ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of Doctor Philosophiæ

BPS States and D1-D5 System in Theories with Sixteen Supercharges

Candidate
Amine B. Hammou

Supervisors
Profs. Edi Gava and
Kumar S. Narain

Accademic year 1999/00

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI

TRIESTE
Via Beirut 2-4

TRIESTE

Acknowledgments

I would like to thank my supervisors Profs. Edi Gava and Kumar S. Narain for introducing me to this very interesting field, and for their generosity with which they always imparted to me with endless enthusiasm. I also thank them for their constant encouragement, support, and enlightening discussions.

Many thanks to Prof. Massimo Bianchi for enlightening discussions, and Profs. Faheem Hussain, George Thompson and Matthias Blau for their encouragement.

I am Very grateful to the Abdus Salam International Center for Theoretical Physics specially the Director, Prof. M. Virasoro, and the High Energy Physics Group for their support.

I would like to thank the International School for Advanced Studies SISSA-ISAS staff specially Riccardo Iancer and Andrea Parma for their assistance and help during my stay at SISSA.

I would like also to thank all the lecturers in the High Energy first year course.

I would like to Thank my collaborator and friend Jose F. Morales for many interesting discussions and encouragement. I would like to thank all my friends G. Bonelli, M. Bertolini, Claudio Scrucca, Marco Serone, Djamel Dou, Noureddine Hambli and Koushik Ray for their encouragement and helpful discussions.

I am grateful to my friend Ali Zaoui for his encouragement and Rula Tabash, Martin O'Loughlin and Avijit Mukherjee for their helpful discussions and to all who made my stay in Trieste unforgettable.

I am grateful to the High Energy Group of the University of Oran Es-Senia specially Professor Mohammed Lagraa for their encouragement.

I am deeply grateful to my wife Faiza Nebia, my family and all my friends in Oran (Algeria) who always had faith in my work. They are with me in my heart.

To my parents....

Contents

Introduction

At present, we know about four fundamental interactions. Three of them, electromagnetism, the weak and the strong interactions are very successfully described by a theory called the "standard model". This theory seems to be in accord with experimental observations. There is so far only one unobserved particle predicted by the standard model and needed for its consistency, namely the Higgs boson. If it is found, which should happen within the next decade, there will be no questions left open within the standard model that require an answer. The fourth interaction, gravity, is in less good shape theoretically. Classically, it is described by Einstein's theory of General Relativity. When one tries to quantize it one gets into serious difficulty. Quantum gravity has serious internal inconsistencies. Indeed, the loop corrections in quantum gravity lead to an infinity of new parameters that cannot be absorbed into the unique parameter of the theory, namely Newton's constant. Strictly speaking that means that any physical quantity we compute in quantum gravity depends on an infinite number of parameters, hence the theory is non-renormalizable. The strongest point in favor of string theory is that it seems to solve these problems. It is the only promising candidate for a consistent theory of quantum gravity, however with the drawback of not being testable within the capacity of present particle accelerators. This does not mean that there is a proof that all theoretical problems of quantum gravity are solved, nor that it is known that no other possibilities exist. There are other problems associated with quantum mechanics and gravity. One of the most intriguing problem is the so called information-loss problem. Indeed, it has been proposed that the existence of black holes would force us to give up unitarity of the S-matrix. A related problem is that of black hole entropy. Bekenstein and Hawking have shown that a black hole has thermodynamics properties: it has black body radiation with a certain temperature, and it behaves as if it has a certain entropy proportional to the area of the horizon. String theory has made interesting contributions to both problems. First of all string theory does indeed change the short distance physics. It has been shown that in certain string theories the unknown or infinite corrections are in fact finite and calculable. Furthermore it has been possible to obtain a description of states of special black holes, so that the correct entropy can be obtained. String theory does not just contain gravity, it comes inevitably with a large number of other particles and interactions. These have the same features as the standard model. Unfortunately, the particles and interactions predicted by string theory are far from unique (for a general review of string theory see for example [1, 2, 3, 4]).

During the last few years considerable progress in the understanding of non-perturbative phenomena in String Theory has been achieved. The key to this development is the use of duality symmetries, which relate the strong coupling regime of one theory to the weak coupling regime of a second (possibly different) theory. This fact, together with the powerful non-renormalization theorems of extended supersymmetry, often allows one to obtain non-perturbative information about one theory by doing perturbative computations in a dual weakly coupled theory.

An extra bonus of these developments has been the realization that all known string theories are just different vacua in the moduli space of a (yet poorly understood) fundamental theory, called M-theory [5, 6].

Most of the dramatic successes of the past fiew years in this field, rely on the observation that certain solitonic objects that carry charges under the so called Ramond-Ramond $(p+1)$-form gauge fields, and that are extended along p spatial directions, have a very simple conformal field theory description as $(p+1)$-dimensional hypersurfaces on which an open string can end (hence the name Dirichlet p-branes) [7]. For the simplest cases in type II theories these configurations preserve half of the 32 spacetime supersymmetries. As a result of the open string quantization, the collective field theory describing the low energy dynamics of these extended solitons turns out to be a maximal supersymmetric Yang-Mills theory in $(p+1)$ dimensions with 16 supercharges. Therefore, questions concerning possible bound states of these solitons (which are BPS states preserving some amount of spacetime supersymmetry) and related multiplicities, can be rephrased as questions about the vacua of the corresponding supersymmetric Yang-Mills theories. It is well known that a good deal of information about these issues is contained in the supersymmetric Witten index, or, for two-dimentional SCFT's, more generally, in the elliptic genus.

A very important application of these ideas has been the microscopic derivation of the Bekenstein-Hawking entropy formula for certain black-holes, precisely in terms of the number of bound states of a system of D-branes [8].

This thesis is devoted to test U-duality [6], a generalization of electromagnetic duality which combine S-duality and T-duality, in string theories with 16 supercharges, obtained by orbifolding/orientifolding IIB string theory. It turns out that there is a variety of models of this type which are related by U-duality, and this puts highly non-trivial constraints on the effective field theory describing bound states of branes in various cases. In these tests a crucial role is played by BPS states [9] (i.e. states preserving half and one quarter of the space-time supersymmetry) in theories with 16 supercharges. They are expected to be stable which allows us to follow them from weak to strong coupling regimes, providing us with highly non trivial consistency checks of the U-duality relation between various models. Typical examples of BPS-states are fundamental strings carrying winding and momentum charges along circles and solitonic configurations representing bound states of D-branes. Below $D = 6$ the spectrum of BPS states include both electric and magnetic charges arising from strings and five-branes wrapping the internal manifold. These are generalizations of dyonic states arising in $\mathcal{N} = 4$ Yang-Mills theory (for groups of rank greater than 1) and generically preserve one quarter of the supersymmetry. In the case where the dyon can be realised in terms of bound states of D1/D5 branes, the spectrum of masses, charges and multiplicities of the dyonic excitation should arise from an index computation in the effective gauge theory describing the system. This project has been recently carried out for type IIB theory on T^4 and $K3$, where the effective field theory of the system is expected to flow in the infrared to a CFT related to the symmetric product of T^4 and $K3$ respectively [10]. In these cases one can also get informations about states which carry longitudinal momentum, preserving 1/8 of the bulk supersymmetry.

This thesis is organized as follows: In the first chapter we briefly review some features of ten dimensional string dualities with one example in six dimensions, namely type IIA on $K3$ and hetrotic on T^4, BPS bounds and D-brane physics which will set the background and notations for the later discussions. In chapter 2 [11] we study the BPS spectrum of D1/D5-brane bound states in several orbifold/orientifold type IIB string vacua with sixteen unbroken supercharges. Orbifold/Orientifold group actions will be always acompained by shifts allowing us (according to the adiabatic argument [12]) to follow carefully BPS states along "U-duality" chains. Orbifold conformal field theories involving symmetric product spaces are proposed as a description of the infrared limit of the corresponding low energy gauge theories describing the D-brane bound state. Bound state degeneracies are then computed by an elliptic genus formula associated to orbifoldings of symmetric product spaces. The results reproduce in each of the cases the multiplicities of fundamental strings carrying both winding and momentum on the U-dual side. In chapter 3 [13] we express the infinite sum of D-fivebrane instanton corrections to $\mathcal{R}^2$ couplings in $\mathcal{N} = 4$ type I string vacua, in terms of an elliptic index counting $\frac{1}{2}$-BPS excitations in the effective $Sp(N)$ brane theory. We compute the index explicitly in the infrared, where the effective theory is argued to flow to an orbifold CFT. The form of the instanton sum agrees completely with the formula predicted from a dual one-loop computation in type IIA theory on $K3 \times T^2$. The proposed CFT provides a proper description of the whole spectrum of masses, charges and multiplicities for $\frac{1}{2}$- and $\frac{1}{4}$- BPS states, associated to bound states of D5-branes and KK momenta. These results are applied to show how fivebrane instanton sums, entering higher derivative couplings which are sensitive to $\frac{1}{4}$-BPS contributions, also match the perturbative results in the dual type IIA theory.

In appendix A we discuss the orientifold projection on Chan-Paton factors for the D1-D5 system in the presence of shifts. In appendix B we derive symmetric product partition functions for free fields acted upon by a Z_2 orbifold. In appendix C we include some details of a genus 1 modular integral relevant to the computation of the low energy couplings in section 6 of chapter 2. Finally some conclusions and discussions are included.

Chapter 1

Low energy effective actions

1.1 Perturbative string theories

String theory is a theory describing the evolution of a one-dimensional object, called "string". As the string propagates in space-time, it sweeps out a world sheet that is the generalization of the world line of a particle. We can have open strings, which have endpoints, and closed strings, which have the topology of a circle. The world sheet is parametrized by two parameters, σ and τ, the latter is a sort of time coordinate for an observer sitting at the position σ along the string. The description requires specifying the position $X^\mu(\sigma,\tau)$ of the string in the target space at given values of σ and τ. Despite all all its beautiful features the bosonic string theory has a number of shortcomings. The most drastic one is the presence of tachyons which indicate the instability of its vacuum. A solution to this problem was provided by the introduction of world sheet supersymmetry that relates the string coordinates $X^\mu(\sigma,\tau)$ to fermionic coordinates $\psi^\mu(\sigma,\tau)$. The latter are two component world-sheet spinors. This theory is called superstring theory and is described by the action

$$S = \frac{T}{2}\int_\Sigma d^2\sigma\sqrt{h}\,\left[h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right], \qquad (1.1)$$

here $\sqrt{h}$ is the square root of the absolute value of the determinant of the world-sheet metric $h_{\alpha\beta}$. T is the constant of proportionality, required to make the action dimensionless. Setting $\hbar = c = 1$ gives T the dimension of $(\text{length})^{-2}$. We can show that T is actually the tension of the string, which is related to the Regge slope parameter of open strings by $2\pi\alpha' T = 1$. The symbol ρ^α represents two-dimensional

Dirac matrices. The action (1.1) is invariant under global Poincare transformations, local two-dimensional reparametrization and under conformal (or Weyl) rescaling of the world-sheet metric $h_{\alpha\beta} \to \Lambda h_{\alpha\beta}$. Due to conformal invariance, the stress energy tensor $T_{\alpha\beta}$ is traceless. Upon quantization one however finds an anomaly $T^{\alpha}_{\alpha} \neq 0$. The cancellation of this conformal anomaly puts strong constraints on the dimension of the space-time in which the string moves, leaving only $D = 26$ and $D = 10$ for the bosonic and fermionic strings respectively. Elementary particles arise as the infinite tower of string vibrational modes. The massless spectrum contains, besides the standard gauge particles and matter, a spin two particle (the graviton). In the low energy limit $\alpha' \to 0$ the massive string states can be integrated out leaving an effective supergravity theory in terms of the massless spectrum of physical states. The interactions are described by joining and splitting of strings during their evolution in space-time. A generic string scattering process is given by a sum over all possible Riemman surfaces weighted by g^{χ}, where g is the string coupling constant and χ is the Euler number of the Riemann surface. There are five known consistent superstring theories in 10-dimensions. They are known as type IIA, type IIB, type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic string theories respectively. Let us give a brief description of the degrees of freedom and the spectrum of massless states in each of these theories. We will work in the light-cone gauge which has the advantage that all the states in the spectrum are physical.

Type II string theories: The world-sheet theory contains eight scalar fields which represent the eight transverse coordinates of a string moving in a nine dimensional space and eight Majorana fermions. It is useful to regard these eight Majorana fermions as sixteen Majorana-Weyl fermions, eight of them having left-handed chirality (left-moving) and the other eight having right-handed chirality (right-moving). The type II string theories contain only closed strings, hence the spatial direction of the world-sheet is a circle. The eight scalar fields have periodic boundary conditions as we go around the circle, whereas the fermions can have either periodic R (Ramond) or antiperiodic NS (Neveu-schwarz) boundary conditions. To get a consistent string theory we need to combine different types of states, with periodic and anti-periodic boundary conditions on the fermions, ending with four sectors. Once all the sectors are included, modular invariance (which is simply the requirement that equivalent Riemman surfaces describing interactions of closed strings, should give the same string answer), requires a suitable projection [14] (GSO) on the spectrum of states, leading to a sum over possible boundary conditions on the fermions. The projection is implemented by keeping only those states in the spectrum constructed from an even number of left-moving fermions and and even number of right-moving fermions. There is an ambiguity in assigning to the ground state either even or odd fermion number. Consistency of string theory leaves us with two possibilities. Depending on whether the GSO projections in the left- and the right-moving sector are identical or different, we have type IIB or type IIA string theory. States from the Ramond sector are in the spinorial representations of SO(1,9), whereas those from the Neveu-Schwartz sector are in the tensorial representations. Since the product of two spinorial representations gives us back a tensor representation, the states from the NS-NS and R-R sectors are bosonic, and those from the NS-R and R-NS sectors are fermionic. Since the two theories differ only in their R-sector, the NS-NS sector states are the same in the two theories. They constitute a symmetric rank two tensor field G_{IJ} (the graviton), an anti-symmetric rank two tensor field B_{IJ}^{NS}, and a scalar field ϕ known as the dilaton. The R-R sector massless states of type IIB string theory consists of a scalar χ, a rank two anti-symmetric tensor field B_{IJ}^{R}, and a rank four anti-symmetric tensor field C_{IJKL}^{+} (with a self dual field strength). On the other hand, the massless states from the R-R sector of type IIA string theory consist of a vector A_I, and a rank three anti-symmetric tensor C_{IJK}. These theories are invariant under ten dimensional $\mathcal{N} = 2$ supersymmetry. For the type IIB theory the supersymmetry generators satisfythe chiral $\mathcal{N} = 2$ superalgebra and for type IIA theory the non-chiral $\mathcal{N} = 2$ superalgebra. These massless content determine completely the low energy effective actions of the theories known as IIA and IIB ten dimensional supergravities.

Heterotic string Theories: The world sheet theory of the heterotic string theories consists of eight scalar fields, eight right-moving Majorana-Weyl fermions and thirty two left-moving Majorana-Weyl fermions. We have NS and R boundary conditions as before, as well as GSO projection involving the right-moving fermions. Unlike in the case of type II string theories, in this case boundary conditions on the left-moving fermions do not affect the Lorentz transformation properties of the states. Thus bosonic states come from states with NS boundary condition on the right-moving fermions and fermionic states come from those with R boundary condition on the right-moving fermions. There are two boundary conditions on the left-moving fermions giving rise to consistent string theories.

• SO(32) heterotic string theory: In this case either all the left-moving fermions have periodic or anti-periodic boundary conditions. The GSO projection keeps only those states in the spectrum which contain even number of left-moving fermions. The massless bosonic states in this theory consist of a symmetric rank two field G_{IJ}, an anti-symmetric rank two tensor field B_{IJ}^{NS}, a scalar field ϕ and a set of 496 gauge fields filling up the adjoint representation of the gauge group SO(32).

• $E_8 \times E_8$ heterotic string theory: In this case we divide the 32 left-moving fermions into two groups of sixteen each and use four possible boundary conditions, 1) all the left-moving fermions have periodic boundary conditions, 2) all the left-moving fermions have anti-periodic boundary conditions, 3) all the left-moving fermions of the first group have periodic boundary conditions and all of the second group have anti-periodic boundary conditions, 4) all the left-moving fermions of the first group have anti-periodic boundary conditions and all of the second group have periodic boundary conditions. In each sector we also have a GSO projection that keeps only those states in the spectrum which contain an even number of left-moving fermions from the first group and even number from the second. The massless bosonic states in this theory consist of a symmetric rank two field G_{IJ}, an anti-symmetric rank two tensor field B^{NS}_{IJ}, a scalar field ϕ and a set of 496 gauge fields filling up the adjoint representation of the gauge group $E_8 \times E_8$. The spectrum of both these theories is invariant under the chiral ten dimensional $\mathcal{N} = 1$ supersymmetry.

Type I string theory: The type IIB superstring, with the same chiralities on both sides, has a symmetry that exchanges the left- and the right-moving sectors in the world sheet theory. This transformation is known as the world sheet parity transformation Ω. Type I theory can be thought as the quotient of type IIB theory by this world sheet parity [15]. We can "gauge" this symmetry by keeping only those states in the spectrum which are invariant under this world sheet transformation to obtain an unoriented closed string theory. In the NS-NS sector, this eliminates the rank two antisymmetric tensor field B^{NS}_{IJ}, leaving the rank two symmetric tensor field G_{IJ} and the scalar field ϕ. The fermionic NS-R and R-NS sectors of type IIB string theory have the same spectra, so the Ω-projection picks out the linear combination (NS-R)+(R-NS). From the R-R sector, the Ω-projection selects the rank two antisymmetric tensor B^{R}_{IJ}, eliminating the scalar χ and the rank four anti-symmetric tensor C^{+}_{IJKL}. However, the resulting theory is not consistent. To have a consistent string theory built from this unoriented closed string, we should include open string states in the spectrum. The end points of the open strings are taken to be free (Neumann boundary conditions) and carry a non-dynamical gauge index in the fundamental representation of U(N) in the oriented string case and the adjoint representation of SO(N) or Sp(N) for the unoriented case. The open string theory that couples to the unoriented closed string theory must also be unoriented for the consistency of the interactions. The tadpole cancellation requirement leaves, however, the SO(32) as the only consistent gauge group. The spectrum of massless bosonic states in the resulting theory, together with G_{IJ}, B^R_{IJ} and ϕ of the closed string sector, consist of 496 gauge fields in the adjoint representation of SO(32) gauge group coming from the open string sector. This spectrum is also invariant under the chiral ten dimensional $\mathcal{N}=1$ supersymmetry.

1.2 String duality conjectures

As was described above, neglecting non supersymmetric projections, there are five consistent string theories in ten space-time dimensions. We can also get many different string theories in lower dimensions by compactifying these five theories on appropriate manifolds $\mathcal{M}$. Each of these theories is parametrized by a set of undetermined scalar expectation values known as moduli [1]. String dualities provide us with equivalence maps between different string theories. These equivalence relations map, in general, the weak coupling region of one theory to the strong coupling region of the second theory and vice versa. If a duality transformation is a symmetry of a string theory, it should than be a symmetry of the corresponding low energy effective action. At two derivative level these actions are completely determined by their supersymmetry and their massless spectrum.

Let us consider in the following some examples of duality conjectures. We will work in the string frame, where a factor of $e^{-2\phi}$, appears in front of the Einstein kinetic term.

1.2.1 Type I-heterotic duality

The massless bosonic states in SO(32) heterotic string theory come from the NS sector, and contains the graviton G_{IJ}, the dilaton ϕ, the rank two antisymmetric tensor field B^{NS}_{IJ}, and the gauge fields A^a_I $(a=1,...,496)$ in the adjoint representation of SO(32). The low energy dynamics involving these bosonic fields is described by the $\mathcal{N}=1$ supergravity coupled to SO(32) super Yang-Mills theory in ten dimensions. The basic action is given by [6, 16, 17, 18]:

$$S^{\text{het}} = \int d^{10}x\sqrt{G}e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \frac{1}{4}F^2 - \frac{1}{12}H^2\right], \tag{1.2}$$

[1]The string coupling constant $g=e^{\phi}$, the shape and size of $\mathcal{M}$, and various other background fields.

where R is the Ricci scalar, F_{IJ} denotes the non-abelian SO(32) gauge field strength, and H_{IJK} is the strength associated with the B_{IJ}^{NS} field.

Let us now turn to type I string theory. The massless bosonic states come from three different sectors. The closed string NS-NS sector gives the graviton G_{IJ}, and the dilaton ϕ. The closed string R-R sector gives an anti-symmetric tensor field B_{IJ}^{R}. Finally, the open string NS sector gives gauge fields A_I^a ($a = 1, ..., 496$) in the adjoint representation of SO(32). The low energy dynamics is again described by the $\mathcal{N} = 1$ supergravity theory coupled to SO(32) super Yang-Mills theory. The action is given by:

$$S^I = \int d^{10}x\sqrt{G}\left[e^{-2\phi}\left(R + 4(\nabla\phi)^2\right) - \frac{1}{4}e^{-\phi}F^2 - \frac{1}{12}\tilde{H}^2\right], \qquad (1.3)$$

where now $\tilde{H}_{IJK}$ is the field strength associated to the B_{IJ}^{R} field.

Notice the absence of any dilaton dependence in the $\tilde{H}$ kinetic term. The latter is coming from the R-R sector of the closed superstring. The dilaton and gauge kinetic terms, on the other hand, are weighted by $e^{-2\phi}$ and $e^{-\phi}$, since they come from the sphere and the disk world-sheet surfaces respectively. It is easy to see that the two actions (1.2) and (1.3) are identical provided we make the identification:

$$\phi_h = -\phi_I, \quad G_{\mu\nu}^h = e^{-\phi_I}G_{\mu\nu}^I, \quad A_\mu^h = A_\mu^I, \quad B_{\mu\nu}^h = B_{\mu\nu}^I. \qquad (1.4)$$

This led to the hypothesis that the type I and SO(32) heterotic string theories in ten dimensions are dual.

Recalling that $g = e^\phi$ is the string coupling, from (1.4) we see that $g_h = 1/g_I$, leading to the hypothesis that the strong coupling limit of type I string theory is related to the weak coupling limit of SO(32) heterotic string theory and vice versa.

1.2.2 IIB self-duality in D=10

The massless bosonic fields in type IIB string theory come from two sectors. The NS-NS sector gives the graviton G_{IJ}, an anti-symmetric tensor field B_{IJ}^{NS}, and the dilaton ϕ. The R-R sector gives a scalar field χ some times called the axion, an anti-symmetric tensor field B_{IJ}^{R}, and a rank four anti-symmetric tensor field C_{IJKL}^{+} whose field strength is self-dual. It turns out that there is no simple covariant action describing the low energy limit of this theory, but there are covariant field equations, which are in fact just the equations of motion of type IIB supergravity. It is often convenient to combine the dilaton and the axion fields into a complex scalar field λ as follows:

$$\lambda = \chi + ie^{-\phi} . \tag{1.5}$$

The equations of motion are covariant (they transform into each other) under $SL(2,R)$ transformations [19, 20, 21, 22]:

$$\begin{aligned} &\lambda \to \frac{a\lambda + b}{c\lambda + d} \;, \quad \begin{pmatrix} B^N_{\mu\nu} \\ B^R_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B^N_{\mu\nu} \\ B^R_{\mu\nu} \end{pmatrix} , \\ &G_{IJ} \to G_{IJ}, \qquad C^+_{IJKL} \to C^+_{IJKL} , \end{aligned} \tag{1.6}$$

where a, b, c, d are real numbers with $ad - bc = 1$. The existence of this $SL(2,R)$ symmetry in the type IIB supergravity theory led to the conjecture that an $SL(2,Z)$ subgroup of it, obtained by restricting a, b, c, d to be integers, is a symmetry of the full type IIB string theory.

Theories obtained by modding out (compactified) type IIB string theory by a discrete symmetry group, where some of the elements of the group involve Ω, are known as orientifolds. The simplest example of an orientifold is type IIB string theory modded out by Ω. This corresponds to type I string theory [15]. The closed string sector of type I theory consist of Ω invariant states of type IIB string theory. The open string states, on the other hand, are the analogs of twisted sector states in an orbifold, which must be added to the theory in order to make it finite.

Type II theories has another discrete symmetry denoted by $(-1)^{F_L}$ (type IIB modded by $(-1)^{F_L}$ coincides with type IIA theory in D=10). It changes the sign of all the Ramond sector states on the left-moving sector of the world-sheet. Acting on the massless bosonic sector fields, it changes the sign of χ, B^R_{IJ} and C^+_{IJKL}, and leaves G_{IJ}, B^{NS}_{IJ} and ϕ invariant.

The combined action $S\,(-1)^{F_L}\,S^{-1}$ where S is the element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $SL(2,Z)$, on the massless fields, turns out to be identical to that of Ω. Since the action of S on the massive fields is not known, it can be defined in such a way that the action of $S\,(-1)^{F_L}\,S^{-1}$ and Ω are identical on all states. This gives [23]

$$S\,(-1)^{F_L}\,S^{-1} = \Omega . \tag{1.7}$$

In the next chapter we will use these symmetries to construct a series of five-dimensional dual pairs by orbifolding/orientifolding type IIB theory.

1.2.3 Heterotic/Type-II duality in D=6

Heterotic string theory compactified to six dimensions on a four dimensional torus T^4 and type IIA string theory compactified on the four dimensional hyperkahler manifold of non trivial $SU(2)$ holonomy $K3$ enjoy both $\mathcal{N}=2$ supersymmetry in six dimensions. At generic points of the moduli space (i.e. $G \to U(1)^{24}$) both theories have the same massless spectrum, containing the $\mathcal{N}=2$ supergravity multiplet and twenty vector multiplets [23].

The six-dimensional tree-level heterotic effective action is given by:

$$S_{D=6}^{\text{heterotic}} = \int d^6x \sqrt{-\det \tilde{G}} e^{-2\phi} \Big[R + 4\partial^\mu\phi\partial_\mu\phi - \frac{1}{12}\hat{H}^{\mu\nu\rho}\hat{H}_{\mu\nu\rho} - \tag{1.8}$$

$$-\frac{1}{4}(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{8}Tr(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\Big] \,,$$

where $i = 1, 2, \ldots, 24$ and

$$\hat{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}L_{ij}A^i_\mu F^j_{\nu\rho} + \text{cyclic.permut.}. \tag{1.9}$$

The moduli scalar matrix $\hat{M}$ is,

$$\hat{M} = \begin{pmatrix} G^{-1} & G^{-1}C & G^{-1}Y^t \\ C^tG^{-1} & G + C^tG^{-1}C + Y^tY & C^tG^{-1}Y^t + Y^t \\ YG^{-1} & YG^{-1}C + Y & \mathbf{1}_{16} + YG^{-1}Y^t \end{pmatrix} \,, \tag{1.10}$$

where $\mathbf{1}_{16}$ is the sixteen-dimensional unit matrix, $G_{\alpha\beta}$ is the internal metric,

$$C_{\alpha\beta} = B_{\alpha\beta} - \frac{1}{2}Y^I_\alpha Y^I_\beta \,. \tag{1.11}$$

with L_{ij} the O(4,20) invariant metric and the Y^I_α moduli are the six dimensional internal gauge fields backgrounds which belong to the Cartan subalgebra of the ten-dimensional gauge group SO(32) (Wilson lines).

The tree-level type IIA effective action is given by [24]:

$$S_{K3}^{IIA} = \int d^6x \sqrt{-\det G_6} e^{-2\phi} \Big[R + 4\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \tag{1.12}$$

$$+\frac{1}{8}Tr(\partial_\mu \hat{M}\partial^\mu \hat{M}^{-1})\Big] - \frac{1}{4}\int d^6x\sqrt{-\det G}(\hat{M}^{-1})_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{16}\int d^6x\epsilon^{\mu\nu\rho\sigma\tau\epsilon}B_{\mu\nu}F^I_{\rho\sigma}\hat{L}_{IJ}F^J_{\tau\epsilon},$$

where $I = 1, 2, \ldots, 24$. $\hat{L}$ is the O(4,20) invariant metric

$$\hat{L} = \begin{pmatrix} 0 & \mathbf{1}_4 & 0 \\ \mathbf{1}_4 & 0 & 0 \\ 0 & 0 & -\mathbf{1}_{16} \end{pmatrix}. \tag{1.13}$$

To see that there is a simple transformation that relates (1.8) and (1.12) it is useful to go to the Einstein frame by performing the Weyl rescaling $G_{\mu\nu} \to e^\phi G_{\mu\nu}$, the heterotic action becomes:

$$S^{\rm het}_{D=6} = \int d^6x\,\sqrt{-G}\,\Big[R - \partial^\mu\phi\partial_\mu\phi - \frac{e^{-\phi}}{6}\hat{H}^{\mu\nu\rho}\hat{H}_{\mu\nu\rho} - \tag{1.14}$$

$$-\frac{e^{-\phi}}{4}(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{8}Tr(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\Big],$$

and the type IIA action in this frame is given by

$$S^{IIA}_{D=6} = \int d^6x\sqrt{-G}\,\Big[R - \partial^\mu\phi\partial_\mu\phi - \frac{1}{12}e^{-2\phi}H^{\mu\nu\rho}H_{\mu\nu\rho} - \tag{1.15}$$

$$-\frac{1}{4}e^\phi(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{8}Tr(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\Big] + \frac{1}{16}\int d^6x\epsilon^{\mu\nu\rho\sigma\tau\epsilon}B_{\mu\nu}F^i_{\rho\sigma}\hat{L}_{ij}F^j_{\tau\epsilon}.$$

Notice that the heterotic $\hat{H}_{\mu\nu\rho}$ contains the Chern-Simons term (1.9), while the type IIA one doesn't. The type IIA action instead contains a parity odd term coupling the gauge fields and $B_{\mu\nu}$. Both effective actions have a continuous $O(4, 20, \mathcal{R})$ symmetry which is broken in string theory to the quantum T-duality group $O(4, 20, \mathcal{Z})$ [25].

The actions (1.14) and (1.15) are identical provided we make the following identifications [26, 27]:

$$\phi^h = -\phi^{IIA} \quad , \quad G^h_{\mu\nu} = G^{IIA}_{\mu\nu} \quad , \quad \hat{M}^h = \hat{M}^{IIA} \quad , \quad A^h_{\mu i} = A^{IIA}_{\mu i}\,, \tag{1.16}$$

$$e^{-2\phi}\hat{H}^{IIA}_{\mu\nu\rho} = \frac{1}{6}\frac{\epsilon_{\mu\nu\rho}{}^{\sigma\tau\epsilon}}{\sqrt{-G}}H^h_{\sigma\tau\epsilon}, \tag{1.17}$$

The effective action of type IIA theory on $K3$ has a string solution which is singular at the core. The zero mode structure of the string is similar to the perturbative type-IIA string [28]. There is also a string solution which is regular at the core. This is a solitonic string and analysis of its zero modes indicates that it has the same (chiral) word-sheet structure as the heterotic string [29]. The string-string duality map (1.17) exchanges the roles of the two strings. The type-IIA string now becomes regular (solitonic), while the heterotic string solution becomes singular.

Upon compactification of the two theories to four dimensions on a two dimensional torus T^2, heterotic-type IIA duality translates into $S \leftrightarrow T$ interchange [30]. As a consequence, fundamental string winding of the type IIA string theory is mapped to NS5-brane charge of the heterotic string and vice versa. Moreover, electrically charged states are interchanged with magnetically charged states.

1.3 BPS states and Bounds

Analysis of the low energy effective action as sketched above, provides us with a crude test of duality. Indeed, most of the duality conjectures were arrived at by analyzing the symmetries of the low energy effective actions. A step further is based on the analysis of the spectrum of BPS states. These are states which are invariant under some fraction of the supersymmetry transformations. These states are very special for various reasons [31]:

i) Although they are massive, they form multiplets under extended supersymmetry which are shorter than the generic massive multiplet.

ii) At generic point in moduli space they are stable because of energy and charge conservation.

iii) Their mass formula is supposed to be exact if on uses "renormalized" values for the charges and moduli.

A common feature to all $\mathcal{N} = 1, 2$ supersymmetries, describing the low energy limits of the discussed string theories, is that they admit "central" extensions in their associated superalgebra [9]. These central charges depend on electric and magnetic charges of the theory as well as on expectation values of some moduli. Some times supersymmetry representations are shorter than usual. This is due to the fact that some of the supersymmetry generators are "null", so that they cannot create new states. The vanishing of some supercharges depends on the relation between the mass of the multiplet and the central charges appearing in the supersymmetry algebra. The BPS states are the lowest lying states and they saturate the so-called BPS bound which, for the point-like states, is of the form:

$$M \geq max(|Z|) \tag{1.18}$$

where $max(|Z|)$ stands for the maximal eigenvalue of the central charge matrix Z.

Consider type IIB string theory compactified on a circle. Since, the total number of supersymmetry generators in this theory is 32, a generic long supermultiplet is $2^{16} = (256)^2$ dimensional. This theory also has BPS states breaking half of the space-time supersymmetry. For these states we have $2^8 = 256$ dimensional representation of the supersymmetry algebra. They are known as ultra-short multiplets. We can also have BPS states breaking 3/4 of the space-time supersymmetry and realizing a $2^{12} = 16 \times 256$ dimensional representation. They are known as short multiplets.

These BPS states can be realized as point-like soliton solutions of the relevant effective supergravity theory. There are also BPS versions of extended objects (BPS p-branes). In the presence of the latter, the supersymmetry algebra can acquire central charges that are not Lorentz scalars [32]. These extended objects are charged under some anti-symmetric tensor $Z_{\mu_1\mu_2\cdots\mu_p}$, and the central charges have values proportional to these charges. The BPS condition would relate these charges with the energy densities (p-brane tensions) of the relevant p-branes. Such p-branes can be viewed as extended soliton solutions of the effective theory. In all cases the BPS condition is the statement that the soliton solution leaves part of the supersymmetry unbroken.

There are several amplitudes that in perturbative string theory obtain contribution from BPS states only. In the case of sixteen conserved supercharges ($\mathcal{N} = 4$ supersymmety in four dimensions), all two derivative terms as well as $\mathcal{R}^2$ terms are of that kind. In the last chapter we will study in some detail the $\mathcal{R}^2$ [13] coupling in the context of type IIA on $K3 \times T^2$-heterotic on T^6 dualtity.

1.4 D-branes and RR charges

Dp-branes are extended objects in p spatial dimensions defined by the property that open strings can end on them [33, 34], see Figure 1.1. Open strings can have two "obvious" kinds of boundary conditions,

$$\begin{aligned} \partial_\sigma X^\mu &= 0 \quad Neumann(N) \\ \partial_\tau X^\mu &= 0 \quad Dirichlet(D), \end{aligned} \tag{1.19}$$

where the Neumann boundary condition amounts to conservation of momentum at the boundary, while the Dirichlet boundary condition means that $X^\mu(0,\tau)$ and $X^\mu(\pi,\tau)$ are fixed. One may consider situations where p space-like coordinates have Dirichlet boundary conditions, and the remaining $9-p$ having Neumann boundary conditions

$$X^I(0,\tau) = X^I(\pi,\tau) = Y^I, I = p+1, \cdots 9,$$

where Y^I is some fixed vector. Hence the open string endpoints are free to move in the $p+1$ directions $X^0, \cdots X^p$, while they are fixed in the remaining directions $X^{p+1} \cdots X^9$. It is obvious that translation invariance is broken in the directions $X^{p+1} \cdots X^9$. This can be understood in terms of objects that are present in space. These objects are called Dp-branes. Note that if we take Neumann boundary conditions in all directions, the endpoints of the open string will be free to move in all space directions. This is what we normally mean by "open strings". From the point of view of perturbative string theory, the positions of the D-branes are fixed, corresponding to a particular string theory background. However the massless modes of the open strings connected to the D-branes can be associated with fluctuation modes of the D-branes themselves, so that in a full non-perturbative context the D-branes are expected to become dynamical p-dimensional branes. In one of the most important papers in the recent string revolution [7], it was pointed out by Polchinski that D-branes are charge carriers for the RR fields of type II superstring theories. Generally, a Dirichlet p-brane couples to RR $p+1$-form field through a term of the form

$$\mu_p \int_{\Sigma_{p+1}} A^{p+1}, \tag{1.20}$$

where the integral is taken over the $p+1$-dimensional word-volume of the p-brane. The D-branes break half of the space-time supersymmetry, considering that ϵ_L, ϵ_R are the space-time supersymmetric parameters in type II string theories, originating in the left-and the right-moving sector of the world-sheet theory respectively. Then ϵ_L and ϵ_R satisfy the chirality constraint

$$\Gamma^0 \cdots \Gamma^9 \epsilon_L = \epsilon_L, \qquad \Gamma^0 \cdots \Gamma^9 \epsilon_R = \xi \epsilon_R. \tag{1.21}$$

Where $\xi = +1(-1)$ in type IIB (IIA) and Γ^μ are the ten dimensional gamma matrices. The open string boundary conditions together with the corresponding boundary conditions on the world-sheet fermions give further restrictions on the supersymmetry parameters of the form [7]:

$$\epsilon_L = \Gamma^{p+1} \cdots \Gamma^9 \epsilon_R \,. \tag{1.22}$$

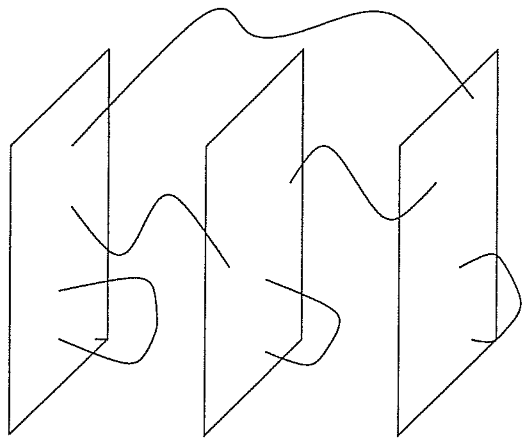

Figure 1.1: D-branes are extended objects on which open strings can end.

The compatibility of (1.21) with (1.22) then implies that p should be odd (even) in type IIB (IIA) string theories. In type IIA theory there are Dirichlet p-branes with $p = 0, 2, 4, 6, 8$ and in type IIB there can be Dirichlet p-branes with $p = -1, 1, 3, 5, 7, 9$. The D-branes with $p > 3$ couple to the duals of the RR fields and are thus magnetically charged under the corresponding RR fields [34].

Let us see now how the dynamical degrees of freedom of a D-brane arise from the massless string spectrum in a fixed D-brane background [35]. In the presence of a D-brane, the open string vector field A_M with $M = 0, \cdots 9$, decomposes into a U(1) gauge field A_μ on the world-volume of the brane with μ a $p+1$-dimensional index, and $9-p$ scalar fields X^a. The fields X^a describe fluctuations of the D-brane world-volume in the transverse directions. In a purely bosonic theory, the equations of motion for a D-brane are precisely those of the action [36] [2]

$$S = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{\det(-G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \qquad (1.23)$$

where G, B and ϕ are the pullbacks of the ten dimensional metric, antisymmetric tensor and dilation to the D-brane world-volume, while $F_{\mu\nu}$ is the field strength of the world-volume U(1) gauge field A_μ. ξ is some coordinate system on the D-brane world-volume, the tension [34] $\tau_p = \frac{T_p}{g} = \frac{1}{g\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p}$ and g the string coupling

[2] In the full supersymmetric string theory, this action must be extended to a supersymmetric Born-Infield action. There are in addition, Chern-Simons terms coupling the D-brane gauge field to the RR fields, of which the leading term is the (1.20) term discussed above.

constant. The inverse string coupling appears because the leading string diagram which contributes to the action (1.23) is the disk diagram. Making a certain number of assumptions, the form of the action (1.23) simplifies considerably. Assume that: i) the background 10-dimensional space-time is flat, so that $g_{MN} = \eta_{MN} = (- + \cdots +)$. ii) the D-brane is approximately flat so that we can identify the world-volume coordinates on the D-brane with $p+1$ of the 10-dimensional coordinates (static gauge assumption) the pullback of the metric to the D-brane world-volume becomes

$$G_{\mu\nu} \simeq \eta_{\mu\nu} + \partial_\mu X^a \partial_\nu X^a + O((\partial X)^4),$$

iii) $B_{\mu\nu}$ vanishes, $2\pi\alpha' F_{\mu\nu}$ and $\partial_\mu X^a$ are small and of the same order. Then we see that the low-energy D-brane world-volume action becomes

$$S = -\tau_p V_p - \frac{1}{4g_{YM}^2} \int d^{p+1}\xi (F_{\mu\nu}F^{\mu\nu} + \frac{2}{(2\pi\alpha')^2}\partial_\mu X^a \partial^\mu X^a) + O(F^4), \qquad (1.24)$$

where V_p is the p-brane world-volume and the Yang-Mills coupling is given by

$$g_{YM}^2 = \frac{g}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-2}.$$

The second term in (1.24) is just the action for a U(1) gauge theory in $p+1$-dimensions with $9-p$ scalar fields. In fact, after including fermions, the low-energy action for a D-brane becomes precisely the U(1) SYM action in $p+1$-dimensions. In the case when we have N parallel Dp-branes, labeling the branes by an index $(i = 1, \cdots N)$, there are massless fields living on each D-brane world-volume, corresponding to a total gauge group $U(1)^N$. In addition, however, there are fields corresponding to strings stretched between each pair of branes A_{ij}^M. Because the strings are oriented, there are $N^2 - N$ such fields. The mass of a field corresponding to a string connecting branes i and j is proportional to the distance between these branes. Witten has pointed out [37] that as the D-branes approach each other, the stretched strings become massless and the fields give the right degrees of freedom of the gauge field components and adjoint scalars of a supersymmetric U(N) gauge theory in $p+1$-dimensions.

Generally, such a SYM theory is described by the dimensional reduction to $p+1$-dimensions of the ten dimensional non-abelian Super Yang-Mills theory where all fields are in the adjoint representation of the gauge group. The $\mathcal{N} = 1$ SYM action is given by

$$S = \frac{1}{g_{YM}^2} \int d^{10}\xi Tr(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\psi}\Gamma^\mu D_\mu \psi),$$

where ψ is a 16-component Majorana-Weyl spinor of $SO(1,9)$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

The covariant derivative D_μ of ψ is given by

$$D_\mu \psi = \partial_\mu \psi - i[A_\mu, \psi].$$

We thus conclude that the low energy physics of N Dirichlet p-branes living in flat space in type II theories is described, in the static gauge, by the dimensional reduction to $p+1$-dimensions of $\mathcal{N}=1$ SYM theory in ten dimensional with gauge group U(N).

Chapter 2

D1/D5 systems in $\mathcal{N} = 4$ string theories

In this chapter we will systematically discuss various aspects of the D1-D5 system in the context of a class of theories with 16 supercharges. These are obtained by orbifolding/orientifolding type IIB theory with freely acting Z_2 actions, which involve shifting along some compact direction together with the action of Z_2 elements such as Ω, the world sheet parity, or the "fermionic parity" $(-)^{F_L}$, with F_L the spacetime left-moving fermion number or I_4, the reflection of the 4 coordinates of a T^4.

The presence of the shift generate theories that, compared with type I theory have the simplifying feature of avoiding the presence of open string sectors. In the case where the Z_2 includes the world-sheet parity operator Ω [38]. Moreover, via the adiabatic argument [12, 23], these actions are expected to commute with S-duality. Therefore some subgroup of the full U-duality group is still at work, relating in a non-perturbative way various brane configurations. In particular we can relate the D1-D5 system in one theory to a fundamental string with winding plus KK momentum in another theory. Also, we obtain non-trivial relations for the three-charge systems in different backgrounds.

We will derive CFT descriptions for the D1-D5 systems in the various cases and test them against the U-duality predictions. We are interested in the counting of BPS states. As we have already mentioned in the introduction the information about multiplicities and masses of these states is encoded in the Witten index or more generally in the elliplic genus. Since the elliptic genus do not depend on the coupling constant, we can take the limit wich is more convenient for our purposes. We will consider the infrared limit of our theories. We will find agreement for the two-charge states but generically disagreement for the three-charge states. For the latter states, we will point out a similar problem in the case of type II on $K3$.

This chapter is organized as follows: in section 2 we will describe the various type IIB orbifold/orientifold backgrounds and the U-duality relations among them and the corresponding charges, that will be relevant to our subsequent analysis. The discussion here will be general for the case where the shift is transverse to the D-branes, but we will partially cover the longitudinal shift case too.

In section 3 we will derive the effective field theory for systems of D1-branes, D5-branes and D1-D5 branes and show that they involve symmetric products of T^4 with appropriate Z_2 orbifold actions, depending on the background under consideration. For theories which involve orientifolding, the resulting D1-D5 CFT is of type $(4,0)$. In section 4 we will derive elliptic genera for symmetric products involving fields with the above extra Z_2 orbifold actions. In section 5 these results will be used to show that the predictions of the proposed CFT's agree with those of the perturbative string partition functions of the U dual theories, for all two-charge cases, in particular the D1-D5 case. In section 6 we compute the moduli dependence of low energy couplings involving the gauge fields arising from KK reduction in various backgrounds.

2.1 U-duality chain of type IIB orbifold/orientifolds

2.1.1 Transversal shift

In this section we construct a series of five-dimensional U-dual models with sixteen unbroken supercharges. We adopt Sen's fiberwise construction procedure [23] to generate lower dimensional dual pairs from the self-dual (under a subgroup of the full five-dimensional U-duality group that we will still call U) type IIB theory on T^5. The compact directions are taken to be 12345. We will further compactify on an additional S^1 of radius R_6 in the 6th direction, to accompany various Z_2 orbifold/orientifold actions with a shift of order two along S^1, denoted by σ_{p_6}. In the case of a geometrical shift by half winding which we will consider first, this will result in a factor $(-1)^{p_6}$ in the corresponding lattice sum, p_6 being the momentum in the X_6 direction. This will make the construction applicable via the adiabatic argument [12].

We start by defining a U-duality chain that maps into each other the various charges in the perturbative and solitonic spectrum of the toroidal type IIB parent theory. Under these duality transformations, perturbative symmetries of the underlying theory such as Ω (worldsheet parity operator), $(-)^{F_L}$ (left moving spacetime fermion number) and I_4 (reflection in the (2345) plane) are mapped into each other. A prototype of such a duality chain is displayed in table 1.1:

A $\xrightarrow{S}$	**B** $\xrightarrow{T_{2345}}$	**C** $\xrightarrow{S}$	**D**
NS_{12345}	D_{12345}	D_1	F_1
F_1	D_1	D_{12345}	NS_{12345}
p_1	p_1	p_1	p_1
$(-)^{F_L} I_4$	ΩI_4	Ω	$(-)^{F_L}$
I_4	I_4	I_4	I_4
$(-)^{F_L}$	Ω	ΩI_4	$(-)^{F_L} I_4$

Table 1.1: D1(D5)-p to fundamental strings

For the time being different columns, labeled by **A**, **B**,..., represent equivalent description of type IIB theory on T^5 but in the future they will stand for a triplet of models obtained by orbifolding/orientifolding the toroidal theory by one of the three perturbative symmetries displayed in each column (accompanied by a Z_2 shift σ_{p_6}). Different columns are connected by S or $T_{ijk...}$ elements (the indices indicating the direction along which T-duality is performed) of the U-duality group. Winding, momentum, NS5-brane and D-brane charges are denoted by $F_i, P_i, NS_{ijklm}, D_{ijk...}$ respectively with the indices specifying the direction along which they are oriented. We will focus on two-charge systems which admits always a U-dual perturbative description in terms of winding-momentum charges.

A bound state of N D1-strings and k units of KK momentum p_1 at step **C**, for example, is mapped through S (step **D**) to a fundamental string wrapped N times on the 1^{st} circle and carrying k units of momentum. Similarly a D5-p bound state at **C** is mapped at step **A** to a fundamental string bound state$F_1 - p_1$. A $D1D5$ bound state, on the other hand, can be mapped again to a fundamental string-momentum bound state through the more involve chain of dualities:

$\mathbf{C}$	$\xrightarrow{S}$	$\mathbf{D}$	$\xrightarrow{T_{15}}$	$\mathbf{E}$	$\xrightarrow{S}$	$\mathbf{F}$	$\xrightarrow{T_{2345}}$	$\mathbf{G}$	$\xrightarrow{S}$	$\mathbf{H}$
D_1		F_1		p_1		p_1		p_1		p_1
D_{12345}		NS_{12345}		NS_{12345}		D_{12345}		D_1		F_1
p_1		p_1		F_1		D_1		D_{12345}		NS_{12345}
Ω		$(-)^{F_L}$		$(-)^{F_L}$		Ω		ΩI_4		$(-)^{F_L} I_4$
I_4		I_4		$(-)^{F_L} I_4$		ΩI_4		Ω		$(-)^{F_L}$
ΩI_4		$(-)^{F_L} I_4$		I_4		I_4		I_4		I_4

Table 1.2: D1-D5 to fundamental strings

Let us now come back to the step **A** in table 1.1, and consider the orbifolding of type IIB on $T^4 \times S_1$ by each of the three Z_2's generated by $(-)^{F_L} I_4 \sigma_{p_6}$, $I_4 \sigma_{p_6}$ and $(-)^{F_L} \sigma_{p_6}$ respectively. We will refer to these theories as I_F, II_F and III_F. Accordingly we will denote by I, II and III the theories obtained by orbifolding $\Omega \sigma_{p_6}$, $I_4 \sigma_{p_6}$ and $\Omega I_4 \sigma_{p_6}$ respectively. In all the above cases the shift along X_6 is transversal to the D1, D5 branes which are wrapped along $X_1, \ldots, X_5$.

The fiberwise construction procedure [23] states that a dual pairs can be defined (under certain adiabatic hypotheses [12]) by modding out the parent theories by two dual actions, i.e. two symmetry elements in the same line in the U-duality chain above. The inclusion of the shift makes the adiabatic argument applicable. Notice also that the shift σ_{p_6} is invariant under all the elements of the U-duality group involved in the above chain.

From table 1.2 we see that D1-D5 bound states in column **C** are mapped to fundamental string states in column **H**, where theories I_F, II_F and III_F can be considered. However, we see that exciting KK momentum on the D1-D5 system amounts to excite NS5-brane charges in **H**.

On the other hand the duality chains above also provide stringent constraints on the three-charge systems $(D1 - D5 - p)$ of our three theories:

- by comparing column **C** with column **B** we see that $D1$ and $D5$ charges are exchanged, with theory II left invariant, while theories I and III are exchanged.
- by comparing column **C** with column **F** we see that $D1$ and p charges are exchanged, with theory I left invariant, while theories II and III are exchanged.

• by comparing column **B** with column **G** we see that $D5$ charges and KK momentum p are exchanged, with theory III left invariant, while theories I and II are exchanged.

As we will see, these relations will put severe constraints on the multiplicity formulae for the three-charge systems and hence on the effective filed theory governing them.

2.1.2 Longitudinal shift

One may ask the question of what happens if the shift is longitudinal to the D-branes, i.e. instead of σ_{p_6} we have σ_{p_1}.

Consider theory II at step **C**. After S duality, at step **D** the twisted sector contains half-integral winding modes localized at fixed points. At step **E** these become, in the twisted sector, half-integral momentum modes localized at fixed points. In going from **E** to **F** we expect a non-perturbative phase σ_{D_1} for the states carrying D_1-brane charge. This cannot be the whole story however. This is clear from a comparison of the perturbative spectrum of states at step **E** and **F**: in the former case, integer (untwisted states) and half-integer (twisted states) momentum modes come with different multiplicities due to the winding shift (see formulas (2.48) below with $F_1 \to p_1$), while in the dual description the distinction between even and odd modes would not exist, since the above non-perturbative phase leaves invariant the whole perturbative spectrum.

An insight about the correct map can be gained from a careful analysis of the fundamental string partition function at step **E**. Model II at this step is type $IIB/(-)^{F_L} I_4 \sigma_{F_1}$. As we mentioned a shift in F_1 implies that states in the twisted sector carry half-integer momenta and are localized at fixed points. Under S duality to step **F**, these are mapped to open string states living on the D5-branes ("twisted sector" with respect to ΩI_4), sitting at orientifold 5-planes at 16 fixed points. There are to begin with 16 pairs of D5-branes, each pair at a fixed point, giving rise to the gauge group $SO(2)^{16}$. However, due to the presence of half-integer momentum modes p_1, we conclude that a Z_2 Wilson line along the circle on X_1 must be turned on at step **F**, thereby breaking completely $SO(2)^{16}$. If we do a further T-duality along X_1 we then have type I' on T^5/Z_2 with 32 4-branes distributed on the 32 fixed points and a completely broken gauge group.

After four T-dualities to step **G** this result, together with the fact that D5 branes sit at sixteen different fixed points of I_4, translates into a type I theory on T^5 with five Wilson lines turned on to break completely the gauge group. Finally under S duality, we get a perturbative description of the D1-D5 system of model II in terms of fundamental heterotic string with a gauge group completely broken by Wilson lines at step **H**. We will refer to this model as model IV.

We will comment later on the difficulties involved in trying to extend the longitudinal shift case to models I and III.

2.2 Effective World Volume CFT's

In this section we will try to obtain the effective field theories for pure D5 branes, pure D1 branes and D1-D5 system for each of the models described in the previous section, as already stated these theories are related by U-duality.

2.2.1 Transversal Shift

We first discuss the case when the shift is transverse to the brane system. In all the models we are considering we have Z_2 orbifolding of tyep IIB theory compactified on $T^4 \times S^1 \times S^1$ where the Z_2 is generated by an element of the form $g\sigma$ with g being a combination of Ω and the reflection of T^4 and σ is a shift along the last S^1 factor. We are considering the system of D1 and D5 branes where the D5 brane is wrapped on the T^4 and the first S^1 factor and the D1 brane is wrapped on the first S^1 factor. If we have Q_1 D1 branes and Q_5 D5 branes in the quotient space, then in the covering space there will be two identical sets of (Q_1, Q_5) systems which are placed at X^6 and $X^6 + \pi R_6$ (see Figure 2.1) with the world volume theory on the two sets being identified via the Z_2 action g. Let Φ_1 and Φ_2 be the world volume fields on the two systems at X^6 and $X^6 + \pi R_6$ respectively, then the identification is given by $\Phi_2 = \hat{g}\Phi_1$, where $\hat{g}$ is the Z_2 action induced by g on the world volume fields. This essentially means that the effective world volume theory is described by just one set of fields (say Φ_1) since the other set is not independent. This would just be the field content of a single set of (Q_1, Q_5) system in type IIB theory compactified on $T^4 \times S^1 \times S^1$ with R_6 being the radius of the last S^1. Where then does one see the effect of Z_2 orbifolding of the underlying IIB theory? To understand this, note that among the fields Φ_1 there is one which corresponds to the center of mass position of D5 branes along the X^6 direction. We shall denote this field by X^6. As one changes

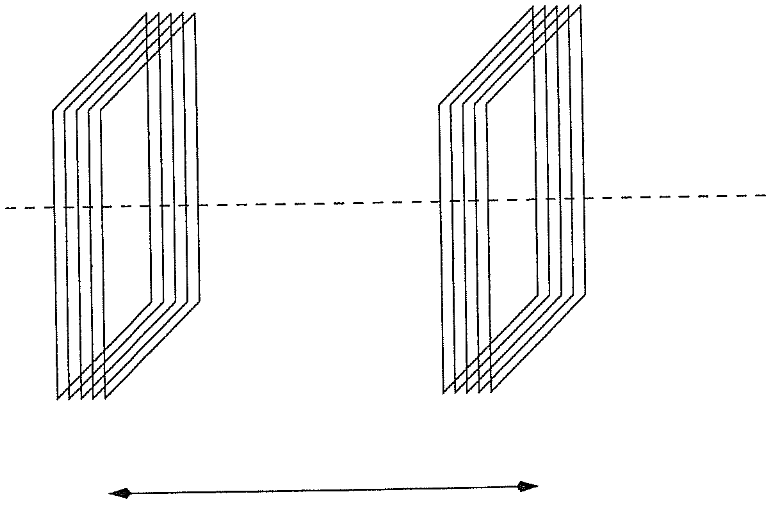

Figure 2.1: two sets of (Q_5, Q_1) placed at X^6 and $X^6 + \pi R_6$.

the value of X^6 the entire system of the two sets of branes moves along this direction. In particular when one moves X^6 all the way to $X^6 + \pi R_6$ and the remaining fields Φ_1 to $\hat{g}\Phi_1$, then this system is equivalent to the original system (in a description where one uses Φ_2 as the independent fields). Thus there is a Z_2 gauging on the effective world volume theory defined by the action $\hat{g}.\sigma$ with σ being the shift on the center of mass world volume field X^6. Now we will apply these general considerations to the cases of pure D5 branes, D1 branes and D1-D5 system for the three models I, II and III respectively:

D5 brane world volume theory

The low energy effective world volume theory on D5 branes in IIB theory is just the 6-dimensional (1,1) supersymmetric $U(Q_5)$ gauge theory. Let $X^0, X^1, ..., X^5$ be the directions along the D5 brane world volume, of which the 4 directions $X^2, ..., X^5$ are compactified on a torus. We can now carry out a dimensional reduction so that the fields depend only on X^0 and X^1 [39] [1]. Let us denote the various sets of indices by $\mu, \nu = 0, 1$, $i, j = 2, ..., 5$ and $a, b = 6, ..., 9$. The world volume fields on the D5 branes

[1]Throughout this paper we will assume that the radii of this torus are of the order of the string scale so that in the low energy effective action one can ignore the KK modes for D5 branes as well as the winding modes for the D1 branes.

are A_μ, A_i, X^a which are in the adjoint representation of $U(Q_5)$ and their fermionic partners ψ. Sometimes for brevity of notation we will denote by A_M, $M = 0, 1, ...9$ all the bosonic fields. Let us denote by g_1, g_2 and g_3 the Z_2 actions Ω, I_{2345} and ΩI_{2345} respectively. The induced action $\hat{g}_2$ on the world volume fields is easily seen to be:

$$\hat{g}_2 : X^6 \rightarrow X^6 + \pi R_6 \mathbf{1}, \quad A_i \rightarrow -A_i, \quad \psi \rightarrow \Gamma_{2345}\psi \tag{2.1}$$

To understand the action $\hat{g}_1$, let us reconsider the system of D5 branes. We have a total of $2Q_5$ D5 branes in the covering space, where Q_5 of them are sitting at the center of mass position X^6 and the remaining Q_5 at $X^6 + \pi R_6$. Thus in the resulting system $U(2Q_5)$ is broken to $U(Q_5) \times U(Q_5)$. The gauge fields can then be represented in terms of $Q_5 \times Q_5$ blocks:

$$\mathcal{A}_M = \begin{pmatrix} A_M & 0 \\ 0 & A'_M \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix} \tag{2.2}$$

where A_M and A'_M are the $U(Q_5)$ gauge fields on the two sets of branes. These two gauge fields are of course not independent of each other; they should be related by the Z_2 action in the underlying string theory. The shift exchanges the two sets of branes and therefore exchanges A_M with A'_M while the Ω projection acts on the Chan Paton indices according to

$$\mathcal{A}_M = \pm \Omega_5 \mathcal{A}_M^t \Omega_5, \qquad \Psi = \Gamma^{(7)} \Omega_5 \Psi^t \Omega_5 \tag{2.3}$$

with Ω_5 in terms of $Q_5 \times Q_5$ blocks is

$$\Omega_5 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4}$$

and $\pm$ sign means $-$ for $M = 0, ..., 5$ and $+$ for $M = 6, ..., 9$. In the above $\Gamma^{(7)} = \Gamma_{012345}$. More explicitly the above projection implies

$$A'_\mu = -A^t_\mu, \tag{2.5}$$

We can now take the independent set of fields to be A_M and ψ, however the Z_2 action which takes the values of these fields to that of A'_M and ψ' will give rise to a configuration which is indistinguishable from the original one. Thus the induced Z_2 gauging on the set of fields A_M and ψ is given by

$$\hat{g}_1 : A_\mu \rightarrow -A^t_\mu, \quad A_i \rightarrow -A^t_i, \quad X_a \rightarrow X^t_a + \pi R_6 \delta_{a6}, \quad \psi \rightarrow -\Gamma^{(7)}\psi^t \tag{2.6}$$

Finally since $g_3 = g_1 \cdot g_2$ it follows that the induced action $\hat{g}_3$ is given by:

$$\hat{g}_3 : A_\mu \to -A^t_\mu, \quad A_i \to +A^t_i, \quad X_a \to X^t_a + \pi R_6 \delta_{a6}, \quad \psi \to -\Gamma_{01}\psi^t \tag{2.7}$$

In the Coulomb branch the moduli space is given by taking diagonal matrices [52] for A_i, X_a and ψ upto the Weyl group which is the permutation group S_{Q_5} acting on the Q_5 eigenvalues. In particular the diagonal entries in X_a describe the transverse positions of each of the D5 branes. We can move any one of these branes along X_6 direction by an amount πR_6 and simultaneously change the field on that brane by the action of $\hat{g}$ and this system would be indistinguishable from the original one. Thus the conformal field theory describing the Coulomb branch is

$$(R^3 \times S^1 \times T^4/Z_2)^{Q_5}/S_{Q_5} \tag{2.8}$$

where S^1 denotes the circle along X^6 direction and T^4 (dual of the original T^4 [43]) appears from the Wilson lines and we will coordinatize it by $A^2, ..., A^5$. Note that the Z_2 action does not commute with the permutation group, and in fact the full orbifold group is the semi-direct product of S_{Q_5} with $Z_2^{Q_5}$. This may seem a bit puzzling since S_{Q_5} is the remnant of the $U(Q_5)$ gauge symmetry of the system before going to the Coulomb branch. The point is that at the level of $U(Q_5)$ gauge theory where one ignores the massive string modes coming from the stretched strings between the D5 branes at X^6 and their images at $X_6 + \pi R_6$, this $Z_2^{Q_5}$ symmetry is broken. However it is easy to see that this should be a symmetry of the theory when one includes these massive states that transform in the $(Q_5, \bar{Q}_5)$ representations of $U(Q_5) \times U(Q_5)$ gauge symmetry. In the infrared limit in the Coulomb branch when one ignores all the massive off-diagonal modes the theory has manifest $Z_2^{Q_5}$ symmetry (see Figure 2.2).

In the three cases the Z_2 actions are:

$$\hat{g}_1 : X^i \to -X^i \tag{2.9}$$

Noting further that Γ_{01} defines the world sheet chirality of the fermions, $\Gamma^{(7)}$ is just $(-1)^{F_L}$ times the reflection along $2, 3, 4, 5$ directions. Thus

$$\hat{g}_1 = (-1)^{F_L} I_{2345}\sigma_{p6} \tag{2.10}$$

Note that for $Q_5 = 1$ the effective theory we have obtained is just the orbifold of IIB by Z_2 generated by $\hat{g}_1$ in the static gauge. Indeed from the U-duality map table (1.1), we recognize this as the fundamental side D5-KK system in string theory defined by Ω projection.

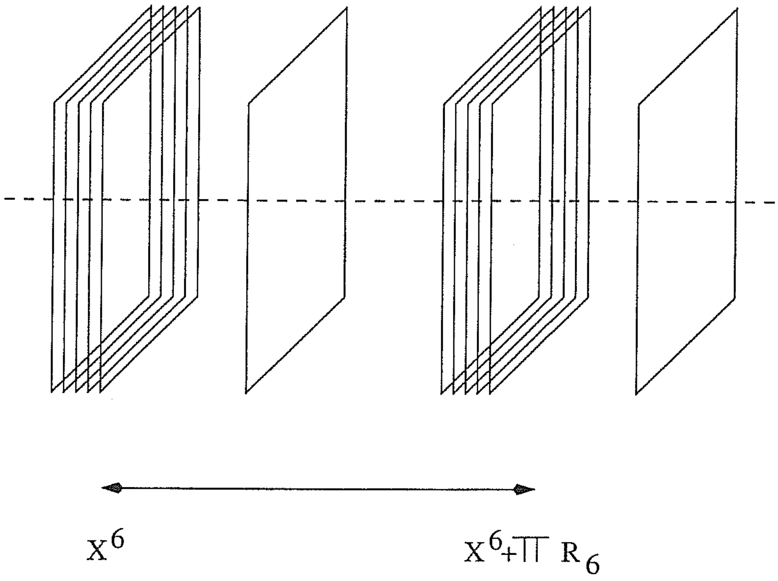

Figure 2.2: we can move independently one brane from one set of branes.

From (2.1) $\hat{g}_2$ is easily seen to be $I_{2345}\sigma_{p_6}$ which again for $Q_5 = 1$ describes the corresponding fundamental theory. And finally $\hat{g}_3$ being given by the products of $\hat{g}_1$ and $\hat{g}_2$ is $(-1)^{F_L}\sigma_{p_6}$.

Note that Z_2 orbifolding breaks the (8,8) supersymmetry of the parent IIB system down to (4,4) for models I and II while it breaks to (8,0) for model III. This is to be expected since by T-dualities along 2,3,4 and 5 directions, D5 brane in the III model is mapped to D1 brane in model I which is just a type I like theory.

D1 brane world volume theory

One can follow the above reasonings also for D1 branes. The only difference is that Ω projection now gives an extra minus sign also for $2, 3, 4$ and 5 directions since they are transversal to the D1 branes, instead of longitudinal as in the D5 brane case. It is easy to see that this just exchanges $\hat{g}_1$ with $\hat{g}_3$ while leaving the $\hat{g}_2$ unchanged. Once again for $Q_1 = 1$ this reproduces the fundamental side (under column D), for these three cases. Moreover this is also consistent with T-duality since by four T-dualities along 2,3,4 and 5 directions the D5 brane is exchanged with D1 brane and model I is exchanged with model III.

We summarize the various Z_2 actions for pure D5 and pure D1 brane systems in the following table:

Model	$D1-KK$	$D5-KK$
I	$(\hat{R}^4 \times \hat{T}^4)^N/S_N$	$(\hat{R}^4 \times \bar{T}^4)^{N'}/S_{N'}$
II	$(R^4 \times \tilde{T}^4)^N/S_N$	$(R^4 \times \tilde{T}^4)^{N'}/S_{N'}$
III	$(\hat{R}^4 \times \bar{T}^4)^N/S_N$	$(\hat{R}^4 \times \hat{T}^4)^{N'}/S_{N'}$

Table 2.1: D1-p, D5-p bound states

where tilde, hat and bar represent the I_4 and $(-)^{F_L}$ and $(-)^{F_L}I_4$ accompanied by a shift in the X_6 direction respectively.

D1-D5 system

Now let us add Q_1 D-strings to the system of Q_5 D5 branes. D1 branes are along X^1 direction while D5 branes are along $X^1, ..., X^5$ directions. In the covering space again this system would split into two sets of D1-D5 system sitting at X^6 and $X^6 + \pi R_6$. For each of these two sets, the 1+1 dimensional common world volume theory is just a supersymmetric sigma model on the moduli space $\mathcal{M}$ of Q_1 instantons of $N = 4$ $U(Q_5)$ gauge theory on T^4 times the center of mass fields corresponding to the common transverse directions $R^3 \times S^1$ [8, 40]. This is exactly the model appearing in the type IIB context. Since each of the Z_2 actions on the D5 brane world volume gauge fields described above leaves the self-duality equations invariant it follows that it induces an action $\hat{g}$ on the instanton moduli space $\mathcal{M}$. Following the logic described above, it then follows that the effective world volume theory is the Z_2 gauging of the theory in the type IIB case where the Z_2 is generated by $\hat{g}.\sigma$.

In the infrared the (4,4) supersymmetric sigma model on $M \times R^3 \times S^1$ would flow to a (4,4) SCFT. It is conjectured that this SCFT is a symmetric product space $R^4 \times T^4 \times (T^4)^N/S_N$ [8, 40]. There has been much critical discussion of this conjecture in the literature [41, 42], and it is generally believed that this is true only for $Q_5 = 1$ case. For other values of Q_5 with Q_1 relatively coprime, this CFT with $N = Q_1.Q_5$ perhaps describes the system at some point in the moduli space of the IIB theory on T^4 (with suitable RR fields turned on), which is related to the trivial point by a U-duality that maps (Q_5, Q_1) system to $(Q_1.Q_5, 1)$ system [41]. Moreover for $Q_5 = 1$ the various factors appearing in the CFT have a clear interpretation: the D-flatness condition sets all the bifundamental fields coming from 1-5 open string states to zero, leaving behind only the Cartan directions of the 1-1 $U(Q_1)$ adjoint states. The latter have the interpretation of the positions of the D-strings inside T^4. Thus the factor $(T^4)^{Q_1}/S_{Q_1}$ represents the positions of the Q_1 instantons, while the center of mass factors R^4 and T^4 represent the transverse position of the D5 brane and its $U(1)$ Wilson lines on the T^4 (so more precisely this should be the dual torus). In our case, of course since the transverse direction along X^6 is compactified on a circle, R^4 should be replaced by $R^3 \times S^1$.

With the physical interpretation of the various fields appearing in the SCFT being clear, we are now in a position to deduce the induced Z_2 action on the instanton moduli space for each of the three models. Let us denote by A_i for $i = 2, ..., 5$ the four $U(1)$ Wilson lines of the D5 brane gauge field and by $X_i^{(\ell)}$ for $\ell = 1, ..., Q_1$ the positions of the Q_1 instantons on T^4. Finally we denote by X_a for $a = 6, ..., 9$ the coordinates of center of mass transverse position $S^1 \times R^3$. The little group $SO(4) \equiv SU(2)_A \times SU(2)_Y$ acts on the tangent space of $S^1 \times R^3$. In the IIB theory the resulting SCFT has (4,4) supersymmetry. The left and right moving supercharges come with definite chiralities with respect to the little group $SO(4)$ [10]. Specifically the left moving supercharges are two $SU(2)_A$ doublets while the right moving ones are two $SU(2)_Y$ doublets. The supermultiplets then are

bosons	Left moving fermions	Right moving fermions
$X_a \equiv X_{AY}$	ψ_A	$\tilde{\psi}_Y$
A_i	ψ_Y	$\tilde{\psi}_A$
$X_i^{(\ell)}$	$\psi_Y^{(\ell)}$	$\tilde{\psi}_A^{(\ell)}$

$\hat{g}_1$ leaves X_a (being the center of mass position) and $X_i^{(\ell)}$ invariant since these are the positions of D1 branes in T^4. It however takes A_i to $-A_i$ since Ω projects out the $U(1)$ gauge field [34]. To understand its action on the fermions, the easiest way is to use the fact that in this theory the D1-D5 system should preserve (4,0) supersymmetry. As a result ψ_A and $\psi_Y^{(\ell)}$ should remain unchanged while ψ_Y must pick a minus sign. On the right-moving fermions the action is exactly the reverse of it i.e. $\tilde{\psi}_Y$ and $\tilde{\psi}_A^{(\ell)}$ should pick a minus sign while $\tilde{\psi}_A$ should remain unchanged. This is because X^a and A_i are D5 brane fields and as we have seen (2.6) that $\hat{g}_1$ acts on the fermions by $\Gamma^{(7)}$ which equivalently measures the chirality with respect to $SO(4)_E$. Thus $SU(2)_A$ and $SU(2)_Y$ doublets must appear with opposite signs. On the other hand $X_i^{(\ell)}$ are the D1 brane fields and on the fermions $\hat{g}_1$ acts as Γ_{01}. Thus the left and right moving fermions appear with opposite signs. To summarize $\hat{g}_1$ maps

$$
\begin{aligned}
(X_a, X_i^{(\ell)}, \psi_A, \psi_Y^{(\ell)}, \tilde{\psi}_A) &\rightarrow (X_a + \pi R_6 \delta_{a6}, X_i^{(\ell)}, \psi_A, \psi_Y^{(\ell)}, \tilde{\psi}_A) \\
(A_i, \psi_Y, \tilde{\psi}_A^{(\ell)}, \tilde{\psi}_Y) &\rightarrow -(A_i, \psi_Y, \tilde{\psi}_A^{(\ell)}, \tilde{\psi}_Y)
\end{aligned}
\tag{2.11}
$$

In model II the induced action is more straightforward to see. In this case D1-D5 system preserves the full (4,4) supersymmetry of the parent IIB system. Thus it is sufficient to specify the $\hat{g}_2$ action on the bosonic fields. Since g_2 is the inversion I_{2345}, it follows that it gives negative sign to A_i and $X_i^{(\ell)}$ and all their fermionic partners. Finally $\hat{g}_3$ is just obtained as product of $\hat{g}_1.\hat{g}_2$. We summarize these different actions in the following table:

Model	D1-D5
I	$\hat{R}^4 \times \bar{T}^4 \times (\hat{T}^4)^N / S_N$
II	$R^4 \times \tilde{T}^4 \times (\tilde{T}^4)^N / S_N$
III	$\hat{R}^4 \times \hat{T}^4 \times (\bar{T}^4)^N / S_N$

Table 2.2: D1-D5 bound states

Here the hat, bar and tilde represent the same actions as in table 2.1, the difference being that these Z_2's act diagonally with respect to S_N (i.e. they commute with S_N).

2.2.2 Longitudinal Shift

We now consider the shift along the common world volume direction X_1. The general discussion in this case is very similar to the transverse shift case. If Φ denotes the set of world volume fields in the type IIB theory case, and $\hat{g}$ the induced Z_2 action then the fields satisfy the condition:

$$\Phi(X_1 + \pi R_1, X_0) = \hat{g}\Phi(X_1, X_0) \tag{2.12}$$

Taking the interval of X_1 to be πR_1, what this condition says is that the fields on which $\hat{g}$ acts as -1 are anti-periodic along X_1 direction and the ones on which the $\hat{g}$ action is +1 are periodic. This means that the SCFT's are again given by tables 2.1 and 2.2, where the tilde, hat and bar refer to the twist along σ direction. These actions are moreover diagonal with respect to the permutation group S_N and S'_N.

Another way to see that the above proposal must be correct is to start from the effective field theories in the transverse case and consider the threshold corrections due to the single (i.e. minimal unit) D1 or D5 instanton obtained by wrapping the time directions of these systems on the X_6 circle by a length πR_6 (i.e. half winding). The resulting amplitude is just the one loop amplitude in the orbifold sector given by the insertion of the operator $\hat{g}$. This is because in the path-integral formulation, it is in this sector that there is a half winding corresponding to the shift along the time direction. On the other hand by a modular transformation, we can exchange τ and σ, and the resulting path integral should be interpreted as that of the field theory living in the single D5 brane aligned along directions 23456 or a single D1 brane along direction 6 with longitudinal shift along X_6. This field theory is just the twisted sector ($\hat{g}$ twist along σ direction) of the Z_2 orbifold of $R^4 \times T^4$. Putting N copies of these together in the Coulomb branch should reproduce the conjecture of the previous subsection. In the case of D1-D5 system the same argument applies in a more straightforward way since there is only a single copy of the center of mass R^4.

There is however one apparent puzzle we would like to discuss here. Consider N D1 brane in model I. The Ω projects the $U(N)$ gauge group to $SO(N)$ so that $SO(N)$ gauge fields are periodic while the remaining ones that are in the symmetric tensor representation of $SO(N)$, are anti-periodic. Now let us put D5 branes. Gimon-Polchinski [44] consistency condition would at first sight imply that the projection should be symplectic. This would produce a doubling phenomenon i.e. one would need even number of type IIB D5 branes.

There are two ways of seeing Gimon-Polchinski consistency condition [44]. One of them involves a consideration of the Dirac charge quantization condition. This was one way to see how in the usual type I theory the presence of a single D-string requires D5-branes to be paired (with respect to IIB counting), i.e. the fact that in the 1-5 sector there is a factor of 1/2 due to the Ω projection. Now let us see what happens in the case under consideration. Consider the D5 brane to be longitudinal to the direction X_1 along which the shift acts. Then the Poincare dual B field which enters in the Dirac quantization condition would refer to D-string which is transversal to X_1. But in this case we have actually 2 D-strings (one sitting at say X_1 and the other at $X_1+\pi R_1$). So the quantization condition is satisfied just with one D5 brane wrapped on the circle with circumference $2\pi R_1$. Similarly if one takes D-string longitudinal to X_1 then its Poincare dual involves D5 brane that is transverse to X_1 in which case again we have 2 D5 branes, showing that it suffices to have just one D-string.

The other way is to consider the action of Ω^2 on the open string states. In the usual Type I theory, Gimon-Polchinski [44] showed that on the 1-5 open strings Ω^2 picks an extra minus sign, due to the fact that these states involve a twist field along the four directions longitudinal to the D5 brane and transversal to the D1 brane. Including the action of Ω on the Chan-Paton indices we have

$$\Omega^2 : |\alpha, \mu > \rightarrow -(\gamma^t \gamma^{-1})_{\alpha\beta} |\beta, \nu > (\gamma'^t \gamma'^{-1})_{\nu\mu} \tag{2.13}$$

where α, β and μ, ν are the Chan-Paton indices on D1 and D5 branes respectively and γ and γ' are Ω actions on the D1 and D5 brane Chan-Paton indices. Due to the extra minus sign above one concludes that if γ is symmetric then γ' must be anti-symmetric and vice versa. So if one system is projected onto Orthogonal group then the other must be projected onto Symplectic group.

In our case however Ω is accompanied by the shift σ_1. Thus $g_1^2 = \Omega^2.\sigma_1^2$ and we can take both systems to have Orthogonal projections provided $\sigma_1^2 = -1$ on the 1-5 string states. This means that 1-5 string states will carry half-integer momenta along the X_1 circle. Since 1-5 states are bi-fundamentals this can be thought of as turning on a Z_2 Wilson line in one of the systems along the circle.

An equivalent way of formulating Gimon-Polchinski condition is perhaps in terms of the consistency of the closed string couplings to the brane system. Here the basic requirement is that in the closed string channel, the sum of Annulus, Mobius strip and Klein bottle should be a perfect square for each closed string state [45]. In Appendix C, we show that even when we take Orthogonal projection for both D1 and D5 branes, this condition is satisfied provided 1-5 string states have half-integer momenta along X_1 circle. This is the same condition we found in the previous paragraph.

This result might seem a little surprising, since D1-D5 system can be thought of as YM instantons in the D5 brane world volume [8, 40]. It is known that in the context of ADHM construction [46, 48], $SO(N)$ instantons have $Sp(k)$ symmetry where k is the instanton number, and vice versa. This is indeed the result for standard Ω projection. In our case however Ω is accompanied with a shift. What this means is that we are looking for $U(N)$ instantons in the 4 directions spanned by $X_2, ...X_5$, and the moduli of the instantons are slowly varying functions of X_1 in such a way that

$$A_\mu(X_1 + \pi R_1) = -g^{-1} A^t_\mu(X_1) g \qquad \mu = 2, 3, 4, 5 \tag{2.14}$$

where $g \in U(N)$ is slowly varying function of X_1. The periodicity condition (upto a possible Wilson line $h \in U(N)$) as $X_1 \to X_1 + 2\pi R_1$ implies

$$g^*.g = h \tag{2.15}$$

For orthogonal and symplectic projections $h = +\mathbf{1}$ and $h = -\mathbf{1}$ respectively and in these two cases we can take g to be $+\mathbf{1}$ and the symplectic matrix $\mathbf{J}$ respectively. In the latter case of course N must be even.

Let us now see how this condition is translated on the ADHM data (here we will take the 4-dimensional space where instanton is sitting to be R^4 since the discussion of the doubling phenomenon should not depend on whether the space is T^4 or R^4). The ADHM data for U(N) [46] consists of a $(N + 2k) \times 2k$ matrix Δ defined as

$$\Delta_{\lambda, i\dot{\alpha}} = a_{\lambda, i\dot{\alpha}} + b^{\alpha}_{\lambda, i} x_{\alpha\dot{\alpha}} \tag{2.16}$$

where $x_{\alpha\dot{\alpha}} = x_\mu \sigma^\mu_{\alpha\dot{\alpha}}$ and the indices $\lambda = u + j\beta$ with u running over N indices and i, j running over k indices. Δ satisfies the quadratic constraint

$$\bar{\Delta}^{\dot{\alpha}}_{i,\lambda} \Delta_{\lambda, j\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} f^{-1}_{ij} \tag{2.17}$$

where f a $k \times k$ hermitian matrix.

The self-dual gauge fields are then given by

$$A_\mu = \bar{U} \partial_\mu U \tag{2.18}$$

where U is an $(N + 2k) \times N$ matrix which satisfies the equations

$$\bar{U}U = \mathbf{1}, \qquad \bar{\Delta}U = \bar{U}\Delta = 0 \tag{2.19}$$

From the above it follows that the projection operator $U\bar{U} = \mathbf{1} - \Delta f \bar{\Delta}$.

The symmetries of these equations are

$$U \to B.U.g, \qquad \Delta \to B.\Delta.(C \times \mathbf{1}_{2\times 2}), \tag{2.20}$$

where g is a local $U(N)$ transformation while B and C are independent of X_μ and are in $U(N + 2k)$ and $GL(k)$ respectively. Using this freedom in defining the data we can set $b^\alpha_{u,i} = 0$ and $b^\alpha_{j\beta,i} = \delta_{ij}\delta^\alpha_\beta$. Then the instanton moduli are contained in the matrix a which, as follows from eq.(2.17), satisfies the constraint that a^μ_{ji} is in the adjoint representation of $U(k)$, where a^μ is defined via $a_{j\beta,i\dot{\alpha}} = a^\mu_{ji}\sigma^\mu_{\beta\dot{\alpha}}$. There are also $3k^2$ D-term constraints quadratic in a that follows from (2.17) but they will not concern us here. With this canonical choice for Δ the global symmetry group $U(N + 2k) \times GL(k)$ reduces to $U(N) \times U(k)$. Explicitely this corresponds to taking C in eq(2.20) to be in $U(k)$ and

$$B = \begin{pmatrix} D & \\ & C^{-1} \times \mathbf{1}_{2\times 2} \end{pmatrix} \tag{2.21}$$

with $D \in U(N)$. Note that the moduli a^μ_{ij} transforming in the adjoint representation of $U(k)$ are part of the 1-1 string states that define the position of D1 brane inside D5 brane, while $a_{u,i\dot{\alpha}} \equiv w_{u,i\dot{\alpha}}$ are the 1-5 string states that are bi-fundamentals of $U(N)$ and $U(k)$. The spinorial index $\dot{\alpha}$ just refers to the fact that the bosonic 1-5 string states are spinors of the $SO(4)$ acting on $X^2, ...X^5$.

However at this point we still have two $U(N)$ actions: the local $U(N)$ action on U on the right and the global $U(N)$ action on the left. The instanton gauge field

A_μ which live in the D5-brane sees only the local action, while the ADHM data w sees only the global action. In order to relate this system to the D1-D5 system we must identify these two $U(N)$ actions. The basic point is to choose a particular gauge for the instanton solution that fixes the local $U(N)$ symmetry. We will choose the singular gauge [47] which is described as follows. Writing U as an $N \times N$ block V and $2k \times N$ block U', the condition $U\bar{U} = \mathbf{1} - \Delta f \bar{\Delta}$ implies $V\bar{V} = \mathbf{1} - wf\bar{w}$. Given a solution for V, $V.g$ will also solve this equation for g being a local $U(N)$ transformation. Choosing the singular gauge amounts to taking V to be one of the 2^N matrix square roots of the right hand side $(\mathbf{1} - wf\bar{w})^{1/2}$. With this choice it is clear that a transformation $w \to Dw$ implies $V \to DVD^{-1}$ and the two $U(N)$'s are identified.

Let us now return to the Z_2 projection condition (2.14). With the two $U(N)$ actions identified, this condion on ADHM data becomes:

$$\begin{aligned} w(X_1 + \pi R_1) &= g w^*(X_1)(C \times \sigma_2) \\ a_\mu(X_1 + \pi R_1) &= C^{-1} a^*_\mu(X_1) C \end{aligned} \tag{2.22}$$

where $C \in U(k)$. Here we have used the fact that $x = \sigma_2 x^* \sigma_2$. Repeating this equation twice we find

$$\begin{aligned} w(X_1 + 2\pi R_1) &= -(g.g^*) w(X_1)(C^* C), \\ a_\mu(X_1 + 2\pi R_1) &= (C^* C)^{-1} a_\mu(X_1)(C^* C) \end{aligned} \tag{2.23}$$

As stated earlier Orthogonal and Symplectic projections of $U(N)$ correspond to $g = \mathbf{1}$ and $g = \mathbf{J}$ respectively. Similarly the Orthogonal and Symplectic projections of $U(k)$ are given by choosing $C = \mathbf{1}$ and $C = \mathbf{J}$ respectively. The above equations show that if both the groups are projected to Orthogonal or Symplectic groups then w which represents 1-5 string states are anti-periodic as $X_1 \to X_1 + 2\pi R_1$, while if they are projected in the opposite ways the w are periodic. This is exactly the condition we found using Gimon-Polchinski consistency condition.

It is instructive to consider $k = 1$ since in this case we can explicitely solve the ADHM constraints. The result is[47]:

$$w_{u\dot{\alpha}} = \rho G \begin{pmatrix} \mathbf{0}_{[N-2]\times[2]} \\ \mathbf{1}_{[2]\times[2]} \end{pmatrix}, \qquad G \in \frac{SU(N)}{SU(N-2)} \tag{2.24}$$

where ρ is the scale of the instanton. One can then solve for U satisfying equation (2.19) and obtain the gauge field as

$$A_\mu = G \begin{pmatrix} 0 & 0 \\ 0 & A^{SU(2)}_\mu \end{pmatrix} G^{-1} \tag{2.25}$$

where $A_\mu^{SU(2)}$ is the standard $SU(2)$ single instanton gauge field in the singular gauge with scale ρ and position a^μ:

$$A_\mu^{SU(2)} = \frac{\rho^2 \bar{\eta}^c_{\mu\nu}(X-a)^\nu \sigma^c}{(X-a)^2((X-a)^2+\rho^2)} \tag{2.26}$$

The moduli of the instanton are the position a_μ, the scale ρ and the gauge orientations contained in G. These moduli are now slowly varying functions of X_1 in such a way that the Z_2 projection condition (2.14) is satisfied. It is easy to see that this condition implies that

$$\begin{aligned} a^\mu(X_1+\pi R_1) &= a^\mu(X_1), \qquad \rho(X_1+\pi R_1) = \rho(X_1), \\ G(X_1+\pi R_1) &= gG^*(X_1)\sigma_2 \end{aligned} \tag{2.27}$$

Repeating this twice and the eq.(2.24), we find the condition $w(X_1+2\pi R_1) = -(gg^*)w(X_1)$.

Taking the orthogonal projection for $U(N)$, namely $h = \mathbf{1}$, we recognize from above that the ADHM data w, which represents the 1-5 string states are anti-periodic as $X_1 \to X_1 + 2\pi R_1$. Note that the above equations also show that the Z_2 projection acts trivially on the instanton position a_μ and scale ρ as expected.

To conclude, assuming the Orthogonal projection on D1 branes, one can get a consistent D5 brane with Orthogonal projection carrying the minimal unit of charge allowed by the Dirac quantization condition.

2.3 Partition functions and symmetric product spaces

In this section we derive a general formula for the character valued string partition function of a symmetric product CFT involving fields carrying non-trivial spin characteristics. More precisely we will consider the orbifold CFT defined as the symmetric product $S_N\mathcal{H} \equiv \mathcal{H}^N/S_N$, with $\mathcal{H}$ describing the Hilbert space of closed string excitations with either periodic or antiperiodic boundary conditions around the two cycles of the worldsheet torus. We label the boundary condition data by the spin characteristic $\left[{g_0 \atop h_0}\right]$, with g_0, h_0 thought as a set of pairs $\{g_\phi, h_\phi = 0, \frac{1}{2}\}$ describing the holonomies of a given field Φ around the worldsheet loops

$$\Phi(\sigma_1+1,\sigma_2) = e^{2\pi i g_\phi}\Phi(\sigma_1,\sigma_2) \qquad \Phi(\sigma_1,\sigma_2+1) = e^{2\pi i h_\phi}\Phi(\sigma_1,\sigma_2) \tag{2.28}$$

The results generalize the more familiar symmetric product formula [49] to the case where some of the fields carry a spin characteristics different from the odd ($g_\phi = h_\phi = 0$ in our notation) and can be associated to sectors of a diagonal Z_2 orbifold of the more familiar symmetric product spaces. As we have seen in the previous section such CFT's naturally arises in the study of D-brane bound state physics for type IIB orbifolds/orientifolds involving Z_2-shifts in the winding-momentum modes.

The derivation of the partition function follow with slight modifications the lines of [49] (see also [51]). We start by specifying the character valued string partition function for single copy of the Hilbert space $\mathcal{H}$

$$\begin{aligned} \mathcal{Z}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix}(\mathcal{H}|q,\bar{q},y) &= \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-c/24}\, \bar{q}^{\bar{L}_0-c/24}\, y^{J_0^3}\, \tilde{y}^{\tilde{J}_0^3} \\ &= \sum C\begin{bmatrix} g_0 \\ h_0 \end{bmatrix}(\Delta,\bar{\Delta},\ell,\tilde{\ell})\, q^{\Delta}\, \bar{q}^{\bar{\Delta}}\, y^{\ell}\, \tilde{y}^{\tilde{\ell}} \end{aligned} \tag{2.29}$$

with the sum running over $\Delta, \bar{\Delta}, \ell, \tilde{\ell}$. The supertrace runs over string states in $\mathcal{H}$ with boundary conditions specified by $\left[\begin{smallmatrix} g_0 \\ h_0 \end{smallmatrix}\right]$, $q = e^{2\pi i \tau}$ describe the genus-one worldsheet modulus, L_0, $\bar{L}_0$ are the Virasoro generators. $J_0^3, \tilde{J}_0^3$ are Cartan generators of a given $SU(2)_L \times SU(2)_R$ current algebra to which y and $\tilde{y}$ couple respectively.

Our task is to evaluate the supertrace (2.29) for a Hilbert space constructed by considering N copies of the Hilbert space $\mathcal{H}$ modding out by the permutation group S_N. This can be done following the standard non-abelian orbifold techniques developed in [49, 50]. The string partition function is written as the double sum

$$Z_{gh} = \sum_{gh=hg} \frac{1}{C_{g,h}}\; [h]\underset{[g]}{\square} \tag{2.30}$$

over orbifold twisted sectors labeled by the conjugacy classes $[g]$ of the permutation group S_N

$$[g] = \prod_{L=1}^{n} (L)^{N_L} \quad \text{with} \quad \sum_{L=1}^{n} L N_L = N, \tag{2.31}$$

and over the conjugacy classes $[h]$ of the centralizer

$$\mathcal{C}_g = \prod_{L=1}^{n} S_{N_L} \ltimes Z_L^{N_L} \tag{2.32}$$

The integers $C_{g,h}$ counts the number of conjugacy classes $[h]$ in $\mathcal{C}_g$ and ensures that the sum over $[h]$ correctly projects onto orbifold group invariant states. In writing (2.30) we have used the fact that traces of elements in the same conjugacy class $[h]$

leads to the same result. We will write elements in $[h]$ as

$$[h]=\prod_{L=1}^{n}\prod_{M=1}^{n_L}(M)^{r^L_M}\,\mathbf{t}\in\prod_{L=1}^{n}\mathcal{S}_{N_L}\ltimes Z_L^{N_L}\qquad\sum_{M=1}^{n_L}Mr^L_M=N_L. \tag{2.33}$$

with $\mathbf{t}$ an element in $\mathbf{Z_L}^{N_L}$ for a given choice of N_L's.

Orbifold group sectors are then parametrized by the integers $\{N_L\}$ (partitions of N), $\{r^L_M\}$ (partitions of N_L) and $\{\mathbf{t}\}$ (elements of $Z_L^{N_L}$). In particular the number of such $[h]$'s is given by

$$\mathcal{C}_{g,h}=\prod_{L,M}L^{r^L_M}M^{r^L_M}r^L_M! \tag{2.34}$$

We can label the N copies of field in $S_N\mathcal{H}$ by the quintuple of integers (L,l,M,m,i) running in the ranges $L=1,2.....n$, $l=0,1,.....L-1$ $M=1,2,....n_L$, $m=0,1,....M-1$ and $i=1,2.....r^L_M$ respectively. Writing Z_L^N elements as $\mathbf{t}=(L)^{s_{m;i}}$ boundary conditions for a field $\Phi_l^{m;i}$ (dependence in L,M are implicitly understood) along the cycles of the worldsheet torus can be written as

$$\begin{aligned}\Phi_l^{m;i}(\sigma_1+1,\sigma_2)&=e^{2\pi i g_\phi}\Phi_{l+1}^{m;i}(\sigma_1,\sigma_2)\\ \Phi_l^{m;i}(\sigma_1,\sigma_2+1)&=e^{2\pi i h_\phi}\Phi_{l+s_{m;i}}^{m+1;i}(\sigma_1,\sigma_2)\end{aligned} \tag{2.35}$$

After iterating (2.35) one is left with the quasiperiodic functions

$$\begin{aligned}\Phi_l^{m;i}(\sigma_1+L,\sigma_2)&=e^{2\pi i L g_\phi}\Phi_l^{m;i}(\sigma_1,\sigma_2)\\ \Phi_l^{m;i}(\sigma_1+s_i,\sigma_2+M)&=e^{2\pi i(sg_\phi+Mh_\phi)}\Phi_l^{m;i}(\sigma_1,\sigma_2)\end{aligned} \tag{2.36}$$

with $s_i=\sum_{m=0}^{M-1}s_{m;i}\ mod\ L$. The contribution to the orbifold string partition function of a given sector specified by $\{N_L,r^L_M,s_{m;i}\}$ can therefore be written in terms of the result for a single copy in a torus with the induced complex structure $\tilde{\tau}=\frac{M}{L}\tau+\frac{s}{L}$ and spin characteristics (2.36), i.e.

$$(M)^{r^L_M}\mathbf{t}\,\underset{(L)^{N_L}}{\square}\;:\prod_{i=1}^{r^L_M}Z\begin{bmatrix}g_0L\\ g_0s_i+Mh_0\end{bmatrix}(\tilde{q}_i,\bar{\tilde{q}}_i,y^M,\bar{y}^M) \tag{2.37}$$

with $\tilde{q}_i=e^{2\pi i\tilde{\tau}_i}=q^{\frac{M}{L}}e^{2\pi i\frac{s_i}{L}}$

Plugging this basic trace result into the sum over orbifold sectors specified by (2.31,2.33), yields0

$$\mathcal{Z}\begin{bmatrix}g_0\\ h_0\end{bmatrix}(S_NM|q,\bar{q},y,\bar{y})= \tag{2.38}$$

$$
\begin{aligned}
&\sum_{\{N_L\},\{r_M^L\},\{s_i^L\}} \prod_{L,M,i} \frac{1}{M^{r_M^L} r_M^L!} \frac{1}{L^{r_M^L}} \mathcal{Z}\begin{bmatrix} g_0 L \\ g_0 s_i^L + M h_0 \end{bmatrix} (\tilde{q}_i, \bar{\tilde{q}}_i, y^M, \tilde{y}^M) \\
&= \sum_{\{N_L\},\{r_M^L\}} \prod_{L,M} \frac{1}{M^{r_M^L} r_M^L!} \\
&\times \left(\frac{1}{L} \sum_{s=0}^{L-1} C\begin{bmatrix} g_0 L \\ g_0 s + M h_0 \end{bmatrix} (\Delta, \bar{\Delta}, \ell, \tilde{\ell})\, q^{\frac{M\Delta}{L}}\, \bar{q}^{\frac{M\bar{\Delta}}{L}}\, e^{2\pi i \frac{s}{L}(\Delta - \bar{\Delta})}\, y^{M\ell}\, \tilde{y}^{M\tilde{\ell}} \right)^{r_M^L}
\end{aligned} \tag{2.39}
$$

with $C\left[\begin{smallmatrix} g_0 \\ h_0 \end{smallmatrix}\right](\Delta, \bar{\Delta}, \ell, \tilde{\ell})$ the expansion coefficients (2.29) defined for a single copy of $\mathcal{H}$.

Before going on it is worth to spend some words on the BPS content of this formula. We have seen in the previous section that quite often the proposed CFT describing excitations of the D-brane bound states involve fermionic zero modes. The trace over these modes leads to the vanishing of the quantity inside the brackets in (2.39) corresponding to the fact that bound state excitations organize themselves into supermultiplets of the unbroken supersymmetry. Sectors with $r_M^L = 1$ correspond then to states in the shortest BPS supermultiplet and the counting formula for these states simplifies to [51]

$$
\begin{aligned}
&\mathcal{Z}_{BPS}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} (S_N M | q, \bar{q}, y, \tilde{y}) = \\
&\quad \frac{1}{N} \sum_{s,L,M} C\begin{bmatrix} g_0 L \\ g_0 s + M h_0 \end{bmatrix} (\Delta, \bar{\Delta}, \ell, \tilde{\ell})\, q^{\frac{M\Delta}{L}}\, \bar{q}^{\frac{M\bar{\Delta}}{L}}\, e^{2\pi i \frac{s}{L}(\Delta - \bar{\Delta})}\, y^{M\ell}\, \tilde{y}^{M\tilde{\ell}}
\end{aligned} \tag{2.40}
$$

with $N = LM$, $s = 0, 1, ..L-1$. This restriction of the more general formula presented below will be enough in most of our future considerations.

Coming back to the general expression (2.39) one can now perform the sum over s in (2.38). It is easy to see (see the appendix B for similar projective sum manipulations) that this leads effectively to a projection onto states satisfying the "level matching condition"

$$
\frac{(\Delta - \bar{\Delta})}{L} \in \mathbf{Z} + \delta. \tag{2.41}
$$

with $\delta = 0, 1/2$ depending on the different orbifold group sectors and boundary conditions. Introducing as in [49] a generating function for the symmetric product formulas (2.38)

$$
\mathcal{Z}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} (p, q, \bar{q}, y, \tilde{y}) = \sum_N p^N \mathcal{Z}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix} (S_N \mathcal{H} | q, \bar{q}, y, \tilde{y}) \tag{2.42}
$$

with $p^N = p^{LMr_M^L}$ and N summing up to infinity, one can write the final result in the compact form

$$\mathcal{Z}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix}(p,q,\bar{q},y,\tilde{y}) = \prod_{\delta=0}(1-p^L q^{\frac{\Delta}{L}}\bar{q}^{\frac{\bar{\Delta}}{L}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{g_0\frac{L}{2}\}(\Delta,\bar{\Delta},\ell,\tilde{\ell})}$$
$$\times\prod_{\delta=g_0}(1-(-)^{2h_0}p^L q^{\frac{\Delta}{L}}\bar{q}^{\frac{\bar{\Delta}}{L}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{g_0\frac{L}{2}\}(\Delta,\bar{\Delta},\ell,\tilde{\ell})} \tag{2.43}$$

The products run over all possible $L, \Delta, \bar{\Delta}, \ell, \tilde{\ell}$ satisfying the level matching condition (2.41) with δ explicitly indicated in (2.43). The coefficients $C_\pm\{g\}$ are defined by

$$C_\pm\{g\} = \frac{1}{2}(C\begin{bmatrix} g \\ 0 \end{bmatrix} \pm C\begin{bmatrix} g \\ \frac{1}{2} \end{bmatrix}) \tag{2.44}$$

and count the number of states in the the g-twisted sector of the original (the single copy) CFT with $\pm$ eigenvalues under the Z_2 orbifold group action. The choice $g_0 = h_0 = 0$ correspond to the case studied in [49] where all fields are periodic in both σ and τ directions and the sum over s results into a projector onto (2.41) with $\delta = 0$.

Specifying to ground states in (let us say) the right moving part of the symmetric product CFT, formula (2.43) reduces to

$$\begin{aligned}
\mathcal{Z}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(p,q,\bar{q},y,\tilde{y}) &= \prod_{L,k}(1-p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \\
\mathcal{Z}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(p,q,\bar{q},y,\tilde{y}) &= \prod_{L,k}(1-p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{0\}}(1+p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{0\}} \\
\mathcal{Z}\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(p,q,\bar{q},y,\tilde{y}) &= \prod_{L,k}(1-p^{2L} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{0\}}(1-p^{2L-1} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{\frac{1}{2}\}} \\
&\quad \times(1-p^{2L} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{0\}}(1-p^{2L-1} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{\frac{1}{2}\}} \\
\mathcal{Z}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(p,q,\bar{q},y,\tilde{y}) &= \prod_{L,k}(1-p^{2L} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{0\}}(1-p^{2L-1} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C_+\{\frac{1}{2}\}} \\
&\quad \times(1+p^{2L} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{0\}}(1+p^{2L-1} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C_-\{\frac{1}{2}\}}
\end{aligned} \tag{2.45}$$

where the arguments of the expansion coefficients $C_\pm\{g\}(\hat{k}\hat{L},\ell,\tilde{\ell})$, with $\hat{k}$ integer or half-integer, L even and odd, appearing in (2.45), have been omitted. The net effect of a non-trivial holonomy $(g_0, h_0) \neq (0,0)$ is then to correlate the parity of excitations in the CFT under the Z_2 orbifold group action with the parity of the permutation group orbifold sector and the level of the SCFT specified by L and $k \equiv \frac{\Delta}{L}$ respectively.

2.4 D1/D5 bound states versus fundamental strings

In this section we evaluate the elliptic genera encoding multiplicities and charges of D1/D5 two-charge bound state systems and compare the result with the ones expected from a U-dual description in terms of winding-momentum modes of fundamental strings. We will also comment about the three-charge system in the three models under consideration. The computation is always performed in the infrared, where the associated gauge theories describing the low energy bound state dynamics are conjectured to flow to one of the orbifold symmetric product CFT's in tables 2.1 and 2.2. There is an important difference between the orbifold CFT's proposals in the two tables for the two and three-charge systems. In the case of bound states of pure $D1(D5)$-branes with KK momentum modes, the position in $\mathbf{R}^4$ is described by the center of mass of N copies of $\mathbf{R}^4$'s in the symmetric product CFT. This leads to a subtlety in the counting of BPS excitations since not all states contributing to the elliptic index correspond to normalizable ground states of the gauge theory [52]. A careful analysis [52] reveals that among all the states with the right supersymmetry structure to reconstruct a short supermultiplet ($r^L_M = 1$ in (2.40)) only involves those coming from the long string sector $[g] = (N)$ that represent truly one-particle states. BPS charges and multiplicities can therefore be read off from formula (2.40) with $M = 1, L = N$. This is not the case for the symmetric products associated to D1D5 bound states (table 2.2), where all intermediate strings contributing to (2.45) are needed in order to reproduce the fundamental string degeneracies. The position of the bound state is now specified by a single coordinate in $\mathbf{R}^4$.

2.4.1 Fundamental string partition functions

Before going on in the study of the spectrum of D-brane bound states let us evaluate the elliptic index in the fundamental sides of the duality chain. Two-charge D-brane bound state will be systematically mapped to a fundamental string carrying both momentum (p_1) and winding modes (F_1) with no extra charges turned on in one of the three type IIB orbifold theories generated by $(-)^{F_L} I_4 \sigma_{p_a}$, $(-)^{F_L} \sigma_{p_a}$ and $I_4 \sigma_{p_a}$. We refer to these theories as I_F, II_F and III_F respectively. σ_{p_a} represents a Z_2-shift in the momentum mode along direction a, with $a = 1, 6$ in the case of a longitudinal and transverse shift respectively.

The partition function for the BPS states of the fundamental string is defined by the supertrace (2.29) restricted to the right moving ground state sector. After performing the spin structure sum these can be written as

$$Z_{I_F}(q,y,\tilde{y}) = \frac{1}{2}\,\frac{\tilde{y}_-^2\vartheta_1^2(y)}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})}\Gamma_{4,4}\left(y_-^2\,\frac{\vartheta_1^2(\tilde{y})}{\eta^6}\Gamma_{1,1}\begin{bmatrix}0\\0\end{bmatrix}\right.$$

$$\left.+y_+^2\,\frac{\vartheta_2^2(\tilde{y})}{\hat{\vartheta}_2^2(0)}\Gamma_{1,1}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} + 16\frac{\vartheta_4^2(\tilde{y})}{\hat{\vartheta}_4^2(0)}\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + 16\frac{\vartheta_3^2(\tilde{y})}{\hat{\vartheta}_3^2(0)}\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right) \tag{2.46}$$

$$Z_{II_F}(q,y,\tilde{y}) = \frac{1}{2}\,\frac{y_-^2\tilde{y}_-^2}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^6}\,\Gamma_{4,4}\left(\vartheta_1^2(y)\vartheta_1^2(\tilde{y})\Gamma_{1,1}\begin{bmatrix}0\\0\end{bmatrix}\right.$$

$$\left.+\vartheta_2^2(y)\vartheta_2^2(\tilde{y})\Gamma_{1,1}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} + \vartheta_4^2(y)\vartheta_4^2(\tilde{y})\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + \vartheta_3^2(y)\vartheta_3^2(\tilde{y})\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right) \tag{2.47}$$

$$Z_{III_F}(q,y,\tilde{y}) = \frac{1}{2}\,\frac{y_-^2\vartheta_1^2(\tilde{y})}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})}\Gamma_{4,4}\left(\tilde{y}_-^2\,\frac{\vartheta_1^2(y)}{\eta^6}\Gamma_{1,1}\begin{bmatrix}0\\0\end{bmatrix}\right.$$

$$\left.+\tilde{y}_+^2\,\frac{\vartheta_2^2(y)}{\hat{\vartheta}_2^2(0)}\Gamma_{1,1}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} + 16\frac{\vartheta_4^2(y)}{\hat{\vartheta}_4^2(0)}\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} + 16\frac{\vartheta_3^2(y)}{\hat{\vartheta}_3^2(0)}\Gamma_{1,1}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right) \tag{2.48}$$

with $y_\pm \equiv y^{\frac{1}{2}} - y^{-\frac{1}{2}}$ and a similar definition for $\tilde{y}_\pm$. $\Gamma_{4,4}$ is the lattice sum over winding-momentum modes on T^4 and

$$\Gamma_{1,1}\begin{bmatrix}g\\h\end{bmatrix} \equiv \sum_{(p_a,w_a)\in(\mathbf{Z},\mathbf{Z}+g)} (-)^{2p_a h} q^{(p_a/R+w_aR)^2}\,\bar{q}^{(p_a/R-w_aR)^2}. \tag{2.49}$$

is the shifted lattice in the direction where we have a shift σ_{p_a}. The hat in the ϑ-functions in the denominators denotes the omission of their zero mode parts, i.e. $\hat{\vartheta}_1(0) \equiv \eta^3$, $\hat{\vartheta}_2(0) \equiv \frac{1}{2}\vartheta_2(0)$ and $\hat{\vartheta}_{3,4}(0) \equiv \vartheta_{3,4}(0)$. The completely untwisted sector, common to all three models, corresponds to type IIB partition function on T^5. In the case of a transversal shift an extra $\Gamma_{1,1}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ lattice sum, common to all orbifold group sectors, should be included.

Multiplicities for fundamental string states carrying k units of momenta and N units of windings can be read from (2.46-2.48) once the level matching condition ($N_R = c_R$)

$$kN = N_L - c_L \tag{2.50}$$

is enforced, with N_L, N_R the oscillator level and c_L, c_R the zero point energies.

The fourth fundamental theory that will be relevant in the longitudinal shift case is the toroidal heterotic string with gauge group $SO(32)$ completely broken by Wilson lines. A possible choice of Wilson lines (in a fermionic representation) can be taken to be

$$A_1 \;:\; (\,(+)^{16}, (-)^{16}\,)$$

$$A_2 \ : \ ((+)^8,(-)^8,(+)^8,(-)^8)$$

$$\vdots$$

$$A_5 \ : \ ((+),(-),(+),(-),....(+),(-)) \qquad (2.51)$$

Alternatively one can represent this model as a Z_2^5 orbifold of the heterotic string on T^5, where the Z_2 generators act simultaneously as a shift in one of the five circles and on the $SO(32)$ lattice by a shift specified by (2.51). The fundamental string partition function can then be written as

$$\begin{aligned} Z_{IV}(q,y,\tilde{y}) &= \frac{1}{2^5}\frac{y_-^2\tilde{y}_-^2}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^{18}}\left(\frac{1}{2}(\vartheta_2^{16}+\vartheta_3^{16}+\vartheta_4^{16})\,\Gamma_{5,5}\begin{bmatrix}0\\0\end{bmatrix}\right.\\ &\left.+\vartheta_3^8\vartheta_4^8\,\Gamma_{5,5}\begin{bmatrix}0\\\epsilon_i\end{bmatrix}+\vartheta_4^8\vartheta_2^8\,\Gamma_{5,5}\begin{bmatrix}0\\\epsilon_i\end{bmatrix}+\vartheta_3^8\vartheta_2^8\,\Gamma_{5,5}\begin{bmatrix}\epsilon_i\\\epsilon_i\end{bmatrix}\right) \end{aligned} \qquad (2.52)$$

where a sum over $\epsilon_i = 0, \frac{1}{2}$ with $i = 1, 2..5$ is always understood. We denote by $\Gamma_{5,5}\left[{g_i \atop h_i}\right]$ the lattice built out from five copies of (2.45) with twists specified by g_i, h_i. Notice that between all the Z_2^5 orbifold group elements only the ϵ_i-projection (for a fixed ϵ_i) leads to a non-trivial result in the ϵ_i-twisted sector.

2.4.2 D1(D5)-momentum bound states

In this subsection we compare the CFT results (long string sector) for multiplicities of BPS excitations, i.e. $\bar{\Delta} = 0$ in the pure D1(D5)-p bound state systems with the predictions from the fundamental string partition functions (2.46-2.48) in the dual theory. The partion function is evaluated using the CFT proposals in table 2.1. As explained before, only the longest string sector, $M = 1$ in (2.40) in the orbifold CFT is relevant to a counting of one-particle states. The results generalize a similar analysis in [51].

In the transversal shift case the CFT proposals for a description of a pure D1-p or D5-p systems are associated to the untwisted sector of diagonal Z_2 orbifolds of $(R^4 \times T^4)^N/S_N$. Specifying to the long string in (2.40), with $g_0 = 0, h_0 = \frac{1}{2}$, one is left with

$$\frac{1}{2}(\mathcal{Z}_{long}\begin{bmatrix}0\\0\end{bmatrix}+\mathcal{Z}_{long}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}) = \sum C_+\{0\}(kN,\ell,\tilde{\ell})\,q^k\,y^\ell\,\tilde{y}^{\tilde{\ell}} \qquad (2.53)$$

where $C_\pm\{g\}(kN,\ell,\tilde{\ell})$ are the expansion coefficients for a single copy ($N = 1$ or $N' = 1$) in table 2.1. We have performed the sum over $s = 0,...N-1$ that projects the sum onto states satisfying the level matching condition $k = \frac{\Delta}{N} \in \mathbf{Z}$. Charges and multiplicites for a bound state of N D1 or D5 branes carrying k units of momenta p_1 are then described in each theory by the corresponding expansion coefficients $C_+\{0\}(kN,\ell,\tilde{\ell})$. Noticing that the $N=1$ or $N'=1$ CFT's, and therefore their $C_+\{0\}$ coefficients, in table 2.1 coincide in each of the cases with their fundamental descriptions. We conclude that the D1(D5)-p proposed CFT's reproduce the multiplicities of untwisted states (even windings) with even momenta in the corresponding fundamental theory (2.46-2.48). This is precisely what one would expect from the duality map, since the image of the pure bound state, $p_6 = F_6 = 0$, carries no windings, no momenta along the shift.

A similar result can be found in the longitudinal shift case σ_{p_1}. Now the long string sector in (2.40), with $g_0 = \frac{1}{2}, h_0 = 0$ leads to

$$\mathcal{Z}_{long}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} = \sum_{k\in\mathbf{Z}} C_+\{\frac{N}{2}\}(kN,\ell,\tilde{\ell})\, q^k\, y^\ell\, \tilde{y}^{\tilde{\ell}} \sum_{k\in\mathbf{Z}+\frac{1}{2}} +C_-\{\frac{N}{2}\}(kN,\ell,\tilde{\ell})\, q^k\, y^\ell\, \tilde{y}^{\tilde{\ell}} \tag{2.54}$$

This is again in complete agreement with (2.46-2.48). Even(odd) fundamental winding states are mapped to bound state involving an even(odd) number N or N' of D-branes and their multiplicities are described by $C_\pm$ according to whether the level k (momentum in the fundamental side) is integer or half-integer.

2.4.3 D1-D5 bound states

In this subsection we consider D1-D5 bound state systems. In order to compare multiplicities of excitations of the bound state with those of fundamental strings (pure $F_1 - p_1$) one should restrict the attention to ground states $\Delta = \bar{\Delta} = 0$ in both left and right moving side of the CFT. Once again the results obtained in the presence of transversal shifts results can be expressed as Z_2 orbifolds

$$Z(p,y,\tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{cm}\begin{bmatrix}0\\h\end{bmatrix}(y,\tilde{y})\, Z_{Sym}\begin{bmatrix}0\\h\end{bmatrix}(p,y,\tilde{y}) \tag{2.55}$$

of the type IIB result

$$Z_{cm}\begin{bmatrix}0\\0\end{bmatrix}(p,y,\tilde{y})\, Z_{Sym}\begin{bmatrix}0\\0\end{bmatrix}(p,y,\tilde{y}) = y_-^2\tilde{y}_-^2\, \frac{\vartheta_1^2(y|p)\vartheta_1^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)\eta^6(p)}. \tag{2.56}$$

associated to the symmetric product space $R^4 \times T^4 \times (T^4)^N/S_N$. We denote by Z_{cm} the contribution coming from the center of mass, while Z_{Sym} will be associated to the symmetric product of the T^4's torii. Plugging the CFT data

$$\begin{aligned}
\hat{T}^4: &\quad \sum_{\ell,\tilde{\ell}} C^I_{\pm}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} = \frac{1}{2}\,(\tilde{y}_-^2\, y_-^2 \pm \tilde{y}_-^2\, y_+^2) \\
\tilde{T}^4: &\quad \sum_{\ell,\tilde{\ell}} C^{II}_{\pm}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} = \frac{1}{2}\,(\tilde{y}_-^2\, y_-^2 \pm \tilde{y}_+^2\, y_+^2) \\
\bar{T}^4: &\quad \sum_{\ell,\tilde{\ell}} C^{III}_{\pm}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} = \frac{1}{2}\,(\tilde{y}_-^2\, y_-^2 \pm \tilde{y}_+^2\, y_-^2)
\end{aligned} \tag{2.57}$$

in the symmetric product formulas (2.45) yields the following for the partition function results

$$\begin{aligned}
Z_I\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,y,\tilde{y}) &= \frac{1}{2}\, y_+^2 \tilde{y}_-^2\, \frac{\vartheta_2^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)}\, \frac{\vartheta_1^2(y|p)}{\vartheta_2^2(0|p)} \\
Z_{II}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,y,\tilde{y}) &= \frac{1}{2}\, y_-^2 \tilde{y}_-^2\, \frac{\vartheta_2^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)}\, \frac{\vartheta_2^2(y|p)}{\eta^6(p)} \\
Z_{III}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,y,\tilde{y}) &= \frac{1}{2}\, y_-^2 \tilde{y}_+^2\, \frac{\vartheta_1^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)}\, \frac{\vartheta_2^2(y|p)}{\vartheta_2^2(0|p)}
\end{aligned} \tag{2.58}$$

where the contributions from the center of mass factors $Z^I_{cm}(y,\tilde{y}) = y_-^4 \tilde{y}_+^4$, $Z^{II}_{cm}(y,\tilde{y}) = Z^{III}_{cm}(y,\tilde{y}) = y_-^2 y_+^2 \tilde{y}_-^2 \tilde{y}_+^2$ have been included.

Together with (2.56) the results (2.58) for the D1-D5 bound state degeneracies (2.58) reproduce the multiplicities (2.81) for untwisted fundamental strings with $p_6 = 0$ as required by the U-duality chain 1.2.

Finally let us compute the D1-D5 bound state spectrum on $T^4 \times S^1/I_4\sigma_{p_1}$ with σ_{p_1} a longitudinal shift. The relevant CFT data is now given in terms of the expansion coefficients for $\tilde{T}^4$ where now the fields are antiperiodic

$$\begin{aligned}
\sum_{\ell,\tilde{\ell}} C^{II}_{\pm}\{0\}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} &= \frac{1}{2}\,(\tilde{y}_-^2\, y_-^2 \pm \tilde{y}_+^2\, y_+^2) \\
\sum_{\ell,\tilde{\ell}} C^{II}_{+}\{\tfrac{1}{2}\}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} &= 16 \\
\sum_{\ell,\tilde{\ell}} C^{II}_{-}\{\tfrac{1}{2}\}(0,0,\ell,\tilde{\ell})\, y^\ell \tilde{y}^{\tilde{\ell}} &= 0
\end{aligned} \tag{2.59}$$

and $Z_{cm} = 16\, y_-^2 \tilde{y}_-^2$. Plugging in (2.45) we are left with

$$\begin{aligned}
Z_{II}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(p,y,\tilde{y}) &= 16\, y_-^2 \tilde{y}_-^2\, \frac{1}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^2}\, \frac{\eta^8}{\vartheta_4^8(0)} \\
&= \frac{1}{2^4} y_-^2 \tilde{y}_-^2\, \frac{1}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^{18}}\, \vartheta_2^8(0)\vartheta_3^8(0)
\end{aligned} \tag{2.60}$$

This is in complete agreement with the degeneracies of heterotic fundamental string states (2.52) coming from the twisted sector once the level matching condition (2.50) is imposed. Notice that the expansion of (2.60) reproduce both signs in (2.52), p_1 even or odd, according to whether we expand in integer or half-integer powers of p. That only states in the twisted sector (odd windings) are relevant to the comparison is due to the fact that the proposed CFT are valid only for a single fivebrane the latter is mapped to a single unit of winding mode in the fundamental string descriptions. One can however, test multiplicities in the untwisted sector (with $p_1 = 1$) by going from step **C** to step **B** after four T-dualities under which the D1 and D5 charges are exchanged within model II. The CFT description of those D1-D5 states is, of course, the same as before and multiplicities are again given by (2.60). The fundamental string multiplicities on the other hand lead to apparently two very different results depending on whether we consider states with F_1 odd (twisted sector) or even (untwisted sector). In the former case one finds again (2.60) in agreement with the duality predictions. The multiplicities for even F_1, on the other hand, can be read from

$$Z^{p_1=1}_{F_1-even}(q, y\ 's) = \frac{1}{2^5} \frac{y_-^2 \tilde{y}_-^2}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^{18}} \left(\frac{1}{2}(\vartheta_2^{16} + \vartheta_3^{16} + \vartheta_4^{16}) - \vartheta_3^8 \vartheta_4^8 \right) \qquad (2.61)$$

Fortunately expression (2.61) coincides with (2.60) after simple manipulations of ϑ-identities. The fact that the heterotic dual model treats on the same footing winding and momentum modes as required by the U-duality chain to models **B** and **C**, can be considered as a further support to the consistency of the whole picture.

One can try to apply a similar analysis to the D1-D5 systems in models I and III with a longitudinal shift, but one immediately runs into problems. The ground states of the natural CFT proposals in table 2.2 are in these cases either tachyonic or massive and a naive application of the elliptic genus formula leads to non-sensible results. A proper description of these (4,0) D1-D5 systems remains as one of the exciting directions for a future research.

2.4.4 Three-charge systems

We will restrict the discussion of three-charge systems to the transversal shift case, since, as we mentioned before, the CFT description of models I and III in the longitudinal shift case is problematic, due to the presence of tachyonic states.

To extract the multiplicities for three-charge systems from our elliptic genera, we restrict the right-moving part on the ground state ($\bar{\Delta} = 0$) and excite the momentum on the left-moving part (which is non-supersymmetric in models I and III). The resulting elliptic genera will be of the form:

$$Z(p,q,y,\tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{cm}\begin{bmatrix}0\\h\end{bmatrix}(q,y,\tilde{y})\, Z_{Sym}\begin{bmatrix}0\\h\end{bmatrix}(p,q,y,\tilde{y}) \tag{2.62}$$

Notice that the q^0 term in Z corresponds to the partition function of the fundamental sides for the D1-D5 systems and it is given in (2.58) for the three theories. Denoting this by $Z_F(p)$, it is convenient to rewrite Z in (2.62) as:

$$Z(p,q,y,\tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{cm}\begin{bmatrix}0\\h\end{bmatrix}(q,y,\tilde{y})\, \hat{Z}_F\begin{bmatrix}0\\h\end{bmatrix}(p,y,\tilde{y})\hat{Z}_{Sym}\begin{bmatrix}0\\h\end{bmatrix}(p,q,y,\tilde{y}) \tag{2.63}$$

where, as before, the hat denotes omission of zero modes.

We have already stressed in section 2 of this chapter that U-duality puts severe constraints on the multiplicities in the three-charge cases. From table 1.2, by comparing columns **F** and **G**, we see that models I and III get exchanged together D1 and D5 charges. The CFT's we have proposed trivially satisfy this symmetry, since in this case $N=1$ in table 2.2 for the two corresponding CFT's and therefore both give $\hat{R}^4 \times \tilde{T}^4 \times \hat{T}^4$.

A less trivial constraint comes from comparison of columns **C** and **F**: in this case the 3-charge system (D1, D5, p) of model II is mapped to (p, D5, D1) in model III.

Let us consider the case of a single D5-brane in theories II and III: we see from table 1.2, looking at columns **C** and **F**, that II is mapped to III and D1 is mapped to KK momentum and viceversa. This means that the full elliptic genera corresponding to models II and III must get exchanged if we exchange q with p. Notice that we are comparing a $\mathcal{N}=(4,4)$ theory (model II) with a $\mathcal{N}=(4,0)$ theory (model III). Although in the previous discussions we have set $\bar{\Delta}=0$ while keeping $y, \tilde{y}$ arbitrary, actually the quantity that is invariant under deformations of the SCFT (the elliptic genus) is obtained by setting $\tilde{y}=1$. However, in order to soak up the fermionic zero modes we will take two derivatives in $\tilde{y}$ and then set $\tilde{y}=1$. This will put the right-moving sector (which is supersymmetric in both theories) on the ground state. The resulting expressions are:

$$Z_{II}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(p,q,y) = \frac{1}{2}y_-^2\, y_+^2 \frac{\eta^6(q)\hat{\vartheta}_2^2(y|q)}{\hat{\vartheta}_1^2(y|q)\hat{\vartheta}_2(0|q)}\, \frac{\hat{\vartheta}_2^2(y|p)\hat{\vartheta}_2(0|p)}{\hat{\vartheta}_1^2(y|p)\eta^6(p)}\, \hat{\mathcal{Z}}_{II}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(p,q,y)\,, \tag{2.64}$$

and

$$Z_{III}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,q,y) = \frac{1}{2}y_-^2\,y_+^2\frac{\hat\vartheta_2^2(y|q)\hat\vartheta_2(0|q)}{\hat\vartheta_1^2(y|q)\eta^6(q)}\,\frac{\eta^6(p)\hat\vartheta_2^2(y|p)}{\hat\vartheta_1^2(y|p)\hat\vartheta_2(0|p)}\,\hat{\mathcal{Z}}_{III}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,q,y)\,, \quad (2.65)$$

with, omitting indices,

$$\hat{\mathcal{Z}}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(p,q,y = \prod_{n,m\geq 1}\left(\frac{1+p^nq^my^l}{1-p^nq^my^l}\right)^{\frac{1}{2}C\left[\substack{0\\ \frac{1}{2}}\right](nm,l)}. \quad (2.66)$$

(2.66) follows from (2.45), using the identities:

$$\begin{aligned}\sum_{\tilde l} C_+0(m,l,\tilde l) &= -\sum_{\tilde l} C_-0(m,l,\tilde l)\\ &= C\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(m,l)\,.\end{aligned}$$

The U-duality requirement that, under (p,q) exchange (2.64) is exchanged with (2.65) implies

$$C_{II}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(m,l) = C_{III}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(m,l)\,, \qquad form \geq 1. \quad (2.67)$$

This requirement is not satisfied by the proposed CFT's for theories II and III.

Notice that in terms of theta functions:

$$\begin{aligned}\sum_{m,l} C_{II}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(m,l)\,q^m\,y^l &= 4\frac{\vartheta_2^2(y|q)}{\hat\vartheta_2^2(0|q)}\,,\\ \sum_{m,l} C_{III}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(m,l)\,q^m\,y^l &= 4\frac{\vartheta_1^2(y|q)}{\hat\vartheta_2^2(0|q)}\,.\end{aligned} \quad (2.68)$$

from which it follows that $C_{II}\left[\substack{0\\ \frac{1}{2}}\right](m,l) = C_{III}\left[\substack{0\\ \frac{1}{2}}\right](m,l)$ for m odd.

Partition functions for symmetric product CFT's have well defined modular properties as functions of q in a power series expansion in p. However, the other way around is in general not true: it is true for model I (and for D1-D5 in the toroidal case), but not for models II and III. So. it seems that the U-duality requirement of mapping the elliptic genus in one theory to the one in the dual theory is not satisfied by our CFT's.

To make this point somewhat more general, let us consider the partition function (2.63), after a (p,q) exchange. We will get an expression $\tilde{Z}$:

$$\tilde{Z}(p,q,y,\tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{cm}\begin{bmatrix}0\\h\end{bmatrix}(p,y,\tilde{y})\,\hat{Z}_F\begin{bmatrix}0\\h\end{bmatrix}(q,y,\tilde{y})\hat{Z}_{SYM}\begin{bmatrix}0\\h\end{bmatrix}(p,q,y,\tilde{y}) \tag{2.69}$$

where we have used the symmetry of $\hat{Z}_{SYM}$ under (p,q) exchange[2] . Now let us expand $\tilde{Z}$ to first order in p: this corresponds in the dual model to $N=1$, and using (2.45), gives:

$$\tilde{Z}(p,q,y,\tilde{y}) = p\cdot\sum_{h=0,\frac{1}{2}} Z_F\begin{bmatrix}0\\h\end{bmatrix}(q,y,\tilde{y})(\hat{C}_{cm}(1,\ell,\tilde{\ell})\,y^\ell\tilde{y}^{\tilde{\ell}}+\sum_{n>0}C(n,\ell,\tilde{\ell})q^n\,y^\ell\tilde{y}^{\tilde{\ell}})+\dots, \tag{2.70}$$

where the C's are as usual the coefficients of the q-expansion of the corresponding partition function. From this equation it is clear that only in the case $\hat{C}_{cm}(1)=C(0)$ the above formula will have a well defined modular properties.

This problem also affects the seemingly well understood case of the (4,4) CFT $\mathcal{R}^4\times(K3)^N/S_N$, describing the D1-D5 system in type IIB on $K3\times S^1$. Using type II/heterotic duality in 6 dimensions, one can relate type IIB on $K3\times S^1$ to type IIB on $S^1\times T^4/\Omega I_4$, while exchanging D1 and KK charges. Thus for D5 charge 1, this amounts to exchanging (p,q) (at order $\bar{q}^0$) in the corresponding elliptic genus. Notice that in this case $Z_F(p)$ is just the bosonic oscillator part of the heterotic string and clearly $\hat{Z}_{SYM}(p,q)$ is symmetric under (p,q) exchange [49, 53, 10]. However it is also easy to see, for instance, that the coefficient of q^1 of the elliptic genus does not have a well defined modular property as a function of p. Finally, the same problem is present in the longitudinal shift case for the model II that we have studied before, although $\hat{Z}(p,q)$ is not (p,q) symmetric in this case.

2.5 One-loop effective gauge couplings

In this section we study the (T,U) moduli dependences of one-loop threshold corrections to $\mathcal{F}^{2k+4}$ gauge couplings in low energy effective action associated to the four dimensional string compactifications with sixteen supercharges under consideration. (T,U) are the Kahler and complex structure moduli of a T^2 along directions 1 and

[2]This symmetry is not there for the case $g_\phi=1/2$, which corresponds to the longitudinal shift case, second and third expressions in (2.45)

6. Aim of this section is to extract this information for some definite combinations of the eight field strengths:

$$\begin{aligned}\mathcal{F}^{\pm}_{Li} &\equiv \partial_{[\mu}(G_{\nu]i} + B_{\nu]i})^{\pm}\\ \mathcal{F}^{\pm}_{Ri} &\equiv \partial_{[\mu}(G_{\nu]i} - B_{\nu]i})^{\pm}\end{aligned} \tag{2.71}$$

arising from KK-reduction of the six-dimensional metric and antisymmetric tensor to $D = 4$, with $i = 1, 6$ and $\pm$ standing for (anti-)self-dual four-dimensional two forms. We will also map the one-loop results to two-charge D-instanton contributions in the non-perturbative descriptions defined by table 1.1.

We will introduce a complex (Euclidean) notation for spacetime super-coordinates

$$\begin{aligned}Z^1 &= \frac{1}{\sqrt{2}}(X^0 + iX^3) \quad Z^2 = \frac{1}{\sqrt{2}}(X^1 + iX^2)\\ \chi^1 &= \frac{1}{\sqrt{2}}(\psi^0 + i\psi^3) \quad \chi^2 = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)\end{aligned} \tag{2.72}$$

with barred quantities given by the complex conjugates.

The moduli dependence will be extracted from the string amplitudes:

$$\mathcal{A}_{2k+4} = \langle \prod_{i=1}^{k+2} V(p_1, \xi_i) V(\bar{p}_2, \tilde{\xi}_i) \rangle \tag{2.73}$$

where for simplicity we choose a kinematical configuration where half of the vertices carry momentum p_1 and the other half $\bar{p}_2$. In addition all the vertices will be chosen with definite (anti-) self-duality properties. The computation and notation follow closely [54].

The vertex operators for the gauge field strengths (2.71) are given by

$$\begin{aligned}V_L(p,\xi) &= \int d^2z\, \xi_{\mu i}(\partial X^\mu - ip\chi\chi^\mu)\,(\bar{\partial}X^i - ip\tilde{\chi}\tilde{\bar{\chi}}^i)\, e^{ipX}\\ V_R(p,\xi) &= \int d^2z \xi_{\mu i}(\partial X^i - ip\chi\chi^i)\,(\bar{\partial}X^\mu - ip\tilde{\chi}\tilde{\chi}^\mu)\, e^{ipX}\end{aligned} \tag{2.74}$$

Notice that each vertex carry at least one power of space-time momentum p_μ, and therefore to the order of momentum we are interested in we can keep only linear terms in p_μ.

More precisely, the representative of such couplings in the three models are indicated in the table below:

II_F	$\xrightarrow{T_{15}ST_{2345}S}$	I_F	$\xrightarrow{T_{15}ST_{2345}S}$	III_F
$(-)^{F_L}\sigma_6$		$(-)^{F_L}I_4\sigma_6$		$I_4\sigma_6$
F_1		P_1		NS_{12345}
NS_{12345}		F_1		P_1
P_1		NS_{12345}		F_1
$(\mathcal{F}_R^+)^2(\mathcal{F}_R^-)^2(\mathcal{F}_L^+)^{2k}$		$(\mathcal{F}_R^+)^2(\mathcal{F}_L^+)^2(\mathcal{F}_L^+)^{2k}$		$(\mathcal{F}_L^-)^2(\mathcal{F}_R^-)^2(\mathcal{F}_L^+)^{2k}$

where $\mathcal{F}_{L,R}^{\pm}$ are defined in (2.71) and σ_6 is a shift of order 2 along the 6th direction..

The couplings in the table are special in the sense that they receive (at each step in the above table) contributions only from right moving ground states ("BPS saturated"). This can be seen by noticing that the insertions exactly soak up the fermionic zero modes in the right moving part of the string amplitude (formulas (2.81) below). This will be understood in most of our discussion.

In the model II_F this corresponds to the case where the eight right moving fermionic zero modes (once the sum over spin structure have been performed) are soaked up by exactly four insertions $(\mathcal{F}_R^+\mathcal{F}_R^-)^2$ of right moving gauge fields. Similarly, in the models I_F and III_F four right-moving insertions of $\mathcal{F}_R^+$ and $\mathcal{F}_R^-$ are needed in order to get a non-trivial result. Vertex operators can therefore be replaced by the effective ones (self-dual components)

$$\begin{aligned}
\vec{V}_L^+(p_1) &= ip_1\vec{P}_L\tau_2\int d^2\sigma(Z^1\bar{\partial}Z^2-\tilde{\chi}^1\tilde{\chi}^2)+\dots\\
\vec{V}_L^+(\bar{p}_2) &= i\bar{p}_2\vec{P}_L\tau_2\int d^2\sigma(\bar{Z}^2\bar{\partial}\bar{Z}^1-\bar{\tilde{\chi}}^2\bar{\tilde{\chi}}^1)+\dots\\
\vec{V}_R^+(p_1) &= ip_1\vec{P}_R\tau_2\int d^2\sigma(Z^1\partial Z^2-\chi^1\chi^2)+\dots\\
\vec{V}_R^+(\bar{p}_2) &= i\bar{p}_2\vec{P}_R\tau_2\int d^2\sigma(\bar{Z}^2\partial\bar{Z}^1-\bar{\chi}^2\bar{\chi}^1)+\dots
\end{aligned}\qquad(2.75)$$

where we have grouped the components $P_L^i=\partial X^i$, $P_R^i=\bar{\partial}X^i$ with $i=1,2$ into a two-dimensional vector and $\partial=\frac{1}{\tau_2}(\partial_{\sigma_2}-\bar{\tau}\partial_{\sigma_1})$. Similar expressions are given for anti-self-dual components, replacing $Z^2,\chi^2,\tilde{\chi}^2$ by their complex conjugates.

From the expressions for the effective vertex operators in (2.75), we see that their insertion in the correlator (2.73) amounts to insert factors $\vec{P}_{L,R}$.

Since the vertices are quadratic in the quantum fluctuations one can exponentiate them into a generating function

$$\mathcal{G}_{\vec{a},\vec{b}}(v,w) = \langle e^{-S_0 - \vec{v}\cdot\vec{V}_L - \vec{w}\cdot\vec{V}_R}\rangle = \int \frac{d^2\tau}{\tau_2^3} \sum_{g,h} q^n \, C\begin{bmatrix} g \\ h \end{bmatrix}(n, \ell\,'s)\, \Gamma_{d,d}\begin{bmatrix} g \\ h \end{bmatrix}(\ell\cdot v, \ell\cdot w) \quad (2.76)$$

with S_0 the free string action, $q = e^{2\pi i\tau}$ (τ the genus-one worldsheet modulus) and $\vec{v}_\pm, \vec{w}_\pm$ are sources for the eight U(1) gauge fields. Scalar product is defined as usual by $\vec{a}\,\vec{b} = a_1 b_1 + a_2 b_2$, while dot product stands for $a\cdot b = a_+ b_+ + a_- b_-$. In the right-hand side we have introduced the notation $v_\pm \equiv \vec{v}_\pm \vec{P}_L$ and a similar definition for $w_\pm$ with $\vec{P}_L$ replaced by $\vec{P}_R$. Vertex insertions are defined by $\vec{v}_\pm, \vec{w}_\pm$-derivatives of (2.76).

Finally we denote as before by $C\left[{g \atop h}\right](n,\ell)$ with $\ell \equiv (\ell, \tilde{\ell}, \ell_*, \tilde{\ell}_*) = (\ell^+, \ell^-, \ell^+_*, \ell^-_*)$ the coefficients in the expansion of the partition function which includes Wilson lines:

$$\begin{aligned} G\begin{bmatrix} g \\ h \end{bmatrix}(q, y\,'s) &= \mathrm{Tr}'_{g-tw}\left[\Theta^h\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24}\, y^{2J_0^3}\, \tilde{y}^{2\tilde{J}_0^3}\, y_*^{2J_0^3}\, \tilde{y}_*^{2\tilde{J}_0^3}\right] \\ &= C\begin{bmatrix} g \\ h \end{bmatrix}(n,\ell)\, q^n\, y^{\ell} \tilde{y}^{\tilde{\ell}}\, y_*^{\ell_*}\, \tilde{y}_*^{\tilde{\ell}_*} \end{aligned} \quad (2.77)$$

Θ is the Z_2 orbifold generator, $y = e^{\pi i v_-}$, $\tilde{y} = e^{\pi i v_+}$, and similarly for $y_*, \tilde{y}_*$ replacing $\vec{v}_\pm$ by $\vec{w}_\pm$. L_0, $\bar{L}_0$ are the Virasoro generators and J_0^3's are four $SU(2)$ Cartan generators to which the corresponding gauge field couples. Therefore we see that (2.77) has the structure of a helicity supertrace [55, 4]. The four possible twists along the σ and τ directions will be denote by $\left[{g \atop h}\right]$ with $g,h = 0, \frac{1}{2}$. Primes denote omission of the bosonic zero mode contributions which have been displayed explicitly in (2.76). $\Gamma_{d,d}$ is a $\Gamma_{2,2}\Gamma_{4,4}$ lattice sum for the completely untwisted sectors in the models I_F, III_F and all sectors in the model II_F, while reduced to a $\Gamma_{2,2}$ lattice for all non-trivial twists in models I_F, III_F. Since we are interested only in the (T,U) moduli dependence of the first torus we will always work in the orbits where neither momentum nor winding modes are excited in the $\Gamma_{4,4}$-lattice. The effects of introducing half-shifts in $\Gamma_{d,d}$ lattices have been extensively studied in [56]. The perturbed lattice sum can be written as the sum

$$\begin{aligned} \Gamma_{2,2}\begin{bmatrix} g \\ h \end{bmatrix}(\ell\cdot\vec{v}, \ell_*\cdot\vec{w}) &= \frac{T_2}{\tau_2}\sum_M \eta\begin{bmatrix} g \\ h \end{bmatrix} e^{-\frac{\pi T_2}{\tau_2 U_2}\left|(1\ U)M\left({\tau \atop -1}\right)\right|^2} \\ &\quad \times e^{2\pi i\left[T\det M + (\ell\cdot v^1\ ,\ell\cdot v^2)M\left({\tau \atop -1}\right) - (\ell_*\cdot w^1\ ,\ell_*\cdot w^2)M\left({\bar{\tau} \atop -1}\right)\right]} \end{aligned} \quad (2.78)$$

over worldsheet instantons

$$\begin{pmatrix} X^1 \\ X^6 \end{pmatrix} = M\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \qquad \vec{m} \in \mathbf{Z} + \vec{b}g,\ \vec{n} \in \mathbf{Z} + \vec{b}h, \quad (2.79)$$

where now the entries in M are integer or half-integer depending on the winding (momentum) shift vector $\vec{a}$ ($\vec{b}$). Finally the lattice sum is weighted by the $\eta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ phase

$$\eta \begin{bmatrix} g \\ h \end{bmatrix} = e^{-4\pi i \vec{a}\vec{b}\, gh - 2\pi i \vec{a}\,(h\vec{m} - g\vec{n})} \tag{2.80}$$

We will consider only shifts involving either a pure momentum or pure winding i.e $\vec{a}\,\vec{b} = 0$.

Evaluating the partition functions (2.77) in the three orbifold models described, above one is left with (after spin structure sums):

$$\begin{aligned}
G_{II_F}\begin{bmatrix} g \\ h \end{bmatrix}(q, \mathbf{y}) &= \frac{\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(y)\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(\tilde{y})}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^6}\,(y_*^{\frac{1}{2}} - y_*^{-\frac{1}{2}})^2(\tilde{y}_*^{\frac{1}{2}} - \tilde{y}_*^{-\frac{1}{2}})^2 \\
G_{I_F}\begin{bmatrix} g \\ h \end{bmatrix}(q, \mathbf{y}) &= \frac{\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(y)\vartheta_1^2(\tilde{y})}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\hat{\vartheta}\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(0)}\,(y_*^{\frac{1}{2}} - y_*^{-\frac{1}{2}})^2 \\
G_{III_F}\begin{bmatrix} g \\ h \end{bmatrix}(q, \mathbf{y}) &= \frac{\vartheta_1^2(y)\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(\tilde{y})}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\hat{\vartheta}\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]^2(0)}\,(\tilde{y}_*^{\frac{1}{2}} - \tilde{y}_*^{-\frac{1}{2}})^2
\end{aligned} \tag{2.81}$$

The hat on the ϑ-functions in the denominators denotes as before the omission of their zero mode parts, i.e. $\hat{\vartheta}_1(v) \equiv \frac{1}{v}\vartheta_1(v)$, $\hat{\vartheta}_2(v) \equiv \frac{1}{2}\vartheta_2(v)$ and $\hat{\vartheta}_{3,4}(v) \equiv \vartheta_{3,4}(v)$.

Notice the modular invariance of $\mathcal{G}(v, w)$ under the SL(2,Z) transformations

$$\tau \to \frac{p\tau + q}{r\tau + s} \qquad v \to \frac{v}{r\bar{\tau} + s} \qquad w \to \frac{w}{r\tau + s} \qquad M \to M \begin{pmatrix} s & q \\ r & p \end{pmatrix} \tag{2.82}$$

The modular integral (2.76) can then be computed following the standard trick [57], that consists in trading the sum over the M_ϵ matrices in (2.78) by sums over $SL(2, Z)$ representatives integrated in unfolded domains. We will concentrate here in the contributions of non-degenerated orbits ($\det M \neq 0$) for which representatives can be chosen to be:

$$M = \begin{pmatrix} m_1 & n_1 \\ 0 & n_2 \end{pmatrix} \tag{2.83}$$

where $m_1 \in \mathbf{Z} + b_1 g$ and $n_1 \in \mathbf{Z} + b_1 h$ The integral (2.76) is then unfolded to the whole upper half plane.

The modular integral (2.76) is evaluated in appendix C. We keep only the leading order in an expansion around $T_2 \to \infty$ of the integral (2.76) associated in the dual picture to the classical contribution of the D-instanton background. Higher orders can in principle be traced as quantum fluctuations around the instanton background as in [58], but this analysis is beyond the scope of this work.

To leading order in $1/T_2$ the result reads

$$I(a,b) = \sum_{n,m_1,n_2,\ell\ 's} e^{2\pi i(m_1 n_2 T + n\frac{n_2 U}{m_1} + n_2 \ell\cdot\hat{v} - n_2 \ell_*\cdot\hat{w})} \mathcal{P}(a,b) + h.c. \tag{2.84}$$

with $T = T_1 + iT_2, U = U_1 + iU_2$, $\hat{v}_\pm = v^2_\pm - Uv^1_\pm$, $\hat{w}_\pm = w^2_\pm - \bar{U}w^1_\pm$. $\mathcal{P}(a,b)$ is a projection factor defined in Appendix B.

The contribution coming from anti-instantons is given by replacing T, U with $\bar{T}, \bar{U}$ and is denoted in (2.84) by $h.c.$.

The final result can then be written as:

$$I(a,b) = \ln \mathcal{Z}(a,b)\bar{\mathcal{Z}}(a,b) \tag{2.85}$$

with

$$\begin{aligned}
\mathcal{Z}(1,0) &= \prod_{\delta=0}(1 - p^{m_1} q^k \hat{y}^\ell \tilde{\hat{y}}^{\tilde{\ell}} \hat{y}_*^{\ell_*} \tilde{\hat{y}}_*^{\tilde{\ell}_*})^{C_+\{\frac{k}{2}\}(km_1,\ell)} \\
&\quad \times \prod_{\delta=\frac{1}{2}}(1 - p^{m_1} q^k \hat{y}^\ell \tilde{\hat{y}}^{\tilde{\ell}} \hat{y}_*^{\ell_*} \tilde{\hat{y}}_*^{\tilde{\ell}_*})^{C_-\{\frac{k}{2}\}(km_1,\ell)} \\
\mathcal{Z}(0,1) &= \prod_{\delta=0}(1 - p^{2m_1} q^{\frac{k}{2}} \hat{y}^\ell \tilde{\hat{y}}^{\tilde{\ell}} \hat{y}_*^{\ell_*} \tilde{\hat{y}}_*^{\tilde{\ell}_*})^{C_+\{\frac{m_1}{2}\}(km_1,\ell)} \\
&\quad \times \prod_{\delta=\frac{1}{2}}(1 - p^{2m_1} q^{\frac{k}{2}} \hat{y}^\ell \tilde{\hat{y}}^{\tilde{\ell}} \hat{y}_*^{\ell_*} \tilde{\hat{y}}_*^{\tilde{\ell}_*})^{C_-\{\frac{m_1}{2}\}(km_1,\ell)}
\end{aligned} \tag{2.86}$$

where $p = e^{2\pi i T}$ and $q = e^{2\pi i U}$ and we have defined the induced sources $\hat{y} = e^{2\pi i \hat{v}}$, with similar definitions for $\tilde{\hat{y}}, \hat{y}_*$ and $\tilde{\hat{y}}_*$. Similar expressions with T, U replaced by $\bar{T}, \bar{U}$ describe the anti-instanton contributions $\bar{\mathcal{Z}}(a,b)$. Notice that $\mathcal{Z}(1,0)$ and $\mathcal{Z}(0,1)$ in (2.86) get exchanged under the simultaneous exchange of the momentum(b_1)-winding(a_1) shifts and k-m_1 modes, as required by T-duality.

This formula is rather more general than what we really need. Still, depending on the model, a certain number of $\hat{y}$-derivatives should be taken and then the right moving source $\hat{w}$ should be set to zero ($\hat{y}_* = \tilde{\hat{y}}_* = 1$). Notice however that already at this stage one can recognize in $\mathcal{Z}(0,1)$ the symmetric product formula (2.45) for the longitudinal shift elliptic genus. We then conclude that the contributions of D-instantons, as computed in [59, 51, 58] always reproduces the threshold corrections of the fundamental string theory, provided that the orbifold CFT describing the D1(D5)-brane- KK momentum bound state is constructed as a symmetric product of N copies of the fundamental theory in the twisted sector (basic unit of winding).

After acting in (2.86) with the appropriate number [3] of w-derivatives (see [10] for

[3] At least one, in order to get a non-trivial result

similar manipulations) one is left with:

$$\begin{aligned}\hat{\mathcal{Z}}(1,0) &= \frac{1}{2}\sum (-)^{m_1 h}\,\hat{C}\begin{bmatrix} g \\ h \end{bmatrix}(km_1,\ell,\tilde{\ell})\,p^{m_1 s}\,q^{ks}\,\hat{y}^{s\ell}\,\tilde{\hat{y}}^{s\tilde{\ell}} \\ \hat{\mathcal{Z}}(0,1) &= \frac{1}{2}\sum (-)^{kh}\,\hat{C}\begin{bmatrix} g \\ h \end{bmatrix}(km_1,\ell,\tilde{\ell})\,p^{m_1 s}\,q^{ks}\,\hat{y}^{s\ell}\,\tilde{\hat{y}}^{s\tilde{\ell}}\end{aligned} \tag{2.87}$$

where

$$\hat{\mathcal{Z}}(a,b) \equiv \frac{1}{m!n!}\frac{\partial^m}{\partial \hat{y}_*^m}\frac{\partial^n}{\partial \tilde{\hat{y}}_*^n}\mathcal{Z}(a,b)|_{\hat{y}_*=\tilde{\hat{y}}_*=1} \tag{2.88}$$

with $m=2, n=0$ for the model I_F, $m=n=2$ for the model II_F and $m=0, n=2$ for the model III_F. The sum run over $\ell,\tilde{\ell}$ integers, $g,h=0,\frac{1}{2}$, $m_1\in\mathbf{Z}+b_1 g$ and $k\in\mathbf{Z}+a_1 g$. Finally, the coefficients $\hat{C}\left[{g \atop h}\right](\Delta,\ell,\tilde{\ell})$ are similarly defined in terms of the expansion coefficients of (2.81) by

$$\hat{C}\begin{bmatrix} g \\ h \end{bmatrix}(\Delta,\ell,\tilde{\ell}) \equiv \sum_{\ell_*,\tilde{\ell}_*}(-)^{kh}\,\ell_*^m\,\tilde{\ell}_*^n\,C\begin{bmatrix} g \\ h \end{bmatrix}(\Delta,\ell,\tilde{\ell},\ell_*,\tilde{\ell}_*)\,, \tag{2.89}$$

and correspond to the expansion coefficients of the chiral supertraces appearing in (2.81)

Chapter 3

Fivebrane instantons and higher derivative couplings in type I theory

In this chapter we will consider instanton corrections to higher derivative couplings in toroidally compactified type I theory down to four dimensions. The only sources of non-perturbative corrections (to the kind of couplings we will consider) in these string vacua are associated to Euclidean D5-branes entirely wrapping the T^6-torus, whose BPS excitations can be properly described, as we will see, by an $\mathcal{N} = (4,4)$ orbifold CFT. Less supersymmetric instanton configurations would have too many fermionic zero modes to be soaked up by the vertex insertions (typically $\mathcal{R}^2$), while D-string instantons leads to $\mathcal{N} = (8,0)$ effective sigma models [52], where the vertex insertions can soak at most four of the eight left moving fermionic zero modes.

We will study in detail the $\mathcal{R}^2$ couplings and comment about the generalizations to other four and higher derivative "BPS" saturated terms. These couplings are special in that they receive corrections only from states saturating the BPS bounds and they have been extensively studied in many contexts [60]-[61]. Here we will concentrate in the non-perturbative part of the corrections to $D = 4$ effective lagrangians in type I vacua with sixteen supercharges. The instanton sums will be always expressed in terms of an elliptic genus in the effective $Sp(N)$ gauge theory, which encodes the information about masses, charges and multiplicities of $\frac{1}{2}$ BPS excitations in the corresponding D5-brane system. We will compute the "quasi" topological index and show that the form of the instanton corrections to the associated couplings agree with the predictions from the duality to type IIA on $K3 \times T^2$. Interestingly, the CFT description of the infrared limit of the D5-brane world-volume theory reproduces the right multiplicities even for $\frac{1}{4}$-BPS states, associated to D5-KK bound states. We exploit this result in order to show that instanton corrections to certain higher derivative couplings, sensitive to these states, agree once again with the fundamental type IIA results[1].

This chapter is organized as follows. In section 2 we compute D5-instanton corrections to $\mathcal{R}^2$-couplings in type I theory on T^6, and compare the result with the perturbative one obtained in the dual type IIA theory. In section 3, we briefly discuss the extension of such results to other higher derivative terms, which are sensitive to $\frac{1}{4}$ BPS contributions.

3.1 $\mathcal{R}^2$ couplings in type IIA on $K3 \times T^2$/type I on $T^4 \times T^2$

Worldsheet/spacetime instantons in type IIA theory on $K3 \times T^2$/type I theory on T^6 are mapped to each other under the triality map[2]:

$$T_{IIA} = S_H = S_I. \tag{3.1}$$

The subscripts IIA, H, I refer to the type IIA, Heterotic and type I theories, while the complex moduli[3]

$$\begin{aligned} T_{IIA} &= B_{45} + i\sqrt{G} \\ S_H &= a + i\frac{\mathcal{V}_6}{g^2} \\ S_I &= a + i\frac{\mathcal{V}_6}{g}. \end{aligned} \tag{3.2}$$

describe the complexified Kahler structure of the T^2 torus in the type IIA side, and the four dimensional complexified string couplings in the heterotic and type I string vacua respectively with $\mathcal{V}_6$ the T^6-volume. More precisely, $a = \tilde{B}_{456789}$, where $\tilde{B}$ is defined by $d\tilde{B} =^* dB$, B being the second rank, antisymmetric tensor in the corresponding string theory, and $g = e^{\phi}$ the ten-dimensional string coupling.

In this section we will show how the perturbative, world-sheet instanton contribution to the $\mathcal{R}^2$ coupling in the type IIA theory, can be directly reproduced in type

[1] The couplings under consideration are closely related to those studied in [61].

[2] see discussion after (1.17) in chapter 1.

[3] All quantities having dimension of a length will be understood in units of $2\pi\sqrt{\alpha'}$.

I theory, using the effective six dimensional $\mathcal{N}=1$ gauge theory of D5 branes, in the limit where it flows to a two dimensional $\mathcal{N}=(4,4)$ orbifold Conformal Field Theory, after dimensional reduction on a four-torus.

3.1.1 One-loop $\mathcal{R}^2$ couplings in type IIA

Let us start by recalling the results for the moduli dependence $\Delta_{\rm gr}(T,U)$ of $\mathcal{R}^2$-couplings in type IIA theory on $K3\times T^2$ [60, 56] [4]. Perturbative corrections to $\mathcal{R}^2$ terms in $(2,2)$ string vacua are expected to arise only at one-loop level, and depend only on the $T(U)$ complex modulus describing the Kahler(complex) structure of the T^2 torus in the type IIA(IIB) string compactifications. The type IIA result can then be written as [60, 56]

$$\partial_T \Delta_{\rm gr}(T) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \partial_T B_4 \,, \tag{3.3}$$

where B_4, is an index counting the number of $\frac{1}{2}$-BPS string states (the helicity supertrace). The index can be defined as [56]

$$B_4 = \left\langle (\lambda_{\rm L}+\lambda_{\rm R})^4 \right\rangle = \binom{4}{2} \frac{1}{(2\pi i)^2}\, \partial_v^2 \chi(v|\tau)\big|_{v=0} \equiv \frac{6}{16\pi^4}\, \partial_v^2 \partial_{\bar v}^2 Z(v,\bar v)\big|_{v=\bar v=0} \,, \tag{3.4}$$

in terms of the helicity supertrace generating function

$$Z(v,\bar v) = {\rm Tr}'\, q^{L_0-\frac{c}{24}}\, \bar q^{\bar L_0-\frac{\bar c}{24}}\, e^{2\pi i (v\lambda_{\rm L}-\bar v\lambda_{\rm R})} \,. \tag{3.5}$$

$\lambda_{L(R)}$ being the left(right) moving helicity operators, and the prime on Tr means the omission of the space-time bosonic zero mode contributions, and $q\equiv e^{2\pi i\tau}$. One can see that $Z(v,\bar v)$ receives in general contributions from both BPS and non-BPS string states. This is not the case for the chiral supertrace $\chi(v|\tau)$, introduced in (3.4). Indeed, the insertion of two λ_R's in (3.5) precisely soaks up four right-moving fermionic zero modes (after spin structure sums), while massive bosonic and fermionic right moving excitations cancel against each other by supersymmetry, giving as a result the holomorphic function $\chi(v|\tau)$ [5]. This chiral supertrace encodes all the information about BPS multiplicities and charges, and we will refer to it as the "elliptic genus" of the corresponding conformal field theory. Specializing to the case of type IIA string

[4] We will follow the notations and normalizations of [56].

[5] The chiral trace $\chi(v|\tau)$ can be considered as the generating function for the asymmetric supertraces introduced in [61].

theory on $K3 \times T^2$ [6], we get, after summing over the spin structures,

$$Z(v, \bar{v}) = 8 \left| \xi(v) \frac{\vartheta_1^2(\frac{v}{2})}{\eta^6} \right|^2 \sum_{i=1}^{4} \left| \frac{\vartheta_i^2(\frac{v}{2})}{\vartheta_i^2(0)} \right|^2 \Gamma_{2,2} \tag{3.6}$$

where

$$\xi(v) \equiv \frac{sin \pi v}{\pi} \frac{\theta_1'(0)}{\theta_1(v)} \tag{3.7}$$

is the contribution of the spacetime boson coupled to the helicity. Substituting (3.6) in (3.4), one finds [56]

$$B_4 = 36 \; \Gamma_{2,2}. \tag{3.8}$$

The $\Gamma_{2,2}$ lattice sum can be written as

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \sum_{M \in GL(2,Z)} e^{2\pi i T \mathrm{det} M} e^{-\frac{\pi T_2}{\tau_2 U_2} \left| (1 \; U) M \binom{\tau}{-1} \right|^2}, \tag{3.9}$$

where the sum runs over all possible world-sheet instantons

$$\begin{pmatrix} X^4 \\ X^5 \end{pmatrix} = M \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}, \tag{3.10}$$

with worldsheet and target space coordinates σ^1, σ^2 and X^4, X^5 respectively. The modular integration in (3.3) can be performed using the standard trick [57], where integral over the fundamental domain are unfolded to the strip (degenerate orbits $\mathrm{Det} M = 0$) and to the whole upper half plane (non-degenerate orbits $\mathrm{Det} M \neq 0$) and the integrand is a certain $SL(2, Z)$ representative. The final result is [60, 56]

$$\Delta_{\mathrm{gr}}(T) = -36 \log \left(T_2 \left| \eta(T) \right|^4 \right). \tag{3.11}$$

Using the duality relations (3.1) we can rewrite this in terms of the type I variables:

$$\begin{aligned} \Delta_{gr}(S) &= -36 \log \left(S_2 \left| \eta(S) \right|^4 \right) = -36 \log S_2 + 12\pi S_2 \\ &\quad +72 \sum_{N=1} \left(\sum_{M|N} \frac{1}{M} \right) [e^{2\pi i N S} + e^{-2\pi i N \bar{S}}]. \end{aligned} \tag{3.12}$$

where $N|M$ stands for the partitions of N ($N = LM$ with $N, L, M \in \mathbf{Z}$). The first term in (3.12) corresponds to a logarithmic divergence in the weak coupling limit $S_2 = \frac{\nu_6}{\lambda} \to \infty$, the second term come from a disk diagram contribution and the rest

[6] In practice, we consider the T^4/Z_2 orbifold limit of $K3$ as the result will be valid for all values of the $K3$ moduli.

is an infinite sum of D-instanton corrections. The logarithmic divergent term in (3.12) is attributed to an IR divergence, but a complete understanding is still missing (see [63, 64] for details).

It has been shown in [65] that $\mathcal{R}^2$ couplings receive a non-vanishing one-loop contributions from the Klein bottle, annulus and Moebius stipin type I theory. The absence of a one-loop term in the perturbative formula (3.12) suggests that the $\mathcal{R}^2$ result in type IIA should correspond to a combination of $\mathcal{R}^2$ together with other four derivative couplings in the type I side. For instance, the authors of [65] have shown that a suitable combination of $\mathcal{R}^2$ and $\mathcal{F}_1^2\mathcal{F}_2^2$ can account for the discrepancy. One can see that four-derivative couplings involving the dilaton field will lead again to one-loop expressions similar to the closely related $\mathcal{R}^2$-terms, that make them sources of new terms which potentially account for the perturbative discrepancy. Moreover, the duality relation $G^{IIA}_{\mu\nu} = \mathcal{V}_4 e^{-\phi} G^{I}_{\mu\nu}$ suggests that, in the translation of the type IIA $\mathcal{R}^2$-term into type I variables, four-derivative couplings constructed out of the dilaton and volume modulus should indeed be relevant. Our instanton computation will not give a definite answer to this question, but it will support the potential relevance of these latter contributions. The comparison with the perturbative formula will then account only for the form of the instanton sums, leaving an overall coefficient to be accounted for by the right combination of four derivative terms in the type I dual to the type IIA $\mathcal{R}^2$ coupling.

A similar formula (with S_I replaced by S_H in (3.12)) describe the instanton corrections in the heterotic side. The correct weight of the NS5-brane instanton action $e^{2\pi i N S_H}$ were reproduced in [63] from the classical action of heterotic NS-fivebranes wrapped on the T^6 torus, however it is hard to see how the determinant factor $\sum_{N|M} \frac{1}{M}$ can be computed with our current understanding of the NS-fivebrane physics. Fortunately, type I spacetime instantons are associated to the more tractable D-branes, for which a CFT description is at our disposal [7].

3.1.2 D5-brane instanton corrections to $\mathcal{R}^2$-couplings

We are interested in computing the two-graviton correlation function

$$\langle V_g V_g \rangle_{\text{D5-inst}} \tag{3.13}$$

in the background of N Euclidean D5 branes wrapping the T^6 spacetime torus. We take for the spacetime torus the limit in which the volume of a T^4 torus in T^6 becomes very small ($R_i \sim \sqrt{\alpha'}$, $i = 6,7,8,9$) keeping R_4, R_5 fixed. In this limit the D5-brane theory decouples from gravity $\alpha' \to 0$. We will reduce the six-dimensional world-volume theory to two dimensions with light cone coordinates $X^{\pm} = X^4 \pm X^5$.

The classical part of the computation in the D5-instanton background closely follows the one for the heterotic NS-fivebrane [63] with obvious modifications. The low energy effective action describing N spinless Euclidean D5-branes wrapping once the T^6 spacetime torus can be written as [34]

$$\begin{aligned} S_{\mathrm{ND5}} &= NT_5 \int d^6\xi \left[e^{-\phi}\sqrt{\det \hat{G}} + i\tilde{B}_{456789}\right] + \cdots = \\ & -2\pi i N S_I + \frac{2\pi \mathcal{V}_4 U_2}{\lambda} \int d^2 z h_{\mu\nu} \sum_{t=1}^{N} D_z X_t^\mu D_{\bar{z}} X_t^\nu + \ldots \end{aligned} \quad (3.14)$$

where $\mathcal{V}_4$ is the volume of the small T^4 torus, U the complex structure of the large T^2-torus, $T_5 = \frac{1}{(2\pi)^5 \alpha'^3}$ the fivebrane tension [7], S_I is the complexified string coupling constant given by (3.2) and $h_{\mu\nu}$ the quantum fluctuation ($G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) around the flat metric, $\mu, \nu = 0, 1, 2, 3$. We choose the static gauge $X^a = \xi^a$ ($a = 4, 5, .., 9$) for the fivebrane coordinates, where the six-dimensional pullback metric $\hat{G}$ is identified with the spacetime torus metric. The last term in (3.14) represents the coupling of the graviton to the instanton background with covariant derivatives

$$\begin{aligned} D_z X_t^\mu &= \partial_z X_t^\mu - \frac{1}{4} p_\nu S_t \gamma^{\nu\mu} S_t \\ D_{\bar{z}} X_t^\mu &= \partial_{\bar{z}} X_t^\mu - \frac{1}{4} p_\nu \tilde{S}_t \gamma^{\nu\mu} \tilde{S}_t \end{aligned} \quad (3.15)$$

in the $(4,5)$ plane with complex coordinates $z = \xi^4 + U\xi^5$, $\partial_z = \frac{1}{U_2}(\partial_{\xi_4} + U\partial_{\xi_5})$[8]. Quantum corrections around this background are described by an $Sp(N)$ gauge theory defined by the quantization of the massless modes of unoriented open strings ending on the D5 branes.

The computation of the scattering amplitude (3.13) is very similar to the perturbative computation in the one-loop worldsheet instanton background of the last section, with the worldsheet parameter τ replaced by the complex structure U of the target space $(4,5)$ torus. Graviton insertions can be expressed as derivatives of the instanton action (3.14) with respect to the metric fluctuations $h_{\mu\nu}$ $\mu = 0, 1, 2, 3$, bringing down the needed $(4,4)$ fermionic zero modes $S_{0\mathrm{cm}}$, the right power of momenta to

[7] Notice that the T_1, T_5 tensions of type I D-branes are $\frac{1}{\sqrt{2}}$-the corresponding tension in type IIB. As already discussed in chapter 2, the electric-magnetic quantization condition implies then that D5 branes come always in pairs to account for the extra factor of $\sqrt{2}$ [44]. N here counts the number of type I D5-branes which halves the number in the parent type IIB theory.

[8] Strictly speaking, our analysis will be performed in the Minkowski world-volume with time like coordinate ξ^5. At the end we will go back to the Euclidean plane by analytic continuation.

reproduce the $\mathcal{R}^2$ kinematics and an overall $\frac{\mathcal{V}_4^2 U_2^2}{N^2\lambda^2}$ factor. Unlike in references [51, 58], we adopt the canonical normalization $S_t = \frac{1}{\sqrt{N}} S_{\rm cm} + ...$ for the fermionic center of mass ($\sum_{t=1}^N S_t \partial S_t = S_{\rm cm}\partial S_{\rm cm} + ...$), which is responsible for the additional factor of $\frac{1}{N^2}$. The final result can then be written as

$$\Delta_{\rm gr}^{\rm inst} = \frac{\mathcal{V}_4^2 U_2^2}{N^2\lambda^2} \left\langle e^{-S_{\rm class} - S_{SYM}} \right\rangle' = \frac{\mathcal{V}_4^2}{\lambda^2} \sum_N B_4^{ND5}(e^{2\pi i N S} + e^{-2\pi i N \bar{S}}) \qquad (3.16)$$

in terms of the $\frac{1}{2}$-BPS index B_4 (3.4) of the $Sp(N)$ gauge theory. The prime in (3.16) means that the trace does not include the fermionic zero mode part already taken into account by the vertex insertions, while the spacetime bosonic zero mode contribution cancel the U_2^2 factor in the numerator. The overall $\frac{1}{N^2}$ factor in (3.16) has been reabsorbed in $B_4^{\rm ND5}$ in order to make more transparent the comparison with the IIA perturbative result. Finally $S_{class} = -2\pi i N S (2\pi i N \bar{S})$ represents the classical (anti)instanton action.

The rest of this section will be devoted to the computation in the B_4^{ND5} index of the $Sp(N)$ gauge theory. For future reference, we will be slightly more general than what we really need, by determining the whole BPS elliptic genus $\chi(v|\tau)$. We will follow the strategy of [52, 51]. As we have already discussed in chapter 2, the elliptic genus, being invariant under any deformation of the gauge theory, in particular under variation of the string coupling constant, can be evaluated in the regime which is more convenient for our purposes. We will compute it explicitly in the infrared limit, where the theory is expected to flow to an orbifold conformal fixed point.

The low energy effective action associated to a system of N parallel D5-branes in type I theory can be obtained from the more familiar $U(2N)$ gauge theory describing $2N$ D5-branes in the parent type IIB theory. Type IIB D5 brane fields Φ are projected onto $2N \times 2N$ matrices satisfying the symplectic condition [66]

$$\Phi = \pm \Omega \Phi^T \Omega^{-1} \qquad\qquad \Omega = \sigma_2 \times \mathbf{I}_N \qquad (3.17)$$

where $+(-)$ stands for the DD(NN) directions and σ_i are the Pauli matrices. In addition, anomaly cancellation requires the inclusion of 32 D9-branes and the corresponding open string sectors. The resulting two-dimensional field content after dimensional reduction on T^4 is defined by all possible open strings ending on the D5-branes, and is given by

Sector	Bosons	Fermions	$Sp(N)\times SO(32)$
5-5 NN	A_α		$N(2N+1),1$
5-5 NN_I	$a^{A'\tilde{A}'}$	$\epsilon_-^{A\tilde{A}'},\epsilon_+^{AA'}$	$N(2N+1),1$
5-5 DD_E	X^{AY}	$\eta_-^{A'Y},\eta_+^{\tilde{A}'Y}$	$N(2N-1),1$
5-9 ND	h^A	$\rho_-^{A'},\rho_+^{\tilde{A}'}$	$2N,32$

Here the subscript E refers to the 4 directions transverse to the D5-brane whereas I refers to the 4 directions corresponding to the T^4 along which the dimensional reduction is performed. The corresponding isometry groups $SO(4)_E$ and $SO(4)_I$ decompose according to $SO(4)_E = SU(2)_A \times SU(2)_Y$ and $SO(4)_I = SU(2)_{A'} \times SU(2)_{\tilde{A}'}$. A_α is a two-dimensional $Sp(N)$ gauge field and $+(-)$ refer to left-(right-) moving fermions. Fields in the same row are related by $\mathcal{N}=(4,4)$ supersymmetry. We will consider a generic type I background involving Wilson lines on T^4, which break $SO(32)$ down to $U(1)^{16}$. In the strong coupling limit $g\to\infty$ the Coulomb branch of the above gauge theory, the off-diagonal fields get infinite masses and can be integrated out (see [52] for a detailed analysis) leaving a free (up-to a Weyl group orbifolding) conformal field theory in terms of the Cartan components[9]

$$
\begin{array}{ll}
a^{A'\tilde{A}'}, \epsilon_-^{A\tilde{A}'},\ \epsilon_+^{AA'} & \sigma_3 \times \lambda_{\rm N} \\
X^{AY}, \eta_-^{A'Y},\ \eta_+^{\tilde{A}'Y} & \mathbf{I}_2 \times \lambda_{\rm N}
\end{array}
$$

with $\lambda_{\rm N}$ an $N\times N$ matrix with diagonal entries. The Weyl group of $Sp(N)$ is given by the semi-direct product $S_N \ltimes Z_2^N$, with S_N permuting the Z_2's factors, and Z_2's acting as σ_2, and therefore reflecting the fields proportional to σ_3 while leaving invariant the ones proportional to the identity $\mathbf{I_2}$. The breaking of the $Sp(N)$ gauge group down to $S_N \ltimes Z_2^N$ can be understood in two steps. First by giving generic expectation values to the diagonal entries in X (D5-brane positions) we break the group to $Sp(1)^N$ with the Weyl group S_N permuting the branes. One can then further break each $Sp(1)$ to its Weyl subgroup Z_2 by turning on the $SU(2)$ Wilson lines $a^{A'\tilde{A}}$. Alternatively, one can start from the type I theory and perform four T-dualites on the small T^4 directions. The Ω-projection goes under the T-duality map to ΩI_4, introducing 16 5-orientifold planes whose charges are locally cancelled after the inclusion of D5-branes symmetrically distributed over the 16 fixed points. Carrying out the same steps as before on this effective $Sp(N)$ gauge theory, we are left with the N D-string sigma model moving on $(\mathbf{R}^4 \times T^4/Z_2)^N/S_N$

[9] We assume to be away from loci in the moduli space where (5-9) fields can become massless due to cancellations between $SO(32)$ and $Sp(N)$ Wilson lines [39].

The resulting CFT can then be written in terms of a second quantized string theory describing N copies of a type IIA string moving on the target space

$$\mathcal{M} = \left[\mathbf{R}^4 \times T^4/Z_2\right]^N /S_N \tag{3.18}$$

The orbifold partition function as we have seen in chapter 2, is given by a sum over twisted sectors labeled by the conjugacy classes of the $S_N \ltimes Z_2^N$ orbifold group. In particular the sum over conjugacy classes in the permutation group S_N runs over the decompositions

$$[g] = (1)^{N_1}(2)^{N_2}......(s)^{N_s} \tag{3.19}$$

with $\sum_s sN_s = N$. However, as it has been shown in [52], only sectors belonging to the conjugacy classes of the kind $[g] = (L)^M$ (with $N = LM$) will lead to a non trivial contribution to $\chi(v|\tau)$. Sectors with strings of different lengths $[g] = (l_1)^{m_1}(l_2)^{m_2}...$ with $l_1 \neq l_2$ will always contain additional right moving fermionic zero modes leading to vanishing contributions to $\chi(v|\tau)$. The elliptic index $\chi(v|\tau)$ can then be computed using the formula [51] for the N-symmetric product: [10]

$$\chi_N(v|U) = \frac{8}{N}\sum_{L,M}\sum_{s=0}^{L-1} M^{-2}\frac{\xi(Mv|\tilde{U})\theta_1^2(\frac{Mv}{2}|\tilde{U})}{\eta^6(\tilde{U})}\sum_{i=2,3,4}\frac{\theta_i^2(\frac{Mv}{2}|\tilde{U})}{\theta_i^2(0|\tilde{U})} \tag{3.20}$$

where all modular functions are evaluated at the induced space-time complex modulus $\tilde{U} = \frac{MU+s}{L}$. The relative M^{-2} factor is determined by modular transformations in the untwisted sector [51, 58], once the overall N^2 factor brought down by the vertex insertions is taken into account. Let us recall how these relative factors arise in the present canonical normalization. One starts from the trace $Tr_{untwisted}(L)^M$ (see Appendix B) in the untwisted sector where one can unambiguously compute the partition function using the operator formalism. After a modular transformation $\tau \to -\frac{1}{\tau}$, to the $g = (L)^M$-twisted sector we are left with the partition function for M copies of strings of length L ($q \to q^{\frac{1}{L}}$) weighted by an L^{2M} factor. The projection by Z_M permutation elements removes the additional fermionic zero modes, apart from the center of mass zero modes, leading finally to an $(\frac{L}{M})^2$ factor from the uncompact bosons and an M^2-factor from the right moving fermionic Z_M-trace. After including the overall N^2 factor we are left with the M^{-2} result claimed above.

Substituting (3.20) in (3.4), one can see that oscillator contributions from massive

[10] Alternatively one can compute the helicity generating function $Z(v,\bar{v})$ using the more general formula [67] by taking the logarithm of the first $\mathcal{Z}\left[{0 \atop 0}\right]$ in (2.45) and then derive the BPS index from (3.4).

fermionic and bosonic modes cancel against each other leaving the $\frac{1}{2}$-BPS index

$$B_4^{ND5} = 36\frac{1}{N}\sum_{N|L} L = 36\sum_{N|M}\frac{1}{M} \qquad (3.21)$$

Apart from a factor of two, the type I result (3.16) in terms of this B_4 index exactly reproduce the instanton sum in the formula (3.12). The extra factor of $\frac{\mathcal{V}_4^2}{\lambda^2}$, coming from the coupling of the metric to the instanton background (3.16), is cancelled by a similar factor coming from the spacetime measure and the metric used in the $\mathcal{R}^2$ contractions, once the duality map $G^{II}_{\mu\nu} = \frac{\mathcal{V}_4 G^I_{\mu\nu}}{\lambda}$ is taken into account.

One can easily extend the previous results to other four derivative couplings in the $D = 4$ type I effective lagrangians. We can consider for example $\mathcal{F}^4$ terms with $\mathcal{F}$ the $U(1)$ gauge fields coming from the reduction of the metric on the T^2 torus ($F_{\mu\nu} = \partial_{[\mu}G_{\nu]i}$, with $i = 4, 5$). The analysis follows closely our previous one for the $\mathcal{R}^2$ computation, with an effective coupling of the gauge field to the instanton background given in this case by

$$S_{\rm ND5} = -2\pi i N S + \frac{2\pi\mathcal{V}_4}{\lambda}\sum_{t=1}^{N}\left[(h_{\mu5} + U h_{\mu4})D_z X_t^\mu + (h_{\mu5} - \bar{U} h_{\mu4})D_{\bar{z}} X_t^\mu\right] + \ldots \quad (3.22)$$

The four vertex insertions provide then the $(4,4)$ fermionic zero modes needed to get a non trivial answer and the four powers of momenta to reproduce the $\mathcal{F}^4$ kinematics. In addition each $\mathcal{F}_4$ ($\mathcal{F}_5$) insertion carries an additional $\frac{\mathcal{V}_4}{\lambda}(\frac{\mathcal{V}_4 U_2}{\lambda})$ factor from (3.22) (for simplicity we take a rectangular spacetime torus, i.e. $U = iU_2$). The result exactly reproduces the contributions from non-degenerate orbits to the one-loop formula for similar $\mathcal{F}^4$ terms in type IIA. In particular, we can see that the N-instanton contribution to $\mathcal{F}_4^2\mathcal{F}_5^2$ is N^2 times the result previously found for $\mathcal{R}^2$ (3.16). Therefore the mixing of $\mathcal{R}^2$ and $\mathcal{F}_4^2\mathcal{F}_5^2$ cannot account for the absence of the one-loop term in (3.12), without destroying the agreement at the non-perturbative level. On the other hand, one can easily see that four derivative couplings involving the dilaton, mentioned in section (2.1), have the same N dependence of the $\mathcal{R}^2$ term, making them potential candidates to account for the perturbative discrepancy and the factor of 2 in the non-perturbative contribution.

Finally, we would like to stress that the result (3.20) is stronger than what we really need. Indeed, as we have discussed, $\mathcal{R}^2$ couplings in type I theory receive contributions only from the $\frac{1}{2}$-BPS states (3.21). However, one can see that even the degeneracies of $\frac{1}{4}$-BPS states, are correctly reproduced by the CFT elliptic genus (3.20). Indeed according to the type IIA/type I duality map, a fundamental type

IIA string with winding N and momentum k is mapped in the type I theory to a bound state of N D5 branes and k Kaluza-Klein momenta. The masses of the two objects agree according to (3.2). Multiplicities and charges for the bound states can be read from the longest string sector in the CFT elliptic genus (the only orbifold sector representing a true one-particle state as shown in [52]). Restricting to this sector, i.e. $L = N, M = 1$ in (3.20), yields

$$\chi_N(v|\tau) = \frac{8}{N}\sum_{s=0}^{N-1}\frac{\xi(v)\theta_1^2(\frac{v}{2})}{\eta^6}\sum_{i=2,3,4}\frac{\theta_i^2(\frac{v}{2})}{\theta_i^2(0)}(q^{\frac{1}{N}}e^{-\frac{2\pi i s}{N}}) \tag{3.23}$$

The sum over s projects onto states satisfying

$$k \equiv \frac{N_L - c_L}{N} \in \mathbf{Z}$$

which reproduce the level matching condition on the fundamental side (3.6) after putting right moving modes in their ground state ($N_R - c_R = 0$) and identifying k with the KK momentum along the direction where the string wraps. The bound state degeneracies are defined by the coefficients in the expansion of (3.23) in powers of $q^{\frac{N_L - c_L}{N}}$. In particular the ground state multiplicities (q^0 order) count the number of ultrashort supermultiplets associated to bound states of N D5-branes, while degeneracies in the excited left moving part of the CFT (coefficient of q^k with $k > 0$) are associated to bound states of N D5 branes with k units of Kaluza-Klein longitudinal momentum, which sit in intermediate supermultiplets. The expansion clearly coincides with the one for the fundamental string (3.6). We then conclude that the whole spectrum of masses, charges and multiplicities of $\frac{1}{2}$- and $\frac{1}{4}$-BPS excitations in the D5-brane system agree with U-duality predictions. In the next section we will give an application of these results.

3.2 $\frac{1}{4}$-BPS saturated couplings in type I theory

In this section we consider higher derivative couplings which are sensitive to $\frac{1}{4}$-BPS contributions. Instanton corrections to type I thresholds will again translate into an infinite sum of one-loop contributions coming from T^2-wrapping modes of type IIA fundamental strings running in the loop, the novelty being in the fact that now $\frac{1}{4}$-BPS fundamental strings will be the relevant ones. With slight modifications the type IIA perturbative computation follows from the ones appearing in [61] that can be consulted for details and a more complete discussion. We will consider in particular the (4+k)-derivative couplings $\partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L\mathcal{F}_L^k$, with $\mathcal{F}_{\mu\nu\,L} \equiv \partial_{[\mu}(G_{\nu]z} + B_{\nu]z})$ the left moving combination of $U(1)$ gauge fields arising from the reduction of the metric and antisymmetric tensor on T^2.

The relevant string amplitude is given by

$$\mathcal{A}_k = \langle V_L^{k+1} V_{\bar{L}} \rangle \tag{3.24}$$

with vertex operators

$$\begin{aligned} V_L &= (G_{\mu z} + B_{\mu z}) \int d^2z \left(\partial X^\mu - \frac{1}{4} p_\rho S \gamma^{\mu\rho} S\right) \left(\bar{\partial} Z - \frac{1}{4} p_\sigma \tilde{S} \gamma^{z\sigma} \tilde{S}\right) e^{ipX} \\ V_{\bar{L}} &= (G_{\mu\bar{z}} + B_{\mu\bar{z}}) \int d^2z \left(\partial X^\mu - \frac{1}{4} p_\rho S \gamma^{\mu\rho} S\right) \left(\bar{\partial} \bar{Z} - \frac{1}{4} p_\sigma \tilde{S} \gamma^{\bar{z}\sigma} \tilde{S}\right) e^{ipX}. \end{aligned} \tag{3.25}$$

At the order in momentum we are interested in, one can see that the left- moving part in the vertices (3.25) enters only through their zero mode part. Indeed, after soaking up the $(4,4)$ fermionic zero modes the remaining extra k powers of momenta are necessarily cancelled out by the right moving pieces of the vertices (3.25) and therefore the left moving part reduces effectively to the $p_L, \bar{p}_L$ bosonic zero modes of $\bar{\partial} Z$ and $\bar{\partial}\bar{Z}$ respectively. The right moving part can then be replaced by the first order in the momentum effective vertex

$$V_{\text{eff}} = \mathcal{F}_{\mu\nu\,L,\bar{L}} \int d^2z \, (X^\mu \partial X^\nu - \frac{1}{4} S \gamma^{\mu\nu} S) \tag{3.26}$$

The $\mathcal{A}_k$ string amplitudes then reduce to a correlation function of k effective vertices (3.26), which exponentiate to [54, 68]

$$\begin{aligned} \mathcal{A}_k &= \partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L \left\langle p_L^k \tau_2^k \frac{\partial^k}{\partial v^k} \left[e^{-S_{\text{free}} + \frac{v}{\tau_2} V_{\text{eff}}} \right] \right\rangle'_{v=0} \\ &= \partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L\mathcal{F}_L^g \int \frac{d^2\tau}{\tau_2} \tau_2^k \sum_{(p_L, p_R)} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2} p_L^k \, f_k(\tau) \end{aligned} \tag{3.27}$$

with $f_k(\tau)$ holomorphic indices generated by the oscillator contribution $\chi_{\text{osc}}(v|\tau)$ to the type IIA elliptic genus $\chi(v|\tau) = \chi_{\text{osc}}(v|\tau)\Gamma_{2,2}$ through

$$f_k(\tau) \equiv \frac{\partial^k}{\partial v^k} \chi_{\text{osc}}(v|\tau)|_{v=0} \tag{3.28}$$

We have again denoted by a prime the omission of the fermionic zero mode trace in (3.27). We can now follow the results of [65, 61], where the modular integral in (3.27) has been performed for an arbitrary holomorphic function $f_g(\tau)$. The result can be written as

$$\mathcal{A}_k = \partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L\mathcal{F}_{\bar{L}}^k\left(\frac{i}{\pi}\sqrt{\frac{2U_2}{T_2}}\right)^k\sum_{m_1,n_2,n} d(n)\frac{(T_2U_2)^k}{\pi^k(m_1T_2+\frac{n}{m_1}U_1)^{2k}}\frac{\partial^k}{\partial\alpha^k}\frac{\partial^k}{\partial\nu^k}I(\alpha,\nu)|_{\nu=0,\alpha=1}$$

where $d(n)$ are the coefficients in the q-expansion $f_k=\sum_n d(n)q^n$ and

$$\begin{aligned} I(\alpha,\nu) &= \frac{2}{\sqrt{b}}e^{-2\pi\sqrt{\alpha b}(m_1T_2+\frac{n}{m_1}U_2)}e^{-2\pi i\phi} \\ \sqrt{b} &= |n_2+\frac{\nu}{4U_2}| \\ \phi &= n_2m_1T_1-n\frac{n_2}{m_1}U_1-i\nu(\frac{n}{m_1}+m_1\frac{T_2}{U_2}) \end{aligned} \quad (3.29)$$

The expansion in T_2^{II} of the above formula translates into a series of perturbative corrections (subleading orders in S_2^I) around the instanton background. We will consider in the following only the leading order in this expansion. The analysis of subleading quantum corrections to this result can be done following the techniques in [58]. At this order ν derivatives hit the terms proportional to νT_2 and set the remaining ν-dependent terms to zero. The α derivatives hit the exponential leading to an overall $(\pi n_2m_1T_2)^k$ factor. Altogether we are left with

$$\begin{aligned} \mathcal{A}_k &= \partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L\mathcal{F}_{\bar{L}}^k\left(\frac{i}{\pi}\sqrt{\frac{2U_2}{T_2}}\right)^k\sum_{m_1,n_1,n_2}\frac{e^{2\pi in_2m_1T}}{n_2m_1}n_2^kf_k\left(\frac{n_2U+n_1}{m_1}\right)+h.c+\ldots \\ &= \partial\mathcal{F}_L\partial\bar{\mathcal{F}}_L\mathcal{F}_{\bar{L}}^k\frac{e^{2\pi in_2m_1T}}{n_2m_1}\frac{\partial^k}{\partial v^k}\chi\left(\frac{i\sqrt{2U_2}}{\pi\sqrt{T_2}}n_2v|\frac{n_2U+n_1}{m_1}\right)\Bigg|_{v=0}+h.c+\ldots \end{aligned} \quad (3.30)$$

where h.c. stands for hermitian conjugation and dots for the higher orders in T_2 expansion. It is now straightforward to compare this result with the corresponding string amplitudes in the instanton background. Indeed, the partition function for the N D5-instanton in the presence of a background (3.26) is described by the elliptic genus (3.20), whose v-derivatives coincide with the result for fundamental strings (3.30) after obvious identifications.

The presence of perturbative corrections around the instanton background is a new feature of these higher derivative couplings, to be contrasted with the $\mathcal{R}^2$ case. It would be interesting to compare (along the lines of [58]) the fundamental and D-instanton results for these quantum corrections, where the Born-Infeld nature of the instanton couplings are strongly tested. Notice in particular that insertions of $F_{\bar{L}}$ gauge fields appears already at a quantum level. A complete analysis of the quantum subleading terms would determine the moduli dependence of the additional couplings in the effective lagrangian.

Also notice that for the case $g = 4$, the result (3.28) is proportional to $\Gamma_{2,2}$. Indeed $f_4\Gamma_{2,2}$ is nothing but the helicity supertrace $B_6 \equiv \mathrm{Str}\lambda^6$ for type IIA string theory on $K3 \times T^2$, where λ is the 4-dimensional physical helicity. B_6 in this case is therefore only sensitive to short multiplets [63], the intermediate ones give vanishing contribution to this helicity supertrace, however this accident will clearly not repeat for the remaining supertraces generated by $\chi(v|\tau)$.

Conclusions

The main goal of this thesis has been the understanding of the CFT's governing D1-D5 brane systems in a class of models with 16 supercharges as well as for type I on T^4. These models were obtained by orbifolding/orientifolding type IIB theory with freely acting Z_2 elements. Our proposed CFT's involved symmetric products of T^4 with additional Z_2 actions, whose precise form depends on the background in consideration. For backgrounds involving Ω projection, the CFT's turn out to be $\mathcal{N} = (4,0)$.

We have worked out elliptic genus formulae for these modified symmetric products and shown that the resulting multiplicities for D1-D5 bound states were in agreement with those of winding-momentum states in U-dually related theories.

There remain however several open problems, which we have already anticipated in the Introduction and discussed in section 5 of chapter 2.

Probably the most challenging one concerns the issue of three-charge systems. As stressed in chapter 2, our CFT's predict multiplicities which agree with U-duality in the two-charge cases, i.e. in the CFT's of D1/D5 brane system when we restrict to the ground state ($\Delta = \bar{\Delta} = 0$) and in the case of D1(D5)-KK momentum. In general, problems arise when we excite momentum, that is when we let $\Delta \neq 0$. This corresponds to exciting states which preserve 1/4 of the 16 bulk supercharges (in the $(4,0)$ case the momentum is excited in the right-moving, non-supersymmetric sector). As we have noticed in Section 5 of chapter 2, U-duality in this case puts constraints which generically are not satisfied by the proposed CFT's, and this happens in the case of the D1-D5-KK system in type IIB on $K3$, too.

Of course, one could cast doubts on the correcteness of our CFT's for the D1-D5 system, though in fact our "derivation" is on the same level of rigour as that giving the more familiar (and well tested) symmetric product of T^4 for the toroidal type II case. Moreover, as repeatedly stressed, the $K3$ case presents similar problems.

Another problem could be related to the peculiarity of the case $Q_5 = 1$. As we noted in Section 3 of chapter 2, for $Q_5 = 1$ the bifundamentals in the D-term equations are vanishing, only the adjoints describing the positions of D1 branes are non trivial, as if one had a system of Q_1 free D1 strings. In that case the moduli space would be rather the symmetric product of $R^4 \times T^4$ (with some additional Z_2 action), as noted also in [42]. However this does not seem to solve our problem and actually is in contradiction with the two-charge counting.

Perhaps there is a more general physical mechanism behind these discrepancies. One may correctly say that, in general, there is no guarantee that 1/4 BPS states in theories with 16 supercharges should be stable throughout the moduli space. Indeed it is believed [71] that in $\mathcal{N} = 4$ Yang-Mills theory in four dimensions 1/4 BPS dyonic states do indeed decay after crossing regions of marginal stability in the moduli space.

One can analyze the problem in string theory also. That is, one can study the locus of marginal stability for 1/4 BPS states using the corresponding mass formula for $\mathcal{N} = 4$, $D = 4$ string theory:

$$M_{1/4}^2 = \frac{1}{\Im S}(P_e + SP_m) \cdot (P_e + \bar{S}P_m) + 2\sqrt{P_e^2 P_m^2 - (P_e \cdot P_m)^2} \qquad (3.31)$$

where P_e, P_m are electric and magnetic charge vectors, whose dimension depends on the model in consideration (its number of vector multiplets), and S is the complexified string coupling constant. It is then easy to see that the condition of marginal stability is given by a single real equation which defines a locus of real codimenson 1 in the full moduli space. The moduli space is thus divided in two regions and in going from one to the other one necessarily crosses the region of marginal stability. Therefore, any U-duality transformation which maps one theory in one region of moduli space to another theory in the other region, is not guarantee to preserve the multiplicity of the corresponding 1/4 BPS states[11]. In addition, one can also see that when decompactifying from $D = 4$ to $D = 5$, one necessarily sits on the locus of marginal stability.

However we are facing a more subtle problem: indeed the contradictions between symmetric product CFT's and U-duality expectations that we have discussed in Section 5 do not arise for generic 1/4-BPS states, but only for those carrrying $D1 - D5 - KK$ charges and in $(4,0)$ cases the two-charge $(D1 - D5)$ systems are themselves 1/4-BPS.

[11] Another well known example where this phenomenon occurs is of course $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theory, where the region of marginal stability is topologically a circle enclosing the strong coupling region in the two-plane of moduli, and perturbative 1/2 BPS states decay when going from the weak coupling to the strong coupling region [72, 73]

It would be very interesting to find a physical explanation of this problem but we leave this for future investigations.

Another very interesting direction would be the study of the D1-D5 brane system in type I string theory.

In the last section of chapter 2 and chapter 3 we have considered instanton corrections to four and higher derivative couplings in $D = 4$ string vacua with sixteen supercharges. We restricted our attention to couplings for which the Euclidean D5-brane wrapping the T^6 torus represents the only source of instanton corrections. D-string instantons have been extensively studied in [74]-[58] and are fairly well understood. The couplings we have considered are also special in the sense that they are sensitive only to states sitting in short and intermediate multiplets of the $\mathcal{N} = 4$ supersymmetry.

We have worked out the details of the instanton sums for $\mathcal{R}^2$ thresholds in toroidal compactifications of type I string theory. The instanton sums translate, under the duality map, into a sum over wrapping modes of fundamental type IIA strings on the T^2 part of $K3 \times T^2$. We argued that the relevant 6-dimensional $Sp(N)$ gauge theory flows in the infrared to an orbifold conformal fixed point, after dimensional reduction to 1+1 dimensions. The elliptic genus for the orbifold CFT was computed in this limit and shown to reproduce correctly the whole spectrum of $\frac{1}{2}$ BPS masses, charges and multiplicities, as required by type I/type IIA duality. As a consequence, the whole infinite sum of instanton corrections to four derivative couplings agree with the expected result from the IIA fundamental string side.

The proposed CFT's reproduce also the right multiplicities for $\frac{1}{4}$-BPS states associated to bound states of D5 branes and KK momenta in the type I theory and D1(D5)-KK in orbifold/orientifold type IIB theories, providing several more examples of higher derivative couplings where the worldsheet/spacetime instanton correspondence works properly.

Appendix A: $SO(N) \times SO(k)$ D1-D5 gauge theory:

In this appendix we determine the spectrum of open string states living on D1-D5 brane intersections in "type I" like vacuum configurations where the orientifold group action is accompanied by a shift longitudinal to the worldvolume system. Our aim is to show that, unlike the more familiar type I theory where consistency of the underlying open string theory requires that Ω projection acts with a relative sign between the D1 and D5 gauge groups [44], in the presence of a longitudinal shift $SO(N) \times SO(k)$ Chan-Paton assignments are allowed. We adopt the open string descendant techniques systematized in [45] [12]. Being interested in open string theories describing excitations of D-brane bound states rather than vacuum configurations we relax (and generically violate) tadpole cancellation conditions. The discussion is illustrated for model I with orientifold group generated by $\Omega\sigma_{p_1}$, but with minor modifications can be adapted to describe the T-dual model III associated to $\Omega I_4 \sigma_{p_1}$. These vacuum configurations are often called "type I theory without open strings", since the Klein bottle tadpole is removed by the presence of the shift and therefore the inclusion of D9(D5)-branes with their corresponding open string excitations is no longer needed [38]. Although we are mainly interested in the study of pure D1D5 systems, with a little of effort, we can (and we will) include also D9-branes in our analysis. Besides aesthetic reasons the inclusion of D9-branes will help the comparison with more familiar results.

We start by describing the D1-D5-D9 system in the presence of the standard type I orientifold (O9)-plane. We orient D1 and D5 branes along (01) and (012345) planes respectively. For an homogeneous notation it will be convenient to start by wrapping the whole system on a $S^1 \times T^4 \times \tilde{T}^4$ torus, with directions $(1) \times (2345) \times (6789)$, and only at the end take the volume of $\tilde{T}^4$ to infinity. The Annulus, Moebius strip and Klein bottle amplitudes associated to such brane configuration can be written as

[12] For a quick review and applications with notations closer to the one presented here see chapter 2 of [69] and chapter 3 of [70].

$\int \frac{dt}{t^{\frac{3}{2}}} \times$

$$
\begin{aligned}
\mathcal{K} &= \frac{1}{2}\,\rho_{00}(2it)P_1(t)P_4(t)\tilde{P}_4(t) \\
\mathcal{A} &= \frac{1}{2}\left[\rho_{00}\left(\frac{it}{2}\right)\left(M^2\,P_4(t)\tilde{P}_4(t) + k^2\,P_4(t)\tilde{W}_4(t) + N^2\,W_4(t)\tilde{W}_4(t)\right)\right. \\
&\quad \left. +2kN\rho_{A0}\left(\frac{it}{2}\right)\tilde{W}_4(t) + 2kM\rho_{B0}\left(\frac{it}{2}\right)P_4(t)2MN\rho_{C0}\left(\frac{it}{2}\right)\right]P_1(t) \qquad \text{(A.1)} \\
\mathcal{M} &= \frac{1}{2}\left[-M\,\rho_{00}\left(\frac{it}{2}+\frac{1}{2}\right)P_4(t)\tilde{P}_4(t) + k\,\rho_{0B}\left(\frac{it}{2}+\frac{1}{2}\right)P_4(t) - N\,\rho_{0C}\left(\frac{it}{2}+\frac{1}{2}\right)\right]P_1(t)
\end{aligned}
$$

where A, B and C in ρ_{gh} refers to h projection of the g-twisted chiral traces with twists oriented along the planes (2345), (6789) and (23456789) respectively. After performing the sum over spin structures these traces can be written as

$$
\begin{aligned}
\rho_{gh} &= \frac{\vartheta_1^2\,\vartheta^2\begin{bmatrix}\frac{1}{2}+g\\ \frac{1}{2}+h\end{bmatrix}}{\eta^6\,\hat{\vartheta}^2\begin{bmatrix}\frac{1}{2}+g\\ \frac{1}{2}+h\end{bmatrix}} \qquad g,h=\frac{1}{2} \quad \text{for} \quad g,h=A,B \\
\rho_{gh} &= \frac{\vartheta^4\begin{bmatrix}\frac{1}{2}+g\\ \frac{1}{2}+h\end{bmatrix}}{\hat{\vartheta}^4\begin{bmatrix}\frac{1}{2}+g\\ \frac{1}{2}+h\end{bmatrix}} \qquad g,h=\frac{1}{2} \quad \text{for} \quad g,h=C \qquad \text{(A.2)}
\end{aligned}
$$

The integers N, k, and M refer to the numbers of D1, D5 and D9 branes respectively. Finally momentum and winding lattice sums are given by

$$P_d(t) = \sum_{m_i\in \mathbf{Z}^d} e^{-\pi t\alpha' \frac{m_i^2}{R_i^2}} \qquad W_d(t) = \sum_{n_i\in \mathbf{Z}^d} e^{-\pi t n_j^2 \frac{R_j^2}{\alpha'}} \qquad \text{(A.3)}$$

The basic requirement satisfied by the string amplitudes (A.1) is that after the exchange of σ and τ directions they admit an interpretation in terms of closed strings, exchanges between boundaries (D-branes) and crosscaps (O9-planes). More precisely, the sum of closed string amplitudes should reconstruct the whole square

$$\langle B|\,e^{-lH}\,|B\rangle \qquad \text{(A.4)}$$

with

$$|B\rangle = |O9\rangle + M\,|D9\rangle + k\,|D5\rangle + N\,|D1\rangle. \qquad \text{(A.5)}$$

and $|O9\rangle$, $|D9\rangle$, $|D5\rangle$ and $|D1\rangle$ representing the "boundary" state for the corresponding branes.

Rewriting (A.1) in terms of the closed string variables $\ell_K = \frac{1}{2t}$, $\ell_A = \frac{2}{t}$, $\ell_M = \frac{1}{2t}$ one is left (at the origin of the $T^4 \times \tilde{T}^4$ lattice sum) with $\int d\ell \times$

$$\begin{aligned}\widetilde{\mathcal{K}}_0 &= \frac{2^5}{2}\left[\chi_O+\chi_V+\chi_S+\chi_C\right](i\ell)\, v_1 v_4 \tilde{v}_4\, W_1^{\rm even}(\frac{\ell}{2}) \\ \widetilde{\mathcal{A}}_0 &= \frac{2^{-5}}{2}\left[\chi_O I_O^2+\chi_V I_V^2+\chi_S I_S^2+\chi_C I_C^2\right](i\ell)\, W_1(\frac{\ell}{2}) \\ \widetilde{\mathcal{M}}_0 &= -\frac{2}{2}\left[\chi_O I_O+\chi_V I_V+\chi_S I_S+\chi_C I_C\right](i\ell)\, \sqrt{v_1 v_4 \tilde{v}_4}\, W_1^{\rm even}(\frac{\ell}{2}) \qquad \text{(A.6)}\end{aligned}$$

$W_1^{\rm even}$ is defined like (A.3) with n_1 restricted to be even. We have introduced the linear combinations of ρ_{gh} traces

$$\begin{pmatrix}\chi_O\\ \chi_V\\ \chi_S\\ \chi_C\end{pmatrix} = \frac{1}{4}\begin{pmatrix}+&+&+&+\\ +&+&-&-\\ +&-&-&+\\ +&-&+&-\end{pmatrix}\begin{pmatrix}\rho_{00}\\ \rho_{0A}\\ \rho_{0B}\\ \rho_{0C}\end{pmatrix} \qquad \text{(A.7)}$$

and of the Chan-Paton dependent combinations

$$\begin{pmatrix}I_O\\ I_V\\ I_S\\ I_C\end{pmatrix} = \begin{pmatrix}+&+&+\\ +&-&-\\ +&-&+\\ +&+&-\end{pmatrix}\begin{pmatrix}M\sqrt{v_1 v_4 \tilde{v}_4}\\ k\sqrt{\frac{v_1 v_4}{\tilde{v}_4}}\\ N\sqrt{\frac{v_1}{v_4 \tilde{v}_4}}\end{pmatrix}. \qquad \text{(A.8)}$$

with $v_1, v_4, \tilde{v}_4$ the volumes of $S^1, T^4, \tilde{T}^4$. The relevant modular transformations connecting the two expressions (A.1) and (A.6) can be easily read from (A.2)

$$\begin{aligned}&\rho_{00}(-\frac{1}{i\alpha\ell}) = (\alpha\ell)^{-4}\rho_{00}(i\alpha\ell) \quad \rho_{00}(\frac{i}{2t}+\frac{1}{2}) = t^{-4}\rho_{00}(\frac{it}{2}+\frac{1}{2})\\ &\rho_{0h}(-\frac{1}{i\alpha\ell}) = 4(\alpha\ell)^{-2}\rho_{h0}(i\alpha\ell) \quad \rho_{0h}(\frac{i}{2t}+\frac{1}{2}) = -t^{-2}\rho_{0h}(\frac{it}{2}+\frac{1}{2})\\ &\rho_{0C}(-\frac{1}{i\alpha\ell}) = 16\rho_{C0}(i\alpha\ell) \quad \rho_{0C}(\frac{i}{2t}+\frac{1}{2}) = \rho_{0C}(\frac{it}{2}+\frac{1}{2})\\ &\qquad P_d(\frac{1}{\alpha\ell}) = v_d(\alpha\ell)^{\frac{d}{2}} W_d(\alpha\ell) \qquad \text{(A.9)}\end{aligned}$$

with $h = A, B$ and α a factor of 2 depending on the one-loop surface.

Rewritten in the basis (A.7) one can easily recognize in $\widetilde{\mathcal{K}}_0 + \widetilde{\mathcal{A}}_0 + \widetilde{\mathcal{M}}_0$ given by (A.6) the different terms in the square (A.4). Notice that unlike tadpole cancellation conditions the requirement that the whole amplitude reconstruct a square is a restriction in the structure of the entire tower of massive closed string amplitudes. Indeed this requirement together with the choice of Orthogonal gauge group for D9-branes is enough to fix completely the Moebius strip amplitudes once the Klein bottle and annulus amplitudes are given [45]. In particular the relative sign between the Ω projection on D5 and D1(D9) Chan-Paton factors is crucial in order to reproduce the square.

Let us consider a similar system in the "type I" theory with the shift. First let us notice that closed string states in (A.6) with odd windings only enter the annulus amplitudes. This can be attributed to the fact that only even winding modes can be reflected by the standard O9-plane. The situation get reversed if we now accompany the worldsheet parity operator with a σ_{p_1} momentum shift along the circle. This is done by replacing the lattice sum $P_1(t)$ in the Klein bottle and Moebius strip amplitudes (A.1) by

$$P_1(t) \to \sum_{m_1 \in \mathbf{Z}} (-)^{m_1} e^{-\pi t \alpha' \frac{m_1^2}{R_1^2}} \tag{A.10}$$

In the closed string channel this translates into the replacement

$$W_1^{\rm even}(\frac{\ell}{2}) \to W_1^{\rm odd}(\frac{\ell}{2}) \tag{A.11}$$

and therefore now only odd winding modes are reflected by the orientifold plane. we can see that combining this with a non-trivial Wilson line turned on in the D5 brane gauge group one gets the desired result. The Wilson line can be included by replacing the lattice sum $P_1(t)$ accompanying the Annulus terms linear in k by

$$W_1(t) \to \sum_{n_1 \in \mathbf{Z}} e^{-\pi t (n_1 - \frac{1}{2})^2 \frac{R_1^2}{\alpha'}} \tag{A.12}$$

The Klein bottle and Annulus amplitudes can then be written in the closed string channel as

$$\begin{aligned}
\widetilde{\mathcal{K}}_0 &= \frac{2^5}{2} [\chi_O + \chi_V + \chi_S + \chi_C] (i\ell)\, v_1 v_4 \tilde{v}_4\, W_1^{\rm odd}(\frac{\ell}{2}) \\
\widetilde{\mathcal{A}}_0 &= \frac{2^{-5}}{2} \left[\chi_O I_O^2 + \chi_V I_V^2 + \chi_S I_S^2 + \chi_C I_C^2\right] (i\ell)\, W_1^{\rm even}(\frac{\ell}{2}) \\
&\quad + \frac{2^{-5}}{2} \left[\chi_O I_S^2 + \chi_V I_C^2 + \chi_S I_O^2 + \chi_C I_V^2\right] (i\ell)\, W_1^{\rm odd}(\frac{\ell}{2})
\end{aligned} \tag{A.13}$$

The complete square is now reconstructed by

$$\widetilde{\mathcal{M}}_0 = -\frac{2}{2} [\chi_O I_S + \chi_V I_C + \chi_S I_O + \chi_C I_V] (i\ell) \sqrt{v_1 v_4 \tilde{v}_4}\, W_1^{\rm odd}(\frac{\ell}{2}) \tag{A.14}$$

which differ from the ones in (A.6) by the parity of closed string windings $W_1^{\rm odd}$ and in an overall flip of the sign of k. This leads in the open string channel to the lattice sum (A.10) and the gauge group $SO(M) \times SO(k) \times SO(N)$.

Appendix B: Symmetric product orbifold CFT:

In this appendix we give an alternative derivation of the elliptic genus formulae (2.45) for the case where the Hilbert space $\mathcal{H}$ involved in the symmetric product admits a free field theory (orbifold) description. We follow closely [51].

The oscillator contributions of a given worldsheet field Φ with boundary conditions (2.28) to the string partition function is given by

$$\mathcal{Z}_{\rm osc}\begin{bmatrix} g_\phi \\ h_\phi \end{bmatrix}(q,y) = \prod_{n=1}^{\infty}(1 - e^{2\pi i h_\phi} y^{\omega_\phi} q^{n-g_\phi})^{\epsilon_\phi} \tag{B.1}$$

with $\epsilon_\phi = -1$ for bosonic components and $\epsilon_\phi = 1$ for fermions. The string partition (2.29), can then be written as a product over fields Φ of such contributions in the left and right moving part of the string, times a (in general non-holomorphic) zero mode contribution

$$\mathcal{Z}\begin{bmatrix} g_0 \\ h_0 \end{bmatrix}(\mathcal{H}|q,\bar{q},y,\bar{y}) = \tau_2^{-\frac{D}{2}} \prod_{\phi,\bar{\phi}} Z_0 Z_{\rm osc}\begin{bmatrix} g_\phi \\ h_\phi \end{bmatrix}(q,y)\bar{Z}_0\bar{Z}_{\rm osc}\begin{bmatrix} g_{\bar{\phi}} \\ h_{\bar{\phi}} \end{bmatrix}(\bar{q},\bar{y}) \tag{B.2}$$

For complex bosonic and fermionic degrees of freedom this zero mode contribution can be written as

$$\begin{aligned}
Z_0\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\rm boson}(q,y) &= q^{-\frac{1}{12}} q^{\frac{1}{2}P_L^2} \\
Z_0\begin{bmatrix} g_\phi \\ h_\phi \end{bmatrix}_{\rm boson}(q,y) &= q^{\chi_\phi} \\
Z_0\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\rm fermi}(q,y) &= q^{\frac{1}{12}}\left(y^{\omega_\phi} + y^{-\omega_\phi} - 2\right) \\
Z_0\begin{bmatrix} g_\phi \\ h_\phi \end{bmatrix}_{\rm fermi}(q,y) &= q^{-\chi_\phi}
\end{aligned} \tag{B.3}$$

with similar expressions for the right moving components in term of bar quantities and P_L replaced by P_R. In the following we will display only holomorphic formulas since the analysis of the antiholomorphic part follows similarly. The boson in (B.3) is understood to be compact, for non compact components we should of course simply omit the lattice sum in (B.3). ω_ϕ stands for the charge of the field ϕ under J_0^3 and χ_ϕ represents the contribution of a complex boson with spin characteristics g_ϕ, h_ϕ to the zero point energy

$$\chi_\phi = -\frac{1}{12} + \frac{1}{2} g_\phi (1 - g_\phi). \tag{B.4}$$

Orbifold group sectors and the N copies of the field Φ are labeled following the notation of section 4. Since the g and h twists commute we can diagonalize them simultaneously. In this basis one can write

$$\begin{aligned} \mathbf{g} &= e^{2\pi i \frac{l}{L}} \\ \mathbf{h} &= e^{2\pi i (-\frac{l s_{m;i}}{ML} + \frac{m}{M})} \end{aligned} \tag{B.5}$$

Let us now evaluate the basic trace (2.37). For the time being we will concentrate on the non-zero left moving contribution $Z_{\text{osc}}\left[\begin{smallmatrix} g_\phi \\ h_\phi \end{smallmatrix}\right](q, y)$. After simple manipulations of the product formulae one is left with

$$\begin{aligned} (M)^{r_M^L} \mathbf{t} \, \square_{(L)^{N_L}} : \quad & \prod_{i,l,m} \prod_{n=1}^{\infty} (1 - e^{2\pi i(-\frac{l s_{m;i}}{ML} + \frac{m}{M} + h_\phi)} \, y^{\omega_\phi} q^{n - g_\phi - l/L})^{\epsilon_\phi} \\ &= \prod_{i} \prod_{n=1}^{\infty} (1 - e^{2\pi i (s_i g_\phi + M h_\phi)} \, y^{M\omega_\phi} (q^{\frac{M}{L}} e^{2\pi i \frac{s_i}{L}})^{n - g_\phi L})^{\epsilon_\phi} \\ &= \prod_i Z_{\text{osc}} \begin{bmatrix} g_\phi L \\ g_\phi s_i + M h_\phi \end{bmatrix} (q^{\frac{M}{L}} e^{2\pi i \frac{s_i}{L}}, y^M) \end{aligned} \tag{B.6}$$

The result is in agreement with (2.37). One can follow similar manipulations to show that the zero mode contribution to $S_N \mathcal{H}$ can again be reexpressed as $Z_0\left[\begin{smallmatrix} g_\phi L \\ g_\phi s_i^L + M h_\phi \end{smallmatrix}\right](q^{\frac{M}{L}} e^{2\pi i \frac{s_i^L}{L}}, y^M)$. This is clear for the lattice sum and the fermionic zero mode trace following similar manipulations as before, while for the zero point energy this can be read from (B.4) (let's say in the $(L)^M$-twisted sector) leading to a contribution $q^{\chi_{L,M}}$ with

$$\begin{aligned} \chi_{L,M} &= M \epsilon_\phi \sum_{l=0}^{L-1} \left[\frac{1}{12} - \frac{1}{2} (g_\phi + \frac{l}{L})(1 - g_\phi - \frac{l}{L}) \right] \\ &= \frac{M}{L} \epsilon_\phi \left[\frac{1}{12} - \frac{1}{2} g_\phi L (1 - g_\phi L) \right] \end{aligned} \tag{B.7}$$

as expected.

This concludes our derivation of (2.37) in this restricted context. Following the subsequent steps in the main text one is lead straight to the symmetric product formulae (2.45).

Appendix C: Modular integral with shifts:

In this appendix we evaluate the modular integral (2.76).

$$\mathcal{G}_{\vec{a},\vec{b}}(v,w) = \frac{1}{2}\int \frac{d^2\tau}{\tau_2^{3+\frac{d}{2}}} \sum_{g,h,n} C\begin{bmatrix} g \\ h \end{bmatrix}(n)\Gamma_{2,2}\begin{bmatrix} g \\ h \end{bmatrix}(v,w)\, e^{2\pi i n\tau} \tag{C.1}$$

with $d=2(6)$ in the case of models I_F,III_F (II_F) and $\Gamma_{2,2}\left[{g \atop h}\right](\ell\cdot\vec{v},\ell_*\cdot\vec{w})$ defined by (2.78).

We start by reabsorbing $(d+2)/2$ powers (the number of $\vec{w}$-insertions) of τ_2 in the rescaling of the left moving source $\vec{w}\to\frac{\vec{w}}{\tau_2}$. Equalities in the next are understood once $d+2/2$ $\vec{w}$-derivatives are applied to the end result and sources $\vec{w}$ are set to zero.

After performing the gaussian integral over τ_1 we are left with

$$\mathcal{G}_{\vec{a},\vec{b}} = \mathcal{C}\,\frac{(U_2T_2)^{\frac{1}{2}}}{|m_1|}\int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}}\, e^{-b_0-\gamma\tau_2-\frac{\beta}{\tau_2}} = \mathcal{C}\,\frac{(U_2T_2)^{\frac{1}{2}}}{|m_1|}\sqrt{\frac{\pi}{\beta}}\, e^{-b_0-2\sqrt{\beta\gamma}} \tag{C.2}$$

with

$$\begin{aligned}
\mathcal{C} &= \frac{1}{2}\sum_M \eta\begin{bmatrix} g \\ h \end{bmatrix} C\begin{bmatrix} g \\ h \end{bmatrix}(n,\ell\,'s) \\
b_0 &= -2\pi i T_1 m_1 n_2 - 2\pi i\frac{n}{m_1}(n_2U_1+n_1) \\
&= 2\pi i\ell\cdot[\vec{n}\vec{v}-v_1(n_2U_1+n_1)] - 2\pi i\ell_*\cdot w_1\left[m_1-\frac{U_2}{m_1T_2}(n+m_1\ell\cdot v_1)\right] \\
\beta &= \pi n_2^2T_2U_2 + 2\pi i n_2\ell_*\cdot(w_1U_1-w_2) + \pi U_2T_2(\ell_*\cdot w_1)^2 \\
\gamma &= m_1^2\frac{\pi T_2}{U_2}\left[1+\frac{U_2}{m_1^2T_2}(n+m_1\ell\cdot v)\right]^2
\end{aligned} \tag{C.3}$$

We are interested in the leading order in a $1/T_2$ expansion of (C.2), associated in the dual theory to the semiclassical approximation around the D-instanton background. In this limit all subleading $\vec{w}$-dependent terms in the exponential can be discarded since they lead to irrelevant contributions once they are hitted upon by $\vec{w}$-derivatives.

To leading order in $1/T_2$ the result can be written as

$$\mathcal{G}_{\vec{a},\vec{b}} = \sum_{m_1,n_2,g} \frac{1}{|n_2|} e^{2\pi i\left[T m_1 n_2 + \frac{n}{m_1} n_2 U + n_2 \ell\cdot\hat{v} - m_1 \ell_* \cdot \hat{w}\right]} \mathcal{J}_{\vec{a},\vec{b}} + h.c. \tag{C.4}$$

where $\hat{v} = v_1 U - v_2$ and $\hat{w} = w_1 \bar{U} - w_2$ are induced source terms and $h.c.$ denotes the hermitian conjugate results coming from anti D-instantons. $\mathcal{J}_{\vec{a},\vec{b}}$ represents the shift dependent sum

$$\mathcal{J}_{\vec{a},\vec{b}} = \frac{1}{2|m_1|} \sum_{n_1,g,h} e^{2\pi i(\frac{n}{m_1} n_1 - 2a_1 b_1 gh - a_1 h m_1 + a_1 g n_1)} C \begin{bmatrix} g \\ h \end{bmatrix} (n, \ell\,'s). \tag{C.5}$$

Our next task is to evaluate (C.5) in the cases of winding ($a_1 = 1$) or momentum ($b_1 = 1$) shifts. The domains of $\vec{m}, \vec{n}$ are specified by (2.79) while n is integer in the untwisted and both integer and half integer in the twisted sectors. Using the identity

$$C \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (n, \ell\,'s) = (-)^{2n}\, C \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (n, \ell\,'s) \tag{C.6}$$

and performing the geometric sums one can write the final result as

$$\begin{aligned} \mathcal{J}_{1,0} &= \frac{1}{2}\left(C \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + (-)^{m_1} C \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)(n, \ell\,'s)\, \mathcal{P}^{\mathbf{Z}}_{\frac{n}{m_1}} + \frac{1}{2} C \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (n, \ell\,'s)\, \mathcal{P}^{\mathbf{Z}+\frac{1}{2}}_{\frac{n}{m_1}} \\ \mathcal{J}_{0,1} &= \frac{1}{2}\left(C \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + (-)^{\frac{n}{m_1}} C \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + 2C \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}\right)(n, \ell\,'s)\, \mathcal{P}^{\mathbf{Z}}_{\frac{n}{m_1}} \end{aligned} \tag{C.7}$$

with $\mathcal{P}^{\mathbf{Z}+\delta}_{\frac{n}{m_1}}$ a projector onto state with $\frac{n}{m_1} \in \mathbf{Z} + \delta$.

Plugging (C.7) in (C.4), introducing the quantum number $k \equiv \frac{n}{m_1} \in \mathbf{Z} + a_1 g$ and performing the remaining m_1, n_2 sums one is left with the final result (2.86).

Bibliography

[1] M. B. Green, J. H. Schwarz, E. Witten, *"Superstring theory"*, Cambridge University Press (1987).

[2] D. Lust, S. Theisen, *"Lectures on string theory"*, Berlin New York Springer-Verlag c1989.

[3] J. Polchinski, *"An introduction to the bosonic string"*, *"Superstring theory and beyond"*, Cambridge monographs on mathematical physics.

[4] E. Kiritsis, *"Introduction to superstring theory"*, hep-th/9709062.

[5] M. J. Duff, P. S. Howe, T. Inami, K. S. Stelle, Phys. Lett. B191:70, 1987;
M. J. Duff, J. X. Lu, Nucl. Phys. B347:394-419, 1990;
M. J. Duff, J. T. Liu, R. Minasian, Nucl. Phys. B452 (1995) 261, hep-th/9506126;
C. M. Hull, P. K. Townsend, Nucl. Phys. B438 (1995) 109, hep-th/9410167;
P. K. Townsend, Phys. Lett. B350 (1995) 184, hep-th/9501068;
C. M. Hull, P. K. Townsend, Nucl. Phys. B451 (1995) 525, hep-th/9505073.

[6] E. Witten, Nucl.Phys. B443 (1995) 85, hep-th/9503124.

[7] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[8] A. Strominger, C. Vafa, Phys.Lett. B379 (1996) 99, hep-th/9601029.

[9] J. Bagger and J. Wess, *"Supersymmetry and supergravity"*, Princeton Series in Physics (1992).

[10] J. Maldacena, G. Moore, A.Strominger, hep-th/9903163.

[11] E. Gava, A. B. Hammou, J. F. Morales, K. S. Narain, work in progress, hep-th/0000000.

[12] C. Vafa, E. Witten, Nucl.Phys.Proc.Suppl. 46 (1996) 225-247, hep-th/9507050.

[13] A. B. Hammou, J. F. Morales, Nucl.Phys. B573 (2000) 335-348, hep-th/9910144.

[14] F. Gliozzi, J. Scherk, D. Olive, Nucl. Phys. B122:253-290, 1977.

[15] A. Sagnotti, in "Non-Perturbative Quantum Field Theory", eds. G. Mack et al (Pergamon Press, 1988), p. 521.
G. Paradisi, A. Sagnotti, Phys. Lett. B216:59, 1989;
M. Bianchi and A. Sagnotti, Phys. Lett. B247 (1990) 517; Nucl. Phys. B361;Phys.Lett. B231:389, 1989;
P. Hořava, Nucl. Phys. B327 (1989) 461; Phys. Lett. B231 (1989) 251.
M. Bianchi, G. Pradisi and A. Sagnotti, Nucl. Phys. B376 (1991) 365.

[16] A. Dabholkar, Phys. Lett. B357 (1995) 307.

[17] C. M. Hull, Phys. Lett. B357 (1995) 545.

[18] J. Polchinski and E. Witten, Nucl. Phys. B460 (1996) 525, hep-th/9511030.

[19] M. B. Green and J. H. Schwarz, Phys. Lett. B122 (1983) 143.

[20] J. H. Schwarz and P. C. West, Phys. Lett. B126 (1983) 301.

[21] J. H. Schwarz, Nucl. Phys. B226 (1983) 269.

[22] P. Howe and P. C. West, Nucl. Phys. B238 (1984) 181.

[23] A. Sen, Lectures given at Isaac Newton Institute and DAMTP, Cambridge, hep-th/9802051.

[24] F. Giani and M. Pernici, Phys. Rev. D30 (1984) 325;
I. C. G. Campbell and P. C. West, Nucl. Phys. B243 (1984) 112;
M. Huq and M. A. Namazie, Class. Quant. Grav. 2 (1985) 293.

[25] P. S. Aspinwall, D. R. Morrison, hep-th/9404151;
Paul S. Aspinwall, hep-th/9611137.

[26] C. M. Hull and P. K. Townsend, Nucl. Phys. B438 (1995) 109, hep-th/9410167.

[27] N. Seiberg, Nucl. Phys. B303(1988) 286.

[28] A. Sen, Nucl. Phys. B450 (1995) 103.

[29] J. Harvey and A. Strominger, Nucl. Phys. B449 (1995) 535.

[30] M.J. Duff and R. Khuri, Nucl. Phys. B411 (1994) 473.

[31] E. Witten, D. Olive, Phys. lett. B78 (1978) 97.

[32] P. K. Townsend, hep-th/9507048.

[33] J. Polchinski, S. Chaudhuri, C. V. Johnson, hep-th/9602052.

[34] J. Polchinski, hep-th/9611050.

[35] J. Dai, R. G. Leigh, J. Polchinski, Mod.Phys.Lett.A4:2073-2083,1989.

[36] R. G. Leigh, Mod.Phys.Lett.A4:2767,1989.

[37] E. Witten, Nucl.Phys.B460:335-350,1996, hep-th/9510135.

[38] M. Bianchi, Nucl.Phys. B528 (1998) 73, hep-th/9711201;
E. Witten, JHEP 9802 (1998) 006, hep-th/9712028;
C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti, Ya.S. Stanev, Phys.Lett. B387 (1996) 743, hep-th/9607229;
A. Dabholkar, J. Park, Nucl.Phys. B477 (1996) 701, hep-th/9604178;
E. G. Gimon, C. V. Johnson, Nucl.Phys. B477 (1996) 715, hep-th/9604129.

[39] E. Witten, Nucl.Phys. B460 (1996) 541-559, hep-th/9511030.

[40] C. Vafa, Nucl.Phys. B463 (1996) 415, hep-th/9511088;
Nucl.Phys. B463 (1996) 435, hep-th/9512078.

[41] F. Larsen, E. Martinec, JHEP 9906 (1999) 019, hep-th/9905064; JHEP 9911 (1999) 002, hep-th/9909088.

[42] N. Seiberg, E. Witten, JHEP 9904 (1999) 017, hep-th/9903224;

[43] K. S. Narain, Phys. Lett. B169 (1986) 41;
K.S. Narain, M.H. Sarmadi and E. Witten, Nucl.Phys. B279 (1987) 369.

[44] E. G. Gimon and J. Polchinski, Phys. Rev. D54 (1996) 1667-1676, hep-th/9601038.

[45] A. Sagnotti, in Non Perturbative Quantum Field Theory, ed. G. Mack et al. (Pergamon, New York, 1988) 521;
M. Bianchi, G. Pradisi, A. Sagnotti, Phys. Lett. B273:389-398, 1991;
D. Fioravanti, G. Pradisi, A. Sagnotti, Phys.Lett. B321 (1994) 349, hep-th/9311183;
G. Pradisi, A. Sagnotti, Ya.S. Stanev, Phys.Lett. B354 (1995) 279, hep-th/9503207; B356 (1995) 230; B381 (1996) 97-104;
M. Bianchi, A. Sagnotti, Phys. Lett. B247:517-524, 1990; Nucl. Phys. B361:519-538, 1991.

[46] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, Yu. I. Manin, Phys. Lett. A65:185-187, 1978.

[47] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis, S. Vandoren, Nucl. Phys. B552, 88, 1999, hep-th/9901128;
E. Gava, K.S.Narain and M.H.Sarmadi, Nucl.Phys.B569, 183, 2000, hep-th/9908125;
T.J. Hollowood, V.V. Khoze and M.P. Mattis, hep-th/9910118.

[48] E. Witten, J. Geom. Phys. 15 (1995) 215, hep-th/9410052.

[49] R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde, Commun.Math.Phys. 185 (1997) 197, hep-th/9608096.

[50] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys B274 (1986) 285.

[51] M. Bianchi, E. Gava, J. F. Morales, K. S. Narain, Nucl.Phys. B547 (1999) 96-126, hep-th/9811013.

[52] E. Gava, J. F. Morales, K. S. Narain, and G. Thompson, Nucl. Phys. B528 (1998) 95.

[53] R. Dijkgraaf, E. Verlinde, H. Verlinde, Nucl.Phys. B486 (1997) 77-88, hep-th/9603126;
Nucl.Phys. B486 (1997) 89-113, hep-th/9604055.

[54] I. Antoniadis, E. Gava, K. S. Narain, T. R. Taylor, Nucl.Phys. B455 (1995) 109-130, hep-th/9507115.

[55] C. Bachas, E. Kiritsis, Nucl. Phys. Proc. Suppl. 55B:194-199, 1997, hep-th/9611205;

E. Kiritsis, hep-th/9708130;
J. A. Harvey, G. Moore, Nucl.Phys.B463:315-368,1996, hep-th/9510182;
S. Ferrara, C.A. Savoy, L. Girardello, Phys.Lett.B105:363,1981;
T. L. Curtright, Phys.Lett.B102:17,1981.

[56] A. Gregori, E. Kiritsis, C. Kounnas, N. A. Obers, P. M. Petropoulos, B. Pioline, Nucl.Phys. **B510** (1998) 423-476, hep-th/9708062.

[57] L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649.

[58] E. Gava, A. B. Hammou, J. F. Morales and K. S. Narain, hep-th/9902202, JHEP 9903 (1999) 023.

[59] E. Kiritsis and N. Obers, J.High Energy Phys. 10 (1997) 004;
C. Bachas, Nucl. Phys. Proc. Suppl. 68 (1998) 348;
N.A. Obers and B. Pioline, Commun. Math. Phys. 209:275-324, 2000, hep-th/9903113.

[60] J. A. Harvey, G. Moore, Phys. Rev. D57 (1998) 2323-2328, hep-th/9610237.

[61] W. Lerche and S. Stierberger, hep-th/9907133.

[62] W. Lerche, S. Stieberger, N.P. Warner hep-th/9811228;
K. Foerger, hep-th/981215.

[63] E. Kiritsis, Lectures given at the 1999 ICTP Spring School, hep-th/9906018.

[64] E. Kiritsis and B. Pioline, Nucl. Phys. B508 (1997) 509, hep-th/9707018.

[65] K. Foerger, S. Stieberger, hep-th/9901020.

[66] S. J. Rey, *Heterotic M(atrix) Strings and Their Interactions*, Nucl.Phys. B502 (1997) 170.

[67] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, Comm. Math. Phys. 185 (1997) 197.

[68] J. F. Morales, M. Serone, Nucl.Phys. B481 (1996) 389-402, hep-th/9607193.

[69] M. Bianchi, J. F. Morales, JHEP 0003 (2000) 030, hep-th/0002149.

[70] M. Bianchi, J. F. Morales, JHEP 0008 (2000) 035, hep-th/0006176.

[71] O. Bergman, Nucl.Phys. B525 (1998) 104-116, hep-th/9712211.

[72] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; Nucl. Phys. B431 (1994) 484;

[73] Frank Ferrari and Adel Bilal, Nucl.Phys. B469 (1996) 387; Nucl.Phys. B480 (1996) 589.

[74] C. Bachas, C. Fabre, E. Kiritsis, N. Obers, and P. Van Hove, Nucl. Phys. B509 (1998) 33.