of a curvature singularity and an event horizon respectively. As long as $r_+ > r_-$ these solutions describe so-called black p-branes, i.e. higher dimensional black holes, whereas when the two parameters coalesce, H_p reduces to the expression above and the extremal solutions are obtained. This helps understanding the causal structure of a p-brane. In fact, in the above coordinate system, there are a horizon and a singularity occurring at $r = 0$.

The 3-brane is special as it is easily seen from (1.23). The dilaton is constant, and the space-time turns out to be totally nonsingular: All curvature invariants are finite everywhere. Let us also notice that, contrary to all other p-brane solutions, the near-horizon limit of a 3-brane is a product space with an AdS factor, namely $AdS_5 \times S^5$.

Although there is no rigorous proof of the conjectured correspondence between D-branes and p-branes, there are a number of compelling arguments in its favor. Let us recall the properties shared by p-branes and D-branes:

1. They are BPS states, preserving $\frac{1}{2}$ supersymmetry

2. They carry the same RR charge

3. They have the same tension $T \sim \frac{1}{g_{st}}$

4. Two parallel branes exert no force on each other .

From the point of view of D-branes, condition 1 follows from fermionic b.c. (most conveniently seen in the Green-Schwarz formalism). The other conditions are instead all gotten from the so-called *annulus* calculation which gives the force between to parallel static D-branes. In order to illustrate this calculation we will use the boundary state formalism.

A boundary state is a coherent ensemble of closed string excitations, constructed applying to the closed string vacuum an operator built out of oscillators, in the same fashion as one constructs the electromagnetic field of a laser starting from the QED Fock vacuum. A D-brane interaction is thus computed inserting the closed string propagator H_c^{-1} between two such states[9].

Enforcing the boundary conditions (1.5 – 1.8) as weak operatorial identities, i.e.

---

[9]Boundary states formalism, first appeared in [41], is treated in many references. We refer to [42, 43] where the construction is carried out in detail, and to [44], where some attention is devoted to Type 0 theories.

acting on a state of the closed string Hilbert space, yields[10]

$$
\begin{aligned}
\text{N}: \quad & (\alpha_n^i + \tilde{\alpha}_{-n}^i)|B\rangle_X = 0 && i = 2, \ldots, p \\
\text{D}: \quad & (\alpha_n^a - \tilde{\alpha}_{-n}^a)|B\rangle_X = 0 && a = p+1, \ldots, 9
\end{aligned}
\tag{1.26}
$$

for bosons, and

$$
\begin{aligned}
\text{N}: \quad & (\psi_n^i + i\eta\tilde{\psi}_{-n}^i)|B,\eta\rangle_\psi = 0 && i = 2, \ldots, p \\
\text{D}: \quad & (\psi_n^a - i\eta\tilde{\psi}_{-n}^a)|B,\eta\rangle_\psi = 0 && a = p+1, \ldots, 9
\end{aligned}
\tag{1.27}
$$

for fermions. $\eta = \pm 1$ corresponds to which spin structure one is choosing or, in other words, to a given GSO projection. The fermion modding can be either integer or half integer, considering the RR and NSNS sectors respectively. The solution to the above equations is easily seen to be, as far as the bosonic sector is concerned

$$
|B\rangle_X = \exp\left\{\sum_{n=1}^{} \frac{1}{n}\left(-\eta_{ij}\alpha_{-n}^i\tilde{\alpha}_{-n}^j + \delta_{ab}\alpha_{-n}^a\tilde{\alpha}_{-n}^b\right)\right\}|0\rangle_X ,
\tag{1.28}
$$

whereas for the fermionic sector it reads

$$
|B,\eta\rangle_\psi = \exp\left\{i\eta\sum_{n>0}\left(-\eta_{ij}\psi_{-n}^i\tilde{\psi}_{-n}^j + \delta_{ab}\psi_{-n}^a\tilde{\psi}_{-n}^b\right)\right\}|0,\eta\rangle_\psi .
\tag{1.29}
$$

The full boundary state is then constructed as the product of the two above

$$
|B,\eta\rangle = |B\rangle_X \otimes |B,\eta\rangle_\psi .
\tag{1.30}
$$

In the NSNS sector the fermionic vacuum $|0,\eta\rangle_{\text{NSNS}}$ is a scalar, while the RR vacuum is projected out of a product representation of $Spin(8)$, according to the zero-mode condition imposed by (1.27), namely [43]

$$
|0,\eta\rangle_{\text{RR}} = \left(\mathcal{C}\Gamma^0\cdots\Gamma^p\frac{1+i\eta\Gamma_\chi}{1+i\eta}\right)_{\alpha\beta}|\alpha\rangle\otimes|\beta\rangle .
\tag{1.31}
$$

The charge conjugation matrix $\mathcal{C}$, and Γ^i are Clifford matrices constructed with linear combinations of fermionic zero modes. Γ_χ is the chirality matrix. A physical D-brane state is obtained as a linear combination of the states (1.30), from different sectors and with different signs of η, which are determined by consistency requirements. Namely, the boundary state must be GSO invariant, with the same GSO projection performed on the closed string spectrum. The open string amplitude

---

[10]We shall use the simpler light-cone gauge construction. Thus avoiding complications coming from the ghost contributions.

obtained by world-sheet duality from the closed one is then an open string partition function, whose states interact consistently with the closed string ones. In the following we will restrict our attention to bosonic closed string exchanges, even if fermionic D-branes can be considered as well.

The appropriate GSO-invariant boundary state describing a Type II Dp-brane turns out to be

$$|Dp\rangle = |B,+\rangle_{\mathrm{NSNS}} + |B,-\rangle_{\mathrm{NSNS}} + |B,+\rangle_{\mathrm{RR}} + |B,-\rangle_{\mathrm{RR}} , \qquad (1.32)$$

therefore, the closed string exchange between two parallel Dp-branes

$$\mathcal{A} = \int \mathrm{d}l \, \langle Dp|e^{-lH_c}|Dp\rangle \qquad (1.33)$$

has ten different contributions. Using the fact that RR and NSNS states do not talk to each other

$$_{\mathrm{NSNS}}\langle B,+|e^{-lH_c}|B,+\rangle_{\mathrm{RR}} = 0 , \qquad (1.34)$$

four of them drop out, while the remaining six can be expressed in terms of open string traces thanks to the following relations holding under world-sheet duality, *i.e.* under the transformation $t = \frac{1}{2l}$ for going to the open string channel [44]

$$\int \langle B,\eta|e^{-lH_c}|B,\eta\rangle_{\mathrm{NSNS}} = \mathcal{N} \int \frac{\mathrm{d}t}{2t}\mathrm{Tr}_{\mathrm{NS}}\left[e^{-tH_o}\right]$$

$$\int \langle B,\eta|e^{-lH_c}|B,-\eta\rangle_{\mathrm{NSNS}} = \mathcal{N} \int \frac{\mathrm{d}t}{2t}\mathrm{Tr}_{\mathrm{R}}\left[e^{-tH_o}\right]$$

$$\int \langle B,\eta|e^{-lH_c}|B,\eta\rangle_{\mathrm{RR}} = -\mathcal{N} \int \frac{\mathrm{d}t}{2t}\mathrm{Tr}_{\mathrm{NS}}\left[(-1)^F e^{-tH_o}\right]$$

$$\int \langle B,\eta|e^{-lH_c}|B,-\eta\rangle_{\mathrm{RR}} = -\mathcal{N} \int \frac{\mathrm{d}t}{2t}\mathrm{Tr}_{\mathrm{R}}\left[(-1)^F e^{-tH_o}\right] . \qquad (1.35)$$

H_c and H_o are the closed and open string light-cone Hamiltonians, and $\mathcal{N}$ is a suitable normalization constant. Setting $q = e^{-\pi t}$, the open string traces over the oscillator part of the Hamiltonian are[11]

$$\mathrm{Tr}_{NS}\left[q^N\right] = \left(\frac{f_3(q)}{f_1(q)}\right)^8$$

$$\mathrm{Tr}_{NS}\left[(-)^F q^N\right] = -\left(\frac{f_4(q)}{f_1(q)}\right)^8$$

---

[11]In standard notation

$$f_1(q) = q^{1/12}\Pi(1 - q^{2n}) \qquad f_2(q) = \sqrt{2}q^{1/12}\Pi(1 + q^{2n})$$
$$f_3(q) = q^{-1/24}\Pi(1 + q^{2n-1}) \qquad f_4(q) = q^{-1/24}\Pi(1 - q^{2n-1}) .$$

$$\text{Tr}_{\text{R}}\left[q^N\right] = \left(\frac{f_2(q)}{f_1(q)}\right)^8$$

$$\text{Tr}_{\text{R}}\left[(-1)^F q^N\right] = 1 \times \text{Tr}\left(\Gamma_\chi\right) = 0 . \tag{1.36}$$

The minus sign in the second of the above equations is due to the fact that in our conventions the open string NS vacuum has fermion number -1

$$(-1)^F |0\rangle_{\text{NS}} = - |0\rangle_{\text{NS}} . \tag{1.37}$$

Using these properties one can easily see that the amplitude takes the form

$$\mathcal{A} = \mathcal{N} \int \frac{dt}{2t} \left(\text{Tr}_{\text{NS}}\left[P_+ e^{-tH_\circ}\right] - \text{Tr}_{\text{R}}\left[P_+ e^{-tH_\circ}\right]\right)$$

$$= \mathcal{N} \int \frac{dt}{2t} t^{-\frac{p+1}{2}} e^{-\frac{d^2 t}{\pi}} \left(\frac{f_3(q)^8 - f_4(q)^8 - f_2(q)^8}{f_1(q)^8}\right) . \tag{1.38}$$

where P_+ and P_- are the open-string GSO projection operators, defined in (1.12).

Finally, thanks to the Jacobi *abstruse* identity, the integrand of (1.38) vanishes, so that parallel Dp-branes in Type II theories do not attract nor repel. This is also the signature for a BPS state, as the vanishing of (1.38) occurs because there is an exact cancellation between dilaton and gravitational attraction and RR repulsion. This result is also expected from the open string point of view, as the vacuum energy of a supersymmetric theory should vanish.

Moreover, as the cylinder has genus $\chi = 0$, it follows that $\mathcal{A} \sim g_{\text{st}}^0 \times 0$, so that the force $F \sim |\mathcal{A}|^2 \sim g_{\text{st}}^0$ as well. Being the Newton constant $G_N \sim g_{\text{st}}^2$, it implies that the D-brane tension goes like $T \sim g_{\text{st}}^{-1}$, thus proving the aforementioned property 3.

We stress that this different description of the same object lays at the heart of many of the most recent developments some of which treated in this thesis, such as the AdS/CFT correspondence and, more generally the gravity/gauge duality.

There is a last remark we want to make before going to the next chapter. One of the most interesting results achieved by the dual description of branes concerns black hole physics. One can in fact explicitly count microstates of black holes, by considering the world-volume theory of the corresponding D-brane. Even if the two descriptions are valid for completely different ranges of parameters, usually BPS properties ensure a smooth transition between them. The procedure is strictly applicable for extremal black holes, of which the p-branes are examples. However, there is evidence that the same should apply also for non-extremal black holes, where supersymmetry is destroyed. One instance are the black p-branes mentioned above, that is, p-branes at some non-zero temperature. Their entropy can be computed at least in two independent ways: Either calculating the area of their horizon, or their free energy, that is the on-shell Euclidean action. The calculations can be compared

with the microscopic ones, as in that limit they reduce to a free particle perfect gas approximation. In Section 3.4 we will apply these techniques in the case of p-brane solutions of effective Type 0 theories.

Chapter 2

Overview of Type 0 string theories

In this chapter we will discuss some general features of Type 0 string theories, with emphasis to their D-brane content and their relation to Type II theories. Despite being purely bosonic string theories they are intimately connected with Type II theories, as they are obtained from the same supersymmetric RNS action (1.1). The Type 0 spectrum is simply gotten performing a different GSO projection. As already noticed, this will include tachyonic excitations. Modular invariance will fix which sectors should be included, eliminating for instance all fermion fields and therefore ruining completely space-time supersymmetry.

As Type 0 string theories contain tachyons, they have been regarded for a long time as unphysical [25, 26]. In general, a tachyonic excitation in the *perturbative* spectrum should not be regarded as an intrinsic illness of a theory, it simply indicates that the vacuum state around which one is perturbing is unstable. In Type 0 string theories this means that Minkowski background is not stable. However, if there were a mechanism such that the tachyon field acquired a nonzero vacuum expectation value, in a background not necessarily Minkowskian, then one could still consider the theory to be physical. Unfortunately this is an extremely hard issue to address in its full generality, as quantizing the string in non flat background is a formidable task, especially in presence of RR fields. However the problem can at least be faced qualitatively at the level of the effective low energy theory. This approach was suggested by Klebanov and Tseytlin in [28], following some suggestions provided by Polyakov in [22]. One has then still to deal with the non trivial problem of computing the effective gravity action for Type 0 low energy fields, including therefore the tachyon, and the ambiguities related to off-shell extrapolations can not be resolved by supersymmetry, or other powerful invariances such as $SL(2, \mathbb{Z})$ symmetry of Type IIB theory. It is then unavoidable, barring any additional progress, to get only qualitative results out of such kind of approach. At the same level of rigor one can go a step further, and embrace the possibility of dealing with string theories in space-time dimensions different from ten. The full non-critical string theory is a

difficult subject in conformal field theory, and not many results are available. For a number of motivation that we will explain in the last section, it is not unnatural trying to extend the effective gravity approach to Type 0 non-critical string theory.

2.1   Closed string spectra

Since one can perform two distinct GSO projections in each given left and right sector, there are altogether sixteen different closed string sectors originating in the RNS formulation. Actually the level-matching condition eliminates six of them, as the NS− sector can not ever be combined with NS+, R+, or R− ones. Some of the remaining ten sectors were combined to get the supersymmetric and tachyon-free Type II theories. However out of the 2^{10} combinations only a few are acceptable. Additional conditions come in fact by requiring that scattering amplitudes involving the vertex operators associated to different sectors have to be well defined, and also, that the OPE must close, that is, poles in scattering amplitudes should correspond to physical states. We will not discuss these requirements here, and refer to [45] for details. In addition, one-loop amplitudes must be modular invariant: We will use this last requirement to see how Type 0 theories arise as (one-loop) modular invariant theories. Modular invariance is a fundamental requirement for a physical string theory, corresponding to the requirement of invariance under "large" reparameterizations of the world-sheet, and it is the main responsible for the absence of UV divergences in string theory.

We will get the closed string spectra of Type 0 theories starting from Type IIA and Type IIB and performing an orbifold procedure, that preserves modular invariance by construction. Let us start from Type IIB theory. The partition function on the torus is

$$Z_{\text{IIB}} = iV_{10} \int \frac{\mathrm{d}^2\tau}{8\pi^2\tau_2^2} Z_X^8 Z_\psi^+ Z_\psi^{+*} , \qquad (2.1)$$

where $Z_X(\tau) = (2\pi^2\tau_2)^{\frac{1}{2}} |\eta(q)|^{-2}$ is the partition function for a single boson[1], whereas the fermionic partition functions read

$$Z_\psi^\pm = \frac{1}{2} \left[Z_0^{0\,4} - Z_1^{0\,4} - Z_0^{1\,4} \mp Z_1^{1\,4} \right] . \qquad (2.2)$$

We have adopted the notation of [45] and defined the fermionic traces for a pair of

---

[1]Here $q = \exp(2\pi i \tau)$.

Majorana-Weyl fermions[2]

$$
\begin{aligned}
Z_0^0(\tau) &= \mathrm{tr}_{NS}\left[q^N\right] \\
Z_1^0(\tau) &= \mathrm{tr}_{NS}\left[(-)^F q^N\right] \\
Z_0^1(\tau) &= \mathrm{tr}_{R}\left[q^N\right] \\
Z_1^1(\tau) &= \mathrm{tr}_{R}\left[(-)^F q^N\right] ,
\end{aligned}
\tag{2.3}
$$

where F is the (left) world-sheet fermion number. We compute now, using standard CFT techniques, the partition function of the theory modded by the space-time fermion number, that is, we consider the following orbifold: $\mathrm{IIB}/(-1)^{\mathbf{F}}$.

Let us briefly recall how the orbifold procedure works [46, 47][3]. Given a discrete symmetry group G of a CFT, one projects out those states that are not invariant under G. After this step the partition function is not modular invariant any more. In order to regain modular invariance, additional states must be included in the theory: They constitute the so-called twisted sector. Finally one has to project out G-noninvariant states from the twisted sector as well. If the symmetry considered has a space-time interpretation, the resulting orbifold is a new space-time in which strings propagate, so that the geometrical meaning is clear. For instance, if in $\mathbb{R}^{d,1}$ one identifies simultaneously $X^1 \simeq -X^1$, and $X^2 \simeq -X^2$, the resulting orbifold space is $\mathbb{R}^{d,1}/\mathbb{Z}_2 = \mathbb{R}^{d-2,1} \times \mathcal{C}$. Twisted states are here small strings localized near the apex of the cone $\mathcal{C}$, that is the fixed point. However, if the symmetry is non geometric, as in our case, one can still apply the procedure, use the machinery of conformal field theory to compute the orbifold partition function, and read the resulting spectrum from it. For our applications the orbifold partition function can be written

$$
Z_{\mathrm{orb}} = \frac{1}{\mathrm{order}(G)} \sum_{g,h \in G} g\,\square_h ,
\tag{2.4}
$$

where the symbol $g\,\square_h$ represents the torus, with the understanding that the horizontal side is the h-twisted σ cycle, whereas the vertical one is the g-twisted τ cycle.

In our case $G = \{e, (-1)^F\}$, so that there four contributions to Z_{orb}, e being the identity element. Using the transformation properties of $g\,\square_h$ under modular transformations

$$
g\,\square_h \rightarrow g^a h^b\,\square_{g^c h^d} \quad \text{under} \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}
\tag{2.5}
$$

---

[2]We use the symbol tr to distinguish it from the trace Tr, used to denote the sum over all eight transverse bosonic and fermionic components.

[3]See [48] for a nice introduction to the subject.

and the relative transformation rules for the symbols (2.3) [45], one easily obtains $(-1)^{\mathbf{F}} \underset{e}{\square}$, $e \underset{(-1)^{\mathbf{F}}}{\square}$, and $(-1)^{\mathbf{F}} \underset{(-1)^{\mathbf{F}}}{\square}$. Putting them together one gets the following fermionic partition function

$$Z_\psi = \frac{1}{2}\left[|Z_0^0|^8 + |Z_1^0|^8 + |Z_0^1|^8 + |Z_1^1|^8\right] . \qquad (2.6)$$

A direct dictionary for reading the spectrum from the partition function is obtained writing Z's in terms of characters of level-one affine $SO(8)$

$$\frac{1}{2}\left[Z_0^{0\,4} + Z_1^{0\,4}\right] = O_8 \quad \rightarrow \quad \text{NS}-$$

$$\frac{1}{2}\left[Z_0^{0\,4} - Z_1^{0\,4}\right] = V_8 \quad \rightarrow \quad \text{NS}+$$

$$\frac{1}{2}\left[Z_0^{1\,4} - Z_1^{1\,4}\right] = C_8 \quad \rightarrow \quad \text{R}-$$

$$\frac{1}{2}\left[Z_0^{1\,4} + Z_1^{1\,4}\right] = S_8 \quad \rightarrow \quad \text{R}+ . \qquad (2.7)$$

Thus (2.6) reads

$$Z_\psi = |O_8|^2 + |V_8|^2 + |S_8|^2 + |C_8|^2 \qquad (2.8)$$

and the resulting closed string spectrum can be read off. What we did is to eliminate all fermions from the closed string spectrum of Type IIB string theory. We lost modular invariance, and in order to regain it we were forced to introduce two twisted sectors, namely the (NS$-$,NS$-$) and (R$-$,R$-$) sectors. Both of them are bosonic. We paid the price of getting a tachyon and no massless states from the first one, and got a doubled RR sector, which will result in a doubled D-brane spectrum as well. These are the sectors of Type 0B string theory. With a sign change in front of the last addendum of (2.6), the same result applies *verbatim* for Type IIA theory. So we have shown that

$$\text{IIA}/(-1)^{\mathbf{F}} = 0\text{A} \qquad (2.9)$$

$$\text{IIB}/(-1)^{\mathbf{F}} = 0\text{B} . \qquad (2.10)$$

In Table 2.1 we summarize the sectors of Type 0 theories, displaying the lowest modes. The partition functions of Type 0 theories do not vanish, in contrast to Type II theories. In the latter supersymmetry implies an exact cancellation of fermionic and bosonic contributions at each level. Conversely in Type 0 theories the lack of supersymmetry spoils this property producing a non-zero cosmological constant at one-loop.

In the remainder of this section we would like to comment on some properties concerning relationships among all possible RNS consistent string theories with $\mathcal{N} =$

	$(NS+, NS+)$	$(NS-, NS-)$	$(R+, R\mp)$	$(R-, R\pm)$
0A	$\Phi, B_{\mu\nu}, g_{\mu\nu}$	T	$A_\mu, A_{\mu\nu\rho}$	$A'_\mu, A'_{\mu\nu\rho}$
0B	$\Phi, B_{\mu\nu}, g_{\mu\nu}$	T	$A, A_{\mu\nu}, A^+_{\mu\nu\rho\sigma}$	$A', A'_{\mu\nu}, A^-_{\mu\nu\rho\sigma}$

Table 2.1: Type 0 sectors and low laying fields.

$(1, 1)$ world-sheet supersymmetry. The list of theories we mentioned so far is almost complete. We have just to recall that Type II theories have a "mirror" theory for each, IIA and IIB. They are gotten by a space-time reflection[4] along a single coordinate, say $X^9 \to -X^9$, and are indistinguishable from their parent theories as they just interchange all R sectors: R+ $\leftrightarrow$ R−. These theories are called IIA′ and IIB′. So there are altogether six modular invariant partitions functions leading, after this equivalence, to four modular invariant string theories.

Let us consider now Type IIB string theory and mod it by the orbifold group $G = \{e, (-1)^{\mathbf{F}_L}\}$, $\mathbf{F}_L$ being the left space-time fermion number. We can apply the orbifold construction described above to get the wanted result, but as a shortcut we observe that we are projecting out the (R+,NS+) and (R+,R+) sectors. It is then clear that in order to achieve modular invariance we must add as twisted sectors the (R−,NS+) and (R+,R−) ones, so getting Type IIA theory. The converse is also true, namely

$$\mathrm{IIA}/(-1)^{\mathbf{F}_L} = \mathrm{IIB} \tag{2.11}$$

$$\mathrm{IIB}/(-1)^{\mathbf{F}_L} = \mathrm{IIA} . \tag{2.12}$$

If we use instead the right space-time fermion number $(-1)^{\mathbf{F}_R}$ to mod out, we end up with Type II′ theories. Finally, in Type 0 theories, one can mod with respect to the left (or right) world-sheet fermion number $(-1)^F$. This symmetry projects out sectors containing NS− and R− states. The orbifold procedure then will reintroduce automatically the NSR fermionic sectors in order to conserve modular invariance, getting back Type II theories

$$0A/(-1)^F = \mathrm{IIA} \tag{2.13}$$

$$0B/(-1)^F = \mathrm{IIB} . \tag{2.14}$$

Somehow surprisingly we see that in an orbifold construction supersymmetry does not need to be lowered. Starting from a purely bosonic string theory we got the supersymmetric Type II theories. The symmetries used in the previous examples

---

[4]Notice that this is not an orbifold, as we are not projecting anything. It is an operation that maps from one theory to another, as T-duality does.

form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, out of which we used some $\mathbb{Z}_2$ subgroups. Using the full group one can go from one theory to another: In Figure 2.1 we illustrate this pictorially.
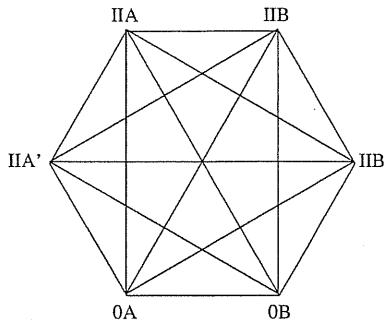

Figure 2.1: Orbifold relationships among RNS theories.

Modding by a $\mathbb{Z}_2$ subgroup one can either change chiralities ($A \leftrightarrow B$) or supersymmetry in target space ($\mathcal{N} = 0 \leftrightarrow \mathcal{N} = 2$). In order to change both of them simultaneously one should mod by the full $\mathbb{Z}_2 \times \mathbb{Z}_2$. There is a slight subtlety arising in this last operation. As there are five disjoint orbits of the modular group, an ambiguity arises in how to define $g_1 \square \atop g_2$, where g_1 and g_2 are two generators of the group. There are two consistent choices differing by a sign (discrete torsion) — accordingly one can go from, say, 0B theory, to Type IIA or Type IIA' respectively.

2.2   Type 0 D-branes

Having at hand the massless RR field content (Table 2.1), and familiarity with notation for the different Type 0 closed sectors, we can now look at the D-branes[5] of these theories and their world-volume excitations. Due to the doubling of RR fields, there are twice as many D-branes than in the corresponding Type II theories. For a given p we have thus four types of elementary Dp-branes (counting the anti-branes) charged with respect to the pair of RR fields as $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, and $(-1, -1)$. Branes charged only with respect to one type of RR field are possible but carry charge $(2n, 0)$ in these units and thus can be built out of the four constituents above.

The boundary state formalism introduced in Chapter 1 turns out to be very convenient for classifying different kinds of D-branes. Let us define the following

---

[5]Papers [49, 50] discuss some aspects of Type 0 D-branes, as well.

boundary states, for a given p

$$
\begin{aligned}
|D,+\rangle &= |B,+\rangle_{\mathrm{NSNS}} + |B,+\rangle_{\mathrm{RR}} & (+1,+1) \\
|\bar{D},+\rangle &= |B,+\rangle_{\mathrm{NSNS}} - |B,+\rangle_{\mathrm{RR}} & (-1,-1) \\
|D,-\rangle &= |B,-\rangle_{\mathrm{NSNS}} + |B,-\rangle_{\mathrm{RR}} & (+1,-1) \\
|\bar{D},-\rangle &= |B,-\rangle_{\mathrm{NSNS}} - |B,-\rangle_{\mathrm{RR}} & (-1,+1) \ .
\end{aligned}
\tag{2.15}
$$

They denote branes and anti-branes corresponding to RR charges listed besides. The signs in kets refer to the phase in the defining equation for the fermionic part of the boundary state and are related to a given spin structure, as discussed in Section 1.2.

A quick way to understand the spectrum of massless excitations living on the world-volume of a stack of branes is to consider the closed string exchange between two such branes, perform a modular transformation, and read off the spectrum of the open string sector. Consider for instance two parallel $|D,+\rangle$ branes. Using Eqs. (1.34) – (1.35) the exchange amplitude can be rewritten in terms of open string traces as follows[6]

$$
\begin{aligned}
\int \mathrm{d}l \, \langle D,+|e^{-lH_c}|D,+\rangle &= \int \mathrm{d}l \, \langle B,+|e^{-lH_c}|B,+\rangle_{\mathrm{NSNS}} + \langle B,+|e^{-lH_c}|B,+\rangle_{\mathrm{RR}} \\
&= \mathcal{N} \int \frac{\mathrm{d}t}{2t} \mathrm{Tr}_{\mathrm{NS}} \left[P_+ e^{-tH_o} \right] \ .
\end{aligned}
\tag{2.16}
$$

From (2.16) one sees that the corresponding open string states are usual bosons of the NS+ sector of Type II theory, the lowest state being the vector A_μ. In particular there is no open string tachyon and no fermions at all. This is the same field content of pure $d = 10$ Yang-Mills theory dimensionally reduced to $p + 1$ dimensions. This observation is crucial for application to AdS/CFT we will consider later.

Using formulae (1.35) one can compute the spectrum of $\langle D,+|D,-\rangle$, $\langle D,+|\bar{D},+\rangle$, $\langle D,+|\bar{D},-\rangle$, etc. There are however only two further independent cases. Consider first the case of branes with charges $(+1,+1)$ and $(+1,-1)$, that is a $\langle D,+|D,-\rangle$ system. Here the closed string exchange gives

$$
\int \mathrm{d}l \, \langle D,+|e^{-lH_c}|D,-\rangle = -\mathcal{N} \int \frac{\mathrm{d}t}{2t} \mathrm{Tr}_{\mathrm{R}} \left[P_+ e^{-tH_o} \right] \ .
\tag{2.17}
$$

From (2.17) we see that this brane configuration has only (left-handed)[7] fermions on the world-volume. This configuration has been studied in [51] in the context of the AdS/CFT correspondence.

---

[6] In (2.16) we have omitted redundant subscripts on bra's.

[7] It turns out that the $\langle D,+|\bar{D},-\rangle$ system contains right-handed fermions instead.

Finally the $\langle D, \pm | \bar{D}, \pm \rangle$ brane/anti-brane configurations correspond to

$$\int dl \, \langle D, \pm | e^{-lH_c} | \bar{D}, \pm \rangle \;=\; \mathcal{N} \int \frac{dt}{2t} \mathrm{Tr_{NS}} \left[P_- e^{-tH_o} \right] \,. \tag{2.18}$$

We see therefore that there is an open string tachyon in this theory, just as in Type II theories, signaling an instability of the system.

We conclude complementing the discussion of the previous section about relationships of different RNS theories from the D-brane point of view. For studying D-branes mapping under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold group discussed in the previous section, it is useful to change basis of boundary states. We can use most conveniently a set of states directly related to their (bosonic) closed string states content. In Type IIB, say, there are two such states

$$|\mathrm{NS+,NS+}\rangle \;=\; |B,+\rangle_{\mathrm{NSNS}} + |B,-\rangle_{\mathrm{NSNS}} \tag{2.19}$$

$$|\mathrm{R+,R+}\rangle \;=\; |B,+\rangle_{\mathrm{RR}} + |B,-\rangle_{\mathrm{RR}} \,, \tag{2.20}$$

in Type 0B there are two more states, corresponding to the twisted sectors

$$|\mathrm{NS-,NS-}\rangle \;=\; |B,+\rangle_{\mathrm{NSNS}} - |B,-\rangle_{\mathrm{NSNS}} \tag{2.21}$$

$$|\mathrm{R-,R-}\rangle \;=\; |B,+\rangle_{\mathrm{RR}} - |B,-\rangle_{\mathrm{RR}} \,. \tag{2.22}$$

Expressing the physical D-brane states in terms of the above basis leads to expressions like

$$|Dp\rangle \;=\; |\mathrm{NS+,NS+}\rangle + |\mathrm{R+,R+}\rangle \tag{2.23}$$

for Type IIB D-branes, and

$$|D,-\rangle \;=\; \frac{1}{2} \Big(|\mathrm{NS+,NS+}\rangle - |\mathrm{NS-,NS-}\rangle + |\mathrm{R+,R+}\rangle - |\mathrm{R-,R-}\rangle \Big) \tag{2.24}$$

for Type 0B D-branes. We wrote here explicitly the $(+1, -1)$ RR charged D-brane. The other three types have similar expressions. The advantage of this form is that one can directly read signs of RR charges, and the correct signs for NSNS states as well.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ group we encountered before will act permuting the four states (2.19) – (2.22) among them. Thus acting on D-branes as a subgroup of S_4. For instance

$$(-1)^{\mathbf{F}_L} |D,+\rangle \;=\; |\bar{D},+\rangle \,,$$
$$(-1)^{F} |D,-\rangle \;=\; |D,+\rangle \,, \tag{2.25}$$

and so on. As a result, one can easily see how D-branes map under the given orbifold projection. As usual, only G-invariant states survive the projection. For instance a

$|D,+\rangle + |D,-\rangle$ configuration in Type 0B is invariant under $(-1)^F$, and it maps to a $|Dp\rangle$ brane of Type IIB, modding by this symmetry.

Looking at the structure of the symmetry group we are considering, it is tempting to ask whether Type 0 string theories posses actually a wider discrete symmetry group. After all, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is already present in Type II theories, where there are half as many RR fields. A natural guess is to define an operator that exchanges the two RR sectors. With this additional operation the group of discrete transformations extends to a $D_4 \simeq O(2,\mathbb{Z})$. This is a potentially interesting group as it is non-Abelian, in contrast to its real counterpart. It has as proper subgroups a $\mathbb{Z}_4$, five $\mathbb{Z}_2$, and two $\mathbb{Z}_2 \times \mathbb{Z}_2$, one of which is the original one.

Unfortunately, the orbifold procedure, applied for some of these subgroups, produces *inconsistent* partition functions, namely with fractional states, thus ruling out this group as a symmetry of the Type 0 string. As the operation we defined is actually a chirality change, it can be implemented by a space-time reflection along an odd number of coordinates, for instance[8] $X^9 \to -X^9$. In Type 0A theory it can be implemented also by the world-sheet parity Ω. In both cases the resulting orbifolds (or orientifolds) are more conventional geometric ones, and have been studied in the literature [52].

2.3   Low energy effective actions

In this section we want to show how the low energy effective actions arise for Type 0 theories. As there is no supersymmetry to guide the derivation, the only available tool is an explicit computation of tree-level string scattering amplitudes. Despite the theory has no space-time supersymmetry, the presence of world-sheet fermions and the orbifold relation to Type II theories ensure a number of simplifying features. It turns out that the structure of Type 0 theories is much more constrained then in conventional bosonic string. Nevertheless, we should point out that whatever accurate the calculations can be, there will always remain a number of ambiguities attached to tachyon couplings. In extrapolating the effective action, that is an off-shell object, from scattering amplitudes, that are on-shell by definitions, it is hard to distinguish between, say, terms[9] of the type $T^2 F^2$ and $F^2 T \nabla^2 T$. The hope is that, as far as application to AdS/CFT are concerned, one can be able to get some results that are robust and do not depend on details of the tachyon couplings. We will achieve partly this goal, and will discuss our results in the next chapter.

In the following we will start by recalling some relevant properties of perturbative world-sheet aspects of Type 0 string theories [28].

---

[8]This has to be accompanied with the reflection $\psi^9 \to -\psi^9$ in order to respect world-sheet supersymmetry.

[9]T denotes the tachyon field, and F a RR field-strength. See below.

1. Looking at Type 0 theories as orbifold of Type II theories (cf. Section 2.1) we can apply the *inheritance principle*, asserting that tree-level amplitudes of untwisted states are the same as in the parent theory. Therefore, in Type 0B say, all tree-level amplitudes of (NS+,NS+) vertex operators with (R+,R+), or (R−,R−), are identical to those of Type IIB — world-sheet supersymmetry is enough to enforce the same form of tree-level effective action as in Type II theories as far as the untwisted sectors are concerned, while novelties can only arise from correlators involving tachyon vertex operators or both kinds of RR fields.

2. The vertex operators for the massless (NS+,NS+) states and for the tachyon in the (NS−,NS−) sector occur in two different forms[10]

$$
\begin{aligned}
\mathcal{V}^{0,0}_{(\mathrm{NS+,NS+})} &= -(i\partial X^\mu + k\cdot\psi\,\psi^\mu)(i\bar\partial X^\nu + k\cdot\tilde\psi\,\tilde\psi^\nu)\mathrm{e}^{ik\cdot X} \\
\mathcal{V}^{-1,-1}_{(\mathrm{NS+,NS+})} &= \mathrm{e}^{-\phi-\tilde\phi}\psi^\mu\,\tilde\psi^\nu\mathrm{e}^{ik\cdot X} \\
\mathcal{V}^{0,0}_{(\mathrm{NS-,NS-})} &= k\cdot\psi\,k\cdot\tilde\psi\,\mathrm{e}^{ik\cdot X} \\
\mathcal{V}^{-1,-1}_{(\mathrm{NS-,NS-})} &= \mathrm{e}^{-\phi-\tilde\phi}\mathrm{e}^{ik\cdot X}\ .
\end{aligned}
\tag{2.26}
$$

ϕ and $\tilde\phi$ are the bosonized ghosts and the superscripts denote their charges. On the sphere we need to take any two of these vertex operators in the $(-1,-1)$ picture and all the rest in the $(0,0)$ picture. Therefore, considering correlation functions involving an *odd* number of tachyons and an arbitrary number of (NS+,NS+) vertex operators, one ends up with an *odd* number of ψ. These vanish as they can not be proportional to any invariant tensor. On the other hand, correlation functions of an *even* number of tachyons lead to an *even* number of ψ, that is in general proportional to a product of η tensors. Thus, there are no T^3 terms in the tachyon potential and in the tachyonic couplings to other (NS+,NS+) fields. We should anticipate here that the mechanism for tachyon stabilization will not rely on T^4 correction to the potential, but instead it will be due to the tachyon coupling to RR fields. As such, it seems a more robust result indeed. Notice also that in this vacuum is $\langle T\rangle = 0$, whereas we expect a tachyon condensate in the physical vacuum.

3. We can always use the vertex operators for the RR fields in the $(-1/2, -1/2)$ picture — written as a bispinor, in terms of (Majorana) spin-fields Θ_α, it reads

$$
\mathcal{V}^{-1/2,-1/2}_{(\mathrm{R\pm,R\pm})} = \mathrm{e}^{-\phi/2-\tilde\phi/2}\,\Theta_\alpha\,\tilde\Theta_\beta\,\mathrm{e}^{ik\cdot X}\ .
\tag{2.27}
$$

Consider a correlation function involving tachyon vertex operators, any number of fields from the (NS+,NS+) and two RR fields. The fermionic correlator

---

[10]In the notation of [45].

for, say, olomorphic fields, gives

$$\langle \Theta_\alpha \Theta_\beta \psi^{\mu_1} \cdots \psi^{\mu_n} \rangle \sim (C\Gamma^{\mu_1 \cdots \mu_n})_{\alpha\beta} , \qquad (2.28)$$

which for n odd commutes with the chirality matrix Γ_χ, whereas for n even anticommutes. Thus for RR fields of the *same* sector, having equal chirality, n must be odd. Since to saturate the ghost charge we need to take only one (NS$\pm$,NS$\pm$) vertex operators in the $(-1,-1)$ picture, it follows that there must be an *even* number of tachyons. This implies that RR kinetic terms have even tachyonic couplings

$$f_{\text{even}}(T)F^2 , \qquad (2.29)$$

and it is consistent with the fact that if the amplitudes with one tachyon did not vanish, there would be a tachyon pole in some tree-level correlation function of the corresponding Type II theory.

4. Conversely, if one considers two RR fields from *different* sectors, an arbitrary number of (NS+,NS+) fields, and some tachyon vertex operators, the non zero correlators must now involve an even number of ψ. Thus, after saturating the ghost charges, one can have only an *odd* number of tachyons. This implies that RR couplings between differen sectors involve odd tachyonic functions

$$f_{\text{odd}}(T)F * F' . \qquad (2.30)$$

As far as Chern–Simons terms are concerned, they do not have tachyon couplings by definition. Namely, they are excluded by gauge invariance. In this case the above property 3 indicates that they involve RR field strengths from the same sector. Thus from the first property it follows that they need to be modified by shifting them with a $B_{\mu\nu}$ dependent transformation just like in Type II supergravity.

The properties listed above are sufficient to perform some explicit tree-level computation which allow to write down, to first non trivial order in α', an effective action up to terms quadratic in the gauge fields. Some of these computations are performed in [28]. Here we will do a sample calculation, whose result are the leading RR-tachyon couplings in the effective Type 0B action[11]. This allows us illustrating specifically the behavior of the five-form, to which these properties do not apply literally. We have grouped the resulting actions for Type 0A and 0B gravities in ten dimensions together with those we will propose for the non-critical Type 0 theories. The relevant formulae are (2.41), (2.43), and (2.44) of the next section.

---

[11]The computation in Sec. 3.1 of [28] uses a slightly different vertex operator for the 5-form.

Three-point RR-RR-Tachyon amplitude on the sphere

We will denote $F = \mathrm{d}A$ RR field-strengths from the (R+,R+) sector and $F' = \mathrm{d}A'$ those from the (R−,R−) one. The full RR vertex operators, contracted with their polarization read

$$\mathcal{V}_F^{-1/2,-1/2} \;=\; \mathrm{e}^{-\phi/2-\tilde{\phi}/2}\Theta\,\mathcal{C}\,\mathcal{F}\frac{1}{2}(1+\Gamma_\chi)\tilde{\Theta}\,\mathrm{e}^{ik\cdot X} \tag{2.31}$$

$$\mathcal{V}_{F'}^{-1/2,-1/2} \;=\; \mathrm{e}^{-\phi/2-\tilde{\phi}/2}\Theta\,\mathcal{C}\,\mathcal{F}'\frac{1}{2}(1-\Gamma_\chi)\tilde{\Theta}\,\mathrm{e}^{ik\cdot X} \;. \tag{2.32}$$

This is all right for the axions and RR two-form fields. However, as both the self-dual and anti-self-dual RR 4-forms are present, they naturally combine in an unconstrained potential, with 5-form field-strength F_5. This will have a *different* vertex operator, namely

$$\mathcal{V}_{F_5}^{-1/2,-1/2} \;=\; \mathrm{e}^{-\phi/2-\tilde{\phi}/2}\Theta\,\mathcal{C}\,\mathcal{F}_5\tilde{\Theta}\,\mathrm{e}^{ik\cdot X} \;. \tag{2.33}$$

Since there are no fields from the (NS+,NS+) sector, the tachyon vertex operator must be taken in the $(-1,-1)$ picture, involving no ψ's. Thus, the relevant spin-fields correlator is

$$\langle\Theta_\alpha(z_1)\tilde{\Theta}_\beta(z_2)\rangle \;=\; z_{12}^{-5/4}\mathcal{C}_{\alpha\beta} \;. \tag{2.34}$$

Using (2.34), since $\{\mathcal{C},\Gamma_\chi\} = 0$ we see that, as far as the one-form and three-form are concerned

$$\frac{1}{2}(1\pm\Gamma_\chi)\,\mathcal{C}\,\frac{1}{2}(1\pm\Gamma_\chi) = 0 \quad\Rightarrow\quad \langle FFT\rangle = 0 = \langle F'F'T\rangle \;, \tag{2.35}$$

whereas

$$\frac{1}{2}(1\pm\Gamma_\chi)\,\mathcal{C}\,\frac{1}{2}(1\mp\Gamma_\chi) = \frac{1}{2}(1\pm\Gamma_\chi)\,\mathcal{C} \quad\Rightarrow\quad \langle FF'T\rangle \neq 0 \;. \tag{2.36}$$

The case of the unconstrained 5-form field strength is special. In fact $\Theta\,\mathcal{C}\,\tilde{\Theta}$ is always non zero, regardless the number of ψ and then the number of tachyon vertex operators.

Combining the above results, the piece of the effective action one gets reads, in an obvious notation

$$S_{\text{RR-RR-T}}^{\text{0B}} \;=\; \int T\left[F_1 * F_1' + F_3 * F_3' + |F_5|^2\right] \;. \tag{2.37}$$

For completeness, let us write down also the contribution following from the RR-RR-graviton amplitude. Using the inheritance principle, those involving the one and three-forms are the same as in Type IIB. The same kind of computation gives a contribution for the 5-form that here is non zero and equal to the others, namely

$$S_{\text{RR-RR-}g}^{\text{0B}} \;=\; \int \frac{1}{2}\left[\,|F_1|^2 + |F_1'|^2 + |F_3|^2 + |F_3'|^2 + |F_5|^2\,\right] \;. \tag{2.38}$$

Therefore, the 5-form will not have definite parity tachyon coupling.

2.4   Non-critical Type 0 effective theories

So far we have discussed a critical string theory, and we have outlined how its low energy effective action is derived. However the original proposal of Polyakov [22] in the context of string/gauge duality, was to consider a theory in $d = 5$ space-time dimensions. The difficulties for treating rigorously such a kind of string theory are well-known, and have to do with nonlinearities introduced by the Liouville mode of the world-sheet metric, that can not be decoupled for $d \neq 10$. A possible way out of this impasse is to resort to an effective field theory description in terms of the lowest modes. In recent years we learned how powerful this approach can be, thanks to the AdS/CFT correspondence and the Holographic principle in general.

Most of the works on Type 0 applications to AdS/CFT have been devoted to the critical case. Following [22], it was however our opinion that the non-critical scenario should be taken seriously and would give rise to additional interesting models that are not accessible at $d = 10$. For this reason, in [53] we have formulated a proposal for an effective description of these conjectured non-critical Type 0 strings. In the following, after motiving why one should hope to be dealing with sensible theories, we will report our proposals for the field content and pertinent effective actions for Type 0 models in $d < 10$. The $d > 10$ cases are ruled by the no-ghost theorem. We shall continue to refer to these lower dimensional theories as Type 0A or 0B in even dimensions, depending on the choice of chirality in the GSO projection. In odd space-time dimensions there is only one such theory due to the lack of chirality. We shall refer to it as Type 0AB. Although the full conformal field theory corresponding to the above non-critical string theory has not yet been constructed, there are some indications that such a construction is indeed possible.

Consider the issue of modular invariance. In the ordinary Type II string theory modular invariance requires that the holomorphic contribution of the fermions to the partition function

$$Z_\psi^\pm = \frac{1}{2}\left[Z_0^{0\,4} - Z_1^{0\,4} - Z_0^{1\,4} \mp Z_1^{1\,4} \right]$$

and the analogous antiholomorphic contribution, be separately modular invariant up to an overall opposite phase. In Type 0 theories, the joined contributions of the holomorphic and the antiholomorphic sectors give rise to

$$Z_\psi^\pm = |Z_0^0(\tau)|^{d-2} + |Z_1^0(\tau)|^{d-2} + |Z_0^1(\tau)|^{d-2} \pm |Z_1^1(\tau)|^{d-2} , \qquad (2.39)$$

which is modular invariant for any d. Of course, for $d \neq 10$ the explicit expressions for the fermionic traces will be modified because of the changes in the spectrum, but we view the above as an indication that the continuation "off-criticality" is more likely to work for the Type 0 string than for the usual Type II.

Another objection that needs to be addressed is the "$d = 2$ barrier". Let us briefly recall the physics behind this problem as presented *e.g.* in [45]. Because we do not yet know how to describe RR fields at the level of the sigma-model we are forced to discuss this argument in the case of the bosonic theory. In this context it is well know that there exists an exact CFT solution in any d in which the only non-zero background fields are a flat metric $g_{\mu\nu}$, a linearly rising dilaton Φ and an exponential tachyon T

$$g_{\mu\nu} = \eta_{\mu\nu}$$
$$\Phi = \sqrt{\frac{26-d}{6}} X^1$$
$$T = \exp\left(\sqrt{\frac{26-d}{6}} - \sqrt{\frac{2-d}{6}}\right) X^1 .$$

For $d \leq 2$ the background tachyon is exponentially rising, preventing the string from entering the region of strong coupling. Moreover, fluctuations around the tachyon background are stable and thus the theory is well defined. On the contrary, for $d > 2$, the background tachyon oscillates. In principle this could still act as a cutoff for the string coupling but the fluctuations have some negative frequency square modes and the theory becomes unstable. As stressed in [22] we should not think of the "$d = 2$ barrier" as a no-go theorem but rather as an indication that solutions for $d > 2$ will necessarily involve a curved space-time metric. This is the type of situation that is of interest in the connection with gauge theory so, in a sense, it is to be expected that flat space-time be ruled out. Unfortunately, we do not yet have an example of an exact CFT of this type and we are forced to work order by order in α' at the level of the effective action. But it should be clear that there are no *a priori* reasons for why there should not exist an exact solution. In fact in [27] Polyakov has recently claimed that it should exist!

Another encouraging sign comes from the analysis of the RR sector performed below. By making some plausible assumptions about the massless degrees of freedom it is possible to construct a rather compelling picture of the RR sectors in various dimensions and their couplings, including Chern–Simons terms. For instance, the necessity of doubling the RR spectrum in $d = 4$ or $d = 8$ Minkowski space-time is seen as coming from the fact that there are no real self-dual forms in these dimensions.

Finally, let us note that considering $d < 10$ from the sigma-model point of view is a very natural thing to do if the string theory has a perturbative tachyon. The only way for a theory with a perturbative tachyon to make sense is if there exists a mechanism through which the tachyon field condenses by acquiring a vacuum expectation value. The tachyon potential at that point will then give rise to a tree-level contribution to the cosmological constant by shifting the central charge. Since

the effective central charge is going to be different from zero anyway[12], one is led to consider the theory with the most general value for the effective central charge

$$c_{\text{eff.}} = 10 - d - \frac{1}{2}V\left(\langle T\rangle\right) .$$

(2.40)

From the target space point of view this acts as a contribution to the cosmological constant and thus shows that, for $d > 2$, one should look at curved space-times.

None of the above points constitute a proof that conformally invariant solutions to the Type 0A/B string exist for arbitrary d but we view them as strong indications that such construction is possible. Having taken this as our basic assumption, we will first construct the field content, and then write down the most general effective actions to one loop in α', in any dimensions – including Chern-Simons couplings – consistent with symmetry properties therein. Lacking a formulation from "prime principles", the identification of the RR sectors and their couplings requires a certain amount of guesswork. The picture that emerges, however, is quite simple and satisfying. We shall see, for instance, that it gives support to the idea that in even dimensions the RR sectors must be doubled compared to the Type II string.

The NSNS sector

The NSNS sector is common to all of these theories and can in principle be obtained from a sigma-model approach. It involves the massless fields of the (NS+, NS+) sector (a dilaton Φ, a graviton $g_{\mu\nu}$ and an antisymmetric tensor $B_{\mu\nu}$) and a tachyon T from the (NS−, NS−) sector. The tachyon potential $V(T)$ is an even function from the property 2 of the previous section, that can be applied for any space-time dimension. The relevant action is thus, in the string frame

$$S_{\text{NSNS}} = \int \mathrm{d}^d x \sqrt{-g}\mathrm{e}^{-2\Phi}\left(R - \frac{1}{2}|\mathrm{d}B|^2 + 4|\mathrm{d}\Phi|^2 - \frac{1}{2}|\mathrm{d}T|^2 - V(T)\right) ,$$

(2.41)

where it is natural to absorb the central charge deficit $10 - d$ into the definition of the tachyon potential, i.e.

$$V(T) = -10 + d - \frac{d-2}{8}T^2 + \cdots .$$

(2.42)

It should be kept in mind that (2.41) is by no means unique. It suffers from the usual ambiguities that come from extrapolating on-shell data. In particular, there could be arbitrary (even) functions of T multiplying the various kinetic terms in the

---

[12]It seems unnatural and there is no symmetry argument for which the tachyon potential should vanish at that point.

Lagrangian. Up to this order in α' this is essentially all that can happen[13]. As shown in the Appendix A of [28] however, precisely because of their ambiguous nature it is possible to redefine away some of them, such as the term RT^2. At the same time, terms of the type $T^2|\mathrm{d}B|^2$ and their counterpart for the RR kinetic terms are needed and should be kept. In the following we will never need the field $B_{\mu\nu}$ and we will set the coefficients of R, $|\mathrm{d}\Phi|^2$ and $|\mathrm{d}T|^2$ as in (2.41), the main conclusions being independent of the presence of such terms.

RR kinetic and Chern-Simons terms

One guiding principle [54] in the identification of the RR sector is the idea that for any d there will still be massless excitations in the R sector of the open string and thus their on-shell degrees of freedom will fall into representations of the little group $SO(d-2)$. The resulting situation is best summarized in Table 2.2.

In ten dimensions one obtains massless RR fields in the Type 0A theory by considering tensor products of spinors of different chiralities $(+-)$ and $(-+)$, and in the 0B theory of same chiralities $(++)$ and $(--)$. This can be readily generalized for any non-critical even dimension, whereas it is not quite clear what the right generalization to odd dimensions is. Note, however, that an odd dimensional bispinor can be decomposed in terms of the lower even dimensional bispinors as $(++) \oplus (+-) \oplus (-+) \oplus (--)$, and that this is the sum of the field contents of the 0A and the 0B theories of one lower dimension. It thus seems reasonable to assume that a sum of two modular invariant sectors should yield a modular invariant theory in one dimension higher without doubling the RR spectrum by hand[14].

To understand the table consider for example $d = 8$. In the Type 0B theory, the bispinor in the $(\mathrm{R}+, \mathrm{R}+)$ sector decomposes into $4 \times 4 = 6 + 10$. These are all complex representations that yield a complex vector and a complex three form with a self dual field strength. Notice that it is possible to construct a self dual form in $d = 8$ only if it is complex because $*^2 = -1$. The bispinor in the $(\mathrm{R}-, \mathrm{R}-)$ sector yields the complex conjugate fields. These two sets of fields can be combined into two real one-forms and one real three-form without any duality constraint. These are the fields written in the last column.

In odd dimensions we have only one version of the theory (0AB) without the restriction on the rank of the forms. In even dimensions the forms come in even or odd rank depending on the RR projection. In $d = 10$ and $d = 6$ the assignment is the familiar one (odd forms for Type A and even for Type B) whereas in $d = 8$ and $d = 4$ it is reversed. Of course, it is possible to dualize some of the fields to obtain the magnetically charged branes. The unique form of degree $d/2 - 1$ for the Type

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[13]Things become even more complex at the next order in α' – for instance there could be terms of the type $T^{2n}R^{\mu\nu}\partial_\mu T\partial_\nu T$.

[14]A different point of view was taken in [54].

d	$SO(d-2)$	Spin reps.		$R \times R$ sector(s)	Real off-shell fields
4	$U(1)$	$\mathbf{1}_{1/2}, \mathbf{1}_{-1/2}$	0A 0B	$\mathbf{1}_0 + \mathbf{1}_0$ $\mathbf{1}_{-1} + \mathbf{1}_1$	$2A$ A_μ
5	$SU(2)$	$\mathbf{2}$	0AB	$\mathbf{1} + \mathbf{3}$	A, A_μ
6	$SU(2)^2$	$(\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1})$	0A 0B	$2(\mathbf{2},\mathbf{2})$ $2(\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1})$	$2A_\mu$ $2A, A_{\mu\nu}$
7	$Sp(4)$	$\mathbf{4}$	0AB	$\mathbf{1} + \mathbf{5} + \mathbf{10}$	$A, A_\mu, A_{\mu\nu}$
8	$SU(4)$	$\mathbf{4}, \bar{\mathbf{4}}$	0A 0B	$2(\mathbf{1} + \mathbf{15})$ $\mathbf{6} + \mathbf{10} + \bar{\mathbf{6}} + \bar{\mathbf{10}}$	$2A, 2A_{\mu\nu}$ $2A_\mu, A_{\mu\nu\rho}$
9	$SO(7)$	$\mathbf{8}$	0AB	$\mathbf{1} + \mathbf{7} + \mathbf{21} + \mathbf{35}$	$A, A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}$
10	$SO(8)$	$\mathbf{8}_s, \mathbf{8}_c$	0A 0B	$2(\mathbf{8} + \mathbf{56})$ $2(\mathbf{1} + \mathbf{28} + \mathbf{35})$	$2A_\mu, 2A_{\mu\nu\rho}$ $2A, 2A_{\mu\nu}, A_{\mu\nu\rho\sigma}$

Table 2.2: Type 0 RR field in any dimension.

0B case admits both electric and magnetic charges.

Notice that if it were not for the doubling of the RR sectors it would be impossible to write off-shell real fields for the 0B theory in $d = 4$ and $d = 8$ because, due to the Minkowski signature, it is impossible to impose either self-duality or anti-self-duality on real forms. We view this fact, together with the argument based on modular invariance, as yet another piece of evidence for the necessity of the presence of both RR sectors.

To obtain the complete form of the RR couplings up to two derivatives we need to address the issue of Chern–Simons terms. The terms of relevance here are those constructed with three gauge fields and two derivatives. Despite the lack of space-time supersymmetry, the Chern–Simons terms are present in $d = 10$, because they are inherited from Type II theories. It thus seems that the appearance of these terms is dictated more by world-sheet supersymmetry than by space-time supersymmetry and it is natural to assume that such terms are also present in lower dimensions.

The presence of so many RR fields may seem to lead to difficulties in determining these terms. However, there are two simplifying features that we infer from the $d = 10$ case: First, there will not be terms involving only RR fields since they correspond to the correlation function of an odd number of spin fields. One of the three gauge fields must therefore be the $(\text{NS}+, \text{NS}+)$ two-form $B_{\mu\nu}$. Second, we can apply the selection rules of the previous section, suitably modified according to dimensionality. In particular, properties 3 and 4 will apply also for $d = 6$, whereas in $d = 4$ and $d = 8$ the opposite is true, since now the conjugation matrix $\mathcal{C}$ commutes with the Clifford chirality matrix Γ_χ. Hence, in general, in $d = 4$ and $d = 8$ we expect to find a non zero coupling only between an *odd* number of tachyons and RR fields from the *same* sector The reverse should hold for correlation functions involving an *even* number of tachyons.

We therefore conclude, that *even* tachyon couplings will involve fields from the *same* RR sector in $d = 6$ and $d = 10$ and from *different* sectors in $d = 4$ and $d = 8$. Conversely, *odd* tachyon couplings will involve fields from the *same* RR sector in $d = 4$ and $d = 8$ and from *different* sectors in $d = 6$ and $d = 10$. The same properties again indicate that there are Chern–Simons terms in the effective actions and that the various field strengths need to be modified by shifting them with a $B_{\mu\nu}$ dependent transformation just as in Type II supergravity. Moreover they will involve RR fields strengths from the *same* sector in $d = 6$, whereas from *different* sectors in $d = 4$ and $d = 8$.

As in the critical case, we parameterize the tachyon couplings in the action with some unspecified functions $f(T)$'s. In $d = 10$, their first few coefficients in the Taylor expansion around zero can in principle be determined by extrapolating from the on-shell computation [28]. However, we shall later see that the detailed form of these functions is not directly relevant for computing properties of the dual field theories.

Let us start with the case of d odd. Here, due to the lack of chirality, there are no obvious selection rules concerning the number of tachyons coupled to two RR fields, thus we do not expect any particular symmetry of the couplings. In this case our proposal for the RR part of the action is

$$S^{0AB}_{d=5} = \int f(T)(F_1 * F_1 + F_2 * F_2) + BF_1F_2$$

$$S^{0AB}_{d=7} = \int f(T)(F_1 * F_1 + F_2 * F_2 + \tilde{F}_3 * \tilde{F}_3) + BF_2F_3$$

$$S^{0AB}_{d=9} = \int f(T)(F_1 * F_1 + F_2 * F_2 + \tilde{F}_3 * \tilde{F}_3 + \tilde{F}_4 * \tilde{F}_4) + BF_3F_4 . \quad (2.43)$$

The forms F_n are the field strengths associated to the RR gauge potentials. A wedge product between forms is always understood. The tilde above the forms always indicates the NSNS–shift, *e.g.* $\tilde{F}_4 = F_4 + BF_2$, with the appropriate modified gauge

transformation just as in Type II supergravity. Notice that, once it is assumed that the Chern–Simons terms are present, the modification in the field strength must also be present for the action to transform correctly under electric/magnetic duality.

To write the actions for the even dimensional cases, we still denote the field strengths from the two RR sectors by F, F' for $d = 6$ and $d = 10$, whereas we use F, $\bar F$ for $d = 4$ and $d = 8$. In the former case the field strengths are real while in the latter they are complex conjugates of each other. The form of highest degree in the Type 0B case is special – it is self dual in the complex case and its vertex operator does not contain the chiral projection.

$$S_{d=4}^{0A} = \int f_{\text{even}}(T)(F_1 * \bar F_1) + f_{\text{odd}}(T)(F_1 * F_1 + \bar F_1 * \bar F_1) + iBF_1\bar F_1$$

$$S_{d=4}^{0B} = \int f_{\text{even}}(T)(F_2 * \bar F_2) + f_{\text{odd}}(T)(F_2 * F_2 + \bar F_2 * \bar F_2)$$

$$S_{d=6}^{0A} = \int f_{\text{even}}(T)(F_2 * F_2 + F_2' * F_2') + f_{\text{odd}}(T)(F_2 * F_2') + B(F_2F_2 + F_2'F_2')$$

$$S_{d=6}^{0B} = \int f_{\text{even}}(T)(F_1 * F_1 + F_1' * F_1' + \tilde F_3 * \tilde F_3) + f_{\text{odd}}(T)(F_1 * F_1' + \tilde F_3 * \tilde F_3)$$
$$+ B(F_1F_3 + F_1'F_3)$$

$$S_{d=8}^{0A} = \int f_{\text{even}}(T)(F_1 * \bar F_1 + \tilde F_3 * \tilde{\bar F}_3) + f_{\text{odd}}(T)(F_1 * F_1 + \bar F_1 * \bar F_1 + \tilde F_3 * \tilde F_3$$
$$+ \tilde{\bar F}_3 * \tilde{\bar F}_3) + iBF_3\bar F_3$$

$$S_{d=8}^{0B} = \int f_{\text{even}}(T)(F_2 * \bar F_2 + \tilde F_4 * \tilde{\bar F}_4) + f_{\text{odd}}(T)(F_2 * F_2 + \bar F_2 * \bar F_2$$
$$+ \tilde F_4 * \tilde F_4 + \tilde{\bar F}_4 * \tilde{\bar F}_4) + B(F_2\bar F_4 + F_4\bar F_2)$$

$$S_{d=10}^{0A} = \int f_{\text{even}}(T)(F_2 * F_2 + F_2' * F_2' + \tilde F_4 * \tilde F_4 + \tilde F_4' * \tilde F_4')$$
$$+ f_{\text{odd}}(T)(F_2 * F_2' + \tilde F_4 * \tilde F_4') + B(F_4F_4 + F_4'F_4')$$

$$S_{d=10}^{0B} = \int f_{\text{even}}(T)(F_1 * F_1 + F_1' * F_1' + \tilde F_3 * \tilde F_3 + \tilde F_3' * \tilde F_3' + \tilde F_5 * \tilde F_5)$$
$$+ f_{\text{odd}}(T)(F_1 * F_1' + \tilde F_3 * \tilde F_3' + \tilde F_5 * \tilde F_5) + B(F_3F_5 + F_3'F_5) . \tag{2.44}$$

Notice that the kinetic terms in the actions can be diagonalized by letting $F_\pm = F \pm F'$ in $d = 6, 10$ and $F_\pm = F \pm i\bar F$ in $d = 4, 8$.

Massive Type 0 gravity

There is still one RR form field that can be added to the actions (2.43) and (2.44). In a d-dimensional space-time it is possible to introduce a rank $d - 1$ gauge potential

coupling to a corresponding extended object. It carries no physical degrees of freedom and therefore it is not visible in the on-shell analysis of the previous subsection. Its rank d field strength, however, carries an energy density and it does affect the physics. This form will be used in the next chapter when addressing the issue of field theory duals. This case will provide the simplest example which displays most of the interesting physics, and it allows one to avoid the complications of disentangling the Kaluza-Klein modes.

In Type IIA supergravity (and thus in $d = 10$ Type 0A for each RR sector) it is well known how to introduce such a field [39, 55]. The required modifications in the bosonic sector are the addition of the terms

$$\int MF_{10} + \frac{1}{2}M^2 * 1 \tag{2.45}$$

to the action, and a further shift of the 2- and the 4-form field strengths by MB and $MB^2/2$, respectively. The gauge transformations are changed accordingly in order to re-ensure gauge invariance. Integrating over the gauge potential of F_{10} imposes the constraint that M be constant. Solving the equation of motion for M establishes a connection between M and F_{10}. In the case $B = 0$ – relevant to our analysis – they are simply Hodge duals of each other as is readily seen from (2.45).

From the string theory point of view [17], the natural generalization of the RR beta function equations implies $\mathrm{d}*F_d = 0$ and $\mathrm{d}F_d = 0$, as the top-form, too, appears in the reduction of an even dimensional Type 0A bispinor into antisymmetric tensor representations. In our case, we must also include the coupling with the tachyon. Thus the relevant addition is

$$-\frac{1}{2}\int f(T)F_d * F_d \ , \tag{2.46}$$

that we assume be present in any dimension.

Chapter 3

AdS/CFT correspondence in Type 0 string theory

The present chapter contains one of the main results collected in this thesis, namely the discussion of certain solutions of (non)-critical Type 0 effective theories, and their dual field theory interpretation [53, 56]. We will be able to show that in any dimension there exists a set of exact solutions of the classical equations of motion, which have AdS metric and involve a non-zero RR field, other than constant dilaton and tachyon. Such solutions depend on a *finite* number of parameters for which a string-theoretical derivation is still lacking.

The solutions support a condensed tachyon and, at least for some specific cases, are stable against quantum fluctuations. Because of mixing of the fields it is not enough to analyze tachyon stability separately: One should disentangle the full set of fluctuations. We have restricted our investigation to the case where the space-time is d-dimensional AdS, so that there are no KK modes to be worried about. There the analysis simplifies considerably, but it is still non trivial because of dilaton-tachyon mixing.

In [22] it was claimed that these gravity solutions may represent an interacting UV conformal point. To address this issue properly one needs to show that they are in fact a point in wider space, that is actually the RG phase diagram of the field theories we have at hand. In this enlarged theory space there may be more than one conformal solution and there can be trajectories which interpolate between these points. These considerations have been previously realized in the framework of various supergravity theories, and have opened a new direction to investigate in the AdS/CFT correspondence, namely the issue of Holographic Renormalization Group. Recently there have been further developments in this direction, after the suggestion in [33] to use Hamilton–Jacobi theory for studying dual RG flows. In Chapters 4 and 5 we will come back to this subject extensively. Here we will show how Type 0 gravity provides an example of this general feature of the AdS/CFT correspondence,

as well. The physics is already captured by a set of solutions involving no compact space. Even without including KK modes, the tachyon will mix with the dilaton field, and generate on the field theory side a RG flow that connects interacting conformal fixed points.

As a complement, we will address the problem from another point of view. Namely, we will find the thermal deformations of the solutions above and compute their entropy. According to [57] these solutions should be dual to the given field theory, in a thermal bath. Therefore, computing the entropy one gets the number of degrees of freedom of the dual theory. In particular we will find a N^2 behavior for any dimension, consistently with the expectation of finding a conformal phase of non-supersymmetric Yang–Mills theories.

In Section 3.5 we have also reported the computation of the Wilson loop and the mass gap, which give results consistent with the whole picture.

3.1   Non-supersymmetric AdS/CFT

Before focusing on the subject mentioned above, we would like to give a brief review of some alternative approaches to non-supersymmetric versions of the AdS/CFT correspondence. The aim is to provide a comparison in order to better appreciate advantages and disadvantages of the Type 0 approach. Furthermore we will also summarize some interesting results in Type 0, contained in [29, 51].

Most of the success of the AdS/CFT correspondence is devoted to those situations where at least some fraction of supersymmetry is preserved. However, an important issue to address is how to embed the physically relevant non-supersymmetric gauge theories in the correspondence, eventually recovering asymptotic freedom and confinement. There are different proposals for giving a holographic description of non-supersymmetric gauge theories, which basically deal with possible mechanisms for breaking supersymmetry.

It was pointed out by Witten [20, 57] that the AdS/CFT correspondence can be formulated at *finite temperature*. In doing so one identifies the Hawking temperature T_H of the supergravity solution with that of the field theory in a thermal bath. The antiperiodic boundary conditions for fermions along the compactified time direction break supersymmetry and give them masses $m_f \sim T_H$ at tree-level. This leads also spin 0 bosons to acquire nonzero masses at one-loop and spoils conformal symmetry because cancellations in the β-function do not occur any more. At large distances ($L \gg R_{\text{comp}} \sim T_H^{-1}$) the infrared effective theory is then expected to be pure YM in one lower dimension. This approach captures some of the expected qualitative features of the quantum field theory, as a *confining* behavior of the Wilson loop, due to the presence of a horizon in the metric, and a *mass-gap* in the glue-ball spectrum. Moreover it provides a Lorentz invariant regularization scheme, as opposed for in-

stance to the lattice regularization. Nonetheless it has some drawbacks: There is still coupling with the physics in higher dimension, as both the masses of glue-balls and of fermions are of the same order of the Hawking temperature. In addition, the p-dimensional 't Hooft coupling obeys $\lambda_p \sim \lambda_{p+1} T_H$, where T_H acts as UV cutoff. As the cutoff is removed letting T_H to infinity, λ_{p+1} should approach zero, but this is opposite to the regime in which the supergravity approximation applies[1].

Among other means to break supersymmetry there are the deformations of supersymmetric solutions. One searches for domain wall interpolating between some AdS vacuum with $\mathcal{N}$ supersymmetries and a different vacuum that can be AdS or not, and has $\mathcal{N}' < \mathcal{N}$. In the field theory side this is interpreted as turning on some relevant operators that drive the initial theory to an effective IR theory via RG flow of the couplings.

We now recall the basic ideas behind applications of the Type 0 construction, with the help of two of the earlier examples provided by Klebanov and Tseytlin in the papers [29, 51]. As in the Type II case, the idea is to consider a setup of D-branes of the theory, work out the gauge theory living on their world-volume, and eventually find the corresponding solution of the gravity theory in order to make comparisons and, hopefully, predictions.

Yang–Mills theory in four dimensions

Consider a stack of N D3-branes of the same type in Type 0B string theory. The configuration is stable and the low energy spectrum on their world-volume consists of $SU(N)$ gauge bosons plus 6 adjoint scalars in $3 + 1$ dimensions. In [29] the authors found two approximate solutions of the gravity equations of motion that involve nontrivial tachyon, dilaton, together with RR 4-form, and whose asymptotics were given in the UV and IR regions[2]. The metric approaches in both cases $AdS_5 \times S^5$ (with different radii) and the behavior of the other fields is as follows. The first one displays a vanishing 't Hooft coupling and a tachyon condensate with $\langle T \rangle \sim -1$, while the second represents an IR conformal point at infinite coupling — in fact the dilaton blows up while the tachyon condensate is $\langle T \rangle \sim 0$. Interestingly, asymptotic freedom is reproduced. It should be noted however that in this situation α' corrections become important, whereas in the IR one has to worry about string loop corrections.

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[1]In this respect the situation is similar to the lattice approach, where computations are possible at strong coupling.

[2]The identifications of these regimes follows from the identification of the radial coordinate transversal to the stack, with energy scale of the gauge theory.

Non-supersymmetric CFT

Another interesting problem that has been studied is the construction of gauge theories that are conformal though non-supersymmetric and their dual gravitational description. Let us illustrate the construction in [51]. In Type 0B theory one starts with a stack of N D3-branes of one type and N of the other. This configuration is argued to be stable in the large N limit[3]. The theory living on the world-volume of this stack is a $SU(N) \times SU(N)$ gauge theory in $3 + 1$ dimensions, whose field content comprises, in addition to gauge bosons, 6 adjoint scalars for each $SU(N)$ factor, 4 bifundamental fermions in the $(\mathbf{N}, \bar{\mathbf{N}})$ and their conjugates. The dual gravity solution found in [51] is again AdS$_5 \times$S^5 with constant dilaton and vanishing tachyon.

String loop suppression requires large N as usual, while tachyon stability translates in a condition on the 't Hooft coupling, namely $\lambda = g_{YM}^2 N \lesssim 100$. Then the dual theory should be a CFT in the large N limit, for not very large coupling.

The gravity solution resembles very much the Type IIB familiar case. In fact, it was pointed out in [58] that the gauge theory belongs to the class of orbifold theories of $\mathcal{N} = 4$ SYM, where the $\mathbb{Z}_2$ projection belongs to the center of the R-symmetry group. Thus it is exactly conformal in the large N limit. This information can be used to reverse the argument, that is, via AdS/CFT the gauge theory may be predictive on the string theory, or gravity, side. In fact stability of the CFT at weak coupling implies that Type 0B theory on AdS$_5 \times$S^5 should be stable for sufficiently small radius. In [59] it was suggested that the instability of the background at large radius translates in a phase transition occurring in the large N CFT at strong coupling, with anomalous dimension of the operator dual to the tachyon field developing a singularity at a critical value λ_c.

3.2   AdS solutions in Type 0 gravity

Below we shall show that Type 0 low energy theories allow Freund–Rubin type solutions [60], where the dilaton and the tachyon are constant, the space-time factorizes into a product of an AdS space and a sphere, and the only nontrivial form-field is a RR field. Such types of solutions are also familiar from the supergravity literature. Lacking of knowledge of the tachyon potential and tachyon couplings to RR fields makes the analysis less quantitative. However we shall point out that relevant features should be independent of details. In particular, computing the area of horizons of the finite temperature version of these solutions, we will extract the number of degrees of freedom which, oppositely to conventional supergravity solutions, scales

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[3]At finite N branes of different types repel each other because of fermionic degrees of freedom in their world-volume.

as N^2 *i.e.* as a gas of weakly interacting YM particles, in any number of space-time dimensions.

The relevant piece of the action discussed in the previous chapter reads

$$
S = \int d^d x \sqrt{-g} \left\{ R - \frac{1}{2}(\partial_M \Phi)^2 - \frac{1}{2}(\partial_M T)^2 - V(T) \, e^{a\Phi} \right.
$$
$$
\left. - \frac{1}{2\,(p+2)!} \, f(T) \, e^{b\Phi} \left(F_{M_1 \cdots M_{p+2}} \right)^2 \right\} .
\tag{3.1}
$$

We have switched to component notation and specified all the appropriate normalizations[4]. Capital letters denote here full d-dimensional range of indices.

Eq. (3.1) is the Einstein frame action, accompanied by the following rescaling of the dilaton

$$
\Phi \;\rightarrow\; \frac{1}{2}\sqrt{\frac{d-2}{2}} \, \Phi \,,
\tag{3.2}
$$

and reduces the ordinary Einstein frame action in $d = 10$. $V(T)$ is the sum of the tachyon potential and the central charge deficit (2.42), and $f(T)$ is the coupling between the $(p+2)$-dimensional RR form F and the tachyon. The RR gauge field is the appropriate linear combination of some of the fields of the previous chapter in such a way that the kinetic terms are diagonal. After diagonalization, $f(T)$ no longer has any particular symmetry property.

The coefficients a and b are

$$
a = \sqrt{\frac{2}{d-2}}
\tag{3.3}
$$

$$
b = \frac{1}{2}(d - 2p - 4)\sqrt{\frac{2}{d-2}} \,.
\tag{3.4}
$$

The field B_{MN} that we are setting to zero here may appear linearly in the full action only in the Chern–Simons term, but in that case multiplied by $F \wedge F$, which will vanish in the Freund–Rubin ansatz.

---

[4]Our normalization of the tachyon differs by a factor of $\sqrt{2}$ from that of most of the recent literature.

The equations of motion can be summarized as follows:

$$\Box\Phi = aV(T)\,e^{a\Phi} + \frac{b}{2}\,f(T)\,e^{b\Phi}\,\frac{1}{(p+2)!}\left(F_{M_1\cdots M_{p+2}}\right)^2 \tag{3.5}$$

$$\Box T = V'(T)\,e^{a\Phi} + \frac{1}{2}\,f'(T)\,e^{b\Phi}\,\frac{1}{(p+2)!}\left(F_{M_1\cdots M_{p+2}}\right)^2 \tag{3.6}$$

$$R_{MN} = \frac{1}{2}\,\partial_M\Phi\,\partial_N\Phi + \frac{1}{2}\,\partial_M T\,\partial_N T$$
$$+\frac{1}{d-2}\,g_{MN}\,V(T)\,e^{a\Phi} + \frac{1}{2}\,f(T)\,e^{b\Phi}\,\tilde{T}_{MN} \tag{3.7}$$

$$0 = \nabla^N\left(f(T)\,e^{b\Phi}\,F_{NM_1\cdots M_{p+1}}\right) . \tag{3.8}$$

The tensor $\tilde{T}_{MN}$ is shorthand for the trace subtracted stress energy tensor

$$\tilde{T}_{MN} = \frac{1}{(p+1)!}\left(F_{MK_1\cdots K_{p+1}}F_N{}^{K_1\cdots K_{p+1}} - \frac{(p+1)\,g_{MN}}{(p+2)(d-2)}\left(F_{K_1\cdots K_{p+2}}\right)^2\right) . \tag{3.9}$$

Again, we have ignored potential contributions from Chern–Simons terms because they vanish for the classical solution. They do contribute to the analysis of the fluctuations in the general case and also for this reason, in the next section, when computing the critical properties of the field theory duals we restrict to the simple case $d = p + 2$ where such complications do not arise.

These equations of motion have a solution with constant dilaton $\Phi = \Phi_0$ and tachyon $T = T_0$ in the gravity background of a product space

$$\mathrm{AdS}_{p+2} \times \mathrm{S}^{d-p-2} . \tag{3.10}$$

The size of the two maximally symmetric spaces is determined by setting[5] (always in units of α')

$$R_{\mu\nu\rho\lambda} = -\frac{1}{R_0^2}\left(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}\right)$$

$$R_{ijkl} = +\frac{1}{L_0^2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) . \tag{3.11}$$

Finally, the RR field is set proportional to the volume-form of the anti-de Sitter space, and hence its only nontrivial components are

$$F_{\mu_1\cdots\mu_{p+2}} = F_0\,\sqrt{-g(\mathrm{AdS}_{p+2})}\,\epsilon_{\mu_1\cdots\mu_{p+2}} , \tag{3.12}$$

---

[5]The Greek indices refer to the AdS space and the Latin indices to the sphere.

where the constant F_0 is related to the conserved charge k by

$$k = f(T_0) \, e^{b\Phi_0} \, F_0 \,.$$

(3.13)

Here k is not to be confused with the number of branes N. The former is computed in the Einstein frame (see above), whereas the latter must be computed in the string frame.

Given the two functions $V(T)$ and $f(T)$, the tachyon and the dilaton vacuum expectation values are determined from Eqs. (3.5), (3.6) and (3.13). The tachyon T_0 can be expressed implicitly, as the solution of an algebraic equation, namely

$$\frac{f'(T_0)}{f(T_0)} = \frac{1}{2}(d - 2p - 4) \, \frac{V'(T_0)}{V(T_0)}.$$

(3.14)

The dilaton Φ_0 can then be readily obtained from

$$e^{(a+b)\Phi_0} = \frac{(d - 2p - 4)}{4} \, \frac{k^2}{f(T_0) \, V(T_0)} \,.$$

(3.15)

The radii[6] of the anti-de Sitter space R_0 and that of the sphere L_0 can be solved from the Einstein equations (3.7)

$$R_0^2 = (p+1)(d - 2p - 4)\frac{e^{-a\Phi_0}}{V(T_0)}$$

(3.16)

$$L_0^2 = (d - p - 3)(d - 2p - 4)\frac{e^{-a\Phi_0}}{V(T_0)} \,.$$

(3.17)

In the derivation we assumed $k \neq 0$. Also three special dimensionalities were excluded for compactness:

a) The case $d = p + 3$ leads to an infinite radius in the AdS space-time, *i.e.* the flat Minkowski space times a circle, and is not considered in what follows.

b) For $d = 2p + 2$ the dilaton becomes a free parameter. Rather than its vacuum expectation value Φ_0, the charge k is determined from the equation of motion (3.5)

$$k^2 = -2V(T_0) \, f(T_0) \,.$$

(3.18)

The rest of the formulae (3.14), (3.16) and (3.17) are still valid.

---

[6]Remember however that, as far as α' corrections are concerned, the radii in the string frame matter.

c) We assumed that $V(T_0) \neq 0$. In addition to some completely Ricci flat solutions this condition also excludes the middle dimensional branes, for which we have $d = 2p + 4$. In these dimensions the radii are

$$R_0^2 = L_0^2 = 4 \, (p+1) \, \frac{f(T_0)}{k^2} \,, \tag{3.19}$$

Now T_0 is determined from $V(T_0) = 0$ (not $V'(T_0) = 0$) and Φ_0 from

$$V'(T_0)\mathrm{e}^{a\Phi_0} - \frac{k^2}{2} \frac{f'(T_0)}{f(T_0)^2} = 0 \,. \tag{3.20}$$

The solutions discussed above are physically acceptable only for $f(T_0) > 0$.

3.3   Dual Type 0 CFT

In this section we make contact with the conjectured gravity/field theory duality by studying the simple case of $d = p + 2$. This case already contains all the relevant qualitative features of the most general one, without the complication of the Kaluza–Klein analysis.

Let us recall the logic of the approach. The classical solutions on the gravity side correspond to fixed points on the field theory side. There is a geodesic flow that relates these classical solutions, which is interpreted as the renormalization group flow connecting different fixed points. Fluctuation modes with positive, vanishing, and negative mass square correspond to irrelevant, marginal, and relevant deformations. The critical exponents can be obtained from these masses and they depend on a finite set of undetermined parameters due to the arbitrariness of the tachyon couplings. These parameters should be fixed by comparing some universal quantities with experiment which leads to a prediction for the remaining quantities.

We will begin by studying under which conditions the fixed point solutions of the previous section are stable. As a byproduct we will get the masses of the fields at these fixed points, that will be used to infer the conformal dimensions of the dual CFT operators.

Classical solutions can serve as sound vacua for a quantum theory only if small fluctuations around the solutions are stable. In Minkowski space this implies that tachyonic fluctuation modes are forbidden. In an AdS background this requirement can be relaxed, and one finds the bound [61, 62]

$$m^2 \geq -\frac{(d-1)^2}{4} \frac{1}{R_0^2} \tag{3.21}$$

for the masses of the scalar fluctuation modes.

The first source of these instabilities near the solutions found in the previous section are obviously fluctuations in the tachyon field T. Tachyonic instabilities may enter also through the various scalar fields that appear on the AdS space, as the fields are compactified on the sphere S^{d-p-2}. In order to show that the theory is stable against these perturbations, one has to linearize the full set of equations of motion around the classical solution, and check that no mode violates the bound (3.21). This can be done, but the physically relevant features already appear in the case where the transverse sphere is absent. We shall discuss this example below in detail.

As $d = p + 2$, the nontrivial RR field is the top-form, dual to a cosmological constant

$$F_{\mu_1 \cdots \mu_{p+2}} = F \sqrt{-g} \, \epsilon_{\mu_1 \cdots \mu_{p+2}} \, . \tag{3.22}$$

Note, that this is no longer the Freund–Rubin ansatz, but the RR field is a priori entirely general and unconstrained. This is the field discussed in Section 2.4 (massive Type 0 gravity). The equation of motion (3.8) becomes in this case a constraint, and it turns out that the conserved charge is

$$k = f(T) \, e^{b\Phi} \, F \, . \tag{3.23}$$

With the help of (3.23), the equations of motion for the other fields reduce to the following form

$$\Box \Phi = -\frac{\partial}{\partial \Phi} \mathcal{V}(\Phi, T) \tag{3.24}$$

$$\Box T = -\frac{\partial}{\partial T} \mathcal{V}(\Phi, T) \tag{3.25}$$

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} \partial_\mu T \partial_\nu T - \frac{1}{d-2} \mathcal{V}(\Phi, T) g_{\mu\nu} \tag{3.26}$$

where the effective potential is

$$\mathcal{V}(\Phi, T) = -V(T) e^{a\Phi} - \frac{1}{2} \frac{k^2}{f(T)} e^{-b\Phi} \, . \tag{3.27}$$

Let us linearize these equations of motion near a classical solution

$$\Phi = \Phi_0 + \varphi \tag{3.28}$$

$$T = T_0 + t \tag{3.29}$$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \, . \tag{3.30}$$

In order to do this, we need some knowledge of the functions $V(T)$ and $f(T)$. The only characteristics of these functions that will enter the stability analysis are the coefficients

$$x = \frac{V'(T_0)}{V(T_0)} \, , \qquad y = \frac{V''(T_0)}{V(T_0)} \, , \qquad \text{and} \qquad z = \frac{f''(T_0)}{f(T_0)} \, . \tag{3.31}$$

Perturbative string theory analysis around $T = 0$ yields [28]

$$V(T) = d - 10 - \frac{d-2}{8} T^2 + \mathcal{O}(T^4) \qquad (3.32)$$

$$f(T) = 1 + T + \frac{1}{2} T^2 + \mathcal{O}(T^3) . \qquad (3.33)$$

This is not enough to determine the coefficients (3.31), and they should indeed be treated as free parameters of the theory. Including other unknown functions would give rise to more than three such parameters, but the analysis performed here would still have the same qualitative features.

The fact that the graviton fluctuations actually decouple completely from those of the scalars simplifies the calculations: The graviton equations of motion can, in fact, be derived to first order from the effective action

$$S_h = \int \mathrm{d}^d x \ \sqrt{-(\hat{g} + h)} \left\{ R(\hat{g}_{\mu\nu} + h_{\mu\nu}) + \mathcal{V}(\Phi_0, T_0) \right\} . \qquad (3.34)$$

The scalar fluctuations obey

$$\left(-\Box + \mathcal{M} \right) \left(\begin{array}{c} \varphi \\ t \end{array} \right) = 0 \qquad (3.35)$$

where the mass matrix is

$$\mathcal{M} = d(d-1) \, R_0^{-2} \left(\begin{array}{cc} 1 & \sqrt{\frac{d-2}{2}} \, x \\ \sqrt{\frac{d-2}{2}} \, x & d \, x^2 - y - \frac{2}{d} z \end{array} \right) . \qquad (3.36)$$

The mass eigenvalues are

$$m_{1,2}^2 = d(d-1) R_0^{-2} \left(1 + \frac{\tau}{2} \pm \frac{1}{2} \sqrt{\tau^2 + (2d-4)x^2} \right) \qquad (3.37)$$

where

$$\tau = d \, x^2 - \frac{2z}{d} - y - 1 . \qquad (3.38)$$

Note that the masses depend only on two independent parameters x and τ.

If we assume, following [29], that $f(T) = \exp(T)$, then the equations of motion give $x = -2/d$, and we can easily extract some interesting qualitative features as the only undetermined parameter is τ.

In this case there turns out to be three different, continuously connected phases: First, there can be two particles, both with positive mass squared. Second, there can be a particle and a tachyon that obeys the bound (3.21). Third, there can be a tachyon that makes the vacuum unstable. In AdS/CFT correspondence this translates into the statement that there can be at most one relevant operator in the infrared near the fixed point described by this theory.

3.4   Yang–Mills conformal phases

Here, with the solutions of the previous section at hands, we can now perform a check of the CFT interpretation given above. In particular, we will compute the number of degrees of freedom pertinent to the given field theory dual, finding that they correspond to ordinary Yang–Mills theory.

Let us start by noticing that, after going to the string frame, the radii of AdS_{p+2} and S^{d-p-2}, and the 't Hooft coupling $\lambda_{p+1} = Ne^{\Phi_0}$ read[7]

$$\lambda_{p+1}^2 = (d-p-3)\left[\frac{2(2p+4-d)(d-p-3)}{c_{\text{eff.}}}\right]^{d-p-3} \tag{3.39}$$

$$b^2 = \frac{2(2p+4-d)(d-p-3)}{c_{\text{eff.}}} \tag{3.40}$$

$$R^2 = \frac{2(p+1)(2p+4-d)}{c_{\text{eff.}}}, \tag{3.41}$$

in terms of the effective central charge (2.40), evaluated at T_0. Note that R and b are fixed and of order $\mathcal{O}(1)$. The relation between the 't Hooft coupling and the radius of curvature then reads

$$\lambda_{p+1}^2 \sim \left(R^2\right)^{d-p-3}. \tag{3.42}$$

We emphasize that it is only in the case of critical Type II theory that one is truly free to vary the parameters in (3.42) although it is tempting to hope that a better control of the tachyon field will allow to give a precise physical meaning to the Type 0 construction as well.

Notice the peculiar behavior of R that scales in a way inversely proportional to the 't Hooft coupling, contrary to the standard situation. Again, this dependence should be interpreted with a grain of salt because at this stage both values are fixed in terms of $c_{\text{eff.}}$ and we cannot take the limit $\lambda_{p+1} \to 0$, as the effective central charge is fixed by the tachyon expectation value.

In order to count the number of microstates living in the dual theory, we need to find the finite temperature version of the solution. The thermal deformation is more easily obtained by going to the gauge

$$ds^2 = \frac{\rho^2 f(\rho)}{R^2}dt^2 + \frac{\rho^2}{R^2}dx_i^2 + \frac{R^2}{\rho^2 f(\rho)}d\rho^2 + b^2 d\Omega^2. \tag{3.43}$$

---

[7]Below we denote with R and b the string frame radii of AdS and compact space respectively, to distinguish them from the Einstein frame ones R_0 and L_0 of Section 3.2. We continue to denote with Φ_0 and F_0 the fixed values of the dilaton and the RR field-strength, even if here they are evaluated in the string frame.

This yields

$$(p+1)f(\rho) + \rho\frac{\mathrm{d}f}{\mathrm{d}\rho} = p+1 , \tag{3.44}$$

whose solution is

$$f(\rho) = 1 - \frac{\rho_T^{p+1}}{\rho^{p+1}} . \tag{3.45}$$

The Hawking temperature for this solution is easily computed to be

$$T_H = \frac{p+1}{4\pi}\frac{\rho_T}{R^2} . \tag{3.46}$$

We are now ready to compute the entropy [63, 57] of the non-critical, non-extremal p-brane solution. We will use two independent methods and find agreement between the results.

Let us recall that for Yang–Mills theory in the weak coupling limit, one can neglect interactions between gluons and compute microscopically thermodynamic quantities using a free Bose gas approximation. In the case of $SU(N)$ gauge theory in $p+1$ dimensions the energy and entropy per unit volume read

$$\frac{E}{V} \sim N^2 T^{p+1} \qquad \frac{S}{V} \sim N^2 T^p \tag{3.47}$$

which are, up to a numerical coefficient, dictated just by dimensional arguments, N^2 being the number of degrees of freedom.

Following [63] and [57], we identify the free energy F of the black-brane as the (subtracted) Euclidean action times the Hawking temperature $\beta = T_H^{-1}$

$$\beta F = I_E[g_{\mu\nu}, \Phi_0, F_0; T_H] - I_E[g_{\mu\nu}, \Phi_0, F_0; 0] \tag{3.48}$$

where we subtract the zero temperature action to get a finite result. Notice that by virtue of equations of motion the Einstein and cosmological terms drop out for constant dilaton and the action gets contribution only from the RR field. In fact

$$I_E = \int \mathrm{d}x^d \sqrt{|g|}\frac{1}{2(p+2)!}F_0^2 = \frac{1}{2}\frac{N^2}{R^{2d-2p-4}}\int \mathrm{d}x^d \sqrt{|g|} . \tag{3.49}$$

After putting a cutoff in the radial integration, one has to evaluate two invariant volumes, where in the black-brane configuration the integration is to be performed in the physical region outside the horizon

$$V(\rho_\infty) = \int_0^\beta \mathrm{d}t \int_{\rho_T}^{\rho_\infty} \mathrm{d}\rho \int \mathrm{d}^p x \frac{\rho^p}{R^p} \int R^{d-p-2}\mathrm{d}\Omega_{d-p-2} \tag{3.50}$$

and

$$V_0(\rho_\infty) = \int_0^{\beta'} dt \int_0^{\rho_\infty} d\rho \int d^p x \frac{\rho^p}{R^p} \int R^{d-p-2} d\Omega_{d-p-2} \qquad (3.51)$$

and let $\rho_\infty \to \infty$ after subtracting them. Here the radius of compactification β' has to be matched with β for the hyper-spheres in the two geometries to be comparable, *i.e.*

$$\beta' \frac{\rho_\infty}{R} = \beta \frac{\rho_\infty}{R} \sqrt{1 - \frac{\rho_T^{p+1}}{\rho_\infty^{p+1}}} \,. \qquad (3.52)$$

The result is

$$\beta F = \frac{N^2}{2R^{2d-2p-4}} \lim_{\rho_\infty \to \infty} (V - V_0) \sim -R^{4+2p-d} \Omega_{d-p-2} N^2 V_p \frac{1}{\beta^p} \qquad (3.53)$$

where Ω_{d-p-2} is the volume of the $(d - p - 2)$-sphere and V_p is the total volume of the p-space. Now recall from (3.41) that $R \sim 1$ so that we finally get the energy

$$E = \frac{\partial}{\partial \beta}(\beta F) \sim N^2 V_p T_H^{p+1} \qquad (3.54)$$

and the entropy

$$S = \beta(E - F) \sim N^2 V_p T_H^p \,. \qquad (3.55)$$

One can also compute the Bekenstein-Hawking entropy. In fact, going back to the Einstein frame, the area of the horizon is easily found to be (recall also that $b \sim R$)

$$A \sim \Omega_{d-p-2} R^{4+2p-d} N^2 V_p T_H^p \,, \qquad (3.56)$$

in agreement with (3.55). Thus we find that the Hawking relation is reproduced and the entropy has the ideal gas scaling behavior.

This result is in agreement with the conclusions of [64] for critical black p-branes, where it is pointed out that constant dilaton is a sufficient condition for such a scaling. Nevertheless, off criticality allows for more general values of p. Note the different scaling powers of N in the analogous relations one gets from evaluating the entropies of black M2 and M5 branes, that are 3/2 and 3 respectively.

3.5   Wilson loop and mass gap

Given the solutions found in Section 3.2, one can also compute the Wilson loop [65, 66] of the dual theory and check whether there is a mass gap in the glue-ball

spectrum [20, 57]. The results are in agreement with expectations, though some differences arise compared to usual the Type IIB case, basically due to the different relations among radii and coupling constants, Eqs. (3.39) – (3.41).

Let us parameterize the world-sheet of the string as $x^1 = \sigma$, $x^2 = \tilde{\sigma}$, $\rho = \rho(\sigma)$ where $-L/2 < \sigma < L/2$, $-\tilde{L}/2 < \tilde{\sigma} < \tilde{L}/2$ and $L << \tilde{L}$. The action to be minimized is

$$S = \frac{\tilde{L}}{2\pi} \int_{-L/2}^{L/2} d\sigma \sqrt{\frac{\rho'^2}{1 - \rho_T^{p+1}/\rho^{p+1}} + \frac{\rho^4}{R^4}} . \tag{3.57}$$

The conserved quantity derived from this action is

$$\frac{1}{\rho^4} \sqrt{\frac{\rho'^2}{1 - \rho_T^{p+1}/\rho^{p+1}} + \frac{\rho^4}{R^4}} = \frac{1}{\rho_0^2 R^2} , \tag{3.58}$$

where $\rho(0) = \rho_0$ and $\rho'(0) = 0$ for symmetry reasons. ρ_0 measures how close the world-sheet approaches the horizon at ρ_T and the behavior of the Wilson loop is governed by the ratio $\epsilon = \rho_T/\rho_0$. For $\epsilon \to 0$ we recover the conformal fixed point, whereas for $\epsilon \to 1$ we should approach the p dimensional theory.

The minimum action is given by the integral (after subtracting the infinite energy of the string)

$$S_{\min} = \frac{\tilde{L}\rho_0}{\pi} \left\{ \epsilon - 1 + \int_1^\infty dy \left[\frac{y^{(p+5)/2}}{\sqrt{(y^4 - 1)(y^{p+1} - \epsilon^{p+1})}} - 1 \right] \right\}, \tag{3.59}$$

where ρ_0 is expressed in terms of R, L and ρ_T by the implicit function

$$\frac{L}{2} = \frac{R^2}{\rho_0} \int_1^\infty dy \frac{y^{(p-3)/2}}{\sqrt{(y^4 - 1)(y^{p+1} - \epsilon^{p+1})}}. \tag{3.60}$$

In the regime $\epsilon \to 0$ we obtain the results of [65, 66]: $S_{\min} \sim R^2 \times (\tilde{L}/L)$. Note that this is the same behavior as in [65, 66] only if expressed in terms of R. The relation between R and the 't Hooft coupling being different, Eq. (3.42), if not in the critical dimension.

The interesting regime is when $\epsilon \to 1$. In this case, both integrals in (3.59), (3.60) scale like $|\log(1 - \epsilon)|$ and we must eliminate the divergence by taking the ratio of the two quantities. This leaves a dependence on ρ_0 but this is easily fixed by realizing that as $\epsilon \to 1$, $\rho_0 \to \rho_T$, yielding

$$S_{\min} = \frac{\rho_T^2}{2\pi R^2} \times \tilde{L}L. \tag{3.61}$$

Eq. (3.61) represents the area law for the p dimensional gauge theory, from which one can read off the bare string tension (always in units of α')

$$\mathcal{T}_{\text{YM}} \;=\; \frac{\rho_T^2}{R^2} \sim T_H^2 \;.\tag{3.62}$$

We immediately see a potentially serious problem with this construction. In the most optimistic scenario, one would like to compute the renormalized string tension by taking the limit $T_H \to \infty$ while the p dimensional coupling $\hat{\lambda}_p$ goes to zero as $1/\log(T_H/\Lambda_{QCD})$. So far, this computation has been out of reach even for the standard Type IIB construction. At least in that case, however, one has two truly independent bare parameters to vary, namely T_H and $\hat{\lambda}_p$. Here instead, the relation $\hat{\lambda}_p = \lambda_{p+1} T_H$ and the fact that λ_{p+1} is fixed to be of order one by the equations of motion forces $\hat{\lambda}_p \sim T_H$.

Finally, we address the question of whether a mass gap will emerge in the p dimensional theory (at zero temperature), consistently with the area law found above. As explained in [20, 57] it will be sufficient to study the equation of motion of a quantum field propagating in the background given by (3.43) and determine its spectrum in the p-dimensional sense. Thus, let us consider the dilaton equation of motion. In spite of the presence of the cosmological constant, the constant dilaton background renders the fluctuation field effectively massless (in d dimensions), so that we must still solve for

$$\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\delta\Phi) \;=\; 0 \;,\tag{3.63}$$

and we search for solution of the form $\delta\Phi = \chi(\rho)e^{ikx}$, $x \in \mathbb{R}^p$.

After defining $y = \frac{\rho}{\rho_T}$ the equation of motion for χ following from (3.63) is

$$\partial_y\left[(y^{p+2} - y)\partial_y\chi\right] + \rho_T^{-2}R^4 M^2 y^{p-2}\chi \;=\; 0 \;,\tag{3.64}$$

$M^2 = -k^2$ being the mass squared of the glue-ball. $M \sim T_H$ as it should, since the bare mass scales with the UV cutoff, the Hawking temperature in this case.

Thus, it is straightforward to repeat the arguments of [57] and conclude that the eigenvalue problem (3.64) has normalizable solutions only for discrete and strictly positive values of M^2. In fact (3.64) actually reduces to the equation appearing in [57] for $p = 3$, while for $p = 4$ we also obtain a mass gap for four dimensional gauge theory, in accordance with the area law.

3.6   Holographic Type 0 RG flows

In this section we anticipate a subject that will be treated later in this thesis. We will use the framework of Type 0 gravity to introduce holographic renormalization group

flows. The construction of gravity solutions describing RG flows is quite general. Given an effective potential for the scalars and the existence of AdS solution (at extrema of the potential), the procedure is simple. In fact the equations of motion for the coupled scalars plus gravity system provide the RG equations for the couplings in the dual theory, once an appropriate ansatz for the metric is inserted in as a physical input. There are some subtleties, concerning acceptability of the solutions if one address the problem in Lagrangian second order language, that we will discuss later.

The main assumption here will be that Eq. (3.14) has more than one solution, say T_1, T_2 at least. This means that exist two AdS solutions, with fixed values for scalars, and radii $R_2 > R_1$, say. The stability analysis as applied to the critical points of the potential yields local information about the behavior of the dual field theory near its fixed points. Depending on the form of the potential, there may exist gravity solutions that interpolate between different critical points. In order to study these interpolating solutions we consider the ansatz

$$ds^2 = dy^2 + A^2(y) \, dx_{\parallel}^2 \tag{3.65}$$

and allow the two scalars to depend on the Liouville coordinate y. We already know from Section 3.2 that there are exact solutions of the form

$$A(y) = e^{y/R_0} , \tag{3.66}$$

where R_0 is the radius of the pertinent AdS space.

The Einstein equation gives rise to two independent equations. Defining the following auxiliary function

$$\gamma(y) = (d-1) \frac{d}{dy} \log(A) , \tag{3.67}$$

the full set of equations (3.5–3.8) takes the form

$$\frac{\ddot{A}}{A} + (d-2) \left(\frac{\dot{A}}{A} \right)^2 = \frac{1}{d-2} \mathcal{V} \tag{3.68}$$

$$\ddot{\vec{\Phi}} + \gamma \dot{\vec{\Phi}} = -\vec{\nabla}\mathcal{V} \tag{3.69}$$

$$\dot{\gamma} = -\frac{d-1}{2(d-2)} (\dot{\vec{\Phi}})^2 \leq 0 . \tag{3.70}$$

Here we denote derivatives with respect to y with a dot, and we have introduced the compact notation $\vec{\Phi} = (\Phi, T)$ for the two scalars.

Provided $\gamma \geq 0$, equation (3.69) has the physical interpretation of a particle moving on a plane in the potential $\mathcal{V}$, subject to a friction force. Let us assume that

the potential has two critical points $\vec{\Phi}_1$ and $\vec{\Phi}_2$ that satisfy $\mathcal{V}(\vec{\Phi}_1) > \mathcal{V}(\vec{\Phi}_2)$, and that there is at least one unstable direction at $\vec{\Phi}_1$ for increasing y and, similarly, a stable direction for $\vec{\Phi}_2$. This can always be arranged by choosing the $\mathcal{O}(T^4)$ part in $V(T)$ suitably. Due to the friction coefficient we expect our particle to roll down starting from the IR fixed point, and to converge in an infinite amount of time towards the lower UV fixed point. This happens, since γ is strictly positive: Indeed, at the critical points γ approaches the values

$$\gamma \;\to\; \frac{d-1}{R_1} \quad \text{for } y \to -\infty \tag{3.71}$$

$$\gamma \;\to\; \frac{d-1}{R_2} \quad \text{for } y \to +\infty \;, \tag{3.72}$$

and the friction coefficient decreases monotonously between them according to (3.70). This is consistent with the fact that

$$R_{1,2}^2 = \frac{(d-1)(d-2)}{\mathcal{V}(\vec{\Phi}_{1,2})} \tag{3.73}$$

as follows from (3.68).

The solution might be oscillatory near the UV critical point. Whether this happens depends on whether the friction is enough to stop the particle as it arrives at the lower point. Clearly, if one wants to interpret the result as an RG flow, the oscillatory behavior would be difficult to accommodate in the field theory picture. These issues can be most properly addressed in the framework of Hamilton–Jacobi theory. In Chapter 4 we will deal with an example where these subtleties are resolved going to Hamiltonian first order equations. Here we just notice that, already at the level of second order Lagrangian equations, sensible consistency conditions arise. Namely, the oscillatory solutions are exactly those that would violate the bound (3.21), which is necessary for consistency of the system on the gravity side. Hence, quite remarkably, the stability in the gravity theory is dual to the consistency of the field theory interpretation.

Some universal information can be read from the local behavior of these solutions. In the spirit of the Wilsonian RG treatment, let us study the critical behavior near the two fixed points in the linearized approximation. We must first identify the appropriate coordinate which in field theory can be interpreted as the energy scale and parameterizes the interpolating solution. Such a coordinate can be chosen to be[8]

$$U \;=\; \frac{A^2}{\dot{A}} \tag{3.74}$$

---

[8]This definition corresponds locally, near the fixed points, to the one used in [18]. However, there are alternative definitions. For instance choosing $U = \dot{A}$ one obtains the holographic relation, cf. [67]. All of these definitions lead to the same universal quantities.

since at the critical points this reduces to $U = R_0 \, e^{y/R_0}$, where the metric takes the standard form

$$\mathrm{d}s^2 = \frac{R_0^2}{U^2}\mathrm{d}U^2 + \frac{U^2}{R_0^2}\mathrm{d}x_\parallel^2 \ . \tag{3.75}$$

Define

$$\vec{\Phi}(U) \;=\; \vec{\Phi}_0 + \delta\vec{\Phi}(U) \ , \tag{3.76}$$

so that (3.35) takes the form

$$\left[-\frac{1}{R_0^2}[d\,U\partial_U + U^2\partial_U^2] + \mathcal{M} \right] \delta\vec{\Phi} \;=\; 0 \ . \tag{3.77}$$

The eigenvalues of $\mathcal{M}$, namely m_i^2 are given in (3.37), and for each of the two eigenvectors we get two linearly independent solutions

$$\delta\tilde{\Phi}_i \;=\; A_i U^{\lambda_+^i} + B_i U^{\lambda_-^i} \ , \tag{3.78}$$

where

$$\lambda_\pm^i \;=\; \frac{-(d-1) \pm \sqrt{(d-1)^2 + 4m_i^2 R_0^2}}{2} \ . \tag{3.79}$$

Notice first that the stability condition (3.21) ensures the reality of the roots and they only depend on the dimensionless parameters x, y, and z.

The IR limit corresponds to taking $U \to 0$, for which there must be at least one positive eigenvalue, say, $(m_1^{IR})^2 > 0$. In order for the solution not to blow up at this point we must choose $B_1^{IR} = B_2^{IR} = 0$. If $(m_2^{IR})^2 < 0$ we must also set $A_2^{IR} = 0$, otherwise, the trajectory may in general start with a linear combination of the two eigenvectors. The trajectory will then evolve to the UV fixed point as $U \to \infty$ where there will be at least one negative mass eigenvalue, say $(m_1^{UV})^2$. Generically, both the coefficients A_1^{UV} and B_1^{UV} will be non-zero and the root λ_+^1 will dominate.

We can now read off the leading order behavior of the β-functions near the fixed points:

$$\beta_i(g_i) \;=\; U\frac{\mathrm{d}g_i}{\mathrm{d}U} = \lambda_+^i(g_i - g_i^*) + \ldots \ , \tag{3.80}$$

where $g_i = \tilde{\Phi}_i$. In particular, the conformal dimension of an operator coupled to the bulk field $\tilde{\Phi}_i(U,x)$ (a linear combination of the original tachyon and dilaton) is $\Delta_i = d - 1 + \lambda_+^i$ and the anomalous dimension is λ_+^i.

Note that, if λ_+ vanishes, then we should have $A = 0$. This represents a vacuum expectation value for the dual operator [68, 69], as will be explained in Chapter 4.

3.7 Conclusions

In this chapter we have illustrated how Type 0 string theory can be used to apply the AdS/CFT correspondence to non-supersymmetric cases. This method offers some advantages compared to others approaches, however, it has some limitations as well. Let us summarize which are the relevant features.

In general, extracting non trivial results when no supersymmetry is present is a difficult task. This is not, however, a drawback peculiar to Type 0. On the other hand, what is still lacking in the Type 0 approach is a satisfactory proof of tachyon condensation. It is interesting to find that, in the presence of RR fields, the equation for its condensation (3.14) has quite different features from the usual condition requiring a minimum of the potential. In particular, (3.14) could also be satisfied if the potential was unbounded from below.

The results discussed are valid to first order in α'. In the case of Type IIB the dilaton and the AdS radius are not fixed, thus allowing one to let R becoming arbitrary large, in units of α', so that higher order corrections to Einstein equations are subleading. On the contrary, the conformal solutions of Type 0 gravity have fixed parameters, of order $\mathcal{O}(1)$ in α', so that corrections are usually important.

As far as the non-critical string is concerned, one should remember that a microscopic description is still to be developed due to the complications caused by the Liouville field in $d < 10$ which induces a two-dimensional gravitational dressing in the vertex operators. Thus computing scattering amplitudes becomes a challenging problem. Nevertheless, from the point of view of Polyakov's path-integral, in computing the partition function with background fields, one can think of the Liouville mode as an additional coordinate in space-time. Some progress in Liouville theory is however desirable, as this should be obviously the most proper framework where to address the problem in its full generality [27].

There are some interesting features of the Type 0 approach, emerging from the analysis we performed. Note for instance that the results are in general disentangled from higher dimensional physics, in contrast to, say, the finite temperature approach where the KK modes are of the same order as the physics scale one is interested in. We have also seen that confining and asymptotically free solutions have been found in the literature [29]. And, most interestingly, we have learned that there exist some new conformal solutions [22, 56, 53, 27]. These solutions could be just an artifact of the approximation, but they could also indicate some novel field theories or strongly coupled conformal fixed points in ordinary Yang–Mills theories as suggested in [22].

Yang-Mills theory in more than four dimensions is perturbatively non renormalizable — however, seen from the point of view of the ϵ-expansion, the $4 + \epsilon$ theory has a phase transition at a finite value of the bare coupling constant, analogous to the three dimensional fixed point of the $O(N)$ non linear sigma model as seen in $d = 2 + \epsilon$ dimensions. In the strong coupling phase, the theory behaves, at low

energies, in a way similar to four dimensional Yang-Mills. It may happen that this is the phase relevant to the string theory description.

Another way to argue the existence of such fixed points, not moving "up" from four dimensions using field theory, but "down" from ten using string theory, is that of [27]: Were it not for the tachyon shift of the central charge, we would have

$$\lambda_{d-1}^2 \quad \sim \quad 10 - d \, . \tag{3.81}$$

Thus, for $d = 10 - \epsilon$, the radius of curvature (3.41) becomes large, and the above approximation is justified. This solution should describe YM theory with an adjoint scalar, in $9 - \epsilon$ dimensions. Restoring the tachyon, one can still hope that $c_{\text{eff.}}$ becomes small decreasing along the RG flow, á la Zamolodchikov.

We also point out that this kind of approach is potentially predictive. Consider in fact formula (3.79) for scaling dimensions of dual operators depending, trough $m^2 R_0^2$, only on x, (3.31) and τ, (3.38). First, note that the tachyon expectation value T_0 behaves as a "bare" quantity: It always appears dressed by some function, trough "universal" ratios and thus it does not enter in determining physical quantities. One can do a field redefinition, shifting T_0, without affecting λ, R_0, Δ.

This is very similar in spirit to what happens in the renormalization group. In studying a physical system near a second order phase transition, the details of the microscopic theory do not really matter. One can have different beta functions depending on its modeling, as well as different numerical values for the coupling constants. Nevertheless, the universal features are encoded in the zero's of the beta functions, some invariant ratios, and a set of critical exponents. If one performs an experiment measuring these quantities, then unambiguously identifies the physical system.

In the same spirit, here, as the scaling dimensions depend on a *finite* number of parameters, one can do an appropriate "experiment" in order to fix them. In fact, one can do a detailed Kaluza–Klein spectroscopy and find relations among the exponents, analogous to the relations among critical exponents following from thermodynamic principles. Thus, one can try to guess a suitable quantum field theory displaying these relations, and in principle predict an infinite tower of scaling dimensions. The drawback of this procedure is that, due to our incomplete knowledge of α' corrections one is only obtaining approximate values for the scaling dimensions and a comparison may well be practically impossible.

Finally, let us notice that, applying the holographic correspondence, some understanding of the dual field theory at this conformal point may shed light on Type 0 and on non-critical string theory.

Chapter 4

From AdS/CFT to Holography

4.1 Generalities on holographic flows

In the present chapter we will see, with the help of explicit examples, how the idea of dual renormalization group flows works. The subject was already touched in Section 3.6, in the context of Type 0 gravity. There, we introduced the setup in its simplest form, showing how, under some circumstances, the second order gravity equations manage to reproduce the first order renormalization group equations. The relevant formulae, for two scalars fields, are Eqs. (3.65) – (3.72), with the understanding that $\mathcal{V}$ can be a generic potential. Here we will extend the discussion on holographic flows, recalling some of their universal features and pointing out some subtleties, clarified by using examples.

In the Type 0 case, due to lack of knowledge of microscopic parameters, we were not able to perform quantitative checks of the duality. On the other hand, if one considers some supersymmetric situation, more detailed comparisons are possible, and in fact have been performed in the literature in various cases. The hope is that one can eventually relax supersymmetry and make contact with the real world. But, as usual, this goal is highly non trivial. A good starting point can be a supersymmetric theory, whose supersymmetry is then somehow broken, making the dual renormalization group flows an interesting possibility.

Soon after the work of Maldacena [18], it was realized that the correspondence between a supergravity solution and a conformal field theory can be naturally extended to non-conformal cases. This feature was in fact already implicit in [19], as the relevant boundary conditions for the fields were given at a finite cutoff scale. Instead of going immediately inside the "throat", one can study the dilaton equation at a finite transverse coordinate, and interpret this as an energy scale of the dual theory [70, 71].

Typically, these non-conformal solutions represent deformations of a given AdS one. In fact, as the latter occur at critical points of some effective potential, the

deformed solutions have instead a non trivial dependence on the bulk fields as this superpotential varies, and approach the AdS one near the critical points. In the dual field theory language the running of the parameters is due to perturbation of the conformal point Lagrangian by some relevant or irrelevant operator. In this respect, supersymmetry, is usually the only tool at disposal to unambiguously identify the operators that drive the flows. They fall into irreducible representations of a superconformal group — the same group that classifies the bulk fields spectra on the supergravity side.

These domain-wall interpolating solutions can preserve some fraction of supersymmetry of the fixed point. From the point of view of the dual theory this means that the perturbation respects some supersymmetry, and it usually reflects in the fact that one can better follow the flow from one conformal theory to another. For some simple cases, namely various gauged supergravities, the vanishing of fermion variations implies a particular form of the effective potential. Namely, it turns out that this can be derived by a superpotential W, via the relation[1]

$$\mathcal{V} = \frac{1}{d-1}\partial_I W \partial^I W - \frac{1}{d-2}W^2 . \tag{4.1}$$

We will see in Sections 4.2 and 4.3 that the converse is not true. Reverting to Hamilton–Jacobi theory one uncovers that the superpotential is just a particular solution of a Hamilton–Jacobi equation. The full set of solutions represents a family of different solutions of the complete gravitational equations of motion.

In this framework, the most extensively studied theory is $d = 5$ $\mathcal{N} = 8$ gauged supergravity [72, 73], see for instance [74, 75, 76][2]. This theory contains forty two scalars parameterizing a $E_{6(6)}/USp(8)$ coset, which have an $SO(6) \times SL(2, \mathbb{R})$ invariant potential. The first factor is the gauged symmetry, whereas the second one is a global one, inherited from Type IIB. This theory is believed to be embedded in Type IIB supergravity, namely it should be a consistent truncation of Type IIB on the sphere S^5. Therefore, by definition, it has AdS$_5$ as a classical solution, and turning on some scalar field, one obtains solutions describing deformations of $\mathcal{N} = 4$ super Yang-Mills.

In general, the supergravity potential maps out the phase diagram of the given CFT under perturbations. Thus, old, and new [80], results on the subject can be readily used to straightforwardly get informations about these CFT deformations. For instance, using the results in [80], one sees the following CFT's be obtained as mass deformations of $\mathcal{N} = 4$ SYM: 1) Three non-supersymmetric theories with global symmetries $SU(3) \times U(1)$, $SO(5)$, and $SU(2) \times U(1)^2$ respectively. The

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[1]Derivatives with indices I are with respect to the scalar fields present in the theory.

[2]Similar techniques have also been applied to deformations of Type IIB solutions itself [77, 78, 79].

corresponding critical points are however unstable. 2) A stable theory with $\mathcal{N} = 1$ and $SU(2) \times U(1)$ global symmetry group.

Among general features of holographic flows there is the existence of a C–theorem. Recall the discussion on page 49 of Section 3.6. We proved that the function γ there defined is strictly positive along the flow from one critical point to another, without assuming any particular form of the potential. Thus, under quite general conditions, there exists a quantity monotonously decreasing towards the infra-red along the flow, as the Zamolodchikov C–function. This observation has been made more precise in [75]. There, it is proven that given a) Poincaré invariance in $d - 1$ dimensions, i.e. the kink ansatz (3.65) for the metric, and b) the "weaker energy condition"[3], then the monotonicity relation follows.

Starting from a CFT, flows towards non-conformal theories are also possible. In the supergravity language this means that the corresponding solution starts from a critical point and evolves indefinitely in some direction, that is a "valley" of the potential. The solution typically develops singularities in going to the IR. Whether these singularities are "good" or "bad" [81], translates to consistency of the quantum field theory.

An example of non-conformal deformation is contained in [76]: the five dimensional supergravity solution therein represents a supersymmetric deformation, in the sense explained above, induced by the mass term of the three fermions in the chiral $\mathcal{N} = 1$ multiplet. In terms of $\mathcal{N} = 1$ superfields this is the operator $\mathrm{Tr}\,(\mathrm{X}_i \mathrm{X}_j)$, belonging to the **6** of $SU(3)$. The corresponding supergravity mode belongs to the **10**→**1**+**3**+**6** decomposition of $SO(6) \simeq SU(4)$ branched to $SU(3) \times U(1)$.

The last issue we discuss here, before going into a more detailed discussion of Hamilton–Jacobi and subsequent explicit examples, is that of flows versus vacuum expectation values [68, 69]. Given a supergravity solution, it turns out that not every gravity solution corresponds to deformations of a CFT by adding an operator to the fixed point Lagrangian. Rather, some of them may correspond to a different vacuum of the same theory, where an operator has acquired a non trivial vacuum expectation value [82, 83]. In general, only a subset of solutions will admit a direct physical interpretation [81], for instance one should rule out flows in which a positive definite operator acquires a negative vacuum expectation value. To ensure a physically acceptable result one specifies the asymptotic behavior of the solutions, near the AdS boundary.

Let us explain how this is realized. Linearizing near an AdS point the second order gravity equations of motion for a scalar field[4], the asymptotic solutions are

$$\phi(y) \quad \sim \quad Ae^{-(d-\Delta_+)y} + Be^{-\Delta_+ y} , \qquad (4.2)$$

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[3]The "weaker energy condition" means the stress-energy tensor satisfies $T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$, where ζ^μ is an arbitrary null vector.

[4]For simplicity we consider the case of only one scalar field.

where

$$\Delta_+ \equiv \frac{d + \sqrt{d^2 + 4m^2}}{2} ,$$
(4.3)

and A, B are numerical constants to be determined somehow. The A-term is always dominant, because $\Delta_+ > d/2$, and when $\Delta_+ = d/2$, A and B are undistinguishable.

In studying the RG flow induced by certain operators from an UV fixed point, one needs to have a dictionary associating each operator to the appropriate field in the bulk. This requires resolving a possible ambiguity arising for a specific range of bulk masses near the stability bound [61], which amounts to making a *choice* between two theories, both in principle described by the same bulk fields. Usually, only one such theory is supersymmetric and the allowed values for the conformal weight Δ can be read off from the representations of supersymmetry. That is, *locally* at the conformal point, one assigns the operator $\mathcal{O}_\Delta$ dual to a field with mass m^2 as follows [69]:

a) If $m^2 > -d^2/2+1$ then the operator is $\mathcal{O}_{\Delta_+}$ with conformal dimension $\Delta = \Delta_+$

b) If $-d^2/4 < m^2 < -d^2/4 + 1$ then one has two options: $\mathcal{O}_{\Delta_+}$ with $\Delta = \Delta_+$ *or* $\mathcal{O}_{\Delta_-}$ with $\Delta = \Delta_-$.

Once this choice is made there is still an identification to be done. Also this left-over ambiguity is resolved by considering the theory *locally*. One in fact studies the modes propagating in AdS [68]: Non-normalizable modes provide non-trivial boundary conditions — they therefore correspond to real sources and do not fluctuate. Conversely, normalizable modes fluctuate in the bulk and thus have a boundary description in terms of quanta of the Hilbert space. They are in fact the states over which one integrates in the path-integral defining the QFT partition function

$$Z[\phi_\infty] = \int \mathcal{D}\delta\phi \ e^{iS[\phi_\infty + \delta\phi]} .$$
(4.4)

The conclusion is that the following correspondence holds

$$\phi(\vec{x}, y) \sim e^{-\Delta_+ y} \ \to \ \text{VEV}$$
$$\phi(\vec{x}, y) \sim e^{-(d-\Delta_+)y} \ \to \ \text{real deformation} .$$
(4.5)

Or, in other words, the practical criterion is simply the following, in (4.2):

$$A = 0 \ \Rightarrow \ \text{VEV of } \mathcal{O}_\Delta$$
$$A \neq 0 \ \Rightarrow \ \text{deformation by } \mathcal{O}_\Delta .$$
(4.6)

This feature can be interpreted also in terms of Hamiltonian formalism. As the QFT generating functional is identified with the on-shell bulk action $F[q]$, it must satisfy the Hamilton–Jacobi equations

$$0 = \mathcal{H}(q, \frac{\delta F[q]}{\delta q}) \tag{4.7}$$

$$p = \frac{\delta F[q]}{\delta q}, \tag{4.8}$$

where $\mathcal{H}$ is the Hamiltonian density and F is evaluated along a classical trajectory $q(t)$. Different F correspond to different solutions of the full set of Hamiltonian equations, thus including the flow equations[5]

$$\dot{q} = \{q, H\}_{\text{DB}}, \tag{4.9}$$

and therefore correspond to different initial momenta. But (4.8) is precisely the definition of the vacuum expectation value of the operator dual to the field q. In fact, to specify the evolution in phase space one must assign the initial coordinate and momenta. This means that, starting from a conformal point, one must specify both bare couplings and VEV's, which will then evolve under the flow, resulting in different end points.

The double way of writing the equations – one as a second order Lagrangian system supplemented by the zero-energy constraint and the other as a first order system written in terms of a "superpotential" – is the origin of some confusion in the implementation of the holographic RG. The name superpotential in this context is somewhat of a misnomer because, whereas the Hamilton–Jacobi equations have a continuum set of solutions parameterized by the constants of motion, only one such solution can be regarded as the superpotential arising in a supersymmetric theory of gravity. We shall reserve the name *superpotential* for the truly supersymmetric one and call all the solutions to the Hamilton–Jacobi equation *generating functions*.

In many circumstances, the few particular generating functions that can be found explicitly are precisely those that can be thought of as true superpotentials. If we are interested in flows between two fixed points of which only one (typically the UV one) appears as an extremum of the known superpotential, we cannot use the first order equations for this purpose and must revert to the Lagrangian system. In this case, it is impossible to obtain an analytical solution and it is impossible to resolve the VEV/deformation ambiguity by asymptotically expanding it near the UV fixed point. How can one decide then which of these two cases is realized? Also, does the fact that the known generating function cannot be used to connect two fixed points mean that it is useless in connection with the RG flow? Or, if it can be used, how can the same boundary operator induce different flows? We will address and solve

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[5]$\{\cdot, \cdot\}_{\text{DB}}$ denote, in general, Dirac brackets.

these problems very explicitly for a particularly simple example – $\mathcal{N} = 1$ $d = 7$ gauged supergravity [84]. This example has all the features we want to study and its simple field content allows for a clear-cut solution[6].

The remainder of the chapter is organized as follows. In the next section we introduce the use of Hamilton–Jacobi theory, by performing a detailed study of holographic flows, with particular care for the treatment of ambiguities and the twofold interpretation of classical solutions discussed above. Section 4.3 is devoted to some comments on three-dimensional CFT's arising from compactification of eleven dimensional supergravity on Einstein seven-manifolds. In Section 4.4 we discuss more extensively the Hamilton–Jacobi formalism and motivate its usefulness in relation to holographic duality. We review the structure of the holographic Callan–Symanzik equation and notice that there exists a distinguished coordinate system in which the flow equations take the familiar form involving beta-functions. We conclude with Section 4.5, where we illustrate how to implement and solve the descent equations that follow from the zero-energy constraint. In the process we will present a novel way of calculating the holographic Weyl anomaly [32]. In preparation to Chapter 5, we make then comments on how holographic Ward identities and anomalies arise.

4.2   Dual flows from 7-dimensional supergravity

The aim of this section [86] is to illustrate the application of Hamilton–Jacobi theory in deriving holographic flows. We want to study the relevant features, discussed in Section 4.1, in Hamiltonian language and explain the relation to the Lagrangian one. This can be obtained looking at the simplest situation of a scalar field coupled to gravity. For concreteness, we have chosen a system arising in seven-dimensional gauged supergravity. Let us start recalling the few facts needed about this theory. For more details see [84, 87].

$\mathcal{N}{=}1$ $d{=}7$ gauged supergravity

Seven-dimensional $\mathcal{N} = 1$ ungauged supergravity has 16 supercharges transforming in the fundamental representation of the $Sp(1) \simeq SU(2)$ R-symmetry group. The gauged extension was constructed in [84], by gauging the $SU(2)$ symmetry therein. The field content of the model consist of, in addition to the graviton and gravitino, a triplet of vectors, a complex $SU(2)$ Majorana spinor, a 3-form potential $A_{\mu\nu\rho}$, and a single scalar. Gauging requires the addition of a cosmological constant, modification of supersymmetry transformation rules, and the introduction of a potential for the scalar field. The full Lagrangian and supersymmetry variations can be found in [84]. At this stage the resulting potential does not have critical points. However,

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[6]The case of $\mathcal{N} = 1$ $d = 7$ gauged supergravity has been recently discussed in [85].

the Lagrangian can be suitably modified, with the introduction of a topological mass term for the three-form

$$\mathcal{L}_{\text{top.}} \quad = \quad -h\,\frac{i}{36}\,A \wedge F\,.$$ (4.10)

Supersymmetry requires h to be constant and an appropriate h-dependent term in the scalar potential. The full two-parameter (h,g) potential reads [87]

$$\mathcal{V}(\sigma) \quad = \quad 16\,h^2\,\sigma^8 - \frac{16}{\sqrt{2}}\,h\,g\,\sigma^3 - g^2\,\sigma^{-2}\,.$$ (4.11)

It has two extrema if $h/g > 0$, one of which supersymmetric and stable for both spin-zero and spin-two fluctuations [87]. For our purposes one combination of h and g can be eliminated by shifting ϕ and the remaining one is an irrelevant overall multiplicative constant in front of the potential. The resulting potential for $\sigma \equiv e^{-\phi/\sqrt{5}}$ reads

$$\mathcal{V}(\phi) \quad = \quad \frac{1}{4}\,e^{-8\phi/\sqrt{5}} - 2\,e^{-3\phi/\sqrt{5}} - 2\,e^{+2\phi/\sqrt{5}}\,.$$ (4.12)

There is a supersymmetric fixed point at $\phi = 0$ and a stable non-supersymmetric one at $\phi = -\log 2/\sqrt{5} \approx -0.3$.
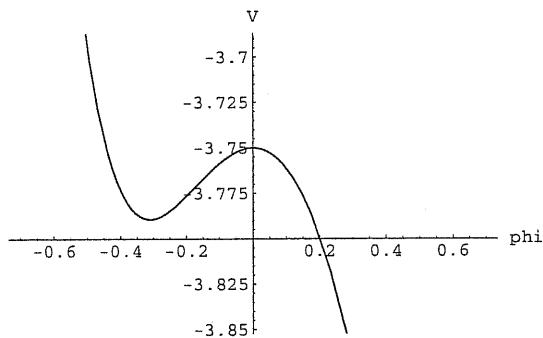

Figure 4.1: The potential $\mathcal{V}(\phi)$ plotted against ϕ.

In dual CFT language, the maximum at $\phi = 0$ corresponds to a supersymmetric UV theory, whereas the minimum at $\phi = -\log 2/\sqrt{5}$, corresponds to a non-supersymmetric but nevertheless stable IR theory. The "tachyonic" excitation near

the UV point has a mass m given, in units of the AdS radius r, by $m^2 r^2 = -8$. The boundary operator corresponding to ϕ is $\mathcal{O}_\phi = \Phi^2$, where Φ is a scalar in the tensor multiplet of the $d = 6$ CFT or, better, its still unknown non-Abelian generalization. Its conformal dimension is fixed by supersymmetry to be $\Delta = 4$. In fact it suffices to look at supermultiplets of extended $\mathcal{N} = 2$ seven-dimensional supersymmetry, say in Table 1 in [88] or Figure 2 in [89], to realize that the other possibility ($\Delta = 2$) is ruled out. In fact, $\Delta = 2$ corresponds to the singleton field Φ itself.

Six-dimensional flows

Setting all fields to zero except for the metric and the scalar ϕ, the action reads

$$S = \int d^7 x \sqrt{g} \left(\frac{1}{2} R - \frac{1}{2}(\partial \phi)^2 - \mathcal{V}(\phi) \right) . \tag{4.13}$$

As here we are interested in studying flows between different fixed point CFT's, and not *local* properties of the dual theories, *e.g.* anomalies, the equations simplify considerably. In fact, using the standard Poincaré invariant domain-wall ansatz

$$ds^2 = dy^2 + e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \text{and} \quad \phi = \phi(y) , \tag{4.14}$$

the Lagrangian equations of motion following from (4.13), can be derived by the following action for a mechanical system

$$S = \int dy \, e^{6A} \left(15\dot{A}^2 - \frac{1}{2}\dot{\phi}^2 - \mathcal{V}(\phi) \right) , \tag{4.15}$$

if we supplement them with the zero-energy constraint (4.7) following from the Hamiltonian reduction of (4.13). The resulting equations read[7]

$$\ddot{\phi} + 6\dot{A}\dot{\phi} = \mathcal{V}'(\phi) \tag{4.16}$$

$$5\ddot{A} + 15\dot{A}^2 + \frac{1}{2}\dot{\phi}^2 = -\mathcal{V}(\phi) \tag{4.17}$$

$$15\dot{A}^2 - \frac{1}{2}\dot{\phi}^2 = -\mathcal{V}(\phi) . \tag{4.18}$$

For this simple case, Eq. (4.17) can be easily shown to follow from (4.16) and (4.18). Notice however that whilst (4.16) and (4.17) correspond to Eq. (4.9), (4.18) expresses actually (4.7).

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[7] The primes denote the derivative with respect to ϕ and the dots the derivative with respect to y.

The equation for Hamilton's characteristic function $F(A, \phi, c)$ generating the canonical transformations to the cyclic coordinates is[8]

$$\frac{1}{60}\left(\frac{\partial F}{\partial A}\right)^2 - \frac{1}{2}\left(\frac{\partial F}{\partial \phi}\right)^2 + e^{12A}\mathcal{V} = 0 . \tag{4.19}$$

By substituting the ansatz $F(A, \phi, c) = e^{6A}W(\phi, c)$ into (4.19) the equation becomes the same as the defining equation for the superpotential. Altogether, expressing the canonical transformation in terms of W we end up with the first order system of Hamilton–Jacobi equations

$$\dot{\phi} = W' \tag{4.20}$$

$$\dot{A} = -\frac{1}{5}W \tag{4.21}$$

$$\mathcal{V} = \frac{1}{2}W'^2 - \frac{3}{5}W^2 . \tag{4.22}$$

Equation (4.22) is obeyed by the superpotential of the theory but it also admits a continuum of solutions, parameterized by c, that have nothing to do with super-symmetry. If one wants to recover all the solutions to the Lagrangian equations this way, one needs to consider all possible solutions to (4.22).

Particularly confusing is the fact that there are different solutions to (4.22) that have an extremum at $\phi = 0$. One solution, W_{susy}, can be easily found by inspection and identified with the superpotential[9]:

$$W_{\text{susy}} = -2e^{\phi/\sqrt{5}} - \frac{1}{2}e^{-4\phi/\sqrt{5}} . \tag{4.23}$$

The flow between the two fixed points is generated by another solution, W_{ir}, not supersymmetric and not analytic at $\phi = 0$ that can only be found numerically. The two functions are plotted for comparison in Figure 4.2. The function W_{ir} is rather tricky to find directly from (4.22) but it can be constructed *a posteriori* once the solution to the Lagrangian system (4.16), (4.17), (4.18) has been found numerically. Such a solution for ϕ_{ir} is presented in Figure 4.3 and can be easily seen to interpolate between the UV and IR fixed points. In fact, once the solution ϕ_{ir} is found, W_{ir} can be defined as

$$\begin{aligned}
W_{\text{ir}}(z) &= \int_{-\log 2/\sqrt{5}}^{z} dw \, \dot{\phi}_{\text{ir}}\left(\phi_{\text{ir}}^{-1}(w)\right) - \frac{5}{2^{1/5}\sqrt{3}} \\
&= \int_{-\infty}^{\phi_{\text{ir}}^{-1}(z)} dy \, \dot{\phi}_{\text{ir}}(y)^2 - \frac{5}{2^{1/5}\sqrt{3}} .
\end{aligned} \tag{4.24}$$

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[8] There is only one constant, c, because the other conjugate variable is set to zero by (4.18).
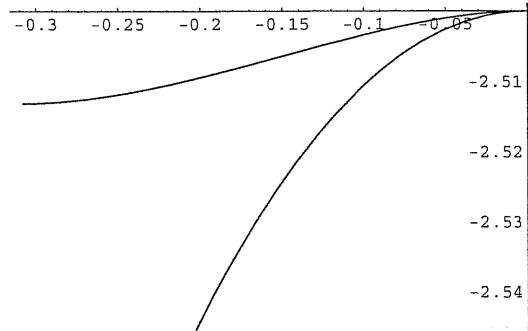[9] This is defined up to an overall unimportant sign.

Figure 4.2: Comparison of the two generating functions W_{ir} and W_{susy} between the IR and UV fixed points. Note that W_{ir} has a second extremum at the IR fixed point, whereas W_{susy} does not.
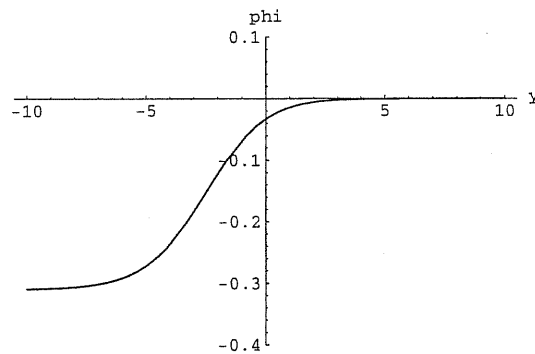

Figure 4.3: The solution ϕ_{ir} connecting the two fixed points plotted against the scale factor y.

The constant in (4.24) is chosen to agree with (4.22) at the IR point. It is interesting to analyze the behaviors of W_{susy} and W_{ir} near the origin. Obviously, W_{susy} is

analytic and

$$W_{\text{susy}}(0) = -\frac{5}{2}, \quad W'_{\text{susy}}(0) = 0, \quad W''_{\text{susy}}(0) \doteq -2, \quad W'''_{\text{susy}}(0) = \frac{6}{\sqrt{5}}, \qquad (4.25)$$

whereas W_{ir} is not analytic, since

$$W_{\text{ir}}(0) = -\frac{5}{2}, \quad W'_{\text{ir}}(0) = 0, \quad W''_{\text{ir}}(0) = -1, \quad W'''_{\text{ir}}(0) = \infty. \qquad (4.26)$$

The two solutions W_{susy} and W_{ir} act as boundaries for a continuum set of solutions that lay between them, all of which have the same behavior as $(4.26)^{10}$.

Since the second derivative of W determines whether the behavior of ϕ at $y \to +\infty$ is square-integrable or not, we see that W_{ir} gives rise to a non-square-integrable behavior, thus corresponding to deforming the fixed point Lagrangian by $\mathcal{O}_\phi$.

In fact *none* of the other solutions (including the superpotential) is physically acceptable in the region $\phi < 0$ because they would correspond to giving a negative VEV to $\mathcal{O}_\phi$, a manifestly positive operator. If we write the asymptotics of ϕ as[11]

$$\phi \sim A e^{-2y/r} + B e^{-4y/r} , \qquad (4.27)$$

the above analysis shows that $A = 0$ for W_{susy} and non-zero for the others. This is shown in Figure 4.4 for the particularly interesting case where the generating function is W_{ir}.
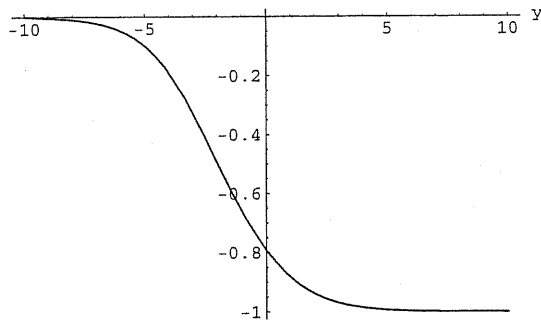

Figure 4.4: The asymptotic behavior of $\dot\phi/\phi$ shows that $\phi \approx e^{-y}$ when $W = W_{\text{ir}}$ corresponding to a true deformation by $\mathcal{O}_\phi$.

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[10]There are also solutions with only an extremum at the IR point which we do not consider.

[11]For simplicity we do not write the polynomial corrections. Also recall that $r^2 = -15/\mathcal{V}(0) = 4$.

If we take the case of W_{susy}, so that the VEV becomes the leading term, we get $B < 0$, since we are studying the region $\phi < 0$. The term B still remains negative by continuity as we use generating functions laying between W_{susy} and W_{ir} and it will reach zero at W_{ir}, precisely as A reaches zero at the opposite end (W_{susy}). B corresponds to a VEV for $\mathcal{O}_\phi$ and therefore a negative value must be excluded.

Since the leading exponential behavior for all generating functions is known explicitly, we can be more precise and analyze the differences between positive and negative ϕ. From (4.20) and (4.21), following [19] and doing the asymptotics for large $|\phi|$, one can easily see that the metric has the following behavior, (shifting the singularity to $y = 0$)

$$\phi > 0 : \quad \mathrm{d}s^2 \;=\; y^2\, \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}y^2 \tag{4.28}$$

$$\phi < 0 : \quad \mathrm{d}s^2 \;=\; y^{1/8}\, \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}y^2 \;. \tag{4.29}$$

Solution (4.29) corresponds to a naked time-like singularity and our analysis says that it should be excluded. On the other hand, the runaway solution for $\phi > 0$ is acceptable and plotted in the neighborhood of the UV point in Figure 4.5.
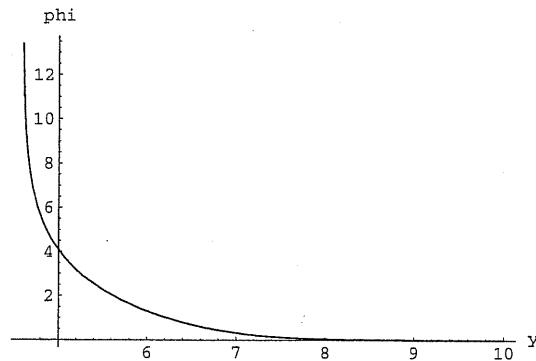
Figure 4.5: The supersymmetric runaway solution ϕ_{susy} corresponding to a non-conformal vacuum where $\langle \mathcal{O}_\phi \rangle > 0$, plotted against the scale factor y in the vicinity of the UV fixed point.

This solution corresponds to going to new non-supersymmetric vacua where $\langle \mathcal{O}_\phi \rangle > 0$, which for the special case of the superpotential corresponds to a supersymmetric vacuum. The reason we have many acceptable generating functions for $\phi > 0$ is that they simply correspond to different VEV's (or vacua), in contrast to the $\phi < 0$ case.

4.3   "Squashed" three-dimensional CFT's

The second example of holographic flows that we want to address involves a two-scalar potential. This potential encodes an effective four-dimensional theory arising in compactifying eleven-dimensional supergravity on a "non-spherical" Einstein manifold. Here the two scalar fields are higher Kaluza–Klein modes, instead of the lowest ones, as in the case of gauged supergravity.

Let us start by recalling some facts about non-standard compactifications and their conjectured dual SCFT's. With the advent of the AdS/CFT correspondence, the issue of supergravity compactifications has seen an unexpected revival. In fact, as symmetries of compactification manifolds correspond to different properties of the CFT, all work on the subject can be used to construct new (super) conformal field theories in various dimensions. The typical instances are product spaces of an AdS with an Einstein manifold, arising in Type IIB or eleven-dimensional supergravities. Compactifications on spheres are the most symmetric and lead to maximally supersymmetric super conformal field theories.

Expanding the original proposal by Maldacena [18], some non-maximally symmetric solutions can be thought of as arising on branes placed at the conical singularity of the cone $C(M)$ over the Einstein manifold M [90]:

$$ds^2|_{C(M)} \;\; = \;\; dr^2 + r^2 ds^2|_M \; . \tag{4.30}$$

This relies on the fact that solutions of the form $AdS_{p+2} \times M$ are near-horizon limits of corresponding p-brane solutions [91]. Thus the correspondence identifies the degrees of freedom living on these branes.

The first example was provided by Klebanov and Witten in [92]. By using Type IIB compactified on the five-dimensional Einstein manifold $T^{1,1} = (SU(2) \times SU(2))/U(1)$, they proposed the dual CFT to be an $\mathcal{N} = 1$ $SU(N) \times SU(N)$ gauge theory, with two chiral fields in $(\mathbf{N},\bar{\mathbf{N}}) \oplus (\bar{\mathbf{N}},\mathbf{N})$, and $U(1)$ R-symmetry.

Let us now focus on the case of M-Theory. Here the maximally symmetric compactification down to four dimensions is $AdS_4 \times S^7$ which then gives rise to an $\mathcal{N} = 8$ three-dimensional SCFT, describing the M2-brane. Supersymmetry constrains the spectrum of chiral primary operators, that can be just read off from tables of supergravity short multiplets [93]. Notice that this theory has a $Spin(8)$ R-symmetry group and its field content consists of eight bosons and eight fermions transforming in the $\mathbf{8}_v$ and $\mathbf{8}_c$ respectively. Starting with a higher dimensional theory, $e.g.$ $\mathcal{N} = 4$ $d = 4$ SYM, and reducing to three dimension, one instead finds seven scalars and eight fermions. Therefore the Yang–Mills Lagrangian can have only a $Spin(7)$ R-symmetry. It is only in the IR that this theory is conjectured to flow to the above SCFT, enhancing its symmetry to $Spin(3,2) \times Spin(8)$ [94].

Considering cases with fewer supersymmetry, as usual, leads to more possibilities, and one needs more guesswork to construct the pertinent SCFT. Such program have

been pursued by a series of papers [95, 96, 97, 98].

An intriguing case arises in considering M-Theory compactified on $N(1,1) \simeq SU(3)/U(1)$ [99]. This is a member of a class of seven-dimensional Einstein manifolds named $N(p,q)$, which has the peculiarity of preserving $\mathcal{N} = 3$ supersymmetries.

By the usual procedure, one can read off global "flavor" and R-symmetries of the dual SCFT from isometries of the corresponding solution. Once an appropriate guess for the gauge group is made, it is possible to get a mapping between CFT operators and fields of the KK spectrum, based on matching supersymmetry representations. Thus, $N(1,1)$ should give rise to a CFT with $SU(3) \times SU(2)$ global symmetries and supercharges transforming in the **3** of $SU(2)$ [12]. Following the ideas used in other cases, a suitable guess for the dual theory could be the following[13]:

- An $SU(N) \times SU(N)$ gauge theory with three copies of chiral fields, belonging to (**3,2**) of $SU(3) \times SU(2)_R$, and transforming in the (**N**,**N̄**) of the gauge group.

- While in four dimensions $\mathcal{N} = 3 \Rightarrow \mathcal{N} = 4$ for the vector supermultiplet[14], in three dimensions this is not true. The three dimensional $\mathcal{N} = 3$ vector multiplet can be obtained by dimensional reduction of the $\mathcal{N} = 2$ $d = 4$ one. It is possible then to reduce supersymmetry including a Chern–Simons interaction [100] (Sec. 5 therein)[15], as in the Abelian theory [101].

Deformations of $\mathcal{N}$=3 and $\mathcal{N}$=8 CFT$_3$

In the remainder, we return to the main subject of this chapter, namely we want to address the issue of supergravity solutions, which are deformation of AdS$_4 \times M$, M being S^7 and $N(1,1)$. The reason why we concentrate on these cases is that these two compactification manifolds exist in two versions, the one with fewer supersymmetry being called a "squashed" manifold. This raises the possibility of having RG flows between the two theories driven by the scalar fields corresponding to the squashing parameter [102, 103, 86].

Squashed metrics are Einstein metrics, obtained by stretching the original one in some directions. Both the squashed 7-sphere $\tilde{S}^7$ and $\tilde{N}(1,1)$ have one or zero supersymmetries, according to their left of right orientation[16].

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[12] See [99] for a comprehensive explanation of properties of seven-dimensional Einstein manifolds.

[13] A proposal for the dual theory is appeared also in [98].

[14] Looking for instance at the tables of 4-dimensional supersymmetry representations (page 18 of [38]), it is seen that the $\mathcal{N} = 3$ CPT-invariant vector multiplet coincides with the $\mathcal{N} = 4$.

[15] Tanks to A. Tomasiello for driving our attention to this paper.

[16] Actually, for $N(1,1)$, the left-handed unsquashed and squashed solutions have $\mathcal{N} = 3$ and $\mathcal{N} = 0$, whereas the right-handed ones have $\mathcal{N} = 0$ and $\mathcal{N} = 1$ respectively [99]. So that interpolating solutions between SCFT with one and three supersymmetries cannot exist.

Applying the AdS/CFT correspondence, these solutions ought to correspond to some conformal limit of three-dimensional QFT, as well. However, for squashed (supersymmetric) solutions, having $\mathcal{N} = 1$ in three dimensions there is no R-symmetry and the usual procedure does not apply straightforwardly. Whilst $\tilde{S}^7$ has $SO(5) \times SO(3)$ isometries, the case of $N(1,1)$ is particularly puzzling as squashed and unsquashed solutions share the same global symmetries.

To study the possibility of having domain-wall solutions interpolating between such theories one considers a truncation of the Kaluza–Klein spectrum and derives an effective four-dimensional action for the non-zero fields. The potential for the sphere is known from the work of [104] in terms of two scalars u and v appearing in the eleven dimensional metric as

$$ds^2 = e^{-7u}ds^2(\text{AdS}_4) + e^{2u+3v}ds^2(\text{base}) + e^{2u-4v}ds^2(\text{fibre}) , \qquad (4.31)$$

where the seven-sphere is thought of as a S^3 fibration over the base S^4. The potential for the squashed $N(1,1)$ has been given in terms of four scalars in [105] but for our purposes it is sufficient to repeat the computation of [104] using an ansatz similar to (4.31) , where now the base manifold is $\mathbb{CP}^2$ and the fiber is $\mathbb{RP}^3$, thus obtaining a potential also dependent only on two scalars. We postpone the explicit calculation until the end of the section, and continue here discussing some of its possible solutions.

It turns out that in both cases the potential can be written as

$$\mathcal{V}(u,v) \;=\; \lambda e^{-9u}\left(\alpha e^{4v} - e^{-3v} - \frac{1}{32\alpha}e^{-10v}\right) + 2Q^2 e^{-21u} , \qquad (4.32)$$

where Q is the Page charge and, for the sphere, $\alpha = -1/8$ and $\lambda = 48$, whereas, for $N(1,1)$, we have $\alpha = -1/16$ and $\lambda = 24$. Amusingly, all the physical quantities, such as the conformal dimensions for the operators, turn out to be independent of α and λ.

The potential (4.32) has two fixed points but the field u always describes a non-renormalizable (irrelevant) operator. From the equivalent mechanical problem, the flow between these two points would have to connect a maximum of $-\mathcal{V}$ to a saddle point of $-\mathcal{V}$, clearly an unstable situation – contrary to the situation occurring for some flows in $d = 5$ gauged supergravity, where the "particle" rolls along a valley from a saddle point to a minimum.

If the RG equations where truly first order, one could argue from general theorems that there must still be a critical line connecting the points. However, the equations expressed in terms of the potential are second order and there is no guarantee that such a solution will survive. After some numerical test we now believe that there is no such flow.

The potential (4.32) has another peculiar property: It is possible to find explicitly one generating function W that has one critical point at the squashed solution[17]. The function is[18]

$$W(u, v) = -\frac{1}{\sqrt{8}}e^{-\frac{9}{2}u}\left(3e^{2v} + 6e^{-5v} - |Q|e^{-6u}\right),$$ (4.33)

which is a solution to

$$\mathcal{V} = \frac{16}{63}(\partial_u W)^2 + \frac{8}{21}(\partial_v W)^2 - 12W^2.$$ (4.34)

At first, it seems rather counterintuitive that the point with *less* supersymmetries should appear as an extremum, but we must remember that we are not dealing with a gauged supergravity, where only low-lying KK excitations are included. Still, it is tempting to believe that the solution associated with W describes different *supersymmetric* vacua of the theory. As a check one can show, expanding W near its critical point, that the solution corresponds to the operator associated to v getting a VEV – more specifically $v \sim \exp(-5y/3r)$. There is a choice between a theory in which the conformal dimension of the operator is $\Delta = 5/3$ or $\Delta = 4/3$, which is also allowed [69]. Finally, one finds that the runaway solution also satisfies the criterion [19] of boundness from above of the potential.

Computation of the effective action

We conclude with the explicit evaluation of (4.32), in the case of $N(1,1)$. We use the notation of [106] where the squashed metric was first found, and follow [104] in computing the effective 4-dimensional action. The starting point are eleven-dimensional supergravity bosonic equations of motion [107, 99]

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3}\left[F_{MPQR}F_N^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}\right]$$ (4.35)

$$\nabla_M F^{MNPQ} = -\frac{1}{576}\epsilon^{M_1...M_8 NPQ}F_{M_1...M_4}F_{M_5...M_8}$$ (4.36)

$$F_{MNPQ} = 4\nabla_{[M}A_{NPQ]}.$$ (4.37)

Choosing the Freund–Rubin ansatz for the field strength, *i.e.* letting it be proportional to the volume form of four dimensional space, and zero all other components

$$F_{\mu\nu\rho\sigma} = Q\,\epsilon_{\mu\nu\rho\sigma},$$ (4.38)

---

[17]W has no fixed point at the unsquashed vacuum, or else a solution connecting the two would exist.

[18]We take the case of the sphere for concreteness. For the $N(1,1)$ case the factor 3 in front of e^{2v} is changed to 3/2.

(4.36) and (4.37) are automatically satisfied, (4.35) gives the following equations for $R_{\mu\nu}$ and R_{mn}

$$R_{\mu\nu} = -\frac{4}{3}Q^2 g_{\mu\nu} \tag{4.39}$$

$$R_{mn} = +\frac{2}{3}Q^2 g_{mn} , \tag{4.40}$$

while $R_{\mu n}$ can be set to zero. Let us make the following ansatz for the metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu + e^{2\xi}\left\{ d\mu^2 + \frac{1}{4}\sin^2\mu\left(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\,\sigma_3^2\right)\right\}$$
$$+ e^{2\eta}\left\{ (\Sigma_1 - \cos\mu\sigma_1)^2 + (\Sigma_2 - \cos\mu\sigma_2)^2 + \left(\Sigma_3 - \frac{1}{2}(1+\cos^2\mu)\sigma_3\right)^2\right\} \tag{4.41}$$

$\{\sigma_i\}$ is a set of one-forms satisfying the $SU(2)$ Lie algebra

$$d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k , \tag{4.42}$$

which together with $d\mu$ parameterize the space $\mathbb{CP}^2$ [108]. Analogously, $\{\Sigma_i\}$ is a set of left-invariant one-forms on the fiber $SO(3) \simeq \mathbb{RP}^3$, such that

$$d\Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma_j \wedge \Sigma_k . \tag{4.43}$$

ξ and η depend only on four-dimensional noncompact coordinates. In the limit in which they are constant one recovers the two Einstein metrics for the compact space, inserting $\xi = 0$ and $e^{2\eta} = 1/2, 1/10$.

Rewriting (4.41) in terms of one-forms $\theta^A = (\theta^\alpha, \theta^{\underline{a}}) \equiv (\theta^\alpha, \theta^a, \theta^i)$

$$ds^2 = \eta_{\alpha\beta}\,\theta^\alpha\,\theta^\beta + \delta_{\underline{ab}}\,\theta^{\underline{a}}\,\theta^{\underline{b}} , \tag{4.44}$$

we compute the connection one-form using first Cartan equation[19]

$$d\theta^A + \omega^{AB} \wedge \theta^B = 0 , \tag{4.45}$$

which splits in

$$d\theta^\alpha + \omega^{\alpha\beta} \wedge \theta^\beta + \omega^{\alpha\underline{b}} \wedge \theta^{\underline{b}} = 0 \tag{4.46}$$

$$d\theta^{\underline{a}} + \omega^{\underline{a}\beta} \wedge \theta^\beta + \omega^{\underline{ab}} \wedge \theta^{\underline{b}} = 0 . \tag{4.47}$$

---

[19]Here we adopt the following notation: $M, N, \ldots$ for 11-dimensional curved indices. $A, B, \ldots$ for 11-dimensional flat indices. $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ for curved and flat 4-dimensional indices respectively. $a, b = 7, \ldots, 10$ and $i, j = 4, 5, 6$ for flat 7-dimensional indices, otherwise altogether denoted with underlined indices $\underline{a}, \underline{b}, \ldots$. Curved seven-dimensional indices are denoted with m, n .

Making the following ansatz for ω, in terms of the four-dimensional vielbein $e^{\alpha}{}_{\mu}$

$$\omega^{\alpha b} = -e^{\alpha}{}_{\mu} \, \partial^{\mu} v \, \theta^{b} \tag{4.48}$$

$$\omega^{\alpha i} = -e^{\alpha}{}_{\mu} \, \partial^{\mu} u \, \theta^{i} \, , \tag{4.49}$$

Eq. (4.46) reduces to an equation for the 4-dimensional connection

$$d\theta^{\alpha} + \omega^{\alpha\beta} \wedge \theta^{\beta} = 0 \, , \tag{4.50}$$

whereas (4.47) gives

$$\omega^{\underline{ab}} \wedge \theta^{\underline{b}} = H^{\underline{abc}} \, \theta^{\underline{b}} \wedge \theta^{\underline{c}} \, , \tag{4.51}$$

from which $\omega^{\underline{ab}}$ is computed as

$$\omega^{\underline{ab}} = (H^{\underline{abc}} - H^{\underline{bac}} - H^{\underline{cab}}) \, \theta^{\underline{c}} \, . \tag{4.52}$$

The tensor $H^{\underline{abc}}$ is defined by

$$e^{\xi} d(e^{-\xi} \theta^{a}) = H^{\underline{abc}} \, \theta^{\underline{b}} \wedge \theta^{\underline{c}} \, .$$
$$e^{\eta} d(e^{-\eta} \theta^{i}) = H^{\underline{ibc}} \, \theta^{\underline{b}} \wedge \theta^{\underline{c}} \, . \tag{4.53}$$

We then compute the curvature two-form R_{AB} using the torsionless second Cartan equation

$$R^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega^{CB} \, , \tag{4.54}$$

which again splits in three equations for $R_{\alpha\beta}$, $R_{\underline{ab}}$, and $R_{\underline{ab}}$ respectively, and extract the Riemann tensor. $R_{\underline{ab}}$ is consistently set to zero and the other components are inserted in (4.39) and (4.40) respectively. In order to put these equations in a particularly simple form we still need to do a Weyl rescaling of the metric

$$\bar{g}_{MN} = e^{2\sigma} \, g_{MN} \, . \tag{4.55}$$

Finally, setting $2\sigma = -3\xi - 4\eta$, the resulting equations read

$$12 \, \partial_{\mu}\eta\partial_{\nu}\eta + 6 \, \partial_{\mu}\xi\partial_{\nu}\eta + \frac{15}{2} \, \partial_{\mu}\xi\partial_{\nu}\xi + \frac{1}{2} \, \bar{g}_{\mu\nu}\mathcal{V} = \bar{R}_{\mu\nu}$$
$$15 \, \Box\xi + 12 \, \Box\eta = \partial_{\xi}\mathcal{V}$$
$$24 \, \Box\eta + 12 \, \Box\xi = \partial_{\eta}\mathcal{V} \, , \tag{4.56}$$

with the effective potential

$$\mathcal{V}(\xi, \eta) = -24e^{-6\eta - 3\xi} + 12e^{-8\eta - \xi} - \frac{3}{2}e^{-4\eta - 5\xi} + 2Q^{2}e^{-12\eta - 9\xi} \, , \tag{4.57}$$

where Q is the conserved Page charge [104]. The above equations can be clearly derived from an action principle for the four-dimensional metric and the two scalar fields. The resulting kinetic term can be diagonalized by a last change of variables

$$\begin{cases} \xi & = & u - 2\,v \\ \eta & = & u + \frac{3}{2}\,v \,. \end{cases}$$

The effective action is finally

$$S \;=\; \int \mathrm{d}^4x \left[\bar{R} - \frac{63}{2}(\partial u)^2 - 21(\partial v)^2 - \mathcal{V}(u,v) \right] , \tag{4.58}$$

with the potential given in (4.32).

4.4 Hamilton–Jacobi theory and Holography

In this section we motivate why using Hamilton–Jacobi treatment can be fruitful for studying some interesting issues such as holographic anomalies and the renormalization program, in addition to holographic flows discussed so far.

The central object in the holographic correspondence is the generator of connected Green functions Z

$$\mathrm{e}^{-Z[g,J]} \;=\; \int \mathcal{D}\Phi \; \mathrm{e}^{-S[g,\Phi]-J\mathcal{O}[\Phi]} \tag{4.59}$$

where J is, on the QFT side, a set of external sources coupled to composite operators $\mathcal{O}$ and g_{ij} is the background metric, which allows one to compute the stress-energy tensor. The conjecture then states that

$$Z[g,J] \;=\; S_{\text{SUGRA}}[g,J] \tag{4.60}$$

where the on-shell supergravity action is evaluated on a classical solution, and it is a functional of initial values of the various fields ϕ appearing in it: $J = \phi\,|_{\text{boundary}}$.

In quantum field theory Z obeys a set of Ward identities, which ensure that classical symmetries are preserved at the quantum level and, consequently, constrain the counter terms that can be used in renormalization. For instance, the Ward identity of (broken) scale invariance is the Callan–Symanzik equation, which is a particular form of RG equation.

As Holography relates theories in d and $d+1$ dimensions it is useful to resort to Hamiltonian formalism, where the transverse "time" coordinate plays a distinguished role. Moreover, as we are particularly interested in the on-shell action, it is rather natural, as proposed in [33], to use Hamilton–Jacobi theory. There are also some conceptual reasons for doing so: On the QFT side, one eventually needs

a formulation in terms of first order equations, as they can be interpreted as RG flow equations. Also, on the gravity side, it is natural to expect the Hamiltonian formulation to arise, as one considers classical gravity as a WKB limit of a quantum theory (String/M-Theory). In fact, deriving the partition function from the vacuum to vacuum amplitude, one gets in the path-integral an exponential factor containing an effective action as follows [109]

$$\mathcal{Z} = \int \mathcal{D}p\mathcal{D}q \; e^{i \int dt \, [p\dot{q} - H(p,q)]} \; . \tag{4.61}$$

Namely, it appears in terms of the phase space variables, and as such it displays automatically the Hamiltonian.

In the Hamiltonian formalism one chooses a coordinate t, with tangent vector field e_0, to parameterize the evolution of the initial value hypersurface, the "boundary". It is sufficient to characterize this hypersurface by giving its normal direction n with norm $n \cdot n = 1$. Writing the full metric

$$ds^2 = (N^2 + N_i N^i)dt^2 + 2N_i dt dx^i + g_{ij}dx^i dx^j \tag{4.62}$$

the canonical ADM gravitational action coupled to matter has the form

$$S = \int dt \left\{ p \, \dot{q} - \int d^d x \; N^\mu \mathcal{H}_\mu + \Theta_A \mathcal{G}^A \right\} \tag{4.63}$$

where $p\dot{q}$ is a shorthand for the kinetic term of the dynamical degrees of freedom, $N^\mu = (N, N^i)$ are the lapse and shift functions, which together with Θ_A are the Lagrange multipliers for the constraints[20]

$$\mathcal{H}^\perp \approx \mathcal{H}^i \approx \mathcal{G}^A \approx 0 \; . \tag{4.64}$$

These constraints generate, in the bulk theory, the flow along the e_0 direction, diffeomorphisms on the initial value hypersurface and possible additional local symmetries, such as gauge, local Lorentz or supersymmetries.

In the Hamilton–Jacobi theory one makes a canonical transformation such that the new phase space coordinates are constants of motion. In gravitational theories the generating functional F of the canonical transformation must satisfy simultaneously the constraints

$$\mathcal{H}^\perp(q, \frac{\delta F}{\delta q}) = 0 \tag{4.65}$$

$$\mathcal{H}^i(q, \frac{\delta F}{\delta q}) = 0 \tag{4.66}$$

$$\mathcal{G}^A(q, \frac{\delta F}{\delta q}) = 0 \; . \tag{4.67}$$

---

[20]These are first class constraints. In Chapter 5 we will consider also second class constraints. They appear in presence of fermionic fields that enter the action linearly.

The Hamilton principal function $F[q]$ is actually just the classical action evaluated at some given time t for fixed boundary values $q(t)$. The momenta can be calculated from

$$p = \frac{\delta F[q]}{\delta q} . \tag{4.68}$$

However, in order to calculate the full equations of motion, we have to consider also the flow equations for the coordinates

$$\dot{q} = \{q, H\}_{\text{DB}} . \tag{4.69}$$

We will see that the holographic correspondence will allow us to give a dual field theory interpretation to the equations above.

The guiding principle that we will follow in implementing the holographic renormalization group is the following. As in quantum field theory, also in gravity there are divergences that one should naturally regularize and subtract, before extracting physical results. It is well-known from explicit calculations that the Einstein–Hilbert action (with the cosmological constant and the Gibbons–Hawking term) is divergent when evaluated on asymptotically AdS solutions. For instance in the coordinate system

$$\mathrm{d}s^2 = \frac{1}{t^2}(\mathrm{d}t^2 + g_{ij}^R \mathrm{d}x^i \mathrm{d}x^j) \tag{4.70}$$

the determinant gives $\sqrt{g} = t^{-d}\sqrt{g^R}$, and $R_{ij} = t^2 R_{ij}^R$, so that divergences arise in the limit $t \to 0$ (asymptotically AdS boundary) for terms containing up to $[\frac{d-1}{2}]$ Ricci's. The existence of such AdS solutions in the context of holographic duality is a physical requirement: It means that the RG has fixed points. In even boundary dimension there can be additional logarithmically divergent terms that, in the Hamilton–Jacobi context, can arise only from *nonlocal* contributions. One should isolate the terms in F that are potentially divergent in the CFT limit: These terms will give rise to the beta functions, while the rest will give the renormalized generating functional.

It will turn out that – in analogy with what is achieved by QFT Ward identities – these local divergent terms, which we will call S_{div}, are fixed by the constraints in the classical theory, and can indeed be determined recursively order by order according to their degree of divergence. In fact (4.66) and (4.67) will produce Ward identities (diffeomorphism and gauge symmetries), while (4.65) provides a recursive equation for S_{div}, that eventually gives rise to a Callan–Symanzik-type equation for Z. This recursive procedure can be expressed as "descent equations" and is computationally equivalent to the counter term generating algorithm proposed in [110, 111] in the context of pure AdS gravity[21].

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[21] In that context form fields were considered in [112].

Let us illustrate how the holographic RG equations arise. For definitiveness, consider the action for scalar fields coupled to gravity[22] [33]

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\left(\frac{1}{\kappa^2}\tilde{R} + \frac{1}{2}G_{IJ}\partial^\mu\phi^I\partial_\mu\phi^J - \mathcal{V}(\phi^I)\right) . \tag{4.71}$$

The Hamiltonian reduction for this system leads to the following form of the zero-energy constraint (4.65)

$$\frac{-\kappa^2}{\sqrt{\hat{g}}}\left(g_{ik}g_{jl} - \frac{1}{d-1}g_{ij}g_{kl}\right)\frac{\delta F}{\delta g_{ij}}\frac{\delta F}{\delta g_{kl}} - \frac{1}{\sqrt{\hat{g}}}\frac{1}{2}G^{IJ}\frac{\delta F}{\delta\phi^I}\frac{\delta F}{\delta\phi^J} = \mathcal{L} , \tag{4.72}$$

where the right hand side reads

$$\mathcal{L} = \sqrt{\hat{g}}\left(\frac{1}{\kappa^2}R + \frac{1}{2}G_{IJ}\partial^i\phi^I\partial_i\phi^J - \mathcal{V}(\phi^I)\right) , \tag{4.73}$$

and depends only on d-dimensional quantities. In particular g_{ij} is defined in (4.62), and the scalar fields, though still denoted with the same symbol, are actually initial values for the bulk scalar fields.

As in [113] we define the following bracket

$$(\cdot,\cdot) \equiv \frac{-\kappa^2}{\sqrt{\hat{g}}}\left(g_{ik}g_{jl} - \frac{1}{d-1}g_{ij}g_{kl}\right)\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}} - \frac{1}{\sqrt{\hat{g}}}\frac{1}{2}G^{IJ}\frac{\delta}{\delta\phi^I}\frac{\delta}{\delta\phi^J} , \tag{4.74}$$

so that, inserting $F = S_{\mathrm{div}} + Z$ in (4.72) one gets

$$(S_{\mathrm{div}}, S_{\mathrm{div}}) + 2(S_{\mathrm{div}}, Z) + (Z, Z) = \mathcal{L} . \tag{4.75}$$

The total on-shell action F has been divided in two terms. S_{div} contains all polynomial divergent terms, whereas Z is the full non-local generating functional, encoding all n-point correlation functions. It may still contain logarithmic divergences.

The second term on the left hand side of (4.75) is a linear operator acting on Z. (Z, Z) can be regarded as a quadratic correction to the linearized RG, whereas the leftover piece gives additional non-homogeneous terms, whose leading contribution will be shown to give the Weyl anomaly. In fact this term can be solved order by order expanding S_{div}, independently of the others. We postpone its discussion until the next section, as it naturally leads to the issue of anomalies and Ward identities.

Because of cancellations achieved in the descent equations (4.88), Eq. (4.75) can be rewritten as

$$2(S_{\mathrm{div}}, Z) + (Z, Z) = \mathcal{O}(1) . \tag{4.76}$$

---

[22] The notation is that adopted in Chapter 5. Namely a tilde and Greek indices indicate always $(d + 1)$-dimensional quantities, while a hat or Latin indices are used for d-dimensional ones.

In order to get a Callan–Symanzik type equation, one can take the following steps [33]: Calculate n variations of (4.76) w.r.t. the sources and drop the contact terms containing more then one delta function, which would not contribute for operators evaluated at different points. Then letting the sources go to constant (in x_i) values, the metric assume the form

$$g_{ij} = \mu(t)\hat{\eta}_{ij} , \qquad (4.77)$$

and finally integrating once over $\mathrm{d}x^d$ one gets expressions of the form

$$\left(\mathcal{U}\mu\frac{\partial}{\partial\mu} + \beta_I\frac{\partial}{\partial\phi_I} \right) \langle \mathcal{O}_{I_1}\mathcal{O}_{I_2} \rangle + \sum_{n=1}^{2} \partial_{I_i}\beta^{J_i} \langle \mathcal{O}_I\mathcal{O}_{J_i} \rangle = \mathcal{O}(t^d \log^2 t) , \qquad (4.78)$$

where

$$\beta_I = \frac{1}{\sqrt{\hat{g}}} G_{IJ} \frac{\delta S_{\mathrm{div}}}{\delta\phi_J} \bigg|_{\substack{\phi_J = \phi_J(t) \\ g = \hat{\eta}}} \qquad (4.79)$$

$$\mathcal{U} \simeq \frac{1}{\sqrt{\hat{g}}} g_{ij} \frac{\delta S_{\mathrm{div}}}{\delta g_{ij}} \bigg|_{\substack{\phi_J = \phi_J(t) \\ g = \hat{\eta}}} . \qquad (4.80)$$

Notice that these equations are taken at a finite cutoff t, *i.e.* for *bare* quantities.

Considering the Hamiltonian flow equations (4.69) it turns out that in the appropriate gauge the transverse coordinate plays the role of a parameter for the scale transformation induced by the metric. As we have seen, in general it is not always true that a gravity solution has the interpretation of a RG flow: It depends on the leading behavior of the bulk fields near the boundary of AdS. In the Hamilton–Jacobi context this means that not every solution S_{div} of the constraints gives rise to physical flows (see Section 4.2). However, for those that are acceptable, the flow equations at leading order read, after fixing the gauge $N_i = 0$

$$\dot{\mu} \simeq N \mathcal{U} \mu \qquad (4.81)$$
$$\dot{\phi}_I \simeq N \beta_I . \qquad (4.82)$$

The scale transformation depends parametrically on the cutoff t. Let us now use (4.81) to express (4.78) in terms of the variation of t, and choose the gauge $N = +\frac{1}{t}$. This yields

$$(t\frac{\partial}{\partial t} + \beta_I\frac{\partial}{\partial\phi_I}) \langle \mathcal{O}_{I_1} \cdots \mathcal{O}_{I_n} \rangle$$

$$+ \sum_{i=1}^{n} \partial_{I_i}\beta^{J_i} \langle \mathcal{O}_{I_1} \cdots \mathcal{O}_{J_i} \cdots \mathcal{O}_{I_n} \rangle = \mathcal{O}(t^d \log^2 t) , \qquad (4.83)$$

and (4.82) can be written as

$$\beta_I \;=\; t\frac{\mathrm{d}}{\mathrm{d}t}\phi_I \;,\tag{4.84}$$

which is consistent with the definition of the beta functions (4.79) and with interpreting (4.83) as a bare RG equation. Furthermore the right hand side represents higher order terms that, in this spirit, are logarithmic corrections to scaling. This choice of gauge differs from the Fefferman–Graham [114] by a sign: The latter corresponds in fact to $N = -\frac{1}{t}$.

As is the case in QFT, the definition of the beta functions is not unique. They are fixed unambiguously near the fixed points, where their behavior is universal, but their extrapolation at intermediate RG steps is not [115]. The definition given in [33] differs from that given above by the ratio of the rate of scale change (4.81). In fact, in [33] $N = 1$ and all quantities depend on μ. However, the fixed points are not affected, because the zeros of the beta function cannot be modified by dividing by $\mathcal{U}$.

4.5   Holographic anomalies

We conclude this chapter discussing how the various Hamiltonian constraints give rise to Ward identities in the dual QFT. We focus in particular on the the zero-energy constraint, showing that it can be solved recursively for the local divergent part of the generating function F, and how this is related to the holographic Weyl anomaly of the dual field theory. We then discuss generic features of holographic Ward identities and anomalies in view of the next chapter, Section 5.5 in particular.

For the purpose of computing the Weyl anomaly it suffices to consider pure gravity with a cosmological constant. The bracket $(\cdot,\cdot)$ is thus simplified as it does not contain variations with respect to scalar fields.

We expand the local part of the generating function, S_{div}, according to the degree of divergence of each term. For the case of pure gravity this is equivalent to a derivative expansion, as is easily seen by a change of coordinates $x_i \to t^{-1}x_i$. The local terms that diverge at the UV fixed point are those that contain up to $d-1$ derivatives. Logarithmically divergent terms may arise in the nonlocal part of Z. However, the most divergent ones come always multiplying a scale invariant term. This is typical of the structure that arises in effective actions that produce a conformal anomaly [116].

Under global scalings the metric behaves as $g_{ij} \to t^{-2} g_{ij}$. Given that[23]

$$(\cdot, \cdot) \to t^d (\cdot, \cdot) \tag{4.85}$$

$$S_n \to t^{-d+n} S_n \tag{4.86}$$

$$Z \to Z - 2 \log t \, S_d \, , \tag{4.87}$$

and $(S_0, S_d) = 0$, the following descent equations have to be satisfied

$$
\begin{aligned}
-\sqrt{\hat{g}} \, \Lambda &= (S_0, S_0) \\
\kappa^{-2} \sqrt{\hat{g}} \, R &= 2 (S_0, S_2) \\
0 &= 2 (S_0, S_4) + (S_2, S_2) \\
&\vdots \\
-2 (S_0, Z) &= 2 (S_2, S_{d-2}) + \cdots + 2 (S_{\frac{d}{2}-2}, S_{\frac{d}{2}+2}) + (S_{\frac{d}{2}}, S_{\frac{d}{2}}) \, .
\end{aligned}
\tag{4.88}
$$

The zeroth order equation fixes the relationship between the cosmological constants in bulk and on the boundary, see (5.59). The n'th order equation is an equation for S_n in terms of $S_2, \ldots, S_{n-2}$. In particular, the last equation[24] gives the trace of the bare stress-energy tensor. This is because the lowest order term S_0 acts through the bracket $(\cdot, \cdot)$ essentially as a scale transformation. The trace anomaly at the conformal points, where the beta functions vanish and couplings go to their fixed point values J^*, is given by

$$\langle T \rangle = \frac{2(d-1)}{\kappa^2 \hat{\Lambda}} \lim_{J \to J^*} \frac{1}{\sqrt{\hat{g}}} (S_0, Z) \, . \tag{4.89}$$

In the UV limit this expression remains finite and reproduces the holographic Weyl anomaly of [32]. The results for $d = 2$ and $d = 4$ are

$$\langle T \rangle = \frac{1}{\kappa^4 \hat{\Lambda}} R \tag{4.90}$$

$$\langle T \rangle = \frac{k_1^2}{\hat{\Lambda}} \left(3 R^{ij} R_{ij} - R^2 \right) \, . \tag{4.91}$$

As far as the other constraints are concerned, the only one that we have to check in this case is diffeomorphism invariance (4.66). This is easily satisfied for the finite (bare) stress-energy tensor, because the counter terms are generally covariant in the boundary metric

$$\nabla_i \frac{\delta Z}{\delta g_{ij}} = -\nabla_i \frac{\delta S_{\mathrm{div}}}{\delta g_{ij}} = 0 \, . \tag{4.92}$$

---

[23]The subscript counts the number of derivatives. See the main text.

[24]In even boundary dimension. In odd dimensions it fixes also higher order terms, and nonlocal logarithmic terms do not arise, as is well known.

This means that the diffeomorphism Ward identity of the dual theory is not anomalous, as one expects.

In presence of non zero matter fields, apart from the additional constraints related to other local symmetries, even the diffeomorphism one will be modified. This corresponds to the fact that with nonzero vacuum expectation values of CFT operators, the stress-energy tensor is not conserved, but obeys rather, for one scalar source, say,

$$\nabla^j \langle T_{ij} \rangle \;=\; - \langle \mathcal{O} \rangle \, \nabla_i \phi \, . \tag{4.93}$$

This should not be considered, however, as a diffeomorphism anomaly, but rather a feature of the spontaneous symmetry breaking induced by the nonzero vacuum expectation values of $\mathcal{O}$. This is in accord with the non conservation of the Brown–York [117] quasi-local stress-energy tensor, as was previously noticed in [118].

The signature of a gravitational anomaly is, instead, stress-energy non conservation with no VEV's turned on, analogously to what happens if there is a holographic chiral anomaly. Let us recall how this one is derived in the AdS/CFT correspondence [20]. After compactifying Type IIB supergravity on the five-sphere S^5, the relevant part of the action, containing the Chern–Simons term, reads[25]

$$S[\hat{A}] \;=\; \int \mathrm{Tr} \left(\frac{1}{2g^2} F \wedge *F + \frac{iN}{16\pi^2} \left(A \wedge \mathrm{d}A \wedge \mathrm{d}A + \cdots \right) \right) \, . \tag{4.94}$$

This is an effective action for the boundary four-dimensional gauge field $\hat{A}$ whose extension in five dimensions is A. Because of gauge invariance, there is no natural choice for A, thus one have to pick just a particular one. This is analogous to the case of the Weyl anomaly, as computed in [32]. There, in fact one has to pick, among the conformal class of boundary metrics $[g_{ij}]$ that extend in a bulk AdS [114], a particular one, thus breaking conformal invariance. The variation of the on-shell action gives the holographic anomaly.

A gauge transformation on $\hat{A}$ induces a gauge transformation on the bulk A. But the action (4.94) is not gauge invariant and picks a boundary term, from the variation of the CS term, which reproduces the chiral anomaly on the boundary theory. We have just seen how in the Hamilton–Jacobi approach, the conformal anomaly is gotten from the descent equation following from the $\mathcal{H}^\perp = 0$ constraints. It turns out that the four-dimensional chiral anomaly can be computed, in this context, from the gauge constraint $\mathcal{G}^A = 0$ [119]. In fact, including the Chern–Simons interaction in the bulk action, the Hamiltonian reduction, gives the following contribution[26]

$$\mathcal{G}_a^{\mathrm{CS}} \;=\; - \frac{iN}{16\pi^2} \frac{1}{\sqrt{\hat{g}}} \left(d_{abc} \, \mathrm{d}\hat{A}^b \wedge \mathrm{d}\hat{A}^c + 2 \, d_{bcd} \, f_{ea}{}^d \hat{A}^b \wedge \mathrm{d}\hat{A}^c \wedge \hat{A}^e \right) \, . \tag{4.95}$$

---

[25] Dots indicate the completion of the CS term.

[26] f_{abc} are the structure constant of $SU(N)$, and $d_{abc} = 2\mathrm{Tr}\left(\{T_a, T_b\} T_c \right)$.

This term, when combined with the full gauge constraint, leads the anomalous Ward identity

$$\nabla_i \frac{\delta Z}{\delta \hat{A}_i^a} \;=\; f_{ab}{}^c \, \hat{A}_i^b \, \frac{\delta Z}{\delta \hat{A}_i^c} - \sqrt{\hat{g}} \, \mathcal{G}_a^{\mathrm{CS}} \, , \tag{4.96}$$

which coincides with the calculation of Witten [20], reproducing the chiral anomaly of the $\mathcal{N} = 4$ SYM at leading order in N.

The previous analysis of the diffeomorphism Ward identities thus shows that in order to obtain an explicit anomaly in pure gravity it would be necessary to consider the appropriate Chern–Simons terms. Let us anticipate that we shall see in Chapter 5 that in presence of fermion fields this structure will be somewhat modified.

Chapter 5

Holographic fermions and forms

This chapter is devoted to the subject of introducing spin-$\frac{1}{2}$ fermions and form fields in a systematic study of Holography using the Hamiltonian formulation of gravity and the Hamilton–Jacobi approach in particular [34]. Before entering the matter, let us discuss the motivations for studying this problem.

5.1 Motivations

One of the most striking features of the AdS/CFT correspondence is the fact that this duality connects a quantum theory with a classical one. In a sense, this gives a shortcut for obtaining results that, on the field theory side, would require heavy computations of loop diagrams.

However, as long as one focuses on highly symmetric cases, there is still the doubt that the success of "high precision" tests of the correspondence can be due to the matching of these symmetries: Compactification of Type IIB supergravity on $AdS_5 \times S^5$ leads to the $SU(2, 2|4)$ superconformal group of isometries. The same group of symmetry, tough differently realized, arises for $\mathcal{N} = 4$ super Yang–Mills. It is a usual property of supersymmetry, the fact that when it is maximal, it constrains much of the structure. The less supersymmetry, the more the correspondence is predictive. We have encountered some examples in Chapter 4 while discussing SCFT's dual to non maximally supersymmetric compactifications of Type IIB or eleven-dimensional supergravity.

A test of the correspondence, going beyond the mapping of algebraic structures, follows for instance from computations of the entropy in the two regimes. This gives agreement of the functional form, even if it leaves a mismatch of numerical coefficients ascribed to the different regimes of computation. Recall also the holographic anomalies, discussed in Section 4.5. In this case the role of any underlying symmetry is not apparent at all. Agreement of anomaly computations have been found

in many cases in addition to $\mathcal{N} = 4$ SYM, for instance in $SO(N)$ or $Sp(N)$ gauge theories, and even for subleading contributions [120].

Let us address also the issue of stability in gravity. Proving positivity of energy for some given background was an urgent problem to solve as soon as the possibility of quantizing gravity was investigated. In [121] Witten was able to prove stability of Minkowski space in classical general relativity. The proof was based on use of spinors with prescribed boundary conditions on an initial value hypersurface ("Witten condition" $\hat{D}^i \epsilon = 0$) which enable writing the energy as a manifestly positive integral. Stability of AdS background was first established for supergravity theories [62, 122, 123], by noticing that the quantum Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{\hbar} \text{tr}\{Q, Q\} , \qquad (5.1)$$

where Q are the supercharges. The corresponding statement for pure classical gravity was then argued by letting $\hbar \to 0$ and simultaneously removing fermions. It was also recognized soon that positivity can also be proven for AdS backgrounds in the framework of classical Hamiltonian supergravity [124, 125]. The anti-commutator in (5.1) becomes a Dirac bracket, but everything else goes trough.

Finally it was discovered that one can prove positivity without referring to supersymmetry, just assuming a special form of the scalar potential [126, 127]. Namely, given a system of scalar fields interacting with gravity, a potential of the form (4.1)

$$\mathcal{V} = \frac{1}{d-1} \partial_I W \partial^I W - \frac{1}{d-2} W^2 . \qquad (5.2)$$

is a sufficient condition for stability of the AdS background, if $\partial_I W = 0$ at that point. Let us notice however, that those arguments still used "Witten spinors".

We have learned that in Hamilton–Jacobi language (5.2) is nothing else than an equation for the "superpotential" W to be solved in terms of the real potential $\mathcal{V}$. Therefore, if such a solution exists, which means that there are flows ending at the fixed point, its stability is automatically satisfied. This surprising fact suggests that there is an interplay between supersymmetry, stability, and holography, which should be clarified.

In order to investigate these possible interconnections it is necessary to consider fermion fields. One can then cover at least two different routes. One can start from a given supergravity model, including spin-$\frac{1}{2}$ fermions and gravitini, and study their role in the Hamilton–Jacobi equations. Or, one can work in a general framework, and inquire whether some restrictions arise, demanding the theory to be holographic. Namely, requiring a mathematical structure that one can interpret in terms of quantum field theory properties. In our approach we have chosen this alternative including, for the time being, spin-$\frac{1}{2}$ fermions interacting with antisymmetric tensor fields. Once framed in a particular context these can be for instance dilatini and RR fields.

So far the analysis of properties of holographic theories has been mostly restricted to the case of gravity coupled to scalars. It is a sensible question to ask whether the emerging structure will persist with the inclusion of different fields. One can check for instance how the holographic Callan–Symanzik equation (4.80) gets changed, and what kind of modifications to the Ward identities the other constraints give. In particular, one can study how the bulk symmetries are realized, via Hamilton–Jacobi, in the holographic theory, and whether they are respected or violated, resulting in anomalies. For instance, including spinors in our theory we must have local Lorentz invariance in addition to diffeomorphism and gauge invariances. Thus our study will give us the opportunity to address the very interesting issue of different holographic anomalies.

Last, but not least, there also some technical reasons justifying our inquiry. The study of spinors in the AdS/CFT correspondence, in particular for calculating fermionic correlation functions, has pointed out some subtleties [128, 109, 129, 130]. In fact these fields have a linear action principle and couple derivatively to the metric. Moreover, they can occur in various species according to dimensionality and signature of space-time: There can be Majorana, Weyl, or Dirac spinors, endowed with a complex structure. It was noticed in [128, 109, 129, 130] that fermion fields split naturally in momentum and coordinate parts, and that the last one are the relevant sources for the boundary operators. From the point of view of Hamiltonian theory this means that one should enforce the second class constraints on fermions in some Lorentz-invariant way. Thus, depending on the specific situation, one can impose different chirality or reality conditions. Although there exist literature on Hamiltonian formulation of supergravity, it seems there are no works where Hamilton–Jacobi theory is used to study general properties and possible solutions to the equations of motion. With our work we have given a contribution for filling this gap, providing equations valid in every space-time dimensions, and embracing different kind of spinors as well.

In the next section we start by showing an example of use of spinors in the AdS/CFT correspondence [128]. Then we explain the Hamiltonian treatment of spin-$\frac{1}{2}$ fermions and form fields. In Section 5.4 we present the bracket structure induced by the Hamiltonian zero-energy constraint, generalizing (4.74), while in Section 5.5 we discuss the other Ward identities. These generalize the Ward identities arising from the scalar (see Eqs. (4.72), (4.83)) and Yang–Mills sectors (see Eq. (4.96)), but display some intriguing novel features, as well. In Section 5.6 we solve, under certain assumptions, the bracket equations, finding agreement with results concerning the pure gravity sector and some interesting constraints in the form field and fermion sectors.

5.2   Spinors in the AdS/CFT correspondence

Consider a free spinor action on Euclidean AdS_{d+1}

$$S_0 = \int d^{d+1}x \sqrt{g}\,\bar{\psi}\,(\slashed{D} - m)\,\psi \,. \tag{5.3}$$

According to the usual prescription, the dual QFT generating function Z is equal to (5.3) evaluated along a classical trajectory satisfying given boundary conditions. However, for spinors, the equation of motion

$$(\slashed{D} - m)\psi = 0 \tag{5.4}$$

makes it vanishing, regardless the boundary condition. The proposal in [128], for reconciling with the AdS/CFT correspondence, was to add a boundary term to the above action, namely

$$S_1 = \alpha \lim_{\epsilon \to 0} \int_{M_\epsilon} d^d x \sqrt{\hat{g}_\epsilon}\,\bar{\psi}\psi \,, \tag{5.5}$$

where M_ϵ is a closed d-dimensional submanifold of AdS_{d+1}, which approaches its boundary as $\epsilon \to 0$. The metric $\hat{g}_\epsilon$ is the metric induced by g on M_ϵ. While not changing the equations of motion, it depends on boundary conditions given on M_ϵ. According to the AdS/CFT prescription this term does contribute to the partition function, and in fact is the only non trivial one.

After fixing a convenient gauge for the vielbein, the equations of motion for the bulk spinors can be solved in terms of arbitrary fields ψ_0 on M_ϵ

$$\psi(x^0, \mathbf{x}) = \int d^d \mathbf{x}' \left(x^0 \Gamma^0 + (\mathbf{x}-\mathbf{x}') \cdot \Gamma\right) \left((x^0)^2 + |\mathbf{x}-\mathbf{x}'|^2\right)^{-\frac{d+1}{2}+m} (x^0)^{\frac{d}{2}-m} \psi_0(\mathbf{x}') \tag{5.6}$$

and an analogous equation for $\bar{\psi}$ in terms of $\bar{\psi}_0$. Decomposing the spinors ψ_0 according to their Γ_0 eigenvalues[1]

$$\psi_0 = \psi_+ + \psi_- \tag{5.7}$$
$$\bar{\psi}_0 = \bar{\psi}_+ + \bar{\psi}_- \,, \tag{5.8}$$

it turns out that, in order for the fields to be square integrable, the following restrictions must be imposed

$$\psi_+(\mathbf{x}') = 0$$
$$\bar{\psi}_-(\mathbf{x}') = 0 \,. \tag{5.9}$$

---

[1] Here $\Gamma_0 \psi_\pm = \pm \psi_\pm$ and $\bar{\psi}_\pm \Gamma_0 = \pm \bar{\psi}_\pm$.

This requirement is the same that we discussed for scalar fields in Section 4.1. In this case it means that, when d is *even* the boundary spinor is chiral, whereas when d is *odd* it is a single Dirac spinor.

The QFT partition function in terms of the sources can be computed to be[2]

$$
Z[\psi_-, \bar{\psi}_+] = \exp\left(-\alpha\, c \int \mathrm{d}^d\mathbf{x}' \int \mathrm{d}^d\mathbf{x}'' \left[\bar{\psi}_+(\mathbf{x}'')|\mathbf{x}''-\mathbf{x}'|^{-(d+1+2m)}(\mathbf{x}''-\mathbf{x}')\cdot\Gamma\right]\psi_-(\mathbf{x}')\right)
$$

(5.10)

This partition function is the correct one for reproducing the correlation function of two free-field spinorial operators with scaling dimension

$$
\Delta = \frac{d}{2} + 1 .
$$

(5.11)

What lessons can we learn from this results? First, boundary terms in the AdS/CFT correspondence are *important*. A careful analysis should always take care of them. Moreover it was later pointed out [109, 130] that the boundary term (5.5), introduced *ad hoc* in [128], has a natural explanation in the Hamiltonian formulation of gravity, where the otherwise arbitrary coefficient α is fixed.

Second, the sources involve only half the number of degrees of freedom of a pair of conjugated bulk Dirac spinors. Above, the other half has been set to zero. However, once the Lorentz invariant split is done, the non-square-integrable fields play a role as well. As we have seen in the previous chapter, they represent vacuum expectation values of holographically dual operators, but *also* conjugated momenta in the Hamiltonian formulation. In this respect, the treatment of spinors is more subtle then for other fields, because the second class constraints enforce relations between coordinates and momenta.

In the following, we will see how these features are achieved in the Hamiltonian approach, and how, using Hamilton–Jacobi theory, the bulk symmetries are realized on the boundary theory as fermionic Ward identities.

5.3   Fermions and forms in Hamiltonian theory

Let us now turn to a generic $(d+1)$-dimensional theory with spinors, form fields and local diffeomorphism invariance. We will consider the most general two-derivative action (neglecting Chern–Simons terms) with quadratic fermion couplings consistent with gauge symmetry, and perform its Hamiltonian reduction. Here we will assume

---

[2]c is a numerical coefficient depending on m.

the initial value hypersurface to be either space-like or time-like, according to the sign of the norm of its normal direction n

$$n \cdot n = \eta = \pm 1 \ , \tag{5.12}$$

with its associated extrinsic curvature

$$K_{ij} = -\frac{1}{2N} \left(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right) \ . \tag{5.13}$$

Therefore the starting point is an action with the following three contributions

$$S_I = \frac{1}{\kappa^2} \int \mathrm{d}^{d+1}x \sqrt{g} \left(\tilde{R} - 2\eta \tilde{\nabla} \cdot \tilde{\nabla}_n n + 2\eta \tilde{\nabla} \cdot (n \mathrm{tr} K) - \kappa^2 \Lambda \right) \tag{5.14}$$

$$S_{II} = \int \mathrm{d}^{d+1}x \sqrt{g} \left(\frac{1}{2\lambda^2} F_A F^A + F_A J^A \right) \tag{5.15}$$

$$S_{III} = \frac{1}{2} \int \mathrm{d}^{d+1}x \ \sqrt{g} \left(\bar{\psi} M \slashed{D} \psi - (\slashed{D}\bar{\psi}) M \psi + 2\bar{\psi} Z_A \Gamma^A \psi \right) \ . \tag{5.16}$$

The fields appearing in these expressions are an Abelian p-form field strength $F = \mathrm{d}A$, which couples to the fermions through $J^A = \bar{\psi}\zeta\Gamma^A\psi$. The capital Latin letters refer to a multi-index of pertinent rank. Summations include division by the factorial of the rank. There can be arbitrarily many fermion flavors, but we always suppress the index that would distinguish them. This action captures, and generalizes, many interesting features of the effective superstring actions. For instance, F could be thought of as a Ramond–Ramond field.

The non-dynamical couplings ζ, Z_A and M mix fermion flavors, and are not assumed to be space-time constants, unless explicitly indicated. They satisfy suitable hermiticity conditions so that the action is always real. They can be thought of as the Yukawa couplings to higgsed scalar fields. As Z_A is not a dynamical field, we have assumed $Z_{0A} = 0$.

In order to go over to the Hamiltonian formalism, we need to choose a particular evolution parameter t whose tangent field we call e_0. The full ADM metric takes then the form

$$\mathrm{d}s^2 = (\eta N^2 + N_i N^i)\mathrm{d}t^2 + 2N_i \mathrm{d}t \mathrm{d}x^i + g_{ij}\mathrm{d}x^i \mathrm{d}x^j \ . \tag{5.17}$$

The bulk vielbein $e_\mu{}^{\underline{\alpha}}$ decomposes into boundary vielbein $L_i{}^{\underline{\alpha}}$, the vector n, and the non-dynamical degrees of freedom N and N^i. The connection on the boundary is obtained (see Appendix A) from that in the bulk by shifting the Christoffel symbols in such a way that the n becomes a covariantly constant vector field on the boundary, and that the boundary basis is covariantly constant in the normal direction, i.e.

$$\nabla_i n = 0 \tag{5.18}$$

$$n \cdot \nabla_i e_j = 0 \ . \tag{5.19}$$

We can now reduce all tensor fields in terms of fields defined on the boundary. Isolating the kinetic terms in the action, *i.e.* terms involving derivatives along e_0, the remainder is by definition the total Hamiltonian

$$S = \int dt \ (p \ \dot{q} - H) \tag{5.20}$$

where

$$p \ \dot{q} = \int d^d x \left(p^{i\underline{\alpha}} \ \partial_t L_{i\underline{\alpha}} + E^{\hat{A}} \ \partial_t A_{\hat{A}} + \bar{\chi} \ \partial_t \psi - \partial_t \bar{\psi} \ \chi \right) \tag{5.21}$$

$$H = \int d^d x \left(N\mathcal{H}_\perp + N^i \mathcal{H}_i + A_0^{\hat{A}} \mathcal{G}_{\hat{A}} + \varepsilon_{\underline{\alpha}\underline{\beta}} \mathcal{J}^{\underline{\alpha}\underline{\beta}} \right) . \tag{5.22}$$

We have added here, by hand, the constraint that guarantees freedom to choose the flat basis for vielbein freely, *i.e.* the generator of local Lorentz transformations in the bulk, $\mathcal{J}^{\underline{\alpha}\underline{\beta}}$. For notation, that is standard, see Appendix A.

The physical phase space can now be easily read off from the above reformulation. It consists of the canonical pairs $(p_{\underline{\alpha}}^i, L_{\hat{j}}^{\underline{\beta}})$, $(E^{\hat{A}}, A_{\hat{B}})$, $(\bar{\chi}^a, \psi_b)$, and $(\bar{\psi}^a, \chi_b)$. The fields $A_{0\hat{A}}$ and N^μ are Lagrange multipliers that correspond to the first class constraints $\mathcal{G}^{\hat{A}}$ and $\mathcal{H}_\mu$, respectively. Due to fermions, having a first order action principle, there is also the second class constraint[3]

$$\chi = \frac{1}{2} \eta \sqrt{\hat{g}} \ \Gamma^n M \psi , \tag{5.23}$$

and its conjugate. In order to solve these constraints, we have to split the fermion phase space in some Lorentz invariant way into two parts, one of which we treat as the configuration space, and hence boundary fields, and the other as the momenta, to be solved as functionals of the boundary fields in Hamilton–Jacobi theory. There are essentially two ways to proceed: Imposing either a Weyl condition or a Majorana condition. We have analyzed extensively the various possibilities in the Appendix C. Here we shall point out some salient features, referring explicitly to chirality conditions[4].

The conjugate spinors obey the following relation

$$\bar{\psi}_\pm \sqrt{\eta} \ \Gamma^n = \pm \ \varepsilon\eta \ \bar{\psi}_\pm, \tag{5.24}$$

where

$$\varepsilon\eta = \begin{cases} (-)^{d-} & d+1 \ \text{odd} \\ \eta & d+1 \ \text{even} \end{cases} \tag{5.25}$$

---

[3]$\Gamma^n \equiv n_{\underline{\alpha}}\Gamma^{\underline{\alpha}}$. See Appendix A for further notations.

[4]With chirality we refer to the eigenvalues ± 1 of $\sqrt{\eta}\,\Gamma^n$: $\sqrt{\eta}\,\Gamma^n\,\psi_\pm = \pm\,\psi_\pm$. This is a covariant generalization of (5.7).

This means that the the Dirac conjugate of a spinor of definite chirality is either of the same or the opposite chirality, and in general it will change the identification of canonical pairs. Interestingly, the kinetic terms always assume the same form, namely

$$\bar{\chi}\,\partial_t\psi - \partial_t\bar{\psi}\,\chi \;=\; \bar{\pi}\dot{\varphi} - \dot{\bar{\varphi}}\pi + \cdots , \tag{5.26}$$

where the symplectic pairs are the following:

$$\begin{cases} (\varphi,\bar{\pi}) &=& (\psi_-, -\sqrt{\eta\hat{g}}\;\bar{\psi}_- M) \\ (\bar{\varphi},\pi) &=& (\bar{\psi}_+, \sqrt{\eta\hat{g}}\;M\psi_+) \end{cases} \qquad \text{if}\;\; \varepsilon\eta = +1 , \tag{5.27}$$

and

$$\begin{cases} (\varphi,\bar{\pi}) &=& (\psi_-, -\sqrt{\eta\hat{g}}\;\bar{\psi}_+ M) \\ (\bar{\varphi},\pi) &=& (\bar{\psi}_-, \sqrt{\eta\hat{g}}\;M\psi_+) \end{cases} \qquad \text{if}\;\; \varepsilon\eta = -1 . \tag{5.28}$$

The leftover terms in (5.26) involve a total time derivative and terms containing derivatives of the boundary metric $\hat{g}$.

- The total time-derivative term $\frac{1}{2}\partial_t(\bar{\varphi}\pi - \bar{\pi}\varphi)$ should be subtracted from the action, as argued in [109, 130]. Generally speaking, the reason for this procedure is that the generating function arises also on the bulk side from a path integral, which is going to be defined in Hamiltonian language, cf. (4.61).

- The terms involving a time derivative of the metric will change the gravitational momentum but in a way that is easily kept track of. In fact, they simply shift it by

$$\pi^{ij} \;\rightarrow\; \Pi_{ij} \equiv \pi^{ij} - \frac{1}{4}\hat{g}^{ij}(\bar{\varphi}\pi - \bar{\pi}\varphi) . \tag{5.29}$$

5.4   The Callan–Symanzik equation

Having split the fermion phase space we are now in the position to write down the Hamilton–Jacobi equations for the full system. We will perform this analysis for Weyl fermions and assume that M is a space-time constant matrix, commenting later on what happens for non constant M.

It turns out that, provided there are no marginal operators Z present in the bulk and that the rank of the tensor field p is *odd*, the Hamilton–Jacobi equation originating from $\mathcal{H}_\perp$ does indeed take the generally expected form

$$(F, F) = \mathcal{L} . \tag{5.30}$$

Where now

$$(F, F) = (F, F)_g + (F, F)_{\hat{A}} + (F, F)_\varphi , \tag{5.31}$$

and the right hand side of (5.30) is

$$\mathcal{L} = \sqrt{\hat{g}} \left(\frac{1}{\kappa^2} R - \Lambda + \frac{1}{2\lambda^2} F_{\hat{A}} F^{\hat{A}} + F_{\hat{A}} \bar{\varphi} \zeta \Gamma^{\hat{A}} \varphi \right.$$
$$\left. + \frac{1}{2} \bar{\varphi} M \hat{\not{D}} \varphi - \frac{1}{2} (\hat{\not{D}} \bar{\varphi}) M \varphi + \bar{\varphi} Z_{\hat{A}} \Gamma^{\hat{A}} \varphi \right) . \tag{5.32}$$

It is useful to define the following operators:

$$\mathcal{D} = \frac{1}{2} \bar{\varphi} \frac{\delta}{\delta\bar{\varphi}} - \frac{1}{2} \frac{\delta}{\delta\varphi} \varphi \tag{5.33}$$

$$\mathcal{D}^{ij} = \frac{\delta}{\delta g_{ij}} - \frac{1}{2} g^{ij} \mathcal{D} \tag{5.34}$$

$$\mathcal{D}^{\hat{A}} = \frac{\delta}{\delta A_{\hat{A}}} + \bar{\varphi} \zeta M^{-1} \Gamma^{\hat{A}} \frac{\delta}{\delta\bar{\varphi}} + \frac{\delta}{\delta\varphi} M^{-1} \zeta \Gamma^{\hat{A}} \varphi . \tag{5.35}$$

Had we also considered p even, the last equation would have been different: The operator $\mathcal{D}^{\hat{A}}$ would have contained terms with either no or two derivatives w.r.t. the fermion fields. Derivatives act from the left, but not on fields included in the same operator. The brackets can be easily written as

$$(F, H)_g = \frac{-\eta\kappa^2}{\sqrt{\hat{g}}} (g_{il} g_{jk} - \frac{1}{d-1} g_{ij} g_{kl}) (\mathcal{D}^{ij} F) (\mathcal{D}^{kl} H) \tag{5.36}$$

$$(F, H)_A = \frac{\eta\lambda^2}{2\sqrt{\hat{g}}} (\mathcal{D}^{\hat{A}} F) (\mathcal{D}_{\hat{A}} H) \tag{5.37}$$

$$(F, H)_\varphi = \frac{-\eta}{2\sqrt{\hat{g}}} \left(\frac{\delta F}{\delta\varphi} M^{-1} \hat{\not{D}} \frac{\delta H}{\delta\bar{\varphi}} - (\hat{\not{D}} \frac{\delta F}{\delta\varphi}) M^{-1} \frac{\delta H}{\delta\bar{\varphi}} \right) . \tag{5.38}$$

The reason for the fact that the fermionic momenta also give rise to a bracket is easily seen in the case for Weyl fermions: The chiralities of the coordinates and the momenta are such that if we insert any operator even in Clifford matrices between them, $\bar{\pi}\mathcal{O}_{\text{even}}\varphi$, the result is nontrivial. Similarly, the nontrivial results for odd operators arise from insertions between either two coordinates, $\bar{\varphi}\mathcal{O}_{\text{odd}}\varphi$, or two momenta, $\bar{\pi}\mathcal{O}_{\text{odd}}\pi$. An analogous structure also arises for Majorana fields.

This structure changes slightly if the bulk mass terms, or couplings to external form fields, Z, are included. If Z is even, e.g. a mass term, there will be an additional term

$$\delta_Z F + (F, F) = \mathcal{L} \tag{5.39}$$

where

$$\delta_Z = \frac{1}{\sqrt{\eta}} \left(\bar{\varphi} Z M^{-1} \frac{\delta}{\delta \bar{\varphi}} - \frac{\delta}{\delta \varphi} M^{-1} Z \varphi \right) , \tag{5.40}$$

however this addition preserves the form of the final Callan–Symanzik equations. Also, if Z is odd or M is not constant, there will be an additional quadratic piece in the fermion momenta, but the basic form of the brackets will remain unchanged.

If there are no Majorana conditions at our disposal and we are forced to give the initial data in terms of Weyl fermions, the requirement that the bulk action produces a QFT generating functional that obeys the Callan–Symanzik equation implies that bulk theories where fermions are coupled to dynamical *even* rank form fields do not posses a simple holographic dual.

Including higher order interactions could be a problem, because they would introduce higher powers of momenta, which would spoil the basic form of the brackets. However, we have seen above an encouraging rearrangement of terms, where the structure of the theory solves a similar problem. For instance, the form field kinetic term absorbs some four fermion couplings in the expression $(\mathcal{D}^{\hat{A}} F)^2$. We can indeed view the fermionic additions in $\mathcal{D}^{\hat{A}}$ as a covariantization of the flat derivative with respect to the form field $\hat{A}$. There is therefore reason to expect that theories that are known to have a holographic dual but which contain four fermion interactions the additional symmetries, such as local supersymmetry or the $SL(2, \mathbb{Z})$ invariance in Type IIB supergravity, might arrange the fermion structure in such a way that the higher fermion derivatives would still be manageable.

5.5 Ward identities

In addition to the zero-energy constraint $\mathcal{H}_\perp$ treated above, we also have to solve the remaining first class constraints $\mathcal{G}^{\hat{A}}$, $\mathcal{H}_i$ and $\mathcal{J}^{\underline{\alpha\beta}}$. They will impose respectively gauge symmetry, diffeomorphism invariance and local Lorentz symmetry on the boundary. This is not so straightforward because in the bulk these constraints generate symmetry transformations through Dirac brackets. Due to the ansatz (4.68) in the Hamilton–Jacobi formalism they will effectively act through Poisson brackets instead, and generate a different action on F. As far as the Lorentz and the gauge symmetries are concerned the geometrically expected actions are obtained. In case the of diffeomorphism invariance some additional constraints arise.

For instance, the fact that $A_{\hat{A}}$ enters the generating functional F only through its field strength is sufficient to guarantee $\mathcal{G}^{\hat{A}} = 0$. This simply reflects the fact that the boundary theory must have the same gauge symmetry as the bulk theory. A similar situation prevails as far as $\mathcal{J}^{\underline{\alpha\beta}}$ is concerned: $n_{\underline{\alpha}} \mathcal{J}^{\underline{\alpha\beta}} = 0$ just because $n_{\underline{\alpha}} L_i{}^{\underline{\alpha}} = 0$, and the rest of the components generate the expected action on vielbein and spinors,

through

$$\Lambda^{jk} \;=\; L^{[j}_{\;\underline{\alpha}} \frac{\delta}{\delta L_{k]\underline{\alpha}}} - \frac{1}{4}\left(\frac{\delta}{\delta\varphi}\Gamma^{jk}\varphi + \bar{\varphi}\Gamma^{jk}\frac{\delta}{\delta\bar{\varphi}}\right) \; . \tag{5.41}$$

This constraint guarantees, therefore, local Lorentz invariance on the boundary. Choosing gauge and Lorentz invariant S_{div} will be enough to avoid anomalies in the boundary theory.

The situation is somewhat more involved when the constraints that guarantee diffeomorphism invariance are considered. This means that the effective action should be invariant under translations generated by a vector field χ. In other words, shifting the fields $\hat{A}$ and φ infinitesimally by their Lie-derivatives

$$\mathcal{L}_\chi \hat{A} \;=\; \iota_\chi \hat{F} + \mathrm{d}\iota_\chi \hat{A} \tag{5.42}$$

$$\mathcal{L}_\chi \varphi \;=\; (\hat{D}_\chi + \frac{1}{4}\nabla_{[i}\chi_{j]}\Gamma^{ij})\varphi \; , \tag{5.43}$$

gives a contribution that combines together with a contribution from the integration measure to a total derivative of the Lagrangian. Note that a general spinorial Lie-derivative does not obey the Leibniz rule[5]. It is therefore useful to restrict to Killing fields $\nabla_{(i}\chi_{j)} = 0$, for which the formula (5.43) applies.

Solving the constraint $\mathcal{H}_i = 0$ we get, again assuming p odd and the Weyl decomposition, that the variation

$$\delta_\chi \;=\; \nabla_k \chi^j \, L_{j\underline{\alpha}} \frac{\delta}{\delta L_{k\underline{\alpha}}} + \chi^i \left(\partial_{[i}A_{\hat{A}]}\mathcal{D}^{\hat{A}} - \frac{\delta}{\delta\varphi}\hat{D}_i\varphi + \hat{D}_i\bar{\varphi}\frac{\delta}{\delta\bar{\varphi}}\right) \tag{5.44}$$

should annihilate the effective action. This differs from a Lie-derivative in two respects: First, the transformation of the form field is accompanied by a gauge transformation

$$\Delta_\chi \hat{A} \;=\; -\,\mathrm{d}\iota_\chi \hat{A} \; . \tag{5.45}$$

Second, its action on fermions is modified by

$$\Delta_\chi \varphi \;=\; -\iota_\chi F_{\hat{A}}\,\Gamma^{\hat{A}}M^{-1}\zeta\varphi \tag{5.46}$$

$$\Delta_\chi \bar{\varphi} \;=\; \iota_\chi F_{\hat{A}}\,\bar{\varphi}\Gamma^{\hat{A}}\zeta M^{-1} \; , \tag{5.47}$$

where $\Delta_\chi = \delta_\chi - \mathcal{L}_\chi$. If we want to restrict to theories where the boundary diffeomorphism invariance still prevails, we have to put this difference to zero. A solution of

$$\Delta_\chi S_{\text{div}} \;=\; 0 \tag{5.48}$$

is ensured imposing the following conditions:

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[5] For a general introduction to spinors and geometry see for instance [131].

1) The Clifford action of any differential form K appearing in the fermion couplings $\bar{\varphi} K_{\hat{A}} \Gamma^{\hat{A}} \varphi$ commutes with that of $\iota_\chi \hat{F}$, i.e.

$$[K_{\hat{A}} \Gamma^{\hat{A}}, \chi^i F_{i\hat{B}} \Gamma^{\hat{B}}] = 0 \ . \tag{5.49}$$

For instance, for $K_{\hat{A}} = F_{\hat{A}}$, $F_{\hat{A}}$ being of odd rank, this is clearly true. This condition means then that $K_{\hat{A}}$ and $F_{\hat{A}}$ should be aligned in a certain way. In addition to this, the couplings should satisfy, in the notation of formula (5.58),

$$\hat{\zeta} M^{-1} \zeta = \zeta M^{-1} \hat{\zeta} \ . \tag{5.50}$$

2) If there is a kinetic term on the boundary, such as $\bar{\varphi} \hat{\slashed{D}} \varphi - (\hat{\slashed{D}} \bar{\varphi}) \varphi$ the following restriction must be true

$$\mathcal{L}_\chi \hat{F} \ = \ 0 \tag{5.51}$$
$$\iota_\chi F^{i\hat{A}} \hat{D}_i \varphi \ = \ 0 \ . \tag{5.52}$$

These are strong requirements, as they concern the boundary fields and not only couplings, and therefore really restrict, from the bulk point of view, the set of acceptable initial conditions. The simplest way to solve them is naturally to exclude the kinetic terms from the action, cf. end of Section 5.6. However, this might be too drastic a solution as, in the above, we are assuming that there exist at least one Killing field χ; after all, we are considering an interacting theory with physical sources. More interestingly, these conditions can be solved by assuming $\iota_\chi \hat{F} = 0$: This means that the Killing isometry only changes the field $\hat{A}$ by generating a gauge transformation. With this understanding it is sufficient to set

$$\mathcal{L}_\chi \hat{A} \ \approx \ 0 \ . \tag{5.53}$$

This would mean that there are no restrictions on the fermion fields, whereas the form field potential is frozen to configurations covariant under flows generated by χ.

Considering diffeomorphism invariant terms in S_{div}, one obtains the Ward identity

$$\mathcal{L}_\chi W + \Delta_\chi \bar{\varphi} \langle \mathcal{O}_{\bar{\varphi}} \rangle - \langle \mathcal{O}_\varphi \rangle \Delta_\chi \varphi \ = \ -\Delta_\chi S_{\text{div}} \ . \tag{5.54}$$

The right hand side is the failure of the counter terms to satisfy the constraint $\mathcal{H}_i = 0$, while the vacuum expectation values would signal the spontaneous symmetry breaking of Lorentz symmetry in the dual QFT. This is analogous to the analysis in Section 4.5.

5.6 Local expansion of the counter term action

Let us consider a particular solution of the classical equations of motion that behaves at the boundary $t \to 0$ as

$$g_{ij}(x,t) = t^{-2}g_{ij}^R(x) + \mathcal{O}(t^{-1}) \tag{5.55}$$

$$A_{\hat{A}}(x,t) = t^\nu A_{\hat{A}}^R(x) + \mathcal{O}(t^{\nu+1}) \tag{5.56}$$

$$\psi(x,t) = t^\sigma \psi^R(x) + \mathcal{O}(t^{\sigma+1}) \tag{5.57}$$

in the coordinate system of (4.70). The quantities with superscript R refer to expressions that have a finite and nonzero limit at $t \to 0$. We will actually not need to show whether the full coupled system has solutions with this particular asymptotic behavior. This would also be quite difficult — it was found, for instance, in [109, 130] that free fermions scale as $\sigma = d/2 - m$, where m is the bulk mass. In our case the coupling of the form field F_A to the fermions gives rise to an effective mass term. We cannot, however, fix the field F_A in any useful way in order to analyze conclusively the scaling behavior in this coupled system.

Instead, we impose the above scaling behavior for some set of "critical exponents" ν and σ and then derive from this – and the assumption that the holographic dual exist at all – consistency conditions for both the bulk and the boundary theories. Let us consider, in particular, the case $\nu = -n + 1$ and $\sigma = 1/2$. This assignment has the virtue that the expansion of S_{div} will look like an expansion in terms of the naive mass dimension relevant to supersymmetric models. This choice will turn out to be a convenient book-keeping device, but the results we eventually get are valid more generally. In order to solve the full bracket equation, we will have to arrange the coefficients of terms that not only have the same scaling behavior, but also the same structure, to cancel. So, in principle, one can check *a posteriori* for which range of scaling exponents the terms we neglected are still subleading and our results continue to be valid. We shall further assume that all couplings such as M and Z are marginal operators, and as such t-independent. This assumption is not very restrictive, not even in the presence of scalar fields, that would render the couplings dynamical. We can now write a local ansatz for S_{div} that is of the same functional form as (5.32):

$$S_{\text{div}} = \int \mathrm{d}^d x \sqrt{\hat{g}} \left(k_1 R - \hat{\Lambda} + \frac{1}{2}k_2 F_{\hat{A}} F^{\hat{A}} + F_{\hat{A}}\bar{\varphi}\hat{\zeta}\Gamma^{\hat{A}}\varphi \right.$$

$$\left. + \frac{1}{2}\bar{\varphi}\hat{M}\hat{\slashed{D}}\varphi - \frac{1}{2}(\hat{\slashed{D}}\bar{\varphi})\hat{M}\varphi + \bar{\varphi}L_{\hat{A}}\Gamma^{\hat{A}}\varphi \right) . \tag{5.58}$$

As we shall later restrict to a scalar coupling L, it turns out that no four-fermion terms are needed. The couplings $k_1, k_2, \hat{\zeta}, \hat{M}$ and $L_{\hat{A}}\Gamma^{\hat{A}}$ need not be constant on the boundary, but they are assumed marginal, *i.e.* time-independent.

At leading order S_{div} will diverge as t^{-d} when $t \to 0$. We shall consider the three lowest order contributions to the equation $\mathcal{H}_\perp = 0$, or, $(S_{\text{div}}, S_{\text{div}}) = \mathcal{L}$.

1) To the leading order we get

$$\Lambda + \frac{\eta\kappa^2}{4}\frac{d}{d-1}\hat{\Lambda}^2 = 0 \ . \tag{5.59}$$

The boundary cosmological constant is therefore essentially the square root of the bulk one and sets the scale. We notice that depending on the sign of the bulk gravitational constant we can choose to look at either time-like or space-like boundary surfaces. However, only the space-like surfaces admit an asymptotically anti de Sitter solution.

2) The $\mathcal{O}(t^{-d+1})$ equations depend on the details of Z. The equations simplify assuming that Z is odd in Clifford matrices and that L is a scalar, *i.e.* a flavor matrix, as we then get

$$Z + \eta LM^{-1}ZM^{-1}L = 0 \ . \tag{5.60}$$

If Z is even, we get

$$ZM^{-1}L + LM^{-1}Z = 0 \ . \tag{5.61}$$

In particular, we could consistently set $L = 0$. This would mean that, for instance, a bulk mass term does not automatically lead to a mass term on the boundary.

3) At the next order $\mathcal{O}(t^{-d+2})$ we get from the Einstein–Hilbert and the form field kinetic terms, conditions for the couplings k_1 and k_2

$$\frac{\eta}{d-1}\kappa^2\hat{\Lambda}(\frac{d}{2} - 1)k_1 - \frac{1}{\kappa^2} = 0 \tag{5.62}$$

$$\frac{\eta}{d-1}\kappa^2\hat{\Lambda}(\frac{d}{2} - p)k_2 - \frac{1}{\lambda^2} = 0 \ . \tag{5.63}$$

For $d > 2$ equation (5.62) allows one to compute the coefficient of the R term, as expected. In $d = 2$ this equation does not arise as $\sqrt{\hat{g}}R$ is marginal, and it is not to be included in S_{div}. Instead, the impossibility of canceling it in the descent equations translates to the Weyl anomaly in $d = 2$ as in Eq. (4.90).

Equation (5.63) gives the value of k_2 for $d \neq 2p$. Note, however, that for middle dimensional form fields $d = 2p$ this equation will not have solutions. In the case of free form fields, as they are always marginal [112], it is just the analog of (5.62) for pure gravity: In this case $F^{\hat{A}}F_{\hat{A}}$ is a marginal operator and

it contributes to the matter part of the Weyl anomaly. In the interacting case the treatment of this term depends on the scaling dimension ν. If $\nu < 0$ then the contribution should really be included in the divergent part, and a middle dimensional form field would not allow consistent solutions for the bracket equation.

4) Assuming that M and L are constants on the boundary, the fermion terms yield the constraints

$$-\frac{p}{2}\frac{\eta}{d-1}\kappa^2\hat{\Lambda}\hat{M} \;=\; M + \eta L M^{-1} L \tag{5.64}$$

$$-\frac{\eta}{2d-2}\kappa^2\hat{\Lambda}\hat{\zeta} \;=\; \zeta + \eta L M^{-1}\zeta M^{-1} L \;. \tag{5.65}$$

In all of the equations (5.59) and (5.62 – 5.65) we see that the relationship between the bulk and the boundary couplings is essentially a scale factor $\kappa^2\hat{\Lambda}$. It is interesting to note that the role of L is to mix bulk fermion flavors into new combinations on the boundary. Assuming L proportional to M would lead to scaling the fermion field. Furthermore, setting $L = \sqrt{-\eta}M$ the dynamical part of the fermion action drops out completely $\hat{M} = \hat{\zeta} = 0$, and the only contribution comes from the L-term itself

$$\sqrt{-\eta}\int \mathrm{d}^d x\sqrt{\hat{g}}\bar{\varphi}M\varphi. \tag{5.66}$$

As discussed in the previous section this solution would not produce a diffeomorphism anomaly, even without restricting to symmetric form fields, cf. (5.53).

5.7 Conclusions

In this chapter we have investigated the relationship between diffeomorphism invariant theories and their holographic duals, showing in particular that, in a theory that contains fermions, nontrivial consistency conditions arise. These conditions restrict, for instance, the couplings of even rank form field strengths to fermions in dimensions where we have to – or choose to – consider only chiral boundary data.

After explaining how the Hamiltonian reduction is performed, with particular attention to fermion fields, in Sections 5.4 and 5.5 we have discussed how the holographic Callan–Symanzik equation and other Ward identities following from first class constraints get modified in presence of fermions and forms. Although the gauge and the Lorentz constraints did not lead to surprises, the Poincaré ones resulted in an anomalous contribution of the diffeomorphism Ward identity on the boundary, cf. Eqs. (5.45) – (5.47). However, one can get rid of these terms, either

imposing conditions on the sources, as for instance the boundary gauge potential to be constant in the sense of (5.53), or by the choice of a suitable counter term (5.66).

We have also derived relationships between the bulk and boundary couplings, finding that the role of the cosmological constant on the boundary is to set the scale of all boundary couplings w.r.t. bulk couplings, as expected. Moreover, we were able to find the first terms of the expansion of the counter term action S_{div}, with couplings fixed in terms of those appearing in the bulk action. This analysis is important, as it shows that one can avoid including fermion kinetic terms in S_{div}, that would have a nonzero anomalous diffeomorphism variation.

The formal structures arising are quite intriguing and we think it would be interesting to investigate the subject more deeply. The zero-energy constraint, for instance, gives rise to a descent equation for solving the holographic trace anomaly. The similarity with the Batalin–Vilkovisky approach is amusing, if one compares the $(\cdot,\cdot)$ bracket defined above, with the BV antibracket. The cohomological nature of QFT anomalies in this formalism is well known[6], and recently the same feature was recognized as far as the holographic Weyl anomaly is concerned [133]. It would be very interesting to understand whether these similarities are coincidences, or there is some deeper reason for them to hold. Finally, in order to address the most fundamental question of the role of supersymmetry in the duality, one should consider Rarita–Schwinger fermions, and repeat an analysis similar to that performed here.

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[6]See for instance [132].

Appendix A

Notation and useful formulae[†]

Greek indices $\mu, \nu, \ldots$ refer to the coordinate directions in the bulk, underlined Greek indices $\underline{\alpha}, \underline{\beta}, \ldots$ are flat indices in the bulk, lower case Latin indices $i, j, \ldots$ refer to coordinate directions in the boundary, upper case Latin indices $A, B, \ldots$ refer to normalized multi-indices in the bulk, and hatted upper case Latin indices $\hat{A}, \hat{B}, \ldots$ refer to normalized multi-indices in the boundary. If there is danger of confusion, symbols with a tilde, such as $\tilde{\nabla}$, are used to refer to bulk quantities and symbols with a hat, such as $\hat{\not{D}}$, to boundary quantities.

We can express the connection between an arbitrary flat coordinate basis $\{e_{\underline{\alpha}}\}$ and a basis $\{e_0, e_i\}$ that involves the direction of the evolution coordinate (bulk direction) e_0 in terms of the vielbein

$$e_i{}^{\underline{\alpha}} = L_i{}^{\underline{\alpha}} \tag{A.1}$$

$$e_0{}^{\underline{\alpha}} = N n^{\underline{\alpha}} + N^i L_i{}^{\underline{\alpha}} . \tag{A.2}$$

Due to the algebraic constraint $L_i{}^{\underline{\alpha}} n_{\underline{\alpha}} = 0$ we have $e_i \cdot n = 0$. For other properties of this frame see for instance [134]. The boundary metric and vielbein are related through

$$g_{ij} = L_i{}^{\underline{\alpha}} L_j{}^{\underline{\beta}} \eta_{\underline{\alpha\beta}} \tag{A.3}$$

$$\eta_{\underline{\alpha\beta}} = L_{i\underline{\alpha}} L^i{}_{\underline{\beta}} + \eta n_{\underline{\alpha}} n_{\underline{\beta}} , \tag{A.4}$$

and the boundary gamma matrices are defined by

$$\Gamma^n = n_{\underline{\alpha}} \Gamma^{\underline{\alpha}} \tag{A.5}$$

$$\Gamma^i = L^i{}_{\underline{\alpha}} \Gamma^{\underline{\alpha}} . \tag{A.6}$$

Given the Levi–Civita connection $\tilde{\nabla}$ in the bulk we can construct a metric connection on the boundary by setting [135]

$$\nabla_X Y = \tilde{\nabla}_X Y + \eta n (Y \cdot \tilde{\nabla}_X n) - \eta \tilde{\nabla}_X n (Y \cdot n) \tag{A.7}$$

[†]The appendices refer to and use notation of, Chapter 5.

for arbitrary vector fields X, Y. This connection enjoys the properties

$$\nabla_i n = 0 \qquad\qquad (A.8)$$

$$n \cdot \nabla_i e_j = 0 . \qquad\qquad (A.9)$$

The spin connection in the bulk can be expressed in terms of that on the boundary using

$$\tilde{\omega}_{i\alpha\beta} = \Gamma_{jik} L^j_{\underline{\alpha}} L^k_{\underline{\beta}} - L^k_{\underline{\beta}} \partial_i L_{k\underline{\alpha}} - \eta\, n_{\underline{\beta}}\, \partial_i n_{\underline{\alpha}}$$
$$+\eta\, K_{ij}\, (n_{\underline{\alpha}}\, L^j_{\underline{\beta}} - n_{\underline{\beta}}\, L^j_{\underline{\alpha}}) \qquad\qquad (A.10)$$

$$\tilde{\omega}_{0\alpha\beta} = (\partial_j N + \eta N^i\, K_{ij})\, (n_{\underline{\alpha}}\, L^j_{\underline{\beta}} - n_{\underline{\beta}}\, L^j_{\underline{\alpha}})$$
$$-\nabla_{[j} N_{k]}\, L^j_{\underline{\alpha}} L^k_{\underline{\beta}} + \eta\, n_{[\underline{\alpha}}\, \partial_t n_{\underline{\beta}]} + L_{j[\underline{\alpha}}\, \partial_t L^j_{\underline{\beta}]} . \qquad\qquad (A.11)$$

The extrinsic curvature is

$$K_{ij} = -\frac{1}{2N}\, (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) . \qquad\qquad (A.12)$$

From the point of view of the Lagrangian formalism, the momenta are just notation for expressions involving fields and their derivatives

$$p^{i\underline{\alpha}} = 2\pi^{ij} L^\alpha_j - \frac{1}{8}\eta\, \sqrt{\hat{g}}\, \bar{\psi} M \{\Gamma^n, \Gamma^{\underline{\alpha\beta}}\} \psi\, L^i_{\underline{\beta}} \qquad\qquad (A.13)$$

$$\pi^{ij} = -\frac{\eta}{\kappa^2}\, \sqrt{\hat{g}}\, (\hat{g}^{ij} \mathrm{tr} K - K^{ij}) \qquad\qquad (A.14)$$

$$E^{\hat{A}} = \sqrt{g}\, \left(\frac{1}{\lambda^2} F^{0\hat{A}} + J^{0\hat{A}}\right) \qquad\qquad (A.15)$$

$$\bar{\chi} = \frac{1}{2}\eta\sqrt{\hat{g}}\, \bar{\psi}\, \Gamma^n M \qquad\qquad (A.16)$$

$$\chi = \frac{1}{2}\eta\sqrt{\hat{g}}\, \Gamma^n M \psi . \qquad\qquad (A.17)$$

The Poincaré constraints consist of three parts $\mathcal{H}^\mu = \mathcal{H}^\mu_I + \mathcal{H}^\mu_{II} + \mathcal{H}^\mu_{III}$. In the pure gravity sector we have

$$\mathcal{H}^\perp_I = \sqrt{\hat{g}}\, \left(-\frac{1}{\kappa^2} R + \Lambda\right) - \frac{\eta\kappa^2}{\sqrt{\hat{g}}}\, \left(\mathrm{tr}\Pi^2 - \frac{1}{d-1}(\mathrm{tr}\Pi)^2\right) \qquad (A.18)$$

$$\mathcal{H}^i_I = -\nabla_j(P^{j\underline{\alpha}} L^i_{\underline{\alpha}}) , \qquad\qquad (A.19)$$

where the gravitational momenta have been shifted according to

$$\Pi^{ij} = \pi^{ij} - \frac{1}{2}\hat{g}^{ij} G \qquad\qquad (A.20)$$

$$P^{j\underline{\alpha}} = p^{j\underline{\alpha}} - L^{i\underline{\alpha}} G . \qquad\qquad (A.21)$$

In the form field sector

$$\mathcal{H}_{II}^{\perp} = \sqrt{\hat{g}}\left(-\frac{1}{2\lambda^2}F_{\hat{A}}F^{\hat{A}} - F_{\hat{A}}\,J^{\hat{A}}\right)$$

$$+\frac{1}{\sqrt{\hat{g}}}\frac{\eta\lambda^2}{2}\left(E_{\hat{A}} - \sqrt{g}J^0_{\hat{A}}\right)\left(E^{\hat{A}} - \sqrt{g}J^{0\hat{A}}\right) \tag{A.22}$$

$$\mathcal{H}_{II}^{i} = F^i_{\hat{A}}(E^{\hat{A}} - \sqrt{g}J^{0\hat{A}}) \,, \tag{A.23}$$

the different signs in front of the fermionic dynamical (A.23) and background (A.25) currents is not a surprise, as a contribution of the first one has been used in the definition of the electric field $E_{\hat{A}}$.

In the fermion sector

$$\mathcal{H}_{III}^{\perp} = \frac{1}{2}\sqrt{\hat{g}}\left(-\bar{\psi}M\hat{\not{D}}\psi + (\hat{\not{D}}\bar{\psi})M\psi - 2\bar{\psi}Z_{\hat{A}}\Gamma^{\hat{A}}\psi\right) \tag{A.24}$$

$$\mathcal{H}_{III}^{i} = \frac{1}{2}\eta\sqrt{\hat{g}}\left(\bar{\psi}\,\Gamma^n M\hat{D}^i\psi - (\hat{D}^i\bar{\psi})\Gamma^n M\psi + 2\bar{\psi}Z^i_{\hat{A}}\Gamma^n\Gamma^{\hat{A}}\psi\right) \tag{A.25}$$

The action of the covariant derivative on spinors is, by definition

$$\not{D}\psi = \Gamma^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu\underline{\alpha\beta}}\Gamma^{\underline{\alpha\beta}})\psi \tag{A.26}$$

$$(\not{D}\bar{\psi}) = (\partial_\mu\bar{\psi} - \frac{1}{4}\bar{\psi}\omega_{\mu\underline{\alpha\beta}}\Gamma^{\underline{\alpha\beta}})\Gamma^\mu \,, \tag{A.27}$$
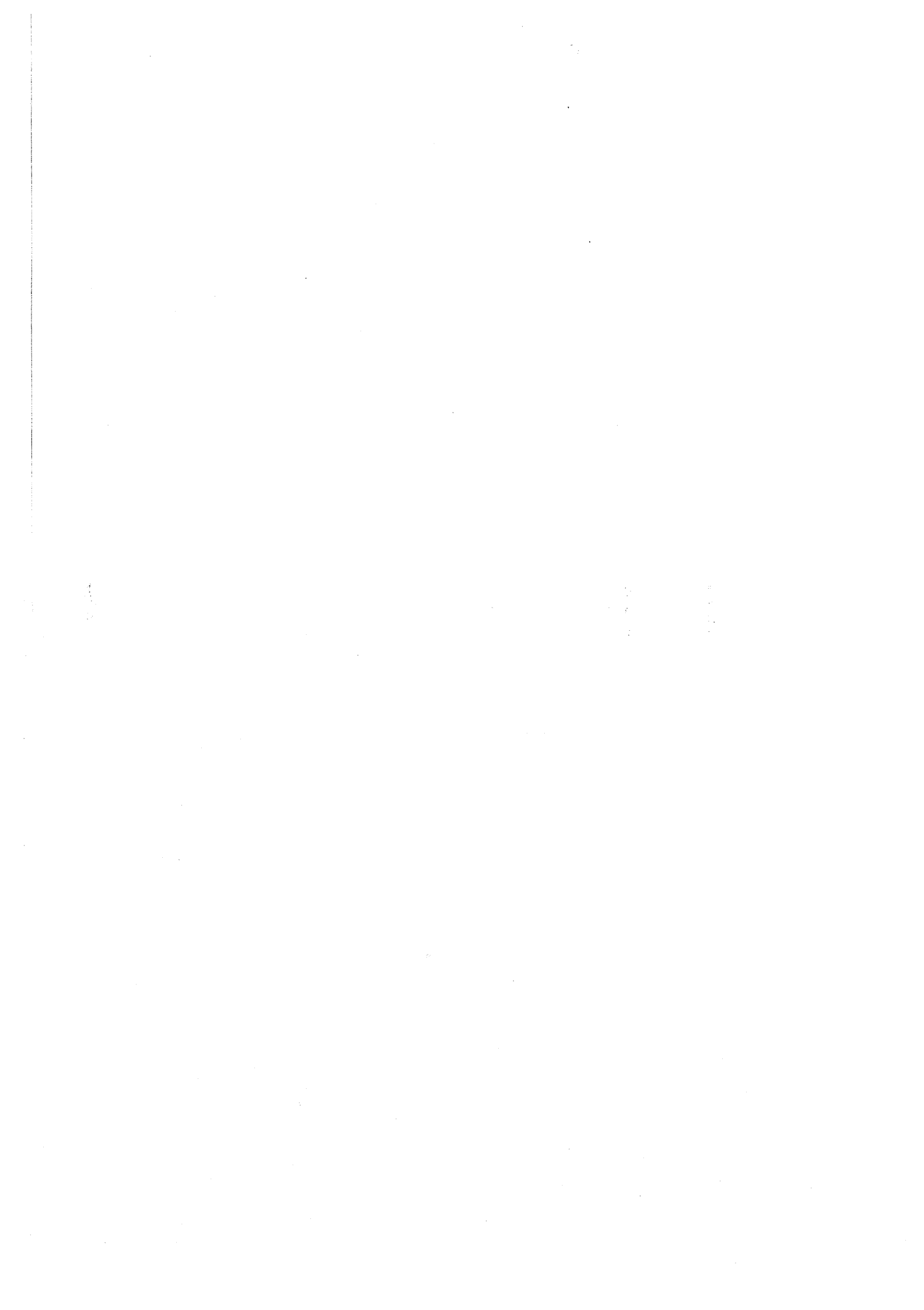
so that

$$\tilde{\nabla}_\mu(\bar{\chi}\Gamma^\mu\psi) = (\not{D}\bar{\chi})\psi + \bar{\chi}\not{D}\psi \,. \tag{A.28}$$

In addition to the Poincaré constraints there are also constraints that generate the gauge transformations $A \rightarrow A + \mathrm{d}B$ and gauge transformations on the frame bundle (local Lorentz transformations)

$$\mathcal{G}^{\hat{A}} = \partial_i E^{i\hat{A}} \tag{A.29}$$

$$\mathcal{J}^{\underline{\alpha\beta}} = p^{i[\alpha}L_i^{\beta]} + \frac{1}{8}\eta\,\sqrt{\hat{g}}\,\bar{\psi}M\{\Gamma^n,\Gamma^{\underline{\alpha\beta}}\} \,. \tag{A.30}$$

Appendix B

Clifford algebra

The formulae in this appendix are taken mostly from [136].

B.1 Even dimensions

Consider a metric with signature $\eta_{ab} = \{(-)^{d_-}, (+)^{d_+}\}$, such that $d = d_- + d_+ = 2m$. The Clifford algebra is span by

$$\{\Gamma_i, \Gamma_j\} = 2g_{ij} . \tag{B.1}$$

All representations are unitarily equivalent, and the intertwining operators are

$$\Gamma_i^\dagger = A\Gamma_i A^{-1} \tag{B.2}$$
$$-\Gamma_i^T = C^{-1}\Gamma_i^T C \tag{B.3}$$
$$-\Gamma_i^* = D^{-1}\Gamma_i^T D \tag{B.4}$$
$$\Gamma_i^* = \tilde{D}^{-1}\Gamma_i^T \tilde{D} \tag{B.5}$$

where $D = CA^T$ and $\tilde{D} = \Gamma CA^T$. The chirality operator is

$$\Gamma = \Gamma^1 \cdots \Gamma^d . \tag{B.6}$$

We have the following phases

$$A = \alpha A^\dagger \tag{B.7}$$
$$C = \tilde{\eta} C^T \tag{B.8}$$
$$DD^{-1} = \delta \tag{B.9}$$
$$\tilde{D}\tilde{D}^{-1} = \tilde{\delta} \tag{B.10}$$

where $|\alpha| = 1$, $\tilde{\eta} = \pm 1$, and $\delta^* = \tilde{\delta}$. The Dirac conjugate is defined as $\bar{\psi} = \psi^\dagger A$, and the charge conjugate as $\psi^c = CA^T\psi^*$, or $\psi^c = \Gamma CA^T\psi^*$. The chirality matrix

satisfies

$$\Gamma^{\dagger} = (-)^m A\Gamma A^{-1} \qquad \qquad \text{(B.11)}$$

$$\Gamma^T = (-)^m C^{-1}\Gamma C . \qquad \qquad \text{(B.12)}$$

B.2   Odd dimensions

In this appendix we build a representation of an odd-dimensional $D = d+1$ Clifford algebra with $(\Gamma^n)^2 = \eta$ starting from a given even dimensional $d = 2m$ Clifford algebra. We actually only need to construct the correct Clifford matrix Γ^n

$$\Gamma^n = \sqrt{(-)^{m+d_-}\eta\hat{g}}\ \Gamma . \qquad \qquad \text{(B.13)}$$

In odd dimensions not all representations are equivalent. Instead, we only have the intertwining operators

$$\Gamma_{\underline{\alpha}}^{\dagger} = (-)^{d_-}\eta\ \tilde{A}\Gamma_{\underline{\alpha}}\tilde{A}^{-1} \qquad \qquad \text{(B.14)}$$

$$\Gamma_{\underline{\alpha}}^T = (-)^m\ \tilde{C}^{-1}\Gamma_{\underline{\alpha}}^T\tilde{C} \qquad \qquad \text{(B.15)}$$

$$\Gamma_{\underline{\alpha}}^* = (-)^{m+d_-}\eta\ \mathcal{D}^{-1}\Gamma_{\underline{\alpha}}\mathcal{D} \qquad \qquad \text{(B.16)}$$

where $\mathcal{D} = \tilde{C}\tilde{A}^T$, not to be confused with (5.33). There are two inequivalent conjugacy classes: Which representations belong to which depends on m, d_-, and η.

We can represent the odd-dimensional intertwiners in terms of their even dimensional counter parts as

$$(-)^{d_-}\eta = \begin{cases} +1, & \tilde{A} = A \\ -1, & \tilde{A} = A\Gamma^n \end{cases} \qquad \qquad \text{(B.17)}$$

$$(-)^m = \begin{cases} +1, & \tilde{C} = \Gamma^n C \\ -1, & \tilde{C} = C \end{cases} \qquad \qquad \text{(B.18)}$$

We sometimes abbreviate the sign $(-)^{d_-}\eta$ by ε. The operators $\mathcal{D}$ can be expressed in terms of boundary operators

$\mathcal{D}$	$(-)^m = 1$	$(-)^m = -1$
$(-)^{d_-}\eta = 1$	$\tilde{D}$	D
$(-)^{d_-}\eta = -1$	ηD	$-\tilde{D}$

$$\text{(B.19)}$$

where $D = CA^T$ and $\tilde{D} = \Gamma^n D$.

Appendix C

Fermion phase space

Fermions can be decomposed in bulk and boundary components in essentially two Lorentz invariant ways, namely by using chirality or reality conditions.

C.1 Chirality conditions

With chirality we refer to the eigenvalues ± 1 of $\sqrt{\eta}\,\Gamma^n$. This is essentially the only Clifford matrix whose eigenvalues we can consider without breaking Lorentz invariance explicitly.

Note, however, that if the bulk dimension $d+1$ is even there is naturally the bulk chirality matrix. We shall not consider this case separately, because it simply reduces to a change of the matrix M in the kinetic term: We can namely reduce the bulk gamma matrices $\Gamma^{\underline{\alpha}}$ to expressions involving only the boundary gamma matrices $\gamma^{\underline{\alpha}}$, pertinent to the one dimension lower Clifford algebra, by setting $\Gamma^{\underline{\alpha}} = \sigma^1 \otimes \gamma^{\underline{\alpha}}$. We can then combine the appearing extra Pauli sigma matrices with the coupling M and proceed as in Appendix C.3.

We have to divide our analysis in two cases depending on the sign η and details of the bulk metric. Defining $\sqrt{\eta}\,\Gamma^n\,\psi_\pm = \pm\,\psi_\pm$ we get

$$\bar{\psi}_\pm \sqrt{\eta}\,\Gamma^n = \pm\,\varepsilon\eta\,\bar{\psi}_\pm \ , \tag{C.1}$$

where ε is the sign that appears in the relation of the Clifford matrices to their Hermitian conjugates (B.14). This means that the the Dirac dual of a spinor of definite chirality is either of the same or the opposite chirality; as a consequence, the Lagrangian fields that correspond to the phase space coordinates will be different. It is useful to note that

$$\varepsilon\eta = \left\{ \begin{array}{ll} (-)^{d-} & d+1 \ \ \text{odd} \\ \eta & d+1 \ \ \text{even} \end{array} \right. \tag{C.2}$$

Case I. Assume $\varepsilon\eta = 1$. The kinetic term separates into

$$-\sqrt{\eta\hat{g}}(\bar{\psi}_- M \dot{\psi}_- + \dot{\bar{\psi}}_+ M \psi_+) - \partial_t G \qquad (C.3)$$

$$-\frac{1}{2}\bar{\psi}_- \partial_t(\sqrt{\eta\hat{g}}M)\,\psi_- - \frac{1}{2}\bar{\psi}_+ \partial_t(\sqrt{\eta\hat{g}}M)\,\psi_+ \qquad (C.4)$$

where

$$G \;=\; -\frac{1}{2}\sqrt{\eta\hat{g}}(\bar{\psi}_- M \psi_- + \bar{\psi}_+ M \psi_+)\,. \qquad (C.5)$$

The fermionic phase space consists therefore of the symplectic pairs $(\varphi, \bar{\pi})$ and $(\bar{\varphi}, \pi)$, where $\varphi = \psi_-$ and $\bar{\varphi} = \bar{\psi}_+$ and

$$\bar{\pi} \;=\; -\sqrt{\eta\hat{g}}\;\bar{\psi}_- M \qquad (C.6)$$

$$\pi \;=\; \sqrt{\eta\hat{g}}\;M\psi_+\,. \qquad (C.7)$$

The last two terms in (C.4) produce a term

$$\frac{1}{2}\hat{g}^{ij}G\;\partial_t g_{ij} \qquad (C.8)$$

in the action, and therefore cause a shift in the gravitational momentum.

Case II. Assume $\varepsilon\eta = -1$. The kinetic term separates into

$$-\sqrt{\eta\hat{g}}(\dot{\bar{\psi}}_- M \psi_+ + \bar{\psi}_+ M \dot{\psi}_-) - \partial_t G \qquad (C.9)$$

$$-\frac{1}{2}\bar{\psi}_- \partial_t(\sqrt{\eta\hat{g}}M)\,\psi_+ - \frac{1}{2}\bar{\psi}_+ \partial_t(\sqrt{\eta\hat{g}}M)\,\psi_- \qquad (C.10)$$

where

$$G \;=\; -\frac{1}{2}\sqrt{\eta\hat{g}}(\bar{\psi}_- M \psi_+ + \bar{\psi}_+ M \psi_-)\,. \qquad (C.11)$$

The configuration space is span by $\varphi = \psi_-$ and $\bar{\varphi} = \bar{\psi}_-$, and the momenta are

$$\bar{\pi} \;=\; -\sqrt{\eta\hat{g}}\;\bar{\psi}_+ M \qquad (C.12)$$

$$\pi \;=\; \sqrt{\eta\hat{g}}\;M\psi_+\,. \qquad (C.13)$$

Notice that, due to (B.17), the Dirac conjugates $\bar{\varphi}$ and $\bar{\psi}_-$ are formed differently: The former using the matrix A and the latter with $\tilde{A}$. This will result in an extra Γ^n everywhere, including the kinetic term, and the resulting extra sign hence cancels out. The gravitational momenta are shifted as in Case I.

C.2 Reality conditions

In this and the following sections, we have included the analysis of reality conditions, although not used in the main text.

Assume

$$\Gamma_\mu^T = \gamma \, \tilde{C}^{-1}\Gamma_\mu\tilde{C} \tag{C.14}$$

$$\Gamma_\mu^\dagger = \varepsilon \, \tilde{A}^{-1}\Gamma_\mu\tilde{A} \, . \tag{C.15}$$

Consider a generic differential form ϖ, and the thereto associated element in Clifford algebra

$$\varpi = \frac{1}{k!}\varpi_{\mu_1\cdots\mu_k} \, \Gamma^{\mu_1\cdots\mu_k} \, . \tag{C.16}$$

In particular, one could assume $\varpi = \mathbf{1}$ or $\varpi = \Gamma^n$, which would lead to the standard Majorana conditions discussed in Appendix B.1. Here we shall just assume

$$\varpi^2 = \eta' = \pm 1 \tag{C.17}$$

$$\hat{\slashed{D}}\varpi = (-)^k\varpi\hat{\slashed{D}} \tag{C.18}$$

$$\varpi\Gamma^n = \sigma\Gamma^n\varpi, \qquad \sigma = \pm 1 \tag{C.19}$$

$$\dot{\varpi}\Gamma^n = -\sigma\Gamma^n\dot{\varpi} \tag{C.20}$$

$$\varpi^T = \gamma' \, \tilde{C}^{-1}\varpi\tilde{C} \tag{C.21}$$

$$\varpi_\mu^\dagger = \varepsilon' \, \tilde{A}\varpi\tilde{A}^{-1} \, . \tag{C.22}$$

We can now define the ϖ-conjugate

$$\psi^\varpi = \varpi\mathcal{D}\psi^* \tag{C.23}$$

and use the decomposition $\psi = \psi_1 + i\psi_2$, where $\psi_{1,2}^\varpi = \psi_{1,2}$. The sign of $(\varpi\mathcal{D})^*(\varpi\mathcal{D}) = \pm 1$ will determine whether the decomposition is possible. For $\varpi = \mathbf{1}, \Gamma^n$ this just reduces to the signs $\delta, \tilde{\delta}$.

The fermion kinetic term decomposes into symmetric and antisymmetric parts

$$\sum_{a=1,2} \psi_a^T \, R\dot{\psi}_a + \psi_2^T B\dot{\psi}_1 - \psi_1^T B\dot{\psi}_2 \tag{C.24}$$

$$+ \sum_{a=1,2} \psi_a^T \, R \, (\tfrac{1}{2}\eta'\varpi\dot{\varpi}) \, \psi_a + \eta'\psi_2^T B\varpi\dot{\varpi}\psi_1 \tag{C.25}$$

where

$$R = \eta\eta'\sqrt{\tilde{g}} \, \tfrac{1}{2}(\gamma'\tilde{\eta}M + \gamma\sigma M^T)\tilde{C}^{-1}\varpi\Gamma^n \tag{C.26}$$

$$B = \eta\eta'\sqrt{\tilde{g}} \, \tfrac{1}{2}(\gamma'\tilde{\eta}M - \gamma\sigma M^T)\tilde{C}^{-1}\varpi\Gamma^n \, . \tag{C.27}$$

A Majorana condition leads to a good separation of the phase space if the diagonal term, and hence R, vanishes. This means that we have achieved separation if

$$M^T = -\sigma\gamma\gamma'\tilde{\eta}M \tag{C.28}$$

In the cases where M is antisymmetric, at least two fermions are needed.

The boundary kinetic term in $\mathcal{H}^\perp$ decomposes similarly

$$\frac{1}{2}\bar{\psi}M\hat{\slashed{p}}\psi - \frac{1}{2}(\hat{\slashed{p}}\bar{\psi})M\psi = \sum_{a=1,2} \psi_a^T \, \mathcal{R}\hat{\slashed{p}}\psi_a + \psi_2^T \, \mathcal{B}\hat{\slashed{p}}\psi_1 - \psi_1^T \, \mathcal{B}\hat{\slashed{p}}\psi_2 \tag{C.29}$$

where

$$\mathcal{R} = \eta'\frac{1}{2}(\gamma'\tilde{\eta}M + (-)^k\epsilon\gamma M^T)C^{-1}\varpi \tag{C.30}$$

$$\mathcal{B} = \eta'\frac{1}{2}(\gamma'\tilde{\eta}M - (-)^k\epsilon\gamma M^T)C^{-1}\varpi \ . \tag{C.31}$$

Provided we choose the global sign of the metric suitably, and that M satisfies (C.28), then $\mathcal{B}$ vanishes, and the boundary kinetic term decomposes to two terms that are quadratic in momenta and coordinates, respectively.

Assuming now (C.28), and that $\dot{M} = 0$ we can arrange the kinetic term into the form

$$2\psi_2 B\dot{\psi}_1 - \partial_t G + t^{i\underline{\alpha}}\dot{L}_{i\underline{\alpha}} \tag{C.32}$$

where

$$t^{i\underline{\alpha}} = L^{i\underline{\alpha}}G - \eta n^{\underline{\alpha}}\psi_2^T B\Gamma^{ni}\psi_1 \tag{C.33}$$

$$G = \psi_2^T B\psi_1 \ . \tag{C.34}$$

The phase space is given, therefore, by the pair $(\varphi, \pi) = (\psi_1, -2B\psi_2)$. The gravitational momenta get shifted by $t^{i\underline{\alpha}}$.

C.3   Symplectic conditions

If the condition (C.28) constrains M to be antisymmetric, we have to have at least two complex fermions in the bulk. We may say that the matrix M determines a symplectic structure in the fermion phase space.

If the bulk fermions are already constrained to be Majorana fermions, we need to proceed somewhat differently. Of course, if the boundary dimension is such that Majorana–Weyl fermions exist, we may resort to the chiral conditions treated in Appendix C.1. Otherwise, we need to have an even number of these real fermions.

We may then assume that ϖ is a matrix that carries both Clifford indices and mixes fermion flavors. The above analysis is true *verbatim*, provided the symplectic structures defined by ϖ and M are compatible, *i.e.* we can still use the property $[\varpi, M] = 0$. As the bulk spinors are already Majorana, we should put $\psi_2 = 0$, and choose the symmetry properties of M such that R in (C.30) does not vanish.

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