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Nonlinear differential equations on non-compact domains

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Simone Secchi

SUPERVISORS
Prof. Antonio Ambrosetti

Thesis submitted for the degree of *Doctor Philosophiae*
Academic Year 2001-2002

SISSA - SCUOLA INTERNAZIONALE SUPERIORE STUDI AVANZATI
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Vorrei esprimere la mia gratitudine a tutti le persone che mi sono state vicine in questi anni: la mia famiglia, i miei amici, e tutti quelli che hanno discusso con me di matematica. Un ringraziamento particolare va al mio supervisore, il prof. Antonio Ambrosetti. Da lui ho imparato tutto quello che so sulla ricerca scientifica, e non solo le tecniche.

Preface

The aim of this Ph.d. thesis is to present some very recent results concerning specific problems that share a common feature: a certain kind of *non-compactness*.

As explained in the first chapter, while *compact* problems have a well-established existence theory, much less is known for non–compact problems which have attracted a great attention in recent years. Compactness usually breaks down for topological reasons, for example because an equation should be solved on an unbounded domain. Hence the elementary tools of variational analysis cannot be applied, and one is compelled to find new ways out.

We present three possible approaches to non–compact problems:

(1) Concentration–Compactness;
(2) Perturbation methods;
(3) Bifurcation theory.

The first one was introduced by P. L. Lions in 1984 ([65, 66]), and immediately adapted to a wide range of differential equations and, more generally, variational problems. As an application, we study here the existence of best constants for a class of Hardy–like inequalities in $\mathbb{R}^n$. See section 1.7. The main result is Theorem 1.11. This result will appear in our forthcoming paper [79].

The second approach was introduced in recent years by Ambrosetti and Badiale (see [3, 4]) as a generalization of previous results by Ambrosetti, Coti Zelati and Ekeland ([7]). It is still a "young" tool, though several applications have already appeared in the literature. We present new results concerning the existence of closed geodesics on infinite cylinders, following our paper [78], and to some Schrödinger–like elliptic partial differential equations, following [10]. Moreover, applications to a different problem where a vector potential function appears as a perturbation term are described, following [32]. The main results are Theorems 21 and 23 in chapter 3, Theorem 25 in chapter 4 and Theorems 30 and 31 in chapter 5. More comments on the history and background material on these topics are contained in the next chapters.

Finally, Bifurcation Theory has also been used to prove existence of non-trivial solutions for some classes of nonlinear elliptic equaitons on $\mathbb{R}^n$. More recently, Fitzpatrick and Pejsachowicz have defined a degree for Fredholm maps to find more general bifurcation results for such equations on $\mathbb{R}^n$. Following [80], we discuss in chapter 6 an application of this degree to hamiltonian systems.

At last, let me acknowledge the support I have been given during the last four years at S.I.S.S.A. and the hospitality of the universities I have been visiting. In particular, I which to thank Dr. Silvia Cingolani (Politecnico di Bari), Prof. Charles Stuart (École Polytechnique Fédérale, Lausanne), and Prof. Michel Willem (Université Catholique de Louvain-la-Neuve).

Trieste

Simone Secchi
August 2002

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Notation

We collect here a list of notation commonly used in this thesis. We have tried to stick to the rule of *...with an obvious meaning of symbols.*

- X will be a real Banach space, and E a real Hilbert space.
- The Fréchet derivative of a differentiable function $J\colon E \to \mathbb{R}$ at a point $u \in E$ will be denoted by $DJ(u)$. Sometimes, to have more readable formulas, we use $J'(u)$, and similarly D^2J or J'' for second-order derivatives. Since $DJ(u)$ is a linear map, we frequently omit brackets, *viz.* $DJ(u)v$ stands for the action of the linear map $DJ(u)$ on the vector v.
- Partial derivatives of functions $u\colon \mathbb{R}^n \to \mathbb{R}$ will be denoted by either $\partial_i u$, or $\partial u/\partial x_i$ when the name of the independent variable is important in the context.[1]
- The symbol Δ stands for the ordinary Laplace operator: if $u\colon \mathbb{R}^n \to \mathbb{R}$ is a regular function, then
$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$
or, equivalently, the trace of the Hessian matrix of u.
- It is always a delicate matter to fix notation for multiplication of real numbers and scalar products in Hilbert spaces. We are not going to say the ultimate word here. Inner products should be written as $\langle u, v \rangle$ whenever possible, but we will use $u \bullet v$ when dealing with geometric problems, and also $u \cdot v$ from time to time.
- The complex conjugate of any number $z \in \mathbb{C}$ will be denoted by $\bar{z}$.
- The real part of a number $z \in \mathbb{C}$ will be denoted by $\operatorname{Re} z$.
- The ordinary inner product between two vectors $a, b \in \mathbb{R}^n$ will be denoted by $a \cdot b$.
- Integration with respect to the ordinary Lebesgue measure is denoted by $\int \ldots dx$. This is a little abuse of notation, since nobody denotes this measure by x! A more correct $\int \ldots d\mathcal{L}$ shoud be used throughout, but we refrain from this "calvinism". Even worse, when no confusion can arise, we omit the symbol dx in integrals over $\mathbb{R}^n$.

[1] Of course, the first choice is more intrinsic, though the second is often more explicit.

- C denotes a generic positive constant, which may vary inside a chain of inequalities.
- We use the Landau symbols. For example $O(\varepsilon)$ is a generic function such that $\limsup_{\varepsilon\to 0}[O(\varepsilon)/\varepsilon] < \infty$, and $o(\varepsilon)$ is a function such that $\lim_{\varepsilon\to 0}[o(\varepsilon)/\varepsilon] = 0$. Of course, a function can depend on several parameters, and its behavior is frequently different according to which parameter is considered. In these cases, we use a subscript to single out the parameter that "tends to..."
- The end of a proof is denoted by the symbol ♣.[2]

[2] This choice is not completely standard. I like it because the symbol ♣ can be immediately seen inside a page.

Part I

The importance of being compact

1 Why compactness?

The rôle of compactness in Modern Analysis can hardly be overrated. From the basic theorem of Weierstrass stating that any continuous function on a compact space attains a maximum and a minimum, to the most advanced results in Calculus of Variations, compactness properties are exploited to prove existence of optimization problems, ordinary and partial differential equations, regularity theory, and much more.

1.1 Variational methods: how do they work?

Before we can actually launch ourselves into the study of non–compact problems, we need to develop some tools of modern[1] analysis. Since this is not a treatise on nonlinear functional analysis, we shall content ourselves with a brief survey of the main ideas. Probably the best example is that of elliptic partial differential equations (PDE's). While a nice feature of ordinary differential equations is that they can often be solved explicitely or numerically, the situation changes dramatically for PDE's. For example, take a bounded domain $\Omega \subset \mathbb{R}^n$, and a right-hand side $f \in C(\overline{\Omega})$. It can be shown that the innocent–looking problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has, in general,[2] no solution of class C^2. And although there are explicit representations of solutions when Ω has a particular shape (a ball, a half-space), nevertheless these formulas are seldom useful for both theoretic and practical purposes.

The natural conclusion is that one should probably decide to weaken the concept of solution: this is the concept of *weak solutions*. Again, instead of presenting a thorough treatment of the general case, we prefer to focus on the

[1] The word "modern" is quite abused. Its meaning depends on the context. Roughly speaking, we might replace it by "functional–analytic", at least in this work.

[2] With respect to the dimension n and to the domain Ω.

previous example. Assume $u \in C^2(\Omega)$ is a classical solution of (1.1). Multiply both sides of the equation by any $\varphi \in C_0^\infty(\Omega)$, and use the Green formula (see Appendix). Since u is zero on the boundary,

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{1.2}$$

Remark that (1.2) makes sense for any $u \in L^1_{loc}(\Omega)$. We could thus define *distributional solutions* $u \in L^1_{loc}(\Omega)$, but we will find the next concept more useful.

Definition 1.1. *We say that* $u \in H^1(\Omega)$ *is a weak solution of* (1.1) *if* (1.2) *is true for all* $\varphi \in H^1(\Omega)$.

Now, we can present the core of the variational approach to problems like (1.1).

(A) Look for a weak solution, and possibly for its uniqueness.
(B) Prove that this solution is actually of class C^2.
(C) Prove that any weak solution of class C^2 is a classical solution, i.e. (1.1) is true pointwise.

Step (C) is usually easy, whereas step (B) is rather difficult for non–linear equations. We won't go into details.

At this stage, how can we treat (A)? At a first glance, (1.2) is worse than the original equation, but we can remark that it can be seen from many viewpoints. For example as an orthogonality relation, or as the fact that the differential of the functional $J \colon H_0^1(\Omega) \to \mathbb{R}$ is zero at u, where

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx.$$

The former viewpoint immediately reminds us of the Lax–Milgram theorem, which is indeed a powerful tool for linear elliptic equations (see [76]); the latter viewpoint opens the way to *critical point theory*, which is the art of finding zeroes of derivatives in Banach or Hilbert spaces. We will say some words in the next section, but we refer to the literature for extensive tretments ([2, 27, 68, 81]). The main advantage of this second approach is that it works for a wide class of semilinear elliptic equations.

1.2 A standard example of the use of compactness

At this stage it is not clear why compactness plays a rôle in all this stuff. To clarify the situation, we present a popular approach to some equations, i.e. constrained minimization.

Theorem 1.1. *Let X be a reflexive Banach space and a functional $J\colon X \to \mathbb{R}$ of class C^1 be given. Let $u \in X$ be a point of absolute minimum of J. Then*

$$DJ(u) = 0 \quad \text{in } X^*,$$

where X^* is the dual space of X.

Definition 1.2. *A functional $J\colon X \to \mathbb{R}$ is called boudned from below on X if there exists a constant $M > 0$ such that*

$$J(u) \geq M$$

for all $u \in X$.

Definition 1.3. *A functional $J\colon X \to \mathbb{R}$ is called coercive on X if*[3]

$$\lim_{\|u\| \to \infty} J(u) = +\infty.$$

Theorem 1.2. *Let X be a reflexive Banach space. Let a functional $J\colon X \to \mathbb{R}$ be*

1. *bounded from below on X;*
2. *weakly lower semicontinuous on X;*
3. *coercive on X.*

Then J has a minimum.

Proof. Let $\{u_n\}$ be a minimizing sequence for J, i.e.

$$\lim_{n \to \infty} J(u_n) = \inf_{u \in X} J(u) =: I.$$

In particular, since J is bounded from below, $I > -\infty$, and there exists a constant C such that $|J(u_n)| \leq C$ for all $n \in \mathbb{N}$. Since J is coercive, $\{u_n\}$ is bounded in the norm of X. Since X is reflexive, bounded sequences in X are weakly relatively compact, so that, up to a subsequence, $u_n \to u$ weakly as $n \to \infty$. By the semicontinuity,

$$I = \lim_{n \to \infty} J(u_n) \geq J(u),$$

thus proving that $u \in X$ is a minimum point. Notice that the compactness assumptions has given us the element that has turned out to be the minimum of J. ♣

[3] More generally, a functional $J\colon X \to \mathbb{R}$ defined on the topological space X is coercive if the set $\{x \in X \mid J(x) \leq t\}$ is relatively compact in X for all $t \in \mathbb{R}$.

Example 1.1. We come back to (1.1). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with smooth boundary $\partial\Omega$. Let $H_0^1(\Omega)$ be the ordinary Sobolev space (see appendix). For a given $f \in L^2(\Omega)$, define

$$\begin{aligned} J\colon H_0^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\mapsto \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx. \end{aligned}$$

It is easy to prove that J is weakly lower semicontinuous. Moreover, from the Poincaré inequality it follows that J is coercive and bounded from below. Hence it attains a minimum at some $\bar{u} \in H_0^1(\Omega)$. J is trivially of class C^1, and so $DJ(\bar{u}) = 0$ by Theorem 1.1. By definition, $\bar{u}$ is a ***weak solution*** of the linear problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, the previous example is almost trivial, but in some sense it is also the starting point of the variational theory of elliptic equations.

1.3 When compactness breaks down

We have seen that weak compactness plays an important rôle in establishing the existence of the minimum point. To get a more precise feeling, consider the following minimization problem:

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx} \tag{1.3}$$

By the Lagrange multiplier rule, any minimizer satisfies an equation of the form

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some $\lambda \in \mathbb{R}$.

We can take a minimizing sequence $\{u_n\}$, and it is not restrictive to assume

$$\int_\Omega |u_n|^2\,dx = 1 \quad \forall n \geq 1.$$

Again, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and so, up to a subsequence, it converges *strongly* in $L^2(\Omega)$ (see appendix) to some $\bar{u}$. Hence

$$\int_\Omega |u_n|^2\,dx \to \int_\Omega |\bar{u}|^2\,dx$$

and by weak lower semicontinuity,

$$\int_\Omega |\nabla \bar{u}|^2\, dx \leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2\, dx.$$

This proves that $\bar{u}$ is actually a minimum point of the *Rayleigh–Ritz* quotient (1.3).

But the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ fails if Ω is unbounded, for instance $\Omega = \mathbb{R}^n$. Essentially, there are only three reasons why a weakly convergent sequence $\{u_n\}$ in an infinite–dimensional space, for example in $L^2(\Omega)$, does not converge strongly[4] (see also [64, 89]):

a) **Oscillation.** If $\{u_n\}$ converges in $L^2(\Omega)$, then it has a subsequence converging almost everywhere in Ω. But the sequence $u_n(x) = e^{2\pi i n x}$ does not converge in $L^2(-1/2, 1/2)$, although it converges weakly to zero. Indeed, $\|u_n\|_{L^2} = 1$.
b) **Concentration.** Let u be the characteristic function of $(-1/2, 1/2)$, and let $u_n(x) = n^{1/2}u(nx)$. For all $g \in C_0^\infty(-1/2, 1/2)$, we have
$$\int_{(-1/2,1/2)} u_n g\, dx \to 0.$$
Since again $\|u_n\|_{L^2} = 1$, this implies that $u_n \rightharpoonup 0$ in $L^2(-1/2, 1/2)$. In the sense of distributions,
$$u_n^2 \to \delta,$$
where δ is the Dirac measure at zero. See [90] for a study of distribution theory. We remark that $u_n(x) \to 0$ for all x but $x = 0$.
c) **Vanishing.** Let u be the characteristic function of $(-1/2, 1/2)$, and let $u_n(x) = u(x - n)$. As before, $u_n \rightharpoonup 0$. This time $u_n(x) \to 0$ for all x.

We shall see later that, roughly speaking, minimizing sequence can be classified according to their behavior like a), b) or c).

In the next section, we present the basis of the modern variational approach to possibly non–compact problems.

1.4 Critical points

As we have seen before, one way to find solutions of differential equations is to find global minima of suitable maps J defined on Banach or Hilbert spaces. But global extrema are very often too much to be sought, and indeed their existence is far from being trivial. Moreover, the key point is to find points where the derivative of J vanishes, i.e. *critical points of J*.

Let E be a real Banach space with norm $\|\cdot\|$ and $J\colon E \to \mathbb{R}$ a C^1 functional.

---

[4] It is trivially true that weakly and strongly convergent sequences coincide in finite dimensional vector spaces.

Definition 1.4. *A critical point of J is a point $u \in E$ such that $DJ(u) = 0$ in E^*. We say that $c \in \mathbb{R}$ is a critical level for J if there exists a critical point u of J such that $J(u) = c$.*

Definition 1.5. *An operator $A\colon E \to E$ is called variational if there exists a functional $J \in C^1(E, \mathbb{R})$ such that $A = DJ$.*

Hence a problem that can be translated into a functional equation like $A(u) = 0$ is called a variational problem if A is variational.

Example 1.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and let $E = H_0^1(\Omega)$. Let $p\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|p(x,s)| \leq a_1 + a_2|s|^\ell \tag{1.4}$$

where $\ell \leq \frac{n+2}{n-2}$ if $n > 2$, ℓ is unrestricted if $n = 1, 2$. We assume, for simplicity, $n > 2$. Let $P(x,s) = \int_0^s p(x,\tau)\,d\tau$. Then

$$|P(x,s)| \leq a_3|s| + a_4|s|^{\ell+1}.$$

Since $\ell + 1 \leq 2^* = \frac{2n}{n-2}$, we have from the Sobolev embedding $E \subset L^{\ell+1}(\Omega)$ and it makes sense to define $\phi\colon E \to \mathbb{R}$ by

$$\phi(u) = \int_\Omega P(x, u(x))\,dx.$$

Moreover, it is easy to check that $\phi \in C^1(E, \mathbb{R})$, and

$$D\phi(u)\colon v \in E \mapsto \int_\Omega p(x, v(x))\,dx.$$

We recall that since Ω is bounded, the embedding $E \subset L^{\ell+1}(\Omega)$ is compact, and hence the gradient of ϕ is a compact map.

Define $J \in C^1(E, \mathbb{R})$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \phi(u).$$

If $u \in E$ is a critical point of J, then

$$\int_\Omega [\nabla u \cdot \nabla v - p(x, u(x))v(x)]\,dx = 0 \quad \forall v \in H_0^1(\Omega).$$

Hence u is a weak solution of the semilinear Dirichlet boundary value problem

$$\begin{cases} -\Delta u = p(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If p is locally Hölder continuous, it can be proved that u is in fact a classical C^2–solution.

A common strategy to find critical points is to look for *constrained extremum points* over some suitable submanifold.

Definition 1.6. *Let M be a C^1 Riemannian manifold modeled on a Hilbert space E, and let $J \in C^1(M, \mathbb{R})$. A critical point of J constrained on M is a point $u \in M$ such that $D_M J(u) = 0$. Here D_M stands for the derivative of J on M.*

Suppose M has codimension 1 in E. This means (see [61]) that there exists a functional $g \in C^1(E, \mathbb{R})$ such that

$$M = \{u \in E \colon g(u) = 0\}$$

and $Dg(u) \neq 0$ for all $u \in M$. The tangent space to M at u is given by

$$T_u M = \{v \in E \mid Dg(u)v = 0\}$$

and a critical point of J on M is a point $u \in M$ such that $DJ(u)v = 0$ for all $v \in T_u M$. Hence u satisfies

$$DJ(u) = \lambda Dg(u)$$

for some $\lambda \in \mathbb{R}$. This is the *Lagrange multiplier rule.*

Example 1.3. By considering $J \colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_\Omega |u|^2 \, dx,$$

we can look for critical points constrained on $\overline{B} = \{u \in H_0^1(\Omega) \mid \int_\Omega |\nabla u|^2 \, dx \leq 1\}$. It is easy to show that $J_{|\overline{B}}$ has a maximum u, which solves by the above

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The number λ is the *first eigenvalue* of $-\Delta$ on $H_0^1(\Omega)$.

1.5 Partial compactness: Palais–Smale and mountain passes

As a rule, to identify a minimum, one constructs a minimizing sequence, and tries to prove that it converges (up to a subsequence, if necessary). In the previuos examples, this was true because of the compact Sobolev embedding. When this tool is no longer available, the approach must be refined. Actually, relative compactness of *all* bounded sequences is too much.

Definition 1.7 (Palais–Smale sequences). *Let E be a Banach space, and let $J \in C^1(E, \mathbb{R})$ be a given functional. We say that J satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, $(PS)_c$ for short, if every sequence $\{u_k\}$ in E such that $J(u_k) \to c$ and $DJ(u_k) \to 0$ in $E^\star$ is relatively compact in E.*

Clearly, this is a weaker considition than compactness itself, but it is often enough in Critical Point Theory. For the sake of completeness, we remind of the fact that several conditions like (PS) have been introduced over the past years, conditions that cover even weker situations. It is not our purpose to write a treatise on this subject, and we refer to the huge literature for details.

In 1973, Ambrosetti and Rabinowitz ([11]) proved the following theorem.

Theorem 1.3 (Mountain pass). *Let J be a C^1 functional on a Banach space X. Suppose*

(i) there exist a neighborhood U of 0 in X and a constant ϱ such that $J(u) \geq \varrho$ for every u on the boundary of U,
(ii) $J(0) < \varrho$ and $J(v) < \varrho$ for some $v \notin U$.

Set

$$c = \inf_{P \in \mathcal{P}} \max_{w \in P} J(w) \geq \varrho,$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to v. The conclusion is: there is a sequence $\{u_j\}$ in X such that $J(u_j) \to c$ and $DJ(u_j) \to 0$ in $X^\star$. If J satisfies $(PS)_c$, then c is a critical level for J.

This result is very useful to find critical points of J even when J is strongly undefined, i.e. when $\inf_E J = -\infty$ and $\sup_E J = +\infty$. The geometry of J is that 0 is local minimum, but not a global one. See the figure below, showing a profile with two maxima, one local minimum, and a path that a clever walker should follow to minimize effort.

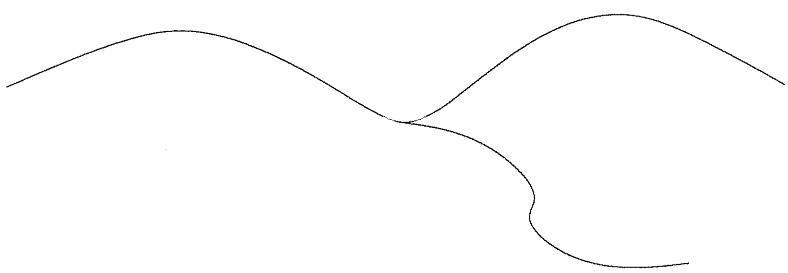

Fig. 1.1. The mountain–pass geometry

Needless to say, the difficult task is to prove that J satisfies the Palais–Smale condition at the "right" level. Good functionals satisfy it all levels, but many others satisfy it only in subregions of $\mathbb{R}$.

1.6 Critical exponents

Apart from unboundedness of the domain, another source of non–compactness is the presence of the *critical Sobolev exponent* (see the Appendix). The embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ fails to be compact even if Ω is bounded.[5] Hence differential equations with a critical behavior appear to be very interesting. Indeed, they can have only trivial solutions, as the following example shows.

Example 1.4. Consider

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$. For $x \in \partial\Omega$, we denote by ν_x the outward normal vector to Ω.

Proposition 1.1. *If $x \cdot \nu_x \geq 0$ on $\partial\Omega$, then* (1.5) *has only the trivial solution $u \equiv 0$, whenever $q \geq \frac{n+2}{n-2}$.*

The proof is an immediate consequence of the next result, due to Pohozaev.

Lemma 1.1. *If u is a smooth solution of*

$$\begin{cases} -\Delta u = p(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

then

$$n \int_\Omega P(u(x))\,dx + \frac{2-n}{2}\int_\Omega u(x)p(u(x))\,dx = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu_x)\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma.$$

Probably the first, but surely the most impressive paper to study exstensively the loss of compactness in differential equations is that by Brezis and Nirenberg ([23]). In a previous section, we showed that equations on unbounded domains usually present additional difficulties, due to the loss of compact embedding in L^p spaces. A different phenomenon leading to "non–compact" problems is that of *limiting Sobolev exponent*, for which the embedding is not compact, even on bounded domains.

In their paper, Brezis and Nirenberg consider a bounded domain $\Omega \subset \mathbb{R}^n$ and the critical elliptic equation

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

[5] See the appendix.

where $p = (n+2)/(n-2)$, $n \geq 3$, and λ is a real constant. We remark explicitely that the same problem with subcritical exponent is completely solvable:

Theorem 1.4. *Assume that Ω is a bounded domain, and $2 < p < 2^\star$. Then the problem*

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

has a nontrivial solution if and only if $\lambda > -\lambda_1(\Omega)$, the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Proof. See [90] ♣

Their main result is

Theorem 1.5. *Assume $n \geq 4$. Then for every $\lambda \in (0, \lambda_1)$, there exists a solution of (1.7). Here λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions on Ω.*

Since their argument has opened the way to hundreds of papers on non-compact problems, we sketch the main ideas. First of all, they choose a variationa approach, and set

$$S_\lambda = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^p}=1}} \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right)\, dx,$$

so that S_0 corresponds to the best constant in the Sobolev embedding theorem $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ When $\Omega = \mathbb{R}^n$, it was proved by Aubin and Talenti that S_0 is attained by the functions

$$U_\varepsilon(x) = C_\varepsilon(\varepsilon + |x|^2)^{-(n-2)/2},$$

where $C_\varepsilon > 0$ is a normalization constant.

The second step is to show a *a priori* existence result:

Lemma 1.2. *If $S_\lambda < S$, then S_λ is attained.*

Proof. Let $\{u_j\}$ be a minimizing sequence in H_0^1, i.e.

$$\int_\Omega |u_j|^{p+1}\, dx = 1, \tag{1.9}$$

$$\int_\Omega |\nabla u_j|^2\, dx - \lambda \int_\Omega |u_j|^2\, dx = S_\lambda + o(1) \quad \text{as } j \to \infty. \tag{1.10}$$

Up to a subsequence,

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$u_j \to u \quad \text{strongly in } L^2(\Omega),$$

$$u_j \to u \quad \text{almost everywhere in } \Omega,$$

with $||u||_{L^{p+1}} \leq 1$. Set $v_j = u_j - u$. Hence we have $||\nabla u_j||_{L^2} \geq S_0$. It follows that $\lambda ||u||_{L^2}^2 \geq S_0 - S_\lambda > 0$, and so $u \not\equiv 0$. By construction,

$$||\nabla u||_{L^2}^2 + ||\nabla v_j||_{L^2}^2 - \lambda ||u||_{L^2}^2 = S_\lambda + o(1) \tag{1.11}$$

since $v_j \rightharpoonup 0$ weakly. By the Brezis–Lieb Lemma (see the next section),

$$||u + v_j||_{L^{p+1}}^{p+1} = ||u||_{L^{p+1}}^{p+1} + ||v_j||_{L^{p+1}}^{p+1} + o(1).$$

Thus we have

$$1 = ||u||_{L^{p+1}}^{p+1} + ||v_j||_{L^{p+1}}^{p+1} + o(1)$$

and therefore

$$1 \leq ||u||_{L^{p+1}}^2 + ||v_j||_{L^{p+1}}^2 + o(1)$$

which leads to

$$1 \leq ||u||_{L^{p+1}}^2 + \frac{1}{S_0} ||\nabla v_j||_{L^{p+1}}^2 + o(1). \tag{1.12}$$

The following claim can be proved by distinguishing the cases (i) $S_\lambda > 0$ and (ii) $S_\lambda \leq 0$:

$$||\nabla u||_{L^2}^2 - \lambda ||u||_{L^2}^2 \leq S_\lambda ||u||_{L^{p+1}}^2.$$

Now the proof is complete, since u is not identically zero. ♣

Hence we should compare S_λ and S_0, and this is done by some *cut-off* procedure:

Lemma 1.3. *We have*

$$S_\lambda < S_0 \qquad \text{for all } \lambda > 0.$$

Proof. The task is to estimate the ratio

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx}{\left(\int_\Omega |u|^{p+1} \, dx\right)^{2/(p+1)}}.$$

This can be done with the functions

$$u(x) = u_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|)^{(n-2)/2}},$$

where φ is C_0^∞ function concentrated near the origin of $\mathbb{R}^n$. After some calculations, it is possible to prove that $Q_\lambda(u_\varepsilon) < 0$ if ε is small enough. ♣

Now, a solution of (1.7) is found by rescaling any minimizer in Lemma 1.2. By the same token, the following result can be established.

Lemma 1.4. *Let*

$$S_0 = \inf\{\|u\|_2^2 \mid (u \in H_0^1(\Omega)) \wedge (\|u\|_{2^\star}^{2^\star} = 1)\}$$

be the best Sobolev constant. Then for any

$$c < \frac{1}{n} S_0^{n/2}$$

the functional

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda |u|^2)\, dx - \frac{1}{2^\star}\int_\Omega |u|^{2^\star}\, dx$$

satisfies the $(PS)_c$ condition.

More general situations can be studied by means of the *mountain–pass* theorem, for example

$$\begin{cases} -\Delta u = u^p + f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{1.13}$$

where the main assumption of $f\colon \Omega \times [0,\infty) \to \mathbb{R}$ is that

$$\lim_{u \to +\infty} \frac{f(x,u)}{u^p} = 0.$$

The case $n = 3$ is more delicate, since the estimates in Lemma 1.3 are no longer true. Anyway, the following result is given.

Theorem 1.6. *Assume Ω is a ball. Then* (1.7) *has a solution if and only if* $\frac{1}{4}\lambda_1 < \lambda < \lambda_1$.

1.6.1 The Brezis–Lieb lemma

In this appendix we present a very useful result about convergence in L^p spaces. The situation is this: a bounded sequence $\{u_n\}$ in L^p tends pointwise to some $u \in L^p$. If $\|u_n\|_p \to \|u\|_p$, and $p > 1$, it is a consequence of the uniform convexity that $u_n \to u$ strongly. Is it possible to precise the behavior of $\|u_n - u\|_p$?

Lemma 1.5 ([22]). *Let Ω be an open subset of $\mathbb{R}^n$, and let $\{u_n\}$ be a bounded sequence of $L^p(\Omega)$ functions tending a.e. to $u \in L^p$. Then*

$$\|u\|_p = \lim_{n \to \infty} (\|u_n\|_p - \|u_n - u\|_p)\,.$$

Proof. Let $M = \sup_n ||u_n||_p < \infty$. By the limit

$$\lim_{|s|\to\infty} \frac{|s+1|^p - |s|^p - 1}{|s|^p} = 0,$$

we see that for every $\varepsilon > 0$ there is a constant C_ε such that

$$||s+1|^p - |s|^p - 1| \leq \varepsilon |s|^p + C_\varepsilon \quad \forall s \in \mathbb{R}.$$

Hence, for all $a,b \in \mathbb{R}$,

$$||a+b|^p - |a|^p - |b|^p| \leq \varepsilon |a|^p + C_\varepsilon |b|^p.$$

Let now $f_n = ||f_n|^p - |f_n - f|^p - |f|^p|$ and $Z_n = \max\{0, f_n - \varepsilon |u_n - u|^p\}$. We know that Z_n tends to zero a.e., and $0 \leq Z_n \leq C_\varepsilon |f|^p$. By the Dominated Convergence theorem, $Z_n \to 0$ in L^1. But $0 \leq f_n \leq \varepsilon |u_n - u|^p + Z_n$, and so

$$||f_n||_1 \leq \varepsilon ||u_n - u||_1 + ||Z_n||_1 \leq 2^p M^p + ||Z_n||_1.$$

Finally, we deduce that $||f_n||_1 \to 0$. ♣

1.7 The Concentration–Compactness principle

As seen before, there are some evident reasons why a bounded sequence in $L^2(\mathbb{R}^n)$ cannot converge strongly. Roughly speaking, we can say $\mathbb{R}^n$ is very large, and sequences can "move off" to infinity, or concentrate at finite points even though they are constrained to minimize some functional. Hence it would be desirable to have some tool to work out these features, to regain compactness. In this section we briefly survey the celebrated Concentration–Compactness principle by P. L. Lions ([65, 66]). A completely different strategy will be one of the most important topics of our thesis in the next part.

Theorem 1.7 (Locally compact case). *Let $\{\varrho_n\}$ be a sequence in $L^1(\mathbb{R}^n)$ such that*

(1) $\varrho_n \geq 0$ almost everywhere
(2) $\int_{\mathbb{R}^n} \varrho_n(x)\, dx = \lambda > 0$.

Then $\{\varrho_n\}$ has a subsequence $\{\varrho_{n_k}\}$ satisfying one of the following properties:

– **Compactness.** *There is a sequence $\{y_k\}$ in $\mathbb{R}^n$ such that*

$$(\forall \varepsilon > 0)(\exists R > 0) \colon \int_{y_k + B_R} \varrho_{n_k}(x)\, dx \geq \lambda - \varepsilon.$$

– **Vanishing.**

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} \varrho_{n_k}(x)\, dx = 0 \quad \forall R > 0.$$

– **Dichotomy.** *There exists* $\alpha \in (0,\lambda)$ *such that for all* $\varepsilon > 0$ *there are an integer* $k_0 \geq 1$ *and* $\rho_k^1, \varrho_k^2 \in L^1(\mathbb{R}^n)$ *such that*
   1. $\varrho_k^1,\ \varrho_k^2 \geq 0$
   2. $\|\varrho_{n_k} - (\varrho_k^1 + \rho_k^2)\|_{L^1} \leq \varepsilon$
   3. $\left|\int_{\mathbb{R}^n} \varrho_k^1(x)\, dx - \alpha\right| \leq \varepsilon$
   4. $\left|\int_{\mathbb{R}^n} \varrho_k^2(x)\, dx - \alpha\right| \leq \varepsilon$
   5. $\operatorname{dist}(\operatorname{supp} \varrho_k^1, \operatorname{supp} \varrho_k^2) \to +\infty$ *as* $k \to +\infty$.

The following result is often useful to exclude the vanishing case.

Lemma 1.6 ([65]). *Let* $1 < p \leq \infty$, $1 \leq q < \infty$, *and* $N \geq 1$ *an integer. When* $N > p$, *we suppose that* $q \neq p^\star$. *Let* $\{u_k\}$ *be a bounded sequence in* $L^q(\mathbb{R}^n)$ *such that* $\{|\nabla u_k|\}$ *is bounded in* $L^p(\mathbb{R}^n)$; *if there is* $R > 0$ *such that*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B(y,R)} |u_k(x)|\, dx = 0,$$

then $u_k \to 0$ *in* $L^r(\mathbb{R}^n)$ *provided that* $\min\{q,p^\star\} < r < \max\{q,p^\star\}$.

To present the result about the non–locally compact case, i.e. when some critical exponent is involved, we need some preliminaries.

Definition 1.8. $\mathcal{M}(\Omega)$ *denotes the space of bounded Radon measures on the open set* $\Omega \subset \mathbb{R}^n$, *endowed with the norm*

$$\|\mu\| = \sup_{\substack{u\in C_0(\Omega)\\ \|u\|_\infty = 1}} |\mu(u)|\,.$$

we say that a sequence $\{\mu_k\}$ *in* $\mathcal{M}(\Omega)$ *converges weakly*[6] *to* μ *if*

$$\lim_{k\to\infty} \mu_k(u) = \mu(u) \quad \forall u \in C_0(\Omega).$$

and $\{\mu_k\}$ *converges tightly to* μ *if*

$$\lim_{k\to\infty} \mu_k(u) = \mu(u) \quad \forall u \in L^\infty(\Omega).$$

Theorem 1.8 (Critical case). *Let* $m \geq 1$ *be an integer,* $p \in [1, n/m)$, *and denote*

$$p^{*m} = \frac{np}{n-mp}$$

$$\theta = 1 - \frac{mp}{n}.$$

Let

[6] The reader may remark that this convergence should be rather called *weak**. Anyway, such a simplification is customary in Probability theory.

$$S(m,p,n) = \inf_{\substack{u\in D^{m,p}(\mathbb{R}^n)\\ \|u\|_{p^{*m}}=1}} \|D^m u\|_p^p.$$

Assume the sequence $\{u_k\}$ *in* $D^{m,p}(\mathbb{R}^n)$ *converges weakly to* $u \in D^{m,p}(\mathbb{R}^n)$, *and that there exist two bounded measures* λ *and* μ *such that*

$$|D^m u_k| \rightharpoonup \mu \quad \text{in } \mathcal{M}(\mathbb{R}^n)$$

$$|u_k|^{p^{*m}} \rightharpoonup \lambda \quad \text{tightly.}$$

Then

1. *There exist a set* J, *at most countable, some points* $\{x_j\}_{j\in J}$, *and some real numbers* $\{a_j\}_{j\in J}$ *such that* $a_j > 0$ *for all* $j \in J$ *and*
$$\lambda = |u|^{p^{*m}} + \sum_{j\in J} a_j \delta_{x_j}.$$
2. *There exist real numbers* $b_j > 0$ *such that* $S(m,p,n)a_j^\theta \le b_j$ *and*
$$|D^m u|^p + \sum_{j\in J} b_j\delta_{x_j} \le \mu,$$
$$\sum_{j\in J} a_j^\theta < \infty.$$
3. *If* $v \in D^{m,p}(\mathbb{R}^n)$ *and* $|D^m(u_k+v)|^p \rightharpoonup \tilde{\mu}$ *in* $\mathcal{M}(\mathbb{R}^n)$, *then* $\tilde{\mu} - \mu \in L^1(\mathbb{R}^n)$ *and*
$$|D^m(u+v)|^p + \sum_{j\in J} b_j\delta_{x_j} \le \tilde{\mu}.$$
4. *If* $u \equiv 0$ *and* $0 < \mu(\mathbb{R}^n) < S(m,p,n)\lambda(\mathbb{R}^n)$, *then* J *is a singleton, and there exist* $C > 0$, $x_0 \in \mathbb{R}^n$ *such that*
$$\lambda = C\delta_{x_0}, \qquad \mu = S(m,p,n)C^{1+\theta}\delta_{x_0}.$$

Finally, we present a recent improvement appearing in [17], which unifies the concentration at infinity and the concentration at finite points. Despite its simple appearence, it can be adapted to a lot of different situations.

Theorem 1.9. *Let* $\Omega \subset \mathbb{R}^n$ *an open set, and let* $1 \le p < \infty$. *Assume that*

a) $\{u_k\}$ *is a bounded sequence in* $L^p(\Omega)$;
b) $u_k \to u$ *a.e. in* Ω,
c) $|u_k - u|^p \rightharpoonup \nu$ *in* $\mathcal{M}(\Omega)$,

and define

$$\nu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{|x|>R} |u_k|^p\,dx.$$

Then it follows that

$$\limsup_{k\to\infty} |u_k|_p^p = |u_k|_p^p + \|\nu\| + \nu_\infty.$$

Proof. 1) We write $v_k = u_k - u$. According to the Brezis–Lieb lemma, we have, for every non-negative $h \in C_0^\infty(\Omega)$,

$$\int h|u|^p\,dx = \lim_{k\to\infty}\left(\int h|u_k|^p\,dx - \int h|v_k|^p\,dx\right).$$

It follows that

$$|u_k|^p \rightharpoonup |u|^p + \nu \quad \text{in } \mathcal{M}(\Omega). \tag{1.14}$$

2) For $R > 1$, let $\Psi_R \in C_0^\infty(\mathbb{R}^n)$ be such that $\Psi_R(x) = 1$ for $|x| > R+1$, $\Psi_R(x) = 0$ for $|x| < R$, and $0 \le \Psi_R \le 1$ on $\mathbb{R}^n$. It is easy to check that

$$\nu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int \Psi_R|u_k|^p\,dx.$$

3) For every $R > 1$, we have by (1.14)

$$\begin{aligned}\limsup_{k\to\infty}|u_k|^p\,dx &= \limsup_{k\to\infty}\left(\int \Psi_R^p|u_k|^p\,dx + \int(1-\Psi_R^p)|u_k|^p\,dx\right)\\ &= \limsup_{k\to\infty}\int \Psi_R^p|u_k|^p\,dx + \int(1-\Psi_R^p)|u|^p\,dx + \int(1-\Psi_R^p)\,d\nu(x).\end{aligned}$$

When $R \to \infty$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\int|u_k|^p\,dx &= \nu_\infty + \int|u|^p\,dx + \int d\nu(x)\\ &= \nu_\infty + |u|_p^p + \|\nu\|.\end{aligned}$$

♣

Remark 1.1. Clearly, the terms $\|\nu\|$ and ν_∞ are responsible for lack of compactness. The rôle of the latter is evident, while the first measures the concentration of the sequence $\{u_k\}$ at finite points.

1.8 A Sobolev-Hardy type inequality

In this section we use Theorem 1.19 to find the best constant of a Hardy–type inequality. We follow [79]. Set $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, with $2 \le k \le N$, and write $x = (x', z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. For given numbers q, s such that $1 < q < N$, $0 \le s \le q$, and $s < k$, set $q_\star(s, N, q) = \frac{q(N-s)}{N-q}$. We also write $q_\star(s)$ instead of $q_\star(s, N, q)$.

We are able to proof the following statement, which extends a previous result contained in [13].

Theorem 1.10. *Let* $2 \le k \le N$, $1 \le p < k$, $\alpha + k > 0$, $u \in C_0^\infty(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} |u|^p |x'|^\alpha \, dx \le \frac{p^p}{(\alpha+k)^p} \int_{\mathbb{R}^n} |\nabla u|^p |x'|^{\alpha+p} \, dx. \tag{1.15}$$

Moreover, the constant $\frac{p^p}{(\alpha+k)^p}$ *is optimal.*

It is convenient to state the following Lemma.

Lemma 1.7. *If* $1 < p < \infty$ *and* $\alpha + N > 0$, *then*

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^p |x|^\alpha \, dx &\le \frac{p^p}{(\alpha+N)^p} \int_{\mathbb{R}^n} |x \cdot \nabla u|^p |x|^\alpha \, dx \\ &\le \frac{p^p}{(\alpha+N)^p} \int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha+N} \, dx\end{aligned}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. *Moreover, the constant* $\frac{p^p}{(\alpha+N)^p}$ *is the best possible.*

Proof. We approximate $x \mapsto |x|^\alpha$ by the "sequence" $x \mapsto (|x|^2 + \varepsilon)^{\alpha/2}$, so that

$$\operatorname{div}[|x|^\alpha x] = (\alpha+N)|x|^\alpha,$$

at least in a weak sense. Moreover, for all $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \operatorname{div}[|u|^p |x|^\alpha x] \, dx = 0.$$

We get from the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^p |x|^\alpha \, dx &\le \frac{p^p}{(\alpha+N)^p} \int_{\mathbb{R}^n} |x \cdot \nabla u| |u|^{p-1} |x|^\alpha \, dx \\ &\le \frac{p^p}{(\alpha+N)^p} \left(\int_{\mathbb{R}^n} |u|^{(p-1)p'} |x|^\alpha \, dx \right)^{1/p'} \left(\int_{\mathbb{R}^n} |x \cdot \nabla u|^p |x|^\alpha \, dx \right)^{1/p} \\ &= \frac{p^p}{(\alpha+N)^p} \left(\int_{\mathbb{R}^n} |u|^p |x|^\alpha \, dx \right)^{1-1/p} \left(\int_{\mathbb{R}^n} |x \cdot \nabla u|^p |x|^\alpha \, dx \right)^{1/p}.\end{aligned}$$

For the optimality of the constant, we refer to [79]. ♣

Proof (of Theorem 10). Consider first the case $p = 2$. Then

$$\begin{aligned}\int_{\mathbb{R}^{N-k} \times \mathbb{R}^k} |\nabla u|^2 |x'|^{\alpha+2} \, dx &\ge \int_{\mathbb{R}^{N-k}} dz \int_{\mathbb{R}^k} |\nabla_{x'} u|^2 |x'|^{\alpha+2} \, dx' \\ &\ge \frac{(\alpha+2)^2}{4} \int_{\mathbb{R}^{N-k}} dz \int_{\mathbb{R}^k} |u|^2 |x'|^\alpha \, dx' \\ &= \frac{(\alpha+2)^2}{4} \int_{\mathbb{R}^n} |u|^2 |x'|^\alpha \, dx\end{aligned}$$

where we have used the previous Lemma. Choose now

$$u\colon (x', z) \mapsto v(x')w(z),$$

where $v \in C_0^\infty(\mathbb{R}^k)$ and $w \in C_0^\infty(\mathbb{R}^{N-k})$. In the rest of the proof, we identify the gradients of v and w with two vectors in $\mathbb{R}^n$:

$$\begin{cases} \nabla v = \left(\frac{\partial v}{\partial x'}, 0\right) \\ \nabla w = \left(0, \frac{\partial w}{\partial z}\right). \end{cases}$$

Hence, $\nabla v \perp \nabla w$. By definition,

$$\begin{aligned} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 |x'|^{\alpha+2}\, dx}{\int_{\mathbb{R}^n} |u|^2 |x'|^\alpha\, dx} &= \frac{\int_{\mathbb{R}^n} |x'|^{\alpha+2} \left(|\nabla v|^2 |w|^2 + |v|^2 |\nabla w|^2\right) dx}{\int_{\mathbb{R}^n} |x'|^\alpha |v|^2 |w|^2\, dx} \\ &= \frac{\int_{\mathbb{R}^k} |x'|^{\alpha+2} |\nabla v|^2\, dx'}{\int_{\mathbb{R}^k} |v|^2 |x'|^\alpha\, dx'} + \frac{\int_{\mathbb{R}^k} |x'|^{\alpha+2} |v|^2\, dx'}{\int_{\mathbb{R}^k} |x'|^\alpha |v|^2\, dx'} \frac{\int_{\mathbb{R}^{N-k}} |\nabla w|^2}{\int_{\mathbb{R}^{N-k}} |w|^2}. \end{aligned}$$

We deduce

$$\begin{aligned} &\inf_{u \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 |x'|^{\alpha+2}\, dx}{\int_{\mathbb{R}^n} |u|^2 |x'|^\alpha\, dx} \\ &\quad \leq \inf_{\substack{v \in C_0^\infty(\mathbb{R}^k)\setminus\{0\} \\ w \in C_0^\infty(\mathbb{R}^{N-k}\setminus\{0\}}} \frac{\int_{\mathbb{R}^n} |x'|^{\alpha+2} \left(|w|^2 |\nabla v|^2 + |v|^2 |\nabla w|^2\right) dx}{\int_{\mathbb{R}^n} |x'|^2 |v|^2 |w|^2\, dx} \\ &\quad \leq \inf_{v \in C_0^\infty(\mathbb{R}^k)\setminus\{0\}} \frac{\int_{\mathbb{R}^k} |x'|^{\alpha+2} |\nabla v|^2\, dx}{\int_{\mathbb{R}^k} |x'|^\alpha |v|^2\, dx} = \frac{(\alpha+2)^2}{4} \end{aligned}$$

because

$$\inf_{w \in C_0^\infty(\mathbb{R}^{N-k})\setminus\{0\}} \frac{\int_{\mathbb{R}^k} |\nabla w|^2\, dx}{\int_{\mathbb{R}^{N-k}} |w|^2\, dx} = 0$$

and the optimality result contained in the previous Lemma.

The general case $p \neq 2$ is slightly more involved, since we work in a non-hilbertian setting. The validity of the inequality is checked as before. Anyway, the optimality of $p^p/(\alpha+k)^p$ requires the following modifications. Split again $u(x) = v(x')w(z)$, so that

$$\frac{\int_{\mathbb{R}^n} |\nabla u|^p |x'|^{\alpha+p}\, dx}{\int_{\mathbb{R}^n} |u|^p |x'|^\alpha\, dx} = \frac{\int_{\mathbb{R}^n} |x'|^{\alpha+p} \left(|\nabla v|^2 |w|^2 + |\nabla w|^2 |v|^2\right)^{p/2} dx}{\int_{\mathbb{R}^n} |x'|^\alpha |v|^p |w|^p\, dx}.$$

We introduce the map

$$\begin{aligned} f\colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (s,t) &\mapsto \left(s^2 + t^2\right)^{p/2}. \end{aligned}$$

Clearly f is a convex function. For each $\varepsilon \in (0,1)$, we see that

$$f(s,t) = f\left(\varepsilon(\frac{s}{\varepsilon},0) + (1-\varepsilon)(0,\frac{t}{1-\varepsilon})\right) \leq$$
$$\varepsilon f\left(\frac{s}{\varepsilon},0\right) + (1-\varepsilon) f\left(0,\frac{t}{1-\varepsilon}\right) = \varepsilon \frac{s^p}{\varepsilon^p} + (1-\varepsilon)\frac{t^p}{(1-\varepsilon)^p}.$$

Hence

$$\frac{\int_{\mathbb{R}^n} |\nabla u|^p |x'|^{\alpha+p}\,dx}{\int_{\mathbb{R}^n} |u|^p |x'|^{\alpha}\,dx} \leq \varepsilon^{1-p} \frac{\int_{\mathbb{R}^k} |x'|^{\alpha+p} |\nabla v|^p\,dx'}{\int_{\mathbb{R}^k} |x'|^{\alpha} |v|^p\,dx'} + (1-\varepsilon)^{1-p} \frac{\int_{\mathbb{R}^{N-k}} |\nabla w|^p\,dz}{\int_{\mathbb{R}^{N-k}} |w|^p\,dz}.$$

We deduce that

$$\inf_{u \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x'|^{\alpha+p}\,dx}{\int_{\mathbb{R}^n} |x'|^{\alpha} |u|^p\,dx} \leq \varepsilon^{1-p} \inf_{v \in C_0^\infty(\mathbb{R}^k)\setminus\{0\}} \frac{\int_{\mathbb{R}^k} |x'|^{\alpha+p} |\nabla v|^p\,dx'}{\int_{\mathbb{R}^k} |x'|^{\alpha} |v|^p\,dx'}.$$

We conclude by letting $\varepsilon \to 0$. ♣

It is a natural question if the best constant C in (1.15) is attained. More precisely, let

$$S = \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^q\,dx \colon u \in D^{1,q}(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|u(x',z)|^{q_*(s)}}{|x'|^s}\,dx'dz = 1 \right\}. \tag{1.16}$$

Here S depends on all the constants of the problem, i.e. k, q, s and N. For $s = 0$, it reduces to the best Sobolev constant, and to the best Hardy constant for $s = q$ and $k = N$. It is well-known that the best Hardy constant is never attained. However, the following theorem states that in the whole family of inequalities between the Sobolev and the Hardy's one, the latter is the only negative case.

Theorem 1.11. *Assume $1 \leq k \leq N$, $1 < q < N$, $x = (x', z)$, $0 \leq s < q$ and $s < k$. Then the extremal problem (1.16) attains its infimum at a function $u \in D^{1,q}(\mathbb{R}^N)$ which satisfies*

$$\int_{\mathbb{R}^N} |\nabla u|^q\,dx = S, \qquad \int_{\mathbb{R}^N} \frac{|u(x',z)|^{q_*(s)}}{|x'|^s}\,dx'dz = 1.$$

Proof. For a standard proof using Theorems 7 and 8, we refer to [14]. Here we give a rather short proof by means of a suitable modification of Theorem 9.

Let $\{u_n\}$ be a minimizing sequence for S. Hence $\{u_n\}$ is bounded in $D^{1,q}(\mathbb{R}^N)$. By Remark 7 in [14], it is known that we can always find sequences $\lambda_n > 0$ and $\zeta_n \in \mathbb{R}^{N-k}$ such that the new sequence $\{v_n\}$ defined by

$$v_n(x',z) = \lambda_n^{\frac{N-q}{q}} u_n(\lambda_n x', \lambda_n(z - \zeta_n))$$

is still a minimizing sequence for S, and moreover

$$\sup_{\zeta\in\mathbb{R}^{N-k}} \int_{|z-\zeta|<1}\int_{|x'|<1} \frac{|v_n|^{q_*(s)}}{|x'|^s}\,dx = \int_{|z|<1}\int_{|x'|<1} \frac{|v_n|^{q_*(s)}}{|x'|^s}\,dx = \frac{1}{2}.$$

Hence, without loss of generality, we can replace u_n by v_n. By the Sobolev embedding theorem, we may also assume, up to a subsequence, that

$$\begin{gathered} v_n \rightharpoonup v \quad \text{in } D^{1,q}(\mathbb{R}^N) \\ |\nabla(v_n - v)|^q \to \mu \quad \text{in } \mathcal{M}(\mathbb{R}^N) \\ \left|\frac{v_n - v}{|x'|^{s/q_*(s)}}\right|^{q_*(s)} \to \nu \quad \text{in } \mathcal{M}(\mathbb{R}^N) \\ v_n \to v \quad \text{a.e. in } \mathbb{R}^N \end{gathered}$$

We introduce now the two quantities

$$\mu_\infty = \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |\nabla v_n|^q\,dx$$

$$\nu_\infty = \lim_{R\to+\infty} \limsup_{n\to\infty} \int_{|x|\geq R} \frac{|v_n|^{q_*(s)}}{|x'|^s}\,dx.$$

From the definition of S we deduce

$$\begin{aligned} \nu_\infty^{q/q_*(s)} &\leq S^{-1}\mu_\infty \\ \|\nu\|^{q/q_*(s)} &\leq S^{-1}\|\mu\| \\ |\nabla v|_q^q + \|\mu\| + \mu_\infty &\leq S \\ \left|\frac{v}{|x'|^{s/q_*(s)}}\right|_{q_*(s)}^{q_*(s)} + \|\nu\| + \nu_\infty &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} S \geq |\nabla v|_q^q + \|\mu\| + \mu_\infty &\geq |\nabla v|_q^q + S\|\nu\|^{q/q_*(s)} + S\nu_\infty^{q/q_*(s)} \\ &\geq S\left|\frac{v}{|x'|^{s/q_*(s)}}\right|_{q_*(s)}^{q_*(s)} + S\|\nu\|^{q/q_*(s)} + S\nu_\infty^{q/q_*(s)}, \end{aligned}$$

and finally

$$\left|\frac{v}{|x'|^{s/q_*(s)}}\right|_{q_*(s)}^{q_*(s)} + \|\nu\|^{q/q_*(s)} + \nu_\infty^{q/q_*(s)} \leq 1.$$

This implies that $\|\nu\|$, ν_∞ and $\left|\frac{v}{|x'|^{s/q_*(s)}}\right|_{q_*(s)}^{q_*(s)}$ are all equal to either 0 or 1. But $\nu_\infty = 0$, since it follows from the choice of λ_n and ζ_n that $\nu_\infty \leq 1/2$. Suppose that $v = 0$. Then

$$1 = \|\nu\|^{q/q_*(s)} = S^{-1}\|\mu\|.$$

Hence, for all open subsets $\omega \subset \mathbb{R}^n$, $\nu(\omega) \leq \nu(\mathbb{R}^n)$, and this immediately implies that ν and μ are concentrated on a single point $x_0 = (x_0', z_0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Once again, by the choice of λ_n and ζ_n, there is a number $r > 0$ such that $|x_0| = \max\{|x_0'|, |z_0|\} \geq r$. We then reach the contradiction

$$0 = \lim_{n\to\infty} \int_{|x-x_0|<r} \frac{|v_n|^{q_*(s)}}{|x'|^s}\, dx = \|\nu\| = 1.$$

The only possibility is that $\|\nu\| = 0$, so that

$$\frac{|v|^{q_*(s)}}{|x'|^s}\, dx = 1.$$

In other words, v is a minimizer for S. ♣

Remark 1.2. The previous theorems are contained in the forthcoming paper [79], where we also prove that all minimizers corresponding to the best constant have a rather precise symmetric shape.

Part II

...provided ε is small enough

2 The perturbation technique

2.1 Where is *Critical point theory* going?

Over the last two decades, a lot of attention has been paid to *Critical point theory*, because of its flexibility and also intuitiveness.

Besides the Concentration–Compactness Principle already described, a new technique to deal with problems without compactness has been introduced in the last years by Ambrosetti and Badiale ([3, 4, 6]), which improves some previous results contained in [7]. It allows us to treat problems in which a "small parameter" appears, and permits to find existence and multiplicity results even in some situations where the concentration-compactness methods fails or requires heavy calculations. For the sake of completeness, we present below this technique in some detail. Roughly speaking, we can say that one exploits some very precise properties of solutions to *unperturbed equations* to find one or more solutions of a *perturbed equation*.

Suppose we are given a family $\{f_\varepsilon\}$ of C^2 functionals over a real Hilbert space, parametrized by $\varepsilon \geq 0$. When $\varepsilon = 0$, we have a "distinguished" functional f_0, and we suppose that f_0 has a whole manifold of critical points.[1] Under suitable assumptions, it is possible to prove the existence of critical points of each f_ε, provided ε is small enough. Let us formalize what we have said.

We want to find critical points of functionals of the form

$$f_\varepsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + G(\varepsilon, u), \tag{2.1}$$

where $\|\cdot\|$ denotes the norm in an Hilbert space E, $F\colon E \to \mathbb{R}$ and $G : \mathbb{R} \times E \to \mathbb{R}$. We assume that $f_0 = \frac{1}{2}\|\cdot\|^2 - F$ has a nondegenerate critical manifold[2] Z, namely we require that

(F0) $F \in C^2$ and $G(0,u) = 0$ for all $u \in E$;
(F1) there exists a d-dimensional C^2 manifold Z at level b of critical points of f_0;

[1] Very recently, this assuption has been highly relaxed, to cover situations where the points of this manifold are far from being critical. We refer to [9].
[2] We always assume manifold to be smooth enough. As a rule, C^2 *is* enough.

(F2) $F''(z)$ is compact, for all $z \in Z$;
(F3) $T_z Z = \ker f_0''(z)$, for all $z \in Z$.

Here $T_z Z$ stands for the tangent space to Z at the point $z \in Z$. Indicating with G' and G'' respectively the first and the second derivative of G with respect to u, on G we assume

(G0) G is continuous in $\mathbb{R} \times E$ and $G(0, u) = 0$, for all $u \in E$;
(G1) G is of class C^2 with respect to $u \in E$;
(G2) the maps $(\varepsilon, u) \longmapsto G'(\varepsilon, u)$ and $(\varepsilon, u) \longmapsto G''(\varepsilon, u)$ are continuous;
(G3) there exist $\alpha > 0$ and a continuous function $\Gamma : Z \to \mathbb{R}$ such that

$$\Gamma(z) = \lim_{\varepsilon \to 0} \frac{G(\varepsilon, z)}{\varepsilon^\alpha}$$

and

$$G'(\varepsilon, z) = o(\varepsilon^{\alpha/2}).$$

To avoid technicalities, we will assume that $Z = \zeta(\mathbb{R}^d)$ where $\zeta \in C^2(\mathbb{R}^d, E)$. We will set $Z^r = \zeta(B_r)$ where $B_r = \{x \in \mathbb{R}^d : |x| \leq r\}$.

The main idea is to use the implicit function theorem to produce a good local deformation of Z in the normal direction. More precisely we have the following.

Lemma 2.1. *Given $R > 0$, there exist $\varepsilon_0 > 0$ and a smooth function*

$$w \colon M = Z^R \times (-\varepsilon_0, \varepsilon_0) \to E$$

such that

(i) $w(z, 0) = 0$ for all $z \in Z^R$;
(ii) $f_\varepsilon'(z + w(z, \varepsilon)) \in T_z Z$ for all $(z, \varepsilon) \in M$;
(iii) $w(z, \varepsilon)$ is orthogonal to $T_z Z$ for all $(z, \varepsilon) \in M$.

Proof. See [3].

Lemma 2.2. *The manifold*

$$Z_\varepsilon = \{z + w(z, \varepsilon) \mid (z, \varepsilon) \in M\}$$

is a natural constraint for f_ε, namely: if $u \in Z_\varepsilon$ and $f_\varepsilon{}'_{|Z_\varepsilon}(u) = 0$, then $f_\varepsilon'(u) = 0$.

Proof. Suppose that $f_\varepsilon{}'_{|Z_\varepsilon}(u) = 0$. Then $f_\varepsilon'(u)$ is orthogonal to $T_u Z_\varepsilon$. On the other hand, $f_\varepsilon(u) \in T_z Z$ and $T_u Z_\varepsilon$ is close to $T_z Z$ for ε small. This implies $f_\varepsilon'(u) = 0$.

We have the following theorem (see Theorem 1.3 of [4]):

Theorem 2.1. *Suppose (F0-3) and (G0-3) hold and assume that there exist $\delta > 0$ and $z^* \in Z$ such that*

$$\text{either} \quad \min_{\|z-z^*\|=\delta} \Gamma(z) > \Gamma(z^*) \quad \text{or} \quad \max_{\|z-z^*\|=\delta} \Gamma(z) < \Gamma(z^*).$$

Then, for ε small, f_ε has a critical point in Z.

Proof. We give only the sketch of the proof divided into three steps.

Step 1 It is not difficult to prove that the function w built in the previous Lemma is such that

$$\|w\| = o(\varepsilon^{\alpha/2}) \qquad \text{and} \qquad f'_\varepsilon(z+w) \in T_z Z. \tag{2.2}$$

Step 2 Using the Taylor expansion, for $u \in Z_\varepsilon$, one can easily see that

$$f_\varepsilon(u) = b + \varepsilon^\alpha \Gamma(z) + o(\varepsilon^\alpha) \qquad (\varepsilon \to 0).$$

Step 3 It readily follows that f_ε has a local constrained minimum (or maximum) on Z_ε at some u_ε. According to Lemma 2.2, such a u_ε is a critical point of f_ε.

Under some additional assumptions, we can have more information about the bifurcation set: we can say that this set is a curve.

Hereafter, we will denote z_θ, $\theta \in \mathbb{R}^d$, the elements of Z. We will indicate with $\partial_i z$ and $\partial_{ij} z$ respectively $\partial z/\partial\theta_i$ and $\partial^2 z/\partial\theta_i\partial\theta_j$.

Now we consider two case:

First case Suppose that

(F4) F is of class C^4;

(G4) G is of class C^4 with respect to $u \in E$, moreover the maps $(\varepsilon, u) \longmapsto G'''(\varepsilon, u)$ is continuous;

(G5) $G'(\varepsilon, z_\theta) = o(\varepsilon^{\alpha/2})$ and there exist continuous functions $\gamma_{ij}, \tilde{\gamma}_{ij} : Z \to \mathbb{R}$ such that if $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$, there results

$$\varepsilon^{-\alpha} \left(G'(\varepsilon, z_{\theta_\varepsilon}) | \partial_{ij} z_{\theta_\varepsilon}\right) \to \gamma_{ij}(\theta), \tag{2.3}$$

$$\varepsilon^{-\alpha} \left(G''(\varepsilon, z_{\theta_\varepsilon}) \partial_i z_\theta | \partial_j z_\theta\right) \to \tilde{\gamma}_{ij}(\theta), \tag{2.4}$$

$$\varepsilon^{-\alpha/2} G''(\varepsilon, z_{\theta_\varepsilon}) \partial_{ij} z_{\theta_\varepsilon} \to 0, \tag{2.5}$$

$$\varepsilon^{-\alpha/2} G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_\theta, \partial_j z_\theta] \to 0, \tag{2.6}$$

as $\varepsilon \to 0$.

Second case Suppose that

(F4)′ F is of class C^3;

(G4)′ G is of class C^3 with respect to $u \in E$ and $G'''(\varepsilon, u) \to 0$ as $\varepsilon \to 0$, uniformly in $u \in Z^r$; moreover the map $(\varepsilon, u) \longmapsto G'''(\varepsilon, u)$ is continuous;

(G5)′ $G'(\varepsilon, z_\theta) = O(\varepsilon^\alpha)$ and there exist continuous functions $\gamma_{ij}, \tilde{\gamma}_{ij} : Z \to \mathbb{R}$ such that if $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$, there results

$$\varepsilon^{-\alpha} \left(G'(\varepsilon, z_{\theta_\varepsilon}) | \partial_{ij} z_{\theta_\varepsilon}\right) \to \gamma_{ij}(\theta),$$

$$\varepsilon^{-\alpha} \left(G''(\varepsilon, z_{\theta_\varepsilon}) \partial_i z_\theta | \partial_j z_\theta\right) \to \tilde{\gamma}_{ij}(\theta),$$

as $\varepsilon \to 0$.

The next theorem shows that the Morse index of the critical points u_ε is related to the nature of u_ε as a critical point of the Melnikov function Γ. Sometimes, this allows us to conclude that a particular critical point could not be found directly by an application of constrained minimization or of the mountain pass theorem. The Morse index of critical points of *mountain pass type* has been calculated. We refer to [27].

Despite its length, we present the proof, since in the original paper [4] there is a mistake. We wish to express our gratitude to A. Pomponio for pointing out this fact to us and supplying us with the correct statement and proof. See also [13].

Theorem 2.2. *Let $F \in C^3$, (F0-3) and (G0-3) hold. Suppose either (F4) and (G4-5) or (F4)′ and (G4-5)′ are satisfied. Let $\theta \in \mathbb{R}^d$ be given and let $u_\varepsilon = z(\theta_\varepsilon) + w(\varepsilon, \theta_\varepsilon) \in Z_\varepsilon$ be a critical point of f_ε on Z_ε with $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$. Assume that $z_\theta = \lim_\varepsilon z(\theta_\varepsilon)$ is nondegenerate for the restriction of f_0 to $(T_{z_\theta} Z)^\perp$, with Morse index equal to m_0, and that the Hessian $D^2\Gamma(\theta)$ is positive (negative) definite.*

Then u_ε in a nondegenerate critical point for f_ε with Morse index equal to m_0, if $D^2\Gamma(\theta) > 0$, and to $m_0 + d$, if $D^2\Gamma(\theta) < 0$. As a consequence, the critical points of f_ε form a continuous curve.

Proof. The first part of the proof works for all the two cases.

We write $E = E^+ \oplus E^0 \oplus E^-$ where $E^0 = T_z Z$, $\dim(E^-) = m_0$ and

$$\begin{cases} D^2 f_0(z)[v, v] > 0 & \forall v \in E^+ \\ D^2 f_0(z)[v, v] < 0 & \forall v \in E^-. \end{cases} \tag{2.7}$$

Moreover we have

$$D^2 f_0(z)[\partial_i z, \partial_j z] = 0.$$

We denote with $\varphi_i^0 = \partial_i z$, for $i = 1, \dots, d$, and with t_i^0, for $i = 1, \dots, m_0$, respectively the orthonormal base of E^0 and E^-.

We call now $E_\varepsilon^0 = T_{u_\varepsilon} Z$ and we will denote with φ_i^ε, for $i = 1, \dots, d$, its orthonormal base.

For $i = 1, \dots, m_0$, we want to find α_j^ε such that if $t_i^\varepsilon = t_i^0 + \alpha_i^\varepsilon$ then $(t_i^\varepsilon | \varphi_j^\varepsilon) = 0$, for all $j = 1, \dots, d$. So we must have

$$0 = (t_i^\varepsilon|\varphi_j^\varepsilon) = (t_i^0 + \alpha_i^\varepsilon|\varphi_j^0 + (\varphi_j^\varepsilon - \varphi_j^0)) = (t_i^0|\varphi_j^\varepsilon - \varphi_j^0) + (\alpha_i^\varepsilon|\varphi_j^\varepsilon),$$

therefore

$$(\alpha_i^\varepsilon|\varphi_j^\varepsilon) = -(t_i^0|\varphi_j^\varepsilon - \varphi_j^0)$$

and we take $\alpha_i^\varepsilon = \sum_{j=1}^d -(t_i^0|\varphi_j^\varepsilon - \varphi_j^0)\varphi_j^\varepsilon$.

Let E_ε^- be the space spanned by $\left\{\frac{t_1^\varepsilon}{|t_1^\varepsilon|}, \ldots, \frac{t_{m_0}^\varepsilon}{|t_{m_0}^\varepsilon|}\right\}$. We have

$$D^2 f_\varepsilon(u_\varepsilon)[t_i^\varepsilon, t_j^\varepsilon] = D^2 f_\varepsilon(u_\varepsilon)[t_i^0 + \alpha_i^\varepsilon, t_j^0 + \alpha_j^\varepsilon] =$$
$$= D^2 f_\varepsilon(u_\varepsilon)[t_i^0, t_j^0] + D^2 f_\varepsilon(u_\varepsilon)[t_i^0, \alpha_j^\varepsilon] + D^2 f_\varepsilon(u_\varepsilon)[\alpha_i^\varepsilon, t_j^0] + D^2 f_\varepsilon(u_\varepsilon)[\alpha_i^\varepsilon, \alpha_j^\varepsilon].$$

Since α_i^ε goes to 0, as $\varepsilon \to 0$, by (2.7) we have

$$D^2 f_\varepsilon(u_\varepsilon)[t_i^\varepsilon, t_j^\varepsilon] < 0 \tag{2.8}$$

and therefore $D^2 f_\varepsilon(u_\varepsilon)$ is negative definite on E_ε^-, for ε small.

Let us put now $E_\varepsilon^+ = (E_\varepsilon^0 \oplus E_\varepsilon^-)^\perp$. Of course we have

$$E = E_\varepsilon^+ \oplus E_\varepsilon^0 \oplus E_\varepsilon^-.$$

We want to show that $D^2 f_\varepsilon(u_\varepsilon)$ is positive definite on E_ε^+. Let $v_\varepsilon \in E_\varepsilon^+$ with $|v_\varepsilon| = 1$. Let P^+ indicate the orthogonal projection on E^+, then

$$D^2 f_\varepsilon(u_\varepsilon)[v_\varepsilon, v_\varepsilon] = D^2 f_\varepsilon(u_\varepsilon)[P^+v_\varepsilon + (v_\varepsilon - P^+v_\varepsilon), P^+v_\varepsilon + (v_\varepsilon - P^+v_\varepsilon)] =$$
$$= D^2 f_\varepsilon(u_\varepsilon)[P^+v_\varepsilon, P^+v_\varepsilon] + 2D^2 f_\varepsilon(u_\varepsilon)[P^+v_\varepsilon, v_\varepsilon - P^+v_\varepsilon] +$$
$$+ D^2 f_\varepsilon(u_\varepsilon)[v_\varepsilon - P^+v_\varepsilon, v_\varepsilon - P^+v_\varepsilon].$$

We claim that there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all $v_\varepsilon \in E_\varepsilon^+$ we have

$$D^2 f_\varepsilon(u_\varepsilon)[v_\varepsilon, v_\varepsilon] \geq \delta. \tag{2.9}$$

By contradiction, suppose the contrary, i.e. for all $k \in \mathbb{N}$ there exist ε_k and $v_{\varepsilon_k} \in E_{\varepsilon_k}^+$ such that

$$D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_{\varepsilon_k}, v_{\varepsilon_k}] < \frac{1}{k}.$$

So we should have

$$\frac{1}{k} > D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+v_{\varepsilon_k}, P^+v_{\varepsilon_k}] + 2D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+v_{\varepsilon_k}, v_{\varepsilon_k} - P^+v_{\varepsilon_k}] +$$
$$+ D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_{\varepsilon_k} - P^+v_{\varepsilon_k}, v_{\varepsilon_k} - P^+v_{\varepsilon_k}]$$

Recall that

$$v_{\varepsilon_k} - P^+v_{\varepsilon_k} = \sum_{i=1}^{m_0} (v_{\varepsilon_k}|t_i^0)t_i^0 + \sum_{i=1}^{d} (v_{\varepsilon_k}|\varphi_i^0)\varphi_i^0.$$

and notice that, since $v_{\varepsilon_k} \in E^+_{\varepsilon_k}$, $(v_{\varepsilon_k}|t_i^\varepsilon) = 0$ and $(v_{\varepsilon_k}|\varphi_i^\varepsilon) = 0$. We have

$$(v_{\varepsilon_k}|t_i^0) = (v_{\varepsilon_k}|t_i^\varepsilon) + (v_{\varepsilon_k}|t_i^0 - t_i^\varepsilon),$$

the first term is zero, the second one goes to zero since $\{v_{\varepsilon_k}\}$ is a bounded sequence. Similarly we have that also $(v_{\varepsilon_k}|\varphi_i^0)$ goes to zero and so

$$v_{\varepsilon_k} - P^+ v_{\varepsilon_k} \to 0.$$

This is a contradiction since, for k sufficiently large, by (2.7) we have

$$D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_{\varepsilon_k}, v_{\varepsilon_k}] \geq \bar{\delta} > 0.$$

At this moment, by (2.8) and (2.9), we have shown that $D^2 f_\varepsilon(u_\varepsilon)$ is negative definite on E^-_ε and positive definite on E^+_ε. We now want to investigate the behavior of $D^2 f_\varepsilon(u_\varepsilon)$ on E^0_ε.

First case We have

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = \\ &\quad = (\partial_i z_{\theta_\varepsilon}|\partial_j z_{\theta_\varepsilon}) - F''(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] + G''(\varepsilon, u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \end{aligned} \tag{2.10}$$

We know by **(F3)** that $\partial_i z_{\theta_\varepsilon} \in \ker[I_E - F''(z_{\theta_\varepsilon})]$ and so recalling that $w(0, z) = 0$ and expanding $F''(u_\varepsilon)$ and $G''(\varepsilon, u_\varepsilon)$, (2.10) becomes

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = \\ &\quad = (\partial_i z_{\theta_\varepsilon}|\partial_j z_{\theta_\varepsilon}) - F''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] - (F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) + \\ &\quad + G''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] + (G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) + O(\|w_\varepsilon\|^2) = \\ &\qquad = -(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) + G''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] + \\ &\qquad\qquad + (G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) + O(\|w_\varepsilon\|^2) \end{aligned}$$

We observe explicitly that, since $w_\varepsilon = o(\varepsilon^{\alpha/2})$ and by (2.6) of **(G5)** we can write

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = \\ &\quad = -(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) + G''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] + o(\varepsilon^\alpha) \end{aligned} \tag{2.11}$$

By (2.2) we have

$$z_{\theta_\varepsilon} + w_\varepsilon - F'(z_{\theta_\varepsilon} + w_\varepsilon) + G'(\varepsilon, z_{\theta_\varepsilon} + w_\varepsilon) = \sum_l a_l \partial_l z_{\theta_\varepsilon},$$

and so a development of $F'(z_{\theta_\varepsilon} + w_\varepsilon)$ and $G'(\varepsilon, z_{\theta_\varepsilon} + w_\varepsilon)$ yields

$$z_{\theta_\varepsilon} + w_\varepsilon - F'(z_{\theta_\varepsilon}) - F''(z_{\theta_\varepsilon})w_\varepsilon + G'(\varepsilon, z_{\theta_\varepsilon}) + G''(\varepsilon, z_{\theta_\varepsilon})w_\varepsilon = \sum_l a_l \partial_l z_{\theta_\varepsilon}. \tag{2.12}$$

Without loss of generality, we can assume that the parametrization of Z is normal to θ, namely that $\partial_{ij} z_{\theta_\varepsilon} \perp T_{z_{\theta_\varepsilon}} Z$. Taking the scalar product of (2.12) with $\partial_{ij} z_{\theta_\varepsilon}$ and, since $z_{\theta_\varepsilon} = F'(z_{\theta_\varepsilon})$, we find

$$(w_\varepsilon|\partial_{ij} z_{\theta_\varepsilon}) - (F''(z_{\theta_\varepsilon})w_\varepsilon|\partial_{ij} z_{\theta_\varepsilon}) + \\ (G'(\varepsilon, z_{\theta_\varepsilon})|\partial_{ij} z_{\theta_\varepsilon}) + (G''(\varepsilon, z_{\theta_\varepsilon})w_\varepsilon|\partial_{ij} z_{\theta_\varepsilon}) = 0 \quad (2.13)$$

By (2.5) of **(G5)**, (2.13) becomes

$$(w_\varepsilon|\partial_{ij} z_{\theta_\varepsilon}) - (F''(z_{\theta_\varepsilon})w_\varepsilon|\partial_{ij} z_{\theta_\varepsilon}) + (G'(\varepsilon, z_{\theta_\varepsilon})|\partial_{ij} z_{\theta_\varepsilon}) + o(\varepsilon^\alpha) = 0 \quad (2.14)$$

From $z_{\theta_\varepsilon} = F'(z_{\theta_\varepsilon})$, it follows

$$\partial_{ij} z_{\theta_\varepsilon} = F''(z_{\theta_\varepsilon})\partial_{ij} z_{\theta_\varepsilon} + F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]. \quad (2.15)$$

The scalar product of (2.15) with w_ε is

$$(\partial_{ij} z_{\theta_\varepsilon}|w_\varepsilon) = (F''(z_{\theta_\varepsilon})\partial_{ij} z_{\theta_\varepsilon}|w_\varepsilon) + (F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon). \quad (2.16)$$

Substituting (2.16) in (2.14), we have

$$-(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) = (G'(\varepsilon, z_{\theta_\varepsilon})|\partial_{ij} z_{\theta_\varepsilon}) + o(\varepsilon^\alpha). \quad (2.17)$$

Dividing by ε^α and passing to the limit for $\varepsilon \to 0$, by (2.3) and (2.4) of **(G5)** we have:

$$-\varepsilon^{-\alpha} (F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]|w_\varepsilon) = \varepsilon^{-\alpha}(G'(\varepsilon, z_{\theta_\varepsilon})|\partial_{ij} z_{\theta_\varepsilon}) + o(1) \to \gamma_{ij}(\theta), \\ \varepsilon^{-\alpha} G''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \to \tilde{\gamma}_{ij}(\theta).$$

Therefore, by (2.11)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = \gamma_{ij}(\theta) + \tilde{\gamma}_{ij}(\theta).$$

On the other side we have

$$\partial_{ij}\Gamma(\theta) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} (G'(\varepsilon, z)|\partial_{ij} z) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} G''(\varepsilon, z)[\partial_i z, \partial_j z] = \gamma_{ij}(\theta) + \tilde{\gamma}_{ij}(\theta).$$

Second case To study the behavior of $D^2 f_\varepsilon(u_\varepsilon)$ on E^0_ε, we note that, by **(G5)**$'$ and recalling that $w_\varepsilon = O(\varepsilon^\alpha)$ (see Lemma 2.2 of [4]), one can show easily that (2.11) and (2.17) still hold and so the proof goes as in our case.

In [3], the case of a perturbation like

$$f_\varepsilon = f_0 + \varepsilon G$$

was considered, leading to to easier statements and proofs. For some applications, we refer to the bibliography.

We present in the next chapters a few applications of the perturbation technique. The reader will see that the strategy is rather clear, despite some modifications due to the specific shape of the perturbation terms. In the case of the *Schrödinger equation*, the perturbation term is not globally smooth in the variables $(\varepsilon, u) \in \mathbb{R} \times H^1(\mathbb{R}^n)$, so that some trick is needed in order to apply the finite–dimensional reduction.

3 Closed geodesics on cylinders

In this chapter, we concentrate on the existence of closed geodesics on a non–compact manifold M. First, we gather some results from Riemannian geometry.

3.1 Tools from differential geometry

We assume some knowledge of what a smooth differentiable manifold is, together with the main structures: tangent and cotangent spaces, inner products, and so on.

Definition 3.1. *Let M be a smooth differentiable manifold. A* Riemannian metric *on M is a tensor field $g\colon C_2^\infty(TM) \to C_2^\infty(TM)$ such that for each $p \in M$ the restriction $g_p = g_{|T_pM\otimes T_pM}\colon T_pM \otimes T_pM \to \mathbb{R}$ with*

$$g_p\colon (X_p, Y_p) \mapsto g(X,Y)(p)$$

is an inner product on the vector space T_pM The pair (M,g) is acalled a Riemannian manifold.

Definition 3.2. *Let (E, M, π) be a smooth vector bundle over M. A* connection *on (E, M, π) is a map $\nabla\colon C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$ such that*

1. $\nabla_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \nabla_X v + \mu \cdot \nabla_X w$;
2. $\nabla_X(f \cdot v) = X(f) \cdot v + f \cdot \nabla_X v$;
3. $\nabla_{(f\cdot X + g\cdot Y)} v = f \cdot \nabla_X v + g \cdot \nabla_Y v$

for all λ, $\mu \in \mathbb{R}$, X, $Y \in C^\infty(TM)$, v, $w \in C^\infty(E)$, and f, $g \in C^\infty(M)$. A section $v \in C^\infty(E)$ is called parallel *with respect to the connection if $\nabla_X v = 0$ for all $X \in C^\infty(TM)$.*

Definition 3.3. *Let M be a smooth manifold, and ∇ be a connection on the tangent bundle (TM, M, π). The* torsion *of ∇ is the map $T\colon C_2^\infty(TM) \to C_1^\infty(TM)$ defined by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[\,,\,]$ *is the Lie bracket on* $C^\infty(M)$. *A connection* ∇ *is* torsion–free *if* $T \equiv 0$.

If g *is a Riemannian metric on* M, *then* ∇ *is said to be* metric *if*

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all X, Y, $Z \in C^\infty(TM)$.

Theorem 3.1. *Let* (M,g) *be a Riemannian manifold. There exists one and only one connection* ∇ *on* M *such that*

$$g(\nabla_X Y, Z) = \frac{1}{2}\left(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + \right. \\ \left. + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])\right)$$

for all X, Y, $Z \in C^\infty(TM)$. *This connection is called the* Levi-Civita connection *on* M.

Definition 3.4. *Let* M *be a smooth manifold and* (TM, M, π) *be its tangent bundle. A* vector field X along a curve $\gamma\colon I \to M$ *is a curve* $X\colon I \to TM$ *such that* $\pi \circ X = \gamma$. *By* $C^\infty_\gamma(TM)$ *we denote the set of all smooth vector fields along* γ.

Definition 3.5. *Let* (M,g) *be a Riemannian manifold and* $\gamma\colon I \to M$ *be a* C^1*-curve. A vector field* X *along* γ *is said to be* parallel *along* γ *if*

$$\nabla_{\dot\gamma} X = 0.$$

A C^2*-curve* $\gamma\colon I \to M$ *is said to be a* geodesic *if the vector field* $\dot\gamma$ *is parallel along* γ, *i.e.*

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

Geodesics possess a variational characterization that we now clarify.

Definition 3.6. *Let* (M,g) *be a Riemannian manifold and* $\gamma\colon I \to M$ *be a* C^2*-curve on* M. *For every compact interval* $[a,b] \subset I$ *we define the energy functional*

$$E_{[a,b]}(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma(t), \dot\gamma(t))\, dt.$$

Theorem 3.2. *A* C^2*-curve* γ *is a critical point for the energy functional if and only if it is a geodesic.*

Definition 3.7. *A geodesic is* closed *if its image is diffeomorphic to* S^1.

When M is compact, closed geodesics can be found by standard variational arguments. For a survey of standard and advanced results on the existence of closed geodesics, we refer to [58].

On the other hand, when M is not compact, the existence of critical points of the energy E is no longer evident. Indeed, very few results are known in the literature, see [16, 59, 85]. In particular, Tanaka deals with the manifod $M = \mathbb{R} \times S^N$, endowed with a metric $g(s,\xi) = g_0(\xi) + h(s,\xi)$, where g_0 is the standard product metric on $\mathbb{R} \times S^N$. Under the assumption that $h(s,\xi) \to 0$ as $|s| \to \infty$, he proves the existence of a closed geodesic, found as a critical point of the energy functional

$$E(u) = \frac{1}{2} \int_0^1 g(u)[\dot{u}, \dot{u}]\, dt, \tag{3.1}$$

defined on the loop space $\Lambda = \Lambda(M) = H^1(S^1, M)$. [1] The lack of compactness due to the unboundedness of M is overcome by a suitable use of the concentration–compactness principle. To carry out the proof, the fact that M has the specific form $M = \mathbb{R} \times S^N$ is fundamental, because this permits to compare E with a *functional at infinity* whose behavior is explicitely known.

Here we consider a perturbed metric $g_\varepsilon = g_0 + \varepsilon h$, and extend Tanaka's result in two directions. First, we show the existence of at least N, in some cases $2N$, closed geodesics on $M = \mathbb{R} \times S^N$, see Theorem 3.5. Such a theorem can also be seen as an extension to cilyndrical domains of the result by Carminati [25]. Next, we deal with the case in which $M = \mathbb{R} \times M_0$ for a general compact N–dimensional manifold M_0. [2] The existence result we are able to prove requires that either M_0 possesses a non–degenerate closed geodesic, see Theorem 3.6, or that $\pi_1(M_0) \neq \{0\}$ and the geodesics on M_0 are *isolated*, see Theorem 3.8.

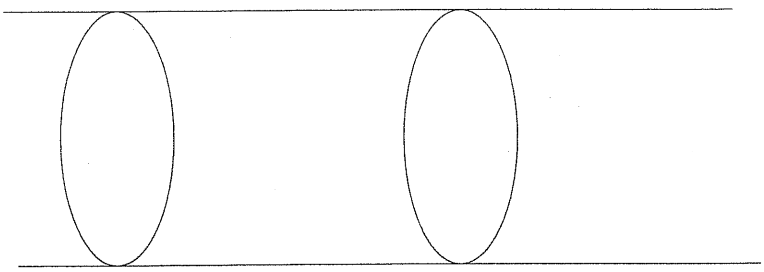

Fig. 3.1. Unperturbed cylinder

The approach we use is different that Tanaka's one, and relies on the perturbation result discussed before. Roughly, the main advantages of using this abstract perturbation method are that

(i) we can obtain sharper results, like the multeplicity ones;

[1] We will identify S^1 with $[0,1]/\{0,1\}$.

[2] By manifold we mean a smooth, connected manifold.

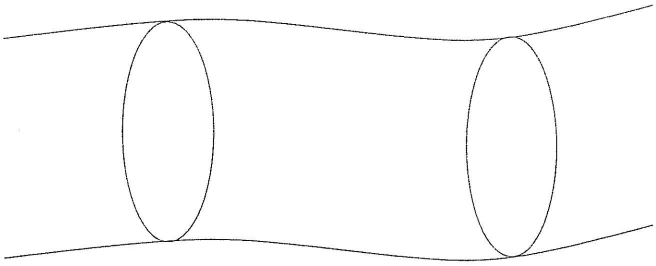

Fig. 3.2. Perturbed cylinder

(ii) we can deal with a general manifold like $M = \mathbb{R} \times M_0$, not only $M = \mathbb{R} \times S^N$, when the results – for the reasons indicated before – cannot be easily obtained by using Tanaka's approach.

3.2 Spheres

In this section we assume $M = \mathbb{R} \times S^N$, where $S^N = \{\xi \in \mathbb{R}^{N+1} : |\xi| = 1\}$ [3]. For $s \in \mathbb{R}$, $r \in T_s\mathbb{R} \approx \mathbb{R}$, $\xi \in S^N$, $\eta \in T_\xi S^N$, let

$$g_0(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + |\eta|^2 \tag{3.2}$$

be the standard product metric on $M = \mathbb{R} \times S^N$. We consider a perturbed metric

$$g_\varepsilon(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + |\eta|^2 + \varepsilon h(s,\xi)((r,\eta),(r,\eta)), \tag{3.3}$$

where $h(s,\xi)$ is a bilinear form, not necessarily positive definite.

Define the space of closed loops

$$\Lambda = \{u = (r,x) \in H^1(S^1,\mathbb{R}) \times H^1(S^1,S^N)\} \tag{3.4}$$

Closed geodesics on (M, g_ε) are the critical points of $E_\varepsilon : \Lambda \to \mathbb{R}$ given by

$$E_\varepsilon(u) = \frac{1}{2}\int_0^1 g_\varepsilon(u)[\dot{u},\dot{u}]\,dt. \tag{3.5}$$

One has that

$$E_\varepsilon(u) = E_\varepsilon(r,x) = E_0(r,x) + \varepsilon G(r,x), \tag{3.6}$$

where

$$E_0(r,x) = \frac{1}{2}\int_0^1 \left(|\dot{r}|^2 + |\dot{x}|^2\right) dt$$

[3] Hereafter, we use the notation $\xi \bullet \eta = \sum_i \xi_i \eta_i$ for the scalar product in $\mathbb{R}^{N+1}$, and $|\xi|^2 = \xi \bullet \xi$.

and

$$G(r,x) = \frac{1}{2}\int_0^1 h(r,x)[(\dot r,\dot x),(\dot r,\dot x)]\,dt. \tag{3.7}$$

In particular, we split E_0 into two parts, namely

$$E_0(r,x) = L_0(r) + E_{M_0}(x), \tag{3.8}$$

where

$$L_0(r) = \frac{1}{2}\int_0^1 |\dot r|^2\,dt, \quad E_{M_0}(x) = \frac{1}{2}\int_0^1 |\dot x|^2\,dt.$$

The form of E_ε suggests to apply the perturbative results of [3] in the following form.

Theorem 3.3. *Let H be a real Hilbert space, $E_\varepsilon \in C^2(H)$ be of the form*

$$E_\varepsilon(u) = E_0(u) + \varepsilon G(u), \tag{3.9}$$

where $G \in C^2(E)$.. Suppose that there exists a finite dimensional manifold Z such that

(AS1) $E_0'(z) = 0$ for all $z \in Z$;
(AS2) $E_0''(z)$ is a compact perturbation of the identity, for all $z \in Z$;
(AS3) $T_zZ = \ker E_0''(z)$ for all $z \in Z$.

There exist a positive number ε_0 and a smooth function $w\colon Z\times(-\varepsilon_0,\varepsilon_0) \to H$ such that the critical points of

$$\Phi_\varepsilon(z) = E_\varepsilon(z + w(z,\varepsilon)), \quad z \in Z, \tag{3.10}$$

are critical points of E_ε.

Moreover, it is possible to show that

$$\Phi_\varepsilon(z) = b + \varepsilon\Gamma(z) + o(\varepsilon), \tag{3.11}$$

where $b = E_0(z)$ and $\Gamma = G_{|Z}$. From this "first order" expansion, one infers

Theorem 3.4 ([3]). *Let H be a real Hilbert space, $E_\varepsilon \in C^2(H)$ be of the form (3.9). Suppose that (AS1)–(AS3) hold. Then any strict local extremum of $G_{|Z}$ gives rise to a critical point of E_ε, for $|\varepsilon|$ sufficiently small.*

In the present situation, the critical points of E_0 are nothing but the great circles of S^N, namely

$$z_{p,q} = p\cos 2\pi t + q\sin 2\pi t, \tag{3.12}$$

where $p, q \in \mathbb{R}^{N+1}$, $p \bullet q = 0$, $|p| = |q| = 1$. Hence E_0 has a "critical manifold" given by

$$Z = \{z(r,p,q) = (r, z_{p,q}(\cdot)) \mid r \in \mathbb{R},\ z_{p,q} \text{ as in (3.12)}\}.$$

Lemma 3.1. *Z satisfies (AS2)–(AS3).*

Proof. The first assertion is known, see for instance [56].

For the second statement, we closely follow [25].

For $z \in Z$, of the form $z(t) = (r, z_{p,q}(t))$, it turns out that

$$E_0''(z)[h,k] = \int_0^1 \left[\dot{h} \bullet \dot{k} - |\dot{z}|^2 h \bullet k\right] dt$$

for any $h, k \in T_z Z$.

Let $e_i \in \mathbb{R}^{N+1}$, $i = 2, \dots, N+1$, be orthonormal vectors such that $\{\frac{1}{2\pi}\dot{z}_{p,q}, e_2, \dots e_{N+1}\}$ is a basis of $T_z Z$, and set

$$e_i(t) = \begin{cases} \dot{z}_{u^1,u^2}(t)/2\pi & \text{if } i = 1 \\ e_i & \text{if } i > 1, \end{cases}$$

Then, for h, k as before, we can write a "Fourier–type" expansion

$$h(t) = h_0(t)\frac{d}{dt} + \sum_{i=1}^{N-1} h_i(t)e_i(t), \quad k(t) = k_0(t)\frac{d}{dt} + \sum_{i=1}^{N-1} k_i(t)e_i(t). \tag{3.13}$$

Assume now that $h \in \ker E_0''(z_{p,q})$, i.e.

$$\int_0^1 \dot{h} \bullet \dot{k}\, dt = \int_0^1 |\dot{z}|^2 h \bullet k\, dt \quad \forall k \in T_{z_{p,q}} Z.$$

We plug (3.13) into this relations, and we get the system

$$\begin{cases} \ddot{h}_1 = 0 \\ \ddot{h}_j + 4\pi^2 h_j = 0 \quad j = 2, \dots, N-1 \\ \ddot{h}_0 = 0. \end{cases} \tag{3.14}$$

Recalling that h_0 and h_1 are periodic, we find

$$\begin{cases} h_0 = \lambda_0, \quad h_1 = \lambda_1 \\ h_j = \lambda_j \cos 2\pi t + \mu_j \sin 2\pi t \quad j = 2, \dots N-1. \end{cases} \tag{3.15}$$

Therefore, $h \in T_z Z$. This shows that $\ker E_0''(z_{p,q}) \subset T_{z_{p,q}} Z$. Since the converse inclusion is is always true, the lemma follows.

Lemma 3.2. *Suppose*

(h1) $h(r, \cdot) \to 0$ *pointwise on* S^N, *as* $|r| \to \infty$,

then

$$\Phi_\varepsilon \to b \equiv E_0(z).$$

Recall that Φ_ε *was defined in* (3.10).

Proof. This is proved as in [8, 19]. We just sketch the argument. The idea is to use the contraction mapping principle to characterize the function $w(\varepsilon, z)$ (see Theorem 1). Indeed, define

$$H(\alpha, w, z_r, \varepsilon) = \begin{pmatrix} E'_\varepsilon(z_r + w) - \alpha \dot{z} \\ w \bullet \dot{z}. \end{pmatrix}$$

So $H = 0$ if and only if $w \in (T_{z_r} Z)^\perp$ and $E'_\varepsilon(z_r + w) \in T_{z_r} Z$. Now,

$$H(\alpha, w, z_r, \varepsilon) = 0 \Leftrightarrow H(0,0,z_r,0) + \frac{\partial H}{\partial(\alpha, w)}(0,0,z_r,0)[\alpha, w] + \\ R(\alpha, w, z_r, \varepsilon) = 0,$$

where $R(\alpha, w, z_r, \varepsilon) = H(\alpha, w, z_r, \varepsilon) - \frac{\partial H}{\partial(\alpha,w)}(0,0,z_r,0)[\alpha, w]$.

Setting

$$R_{z_r,\varepsilon}(\alpha, w) = -\left[\frac{\partial H}{\partial(\alpha, w)}(0,0,z_r,0)\right]^{-1} R(\alpha, w, z_r, \varepsilon),$$

one finds that

$$H(\alpha, w, z_r, \varepsilon) = 0 \Leftrightarrow (\alpha, w) = R_{z_r,\varepsilon}(\alpha, w).$$

By the Cauchy–Schwarz inequality, it turns out that $R_{zr,\varepsilon}$ is a contraction mapping from some ball $B_{\rho(\varepsilon)}$ into itself. If $|\varepsilon|$ is sufficiently small, we have proved the existence of (α, w) uniformly for $z_r \in Z$. We want to study the asymptotic behavior of $w = w(\varepsilon, z_r)$ as $|r| \to +\infty$. We denote by R^0_ε the functions $R_{z_r,\varepsilon}$ corresponding to the unperturbed energy functional $E_0 = E_{M_0}$. It is easy to see ([19], Lemma 3) that the function w^0 found with the same argument as before satisfies $\|w^0(z_r)\| \to 0$ as $|r| \to +\infty$. Thus, by the continuous dependence of $w(\varepsilon, z_r)$ on ε and the characterization of $w(\varepsilon, z_r)$ and w^0 as fixed points of contractive mappings, we deduce as in [8], proof of Lemma 3.2, that $\lim_{r\to\infty} w(\varepsilon, z_r) = 0$. In conclusion, we have that $\lim_{|r|\to+\infty} \Phi_\varepsilon(z_r + w(\varepsilon, z_r)) = E_{M_0}(z_0)$.

Remark 3.1. There is a natural action of the group $O(2)$ on the space Λ, given by

$$\{\pm 1\} \times S^1 \times \Lambda \longrightarrow \Lambda \\ (\pm 1, \theta, u) \mapsto u(\pm t + \theta),$$

under which the energy E_ε is invariant. Since this is an isometric action under which Z is left unchanged, it easily follows that the function w constructed in Theorem 3.3 is invariant, too.

Theorem 3.5. *Assume that the functions $h_{ij} = h_{ji}$'s are smooth, bounded, and (h1) holds Then $M = \mathbb{R} \times M_0$ has at least N non-trivial closed geodesics, distinct modulo the action of the group $O(2)$. Furthermore, if* *(h2) the matrix $[h_{ij}(p,\cdot)]$ representing the bilinear form h is positive definite for $p \to +\infty$, and negative definite for $p \to -\infty$,*

then M possesses at least $2N$ non-trivial closed geodesics, geometrically distinct.

Proof. Observe that $Z = \mathbb{R} \times Z_0$, where $Z_0 = \{z_{p,q} \mid |p| = |q| = 1,\ p \bullet q = 0\}$. According to Theorem 2.1, it suffices to look for critical points of Φ_ε. From Lemma 3.2, it follows that either $\Phi_\varepsilon = b$ everywhere, or it has a critical point $(\bar{r}, \bar{p}, \bar{q})$. In any case such a critical point gives rise to a (non-trivial) closed geodesic of (M, g_ε).

From Remark 3.1, we know that Φ_ε is $O(2)$–invariant. This allows us to introduce the $O(2)$–category $\mathrm{cat}_{O(2)}$. One has

$$\mathrm{cat}_{O(2)}(Z) \ge \mathrm{cat}(Z/O(2)) \ge \mathrm{cuplength}(Z/O(2)) + 1.$$

Since $\mathrm{cuplength}(Z/O(2)) \ge N - 1$, (see [77]), then $\mathrm{cat}_{O(2)}(Z) \ge N$. Finally, by the Lusternik–Schnirel'man theory, M carries at least N closed geodesics, distinct modulo the action $O(2)$. This proves the first statement.

Next, let

$$\Gamma(r,p,q) = G((r, z_{p,q})) = \frac{1}{2}\int_0^1 h(r, z_{p,q}(t))[\dot{z}_{p,q}, \dot{z}_{p,q}]\, dt \tag{3.16}$$

Then (h) immediately implies that

$$\Gamma(r,p,q) \to 0 \quad \text{as } |r| \to \infty, \tag{3.17}$$

Moreover, if (h2) holds, then $\Gamma(r,p,q) > 0$ for $r > r_0$, and $\Gamma(r,p,q) < 0$ for $r < -r_0$. Since (recall equation (3.11))

$$\Phi_\varepsilon(r,p,q) = b + \varepsilon\Gamma(r,p,q) + o(\varepsilon), \tag{3.18}$$

it follows that

$$\begin{cases} \Phi_\varepsilon(r,p,q) > b & \text{for } r > r_0 \\ \Phi_\varepsilon(r,p,q) < b & \text{for } r < -r_0. \end{cases}$$

We can now exploit again the $O(2)$ invariance.

By assumption, and a simple continuity argument, $\{\Phi_\varepsilon > b\} \supset [R_0, \infty) \times Z_0$, and similarly $\{\Phi_\varepsilon < b\} \supset [-\infty, -R_0) \times Z_0$, for a suitably large $R_0 > 0$. Hence $\mathrm{cat}_{O(2)}(\{\Phi_\varepsilon > b\}) \ge \mathrm{cat}_{O(2)}(Z_0) = N$. The same argument applies to $\{\Phi_\varepsilon < b\}$. This proves the existence of at least $2N$ closed geodesics.

Remark 3.2. (i) In [25], the existence of N closed geodesics on S^N endowed with a metric close to the standard one is proved. Such a result does not need any study of Φ_ε and its behavior. The existence of $2N$ geodesics is, as far as we know, new. We emphasize that it strongly depends on the *form* of $M = \mathbb{R} \times M_0$.

(ii) In [84], the metric g on M is possibly not perturbative. No multiplicity result is given.

3.3 The general case

In this section we consider a compact riemannian manifold (M_0, g_0), and in analogy to the previous section, we put

$$g_\varepsilon(s,\xi)((r,\eta),(r,\eta)) = |r|^2 + g_0(\xi)(\eta,\eta) + \varepsilon h(s,\xi)((r,\eta),(r,\eta)). \tag{3.19}$$

Again, we define $\Lambda = \{u = (r,x) \mid r \in H^1(S^1,\mathbb{R}),\ x \in H^1(S^1, M_0)\}$,

$$E_{M_0}(x) = \frac{1}{2}\int_0^1 g_0(x)(\dot{x},\dot{x})\,dt, \quad E_0(r,x) = \frac{1}{2}\int_0^1 |\dot{r}|^2\,dt + E_{M_0}(x),$$

and finally

$$E_\varepsilon(r,x) = E_0(r,x) + \varepsilon G(r,x),$$

with G as in (3.7). It is well known ([57]) that M_0 has a closed geodesic z_0. The functional E_{M_0} has again a critical manifold Z given by

$$Z = \{u(\cdot) = (\rho, z_0(\cdot + \tau)) \mid \rho \ \text{ constant},\ \tau \in S^1\}.$$

Let $Z_0 = \{z_0(\cdot + \tau) \mid \tau \in S^1\}$. It follows that $Z \approx \mathbb{R} \times Z_0$. The counterpart of Γ in (3.11) is

$$\Gamma(r,\tau) = \frac{1}{2}\int_0^1 h(r, z_\tau)[\dot{z}_\tau, \dot{z}_\tau]\,dt. \tag{3.20}$$

Let us recall some facts from [56].

Remark 3.3. There is a linear operator $A_z : T_z\Lambda(M_0) \to T_z\Lambda$, which is a compact perturbation of the identity, such that

$$E''_{M_0}(z)[h,k] = \langle A_z h \mid k\rangle_1 = \int_0^1 \widehat{A_z h} \bullet \dot{k}\,dt.$$

In particular, E_0 satisfies (AS2).

Definition 3.8. *Let*

$$\ker E''_{M_0}(z_0) = \{h \in T_{z_0}\Lambda(M_0) \mid \langle A_{z_0} h \mid k\rangle_1 = 0 \quad \forall k \in T_{z_0}\Lambda(M_0)\}.$$

We say that a closed geodesic z_0 of M_0 is non–degenerate, *if*

$$\dim \ker E''_{M_0}(z_0) = 1.$$

Remark 3.4. For example, it is known that when M_0 has negative sectional curvature, then all the geodesics of M_0 are non–degenerate. See [39]. Moreover, it is easy to see that the existence of non–degenerate closed geodesics is a *generic* property.

Lemma 3.3. *If z_0 is a non–degenerate closed geodesic of M_0, then Z satisfies (AS2).*

Proof. It is always true that $T_{z_r}Z \subset \ker E_0''(z_r)$. By (3.26), we have that $\dim T_{z_r}Z = \dim\ker E_0''(z_r)$. This implies that $T_{z_r}Z = \ker E_0''(z_r)$. A generic element of Z has the form (ρ, z^τ) for $\rho \in \mathbb{R}$ and $z^\tau = z(\cdot + \tau)$; then

$$T_{(\rho,z^\tau)}M = \mathbb{R} \times T_{z^\tau}M_0,$$

and any two vector fields Y and W along a curve on $M = \mathbb{R} \times M_0$ can be decomposed into

$$Y = h(t)\frac{d}{dt} + y(t) \in \mathbb{R} \oplus T_{z^\tau}Z_0, \tag{3.21}$$

$$W = k(t)\frac{d}{dt} + w(t) \in \mathbb{R} \oplus T_{z^\tau}Z_0. \tag{3.22}$$

In addition, there results (see [49])

$$E''_{M_0}(z_0)[y,w] = \int_0^1 [g_0(D_t y, D_t w) - g_0(R_{M_0}y(t), \dot{z}_0(t))\dot{z}_0(t) \mid w(t))]\, dt, \tag{3.23}$$

and

$$R_M(r,z) = R_{\mathbb{R}}(r) + R_{M_0}(z) = R_{M_0}(z), \tag{3.24}$$

where R_M, R_{M_0}, etc. stand for the curvature tensors of M, M_0, etc. By (3.23), (3.21) and (3.22), as in the previous section, $E_0''(\rho, z_\tau)[Y,W] = 0$ is equivalent to the system

$$\begin{cases} \ddot{h} = 0 \\ \int_0^1 g_0(z)[D_t y, D_t w] - \langle R_{M_0}(y(t), \dot{z}_r(t))\dot{z}_r(t) \mid w(t)\rangle\, dt = 0. \end{cases} \tag{3.25}$$

As in the case of the sphere, the first equation implies that h is constant. The second equation in (3.25) implies that $y \in \ker E''_{M_0}(z^\tau) = \ker E''_{M_0}(z_0)$. Hence,

$$\ker E_0''(z_r) = \{(h,y) \mid h \text{ is constant, and } y \in \ker E''_{M_0}(z_0)\}. \tag{3.26}$$

This completes the proof.

Theorem 3.6. *Let M_0 be a compact, connected manifold of dimension $N < \infty$. Assume that M_0 admits a non-degenerate closed geodesic z, and that (in local coordinates) $h_{ij}(p,\cdot) \to a_-$ as $p \to -\infty$, and $h_{ij}(p,\cdot) \to a_+$ as $p \to +\infty$.*

1. *If $a_- = a_+$ and $h_{ij}(p,\cdot)$ satisfies (h2), then M has at least one closed geodesic.*
2. *If $a_- \leq a_+$ and $h_{ij}(p,\cdot)[u,v] - a_-(u \mid v)$ is negative definite for $p \to -\infty$ and $h_{ij}(p,\cdot)[u,v] - a_+(u \mid v)$ is positive definite for $p \to +\infty$, then M has at least two non-trivial closed geodesic.*

Proof. Lemma 3.3 allows us to repeat all the argument in Theorem 3.5, and the result follows immediately.

3.4 Isolated geodesics

In this final section, we discuss one situaion where the critical manifold Z may be degenerate. Here, the non–degeneracy condition (AS3) fails, and $T_z Z \subset \ker E_0''(z)$ strictly. Fix a closed geodesic Z_0 for M_0, and put $\tilde{W} = (T_{z_0} Z)^\perp$. Since $T_z Z \subset \ker E_0''(z)$ strictly, there exists $k > 0$ such that $\tilde{W} = (\ker E_0''(z_0))^\perp \oplus \mathbb{R}^k$. Repeating the preceding finite dimensional reduction, one can find again a unique map $\tilde{w} = \tilde{w}(z, \zeta)$, where $z \in Z$ and $\zeta \in \mathbb{R}^k$, in such a way that $E_\varepsilon' = 0$ reduces to an equation like

$$\nabla A(z + \zeta + \tilde{w}(z, \zeta)) = 0.$$

If z_0 is an isolated minimum of the energy E_{M_0} over some connected component of $\Lambda(M_0)$, then it is possible to show that there exists again a function $\Gamma \colon Z \to \mathbb{R}$ such that

$$\nabla A(z + \zeta + \tilde{w}(z, \zeta)) = 0 \iff \frac{\partial \Gamma}{\partial r}(-R, \tau) \frac{\partial \Gamma}{\partial r}(R, \tau) \neq 0$$

for some $R \in \mathbb{R}$ and all $\tau \in S^1$. For more details, see [18]. In particular, we will use the following result.

Theorem 3.7. *Let H be a real Hilbert space, $f_\varepsilon \colon H \to \mathbb{R}$ is a family of C^2–functionals of the form $f_\varepsilon = f_0 + \varepsilon G$, and that:*

(f0) f_0 has a finite dimensional manifold Z of critical points, each of them being a minimum of f_0;
(f1) for all $z \in Z$, $f_0''(z)$ is a compact perturbation of the identity.

Fix $z_0 \in Z$, put $W = (T_{z_0} Z)^\perp$, and suppose that $(f_0)_{|W}$ has an isolated minimum at z_0. Then, for ε sufficiently small, f_ε has a critical point, provided $\deg(\Gamma', B_R, 0) \neq 0$.

Remark 3.5. Theorem 3.7 has been presented in a linear setting. For Riemannian manifold, we can either reduce to a *local* situation and then apply the exponential map, or directly resort to the slightly more general degree theory on Banach manifold developed in [41].

Theorem 3.8. *Assume that $\pi_1(M_0) \neq \{0\}$, and that all the critical points of E_0, the energy functional of M_0, are isolated. Suppose the bilinear form h satisfies (h1), and*

(h3) $\dfrac{\partial h}{\partial r}(R, \xi) \dfrac{\partial h}{\partial r}(-R, \xi) \neq 0$ for some $R > 0$ and all $\xi \in S^1$.

Then, for $\varepsilon > 0$ sufficiently small, the manifold $M = \mathbb{R} \times M_0$ carries at least one closed geodesic.

Proof. We wish to use Theorem 3.7. Since $\pi_1(M_0) \neq \{0\}$, then E_0 has a geodesic z_0 such that $E_0(z_0) = \min E_0$ over some component C of $\Lambda(M_0)$. See [57].

We consider the manifold

$$Z = \{u \in \Lambda \mid u(t) = (\rho, z_0(t+\tau)),\ \rho \ \text{constant},\ \tau \in S^1\}.$$

Here we do not know, a priori, if Z is non–degenerate in the sense of condition (AS2). But of course $(E_0)_W$ has a minimum at the point (ρ, z_0), where $W = (T_{\rho,z^\tau)}Z)^\perp$. We now check that it is isolated for $(E_0)_W$. We still know that $Z = \mathbb{R} \times Z_0$. Take any point $(\rho, z_\tau) \in Z$, and observe that $T_{(\rho,z^\tau)}Z = \{(r,y) \mid r \in \mathbb{R},\ y \in T_{z^\tau}Z_0\}$. For all $(r,y) \in W$ sufficiently close to (ρ, z_0), it holds in particular that $y \perp z_\tau$. Hence

$$E_0(r,y) = L_0(r) + E_{M_0}(y) \geq E_{M_0}(y) > E_{M_0}(z^\tau) = E_{M_0}(z_0) = E_0(\rho, z_0)$$

since $L_0 \geq 0$ and z_0 (and hence z^τ) is a minimum of E_{M_0} by assumption.

Finally, thanks to assumption (h3), $\frac{\partial \Gamma}{\partial r}(-R,\tau)\frac{\partial \Gamma}{\partial r}(R,\tau) \neq 0$.

This concludes the proof.

4 Semilinear Schrödinger equations

In this chapter, following closely [10], we present another application of the perturbation method to the existence of multiple positive solutions for a class of nonlinear Schrödinger Equation (NLS in short)

$$\begin{cases} -\varepsilon^2 \Delta u + u + V(x)u = K(x)u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0 \end{cases} \tag{NLS}$$

where Δ denotes the Laplace operator and

$$1 < p < \begin{cases} \frac{n+2}{n-2}, & \text{if } n \geq 3, \\ +\infty, & \text{if } n = 2. \end{cases}$$

This equation has been studied under several viewpoints, and for a good review of methods and results we refer to the monograph [26] and its references.

As a first step, we introduce some motivation for the study of the *stationary Schrödinger equation*, which will be the object of our interest in the next chapters. We shall assume some knowledge of linear functional analysis and operator theory. We refer to [88] for details.

Definition 4.1. *Given $V \in L^\infty(\mathbb{R}^n)$, we define the Schrödinger operator $S\colon D(S) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ generated by the potential V by*

$$\begin{cases} D(S) = H^2(\mathbb{R}^n) \\ Su = -\Delta u + Vu. \end{cases}$$

Lemma 4.1. *1. S is self-adjoint.*
2. $\lambda \in \rho(S)$ if and only if (a) $S_\lambda I$ is injective and (b) $S - \lambda I)^{-1}\colon R(S-\lambda I) \subset L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ is bounded.
3. S has no eigenvalues if $V \equiv 0$.

We now give some properties of the first eigenvalue of S. Let

$$\Lambda = \inf \left\{ \int_{\mathbb{R}^n} (|\nabla u|^2 + Vu)\, dx \mid u \in H^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} |u|^2 = 1 \right\}. \tag{4.1}$$

Definition 4.2. *Any minimizer of the right-hand side of* (4.1) *is called a* ground state *of the Schrödinger equation*

Lemma 4.2. *Let $V \in L^\infty(\mathbb{R}^n)$. Then*

1. $\Lambda \geq -\|V\|_\infty > -\infty$.
2. $\Lambda = \inf\{\int_{\mathbb{R}^n}(|\nabla u|^2 + Vu)\,dx \mid u \in D(S)$ *and* $\int_{\mathbb{R}^n}|u|^2 = 1\}$ *and so we also have*

$$\Lambda = \inf\left\{\int_{\mathbb{R}^n}(Su)u\,dx \mid u \in H^1(\mathbb{R}^n) \quad \text{and} \int_{\mathbb{R}^n}|u|^2 = 1\right\}.$$

3. *If* $u \in H^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n}|u|^2\,dx = 1$ *and* $\int_{\mathbb{R}^n}(|\nabla u^2 + Vu)\,dx = \Lambda$, *then*

$$u \in H^2(\mathbb{R}^n),\ u \in \ker(S - \Lambda I) \quad \text{and } \Lambda \in \sigma_p(S).$$

Lemma 4.3 (Properties of eigenfunctions).

1. *Let* $V \in L^\infty(\mathbb{R}^n)$ *and consider* $u \in \ker(S - \lambda I)$ *for some* $\lambda \in \mathbb{R}^n$. *Then* $u \in C(\mathbb{R}^n) \cap W^{1,s}(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$ *for all* $2 \leq s \leq \infty$ *and*

$$\lim_{|x|\to\infty} u(x) = 0.$$

2. *Let* $V \in L^\infty(\mathbb{R}^n)$ *and choose* $\xi < l = \lim_{R\to\infty} \operatorname{ess\,inf}_{|x|\geq R}|V(x)|$. *For any* $\mu \in (0, \sqrt{l - \xi})$, *there is a constant* C, *depending only on* ξ *and* μ, *such that*

$$|u(x)| \leq C\|u\|_\infty\, \mathrm{e}^{-\mu|x|} \quad \text{for all } x \in \mathbb{R}^n$$

provided that $u \in \ker(S - \lambda I)$ *for some* $\lambda \leq \xi$.

In the sequel we will always assume that $V, K \colon \mathbb{R}^n \to \mathbb{R}$ satisfy

$(V1)$ $V \in C^2(\mathbb{R}^n)$, V and D^2V are bounded;
$(V2)$ $\inf_{x\in\mathbb{R}^n}(1 + V(x)) > 0$;
$(K1)$ $K \in C^2(\mathbb{R}^n)$, K is bounded and $K(x) > 0$ for all $x \in \mathbb{R}^n$.

We seek solutions u_ε of (NLS) that concentrate, as $\varepsilon \to 0$, near some point $x_0 \in \mathbb{R}^n$ (semiclassical standing waves). By this we mean that for all $x \in \mathbb{R}^n \setminus \{x_0\}$ one has that $u_\varepsilon(x) \to 0$ as $\varepsilon \to 0$.

When K equals a positive constant, say $K(x) \equiv 1$, (NLS) has been widely investigated, see [6, 5, 36, 46, 63, 69, 86] and references therein. Moreover, the existence of multibump solutions has also been studied in [31, 48], see also [5] where solutions with infinitely many bumps has been proved. It has been also pointed out, see e.g. [6, Section 6], that the results contained in the forementioned papers can be extended to equations where u^p is substituted by a function $g(u)$, which behaves like u^p. Nonlinearities depending upon x have been handled in [47, 87] where the existence of one-bump solutions is proved.

In a group of papers Equation (NLS) is studied by perturbation arguments. For example, in [6] a Liapunov-Schmidt type procedure is used to reduce, for ε small, (NLS) to a finite dimensional equation, that inherits the variational structure of the original problem. So, one looks for the critical points of a finite dimensional functional, which leading term is strictly related to the behaviour of V near its stationary points. For instance, by a direct application of the general theory described in Chapter 3, Ambrosetti, Badiale and Cingolani ([6]) obtained the following result, in which one solution is found whenever $K \equiv 1$ and V has a proper extremum at some point:

Theorem 4.1. *Let $K \equiv 1$, and assume that $V\colon \mathbb{R}^n \to \mathbb{R}$ is of class C^2 and possesses a non-degenerate maximum (or minimum) at a pont $x_0 \in \mathbb{R}^n$. Then the equation*

$$-\Delta u + V(\varepsilon x)u = |u|^{p-1}u$$

has at least a positive solution ocncentrating near x_0.

It should be clear that more than one solution can be found under the assumption that V has well–separated proper extrema. One can indeed apply a rather standard gluing technique, similar to that appearing in [5].

As we shall see in the next sections, our approach is slightly different, and in some sense more general, since we can also find work in the "non generic" situation in which the critical points of V arise in a compact manifold like a torus or a sphere. The price we pay is that the technical framework gets a bit more involved. As a counterpart, Theorem 4.1 can be considered as a corollary or a special case of our main results.

A different approach has been carried out in another group of papers, like [36, 86, 87]. First, one finds a solution u of (NLS) as a Mountain Pass critical point of the Euler functional; after, one proves the concentration as $\varepsilon \to 0$. Although this second approach allows to deal with potentials V such that $\sup V = +\infty$, it works, roughly speaking, near minima of V, only. Actually, as pointed out in [6], see also [3, Theorem 3.2], the Morse index of u is equal to $1+$ the Morse index of the stationary point V where concentration takes place. Hence u can be found as a Mountain Pass only in the case of minima of V. Such a severe restriction is even more apparent when one deals with nonlinearities depending upon x, see [87]. A case dealing with saddle-like potentials is studied in [37], but under some technical conditions involving the saddle level and the supremum of V.

The second approach sketched above has been used in [29] to obtain the existence of multiple *one–bump* solutions. Letting $V_0 = \inf[1 + V(x)]$, it is assumed that $K(x) \equiv 1$ and $\liminf_{|x|\to\infty}(1 + V(x)) > V_0 > 0$. It has been shown that, for $\varepsilon > 0$ small, (NLS) has at least $\mathrm{cat}(M, M_\delta)$ solutions, where $M = \{x \in \mathbb{R}^n \mid 1 + V(x) = V_0\}$, M_δ is a δ-neighbourhood of M and cat denotes the Lusternik-Schnirelman category. This result has been extended in [30] to nonlinearities like $K(x)u^p + Q(x)u^q$, with $1 < q < p$.

In the the present paper we will improve the above multiplicity result. For example, when $K(x) \equiv 1$, we can obtain the results summarized below.

(i) suppose that V has a manifold M of stationary points, and that M is nondegenerate in the sense of Bott, see [21]. We prove, see Theorem 4.3, that (NLS) has at least $l(M)$ critical points concentrating near points of M. Here $l(M)$ denotes the cup long of M. This kind of result is new because [29, 30] deal only with (absolute) minima.

(*ii*) If the points of M are local minima or maxima of V the above result can be sharpened because M does not need to be a manifold and $l(M)$ can be subsituted by $\mathrm{cat}(M, M_\delta)$, see Theorem 4.5. Unlike [29, 30], we can handle sets of *local* minima. Furthermore, we do not require any condition at infinity. The case of maxima is new.

(*iii*) When $K(x)$ is not identically constant, the preceding result holds true provided that V is substituted by $A = (1+V)^\theta K^{-2/(p-1)}$, where $\theta = -(n/2) + (p+1)/(p-1)$, see Theorem 4.4.

A similar multiplicity result holds for problems involving more general non-linearities, as in [47], see Remark 4.3.

Our approach is based on the perturbation technique introduced above. However, a straight application of the general arguments, would only provide the existence of one solution of (NLS) because that procedure leads to finding critical points of a finite dimensional functional which does not inherit the topological features of M. So, in order to find multiplicity results like those described above, it is necessary to use a different finite dimensional reduction. With this new approach one looks for critical points of a finite dimensional functional Φ_ε which is defined in a δ–neighbourhood M_δ of M and is close to V (or A) in the C^1 norm. This readily permits us to prove the multiplicity results in the case of maxima or minima, using the Lusternik–Schnirelman category. In the general case, we can use Conley theoretical arguments, see [33]. Rouglhy, it turns out that M_δ is an isolating block for the flow of $\nabla\Phi_\varepsilon$ and we can apply an abstract critical point result of [27] to find $l(M)$ solutions.

Notation

- $W^{1,2}(\mathbb{R}^n)$ denotes the usual Sobolev space, endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} u^2 dx.$$

- $o_h(1)$ denotes a function that tends to 0 as $h \to 0$.
- c, C, c_i etc. denote (possibly different) positive constants, independent of the parameters.

4.1 Preliminaries

In this section we will collect some preliminary material which will be used in the rest of the paper. In order to semplify the notation, we will discuss in detail equation (NLS) when $K \equiv 1$. The general case will be handled in Theorem 4.4 and requires some minor changes, only.

Without loss of generality we can assume that $V(0) = 0$. Performing the change of variable $x \mapsto \varepsilon x$, equation (NLS) becomes

$$-\Delta u + u + V(\varepsilon x)u = u^p. \tag{P_ε}$$

Solutions of (P_ε) are the critical points $u \in W^{1,2}(\mathbb{R}^n)$ of

$$f_\varepsilon(u) = f_0(u) + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)u^2 dx,$$

where

$$f_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1} dx.$$

The solutions of (P_ε) will be found near a solution of

$$-\Delta u + u + V(\varepsilon\xi)u = u^p, \tag{4.2}$$

for an appropriate choice of $\xi \in \mathbb{R}^n$. The solutions of (4.2) are critical points of the functional

$$F^{\varepsilon\xi}(u) = f_0(u) + \frac{1}{2}V(\varepsilon\xi)\int_{\mathbb{R}^n} u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1} dx \tag{4.3}$$

and can be found explicitly. Let U denote the unique, positive, radial solution of

$$-\Delta u + u = u^p, \qquad u \in W^{1,2}(\mathbb{R}^n). \tag{4.4}$$

Then a straight calculation shows that $\alpha U(\beta x)$ solves (P_ε) whenever

$$\beta = \beta(\varepsilon\xi) = [1 + V(\varepsilon\xi)]^{1/2} \quad \text{and} \quad \alpha = \alpha(\varepsilon\xi) = [\beta(\varepsilon\xi)]^{2/(p-1)}.$$

We set

$$z^{\varepsilon\xi}(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x) \tag{4.5}$$

and

$$Z^\varepsilon = \{z^{\varepsilon\xi}(x-\xi) \mid \xi \in \mathbb{R}^n\}.$$

When there is no possible misunderstanding, we will write z, resp. Z, instead of $z^{\varepsilon\xi}$, resp Z^ε. We will also use the notation z_ξ to denote the function $z_\xi(x) := z^{\varepsilon\xi}(x-\xi)$. Obviously all the functions in $z_\xi \in Z$ are solutions of (4.4) or, equivalently, critical points of $F^{\varepsilon\xi}$. For future references let us point out some estimates. First of all, we evaluate:

$$\begin{aligned}\partial_\xi z^{\varepsilon\xi}(x-\xi) &= \partial_\xi\left[\alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi))\right] = \\ &= \varepsilon\alpha'(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi)) + \varepsilon\alpha(\varepsilon\xi)\beta'(\varepsilon\xi)U'(\beta(\varepsilon\xi)(x-\xi)) - \\ &\qquad \alpha(\varepsilon\xi)U'(\beta(\varepsilon\xi)(x-\xi)).\end{aligned}$$

Recalling the definition of α, β one finds:

$$\partial_\xi z^{\varepsilon\xi}(x-\xi) = -\partial_x z^{\varepsilon\xi}(x-\xi) + O(\varepsilon|\nabla V(\varepsilon\xi)|). \tag{4.6}$$

The next Lemma shows that $\nabla f_\varepsilon(z_\xi)$ is close to zero when ε is small.

Lemma 4.4. *For all $\xi \in \mathbb{R}^n$ and all $\varepsilon > 0$ small, one has that*

$$\|\nabla f_\varepsilon(z_\xi)\| \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right), \quad C > 0.$$

Proof. From

$$f_\varepsilon(u) = F^{\varepsilon\xi}(u) + \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, u^2 dx$$

and since z_ξ is a critical point of $F^{\varepsilon\xi}$, one has

$$\begin{aligned}(\nabla f_\varepsilon(z_\xi)|v) = (\nabla F^{\varepsilon\xi}(z_\xi)|v) &+ \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi v\, dx \\ &= \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi v\, dx.\end{aligned}$$

Using the Hölder inequality, one finds

$$|(\nabla f_\varepsilon(z_\xi)|v)|^2 \le \|v\|_{L^2}^2 \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx. \tag{4.7}$$

From the assumption that $|D^2V(x)| \le$ const. one infers

$$|V(\varepsilon x) - V(\varepsilon\xi)| \le \varepsilon|\nabla V(\varepsilon\xi)| \cdot |x - \xi| + c_1\varepsilon^2|x - \xi|^2.$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx \le \\ &\quad c_1\varepsilon^2|\nabla V(\varepsilon\xi)|^2 \int_{\mathbb{R}^n} |x-\xi|^2 z^2(x-\xi)dx + c_2\varepsilon^4 \int_{\mathbb{R}^n} |x-\xi|^4 z^2(x-\xi)dx.\end{aligned}$$

Recalling (4.5), a direct calculation yields

$$\begin{aligned}\int_{\mathbb{R}^n} |x-\xi|^2 z^2(x-\xi)dx &= \alpha^2(\varepsilon\xi) \int_{\mathbb{R}^n} |y|^2 U^2(\beta(\varepsilon\xi)y)dy \\ &= \alpha^2\beta^{-n-2} \int_{\mathbb{R}^n} |y'|^2 U^2(y')dy' \le c_3.\end{aligned}$$

From this (and a similar calculation for for the last integral in the above formula) one infers

$$\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 z_\xi^2 dx \le c_4\varepsilon^2|\nabla V(\varepsilon\xi)|^2 + c_5\varepsilon^4. \tag{4.8}$$

Putting together (4.8) and (4.7), the Lemma follows.

4.2 Invertibility of $D^2 f_\varepsilon$ on $TZ^\perp$

In this section we will show that $D^2 f_\varepsilon$ is invertible on $TZ^\perp$. This will be the main tool to perform the finite dimensional reduction, carried out in Section 4.3.

Let $L_{\varepsilon,\xi} : (T_{z_\xi} Z^\varepsilon)^\perp \to (T_{z_\xi} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,\xi} v|w) = D^2 f_\varepsilon(z_\xi)[v,w]$. We want to show

Lemma 4.5. *Given $\overline{\xi} > 0$ there exists $C > 0$ such that for ε small enough one has that*

$$|(L_{\varepsilon,\xi} v|v)| \geq C\|v\|^2, \qquad \forall\, |\xi| \leq \overline{\xi},\ \forall\, v \in (T_{z_\xi} Z^\varepsilon)^\perp. \tag{4.9}$$

Proof. From (4.6) it follows that every element $\zeta \in T_{z_\xi} Z$ can be written in the form $\zeta = z_\xi - \partial_x z^{\varepsilon\xi}(x-\xi) + O(\varepsilon)$. As a consequence,

(∗) it suffices to prove (4.9) for all $v \in \operatorname{span}\{z_\xi, \phi\}$, where ϕ is orthogonal to $\operatorname{span}\{z_\xi, \partial_x z^{\varepsilon\xi}(x-\xi)\}$.

Precisely we shall prove that there exist $C_1, C_2 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \leq |\overline{\xi}|$ one has:

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) \leq -C_1 < 0. \tag{4.10}$$

$$(L_{\varepsilon,\xi} \phi | \phi) \geq C_2 \|\phi\|^2. \tag{4.11}$$

It is clear that the Lemma immediately follows from (∗), (4.10) and (4.11).

Proof of (4.10). First, let us recall that, since z_ξ is a Mountain Pass critical point of F, then given $\overline{\xi}$ there exists $c_0 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \leq |\overline{\xi}|$ one finds:

$$D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] < -c_0 < 0. \tag{4.12}$$

One has:

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) = D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z_\xi^2 dx.$$

The last integral can be estimated as in (4.8) yielding

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) \leq D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] + c_1 \varepsilon |\nabla V(\varepsilon\xi)| + c_2 \varepsilon^2. \tag{4.13}$$

From (4.12) and (4.13) it follows that (4.10) holds.

Proof of (4.11). As before, the fact that z_ξ is a Mountain Pass critical point of F implies that

$$D^2 F^{\varepsilon\xi}(z_\xi)[\phi, \phi] > c_1 \|\phi\|^2. \tag{4.14}$$

Let $R \gg 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^n \to \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \le R; \qquad \chi_1(x) = 0, \quad \text{for } |x| \ge 2R; \tag{χ'}$$

$$|\nabla \chi_1(x)| \le \frac{2}{R}, \quad \text{for } R \le |x| \le 2R. \tag{χ''}$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

A straight computation yields:

$$\int_{\mathbb{R}^n} \phi^2 = \int_{\mathbb{R}^n} \phi_1^2 + \int_{\mathbb{R}^n} \phi_2^2 + 2 \int_{\mathbb{R}^n} \phi_1 \, \phi_2,$$

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 = \int_{\mathbb{R}^n} |\nabla \phi_1|^2 + \int_{\mathbb{R}^n} |\nabla \phi_2|^2 + 2 \int_{\mathbb{R}^n} \nabla \phi_1 \cdot \nabla \phi_2,$$

and hence

$$||\phi||^2 = ||\phi_1||^2 + ||\phi_2||^2 + 2 \int_{\mathbb{R}^n} [\phi_1 \, \phi_2 + \nabla \phi_1 \cdot \nabla \phi_2] \, .$$

Letting I denote the last integral, one immediately finds:

$$I = \underbrace{\int_{\mathbb{R}^n} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2)}_{I_\phi} + \underbrace{\int_{\mathbb{R}^n} \phi^2 \nabla \chi_1 \cdot \nabla \chi_2}_{I'} + \underbrace{\int_{\mathbb{R}^n} \phi_1 \nabla \phi \cdot \nabla \chi_2 + \phi_2 \nabla \phi \cdot \nabla \chi_1}_{I''} .$$

Due to the definition of χ, the two integrals I' and I'' reduce to integrals from R and $2R$, and thus they are $o_R(1)||\phi||^2$. As a consequence we have that

$$||\phi||^2 = ||\phi_1||^2 + ||\phi_2||^2 + 2I_\phi + o_R(1)||\phi||^2, \tag{4.15}$$

After these preliminaries, let us evaluate the three terms in the equation below:

$$(L_{\varepsilon,\xi}\phi|\phi) = \underbrace{(L_{\varepsilon,\xi}\phi_1|\phi_1)}_{\alpha_1} + \underbrace{(L_{\varepsilon,\xi}\phi_2|\phi_2)}_{\alpha_2} + 2\underbrace{(L_{\varepsilon,\xi}\phi_1|\phi_2)}_{\alpha_3} .$$

One has:

$$\alpha_1 = (L_{\varepsilon,\xi}\phi_1|\phi_1) = D^2 F^{\varepsilon\xi}[\phi_1, \phi_1] + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] \, \phi_1^2. \tag{4.16}$$

In order to use (4.14), we introduce the function $\overline{\phi}_1 = \phi_1 - \psi$, where

$$\psi = (\phi_1|z_\xi)z_\xi + (\phi_1|\partial_x z_\xi)\partial_x z_\xi.$$

Then we have:

$$D^2F^{\varepsilon\xi}[\phi_1,\phi_1] = D^2F^{\varepsilon\xi}[\overline{\phi}_1,\overline{\phi}_1] + D^2F^{\varepsilon\xi}[\psi,\psi] + 2D^2F^{\varepsilon\xi}[\overline{\phi}_1,\psi] \tag{4.17}$$

Let us explicitely point out that $\overline{\phi}_1 \perp \mathrm{span}\{z_\xi, \partial_x z^{\varepsilon\xi}(x-\xi)\}$ and hence (4.14) implies

$$D^2F^{\varepsilon\xi}[\overline{\phi}_1,\overline{\phi}_1] \geq c_1\|\overline{\phi}_1\|^2. \tag{4.18}$$

On the other side, since $(\phi|z_\xi) = 0$ it follows:

$$\begin{aligned}(\phi_1|z_\xi) &= (\phi|z_\xi) - (\phi_2|z_\xi) = -(\phi_2|z_\xi)\\ &= -\int_{\mathbb{R}^n} \phi_2 z_\xi dx - \int_{\mathbb{R}^n} \nabla z_\xi \cdot \nabla\phi_2 dx\\ &= -\int_{\mathbb{R}^n} \chi_2(y)z(y)\phi(y+\xi)dy - \int_{\mathbb{R}^n} \nabla z(y)\cdot\nabla\chi_2(y)\phi(y+\xi)dy.\end{aligned}$$

Since $\chi_2(x) = 0$ for all $|x| < R$, and since $z(x) \to 0$ as $|x| = R \to \infty$, we infer $(\phi_1|z_\xi) = o_R(1)\|\phi\|$. Similarly one shows that $(\phi_1|\partial_x z_\xi) = o_R(1)\|\phi\|$ and it follows that

$$\|\psi\| = o_R(1)\|\phi\|. \tag{4.19}$$

We are now in position to estimate the last two terms in Eq. (4.17). Actually, using (4.19) we get

$$D^2F^{\varepsilon\xi}[\psi,\psi] = \|\psi\|^2 + V(\varepsilon\xi)\int_{\mathbb{R}^n}\psi^2 - p\int_{\mathbb{R}^n} z_\xi^{p-1}\psi^2 = o_R(1)\|\phi\|^2. \tag{4.20}$$

The same arguments readily imply

$$D^2F^{\varepsilon\xi}[\overline{\phi}_1,\psi] = (\overline{\phi}_1|\psi) + V(\varepsilon\xi)\int_{\mathbb{R}^n}\overline{\phi}_1\psi - p\int_{\mathbb{R}^n} z_\xi^{p-1}\overline{\phi}_1\psi = o_R(1)\|\phi\|^2. \tag{4.21}$$

Putting together (4.18), (4.20) and (4.21) we infer

$$D^2F^{\varepsilon\xi}[\phi_1,\phi_1] \geq \|\phi_1\|^2 + o_R(1)\|\phi\|^2. \tag{4.22}$$

Using arguments already carried out before, one has

$$\begin{aligned}\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|\phi_1^2 dx &\leq c_2\int_{\mathbb{R}^n} |x-\xi|\chi^2(x-\xi)\phi^2(x)dx\\ &\leq \varepsilon c_3\int_{\mathbb{R}^n} y\chi(y)\phi^2(y+\xi)dy\\ &\leq \varepsilon c_4\|\phi\|^2.\end{aligned}$$

This and (4.22) yield

$$\alpha_1 = (L_{\varepsilon,\xi}\phi_1|\phi_1) \geq c_5\|\phi_1\|^2 - \varepsilon c_4\|\phi\|^2 + o_R(1)\|\phi\|^2. \tag{4.23}$$

Let us now estimate α_2. One finds

$$\alpha_2 = (L_{\varepsilon,\xi}\phi_2|\phi_2) = \int_{\mathbb{R}^n} |\nabla\phi_2|^2 + \int_{\mathbb{R}^n} (1 + V(\varepsilon x))\phi_2^2 - p\int_{\mathbb{R}^n} z_\xi^{p-1}\phi_2^2$$

and therefore, using $(V2)$,

$$\alpha_2 \geq c_6\|\phi_2\|^2 - p\int_{\mathbb{R}^n} z_\xi^{p-1}\phi_2^2.$$

As before, $\phi_2(x) = 0$ for all $|x| < R$ and $z(x) \to 0$ as $|x| = R \to \infty$ imply that

$$\alpha_2 \geq c_6\|\phi_2\|^2 + o_R(1)\|\phi\|^2. \tag{4.24}$$

In a quite similar way one shows that

$$\alpha_3 \geq c_7 I_\phi + o_R(1)\|\phi\|^2. \tag{4.25}$$

Finally, (4.23), (4.24), (4.25) and the fact that $I_\phi \geq 0$, yield

$$\begin{aligned}(L_{\varepsilon,\xi}\phi|\phi) &= \alpha_1 + \alpha_2 + 2\alpha_3 \\ &\geq c_8\left[\|\phi_1\|^2 + \|\phi_2\|^2 + 2I_\phi\right] - c_9\varepsilon\|\phi\|^2 + o_R(1)\|\phi\|^2.\end{aligned}$$

Recalling (4.15) we infer that

$$(L_{\varepsilon,\xi}\phi|\phi) \geq c_{10}\|\phi\|^2 - c_9\varepsilon\|\phi\|^2 + o_R(1)\|\phi\|^2.$$

Taking ε small and R large, eq. (4.11) follows. This completes the proof of Lemma 4.5.

4.3 The finite dimensional reduction

In this Section we will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional. The proof will be carried out in 2 subsections dealing, respectively, with a Liapunov-Schmidt reduction, and with the behaviour of the auxiliary finite dimensional functional. In a final subsection we handle the general case in which K is not identically constant.

4.4 A Liapunov-Schmidt type reduction

The main result of this section is the following lemma.

Lemma 4.6. *For $\varepsilon > 0$ small and $|\xi| \le \overline{\xi}$ there exists a unique $w = w(\varepsilon, \xi) \in (T_{z_\xi} Z)^\perp$ such that $\nabla f_\varepsilon(z_\xi + w) \in T_{z_\xi} Z$. Such a $w(\varepsilon, \xi)$ is of class C^2, resp. $C^{1,p-1}$, with respect to ξ, provided that $p \ge 2$, resp. $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$ has the same regularity of w and satisfies:*

$$(\nabla \Phi_\varepsilon(\xi_0) = 0) \Rightarrow (\nabla f_\varepsilon(z_{\xi_0} + w(\varepsilon, \xi_0)) = 0)$$

Proof. Let $P = P_{\varepsilon\xi}$ denote the projection onto $(T_{z_\xi} Z)^\perp$. We want to find a solution $w \in (T_{z_\xi} Z)^\perp$ of the equation $P \nabla f_\varepsilon(z_\xi + w) = 0$. One has that $\nabla f_\varepsilon(z + w) = \nabla f_\varepsilon(z) + D^2 f_\varepsilon(z)[w] + R(z, w)$ with $\|R(z, w)\| = o(\|w\|)$, uniformly with respect to $z = z_\xi$, for $|\xi| \le \overline{\xi}$. Using the notation introduced in the preceding Section 4.2, we are led to the equation:

$$L_{\varepsilon,\xi} w + P \nabla f_\varepsilon(z) + P R(z, w) = 0.$$

According to Lemma 4.5, this is equivalent to

$$w = N_{\varepsilon,\xi}(w), \quad \text{where} \quad N_{\varepsilon,\xi}(w) = -L_{\varepsilon,\xi}^{-1} \left(P \nabla f_\varepsilon(z) + P R(z, w) \right).$$

From Lemma 4.4 it follows that

$$\|N_{\varepsilon,\xi}(w)\| \le c_1 (\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon^2) + o(\|w\|). \tag{4.26}$$

Then one readily checks that $N_{\varepsilon,\xi}$ is a contraction on some ball in $(T_{z_\xi} Z)^\perp$ provided that $\varepsilon > 0$ is small enough and $|\xi| \le \overline{\xi}$. Then there exists a unique w such that $w = N_{\varepsilon,\xi}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon, \xi)$, because the map $(\varepsilon, u) \mapsto P \nabla f_\varepsilon(u)$ fails to be C^2. However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(\xi, w) \mapsto P \nabla f_\varepsilon(z_\xi + w)$. Then, in particular, the function $w(\varepsilon, \xi)$ turns out to be of class C^1 with respect to ξ. Finally, it is a standard argument, see [3, 6], to check that the critical points of $\Phi_\varepsilon(\xi) = f_\varepsilon(z + w)$ give rise to critical points of f_ε.

Remark 4.1. From (4.26) it immediately follows that:

$$\|w\| \le C \left(\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon^2 \right), \tag{4.27}$$

where $C > 0$.

For future references, it is convenient to estimate the derivative $\partial_\xi w$.

Lemma 4.7. *One has that:*

$$\|\partial_\xi w\| \le c \left(\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2) \right)^\gamma, \qquad c > 0,\ \gamma = \min\{1, p-1\}. \tag{4.28}$$

Proof. In the proof we will write $\dot{w}_i$, resp $\dot{z}_i$, to denote the components of $\partial_\xi w$, resp. $\partial_\xi z$; moreover we will set $h(z,w) = (z+w)^p - z^p - pz^{p-1}w$. With these notations, and recalling that $L_{\varepsilon,\xi} w = -\Delta w + w + V(\varepsilon x)w - pz^{p-1}w$, it follows that w satisfies $\forall\, v \in (T_{z_\xi} Z)^\perp$:

$$(w|v) + \int_{\mathbb{R}^n} V(\varepsilon x) w v dx - p \int_{\mathbb{R}^n} z^{p-1} w v dx + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)] z v dx - \int_{\mathbb{R}^n} h(z,w) v dx = 0.$$

Hence $\dot{w}_i$ verifies

$$(\dot{w}_i|v) + \int_{\mathbb{R}^n} V(\varepsilon x)\dot{w}_i v dx - p \int_{\mathbb{R}^n} z^{p-1}\dot{w}_i v dx - p(p-1)\int_{\mathbb{R}^n} z^{p-2}\dot{z}_i w v dx + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\dot{z}_i v dx - \varepsilon \partial_{x_i} V(\varepsilon\xi) \int_{\mathbb{R}^n} z v dx - \int_{\mathbb{R}^n} (h_z \dot{z}_i + h_w \dot{w}_i) v dx = 0. \tag{4.29}$$

Let us set $L' = L_{\varepsilon,\xi} - h_w$. Then (4.29) can be written as

$$(L'\dot{w}_i|v) = p(p-1)\int_{\mathbb{R}^n} z^{p-2}\dot{z}_i w v - \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\dot{z}_i v + \varepsilon\partial_{x_i} V(\varepsilon\xi) \int_{\mathbb{R}^n} z v + \int_{\mathbb{R}^n} h_z \dot{z}_i v, \tag{4.30}$$

and hence one has

$$|(L'\dot{w}_i|v)| \le c_1 \|w\| \cdot \|v\| + \left| \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\dot{z} v dx \right| + c_2 \varepsilon |\nabla V(\varepsilon\xi)| \cdot \|v\| + \left| \int_{\mathbb{R}^n} h_z \dot{z} v dx \right|.$$

As in the proof of Lemma 4.4 one gets

$$\left| \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\dot{z} v dx \right| \le \left(c_3 \varepsilon |\nabla V(\varepsilon\xi)| + c_4 \varepsilon^2 \right) \|v\|.$$

Furthermore, the definition of h immediately yields

$$\left| \int_{\mathbb{R}^n} h_z \dot{z} v dx \right| \le c_5 ||w||^\gamma ||v||, \quad \text{where} \quad \gamma = \min\{1, p-1\}.$$

Putting together the above estimates we find

$$|(L'\dot{w}_i|v)| \le \left[c_6\varepsilon|\nabla V(\varepsilon\xi)| + c_4\varepsilon^2 + c_6||w||^\gamma\right] ||v||$$

Since $h_w \to 0$ as $w \to 0$, the operator L', likewise L, is invertible for $\varepsilon > 0$ small and therefore one finds

$$||\dot{w}|| \le \left(c_7\varepsilon|\nabla V(\varepsilon\xi)| + c_8\varepsilon^2\right) + c_9||w||^\gamma,$$

Finally, using (4.27) the Lemma follows.

4.5 The finite dimensional functional

The main purpose of this subsection is to use the estimates on w and $\partial_\xi w$ estabilished above to find an expansion of $\nabla\Phi_\varepsilon(\xi)$, where $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$. In the sequel, to be short, we will often write z instead of z_ξ and w instead of $w(\varepsilon, \xi)$. It is always understood that ε is taken in such a way that all the results discussed in the preceding Section hold.

For the reader's convenience we will divide the arguments in some steps.

Step 1. We have:

$$\Phi_\varepsilon(\xi) = \frac{1}{2}||z + w||^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)(z + w)^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} (z + w)^{p+1}.$$

Since $-\Delta z + z + V(\varepsilon\xi)z = z^p$ we infer that

$$||z||^2 = -V(\varepsilon\xi)\int_{\mathbb{R}^n} z^2 + \int_{\mathbb{R}^n} z^{p+1},$$

$$(z|w) = -V(\varepsilon\xi)\int_{\mathbb{R}^n} zw + \int_{\mathbb{R}^n} z^p w,$$

Then we find :

$$\Phi_\varepsilon(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} z^{p+1} + \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2$$
$$+ \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x) w^2$$
$$+ \frac{1}{2}||w||^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} \left[(z + w)^{p+1} - z^{p+1} - (p+1)z^p w\right].$$

Since $z(x) = \alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)x)$, where $\alpha = (1+V)^{1/(p-1)}$ and $\beta = (1+V)^{1/2}$, see (4.5), it follows

$$\int_{\mathbb{R}^n} z^{p+1}dx = C_0\,(1+V(\varepsilon\xi))^{\theta}, \quad C_0 = \int_{\mathbb{R}^n} U^{p+1}; \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

Letting $C_1 = C_0[1/2 - 1/(p+1)]$ one has

$$\begin{aligned}\Phi_\varepsilon(\xi) = C_1\,(1+V(\varepsilon\xi))^{\theta} + \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2 + \\ \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)w^2 \\ + \frac{1}{2}\|w\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} \left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right]. \quad (4.31)\end{aligned}$$

Step 2. Let us now evaluate the derivative of the right hand side of (4.31). For this, let us write:

$$\Phi_\varepsilon(\xi) = C_1(1+V(\varepsilon\xi))^{\theta} + \Lambda_\varepsilon(\xi) + \Psi_\varepsilon(\xi), \quad (4.32)$$

where

$$\Lambda_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2 + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw$$

and

$$\Psi_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)w^2 + \frac{1}{2}\|w\|^2 - \int_{\mathbb{R}^n} \left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right].$$

It is easy to check, by a direct calculation, that

$$|\nabla\Psi_\varepsilon(\xi)| \le c_1\|w\|\,\|\dot{w}\|. \quad (4.33)$$

Furthermore, since $V(\varepsilon x) - V(\varepsilon\xi) = \varepsilon\nabla V(\varepsilon\xi)\cdot(x-\xi) + \varepsilon^2 D^2V(\varepsilon\xi + \tau\varepsilon(x-\xi))[x-\xi, x-\xi]$ for some $\tau \in [0,1]$ we get:

$$\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2 dx = \varepsilon \int_{\mathbb{R}^n} \nabla V(\varepsilon\xi) \cdot (x-\xi) z^2 dx$$
$$+\varepsilon^2 \int_{\mathbb{R}^n} D^2 V(\varepsilon\xi + \tau\varepsilon(x-\xi))[x-\xi, x-\xi] z^2 dx$$
$$= \varepsilon \int_{\mathbb{R}^n} \nabla V(\varepsilon\xi) \cdot y\, z^2(y) dy$$
$$+\varepsilon^2 \int_{\mathbb{R}^n} D^2 V(\varepsilon\xi + \tau\varepsilon y)[y,y] z^2(y) dy$$
$$= \varepsilon^2 \int_{\mathbb{R}^n} D^2 V(\varepsilon\xi + \tau\varepsilon y)[y,y] z^2(y) dy.$$

Similarly, from $V(\varepsilon x) - V(\varepsilon\xi) = \varepsilon \nabla V(\varepsilon\xi + \tau\varepsilon(x-\xi)) \cdot (x-\xi)$ one finds

$$\int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw = \varepsilon \int_{\mathbb{R}^n} \nabla V(\varepsilon\xi + \tau\varepsilon(x-\xi)) \cdot (x-\xi) zw$$

The preceding equations imply:

$$|\nabla \Lambda_\varepsilon(\xi)| \le c_2\varepsilon^2 + c_3\varepsilon\|w\|. \tag{4.34}$$

Taking the gradients in (4.32), using (4.33-4.34) and recalling the estimates (4.27) and (4.28) on w and $\dot{w}$, respectively, we readily find:

Lemma 4.8. *Let $a(\varepsilon\xi) = \theta C_1 (1 + V(\varepsilon\xi))^{\theta-1}$. Then one has:*

$$\nabla \Phi_\varepsilon(\xi) = a(\varepsilon\xi)\varepsilon \nabla V(\varepsilon\xi) + \varepsilon^{1+\gamma} R_\varepsilon(\xi),$$

where $|R_\varepsilon(\xi)| \le const$ and $\gamma = \min\{1, p-1\}$.

Remark 4.2. Using similar arguments one can show that

$$\Phi_\varepsilon(\xi) = C_1 \left(1 + V(\varepsilon\xi)\right)^\theta + \rho_\varepsilon(\xi), \quad C_1 > 0, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2},$$

where $|\rho_\varepsilon(\xi)| \le \text{const.} \left(\varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon^2\right)$.

4.6 The general case

Let us indicate the counterpart of the above results in the case in which K is not constant. Since the arguments are quite similar, we will only outline the main modifications that are needed.

After rescaling, the solutions of (NLS) are the critical points of

$$\widetilde{f}_\varepsilon(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)u^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon x)u^{p+1}.$$

The solutions of (NLS) will be found near solutions of

$$-\Delta u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)u^p, \tag{4.35}$$

namely near critical points of

$$\widetilde{F}^{\varepsilon\xi}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}V(\varepsilon\xi)\int_{\mathbb{R}^n} u^2 - \frac{1}{p+1}K(\varepsilon\xi)\int_{\mathbb{R}^n} u^{p+1}.$$

If $\widetilde{z} = \widetilde{z}^{\varepsilon\xi}$ is a solution of (4.35), then $\widetilde{z}(x) = \tilde{\alpha}(\varepsilon\xi)U(\tilde{\beta}(\varepsilon\xi)x)$, where

$$\tilde{\alpha}(\varepsilon\xi) = (1+V(\varepsilon\xi))^{1/2}, \quad \tilde{\beta}(\varepsilon\xi) = \left(\frac{1+V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{1/(p-1)}.$$

This implies that

$$\int_{\mathbb{R}^n} \widetilde{z}^{p+1} = C_0 A(\varepsilon\xi), \tag{4.36}$$

where

$$A(x) = (1+V(x))^\theta\,[K(x)]^{-2/(p-1)}, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2}.$$

Define $\widetilde{L} = \widetilde{L}_{\varepsilon,\xi}$ on $(T\widetilde{Z})^\perp$ by setting $(\widetilde{L}v|w) = D^2\widetilde{f}_\varepsilon(\widetilde{z}_\xi)[v,w]$. As in Lemma 4.5 one shows that $\widetilde{L}$ is invertible for ε small. Furthermore, one has

$$\widetilde{f}_\varepsilon(u) = \widetilde{F}^{\varepsilon\xi}(u) + \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]u^2 - \frac{1}{p+1}\int_{\mathbb{R}^n}[K(\varepsilon x) - K(\varepsilon\xi)]u^{p+1}$$

and, as in Lemma 4.4 one finds that

$$\|\widetilde{f}_\varepsilon(u)\| \le c_1\left(\varepsilon(|\nabla V(\varepsilon\xi)| + |\nabla K(\varepsilon\xi)|) + \varepsilon^2\right).$$

This and the invertibility of $\widetilde{L}$ imply, as in Lemma 4.6, the existence of $\widetilde{w}(\varepsilon,\xi)$ such that the critical points of the finite dimensional functional

$$\widetilde{\Phi}_\varepsilon(\xi) = \widetilde{f}_\varepsilon(\widetilde{z}_\xi + \widetilde{w}(\varepsilon,\xi))$$

give rise to critical points of $\widetilde{f}_\varepsilon(u)$. Such a $\widetilde{w}$ is C^1 with respect to ξ and, as in Lemma 4.7, the following estimate holds

$$\|\partial_\xi\widetilde{w}\| \le c_2\left(\varepsilon(|\nabla V(\varepsilon\xi)| + |\nabla K(\varepsilon\xi)|) + O(\varepsilon^2)\right)^\gamma.$$

It remains to study the finite dimensional functional $\widetilde{\Phi}_\varepsilon(\xi)$,

$$\widetilde{\Phi}_\varepsilon(\xi) = \frac{1}{2}\|\widetilde{z}+\widetilde{w}\|^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)(\widetilde{z}+\widetilde{w})^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon x)(\widetilde{z}+\widetilde{w})^{p+1}.$$

Since now $\|\widetilde{z}\|^2 = -V(\varepsilon\xi)\int \widetilde{z}^2 + K(\varepsilon\xi)\int \widetilde{z}^{p+1}$ and $(\widetilde{z}|\widetilde{w}) = -V(\varepsilon\xi)\int \widetilde{z}\widetilde{w} + K(\varepsilon\xi)\int \widetilde{z}^p\widetilde{w}$, one gets

$$\begin{aligned}\widetilde{\Phi}_\varepsilon(\xi) = &\int_{\mathbb{R}^n}\left(\frac{1}{2}K(\varepsilon x) - \frac{1}{p+1}K(\varepsilon\xi)\right)\widetilde{z}^{p+1} + \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]\,\widetilde{z}^2\\ &+\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]\,\widetilde{z}\widetilde{w} + \frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)\widetilde{w}^2 + \frac{1}{2}\|w\|^2\\ &+K(\varepsilon\xi)\int_{\mathbb{R}^n}\widetilde{z}^p\widetilde{w} - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon x)\left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right]\end{aligned}$$

From $K(\varepsilon x) = K(\varepsilon\xi) + \varepsilon\nabla K(\varepsilon\xi)\cdot(x-\xi) + O(\varepsilon^2)$ and since $\int \nabla K(\varepsilon\xi)\cdot y\widetilde{z}^{p+1} = 0$ we infer:

$$\int_{\mathbb{R}^n}\left(\frac{1}{2}K(\varepsilon x) - \frac{1}{p+1}K(\varepsilon\xi)\right)\widetilde{z}^{p+1} = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} K(\varepsilon\xi)\widetilde{z}^{p+1} + O(\varepsilon^2).$$

Using (4.36), one finds the counterpart of (4.31), namely

$$\begin{aligned}\widetilde{\Phi}_\varepsilon(\xi) = C_1 A(\varepsilon\xi) &+ \frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]\,\widetilde{z}^2 + \int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]\,\widetilde{z}\widetilde{w}\\ &+\frac{1}{2}\int_{\mathbb{R}^n} V(\varepsilon x)\widetilde{w}^2 + \frac{1}{2}\|w\|^2 + K(\varepsilon\xi)\int_{\mathbb{R}^n}\widetilde{z}^p\widetilde{w}\\ &-\frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon x)\left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right] + O(\varepsilon^2).\end{aligned}$$

Taking the derivative of the above equation and using the preceding estimates, one finally yields

$$\nabla\widetilde{\Phi}_\varepsilon(\xi) = C_1\nabla A(\varepsilon\xi) + \varepsilon^{1+\gamma}\widetilde{R}_\varepsilon(\xi), \quad \gamma = \min\{1, p-1\}, \tag{4.37}$$

where $|\widetilde{R}_\varepsilon(\xi)| \le \text{const}$, which is the counterpart of Lemma 4.8. Let us also point out that, like in Remark 4.2, one has:

$$\widetilde{\Phi}_\varepsilon(\xi) = C_1 A(\varepsilon\xi) + O(\varepsilon). \tag{4.38}$$

4.7 Main results

In this section we will prove the main results of the present paper. First, some preliminaries are in order.

Given a set $M \subset \mathbb{R}^n$, $M \neq \emptyset$, we denote by M_δ its δ neighbourhood. If $M \subset N$, $cat(M,N)$ denotes the Lusternik-Schnirelman category of M with respect to N, namely the least integer k such that M can be covered by k closed subsets of N, contractible to a point in N. We set $cat(M) = cat(M,M)$.

The *cup long* $l(M)$ of M is defined by

$$l(M) = 1 + \sup\{k \in \mathbb{N} : \exists \alpha_1, \ldots, \alpha_k \in \check{H}^*(M) \setminus 1,\ \alpha_1 \cup \ldots \cup \alpha_k \neq 0\}.$$

If no such class exists, we set $l(M) = 1$. Here $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients and $\cup$ denotes the cup product. For example, if $M = S^{n-1}$, the $(n-1)$–dimensional sphere in $\mathbb{R}^n$, then $l(M) = \text{cat}(M) = 2$; however, in general, $l(M) \leq \text{cat}(M)$.

Let us suppose that V has a smooth manifold of critical points M. According to Bott [21], we say that M is nondegenerate (for V) if every $x \in M$ is a nondegenerate critical point of $V_{|M^\perp}$. The Morse index of any $x \in M$, as a critical point of $V_{|M^\perp}$, is constant and is, by definition, the Morse index of M.

We first recall a result from [27], namely Theorem 6.4 in Chapter II, adapted our notatin and purposes. See also [33].

Theorem 4.2. *Let $h \in C^2(\mathbb{R}^n)$ and let $\Sigma \subset \mathbb{R}^n$ be a smooth compact nondegenerate manifod of critical points of h. Let U be a neighborhood of Σ and let $\ell \in C^1(\mathbb{R}^n)$. Then, if $\|\ell - h\|_{C^1(U)}$ is sufficiently small, the function ℓ possesses at least $l(\Sigma)$ critical points in U.*

Roughly, the previuos theorem fits into the frame of Conley theory and extends an older result by Conley and Zehnder [34].

We are now ready to state our multiplicity results. Our first result deals with (NLS) with $K(x) \equiv 1$, namely with the equation (P_ε) introduced in Section 4.1.

Theorem 4.3. *Let $(V1-2)$ hold and suppose V has a nondegenerate smooth manifold of critical points M. Then for $\varepsilon > 0$ small, (P_ε) has at least $l(M)$ solutions that concentrate near points of M.*

Proof. First of all, we fix $\bar{\xi}$ in such a way that $|x| < \bar{\xi}$ for all $x \in M$. We will apply the finite dimensional procedure with such $\bar{\xi}$ fixed. Since M is a nondegenerate smooth manifold of critical points of V, it is obviuosly a nondegenerate manifold of critical points of $C_1(1+V(\cdot))^\theta$ as well.

In order to use Theorem 20 we set $h(\xi) = C_1(1+V(\xi))^\theta$, $\Sigma = M$ and $\ell(\xi) = \Phi_\varepsilon(\xi/\varepsilon)$. Fix a δ-neighborhood M_δ of M in such a way that $M_\delta \subset \{|x| < \bar{\xi}\}$ and the only critical points of V in M_δ are those of M, and let

$U = M_\delta$. From Lemma 19, the function $\Phi_\varepsilon(\cdot/\varepsilon)$ converges to $C_1(1+V(\cdot))^\theta$ in $C^1(U)$ when $\varepsilon \to 0$. Hence Theorem 20 applies and we can infer the existence of at least $l(M)$ critical points of ℓ, provided $\varepsilon > 0$ is small enough. Let $\xi_i \in M_\delta$ be any of those critical points. Then ξ_i/ε is a critical point of Φ_ε and so $u_{\varepsilon,\xi_i}(x) = z^{\xi_i}(x - \xi_i/\varepsilon) + w(\varepsilon, \xi_i)$ is a critical point of f_ε and hence a solution of (P_ε). It follows that

$$u_{\varepsilon,\xi_i}(x/\varepsilon) \approx z^{\xi_i}\left(\frac{x-\xi_i}{\varepsilon}\right)$$

is a solution of (NLS), with $K \equiv 1$. Any ξ_i converges to some $\xi_i^* \in M_\delta$ as $\varepsilon \to 0$ and, using again Lemma 19, it is easy to see that $\xi_i^* \in M$. This shows that $u_{\varepsilon,\xi_i}(x/\varepsilon)$ concentrates near a point of M.

The next result deals with the more general equation (NLS). The results will be given using the auxiliary function

$$A(x) = (1+V(x))^\theta \, [K(x)]^{-2/(p-1)}, \quad \theta = \frac{p+1}{p-1} - \frac{n}{2},$$

introduced in Subsection 4.6.

Theorem 4.4. *Let $(V1-2)$ and $(K1)$ hold and suppose A has a nondegenerate smooth manifold of critical points $\widetilde{M}$. Then for $\varepsilon > 0$ small, (NLS) has at least $l(\widetilde{M})$ solutions that concentrate near points of $\widetilde{M}$.*

Proof. The proof is quite similar to the preceding one, by using the results discussed in Subsection 4.6.

Remark 4.3. The preceding results can be extended to cover a class of nonlinearities $g(x,u)$ satisfying the same assumptions of [47]. In addition to some techenical conditions, one roughly assumes that the problem

$$-\Delta u + u + V(\varepsilon\xi)u = g(\varepsilon\xi, u), \quad u > 0, \quad u \in W^{1,2}(\mathbb{R}^n), \tag{4.39}$$

has a unique radial solution $z = z^{\varepsilon\xi}$ such that the linearized problem at z is invertible on $(T_z Z)^\perp$. In such a case one can obtain the same results as above with the auxiliary function A substituted by

$$\mathcal{A}(\varepsilon\xi) = \int_{\mathbb{R}^n} \left[\frac{1}{2} g(\varepsilon\xi, z^{\varepsilon\xi}(x-\xi)) - G(\varepsilon\xi, z^{\varepsilon\xi}(x-\xi))\right] dx,$$

where $\partial_u G = g$. When $g(x,u) = K(x)u^p$ one finds that $\mathcal{A} = A$. But, unlike such a case, the function $\mathcal{A}$ cannot be written in an explicit way. This is the reason why we have focused our study to the model problem (NLS) when the auxiliary function A has an explicit and neat form. Let us also point out that the class of nonlinearities handled in [30] does not even require that (4.39) has a unique solution.

When we deal with local minima (resp. maxima) of V, or A, the preceding results can be improved because the number of positive solutions of (NLS) can be estimated by means of the category and M does not need to be a manifold.

Theorem 4.5. *Let $(V1-2)$ and $(K1)$ hold and suppose A has a compact set X where A achieves a strict local minimum, resp. maximum.*

Then there exists $\varepsilon_\delta > 0$ such that (NLS) has at least $cat(X, X_\delta)$ solutions that concentrate near points of X_δ, provided $\varepsilon \in (0, \varepsilon_\delta)$.

Proof. Let $\delta > 0$ be such that

$$b := \inf\{A(x) : x \in \partial X_\delta\} > a := A_{|X}$$

and fix again $\bar{\xi}$ in such a way that X_δ is contained in $\{x \in \mathbb{R}^n : |x| < \bar{\xi}\}$. We set $X^\varepsilon = \{\xi : \varepsilon\xi \in X\}$, $X_\delta^\varepsilon = \{\xi : \varepsilon\xi \in X_\delta\}$ and $Y^\varepsilon = \{\xi \in X_\delta^\varepsilon : \widetilde{\Phi}_\varepsilon(\xi) \le C_1(a+b)/2\}$. From (4.38) it follows that there exists $\varepsilon_\delta > 0$ such that

$$X^\varepsilon \subset Y^\varepsilon \subset X_\delta^\varepsilon, \tag{4.40}$$

provided $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, if $\xi \in \partial X_\delta^\varepsilon$ then $V(\varepsilon\xi) \ge b$ and hence

$$\widetilde{\Phi}_\varepsilon(\xi) \ge C_1 V(\varepsilon\xi) + o_\varepsilon(1) \ge C_1 b + o_\varepsilon(1).$$

On the other side, if $\xi \in Y^\varepsilon$ then $\widetilde{\Phi}_\varepsilon(\xi) \le C_1(a+b)/2$. Hence, for ε small, Y^ε cannot meet $\partial X_\delta^\varepsilon$ and this readily implies that Y^ε is compact. Then $\widetilde{\Phi}_\varepsilon$ possesses at least $cat(Y^\varepsilon, X_\delta^\varepsilon)$ critical points in X_δ. Using (4.40) and the properties of the category one gets

$$\mathrm{cat}(Y^\varepsilon, Y^\varepsilon) \ge \mathrm{cat}(X^\varepsilon, X_\delta^\varepsilon) = \mathrm{cat}(X, X_\delta),$$

and the result follows.

Remark 4.4. The approach carried out in the present paper works also in the more standard case when V, or A, has an isolated set S of stationary points and $\deg(\nabla V, \Omega, 0) \neq 0$ for some neighbourhood Ω of S. In this way we can, for example, recover the results of [47].

5 Magnetic fields

Let us consider the nonlinear Schrödinger equation

$$\mathrm{i}h\frac{\partial\psi}{\partial t} = \left(\frac{h}{\mathrm{i}}\nabla - A(x)\right)^2 \psi + U(x)\psi - f(x,\psi), \quad x \in \mathbb{R}^n \tag{5.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ $(n \geq 2)$. The function $\psi(x,t)$ takes on complex values, h is the Planck constant, i is the imaginary unit. Here $A\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes a magnetic potential and the Schrödinger operator is defined by

$$\left(\frac{h}{\mathrm{i}}\nabla - A(x)\right)^2 \psi := -h^2\Delta\psi - \frac{2h}{\mathrm{i}}A\cdot\nabla\psi + |A|^2\psi - \frac{h}{\mathrm{i}}\psi \operatorname{div} A\,.$$

Actually, in general dimension $n \geq 2$, the magnetic field B is a 2-form where $B_{i,j} = \partial_j A_k - \partial_k A_j$; in the case $n = 3$, $B = \operatorname{curl} A$. The function $U\colon \mathbb{R}^n \to \mathbb{R}$ represents an electric potential. In the sequel, for the sake of simplicity, we limit ourselves to the particular case in which $f(x,t) = K(x)|t|^{p-1}t$, with $p > 1$ if $n = 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$.

It is now well known that the nonlinear Schrödinger equation (5.1) arises from a perturbation approximation for strongly nonlinear dispersive wave systems. Many papers are devoted to the nonlinear Schrödinger equation and its solitary wave solutions.

In this section we seek for standing wave solutions to (5.1), namely waves of the form $\psi(x,t) = e^{-iEth^{-1}}u(x)$ for some function $u\colon \mathbb{R}^n \longrightarrow \mathbb{C}$. Substituting this *ansatz* into (5.1), and denoting for convenience $\varepsilon = h$, one is led to solve the complex equation in $\mathbb{R}^n$

$$\left(\tfrac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2 u + (U(x) - E)u = K(x)|u|^{p-1}u\,. \tag{NLS}$$

Renaming $V(x)+1 = U(x)-E$, we assume from now on that $1+V$ is strictly positive on the whole $\mathbb{R}^n$. Moreover, by an obvious change of variables, the problem becomes that of finding some function $u\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\left(\tfrac{\nabla}{\mathrm{i}} - A(\varepsilon x)\right)^2 u + u + V(\varepsilon x)u = K(\varepsilon x)|u|^{p-1}u, \quad x \in \mathbb{R}^n\,. \tag{S_ε}$$

Concerning nonlinear Schrödinger equation with external magnetic field, we firstly quote a paper by Esteban and Lions [42], where concentrations and compactness arguments are applied to solve some minimization problems associated to (S_ε) under suitable assumptions on the magnetic field.

The purpose of this section is to study the time–independent nonlinear Schrödinger equation (S_ε) in the semiclassical limit. This seems a very interesting problem since the Correspondence's Principle establishes that Classical Mechanics is, roughly speaking, contained in Quantum Mechanics. The mathematical transition is obtained letting to zero the Planck constant $(\varepsilon \to 0)$ and solutions $u(x)$ of (S_ε) which exist for small value of ε are usually referred as semi-classical ones (see [51]).

We remark that in the linear case, Helffer *et al.* in [50, 52] have studied the asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit. Note that in these papers the *wells* of the Schrödinger operators with magnetic fields are the same as those without magnetic field, so that one doesn't 'see' the magnetic field in the definition of the *well*. See also [24] for generalization of the results by [52] for potentials which degenerate at infinity.

In the case $A = 0$, (no magnetic field), a recent extensive literature is devoted to study the time–independent nonlinear Schrödinger equation (S_ε) in the semi-classical limit. We shortly recall the main results in literature. The first paper is due to Floer and Weinstein which investigated the one-dimensional nonlinear Schrödinger equation (with $K(x) = 1$) and gave a description of the limit behavior of $u(x)$ as $\varepsilon \to 0$. Really they proved that if the potential V has a non-degenerate critical point, then $u(x)$ concentrates near this critical point, as $\varepsilon \to 0$.

Later, other authors proved that this problem is really local in nature and the presence of an isolated critical point of the potential V (in the case $K(x) = 1$) produces a semi-classical solution $u(x)$ of (S_ε) which concentrates near this point. Different approaches are used to cover different cases (see [6, 36, 63, 69, 87]). Moreover when V oscillates, the existence of multibumps solutions has also been studied in [5, 31, 37, 48]. Furthermore multiplicity results are obtained in [29, 30] for potentials V having a set of degenerate global minima and recently in [10], for potentials V having a set of critical points, not necessarily global minima.

A natural answer arises: *how does the presence of an external magnetic field influence the existence and the concentration behavior of standing wave solutions to* (5.1) *in the semi-classical limit?*

A first result in this direction is contained in [59] where Kurata has proved the existence of least energy solutions to (S_ε) for any $\varepsilon > 0$, under some assumptions linking the magnetic field $B = (B_{i,j})$ and the electric potential $V(x)$. The author also investigated the semi-classical limit of the found least energy solutions and showed a concentration phenomenon near global minima of the electric potential in the case $K(x) = 1$ and $|A|$ is small enough.

Recently in [28], Cingolani obtained a multiplicity result of semi-classical standing waves solutions to (S_ε), relating the number of solutions to (S_ε) to the *richness* of a set M of global minima of an auxiliary function A defined by setting

$$A(x) = \frac{(1+V(x))^\theta}{K(x)^{-2/(p-1)}}, \qquad \theta = \frac{p+1}{p-1} - \frac{n}{2},$$

(see (5.41) in section 4 for details) depending on $V(x)$ and $K(x)$. We remark that, if $K(x) = 1$ for any $x \in \mathbb{R}^n$, global minima of A coincides with global minima of V. The variational approach, used in [28], allows to deal with unbounded potential V and does not require assumptions on the magnetic field. However this approach works only near global minima of A.

In the present section we deal with the more general case in which the auxiliary function A has a manifold M of stationary points, not necessarily global minima. For bounded magnetic potentials A, we are able to prove a multiplicity result of semi-classical standing waves of (S_ε), following the perturbation approach described in the previous chapter.

Now we briefly describe the proof of the result. First of all, we highlight that solutions of (S_ε) naturally appear as *orbits*: in fact, equation (S_ε) is invariant under the multiplicative action of S^1. Since there is no danger of confusion, we simply speak about solutions. The complex–valued solutions to (S_ε) are found near least energy solutions of the equation

$$\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon\xi)\right)^2 u + u + V(\varepsilon\xi)u = K(\varepsilon\xi)|u|^{p-1}u. \tag{5.2}$$

where $\varepsilon\xi$ is in a neighborhood of M. The least energy of (5.2) have the form

$$z^{\varepsilon\xi,\sigma} : x \in \mathbb{R}^n \mapsto e^{\mathrm{i}\sigma + \mathrm{i}A(\varepsilon\xi)\cdot x} \left(\frac{1+V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{\frac{1}{p-1}} U((1+V(\varepsilon\xi))^{1/2}(x-\xi)) \tag{5.3}$$

where $\varepsilon\xi$ belongs to M and $\sigma \in [0, 2\pi]$. As before, the proof relies on a suitable finite dimensional reduction, and critical points of the Euler functional f_ε associated to problem (S_ε) are found near critical point of a finite dimensional functional Φ_ε which is defined on a suitable neighborhood of M. This allows to use Lusternik-Schnirelman category in the case M is a set of local maxima or minima of A. We remark that the case of maxima cannot be handled by using direct variational arguments as in [28].

Again, under suitable assumptions on M, more than one solution can be found.

Firstly we present a special case of our results.

Theorem 5.1. *Assume that*

$(K1)$ $K \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *is strictly positive and* K'' *is bounded;*
$(V1)$ $V \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *satisfies* $\inf_{x\in\mathbb{R}^n}(1 + V(x)) > 0$, *and* V'' *is bounded;*
$(A1)$ $A \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n, \mathbb{R}^n)$, *and the jacobian* J_A *of* A *is globally bounded in* $\mathbb{R}^n$.

If the auxiliary function A has a non-degenerate critical point $x_0 \in \mathbb{R}^n$, then for $\varepsilon > 0$ small, the problem (S_ε) has at least a (orbit of) solution concentrating near x_0.

Actually, we are able to prove the following generalization.

Theorem 5.2. *As in Theorem 1.1, assume again (K1), (V1) and (A1) hold. If the auxiliary function A has a smooth, compact, non-degenerate manifold of critical points M, then for $\varepsilon > 0$ small, the problem (S_ε) has at least $\ell(M)$ (orbits of) solutions concentrating near points of M.*

Finally we point out that the presence of an external magnetic field produces a phase in the complex wave which depends on the value of A near M. Conversely the presence of A does not seem to influence the location of the peaks of the modulus of the complex wave. Although we will not deal with this problem, we believe that in order to have a local C^2 convergence of the solutions, some assumption about the smallness of the magnetic potential A should be added, as done in [59] for minima of V.

Finally we point out that Theorem 5.1 and Theorem 5.2 hold for problems involving more general nonlinearities. See Remark 5.3 in the last section.

5.1 The variational framework

We work in the real Hilbert space E obtained as the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm associated to the inner product

$$\langle u \mid v \rangle = \mathrm{Re} \int_{\mathbb{R}^n} \nabla u \cdot \overline{\nabla v} + u\bar{v}..$$

Solutions to (S_ε) are, under some conditions we are going to point out, critical points of the functional formally defined on E as

$$f_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(\left| \left(\frac{1}{i} \nabla - A(\varepsilon x) \right) u \right|^2 + |u|^2 + V(\varepsilon x)|u|^2 \right) dx \\ - \frac{1}{p+1} \int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}\, dx. \quad (5.4)$$

In what follows, we shall assume that the functions V, K and A satisfy the following assumptions:

(K1) $K \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ is strictly positive and K'' is bounded;
(V1) $V \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies $\inf_{x \in \mathbb{R}^n}(1 + V(x)) > 0$, and V'' is bounded;

(A1) $A \in L^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n, \mathbb{R}^n)$, and the jacobian J_A of A is globally bounded in $\mathbb{R}^n$.

Indeed,

$$\int_{\mathbb{R}^n} \left(\left| \left(\frac{1}{\mathrm{i}} \nabla - A(\varepsilon x) \right) u \right|^2 \right) dx =$$
$$= \int_{\mathbb{R}^n} \left(|\nabla u|^2 + |A(\varepsilon x) u|^2 - 2\,\mathrm{Re}\big(\tfrac{\nabla u}{\mathrm{i}} \cdot A(\varepsilon x)\overline{u}\big) \right) dx,$$

and the last integral is finite thanks to the Cauchy–Schwartz inequality and the boundedness of A.
It follows that f_ε is actually well-defined on E.

In order to find possibly multiple critical points of (5.4), we follow the approach already presented in the previous chapter. In our context, we need to find complex–valued solutions, and so some further remarks are due.

Let $\xi \in \mathbb{R}^n$, which will be fixed suitably later on: we look for solutions to (S_ε) "close" to a particular solution of the equation

$$\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon \xi) \right)^2 u + u + V(\varepsilon\xi) u = K(\varepsilon\xi)|u|^{p-1}u. \tag{5.5}$$

More precisely, we denote by $U_c \colon \mathbb{R}^n \to \mathbb{C}$ a least–energy solution to the scalar problem

$$-\Delta U_c + U_c + V(\varepsilon\xi) U_c = K(\varepsilon\xi)|U_c|^{p-1}U_c \quad \text{in } \mathbb{R}^n. \tag{5.6}$$

By energy comparison (see [59]), one has that

$$U_c(x) = e^{\mathrm{i}\sigma} U^\xi(x - y_0)$$

for some choice of $\sigma \in [0, 2\pi]$ and $y_0 \in \mathbb{R}^n$, where $U^\xi \colon \mathbb{R}^n \to \mathbb{R}$ is the unique solution of the problem

$$\begin{cases} -\Delta U^\xi + U^\xi + V(\varepsilon\xi) U^\xi = K(\varepsilon\xi)|U^\xi|^{p-1}U^\xi \\ U^\xi(0) = \max_{\mathbb{R}^n} U^\xi \\ U^\xi > 0. \end{cases} \tag{5.7}$$

If U denotes the unique solution of

$$\begin{cases} -\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \\ U(0) = \max_{\mathbb{R}^n} U \\ U > 0, \end{cases} \tag{5.8}$$

then some elementary computations prove that $U^\xi(x) = \alpha(\varepsilon\xi) U(\beta(\varepsilon\xi)x)$, where

$$\alpha(\varepsilon\xi) = \left(\frac{1+V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{\frac{1}{p-1}}$$
$$\beta(\varepsilon\xi) = (1+V(\varepsilon\xi))^{1/2}.$$

It is easy to show, by direct computation, that the function $u(x) = e^{iA(\varepsilon\xi)\cdot x}U_c(x)$ actually solves (5.5).

For $\xi \in \mathbb{R}^n$ and $\sigma \in [0, 2\pi]$, we set

$$z^{\varepsilon\xi,\sigma} : x \in \mathbb{R}^n \mapsto e^{\mathrm{i}\sigma + \mathrm{i}A(\varepsilon\xi)\cdot x}\alpha(\varepsilon\xi)U(\beta(\varepsilon\xi)(x-\xi)). \tag{5.9}$$

Sometimes, for convenience, we shall identify $[0, 2\pi]$ and $S^1 \subset \mathbb{C}$, through $\eta = e^{i\sigma}$.

Introduce now the functional $F^{\varepsilon\xi,\sigma} : E \to \mathbb{R}$ defined by

$$F^{\varepsilon\xi,\sigma}(u) = \frac{1}{2}\int_{\mathbb{R}^n}\left(\left|\left(\frac{\nabla u}{\mathrm{i}} - A(\varepsilon\xi)u\right)\right|^2 + |u|^2 + V(\varepsilon\xi)|u|^2\right)dx$$
$$- \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon\xi)|u|^{p+1}\,dx,$$

whose critical point correspond to solutions of (5.5).

The set

$$Z^\varepsilon = \{z^{\varepsilon\xi,\sigma} \mid \xi \in \mathbb{R}^n \ \wedge \ \sigma \in [0,2\pi]\} \simeq S^1 \times \mathbb{R}^n$$

is a regular manifolds of critical points for the functional $F^{\varepsilon\xi,\sigma}$.

It follows from elementary differential geometry that

$$T_{z^{\varepsilon\xi,\eta}}Z^\varepsilon = \mathrm{span}_{\mathbb{R}}\{\tfrac{\partial}{\partial\sigma}z^{\varepsilon\xi,\sigma}, \tfrac{\partial}{\partial\xi_1}z^{\varepsilon\xi,\sigma}, \ldots, \tfrac{\partial}{\partial\xi_n}z^{\varepsilon\xi,\sigma}\} =$$
$$= \mathrm{span}_{\mathbb{R}}\{\mathrm{i}z^{\varepsilon\xi,\sigma}, \tfrac{\partial}{\partial\xi_1}z^{\varepsilon\xi,\sigma}, \ldots, \tfrac{\partial}{\partial\xi_n}z^{\varepsilon\xi,\sigma}\},$$

where we mean by the symbol $\mathrm{span}_{\mathbb{R}}$ that all the linear combinations must have real coefficients.

We remark that, for $j = 1, \ldots, n$,

$$\frac{\partial}{\partial\xi_j}z^{\varepsilon\xi,\sigma} = -\frac{\partial}{\partial x_j}z^{\varepsilon\xi,\sigma} + O(\varepsilon|\nabla V(\varepsilon\xi)|) +$$
$$+ \mathrm{i}\alpha(\varepsilon\xi)e^{iA(\varepsilon\xi)\cdot x + \mathrm{i}\sigma}U(\beta(\varepsilon\xi)(x-\xi))\left(\frac{\partial}{\partial\xi_j}(A(\varepsilon\xi)\cdot x) + A_j(\varepsilon\xi)\right) =$$
$$= -\frac{\partial}{\partial x_j}z^{\varepsilon\xi,\sigma} + O(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)|) + \mathrm{i}z^{\varepsilon\xi,\sigma}A_j(\varepsilon\xi),$$

so that

$$\frac{\partial}{\partial\xi_j}z^{\varepsilon\xi,\sigma} = -\frac{\partial}{\partial x_j}z^{\varepsilon\xi,\sigma} + \mathrm{i}z^{\varepsilon\xi,\sigma}A_j(\varepsilon\xi) + O(\varepsilon).$$

Collecting these remarks, we get that any $\zeta \in T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon$ can be written as

$$\zeta = \mathrm{i}\ell_1 z^{\varepsilon\xi,\sigma} + \sum_{j=2}^{n+1} \ell_j \frac{\partial}{\partial x_{j-1}} z^{\varepsilon\xi,\sigma} + O(\varepsilon)$$

for some real coefficients $\ell_1, \ell_2, \dots, \ell_{n+1}$.

The next lemma shows that $\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma})$ gets small when $\varepsilon \to 0$.

Lemma 5.1. *For all $\xi \in \mathbb{R}^n$, all $\eta \in S^1$ and all $\varepsilon > 0$ small, one has that*

$$\|\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma})\| \le C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|\nabla K(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)| + \varepsilon|\operatorname{div} A(\varepsilon\xi)| + \varepsilon^2\right),$$

for some constant $C > 0$.

Proof. From

$$f_\varepsilon(u) = F^{\varepsilon\xi,\eta}(u) + \frac{1}{2}\int_{\mathbb{R}^n}\left(\left|\frac{\nabla u}{\mathrm{i}} - A(\varepsilon x)u\right|^2 - \left|\frac{\nabla u}{\mathrm{i}} - A(\varepsilon\xi)u\right|^2\right) + \\ + \frac{1}{2}\int_{\mathbb{R}^n}\left[V(\varepsilon x) - V(\varepsilon\xi)\right]u^2 - \frac{1}{p+1}\int_{\mathbb{R}^n}[K(\varepsilon x) - K(\varepsilon\xi)]|u|^{p+1} \tag{5.10}$$

and since $z^{\varepsilon\xi,\eta}$ is a critical point of $F^{\varepsilon\xi,\eta}$, one has (with $z = z^{\varepsilon\xi,\eta}$)

$$\begin{aligned}
\langle\nabla f_\varepsilon(z) \mid v\rangle &= \operatorname{Re}\int_{\mathbb{R}^n}\left(\frac{1}{\mathrm{i}}\nabla - A(\varepsilon\xi)\right)z\cdot(A(\varepsilon\xi) - A(\varepsilon x))\bar{v} \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^n}(A(\varepsilon\xi) - A(\varepsilon x))z\cdot\overline{\left(\frac{1}{\mathrm{i}}\nabla - A(\varepsilon\xi)\right)v} + \\
&\quad \operatorname{Re}\int_{\mathbb{R}^n}(A(\varepsilon\xi) - A(\varepsilon x))z\cdot(A(\varepsilon\xi) - A(\varepsilon x))\bar{v} \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^n}(V(\varepsilon x) - V(\varepsilon\xi))z\bar{v} - \operatorname{Re}\int_{\mathbb{R}^n}(K(\varepsilon x) - K(\varepsilon\xi))|z|^{p-2}z\bar{v} \\
&= \varepsilon\operatorname{Re}\int_{\mathbb{R}^n}\frac{1}{\mathrm{i}}(\operatorname{div} A(\varepsilon x))z\bar{v} + 2\operatorname{Re}\int_{\mathbb{R}^n}(A(\varepsilon\xi) - A(\varepsilon x))z\cdot\overline{\left(\frac{1}{\mathrm{i}}\nabla - A(\varepsilon\xi)\right)v} \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^n}(V(\varepsilon x) - V(\varepsilon\xi))z\bar{v} - \operatorname{Re}\int_{\mathbb{R}^n}(K(\varepsilon x) - K(\varepsilon\xi))|z|^{p-2}z\bar{v}
\end{aligned}$$

From the assumption that $|D^2V(x)| \le$ const. one infers

$$|V(\varepsilon x) - V(\varepsilon\xi)| \le \varepsilon|\nabla V(\varepsilon\xi)|\cdot|x - \xi| + c_1\varepsilon^2|x - \xi|^2.$$

This implies

$$\int_{\mathbb{R}^n}|V(\varepsilon x) - V(\varepsilon\xi)|^2 z^{\varepsilon\xi,\sigma^2} \le c_1\varepsilon^2|\nabla V(\varepsilon\xi)|^2\int_{\mathbb{R}^n}|x - \xi|^2 z^2(x - \xi) + \\ c_2\varepsilon^4\int_{\mathbb{R}^n}|x - \xi|^4 z^2(x - \xi). \tag{5.11}$$

A direct calculation yields

$$\int_{\mathbb{R}^n} |x-\xi|^2 z^2(x-\xi) = \alpha^2(\varepsilon\xi) \int_{\mathbb{R}^n} |y|^2 U^2(\beta(\varepsilon\xi)y)dy$$
$$= \alpha(\varepsilon\xi)^2 \beta(\varepsilon\xi)^{-n-2} \int_{\mathbb{R}^n} |y'|^2 U^2(y')dy' \leq c_3.$$

From this (and a similar calculation for the last integral in the above formula) one infers

$$\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|^2 |z^{\varepsilon\xi,\sigma}|^2 \leq c_4 \varepsilon^2 |\nabla V(\varepsilon\xi)|^2 + c_5 \varepsilon^4. \tag{5.12}$$

Of course, similar estimates hold for the terms involving K. It then follows that

$$\|\nabla f_\varepsilon(z^{\varepsilon\xi,\eta})\| \leq C(\varepsilon|\operatorname{div} A(\varepsilon\xi)| + \varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)| + \varepsilon^2),$$

and the lemma is proved.

5.2 The invertibility of $D^2 f_\varepsilon$ on $(TZ^\varepsilon)^\perp$

To apply the perturbative method, we need to exploit some non–degeneracy properties of the solution $z^{\varepsilon\xi,\sigma}$ as a critical point of $F^{\varepsilon\xi,\sigma}$.

Let $L_{\varepsilon,\sigma,\xi} \colon (T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp \longrightarrow (T_{z^{\varepsilon\xi,\sigma}} Z)^\perp$ be the operator defined by

$$\langle L_{\varepsilon,\sigma,\xi} v \mid w \rangle = D^2 f_\varepsilon(z^{\varepsilon\xi,\sigma})(v,w)$$

for all $v, w \in (T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$.

The following elementary result will play a fundamental rôle in the present section.

Lemma 5.2. *Let $M \subset \mathbb{R}^n$ be a bounded set. Then there exists a constant $C > 0$ such that for all $\xi \in M$ one has*

$$\int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{\mathrm{i}} - A(\xi) \right) u \right|^2 + |u|^2 \geq C \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2) \qquad \forall u \in E. \tag{5.13}$$

Proof. To get a contradiction, we assume on the contrary the existence of a sequence $\{\xi_n\}$ in M and a sequence $\{u_n\}$ in E such that $\|u_n\|_E = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \left[\int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{\mathrm{i}} - A(\xi) \right) u_n \right|^2 + \int_{\mathbb{R}^n} |u_n|^2 \right] = 0. \tag{5.14}$$

In particular, $u_n \to 0$ strongly in $L^2(\mathbb{R}^n, \mathbb{C})$. Moreover, since M is bounded, we can assume also $\xi_n \to \xi^* \in \overline{M}$ as $n \to \infty$. From

$$\int_{\mathbb{R}^n}\left|\left(\frac{\nabla}{\mathrm{i}} - A(\xi_n)\right)u_n\right|^2 = \int_{\mathbb{R}^n}\left(|\nabla u_n|^2 + |A(\xi_n)|^2|u_n|^2 - 2\,\mathrm{Re}\,\frac{1}{\mathrm{i}}\nabla u_n \cdot A(\xi_n)\overline{u_n}\right)$$

we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^n}|\nabla u_n|^2 = 1$$
$$\lim_{n\to+\infty}\mathrm{Re}\int_{\mathbb{R}^n}\tfrac{1}{\mathrm{i}}\nabla u_n \cdot A(\xi_n)\overline{u_n} = \frac{1}{2}\,.$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^n}|\nabla u_n|\,|A(\xi_n)|\,|u_n| &\geq \limsup_{n\to\infty}\left|\int_{\mathbb{R}^n}\tfrac{1}{\mathrm{i}}\nabla u_n \cdot A(\xi_n)\overline{u_n}\right| \\ &\geq \limsup_{n\to\infty}\mathrm{Re}\int_{\mathbb{R}^n}\tfrac{1}{\mathrm{i}}\nabla u_n \cdot A(\xi_n)\overline{u_n} = \frac{1}{2}\,.\end{aligned}$$

From this we conclude that

$$\begin{aligned}\frac{1}{2} &\leq |A(\xi^*)|\limsup_{n\to\infty}\|\nabla u_n\|_{L^2}\|u_n\|_{L^2} \leq \\ &\leq |A(\xi^*)|\limsup_{n\to\infty}\|u_n\|_{L^2} = 0,\end{aligned}$$

which is clearly absurd. This completes the proof of the lemma.

At this point we shall prove the following result:

Lemma 5.3. *Given $\bar{\xi} > 0$, there exists $C > 0$ such that for ε small enough one has*

$$|\langle L_{\varepsilon,\sigma,\xi}v \mid v\rangle| \geq C\|v\|^2, \quad \forall|\xi| \leq \bar{\xi},\ \forall\sigma\in[0,2\pi],\ \forall v \in (T_{z^{\varepsilon\xi,\sigma}}Z^\varepsilon)^\perp. \tag{5.15}$$

Proof. We follow the arguments in [10], with some minor modifications due to the presence of A. Recall that

$$T_{z^{\varepsilon\xi,\sigma}}Z = \mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial\xi_1}z^{\varepsilon\xi,\sigma},\ldots,\frac{\partial}{\partial\xi_n}z^{\varepsilon\xi,\sigma}, iz^{\varepsilon\xi,\sigma}\}.$$

Define

$$\mathcal{V} = \mathrm{span}_{\mathbb{R}}\{\tfrac{\partial}{\partial x_1}z^{\varepsilon\xi,\sigma},\ldots,\tfrac{\partial}{\partial x_n}z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}, iz^{\varepsilon\xi,\sigma}\}.$$

As in the previous chapter, it suffices to prove (5.15) for all $v \in \mathrm{span}_{\mathbb{R}}\{z^{\varepsilon\xi,\sigma},\phi\}$, where $\phi \perp \mathcal{V}$. More precisely, we shall prove that for some constants $C_1 > 0$, $C_2 > 0$, for all ε small enough and all $|\xi| \leq \bar{\xi}$ the following hold:

$$\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle \leq -C_1 < 0, \tag{5.16}$$

$$\langle L_{\varepsilon,\sigma,\xi} \phi \mid \phi \rangle \geq C_2 \|\phi\|^2 \quad \forall \phi \perp \mathcal{V}. \tag{5.17}$$

For the reader's convenience, we reproduce here the expression for the second derivative of $F^{\varepsilon\xi,\sigma}$:

$$D^2 F^{\varepsilon\xi,\sigma}(u)(v,v) = \int_{\mathbb{R}^n} \left| \left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon\xi) \right) v \right|^2 + |v|^2 + V(\varepsilon\xi)|v|^2$$
$$- K(\varepsilon\xi) \left[(p-1) \operatorname{Re} \int_{\mathbb{R}^n} |u|^{p-3} \operatorname{Re}(u\bar{v}) u\bar{v} + \int_{\mathbb{R}^n} |u|^{p-1} |v|^2 \right].$$

Moreover, since $z^{\varepsilon\xi,\sigma}$ is a solution of (5.5), we immediately get

$$\int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon\xi) \right) z^{\varepsilon\xi,\sigma} \right|^2 + V(\varepsilon\xi) |z^{\varepsilon\xi,\sigma}|^2 + |z^{\varepsilon\xi,\sigma}|^2 \right) = K(\varepsilon\xi) \int_{\mathbb{R}^n} |z^{\varepsilon\xi,\sigma}|^{p+1}.$$

From this it follows readily that we can find some $c_0 > 0$ such that for all $\varepsilon > 0$ small, all $|\xi| \leq \bar{\xi}$ and all $\sigma \in [0, 2\pi]$ it results

$$D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) < c_0 < 0. \tag{5.18}$$

Recalling (5.10), we find

$$\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle = D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) +$$
$$+ \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon\xi)] |z^{\varepsilon\xi,\sigma}|^2 - \int_{\mathbb{R}^n} [K(\varepsilon x) - K(\varepsilon\xi)] |z^{\varepsilon\xi,\sigma}|^{p+1} +$$
$$\int_{\mathbb{R}^n} \left(\left| \left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon x) \right) z^{\varepsilon\xi,\sigma} \right|^2 - \left| \left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon\xi) \right) z^{\varepsilon\xi,\sigma} \right|^2 \right).$$

It follows that

$$\langle L_{\varepsilon,\sigma,\xi} z^{\varepsilon\xi,\sigma} \mid z^{\varepsilon\xi,\sigma} \rangle \leq D^2 F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(z^{\varepsilon\xi,\sigma}, z^{\varepsilon\xi,\sigma}) +$$
$$+ c_1 \varepsilon |\nabla V(\varepsilon\xi)| + c_2 \varepsilon |\nabla K(\varepsilon\xi)| + c_3 \varepsilon |J_A(\varepsilon\xi)| + c_4 \varepsilon^2. \tag{5.19}$$

Hence (5.16) follows. The proof of (5.17) is more involved. We first prove the following claim.

Claim. There results

$$D^2 F^{\varepsilon\xi}(z^{\varepsilon\xi,\sigma})(\phi, \phi) \geq c_1 \|\phi\|^2 \quad \forall \phi \perp \mathcal{V}. \tag{5.20}$$

Recall that the complex ground state U_c introduced in (5.6) is a critical point of mountain-pass type for the corresponding energy functional $J \colon E \longrightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2 + V(\varepsilon\xi)|u|^2) - \frac{1}{p+1}\int_{\mathbb{R}^n} K(\varepsilon\xi)|u|^{p+1}. \quad (5.21)$$

Let

$$\mathcal{M} = \{u \in E\colon \textstyle\int_{\mathbb{R}^n}(|\nabla u|^2 + |u|^2 + V(\varepsilon\xi)|u|^2) = \int_{\mathbb{R}^n} |u|^{p+1}\}$$

be the Nehary manifold of J, which has codimension one. Let

$$\mathcal{N} = \left\{u \in E\colon \textstyle\int_{\mathbb{R}^n} \left(\left|\left(\frac{\nabla}{i} - A(\varepsilon\xi)\right) u\right|^2 + |u|^2 + V(\varepsilon\xi)|u|^2\right) = \int_{\mathbb{R}^n} |u|^{p+1} \right\}$$

be the Nehari manifold of $F^{\varepsilon\xi,\sigma}$. One checks readily that $\operatorname{codim}\mathcal{N} = 1$. Recall ([59]) that U_c is, up to multiplication by a constant phase, the *unique* minimum of J restricted to $\mathcal{M}$. Now, for every $u \in \mathcal{M}$, the function $x \mapsto e^{iA(\varepsilon\xi)\cdot x}u(x)$ lies in $\mathcal{N}$, and viceversa. Moreover

$$J(u) = F^{\varepsilon\xi,\sigma}(e^{iA(\varepsilon\xi)\cdot x}u).$$

This immediately implies that $\min_{\mathcal{N}} F^{\varepsilon\xi,\sigma}$ is achieved at a point which differs from $e^{iA(\varepsilon\xi)\cdot x}U_c(x)$ at most for a constant phase. In other words, $z^{\varepsilon\xi,\sigma}$ is a critical point for $F^{\varepsilon\xi,\sigma}$ of mountain-pass type, and the claim follows by standard results (see [27]).

Let $R \gg 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \le R; \qquad \chi_1(x) = 0, \quad \text{for } |x| \ge 2R; \quad (5.22)$$

$$|\nabla\chi_1(x)| \le \frac{2}{R}, \quad \text{for } R \le |x| \le 2R. \quad (5.23)$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

A straightforward computation yields:

$$\int_{\mathbb{R}^n} |\phi|^2 = \int_{\mathbb{R}^n} |\phi_1|^2 + \int_{\mathbb{R}^n} |\phi_2|^2 + 2\,\mathrm{Re}\int_{\mathbb{R}^n} \phi_1\,\bar{\phi}_2,$$

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 = \int_{\mathbb{R}^n} |\nabla\phi_1|^2 + \int_{\mathbb{R}^n} |\nabla\phi_2|^2 + 2\,\mathrm{Re}\int_{\mathbb{R}^n} \nabla\phi_1 \cdot \overline{\nabla\phi_2},$$

and hence

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2\,\mathrm{Re}\int_{\mathbb{R}^n} \left[\phi_1\,\overline{\phi}_2 + \nabla\phi_1 \cdot \overline{\nabla\phi_2}\right].$$

Letting I denote the last integral, one immediately finds:

$$I = \underbrace{\int_{\mathbb{R}^n} \chi_1\chi_2(\phi^2 + |\nabla\phi|^2)}_{I_\phi} + \underbrace{\int_{\mathbb{R}^n} \phi^2\nabla\chi_1 \cdot \nabla\chi_2}_{I'} + \underbrace{\int_{\mathbb{R}^n} (\phi_2\nabla\chi_1 \cdot \overline{\nabla\phi} + \overline{\phi_1}\,\nabla\phi \cdot \nabla\chi_2)}_{I''}.$$

Due to the definition of χ, the two integrals I' and I'' reduce to integrals from R and $2R$, and thus they are $o_R(1)||\phi||^2$, where $o_R(1)$ is a function which tends to 0, as $R \to +\infty$. As a consequence we have that

$$||\phi||^2 = ||\phi_1||^2 + ||\phi_2||^2 + 2I_\phi + o_R(1)||\phi||^2 . \tag{5.24}$$

After these preliminaries, let us evaluate the three terms in the equation below:

$$(L_{\varepsilon,\sigma,\xi}\phi|\phi) = \underbrace{(L_{\varepsilon,\sigma,\xi}\phi_1|\phi_1)}_{\alpha_1} + \underbrace{(L_{\varepsilon,\sigma,\xi}\phi_2|\phi_2)}_{\alpha_2} + 2\underbrace{(L_{\varepsilon,\sigma,\xi}\phi_1|\phi_2)}_{\alpha_3}.$$

One has:

$$\alpha_1 = \langle L_{\varepsilon,\sigma,\xi}\phi_1 \mid \phi_1\rangle = D^2F^{\varepsilon\xi,\sigma}(z^{\varepsilon\xi,\sigma})(\phi_1,\phi_1) + \int_{\mathbb{R}^n}[V(\varepsilon x) - V(\varepsilon\xi)]|\phi_1|^2$$
$$-\int_{\mathbb{R}^n}[K(\varepsilon x) - K(\varepsilon\xi)]|\phi_1|^{p+1} + \int_{\mathbb{R}^n}\left(\left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon x)\right)\phi_1\right|^2 - \left|\left(\frac{\nabla}{\mathrm{i}} - A(\varepsilon\xi)\right)\phi_1\right|^2\right).$$

In order to use (5.20), we introduce the function $\phi_1^* = \phi_1 - \psi$, where ψ is the projection of ϕ_1 onto $\mathcal{V}$:

$$\psi = (\phi_1|z^{\varepsilon\xi,\sigma})z^{\varepsilon\xi,\sigma}||z^{\varepsilon\xi,\sigma}||^{-2} + (\phi_1|iz^{\varepsilon\xi,\sigma})iz^{\varepsilon\xi,\sigma}||z^{\varepsilon\xi,\sigma}||^{-2}$$
$$+ \sum(\phi_1|\partial_{x_i}z^{\varepsilon\xi,\sigma})\partial_{x_i}z^{\varepsilon\xi,\sigma}||\partial_{x_i}z^{\varepsilon\xi,\sigma}||^{-2}.$$

Then we have:

$$D^2F^{\varepsilon\xi}[\phi_1,\phi_1] = D^2F^{\varepsilon\xi}[\phi_1^*,\phi_1^*] + D^2F^{\varepsilon\xi}[\psi,\psi] + 2\,\mathrm{Re}\,D^2F^{\varepsilon\xi}[\phi_1^*,\psi]. \tag{5.25}$$

Since $z^{\varepsilon\xi,\sigma}$ is orthogonal to $\partial_{x_i}z^{\varepsilon\xi,\sigma}$, $i = 1,\dots,n$, then one readily checks that $\phi_1^* \perp \mathcal{V}$ and hence (5.20) implies

$$D^2F^{\varepsilon\xi}[\phi_1^*,\phi_1^*] \ge c_1||\phi_1^*||^2. \tag{5.26}$$

On the other side, since $(\phi|z^{\varepsilon\xi,\sigma}) = 0$ it follows:

$$\begin{aligned}(\phi_1|z^{\varepsilon\xi,\sigma}) &= (\phi|z^{\varepsilon\xi,\sigma}) - (\phi_2|z^{\varepsilon\xi,\sigma}) = -(\phi_2|z^{\varepsilon\xi,\sigma})\\ &= -\mathrm{Re}\int_{\mathbb{R}^n}\phi_2 z^{\varepsilon\xi,\sigma} - \mathrm{Re}\int_{\mathbb{R}^n}\nabla z^{\varepsilon\xi,\sigma}\cdot\nabla\phi_2\\ &= -\mathrm{Re}\int_{\mathbb{R}^n}\chi_2(y)z(y)\phi(y+\xi)dy - \mathrm{Re}\int_{\mathbb{R}^n}\nabla z(y)\cdot\nabla\chi_2(y)\phi(y+\xi)dy.\end{aligned}$$

Since $\chi_2(x) = 0$ for all $|x| < R$, and since $z(x) \to 0$ as $|x| = R \to \infty$, we infer $(\phi_1 | z^{\varepsilon\xi,\sigma}) = o_R(1)||\phi||$. Similarly one shows that $(\phi_1 | \partial_x z^{\varepsilon\xi,\sigma}) = o_R(1)||\phi||$ and it follows that

$$||\psi|| = o_R(1)||\phi||. \tag{5.27}$$

We are now in position to estimate the last two terms in equation (5.25). Actually, using Lemma 3.1 we get

$$\begin{aligned} D^2 F^{\varepsilon\xi}[\psi,\psi] &\geq C||\psi||^2 + V(\varepsilon\xi) \int_{\mathbb{R}^n} \psi^2 \\ &- K(\varepsilon\xi)\Big[\operatorname{Re}(p-1)\int_{\mathbb{R}^n} |z^{\varepsilon\xi,\sigma}|^{p-3}\operatorname{Re}(z^{\varepsilon\xi,\sigma}\bar{\psi}) z^{\varepsilon\xi,\sigma}\bar{\psi} \\ &+ \int_{\mathbb{R}^n} |z^{\varepsilon\xi,\sigma}|^{p-1}|\psi|^2\Big] = o_R(1)||\phi||^2. \end{aligned} \tag{5.28}$$

The same arguments readily imply

$$\operatorname{Re} D^2 F^{\varepsilon\xi}[\phi_1^*, \psi] = o_R(1)||\phi||^2. \tag{5.29}$$

Putting together (5.26), (5.28) and (5.29) we infer

$$D^2 F^{\varepsilon\xi}[\phi_1, \phi_1] \geq C||\phi_1||^2 + o_R(1)||\phi||^2. \tag{5.30}$$

Using arguments already carried out before, one has

$$\begin{aligned} \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon\xi)|\phi_1^2 &\leq \varepsilon c_2 \int_{\mathbb{R}^n} |x-\xi|\chi_1^2(x-\xi)\phi^2(x) \\ &\leq \varepsilon c_3 \int_{\mathbb{R}^n} |y|\chi_1^2(y)\phi^2(y+\xi)dy \\ &\leq \varepsilon c_4 R||\phi||^2, \end{aligned}$$

and similarly for the terms containing K. This and (5.30) yield

$$\alpha_1 = (L_{\varepsilon,\sigma,\xi}\phi_1|\phi_1) \geq c_5||\phi_1||^2 - \varepsilon c_4 R||\phi||^2 + o_R(1)||\phi||^2. \tag{5.31}$$

Let us now estimate α_2. One finds

$$\alpha_2 = \langle L_{\varepsilon,\sigma,\xi}\phi_2 \mid \phi_2\rangle \geq c_6||\phi_2||^2 + o_R(1)||\phi||^2. \tag{5.32}$$

In a quite similar way one shows that

$$\alpha_3 \geq c_7 I_\phi + o_R(1)||\phi||^2. \tag{5.33}$$

Finally, (5.31), (5.32), (5.33) and the fact that $I_\phi \geq 0$, yield

$$
\begin{aligned}
(L_{\varepsilon,\sigma,\xi}\phi|\phi) &= \alpha_1 + \alpha_2 + 2\alpha_3 \\
&\geq c_8 \left[||\phi_1||^2 + ||\phi_2||^2 + 2I_\phi\right] - c_9 R\varepsilon||\phi||^2 + o_R(1)||\phi||^2.
\end{aligned}
$$

Recalling (5.24) we infer that

$$(L_{\varepsilon,\sigma,\xi}\phi|\phi) \geq c_{10}||\phi||^2 - c_9 R\varepsilon||\phi||^2 + o_R(1)||\phi||^2.$$

Taking $R = \varepsilon^{-1/2}$, and choosing ε small, equation (5.17) follows. This completes the proof.

5.3 The finite dimensional reduction

In this Section we will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional. The proof will be carried out in two subsections dealing, respectively, with a Liapunov-Schmidt reduction, and with the behaviour of the auxiliary finite dimensional functional.

5.4 A Liapunov-Schmidt type reduction

The main result of this section is the following lemma.

Lemma 5.4. *For $\varepsilon > 0$ small, $|\xi| \leq \overline{\xi}$ and $\sigma \in [0, 2\pi]$, there exists a unique $w = w(\varepsilon, \sigma, \xi) \in (T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma} + w) \in T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon$. Such a $w(\varepsilon, \sigma, \xi)$ is of class C^2, resp. $C^{1,p-1}$, with respect to ξ, provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\sigma, \xi) = f_\varepsilon(z^{\varepsilon\xi,\sigma} + w(\varepsilon, \sigma, \xi))$ has the same regularity as w and satisfies:*

$$\nabla\Phi_\varepsilon(\sigma_0, \xi_0) = 0 \quad \Longleftrightarrow \quad \nabla f_\varepsilon\left(z_{\xi_0} + w(\varepsilon, \sigma_0, \xi_0)\right) = 0.$$

Proof. Let $P = P_{\varepsilon\xi,\sigma}$ denote the projection onto $(T_{z^{\varepsilon\xi,\sigma}} Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{z^{\varepsilon\xi,\sigma}} Z)^\perp$ of the equation $P\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma} + w) = 0$. One has that $\nabla f_\varepsilon(z + w) = \nabla f_\varepsilon(z) + D^2 f_\varepsilon(z)[w] + R(z, w)$ with $||R(z, w)|| = o(||w||)$, uniformly with respect to $z = z^{\varepsilon\xi,\sigma}$, for $|\xi| \leq \overline{\xi}$. Using the notation introduced in the previous section, we are led to the equation:

$$L_{\varepsilon,\sigma,\xi} w + P\nabla f_\varepsilon(z) + PR(z, w) = 0.$$

According to Lemma 5.3, this is equivalent to

$$w = N_{\varepsilon,\xi,\sigma}(w), \quad \text{where} \quad N_{\varepsilon,\xi,\sigma}(w) = -L^{-1}_{\varepsilon,\sigma,\xi}\left(P\nabla f_\varepsilon(z) + PR(z, w)\right).$$

From Lemma 5.1 it follows that

$$\|N_{\varepsilon,\xi,\sigma}(w)\| \leq c_1(\varepsilon|\nabla V(\varepsilon\xi)|++\varepsilon|\nabla K(\varepsilon\xi)|+\varepsilon|J_A(\varepsilon\xi)|+\varepsilon^2)+o(\|w\|). \quad (5.34)$$

Then one readily checks that $N_{\varepsilon,\xi,\sigma}$ is a contraction on some ball in $(T_{z^{\varepsilon\xi,\sigma}}Z^{\varepsilon})^{\perp}$ provided that $\varepsilon > 0$ is small enough and $|\xi| \leq \bar{\xi}$. Then there exists a unique w such that $w = N_{\varepsilon,\xi,\sigma}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon,\xi,\sigma)$, because the map $(\varepsilon,u) \mapsto P\nabla f_\varepsilon(u)$ fails to be C^2. However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(\xi,\sigma,w) \mapsto P\nabla f_\varepsilon(z^{\varepsilon\xi,\sigma}+w)$. Then, in particular, the function $w(\varepsilon,\xi,\sigma)$ turns out to be of class C^1 with respect to ξ and σ. Finally, it is a standard argument, see [3, 6], to check that the critical points of $\Phi_\varepsilon(\xi,\sigma) = f_\varepsilon(z+w)$ give rise to critical points of f_ε.

Remark 5.1. Since $f_\varepsilon(z^{\varepsilon\xi,\sigma})$ is actually independent of σ, the implicit function w is constant with respect to that variable. As a result, there exists a functional $\Psi_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\Phi_\varepsilon(\sigma,\xi) = \Psi_\varepsilon(\xi), \qquad \forall\sigma \in [0,2\pi], \quad \forall\xi \in \mathbb{R}^n.$$

In the sequel, we will omit the dependence of w on σ, even it is defined over $S^1 \times \mathbb{R}^n$.

Remark 5.2. From (5.34) it immediately follows that:

$$\|w\| \leq C\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|\nabla K(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)| + \varepsilon^2\right), \qquad (5.35)$$

where $C > 0$.

The following result can be proved by adapting the same argument as in [10].

Lemma 5.5. *One has that:*

$$\|\nabla_\xi w\| \leq c\left(\varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon|\nabla K(\varepsilon\xi)| + \varepsilon|J_A(\varepsilon\xi)| + O(\varepsilon^2)\right)^\gamma, \qquad (5.36)$$

where $\gamma = \min\{1, p-1\}$ *and* $c > 0$ *is some constant.*

5.5 The finite dimensional functional

The purpose of this subsection is to give an explicit form to the finite-dimensional functional $\Phi_\varepsilon(\sigma,\xi) = \Psi_\varepsilon(\xi) = f_\varepsilon(z^{\varepsilon\xi,\sigma} + w(\varepsilon,\xi))$.

Recall the precise definition of $z^{\varepsilon\xi,\sigma}$ given in (5.9). For brevity, we set in the sequel $z = z^{\varepsilon\xi,\sigma}$ and $w = w(\varepsilon,\xi)$.

Since z satisfies (5.5), we easily find the following relations:

$$\int_{\mathbb{R}^n} \left|\left(\frac{\nabla}{i} - A(\varepsilon\xi)\right)z\right|^2 + |z|^2 + V(\varepsilon\xi)|z|^2 = \int_{\mathbb{R}^n} K(\varepsilon\xi)|z|^{p+1} \qquad (5.37)$$

$$\operatorname{Re}\int_{\mathbb{R}^n}\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon\xi)\right)z\cdot\overline{\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon\xi)\right)w}+\operatorname{Re}\int_{\mathbb{R}^n}z\bar{w}$$
$$+\operatorname{Re}\int_{\mathbb{R}^n}V(\varepsilon\xi)z\bar{w}=\operatorname{Re}\int_{\mathbb{R}^n}K(\varepsilon\xi)|z|^{p-1}z\bar{w}\,. \quad (5.38)$$

Hence we get

$$\begin{aligned}
\Phi_\varepsilon(\sigma,\xi) &= f_\varepsilon(z^{\varepsilon\xi,\sigma}+w(\varepsilon,\sigma,\xi)) = \\
&= K(\varepsilon\xi)\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}|z|^{p+1}+\frac{1}{2}\int_{\mathbb{R}^n}|A(\varepsilon\xi)-A(\varepsilon x)|^2z^2+ \\
&\operatorname{Re}\int_{\mathbb{R}^n}(A(\varepsilon\xi)-A(\varepsilon x))z\cdot(A(\varepsilon\xi)-A(\varepsilon x))\bar{w}+\varepsilon\operatorname{Re}\int_{\mathbb{R}^n}\frac{1}{\mathrm{i}}z\bar{w}\operatorname{div}A(\varepsilon x) \\
&+\frac{1}{2}\int_{\mathbb{R}^n}\left|\left(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\right)w\right|^2+\operatorname{Re}\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]z\bar{w} \\
&+\frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]|w|^2+\frac{1}{2}\int_{\mathbb{R}^n}[V(\varepsilon x)-V(\varepsilon\xi)]z^2 \\
&+\frac{1}{2}V(\varepsilon\xi)\int_{\mathbb{R}^n}|w|^2-\frac{1}{p+1}\operatorname{Re}\int_{\mathbb{R}^n}K(\varepsilon x)(|z+w|^{p+1}-|z|^{p+1}-(p+1)|z|^{p-1}z\bar{w}) \\
&+\operatorname{Re}K(\varepsilon\xi)\int_{\mathbb{R}^n}|z|^{p-1}z\bar{w}+O(\varepsilon^2)\,. \quad (5.39)
\end{aligned}$$

Here we have used the estimate

$$\int_{\mathbb{R}^n}\left(\frac{1}{2}K(\varepsilon x)-\frac{1}{p+1}K(\varepsilon\xi)\right)|z|^{p+1}=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}K(\varepsilon\xi)|z|^{p+1}+O(\varepsilon^2),$$

which follows from the boundedness of K''. Since we know that

$$\alpha(\varepsilon\xi)=\left(\frac{1+V(\varepsilon\xi)}{K(\varepsilon\xi)}\right)^{\frac{1}{p-1}}$$
$$\beta(\varepsilon\xi)=(1+V(\varepsilon\xi))^{\frac{1}{2}}\,,$$

we get immediately

$$\int_{\mathbb{R}^n}|z^{\varepsilon\xi,\sigma}|^{p+1}=C_0\,\Lambda(\varepsilon\xi)[K(\varepsilon\xi)]^{-1}, \quad (5.40)$$

where we define the auxiliary function

$$\Lambda(x)=\frac{(1+V(x))^\theta}{K(x)^{2/(p-1)}},\qquad \theta=\frac{p+1}{p-1}-\frac{n}{2}, \quad (5.41)$$

and $C_0=\|U\|_{L^2}$. Now one can estimate the various terms in (5.39) by means of (5.35) and (5.36), to prove that

$$\Phi_\varepsilon(\sigma,\xi) = \Psi_\varepsilon(\xi) = C_1\,A(\varepsilon\xi) + O(\varepsilon). \tag{5.42}$$

Similarly,

$$\nabla\Psi_\varepsilon(\xi) = C_1 \nabla A(\varepsilon\xi) + \varepsilon^{1+\gamma} O(1), \tag{5.43}$$

where $C_1 = \left(\frac{1}{2} - \frac{1}{p+1}\right) C_0$. We omit the details, which can be deduced without effort from [10].

5.6 Statement and proof of the main results

In this section we exploit the finite-dimensional reduction performed in the previous section to find existence and multiple solutions of (NLS). Recalling Lemma 5.4, we have to look for critical points of Φ_ε as a function of the variables $(\sigma,\xi) \in [0, 2\pi] \times \mathbb{R}^n$ (or, equivalently, $(\eta,\xi) \in S^1 \times \mathbb{R}^n$).

In what follows, we use the following notation: given a set $\Omega \subset \mathbb{R}^n$ and a number $\rho > 0$,

$$\Omega_\rho \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,\Omega) < \rho\}.$$

We start with the following result, which deals with local extrema.

Theorem 5.3. *Suppose that (K1), (V1) and (A1) hold. Assume moreover that there is a compact set $M \subset \mathbb{R}^n$ over which A achieves an isolated strict local minimum with value a. By this we mean that for some $\delta > 0$,*

$$b \stackrel{\text{def}}{=} \inf_{x \in \partial M_\delta} A(x) > a. \tag{5.44}$$

Then there exists $\varepsilon_\delta > 0$ such that (S_ε) has at least $\operatorname{cat}(M, M_\delta)$ (orbits of) solutions concentrating near M_δ, for all $0 < \varepsilon < \varepsilon_\delta$.

Conversely, assume that K is a compact set $\mathbb{R}^n$ over which A achieves an isolated strict local maximum with value b, namely for some $\gamma > 0$,

$$a \stackrel{\text{def}}{=} \inf_{x \in \partial K_\gamma} A(x) < b. \tag{5.45}$$

Then there exists $\varepsilon_\gamma > 0$ such that (S_ε) has at least $\operatorname{cat}(K, K_\gamma)$ (orbits of) solutions concentrating near K_γ, for all $0 < \varepsilon < \varepsilon_\gamma$.

Proof. As in the previous theorem, one has $\Phi_\varepsilon(\eta,\xi) = \Psi_\varepsilon(\xi)$. Now choose $\bar\xi > 0$ in such a way that $M_\delta \subset \{x \in \mathbb{R}^n \mid |x| < \bar\xi\}$. Define again $\overline{A}$ as in the proof of Theorem 5.4. Let

$$\begin{aligned} N^\varepsilon &= \{\xi \in \mathbb{R}^n \mid \varepsilon\xi \in M\} \\ N^\varepsilon_\delta &= \{\xi \in \mathbb{R}^n \mid \varepsilon\xi \in M_\delta\} \\ \Theta^\varepsilon &= \{\xi \in \mathbb{R}^n \mid \Psi_\varepsilon(\xi) \le C_1 \tfrac{a+b}{2}\}. \end{aligned}$$

From (5.42) we get some $\varepsilon_\delta > 0$ such that

$$N^\varepsilon \subset \Theta^\varepsilon \subset N^\varepsilon_\delta, \tag{5.46}$$

for all $0 < \varepsilon < \varepsilon_\delta$. To apply standard category theory, we need to prove that Θ^ε is compact. To this end, as can be readily checked, it suffices to prove that Θ^ε cannot touch $\partial N^\varepsilon_\delta$. But if $\varepsilon\xi \in \partial M$, one has $\overline{\Lambda}(\varepsilon\xi) \geq b$ by the very definition of δ, and so

$$\Psi_\varepsilon(\xi) \geq C_1\overline{\Lambda}(\varepsilon\xi) + o_\varepsilon(1) \geq C_1 b + o_\varepsilon(1).$$

On the other hand, for all $\xi \in \Theta^\varepsilon$ one has also $\Psi_\varepsilon(\xi) \leq C_1\frac{a+b}{2}$. We can conclude from (5.46) and elementary properties of the Lusternik–Schnirel'man category that Ψ_ε has at least

$$\mathrm{cat}(\Theta^\varepsilon, \Theta^\varepsilon) \geq \mathrm{cat}(N^\varepsilon, N^\varepsilon_\delta) = \mathrm{cat}(N, N_\delta)$$

critical points in Θ^ε, which correspond to at least $\mathrm{cat}(M, M_\delta)$ orbits of solutions to (S_ε). Now, let $(\eta^*, \xi^*) \in S^1 \times M_\delta$ a critical point of Φ_ε. Hence this point (η^*, ξ^*) localizes a solution $u_{\varepsilon,\eta^*,\xi^*}(x) = z^{\varepsilon\xi^*,\eta^*}(x) + w(\varepsilon, \eta^*, \xi^*)$ of (S_ε). Recalling the change of variable which allowed us to pass from (NLS) to (S_ε), we find that

$$u_{\varepsilon,\eta^*,\xi^*}(x) \approx z^{\varepsilon\xi^*,\eta^*}(\tfrac{x-\xi^*}{\varepsilon}).$$

solves (NLS). The concentration statement follows from standard arguments ([6, 10]). The Proof of the second part of Theorem 5.4 follows with analogous arguments.

Theorem 5.1 is an immediate corollary of the previous one when x_0 is either a nondegenerate local maximum or minimum for Λ. We remark that the case in which Λ has a maximum cannot be handled using a direct variational approach and the arguments in [28] cannot be applied.

To treat the general case, we need some more work. In order to present our main result, we need to introduce some topological concepts.

Given a set $M \subset \mathbb{R}^n$, the *cup long* of M is by definition

$$\ell(M) = 1 + \sup\{k \in \mathbb{N} \mid (\exists \alpha_1, \ldots, \alpha_n \in \check{H}^*(M) \setminus \{1\})(\alpha_1 \cup \cdots \cup \alpha_k \neq 0)\}.$$

If no such classes exists, we set $\ell(M) = 1$. Here $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients, and $\cup$ denotes the cup product. It is well known that $\ell(S^{n-1}) = \mathrm{cat}(S^{n-1}) = 2$, and $\ell(T^n) = \mathrm{cat}(T^n) = n+1$, where T^n is the standard n–dimensional torus. But in general, one has $\ell(M) \leq \mathrm{cat}(M)$.

The following definition dates back to Bott ([20]).

Definition 5.1. *We say that M is non-degenerate for a C^2 function $I\colon \mathbb{R}^N \to \mathbb{R}$ if M consists of Morse theoretically non-degenerate critical points for the restriction $I_{|M^\perp}$.*

To prove our existence result, we use again Theorem 4.2.

We are now ready to prove an existence and multiplicity result for (NLS).

Theorem 5.4. *Let (V1), (K1) and (A1) hold. If the auxiliary function* Λ *has a smooth, compact, non-degenerate manifold of critical points* M*, then for* $\varepsilon > 0$ *small, the problem* (S_ε) *has at least* $\ell(M)$ *(orbits of) solutions concentrating near points of* M*.*

Proof. By Remark 5.1, we have to find critical points of $\Psi_\varepsilon = \Psi_\varepsilon(\xi)$. Since M is compact, we can choose $\bar{\xi} > 0$ so that $|x| < \bar{\xi}$ for all points $x \in M$. From this moment, $\bar{\xi}$ is kept fixed. form $\{\eta^*\} \times M$ is obviously a non-degenerate critical manifold We set now $V = \mathbb{R}^n$, $J = \Lambda$, $\Sigma = M$, and $I(\xi) = \Psi_\varepsilon(\eta, \xi/\varepsilon)$. Select $\delta > 0$ so that $M_\delta \subset \{x \colon |x| < \bar{\xi}\}$, and no critical points of Λ are in M_δ, except fot those of M. Set $\mathcal{U} = M_\delta$. From (5.42) and (5.43) it follows that I is close to J in $C^1(\overline{\mathcal{U}})$ when ε is very small. We can apply Theorem 4.2 to find at least $\ell(M)$ critical points $\{\xi_1, \ldots, \xi_{\ell(M)}\}$ for Ψ_ε, provided ε is small enough. Hence the orbits $S^1 \times \{\xi_1\}, \ldots, S^1 \times \{\xi_{\ell(M)}\}$ consist of critical points for Φ_ε which produce solutions of (S_ε). The concentration statement follows as in [10].

Remark 5.3. We point out that Theorem 1.1, Theorem 5.3 and Theorem 5.4 hold for problems involving more general nonlinearities $g(x,u)$ satisfying the same assumptions in [47] (see also Remark 5.4 in [10]). For our approach, we need the uniqueness of the radial solution z of the corresponding scalar equation

$$-\Delta u + u + V(\varepsilon\xi)u = g(\varepsilon\xi, u), \quad u > 0, \quad u \in W^{1,2}(\mathbb{R}^n)\,. \tag{5.47}$$

Let us also remark that in [28] the class of nonlinearities handled does not require that equation (5.47) has a unique solution.

6 Homoclinic solutions of Hamiltonian systems

A different tool for analyzing non–compact problems has been studied by Rabier and Stuart ([72, 73]). It makes use of a recent notion of topological degree, introduced by Fitzpatrick, Pejsachowicz and Rabier, for C^1 Fredholm maps between Banach spaces. In this chapter we summarize some results obtained in the joint paper [80].

6.1 Basic definitions of the new topological degree

Consider two real Banach spaces X and Y. First of all, one defines the **parity** of a continuous path $\lambda \in [a,b] \mapsto A(\lambda)$ of bounded linear Fredholm operators with index zero from X into Y. It is always possible to find a **parametrix** for this path, namely a continuous function $B\colon [a,b] \to GL(Y,X)$ such that the composition $B(\lambda)A(\lambda)\colon X \to X$ is a compact perturbation of the identity for every $\lambda \in [a,b]$. If $A(a)$ and $A(b)$ belong to $GL(X,Y)$, then the parity of the path A on $[a,b]$ is by definition

$$\pi(A(\lambda) \mid \lambda \in [a,b]) = \deg(B(a)A(a)) \deg(B(b)A(b)).$$

This is a good definition in the sense that it is independent of the parametrix B. The following criterion can be useful for evaluating the parity of an admissible path.

Proposition 6.1. *Let $A\colon [a,b] \to B(X,Y)$ be a continuous path of bounded linear operators having the following properties.*

(i) $A \in C^1([a,b], B(X,Y))$.
(ii) $A(\lambda)\colon X \to Y$ is a Fredholm operator of index zero for each $\lambda \in [a,b]$.
(iii) There exists $\lambda_0 \in (a,b)$ such that

$$A'(\lambda_0)[\ker A(\lambda_0)] \oplus \operatorname{rge} A(\lambda_0) = Y \qquad (6.1)$$

in the sense of a topological direct sum.

Then there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset [a, b]$,

$$A(\lambda) \in GL(X, Y) \text{ for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon] \tag{6.2}$$

and

$$\pi(A(\lambda) \mid \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^k \tag{6.3}$$

where $k = \dim \ker A(\lambda_0)$.

The proof of this proposition is essentially contained in [43, 44].

We remark that given a continuous path $A\colon [a, b] \to \Phi_0(X, Y)$ and any $\lambda_0 \in [a, b]$ such that $A(\lambda) \in GL(X, Y)$ for all $\lambda \neq \lambda_0$, the parity $\pi(A(\lambda) \mid \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon])$ is the same for all $\varepsilon > 0$ sufficiently small. This number is then called the parity of A across λ_0.

This degree has many features of the ordinary Leray–Schauder degree, but the homotopy invariance is only up to a $\pm$ sign. We refer to [80] for a short review of this concept.

We recall here that a bounded linear operator $L\colon X \to Y$ acting between two Banach spaces X and Y is said to be *Fredholm of index zero* if its range $L(X)$ is closed in Y, $\ker L$ is finite–dimensional and $\dim \ker L = \operatorname{codim} L(X)$.

Definition 6.1. *Let X and Y be real Banach spaces and consider a function $F \in C^1(\Lambda \times X, Y)$, where Λ is an open interval. Let $P(\lambda, x) = \lambda$ be the projection of $\mathbb{R} \times X$ onto $\mathbb{R}$. We say that Λ is an admissible interval for F provided that*

(i) for all $(\lambda, x) \in \Lambda \times X$, the bounded linear operator $\partial_x F(\lambda, x)\colon X \to Y$ is a Fredholm operator of index zero;
(ii) for any compact subset $K \subset Y$ and any closed bounded subset W of $\mathbb{R} \times X$ such that

$$\inf \Lambda < \inf PW \leq \sup PW < \sup \Lambda.$$

it results that $F^{-1}(K) \cap W$ is a compact subset of $\mathbb{R} \times X$.

In figure 6.1, the set $F^{-1}(K)$ is not compact. Nevertheless, the shaded region represents the compact set $F^{-1}(K) \cap W$.

The following theorem shows that the new degree π for Fredholm maps of index zero allows us to recast a bifurcation theorem like the one due to Rabinowitz.

Theorem 6.1. *Let X and Y be real Banach spaces and consider a function $F \in C^1(\Lambda \times X, Y)$ where Λ is an admissible open interval for F. Suppose that $\lambda_0 \in \Lambda$ and that there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset \Lambda$,*

$$D_x F(\lambda, 0) \in GL(X, Y) \text{ for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$$

and

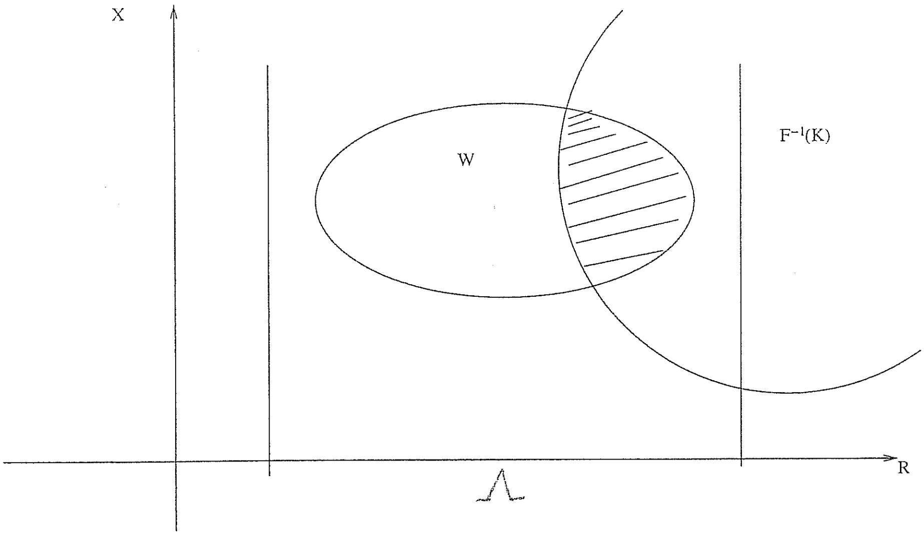

Fig. 6.1. Properness on bounded subsets

$$\pi(D_xF(\lambda,0) \mid [\lambda_0-\varepsilon,\lambda_0+\varepsilon]) = -1.$$

Let $Z = \{(\lambda,u) \in \Lambda \times X \mid u \neq 0$ *and* $F(\lambda,u) = 0\}$ *and let* C *denote the connected component of* $Z \cup \{(\lambda_0,0)\}$ *containing* $(\lambda_0,0)$.

Then C *has at least one of the following properties:*

(1) C *is unbounded.*
(2) The closure of C *contains a point* $(\lambda_1,0)$ *where* $\lambda_1 \in \Lambda \setminus [\lambda_0-\varepsilon,\lambda_0+\varepsilon]$ *and* $D_xF(\lambda_1,0) \notin GL(X,Y)$.
(3) The closure of PC *intersects the boundary of* Λ.

6.2 A concrete example

In practice, the application of Theorem 6.1 consists of this: first, one identifies solutions of a given problem as the zeroes of a C^1 map F depending on a parameter λ. Then one tries to single out admissible intervals for this map F. To this aim, one often uses the asymptotic behavior of the problem. Roughly speaking, if F "tends" at infinity in a suitable sense to another map with a simpler structure (for example an autonomous structure), and this limit map has no non–trivial zeroes for a certain range Λ of the parameter λ, then Λ is admissible for F. The last step is to find a suitable value $\lambda_0 \in \Lambda$ from which a global branch of solutions can bifurcate.

These ideas have been applied to nonlinear elliptic differential equations ([73]), and we focus here on the case of hamiltonian systems, which present an easier technical framework.

Precisely, let us introduce the hamiltonian $H\colon \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ as

$$H(t,u,v,\lambda)=\frac{1}{2}\left\{v^2+\lambda u^2+a(t)u^2\right\}-\frac{|u|^{\sigma+2}}{(\sigma+2)(1+\mathrm{e}^{-t})}+\frac{u^2v^2}{2(\mathrm{e}^t+1)}, \quad (6.4)$$

where $\sigma>0$, $a\in C(\mathbb{R})$ is non-negative, not identically zero, and

$$\lim_{|t|\to+\infty} a(t)=0. \quad (6.5)$$

We look for solutions of the system

$$\begin{cases} Jx'=\nabla H(t,x,\lambda)\\ \lim_{|t|\to\infty}|x(t)|=0 \end{cases} \quad (6.6)$$

where $x=(u,v)\in H^1(\mathbb{R}^2)$, or, more explicitely,

$$\begin{cases} -v'=\lambda u+a(t)u-\frac{1}{1+\mathrm{e}^{-t}}|u|^\sigma u\\ \ u'=v+\frac{u^2v}{1+\mathrm{e}^t}. \end{cases}$$

Define

$$\begin{aligned} H^+(t,u,v,\lambda)&=\frac{1}{2}\{v^2+\lambda u^2\}-\frac{|u|^{\sigma+2}}{\sigma+2},\\ H^-(t,u,v,\lambda)&=\frac{1}{2}\{v^2+\lambda u^2\}+\frac{u^2v^2}{2},\\ g^+(u,v,\lambda)&=(\lambda u-|u|^\sigma u,v)\\ g^-(u,v,\lambda)&=(\lambda u+uv^2,v+u^2v)\\ A_\lambda(t)=D^2_{(u,v)}H(t,0,0,\lambda)&=\begin{pmatrix}\lambda+a(t) & 0\\ 0 & 1\end{pmatrix}\\ A^\pm_\lambda=D_{(u,v)}g^\pm(0,0,\lambda)&=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}. \end{aligned}$$

The $g^\pm$ and $H^\pm$ verify

(1) $g^+(t,0,\lambda)=g^-(t,0,\lambda)=0$ for all $t,\lambda\in\mathbb{R}$
(2) $\lim_{t\to\infty}\{D^2_\xi H(t,\xi,\lambda)-D_\xi g^+(t,\xi,\lambda)\}=\lim_{t\to-\infty}\{D^2_\xi H(t,\xi,\lambda)-D_\xi g^-(t,\xi,\lambda)\}=0$, uniformly for $\xi=(u,v)$ in bounded subsets of $\mathbb{R}^2$
(3) $\lim_{t\to\infty}\{H(t,\xi,\lambda)-H^+(t,\xi,\lambda)\}=\lim_{t\to-\infty}\{H(t,\xi,\lambda)-H^-(t,\xi,\lambda)\}=0$ uniformly for ξ in bounded subsets of $\mathbb{R}^2$

By (6.5), $H^\pm$ play the rôle of the *problems at infinity* we described above.

We will apply Theorem 6.1 with $X=H^1(\mathbb{R},\mathbb{R}^2)$, $Y=L^2(\mathbb{R},\mathbb{R}^2)$,

$$F(\lambda,x)=Jx'-\nabla H(\cdot,x(\cdot),\lambda).$$

We will also check that an admissible interval for this F is $\Lambda=(-\infty,0)$.

We state and prove the main theorem concerning the system (6.6).

Theorem 6.2. *Define*

$$\lambda_0 = \inf\left\{\int_{\mathbb{R}} |\varphi'|^2 - a(t)|\varphi|^2 \mid \varphi \in H^1(\mathbb{R}) \wedge \int_{\mathbb{R}} |\varphi|^2 = 1\right\}. \tag{6.7}$$

Then a global branch of homoclinic solutions of (6.6) *bifurcates at* λ_0 *in the sense of Theorem 6.1 with* $\Lambda = (-\infty, 0)$.

Proof. We will apply the criterion of Proposition 6.1. It is immediate to prove that $F \in C^1(\mathbb{R} \times H^1, L^2)$, from the very definition of the Fréchet derivative.

We define $F^{\pm} \colon \mathbb{R} \times H^1 \to L^2$ as

$$F^{\pm}(\lambda, x) = Jx' - \nabla H^{\pm}(\cdot, x(\cdot), \lambda),$$

where, as above, $x = (u, v)$. Similarly, we define

$$F^{\pm}(\lambda, x) = Jx' - \nabla H^{\pm}(\cdot, x(\cdot), \lambda).$$

We claim that $\Lambda = (-\infty, 0)$ *is an admissible interval for* F.

First step: $D_x F(\lambda, 0) \in \Phi_0(H^1, L^2)$ for all $\lambda < 0$.

Let $L = D_x F(\lambda, 0)$, namely

$$Lx(t) = Jx'(t) - A_\lambda(t)x(t)$$

The linear operator L is trivially bounded and also self–adjoint. From well–known results of linear functional analysis, L is Fredholm of index zero provided $\mathrm{rge}(L)$ is closed. With this in mind, let $f \in L^2$ and suppose that there exists a sequence $\{f_n\} \subset \mathrm{rge}\, L$ such that $\|f - f_n\|_2 \to 0$. Let $\{x_n\} \subset H^1$ be such that $Lx_n = f_n$.

Let P denote the orthogonal projection of L^2 onto $\ker L$ and set $Q = I - P$. Then $Qx_n = x_n - Px_n \in H^1$ and $Qx_n \in [\ker(L)]^{\perp}$, the orthogonal complement of $\ker L$ in L^2. Setting

$$u_n = Qx_n \text{ we have that } u_n \in H^1 \cap [\ker(L)]^{\perp} \text{ and } Lu_n = f_n.$$

Let us prove that the sequence $\{u_n\}$ is bounded in H^1. For this we use $S^{\pm} : H^1 \to L^2$ to denote the bounded linear operators defined by

$$S^{\pm}x = Jx' - A_\lambda^{\pm}x \text{ for all } x \in H^1.$$

The two operators $S^+ : H^1 \to L^2$ and $S^- : H^1 \to L^2$ are isomorphisms because

$$A_\lambda^{\pm} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

and the spectrum of the matrix

$$J\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$$

is

$$\sigma\left(J\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\right)=\begin{cases}\{\pm\mathrm{i}\sqrt{\lambda}\} & \text{if } \lambda>0,\\ \{0\} & \text{if } \lambda=0,\\ \{\pm\sqrt{|\lambda|}\} & \text{if } \lambda<0.\end{cases}$$

Hence this spectrum is, for $\lambda < 0$, disjoint from $\mathrm{i}\mathbb{R}$, the imaginary axis; by a standard result (see [54, 60]), this implies that $S^{\pm}$ are both isomorphisms.

So there exists a constant k such that

$$\left\|S^{\pm}x\right\|_2 \geq k\,\|x\| \text{ for all } x \in H^1. \tag{6.8}$$

Supposing that $\|u_n\| \to \infty$, we set $w_n = \frac{u_n}{\|u_n\|}$. Then $\{w_n\} \subset H^1$ with $\|w_n\| = 1$ for all $n \in \mathbb{N}$. Passing to a subsequence, we can suppose that $w_n \rightharpoonup w$ weakly in H^1, and hence that $Lw_n \rightharpoonup Lw$ weakly in L^2. Furthermore,

$$Lw_n = \frac{Lu_n}{\|u_n\|} = \frac{f_n}{\|u_n\|} \text{ and so } \|Lw_n\|_2 = \frac{\|f_n\|_2}{\|u_n\|} \to 0$$

since $\|f_n\|_2 \to \|f\|_2$ and $\|u_n\| \to \infty$. Thus $Lw = 0$. But $w_n \in H^1 \cap [\ker(L)]^{\perp}$ for all $n \in \mathbb{N}$, from which it follows that $w \in H^1 \cap [\ker(L)]^{\perp}$. Thus $w = 0$ and $w_n \rightharpoonup 0$ weakly in H^1. Consequently,

$$w_n \to 0 \text{ uniformly on } [-R, R] \text{ for any } R \in (0, \infty).$$

But, for all $t \in \mathbb{R}$,

$$Jw_n'(t) = A_\lambda(t)w_n(t) + \frac{f_n(t)}{\|u_n\|}$$

and so

$$\|w_n'\|_{L^2(-R,R)} \leq \sup_{t\in\mathbb{R}} \|A_\lambda(t)\|\,\|w_n\|_{L^2(-R,R)} + \frac{\|f_n\|_2}{\|u_n\|},$$

showing that $\|w_n'\|_{L^2(-R,R)} \to 0$ as $n \to \infty$, for all $R \in (0, \infty)$. In particular,

$$\|w_n\|_{H^1(-R,R)} \to 0 \text{ as } n \to \infty.$$

Now choose any $\varepsilon > 0$. There is a constant $r \in (0, \infty)$ such that

$$\left|A_\lambda(t) - A_\lambda^+\right| \leq \varepsilon \text{ for all } t \geq r \text{ and } \left|A_\lambda(t) - A_\lambda^-\right| \leq \varepsilon \text{ for all } t \leq -r.$$

Choose a constant $R > r + \frac{1}{\varepsilon}$ and a function $\varphi \in C^1(\mathbb{R})$ such that
$0 \leq \varphi(t) \leq 1$ for all $t \in \mathbb{R}$, $\varphi(t) = 0$ for $t \leq r$, $\varphi(t) = 1$ for $t \geq R$ and $|\varphi'(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$.

Consider now the function $z_n(t) = \varphi(t)w_n(t)$. Clearly $z_n \in H^1$ and

$$\begin{aligned}S^+z_n(t) &= \varphi'(t)Jw_n(t) + \varphi(t)Jw_n'(t) - A_\lambda^+ z_n(t)\\ &= \varphi'(t)Jw_n(t) + \varphi(t)Lw_n(t) + \varphi(t)\{A_\lambda(t) - A_\lambda^+\}w_n(t)\\ &= \varphi'(t)Jw_n(t) + \varphi(t)\frac{f_n(t)}{\|u_n\|} + \varphi(t)\{A_\lambda(t) - A_\lambda^+\}w_n(t).\end{aligned}$$

Thus,

$$\begin{aligned}\left\|S^{+}z_n\right\|_2 &\le \varepsilon\left\|w_n\right\|_2 + \frac{\|f_n\|_2}{\|u_n\|} + \sup_{t\ge r}\left|A_\lambda(t) - A_\lambda^{+}\right|\|w_n\|_2\\ &\le \varepsilon + \frac{\|f_n\|_2}{\|u_n\|} + \varepsilon\end{aligned}$$

since $\|w_n\|_2 \le 1$. Hence, by (6.8)

$$\|w_n\|_{H^1(R,\infty)} = \|z_n\|_{H^1(R,\infty)} \le \|z_n\| \le \frac{1}{k}\{2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\}.$$

A similar argument, using $\varphi(-t)w_n(t)$ instead of z_n, shows that

$$\|w_n\|_{H^1(-\infty,-R)} \le \frac{1}{k}\{2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\}.$$

Finally, we have shown that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|w_n\|^2 &= \|w_n\|^2_{H^1(-\infty,-R)} + \|w_n\|^2_{H^1(-R,R)} + \|w_n\|^2_{H^1(R,\infty)}\\ &\le \frac{2}{k^2}\{2\varepsilon + \frac{\|f_n\|_2}{\|u_n\|}\}^2 + \|w_n\|^2_{H^1(-R,R)}\end{aligned}$$

and, letting $n \to \infty$,

$$\limsup_{n\to\infty}\|w_n\|^2 \le \frac{2}{k^2}\{2\varepsilon\}^2$$

since $\|w_n\|^2_{H^1(-R,R)} \to 0, \|f_n\|_2 \to \|f\|_2$ and $\|u_n\| \to \infty$. But $\|w_n\| \equiv 1$ and $\varepsilon > 0$ can be chosen so that $\frac{2}{k^2}\{2\varepsilon\}^2 < 1$. This contradiction establishes the boundedness of the sequence $\{u_n\}$ in H^1.

By passing to a subsequence, we can now suppose that $u_n \rightharpoonup u$ weakly in H^1, and consequently that $Lu_n \rightharpoonup Lu$ weakly in L^2. However, $Lu_n = f_n$ and $\|f_n - f\|_2 \to 0$, showing that $Lu = f$. This proves that $\operatorname{rge} L$ is a closed subspace of L^2 and we have shown that $D_xF(\lambda, 0) \in \Phi_0(H^1, L^2)$.

Second step: the properness of F.

In the sequel, we will use the following statement, whose proof is omitted and can be found in our paper [80]: *if* $D_xF(\lambda,0) \in \Phi_0(H^1, L^2)$, *then* $D_xF(\lambda, x) \in \Phi_0(H^1, L^2)$ *for all* $x \in H^1$.

The next step is to show that F is a proper map on the class of closed and bounded subsets of $\Lambda \times H^1$. Hence, let us fix a compact subset $K \subset L^2$ and a closed, bounded subset $W \subset \Lambda \times H^1$, in such a way that

$$-\infty < \inf PW \le \sup PW < 0.$$

Choose any sequence $\{(\lambda_n, x_n)\}$ from $F^{-1}(K) \cap W$; we may assume that

$$\lambda_n \to \lambda \le \sup PW < 0,$$

$$||F(\lambda_n, x_n) - y||_2 \to 0$$

for some $y \in L^2$. Now, take any $x \in W$ and any λ, μ in Λ. Then

$$||F(\lambda, x) - F(\mu, x)||_2 = ||D_x H(\cdot, x, \lambda) - D_x H(\cdot, x, \mu)||_2$$

and

$$D_x H(\cdot, x, \lambda) - D_x H(\cdot, x, \mu) = (\lambda - \mu) \int_0^1 D_\lambda D_{(u,v)} H(\cdot, x, s\lambda + (1-s)\mu)\, ds.$$

Set $L = \sup_{x \in W} ||x||_\infty < \infty$, since W is bounded in H^1. If $K = \sqrt{L + |\lambda| + 1}$, then

$$||F(\lambda, x) - F(\mu, x)||_2 \leq C(K)||x||_2 |\lambda - \mu|$$

for some constant $C(K) > 0$ depending only on K. We have thus proved that the family $\{F(\cdot, x_n)\}_{n \geq 1}$ is equicontinuous at every point $\lambda \in \Lambda$. This immediately implies that

$$||F(\lambda, x_n) - y||_2 \to 0.$$

Now we "freeze" λ. Since W is bounded, we may assume that

$$x_n \rightharpoonup x \in H^1.$$

We claim that: for all $\varepsilon > 0$ there exists $R > 0$ such that for all integers $n \geq 1$ and all $t \in \mathbb{R}$ such that $|t| \geq R$, there results

$$|x_n(t)| \leq \varepsilon.$$

Suppose this is not true. By a simple reasoning, the sequence $\{x_n\}$ must slip off to infinity. More precisely, one of the following conditions must be true:

(1) There is a sequence $\{l_k\} \subset \mathbb{Z}$ with $\lim_{k\to\infty} l_k = \infty$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\widetilde{x_k} = \tau_{l_k}(x_{n_k}) = x_{n_k}(\cdot + l_k)$ converges weakly in H^1 to an element $\widetilde{x} \neq 0$
(2) There is a sequence $\{l_k\} \subset \mathbb{Z}$ with $\lim_{k\to\infty} l_k = -\infty$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\widetilde{x_k} = \tau_{l_k}(x_{n_k}) = x_{n_k}(\cdot + l_k)$ converges weakly in H^1 to an element $\widetilde{x} \neq 0$

To fix ideas, let us suppose that $\{x_n\}$ has the property (1) . The invariance by translation of the Lebesgue measure implies that $||F(\lambda, x_n) - y||_2 = ||\tau_{l_k}(F(\lambda, x_n) - y)||_2$ so

$$||\tau_{l_k}(F(\lambda, x_{n_k})) - \widetilde{y_k}||_2 \to 0 \text{ where } \widetilde{y_k}(t) = y(t + l_k)$$

For any $\omega \in (0, \infty)$, it is easy to show that

$$\left\|\tau_{l_k}(F(\lambda, x_{n_k})) - \tau_{l_k}(F^+(\lambda, x_{n_k}))\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty.$$

Hence

$$\left\|\tau_{l_k}(F^+(\lambda, x_{n_k})) - \widetilde{y_k}\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty.$$

But

$$\begin{aligned}\tau_{l_k}(F^+(\lambda, x_{n_k}))(t) &= Jx'_{n_k}(t + l_k) - g^+(x_{n_k}(t + l_k), \lambda) \\ &= J\widetilde{x'_k}(t) - g^+(\widetilde{x_k}(t), \lambda) = F^+(\lambda, \widetilde{x_k})(t)\end{aligned}$$

because g^+ is independent of t. Consequently,

$$\left\|F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k}\right\|_{L^2(-\omega,\omega)} \to 0 \text{ as } k \to \infty,$$

for all $\omega \in (0, \infty)$. Since the sequence $\{F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k}\}$ is bounded in L^2, this implies that $F^+(\lambda, \widetilde{x_k}) - \widetilde{y_k} \rightharpoonup 0$ weakly in L^2. But $\widetilde{y_k} \rightharpoonup 0$ weakly in L^2 by standard results in functional analysis, so we now have that $F^+(\lambda, \widetilde{x_k}) \rightharpoonup 0$ weakly in L^2. However, the weak sequential continuity of $F^+(\lambda, \cdot) : H^1 \to L^2$ implies that

$$F^+(\lambda, \widetilde{x_k}) \rightharpoonup F^+(\lambda, \widetilde{x}) \text{ weakly in } L^2,$$

so we must have $F^+(\lambda, \widetilde{x}) = 0$. We now show that this implies $x \equiv 0$, the desired contradiction. Notice that $x \in C^1(\mathbb{R})$, and set

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\frac{d}{dt}\langle Cx(t), x(t)\rangle = 2\langle Cx(t), x'(t)\rangle = \lambda u(t)^2 - |u|^{\sigma+2} + v^2 > 0$$

whenever $x = (u, v)$ is not identically zero. Since $\lim_{|t|\to\infty} |x(t)| = 0$ because $x \in H^1$, then $\langle Cx(t), x(t)\rangle \equiv 0$. This implies that $\langle JCx(t), g^+(x(t), \lambda)\rangle = 0$ for all $t \in \mathbb{R}$. Finally, $x(t) = 0$ for all t.[1]

By combining the Sobolev imbedding $H^1 \subset L^\infty$ on bounded sets and the claim we have just proved, it is immediate to obtain that

$$\|x_n - x\|_\infty \to 0.$$

Hence we have proved that $F(\lambda, \cdot)$ is a proper map on the bounded subsets of H^1. Hence there must be a subsequence, still denoted by $\{x_n\}$, such that

$$x_n \to x \text{ strongly in } H^1.$$

[1] Of course, the same reasoning would apply to case (2).

But then $(\lambda_n, x_n) \to (\lambda, x)$ strongly, and in particular $(\lambda, x) \in W$ because W is closed. This finally proves that $F^{-1}(K) \subset W$ is compact in H^1, and also that $A = (-\infty, 0)$ is admissible for F.

Third step: the choice of λ_0.

It is well-known (see [64], theorem 11.5) that λ_0 defined in (6.7) belongs to $(-\infty, 0)$ and that there exists an element $\varphi_0 \in H^1(\mathbb{R})$ such that

$$\varphi_0(t) > 0 \text{ for all } t \in \mathbb{R} \text{ and } \lambda_0 \int_{-\infty}^{\infty} \varphi(t)_0^2 dt = \int_{-\infty}^{\infty} \varphi_0'(t)^2 - a(t)\varphi_0(t)^2 dt.$$

Furthermore, $\varphi_0 \in H^2(\mathbb{R}) \cap C^2(\mathbb{R})$ and satisfies the equation

$$\varphi''(t) + \{\lambda_0 + a(t)\}\varphi(t).$$

Setting $x_0 = (\varphi_0, \varphi_0')$, we find that $\ker D_x F(\lambda_0, 0) = \operatorname{span}\{x_0\}$.

Finally we observe that

$$D_\lambda D^2_{(u,v)} H(t, 0, 0, \lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We are now ready to prove that the statements of Proposition 6.1 apply with $A(\lambda) = D_x F(\lambda, 0)$ and the selected value λ_0. Indeed, since $\ker A(\lambda_0)$ is one–dimensional,

$$A'(\lambda_0)[\ker A(\lambda_0) \cap \operatorname{rge} A(\lambda_0) = \emptyset.$$

Moreover, $\operatorname{codim} \operatorname{rge} A(\lambda_0) = \dim \ker A(\lambda_0)$ because $A(\lambda_0)$ is Fredholm of index zero. Since

$$A'(\lambda_0)[\ker D_x F(\lambda_0, 0)] = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u \mid u \in \ker D_x F(\lambda_0, 0) \right\},$$

$A'(\lambda_0)[\ker A(\lambda_0)]$ and $\operatorname{rge} A(\lambda_0)$ are complementary in L^2:

$$A'(\lambda_0)[\ker A(\lambda_0)] \oplus \operatorname{rge} A(\lambda_0) = L^2.$$

In our situation, $\pi(D_x F(\lambda_0, 0) \mid [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) = (-1)^1 = -1$, and Theorem 6.1 applies. ♣

6.3 The general result

The topological degree allows us to handle more general hamiltonian systems like

$$Jx'(t) = \nabla H(t, x(t), \lambda) \tag{6.9}$$

where $x \in H^1(\mathbb{R}, \mathbb{R}^{2N})$, and $H \colon \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

(H1) $H \in C(\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R})$ with $H(t,\cdot,\lambda) \in C^2(\mathbb{R}^{2N})$ and $D_\xi H(t,0,\lambda) = 0$ for all $t, \lambda \in \mathbb{R}$

(H2) The partial derivatives $D_\xi H, D_\xi^2 H, D_\lambda D_\xi H, D_\lambda D_\xi^2 H$ and $D_\xi D_\lambda D_\xi H$ exist and are continuous on $\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}$.

(H3) For each $\lambda \in \mathbb{R}$, $D_\xi H(\cdot,\cdot,\lambda) : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a C_ξ^1-bundle map, and $D_\lambda D_\xi^2 H : \mathbb{R}\times(\mathbb{R}^{2N} \times \mathbb{R}) \to \mathbb{R}$ is a $C^0_{(\xi,\lambda)}-$bundle map.

(H4) $D_\xi^2 H(\cdot,0,0)$ and $D_\lambda D_\xi^2 H(\cdot,0,0) \in L^\infty(\mathbb{R})$.

(H^∞) For all $\lambda \in \mathbb{R}$, there exist two C_ξ^1-bundle maps $g^+(\cdot,\cdot,\lambda)$ and $g^-(\cdot,\cdot,\lambda) : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that

(1) $g^+(t,0,\lambda) = g^-(t,0,\lambda) = 0$ for all $t, \lambda \in \mathbb{R}$

(2) $\lim_{t\to\infty}\{D_\xi^2 H(t,\xi,\lambda) - D_\xi g^+(t,\xi,\lambda)\} = \lim_{t\to-\infty}\{D_\xi^2 H(t,\xi,\lambda) - D_\xi g^-(t,\xi,\lambda)\} = 0$, uniformly for ξ in bounded subsets of $\mathbb{R}^{2N}$

(3) $g^+(t+T^+,\xi,\lambda) - g^+(t,\xi,\lambda) = g^-(t+T^-,\xi,\lambda) - g^-(t,\xi,\lambda) = 0$ for some $T^+, T^- > 0$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^{2N}$.

We point out that (H^∞) describes again the "problems at infinity".

Theorem 6.3 ([80]). *Suppose that (H1) to (H4) and (H^∞) are satisfied. An open interval Λ is admissible provided that, for all $\lambda \in \Lambda$, the following conditions are satisfied.*

(1) The periodic, linear Hamiltonian systems

$$Jx' - A_\lambda^+(t)x = 0 \text{ and } Jx' - A_\lambda^-(t)x = 0$$

have no characteristic multipliers on the unit circle.

(2) The asymptotic limit g^+ satisfies the following condition: If there is a real, symmetric $2N \times 2N-$matrix C such that

$$\langle g^+(t,\xi,\lambda), JC\xi\rangle > 0 \text{ for all } \xi \in \mathbb{R}^{2N}\backslash\{0\},$$

then $\{x \in H^1 : F^+(\lambda,x) = 0\} = \{0\}$.

(3) The asymptotic limit g^- satisfies the following condition: If there is a real, symmetric $2N \times 2N-$matrix C such that

$$\langle g^-(t,\xi,\lambda), JC\xi\rangle > 0 \text{ for all } \xi \in \mathbb{R}^{2N}\backslash\{0\},$$

then $\{x \in H^1 : F^-(\lambda,x) = 0\} = \{0\}$.

(4) There is a point $\lambda_0 \in \Lambda$ such that

(i) $k = \dim N(\lambda_0)$ is odd where $N(\lambda) = \{u \in C^2(\mathbb{R},\mathbb{R}^{2N}) : Ju'(t) - A_\lambda(t)u(t) \equiv 0$ and $\lim_{|t|\to\infty} u(t) = 0\}$,

(ii) for every $u \in N(\lambda_0)\backslash\{0\}$ there exists $v \in N(\lambda_0)$ such that

$$\int_{-\infty}^{\infty} \langle T_{\lambda_0}(t)u(t), v(t)\rangle\, dt \neq 0 \text{ where } T_\lambda(t) = D_\lambda D_\xi^2 H(t,0,\lambda) \text{ and}$$

(iii) $\dim\{T_{\lambda_0}(\cdot)u : u \in N(\lambda_0)\} = k$.

Then a global branch of homoclinic solutions of (6.9) bifurcates at λ_0 in the sense of Theorem 6.1 with $X = H^1$ and $Y = L^2$.

The proof of this theorem follows the same lines of the one presented for the concrete example. We refer to [80] for the details.

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