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Strong coupling analysis of $D = 2$ and $D = 4$ maximally supersymmetric YM theories

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Contents

Introduction

Supersymmetric instantons, i.e. topologically nontrivial classical solutions of supersymmetric Yang Mills theory which preserve some fraction of supersymmetry, have been widely studied as they provide a mechanism of supersymmetry dynamical breaking, which in turn can give important contributions to the solution of the Hierarchy Problem.

It is well known that a perturbative expansion around such an instanton trivializes, since quantum corrections cancel among each other, see [46] and references therein. Supersymmetric instantons do generate effects which are leading with respect to the perturbative ones. These striking properties are very well established as far as the weak YM coupling regime is concerned.

But what about the strong coupling regime? Is it possible to recover the abovementioned characteristics of a SYM theory in instanton background, also in the limit of strong coupling? Moreover, what is the meaning of such classical objects? Should it be possible that they become so relevant as to mediate almost all the interaction, regardless of whatever perturbative contributions the underlying quantum field theory provides?

It is indeed true that topologically non trivial classical solutions can be found in the $D = 4$, $\mathcal{N} = 4$ SYM theory, to fit nicely with the strong coupling limit and that they constitute, in such a regime, the driving sector of the theory.

Such solutions are very well known in $D = 2$, [6, 7, 8, 9]. They play a fundamental role in providing a precise connection between $D = 2$, $\mathcal{N} = (8, 8)$ and type IIA string theory, in the scenario of Matrix String Theory (MST). These solutions, which are called *stringy instantons*, can be in fact recognized to become, in the strong coupling limit, Riemann surfaces, which in turn are the candidates to represent closed string scatterings.

Let us see how this connection can be set up in the framework of M-theory, so that we can then extend it to obtain a four dimensional SYM theory.

M–theory can be represented, relying on the Matrix Theory Conjecture, [34, 36, 37, 38, 39], by the supersymmetric quantum mechanics of matrices ($U(N)$ SYM theory in $0+1$ dimensions, with 16 supercharges) in the large N limit. Matrix theory is in fact based on the idea that a system of N D0–branes infinitely boosted along a fixed direction, say the 11–th, describes the essential features of M theory. Each D0–brane has the 11-th component of the momentum $p_{11} \sim 1/R_{11}$ far larger than the transverse components (where R_{11} is interpreted as a large compactification radius for M theory). It is expected that the Matrix Theory description of M theory becomes more and more faithful as N becomes larger and larger. Compactification of M–theory on a circle of radius, say, R_9 is expected to lead to IIA theory in the $R_9 \to 0$ limit. In Matrix Theory the corresponding operation consists in compactifying the base manifold of SYM theory on the dual radius, so that one ends up with 1+1 dimensional SYM theory with $U(N)$ gauge group and $\mathcal{N} = (8,8)$ supersymmetry: this is called Matrix String Theory.

With the aim of obtaining a four dimensional SYM theory, let us push forward this procedure, simply compactifying the base manifold of SYM theory along some additional dimensions, [41] (see also [42, 43]). Let us denote by $\tilde{R}_i$ the field theory compactification radii and by R_i the corresponding M–theory radii. They are related by, see [41],

$$\tilde{R}_i = \frac{\ell_{11}^3}{R_i R_{11}}, \qquad i = 9, 8, ... \tag{1}$$

where ℓ_{11} is the 11–th dimensional length scale. For example, if one compactifies on a two–torus ($i = 9, 8$) and takes the limit $R_9, R_8 \to 0$, one can convince himself that one series of massless states is produced, which is interpreted as a new dimension that opens up. This new dimension plus the one implicit in the large N limit lead us back to 10 dimensions in a type IIB framework. If, instead, we compactify on a three–torus ($i = 9, 8, 7$) and take the limit $R_9, R_8, R_7 \to 0$ we can see that three new dimensions open up: in this case dimensions and context are those of M–theory. The latter however is not the limit we are interested in here. We will rather consider the limit

$$R_7, R_8 \to 0, \qquad R_9 \to \infty. \tag{2}$$

It is easy to see that in this case only one series of massless states is produced, i.e. only one new dimension opens up (instead of three). Naturally we have to add the decompactification related to R_9 being very large and the new dimension implicit in the large N limit. Therefore the context of (2) is that of a 10 dimensional type IIB theory. Moreover it describes D3–branes, as can be seen by starting from Matrix Theory, which is a theory of D0–branes, compactifying along the 7,8,9–th directions and t–dualizing the three circles. The D0–branes become D3–branes wrapped around the three–torus. Finally one takes the limit (2) or the corresponding one in the dual variables according to (1).

The YM theory we obtain is exactly $D = 4$, $\mathcal{N} = 4$ SYM. In fact the dependence on the compactification radii can be entirely collected in the dimensionless coupling constant

$$g_{YM}^2 = \frac{\ell_{11}^3}{R_7 R_8 R_9}. \tag{3}$$

It is clear that the compactification regime (2) allows us to recover the strong coupling limit in the SYM theory.

The abovementioned argument suggests how, similarly to the MST case, there should be room in $\mathcal{N} = 4$ SYM for the existence of classical solutions, analogous to two dimensional stringy instantons, which could possibly be interpreted, at strong coupling, as carriers of the interaction between D3–branes.

The first four chapters of this dissertation will be devoted to find such solutions, which will be called *ic-instantons*, and to show how the $\mathcal{N} = 4$ action, expanded around them, reduces at strong coupling to the world-volume theory for D3–branes. The ic-instantons are found to be maps which represent spectral coverings of the base manifold.

In the subsequent chapters we will concentrate on the possible quantum corrections which can arise, up to the second order in the inverse YM coupling, to the world-volume theory, from the perturbative expansion of the $\mathcal{N} = 4$. All the arguments here will be carried out in parallel, both in two and four dimensions, in order to give a complete description of the problem. We will find out that, as far as the perturbative sector is concerned, really no substantial differences are present between the two cases, as a consequence of the fact that they both originate from the ten dimensional SYM theory.

As a byproduct of this analysis, we will get some insights about the abelianization procedure of maximally supersymmetric YM theories. Indeed ic-instantons can be the right solutions to be used in the abelianization. This subject has been firstly studied in [47], where a $U(N)$ YM theory is shown to become an abelian one, with a suitable gauge choice, and with the appearance of monopoles manifesting topological properties which are similar to the ones of our ic-instantons.

Chapter 1

A new kind of instantons

1.1 Introduction

Starting from the $D = 4$, $\mathcal{N} = 4$, SYM action we derive the equations for 1/4 supersymmetry preserving classical solutions. We then sketch the basic properties of such solutions. Finally, relying on the properties of branched coverings, a particular form for this solutions is derived, in which it is clear how its coupling dependence is connected with its worldsheet singularities.

The material of the present and the following chapter is mainly based on [10].

1.2 Interaction–carrying classical solutions

The Minkowski action of $\mathcal{N} = 4$ SYM theory in 4D is

$$\begin{aligned} S &= \int_{\mathcal{X}} d^4x \,\mathrm{Tr}\,\left(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu X^I D^\mu X_I + \frac{g^2}{4}\left[X^I, X^J\right]^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu \lambda \right. \\ &\left. + \frac{g}{4}\left(\lambda^T C \gamma^{I\dagger}\left[X^I, \lambda\right] - \lambda^\dagger C \gamma^I \left[X^I, \lambda^*\right]\right)\right) \end{aligned} \qquad (1.1)$$

where $i = 1, \ldots, 6$. $\mathcal{X}$ is a four dimensional manifold of the type $\mathcal{X} = \mathbb{R} \times M_3$, where M_3 is a three–dimensional compact manifold and $\mathbb{R}$ is the line $-\infty < x^0 < \infty$. Although the action (1.1) can be studied on more general manifolds, we will consider in the following essentially two examples: $M_3 = S^3$, the 3–sphere, and $M_3 = \mathbb{T}^3$, the 3–torus defined by periodic x^1, x^2, x^3. We always suppose that $\mathcal{X}$ admit complex structures.

$F^{\mu\nu}$ is the field strength of the gauge field A_μ, the X^I are $N \times N$ hermitean matrices in the adjoint of $U(N)$; from a geometrical point of view, we understand the existence of a vector bundle E, with structure group $U(N)$, so that X^I are sections of $EndE$. λ is an $N \times N$ matrix whose entries are both Weyl spinors of $SO(1,3)$ and vectors in the fundamental of $SU(4)$: namely the γ^μ's will act on the $SO(1,3)$ spinorial indices, while the γ^I's on the $SU(4)$ ones. Since we make explicit use of them in the following, we write down our definitions for the gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \gamma^{I\dagger} & 0 \end{pmatrix}$$

where $\sigma^0 = -\bar{\sigma}^0 = \mathbf{1}$, $\sigma^i = \bar{\sigma}^i$ are the Pauli matrices; the Γ^I are the 8×8 6D gamma matrices as in [13] and C is the 4D charge conjugation matrix; they satisfy the usual anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \{\Gamma^I, \Gamma^J\} = 2\delta^{ij}. \tag{1.2}$$

The supersymmetric transformations are

$$\begin{aligned} \delta X^I &= \frac{i}{g}\left(\epsilon^T C \gamma^{I\dagger} \lambda - \epsilon^\dagger C \gamma^I \lambda^*\right) \\ \delta A_\mu &= -i\left(\bar{\epsilon}\gamma^\mu\lambda - \bar{\lambda}\gamma^\mu\epsilon\right) \\ \delta\lambda &= -\frac{1}{g^2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon - i\left[X_I, X_J\right]\gamma^{ij}\epsilon + \frac{2}{g}D_\mu X_I \gamma^\mu\gamma^0 C\gamma^I\epsilon^*. \end{aligned} \tag{1.3}$$

Ic–instantons in 2D can appear either as BPS classical solutions of the SYM theory or as 4D self–dual systems reduced to 2 dimensions. Analogously, in 4D, ic–instantons can be seen either as classical solutions that preserve part of the supersymmetry or as 8D self–dual systems reduced to 4D.

In the next subsection we will discuss classical solutions that preserve some supersymmetry. We can follow two courses: either we use the Minkowski supersymmetric transformations (1.3) above, find classical equations whose solutions preserve a fraction of supersymmetry and Wick–rotate such equations to their Euclidean form; or we can use the Euclidean version of the supersymmetry transformations and find equations whose solutions preserve some supersymmetry. For simplicity we take the second course but the result is the same in both cases. However passing to the Euclidean formulation introduces a well–known problem in supersymmetric theories. The Euclidean transcription of a supersymmetric Minkowski theory may considerably modify the supersymmetric properties of the latter if Weyl/Majorana fermions are involved, which is the case here. There are several recipes to deal with this problem, see [14] and references therein. We will follow [14]: such an approach amounts to an effective doubling of the degrees of freedom of the Euclidean version with respect to the Minkowski one.

With some abuse of language we will call the above solutions BPS solutions, in the sense of supersymmetry preserving solutions. This is substantially motivated by the fact that the final theory on the covering space at strong coupling will turn out to be supersymmetric (see below).

1.2.1 Ic–instanton equations as BPS solutions

We write first the Euclidean action in terms of the complex coordinates $v = \frac{1}{2}(x^1 + ix^2)$, $w = \frac{1}{2}(x^3 + ix^4)$,

$$\begin{aligned} S \;=\; & \int_{\mathcal{X}} d^2v d^2w \;\mathrm{Tr}\, \Big(D_v X^I D_{\bar{v}} X^I + D_w X^I D_{\bar{w}} X^I - \frac{g^2}{2}[X^I, X^J]^2 \\ & -\frac{1}{4g^2}(F_{v\bar{v}}^2 + F_{w\bar{w}}^2 - 2F_{vw}F_{\bar{v}\bar{w}} - 2F_{v\bar{w}}F_{\bar{v}w}) \\ & -2\left(\lambda_1^* D_{\bar{v}}\lambda_1 + \lambda_2^* D_v \lambda_2\right) - 2\left(\lambda_1^* D_{\bar{w}}\lambda_2 - \lambda_2^* D_w \lambda_1\right) \\ & -\frac{g}{2}\left(\lambda^T C \gamma^{I\dagger}\left[X^I, \lambda\right] - \lambda^\dagger C \gamma^I \left[X^I, \lambda^*\right]\right)\Big) \end{aligned} \qquad (1.4)$$

Next we write the Euclidean version of the $\mathcal{N} = 4$ supersymmetric transformations

$$\begin{aligned} \delta X^I &= \frac{i}{g}\left(\epsilon^T C \gamma^{I\dagger}\lambda - \epsilon^\dagger C \gamma^I \lambda^*\right) \\ \delta A_\mu &= -\left(\epsilon^\dagger \gamma^\mu \lambda - \lambda^\dagger \gamma^\mu \epsilon\right) \\ \delta\lambda &= -\frac{1}{g^2} F_{\mu\nu}\gamma^{\mu\nu}\epsilon - i\left[X_I, X_J\right]\gamma^{ij}\epsilon - \frac{2i}{g} D_\mu X_I \gamma^\mu C \gamma^I \epsilon^* \end{aligned} \qquad (1.5)$$

where, according to [14], we consider the variables λ^* and ϵ^* as independent from λ and ϵ, respectively. The superscript ^{T} represents the transpose matrix and † stands for ^{*T}.

We look for solutions that preserve $\frac{1}{4}$ supersymmetry, by setting all fermions and all X^I, with $i = 3, .., 6$, to zero, and defining $X = X^1 + iX^2$ and $\bar{X} = X^\dagger$. The equations that define such solutions are

$$F_{v\bar{v}} + F_{w\bar{w}} - ig^2[X, \bar{X}] = 0 \qquad (1.6)$$

$$F_{vw} = 0, \quad F_{\bar{v}\bar{w}} = 0, \qquad (1.7)$$

$$D_{\bar{v}}X = 0 = D_v\bar{X}, \quad D_w\bar{X} = 0 = D_{\bar{w}}X \qquad (1.8)$$

We will refer to the solutions of these equations as ic–instantons. Analogous equations for ic–anti–instantons can be obtained by an anti–holomorphic involution. Similar equations were previously discussed, in the context of $\mathcal{N} = 4$ theory, for compact manifolds by [15].

1.2.2 General properties of ic–instantons

The system of equations (1.6,1.7,1.8) has various types of solutions. Notice that, if we set $X = 0$, they becomes the usual self–duality condition in 4D. Therefore the set of solutions of (1.6,1.7,1.8) will include in particular all the ordinary instantons of YM in 4D, compatible with the topology of the base manifold. In principle we could consider solutions with $X \neq 0$ and nonvanishing instanton number. However the vector bundle E is such that $c_2(E)$ is trivial, therefore we only consider solutions with vanishing instanton number.

Let us now compute the action of a configuration that satisfies (1.6,1.7,1.8). Starting from (1.4) we get

$$S_{inst} = \frac{1}{2}\int_{\mathcal{X}} d^2v d^2w \quad \mathrm{Tr} \quad \Big(D_v X D_{\bar{v}} \bar{X} + D_v \bar{X} D_{\bar{v}} X + D_w X D_{\bar{w}} \bar{X} + D_w \bar{X} D_{\bar{w}} X + \frac{g^2}{2}[X, \bar{X}]^2 - \frac{1}{2g^2}(F_{v\bar{v}}^2 + F_{w\bar{w}}^2 - 2F_{vw}F_{\bar{v}\bar{w}} - 2F_{v\bar{w}}F_{\bar{v}w})\Big) \tag{1.9}$$

This can be rewritten as

$$\begin{aligned} S_{inst} &= S_{bulk} + S_{boundary} \\ S_{bulk} &= \int d^2v d^2w \, \mathrm{Tr}\Big(D_v \bar{X} D_{\bar{v}} X + D_w \bar{X} D_{\bar{w}} X + \frac{1}{g^2} F_{vw} F_{\bar{v}\bar{w}} \\ &\quad - \frac{1}{4g^2}(F_{v\bar{v}} + F_{w\bar{w}} - ig^2[X, \overline{X}])^2\Big) \\ S_{boundary} &= \int d^2v d^2w \, \mathrm{Tr}\Big(D_v(X D_{\bar{v}} \bar{X}) + D_w(X D_{\bar{w}} \bar{X}) + \frac{1}{4g^2} dK(A_v, A_w)\Big), \end{aligned} \tag{1.10}$$

where K is the Chern–Simons term corresponding to $F \wedge F$. More explicitly

$$(F_{v\bar{w}} F_{\bar{v}w} - F_{vw} F_{\bar{v}\bar{w}} + F_{v\bar{v}} F_{w\bar{w}})\, d^2v d^2w = dK(A_v, A_w)$$

where d is the exterior derivative in 4D. One sees immediately that for an ic–instanton $S_{bulk} = 0$. It is well–known that the Chern–Simons term in $S_{boundary}$ is equal to the instanton number. Therefore, since in this paper we only consider solutions with instanton number 0, the Chern–Simons term does not contribute. The other term in $S_{boundary}$ is usually divergent for the ic–instantons solutions and apparently one cannot attach any geometric meaning to it. On the other hand it is very easy to get rid of it by simply saying that our starting action is (1.4), in which the first two terms have been modified to $-\frac{1}{2}\Big(X^I\{D_v, D_{\bar{v}}\}X^I + X^I\{D_w, D_{\bar{w}}\}X^I\Big)$. With these provisos the action of the ic–instantons considered in this paper vanishes. A similar conclusion holds for ic–anti–instantons.

1.3 Ic–instantons

Our purpose is to find solutions (A, X) of (1.6,1.7,1.8). For definiteness let us consider a concrete case, say $\mathcal{X} = \mathbb{R} \times \mathbb{T}^3$. The construction is parallel to the one carried out in [6, 8]. In the following we stick to the complex structure of the punctured sphere $\mathbb{P}^1$ times $\mathbb{T}^2$, with local coordinates v and w. At times it is convenient to use the coordinate $z = e^v$. We start from the (simple) ansatz

$$A_v = i\partial_v Y^\dagger (Y^{-1})^\dagger, \qquad A_w = i\partial_w Y^\dagger (Y^{-1})^\dagger, \qquad X = Y^{-1} M Y \tag{1.11}$$

where Y is a generic element in the complex group $SL(N, \mathbb{C})$ and M specifies a branched covering of the base manifold. A more general ansatz will be considered later on. As a consequence of (1.11) the equations $D_{\bar{v}} X = 0 = D_v \bar{X}$ are equivalent to

$$\partial_{\bar{v}} M = 0 = \partial_{\bar{w}} M \tag{1.12}$$

which means that the matrix M is holomorphic in v, w. Eq.1.12 guarantees that eqs.(1.8) are satisfied.

The ansatz (1.11) is given in terms of two matrices, Y and M. Y will be called the *group theoretical factor*, while M defines a general *branched covering* of the base manifold, i.e. a two dimensional complex manifold. The factor Y will be discussed below, while branched coverings will be discussed later on. For the time being let us give some essential information. Let us consider the polynomial

$$P_X(y) = \det(y - X) = y^N + \sum_{i=0}^{N-1} y^i a_i \,,$$

where y is a complex indeterminate. The equation

$$P_X(y) = 0 \tag{1.13}$$

can also be written as the matrix equation

$$X^N + a_{N-1}X^{N-1} + \cdots + a_0 = 0\,. \tag{1.14}$$

A diagonalizable matrix, which is solution of eq. (1.14), can always be cast in the canonical form

$$M = \begin{pmatrix} -a_{N-1} & -a_{N-2} & \dots & \dots & -a_0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{1.15}$$

Due to (1.12), we have $\partial_{\bar{v}} a_i = 0 = \partial_{\bar{w}} a_i$, which means that the set of functions $\{a_i\}$ are holomorphic in v, w, although they are allowed to have poles at $z = 0$ and $z = \infty$. The point is that, as we shall see in many examples, Eq.(1.13) identifies in the (y, z, w) space a complex 2–manifold (a surface) Σ, which is an N–sheeted branched covering of the base manifold. The explicit form of the covering is given by the set $\{x^{(1)}(z,w), \dots, x^{(N)}(z,w)\}$ of eigenvalues of X. Each eigenvalue spans a sheet. The projection map to the base cylinder $\mathcal{X}$ will be denoted $\pi : \Sigma \to \mathcal{X}$. The divisor (complex 1–submanifold) where two eigenvalues coincide is the branch locus. We can also define branch cuts: they are 3d manifolds that connect disconnected components of the branch locus.

We stress that the covering is independent of the coupling g.

1.3.1 Explicit construction of ic–instantons

The aim of the present subsection is to construct the group theoretical factor corresponding to the most general covering. The construction is close to the one in [8], so we will be brief.

Let us recall our ansatz (1.11). The group theoretical factor Y takes values in the complex group $SL(N, \mathbb{C})$, while the matrix M determines the branched covering. The dependence on the Yang-Mills coupling constant g is contained in the Y factor, while M does not depend on g. We set $Y = KL$ where L, *the dressing factor*, is expected to tend to 1 in the strong coupling limit outside the branch locus, while K is a special matrix, independent of g, endowed with the property that $K^{-1}MK$ and $K^{\dagger}M^{\dagger}(K^{\dagger})^{-1}$ are simultaneously diagonalizable.

It is well-known, [4], that the matrix M can be diagonalized

$$M = S\hat{M}S^{-1}, \qquad \hat{M} = \mathrm{Diag}(\lambda_1, \dots, \lambda_N) \tag{1.16}$$

by means of the following matrix $S \in SL(N, \mathbb{C})$:

$$S = \Delta^{-\frac{1}{N}} \begin{pmatrix} \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \dots & \lambda_N^{N-1} \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \dots & \lambda_N^{N-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1 \end{pmatrix}, \tag{1.17}$$

where

$$\Delta = \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)\,. \tag{1.18}$$

Δ vanishes whenever two eigenvalues coincide. Two coincident eigenvalues define a component of the branch locus of the covering. Going around a branch locus and crossing a branch cut in the v, w–plane, produces a reshuffling of the eigenvalues that can be represented via a monodromy matrix Λ: $\hat{M} \to \Lambda\hat{M}\Lambda^{-1}$. Correspondingly we have $S \to S\Lambda^{-1}$, so that the single–valuedness of M is preserved.

The explicit construction of K and L in the general case is given in [8] and will not be reported here. The qualitative features are as follows. First one introduces a monodromy–invariant K such that $K^{-1}S = U$ be unitary. To this end one sets $K = \sqrt{SS^{\dagger}}$ and easily verifies that U is unitary. As it turns out, K may have singularities at the points of $\mathcal{X}$ where any two eigenvalues of M coincide, i.e. at the branch locus of the spectral covering (the elements of K contains as factors fractional powers of $|\Delta|$). Therefore $K^{-1}MK$ is in general singular at these points. That is why we must introduce into the game a new monodromy invariant matrix L, with the purpose of canceling the singularities of $K^{-1}MK$ in such a way that $L^{-1}K^{-1}MKL$ be smooth and satisfy (1.6,1.7). Let us denote again by ϕ the generic entry of L. For (1.6) to be satisfied ϕ must satisfy, [8], an equation of the WZNW type with the following general structure

$$(\partial_v\partial_{\bar{v}} + \partial_w\partial_{\bar{w}})\phi + \dots \sim (\partial_v\partial_{\bar{v}} + \partial_w\partial_{\bar{w}}) \ln|\Delta| = \pi\Big(\partial_v\Delta\partial_{\bar{v}}\bar{\Delta} + \partial_w\Delta\partial_{\bar{w}}\bar{\Delta}\Big)\delta(\Delta)\,, \tag{1.19}$$

where dots represent all the other terms, which are irrelevant in the cancellation of singularities. In some equations (but not in all) the coefficients in front of the delta–function terms may vanish. This term has support at the zeroes of Δ, i.e. at the branch locus. The equation $F_{vw} = 0$ in (1.7), on the other hand, does not give rise to delta function terms:

$$\partial_v \partial_w \phi + \dots = 0 \tag{1.20}$$

Let us refer to the above equations collectively as the 'dressing equations'.

By construction K is independent of g while L does depend on g. One can show that in fact $L \to 1$ as $g \to \infty$, outside the zeroes of the discriminant. Let us present a simple argument in this sense.

The solution X exists with the required properties only if the 'dressing equations' admit solutions that vanish at $v = \pm\infty$ (to this end, of course, we have to exclude possible branch locus components at $t = \pm\infty$ from the right hand side of eq.(1.19). To our best knowledge, not much is known in the literature concerning the existence of such solutions. Based on the analysis of [7], we assume that the 'dressing equations' do admit solutions that vanish at $v = \pm\infty$. Once one assumes this, it is rather easy to argue, on a completely general ground, that in the strong coupling limit, $g \to \infty$, such solutions vanish outside the zeroes of the discriminant. The argument goes as follows. Consider a candidate solution of (1.6,1.7) in which $\phi = 0$ outside the zeroes of the discriminant, for all the ϕ's. Then, there, $L = 1$, and $X = K^{-1}MK$. As noted previously, in such a situation $[X, \bar{X}] = 0$, since both X and $\bar{X}$ are simultaneously diagonalized by the matrix $U = K^{-1}S$. Now we have to show that also $F_{v\bar{v}}$ and $F_{w\bar{w}}$ vanish outside the zeroes of the discriminant if $L = 1$. In fact when $L = 1$,

$$A_{\bar{v}} = -iK^{-1}\partial_{\bar{v}}K = -i(K^{-1}SS^{-1})\partial_{\bar{v}}(SS^{-1}K) = -iU(\partial_{\bar{v}} + \tilde{A}_{\bar{v}})U^{-1}\,,$$

where $\tilde{A}_{\bar{v}} = S^{-1}\partial_{\bar{v}}S$. But $\partial_{\bar{v}}S \equiv 0$ due to holomorphicity of the eigenvalues of M. Therefore $F_{v\bar{v}} = 0$. The same can be done for A_w, therefore $F_{w\bar{w}} = F_{vw} = 0$. In conclusion (1.6,1.7) is identically satisfied by the ansatz $L = 1$ outside the zeroes of the discriminant. Since the solutions are uniquely determined by their boundary conditions, we can conclude that, as $g \to \infty$, the only solution of the dressing equations outside the zeroes of the discriminant, is the identically vanishing solution. We infer from this argument that the solutions of the dressing equations for large g are concentrated around the branch locus and become more and more spiky as g grows larger and larger. Therefore the matrix L has the properties we expect.

The previous argument hinges on the occurrence that, as $g = \infty$, we have both $[X^\infty, \bar{X}^\infty] = 0$ and $F^\infty_{v\bar{v}} = F^\infty_{w\bar{w}} = 0$ (the superscript $^\infty$ obviously represents the strong coupling value of a field). Other types of solutions can be envisaged, see below and [8].

1.3.2 Generalized ic–instantons

In this paper we will have to take into consideration more general solutions than those just studied. Instead of (1.11) let us start from

$$A_v = i\mathcal{D}_v Y^\dagger (Y^{-1})^\dagger, \qquad A_w = i\mathcal{D}_w Y^\dagger (Y^{-1})^\dagger, \qquad X = Y^{-1} M Y \tag{1.21}$$

where Y is as before and the covariant derivative $\mathcal{D}$ is relative to a connection $\mathcal{A}$ which commutes with M. As a consequence of this, we have again that the equations $D_{\bar{v}} X = 0 = D_v \bar{X}$ imply (1.12). Moreover, the connection $\mathcal{A}$ is diagonalized by S and

$$\mathcal{A} = S\hat{A}S^{-1}$$

Since any solution A, X must be smooth, the monodromy of $\hat{A}$ must be the same as the monodromy of $\hat{M}$, i.e. going around a branch locus produces a reshuffling of the eigenvalues that can be represented via the same monodromy matrix Λ: $\hat{A} \to \Lambda \hat{A} \Lambda^{-1}$.

The construction of such ic–instantons carries through as before. The only remarkable difference is that in the strong coupling limit the connection $\mathcal{A}^\infty = U\hat{A}U^{-1}$ does not evaporate into a pure gauge as before. Not only do we have a covering described by $\hat{X}$, but also a connection $\hat{A}$ valued in the Cartan subalgebra.

In the strong coupling limit, instead of (1.6), in this case we find

$$\begin{aligned} &[X^\infty, \bar{X}^\infty] = 0 \\ &F^\infty_{v\bar{v}} + F^\infty_{w\bar{w}} = 0 \end{aligned} \tag{1.22}$$

i.e. we obtain a non–trivial self–dual connection. Of course we can do the same with anti–self–dual ic–instantons and obtain strong coupling antiself–dual connections

An important proviso: a basic condition for us to call all the above solutions *ic–instantons* is that $[X, \bar{X}] \neq 0$ for finite g: only in this case do they represent interpolating solutions between genuine initial and final brane configurations.

Chapter 2

The expansion of $D = 4$, $\mathcal{N} = 4$ SYM in ic-instanton background

2.1 Introduction

Our purpose in this section is to expand the action about a classical ic–instanton solution. For definiteness we choose an instanton rather than an anti–instanton, but everything can be repeated for the latter. The analysis is along the lines of [7], but there are important differences which we will try to emphasize while we will rapidly go through the repetitive aspects.

2.2 The background

As a first step let us analyze the background part. The dependence on the coupling is entirely contained in the factor L. We have seen that in the strong coupling limit $L \to 1$ outside the branch locus of the covering. Since here we are interested in expanding the action (1.4) in inverse powers of $1/g$, and actually in singling out the dominant term in this expansion (see below), we will consider the action (1.4) around a given classical solution stripped of the above dressing factor, and exclude from the integration region the branch locus on the base manifold, $\mathcal{X}$. In other words we will consider from now on the action (1.4) in which the relevant Y is replaced by K and the integral extends over $\mathcal{X}_0$ which is the initial $\mathcal{X}$ from which small tubular neighborhoods have been cut out around the branch locus. Said otherwise, we introduce in our integrated action a regulator (which will eventually be removed).

After getting rid of the dressing factor, the classical background configuration is specified by X^∞ and A^∞ (see section 3.2). As expected, this configuration is singular exactly at the branch locus. We have seen that $M = S\hat{M}S^{-1}$. $\hat{M}$ is the matrix of eigenvalues of M and of X, so we denote it equivalently by $\hat{X}$. In the strong coupling limit $X \to U\hat{X}U^{-1}$, where $U = K^{-1}S$ is a unitary matrix and therefore simultaneously diagonalizes X and $\bar{X}$. Corresponding to $\hat{X}$ we have $\hat{A}_v, \hat{A}_w$.

U is finite in $\mathcal{X}_0$. Therefore, with a gauge transformation, we can remove it from the action defined in $\mathcal{X}_0$. This leads us to

- $\hat{X}$ and $\hat{A}$ diagonal,

for the classical background in the strong coupling limit.

Let us return now to the bosonic action (1.4) (the fermionic part will be analyzed later on) with the above understanding of the background part. To extract the strong coupling effective theory, we first rewrite the action in the following useful form

$$\begin{aligned} S^{(b)} &= \int d^2v d^2w \,\mathrm{Tr}\Big(D_v X^I D_{\bar{v}} X^I + D_w X^I D_{\bar{w}} X^I - \frac{g^2}{2}[X^I, X^J]^2 - g^2[X^I, X][X^I, \overline{X}] \\ &\quad + D_v \bar{X} D_{\bar{v}} X + D_w \bar{X} D_{\bar{w}} X + \frac{1}{g^2} F_{vw} F_{\bar{v}\bar{w}} - \frac{1}{4g^2}(F_{v\bar{v}} + F_{w\bar{w}} - ig^2[X, \overline{X}])^2 \Big) , \end{aligned}$$

where (only in the present chapter) $I = 3, ..., 6$. We now expand the action around a generic ic–configuration as follows

$$\Phi = \Phi^{(b)} + \phi^{\mathfrak{t}} + \phi^{\mathfrak{n}} \equiv \Phi^{(b)} + \phi \equiv \Phi^{\circ} + \phi^{\mathfrak{n}} , \tag{2.1}$$

where $\Phi^{(b)}$ is the background value of the field at infinite coupling, $\phi^{\mathfrak{t}}$ are the fluctuations along the Cartan directions and $\phi^{\mathfrak{n}}$ are the fluctuations along the complementary directions in the Lie algebra $\mathfrak{u}(N)$. In the following we suppose we have carried out the operation described above and by background value we refer to the diagonal representation.

The expansion of the action starts with quadratic terms in the fluctuations and $\hat{A}$ drops out from all the terms, except from the kinetic energy term of the (diagonal) Yang–Mills field. To simplify the subsequent formulas we will drop $\hat{A}$ for the time being and resume it later on.

To proceed further let us fix the gauge. We use, in the strong coupling limit, the following gauge–fixing term

$$S_{gf} = \frac{1}{4\pi g^2} \int d^2v d^2w \ \mathrm{Tr}\ \mathcal{G}^2 \tag{2.2}$$

where

$$\mathcal{G} = D^\circ_v a_{\bar{v}} + D^\circ_{\bar{v}} a_v + D^\circ_w a_{\bar{w}} + D^\circ_{\bar{w}} a_w + ig^2([X^\circ, \bar{x}] + [\bar{X}^\circ, x]) + 2ig^2[X^{\circ I}, x^I], \tag{2.3}$$

and D° is the covariant derivative with respect to A°. Next we introduce the Faddeev–Popov ghost and antighost fields c and $\bar{c}$ and expand them like all the other fields and add to the action the corresponding Faddeev–Popov ghost term

$$S_{ghost} = -\frac{1}{2\pi g^2} \int d^2v d^2w \ \mathrm{Tr}\ \left(\bar{c} \frac{\delta \mathcal{G}}{\delta c} c \right), \tag{2.4}$$

where δ represents the gauge transformation with parameter c.

At this point, to single out the strong coupling limit of the action, we rescale the fields in appropriate manner. Precisely, we redefine our fields as follows

$$A_v = g a^{\mathrm{t}}_v + a^{\mathrm{n}}_v, \quad A_w = g a^{\mathrm{t}}_w + a^{\mathrm{n}}_w, \quad X = \hat{X} + x^{\mathrm{t}} + \frac{1}{g} x^{\mathrm{n}}, \quad X^I = x^{I\mathrm{t}} + \frac{1}{g} x^{I\mathrm{n}}$$

and likewise for the conjugate variables. For the ghosts we set

$$c = g c^{\mathrm{t}} + \sqrt{g} c^{\mathrm{n}}, \quad \bar{c} = g \bar{c}^{\mathrm{t}} + \frac{1}{\sqrt{g}} \bar{c}^{\mathrm{n}}.$$

After these rescalings the action becomes

$$S^{(b)} = S^{(b)}_{sc} + S^{(b)}_{\mathrm{n}} + o\left(\frac{1}{\sqrt{g}}\right),$$

where

$$\begin{aligned} S^{(b)}_{sc} &= \int_{\mathcal{X}_0} d^2v d^2w \ \mathrm{Tr}\ \big[\partial_v x^{I\mathrm{t}} \partial_{\bar{v}} x^{I\mathrm{t}} + \partial_w x^{I\mathrm{t}} \partial_{\bar{w}} x^{I\mathrm{t}} + \partial_v x^{\mathrm{t}} \partial_{\bar{v}} \bar{x}^{\mathrm{t}} + \partial_w x^{\mathrm{t}} \partial_{\bar{w}} \bar{x}^{\mathrm{t}} \\ &+ \partial_v \bar{c}^{\mathrm{t}} \partial_{\bar{v}} c^{\mathrm{t}} + \partial_w \bar{c}^{\mathrm{t}} \partial_{\bar{w}} c^{\mathrm{t}} + \partial_v a^{\mathrm{t}}_{\bar{v}} \partial_{\bar{v}} a^{\mathrm{t}}_v + \partial_w a^{\mathrm{t}}_{\bar{w}} \partial_{\bar{w}} a^{\mathrm{t}}_w + \partial_v a^{\mathrm{t}}_{\bar{w}} \partial_{\bar{v}} a^{\mathrm{t}}_w + \partial_w a^{\mathrm{t}}_{\bar{v}} \partial_{\bar{w}} a^{\mathrm{t}}_v\big] \end{aligned} \tag{2.5}$$

$S^{(b)}_K$ is the purely quadratic term in the ϕ^{n} fluctuations. Let us see this in detail.

$S_K^{(b)}$ has the form

$$S_{\rm n} = \int d^2v d^2w \ {\rm Tr} \ \left[\bar{x}^{\rm n} \mathcal{Q} x^{\rm n} + x^{I{\rm n}} \mathcal{Q} x^{I{\rm n}} + a_{\bar{v}}^{\rm n} \mathcal{Q} a_v^{\rm n} + a_{\bar{w}}^{\rm n} \mathcal{Q} a_w^{\rm n} + \bar{c}^{\rm n} \mathcal{Q} c^{\rm n}\right] , \tag{2.6}$$

where

$$\mathcal{Q} = {\rm ad}_{\bar{X}^\circ} \cdot {\rm ad}_{X^\circ} + {\rm ad}_{a_{\bar{v}}^{\rm t}} \cdot {\rm ad}_{a_v^{\rm t}} + {\rm ad}_{a_{\bar{w}}^{\rm t}} \cdot {\rm ad}_{a_w^{\rm t}} + {\rm ad}_{x^{I{\rm t}}} \cdot {\rm ad}_{x^{I{\rm t}}} \tag{2.7}$$

There are no zero modes involved; therefore the integration gives a certain power of the determinant of $\mathcal{Q}$. This has to be compared with the fermionic part of the action. So let us look at the latter. After the rescaling $\lambda = \lambda^{\rm t} + \frac{1}{\sqrt{g}}\lambda^{\rm n}$, we have analogously

$$S^{(f)} = S_{sc}^{(f)} + S_{\rm n}^{(f)} + o\left(\frac{1}{\sqrt{g}}\right) \tag{2.8}$$

where, ,

$$S_{sc}^{(f)} \ = \ \int_{\mathcal{X}_0} d^2v d^2w \left[-2\left(\lambda_1^{{\rm t}*}\partial_{\bar{v}}\lambda_1^{\rm t} + \lambda_2^{{\rm t}*}\partial_v\lambda_2^{\rm t}\right) - 2\left(\lambda_1^{{\rm t}*}\partial_{\bar{w}}\lambda_2^{\rm t} - \lambda_2^{{\rm t}*}\partial_w\lambda_1^{\rm t}\right)\right] \tag{2.9}$$

The fermionic off–diagonal fluctuations contribute quadratically in the following way. We arrange the $\lambda_\alpha^{\rm n}$ and $\lambda_\alpha^{{\rm n}*}$ in a unique "spinor" $\psi^{{\rm n}T} = (\lambda_1^{\rm n}, \lambda_2^{\rm n}, \lambda_1^{{\rm n}*}, \lambda_2^{{\rm n}*})$,

$$S_K^{(f)} = \int d^2v d^2w \ \psi^{{\rm n}T} \mathcal{A} \ \psi^{\rm n} \ , \tag{2.10}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & \gamma^{I\dagger}{\rm ad}_{X_I^\circ} & -i{\rm ad}_{a_{\bar{v}}^{\rm t}} & -i{\rm ad}_{a_w^{\rm t}} \\ -\gamma^{I\dagger}{\rm ad}_{X_I^\circ} & 0 & i{\rm ad}_{a_{\bar{w}}^{\rm t}} & -i{\rm ad}_{a_v^{\rm t}} \\ -i{\rm ad}_{a_{\bar{v}}^{\rm t}} & i{\rm ad}_{a_{\bar{w}}^{\rm t}} & 0 & -\gamma^I{\rm ad}_{X_I^\circ} \\ -i{\rm ad}_{a_w^{\rm t}} & -i{\rm ad}_{a_v^{\rm t}} & \gamma^I{\rm ad}_{X_I^\circ} & 0 \end{pmatrix} \tag{2.11}$$

Now let us observe that the components of this matrix commute with respect to the action of the adjoint, and to the $SU(4)$ indices, so that we can directly compute the determinant looking at it as a 4×4 matrix. Taking into account the $SU(4)$ and Lorentz indices, we get

$${\rm Det}\mathcal{A} = ({\rm Det}\mathcal{Q})^8 \tag{2.12}$$

As this is precisely the determinant provided by the path integration on fermions, we now have to compare it with the bosonic one. This last turns out to be $(\mathrm{Det}\mathcal{Q})^{-8}$, obtained counting 6 scalars plus 4 gauge bosons minus 2 ghosts, and taking into account that the number of bosons too has been doubled, as an effect of the Wick rotation. So the final net contribution of the $\mathfrak{n}$ fields to the partition function is 1.

As it was pointed out in [7], each separate entry of the diagonal matrix fields appearing in (4.5) is not a true free field, as it is not single–valued. However each diagonal matrix field defines a unique (single–valued) field on the covering surface Σ of $\mathcal{X}$ (see Appendix). For example the matrices x^{It} represent scalar fields $\mathbf{x}^{\mathbf{I}}$, the matrix a^t represents a one–form field $\mathbf{a}$ on Σ and so on. A boldface letter will be henceforth the hallmark of a well–defined bosonic field on Σ. As for λ its global existence on Σ understands that the latter is a spin manifold.

In conclusion the strong coupling theory represents a free $U(1)$ gauge theory with matter on Σ:

$$S = \frac{1}{2}\int_{\Sigma} d^4\xi \left(\frac{1}{2}\partial_\mu \mathbf{x}^I \partial^\mu \mathbf{x}^I + \frac{1}{2}\partial_\mu \mathbf{a}_\nu \partial^\mu \mathbf{a}^\nu + \frac{1}{2}\partial_\mu \bar{\mathbf{c}} \partial^\mu \mathbf{c} - \frac{1}{2}\lambda^\dagger \gamma^\mu \partial_\mu \lambda\right) \tag{2.13}$$

where ξ are local coordinates on Σ (for example, z and w). The expression of the strong coupling (2.13) is only symbolic. It is in fact strictly valid only if Σ is a flat manifold, in which case we recover full $\mathcal{N}=4$ supersymmetry. But of course in general Σ will not be flat. In the non–flat cases (2.13) will only hold outside a neighborhood of the ramification locus, in which the curvature is concentrated. The problem of course is not how to extend the action (2.13) in such a way as to incorporate a non–trivial metric, which is straightforward, but rather how to do it in a supersymmetric way, so as to obtain an $\mathcal{N}=4$ supersymmetric theory. This problem is analogous to the covariant formulation of Green–Schwarz superstring theory on a generic Riemann surface, met in MST. The difficulty of such problems stems from the fact that, at first sight, it would seem inevitable to introduce supergravity on the world–volume in order to guarantee supersymmetry. However this is not necessary. In fact both these problems, as as well as other similar problems concerning D–brane actions embedded in space–time, have been solved using the superembedding principle. An essential role is played by κ symmetry, and the above mentioned difficulty is overcome by pulling back the (possibly trivial) metrics and gravitinos from the ambient space, which are therefore non–dynamical. All this fits very well in our approach, and we limit ourselves to relying on the literature: the action will be an extension of (2.13) to include the branch locus – possibly substituting the SYM action with the corresponding DBI one. In our specific case we have in mind [30] (for a review of this and related problems, see [31]).

We remark that (2.13) contains the fields which are expected to live on a D3–brane and it is itself the low energy and low curvature action for a D3–brane. We will further comment on it later.

In (2.13) the gauge coupling constant is 1. However, as shown in [7], in the path integral there is a non–trivial dependence on the original gauge coupling g which is due to the integration over the zero modes. For our previous rescaling of the various fields by powers of g involves, in particular, a rescaling of both the gauge and ghost diagonal degrees of freedom. When defining the path integral we have to take this fact into account, which amounts to rescaling it by an overall factor for any given instanton. This factor is a power of g, the exponent being the number of zero modes for each rescaled field with the appropriate sign. It would seem therefore that we have to count the number of ghost and gauge zero modes. However this would lead us to a wrong result for the reason explained below.

2.2.1 Summing over line bundles

Eq. (2.13) does not tell the whole story. In fact in the previous subsection we have dropped the diagonal connection $\hat{A}$. Reintroducing now this connection amounts to replacing $\mathbf{a}$ with $\mathbf{A} + \mathbf{a}$, where $\mathbf{A}$ is a non–trivial self–dual or anti–self–dual connection. Since self–dual and anti–self–dual instantons lead to the same coverings, when selecting a definite interpolating surface Σ (to represent a given scattering process) we have to allow for (i.e. to sum over) all the ic–instanton solutions that contain such a surface as a covering, both self–dual and anti–self–dual, and with all the possible non–trivial connections $\mathbf{A}$. These are line–bundle connections (it is useful to clarify that the fluctuation $\mathbf{a}$ is a 1–form: added to a line bundle connection it supplies another connection; in the treatment of the previous subsection it was supposed to be added to the 0 connection, i.e. to represent a connection in the trivial line bundle over Σ; in turn the fluctuations $\mathbf{x}^I$ as well as all the other fluctuating fields are section of trivial line bundles). In conclusion we have to sum over all line bundles on Σ and integrate over all the connections in each line bundle.

There is another way one can view the same problem: we have to admit on Σ any line bundle whose direct image under π coincides with the initial vector bundle E on $\mathcal{X}$. The construction is, roughly speaking, as follows. In a covering with N sheets, any line bundle

L generates an N–component 'vector' on $\mathcal{X}$: its components are just the N lines that lie over the same point of $\mathcal{X}$. Therefore to any line bundle over Σ there corresponds a vector bundle E over $\mathcal{X}$. The Chern classes $c_1(E), c_2(E)$ are connected to the Chern class of L via the Grothendieck–Riemann–Roch theorem, but, in the case of a noncompact manifold like $\mathcal{X}$, these constraints may become irrelevant.

A clarification is in order concerning $c_1(E)$ ($c_2(E)$ is trivial, therefore $c_2(E)$ does not need a comment). A non–trivial first Chern class on a brane world–volume is usually interpreted as the signal of the presence of a membrane. According to our interpretation membranes are not present in the theory, and $c_1(E)$ is a pure geometrical feature of the base manifold, which must be considered on the same footing as the complex structure and the like. It is only if we sum over all non–trivial $c_1(E)$'s on the base that we are allowed to sum over all the line bundles on the covering.

Summarizing, the spirit of our approach implies that we have to allow for anything in Σ can be lifted from $\mathcal{X}$, or, equivalently, for anything in Σ can be projected down to something that lives in $\mathcal{X}$. Therefore, on Σ, we have to allow for all possible non–trivial line bundles. The path integral must include the sum over such line bundles over Σ, as well as the path integration over all the connections on such line bundles.

A path integration with sum over all line bundles in a Maxwell theory has already been carried out in [32], see also [33], and we follow this calculation. For the sake of clarity we partially reproduce it here.

The trick consists in passing to a dual formulation by introducing auxiliary fields and enlarging the gauge symmetry. Given a connection $\mathbf{A}$ on a line bundle $\mathcal{L}$, one first introduces the auxiliary field $\mathbf{G}_{\mu\nu}$, and requires the theory to be invariant under an extended gauge symmetry whose local version is

$$\mathbf{A} \rightarrow \mathbf{A} + \mathbf{\Omega}, \qquad \mathbf{G} \rightarrow \mathbf{G} + d\mathbf{\Omega} \tag{2.14}$$

where $\mathbf{\Omega}$ is a local one–form. In addition (global version) one requires that $\mathbf{G}$ be defined up to closed two–forms. This is tantamount to asking that the integrals of $\mathbf{G}$ over two–cycles be defined up to integers. Then, if $\mathbf{F}$ is the curvature of $\mathbf{A}$, one defines the combination $\mathfrak{F}_{\mu\nu} = \mathbf{F}_{\mu\nu} - \mathbf{G}_{\mu\nu}$. $\mathfrak{F}$ is clearly invariant under the generalized gauge transformation (2.14) because $\mathbf{F}$ integrated over a two–cycle gives an integer.

Now one considers the series of dual line bundles $\tilde{\mathcal{L}}$ with dual connection $\mathbf{V}_\mu$ and curvature $\mathbf{W}_{\mu\nu}$, and writes the action

$$I = \frac{1}{2}\int_{\Sigma} d^4\xi\sqrt{h}\left(\frac{i}{4\pi}\epsilon^{\mu\nu\lambda\rho}\mathbf{W}_{\mu\nu}\mathbf{G}_{\lambda\rho} + \mathfrak{F}^{+}_{\mu\nu}\mathfrak{F}^{+\mu\nu} + \mathfrak{F}^{-}_{\mu\nu}\mathfrak{F}^{-\mu\nu}\right) \tag{2.15}$$

where + and – denotes the self–dual and anti–self–dual part of a two form, respectively. There are two alternatives. On the one hand, integrating over $\mathbf{V}$ one obtains that $d\mathbf{G} = 0$ and $\mathbf{G}$ has integral periods, which allows us to set $\mathbf{G} = 0$, in view of (2.14); in this way we get back the original lagrangian for $\mathbf{A}$. On the other hand, using (2.14) one can simply set $\mathbf{A} = 0$, and end up with the dual formulation, where the basic fields are the connection $\mathbf{V}$ and the two–form $\mathbf{G}$.

2.2.2 Counting zero modes

The dual formulation has the virtue of transforming the discrete summation over line bundles into an integration over continuous fields. Now we can see that the relevant zero modes are those of the one–forms $\mathbf{V}$ together with the corresponding ghosts (the dual of $\mathbf{c}$), and the two forms $\mathbf{G}$. Looking at (2.15) and at the definition of $\mathfrak{F}$ one sees that $\mathbf{V}$ rescales inversely with respect to $\mathbf{A}$, while $\mathbf{G}$ rescales in the same way. Therefore the zero modes of $\mathbf{G}$ and the ghosts of $\mathbf{V}$ will contribute with the same sign, while the zero modes of $\mathbf{V}$ will contribute with the opposite sign to the overall factor in front of the path integral. We are now ready to compute the latter.

Let us recall that our surface Σ is a two–dimensional complex variety with n punctures (see section 4). If it were a compact surface we would say that there are $2b_1$ zero modes of $\mathbf{V}$, two zero modes of the ghosts and b_2 zero modes of $\mathbf{G}$, where b_1, b_2 are Betti numbers of Σ. In conclusion we would get an overall factor $g^{2b_1-b_2-2}$. Now since $2-2b_1+b_2$ is the Euler characteristics of a compact 4D manifold, and since a puncture takes away one unit of Euler characteristics (as can be seen for example by triangulating the manifold), we are led to the conclusion that the overall factor in the presence of n puncture is $g^{-\chi}$ where $\chi = 2-2b_1+b_2-n$. This is the correct result and there are several ways one can convince oneself of it. The easiest one is probably by use of a doubling construction, as in [7]. Let us first make more precise the concept of puncture. The open surface Σ has boundaries B_i with $i = 1, \ldots, n$, which are 3–manifolds. For each B_i let us consider the cone C_i, which is obtained from the 'cylinder' $B_i \times I$, where I is a finite interval, by 'squeezing' to a point one of the boundaries of the cylinder. We can attach the boundaries of these cones to the corresponding boundaries B_i of Σ and obtain a compact surface Σ_c. Since each B_i has Euler characteristic 1, using additivity of the latter, we get $\chi \equiv \chi_\Sigma = \chi_c - n$, where χ_c is the Euler characteristic of the compact surface, i.e. $\chi_c = 2 - 2b_1 + b_2$. Now let us turn to the double of Σ: one constructs a complex surface $\hat{\Sigma}$ endowed with an antianalytic involution (locally this is $z \to \bar{z}$). Roughly speaking the double is made of two copies of Σ attached by the boundaries to form a compact surface: each boundary of one copy is attached to the corresponding boundary of the other copy. Denoting by a hat the quantities relevant to the double and using again additivity of the Euler characteristic, we have

$$\hat{\chi} = 2 - 2\hat{b}_1 + \hat{b}_2 = 2\chi = 4 - 4b_1 + 2b_2 - 2n$$

Now the number $2 - 2\hat{b}_1 + \hat{b}_2$ is the total alternating sum of zero modes on the double. Since it can be expressed via the Gauss–Bonnet theorem as an integral over $\hat{\Sigma}$, we expect that the same integral over Σ would yield half of it, i.e. χ, thanks to the symmetry implied by the anti–involution. We expect therefore that the total number of zero modes on Σ with the appropriate sign be given by χ, which is the result anticipated above.

Now one can see that for all Σ's considered in this paper as branched coverings of $\mathcal{X}$, $\chi \geq 0$, therefore the sum over ic–instantons gives rise to a series of non–negative powers [1] of $1/g$. This suggests that we interpret $g_3 = 1/g$ as the D3–brane perturbative interaction coupling, analogous to the string coupling of [7].

What we have shown so far is not enough to draw definite and unambiguous conclusions, however we have seen that in the strong coupling limit of 4D SYM theory there is room to describe scattering processes of D3–branes. Let us discuss a few general aspects of these processes. Ic–instantons become four–manifolds with boundaries consisting of three–manifolds. These geometrical configurations lend themselves to an interpretation in terms of three–brane scattering. In turn this interpretation fits very well in the path integral formalism, since it gives rise to a perturbative series in $1/g$. It remains for us to specify what are the amplitudes involved in these scattering of branes. Like in the case of string scattering, we will not really mean scattering of full branes but rather scattering of particle states which represent brane excitations. Although we do not know the spectrum of states of a D3–brane, in the case at hand we know plenty of such states: all the fields $\mathbf{x}^I, \mathbf{a}_\mu$ as well as the fermions which appear in (2.13), together with their derivatives and products are eligible to create such states. It is clear how to proceed: whenever we want to represent scattering of 3–branes excitations with given incoming and outgoing states, we

[1] In MST the corresponding series is in terms of g^χ; however Riemann surfaces with at least two punctures have negative χ

have to insert in the path integral the appropriate fields and evaluate the corresponding amplitudes. Of course this is not the end of the story, since one should then sum over all the instantons that interpolate between the same initial and final states, which means a sum over the appropriate instanton topologies and, at fixed topology, an integral over the appropriate moduli space.

The states we have just mentioned are local states, i.e. they should be associated to points of Σ, not to boundary 3–manifolds B_i. Suitable 4–manifolds are obtained by attaching cones C_i to B_i, as explained above, and smoothing out the result. Alternatively, we can imagine diffeomorphisms that deform the boundaries to points, and restrict ourselves to such configurations.

To end this section let us remark a difference with MST. While in MST the Euler characteristics entirely determines the instanton topology, that is not so in the present case. In fact since $\chi = 2 - 2b_1 + b_2 - n$, there may be and in fact there are manifolds with different b_1 and b_2 but the same χ. Therefore each term in the perturbative expansion in $1/g$ consist in general of a sum over different topologies. This sum is potentially infinite. However, as long as N is finite, the number of topologies which is possible to realize as branched coverings will be finite. Therefore N may be considered as a regulator for these sums. In this regard there is an interesting possibility: it is possible to introduce a parameter that allows us to discriminate among the various terms of a sum corresponding to a given χ. This is θ, the angle in front of the topological *theta term*, which can be introduced in the theory in the usual way (see [32] for an analogous context). We will not do it here.

Chapter 3

Cartan effective theory from $U(N)$ $D = 10$ $\mathcal{N} = 1$ SYM

3.1 Introduction

Once we have understood the non perturbative dynamics provided by the ic-instantons, our next task is to investigate the interference which can arise from the perturbative sector, i.e. the quantum corrections originating from the non-Cartan fields dynamics.

In so doing we would like to know as much as possible from supersymmetry, possibly extracting all the possible a priori informations, which could somehow constrain our future loop computations.

Surprisingly enough, we will get some insights on the interplay between the abelianization procedure of a non-abelian theory and the supersymmetric properties of the corresponding abelian theory.

The problem of the abelianization of a $U(N)$ four dimensional Yang Mills theory, i.e. the possibility of rewriting a non-abelian gauge theory (with a suitable gauge choice) in terms of some abelian one, has been firstly studied in the work [47], in the context of quark confinement.

As will be explained in detail in the subsequent sections, we are interested in getting rid of the non Cartan degrees of fredom of a SYM theory: the effective theory containing the dynamics for the Cartan degrees of freedom is obviously expected to be an abelian one.

The main purpose of this section is to obtain a deep understanding of the behavior of the supersymmetric properties in moving from a non abelian (Cartan and non-Cartan dof) to an abelian (only Cartan dof) SYM theory.

The subsequent three chapters mostly rely on [11].

3.1.1 Cartan - non-Cartan decomposition of the fields

Starting from the $U(N)$, $D=10$, $\mathcal{N}=1$ SYM action,

$$S[A,\theta]=\int d^{10}x\mathrm{Tr}\left(\frac{1}{4}F_{MN}F^{MN}+\frac{i}{2}\bar{\theta}\Gamma^{M}D_{M}\theta\right) \tag{3.1}$$

and its supersymmetry tranformations,

$$\begin{aligned}\delta A_M &= \frac{i}{2}\bar{\epsilon}\Gamma_M\theta \\ \delta\theta &= -\frac{1}{4}\Gamma^{MN}F_{MN}\epsilon \quad ,\end{aligned} \tag{3.2}$$

we now separate the degrees of freedom of the fields, which are valued in the $u(N)$ algebra, in Cartan and non Cartan fields:

$$\begin{aligned}A_M &= A_M^{\mathfrak{t}}+A_M^{\mathfrak{n}} \\ \theta &= \theta^{\mathfrak{t}}+\theta^{\mathfrak{n}} \quad ,\end{aligned}$$

where Cartan fields are cast in a diagonal form, while in the same basis the non Cartan fields are matrices with all zeroes on the diagonal.

Under this splitting the action decomposes as follows:

$$\begin{aligned}S[A^{\mathfrak{t}},A^{\mathfrak{n}},\theta^{\mathfrak{t}},\theta^{\mathfrak{n}}] = \int d^{10}x\mathrm{Tr}\Big\{&\frac{1}{2}\partial_M A_N^{\mathfrak{t}}(\partial_M A_N^{\mathfrak{t}}-\partial_N A_M^{\mathfrak{t}})+\frac{1}{2}\partial_M A_N^{\mathfrak{n}}(\partial_M A_N^{\mathfrak{n}}-\partial_N A_M^{\mathfrak{n}}) \\ &+i\partial_M A_N^{\mathfrak{t}}\left[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}\right]+i\partial_M A_N^{\mathfrak{n}}\left[A_M^{\mathfrak{t}},A_N^{\mathfrak{n}}\right]+i\partial_M A_N^{\mathfrak{n}}\left[A_M^{\mathfrak{n}},A_N^{\mathfrak{t}}\right] \\ &-\frac{1}{2}\left[A_M^{\mathfrak{t}},A_N^{\mathfrak{n}}\right]\left(\left[A_M^{\mathfrak{t}},A_N^{\mathfrak{n}}\right]-\left[A_N^{\mathfrak{t}},A_M^{\mathfrak{n}}\right]\right) \\ &-\left[A_M^{\mathfrak{t}},A_N^{\mathfrak{n}}\right]\left[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}\right] \\ &-\frac{1}{4}[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}]^{\mathfrak{n}}[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}]^{\mathfrak{n}}-\frac{1}{4}[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}]^{\mathfrak{t}}[A_M^{\mathfrak{n}},A_N^{\mathfrak{n}}]^{\mathfrak{t}} \\ &+\frac{i}{2}\bar{\theta}^{\mathfrak{t}}\Gamma^M\partial_M\theta^{\mathfrak{t}}+\frac{i}{2}\bar{\theta}^{\mathfrak{n}}\Gamma^M\partial_M\theta^{\mathfrak{n}} \\ &-\bar{\theta}^{\mathfrak{n}}\Gamma^M\left[A_M^{\mathfrak{n}},\theta^{\mathfrak{t}}\right] \\ &-\frac{1}{2}\bar{\theta}^{\mathfrak{n}}\Gamma^M\left[A_M^{\mathfrak{t}},\theta^{\mathfrak{n}}\right]-\frac{1}{2}\bar{\theta}^{\mathfrak{n}}\Gamma^M\left[A_M^{\mathfrak{n}},\theta^{\mathfrak{n}}\right]\Big\} \quad .\end{aligned} \tag{3.3}$$

While we get for the supersymmetry transformations:

$$\begin{aligned}
\delta A^{\mathrm{t}}_M &= \frac{i}{2}\bar{\epsilon}\Gamma_M\theta^{\mathrm{t}} \\
\delta\theta^{\mathrm{t}} &= -\frac{1}{2}\partial_M A^{\mathrm{t}}_N\Gamma^{MN}\epsilon - \frac{i}{4}[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}}\Gamma^{MN}\epsilon \\
\delta A^{\mathrm{n}}_M &= \frac{i}{2}\bar{\epsilon}\Gamma_M\theta^{\mathrm{n}} \\
\delta\theta^{\mathrm{n}} &= -\frac{1}{2}\partial_M A^{\mathrm{n}}_N\Gamma^{MN}\epsilon - \frac{i}{4}[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{n}}\Gamma^{MN}\epsilon - \frac{i}{4}\left[A^{\mathrm{t}}_M, A^{\mathrm{n}}_N\right]\Gamma^{MN}\epsilon \quad . \qquad (3.4)
\end{aligned}$$

We are actually interested in computing the effective theory for the Cartan dof, after having integrated away the non Cartan fields; let us define

$$Z[A^{\mathrm{t}}, \theta^{\mathrm{t}}] = \int \mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}} e^{-S[A^{\mathrm{t}}, A^{\mathrm{n}}, \theta^{\mathrm{t}}, \theta^{\mathrm{n}}]} \quad . \qquad (3.5)$$

Moreover we would like to know how much of supersymmetry is left in the effective action after the removal of the non-Cartan sector.

We start from (3.4), and immediately recognize that the Cartan fields tranformations do not close, since a term proportional to $[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}}$ arises from $\delta\theta^{\mathrm{t}}$, forbidding us to draw any distinction between the Cartan and non-Cartan sectors. So, it turns out at this stage that we can not even define a transformation of $Z[A^{\mathrm{t}}, \theta^{\mathrm{t}}]$ with respect to the Cartan fields, since we would obtain non-Cartan dependent fields, upon which we are interating.

To proceed further we need to disentangle the $^{\mathrm{t}}$ and $^{\mathrm{n}}$ supersymmetry transformations.

3.1.2 The Lagrange multiplier trick

Since it is by now clear that the supersymmetry transformations of the Cartan and non-Cartan fields are strictly interconnected by means of the $^{\mathrm{t}}$ component of the commutator of two A^{n} fields, i.e. $[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}}$, let us try to get rid of this term directly from the action!

For this purpose let us insert in the action (3.3) a set of Lagrange multipliers D^{t}_{MN} living in the Cartan of $u(N)$, which should reproduce, upon solution of their equations of motion, the term

$$-\frac{1}{4}[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}}[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}} \quad ,$$

which hence get substituted by

$$-\frac{1}{4}D^{\mathrm{t}}_{MN}D^{\mathrm{t}MN} + \frac{i}{2}D^{\mathrm{t}MN}[A^{\mathrm{n}}_M, A^{\mathrm{n}}_N]^{\mathrm{t}} \quad .$$

So, we end up with the new action

$$S_D[A^{\rm t}, A^{\rm n}, \theta^{\rm t}, \theta^{\rm n}, D^{\rm t}] = S[A^{\rm t}, A^{\rm n}, \theta^{\rm t}, \theta^{\rm n}] + \frac{1}{4}\int d^{10}{\rm Tr}[A^{\rm n}_M, A^{\rm n}_N]^{\rm t}[A^{\rm n}_M, A^{\rm n}_N]^{\rm t}$$
$$- \int d^{10}x{\rm Tr}\left\{\frac{1}{4}D^{\rm t}_{MN}D^{{\rm t}MN} - \frac{i}{2}D^{{\rm t}MN}[A^{\rm n}_M, A^{\rm n}_N]^{\rm t}\right\} \quad . \tag{3.6}$$

Since the Lagrange multiplier appears quadratically in the action, path integration over it exactly coincides (up to a numerical factor) with solving the equations of motion, and substituting back the result in the action, we have

$$Z[A^{\rm t}, \theta^{\rm t}] = \int \mathcal{D}D^{\rm t} \int \mathcal{D}A^{\rm n}\mathcal{D}\theta^{\rm n} e^{-S_D[A^{\rm t}, A^{\rm n}, \theta^{\rm t}, \theta^{\rm n}, D^{\rm t}]} \tag{3.7}$$

which allows us to compute the same effective action by means of S_D instead of S; for future purposes let us stress the we are not at all interested in the dependence of the effective action on the Lagrange multiplier, being the latter of no physical meaning.

Now we turn our attention to the SUSY transformations, and check how they get modified by the insertion of the Lagrange multiplier. It turns out that the $^{\rm n}$ transformations remain unchanged, while the $^{\rm t}$ ones become like the following:

$$\begin{aligned}
\delta A^{\rm t}_M &= \frac{i}{2}\bar{\epsilon}\Gamma_M\theta^{\rm t} \\
\delta_1\theta^{\rm t} &= -\frac{1}{2}\partial_M A^{\rm t}_N\Gamma^{MN}\epsilon \\
\delta_2\theta^{\rm t} &= -\frac{1}{4}D^{\rm t}_{MN}\Gamma^{MN}\epsilon \\
\delta\theta^{\rm t} &= \delta_1\theta^{\rm t} + \delta_2\theta^{\rm t} \\
\delta D^{\rm t}_{MN} &= \frac{i}{2}\partial_R\bar{\theta}^{\rm t}\Gamma^R\Gamma_{MN}\epsilon \quad .
\end{aligned} \tag{3.8}$$

This new set of transformations leave S_D invariant only using the equations of motion of $D^{\rm t}$, as we are going to explain.

Let us compute the variation of the action S_D:

$$\begin{aligned}
\delta S_D = {} & S_D[A^{\rm t} + \delta A^{\rm t}, A^{\rm n} + \delta A^{\rm n}, \theta^{\rm t} + \delta\theta^{\rm t}, \theta^{\rm n} + \delta\theta^{\rm n}, D^{\rm t} + \delta D^{\rm t}] \\
& - S_D[A^{\rm t}_M, A^{\rm n}_M, \theta^{\rm t}, \theta^{\rm n}, D^{\rm t}] \quad .
\end{aligned} \tag{3.9}$$

Among all the contributions present here, we are interested in those which are modified, with respect to δS, by the presence of the Lagrange multiplier; these terms can be read from (3.6) and (3.3) (a trace over $U(N)$ indices is understood):

$$\begin{aligned} & i\bar{\theta}^{\mathfrak{t}}\Gamma^M\partial_M\delta_2\theta^{\mathfrak{t}} - \frac{1}{2}D^{\mathfrak{t}MN}\delta D^{\mathfrak{t}}_{MN} = 0 \\ & -\bar{\theta}^{\mathfrak{t}}\Gamma^M\left[A^{\mathfrak{n}}_M, \delta\theta^{\mathfrak{n}}\right] + \frac{i}{2}\delta D^{\mathfrak{t}MN}[A^{\mathfrak{n}}_M, A^{\mathfrak{n}}_N]^{\mathfrak{t}} \\ & +i\partial_M\delta A^{\mathfrak{t}}_N\left[A^{\mathfrak{n}}_M, A^{\mathfrak{n}}_N\right] + i\partial_M A^{\mathfrak{n}}_N\left[\delta A^{\mathfrak{t}}_M, A^{\mathfrak{n}}_N\right] + i\partial_M A^{\mathfrak{n}}_N\left[A^{\mathfrak{n}}_M, \delta A^{\mathfrak{t}}_N\right] = 0 \quad . \end{aligned} \tag{3.10}$$

So far the cancellations take place without referring to the equations of motions for $D^{\mathfrak{t}}$; the difficulty arises in the next variation:

$$\begin{aligned} & -\bar{\theta}^{\mathfrak{n}}\Gamma^M\left[A^{\mathfrak{n}}_M, \delta_2\theta^{\mathfrak{t}}\right] + \frac{i}{2}D^{\mathfrak{t}MN}\delta\left([A^{\mathfrak{n}}_M, A^{\mathfrak{n}}_N]^{\mathfrak{t}}\right) \\ & -\bar{\theta}^{\mathfrak{n}}\Gamma^M\left[A^{\mathfrak{n}}_M, \delta_2\theta^{\mathfrak{n}}\right] - \frac{1}{4}\delta\left([A^{\mathfrak{n}}_M, A^{\mathfrak{n}}_N]^{\mathfrak{n}}[A^{\mathfrak{n}}_M, A^{\mathfrak{n}}_N]^{\mathfrak{n}}\right) \\ = \; & \frac{1}{4}\bar{\theta}^{\mathfrak{n}}\Gamma^{MRN}\epsilon\left[A^{\mathfrak{n}}_M, D^{\mathfrak{t}}_{RN}\right] + \frac{i}{4}\bar{\theta}^{\mathfrak{n}}\Gamma^{MRN}\epsilon\left[A^{\mathfrak{n}}_M, [A^{\mathfrak{n}}_R, A^{\mathfrak{n}}_N]^{\mathfrak{n}}\right] \\ = \; & H^{RN}(A^{\mathfrak{n}}, \theta^{\mathfrak{n}})\epsilon D^{\mathfrak{t}}_{RN} + G(A^{\mathfrak{n}}, \theta^{\mathfrak{n}})\epsilon \quad . \end{aligned} \tag{3.11}$$

where the last equality defines H^{RN} and G as functions of the non-Cartan fields. This variation is clearly non zero unless we substitute $D^{\mathfrak{t}}_{RN} = i[A^{\mathfrak{n}}_R, A^{\mathfrak{n}}_N]^{\mathfrak{t}}$, since in this case Jacoby identity will ensure the cancellation.

3.1.3 Supersymmetry transformation of the partition function

To end this section we study what happens to $Z[A^{\mathfrak{t}}, \theta^{\mathfrak{t}}]$ if the fields undergo a SUSY transformation. We obtain a result which is somewhat weaker than the computation of the following transformation

$$\delta Z[A^{\mathfrak{t}}, \theta^{\mathfrak{t}}] \tag{3.12}$$

since our result will hold only under integration on the Lagrange multiplier; but let us stress that, at this stage, we do not even know the meaning of (3.12) since the "effective" transformations for the Cartan fields are not yet defined.

Let us denote by $\delta^{\mathfrak{t}}$ the supersymmetry transformations acting only on the $A^{\mathfrak{t}}$ and the $\theta^{\mathfrak{t}}$ fields; it turns out, looking at the (3.8), that these transformations depend on the Lagrange multiplier, while they do not act on it.

Now, compute the following transformation:

$$\begin{aligned}
&\int \mathcal{D}D^{\mathfrak{t}}\left(1+\delta^{\mathfrak{t}}\right)\int \mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S_D[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]} \\
=& \int \mathcal{D}D^{\mathfrak{t}}\mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S_D[A^{\mathfrak{t}}+\delta A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}}+\delta\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]} \qquad (3.13)\\
=& \int \mathcal{D}(D^{\mathfrak{t}}+\delta D^{\mathfrak{t}})\mathcal{D}(A^{\mathrm{n}}+\delta A^{\mathrm{n}})\mathcal{D}(\theta^{\mathrm{n}}+\delta\theta^{\mathrm{n}})e^{-S_D[A^{\mathfrak{t}}+\delta A^{\mathfrak{t}},A^{\mathrm{n}}+\delta A^{\mathrm{n}},\theta^{\mathfrak{t}}+\delta\theta^{\mathfrak{t}},\theta^{\mathrm{n}}+\delta\theta^{\mathrm{n}},D^{\mathfrak{t}}+\delta D^{\mathfrak{t}}]} \\
=& \int \mathcal{D}D^{\mathfrak{t}}\mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S_D[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]-\int d^{10}x\mathrm{Tr}\left\{H^{RN}(A^{\mathrm{n}},\theta^{\mathrm{n}})\epsilon D^{\mathfrak{t}}_{RN}+G(A^{\mathrm{n}},\theta^{\mathrm{n}})\epsilon\right\}} \\
=& \int \mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}}]}
\end{aligned}$$

the last equality holds since, once we have read the equations of motion for $D^{\mathfrak{t}}$ as they result after the ϵ variation,

$$e.o.m. = D^{\mathfrak{t}}_{RN} - i[A^{\mathrm{n}}_R, A^{\mathrm{n}}_N]^{\mathfrak{t}} - 2H_{RN}(A^{\mathrm{n}},\theta^{\mathrm{n}})\epsilon = 0$$

we can plug them back in the transformed S_D action, getting:

$$\begin{aligned}
& S_D[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]\big|_{e.o.m.=0} \\
& + \int d^{10}x\mathrm{Tr}\left\{H^{RN}(A^{\mathrm{n}},\theta^{\mathrm{n}})\epsilon D^{\mathfrak{t}}_{RN} + G(A^{\mathrm{n}},\theta^{\mathrm{n}})\epsilon\right\}\Big|_{e.o.m.=0} \\
=\;& S[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}}] + \frac{i}{4}\int d^{10}\mathrm{Tr}\left\{\bar{\theta}^{\mathrm{n}}\Gamma^{MRN}\epsilon\left[A^{\mathrm{n}}_M,[A^{\mathrm{n}}_R,A^{\mathrm{n}}_N]\right]\right\} + O(\epsilon^2) \\
=\;& S[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}}] + O(\epsilon^2) \quad ,
\end{aligned}$$

where in the last equality we invoked the Jacoby identity to remove the ϵ term.

Keeping in mind that what we have done here is nothing but the path integration over $D^{\mathfrak{t}}$ (up to a numerical factor), the proof of (3.13) is complete.

Henceforth this computation results in the following invariance for the partition function:

$$\int \mathcal{D}D^{\mathfrak{t}}\,\delta^{\mathfrak{t}}\int \mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S_D[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]} = 0 \quad , \qquad (3.14)$$

and, introducing the Cartan effective action $\tilde{S}$

$$e^{-\tilde{S}_D[A^{\mathfrak{t}},\theta^{\mathfrak{t}},D^{\mathfrak{t}}]} = \int \mathcal{D}A^{\mathrm{n}}\mathcal{D}\theta^{\mathrm{n}}e^{-S_D[A^{\mathfrak{t}},A^{\mathrm{n}},\theta^{\mathfrak{t}},\theta^{\mathrm{n}},D^{\mathfrak{t}}]} \quad ,$$

we have achieved the task of extracting its supersymmetric properties from the global theory, since (3.14) immediately results in

$$\int \mathcal{D}D^{\mathfrak{t}}\, \delta^{\mathfrak{t}}\, e^{-\tilde{S}_D[A^{\mathfrak{t}},\theta^{\mathfrak{t}},D^{\mathfrak{t}}]} = 0 \quad . \tag{3.15}$$

3.1.4 Dimensional reduced versions of $D = 10$ SYM

We are actually interested in the $D = 2$ $\mathcal{N} = (8,8)$ and $D = 4$ $\mathcal{N} = 4$ SYM theories and since these can be obtained by dimentional reduction from the ten dimentional SYM, we spend now a few words about the behavior of the previous arguments under dimensional reduction itself.

Our arguments to prove (3.14) are entirely based on the non derivative gauge interactions in the action. These terms remain unchanged under dimensional reduction, the only difference being a renaming of some components of the gauge field, which become scalars transforming in the adjoint of $U(N)$.

According to the decomposition

$$A_M \to (A_\mu, X_i) \text{ with } (\mu = 1,2\ ;\ i = 1,\ldots,8) \text{ or } (\mu = 1,\ldots,4\ ;\ i = 1,\ldots,6) \quad ,$$

the Lagrange multiplier decomposes into worldsheet or worldvolume and transverse directions. In particular the fully transverse components $D^{\mathfrak{t}}_{ij}$ can be identified, at least in $D = 4$, with the Cartan components of the auxiliary fields, appearing in the six chiral and one vector supermultiplets.

As far as For the cancellations taking place in (3.10) are concerned, dimensional reduction coincides with taking $\partial_M \phi = 0$ when $M \neq \mu$, and expressing the ten dimensional gamma matrices as the direct product

$$\Gamma^\mu_{10} = \gamma^\mu_4 \otimes 1_6 \quad ; \tag{3.16}$$

so, dimensional reduction introduces nothing but technical difficulties, as can be verified by direct computation.

Hence the same conclusions as (3.14) hold in $D = 2$ and $D = 4$, where S_D has to be intended as the usual action where the terms

$$[A^{\mathfrak{n}}_\mu, A^{\mathfrak{n}}_\nu]^{\mathfrak{t}}[A^{\mathfrak{n}}_\mu, A^{\mathfrak{n}}_\nu]^{\mathfrak{t}} + [X^{\mathfrak{n}}_I, X^{\mathfrak{n}}_J]^{\mathfrak{t}}[X^{\mathfrak{n}}_I, X^{\mathfrak{n}}_J]^{\mathfrak{t}} + [A^{\mathfrak{n}}_\mu, X^{\mathfrak{n}}_I]^{\mathfrak{t}}[A^{\mathfrak{n}}_\mu, X^{\mathfrak{n}}_I]^{\mathfrak{t}} \tag{3.17}$$

have been substituted by the corresponding Lagrange multiplier terms.

Chapter 4

Non Cartan dynamics

4.1 Introduction

So far our analysis has been mostly concerned with the non perturbative sector which arises expanding the theory $D = 4$, $\mathcal{N} = 4$ around the instanton solutions which preserve some fraction of supersimmetry.Similar result holds of course for $D = 2$, $\mathcal{N} = (8,8)$ This lead to the perturbative (small $1/g$) theories of interacting fundamental strings of the IIA or of interacting D3-branes of the IIB.

In this framework, we would like to know which contributions to the interactions of such objects can arise from the perturbative expansion in $1/g$ of the underlying SYM theories.

For this purpose, let us recall how the actions can be rewritten after the Cartan and non-Cartan splitting and a suitable rescaling of the fields, such that we can achieve a sensible strong coupling limit (see eq. (2.5-2.10) for $D = 4$ and [7] for $D = 2$):

$$S[\phi^{\rm t}, \phi^{\rm n}] = S_{sc}[\phi^{\rm t}] + S_K[\phi^{\rm t}, \phi^{\rm n}] + o\,(1/g) \quad , \tag{4.1}$$

where at this stage we are not making any distinction between two and four dimensions.

The effective theory for the $\phi^{\rm t}$ can be obtained integrating the exponential of (4.1) along the $\phi^{\rm n}$ directions. As previously shown $S_K[\phi^{\rm t}, \phi^{\rm n}]$ vanishes exactly, as a result of the cancellation between bosonic and fermionic determinants. Hence, a first exact result is that at the zeroth order in $1/g$ the effective action precisely coincides with $S_{sc}[\phi^{\rm t}]$.

Moving to the $o\,(1/g)$ terms, one can naively argue that such interactions are strongly suppressed at high values of g, and that an eventual effective potential arising from them will yield negligible corrections to the dynamics described by $S_{sc}[\phi^{\rm t}]$.

To see how this argument can be measleading let us consider a possible interaction term for the non-Cartan fields, which will contribute perturbavely to the effective potential for the Cartan ones. A detailed classification of all such terms will be given in this chapter. For the moment, we just concetrate on the following term (in two dimensions):

$$S_{int} = \frac{1}{g^2} \int d^2w \mathrm{Tr} \left(\partial x^{\mathrm{n}i} \bar{\partial} x^{\mathrm{n}i} \right) \quad .$$

This term will contribute to the effective potential at the second order in $1/g$, via the following vacuum to vacuum amplitude:

$$e^{V_{eff}[\phi^{\rm t}]} = 1 + \frac{1}{g^2} \int \mathcal{D}x^{\rm n} \int d^2w \mathrm{Tr} \left(\partial x^{\mathrm{n}i} \bar{\partial} x^{\mathrm{n}i} \right) e^{-S_K[\phi^{\rm t}, \phi^{\rm n}]} \tag{4.2}$$

It can be immediately recognized that this correction is proportional to a term such as $\partial \bar{\partial} \delta^{(2)}(w)$ evaluated in the origin, and so it is badly divergent. Hence the need of regularizing this divergence; let us choose the gaussian smoothing for the delta function:

$$\delta(w) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} e^{-w\bar{w}/\epsilon^2} \quad . \tag{4.3}$$

With this choice (4.2) becomes

$$e^{V_{eff}[\phi^{\rm t}]} = 1 + \frac{1}{g^2 \epsilon^4} \alpha[\phi^{\rm t}]$$

and it is evident that we can not neglect the correction at big values of g unless ϵ is fixed and small compared to it: hence we should introduce a dependence between the coupling g and the cutoff $1/\epsilon$, or, in other words, our strong coupling limit would hold only once we have fixed the energy scale.

The existence of such divergent corrections to the effective action S_{sc} would be rather peculiar: one would have to understand which kind of physical objects really interact with this new potential; and moreover a renormalization procedure would be necessary to absorb the infinities arising in the action.

Fortunately it happens that this corrections cancel exactly, at least at the second order in $1/g$, resulting only in the renormalization of a Lagrange multiplier, which can then be removed by means of its equation of motion.

The proof of such cancellations is rather lenghty and full of technical difficulties; it will be the main subject of the following chapter.

In view of this we will concentrate at this point on the non-Cartan theories, both in two and four dimensions, as they emerge from the Cartan-non-Cartan splitting of the fields. We will try to learn as much as possible about the non-Cartan dynamics. For this purposes, in the following the Cartan components of the fields will be considered as non dynamical, "frozen" in some sense; they will be treated as worldsheet or worldvolume-dependent parameters. This parameters are going to occur both in the propagators of the non-Cartan fields and in their interaction terms.

4.2 Non Cartan sector of $D = 2$ $\mathcal{N} = (8, 8)$ SYM

Let us recall the form of the $D = 2$ $\mathcal{N} = (8, 8)$ SYM action, as it results from [7]:

$$S = \int d^2w \quad \mathrm{Tr} \quad \Big(D_w X^I D_{\bar{w}} X^I - \frac{g^2}{2}[X^I, X^J]^2 - g^2[X^I, X][X^I, \overline{X}] + D_w X D_{\bar{w}} \overline{X} - \frac{1}{4g^2}\left(F_{w\bar{w}} + ig^2[X, \overline{X}]\right)^2 + i(\theta_s^- D_{\bar{w}} \theta_s^- + \theta_c^+ D_w \theta_c^+) + ig\theta^T \Gamma_i [X^i, \theta] \Big) \tag{4.4}$$

where $I = 3, \ldots, 8$, and then expand it like the following:

$$S = S_{sc} + S_K + o\left(\frac{1}{\sqrt{g}}\right) ,$$

where

$$S_{sc} = \int d^2w \quad \mathrm{Tr} \quad \Big[\partial_w x^{I\mathrm{t}} \partial_{\bar{w}} x^{I\mathrm{t}} + \partial_w x^{\mathrm{t}} \partial_{\bar{w}} \bar{x}^{\mathrm{t}} + i(\theta_s^{\mathrm{t}} \partial_{\bar{w}} \theta_s^{\mathrm{t}} + \theta_c^{\mathrm{t}} \partial_w \theta_c^{\mathrm{t}}) + \partial_w a_{\bar{w}}^{\mathrm{t}} \partial_{\bar{w}} a_w^{\mathrm{t}} + \partial_w \bar{c}^{\mathrm{t}} \partial_{\bar{w}} c^{\mathrm{t}}\Big] \quad , \tag{4.5}$$

and S_K is the term quadratic in the $^{\mathrm{n}}$ fields, and will be defined in the nenxt section.

Let us now extract the dynamics for the non-Cartan components of the fields:

$$\begin{aligned} Z^{D=2} &= \int \mathcal{D}\phi \mathcal{D}\theta \mathcal{D}c \; e^{-S^{D=2}[\phi,\theta,c]} \\ &= \int \mathcal{D}\phi^{\mathrm{t}} \mathcal{D}\theta^{\mathrm{t}} \mathcal{D}c^{\mathrm{t}} \; e^{-S_{sc}^{D=2}[\phi^{\mathrm{t}},\theta^{\mathrm{t}},c^{\mathrm{t}}]} \\ &\times \int \mathcal{D}\phi^{\mathrm{n}} \mathcal{D}\theta^{\mathrm{n}} \mathcal{D}c^{\mathrm{n}} \; e^{-S_K^{D=2}[\phi^{\mathrm{n}},\theta^{\mathrm{n}},c^{\mathrm{n}},\phi^{\mathrm{t}}] - S_{int}^{D=2}[\phi^{\mathrm{n}},\theta^{\mathrm{n}},c^{\mathrm{n}},\phi^{\mathrm{t}},\theta^{\mathrm{t}},c^{\mathrm{t}}]} \\ &= \int \mathcal{D}\phi^{\mathrm{t}} \mathcal{D}\theta^{\mathrm{t}} \mathcal{D}c^{\mathrm{t}} \; e^{-S_{sc}^{D=2}[\phi^{\mathrm{t}},\theta^{\mathrm{t}},c^{\mathrm{t}}]} \; Z_{\mathrm{n}}^{D=2}[\phi^{\mathrm{t}}, \theta^{\mathrm{t}}, c^{\mathrm{t}}, g] \end{aligned} \tag{4.6}$$

where ϕ collects together all the bosonic (gauge and scalars) degrees of freedom, θ are the fermions and c the ghosts.

The partition function of our interest is $Z_{\mathfrak{n}}^{D=2}[\phi^{\mathfrak{t}}, \theta^{\mathfrak{t}}, c^{\mathfrak{t}}, \bar{c}^{\mathfrak{t}}, g]$, where all the dynamics of the $\mathfrak{n}$ fields is encoded, and which depends on the $\mathfrak{t}$ "parameters".

$S_K^{D=2}$ can be recognized to be the kinetic term, since it is quadratic in the non-Cartan fields and does not depend on the coupling; $S_{int}^{D=2}$ will provide the perturbative corrections to the non interacting theory, as usual.

4.2.1 The propagators for the non-Cartan fields

Let us now look at the kinetic term in details, so that we can extract the propagators:

$$S_K^{D=2} = \int d^2w \ \mathrm{Tr} \left[x^{\mathfrak{n}I} \mathcal{Q} x^{\mathfrak{n}I} + a_{\bar{w}}^{\mathfrak{n}} \mathcal{Q} a_w^{\mathfrak{n}} + \bar{c}^{\mathfrak{n}} \mathcal{Q} c^{\mathfrak{n}} + i \left(\theta_s^{\mathfrak{n}}, \theta_c^{\mathfrak{n}}\right) \mathcal{A} \begin{pmatrix} \theta_s^{\mathfrak{n}} \\ \theta_c^{\mathfrak{n}} \end{pmatrix} \right], \tag{4.7}$$

where

$$\mathcal{Q} = \mathrm{ad}_{a_{\bar{w}}^{\mathfrak{t}}} \cdot \mathrm{ad}_{a_w^{\mathfrak{t}}} + \mathrm{ad}_{X^{\circ I}} \cdot \mathrm{ad}_{X^{\circ I}}$$

and

$$\mathcal{A} = \begin{pmatrix} i\mathrm{ad}_{a_{\bar{w}}^{t}} & \gamma_i \mathrm{ad}_{X^{\circ I}} \\ \tilde{\gamma}_i \mathrm{ad}_{\bar{X}^{\circ I}} & i\mathrm{ad}_{a_w^{t}} \end{pmatrix}.$$

$\mathcal{Q}$ and $\mathcal{A}$ are operators acting on the $u(N)$ algebra, and depending on the Cartan fields. Since the inverse of such operators can not be represented in terms of some matrix living in the algebra, it turns out to be much more useful, although a bit lengthy, to write down explicitely the matrix components of the fields. For more details about these conventions and properties, we refer the reader to Appendix A.

We compute, as an example, the propagator for the $x^{\mathfrak{n}I}$ scalar fields, since the other correlators can be carried out in the same way. Expanding the non diagonal matrices on the standard basis, we find

$$\mathrm{Tr}\left(x^{\mathfrak{n}I} \mathcal{Q} x^{\mathfrak{n}I}\right) = \sum_{p \neq q}^{N} x_{qp}^{I} Q_{pq} x_{pq}^{I} \tag{4.8}$$

where Q_{pq} are the eigenvalues of $\mathcal{Q}$, the eigenvectors being the elements of the standard basis: $\mathcal{Q} e_{(pq)} = Q_{pq} e_{(pq)}$, with, (see appendix A), $Q_{qp} = a_{w\,(qp)}^{\mathfrak{t}} a_{\bar{w}\,(qp)}^{\mathfrak{t}} + (X_{(qp)}^{\circ I})^2$.

Next we introduce $8N(N-1)$ sources, j^I_{pq}, associated to each component x^I_{qp} of x^{nI}, and write the generating functional of the free theory n-points function:

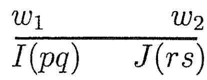
$$
\begin{aligned}
Z_K^{D=2}[j] &= \frac{1}{Z_K^{D=2}[0]} \int \prod_J^8 \prod_{r\neq s}^N \mathcal{D}x^J_{rs} e^{-\sum_{p\neq q}^N \int d^2w \left(x^I_{qp} Q_{pq} x^I_{pq} + x^I_{qp} j^I_{pq}\right)} \\
&= e^{\frac{1}{4}\sum_{p\neq q} \int d^2w \; j^I_{pq} \frac{1}{Q_{pq}} j^I_{qp}}
\end{aligned} \tag{4.9}
$$

from which it is straightforward to obtain the propagator:

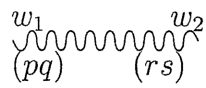
$$
\begin{aligned}
\left\langle \left(x^{nI}(w_1)\right)_{pq} \; \left(x^{nJ}(w_2)\right)_{rs} \right\rangle &= \left. \frac{\delta}{\delta j^I_{qp}} \frac{\delta}{\delta j^J_{sr}} Z_K^{D=2}[j] \right|_{j=0} \\
&= \frac{1}{2} \delta^{IJ} \delta^{ps} \delta^{qr} \frac{1}{Q_{pq}(w_1)} \delta^{(2)}(w_1, w_2)
\end{aligned}
$$

Following the same procedure all the propagators can be computed, leading to:

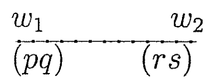
$$\left\langle \left(x^{nI}(w_1)\right)_{pq} \; \left(x^{nJ}(w_2)\right)_{rs} \right\rangle = \tfrac{1}{2}\delta^{IJ}\delta^{ps}\delta^{qr}\frac{1}{Q_{pq}(w_1)}\delta(w_1, w_2) =$$



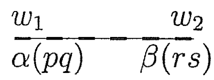
$$\left\langle \left(a^n_{\bar{w}}(w_1)\right)_{pq} \; \left(a^n_{w}(w_2)\right)_{rs} \right\rangle = \delta^{ps}\delta^{qr}\frac{1}{Q_{pq}(w_1)}\delta(w_1, w_2) =$$



$$\left\langle \left(\bar{c}^n(w_1)\right)_{pq} \; \left(c^n(w_2)\right)_{rs} \right\rangle = \delta^{ps}\delta^{qr}\frac{1}{Q_{pq}(w_1)}\delta(w_1, w_2) =$$



$$\left\langle \left(\theta^n_{\alpha}(w_1)\right)_{pq} \; \left(\theta^n_{\beta}(w_2)\right)_{rs} \right\rangle = -\tfrac{1}{2}\delta^{ps}\delta^{qr}\left(A^{-1}_{pq}(w_1)\right)_{\alpha\beta}\delta(w_1, w_2) =$$



where we have grouped together the two spinorial rapresentations of $SO(8)$ in the following way:

$$
\theta_\alpha = \begin{cases} \theta_{s,\alpha} \text{ if } \alpha = 1, \ldots, 8 \\ \theta_{c,(\alpha-8)} \text{ if } \alpha = 9, \ldots, 16 \end{cases},
$$

and, moreover, A^{-1}_{pq} are the eigenvalues of the inverse operator

$$
\mathcal{A}^{-1} e_{(pq)} = \mathcal{Q}^{-1} \begin{pmatrix} -i\mathrm{ad}_{a^t_w} & \gamma_I \mathrm{ad}_{X^{\circ I}} \\ \tilde{\gamma}_I \mathrm{ad}_{X^{\circ I}} & -i\mathrm{ad}_{a^t_{\bar{w}}} \end{pmatrix} e_{(pq)} = A^{-1}_{pq} e_{(pq)} \quad . \tag{4.10}
$$

Behavior of the propagators near the boundary

An important remark has to be done at this point. Since the ϕ^{t} are quantum fluctuations in the full theory, they approach zero near the boundary of the integration manifold, i.e. when $Re(w) \to \pm\infty$. This causes the functions present in the propagators to diverge, in fact

$$\frac{1}{Q_{pq}} \approx \frac{1}{(\phi^{\mathrm{t}}_{(p)} - \phi^{\mathrm{t}}_{(q)})^2}$$
$$A^{-1}_{pq} \approx \frac{1}{(\phi^{\mathrm{t}}_{(p)} - \phi^{\mathrm{t}}_{(q)})}$$

To regularize such divergences we integrate over a finite cylinder which will be denoted by $\mathcal{C}_T$ (where T is the length of the cylinder), such that $Re(w)$ is forbidden to reach infinity.

This regulator will be removed only after that any manipulation on the integrals has been carried out.

Of course this means that we should take a great care about the boundary terms appearing in the integrals, since they seem to be quite far from vanishing, as they are made up by Q^{-1} and A^{-1}.

4.2.2 The non-Cartan interactions

It is time to zoom on the interactions between the non-Cartan fields, so that we can analyze them in detail.

Their dependence on the coupling is of the kind g^{-k} with $k = \frac{1}{2}, 1, \frac{3}{2}, 2$, and they can be divided in two different kinds, derivative and polynomial interaction terms:

$$S^{D=2}_{int}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}}, c^{\mathrm{t}}] = S^{D=2}_{der}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, c^{\mathrm{t}}] + S^{D=2}_{pol}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}}, c^{\mathrm{t}}].$$

We will treat them separately for two main reasons. The first is that, at least at one loop, these two sectors are not connected and they can be computed independently. The second concerns the great care which has to be taken in dealing with regularized integrals arising from derivative interactions: since boundary terms are strongly involved here, we have to check they do not contribute. Evidently this is not for the polynomial interactions, which, moreover, are quite easy to deal with, since they are nothing but the same interactions present in the ten dimensional theory.

A simplifying trick

The supersymmetric properties of the effective theory we will compute, which have been proven in the previous chapter, can now be invoked to introduce a simplification in our work.

To do so we need to take a step back with respect to the action appearing in (4.6). In fact having we fixed the gauge with a evidently non supersymetric gauge fixing, we can not apply the argument of the previous chapter straightforwardly to this action.

Here is how to proceed: we start from the non gauge fixed theory; applying the argument stated at the end of section 3.1.3, we can just set to zero the Cartan components of the fermions, being guaranteed by δ^{t} invariance that we will be able to come back to nonzero θ^{t} without the arising of extra bad terms.

For this purpose it is necessary to insert in the action the Lagrange multipliers D^{t}. Only at this stage we add the gauge fixing and Faddeev-Popov terms. In so doing we substitute a scalar-ghost interaction term with the corresponding Lagrange multiplier one.

This same procedure will be followed backward on the final effective theory in order to show that it possesses the same supersymmetric properties.

Notice that neither the introduction of D^{t}, nor the choice $\theta^{\mathrm{t}} = 0$ are going to affect the derivative interactions, since the Lagrange multipliers are connected with the polynomial interaction, and $S^{D=2}_{der}$ does not depend on θ^{t} by definition. For the same reason the propagators are left unchanged. Hence the only substitution we have to perform is

$$S^{D=2}_{pol}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}}, c^{\mathrm{t}}] \to S^{D=2}_{pol,D}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}}, c^{\mathrm{t}}, D^{\mathrm{t}}] \quad ,$$

and we end up with the following modified action:

$$\begin{aligned} S^{D=2}[\phi, \theta^{\mathrm{n}}, \theta^{\mathrm{t}} = 0, c] &= S^{D=2}_{sc,D}[\phi^{\mathrm{t}}, \theta^{\mathrm{t}} = 0, c^{\mathrm{t}}, D^{\mathrm{t}}] + S^{D=2}_{K}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}] \\ &\quad + S^{D=2}_{der}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, c^{\mathrm{t}}] + S^{D=2}_{pol,D}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}} = 0, c^{\mathrm{t}}, D^{\mathrm{t}}]. \end{aligned}$$

4.2.3 Derivative interactions and their vertices

Expanding the action (4.4) and selecting the terms containing at least one derivative we obtain

$$\begin{aligned} S^{D=2}_{der} &= V^{(b)}_1[x^{\mathrm{n}}] + V^{(g)}_1[a^{\mathrm{n}}] + V^{(b,g)}_1[x^{\mathrm{n}}, a^{\mathrm{n}}] + V^{(b)}_2[x^{\mathrm{n}}] + V^{(g)}_2[a^{\mathrm{n}}] \\ &\quad + V^{(f)}_1[\Theta^{\mathrm{n}}] \\ &\quad + V^{(c)}_1[c^{\mathrm{n}}] + V^{(c,g)}_1[c^{\mathrm{n}}, a^{\mathrm{n}}] + V^{(c)}_2[c^{\mathrm{n}}] \quad , \end{aligned} \tag{4.11}$$

where the suffixes stand for the order of the derivatives involved. We explicitly have:

$$
\begin{aligned}
V_1^{(b)}[x^{\rm n}] &= -\frac{i}{g}\int d^2w \mathrm{Tr}\left[x^{{\rm n}I}\left(\mathrm{ad}_{a^{\rm t}_{\bar w}}\partial_w + \mathrm{ad}_{a^{\rm t}_w}\partial_{\bar w}\right)x^{{\rm n}I}\right] \\
V_2^{(b)}[x^{\rm n}] &= \frac{1}{g^2}\int d^2w \mathrm{Tr}\left(\partial_w x^{{\rm n}I}\ \partial_{\bar w}x^{{\rm n}I}\right) \\
V_1^{(g)}[a^{\rm n}] &= \frac{i}{2g}\int d^2w \mathrm{Tr}\left[a^{\rm n}_{\bar w}\left(\mathrm{ad}_{\partial_{\bar w}a^{\rm t}_w} - \mathrm{ad}_{\partial_w a^{\rm t}_{\bar w}}\right)a^{\rm n}_w + 2\ \partial_w a^{\rm n}_{\bar w}\mathrm{ad}_{a^{\rm t}_{\bar w}}a^{\rm n}_w + 2\ \partial_{\bar w}a^{\rm n}_w \mathrm{ad}_{a^{\rm t}_w}a^{\rm n}_{\bar w}\right] \\
V_2^{(g)}[a^{\rm n}] &= \frac{1}{g^2}\int d^2w \mathrm{Tr}\left(\partial_w a^{\rm n}_{\bar w}\ \partial_{\bar w}a^{\rm n}_w\right) \\
V_1^{(b,g)}[x^{\rm n},a^{\rm n}] &= -\frac{2i}{g}\int d^2w \mathrm{Tr}\left[a^{\rm n}_{\bar w}\mathrm{ad}_{\partial_w X^{\circ I}}x^{{\rm n}I} + a^{\rm n}_w \mathrm{ad}_{\partial_{\bar w}X^{\circ I}}x^{{\rm n}I}\right] \qquad (4.12) \\
V_1^{(f)}[\Theta^{\rm n}] &= \frac{i}{g}\int d^2w \mathrm{Tr}\left[\Theta^{{\rm n}T}(\bar\rho\partial_w + \rho\partial_{\bar w})\Theta^{\rm n}\right] \\
V_1^{(c)}[c^{\rm n}] &= -\frac{i}{2g}\int d^2w \mathrm{Tr}\left[\bar c^{\rm n}\left(2\mathrm{ad}_{a^{\rm t}_w}\partial_{\bar w} + 2\mathrm{ad}_{a^{\rm t}_{\bar w}}\partial_w + \mathrm{ad}_{\partial_{\bar w}a^{\rm t}_w} + \mathrm{ad}_{\partial_w a^{\rm t}_{\bar w}}\right)c^{\rm n}\right] \\
V_2^{(c)}[c^{\rm n}] &= \frac{1}{g^2}\int d^2w \mathrm{Tr}\left(\partial_w \bar c^{\rm n}\ \partial_{\bar w}c^{\rm n}\right) \\
V_1^{(c,g)}[c^{\rm n},a^{\rm n}] &= -\frac{i}{2\sqrt{g}}\int d^2w \mathrm{Tr}\left[a^{\rm n}_{\bar w}\mathrm{ad}^{\{\}}_{\partial_w \bar c^{\rm t}}c^{\rm n} + a^{\rm n}_w \mathrm{ad}^{\{\}}_{\partial_{\bar w}\bar c^{\rm t}}c^{\rm n}\right] + \\
&\quad + \frac{i}{2g^{3/2}}\int d^2w \mathrm{Tr}\left[\bar c^{\rm n}\mathrm{ad}_{c^{\rm t}}\partial_w a^{\rm n}_{\bar w} + \bar c^{\rm n}\mathrm{ad}_{c^{\rm t}}\partial_{\bar w}a^{\rm n}_w\right] + \\
&\quad - \frac{i}{2g^2}\int d^2w \mathrm{Tr}\left[\bar c^{\rm n}\partial_w\left[a^{\rm n}_{\bar w}, c^{\rm n}\right] + \bar c^{\rm n}\partial_{\bar w}\left[a^{\rm n}_w, c^{\rm n}\right]\right] \quad , \qquad (4.13)
\end{aligned}
$$

where $\mathrm{ad}^{\{\}}_M \cdot \equiv \{M, \cdot\}$.

Since these are all quadratic terms in the $^{\rm n}$ fields, one may wonder why we are considering them as interaction terms instead of kinetic ones: of course in principle this could be done as well, but we are interested in a perturbative expantion, and we don't want any dependence of the propagators on the coupling, at least untill we will deal with cutoffs.

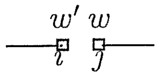
To extract the vertices from these interactions, the following scheme can be used, shown here in the case of $V_1^{(b)}(w)$:

$$
\begin{aligned}
\int_{C_T} d^2w \mathrm{Tr}\left[x^{{\rm n}i}\ \mathrm{ad}_{a^{\rm t}_{\bar w}}\partial_w x^{{\rm n}i}\right] &= \int_{C_T} d^2w d^2w' \mathrm{Tr}\left[x^{{\rm n}i}(w')\ \mathrm{ad}_{a^{\rm t}_{\bar w}(w)}\partial_w x^{{\rm n}i}(w)\right]\delta(w-w') \\
&= -\int_{C_T} d^2w d^2w' \mathrm{Tr}\left[x^{{\rm n}i}(w')\ \partial_w\left(\mathrm{ad}_{a^{\rm t}_{\bar w}(w)}\delta(w-w')\right)x^{{\rm n}i}(w)\right]
\end{aligned}
$$

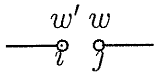
$$
\begin{aligned}
&+\int_{C_T} d^2w \oint_{\partial C_T} dw' \mathrm{Tr}\left[x^{\mathrm{n}i}(w')\ \mathrm{ad}_{a^{\mathrm{t}}_{\bar{w}}(w)} x^{\mathrm{n}i}(w)\right] \delta(w-w') \\
=\ &-\int_{C_T} d^2w d^2w' \mathrm{Tr}\left[x^{\mathrm{n}i}(w')\ \mathrm{ad}_{a^{\mathrm{t}}_{\bar{w}}(w)} \partial_w \delta(w-w') x^{\mathrm{n}i}(w)\right]
\end{aligned}
$$

due to the trace and the delta function.

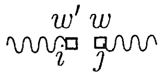
So we are left with vertices which can be diagrammatically described as follows:

from $V_1^{(b)}[x^{\mathrm{n}}]$ $\Rightarrow$ 

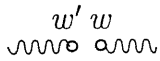
$= \frac{i}{g}\delta^{ij}\ K(w,w')$

from $V_2^{(b)}[x^{\mathrm{n}}]$ $\Rightarrow$ 

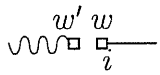
$= \frac{1}{g^2}\delta^{ij} \int d^2w'' \partial_w \delta(w-w'') \partial_{\bar{w}'} \delta(w'-w'')$

from $V_1^{(g)}[a^{\mathrm{n}}]$ $\Rightarrow$ 

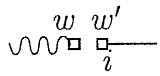
$= \frac{i}{g}\left[K(w,w') + L(w,w')\right]$

from $V_2^{(g)}[a^{\mathrm{n}}]$ $\Rightarrow$ 

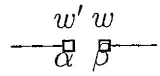
$= \frac{1}{g^2} \int d^2w'' \partial_w \delta(w-w'') \partial_{\bar{w}'} \delta(w'-w'')$

from $V_1^{(b,g)}[x^{\mathrm{n}}, a^{\mathrm{n}}]$ $\Rightarrow$ 

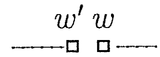
$= -\frac{2i}{g}\delta(w-w')\mathrm{ad}_{\partial_w X^{\circ i}}$ (4.14)



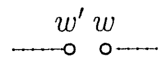
$= -\frac{2i}{g}\delta(w-w')\mathrm{ad}_{\partial_{\bar{w}} X^{\circ i}}$

from $V_1^{(f)}[\Theta^{\mathrm{n}}]$ $\Rightarrow$ 

$= -\frac{i}{g}\left[(\rho^0\bar{\rho})_{\alpha\beta}\, \partial_w \delta(w-w') - (\rho^0\rho)_{\alpha\beta}\, \partial_{\bar{w}'}\delta(w-w')\right]$

from $V_1^{(c)}[c^{\mathrm{n}}]$ $\Rightarrow$ 

$= \frac{i}{g}\left[K(w,w') - L(w,w')\right]$

from $V_2^{(c)}[c^{\mathrm{n}}]$ $\Rightarrow$ 

$= \frac{1}{g^2} \int d^2w'' \partial_w \delta(w-w'') \partial_{\bar{w}} \delta(w'-w'')$

where $V_i^{(\cdot)}$ is represented by a square if $i=1$, or by a circle if $i=2$; moreover the following distributions have been defined for simplicity (notice that they act on the matrix structure of the fields):

$$
\begin{aligned}
K(w,w') &= \mathrm{ad}_{a^{\mathrm{t}}_{\bar{w}}} \partial_w \delta(w-w') - \mathrm{ad}_{a^{\mathrm{t}}_{w'}} \partial_{\bar{w}'} \delta(w-w') \\
L(w,w') &= \frac{1}{2}\left[\mathrm{ad}_{\partial_{\bar{w}'} a^{\mathrm{t}}_{w'}} \delta(w-w') - \mathrm{ad}_{\partial_w a^{\mathrm{t}}_{\bar{w}}} \delta(w-w')\right].
\end{aligned}
\qquad (4.15)
$$

4.2.4 Polynomial interactions

Neither can dimensional reduction affect the polynomial potentials in the $D = 10$ theory, nor can it originate new of ones in the $D = 2$ theory. This is why the interactions we want to describe in this section precisely look like the ones in the ten dimensional theory, apart from the dimensionality of the worldsheet, of course, and a renaming of the components of the $10D$ gauge field. But if we group the $2D$ vector and the scalars in the unique field $\phi_M = (A_0, A_1, X^I)$ of section 4.2.2, we come back to the ten dimensional situation.

Remembering the presence of the Lagrange multiplier, and setting $\theta^{\mathrm{t}} = 0$ we have the following interactions:

$$\begin{aligned} S^{D=2}_{pol,D}[\phi^{\mathrm{n}}, \theta^{\mathrm{n}}, \phi^{\mathrm{t}}, \theta^{\mathrm{t}}, D^{\mathrm{t}}, C^{\mathrm{t}}] \;=\; & \int d^2w \mathrm{Tr}\left\{ -\frac{1}{2g^2}[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N]^{\mathrm{n}}[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N]^{\mathrm{n}} \right. \\ & -\frac{2}{g}\left[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N\right]\left[\phi^{\mathrm{t}}_M, \phi^{\mathrm{n}}_N\right]] + \frac{i}{g} D^{\mathrm{t}MN}[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N]^{\mathrm{t}} \\ & +\frac{i}{g}\theta^{\mathrm{n}T} G^{(2)}_M \left[\phi^{\mathrm{n}}_M, \theta^{\mathrm{n}}\right] + \frac{1}{g}\left[\bar{c}^{\mathrm{n}}, \phi^{\mathrm{t}}_M\right]\left[\phi^{\mathrm{n}}_M, c^{\mathrm{n}}\right] \\ & \left. +\frac{i}{g}\bar{C}^{\mathrm{t}}_M[\phi^{\mathrm{n}}_M, c^{\mathrm{n}}]^{\mathrm{t}} + \frac{i}{g}[\bar{c}^{\mathrm{n}}, \phi^{\mathrm{n}}_M]^{\mathrm{t}} C^{\mathrm{t}}_M \right\} \quad , \end{aligned} \tag{4.16}$$

where the $G^{(2)}_M$ are 16 by 16 matrices, defined in Appendix A. while C^{t} and $\bar{C}^{\mathrm{t}}$ are the Lagrange multipliers through which we rewrite the $1/g^2$ boson-ghost interaction. It is worth stressing that, by taking $\theta^{\mathrm{t}} = 0$, we are neglecting the interaction term involving the Cartan fermions, which turns out to be

$$\equiv \frac{1}{\sqrt{g}}\theta^{\mathrm{n}T} G^{(2)}_M \left[\theta^{\mathrm{t}}, \phi^{\mathrm{n}}_M\right] \quad , \tag{4.17}$$

and which would be responsible of the presence of a divergent correction in the effective action, since there are no terms which can possibly cancel its contribution by means of supersymmetry. A further and strong indication that this is a sensible choice will be given at the end of the next chapter, mentioning the superfields approach.

Now let us extract the vertices.

Quartic potential for the ϕ^{n}

Let us write the quartic potential as

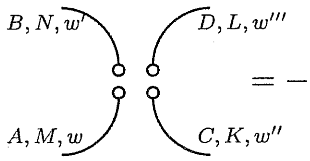
$$V^{(4)}[\phi^{\mathrm{n}}] \;=\; -\frac{1}{2g^2}\int d^2w \mathrm{Tr}\left\{[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N]^{\mathrm{n}}[\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N]^{\mathrm{n}}\right\}$$

$$
\begin{aligned}
&= -\frac{1}{2g^2} T^{(4)}_{AB,CD} \delta^{MK} \delta^{NL} \int d^2w d^2w' d^2w'' d^2w''' \delta(w,w'')\delta(w,w')\delta(w'',w''') \\
&\times \left[\phi^{\mathrm{n}M}_{\ A}(w)\ \phi^{\mathrm{n}N}_{\ B}(w')\ \phi^{\mathrm{n}K}_{\ C}(w'')\ \phi^{\mathrm{n}L}_{\ D}(w''') \right]
\end{aligned}
$$

where the group structure is encoded in $T^{(4)}_{ABCD}$, and we have expanded the fields in the standard basis (see Appendix A):

$$
T^{(4)}_{AB,CD} = \mathrm{Tr}\left\{ [e_A, e_B]^{\mathrm{n}} [e_C, e_D]^{\mathrm{n}} \right\} \quad ,
$$

where the indices A, B, C are defined in Appendix A. So the vertex will look like



$$
= -\tfrac{1}{2g^2} T^{(4)}_{AB,CD} \delta^{MK} \delta^{NL} \int d^2w d^2w' d^2w'' d^2w''' \delta(w,w'')\delta(w,w')\delta(w'',w''') \ . \tag{4.18}
$$

Cubic potential for the ϕ^{n}

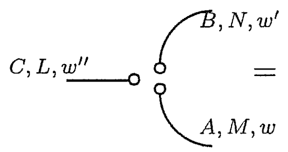
Starting from the cubic potential we get:

$$
\begin{aligned}
V^{(3)}[\phi^{\mathrm{n}}] &= -\frac{2}{g} \int d^2w \mathrm{Tr}\left\{ [\phi^{\mathrm{n}}_M, \phi^{\mathrm{n}}_N] \, [\phi^{\mathrm{t}}_M, \phi^{\mathrm{n}}_N] \right\} \\
&= -\frac{2}{g} T^{(3)}_{AB,C} \delta^{MK} \delta^{NL} \int d^2w d^2w' d^2w'' \delta(w,w')\delta(w,w'') \\
&\times \left[\phi^{\mathrm{t}K}_{\ C}(w'')\ \phi^{\mathrm{n}M}_{\ A}(w)\ \phi^{\mathrm{n}N}_{\ B}(w')\ \phi^{\mathrm{n}L}_{\ C}(w'') \right]
\end{aligned}
$$

where

$$
T^{(3)}_{AB,C} = \mathrm{Tr}\left([e_A, e_B] e_C \right)
$$

and we recall that $\phi^{\mathrm{t}K}_{\ C} = \phi^{\mathrm{t}K}_{\ (pq)} = \phi^{\mathrm{t}K}_{\ (p)} - \phi^{\mathrm{t}K}_{\ (q)}$. We define the vertex to be:



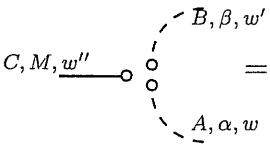
$$
= -\tfrac{2}{g} T^{(3)}_{AB,C} \delta^{NL} \int d^2w d^2w' d^2w'' \delta(w,w')\delta(w,w'') \phi^{\mathrm{t}M}_{\ C}(w'') \quad . \tag{4.19}
$$

The fermions vertex

Following the same scheme, the fermionic potential can be manipulated to give:

$$
\begin{aligned}
V^{(f)}[\theta^{\mathrm{n}}] &= \frac{i}{g}\int d^2w \mathrm{Tr}\left\{\theta^{\mathrm{n}T} G^{(2)}_M \left[\phi^{\mathrm{n}}_M, \theta^{\mathrm{n}}\right]\right\} \\
&= -\frac{i}{g}\left(G^{(2)}_M\right)_{\alpha\beta} T^{(3)}_{AB,C} \int d^2w d^2w' d^2w'' \delta(w,w')\delta(w,w'') \left[\theta^{\mathrm{n}}_{\alpha,A}(w)\ \theta^{\mathrm{n}}_{\beta,B}(w')\ \phi^{\mathrm{n}M}_{C}(w'')\right] \quad .
\end{aligned}
$$

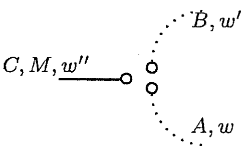
The vertex is



$$
= -\frac{i}{g}\left(G^{(2)}_M\right)_{\alpha\beta} T^{(3)}_{AB,C} \int d^2w d^2w' d^2w'' \delta(w,w')\delta(w,w'') \quad . \tag{4.20}
$$

The $1/g$ ghost interaction

Exactly in the same way the cubic scalar vertex has been carried out, we obtain for the boson-ghost vertex:



$$
= \frac{1}{g} T^{(3)}_{AB,C} \int d^2w d^2w' d^2w'' \delta(w,w')\delta(w,w'')\phi^{\mathrm{t}M}_{C}(w'') \quad . \tag{4.21}
$$

The terms involving Lagrange multipliers

From the point of view of the non-Cartan fields these potentials correspond to two legs vertices, as it can be seen from the polynomial action (4.16), leading to simple interactions; we will deal straightforwardly with them when computing the perturbative corrections.

This ends our introductive considerations about the two dimensional non-Cartan theory. All this material will be intensively used in showing the cancellation of the perturbative corrections.

It is now time to turn to the four dimensional SYM theory.

4.3 Non Cartan sector of $D = 4$ $\mathcal{N} = 4$ SYM

All the general arguments carried out in the previous sections for what concerns the two dimensional case, extend straightforwardly in $D = 4$

Let us follow the same scheme, starting from the $D = 4$ $\mathcal{N} = 4$ SYM theory. The effective theory for the Cartan fields can be extracted as usual:

$$\begin{aligned} Z^{D=4} &= \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}c \; e^{-S^{D=4}[\phi,\psi,c]} \\ &= \int \mathcal{D}\phi^{\mathrm{t}} \mathcal{D}\psi^{\mathrm{t}} \mathcal{D}c^{\mathrm{t}} \; e^{-S_{sc}^{D=4}[\phi^{\mathrm{t}},\psi^{\mathrm{t}},c^{\mathrm{t}}]} \\ &\quad \times \int \mathcal{D}\phi^{\mathrm{n}} \mathcal{D}\psi^{\mathrm{n}} \mathcal{D}c^{\mathrm{n}} \; e^{-S_K^{D=4}[\phi^{\mathrm{n}},\psi^{\mathrm{n}},c^{\mathrm{n}},\phi^{\mathrm{t}}] - S_{int}^{D=4}[\phi^{\mathrm{n}},\psi^{\mathrm{n}},c^{\mathrm{n}},\phi^{\mathrm{t}},\psi^{\mathrm{t}},c^{\mathrm{t}}]} \\ &= \int \mathcal{D}\phi^{\mathrm{t}} \mathcal{D}\psi^{\mathrm{t}} \mathcal{D}c^{\mathrm{t}} \; e^{-S_{sc}^{D=4}[\phi^{\mathrm{t}},\psi^{\mathrm{t}},c^{\mathrm{t}}]} \; Z_{\mathrm{n}}^{D=4}[\phi^{\mathrm{t}}, \psi^{\mathrm{t}}, c^{\mathrm{t}}, g] \end{aligned} \tag{4.22}$$

hence the need for analyzing in detail the non-Cartan dynamics encoded in $Z_{\mathrm{n}}^{D=4}$.

4.3.1 The propagators

We start from

$$S_K^{D=4}[\phi^{\mathrm{n}}, \psi^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}] = \int d^2w d^2v \; \mathrm{Tr} \; \left[x^{\mathrm{n}I} \mathcal{Q} x^{\mathrm{n}I} + a^{\mathrm{n}}_{\bar{z}_i} \mathcal{Q} a^{\mathrm{n}}_{z_i} + \bar{c}^{\mathrm{n}} \mathcal{Q} c^{\mathrm{n}} + \psi^{\mathrm{n}T} \mathcal{A} \, \psi^{\mathrm{n}} \right]$$

Notice that $\mathcal{Q}$ and $\mathcal{A}$ are different from the two dimensional ones; we omit the suffix $^{D=4}$ for simplicity.

We can work out the propagators for the non-Cartan components:

$$\left\langle \left(x^{\mathrm{n}I}(z_1)\right)_{pq} \; \left(x^{\mathrm{n}J}(z_2)\right)_{rs} \right\rangle = \tfrac{1}{2} \delta^{IJ} \delta^{ps} \delta^{qr} \frac{1}{Q_{pq}(z_1)} \delta(z_1, z_2) = \overset{z_1}{I(pq)} \;\; \overset{z_2}{J(rs)}$$

$$\left\langle \left(a^{\mathrm{n}}_{\bar{z}_i}(z_1)\right)_{pq} \; \left(a^{\mathrm{n}}_{z_j}(z_2)\right)_{rs} \right\rangle = \delta^{ij} \delta^{ps} \delta^{qr} \frac{1}{Q_{pq}(z_1)} \delta(z_1, z_2) = \overset{z_1}{i(pq)} \;\; \overset{z_2}{j(rs)}$$

$$\left\langle \left(\bar{c}^{\mathrm{n}}(z_1)\right)_{pq} \; \left(c^{\mathrm{n}}(z_2)\right)_{rs} \right\rangle = \delta^{ps} \delta^{qr} \frac{1}{Q_{pq}(z_1)} \delta(z_1, z_2) = \overset{z_1}{(pq)} \;\; \overset{z_2}{(rs)}$$

$$\left\langle \left(\psi^{\mathrm{n}}_{\alpha}(z_1)\right)_{pq} \; \left(\psi^{\mathrm{n}}_{\beta}(z_2)\right)_{rs} \right\rangle = -\tfrac{1}{2} \delta^{ps} \delta^{qr} \left(A^{-1}_{pq}(z_1)\right)_{\alpha\beta} \delta(z_1, z_2) = \overset{z_1}{\alpha(pq)} \;\; \overset{z_2}{\beta(rs)}$$

where

$$\mathcal{A}^{-1} = \mathcal{Q}^{-1}\begin{pmatrix} 0 & -\gamma^I \mathrm{ad}_{X_I^\circ} & i\mathrm{ad}_{a_v^t} & i\mathrm{ad}_{a_{\bar{w}}^t} \\ \gamma^I \mathrm{ad}_{X_I^\circ} & 0 & -i\mathrm{ad}_{a_w^t} & i\mathrm{ad}_{a_{\bar{v}}^t} \\ i\mathrm{ad}_{a_v^t} & -i\mathrm{ad}_{a_w^t} & 0 & \gamma^{I\dagger}\mathrm{ad}_{X_I^\circ} \\ i\mathrm{ad}_{a_{\bar{w}}^t} & i\mathrm{ad}_{a_{\bar{v}}^t} & -\gamma^{I\dagger}\mathrm{ad}_{X_I^\circ} & 0 \end{pmatrix} .$$

Like in the two dimensional case the present propagators suffer from the divergences at the boundary of our integration manifold $\mathcal{X} = M_3 \times \mathbb{R}$. The recipe is again to substitute $\mathcal{X} \to \mathcal{X}_T$, and then remove it, sending $T \to \infty$, after the manipulations have been carried out.

4.3.2 The interaction terms

We prepare the $D = 4$ SYM action by setting $\psi^{\mathrm{t}} = 0$ and by inserting the Lagrange multipliers. Next we fix the gauge and add the Faddeev-Popov ghost term, taking into account also the insertion of Lagrange multipliers for the ghost interaction. We obtain the following interacting action:

$$\begin{aligned} S^{D=4}[\phi, \psi^{\mathrm{n}}, \psi^{\mathrm{t}} = 0, c] &= S^{D=4}_{sc,D}[\phi^{\mathrm{t}}, \psi^{\mathrm{t}} = 0, c^{\mathrm{t}}, D^{\mathrm{t}}] + S^{D=4}_{K}[\phi^{\mathrm{n}}, \psi^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}] \\ &+ S^{D=4}_{der}[\phi^{\mathrm{n}}, \psi^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, c^{\mathrm{t}}] + S^{D=4}_{pol,D}[\phi^{\mathrm{n}}, \psi^{\mathrm{n}}, c^{\mathrm{n}}, \phi^{\mathrm{t}}, \psi^{\mathrm{t}} = 0, c^{\mathrm{t}}, D^{\mathrm{t}}]. \end{aligned}$$

4.3.3 Derivative interaction terms

The terms containing derivatives are the most affected by dimensional reduction. Hence we will find here the major differences between the two theories we are studying.

Without the risk of getting confused, let us use the same names, as in the $D = 2$ case, for the single potentials arising from the original derivative terms in the theory:

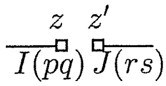
$$\begin{aligned} S^{(der)}_{int} &= V_1^{(b)}[x^{\mathrm{n}}] + V_1^{(g)}[a^{\mathrm{n}}] + V_1^{(c)}[c^{\mathrm{n}}] + V_1^{(f)}[\psi^{\mathrm{n}}] + V_1^{(b,g)}[x^{\mathrm{n}}, a^{\mathrm{n}}] \\ &+ V_2^{(b)}[x^{\mathrm{n}}] + V_2^{(g)}[a^{\mathrm{n}}] + V_2^{(c)}[c^{\mathrm{n}}] \\ &+ V_1^{(c,g)}[c^{\mathrm{n}}, a^{\mathrm{n}}] \end{aligned} \tag{4.23}$$

where the suffixes stand for the number of involved derivatives (and, apart from the last one, this is the same as the coupling power k); we have explicitely:

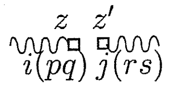
$$V_1^{(b)}[x^{\mathrm{n}}] = -\frac{i}{g}\int d^2w d^2v \mathrm{Tr}\left[x^{\mathrm{n}I}\left(\mathrm{ad}_{a^{\mathrm{t}}_{\bar{z}_i}}\partial_{z_i} + \mathrm{ad}_{a^{\mathrm{t}}_{z_i}}\partial_{\bar{z}_i}\right)x^{\mathrm{n}I}\right]$$

$$
\begin{aligned}
V_1^{(g)}[a^{\mathrm{n}}] &= \frac{i}{2g}\int d^2w d^2v \mathrm{Tr}\left[a^{\mathrm{n}}_{\bar{z}_j}\left(\mathrm{ad}_{\partial_{\bar{z}_i}a^{\mathrm{t}}_{z_j}} - \mathrm{ad}_{\partial_{z_j}a^{\mathrm{t}}_{\bar{z}_i}}\right)a^{\mathrm{n}}_{z_i} + \right.\\
&\qquad\qquad \left. + 2\,\partial_{z_i}a^{\mathrm{n}}_{\bar{z}_j}\mathrm{ad}_{a^{\mathrm{t}}_{\bar{z}_i}}a^{\mathrm{n}}_{z_j} + 2\,\partial_{\bar{z}_i}a^{\mathrm{n}}_{z_j}\mathrm{ad}_{a^{\mathrm{t}}_{z_i}}a^{\mathrm{n}}_{\bar{z}_j}\right]\\
V_1^{(c)}[c^{\mathrm{n}}] &= -\frac{i}{2g}\int d^2w d^2v \mathrm{Tr}\left[\bar{c}^{\mathrm{n}}\left(2\mathrm{ad}_{a^{\mathrm{t}}_{z_i}}\partial_{\bar{z}_i} + 2\mathrm{ad}_{a^{\mathrm{t}}_{\bar{z}_i}}\partial_{z_i} + \mathrm{ad}_{\partial_{\bar{z}_i}a^{\mathrm{t}}_{z_i}} + \mathrm{ad}_{\partial_{z_i}a^{\mathrm{t}}_{\bar{z}_i}}\right)c^{\mathrm{n}}\right]\\
V_1^{(f)}[\psi^{\mathrm{n}}] &= \frac{i}{g}\int d^2w d^2v \mathrm{Tr}\left[\psi^{\mathrm{n}T}(\bar{\gamma}_i\partial_{z_i} + \gamma_i\partial_{\bar{z}_i})\psi^{\mathrm{n}}\right]\\
V_1^{(b,g)}[x^{\mathrm{n}}, a^{\mathrm{n}}] &= -\frac{2i}{g}\int d^2w d^2v \mathrm{Tr}\left[a^{\mathrm{n}}_{\bar{z}_i}\mathrm{ad}_{\partial_{z_i}X^{\circ I}}x^{\mathrm{n}I} + a^{\mathrm{n}}_{z_i}\mathrm{ad}_{\partial_{\bar{z}_i}X^{\circ I}}x^{\mathrm{n}I}\right] \qquad (4.24)\\
V_2^{(b)}[x^{\mathrm{n}}] &= \frac{1}{g^2}\int d^2w d^2v \mathrm{Tr}\left(\partial_w x^{\mathrm{n}I}\ \partial_{\bar{w}}x^{\mathrm{n}I}\right)\\
V_2^{(g)}[a^{\mathrm{n}}] &= \frac{1}{g^2}\int d^2w d^2v \mathrm{Tr}\left(\partial_{z_i}a^{\mathrm{n}}_{\bar{z}_j}\ \partial_{\bar{z}_i}a^{\mathrm{n}}_{z_j}\right)\\
V_2^{(c)}[c^{\mathrm{n}}] &= \frac{1}{g^2}\int d^2w d^2v \mathrm{Tr}\left(\partial_{z_i}\bar{c}^{\mathrm{n}}\ \partial_{\bar{z}_i}c^{\mathrm{n}}\right)\\
V_1^{(c,g)}[c^{\mathrm{n}}, a^{\mathrm{n}}] &= -\frac{i}{2\sqrt{g}}\int d^2w d^2v \mathrm{Tr}\left[a^{\mathrm{n}}_{\bar{z}_i}\mathrm{ad}^{\{\}}_{\partial_{z_i}\bar{c}^{\mathrm{t}}}c^{\mathrm{n}} + a^{\mathrm{n}}_{z_i}\mathrm{ad}^{\{\}}_{\partial_{\bar{z}_i}\bar{c}^{\mathrm{t}}}c^{\mathrm{n}}\right] +\\
&\quad + \frac{i}{2g^{3/2}}\int d^2w d^2v \mathrm{Tr}\left[\bar{c}^{\mathrm{n}}\mathrm{ad}_{c^{\mathrm{t}}}\partial_{z_i}a^{\mathrm{n}}_{\bar{z}_i} + \bar{c}^{\mathrm{n}}\mathrm{ad}_{c^{\mathrm{t}}}\partial_{\bar{z}_i}a^{\mathrm{n}}_{z_i}\right] +\\
&\quad - \frac{i}{2g^2}\int d^2w d^2v \mathrm{Tr}\left[\bar{c}^{\mathrm{n}}\partial_{z_i}\left[a^{\mathrm{n}}_{\bar{z}_i}, c^{\mathrm{n}}\right] + \bar{c}^{\mathrm{n}}\partial_{\bar{z}_i}\left[a^{\mathrm{n}}_{z_i}, c^{\mathrm{n}}\right]\right]
\end{aligned}
$$

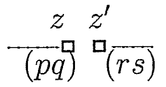
To extract the vertices from these potentials we proceed like in the two dimensional case, obtaining:

from $V_1^{(b)}[x^{\mathrm{n}}]$ $\Rightarrow$ 

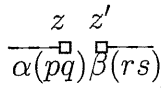
$= \frac{i}{g}\delta^{IJ}\delta^{ps}\delta^{qr}\ B_{pq}(z, z')$

from $V_1^{(g)}[a^{\mathrm{n}}]$ $\Rightarrow$ 

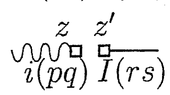
$= \frac{i}{g}\delta^{ps}\delta^{qr}\left[\delta^{ij}B_{pq}(z, z') + \bar{M}_{i,pq}[a^{\mathrm{t}}_{z_j}] + M_{j,pq}[a^{\mathrm{t}}_{\bar{z}_i}]\right]$

from $V_1^{(c)}[c^{\mathrm{n}}]$ $\Rightarrow$ 

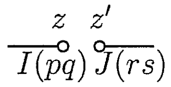
$= \frac{i}{g}\delta^{ps}\delta^{qr}\left[B_{pq}(z, z') + \bar{M}_{i,pq}[a^{\mathrm{t}}_{z_i}] + M_{i,pq}[a^{\mathrm{t}}_{\bar{z}_i}]\right]$

from $V_1^{(f)}[\psi^{\mathrm{n}}]$ $\Rightarrow$ 

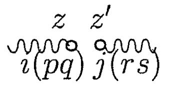
$= -\frac{1}{g}\delta^{ps}\delta^{qr}F_{\alpha\beta}(z, z')$

from $V_1^{(b,g)}[x^{\mathrm{n}}, a^{\mathrm{n}}]$ $\Rightarrow$ 

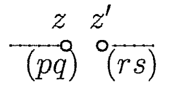
$= -\frac{2i}{g}\delta^{ps}\delta^{qr}\left(M_{i,pq}[X^{\circ I}] + \bar{M}_{i,pq}[X^{\circ I}]\right)$

from $V_2^{(b)}[x^{\rm n}] \Rightarrow$ 

$= \frac{1}{g^2}\delta^{IJ}\delta^{ps}\delta^{qr}\int d^4z''\partial_{z_i}\delta(z,z'')\partial_{\bar{z}'_i}\delta(z',z'')$

from $V_2^{(g)}[a^{\rm n}] \Rightarrow$ 

$= \frac{1}{g^2}\delta^{ij}\delta^{ps}\delta^{qr}\int d^4z''\partial_{z_k}\delta(z,z'')\partial_{\bar{z}'_k}\delta(z',z'')$

from $V_2^{(c)}[c^{\rm n}] \Rightarrow$ 

$= \frac{1}{g^2}\delta^{ps}\delta^{qr}\int d^4z''\partial_{z_i}\delta(z,z'')\partial_{\bar{z}'_i}\delta(z',z'')$,

where we have defined the following objects:

$$\begin{aligned}
B_{pq}(z,z') &= \partial_{z_i}\delta(z,z')\left(\mathrm{ad}_{a^{\rm t}_{\bar{z}_i}}(z)\right)_{pq} - \partial_{\bar{z}'_i}\delta(z,z')\left(\mathrm{ad}_{a^{\rm t}_{z_i}}(z')\right)_{pq} \\
M_{i,pq}[\Phi^{\rm t},z,z'] &= \partial_{z_i}\delta(z,z')\left(\mathrm{ad}_{\Phi^{\rm t}(z)}\right)_{pq} + \partial_{z'_i}\delta(z,z')\left(\mathrm{ad}_{\Phi^{\rm t}(z')}\right)_{pq} \\
\bar{M}_{i,pq}[\Phi^{\rm t},z,z'] &= \partial_{\bar{z}_i}\delta(z,z')\left(\mathrm{ad}_{\Phi^{\rm t}(z)}\right)_{pq} + \partial_{\bar{z}'_i}\delta(z,z')\left(\mathrm{ad}_{\Phi^{\rm t}(z')}\right)_{pq} \\
F_{\alpha\beta}(z,z') &= (\gamma_i)_{\alpha\beta}\,\partial_{\bar{z}_i}\delta(z,z') + (\bar{\gamma}_i)_{\alpha\beta}\,\partial_{z_i}\delta(z,z').
\end{aligned} \tag{4.25}$$

4.3.4 Polynomial interactions

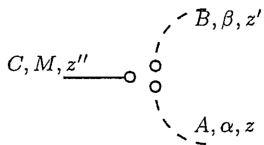
The bosonic, ghost and Lagrange multiplier interactions, contained in $S^{D=4}_{pol,D}[\phi^{\rm n},\psi^{\rm n},c^{\rm n},\phi^{\rm t},\psi^{\rm t}=0,c^{\rm t},D^{\rm t}]$, are the same as the two dimensional case, the only difference being the integration domain, which is four dimensional. Hence for these interactions and their vertices we refer the reader to equations (4.18), (4.19) and (4.21), where one has to substitute $d^2w \to d^4z$ and $\delta^{(2)}(w,w') \to \delta^{(4)}(z,z')$.

Care has to be taken with fermions, since they are completely different from the two dimensions. We start from the action term:

$$\begin{aligned}
V^{(f)}[\psi^{\rm n}] &= \frac{i}{g}\int d^4z\mathrm{Tr}\left\{\psi^{{\rm n}T}G^{(4)}_M\left[\phi^{\rm n}_M,\psi^{\rm n}\right]\right\} \\
&= -\frac{i}{g}(g_M)_{\alpha\beta}\,T^{(3)}_{AB,C}\int d^4zd^4z'd^4z''\delta(z,z')\delta(z,z'')\left[\psi^{\rm n}_{\alpha,A}(z)\;\psi^{\rm n}_{\beta,B}(z')\;\phi^{{\rm n}M}_{C}(z'')\right]
\end{aligned}$$

where $G^{(4)}_M$ is defined in the Appendix A.

The vertex is



$$= -\frac{i}{g}\left(G^{(4)}_M\right)_{\alpha\beta}T^{(3)}_{AB,C}\int d^4zd^4z'd^4z''\delta(z,z')\delta(z,z'') \quad . \tag{4.26}$$

4.4 About the need of a regularization

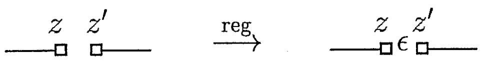
To end this chapter let us come back to the introduction, where we anticipated with an example the problem of the divergencies arising in computing perturbative corrections. The naive regularization introduced there can now be given a rigorous definition.

It is by now clear that the presence of divergences is strictly connected with the nature of the propagators, which are contact propagators: this implies that any time we compute a loop diagram we should expect to find products of delta functions evaluated on the same point, this in turn leading to bad divergences and ambiguities, which must be regularized and fixed before one goes on performing any kind of formal manipulations. If the regulator can be inoffensively removed then one is sure that the divergences will not affect the final result.

We choose a *point-splitting* regularization, which consists of picking up a vertex, once the graph is given, and "opening" it, the result being of course independent of the choice of the particular vertex. Since we have already performed a formal point splitting to define the vertices, now it is enough to smooth the deltas which are there with some regulator ϵ, or in other words substitute them with a function $\delta_\epsilon(w)$ that must undergo the condition $\delta_\epsilon(w) \to \delta(w)$ as $\epsilon \to 0$ in the distribution sense.

The most tricky vertices to be regularized are the derivative ones, since in this case the ambiguities deriving from the ill-defined expressions can be very misleading, due to the big deal of by part integrations we have to perform.

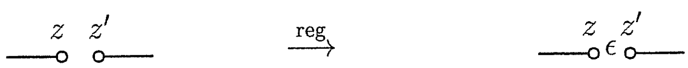
Let us sketch , for example, how the regularization for the vertex $V_1^{(b)}$ works:



$$\frac{i}{g} B_{pq}(z,z') \quad \xrightarrow{\text{reg}} \quad \frac{i}{g} B^\epsilon_{pq}(z,z')$$

where $B^\epsilon_{pq}(z,z')$ stands for $B_{pq}(z,z')$ where $\delta(z,z')$ has been replaced by $\delta_\epsilon(z,z')$.

Coherently with this scheme we will have for the two derivatives vertices the simmetrized expression (necessary for the regularization not to break the simmetry of this vertices under the exchange of z and z'):



$$\int d^4z'' \partial_w \delta(z,z'') \partial_{\bar{z}'} \delta(z',z'') \quad \xrightarrow{\text{reg}} \quad \frac{1}{2} \int d^4z'' \left[\partial_w \delta_\epsilon(z,z'') \partial_{\bar{z}'} \delta(z',z'') + \right.$$
$$\left. + \partial_w \delta(z,z'') \partial_{\bar{z}'} \delta_\epsilon(z',z'') \right] .$$

As far as the polynomial interactions are concerned, they contribute at least at two loops to the vacuum diagrams we will compute. This is why the regularization here concerns two delta functions in the vertex, otherwise it would not be sufficient to smooth the divergences. For the three legs vertices, which have been defined by means of two delta funcions, we simply get:

$$\delta(z,z')\delta(z,z'') \xrightarrow{\text{reg}} \delta_\epsilon(z,z')\delta_\epsilon(z,z'') \tag{4.27}$$

while for the four legs vertex we need a simmetrized regularization, as in the derivative case:

$$\delta(z,z'')\delta(z,z')\delta(z'',z''') \xrightarrow{\text{reg}} \frac{1}{2}\delta_\epsilon(z,z'')\left[\delta_\epsilon(z,z')\delta(z'',z''') + \delta(z,z')\delta_\epsilon(z'',z''')\right] \tag{4.28}$$

We adopt the following notation for the two dimensional regularized delta:

$$\begin{aligned} h(\epsilon) &= \delta_\epsilon(v_1,v_2)\big|_{v_1=v_2} \\ h_v(\epsilon) &= \partial_{v_1}\delta_\epsilon(v_1,v_2)\big|_{v_1=v_2} \\ h_{v\bar{v}}(\epsilon) &= \partial_{v_1}\partial_{\bar{v}_2}\delta_\epsilon(v_1,v_2)\big|_{v_1=v_2} \\ &\dots \end{aligned}$$

Of course these functions diverge as ϵ approaches zero. Since $\delta^{(4)}(z) = \delta^{(2)}(v)\ \delta^{(2)}(w)$ we ask the same propertyto be true for $\delta_\epsilon^{(4)}(z)$.

We want to stress an important point: we are not constraining the explicit form of the function $\delta_\epsilon(w)$ in any way; this is necessary in order our results do not depend on the particular regularization we are choosing.

However, for practical purposes, once we have verified any smoothing can be choosen, we will adopt functions which possess the same symmetry of the original delta, i.e. we will ask that $\delta_\epsilon(v) = \delta_\epsilon(|v|)$. This will lead to out simplification in the computations.

Chapter 5

Corrections to the second order in $1/g$

5.1 Introduction

At this point we have at our disposal all the necessary tools to analyze the perturbative structure of both $D = 2$ $\mathcal{N} = (8,8)$ and $D = 4$ $\mathcal{N} = 4$, in the regime of strong YM coupling.

The zero modes dependence on the YM coupling leads to the interpretation of such strong coupling limit in terms of perturbative weak coupling (in g_s and g_3) interactions of IIA strings (two dimensional case) and IIB D3-branes (four dimensional case).

Looking at (4.6) and (4.22), it is immediately clear how any perturbative contribution to the effective action coming from $Z_{\mathfrak{n}}^{D=2,4}[\phi^{\mathfrak{t}}, \psi^{\mathfrak{t}}, c^{\mathfrak{t}}, g]$ can dramatically modify the interpretation of the strong coupling limit; in fact one should have to combine, order by order, the non perturbative (topological) expansion in $1/g$ with the perturbatively originated one; in so doing we would be lead to a radically different dynamics.

But, as already anticipated in the previous chapter, this is not the whole story: the non-Cartan field interactions would provide to the effective theory divergent contributions, making hard the task of interpreting them in terms of physical interactions.

This justifies enough our efforts in deeply understanding the perturbative behavior of the non-Cartan theories. Such an analysis will lead us to obtain an exact cancellation of the divergent contributions, thanks to supersymmetry, up to the second order in $1/g$.

5.2 The perturbative expansion

We use as the starting point for the computations throughout this chapter, the non-Cartan partition functions appearing in (4.6) and (4.22), with the already explained insertion of the Lagrange multipliers, and switching off the Cartan components of the fermions (see section 4.2.2). Let us call $\Phi^{\mathfrak{t}}$ the remaining Cartan fields, i.e. $\Phi^{\mathfrak{t}} = (\phi^{\mathfrak{t}}, c^{\mathfrak{t}}, D^{\mathfrak{t}}, C^{\mathfrak{t}})$. Moreover we adopt a suffix $_D$ to distinguish between objects where the Lagrange multipliers have been inserted and whose where not.

Now we expand it, both in two and four dimensions, up to the second order:

$$\begin{aligned} Z_{\mathfrak{n},D}^{D=2,4}[\Phi^{\mathfrak{t}}, g] &= \int \mathcal{D}\phi^{\mathfrak{n}}\mathcal{D}\theta^{\mathfrak{n}}\mathcal{D}c^{\mathfrak{n}}\; e^{-S_K^{D=2,4}[\phi^{\mathfrak{n}},\theta^{\mathfrak{n}},c^{\mathfrak{n}},\phi^{\mathfrak{t}}]-S_{der}^{D=2,4}[\phi^{\mathfrak{n}},\theta^{\mathfrak{n}},c^{\mathfrak{n}},\Phi^{\mathfrak{t}}]-S_{pol,D}^{D=2,4}[\phi^{\mathfrak{n}},\theta^{\mathfrak{n}},c^{\mathfrak{n}},\Phi^{\mathfrak{t}}]} \\ &= \int \mathcal{D}\phi^{\mathfrak{n}}\mathcal{D}\theta^{\mathfrak{n}}\mathcal{D}c^{\mathfrak{n}} \\ &\quad \times \left[1 - S_{der}^{D=2,4} + \frac{1}{2}\left(S_{der}^{D=2,4}\right)^2 - S_{pol,D}^{D=2,4} + \frac{1}{2}\left(S_{pol,D}^{D=2,4}\right)^2\right] e^{-S_K^{D=2,4}} \\ &= 1 + \frac{1}{g}\alpha_{der,1}^{D=2,4}[\Phi^{\mathfrak{t}}] + \frac{1}{g^2}\alpha_{der,2}^{D=2,4}[\Phi^{\mathfrak{t}}] + \frac{1}{g}\alpha_{pol,1}^{D=2,4}[\Phi^{\mathfrak{t}}] + \frac{1}{g^2}\alpha_{pol,2}^{D=2,4}[\Phi^{\mathfrak{t}}] \end{aligned} \quad (5.1)$$

where we used the fact that, at the second order, the derivative and polynomial sectors are not connected, as can be easily seen from the vertices, this allowing us to compute the corrections separately. We immediately learn from (5.1) that we have to multiply each diagram by the factor $(-1)^V/V!$, where V is the number of vertices involved.

We now start to compute the α_n for $n = 1, 2$, separately for the two and four dimensional cases, and for the derivative and polynomial interactions.

5.3 Derivative corrections in $D = 2$

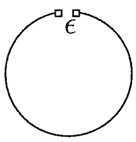
With the aim of computing

$$\frac{1}{g}\alpha_{der,1}^{D=2}[\Phi^{\mathfrak{t}}] + \frac{1}{g^2}\alpha_{der,2}^{D=2}[\Phi^{\mathfrak{t}}] \quad (5.2)$$

we give here a very detailed proof of their vanishing, taking into account also the boundary terms emerging in such corrections. Since, as we will see, the choice $h_v(\epsilon) = 0$ precisely kills the boundary terms arising in the integrations by part, we simply do not impose such a condition, although this would be a very natural choice, since this property is conserved in the limit $\epsilon \to 0$, being in fact a property of the delta function itself. This results in a computation were we have a full control of the cancellations.

5.3.1 The computation to the first order in g^{-1}

Everything vanishes here thanks mainly to the antisymmetry of the operator ad_{Φ^t} coming always with the same power of the coupling, i.e. one. Let us see this in an example, computing the $1/g$ vacuum diagram for the scalars; it is easily seen that only the square vertices can contribute to the first order, in the following way:



$$\approx \sum_{p \neq q} K_{pq} \, Q^{-1}_{pq} = 0 \tag{5.3}$$

due to the antisymmetry of K_{pq}, defined in (4.15).
For fermions the same happens thanks to the antisymmetry of the propagator A^{-1}_{pq}.

So a purely algebraic argument leads us to conclude that

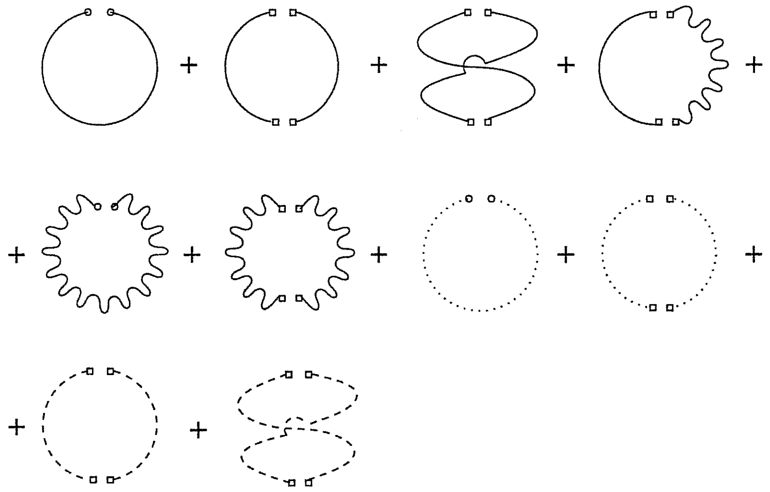
$$\alpha^{D=2}_{der,1}[\Phi^t] = 0 \tag{5.4}$$

and, more generally, that any loop diagram made up by an odd number of square vertices (either bosonic or fermionic or gauge) vanishes.

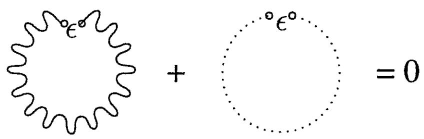
5.3.2 The computation to the second order in g^{-1}

If we combine the vertices (4.14) in all the possible ways, consistently with the g^{-2}, we obtain the following:

$\frac{1}{g^2}\alpha^{D=2}_{der,2}[\Phi^{\mathrm{t}}] =$



Let us start with the gauge and ghosts contributions, recalling that every ghost loop brings a minus sign (to take into account the fermionic statistic they have) and noticing that the propagators are exactly the same. So we can straightforwardly compute:



which can also be thought of as the ratio of determinants:

$$\frac{\mathrm{Det}(\partial\bar{\partial})}{\mathrm{Det}(\partial\bar{\partial})} = 1 \tag{5.5}$$

since the g^{-2} vertices are the same. Moreover the potential $V_1^{(c,g)}[c^{\mathrm{n}}, a^{\mathrm{n}}]$ will not contribute to the second order; in fact to achieve a g^{-2} contribution from a $g^{-1/2}$ and a $g^{-3/2}$ terms the final expression must contain for example

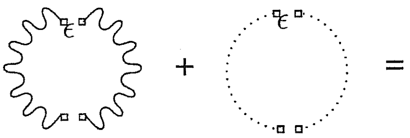
$$\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}\mathrm{ad}^{\{\}}_{\partial_w\bar{c}^{\mathrm{t}}}\mathrm{ad}_{c^{\mathrm{t}}}\right] = 0$$

thanks to the antisymmetry of the ad operator and to the fact that $\mathrm{ad}^{\{\}}_{\partial_w\bar{c}^{\mathrm{t}}}N = \{\partial_w\bar{c}^{\mathrm{t}}, N\}$ and so it is symmetric. For the defininition of $\mathrm{Tr}_{\mathcal{O}}$ see Appendix A.

We will now go into some details of the computation, selecting the sector concerning $h_w(\epsilon)$ and the products $\mathrm{ad}_{a^{\mathrm{t}}_w}\cdot\mathrm{ad}_{a^{\mathrm{t}}_{\bar{w}}}$ and $\mathrm{ad}_{X^{\circ I}}\mathrm{ad}_{X^{\circ I}}$. The following notation will be adopted:

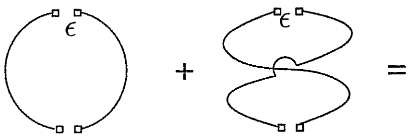
$A_1 = \mathrm{ad}_{a^t_{w_1}}$, $A_{\bar{z}1} = \mathrm{ad}_{a^t_{\bar{w}_1}}$, $X^I_1 = \mathrm{ad}_{X^{\circ I}(w_1)}$. For compactness we will also write $f(1) = f(w_1)$, where f is whatever function involved.

We have:



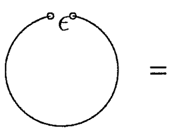
$$
\begin{aligned}
&= 2\left(\frac{i}{g}\right)^2 \int d^2w_1 d^2w_2 \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}_1^{-1}\cdot \mathcal{Q}_2^{-1}\bar{\partial}A_2 A_{\bar{z}2}\right]\delta(1,2)\partial_2\delta_\epsilon(1,2) + (w \leftrightarrow \bar{w}) \\
&= -\frac{2}{g^2}h_w(\epsilon)\int d^2w \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}\bar{\partial}AA_{\bar{z}}\right] + (w \leftrightarrow \bar{w})
\end{aligned}
$$

Next we continue with the bosonic sector, obtaining:



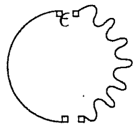
$$
\begin{aligned}
&= 2\left(\frac{i}{g}\right)^2 \int d^2w_1 d^2w_2 \mathrm{Tr}_{\mathcal{O}}\left\{\mathcal{Q}_1^{-1}\cdot \mathcal{Q}_2^{-1}\times\right. \\
&\qquad \times\left.\left[-A_{\bar{z}2}\partial_2\delta_\epsilon(2,1)A_2\bar{\partial}_2\delta(1,2) + A_{\bar{z}1}\partial_1\delta_\epsilon(1,2)A_2\bar{\partial}_2\delta(1,2)\right]\right\} + (w \leftrightarrow \bar{w}) \\
&= \frac{4}{g^2}h_{w\bar{w}}(\epsilon)\int d^2w \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}AA_{\bar{z}}\right] + \frac{2}{g^2}h_w(\epsilon)\int d^2w \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}\bar{\partial}AA_{\bar{z}}\right] \\
&\quad + \frac{4}{g^2}h_w(\epsilon)\int d^2w \mathrm{Tr}_{\mathcal{O}}\left[\bar{\partial}\mathcal{Q}^{-1}\mathcal{Q}^{-1}AA_{\bar{z}}\right] - \frac{4}{g^2}h_w(\epsilon)\oint dw \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}A_{\bar{z}}A\right] + (w \leftrightarrow \bar{w})
\end{aligned}
$$

Next we are faced with the computation of the 1-vertex bosonic graph:



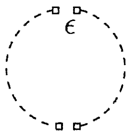
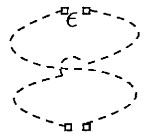
$$
\begin{aligned}
&= \frac{2}{g^2}\int d^2w_1 d^2w_2 \partial_1\delta_\epsilon(1,2)\bar{\partial}_1\delta(1,2)\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}_1^{-1}\right] + (w \leftrightarrow \bar{w}) \\
&= +\frac{2}{g^2}h_{w\bar{w}}(\epsilon)\int d^2w \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-1}\right] + (w \leftrightarrow \bar{w})
\end{aligned}
$$

The gauge-boson interaction leads to the following:

 $=$

$$= \frac{2}{g^2}\int d^2w_1 d^2w_2 \mathrm{Tr}_{\mathcal{O}}\left\{\mathcal{Q}_1^{-1}\mathcal{Q}_2^{-1}\left[X_1^I\partial_2\delta_\epsilon(1,2)+X_2^I\partial_1\delta_\epsilon(2,1)\right]\bar{\partial}X_2^I\right\}\delta(1,2)+(w\leftrightarrow\bar{w})$$
$$= -\frac{4}{g^2}h_w(\epsilon)\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^I\bar{\partial}X^I\right]+(w\leftrightarrow\bar{w})$$

And finally the fermionic contribution:

 $+$  $= \frac{1}{g^2}\alpha^{(f)}_{der,2} =$

$$= -4\left(\frac{i}{g}\right)^2\int d^2w_1 d^2w_2 \mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}_1^{-1}\mathcal{Q}_2^{-1}X_1^IX_2^I\right]\partial_1\delta_\epsilon(1,2)\bar{\partial}_1\delta(1,2)+(w\leftrightarrow\bar{w})$$
$$\begin{aligned} &= \frac{4}{g^2}h_{w\bar{w}}(\epsilon)\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^IX^I\right] \\ &\quad+\frac{4}{g^2}h_w(\epsilon)\left\{\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\bar{\partial}\mathcal{Q}^{-1}\mathcal{Q}^{-1}X^IX^I+\mathcal{Q}^{-2}X^I\bar{\partial}X^I\right]-\oint dw\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^IX^I\right]\right\} \\ &\quad+(w\leftrightarrow\bar{w}) \end{aligned}$$

Summing up all the bosonic and ghost contributions in $\frac{1}{g^2}\alpha^{(b)}_{der,2}$, we have:

$$\begin{aligned} \frac{1}{g^2}\alpha^{(b)}_{der,2} &= \frac{2h_w(\epsilon)}{g^2}\left\{\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\bar{\partial}\mathcal{Q}^{-1}\mathcal{Q}^{-1}AA_{\bar{z}}\right]-\oint dw\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}A_{\bar{z}}A\right]\right. \\ &\qquad\left.-\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^I\bar{\partial}X^I\right]\right\}-\frac{2h_{w\bar{w}}(\epsilon)}{g^2}\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^IX^I\right] \\ &\quad+(w\leftrightarrow\bar{w}) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{g^2}\alpha^{(f)}_{der,2} &= \frac{2h_w(\epsilon)}{g^2}\left\{\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\bar{\partial}\mathcal{Q}^{-1}\mathcal{Q}^{-1}X^IX^I\right]-\oint dw\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^IX^I\right]\right. \\ &\qquad\left.+\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^I\bar{\partial}X^I\right]\right\}+\frac{2h_{w\bar{w}}(\epsilon)}{g^2}\int d^2w\mathrm{Tr}_{\mathcal{O}}\left[\mathcal{Q}^{-2}X^IX^I\right] \\ &\quad+(w\leftrightarrow\bar{w}) \end{aligned}$$

from which we can conclude that

$$\frac{1}{g^2}\alpha^{D=2}_{der,2}[\Phi^t] = \frac{1}{g^2}\alpha^{(f)}_{der,2} + \frac{1}{g^2}\alpha^{(b)}_{der,2} = 0 \tag{5.6}$$

Since these are all the terms involving derivatives at one loop, either they cancel among themselves or they do not cancel at all. Fortunately they do.

5.4 Derivative corrections in $D = 4$

With the two dimensional computation we have definitely proved the independence of the derivative corrections with respect to the particular choice of the smoothing for the delta function, since our arguments are not based on any particular property of $\delta_\epsilon(w)$.

Let us put ourseslves in a simpler situation for the $D = 4$ check. It turns out to be very natural (and useful in simplifying the computations) to ask, for the smoothing, the rotational invariance, or in other words to ask that $\delta_\epsilon(v) = \delta_\epsilon(|v|)$, since we are just requiring that the regularized delta enjoys the same simmetric properties as the original one. An immediate consequence of such a choice is obviously that

$$h_v(\epsilon) = h_{vv}(\epsilon) = h_{\bar{v}\bar{v}}(\epsilon) = 0 \tag{5.7}$$

and, more than this, we have in 4D:

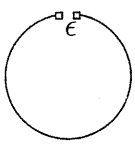
$$\partial_{z_i}\partial_{\bar{z}_j}\delta_\epsilon(z) = \delta^{ij}h_{v\bar{v}}(\epsilon)h(\epsilon).$$

We are now ready to compute the derivative corrections up to second order:

$$\frac{1}{g}\alpha^{D=4}_{der,1}[\Phi^t] + \frac{1}{g^2}\alpha^{D=4}_{der,2}[\Phi^t] \tag{5.8}$$

5.4.1 $1/g$ diagrams

The argument here goes through exactly in the same way as it has been carried out in two dimensions. Let us give again an example:



$$\approx \textstyle\sum_{p\neq q} B_{pq}\, Q^{-1}_{pq} = 0 \tag{5.9}$$

due to the antisymmetry of B_{pq}, defined in (4.25).

We conclude that

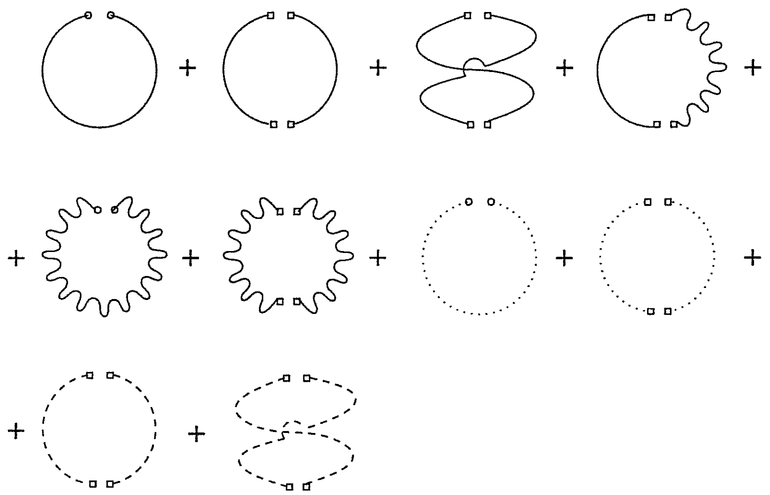
$$\alpha^{D=4}_{der,1}[\Phi^{t}] = 0 \tag{5.10}$$

and, more generally, that any loop diagram made up by an odd number of square vertices (either bosonic or fermionic or gauge) vanishes.

5.4.2 $1/g^2$ diagrams

If we combine the vertices in all the possible ways, consistently with the second order, we find the following:

$$\frac{1}{g^2}\alpha^{D=4}_{der,2}[\Phi^{t}] =$$



M and $\bar{M}$ terms

Let us restrict for a moment on the two-vertices bosonic, gauge or ghost diagrams: these will contain contributions of the kind

$$a_2 = \int d^4z_1 d^4z_2 \; B^{\epsilon}_{pq}(z_1, z_2) M_{i,rs}[\Phi^{t}, z_1, z_2] Q^{-1}_{pq}(z_1) Q^{-1}_{rs}(z_2) \tag{5.11}$$

or

$$b_2 = \int d^4z_1 d^4z_2 \; \bar{M}^{\epsilon}_{i,pq}[\Phi^{t}, z_1, z_2] M_{i,rs}[\Phi^{t}, z_1, z_2] Q^{-1}_{pq}(z_1) Q^{-1}_{rs}(z_2) \quad . \tag{5.12}$$

Take the first one and make it more explicit:

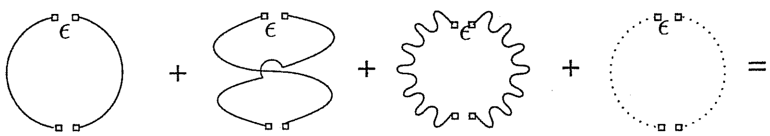
$$
\begin{aligned}
a_2 &= \int d^4z_1 d^4z_2 \; B^{\epsilon}_{pq}(z_1, z_2) \\
&\times \; \left[\partial_{z_{1i}}\delta(z_1, z_2)\left(\mathrm{ad}_{\Phi^{\dagger}(z_1)}\right)_{rs} + \partial_{z_{2i}}\delta(z_1, z_2)\left(\mathrm{ad}_{\Phi^{\dagger}(z_2)}\right)_{rs}\right] Q^{-1}_{pq}(z_1)Q^{-1}_{rs}(z_2) = \\
&= \int d^4z_1 d^4z_2 \; \partial_{z_{1i}} B^{\epsilon}_{pq}(z_1, z_2)\left[\left(\mathrm{ad}_{\Phi^{\dagger}(z_1)}\right)_{rs} - \left(\mathrm{ad}_{\Phi^{\dagger}(z_2)}\right)_{rs}\right]\delta(z_1, z_2)Q^{-1}_{pq}(z_1)Q^{-1}_{rs}(z_2) \\
&= 0 \quad .
\end{aligned}
$$

where we have used the fact that the integration by part moves the derivatives only from $\delta(z_1, z_2)$ in $M_{i,pq}$ to $\delta_{\epsilon}(z_1, z_2)$ in B^{ϵ}_{pq} , since any other term (as well as the boundary terms) should be proportional to $h_v(\epsilon)$ or $h_{\bar{v}}(\epsilon)$, and so gets killed by our choice of the smoothing.

So the $M_{i,pq}$ and $\bar{M}_{i,pq}$ in the vertices can be neglected, up to the second order.

Two-vertices bosonic gauge and ghost graphs

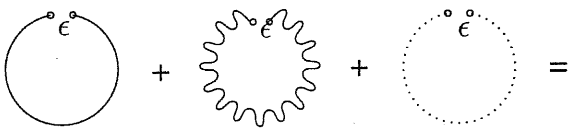
We now analyze the non zero contributions coming from these diagrams:



$$
\begin{aligned}
&= -\frac{1}{g^2}\left(\frac{\delta^{II}}{2} + \delta^{ii} - 1\right)\sum_{p\neq q}\int d^4z_1 d^4z_2 B^{\epsilon}_{pq}(z_1, z_2)B_{pq}(z_2, z_1)Q^{-1}_{pq}(z_1)Q^{-1}_{pq}(z_2) = \\
&= -\frac{8}{g^2}h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z \; Q^{-1}_{pq}(z)Q^{-1}_{pq}(z)\left(\mathrm{ad}_{a^{\dagger}_{\bar{z}_j}}\right)_{pq}\left(\mathrm{ad}_{a^{\dagger}_{z_j}}\right)_{pq}
\end{aligned}
\tag{5.13}
$$

One-vertex bosonic gauge and ghost graphs

From the circle vertices the following contribution arise:



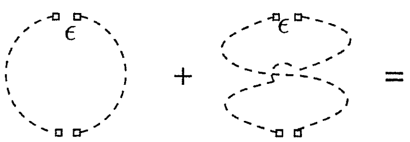
$$
= \frac{1}{g^2}\left(\frac{\delta^{II}}{2} + \delta^{ii} - 1\right)\sum_{p\neq q}\int d^4z d^4z' d^4z'' \; \partial_{z_j}\delta_{\epsilon}(z, z'') \; \partial_{\bar{z}'_j}\delta(z', z'') \; \delta(z, z')Q^{-1}_{pq}(z) =
$$

$$= \ -\frac{8}{g^2}h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z \ Q^{-1}_{pq}(z) \tag{5.14}$$

The last two results were easily predictable in what concerns the coefficient, since the interaction terms we are dealing with are nothing but the dimensionally reduced 10D derivative terms for the, say, A^M, $M = 1,\ldots,10$ gauge field; hence although the dof are organized in a different way (according to the dimensionality of the worldsheet) their counting is always equivalent to the one in 10D.

Fermions

Fermions can interact only via the square vertex:



$$\begin{aligned}
&= \ -\frac{1}{2g^2}\sum_{p\neq q}\int d^4z_1d^4z_2\mathrm{Tr}\left[F^{\epsilon}(z_1,z2)A^{-1}_{pq}(z_1)F(z_2,z1)A^{-1}_{pq}(z_2)\right] = \\
&= \ -\frac{1}{g^2}h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z\mathrm{Tr}\left(\gamma_iA^{-1}_{pq}(z)\bar{\gamma}_iA^{-1}_{pq}(z)\right) \\
&= \ -\frac{8}{g^2}h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z\left[Q^{-1}_{pq}+Q^{-2}_{pq}\left(\mathrm{ad}_{X^{\circ I}}\right)_{pq}\left(\mathrm{ad}_{X^{\circ I}}\right)_{pq}\right]
\end{aligned} \tag{5.15}$$

where the trace acts on the spinorial indices.

Result

Summing up the results of (5.13), (5.14), (5.15) we end with the coefficient of the second order derivative correction to the partition function:

$$\begin{aligned}
\alpha^{D=4}_{der,2}[\Phi^{\mathfrak{t}}] &= h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z\left[-4Q^{-2}_{pq}\left(\mathrm{ad}_{a^{\mathfrak{t}}_{\bar{z}_j}}\right)_{pq}\left(\mathrm{ad}_{a^{\mathfrak{t}}_{z_j}}\right)_{pq}+8Q^{-1}_{pq}+\right. \\
&\qquad\qquad \left. -4Q^{-1}_{pq}-4Q^{-2}_{pq}\left(\mathrm{ad}_{X^{\circ I}}\right)_{pq}\left(\mathrm{ad}_{X^{\circ I}}\right)_{pq}\right] = \\
&= h_{v\bar{v}}(\epsilon)h(\epsilon)\sum_{p\neq q}\int d^4z\left[4Q^{-1}_{pq}-4Q^{-1}_{pq}\right]=0 \quad .
\end{aligned}$$

5.5 Polynomial corrections in $D = 2$

Starting from the vertices defined in (4.18), (4.19), (4.20) and (4.21) we can compute the first and second order perturbative corrections provided to the Cartan effective action by the non Cartan fields:

$$\frac{1}{g}\alpha^{D=2}_{pol,1}[\Phi^{\mathfrak{t}}] + \frac{1}{g^2}\alpha^{D=2}_{pol,2}[\Phi^{\mathfrak{t}}] \quad . \tag{5.16}$$

5.5.1 $1/g$ contributions

As well as in the derivative vertices these contributions are not present; the $1/g$ vertices have three legs, and cannot generate any self interaction at one loop. Hence

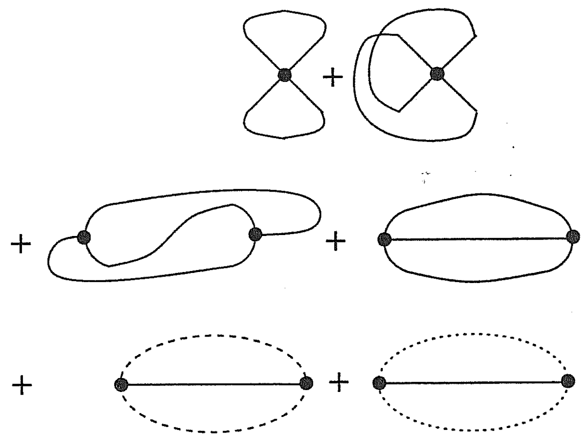
$$\frac{1}{g}\alpha^{D=2}_{pol,1}[\Phi^{\mathfrak{t}}] = 0 \tag{5.17}$$

5.5.2 $1/g^2$ contributions

These corrections arise from two loops diagrams. Nevertheless they do not exhaust all the two loops diagrams, since at higher perturbative order these are allowed, originating, for example, from one quartic vertex and two derivative ones leading to a $1/g^4$ contribution.

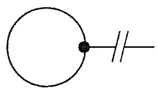
The global contribution from polynomial interactions at $1/g^2$ originates from the following perturbative expansion:

$\frac{1}{g^2}\alpha^{D=2}_{pol,2}[\Phi^{\mathfrak{t}}] =$



+(lagrange multiplier terms)

where we sketched the different topologies, neglecting to draw all the possible graph which can be generated once the topology is fixed, these amounting simply in combinatoric factors. Moreover we get rid of certain diagrams, since we already know they vanish. Let us see why. Any diagram containing a factor of the kind



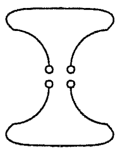
is automatically zero, since the loop is made up of a propagator, which contains the symmetric factor P_{AB} and a three legs vertex, which yields a $T^{(3)}_{AB,C}$ factor, antisymmetric in AB. Notice that bosons, fermions or ghostscan circulate in the loop. The symbol P is defined in Appendix A.

One vertex diagram

The only vertex which can contribute by itself to $1/g^2$ is the bosonic four legs one, defined in (4.18).

Let us connect its legs in all the possible ways; among the three graphs which arise, one vanishes immediately, since it is proportional to $T^{(4)}_{AB,CD}P_{AB}P_{CD} = 0$ for symmetry properties of $T^{(4)}$.

If we take into account the regularization scheme proposed in section 4.4 we obtain, for the non vanishing diagrams:



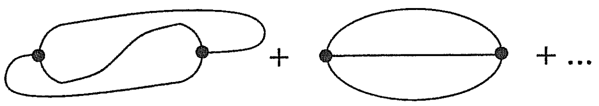
$$
\begin{aligned}
&= -\frac{1}{8g^2}h(\epsilon)^2\left(\delta^{MM}\delta^{NN} - \delta^{MN}\delta^{MN}\right)\sum_{ABCD}\int d^2w T^{(4)}_{AB,CD}P_{AC}P_{BD}Q_A^{-1}(w)Q_B^{-1}(w) \\
&= \frac{45}{2g^2}h(\epsilon)^2 \sum_{\substack{q\neq l\\ l\neq m\\ q\neq m}} \int d^2w\ Q_{ql}^{-1}(w)Q_{lm}^{-1}(w) \qquad (5.18)
\end{aligned}
$$

where it is worth stressing that the summation constraints have different origins: $q \neq l$ and $l \neq m$ originate from the non Cartan form of the fields, while $q \neq m$ originates from the non Cartan form of the commutators in the quartic potential. This is due to the fact that we have eliminated the Cartan components of the commutators in this potential by means of the lagrange multipliers, which in turn were introduced to disentangle the supersymmetry transformations along the Cartan - non Cartan decomposition. The $q \neq m$ constraint is pretty important, since the candidates to cancel such contribution possess it for other independent reasons.

Two loops diagrams from three legs bosonic vertex

We proceed computing the contribution arising from all the possible connections of two $1/g$ bosonic vertices, which can be read in (4.19). Coherently with our regularization scheme we "open" one of the two vertices with the insertion of the regulator ϵ.

Let us define, for the moment, $V^M_{AB,C} = \phi^{\mathrm{t}M}_{\ C} T^{(3)}_{AB,C}$. We have:

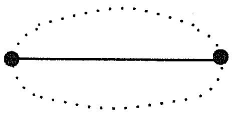


$$
\begin{aligned}
&= \frac{1}{2g^2} h(\epsilon)^2 \sum_{ABCDEF} \int d^2w V^M_{AB,C} V^N_{DE,F} \\
&\quad \times \left\{ P_{AD} P_{EB} P_{CF} \left(\delta^{MN}\delta^{LK}\delta^{LK} - \delta^{LK}\delta^{KN}\delta^{LM}\right) Q^{-1}_{\bar{A}}(w) Q^{-1}_{\bar{B}}(w) Q^{-1}_{\bar{C}}(w) \right. \\
&\quad + P_{CD} P_{EB} P_{AF} \left(\delta^{MN}\delta^{LK}\delta^{LK} - \delta^{LK}\delta^{KN}\delta^{LM}\right) Q^{-1}_{\bar{A}}(w) Q^{-1}_{\bar{B}}(w) Q^{-1}_{\bar{C}}(w) \\
&\quad + P_{AD} P_{CB} P_{EF} \left(\delta^{MN}\delta^{KK}\delta^{LL} - \delta^{ML}\delta^{KK}\delta^{NL} \right. \\
&\quad \left. \left. + \delta^{MK}\delta^{KL}\delta^{NL} - \delta^{MK}\delta^{KN}\delta^{LL}\right) Q^{-1}_{\bar{A}}(w) Q^{-1}_{\bar{B}}(w) Q^{-1}_{\bar{E}}(w) \right\} \\
&= \frac{9}{g^2} h(\epsilon)^2 \sum_{\substack{q \neq l \\ l \neq m \\ q \neq m}} \int d^2w \left(\phi^{\mathrm{t}M}_{(lq)} \phi^{\mathrm{t}M}_{(lq)} + \phi^{\mathrm{t}M}_{(ml)} \phi^{\mathrm{t}M}_{(ql)} \right) Q^{-1}_{ml} Q^{-1}_{lq} Q^{-1}_{qm} \\
&= \frac{27}{2g^2} h(\epsilon)^2 \sum_{\substack{q \neq l \\ l \neq m \\ q \neq m}} \int d^2w \; Q^{-1}_{ql}(w) Q^{-1}_{\bar{l}\bar{m}}(w) \quad , \qquad (5.19)
\end{aligned}
$$

where we have used the definition of $\mathcal{Q}$.

Diagrams involving ghosts

Since the structure of the boson-ghost vertex is similar to the bosonic cubic one, and the propagator is the same, we straightforwardly obtain:

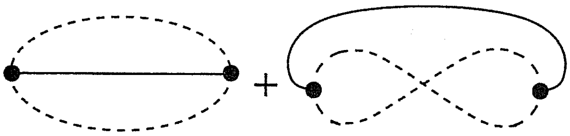


$$= -\frac{1}{2g^2}h(\epsilon)^2 \sum_{\substack{q\neq l\\ l\neq m\\ q\neq m}} \int d^2w\ Q^{-1}_{ql}(w)Q^{-1}_{lm}(w) \tag{5.20}$$

where the minus sign takes into account the fermionic statistics of ghosts.

Diagrams with one fermionic loop

The contributions involving fermions can be carried out with a bit of care about the gamma matrices:



$$\begin{aligned} &= \frac{1}{4g^2}h(\epsilon)^2 \sum_{ABCDEF} T^{(3)}_{AB,C}T^{(3)}_{DE,F}P_{AD}P_{EB}P_{CF} \\ &\times \int d^2wQ^{-1}_{A}(w)Q^{-1}_{E}(w)Q^{-1}_{C}(w)\mathrm{Tr}\left[G^{(2)}_{M}A^{-1}_{A}G^{(2)}_{M}A^{-1}_{E}\right] \\ &= \frac{32}{g^2}h(\epsilon)^2 \sum_{\substack{q\neq l\\ l\neq m\\ q\neq m}} \int d^2wQ^{-1}_{ql}(w)Q^{-1}_{lm}(w) \end{aligned} \tag{5.21}$$

where, to obtain the last equality, we needed to compute the trace over the spinorial indices, getting:

$$\mathrm{Tr}\left[G^{(2)}_{M}A^{-1}_{A}G^{(2)}_{M}A^{-1}_{B}\right] = -64\left(2X^{\circ I}{}_{A}X^{\circ I}{}_{B} + a^{\mathrm{t}}_{\bar{w}A}a^{\mathrm{t}}_{wB} + a^{\mathrm{t}}_{wA}a^{\mathrm{t}}_{\bar{w}B}\right). \tag{5.22}$$

Diagrams containing the lagrange multipliers

We are only interested in the qualitative behavior of the contributions tied with the lagrange multiplier fields. So let us sketch the result of the one loop diagram computation:

$$\frac{1}{g^2}\alpha_D^{D=2}[\phi^t] = \frac{1}{g^2}h(\epsilon)^2 \sum_{p,q} \int d^2w d_{pq}^{MNKL}(\phi^t) D^{tMN}_{(p)} D^{tKL}_{(q)}$$

where $d(\phi^t)$ is a function on the Cartan bosonic fields, as it results from the propagators circulating in the loop.

The same situation is reproduced for the ghost lagrange multipliers, leading to:

$$\frac{1}{g^2}\alpha_C^{D=2}[\phi^t] = \frac{1}{g^2}h(\epsilon)^2 \sum_{p,q} \int d^2w \bar{C}^t_{M(p)} f_{pq}^{MN}(\phi^t) C^t{}_{N(q)}.$$

The final result

Just collect together the partial results conteined in (5.18), (5.19), (5.20) and (5.21) to obtain:

$$\begin{aligned}\frac{1}{g^2}\alpha_{pol,2}^{D=2}[\Phi^t] &= \frac{1}{g^2}\left(-\frac{45}{2}+\frac{27}{4}+16-\frac{1}{4}\right) h(\epsilon)^2 \sum_{\substack{q\neq l\\ l\neq m\\ q\neq m}} \int d^2w Q_{ql}^{-1}(w) Q_{lm}^{-1}(w)\\ &+\frac{1}{g^2}\alpha_D^{D=2}[\phi^t] + \frac{1}{g^2}\alpha_C^{D=2}[\phi^t] \quad .\end{aligned}$$

Surprisingly enough, we have

$$-\frac{45}{2}+\frac{27}{4}+16-\frac{1}{4}=0$$

which brings us to the important conclusion:

$$\alpha_{pol,2}^{D=2}[\Phi^t] = \alpha_D^{D=2}[\phi^t] + \alpha_C^{D=2}[\phi^t] \tag{5.23}$$

We can see the exact cancellation to appear between bosonic, fermionic and ghost contributions, before we send $\epsilon \to 0$.

Hence one has to be worried only about the lagrange multipliers contributions, which amount in a divergent correction to the effective action. This result is quite important, since it shows how the presence of divergences is in one to one correspondence with the impossibility of disentangling supersymmetry in the two sectors, Cartan and non Cartan, of the theory; indeed the lagrange multipliers have their origin exactly in this effort of separating a supersymmetry for the Cartan fields; and, at the same time, they are responsible for the appearance of the divergences in the effective action.

5.6 Polynomial corrections in $D = 4$

The functional defining the bosonic polynomial potential has exactly the same form both in two and four dimensions, as well as in ten. The only difference, as already stressed, is in the manifold of integration, its dimensionality, and the delta functions with their smoothing. Since our previous arguments to compute the two dimensional corrections were not based on the integrals, being these just formal (since we do not even know the explicit form of what we integrate), we arrive to the conclusion that the bosonic and ghost coefficients of the corrections are the same as in two dimensions. Nevertheless the explicit $4D$ computation has been carried out, leading to the very same result as the $2D$ one.

Some care has, instead, to be taken for what concerns the fermionic vertex (4.26), since dimensional reduction acts in a non trivial way on fermions leading to a different situation from the two dimensional one.

What has to be checked is in particular the fermionic trace appearing in the two loop graph (5.21), which in four dimensions turns out to be:

$$\mathrm{Tr}\left[G^{(4)}A_A^{-1}G^{(4)}A_B^{-1}\right] = -128\; Q_A^{-1}Q_B^{-1}\phi^{tM}_{\ A}\phi^{tM}_{\ B} \quad , \tag{5.24}$$

which can be carried out expanding the propagator $\mathcal{A}^{-1}$ in terms of the $G^{(4)}$, as follows:

$$\mathcal{A}^{-1} = \mathcal{Q}^{-1}\mathrm{ad}_{\phi^t_M} G^{(4)\dagger}_M \quad .$$

Comparing such result with the one of (5.22), we recognize they are equal. The cancellation between the divergent perturbative contributions is ensured also in $D = 4$.

5.7 $D = 2$ strong coupling effective theory

We are going to show how the divergences present in (5.23) result, at the end, in a trivial renormalization of the lagrange multipliers.

For this purpose we insert the results (5.4), (5.6), (5.17) and (5.23) in (5.1), to get:

$$\begin{aligned} Z_{\mathrm{n},D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,D^{\mathrm{t}},C^{\mathrm{t}},g] &= 1+\frac{1}{g^2}\alpha_D[\phi^{\mathrm{t}}]+\frac{1}{g^2}\alpha_C[\phi^{\mathrm{t}}] \\ &= \exp\left\{\frac{1}{g^2}\alpha_D[\phi^{\mathrm{t}}]+\frac{1}{g^2}\alpha_C[\phi^{\mathrm{t}}]\right\}+o(\frac{1}{g^3}) \end{aligned} \tag{5.25}$$

then recall the form of the strong coupling cartan action:

$$\begin{aligned} S^{gf}_{sc,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,c^{\mathrm{t}},D^{\mathrm{t}},C^{\mathrm{t}}] &= \int d^2w \mathrm{Tr}\left[\partial_w x^{\mathrm{t}I}\partial_{\bar{w}}x^{\mathrm{t}I}+\partial_w a^{\mathrm{t}}_{\bar{w}}\partial_{\bar{w}}a^{\mathrm{t}}_w+\partial_w\bar{c}^{\mathrm{t}}\partial_{\bar{w}}c^{\mathrm{t}}\right. \\ &\quad \left. +D^{\mathrm{t}}{}_{MN}D^{\mathrm{t}}{}_{MN}+\bar{C}^{\mathrm{t}}_M C^{\mathrm{t}}{}_M\right] \end{aligned} \tag{5.26}$$

where ^{gf} stands for "gauge fixed", ${}_{sc}$ for "strong coupling" and ${}_D$ means that the action is modified by the lagrange multipliers.

Looking at (4.6), we are now able to see which corrections are yielded to the strong coupling action by $Z_{\mathrm{n},D}$; defining by $\tilde{S}^{gf}_{2,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,D^{\mathrm{t}},C^{\mathrm{t}}]$ the Cartan effective action up to the second order in $1/g$, we get:

$$e^{-\tilde{S}^{gf}_{2,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,c^{\mathrm{t}},D^{\mathrm{t}},C^{\mathrm{t}}]}=e^{-S^{gf}_{sc,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,c^{\mathrm{t}},D^{\mathrm{t}},C^{\mathrm{t}}]}Z_{\mathrm{n},D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,D^{\mathrm{t}},C^{\mathrm{t}},g] \quad ,$$

which can be solved for $\tilde{S}^{gf}_{2,D}$ giving:

$$\begin{aligned} \tilde{S}^{gf}_{2,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,c^{\mathrm{t}},D^{\mathrm{t}},C^{\mathrm{t}}] &= S^{gf}_{sc,D}[\phi^{\mathrm{t}},\theta^{\mathrm{t}}=0,c^{\mathrm{t}},D^{\mathrm{t}},C^{\mathrm{t}}] \\ &\quad -\frac{1}{g^2}h(\epsilon)^2\sum_{p,q}\int d^2w\left\{d^{MNKL}_{pq}(\phi^{\mathrm{t}})D^{\mathrm{t}MN}_{(p)}D^{\mathrm{t}KL}_{(q)}\right. \\ &\quad \left.+\bar{C}^{\mathrm{t}}_{M(p)}f^{MN}_{pq}(\phi^{\mathrm{t}})C^{\mathrm{t}}{}_{N(q)}\right\} \quad . \end{aligned}$$

Since we are not interested in the lagrange multipliers dependence of the effective action, at this point one could naively integrate over them: this would nicely lead to a numerical factor in front of the partition function; alternatively one could solve the equations of motion for D^{t} and C^{t}, and since they appear quadratically in the action, this would simply give $D^{\mathrm{t}}=0$ and $C^{\mathrm{t}}=0$. In fact, this is the main reason for the decoupling of the lagrange multipliers in the effective theory, and one can simply forget about them.

To see why, for what concerns D^{t}, this is not the whole story, remember that the effective action we have computed so far is evaluated at $\theta^{\mathrm{t}}=0$, as it was allowed to us by supersymmetry. However, recalling (3.14), let us stress that our argument to show the invariance of the partition function under δ^t is strongly based on the path integration over the lagrange multiplier for the bosonic fields. We cannot integrate them away at this stage if we want, by means of supersymmetry, to rebuild the $\theta^t \neq 0$ terms in the effective theory.

Hence, as anticipated in section 4.2.2, we now proceed in the following way. Firstly we get rid of C^t by means of its equations of motion. Then we remove the gauge fixing as well as the Faddeev-Popov ghost term, postulating (naturally enough) that we will land to a non gauge-fixed two dimensional Maxwell theory, whose action can be written as:

$$S_{Maxwell} = -\frac{1}{4}\int d^2w \text{Tr}\left[\left(\partial_w a^t_{\bar{w}} - \partial_{\bar{w}} a^t_w\right)\left(\partial_w a^t_{\bar{w}} - \partial_{\bar{w}} a^t_w\right)\right] \quad ,$$

by means of which the strong coupling non gauge-fixed action turns out to be:

$$S_{sc,D}[\phi^t, \theta^t = 0, D^t] \;=\; S_{Maxwell} + \int d^2w \text{Tr}\left[\partial_w x^{tI} \partial_{\bar{w}} x^{tI} + D^t{}_{MN} D^t{}_{MN}\right] \quad .$$

Finally the non gauge-fixed effective action at the second order can be given:

$$\begin{aligned}\tilde{S}_{2,D}[\phi^t, \theta^t = 0, D^t] \;&=\; S_{sc}[\phi^t, \theta^t = 0] \\ &\quad + \sum_{p,q} \int d^2w D^{tMN}_{(p)} O^{MNKL}_{pq} D^{tKL}_{(q)} \quad ,\end{aligned} \tag{5.27}$$

where the D^t dependance of $S_{sc,D}$ has been moved to the tensor O defined by the following:

$$O^{MNKL}_{pq} = \delta^{MK}\delta^{NL}\delta^{pq} - \frac{1}{g^2} h(\epsilon)^2 d^{MNKL}_{pq}(\phi^t) \quad .$$

This operation of gauge fixing removal is necessary since we would like to check the supersymmetry of the effective action, and the gauge fixing manifestly breaks it. An alternative would consist in inserting a supersimmetric gauge fixing, and consequently the super-ghosts and their supersymmetry transformations; we will mention this approach at the end of the chapter.

So far we learned the precise form of the effective action at $\theta^t = 0$; in order to compute the fermionic sector we recall that the effective action should satisfy, at all perturbative orders, the following condition, which has benn proven in section 3.1.3:

$$\int \mathcal{D}D^t \, \delta^t \, e^{-\tilde{S}_D[\phi^t, \theta^t, D^t]} = 0 \quad , \tag{5.28}$$

which, hence, should be true also for $\tilde{S}_{2,D}$. Once the bosonic part of the effective action is given, this condition is rather stringent, since it turns out that the only fermionic term, which is going to fulfill it, is of the kind $\theta^{\rm t}\not\partial\theta^{\rm t}$. Any other one would, in fact, break supersymmetry, as can be verified by directly computing the variation (5.28).

So we end up with the definitive expression for the effective action at the second order:

$$\begin{aligned} \tilde{S}_{2,D}[\phi^{\rm t},\theta^{\rm t},D^{\rm t}] &= S_{Maxwell} \\ &+\int d^2w Tr\left[\partial_w x^{{\rm t}I}\partial_{\bar{w}}x^{{\rm t}I} + i\left(\theta^{\rm t}_s\partial_{\bar{w}}\theta^{\rm t}_s + \theta^{\rm t}_c\partial_w\theta^{\rm t}_c\right)\right] \\ &+\sum_{p,q}\int d^2w D^{{\rm t}MN}_{(p)} O^{MNKL}_{pq} D^{{\rm t}KL}_{(q)} \quad . \end{aligned} \tag{5.29}$$

So we are done. We computed perturbatively the bosonic sector of the effective action; next we rebuilt the fermionic sector by means of supersymmetry. What is left to do now is to integrate away the lagrange multipliers, and let the infinities in the action disappear with them. We have up to second order:

$$\begin{aligned} e^{-\tilde{S}^{D=2}[\phi^{\rm t},\theta^{\rm t}]} &= \int \mathcal{D}D^{\rm t} e^{-\tilde{S}_{2,D}[\phi^{\rm t},\theta^{\rm t},D^{\rm t}]} = \int \mathcal{D}D^{\rm t}\int \mathcal{D}\phi^{\rm n}\mathcal{D}\theta^{\rm n} e^{-S^{D=2}_D[\phi^{\rm n},\theta^{\rm n},\phi^{\rm t},\theta^{\rm t},D^{\rm t}]} \\ &= e^{-S_{sc}[a^{\rm t},x^{\rm t},\theta^{\rm t}]} \quad . \end{aligned}$$

5.7.1 $D=4$ theory

Starting from the following perturbative result:

$$Z^{D=4}_{{\rm n},D}[\Phi^{\rm t},g] = 1 + \frac{1}{g^2}\alpha^{D=4}_D[\Phi^{\rm t}] + \frac{1}{g^2}\alpha^{D=4}_C[\Phi^{\rm t}] \quad , \tag{5.30}$$

all the arguments carried out in this section can be repeated for the four dimensional theory as well, hence leading to the following strong coupling effective action:

$$\begin{aligned} e^{-\tilde{S}^{D=4}[\phi^{\rm t},\psi^{\rm t}]} &= \int \mathcal{D}D^{\rm t}\int \mathcal{D}\phi^{\rm n}\mathcal{D}\psi^{\rm n} e^{-S^{D=4}_D[\phi^{\rm n},\psi^{\rm n},\phi^{\rm t},\psi^{\rm t},D^{\rm t}]} \\ &= e^{-S_{sc}[a^{\rm t},x^{\rm t},\psi^{\rm t}]}\left[1 + O\left(\frac{1}{g^3}\right)\right] \quad , \end{aligned}$$

where

$$\begin{aligned} S_{sc}[a^{\rm t},\bar{a}^{\rm t},x^{\rm t},\psi^{\rm t}] &= S^{D=4}_{Maxwell} + \int d^4z Tr\left[\partial_w x^{{\rm t}I}\partial_{\bar{w}}x^{{\rm t}I}\right] \\ &-2\int d^4z\left[\lambda^{{\rm t}\star}_1\partial_{\bar{v}}\lambda^{\rm t}_1 + \lambda^{{\rm t}\star}_2\partial_v\lambda^{\rm t}_1 + \lambda^{{\rm t}\star}_1\partial_{\bar{w}}\lambda^{\rm t}_2 - \lambda^{{\rm t}\star}_2\partial_w\lambda^{\rm t}_1\right] \quad . \end{aligned}$$

This is quite an important result, since it deals with the abelianization of the $U(N)$ maximally supersymmetric gauge theories, both in $D = 2$ and $D = 4$.

Our result holds in the strong coupling limit, and we have checked it up to the second order. However all the cancellations obtained rely on identities between non-Cartan bosonic and fermionic propagators, such as the traces (5.22) and (5.24) or the determinant (2.12); this suggests the presence of a cancellation mechanism which should hold at all perturbative orders.

Let us now switch for a while to a different approach which could possibly be applied to the task of extracting the strong coupling effective theory from $D = 4$, $\mathcal{N} = 4$ SYM.

5.8 Superfield formulation

We concentrate here on the four dimensional theory, since the formulation of $D = 2$, $\mathcal{N} = (8,8)$ in term of superfields would require the introduction of the twisted chiral superfields, as it was found out in [48], bringing us far from the present brief discussion on superspace formalism.

It is by now well established that a correct cancellation of perturbative divergent contributions in the $D = 4$, $\mathcal{N} = 4$ SYM theory, can be achieved starting from the superfield formulation of this theory avoiding to use the Wess Zumino gauge, when the superfields components should be written down; for a detailed discussion on this subject we refer the reader to [49]. This means that the computations should be carried out taking into account all the auxiliary fields present in the supermultiplets, and that the action is no more polynominal, being instead an infinite series of terms.

Our trick of introducing the Lagrange multipliers as to disentangle supersymmetry and integrate away the divergences, is somehow founded on the fundamental role played by the auxiliary fields in the dynamics of the $\mathcal{N} = 4$ theory. In fact the transverse components D^t_{IJ} of the Lagrange multipliers are nothing but the auxiliary fields arising in the vector and chiral supermultiplets.

Moreover we needed to remove the gauge fixing and ghost terms from the action before we were able to implement on it the supersymmetry transformations (see sections 4.2.2 and 5.7): in the superfields formulation, instead, the gauge fixing and Faddeev Popov terms arise in an automatically supersymmetric form.

Let us now go into some details of this formulation, to see how it should be possible to implement the Cartan non-Cartan decomposition on the $\mathcal{N} = 4$ SYM in the $\mathcal{N} = 1$

superfield formulation, which turns out to be:

$$S = \int d^4x \mathrm{Tr}\left\{\left[d^4\theta\, e^{-V}\Phi_I^\dagger e^V \Phi^I + \frac{1}{4g^2}\left(\int d^2\theta \frac{1}{4} W^\alpha W_\alpha + \text{h.c.}\right) + \right.\right.$$
$$\left.\left. + ig\frac{\sqrt{2}}{3!}\left(\int d^2\theta\, \varepsilon_{IJK}\Phi^I[\Phi^J,\Phi^K] + \int d^2\overline{\theta}\, \varepsilon^{IJK}\Phi_I^\dagger[\Phi_J^\dagger,\Phi_K^\dagger]\right)\right]\right\} \quad , \quad (5.31)$$

where V is the vector superfield, while Φ^I are three chiral superfields; all of them are understood to be in the adjoint of $U(N)$.

To perform the Cartan non-Cartan splitting we rely on the background field method, for which we refer to [50], somehow extending it to our purposes. Since we are going to consider the Cartan components of the field as frozen, from the point of view of the non-Cartan dynamics, we can just assume these behave in the theory like background classical fields. Hence we get the following splitting of the fields:

$$\begin{aligned} e^V &= e^{V^\mathfrak{t}/2} e^{V^\mathfrak{n}} e^{V^\mathfrak{t}/2} \quad , \\ \Phi &= \Phi^\mathfrak{t} + \Phi^\mathfrak{n} \quad , \end{aligned}$$

where we truncate the formal expansion of the exponential at the second power, getting

$$e^V = e^{V^\mathfrak{t}/2} e^{V^\mathfrak{n}} e^{V^\mathfrak{t}/2} = e^{V^\mathfrak{t}+V^\mathfrak{n}} + O\left((V^\mathfrak{t}+V^\mathfrak{n})^3\right) \quad .$$

Moreover we perform the following rescalings

$$\begin{aligned} e^V &= e^{gV^\mathfrak{t}/2} e^{V^\mathfrak{n}} e^{gV^\mathfrak{t}/2} \quad , \\ \Phi &= \Phi^\mathfrak{t} + \frac{1}{g}\Phi^\mathfrak{n} \quad . \end{aligned} \quad (5.32)$$

Upon substitution of the latter in the vector superfield action, we get:

$$\begin{aligned} S[V^\mathfrak{n}, V^\mathfrak{t}] = \int & d^4x d^4\theta \mathrm{Tr}\left\{D_A V^\mathfrak{t} \overline{DD} D^A V^\mathfrak{t} \right. \\ & + \frac{1}{4}\left[\left[D_A V^\mathfrak{t}, V^\mathfrak{n}\right]\overline{DD}\left[D^A V^\mathfrak{t}, V^\mathfrak{n}\right] + \left[D_A V^\mathfrak{n}, V^\mathfrak{t}\right]\overline{DD}\left[D^A V^\mathfrak{t}, V^\mathfrak{n}\right]\right. \\ & + \left[D_A V^\mathfrak{t}, V^\mathfrak{n}\right]\overline{DD}\left[D^A V^\mathfrak{n}, V^\mathfrak{t}\right] + \left.\left[D_A V^\mathfrak{n}, V^\mathfrak{t}\right]\overline{DD}\left[D^A V^\mathfrak{n}, V^\mathfrak{t}\right]\right] \\ & \left. + \text{infinitely many terms}\right\} \quad , \end{aligned}$$

where here D stands for the spinorial covariant derivative, with respect to θ and $\bar{\theta}$, while A is the spinorial index.

We can immediately recognize the abelian theory realized through the Cartan superfields, since an abelian supersymmetric gauge theory is really realized without relying on an exponential action:

$$S_{V^{\mathfrak{t}}} = \int d^4x d^4\theta \mathrm{Tr}\left[D_A V^{\mathfrak{t}} \overline{D}\overline{D} D^A V^{\mathfrak{t}}\right] \quad .$$

A similar result can be also found as far as the chiral superfields are involved.

Without the claim of rigorousness, our purpose was here to show that, in the superfield approach, there is in fact room enough to set up the abelianization of the $\mathcal{N} = 4$ SYM theory. Once we know the action the computations of the contributions arising from the dynamics of $V^{\mathfrak{n}}$ and $\Phi^{\mathfrak{n}}$ can be worked out with the powerful tool of Feynman supergraphs.

Let us devote a few words about how the gauge fixing is described in this framework. Skipping for the moment the transverse directions, the gauge fixing was written, in components, like $G = D^{\circ}_{\overline{w}} a^{\mathfrak{n}}_{w}$, and can be substituted, by means of the background field method, by the following:

$$G^2 = \mathcal{D}\mathcal{D}V^{\mathfrak{n}}\overline{\mathcal{D}}\overline{\mathcal{D}}V^{\mathfrak{n}} \quad , \tag{5.33}$$

where $\mathcal{D}_A = e^{-V^{\mathfrak{t}}/2} D_A e^{V^{\mathfrak{t}}/2}$ is the background-covariant spinorial derivative. If we expand G^2 by means of (5.32), and then in component fields, we get, among the others, the following term:

$$\begin{aligned} \int d^4x d^2\theta d^2\bar{\theta}\ G^2\big|_{(ferm)} &= \ldots + \int d^4x d^2\theta d^2\bar{\theta}\ \overline{D}\overline{D}V^{\mathfrak{n}} D_A V^{\mathfrak{t}} D^A V^{\mathfrak{n}}\big|_{(ferm)} + \ldots \\ &= \ldots + \int d^4x\ \bar{\lambda}^{\mathfrak{n}}\sigma^{\mu}\lambda^{\mathfrak{t}} a^{\mathfrak{n}}_{\mu} + \cdots \quad , \end{aligned}$$

finding out that this is nothing but one of the terms (4.17) we got rid of, choosing $\lambda^{\mathfrak{t}} = 0$ to avoid divergences. This again shows how our orguments to compute the effective action can be made even more precise, by following the superfield approach; in fact it just provides us with the appropriate gauge fixing term, to cancel the interaction, which, otherwise, would generate divergences in the effective action. Moreover this is a strong indication that the somehow heuristic argument, that we sketched in section 5.7, is however correct.

Appendix A

Conventions and definitions

A.1 Notations

A.1.1 Indices

- w is the complex variable spanning the cilinder $\mathcal{C}$ in the two dimensional theory; primed w denote different points on the worldsheet;

- z_i with $i = 1, 2$ and $z_1 = v$, $z_2 = w$ are the two complex variables spanning the 4D worldsheet, which is $\mathcal{X} = \mathbb{R} \times M_3$ where M_3 is a three-dimensional compact manifold (we can think of $\mathcal{X}$ as the cone over M_3, witn the vertex excluded, and the radial coordinate identified with time); z itself indicates a generic point in $\mathcal{X}$. If there is no chance to get confused z_1 and z_2 are also used to identify different points in $\mathcal{X}$, each one with two complex components z_{1i} and z_{2i}. Also primed z denote different points.

- Capital indices I, J, K will be used to refer to the transverse directions X^I; their range is context depending: $I = 1 \ldots 8$ in $D = 2$ and $I = 1 \ldots 6$ in $D = 4$;

- Capital indices L, M, N are both the ten dimensional Lorentz indices, or the indices of the field ϕ_M grouping vector and scalars deriving from dimensional reduction of $D = 10$;

- Couples of small indices $p, q, r, s = 1, \ldots, N$ will indicate elements of the matrices in the adjoint of $U(N)$; it is useful sometimes to group this indices in couples, for which we use capital indices from the beginning of the alphabet: $A = (pq)$.

A.1.2 Fermionic matrices

We rearrange Dirac matrices in the form which is most useful for our computations. We are interested in the Wick rotated form of the gamma matrices, since both the instanton computations and the perturbative analysis are better performed in the euclidean world-sheet or world-volume. In $D = 2$ we get $G_M^{(2)}$ defined as follows:

$$G_M^{(2)} = \left\{ i\mathbf{1}_{16}, -\gamma_9, \begin{pmatrix} 0 & \gamma_I \\ \tilde{\gamma}_I & 0 \end{pmatrix} \right\} \quad , \tag{A.1}$$

where the $G_I^{(2)}$ are gamma matrices of $SO(8)$.

In $D = 4$ we get $G_M^{(4)}$ defined as follows:

$$G_1^{(4)} = \begin{pmatrix} 0 & 0 & -i\mathbf{1} & 0 \\ 0 & 0 & 0 & -i\mathbf{1} \\ -i\mathbf{1} & 0 & 0 & 0 \\ 0 & -i\mathbf{1} & 0 & 0 \end{pmatrix} \quad G_2^{(4)} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & -\mathbf{1} & 0 & 0 \end{pmatrix}$$

$$G_3^{(4)} = \begin{pmatrix} 0 & 0 & 0 & -i\mathbf{1} \\ 0 & 0 & i\mathbf{1} & 0 \\ 0 & i\mathbf{1} & 0 & 0 \\ -i\mathbf{1} & 0 & 0 & 0 \end{pmatrix} \quad G_4^{(4)} = \begin{pmatrix} 0 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & -\mathbf{1} & 0 & 0 \\ -\mathbf{1} & 0 & 0 & 0 \end{pmatrix}$$

$$G_I^{(4)} = \begin{pmatrix} 0 & \gamma^{I\dagger} & 0 & 0 \\ -\gamma^{I\dagger} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma^I \\ 0 & 0 & \gamma^I & 0 \end{pmatrix}$$

where the blocks are 4 by 4, and the $G_I^{(4)}$ are made up of $SO(6)$ gamma matrices.

A.2 The adjoint operator

We would like to describe the adjoint action of the Cartan subalgebra on the non-Cartan subspace in an operator form, without referring to it as a commutator. For this purpose let us take the standard basis for the $N \times N$ matrices:

$$\left(e_{(pq)}\right)^{rs} = \delta^{rp}\delta^{sq} \quad . \tag{A.2}$$

It turns out that $\mathrm{Tr}[e_{(pq)}e_{(rs)}] = \delta^{ps}\delta^{qr} \equiv P_{pq,rs} = P_{AB}$.

Now consider a Cartan field $\Phi^{\mathfrak{t}}$ with diagonal elements $\Phi^{\mathfrak{t}}_{(p)}$, its adjoint action on the standard basis being:

$$\mathrm{ad}_{\Phi^{\mathfrak{t}}} e_{(pq)} = \left(\Phi^{\mathfrak{t}}_{(p)} - \Phi^{\mathfrak{t}}_{(q)}\right) e_{(pq)} \equiv \Phi^{\mathfrak{t}}_{(pq)} e_{(pq)} \quad ,$$

from which it is clear the antisymmetric nature of the ad.

If we take an operator (like $\mathcal{Q}$, see (2.7)) which is a product of some adjoints, then this will be diagonal in the same basis as the adjoint itself, i.e. the standard basis; if $\mathcal{K} = \mathrm{ad}_{\Phi^{\mathfrak{t}}} \cdot \mathrm{ad}_{\Phi^{\mathfrak{t}}}$, then

$$\mathcal{K} e_{(pq)} = K_{pq} e_{(pq)} \quad , \tag{A.3}$$

with $K_{pq} = \Phi^{\mathfrak{t}}_{pq} \Phi^{\mathfrak{t}}_{pq}$.

We are now able to define the inverse $\mathcal{K}^{-1} \cdot \mathcal{K}\ e_{(pq)} = e_{(pq)}$ through its eigenvalues:

$$\mathcal{K}^{-1}\ e_{(pq)} = \frac{1}{K_{pq}}\ e_{(pq)} \quad . \tag{A.4}$$

It will be useful in some cases, for compactness, to use the trace of such operators, defined as follows:

$$\mathrm{Tr}_{\mathcal{O}}[\mathcal{K}] = \sum_{p \neq q} K_{pq}$$

As a consequence of the adjoint antisymmetry the following identities are satisfied:

$$\sum_{p,q} \left(\mathrm{ad}_{\Phi^{\mathfrak{t}}}\right)_{pq} = 0$$

$$\sum_{p,q} \mathcal{K}_{pq} \left(\mathrm{ad}_{\Phi^{\mathfrak{t}}}\right)_{pq} = 0 \text{ if } K_{pq} = K_{qp}$$

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