ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Capacity Methods in the Study of Convergence of Diffusions

Thesis submitted for the degree of

"Doctor Philosophiæ"

CANDIDATE

Andrea Posilicano

SUPERVISOR

Prof. Gianfausto Dell'Antonio

October 1991

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Introduction

The aim of this work is the study of the continuity of the map which assigns to a drift vector field b_t the probability measure P, weak solution, if it exists, of the stochastic differential equation

$$X_t = X_0 + \int_0^t \mathrm{b}_s(X_s)ds + B_t, \tag{1}$$

where $X_t(\gamma) := \gamma(t)$ is the canonical stochastic process on the path space $C(\mathbb{R}_+;\mathbb{R}^d)$. We will interested in the case where the drift b is singular and unbounded since this is the case in most of the physical interesting situations. We will study therefore the situation where is not possible to apply the usual convergence theorems for diffusion processes. To handle this pathological situation we give a criterion for convergence of weak solutions of stochastic differential equations which only require to control the convergence of the corresponding drifts outside "bad" sets for which the probability to be attained can be made arbitrary small (thm.3.1). Making use of relative entropy estimates (lemma 2.3) all our convergence results are given in terms of total variation norm. Our convergence criterion, which contains the previously known results (see §4), and can be easily applied, via classical potential theory, to the case in which the drifts are in $H^1(\mathbb{R}^d)$ (see §5), becomes really useful when one has an explicit knowledge of the densities of the processes. We will therefore concentrate on the case where

$$\mathrm{b} = v + u, \quad u_t(x) := \frac{1}{2}\frac{\nabla\rho_t(x)}{\rho_t(x)},$$

if $\rho_t(x) \neq 0$, $u = 0$ otherwise, where $\rho(t,x) = \rho_t(x)$ is a given family of probability densities, and v satisfies the Fokker–Planck equation

$$\int_{\mathbb{R}^d} f(x,T)\rho(x,T)dx - \int_{\mathbb{R}^d} f(x,0)\rho(x,0)dx = \int_0^T \int_{\mathbb{R}^d} (v\cdot\nabla f)\rho(x,t)dxdt\ ,$$

for all $T \geq 0$, and all $f \in C_0^\infty(\mathbb{R}^{d+1})$. We will require moreover that the drift satisfies the "finite energy condition"

$$\int_0^T \int_{\mathbb{R}^d} \left(\|u_t(x)\|^2 + \|v_t(x)\|^2\right)\rho_t(x)dxdt < +\infty\ .$$

Within these hypotheses always there exists an unique P solving (1) w.r.t. which X_t is a Markov process with density ρ_t (see [C1], [C2], [C3]). Therefore in this setting our problem is well posed and we have a well defined map $\mathcal{C}$ from the set $\mathcal{P}$ of pairs (v,ρ) into $\mathcal{M}_1(\Omega)$, the space of probability measures on Ω. The elements in the range of this map are called Nelson Diffusions. An important class of Nelson Diffusions is the one corresponding to the pairs $(\psi_t\overline{\psi}_t, \Im(\nabla\psi_t/\psi_t))$, where ψ_t is a solution of a Schrödinger equation (observe that here the drift is unbounded on the nodes of ψ_t). We have therefore that in this case the stochastic process X_t gives us a pathwise representation of quantum phenomena (see [N1], [N2] for this non trivial fact).

In order to maximize the applicability of our convergence criterion we introduce an appropriate parabolic capacity (§8), and using nonattainability results (§9), in §10 we prove that the map $\mathcal{C}$ is continuous with respect to a natural metric (the Guerra metric) on $\mathcal{P}$ (thm.10.2). Moreover in §12, after giving a result on the continuity, in H^1–norm, of solutions of Schrödinger equation with respect to the choice of Kato–class potentials (lemma 12.1), we prove convergence of Nelson Diffusions corresponding to Schrödinger operators in terms of H^1–convergence of initial data and convergence in Kato–norm of the potentials. This also shows that the Guerra metric is too strong with respect to convergence of solutions of Schrödinger equations. In fact in this case our convergence result is not a corollary of the theorem on convergence of general Nelson Diffusions (even if the proof is essentially the same).

We conclude remarking that these results on convergence of diffusion processes with singular drifts seems also to be useful in a pure functional analytic contest. Indeed in §7 we give a convergence theorem for diffusion corresponding to energy forms (a particular class of Nelson Diffusions corresponding to the case $v = 0$), and we use it to give a "regularization" theorem for "generalized" Schrödinger operators in which the potential should not necessarily exists as a measurable function.

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Acknowledgements

References

1. Topologies on $\mathcal{M}_1(\Omega)$

We will work on the measurable space $(\Omega, \mathcal{F})$, where $\Omega = C(\mathbb{R}_+; \mathbb{R}^d)$ is the space of continuous paths in $\mathbb{R}^d$, and $\mathcal{F}$ is the Borel σ–algebra with respect to the topology on Ω given by the metric

$$m(\gamma, \gamma') := \sum_{n \geq 1}^{+\infty} \frac{1}{2^n} \frac{\sup_{0 \leq t \leq n} \|\gamma(t) - \gamma'(t)\|}{1 + \sup_{0 \leq t \leq n} \|\gamma(t) - \gamma'(t)\|} .$$

Let

$$X_t \; : \; \Omega \; \to \; \mathbb{R}^d \qquad X_t(\gamma) := \gamma(t)$$

be the evaluation stochastic process . Since X_t is continuous, and therefore measurable, it is easy to see ([BV], §1.3) that

$$\mathcal{F} = \sigma \left(X_t \; : \; t \geq 0 \right) .$$

Moreover we will consider the increasing filtration $\{\mathcal{F}_t\}_{t \geq 0}$

$$\mathcal{F}_t := \sigma \left(X_s \; : \; 0 \leq s \leq t \right) .$$

Obviously (see [SV], §1.3)

$$\mathcal{F}_t = \sigma \left(\bigcup_{s<t} \mathcal{F}_s \right) \qquad \forall t > 0 ,$$

and

$$\mathcal{F} = \sigma \left(\bigcup_{t \geq 0} \mathcal{F}_t \right) .$$

Moreover, if τ is a $\mathcal{F}_t$–stopping time, then ([SV], lemma 1.3.3)

$$\mathcal{F}_\tau := \{ \; E \in \mathcal{F} \; : \; E \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0 \; \} \equiv \sigma \left(X_{s \wedge \tau} \; : \; s \geq 0 \right) .$$

We will denote with $\mathcal{M}_1(\Omega)$ the space of probability measures on $(\Omega, \mathcal{F})$. If $\mathcal{B}$ is some sub–σ–algebra of $\mathcal{F}$, then $\mathcal{M}_1(\Omega, \mathcal{B})$ will denote the corresponding space of probability measures.

Let $C_b(\Omega)$ denote the set of all bounded continuous functions on Ω. Viewing $\mathcal{M}_1(\Omega)$ as a subset of the dual space of $C_b(\Omega)$, $\mathcal{M}_1(\Omega)$ inherits the weak–$*$ topology. Since Ω is a complete metric space, $\mathcal{M}_1(\Omega)$ will admit a complete metric inducing the weak–$*$ topology (see [DM], chap.III, no.60). Such metric is the Prohorov metric p so defined (see [B], appendix III):

$$p(P,Q) := \inf\{\epsilon \geq 0 \ : \ Q(A) \leq P(A^\epsilon) + \epsilon, \ P(A) \leq Q(A^\epsilon) + \epsilon, \ \forall A \in \mathcal{F}\} \ ,$$

where A^ϵ is the ϵ–nbh. of A w.r.t. the metric m.

The following theorem characterizes convergence w.r.t. the Prohorov metric (see [SV], thm.1.1.1 or [B], thm.2.1, chap. 2):

Theorem 1.1. *Let P^n, P, $n \geq 1$ be probability measures in $\mathcal{M}_1(\Omega)$. These five conditions are equivalent:*

i)

$$\lim_{n\to+\infty} p(P^n, P) = 0;$$

ii)

$$\lim_{n\to+\infty} \int_\Omega f dP^n = \int_\Omega f dP \quad \forall f \in C_b(\Omega);$$

iii)

$$\limsup_{n\to+\infty} P^n(C) \leq P(C) \quad \forall C \ closed;$$

iv)

$$\liminf_{n\to+\infty} P^n(G) \geq P(G) \quad \forall G \ open;$$

v)

$$\lim_{n\to+\infty} P^n(B) = P(B) \quad \forall B \in \mathcal{F} \ s.t. \ P(\partial B) = 0.$$

We will moreover consider on $\mathcal{M}_1(\Omega)$ the stronger topology induced by the metric

$$\|P - Q\| = \sup_{\{E_k\}_{k\in\mathbb{N}}} \sum_k |P(E_k) - Q(E_k)| \quad ,$$

where the sup is taken over all measurable partitions of Ω. This is the topology that $\mathcal{M}_1(\Omega)$ inherits as subset of the Banach lattice of bounded signed measures on Ω, normed with the total variation norm. A more workable equivalent definition is (see [JS], lemma 4.3, chap.V)

$$\|P - Q\| := 2 \sup_{A\in\mathcal{F}} |P(A) - Q(A)| \ .$$

By v) in the preceeding theorem we see that the topology induced by convergence in total variation is strictly stronger than the weak–$*$ one.

If the supremum in the above definitions is taken over a sub–σ– algebra $\mathcal{B}$, then we will denote the corresponding metric on $\mathcal{M}_1(\Omega, \mathcal{B})$ by $\| \cdot \|_{\mathcal{B}}$.

Since the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing one, we can introduce another topology on $\mathcal{M}_1(\Omega)$, stronger than the weak–$*$ one but weaker than the one given by the total variation norm. Such topology is the topology of "local" convergence in total variation, i.e. a sequence $\{P^n\}_{n\geq 1}$ will converge to P iff $\|P^n - P\|_{\mathcal{F}_t} \to 0$, $\forall t \geq 0$. This topology is obviously induced by the metric

$$d(P,Q) := \sum_{n\geq 0}^{+\infty} \frac{1}{2^n} \frac{\|P-Q\|_{\mathcal{F}_n}}{1+\|P-Q\|_{\mathcal{F}_n}} \ .$$

Since $\cup_{t\geq 0}\mathcal{F}_t$ generates the Borel σ–algebra, $\mathcal{M}_1(\Omega)$ is a complete metric space w.r.t. d. Such topology is stronger than the weak–$*$ one since weak–$*$ convergence on $C([0,T];\mathbb{R}^d)$ $\forall T \geq 0$, implies weak–$*$ convergence on Ω (see [W], thm.5 or alternatively use lemma 11.1.1 in [SV] applied to the sequence of trivial stopping times $\tau_k = k$).

2. Relative Entropy and the distance between Diffusion Processes

Let $P, Q \in \mathcal{M}_1(\Omega)$, and let $\mathcal{B}$ be a sub–σ–algebra. We define the relative entropy of $P_{\mathcal{B}}$ w.r.t. $Q_{\mathcal{B}}$ as

$$H_{\mathcal{B}}(P;Q) := \int_\Omega \log \left. \frac{dP}{dQ} \right|_{\mathcal{B}} dP$$

if $P \ll Q$ on $\mathcal{B}$, and $H_{\mathcal{B}}(P;Q) = +\infty$ otherwise.

The total variation distance can be estimated in terms of relative entropy by the Csizlár–Kullback inequality (see e.g. [Fö2], remark 3.2)

$$\|P-Q\|_{\mathcal{B}}^2 \leq 2H_{\mathcal{B}}(P;Q) \ .$$

In order to give estimates on the distance between weak solutions of stochastic differential equations we therefore need to write relative entropy of such measures in term of the corresponding infinitesimal characteristics. This is given in the following lemma which follows from Girsanov's theorem (see [H] or [JS], thm.4.23, chap. IV for i), and see [Fö1], prop. 2.11, or [Fö2], remark 1.3 for ii)).

Lemma 2.1. *Let B_t be a $\mathcal{F}_t$-Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$, and let $P \in \mathcal{M}_1(\Omega)$. Then* *i)*

$$P_{|\mathcal{F}_t} \ll Q_{|\mathcal{F}_t} \quad \forall t \geq 0$$

if and only if there exists an adapted process b_t*, with*

$$\int_0^t \|\mathrm{b}_s\|^2 < +\infty \quad P-a.s. \quad \forall t \geq 0,$$

such that

$$B_t - B_0 - \int_0^t \mathrm{b}_s ds \quad \textit{is a } P - \textit{Brownian motion.}$$

In this case, for any $\mathcal{F}_t$*– stopping time* τ*, one has*

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_{t\wedge\tau}} = \left.\frac{dP}{dQ}\right|_{\mathcal{F}_0} \exp\left(\int_0^{t\wedge\tau} \mathrm{b}_s dB_s - \frac{1}{2}\int_0^{t\wedge\tau} \|\mathrm{b}_s\|^2 ds\right).$$

ii)

$$H_{\mathcal{F}_t}(P;Q) < +\infty \quad \forall t \geq 0$$

if and only if

$$E_P \int_0^t \|\mathrm{b}_s\|^2 ds < +\infty \quad \forall t \geq 0.$$

In this case, for any $\mathcal{F}_t$*– stopping time* τ*, one has*

$$H_{\mathcal{F}_{t\wedge\tau}}(P;Q) = H_{\mathcal{F}_0}(P;Q) + \frac{1}{2} E_P \int_0^{t\wedge\tau} \|\mathrm{b}_s\|^2 ds.$$

The adapted process b_t in the previous lemma can be computed as a stochastic forward derivative in the sense of Nelson [N]:

Theorem 2.2. ([Fö1],[Fö2]) *If* $H_{\mathcal{F}_t}(P;Q) < +\infty$*,* $\forall t \geq 0$ *then, for almost every* t

$$\mathrm{b}_t = L^2 - \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} E_P(B_{t+\epsilon} - B_t|\mathcal{F}_t) .$$

From the previous lemma, and the Csizlár–Kullback inequality, one can estimate the distance in total variation of two weak solutions of stochastic differential equations:

Lemma 2.3. *If* P *and* P' *are probability measures on* Ω *given by weak solutions of stochastic differential equations*

$$X_t(\gamma) = x + \int_0^t \mathrm{b}_s(\gamma) ds + B_t(\gamma)$$

$$X_t(\gamma) = x + \int_0^t \mathrm{b}'_s(\gamma) ds + B'_t(\gamma) ,$$

with drifts b *and* b′ *such that*

$$\int_0^t \|b_s\|^2 ds < +\infty \quad P-a.s., \qquad \int_0^t \|b'_s\|^2 ds < +\infty \quad P'-a.s., \qquad \forall t \geq 0\ ,$$

then

$$\|P-P'\|^2_{\mathcal{F}_t} \leq E_P \int_0^t \|b_s - b'_s\|^2 ds \quad \forall t \geq 0.$$

Proof: since

$$E_P \int_0^t \|b_s - b'_s\|^2 ds < +\infty \qquad \Rightarrow \qquad P_{|\mathcal{F}_t} \ll Q_{|\mathcal{F}_t}$$

(see [JS], theorem 4.23, chap.IV), we have

$$B_t = \int_0^t (b_s - b'_s) ds + B'_t \qquad P_{|\mathcal{F}_t}\text{-a.s.} \quad .$$

Therefore, by the previous lemma,

$$H_{\mathcal{F}_t}(P;P') = \frac{1}{2} E_P \int_0^t \|b_s - b'_s\|^2 ds \quad ,$$

and, by the Csizlár–Kullback inequality it follows

$$\|P-P'\|^2_{\mathcal{F}_t} \leq E_P \int_0^t \|b - b'\|^2(s) ds \quad .$$

Remark 2.4. An estimate of the same kind as the one given in the previous lemma can be obtained as a corollary of more general results giving estimates of the total variation distance in terms of the expectation of the Hellinger process corresponding to the pair (P, P') (see [KLS], thm.5.1 or [JS], chap.V, §4d). However, in the case of measures corresponding to diffusion processes, such generality is unnecessary.

3. A criterion for convergence of weak solutions of stochastic differential equations

In this chapter we will give a criterion for convergence of probability measures which will be particularly useful when applied to Nelson diffusions.

Theorem 3.1. *Let P, P^n, $n \geq 1$, be probability measures on $(\Omega, \mathcal{F})$, weak solutions of the stochastic differential equations*

$$X_t = X_0 + \int_0^t \mathrm{b}_n(s, X_s)ds + B_t^n \quad n \geq 1 \ ,$$

$$X_t = X_0 + \int_0^t \mathrm{b}(s, X_s)ds + B_t \ .$$

Let $\mu_t = P \circ X_t^{-1}, \mu_t^n = P^n \circ X_t^{-1}$ the corresponding one dimensional marginals. Assume the following holds true:
i)

$$\int_0^t \|\mathrm{b}_n\|^2(s)ds < +\infty \quad P^n - a.s., \qquad \int_0^t \|\mathrm{b}\|^2(s)ds < +\infty \quad P - a.s., \qquad \forall n \geq 1 \quad \forall t \geq 0 \ ,$$

ii) there exists a decreasing sequence $\{D_k\}_{k\geq 1}$ of Borel-measurable subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$\forall k \geq 0$, for a.e. $t \geq 0$, there exist subsequences $\{\mathrm{b}_{n_j^k}\}_{j\geq 1}$, $\{\mu_t^{n_j^k}\}_{j\geq 1}$ such that

$$\lim_{j\to+\infty} \int_{D_k^c} \|\mathrm{b}_{n_j^k}(t,x) - \mathrm{b}(t,x)\|^2 d\mu_t^{n_j^k}(x)dt = 0 \quad ,$$

iii)

$$\lim_{k\to+\infty} P(\tau_k < t) = 0 \quad \forall t > 0,$$

where τ_k denotes the first hitting time of $Y_t = (t, X_t)$ to the set D_k,
iv)

$$\lim_{n\to+\infty} \|\mu_0^n - \mu_0\| = 0 \quad .$$

Then there exists a subsequence $\{P^{n_j}\}_{j\geq 1}$ such that

$$\lim_{j\to+\infty} d(P^{n_j}, P) = 0 \ .$$

Remark 3.2. By lemma 2.1, hypothesis i) is equivalent to
i')

$$P^n_{|\mathcal{F}_t} \ll W^n_{|\mathcal{F}_t} \quad P_{|\mathcal{F}_t} \ll W_{|\mathcal{F}_t} \quad \forall t \geq 0 \quad , \forall n \geq 1 \ ,$$

where $W^n := \int_{\mathbb{R}^d} W_x d\mu_0^n(x)$, $W := \int_{\mathbb{R}^d} W_x d\mu_0(x)$, and W_x is the standard Wiener measure supported on Ω_x, the space of continuous paths starting at x. This obviously implies that

$$\mu_t, \mu_t^n \ll m \quad \forall t > 0, \ \forall n \geq 1,$$

where m denotes Lebesgue measure.

Remark 3.3. According to [BG], theorem 10.7, definition 10.21, theorem 3.1 holds true with the weaker hypothesis that the sets D_k are "nearly Borel set", i.e. there exist Borel sets B_k and B'_k such that

$$B_k \subset D_k \subset B'_k,$$

and the set $B'_k \cap B^c_k$ is polar for Y_t w.r.t. P^n, P, for all n. We will make use of this later (see lemma 9.1).

For the proof of theorem 3.1 we need the two following lemmas.

Lemma 3.4. *Let P, P^n, $n \geq 1$, be probability measures on Ω, and let $P = \int P_x d\mu(x)$, $P^n = \int P^n_x d\mu_n(x)$ be the disintegration of such measures w.r.t. X_0, $\mu = P \circ X_0^{-1}, \mu_n = P^n \circ X_0^{-1}$. Let $\{\tau_k\}_{k\geq 1}$ be a non-decreasing sequence of $\mathcal{F}_t$-stopping times, and let $\{P^k_x\}_{k\geq1}, \{P^{n,k}_x\}_{k\geq 1}$ be sequences of probability measures such that, for each $k \geq 1$, $P^k_{x|\mathcal{F}_k} = P_{x|\mathcal{F}_k}, P^{n,k}_{x|\mathcal{F}_k} = P^n_{x|\mathcal{F}_k}$, with $\mathcal{F}_k \equiv \mathcal{F}_{\tau_k}$. Finally assume that, for each $k \geq 1$, there exist subsequences $\{P^{n^k_j,k}_x\}_{j\geq 1}$, $\{\mu_{n^k_j}\}_{j\geq1}$ such that*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^d} \|P^{n^k_j,k}_x - P^k_x\| d\mu_{n^k_j}(x) = 0 \quad .$$

If

$$\lim_{k\to+\infty} P(\tau_k < t) = 0 \quad \forall t > 0, \quad \text{and} \quad \lim_{n\to+\infty} \|\mu_n - \mu\| = 0 \quad ,$$

then there exists a subsequence $\{P^{n_j}\}_{j\geq 1}$ such that

$$\lim_{j\to+\infty} d(P^{n_j}, P) = 0 \quad \forall t \geq 0.$$

Remark 3.5. The above lemma is similar to lemma 11.1.1 in [SV], with weak convergence replaced by convergence in variation. We don't need here any hypothesis of lower semi-continuity for the stopping times. Moreover we remark that the disintegration of measures assumed in lemma 3.4 always exists since Ω is a Polish space (see [DM], chap.III, nos70 to 74).

Proof of lemma 3.4: let $\nu_{xnt} = \nu^+_{xnt} - \nu^-_{xnt}$ be the Jordan decomposition of the signed measure $\nu_{xnt} := P^n_{x\,|\mathcal{F}_t} - P_{x|\mathcal{F}_t}$. We have

$$\|P^n_x - P_x\|_{\mathcal{F}_t} = |\nu_{xnt}|(\Omega) = \nu^+_{xnt}(\Omega) + \nu^-_{xnt}(\Omega) \geq |P^n_x(E) - P_x(E)| \quad \forall E \in \mathcal{F}_t \quad .$$

For each n, let $\{A_x^{tn}, B_x^{tn}\}$, $A_x^{tn} \cup B_x^{tn} = \Omega$, be a Hahn decomposition of Ω for ν_{xnt}. Then

$$\|P_x^n - P_x\|_{\mathcal{F}_t} = \nu_{xnt}^+(A_x^{tn}) + \nu_{xnt}^-(B_x^{tn}) \quad .$$

Since $\{\tau_k < t\}$, and $E \cap \{\tau_k \geq t\}$, are $\mathcal{F}_k$–measurable for each $\mathcal{F}_t$–measurable E, and $P^{n,k}_{x|\mathcal{F}_k} = P^n_{x|\mathcal{F}_k}$, we have

$$\begin{aligned} |P_x^n(E) - P_x(E)| &= |P_x^n(E \cap \{\tau_k < t\}) + P_x^{n,k}(E) - P_x^{n,k}(E \cap \{\tau_k < t\}) - P_x(E)| \\ &\leq |P_x^{n,k}(E) - P_x(E)| + |P_x^{n,k}(E \cap \{\tau_k < t\}) - P_x^n(E \cap \{\tau_k < t\})| \\ &\leq |P_x^{n,k}(E) - P_x(E)| + 2P_x^{n,k}(\tau_k < t) \quad . \end{aligned}$$

Analogously we have

$$|P_x^k(E) - P_x(E)| \leq 2P_x(\tau_k < t) \quad ,$$

so that

$$\begin{aligned} \nu_{xnt}^+(A_x^{tn}) &= P_x^n(A_x^{tn}) - P_x(A_x^{tn}) \\ &\leq |P_x^{n,k}(A_x^{tn}) - P_x^k(A_x^{tn})| + 2P_x^{n,k}(\tau_k < t) + 2P_x(\tau_k < t) \\ &\leq \|P_x^{n,k} - P_x^k\| + 2P_x^{n,k}(\tau_k < t) + 2P_x(\tau_k < t) \\ &\leq 3\|P_x^{n,k} - P_x^k\| + 4P_x(\tau_k < t) \quad . \end{aligned}$$

An analogous estimate holds for $\nu_{xnt}^-(B_x^{tn})$. From

$$\|P^n - P\| \leq \int_{\mathbb{R}^d} \|P_x^n - P_x\| d\mu_n(x) + \|\mu_n - \mu\| \quad ,$$

one derives

$$\begin{aligned} \|P^n - P\|_{\mathcal{F}_t} &\leq 6 \int_{\mathbb{R}^d} \|P_x^{n,k} - P_x^k\| d\mu_n(x) \\ &\quad + 8 \int_{\mathbb{R}^d} P_x(\tau_k < t) d\mu_n(x) + \|\mu_n - \mu\| \\ &\leq 6 \int_{\mathbb{R}^d} \|P_x^{n,k} - P_x^k\| d\mu_n(x) + 8P(\tau_k < t) + 9\|\mu_n - \mu\| \quad . \end{aligned}$$

This implies that $P_{|\mathcal{F}_t}$ is a limit point of $\{P^n_{|\mathcal{F}_t}\}_{n \geq 1}$, and our thesis follows.

Lemma 3.6. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, B_t, P)$ be a weak solution of the stochastic differential equation*

$$X_t = X_0 + \int_0^t \mathrm{b}(s, X_s) ds \; + \; B_t,$$

and let $P = \int P_x d\mu(x)$ *be the disintegration of* P *w.r.t. the random variable* X_0, $\mu = P \circ X_0^{-1}$. *Let* τ *be a* $\mathcal{F}_t$*–stopping time. Define* $P_x^\tau := P_x \circ X_\tau^{-1}$, *with*

$$X_\tau \ : \ \Omega \to \Omega \qquad\qquad X_\tau(\gamma)(t) := X_{t\wedge\tau(\gamma)}(\gamma) \quad .$$

Then

$$B_t^{x,\tau} := X_t - x - \int_0^{t\wedge\tau} \mathrm{b}(s, X_s)ds$$

is a P_x^τ*–Brownian motion for* μ*–a.e.* $x \in \mathbb{R}^d$.

Proof : by our hypotheses

$$M_t^f := f(X_t) - \int_0^t L_s f(X_s)ds \quad ,$$

with $L_s := \frac{1}{2}\Delta + \mathrm{b}_s \cdot \nabla$, is a P–martingale for each $f \in C_c^2(\mathbb{R}^d)$, so that

$$\int_A M_s^f(\gamma)dP(\gamma) = \int_A M_t^f(\gamma)dP(\gamma) \qquad \forall\ A \in \mathcal{F}_s \quad \forall\ s \leq t \ .$$

From the definition of disintegration of a measure it follows that $\gamma \mapsto P_{X_0(\gamma)}(\ \cdot\ )$ is a version of the conditional probability $P(\ \cdot\ |\sigma(X_0))$ (see [DM], chap.III, no.70), so that, $\forall\ B \in \mathcal{F}_0, \forall s \leq t$,

$$\int_B \int_A M_s^f(\gamma')dP_{X_0(\gamma)}(\gamma')dP(\gamma) = \int_{A\cap B} M_s^f(\gamma)dP(\gamma)$$
$$= \int_{A\cap B} M_t^f(\gamma)dP(\gamma) = \int_B \int_A M_t^f(\gamma')dP_{X_0(\gamma)}(\gamma')dP(\gamma) \quad .$$

Since $B \in \mathcal{F}_0$ is arbitrary, and $\mu = P \circ X_0^{-1}$, we have that M_t^f is a P_x–martingale for μ–a.e. $x \in \mathbb{R}^d$. From this, and the definition of P_x^τ, we have that

$$f(X_t) - \int_0^{t\wedge\tau} L_s f(X_s)ds$$

is a P_x^τ–martingale, and the lemma now follows from the equivalence between existence of solutions of martingale problem and existence of weak solutions of stochastic differential equations (see [St], theorem 2.6, chap.3).

Proof of theorem 3.1: let $P_x^k \equiv P_x^{\tau_k'}, P_x^{n,k} \equiv P_x^{n,\tau_k'}$, $\tau_k' := \tau_k \wedge k$, be defined as in lemma 3.6. Then there exist sets A, A_n, with $\mu_0(A) = \mu_0^n(A_n) = 1$, such that, $\forall x \in \tilde{A}_n := A \cap A_n$,

$$B_t^k = X_t - x - \int_0^{t\wedge\tau_k'} \mathrm{b}(s, X_s)ds \ ,$$

and

$$B_t^{n,k} = X_t - x - \int_0^{t\wedge\tau_k'} \mathrm{b}_n(s, X_s)ds$$

are Brownian motions w.r.t. P_x^k and $P_x^{n,k}$ respectively (since $\mu_0^n \to \mu_0$ $\exists$ $\bar{n}$ s.t. $\tilde{A}_n \neq \emptyset$ $\forall$ $n \geq \bar{n}$). Since, by our hypotheses i),

$$\int_{\mathbb{R}_+} \chi_{[0,\tau_k')}(s)\|\mathrm{b}\|^2(s)ds < +\infty \quad P_x^k - \text{a.s.},$$

$$\int_{\mathbb{R}_+} \chi_{[0,\tau_k')}(s)\|\mathrm{b}_n\|^2(s)ds < +\infty \quad P_x^{n,k} - \text{a.s.},$$

for all $x \in \tilde{A}_n$, by lemma 2.3 we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} \|P_x^{n,k} - P_x^k\| d\mu_0^n(x)\right)^2 \\
&\leq \int_{D_k^c} \|\mathrm{b}_n(t,y) - \mathrm{b}(t,y)\|^2 d\mu_t^n(y)dt \\
&\quad + \int_{\tilde{A}_n^c} \|P_x^{n,k} - P_x^k\|^2 d\mu_0^n(x) \ .
\end{aligned}$$

Since $\mu_0^n(\tilde{A}_n^c) \to 0$, lemma 3.4 implies our thesis.

Remark 3.7. Since one can exchange P with P' in lemma 2.3, the statement of theorem 3.1 does nor change if we substitute hypothesis ii) with the following one:
ii') there exists a decreasing sequence $\{D_k\}_{k\geq 1}$ of Borel–measurable subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ such that $\forall k \geq 0$, there exists a subsequence $\{\mathrm{b}_{n_j^k}\}_{j\geq 1}$, such that

$$\lim_{j\to+\infty} \int_{D_k^c} \|\mathrm{b}_{n_j^k}(t,x) - \mathrm{b}(t,x)\|^2 d\mu_t(x)dt = 0 \quad .$$

4. The case of drifts in $L^1_{loc}(\mathbb{R}; L^p(\mathbb{R}^d)) + L^\infty(\mathbb{R}^{d+1})$, $p > d+2$.

In this paragraph we will show that theorem 3.1 contains the results previously obtained by various authors (see [P2], chap.I, §6, for the L^p case, and [SV], chap.11, [JS], chap.V, §4d, for the L^∞ case). To this end, in order to apply theorem 3.1, we recall the following

Theorem 4.1.([P1]) *Let* $\mathrm{b} = \mathrm{b}^p + \mathrm{b}^\infty$, *with* $\mathrm{b}^p \in L^1_{loc}(\mathbb{R}; L^p(\mathbb{R}^d))$, $p > d+2$, *and* $\mathrm{b}^\infty \in L^\infty(\mathbb{R}^{d+1})$. *Then there exists an unique* $P \in \mathcal{M}_1(\Omega)$, $P \ll W$ *on* $\mathcal{F}_t$ $\forall t \geq 0$, *which solves, as a weak solution, the stochastic differential equation*

$$X_t = X_0 + \int_0^t \mathrm{b}_s ds + B_t \ .$$

Remark 4.2. Since existence of weak solutions is equivalent to existence of the corresponding martingale problem (see e.g. [St], theorem 2.6, chap. 3) by the results in [SV], §10.0, it follows that thm.4.1 implies that $\lim_{n\to+\infty} P(\tau_n < t) = 0$, $\forall t > 0$, where τ_n denotes the first hitting time to the set $\{\|(t,x)\| > n\}$, i.e. the process does not "explode".

By the previous theorem we can apply theorem 3.1 to the situation in which the drifts are in $L^1_{loc}(\mathbb{R}; L^p(\mathbb{R}^d)) + L^\infty(\mathbb{R}^{d+1})$, $p > d+2$:

Theorem 4.3. *Let* $\mathrm{b}, \mathrm{b}_n, n \geq 1$, *be time-dependent drift vector fields in* $L^1_{loc}(\mathbb{R}; L^p(\mathbb{R}^d)) + L^\infty(\mathbb{R}^{d+1})$, $p > d+2$, *and let* $P, P^n, n \geq 1$ *be the corresponding sequence of weak solutions obtained solving the stochastic differential equations with drifts* b, b_n *and initial distributions* μ, μ_n. *If*

$$\lim_{n\to+\infty} \int_0^T \int_{\|x\|\leq R} \|\mathrm{b}^p_n(t,y) - \mathrm{b}^p(t,y)\|^p dy dt = 0 \quad \forall T \geq 0,\ \forall R \geq 0\ ,$$

$$\lim_{n\to+\infty} ess \sup_{|t|\leq T, \|x\|\leq R} \|\mathrm{b}^\infty_n(t,x) - \mathrm{b}^\infty(t,x)\| = 0 \quad \forall T \geq 0,\ \forall R \geq 0\ ,$$

and

$$\lim_{n\to+\infty} \|\mu_n - \mu\| = 0\ ,$$

then

$$\lim_{n\to+\infty} d(P^n, P) = 0 \quad .$$

Proof: By [P2], lemma 1.8, chap.I, we have

$$\int_0^T \int_{\|x\|\leq R} \|\mathrm{b}^p_n(t,y) - \mathrm{b}^p(t,y)\|^2 d\mu_t(y) dt \leq C_T \left(\int_0^T \int_{\|x\|\leq R} \|\mathrm{b}^p_n(t,y) - \mathrm{b}^p(t,y)\|^p dy dt \right)^{\frac{1}{p}} ,$$

Then convergence follows from theorem 3.1 and remark 3.7 using the sequence of sets $D_k := \{\ |t| + \|x\| \geq k\ \}$, and observing that iii) holds true because this is equivalent to the non explosion of the process.

Remark 4.4. If the drifts are assumed to be all time–independent then the previous theorem holds true with $p > d$ (see [P2], chap.2, §5).

5. The role of capacity: the case of drifts in $H^1(\mathbb{R}^d)$.

Up to now the choice of the sets D_k we made was trivial. We simply use them to avoid to control the convergence of the sequence of drifts at infinity. Now we will show that a more clever choice of the sets D_k will allow us to treat more singular cases. To this end we premise the following key remark.

Remark 5.1. Let B be a Borel (or analytic) set such that $cap(B) = 0$. Here cap denotes the Choquet capacity defined for an open set B by

$$cap(B) := \inf\{ \ \|\phi\|^2_{H^1} \ : \ \phi \in H^1(\mathbb{R}^d), \ \phi \geq \chi_B \ a.e. \ \} \quad ,$$

where χ_B is the characteristic function of B, and by

$$cap(E) := \inf_{B \ open, \ B \supset E} cap(B)$$

for any set E. Let P a probability measure such that $P_{|\mathcal{F}_t} \ll W_{|\mathcal{F}_t}$, $\forall t \geq 0$. Here $W = \int W_x d\mu(x)$, and μ is the initial distribution of P. Suppose moreover that $cap(B) = 0$ implies $\mu(B) = 0$ (for example this is true if $\mu \ll m$, where m denotes the Lebesgue measure). Since $cap(B) = 0 \iff W_x(\tau_B < T) = 0 \quad \forall T > 0, \quad \forall x \in B^c$, where τ_B denotes the first hitting time to the set B (see [Fu1]), we have that $cap(B) = 0$ implies $P(\tau_B < T) = 0$, i.e. capacity zero sets will be polar for the process X_t w.r.t. the probability measure P. Moreover, since (see [Fu1])

$$cap(B_k) = \|e_k\|^2_{H^1}, \quad e_k(x) := E_{W_x}(e^{-\tau_{B_k}}) \quad ,$$

if $\{B_k\}_{k \geq 0}$ is a decreasing sequence of open subsets of $\mathbb{R}^d$ such that $cap(B_k) \downarrow 0$, we have $W(\tau_{B_k} < T) \downarrow 0 \quad \forall T > 0$, and consequently $P(\tau_{B_k} < T) \downarrow 0 \quad \forall T > 0$.

From the previous remark it is obvious that a good choice for the sequence of sets D_k appearing in theorem 3.1 is a sequence of sets with capacity decreasing to zero. Therefore condition ii) in theorem 3.1 forces us to look for the right functions with respect to Newtonian capacity. The following theorem is a well know results of potential theory (see e.g. [Fu1], §3.1)

Theorem 5.2. *i) Let u be in $H^1(\mathbb{R}^d)$. Then there exists a decreasing sequence of open sets $D_k \subset \mathbb{R}^d$ such that $cap(D_k) \downarrow 0$, and the restriction of u to D_k^c is continuous for all k.*
ii) Let $\{u_n\}_{n\geq 1} \subset H^1(\mathbb{R}^d)$ be a Cauchy sequence converging to u. Then there exist a decreasing sequence of open sets $D_k \subset \mathbb{R}^d$ such that $cap(D_k) \downarrow 0$, and a subsequence $\{u_{n_j}\}_{j\geq 1}$ converging pointwise and uniformly to u on D_k^c, for all k.

Now the following theorem easily follows:

Theorem 5.3. *$P, P^n, n \geq 1$ be a sequence of weak solutions of stochastic differential equations with drifts b, b_n in $H^1(\mathbb{R}^d)$, and initial distributions μ, μ_n. Suppose that μ charges no zero (Newtonian) capacity set, and that the hypotheses i) of theorem 3.1 holds true. If*

$$\lim_{n\to+\infty} \|\mathrm{b}_n - \mathrm{b}\|_{H^1} = 0 \ ,$$

$$\lim_{n\to+\infty} \|\mu_n - \mu\| = 0 \ ,$$

then

$$\lim_{n\to+\infty} d(P^n, P) = 0 \quad .$$

Proof: by the hypotheses i) in theorem 3.1 it follows that $P_{|\mathcal{F}_t} \ll W_{|\mathcal{F}_t}$, $\forall t \geq 0$ (see remark 3.2). Moreover, by the remark 5.1, hypothesis iii) in theorem 3.1 holds true for any sequence of sets D_k such that $cap(D_k) \downarrow 0$. The hypotheses on our sequence of drifts imply that, by the preceeding theorem, there exists a sequence $\{D_k\}_{k\geq 1}$ of open subsets of $\mathbb{R}^d$, with $cap(D_k) \downarrow 0$, such that

$$\lim_{j\to+\infty} \|\mathrm{b}_{n_j} - \mathrm{b}\|_{L^\infty(\hat{D}_k^c)} = 0, \quad \forall k \geq 1,$$

for some subsequence $\{b_{n_j}\}_{j\geq 1}$, where $\hat{D}_k := D_k \cup \{\|x\| > k\}$. Proceeding as in theorem 4.3, by using the sets $\tilde{D}_k := D_k \cup \{\|x\| > k\} \cup \{|t| > k\}$, one proves the existence of a converging subsequence $\{P^{n_j}_{|\mathcal{F}_t}\}_{j\geq 1}$. Suppose now that the whole sequence $\{P^n_{|\mathcal{F}_t}\}_{n\geq 1}$ does not converge. Then there exist an $\epsilon > 0$, a $T \geq 0$, and a subsequence $\{P^{n_k}\}_{k\geq 1}$ and such that $\|P^{n_k}_{|\mathcal{F}_t} - P_{|\mathcal{F}_t}\| > \epsilon$ for all k, and for all $t \geq T$. But by the above reasoning applied to the convergent sequence $\{\mathrm{b}_{n_k}\}_{k\geq 1}$ we get a further subsequence along which the measures converge to P, which would be a contradiction, so $\{P^n_{|\mathcal{F}_t}\}_{n\geq 1}$ converges to $P_{|\mathcal{F}_t}$ for all $t \geq 0$.

6. Nelson Diffusions.

Theorem 5.3 in the preceeding section is not satisfying since we have to assume the existence of weak solutions of the stochastic differential equations corresponding to the given sequence of drifts, and we have to explicitly require that all such probability measures are absolutely continuous w.r.t. the Wiener measure, i.e. that they satisfy i) in theorem 3.1. Moreover we have not yet fully used theorem 3.1, in the sense that this theorem is especially powerful when one has an explicit knowledge of the densities of the processes. This will allow us to treat more singular drifts than the ones allowed in theorem 5.3. To this end we introduce the class of Nelson Diffusions. Such diffusions are defined in term of their proper infinitesimal characteristics (see [C3]):

The space $\mathcal{P}$ of proper infinitesimal characteristics is defined to be the set of pairs (v, ρ) with ρ_t a time dependent probability density on $\mathbb{R}^d$ and v_t a time dependent vector field on $\mathbb{R}^d$ defined $\rho_t(x)dxdt$–a.e. so that

$$\int_{\mathbb{R}^d} f(x,T)\rho(x,T)dx - \int_{\mathbb{R}^d} f(x,0)\rho(x,0)dx = \int_0^T \int_{\mathbb{R}^d} (v \cdot \nabla f)\rho(x,t)dxdt \ , \qquad (F.P.)$$

and

$$\int_0^T \left(\|\nabla \sqrt{\rho_t}\|_{L^2}^2 + \|v_t \sqrt{\rho_t}\|_{L^2}^2 \right) dt < +\infty \ , \qquad (F.E.C.)$$

for all $T \geq 0$, and all $f \in C_0^\infty(\mathbb{R}^{d+1})$.

The following theorem ([C1], [C2], [C3]) give us a map from $\mathcal{P}$ into $\mathcal{M}_1(\Omega)$. The probability measures in the range of such map will be called Nelson Diffusions.

Theorem 6.1. *Consider the measurable space $(\Omega, \mathcal{F})$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathcal{F}$ the Borel σ–algebra. Let (v, ρ) be a proper infinitesimal characteristic, define* $\mathrm{b} := u + v$, *where*

$$u := \frac{1}{2} \frac{\nabla \rho(x)}{\rho(x)} \ ,$$

if $\rho(x) \neq 0$, $u(x) := 0$ otherwise, and let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t)$ be the evaluation stochastic process $X_t(\gamma) := \gamma(t)$, with $\mathcal{F}_t = \sigma(X_s, s \leq t)$ the natural filtration. Then there exists a unique Borel probability measure P on Ω such that :

i) $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P)$ is a Markov process;

ii) the image of P under X_t has density $\rho(t, x)$;

iii)

$$B_t := X_t - X_0 - \int_0^t \mathrm{b}(s, X_s)ds$$

is a P-Brownian motion.

Remark 6.2. From the definition of proper infinitesimal characteristic, ii) in theorem 6.1, and iv) in lemma 2.1, it follows

$$H_{\mathcal{F}_t}(P;W) = \frac{1}{2}E\int_0^T \|\mathrm{b}(s)\|^2 ds < \infty \qquad \forall T \geq 0$$

(E denotes the expectation w.r.t. P), so that (F.E.C.) is a finite entropy condition. This moreover implies

$$P_{|\mathcal{F}_t} \ll W_{|\mathcal{F}_t} \quad \forall t \geq 0 \ ,$$

so that, by remark 3.2, the hypothesis i) in theorem 3.1 holds true.

Since $\sigma(\cup_{t\geq 0}\mathcal{F}_t) = \mathcal{F}$, by [JS], theorem 4.23, and corollary 2.8, chap.IV, if

$$E\int_{\mathbb{R}_+} \|\mathrm{b}(s)\|^2 ds < \infty \quad ,$$

then $P \ll W$, and, in this case,

$$H(P;W) = \sup_{T\in\mathbb{R}_+} H_{\mathcal{F}_T}(P;W) = \frac{1}{2}E\int_{\mathbb{R}_+} \|\mathrm{b}(s)\|^2 ds \quad .$$

7. Convergence of Diffusions corresponding to Energy Forms and regularization of Hamiltonians

Let ρ be a probability density such that $\sqrt{\rho} \in H^1(\mathrm{IR}^d)$. Then the pair $(0,\rho)$ is obviously a proper infinitesimal characteristic for a Nelson Diffusion P. Theorem 6.1 implies that P solves, as a weak solution, a stochastic differential equation with drift $\frac{1}{2}\nabla\log\rho$. Moreover P is a stationary probability measure with density given by ρ. Such diffusion process may be alternatively characterized (see [Fu2]) as the one associated to the Dirichlet Form (sometime in this case one speaks of Energy Form)

$$\mathcal{E}_\rho(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g(x)\rho(x)dx \qquad f,g \in C_c^\infty(\mathrm{IR}^d).$$

We will show now that theorem 3.1 gives us a convergence theorem for diffusion processes associated to energy forms within natural hypotheses (compare with the results in [AH-KS2] and [AKS] where only convergence of the finite dimensional distributions is given).

Theorem 7.1. *Let $\{\rho_n\}_{n\geq 1}$ be a sequence of probability densities such that $\{\sqrt{\rho}_n\}_{n\geq 1}$ is a Cauchy sequence in $H^1(\mathbb{R}^d)$. Let $\{P^n\}_{n\geq 1}$ the sequence of probability measures on the path space $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ corresponding to the sequence of energy forms $\{\mathcal{E}_{\rho_n}\}_{n\geq 1}$, as given by theorem 6.1. If*

$$\lim_{n\to+\infty} \|\sqrt{\rho}_n - \sqrt{\rho}\|_{H^1} = 0 \ ,$$

then

$$\lim_{n\to+\infty} d(P^n, P) = 0 \quad .$$

where P is the probability measure corresponding to $\mathcal{E}_\rho$.

Proof : for simplicity we will suppose that $\forall R > 0$ there exists a $\delta_R > 0$ such that the infimum of ρ on $\{\|x\| \leq R\}$ is greater than δ_R; the general case will be contained in theorem 10.1.

We essentially proceed as in the proof of theorem 5.3. By remarks 3.2 and 6.2 it follows that the hypothesis i) in theorem 3.1 holds true. Moreover from remark 5.1 and thm.5.2 it follows that hypothesis iii) in thm.3.1 holds true for a sequence of sets D_k such that

$$\lim_{j\to+\infty} \|\sqrt{\rho_{n_j}} - \sqrt{\rho}\|_{L^\infty(\hat{D}_k^c)} = 0, \quad \forall k \geq 1,$$

for some subsequence $\{\rho_{n_j}\}_{j\geq 1}$, where $\hat{D}_k := D_k \cup \{\|x\| > k\}$. Let us now consider the sets $\tilde{D}_k := D_k \cup \{\|x\| > k\} \cup \{|t| > k\}$. From the definitions of b_n and ρ_n, we have

$$\begin{aligned}
&\int_{\tilde{D}_k^c} \|\mathrm{b}_n - \mathrm{b}\|^2 \rho_n dy dt \\
&= \int_{\tilde{D}_k^c} \left\| \nabla\sqrt{\rho_n}(y) - \sqrt{\frac{\rho_n}{\rho}} \nabla\sqrt{\rho}(y) \right\|^2 dy dt \\
&\leq \int_0^k \int_{\mathbb{R}^d} \|\nabla\sqrt{\rho_n}(y) - \nabla\sqrt{\rho}(y)\|^2 dy dt + \int_{\tilde{D}_k^c} \frac{\|\nabla\sqrt{\rho}\|^2}{\rho} \|\sqrt{\rho_n}(y) - \sqrt{\rho}(y)\|^2 dy dt \\
&\leq k\|\nabla\sqrt{\rho_n} - \nabla\sqrt{\rho}\|_{L^2}^2 + k\delta_k^{-1}\|\nabla\sqrt{\rho}\|_{L^2}^2 \|\sqrt{\rho_n} - \sqrt{\rho}\|_{L^\infty(\hat{D}_k^c)}^2 \ ,
\end{aligned}$$

and so we have proved the existence of a converging subsequence $\{P^{n_j}_{|\mathcal{F}_t}\}_{j\geq 1}$. The convergence of the whole sequence now follows proceeding as in the proof of thm.5.3.

Remark 7.2. For the case of diffusions corresponding to energy forms, thm.6.1 can be improved in the following sense (see [RT–S], §6):
Conclusions i)–iii) of thm.6.1 hold true if $\rho \in L^1(\mathbb{R}^d)$, $\rho > 0$ a.e., and $\sqrt{\rho} \in H^1_{loc}(N^c)$, with $cap_\rho(N) = 0$, N closed.

Here cap_ρ denotes the capacity corresponding to the energy form $\mathcal{E}_\rho$, i.e. (see e.g. [Ful])

$$cap_\rho(E) := \inf\{ \ \mathcal{E}_\rho(u,u) + \langle u\sqrt{\rho}|u\sqrt{\rho}\rangle_{L^2} \ : \ u \in \mathcal{D}(\mathcal{E}_\rho), \ u \geq \chi_E \ a.e. \ \}.$$

Since $cap_\rho(N) = 0$ is equivalent to nonattainability (see [Ful]), and since an analogous of thm.5.2 holds for Newtonian capacity relative to an open subset $U \subseteq \mathbb{R}^d$

$$cap(E;U) := \inf\{ \ \|u\|^2_{H^1(U)} \ : \ u \in H^1(U), \ u \geq \chi_E \ a.e. \ \}$$

(see e.g. [Fr]), it is easy to generalize thm.7.1 to obtain convergence if the limiting density satisfies the hypotheses of this remark. Let us remark that, since N is nonattainable, denoting by τ_n the first hitting time to the set $N \cup \{\|x\| \geq n\}$, $P_x\{\tau_n < t\} \downarrow 0$, $\forall t > 0$, implies

$$H_{\mathcal{F}_{t\wedge\tau_n}}(P_x; W_x) < +\infty, \quad \forall t \geq 0, \ \forall n \geq 0 \quad \Rightarrow \quad P_{x|\mathcal{F}_t} \ll W_{x|\mathcal{F}_t}, \quad \forall t \geq 0.$$

Therefore i) in thm.3.1 holds true. This also implies

$$cap(E) = 0 \quad \Rightarrow \quad cap_\rho(E) = 0.$$

Remark 7.3. If ρ satisfies the hypotheses in the preceeding remark then the corresponding energy form is closable, and, if A denotes the symmetric operator on $L^2(\rho dx)$ corresponding to $\mathcal{E}_\rho$, then A is essentially self–adjoint on $C_c^\infty(\mathbb{R}^d)$ (see [RT–S], §4). Moreover A is unitary equivalent to a "generalized" Schrödinger operator on $L^2(\mathbb{R}^d)$. In fact, setting $\phi = \sqrt{\rho}$, one can define a self–adjoint operator on $L^2(\mathbb{R}^d)$ by $H := \phi H \phi^{-1}$, and, if $\nabla\phi, \phi^{-1}\nabla\phi \in L^2_{loc}(\mathbb{R}^d)$, then the distribution $V := \frac{1}{2}\phi^{-1}\Delta\phi$ is a continuous linear functional on $C_c^1(\mathbb{R}^d)$ and the quadratic form defined by H is given, on $C_c^1(\mathbb{R}^d)$, by (see [AH–KS1], thm.2.3)

$$\frac{1}{2}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 dx + V(f^2) \ .$$

Let H be defined as in the preceeding remark. In general it is not possible to write H as the perturbation of the Laplacean by a potential V, which need not exists as a measurable function (see [AH–KS1] for some examples). It is however natural to ask if it is possible to find approximations of the "generalized" Schrödinger operator H by "smoother" ones which are perturbations of $-\frac{1}{2}\Delta$ by smooth potentials. Theorem 7.1 implies that this is the case (see [AH–KS2], [AKS] for results of this kind obtained under other conditions):

Theorem 7.4. *Let $\{\rho_n\}_{n\geq 1}$ be a sequence of a.e. positive probability densities such that $\sqrt{\rho_n} \in H^1(\mathbb{R}^d)$, $\forall n \geq 1$, and let $\{H_n\}_{n\geq 1}$ be the sequence of generalized Schrödinger* *operators corresponding to the energy forms $\mathcal{E}_{\rho_n}$. If there exists an a.e. positive probability density ρ, $\sqrt{\rho} \in H^1_{loc}(N^c)$, such that*

$$\lim_{n \to +\infty} \|\rho_n - \rho\|_{L^1} = 0,$$
$$\lim_{n \to +\infty} \|\sqrt{\rho_n} - \sqrt{\rho}\|_{H^1_{loc}(N^c)} = 0,$$

where N is a closed set such that $cap_\rho(N) = 0$, then $H_n \to H$ in strong resolvent sense, where H is the self-adjoint operator corresponding to $\mathcal{E}_\rho$.

Proof: let $\rho_n^x(t,y)$, $t > 0$, be the densities of the diffusion processes starting at x, corresponding to the energy forms $\mathcal{E}_{\rho_n}$. By Markovianicity one has $\rho_n^x(t,y) \leq \rho_n(y)$, $\forall t > 0$, for $\rho_n(x)dx$–a.e. x. Therefore by thm.7.1 and remark 7.2 one has

$$\lim_{n \to +\infty} ess \sup_{x \in A_n} \|P_x^n - P_x\| = 0, \qquad \lim_{n \to +\infty} \int_{A_n^c} \rho_n(x)dx = \lim_{n \to +\infty} \int_{A_n^c} \rho(x)dx = 0, \quad (\circ)$$

where P_x is the measure corresponding to $\mathcal{E}_\rho$. Let

$$T_t^n := e^{-tH_n}, \quad T_t := e^{-tH}.$$

By the definition of H_n, H, one has, putting $\phi_n^2 = \rho_n, \phi^2 = \rho$,

$$T_t^n f(x) = \phi_n(x) E_x^n[(f/\phi_n)(X_t)], \quad T_t f(x) = \phi(x) E_x[(f/\phi)(X_t)], \quad f \in L^2(\mathbb{R}^d).$$

Let $f \in L^\infty(\mathbb{R}^d)$, then

$$\begin{aligned}
&\|T_t^n(f\phi_n) - T_t(f\phi)\|_{L^2}^2 \\
\leq &\int_{A_n} |\phi_n(x) E_x^n(f \circ X_t) - \phi(x) E_x(f \circ X_t)|^2 dx \\
+ &\int_{A_n^c} |\phi_n(x) E_x^n(f \circ X_t) - \phi(x) E_x(f \circ X_t)|^2 dx \\
\leq &\|\phi_n - \phi\|_{L^2}^2 \; ess \sup_{x \in A_n} |E_x(f \circ X_t)|^2 + \|\phi_n\|_{L^2}^2 \; ess \sup_{x \in A_n} |E_x^n(f \circ X_t) - E_x(f \circ X_t)|^2 \\
+ &\int_{A_n^c} |\phi_n(x) E_x^n(f \circ X_t) - \phi(x) E_x(f \circ X_t)|^2 dx,
\end{aligned}$$

so that, by $(\circ)$, $\forall f \in L^\infty(\mathbb{R}^d)$,

$$\lim_{n \to +\infty} \|T_t^n(f\phi) - T_t(f\phi)\|_{L^2}^2 \leq \lim_{n \to +\infty} \|T_t^n(f\phi_n) - T_t(f\phi)\|_{L^2}^2 + \|\phi_n - \phi\|_{L^2}^2 = 0.$$

Since $\phi > 0$ a.e., our thesis follows then by density.

8. A Parabolic Capacity

In order to extend theorem 7.1 to the case of general Nelson diffusions we need to introduce a capacity adapted to the space–time process $Y_t = (t, X_t)$, and to functions with the same regularity of time–dependent probability densities corresponding to infinitesimal proper characteristics. Let us consider the Banach space $(\mathcal{W}_T, \|\cdot\|_{\mathcal{W}_T})$, where

$$\mathcal{W}_T := C([0,T], H^1(\mathbb{R}^d)) \qquad \text{and} \qquad \|u\|_{\mathcal{W}_T} := \sup_{0\leq t\leq T} \|u_t\|_{H^1} \quad .$$

We will consider real–valued functions only; considering real and imaginary parts separately, theorem 8.3 below holds for complex–valued functions as well. We need "good" pointwise properties of functions belonging to $\mathcal{W}_T$. To this end we will introduce a sort of parabolic capacity on subsets of $[0,T] \times \mathbb{R}^d$, and we will study properties of elements of $\mathcal{W}_T$ up to sets of arbitrary small capacity. Following the general procedure in reference [AS], we define a set function on subsets of $[0,T] \times \mathbb{R}^d$

$$\Gamma_T(E) := \inf_{\{E_k\}_{k\in\mathbb{N}},\ E_k\ open,\ \cup_k E_k \supset E} \sum_k \delta_T(E_k) \quad ,$$

where, for an open set E

$$\delta_T(E) := \inf\{ \ \|u\|^2_{\mathcal{W}_T} \ : \ u \in \mathcal{W}_T,\ u \geq \chi_E \ a.e. \ \} \quad .$$

The set function Γ_T has the following properties ([AS], pg.146):

P_1:

$$\Gamma_T(\emptyset) = 0;$$

P_2:

$$E \subset E' \Rightarrow \Gamma_T(E) \leq \Gamma_T(E');$$

P_3:

$$\Gamma_T(\cup_k E_k) \leq \sum_k \Gamma_T(E_k);$$

P_4:

$$\forall\ \epsilon > 0 \quad \exists\ \delta > 0 \quad s.t. \quad \delta_T(E) \leq \delta \Rightarrow \Gamma_T(E) \leq \epsilon.$$

From the above definition it is also obvious that there exists a relation between Γ_T and the Choquet capacity defined in remark 5.1: :

Lemma 8.1.

$$\Gamma_T(E) \geq cap(E_t) \qquad \forall t \in [0,T] \quad ,$$

where $E_t := \{\ x \in \mathbb{R}^d \ :\ (t,x) \in E\ \}$.

Proof : from the definitions of Γ_T, *cap*, and by countable subadditivity of *cap* (see [Fu1]), it follows

$$\begin{aligned}\Gamma_T(E) &\geq \inf_{\{E_k\}_{k\in\mathbb{N}},\ E_k\ open,\ \cup_k E_k \supset E} \sum_k cap((E_k)_t) \\ &\geq \inf_{\{E_k\}_{k\in\mathbb{N}},\ E_k\ open,\ \cup_k E_k \supset E} cap(\cup_k (E_k)_t) \\ &\geq \inf_{A\supset E} cap(A_t) = cap(E_t) \quad .\end{aligned}$$

Remark 8.2. Let D be an open subset of $\mathbb{R}_+ \times \mathbb{R}^d$. Since D is open, and $X_{(\cdot)}$ is continuous, we have

$$\{\tau_D < T\} = \bigcup_{q\in Q\cap[0,T)} \{\ X_q \in D_q\ \} \quad ,$$

where Q is any dense denumerable subset of $\mathbb{R}$. By lemma 8.1 and remark 5.1 it follows that if $\Gamma_T(D \cap [0,T] \times \mathbb{R}^d) = 0 \quad \forall T \geq 0$, then D is polar for the process $Y_t := (t, X_t)$ w.r.t. P. Moreover, if $\{D_k\}_{k\geq 1}$ is a decreasing sequence of open subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ such that $\Gamma_T(D_k \cap [0,T] \times \mathbb{R}^d) \downarrow 0 \quad \forall T \geq 0$, then, by lemma 8.1, $cap((D_k)_t) \downarrow 0 \quad \forall t \geq 0$, and, by remark 2.4, $W(\tau_{(D_k)_t} < T) \downarrow 0 \quad \forall T > 0,\ \forall t \geq 0$. Therefore $W(\tau_{D_k} < T) \downarrow 0 \quad \forall T > 0$, and $P(\tau_{D_k} < T) \downarrow 0 \quad \forall T > 0$, by remark 5.1.

We state now the main result of this paragraph. This result does not depend on our particular definition of Γ_T but holds for any capacity defined by means of a "good" functional space (see [AS]).

Theorem 8.3. *i) Let u be in $\mathcal{W}_T$. Then there exists a decreasing sequence of open sets $D_{T,k} \subset [0,T] \times \mathbb{R}^d$ such that $\Gamma_T(D_{T,k}) \downarrow 0$, and the restriction of u to $D^c_{T,k} \cap [0,T] \times \mathbb{R}^d$ is continuous for all k.*
ii) Let $\{u_n\}_{n\geq 1} \subset \mathcal{W}_T$ be a sequence such that $\mathcal{W}_T$-$\lim_{n\to+\infty} u_n = u \in \mathcal{W}_T$. Then there exist a decreasing sequence of open sets $D_{T,k} \subset [0,T] \times \mathbb{R}^d$ such that $\Gamma_T(D_{T,k}) \downarrow 0$, and a subsequence $\{u_{n_j}\}_{j\geq 1}$ converging pointwise and uniformly to u on $D^c_{T,k} \cap [0,T] \times \mathbb{R}^d$ for all k.

Proof : i) first of all we note that $\mathcal{W}_T \cap C([0,T] \times \mathbb{R}^d)$ is dense in $\mathcal{W}_T$. This can be seen considering, for each $u \in \mathcal{W}_T$, the approximating sequence of continuous functions

$u_n(t,x) := (J_{1/n} * u_t)(x)$, where

$$J_{1/n} \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}(J_{1/n}) \subset \{ \ x \ : \ \|x\| \leq 1/n \ \} \quad ,$$

is a molliflier, and then proceeding in the same way as in [LSU], lemma 4.8, chap.II.

From the definition of δ_T we have

$$\delta_T(\{ \ (t,x) \ : \ |u(t,x)| > \lambda \ \}) \leq \frac{1}{\lambda^2}\|u\|^2_{\mathcal{W}_T} \qquad \forall u \in \mathcal{W}_T \cap C([0,T] \times \mathbb{R}^d) \quad . \qquad (*)$$

Now we proceed as in [AS], pp. 148–149 (see also [Fu1], theorem 3.1.3.): let $\{u_n\}_{n\geq 1} \subset \mathcal{W}_T \cap C([0,T] \times \mathbb{R}^d)$ be a sequence such that $\mathcal{W}_T$–$\lim_{n\to+\infty} u_n = u \in \mathcal{W}_T$. Since $\{u_n\}_{n\geq 1}$ is a Cauchy sequence, by (*) and P_4, there exists a subsequence $\{u_{n_j}\}_{j\geq 1}$ such that

$$\Gamma_T(A_j) < 2^{-j} \ ,$$

where the sequence of open sets $\{A_j\}_{j\geq 1}$ is defined by

$$A_j := \{ \ (t,x) \ : \ |u_{n_{j+1}}(t,x) - u_{n_j}(t,x)| > 2^{-j} \ \} \ .$$

If $(t,x) \in (\cup_{j\geq J} A_j)^c$, then $\forall j \geq J$, $\forall p$ we have

$$|u_{n_{j+p}}(t,x) - u_{n_j}(t,x)| \leq \sum_{k=j+1}^{j+p} |u_{n_k}(t,x) - u_{n_{k-1}}(t,x)| \leq 2^{-j} \ ,$$

so that $\{u_{n_j}\}_{j\geq 1}$ uniformly converges on $(\cup_{j\geq J} A_j)^c$. This implies the continuity of u on $(\cup_{j\geq J} A_j)^c$. By P_3 we have

$$\Gamma_T(\cup_{j\geq J} A_j) \leq \sum_{j\geq J} \Gamma_T(A_j) \leq 2^{1-J} \ .$$

Since J is arbitrary, 1) is proven.
ii) by i), proceeding as in [Fu1], lemma 3.1.5., we have

$$\Gamma_T(\{ \ (t,x) \ : \ |u(t,x)| > \lambda \ \}) \leq \frac{1}{\lambda^2}\|u\|^2_{\mathcal{W}_T} \qquad \forall u \in \mathcal{W}_T \quad ,$$

so that, by our hypotheses, u_n converges to u in capacity, i.e.

$$\lim_{n\to+\infty} \Gamma_T(\{ \ (t,x) \ : \ |u_n(t,x) - u(t,x)| > \epsilon \ \}) = 0 \qquad \forall \epsilon > 0 \ .$$

Then one proceeds in essentially the same way as in i).

9. Stopping times and nonattainability

We now define the stopping times we will need for the proof of theorem 10.1 in order to apply theorem 3.1. Let ψ, ψ^n, $n \geq 1$, be functions belonging to $\mathcal{W}_T \quad \forall T \geq 0$. Assume

$$\lim_{n \to +\infty} \|\psi^n - \psi\|_{\mathcal{W}_T} = 0 \qquad \forall T \geq 0 \quad ,$$

and define, $\forall k \geq 1$,

$$\tau_k^j(\gamma) := \inf \{ \, t \geq 0 \; : \; (t, X_t(\gamma)) \in D_k^j \} \qquad j = 1, 2 \; ,$$

where

$$D_k^1 := \{ \, (t, x) \; : \; \|(t, x)\| > k \, \} \cup \{ \, (t, x) \; : \; |\psi(t, x)| < 1/k \, \} \quad ,$$

and the D_k^2's are the open subsets of $\mathbb{R}_+ \times \mathbb{R}^d$

$$D_k^2 := \bigcup_{T \in \mathbb{R}_+} D_{T,k} \quad ,$$

where the sets $D_{T,k}$ are given in theorem 4.3. Define $D_k := D_k^1 \cup D_k^2$; by construction the following holds:

i)

$$\Gamma_T(D_k^2 \cap [0, T] \times \mathbb{R}^d) \downarrow 0 \quad \forall T \geq 0;$$

ii)

$$\psi, \psi^n \in L^\infty(D_k^c) \quad \forall k \geq 1, \quad \forall n \geq 1;$$

iii) there exists a subsequence $\{\psi^{n_j}\}_{j \geq 1}$ such that

$$\lim_{j \to +\infty} \|\psi^{n_j} - \psi\|_{L^\infty(D_k^c)} = 0 \quad \forall k \geq 1.$$

We remark that, since $\| \cdot \|_{\mathcal{W}_T} \leq \| \cdot \|_{\mathcal{W}_{T'}}$, if $T \leq T'$, the $D_{T,k}$'s may be chosen in such a way that $D_{T,k} \subseteq D_{T',k}$, if $T \leq T'$.

In order to apply lemma 3.1 we need to prove that τ_k^1, and τ_k^2, are $\mathcal{F}_t$–stopping times, and that $P(\tau_k^1 \wedge \tau_k^2 < T) \downarrow 0$. This is the content of the following two lemmas.

Lemma 9.1. *τ_k^1 and τ_k^2 are $\mathcal{F}_t$–stopping times.*

Proof : 1) by remark 4.2, and [BG], theorem 10.7, definition 10.21, τ_k^1 is a $\mathcal{F}_t$–stopping time if D_k^1 is a "nearly Borel set", i.e. if there exist Borel sets B_k and B_k' such that

$$B_k \subset D_k^1 \subset B_k', \quad \text{and} \quad \Gamma_T(B_k' \cap B_k^c \cap [0, T] \times \mathbb{R}^d) = 0 \quad \forall T \geq 0 \quad .$$

Since the class of nearly Borel sets is a σ–algebra, it will suffice to prove that $(D_k^1)^c$ is a nearly Borel set.

We have $\psi \in \mathcal{W}_T \quad \forall T \geq 0$, so that, by theorem 8.3, there exists a decreasing sequence of open sets $\{U_m\}_{m\geq 1}$, $U_m \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\Gamma_T(U_m \cap [0,T] \times \mathbb{R}^d) \downarrow 0 \quad \forall T \geq 0$, such that ψ is continuous on $U_m^c \cap [0,T] \times \mathbb{R}^d \quad \forall T \geq 0 \quad \forall m \geq 1$. This implies that

$$|\psi|^{-1}[1/k, +\infty) \cap \{ \ \|(t,x)\| \leq k \ \} \cap \bigcup_{m\geq 1} U_m^c$$

is a Borel set. Since

$$\Gamma_T(\cap_{m\geq 1} U_m \cap [0,T] \times \mathbb{R}^d) \leq \inf_m \Gamma_T(U_m \cap [0,T] \times \mathbb{R}^d) = 0 \quad \forall T \geq 0 \quad ,$$

D_k^1 is a nearly Borel set.
2) τ_k^2 is a $\mathcal{F}_t$–stopping time since D_k^2 is an open set.

Lemma 9.2. *Let ρ_t, ρ_t^n, $n \geq 1$, be continuous families of probability densities. Suppose $\sqrt{\rho_t^n} \to \sqrt{\rho_t}$ in $\mathcal{W}_T$ $\forall T > 0$, define D_k^1, and D_k^2, as above and let P be the probability measure corresponding, according to thm.6.1, to the proper infinitesimal characteristic (v, ρ). Then $P(\tau_k^1 \wedge \tau_k^2 < T) \downarrow 0 \quad \forall T > 0$.*

Proof : since $P(\tau_k^1 \wedge \tau_k^2 < T) \leq P(\tau_k^1 < T) + P(\tau_k^2 < T)$, we will prove $P(\tau_k^1 < T) \downarrow 0$ and $P(\tau_k^2 < T) \downarrow 0$ separately:
1) let us denote by $\tau_k^{1,1}$ and $\tau_k^{1,2}$ the first hitting times to the sets

$$\{ \ (t,x) \ : \ \|(t,x)\| > k \ \} \ , \quad \text{and} \quad \{ \ (t,x) \ : \ \sqrt{\rho}(t,x) < 1/k \ \}$$

respectively. Then

$$P(\tau_k^1 < T) \leq P(\tau_k^{1,1} < T) + P(\tau_k^{1,2} < T) \ .$$

One has $P(\tau_k^{1,1} < T) \downarrow 0 \quad \forall T > 0$ by theorem 6.1, since this is equivalent to the non–explosion of the process $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P)$ (see remark 4.2). From theorem 6.1 one has also $P(\tau_k^{1,2} < T) \downarrow 0 \quad \forall T > 0$, since ρ_t is the density of the process X_t w.r.t P. Indeed, since $\sqrt{\rho}$ is continuous up to a set of arbitrary small Γ–capacity by thm.4.3, we can suppose, without loss of generality, that $\{\sqrt{\rho}(t,x) < 1/k\}$ is an open set. Therefore $\{\tau_k^{1,2} < T\} = \cup_{q \in Q\cap[0,T)} \{\sqrt{\rho_q(X_q)} < 1/k\}$, and the thesis follows.
2) $P(\tau_k^2 < T) \downarrow 0$ by remark 8.2, and the definition of D_k^2.

10. Convergence of Nelson Diffusions

We have now at our disposal all the ingredients to prove our main result. In order to use the results of the preceeding sections we need to consider time–continuous proper infinitesimal characteristic, i.e. we suppose that the family of probability densities is continuous w.r.t. the time parameter. On the space of time–continuous proper infinitesimal characteristics we put a metric. We will use the letter g to denote such metric (and we will call it the Guerra metric since a very similar metric was used by Guerra in [G]). g is so defined:

$$g((v,\rho),(v',\rho')) := \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{g_n((v,\rho),(v',\rho'))}{1+g_n((v,\rho),(v',\rho'))},$$

where

$$g_n((v,\rho),(v',\rho')) := \sup_{t\in[0,n]} \|\sqrt{\rho_t} - \sqrt{\rho'_t}\|_{H^1} + \left(\int_0^n \|v_t\sqrt{\rho_t} - v'_t\sqrt{\rho'_t}\|^2_{L^2}\, dt\right)^{\frac{1}{2}} .$$

Lemma 10.1. *Let $\mathcal{P}_{tc}$ be the space of time continuous proper infinitesimal characteristics. Then $(\mathcal{P}_{tc}, g)$ is a complete metric space.*

Proof: (we will proceed as in [G], thm.1) Let $\{(v_n,\rho_n)\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{P}_{tc}, g)$. Then obviously there exist ρ_t and $w \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^d))$ such that, $\forall T \geq 0$,

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \|\sqrt{(\rho_n)_t} - \sqrt{\rho_t}\|_{H^1} = 0,$$

$$\lim_{n\to+\infty} \int_0^T \|(v_n)_t\sqrt{(\rho_n)_t} - w_t\|^2_{L^2}\, dt = 0.$$

Let us define the functional

$$l_w(f) := \int_0^T \int_{\mathbb{R}^d} w f \sqrt{\rho}\; dx dt \quad f \in L^2([0,T]\times\mathbb{R}^d; \rho dx dt).$$

Since l_w is continuous, then there exists $v \in L^2([0,T]\times\mathbb{R}^d; \rho dx dt)$ realizing such functional. Therefore $v\rho = w\sqrt{\rho}$, so that $v_t := w_t/\sqrt{\rho_t}$ is $\rho_t(x)dxdt$ –a.e. well defined and $v_n\sqrt{\rho}_n \to v\sqrt{\rho}$ in L^2. Since

$$v_n\rho_n - v\rho = v_n\sqrt{\rho_n}(\sqrt{\rho_n} - \sqrt{\rho}) + (v_n\sqrt{\rho_n} - w)\sqrt{\rho},$$

$v_n\rho_n \to v\rho$ in L^1. Therefore (F.P.) in §6 holds in the limit and the proof is completed.

The following theorem shows that the map given in theorem 6.1 is continuous w.r.t. g:

Theorem 10.2. *Let $\{(v_n,\rho_n)\}_{n\geq 1}$ be a sequence of time continuous proper infinitesimal characteristics, and let $\{P^n\}_{n\geq 1}$ the corresponding sequence of probability measures on the path space $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ given by theorem 6.1. If*

$$\lim_{n\to+\infty} g((v_n,\rho_n),(v,\rho)) = 0 \ ,$$

then

$$\lim_{n\to+\infty} d(P^n, P) = 0 \quad .$$

where P is the probability measure corresponding to (v,ρ).

Proof : let $\tau_{D_k} = \tau_k^1 \wedge \tau_k^2 \equiv \tau_k$, where τ_k^1, τ_k^2, and D_k, are defined in §9. We have proven in lemma 9.2 that $P(\tau_k < T) \downarrow 0 \quad \forall T > 0$. Moreover $\|\rho_n(0,\cdot) - \rho(0,\cdot)\|_{L^1}$ converges to zero by our hypotheses. Therefore, in order to use theorem 3.1, we have to prove that there exists a subsequence $\{\mathrm{b}_{n_j}\}_{j\geq 1}$, such that

$$\lim_{j\to+\infty} \int_{D_k^c} \|\mathrm{b}_{n_j} - \mathrm{b}\|^2 \rho_{n_j}(t,y)dydt = 0 \qquad \forall k \geq 1 \quad .$$

From the definitions of b_n and ρ_n, we have

$$\begin{aligned}
&\int_{D_k^c} \|\mathrm{b}_n - \mathrm{b}\|^2 \rho_n dydt \\
&\leq \int_{D_k^c} \|u_n - u\|^2 \rho_n dydt + \int_{D_k^c} \|v_n - v\|^2 \rho_n dydt \\
&= \int_{D_k^c} \left\| \nabla\sqrt{\rho_n}(t,y) - \sqrt{\frac{\rho_n}{\rho}} \nabla\sqrt{\rho}(t,y) \right\|^2 dydt + \int_{D_k^c} \left\| v_n\sqrt{\rho_n}(t,y) - \sqrt{\frac{\rho_n}{\rho}} v\sqrt{\rho}(t,y) \right\|^2 dydt \\
&\leq \int_0^k \int_{\mathbb{R}^d} \|\nabla\sqrt{\rho_n}(t,y) - \nabla\sqrt{\rho}(t,y)\|^2 dydt + \int_0^k \int_{\mathbb{R}^d} \|v_n\sqrt{\rho_n}(t,y) - v\sqrt{\rho}(t,y)\|^2 dydt \\
&+2\int_{D_k^c} \frac{\|\nabla\sqrt{\rho}(t,y)\|^2}{\|\rho(t,y)\|^2} \|\sqrt{\rho_n}(t,y) - \sqrt{\rho}(t,y)\|^2 dydt \\
&\leq k \sup_{0\leq t\leq k} \|\nabla\sqrt{\rho_n}(t,\cdot) - \nabla\sqrt{\rho}(t,\cdot)\|_{L^2}^2 + \int_0^k \int_{\mathbb{R}^d} \|v_n\sqrt{\rho_n}(t,y) - v\sqrt{\rho}(t,y)\|^2 dydt \\
&+2k^3 \sup_{0\leq t\leq k} \|\nabla\sqrt{\rho}(t,\cdot)\|_{L^2}^2 \|\sqrt{\rho_n} - \sqrt{\rho}\|_{L^\infty(D_k^c)}^2 \ ,
\end{aligned}$$

and so we have proved the existence of a converging subsequence $\{P^{n_j}_{|\mathcal{F}_t}\}_{j\geq 1}$. Convergence of the whole sequence now follows proceeding as in the proof of theorem 5.3.

Remark 10.3. If the sequence $\{(v_n, \rho_t^n\}_{n\geq 1}$ given in theorem 10.2 does not converge but it is only bounded w.r.t. the energy norm, i.e. if

$$\sup_{n\in\mathbb{N}} \int_0^T \left(\|\nabla\sqrt{\rho_t^n}\|_{L^2}^2 + \|v_n\sqrt{\rho_n}(t,\cdot)\|_{L^2}^2 \right) dt < +\infty \quad \forall T \geq 0 \ ,$$

then, by remark 2.3,

$$\sup_{n\in\mathbb{N}} H_{\mathcal{F}_T}(P^n; W^n) < +\infty \qquad \forall T \geq 0 \ ,$$

where $W^n := \int W_x \rho_n(0,x)dx$. Suppose moreover that the sequence $\{\rho_n(0,\cdot)dx\}_{n\geq 1}$ is precompact w.r.t. the weak-$*$ topology on $\mathcal{M}_1(\mathbb{R}^d)$. Then, by [Z], theorem 5, the sequence $\{P^n\}_{n\geq 1}$ is precompact with respect to the weak-$*$ topology on $\mathcal{M}_1(\Omega)$, and

$$H_{\mathcal{F}_T}(Q; W) < +\infty \qquad \forall T \geq 0 \quad ,$$

where $W := \int W_x d\mu(x)$, and (μ, Q) is any limit point of $\{\rho_n(0,\cdot)dx\}_{n\geq 1}$.

11. Nelson diffusion corresponding to Schrödinger operators with potentials form–bounded by $-\frac{1}{2}\Delta$

Let K denote the self–adjoint representation of $-\frac{1}{2}\Delta$ on $L^2(\mathbb{R}^d)$, and let V be a real-valued measurable function on $\mathbb{R}^d$ s.t. V is K–form–bounded, with relative bound smaller than one, i.e. $\exists\ a \in [0,1)$, $\exists\ b \geq 0$ s.t.

$$|\langle\psi|V\psi\rangle_{L^2}| \ \leq \ a\langle\psi|K\psi\rangle_{L^2} + b\langle\psi|\psi\rangle_{L^2} \qquad \forall\psi \in H^1(\mathbb{R}^d) \ .$$

We shall discuss only the case of time–independent potentials. The extension to the time–dependent case is immediate at the expense of heavier notation.

Let H be the unique self–adjoint operator associated to the sum of the quadratic forms of K and V. Such H exists by the KLNM theorem (see [RS], theorem X.17). Moreover one has

$$H^2(\mathbb{R}^d) \cap \mathcal{D}(V) \subset \mathcal{D}(H) \subset H^1(\mathbb{R}^d) \quad ,$$

$$\langle\phi|H\psi\rangle_{L^2} \ = \ \langle\phi|K\psi\rangle_{L^2} + \langle\phi|V\psi\rangle_{L^2} \qquad \forall\phi \in H^1(\mathbb{R}^d) \quad \forall\psi \in \mathcal{D}(H) \quad ,$$

and

$$\|\psi\|_{H^1}^2 \leq (1-a)^{-1}(\langle\psi|H\psi\rangle_{L^2} + (b+1)\langle\psi|\psi\rangle_{L^2}) \leq (2(b+1)+a)(1-a)^{-1}\|\psi\|_{H^1}^2 \quad .$$

Let e^{-itH} be the one parameter unitary group generated by H. By the above relation, it follows that e^{-itH} maps $H^1(\mathbb{R}^d)$ into itself, with

$$\|e^{-itH}\|_{H^1,H^1} \leq (2(b+1)+a)(1-a)^{-1} \quad .$$

Moreover, since

$$\lim_{t\to 0_+} \|(H+(b+1)I)^{1/2}(e^{-itH}\phi - \phi)\|_{L^2} = \lim_{t\to 0_+} \|(e^{-itH} - I)(H+(b+1)I)^{1/2}\phi\|_{L^2} = 0 \quad ,$$

e^{-itH} is a continuous one parameter group of bounded linear operators on $H^1(\mathbb{R}^d)$.

By the above discussion, proceeding in the same way as in [C1], we have the following analogue of theorem 2.1 in [C1]:

Theorem 11.1. *Let V be a K-form-bounded potential, with relative bound smaller than one, let ψ_0 be in $H^1(\mathbb{R}^d)$, and let $H = K + V$ be defined as a sum of quadratic forms. Then*
i) e^{-itH} is a continuous one parameter group of bounded linear operators from $H^1(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$;
ii) there are unique jointly measurable functions $\psi(t,x)$ and $\nabla\psi(t,x)$ such that $\psi(t,x) = e^{-itH}\psi_0(x)$, and $\nabla\psi(t,x) = \nabla e^{-itH}\psi_0(x)$;
iii) defining $\rho(t,x) := \psi(t,x)\overline{\psi}(t,x)$, and

$$v(t,x) := \Im\,(\nabla\psi(t,x)/\psi(t,x)),$$

if $\psi(t,x) \neq 0$, $u(t,x) = v(t,x) = 0$ otherwise, one has

$$\int_0^T \int_{\mathbb{R}^d} \|v\|^2 \rho\, dx dt < +\infty \qquad \forall T > 0;$$

iv) $\forall f \in C_b^1(\mathbb{R}^d)$ the function $t \mapsto \int_{\mathbb{R}^d} f(x)\rho(t,x)dx$ is differentiable, and

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(x)\rho(t,x)dx = \int_{\mathbb{R}^d} v(t,x)\cdot\nabla f(x)\ \rho(t,x)dx \quad .$$

From the preceeding theorem it follows that we can associate to the pair (ψ_0, V) an element in $\mathcal{P}$ and so, by theorem 6.1, a probability measure in $\mathcal{M}_1(\Omega)$. We summarize this results in the following

Theorem 11.2. *Consider the measurable space $(\Omega, \mathcal{F})$, with $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, $\mathcal{F}$ the Borel σ-algebra. Let V be a K-form-bounded potential, with relative bound smaller than one,* *let* ψ_0 *be in* $H^1(\mathbb{R}^d)$*, and let* $H = K + V$ *be defined as a sum of quadratic forms. Let* $\psi_t = e^{-itH}\psi_0$*,* $\rho_t := \psi_t\overline{\psi}_t$*, and define*

$$\mathrm{b}(t,x) := \Re + \Im\ (\nabla\psi(t,x)/\psi(t,x)),$$

if $\psi(t,x) \neq 0$*,* $\mathrm{b}(t,x) = 0$ *otherwise. Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t)$ *be the evaluation stochastic process* $X_t(\gamma) := \gamma(t)$*, with* $\mathcal{F}_t = \sigma(X_s, s \leq t)$ *the natural filtration. Then there exists a unique Borel probability measure* P *on* Ω *such that :*

i) $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P)$ *is a Markov process;*

ii) the image of P *under* X_t *has density* $\rho(t,x)$*;*

iii)

$$B_t := X_t - X_0 - \int_0^t \mathrm{b}(s, X_s)ds$$

is a P*–Brownian motion.*

12. Convergence of Nelson Diffusions corresponding to Schrödinger operators with Kato–class potentials

In the light of theorems 10.2 and 11.2 it will be interesting to find conditions on the potentials and on the initial data which will guarantee the convergence of the Nelson Diffusions associated to the corresponding Schrödinger operators. To this end we now suppose that the potentials are in the Kato–class K_d, where

$$K_d := \left\{ V \ : \ \lim_{\alpha\downarrow 0} \sup_x \int_{\|x-y\|\leq\alpha} \|x-y\|^{2-d}|V(y)|\ dy = 0 \right\} \qquad d \geq 3\ ,$$

$$K_2 := \left\{ V \ : \ \lim_{\alpha\downarrow 0} \sup_x \int_{\|x-y\|\leq\alpha} \log\|x-y\|^{-1}|V(y)|\ dy = 0 \right\} \quad ,$$

$$K_1 := \left\{ V \ : \ \sup_x \int_{\|x-y\|\leq 1} |V(y)|\ dy < +\infty \right\}$$

(see [CFKS], §1.2, [Si2], §A2). We also define a K_d–norm by

$$\|V\|_{K_d} := \sup_x \int_{\|x-y\|\leq 1} Q(x-y;d)|V(y)|\ dy \quad ,$$

where Q is the kernel in the above definition. One has the following inclusions:

$$L^p(\mathbb{R}^d) \subset L^p_{unif}(\mathbb{R}^d) \subseteq K_d \subseteq L^1_{unif}(\mathbb{R}^d) \quad ,$$

with $p > d/2$ if $d \geq 2$, $p = 2$ otherwise, where

$$L^p_{unif}(\mathbb{R}^d) := \left\{ V \ : \ \sup_x \int_{\|x-y\|\leq 1} |V(y)|^p dy < +\infty \right\}$$

(see [CFKS], §1.2), and

$$\|V\|_{K_d} \leq \|Q\|_{L^q_{unif}} \|V\|_{L^p_{unif}} \ , \quad 1/q + 1/p = 1 \ .$$

By [CFKS], §1.2, if $V \in K_d$, then V is K–form–bounded, with relative bound zero, so that we can apply theorem 11.2 to obtain a Nelson diffusion. In this way we have a well defined map from $H^1(\mathbb{R}^d) \times K_d$ into $\mathcal{M}_1(\Omega)$. We now prove that this map is a continuous one. To this end we need the following

Lemma 12.1.*Let $V, V_n \in K_d$, $n \geq 1$, $\psi_0, \psi_0^n \in H^1(\mathbb{R}^d)$, $n \geq 1$. If $\psi_t := e^{-itH}\psi_0$, $\psi_t^n := e^{-itH_n}\psi_0^n$, $H = K + V, H_n = K + V_n$, and if*

$$\lim_{n\to+\infty} \|\psi_0^n - \psi_0\|_{H^1} = 0, \quad \textit{and} \quad \lim_{n\to+\infty} \|V_n - V\|_{K_d} = 0 \quad ,$$

then

$$\lim_{n\to+\infty} \sup_{0\leq t\leq T} \|\psi_t^n - \psi_t\|_{H^1} = 0 \quad \forall T \geq 0 \ .$$

Proof: we will prove the case $d \geq 3$, for the other cases the proof is analogous. Since

$$\|e^{-itH_n}\psi_0^n - e^{-itH}\psi_0\|_{H^1} \leq \|(e^{-itH_n} - e^{-itH})\psi_0\|_{H^1} + \|e^{-itH_n}\|_{H^1,H^1}\|\psi_0^n - \psi_0\|_{H^1} \ ,$$

in order to apply theorem 6.1 we have to prove

1) $$\sup_{n\in\mathbb{N}} \|e^{-itH_n}\|_{H^1,H^1} < +\infty \ ,$$

2) $$\lim_{n\to+\infty} \sup_{0\leq t\leq T} \|(e^{-itH_n} - e^{-itH})\psi\|_{H^1} = 0 \quad \forall \psi \in H^1(\mathbb{R}^d) \ .$$

From the Kato–Trotter theorem (see [K], theorem 2.16, chap.IX) 2) is implied by 1) and

2') there exists a complex number z, $\Im z > 0$ s.t.

$$\lim_{n\to+\infty} \|(H_n - zI)^{-1}\psi - (H - zI)^{-1}\psi\|_{H^1} = 0 \qquad \forall \psi \in H^1(\mathbb{R}^d) .$$

Let us at first show that $\forall \epsilon > 0$ $\exists \gamma_\epsilon > 0$, $\exists n_\epsilon > 0$ such that

$$\|(K + \gamma I)^{-1} \; |V_n| \; \|_{L^\infty, L^\infty} < \epsilon \qquad \forall \gamma \geq \gamma_\epsilon, \quad \forall n \geq n_\epsilon .$$

We will proceed as in [CFKS], §1.2. By [RS], theorem IX.29, $(K + \gamma I)^{-1}$ is a convolution operator with an explicit kernel $G(x - y; \gamma)$, so that we may write, using the known properties of G (see [Sc], theorem 3.1, chap.6), and lemma 2.6 in [Sc], chap.5,

$$\begin{aligned}
&\|(K + \gamma I)^{-1} \; |V_n| \; \|_{L^\infty} \\
&\leq \sup_x \int_{\|x-y\|\leq 1/\sqrt{\gamma}} G(x - y; \gamma)|V_n(y)| \; dy + \sup_x \int_{\|x-y\|> 1/\sqrt{\gamma}} G(x - y; \gamma)|V_n(y)| \; dy \\
&\leq c_1 \sup_x \int_{\|x-y\|\leq 1/\sqrt{\gamma}} \|x - y\|^{2-d}|V_n(y)| \; dy + \frac{c_2}{\sqrt{\gamma}} \sup_x \int_{\|x-y\|< 1/\sqrt{\gamma}} |V_n(y)| \; dy \\
&\leq c_1 \|V_n - V\|_{K_d} + c_1 \sup_x \int_{\|x-y\|\leq 1/\sqrt{\gamma}} \|x - y\|^{2-d}|V(y)| \; dy + \frac{c_2}{\sqrt{\gamma}} \|V_n\|_{K_d} \quad .
\end{aligned}$$

Since $\|V_n - V\|_{K_d} \to 0$, and $V \in K_d$, $\forall \epsilon > 0$ $\exists \gamma_\epsilon$, $\exists n_\epsilon$ s.t.

$$\|(K + \gamma I)^{-1} \; |V_n| \; \|_{L^\infty} < \epsilon \qquad \forall \gamma \geq \gamma_\epsilon, \; \forall n \geq n_\epsilon .$$

This gives the result, since $G(\cdot \; - y; \gamma)|V_n|$ is a positive integral kernel, and $\|A\|_{L^\infty, L^\infty} = \|A1\|_{L^\infty}$ for any A with positive integral kernel. From the above result, by duality, and by Stein interpolation theorem, proceeding in the same way as in [CFKS], corollary 2.8., it follows that $\forall \epsilon > 0$ $\exists \gamma_\epsilon$, $\exists n_\epsilon$ s.t.

$$\| \; |V_n|^{1/2}(K + \gamma I)^{-1/2}\|_{L^2, L^2} < \epsilon \qquad \forall \gamma \geq \gamma_\epsilon, \; \forall n \geq n_\epsilon .$$

Since

$$|\langle \psi | V_n \psi \rangle_{L^2}| \leq \| \; |V_n|^{1/2}(K + \gamma I)^{-1/2}\|^2_{L^2, L^2}(\langle \psi | K \psi \rangle_{L^2} + \gamma \|\psi\|^2_{L^2}) \quad ,$$

we have that, $\forall n \geq n_1$, choosing $\gamma \geq \gamma_1$, all the V_n's are K–form–bounded with the same bound

$$a = \sup_{n \geq n_1} \| \; |V_n|^{1/2}(K + \gamma I)^{-1/2}\|^2_{L^2, L^2} < 1, \quad b = \gamma a \quad .$$

Since

$$\|e^{-itH_n}\|_{H^1, H^1} \leq (2(b + 1) + a)(1 - a)^{-1}$$

(see §2), we have that 1) holds true.

Let us now consider the operator

$$A_n(z) := (K + zI)^{-1/2} V_n (K + zI)^{-1/2} \quad .$$

Since V_n is K–form–bounded with relative bound 0, by [CFKS], proposition 1.3, $A_n(i\gamma)$ is a bounded operator with

$$\lim_{\gamma \to +\infty} \|A_n(i\gamma)\|_{L^2,L^2} = 0 \quad .$$

From the definition of A_n it follows, if $\gamma > 0$,

$$\|A_n(i\gamma)\|_{L^2,L^2} \leq c_3 \ \|A_n(\gamma)\|_{L^2,L^2} \leq c_3 \ \| \ |V_n|^{1/2}(K + \gamma I)^{-1/2}\|^2_{L^2,L^2} \quad ,$$

so that

$$\|A_n(i\gamma)\|_{L^2,L^2} < 1 \qquad \forall \gamma \geq \gamma_{1/c_3}, \ \forall n \geq n_{1/c_3} \ ,$$

and the Tiktopoulos' formula holds:

$$(H_n + i\gamma I)^{-1} = (K + i\gamma I)^{-1/2}(I + A_n(i\gamma) \ )^{-1}(K + i\gamma I)^{-1/2}$$

(see [Si1], §II3). Therefore we have

$$\begin{aligned} &\|((H_n + i\gamma)^{-1} - (H + i\gamma)^{-1})\psi\|_{H^1} \\ \leq &\|((I + A_n(i\gamma))^{-1} - (I + A(i\gamma))^{-1})(K + i\gamma I)^{-1/2}\psi\|_{L^2} \ . \end{aligned}$$

Since

$$\begin{aligned} \|A_n(i\gamma) - A(i\gamma)\|_{L^2,L^2} &\leq c_3 \|(K + \gamma I)^{-1/2}(V_n - V)(K + \gamma I)^{-1/2}\|_{L^2,L^2} \\ &\leq c_3 \| \ |V_n - V|^{1/2}(K + \gamma I)^{-1/2}\|^2_{L^2,L^2} \quad , \end{aligned}$$

and

$$\| \ |V_n - V|^{1/2}(K + \gamma I)^{-1/2}\|^2_{L^2,L^2} \leq c_4 \ \|V_n - V\|_{K_d}$$

(see [Sc], theorem 2.2, chap.5, theorem 3.1, chap.6), 2') follows, and the proof of the lemma is complete.

The following theorem gives us a criterion for convergence of Nelson Diffusions in terms of convergence of the physical data that generate them:

Theorem 12.2. *Let $V, V_n \in K_d$, $n \geq 1$, $\psi_0, \psi_0^n \in H^1(\mathbb{R}^d)$, $n \geq 1$. If P, P^n, $n \geq 1$ are the probability measures on $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$ which correspond, according to theorem 11.2, to $\psi_t = e^{-itH}\psi_0, \psi_t^n = e^{-itH_n}\psi_0^n$, $H = K + V, H_n = K + V_n$, and if*

$$\lim_{n \to +\infty} \|\psi_0^n - \psi_0\|_{H^1} = 0, \quad \textit{and} \quad \lim_{n \to +\infty} \|V_n - V\|_{K_d} = 0 \quad ,$$

then

$$\lim_{n\to+\infty} d(P^n, P) = 0 \quad .$$

Proof: from the previous lemma it follows that

$$\lim_{n\to+\infty} \sup_{0\le t\le T} \|\psi_t^n - \psi_t\|_{H^1} = 0 \quad \forall T \ge 0 \, .$$

Moreover we know that in our case

$$\rho_t = |\psi_t|^2, \quad v_t = \Im\left(\frac{\nabla\psi_t}{\psi_t}\right) \, .$$

Therefore we have, if the sequence of sets D_k is defined according to the results of §9,

$$\begin{aligned}
&\int_{D_k^c} \|\mathrm{b}_n - \mathrm{b}\|^2 \rho_n dy dt \\
=&\int_{D_k^c} \left\| \Re\left(\frac{\nabla\psi_t^n}{\psi_t^n} - \frac{\nabla\psi_t}{\psi_t}\right) + \Im\left(\frac{\nabla\psi_t^n}{\psi_t^n} - \frac{\nabla\psi_t}{\psi_t}\right)\right\|^2 \psi_t^n \overline{\psi_t^n} dy dt \\
\le& 2\int_{D_k^c} \left\| \nabla\psi_t^n - \frac{\psi_t^n}{\psi_t}\nabla\psi_t \right\|^2 dy dt \\
\le& 2\int_0^k \int_{\mathbb{R}^d} \|\nabla\psi_t^n - \nabla\psi_t\|^2 dy dt + 2\int_{D_k^c} \frac{\|\nabla\psi_t\|^2}{\|\psi_t\|^2} \|\psi_t^n - \psi_t\|^2 dy dt \\
\le& 2k \sup_{0\le t\le k} \|\nabla\psi_t^n - \nabla\psi_t\|_{L^2}^2 + 2k^3 \sup_{0\le t\le k} \|\nabla\psi_t\|_{L^2}^2 \|\psi^n - \psi\|_{L^\infty(D_k^c)}^2 \, ,
\end{aligned}$$

and the theorem follows proceeding in the same way as in theorem 10.2.

Remark 12.3. In theorem 7.1 one can replace K_3 with

$$R := \left\{ \; V \; : \; \|V\|_R^2 := \int_{\mathbb{R}^6} \frac{|V(x)| \; |V(y)|}{\|x-y\|^2} dx dy < +\infty \; \right\} \, ,$$

the Banach space of Rollnik–class potentials, and Kato–convergence of potentials with convergence w.r.t. Rollnik norm $\| \cdot \|_R$. The proof proceeds in an analogous way, using theorems I.21 and II.13 in [S1].

Remark 12.4. It may appear that convergence of initial data in $H^1(\mathbb{R}^d)$ be an unnecessary strong assumption; since one can disintegrate w.r.t. the initial distributions, one may expect that L^2–convergence be sufficient. However, suppose that, for every $\psi_0 \in H^1(\mathbb{R}^d)$, $T > 0$, and for some $M > 1$

$$\sup_{n\in\mathbb{N}} \|e^{-itH_n}\|_{H^1,H^1} \le M \quad \forall t \in \mathbb{R} \, ,$$

$$\lim_{n\to+\infty}\sup_{0\le t\le T}\|(e^{-itH_n}-e^{-itH})\psi_0\|_{H^1}=0\ ,$$

as is the case by our assumptions $\|V_n-V\|_{K_d}\to 0$. Suppose moreover that $\|\psi_0^n-\psi_0\|_{L^2}\to 0$. Then

$$\|\psi_0^n-\psi_0\|_{H^1}\to 0\qquad\Longleftrightarrow\qquad\int_0^T\|\nabla\psi_t^n-\nabla\psi_t\|_{L^2}^2dt\to 0\ .$$

Indeed by our hypotheses $\psi_t^n\to\psi_t$ in energy norm is equivalent to

$$\int_0^T\|\nabla e^{-itH_n}(\psi_0^n-\psi_0)\|_{L^2}^2dt\to 0\ .$$

From the group property one has

$$\begin{aligned}\|\psi_0^n-\psi_0\|_{H^1}^2&\le\|\psi_0^n-\psi_0\|_{L^2}^2+\sup_{0\le t\le T}\|\nabla e^{-itH_n}(\psi_0^n-\psi_0)\|_{L^2}^2\\&\le\|\psi_0^n-\psi_0\|_{L^2}^2+M^2\inf_{0\le t\le T}\|\nabla e^{-itH_n}(\psi_0^n-\psi_0)\|_{L^2}^2\\&\le\|\psi_0^n-\psi_0\|_{L^2}^2+\frac{M^2}{T}\int_0^T\|\nabla e^{-itH_n}(\psi_0^n-\psi_0)\|_{L^2}^2dt\\&\le\frac{M^2}{T}\int_0^T\|e^{-itH_n}(\psi_0^n-\psi_0)\|_{H^1}^2dt\\&\le M^4\|\psi_0^n-\psi_0\|_{H^1}^2,\end{aligned}$$

and our thesis follows.

13. Nelson Diffusions corresponding to Schrödinger Operators with potentials given by measures

As theorem 12.2 shows, the conditions for convergence given in thm.10.2 are not optimal in the sense that the Guerra metric is too strong with respect to convergence of solutions of Schrödinger equations. In fact from $\psi_n\to\psi$ in $\mathcal{W}_T$ $\forall T\ge 0$ in general it does not follows $v_n\sqrt{\rho_n}\to v\sqrt{\rho}$ in L^2 if we define $v_t:=\Im(\nabla\psi_t/\psi_t)$. As clearly follows from the proof of thm.10.2 to obtain convergence of the processes it is sufficient to require that the sequence $\{v_n\sqrt{\rho_n}\}_{n\ge1}$ converges to $v\sqrt{\rho}$ on the complement of the zero set of ρ up to a set of arbitrary small Γ–capacity, and this is exactly what $\psi_n\to\psi$ implies.

Let us now consider a Cauchy sequence $\{(\psi_0^n,V_n)\}_{n\ge1}$ in $H^1(\mathbb{R}^d)\times K_d$, and let $\{\psi_t^n\}_{n\ge1}$ the sequence of solutions of the corresponding Schrödinger equations. By the proof of lemma 12.1 it follows that this sequence is a Cauchy one in $\mathcal{W}_T$ $\forall T \geq 0$. Let ψ_t be the corresponding limit. Let μ the limit of the sequence $\{V_n\}_{n\geq 1}$. Such limit will not be necessarily a measurable function (see [Si2], §A2, example I, for an example of a measure contained in the closure of K_d). We will therefore think of ψ_t as the solution of the Schrödinger equation corresponding to the operator "$K + \mu$", with initial data ψ_0, the limit point of $\{\psi_0^n\}_{n\geq 1}$.

The remarkable fact is that also to ψ_t correspond a Nelson Diffusion which can be approximated by "more regular" ones. In fact, defining $v_t := \Im(\nabla\psi_t/\psi_t)$, $\rho_t := \psi_t\overline{\psi}_t$, obviously $v_n\rho_n \rightarrow v\rho$ in L^1, so (F.P) in §6 holds in the limit, and the pair (v, ρ) is a proper infinitesimal characteristics to which corresponds, by thm.6.1, a Nelson Diffusion P. Moreover, by thm.12.2, $d(P^n, P) \rightarrow 0$, where P^n are the Nelson Diffusion corresponding to (ψ_0^n, V_n).

It will therefore interesting to characterize the closure of K_d. Do this closure coincide with the space of Kato measures as defined in [BM]?

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References

[AH–KS1] Albeverio, S., Høegh–Krohn, R., Streit, L.: Energy Forms, Hamiltonians, and distorted Brownian Paths. J. Math. Phys. **18**, 907–917 (1977)

[AH–KS2] Albeverio, S., Høegh–Krohn, R., Streit, L.: Regularizations of Hamiltonians and Processes. J. Math. Phys. **21**, 1636–1642 (1980)

[AKS] Albeverio, S., Kusuoka, S., Streit, L.: Convergence of Dirichlet forms and associated Schrödinger Operators, J. Funct. Anal. **68**, 130–148 (1986)

[AS] Aronszajn, N., Smith, K.T.: Functional Spaces and Functional Completion. Ann. Inst. Fourier **6**, 125–185 (1955/56)

[B] Billingsley, P.: Convergence of Probability Measures. New York: John Wiley 1968

[BM] Blanchard, Ph., Ma, Z.M.: Semigroup of Schrödinger Operators with potentials given by Radon Measures. In: Albeverio, S., Casati, G., Cattaneo, U., Merlini, D., Moresi, R. (eds.) Stochastic Processes–Physics and Geometry. Singapore: World Scientific 1989

[BG] Blumenthal, R.M., Getor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968

[C1] Carlen, E.: Conservative Diffusions. Commun. Math. Phys. **94**, 293–315 (1984)

[C2] Carlen, E.: Existence and Sample Path Properties of the Diffusions in Nelson's Stochastic Mechanics. In: Albeverio, S., Blanchard, P., Streit, L. (eds.) Stochastic Processes – Mathematics and Physics. Lecture notes in Mathematics, Vol.1158, pp.25–51. Berlin, Heidelberg, New York: Springer 1985

[C3] Carlen, E.: Progress and Problems in Stochastic Mechanics. In: Gielerak, R., Karwowski, W. (eds.) Stochastic Methods in Mathematical Physics, pp.3–31. Singapore: World Scientific 1989

[CFKS] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger Operators. Berlin, Heidelberg, New York: Springer 1987

[DL] Deny, J., Lions, J.L.: Les Espaces du type de Beppo Levi. Ann. Inst. Fourier **5**, 305–370 (1953/54)

[DM] Dellacherie, C., Meyer, P.–A.: Probabilities and Potential. Amsterdam: North–Holland 1978

[E] Ershov, M.: On the Absolute Continuity of Measures Corresponding to Diffusion Processes. Theory of Prob. and Appl. **17**, 169–174 (1972)

[Fö1] Föllmer, H.: Time reversal on Wiener Spaces. In: Albeverio, S., Blanchard, P., Streit, L. (eds.) Stochastic Processes – Mathematics and Physics. Lecture Notes in Mathematics, Vol.1158, pp.119–129. Berlin, Heidelberg, New York: Springer 1985

[Fö2] Föllmer, H.: Random Fields and Diffusion Processes. In: Hennequin, P.–L. (ed.)

Ecole d'Eté de Probabilities XV–XVII, 1985–87. Lecture Notes in Mathematics, Vol. 1362, pp. 101–203. Berlin, Heidelberg, New York: Springer 1988

[Fr] Frese, J.: Capacity Methods in the Theory of Partial Differential Equations. Jber. d. Dt. Math.-Verein **84**, 1–44 (1982)

[Fu1] Fukushima, M.: Dirichlet Forms and Markov Processes. Amsterdam: North–Holland 1980

[Fu2] Fukushima, M.: Energy Forms and Diffusion Processes. In: Streit, L. (ed.) Mathematics and Physics, Lectures on Recent Results, pp.65–97. Singapore: World Scientific 1984

[G] Guerra, F.: Carlen Processes: a new Class of Diffusions with Singular Drifts. In: Accardi, L., Von Waldenfelds, W. (eds.) Quantum Probability and Applications. Lecture Notes in Mathematics, Vol.1136, pp.259–267. Berlin, Heidelberg, New York: Springer 1985

[JS] Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Berlin, Heidelberg, New York: Springer 1987

[K] Kato, T.: Perturbation Theory for Linear Operators, 2nd. edition. Berlin, Heidelberg, New York: Springer 1976

[KLS] Kabanov, Yu.M., Lipster, R.Sh., Shiryaev, A.N.: On the Variation Distance for Probability Measures Defined on a Filtered Space. Probab. Th. Rel. Fields **71**, 19–35 (1986)

[LSU] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasilinear Equations of Parabolic Type. Providence, R.I.: Amer. Math. Soc. 1968

[N1] Nelson, E.: Dynamical Theories of Brownian Motion. Princeton: Princeton Univ. Press 1967

[N2] Nelson, E.: Quantum Fluctuations. Princeton: Princeton Univ. Press 1985

[P1] Portenko, N.I.: On the Existence of Solutions of Stochastic Differential Equations with Integrable Drift Coefficients. Theory of Prob. and Appl. **19**, 552–557 (1974)

[P2] Portenko, N.I.: Generalized Diffusion Processes. Providence, R.I.: Amer. Math. Soc. 1990

[RS] Reed, M.R., Simon, B.: Fourier Analysis, Self–Adjointness. New York: Academic Press 1975

[RT–S] Röckner, M., Tu–Sheng, Z.: Uniqueness of Generalized Schrödinger Operators and Applications. Bonn Universität preprint no. 163, 1991

[Sc] Schechter, M.: Spectra of Partial Differential Operators. Amsterdam: North–Holland 1971

[Si1] Simon, B.: Quantum Mechanics for Hamiltonians defined as Quadratic Forms. Princeton: Princeton Univ. Press 1971

[Si2] Simon, B.: Schrödinger Semigroups. Bull. AMS **7**, 447–526 (1982)

[St] Stroock, D.W.: Lectures on Stochastic Analysis: Diffusion Theory. Cambridge: Cambridge Univ. Press 1987

[SV] Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes. Berlin, Heidelberg, New York: Springer 1979

[W] Witt, W.: Weak Convergence of Probability Measures on the Function Space $C[0,\infty)$. Ann. Math. Stat. **41**, 939–944 (1970)

[Z] Zheng, W.A.: Tightness results for laws of Diffusion Processes. Ann. Inst. Henri Poincaré **21**, 103–124 (1985)