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Freely acting orbifolds

Thesis submitted for the degree of
"Doctor Philosophiæ"
In
Elementary Particle Theory

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Trieste, October 2003

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Chapter 1

Introduction

String theory represents maybe one of the most attractive scenarios to build suitable extensions of the Standard Model of high energy physics.

In string theory gauge and gravity interactions are naturally arranged together, offering one of the most clear examples, maybe the only one, of a finite theory of quantum gravity. This result has great relevance both from a theoretical and from a phenomenological point of view, since the importance of its cosmological implications. It may give the possibility, for example, of analyzing with a unified language phenomena in which gauge interactions and gravity are both non-negligible, as in the case of the study of the early universe.

String theory has also become a source of ideas to be extrapolated and used in other fields of high energy physics. The most evident example is the new interest observed in the last few years in the so called *"physics of extra dimensions"*, consisting in an embedding of the standard model of particle physics, or a suitable extension of it, in a space-time with one or more extra dimensions.

From the original old idea of Kaluza and Klein [1], revisited after string theory gave it new resonance, a large class of models were built ([2, 3, 4], for a recent review see, for example, [5] and references therein), obtaining an answer, for example, to the so called hierarchy problem, or a way to reduce the huge number of free parameters of the standard model.

This *"bottom-up"* approach, due to its nature, has the good feature of being in strict contact with the observed phenomenology. On the other hand very often it is not possible to obtain models valid at all energy level, but there exists an energy scale where some new *"new physics"* is expected.

In this sense string theory, being naturally a *theory of everything*, offers a more effective way to build models valid at every energy scale, free from anomalies and with a natural coupling to gravity.

String models are obtained from string theory by dimensional reduction, i.e. a non-trivial choice for the (compact) background geometry of the extra dimensions.

Since in string theory the target-space coordinates are fields on the world sheet, it is equivalent to a non trivial choice for the vacuum expectation values of the extra dimension fields.

The choice of the background is essential since it defines the main properties of the final spectrum, such as supersymmetry (SUSY) or the gauge group. Clearly there are conditions imposed on the background to ensure that it is acceptable, at the lowest level, for example, it must satisfy the equation imposed by supergravity (SUGRA) that are an extension of the usual Einstein equations.

One of the most useful and powerful class of backgrounds is given by the orbifolds of flat trivial spaces. Taken a flat background M it is possible to build an orbifold modding out M by a symmetry group G of it. The obtained space is locally flat and the SUGRA e.o.m. are satisfied away from the fixed points of the G-action, but its topology is non-trivial: depending on the form of the symmetry group, the holonomy and homology groups are different from those of the initial space. Quantization of closed string theory on an orbifold give a model with properties governed by the topological properties of the orbifold: SUSY is defined by holonomy, the form of the massless spectrum by homology. Since the first introduction of orbifold compactifications [6] many orbifold models were built starting from the heterotic theory, some with interesting phenomenological features (see for example [7]).

The same possibility has been introduced also for open string theory, where things are more subtle, since in the case of open strings the choice of a non-trivial background is mapped also in the form of the gauge sector (open sector) in a non trivial way. This was introduced and developed in [8, 9]. The crucial point can be viewed in different ways: essentially an open string orbifold can be seen as coming from a closed string but with the insertion, in the symmetry group G, of the world-sheet orientation reversal Ω, and usually the new orbifold is called *orientifold*. This insertion modifies modular invariance in the closed loop amplitude and a new open string sector must be inserted to restore it. The right choice of the open sector makes the model anomaly-free and acceptable (for a review of the subject see [10]).

Through the introduction of the language of D-branes [11, 12, 13] new open string models were built (for a review of open string models with 4 extended dimensions and N=1 SUSY see [14]).

Building string models through orbifold/orientifold compactification spread new light on the rôle played by SUSY in string theory, where its presence is crucial in order to have theories free from instabilities like tachyons. It was through the introduction of SUSY by means of the Gliozzi Scherk and Olive (GSO) [15, 16] projection that it was possible to build a tachyon-free SUSY theory. The orientifold procedure gives the possibility of breaking some of the them but it is possible to see that no stable non-SUSY vacuum is achieved if all the operators acts in a non-free way.

SUSY plays an important rôle also in ordinary field theory, since it provides an elegant solution to the hierarchy problem if it is broken at a sufficiently low energy scale ($\sim TeV$) and so a phenomenologically appealing model representing an extension of the ordinary Standard Model must be non-SUSY at low energy, and some new technique must be used to obtain a stable non-SUSY model. In this sense a way out comes from a well-known field theory mechanism, the so-called Scherk–Schwarz (SS) symmetry-breaking mechanism [17]. It is one of the most interesting and promising mechanisms of symmetry breaking in theories with compact extra-dimensions, such as string theory; it consists in suitably twisting the periodicity conditions of each field along some compact directions. In this way, one obtains a non-local, perturbative and calculable symmetry-breaking mechanism. String models of this type can be constructed by deforming supersymmetric orbifold [6] models, and a variety of four-dimensional (4D) closed string models, mainly based on $\mathbf{Z}_2$ orbifolds, have been constructed in this way [18, 19, 20].

More in general, SS symmetry breaking can be achieved through a particular extension of the orientifold procedure described previously, when the symmetry group contains some freely acting operators, such as translations in some compact direction, called SS direction. The effects of freely acting orbifold projections [21] has recently been exploited in [22] to construct a novel class of closed string examples, including a model based on the $\mathbf{Z}_3$ orbifold. Unfortunately, a low compactification scale is quite unnatural for closed string models, where the fundamental string scale M_s is tied to the Planck scale, and can be achieved only in very specific situations [23] (see also [24]). The situation is different for open strings, where M_s can be very low [25], and interesting open string models with SS SUSY breaking have been derived in [26, 27, 28].

Recently, the SS mechanism has been object of renewed interest also from a more phenomenological bottom-up viewpoint, where it has been used in combination with orbifold projections to construct realistic 5D non-SUSY extensions of the SM [29, 4], confirming how the subject of string phenomenology is active and fruitful also from other points of view.

Building a non-SUSY model through the compactification on a freely acting orbifold ensures the absence of tachyons and the stability of the model at least at tree level in perturbation theory. On the other hand the absence of SUSY makes non-zero most of the loop correction usually taken to be zero and so under control. In particular loop corrections may generate a non-trivial potential for the compactification moduli, usually flat direction for the tree-level potential.

Since the value assumed by moduli has great phenomenological relevance it is crucial to understand if they are stabilized and to which value, or, if not, if there is some cosmological implication to this.

This represents a pressing problem in model building since the final task of a model embedding and reproducing the standard model imposes anyway a complete SUSY breaking. In a context of non-SUSY models with SUSY broken through the SS mechanism the relevance of loop corrections is maybe more urgent, since the dimension of the SS radius is exactly the scale of SUSY breaking and since all these model have a critical value for the SS radius under which a tachyon is introduced in the spectrum. It is crucial to understand if the induced potential has a shape leading to the tachyonic regime [18] or to the decompactification limit [30][1].

From another point of view the presence of such a potential for moduli fixes the open problem of the stabilization of the compactification moduli. The presence of loop corrections, in some sense, reintroduce predictivity in string model building, since they give a way to rule out a model, if the correction lead to physically unacceptable values of the compactification radii, or to accept it fixing the moduli to the "right" value.

It is also of interest to perform a study of the anomaly cancellation in this class of models. Quantum anomalies represent how quantum effects can spoil some property of a theory. In particular a failure of gauge symmetry due to quantum corrections means a failure of unitarity of the theory, making it useless.

Quantum anomalies have played a crucial rôle in the story of string theory, since it has been clear very soon that many low energy spectra are anomalous. In 1984 it was understood how to take care of them [34], opening the so-called *"string revolution"*. The cancellation of anomalies in string theory is guaranteed by the Green Schwarz (GS) [34] mechanism, ensuring that $SO(32)$ open string and heterotic string are anomaly-free. This property is reflected also on the string models one can build from these theories: the GS mechanism ensures in particular that all tadpole-free open models are also anomaly free.

In some sense this seems to be the end of the story, but it is remarkable that the mechanism of anomaly cancellation can modify some of the properties of the gauge group, for example many of the anomalous $U(1)$ bosons of the 4D reduced theory can take a mass [35], the gauge symmetry being realized in a non-linear way. Moreover, as said, the last years have witnessed a great interest for the bottom-up approach. A study of how the anomaly cancellation mechanism acts in string models from the 4D point of view is of great relevance since it may be exported to take care of the problem of anomaly cancellation in models built starting from this different approach.

Furthermore the study of the anomaly cancellation in models with extended and compact dimensions have shown how it can be dangerous to consider "physically meaningful" only quantities depending on the extended space time dimensions, the

[1]See [31] for a similar analysis performed in an M-theory context and [32] for a nice general analysis in non-SUSY heterotic models. See also [33] for an analysis of the stability of a certain class of non-SUSY 6D orientifolds.

internal ones having been integrated out. The presence of non-zero anomalies localized in the extra dimensions and such that they are zero if the extra dimensions are integrated out, as shown, for example, in [36], is dangerous also from the point of view of an effective 4D field theory, even though they are related to anomalous diagrams with heavy Kaluza Klein (KK) modes. This has the clear consequence of enhancing greatly the interest for the study of anomaly cancellation in string models, where the analysis is always performed starting from the 10D theory to be reduced to four dimensions, ensuring that also localized 4D anomalies are canceled.

Freely acting orbifolds offer also a new perspective on the study of the quantization of string theory over background including non-trivial fluxes for the Neveu-Schwarz Neveu-Schwarz (NSNS) three-form field (torsion). Due to the presence of an operator acting at the same time as a translation and as a rotation the orbifold shows non-trivial fibrations between the extra dimensions. It is possible to reparametrize the manifold introducing new off-diagonal terms for the metric, that have a final form resembling an extension of the Melvin background [37, 38, 39, 40]. T-duality maps these terms to a B-field, with non-trivial flux for $H = dB$ [41, 42].

The connection with flux-quantization is particularly interesting since the great revival of the subject in the last few years due to the discovery of its good phenomenological properties. The presence of background fluxes was investigated for the first time in the heterotic context in [43], where the conditions for N=1 SUSY in 4D were analyzed in detail after their first introduction in [44], and in [45, 46, 47]. In a background without torsion and warp these conditions essentially implies that the 4D space-time is Minkowski, while the internal manifold M admits a covariantly constant spinor $\nabla^{(M)}\epsilon = 0$. The existence of such a spinor induces directly the existence of a complex structure and Kähler form for M and so the manifold is complex and Kähler. Furthermore it implies that the manifold has vanishing Ricci tensor, and this kind of manifolds are known and classified as Calabi-Yau manifolds.

The presence of warp and torsion modifies this scenario. In particular the new conditions imply that the 4D space-time manifold has a metric given by the usual Minkowski metric times the warp factor, to be equal to the dilaton. On the internal space the conditions allow the existence of a complex structure that now is invariant under the action of a covariant derivative computed from the usual Christoffel symbols plus a torsion term. This manifold is no more Kähler since the Kähler form is no more closed, the measure of the non-Kählerity given exactly by the derivative of the dilaton/warp.

The presence of a warp modifies crucially also the phenomenology of the models. As pointed out in [3] it can be a solution of the hierarchy problem. Moreover a quantization of string theory over these backgrounds implies a non-trivial potential for many of the compactification moduli, stabilizing them and clarifying the strict relation with SS compactification, where all the twisted moduli are massive and so stabilized. The moduli stabilization, together with the presence of the warp, has given new resonance to this kind of models. The subject has been reanalyzed recently also in an open/IIB context in [48, 49, 50, 51, 52], and many models have been built [53, 54, 55, 56, 57].

Also in the M-theory/heterotic context the subject has been revisited [58, 59, 60, 61, 62, 63, 64] and also a deep analysis from a geometric point of view, in order to give a classification of the new backgrounds, has been performed [65, 66, 67, 68, 69, 70]. Also the approach through the gauged supergravity has been undertaken [71].

In this thesis we study in detail freely acting orbifold/orientifold models of type IIB string theory. In the second chapter we describe in a review section how an orbifold of type IIB closed string theory can be built, then we describe briefly the orientifold procedure, i.e. the orbifold compactification of type I open string theory, with particular care for the tadpole cancellation conditions. Then we review how to introduce the SS mechanism as a freely acting orbifold along the lines of [22]. The final part of the chapter, containing the original results of [72], is devoted to the construction of chiral IIB compact orientifold models with SS supersymmetry breaking. We derive a new $\mathbf{Z}_3 \times \mathbf{Z}'_3$ orientifold by applying a freely acting $\mathbf{Z}'_3$ projection defined as a translation of order 3 and a non-SUSY twist to the known SUSY $\mathbf{Z}_3$ orientifold [73, 14]. The model turns out to be chiral and extremely simple, since only $D9$-branes are present. It exhibits SS SUSY breaking in both the closed and open string sectors. All the gauginos are massive, but there is an anomalous spectrum of massless charginos. The model is classically stable, since all massless Neveu–Schwarz–Neveu–Schwarz (NSNS) and Ramond–Ramond (RR) tadpoles vanish, and potential tachyons can be avoided by taking a sufficiently large volume for the SS torus, i.e. the torus where the translation acts. We also rederive from a more geometrical perspective the $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ model of [27] (see also [74]), by applying to the SUSY $\mathbf{Z}'_6$ model of [14] a freely acting $\mathbf{Z}'_2$ projection generated by a translation of order 2 along a circle combined with a $(-)^F$ operation, where F is the 4D space-time fermion number operator. We then discuss in some detail its rich structure involving $D9$, $D5$ and $\bar{D}5$ branes.

Chapter 3 is devoted to the study of anomaly cancellation in string theory. We begin with a review section where we describe how the GS mechanism acts in type I open string theory, we show the anomalous one-loop open-string amplitudes and we show how they can be canceled through suitable tree-level closed string amplitudes. Then we show the form of the anomalous couplings between Dirichlet branes (D-branes) and orientifold planes (O-planes) with closed string Ramond-Ramond (RR) states, responsible for the cancellation.

Having these, along the path of [75, 76], we show the computation of the corresponding anomalies in orbifold models and the related new anomalous coupling. Then we introduce localized anomalies following [36]. In the last section containing the original results of [72] we perform a detailed study of localized anomaly cancellation for the freely acting orbifold models introduced in the second chapter. To this aim, we will extend the approach that has been followed in [75] for 4D SUSY orientifolds to distinguish between different points in the internal space. We find that all anomalies cancel locally, thanks to an interesting Green–Schwarz (GS) mechanism [34] involving twisted RR axions belonging to 4D sectors localized at fixed points or 6D sectors localized at fixed-planes, as found in [75, 77], but also 4-forms coming from 6D sectors localized at fixed planes. The latter effect arises whenever RR tadpoles are canceled globally but not locally[2], and involves only heavy KK modes of the 4-forms. In non-compact string vacua, such as intersecting branes, this kind of effect is already included in the usual anomaly inflow of [79]. Global irreducible anomalies can arise in this case, since there is no constraint on the global RR flux; they are canceled thanks to RR forms propagating in more than 4D. This shows once again the very close relation between the GS mechanism and the inflow mechanism of [80], even for irreducible terms.

Our results reveal an important distinction between anomalies appearing through a 6-form in the anomaly polynomial and anomalies appearing through the product of a 2-form and a 4-form. In the former case, the GS mechanism is mediated by twisted RR 4-forms and the corresponding symmetry is linearly realized. In the latter, instead, anomalies are canceled by a GS mechanism mediated by twisted RR axions, and the symmetry is realized only non-linearly. When applied to a $U(1)$ factor with an anomaly that is globally but not locally vanishing, these two situations lead respectively to a massless and massive 4D photon[3]. This leads to the important conclusion that the number of spontaneously broken $U(1)$ gauge factors is in general *greater* than what is expected from a global analysis of anomalies. This fact, which has not been appreciated so far in the literature, could have an important impact in the context of open string phenomenology. The difference between the two mechanisms involving axions and 4-forms is particularly striking from a 4D low-energy effective field theory point of view, where heavy KK modes are integrated out. The axions remain dynamical, but the 4-forms must be integrated out, and we will show that their net effect then amounts to a local 6D Chern–Simons counterterm with a discontinuous coefficient, jumping at the fixed points; this counterterm thus occurs in a way that is manifestly compatible with local supersymmetry and falls in the category of terms discussed in [81] (see also [82]). This realizes a 6D version of the possibility of canceling

[2]The global cancellation of RR tadpoles ensures only the global cancellation of cubic irreducible anomalies; see e.g. [78].

[3]As in the standard case [35], a pseudo-anomalous photon can become massive by eating an axion through a Higgs mechanism.

globally vanishing anomalies through a dynamically generated Chern–Simons term [36]. It also confirms in a string context that operators that are odd under the orbifold projection can and do in general occur in the 4D effective theory with odd coefficients, as emphasized in [83].

In chapter 4, based on the original results of [84], we undertake the task of studying the quantum stability of non-SUSY string models, with particular attention for the freely acting orbifolds introduced in chapter 2. Along the lines of [18, 30], we compute the one-loop induced vacuum energy density (cosmological constant) as a function of the radius of the twisted direction. This is obtained by computing the one-loop partition function on the relevant world-sheet surfaces: the torus for IIB orbifolds, and the torus, Klein bottle, annulus and Möbius strip for IIB orientifolds.

The first model we consider is a simple 9-dimensional (9D) $\mathbf{Z}_2$ IIB orientifold, whose vacuum energy density has already been considered in [30]. In agreement with [30], we show how the cosmological constant of this model crucially depends on the choice made for the $\mathbf{Z}_2$ Chan-Paton twist matrix γ_g. Depending on γ_g, the model evolves either toward the tachyonic regime or to the decompactification limit (see figure 4.1).

Orbifolds and orientifolds on twisted Asymptotically Locally Euclidean (ALE) spaces of the form $(\mathbb{C}\times S^1)/\mathbf{Z}_N$, where $\mathbf{Z}_N$ (N odd) acts simultaneously as a rotation on $\mathbb{C}$ and a translation on S^1, are studied next. Such spaces upon reduction on S^1 give rise to Melvin backgrounds [37], and have recently received renewed interest. We show that for any N odd, with or without open strings, the vacuum energy density $\rho^N(R)$ is a negative monotonic function, reaching zero as $R \to \infty$. In the orbifold models we find that $\rho^N(R) > \rho^M(R)$ for $N < M$, for any value of R, and for the lower $N, M = 3, 5, 7, 9$ considered (see figure 4.2). In the orientifold models, with a suitable choice of twist matrices, we find the same behavior (see figure 4.3). In both cases, then, it is reasonable to assume that $\rho^N(R) > \rho^M(R)$ for $N < M$, $\forall N, M$. The perturbative quantum fate of these models is then clear. For any initial value of R and N, R will shrink until $R = R_T$, the critical radius where some twisted string mode becomes tachyonic. The dynamics is now governed by the classical tachyonic instability and, according to the analysis of [85, 86, 87], the model will undergo a phase transition toward another $(\mathbb{C}\times S^1)/\mathbf{Z}'_N$ model, with $N' < N$, with R increasing along the transition [87]. The $\mathbf{Z}_{N'}$ model will also undergo a phase transition and so on, until the twist vanishes and a flat SUSY space-time is recovered. Quite interestingly, we get exactly the same energy density $\rho^N(R)$ for non-compact $(\mathbb{C} \times S^1)/\mathbf{Z}_N$ or compact $(T^2 \times S^1)/\mathbf{Z}_N$ models[4], although the fate of the two twisted directions seem to crucially depend on whether they are compact or not [85, 88].

The last model we consider is the 4D chiral orientifold model, discussed in chapter

[4]In the compact case, N has to be odd and has to preserve the lattice structure of the torus.

2. We first study ρ as a function of the single modulus R, fixing the remaining 5 other compactified directions. For a proper choice of the Chan-Paton twist matrix associated to the $\mathbf{Z}_2'$ element in the $D9$ sector, we find a minimum of $\rho(R)$ at a finite value R_0, where $\rho(R_0) < 0$, providing in this way an interesting non-trivial stabilization mechanism for the SS direction R (see figure 4.4). In order to decide whether this minimum is actually an absolute minimum or not, we also study the dependence of R on some other compact directions, taken to be equal for simplicity. We are not able to give a definite answer to whether R_0 is an absolute minimum or not, but our analysis seems to suggest that for finite values of the remaining compact directions, the minimum is always present, but it could run-away to infinity or toward the tachyonic instability due to the dynamics along the other directions (see figure 4.5).

An interesting by-product of our studies is provided by understanding which string states contribute to $\rho(R)$. This is done by generalizing the known technique of unfolding the fundamental domain of the torus [89, 90], in such a way that it can be computed as an amplitude over the strip $\sigma \in [-\pi, \pi]$, $\tau > 0$. The technique is developed in appendix, mainly based on the original results of [91]. It is studied in the most general case of a closed loop amplitude and then specialized to the case of freely acting orbifold. The final result, in this case, is that the closed string contribution to $\rho(R)$, in both orbifold and orientifolds, can be analytically computed and is given by untwisted closed strings only, where no winding modes along the SS direction appear. As far as the R dependence is concerned, the whole one-loop string partition function looks effectively like that of a purely quantum field theory. This provides a generalization of what is well-known to happen to strings at finite temperature [89, 92] and to $\mathbf{Z}_2$ SS twists [93]. For open strings, it is important to distinguish between longitudinal and transverse SS breaking, depending on whether open strings can propagate or not along the SS direction [26].

In the last chapter we introduce the subject of the compactification of string theory over backgrounds with non-trivial fluxes. In a review section we show the main results of [43], where the it has been discovered that a torsionfull background is admissible if the spacetime metric has a warp factor proportional to the dilaton and the internal space is no more Kähler.

In a following section we introduce the relation between freely-acting orbifolds and a background very similar to Melvin background. Then we show the action of a T-duality along the SS dimensions. We observe that the duality is acceptable in the case in which the dimensions where the SS operator acts as a rotation are non-compact. In the compact case the dimension along which we are going to compactify is a Killing vector for the metric but is present in a non-trivial way in the periodicity conditions of the compact dimensions. This seems to be a "global" obstruction to the

T-duality and the compact case is, at this stage, discarded.

Finally we collect the original results of [94]. We show the duality for a (4D) N=2 heterotic model, built compactifying heterotic string theory on $S^2 \times \mathbb{C}^2/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ operator acts as a reflection in the extended dimensions and as a translation in one of the two circles. We compute the related dual model, building an explicit example of a background with torsion satisfying the requirements imposed in [43]. A detailed study of the background geometry is performed, to classify the background according to [66].

Chapter 2

Orbifold compactifications

Given a smooth manifold M and a discrete symmetry group G with action on M, the space O obtained modding out G from M, is called *"Orbifold"*

$$\mathrm{O} = \mathrm{M}/\mathrm{G}. \tag{2.1}$$

If M is a flat n-dimensional compact manifold and G acts non-freely on it, the resulting orbifold is locally flat with singularities at the fixed points of the action of G. The presence of these singularities modifies the topology of the original space depending on the form of G.

Quantization of string theory on an orbifold is always possible since, being locally flat, it satisfies trivially the equation of motion of the low-energy SUGRA away from the singularities. On the other hand, the properties of the obtained model, bound to the topology of the space, can be tuned by choosing suitably G.

In this chapter we show how it is possible to build a model by quantizing string theory on an orbifold.

2.1 Closed strings

In an orbifold O the symmetry group identifies a a point X with its symmetric X'. The identification has two effects on closed string theory quantized on O. The first is that the string wave function must be invariant under the action of G, and so only invariant states are retained in the spectrum. The second, peculiar of closed strings, is that a string beginning on X and ending on X' is still a closed string. This implies that the closed string sector, originally containing only strings beginning and ending in the same point, must be enlarged to contain also strings beginning in a point and ending on its symmetric under the action of each of the elements g_i of G, namely the g_i-twisted sectors.

These effects have an action essentially on the bosonic degrees of freedom that identify the position of a string; modular invariance and world-sheet SUSY impose

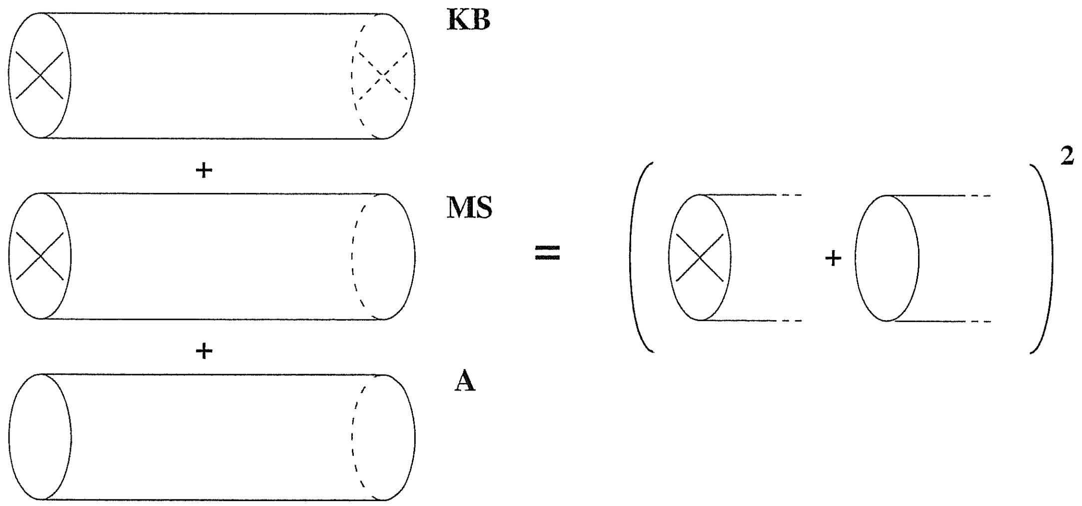

Figure 2.1: *The Klein bottle (KB), Möbius strip (MS) and Annulus (A) amplitudes seen as tree level exchange of closed strings between a crosscap (orientifold) and a crosscap, a crosscap and a brane, a brane and a brane respectively (left side). The sum of the three amplitude can be seen as square of the sum of the crosscap plus brane vertex, times the propagator for the closed state that is the same for all the amplitudes and is dropped (right side) Tadpole cancellation means that the total charge of the extended objects is zero (the sum of the vertexes is zero).*

that they are mapped also on the fermionic degrees of freedom in a suitable way. In the case of type IIB string this is obtained by introducing a projection and a twist on world sheet fermions that is equal to that introduced for bosons, in the heterotic case it is necessary to have an embedding of the action of the orbifold group also on the gauge degrees of freedom. The action is not completely constrained, in particular one of the most common embeddings, known as standard embedding, is obtained taking as many "gauge" fermions as the dimension of $\mathbf{O}$ and requiring that the action on them is the same as the action on the world sheet bosons.

2.2 Orbifolds of type I superstring theory

Type I superstring theory contains closed and open strings. The quantization of the closed sector has already been explained. The open sector case is more subtle, since the open strings carry also gauge degrees of freedom and the orbifold operators must have an action also on them. This action modifies the main condition ensuring the well definition of the model itself, the so called *"tadpole cancellation condition"*. We briefly review here the condition in the simplest case of type I theory on a trivial background, then we see how the condition is modified by the new background and finally we describe how to work out the main properties of a string model.

2.2.1 Tadpole cancellation as a constraint on the open spectrum of type I theory

As known type I string theory contains a closed sector. It is given by the part of type IIB spectrum that is invariant under the action of the world-sheet orientation-reversal operator Ω, having the following action on the world sheet coordinates

$$\Omega : \sigma \rightarrow 2\pi - \sigma,$$
$$\Omega : \tau \rightarrow \tau.$$

If we take into account the closed string loop amplitude without any insertion, namely the partition function, the presence of such a projection is mapped in the insertion in the trace of $(1+\Omega)/2$. The term proportional to 1 is half of the usual torus amplitude, it is modular invariant and the contributions from bosons and fermions are equal to zero. The term with the insertion of Ω, instead, is new, and since Ω acts on the world sheet in a non trivial way, its insertion in the trace modifies it crucially. As shown by [9, 13], the amplitude with the Ω insertion is related to a world-sheet having the shape of a Klein bottle (KB), without edges and without any modular invariance. Furthermore the presence of Ω has also the effect that only states with the same left and right part contribute to the amplitude, so that the trace is independent from the real part of the modular parameter τ. This reflects the fact that the Klein bottle has only one real modular parameter. It is possible to rearrange the form of the world sheet as in [9, 13] and read the amplitude as in fig. (2.1): a cylinder with the two edges replaced by two crosscaps, with the imaginary part of τ being proportional to the radius of the cylinder.

This amplitude is zero by SUSY, but the bosonic and fermionic modes are not zero independently, like in the case of the torus amplitude. Moreover in the new rearrangement the amplitude is the 1-loop one of an open string stretched between the two crosscaps. Open/closed string duality allows us to read it as a tree level closed string exchange between two charged space-time objects called Orientifold planes (O-planes).

The presence of such objects is potentially dangerous, since they are source for closed particles to be originated from the vacuum. They represent an instability of the vacuum itself, a tadpole.

The insertion of an open sector can reabsorb the instabilities, since its presence brings new loop amplitudes in the model. If the open spectrum is chosen in a suitable way, these amplitudes can cancel the KB amplitude and restore the modular invariance originally spoiled by the Ω projection. In another interesting way it is possible to think the Ω projection as a special orbifold of type IIB string theory, the open strings being the Ω-twisted sector.

Also the open loop amplitude contains a projection $(1+\Omega)/2$. The term proportional to 1 is the usual annulus/cylinder (A) term (see 2.1), while the other term can be rearranged obtaining the so called Möbius Strip (MS) amplitude. This last manifold can be described as a cylinder with one edge replaced by a crosscap as in fig. (2.1).

At this stage it is easy to interpret the various amplitudes as an exchange of closed strings between two crosscap (KB), two D-branes (A) a D-brane and a crosscap (MS), and recognize in the tadpole cancellation the cancellation of the total charge between the O-planes and the D-branes.

To obtain the conditions on the open spectrum it is necessary to introduce explicitly the various traces

$$\text{Closed}: \quad \text{Tr}_C\left[\frac{1+\Omega}{2}\frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}e^{2\pi i\tau {L_0}^c}e^{-2\pi i\bar{\tau}\tilde{L_0}^c}\right], \tag{2.2}$$

$$\text{Open}: \quad \text{Tr}_O\left[\frac{1+\Omega}{2}\frac{1+(-1)^F}{2}e^{2\pi i t {L_0}^o}\right], \tag{2.3}$$

where F and $\tilde{F}$ are the left and right fermion numbers and the insertion of $(1+(-1)^F)$ is the usual type IIB GSO projection. ${L_0}^c$ and $\tilde{L_0}^c$ are the generators of translations along $\sigma-\tau$ and $\sigma+\tau$ in the world sheet of closed strings respectively, while ${L_0}^o$ is the generator of translations in $\sigma-\tau$, essentially the Hamiltonian, for open strings.

It is not difficult to evaluate the various traces in the case of type I theory, (for a reference see for example [95, 96]), we review briefly the Chan-Paton sector and then we give the result.

The gauge degrees of freedom of a generic open string state can be described as $|a\rangle = |i,j\rangle\lambda^a_{i,j}$, where $\lambda^a_{i,j}$ is a generic real matrix. The indexes i,j refers to the two endpoints of the open string. Since the spectrum contains only Ω-even states, the Ω-action imposes a restriction. Ω acts on the Chan-Paton degrees of freedom in the following way

$$\Omega : \lambda \rightarrow \gamma_\Omega \lambda^T \gamma_\Omega^{-1}. \tag{2.4}$$

The exchange of the string endpoints due to the world-sheet reversal is encoded in the transposition of λ. Furthermore the Ω action on the oscillators is -1 for the massless excitations. Joining these informations it is easy to see that the allowed λ matrices are of the form

$$\lambda = -\gamma_\Omega \lambda^T \gamma_\Omega^{-1}. \tag{2.5}$$

The choice of γ_Ω fixes the maximal gauge group, the usual choice $\gamma_\Omega = I$, compatible with tadpole cancellation condition we are going to discuss, imposes $SO(n)$ as gauge group, n being the dimensions of the original λ matrix.

We can now take in account the amplitude. The Chan-Paton sector for the Möbius Strip is:

$$\sum_{i,j=1}^{n} \langle i,j|\Omega|i,j\rangle = \langle i,j|\gamma_{\Omega j,h}\gamma_{\Omega}^{-1}{}_{i,k}|h,k\rangle = \mathrm{Tr}[\gamma_{\Omega}^{T}\gamma_{\Omega}^{-1}]. \tag{2.6}$$

The Chan-Paton sector for the annulus, where the identity is acting rather than Ω, is instead

$$\sum_{i,j=1}^{n} \langle i,j|I|i,j\rangle = \langle i,j|\gamma_{I i,h}\gamma_{\Omega}^{-1}{}_{j,k}|h,k\rangle = \mathrm{Tr}[\gamma_I]^2, \tag{2.7}$$

where γ_I is the embedding of the identity in the Chan-Paton degrees of freedom and is, usually, the identity matrix.

This concludes the study of the Chan-Paton sector. The study of the zero-modes part is easily done in the case of n non-compact dimensions with the following result

$$Z_{0\ modes} = \frac{v_n}{t^{n/2}} = \frac{V_n}{(4\pi^2\alpha' t)^{n/2}}, \tag{2.8}$$

where V_n is the "volume" of the n-dimensional space and t is the modular parameter. In the case of compact dimensions this term is replaced by a lattice sum, see the Appendix A for further details. The oscillator part is expressed in terms of the well-known Dirichlet theta functions defined as in [10, 13]. Combining the various terms one obtains, in the open channel,

$$\begin{aligned} Z_K &= \frac{v_{10}}{2}\int_0^\infty \frac{dt}{2t^6}\frac{V_8 - S_8}{\eta^8}(2it), \\ Z_A &= \frac{v_{10}}{2^6}\int_0^\infty \frac{dt}{2t^6}\mathrm{Tr}[\gamma_I]^2\frac{V_8 - S_8}{\eta^8}(it), \\ Z_{MS} &= -\frac{v_{10}}{2^6}\int_0^\infty \frac{dt}{2t^6}\mathrm{Tr}[\gamma_\Omega^T\gamma_\Omega^{-1}]\frac{V_8 - S_8}{\eta^8}(it - \frac{1}{2}), \end{aligned} \tag{2.9}$$

where

$$V_{2n} = \frac{\theta_3^n - \theta_4^n}{2\eta^n}, \quad S_{2n} = \frac{\theta_2^n + \theta_1^n}{2\eta^n}. \tag{2.10}$$

The closed massless tadpole contribution, relevant for the cancellation, is obtained passing to the closed channel. This is obtained by a simple S transformation ($\tau \to -1/\tau$)[1] in the A and KB amplitude, while in the MS case the right transformations is $P = TST^2ST$ involving also the operator $T: \tau \to \tau + 1$. In the closed string channel the corresponding modular parameters are $l_A = 1/(2t)$, $l_{MS} = 1/(8t)$ and

[1] τ is the complex modular parameter, $\tau = it$ in the annulus case, $\tau = it - \frac{1}{2}$ in the MS case, $\tau = 2it$ in the KB case.

$l_{KB} = 1/(4t)$. Given the amplitude in the closed channel the massless contribution is obtained by taking the $l \to \infty$ limit. This gives the result

$$Z \sim \left\{32^2 - 64\mathrm{Tr}\left[\gamma_\Omega^T \gamma_\Omega^{-1}\right] + \mathrm{Tr}[\gamma_I]^2\right\}. \tag{2.11}$$

With the choices $\gamma_I = \gamma_\Omega = I$ one obtains

$$Z \sim \left\{32^2 - 64n + n^2\right\}, \tag{2.12}$$

that is equivalently zero only for $n = 32$. This means that in the model there are O9-planes (KB amplitude) with total charge equal to 32 times the charge of a D9-brane. Their contribution can be canceled introducing in the theory exactly $n = 32$ D9-branes, and, since no other constraint is present, that the gauge group can be the maximal one and so $SO(32)$.

2.2.2 Type I theory compactified on orbifolds

Having introduced the concept of orbifold compactification and the condition to have a well defined open string model we can study type I theory compactified on orbifolds.

Let us reduce our study to the case when the compact manifold M is a 6D torus, factorisable in the direct product of three 2D tori, namely $M = T_6 = T_2 \otimes T_2 \otimes T_2$. We take the orbifold group G of finite order N and acting in a diagonal way on the three tori, namely $G \subset SO(2)^3$. We take the three tori to be parametrized through three complex coordinates z_i, $i \in \{1, 2, 3\}$. With this choice the action of the operator g that spans the full group G is identified by three angles

$$g \,:\, z_i \to e^{2\pi i a_i} z_i, \tag{2.13}$$

and can be summarized by the vector $v = (a_1, a_2, a_3)$.

Supersymmetry

The original type IIB theory has $N = 2$ SUSY in ten dimensions. The Ω projection selects only one chirality of the original SUSY parameter Ψ_{10} so that type I theory has $N = 1$ SUSY in ten dimensions.

The question of the presence of supersymmetries in an orbifold of type I theory is mapped in the question of the presence of a 4D spinor Ψ_4 built by reduction from Ψ_{10} that is left invariant by the action of g.

The original $SO(10)$ Lorentz group is splitted in the usual four dimensional Lorentz group times the internal $SO(6)$. The reduction of Ψ_{10} generates 4 four-dimensional spinors of $SO(6)$ weights $w_1 = (+,+,+)$, $w_2 = (-,-,+)$, $w_3 = (-,+,-)$

and $w_4 = (+,-,-)$. The action of g on each of the four spinors is summarized by the phase $\phi_i = \pi w_i \cdot v$. There is some SUSY if

$$\frac{\phi_i \cdot v}{\pi} = 0 \mod 2 \tag{2.14}$$

for some i.

The analysis of the residual supersymmetries is interesting since, on a smooth manifold, it is related to a well defined geometric property: the holonomy group of the manifold itself. In particular a 6-dimensional torsionless manifold preserving exactly $N = 1$ SUSY in 4 dimensions has $SU(3)$ holonomy, and it is a Kähler 3-fold. A space with these properties is known as a Calabi-Yau space. The case of a 4-dimensional manifold preserving $N = 1$ in 6 dimensions is also related to a Calabi-Yau 2-fold. It is possible to show that orbifolds that can be built and preserve the described number of SUSY are singular limits of the corresponding smooth manifold, and inherit also the other topological properties of these manifolds.

Closed sector

The closed sector is easily built following the recipe given before. Essentially it is made of an untwisted part, containing the Ω- and g- invariant part of the original type IIB spectrum, and by $N-2$ twisted sectors, one for each non-trivial element of G. The m-th twisted sector can be built quantizing type IIB string theory with the condition for the world sheet fields

$$\phi(\sigma + 2\pi, \tau) \longrightarrow g^m \phi(\sigma, \tau) \tag{2.15}$$

and taking only the Ω- and g- invariant states.

Also in this case there is a strict relation with geometry: the number of massless closed particles of the various kind is related to the homology of the space under consideration, that can be obtained recognizing the smooth manifold of which the orbifold is the singular limit.

Some more remarks can be made about closed twisted sectors, in particular the analysis of the zero-modes gives a way to understand in which dimensions these states can propagate.

Let us define the torus parameter as usual as a world sheet field $z_i(\sigma,\tau) = X_i(\sigma,\tau)+iY_i(\sigma,\tau)$ defined with the torus periodicity: $z_i(\sigma,\tau) \sim z_i(\sigma,\tau)+R_i(m+n\tilde{\tau}_i)$, where R_i is the radial dimension of the i-th torus and $\tilde{\tau}_i$ its complex structure.

As said the action of g on $z_i(\sigma,\tau)$ is simply a phase shift. Dropping the i index, useless for the moment, and taking into account the zero modes $z(\sigma,\tau) = z_0 + p_\sigma \sigma + p_\tau \tau$, where z_0 and the p's are complex numbers, we obtain that a g-twisted state is defined as

$$z_0 + p_\sigma(\sigma + 2\pi) + p_\tau \tau = e^{2\pi i a}(z_0 + p_\sigma \sigma + p_\tau \tau). \tag{2.16}$$

If the phase is non-trivial the condition is solved only if $p_\sigma = p_\tau = 0$ and if z_0 is a fixed point of the action of g. This simply means that g-twisted states cannot propagate along dimensions where g acts non-trivially and that they are confined in the fixed points of g.

Open sector

The open sector deserves more care, since it is not simply the Ω- and g- invariant part of the original type I open sector, and a new study of tadpole cancellation must be undertaken. This gives additional constraints on the form of the gauge group. Following the procedure discussed previously the following amplitudes must be canceled

$$\text{Closed}: \quad \frac{1}{N}\sum_{m=0}^{N-1} \mathrm{Tr}_C \left[g^m \frac{1+\Omega}{2} \frac{1+(-1)^F}{2} \frac{1+(-1)^{\bar{F}}}{2} e^{2\pi i \tau L_0{}^c} e^{-2\pi i \bar{\tau} \bar{L_0}{}^c} \right], \tag{2.17}$$

$$\text{Open}: \quad \frac{1}{N}\sum_{m=0}^{N-1} \mathrm{Tr}_O \left[g^m \frac{1+\Omega}{2} \frac{1+(-1)^{\bar{F}}}{2} e^{2\pi i t L_0{}^o} \right], \tag{2.18}$$

where the closed trace is performed over untwisted and twisted states. The handling of these formulas has been studied in different contexts, for example in the case of $N = 1$ supersymmetries a good review is [14]. Through the formulas therein it is possible to compute the modification to the open loop-amplitudes introduced before in the most simple case. The passage to the closed string channel is done exactly as before, and so the extraction of the massless contributions. We do not report here the details of the derivation, we only summarize some model-independent feature.

From the form of the KB amplitude we can extract informations on the kind and number of D-branes present, since the KB amplitude gives in a unique way the dimension and number of O-planes present. It is interesting to note that performing an orbifold of type I theory without an action of the orbifold group in the world sheet the amplitude will always contain a term $\mathrm{Tr}_C\left[g^0\Omega\ldots\right]$ that is equal to the term obtained for type I theory. This term is related to a coupling with a 9-dimensional O-plane, that is always present. The dimensionality of this O-plane can be deduced from the fact that in the trace all the directions momenta are switched on, so that the amplitude itself is proportional to the 10D volume and the amplitude must be read as an interaction with a 10D object. It is possible to conclude that models of this kind contain always a D9-brane sector with gauge group being a subgroup of $SO(32)$. The exact subgroup is obtained from other terms in the amplitude. To avoid the presence of such objects it is necessary to introduce a new world-sheet parity reversal operator $\Omega' = \Omega \times g'$ where g' is an operator acting on the target space. Taking for example g' to be equal to a reflection in two compact dimensions, the volume dependence is reduced to the eight dimensions where g' acts trivially and this means that instead of O9-planes O7-planes are present.

From the volume-dependence analysis is also possible to realize that in the case that O9-branes are presents the only other objects that can appear compatible with unbroken SUSY are O5-planes, and only in the case that the orbifold group is even-order. In presence of O7-planes, instead, only O3-planes can be present.

The tadpole cancellation conditions fix the number and kind of D-branes present in the open sector, in the case of interest 32 D9-branes and no D5-branes if the orbifold group is odd-order, 32 D9-branes and 32 D5-branes if the orbifold group is even-order. Moreover the conditions give constraints on the form of γ_Ω and γ_g the matrices that embed the action of Ω and g on the Chan-Paton degrees of freedom. These conditions depends strictly on the form of the orbifold group.

Given the number of D-branes and the matrices γ_Ω and γ_g the open bosonic spectrum is obtained as follows:

99 States The NS spectrum contains only a 10-dim vector that is reduced by the compactification to a 4-dim vector A and 3 complex scalars S_i. The Ω action on these states is the same and is a phase -1, the g action is trivial on the vector and is a phase $\phi_i = v_i$ on the i-th scalar, so the invariant states are those with Chan-Paton matrices satisfying

$$\lambda_A = + \quad \gamma_{g,9}\, \lambda_A\, \gamma_{g,9}^{-1}, \tag{2.19}$$
$$\lambda_A = - \quad \gamma_{\Omega,9}\, \lambda_A^T\, \gamma_{\Omega,9}^{-1}, \tag{2.20}$$
$$\lambda_{S_i} = e^{2\pi a_i}\, \gamma_{g,9}\, \lambda_{S_i}\, \gamma_{g,9}^{-1}, \tag{2.21}$$
$$\lambda_{S_i} = - \quad \gamma_{\Omega,9}\, \lambda_{S_i}^T\, \gamma_{\Omega,9}^{-1}. \tag{2.22}$$

55 States The NS spectrum has the same content as the 99 sector, the only difference being that the scalars related to direction transverse to the brane have the opposite Ω projection due to the boundary conditions that are Dirichlet rather then Neumann. Taken the complex direction 1 to be parallel and the other to be transverse one obtains

$$\lambda_A = + \quad \gamma_{g,5}\, \lambda_A\, \gamma_{g,5}^{-1}, \tag{2.23}$$
$$\lambda_A = - \quad \gamma_{\Omega,5}\, \lambda_A^T\, \gamma_{\Omega,5}^{-1}, \tag{2.24}$$
$$\lambda_{S_i} = e^{2\pi a_i}\, \gamma_{g,5}\, \lambda_{S_i}\, \gamma_{g,5}^{-1}, \tag{2.25}$$
$$\lambda_{S_1} = - \quad \gamma_{\Omega,5}\, \lambda_{S_1}^T\, \gamma_{\Omega,5}^{-1}, \tag{2.26}$$
$$\lambda_{S_{2/3}} = + \quad \gamma_{\Omega,5}\, \lambda_{S_{2/3}}^T\, \gamma_{\Omega,5}^{-1}. \tag{2.27}$$

95 States In this case the Ω projection, exchanging the string endpoints, relates the 95 sector to the 59 one, without any further condition on the λ matrices. The coordinates orthogonal to the 5-brane obey mixed boundary condition, implying half-integer modded creation operators, so that the zero modes are fermions in the internal space and transform under the action of g with a phase $\phi = 2\pi i(a_2 s_2 + a_3 s_3)$ and the invariance condition implies

$$\lambda = e^{\phi}\, \gamma_{g,5}\, \lambda\, \gamma_{g,9}^{-1}. \tag{2.28}$$

2.3 The Scherk-Schwarz mechanism and freely acting orbifolds

The Scherk-Schwarz symmetry breaking mechanism is introduced in a natural way in field theories with some compact dimensions. Twisting the boundary conditions for a field along a compact dimension ensures that the dimensionally-reduced theory shows a breaking of all the symmetries that do not commute with the twist operator, with a scale of breaking proportional to the length of the compact dimension. The mechanism can be implemented also to break SUSY. In this case the twist is performed by mean of an R-symmetry, i.e. by imposing different boundary conditions for bosons and fermions. As an example we can consider a SUSY theory defined on four extended plus one compact dimensions, where the compact dimension has length $2\pi R$. The theory contains a massless fermion and massless scalars. In the most simple formulation of the mechanism we can consider the twist to be simply a phase, in such a way that a given field is identified as

$$\phi(x^\mu, y + 2\pi R) = e^{2\pi i\alpha}\phi(x^\mu, y). \tag{2.29}$$

The usual Kaluza-Klein reduction for ϕ then reads as

$$\phi(x^\mu, y) = \sum_n \tilde{\phi}(x^\mu) e^{2\pi i y(n+\frac{\alpha}{R})}, \tag{2.30}$$

where now the $\tilde{\phi}(x^\mu)$ represent a tower of states of mass $m_n^2 = (n + \alpha/R)^2$. Clearly the reduction is α-dependent and choosing different phases for bosons and fermions means a SUSY breaking in the reduct 4D theory. The scale of breaking is of order $1/R^2$ and, clearly, SUSY is restored in the decompactification limit.

The Scherk-Schwarz mechanism can be implemented in string theory by means of the so-called freely-acting orbifolds. A freely acting orbifold is an orbifold with some of the elements of the orbifold group acting freely, i.e. without fixed points.

It is very useful to explore the possibilities given by the technique by analyzing a simple toy model, where type I theory is compactified on $S^1 \times T^2/\mathbf{Z}_2$. The Z_2 group is generated by $g = \theta\beta$, with β acting as a translation of order two along S^1 and θ as a rotation of 2π in T^2. It is useful to note that θ has as eigenvalue $+1$ for bosonic states -1 for fermionic states, so its action is the same of an operator $(-1)^F$, F being the fermion number.

Supersymmetry Following the procedure described previously it is easy to note that the vector v has only one non zero component and so the final model is clearly non-SUSY.

Closed sector The closed untwisted sector contains the g-invariant states of type I string theory. The action of the translation is read out from the compact momentum excitations. The operator β is a translation operator

$$\beta = e^{2\pi i PR/2} \tag{2.31}$$

where P is the generator of translations along S^1 and R is the radius of S^1. The action on a state $|n\rangle$ of momentum n/R is then diagonal and given by a phase $e^{i\pi m}$. Due to the phase only the even-m bosonic states survives the projection and so the massless bosons are present in the final spectrum. The massless fermions are instead projected out since only the odd-m states survive. In the model, then, SUSY is completely broken with a scale of breaking proportional to $1/R^2$.

The twisted sector is given by the the states that are identified through the action of g. The effects of the translation are non-trivial only on the zero-modes of the S^1 direction. Called $X(\sigma, \tau)$ the compact S^1 dimension, its zero modes are given by

$$X(\sigma, \tau) = x_0 + p_\sigma \sigma + p_\tau \tau. \tag{2.32}$$

The fact that the S^1 is compact usually means that $X(\sigma, \tau) \sim X(\sigma, \tau) + 2m\pi R$, so the usual boundary condition is

$$x_0 + p_\sigma(\sigma + 2\pi) + p_\tau \tau = x_0 + p_\sigma \sigma + p_\tau \tau + 2m\pi R, \tag{2.33}$$

and implies that $p_\sigma = mR$ are the usual winding modes. A g-twist, instead, implies

$$x_0 + p_\sigma(\sigma + 2\pi) + p_\tau \tau = x_0 + p_\sigma \sigma + p_\tau \tau + 2m\pi R + \pi R, \tag{2.34}$$

where the last term is exactly the effect of translation, and the winding modes are shifted to $p_\sigma = (m + 1/2)R$. This implies that all the twisted states receive a mass shift. This is clearly understood in the philosophy of a twisted state that closes on itself after a twist. If the twist is a translation of length submultiple of the lattice length this means a minimal non-zero length for the string, as shown in fig.(2.2)

The absence of fixed points in the action of the translation ensures that these states can propagate also along the dimensions in which they are twisted, unlike the states that are twisted under the action of a rotation or, as we will see, a rototranslation. The effect of a rotation-twist, or $(-1)^F$, can be read by modular invariance by the action of the same operator in a trace, and it is a reversion of the usual GSO projection. The total effect on fermions is then an inversion of the chirality and a mass shift, so we can conclude that the theory do not contain any massless fermion.

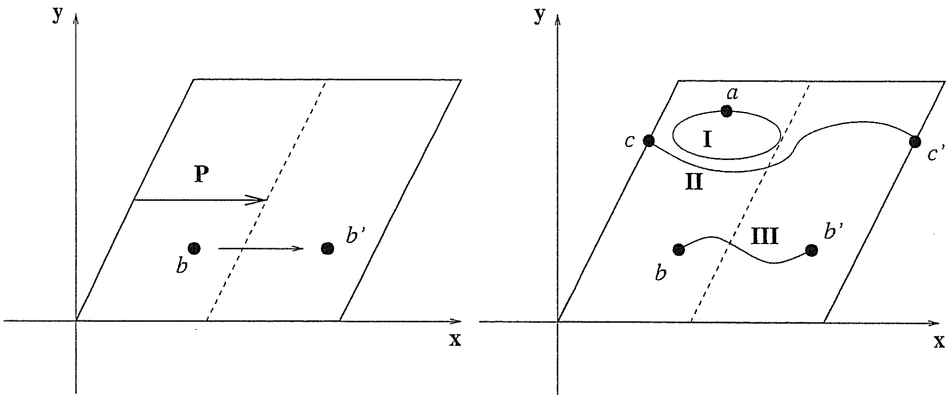

Figure 2.2: *In the picture, on the left side, the action of an order two translation P along the coordinate X is shown, and how a point "b" is mapped under the action of the translation. On the right side, curve I represents an untwisted closed string unwrapped along the torus and so having winding number 0 along all the dimensions, curve II represents an untwisted closed string wrapped one time along the X dimension, while curve III represents a P-twisted closed string, beginning at b and ending in its symmetric b'. As it is easy to see it is wrapped 1/2 times along the X dimension.*

The action of the reversion of the GSO on bosons, instead, project away all the massless states and produces a potential problem because the closed string tachyon is reintroduced in the spectrum. This could be a problem and is a quite usual effect of a mechanism that breaks completely the SUSY of a model. In absence of translation the simple action of the rotation, in fact, would map the original stable theory to the well known type 0 theory. The presence of a translation, instead, ensures a mass shift that makes the tachyon a massive particle for R sufficiently high.

Open sector To analyze the open sector it is necessary to rederive the tadpole cancellation conditions. It is not difficult to note that all the insertions/twists of g, due to the action on the lattice, produce tadpoles for massive fields, while the only massless tadpole to be canceled is exactly 1/2 of the one computed in the case of type I theory. There are no new conditions, and we are free to choose γ_g to be the identity, so that the spectrum is made of the g-invariant states of type I theory. As in the closed case all the even-m bosons survive the projection, and so the model has the same massless content as before, while only odd-m fermions survive and so there are no massless fermions.

In the most general case we introduce a freely-acting group generated, as in the toy model, by an operator acting at the same time as a translation along some compact directions and as a rotation along some other directions. All the computations can be done as in the case of ordinary orbifold, since the only novelty, the translation,

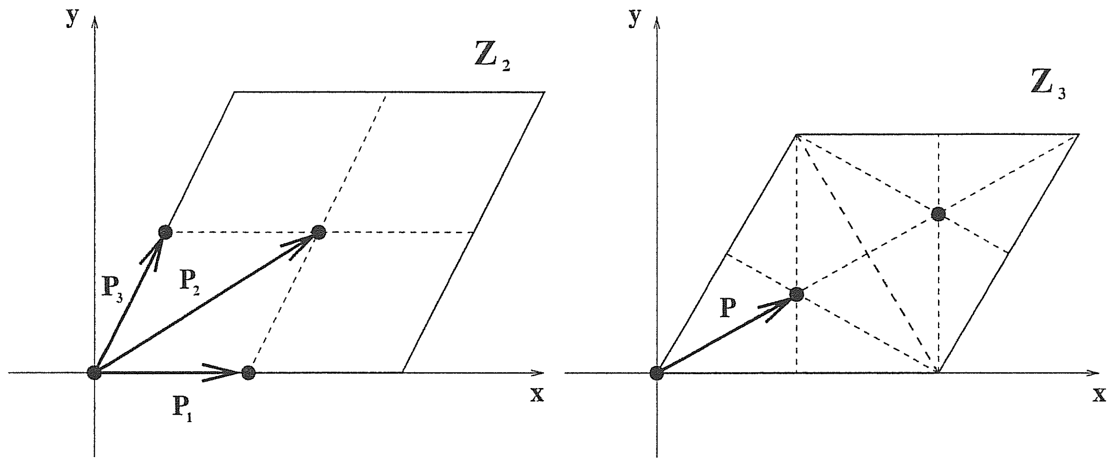

Figure 2.3: *In the picture, on the left side, the finite order translation that commute with an order two rotation. The bullets are placed on the fixed points of the rotation, while the three possible translation are given by the translation along the X direction only, P_1, the translation along the Y direction only, P_2, and the diagonal translation. All of them are of order two. On the right side the same construction in the order three rotation case. Due to the action of the rotation the torus complex structure is fixed to be $Exp[i\pi/3]$ and the torus itself is given by the sum of two equilateral triangles, the rotation acting by rotating each of them by $2\pi/3$ around its center. The fixed points are obviously the centers of the triangles and the zero point. Due to this the only possible translation, namely P, is of order three and is diagonal.*

can always be treated separately and its action is non-trivial only on the zero-modes. In this way the only novelty is a shift in the lattice sums, as shown previously and as summarized in Appendix A.

2.4 Chiral four dimensional models

We have introduced two ways to compactify open string theory breaking some of the supersymmetries. Even though they are essentially obtained in the same way, i.e. compactifying on an orbifold, they have crucial differences in their effects that makes them complementary on the way of building a suitable phenomenologically appealing model. In particular a compactification on a non-freely acting orbifold is dangerous when it breaks SUSY completely, since it introduces tachyons in the spectrum. On the other hand it offer a way of breaking to $N = 1$ SUSY in 4 dimensions without any scale of breaking and with the possibility of a chiral spectrum, that is crucial for phenomenology. Reference [14] is a good review of all the models that can be built in this way.

Since the low-energy physics is non-supersymmetric we need also a mechanism ensuring a complete SUSY breaking, but with a suitable energy scale. A compactification over a freely-acting orbifold gives us the possibility of doing this.

In the next sections we will discuss some examples of orbifold models where both the mechanisms are used to obtain a chiral spectrum and a complete SUSY breaking with a given scale of breaking λ, in such a way that $N = 1$ SUSY is restored at high energy.

The models we are going to build are compactification of type I theory on $T^6/Z_N \times Z'_M$, where Z_N an orbifold group that ensures an $N = 1$ SUSY breaking and Z'_M is the freely acting group ensuring the complete SUSY breaking. It is easy to see from eq. (2.14) that recognize which are the supersymmetries that survive the projection, that to have $N = 1$ is necessary that the rotation operator acts in all the compact dimensions, in particular also in the dimensions where the translation act. Moreover, the freely acting operator must contain, together with the translation, a non-susy rotation in some dimension. In the case of Z_2 this is essentially a $(-1)^F$ operator but in other cases it must be a geometric non-trivial operation. Under these conditions it is clear that different kind of operators can act in the same space, and it is necessary to understand if there are constraint on their action.

We require the full group to be Abelian. Since the rotations are in a discrete subgroup of $SO(2)^3$ their combined action is always Abelian. The only point, so, is the matching between the translation and the rotation. The two actions commute only if translation maps fixed point of rotation in fixed points. On the other hand we demand that the translation is of order greater than 1, and so we require a rotation having more than one fixed point. This is obtained for non-freely acting orbifold elements acting as an order two or three rotation in the torus where the translation acts. Matching an order two rotation with translation means that the translation must be of order two, in the "three" case the rotation must be diagonal and of order three. This is summarized in fig (2.3).

All the other orders are excluded. This gives a stringent bound on the range of orbifold that can be taken in exam. In the next sections we show an order two and an order three example.

We also note that the possibility of performing in the same time a rotation and a translation includes in the closed spectrum rototransled-twisted sectors. Since a rototranslation is not a free operation it has fixed points, the twisted modes are fixed to live in the fixed points and all the windings must be integer. This means that there is no mass shift due to the absence of the zero-winding mode. On the other hand the fact that the freely acting orbifold acts also as a rotation in another torus makes the new sector to be tachyon free and the model to be stable. These new sectors are bound to live on the fixed points of the rototranslation and have reversed GSO with respect to the usual roto-twisted sectors.

2.4.1 The $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model

The $\mathbf{Z}_6' \times \mathbf{Z}_2'$ orientifold of [27] is obtained by applying a SUSY-breaking $\mathbf{Z}_2'$ projection to the SUSY $\mathbf{Z}_6'$ model of [14]. The $\mathbf{Z}_6'$ group is generated by θ, acting as rotations of angles $2\pi v_i^\theta$ in the three internal tori T_i^2 $(i = 1, 2, 3)$, with $v_i^\theta = 1/6(1, -3, 2)$. The $\mathbf{Z}_2'$ group is exactly that described in the previous toy model, and is generated by β, acting as a translation of length πR along one of the radii of T_2^2 (that we shall call SS direction in the following), combined with a sign $(-)^F$, where F is the 4D space-time fermion number. Beside the $O9$-plane, the model contains $O5$-planes at $y = 0$ and $y = \pi R$ along the SS direction (as the corresponding SUSY model [14]) and $\bar{O}5$-planes at $y = \pi R/2$ and $y = 3\pi R/2$ along the SS direction (see Figs. 2.4 and 2.5), corresponding to the two elements of order 2, θ^3 and $\theta^3\beta$. These last objects are a novelty and are related to the rototransled twisted states, that are bound to live in the fixed points of the rototranslation and have opposite GSO projection. In order to cancel both NSNS and RR massless tadpoles, $D9$, $D5$ and $\bar{D}5$-branes must be introduced.

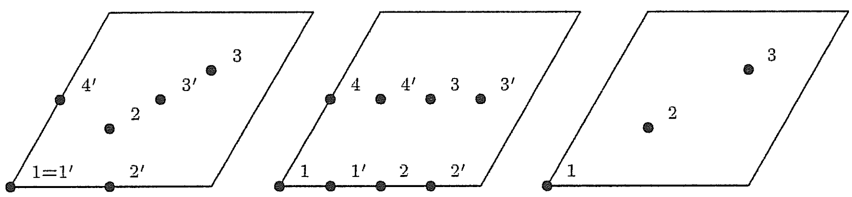

Figure 2.4: *The fixed-points structure in the $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model. We label the 12 θ fixed points with P_{1bc} and the 12 $\theta\beta$ fixed points with $P_{1bc'}$, each index referring to a T^2, ordered as in the figure. Similarly, we denote with $P_{a\bullet c}$ the 9 θ^2 fixed planes filling the second T^2, and respectively with $P_{a'b\bullet}$ and $P_{a'b'\bullet}$ the 16 θ^3 fixed and $\theta^3\beta$ fixed planes filling the third T^2. The 32 $D5$-branes and the 32 $\bar{D}5$-branes are located at point 1 in the first T^2, fill the third T^2, and sit at the points* 1 *and* 1′ *respectively in the second T^2.*

Closed string spectrum

The main features of the closed string spectrum of the $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model can be deduced from those of the $\mathbf{Z}_6'$ model, which can be found in [14]. The only SUSY-breaking generators are β, $\theta^2\beta$ and $\theta^4\beta$; all the other elements preserve some SUSY (generically different from sector to sector). The $\mathbf{Z}_2'$ projection acts therefore in a SUSY-breaking way in the untwisted and $\theta^{2,4}$ twisted sectors, and in a SUSY-preserving way in the remaining $\theta^{1,5}$ and θ^3 twisted sectors of the $\mathbf{Z}'_6$ model. In addition, we must consider the new $\theta^k\beta$ twisted sectors.

Consider first the θ^k sectors already present in the $\mathbf{Z}'_6$ model. In the untwisted sector, one gets a gravitational multiplet and 5 chiral multiplets of $N=1$ SUSY, and the $\mathbf{Z}'_2$ projection eliminates all the fermions. In the $\theta^{2,4}$ twisted sectors, one gets 9 hypermultiplets of $N = 2$ SUSY, and the $\mathbf{Z}'_2$ projection again eliminates all the fermions. Finally, the $\theta^{1,5}$ and θ^3 twisted sectors give each 12 chiral multiplets of $N=1$ SUSY, and the $\mathbf{Z}'_2$ action reduces this number to 6, since it identifies sectors at fixed points that differ by a πR shift in the position along the SS direction.

Consider next the new $\theta^k\beta$ sectors emerging in the $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ model. The β twisted sector yields one real would-be tachyon of mass $\alpha' m^2 = -2 + R^2/(2\alpha')$. Similarly, the $\theta^2\beta$ sectors yield 6 complex would-be tachyons of mass $\alpha' m^2 = -2/3 + R^2/(2\alpha')^2$. Finally, the $\theta^{1,5}\beta$ twisted and $\theta^3\beta$ twisted sectors each give 6 chiral multiplets of $N=1'$ SUSY, which have opposite chirality and a different unbroken SUSY compared to those arising in the $\theta^{1,5}$ and θ^3-sectors, because β changes the GSO projection due to the $(-1)^F$ operation that it involves.

The closed string spectrum that we have just derived is summarized for convenience in Table 2.1.

Tadpole cancellation

As mentioned above, the Ω-projection in the closed string sector introduces $O9$, $O5$, $\bar{O}5$ planes and hence a non-vanishing number n_9, n_5 and $n_{\bar{5}}$ of $D9$, $D5$ and $\bar{D}5$ branes is needed to cancel all massless tadpoles.

The computation of the partition functions on the annulus (A), Möbius strip (M) and Klein bottle (K) surfaces and the extraction of the tadpoles can be done along the lines of the previous sections. The only novelty with respect to the $\mathbf{Z}'_6$ model are the non-SUSY sectors that arise when the $\mathbf{Z}'_2$ generator β enters as twist or insertion in the trace defining the partition function. The corresponding contributions to the partition functions can be easily deduced from their analogues in the $\mathbf{Z}'_6$ model. In the K amplitude, owing to the presence of Ω, the insertion of β acts only in the lattice contribution, as reported in eq. (A.13). As a twist, β inverts the GSO projection and acts in the lattice. This implies that the β twisted contribution, after the S modular transformation, will be the same as the SUSY untwisted sector contribution, but proportional to $(1_{NSNS} + 1_{RR})$ instead of the usual $(1_{NSNS} - 1_{RR})$, and with some terms dropped due to the vanishing of the lattice contribution as in (A.13). This represents a non-vanishing tadpole for the untwisted RR six-form, and reflects the

[2]These are clearly the lightest would-be tachyons in both the β and $\theta^2\beta$ twisted sectors, but it should be recalled that there is actually an infinite tower of such states, with increasing winding mode.

presence of $\bar{O}5$-planes (beside $O5$-planes) in this model. On the A and M surfaces, the insertion of β acts in the lattices as discussed in Appendix A. Apart from that, it simply reverts the R contribution to the partition function. This simple sign flip has, however, different consequences in the two surfaces when analyzing the closed string channel, because of the two different modular transformations (S and P) that are involved. For the M amplitude, the result is obtained from its SUSY analogue by replacing the factor $(1_{NSNS} - 1_{RR})$ by $(1_{NSNS} + 1_{RR})$, and has a clear interpretation as D-branes/$\bar{O}$-planes and $\bar{D}$-branes/O-planes interactions. For the A amplitude, the action of β in the closed string channel reverses the GSO projection and, depending on which $\mathbf{Z}_6'$ generator is inserted (and which boundary conditions are considered), this can lead to an exchange of would-be tachyons.

The group action on the Chan–Paton degrees of freedom is encoded in the twist matrices γ and δ, respectively for the $\mathbf{Z}_6'$ and $\mathbf{Z}_2'$ generators. The group algebra, as usual, allows us to write the Chan–Paton contribution of a M amplitude with the insertion of $\theta^n\beta^m$ as $\pm\mathbf{Tr}(\gamma^n\delta^m)^2$, the freedom of sign being fixed by tadpole cancellation and by the relative action of Ω on 5 and 9 branes, as studied by Gimon and Polchinski (see for example [13, 14]). Tadpole cancellation and the Ω action fix $\gamma^6 = -I$ in the 9, 5 and $\bar{5}$ sector, as in [14], and $\delta^2 = -I$ in the 5 and $\bar{5}$ sectors, with the further condition $\{\gamma, \delta\} = 0$. We also impose $\delta^2 = I$ in the 9 sector; the case $\delta^2 = -I$ will be considered later on. To be fully general, we will use, for the twist matrices γ in the 5 and $\bar{5}$ sectors, an extra index that distinguishes between distinct θ^k fixed points (or fixed planes). Similarly, an other extra index is needed also for the matrices $\gamma\delta$ in the 5 and $\bar{5}$ sectors, running over the $\theta^k\beta$ fixed points.

The final form of the massless tadpoles is most conveniently presented by distinguishing the two closed string sectors with a sign η equal to $+1$ for the NSNS sector and -1 for the RR sector. The result is given by $v_4/12\int dl$ times

$$I: \frac{v_1v_2v_3}{8}\,\eta\left[2^5 - n_9\right]^2 + \frac{v_3}{8v_1v_2}\,\eta\left[2^6\delta_{\eta,1} - n_5 - \eta\, n_{\bar{5}}\right]^2 , \tag{2.35}$$

$$\theta: \frac{\sqrt{3}}{6}\sum_{c=1}^{3}\sum_{b=1}^{4}\eta\left[2^{-1}\,\mathbf{Tr}\,\gamma_9 - \mathbf{Tr}\,\gamma_{5b} - \eta\,\mathbf{Tr}\,\gamma_{\bar{5}b}\right]^2 , \tag{2.36}$$

$$\theta\beta: \frac{\sqrt{3}}{6}\sum_{c=1}^{3}\sum_{b'=1}^{4}\eta\left[2^{-1}\,\mathbf{Tr}\,\gamma_9\delta_9 - \eta\,\mathbf{Tr}\,\gamma_{5b'}\delta_5 - \mathbf{Tr}\,\gamma_{\bar{5}b'}\delta_{\bar{5}}\right]^2 , \tag{2.37}$$

$$\theta^2: \frac{1}{4v_2}\sum_{a,c=1}^{3}\eta\left[2^4\,\delta_{a,1}\,\delta_{\eta,1} + \mathbf{Tr}\,\gamma_{5ac}^2 + \eta\,\mathbf{Tr}\,\gamma_{\bar{5}ac}^2\right]^2 + \frac{v_2}{12}\sum_{a,b=1}^{3}\eta\left[2^3 + \mathbf{Tr}\,\gamma_9^2\right]^2 \tag{2.38}$$

$$\theta^3: v_3\sum_{a',b=1}^{4}\eta\left[2^{-2}\,\mathbf{Tr}\,\gamma_9^3 + \mathbf{Tr}\,\gamma_{5b}^3 + \eta\,\mathbf{Tr}\,\gamma_{\bar{5}b}^3\right]^2 , \tag{2.39}$$

$$\theta^3\beta: v_3\sum_{a',b'=1}^{4}\eta\left[2^{-2}\,\mathbf{Tr}\,\gamma_9^3\delta_9 + \eta\,\mathbf{Tr}\,\gamma_{5b'}^3\delta_5 + \mathbf{Tr}\,\gamma_{\bar{5}b'}^3\delta_{\bar{5}}\right]^2 , \tag{2.40}$$

where we denoted by $\theta^n\beta^m$ the tadpole contribution of the $\theta^n\beta^m$ twisted closed string states, summed over the various fixed points or planes; for convenience we have taken the sums in eqs. (2.36), (2.37), (2.39) and (2.40) to run over closed string twisted states and their images under some orbifold elements. Moreover, $v_4 = V_4/(4\pi^2\alpha')^2$, $v_i = V_i/(4\pi^2\alpha')$ $(i = 1,2,3)$, with V_4 being the volume of the four-dimensional space-time and V_i the volume of the T_i^2.

The NSNS and RR tadpoles differ mainly through relative signs between the contribution from $D5$ and $\bar{D}5$ branes. In addition, there are cross-cap contributions to the NSNS tadpoles in the I and θ^2 sectors that have no analogue in the RR sector (the terms involving $\delta_{\eta,1}$).

We also report the lightest massive NSNS tadpoles, where would-be tachyons can develop:

$$\beta : \frac{v_1v_2v_3}{64}\, q^{-\frac{1}{2}}\hat{\Lambda}\left(\frac{1}{2}\right)[\mathbf{Tr}\,\delta_9]^2 \ , \tag{2.41}$$

$$\theta^2\beta : \frac{v_2}{24}\, q^{-\frac{1}{6}}\hat{\Lambda}\left(\frac{1}{2}\right)\sum_{a,b=1}^{3}\left[\mathbf{Tr}\,\gamma_9^2\delta_9\right]^2 \ . \tag{2.42}$$

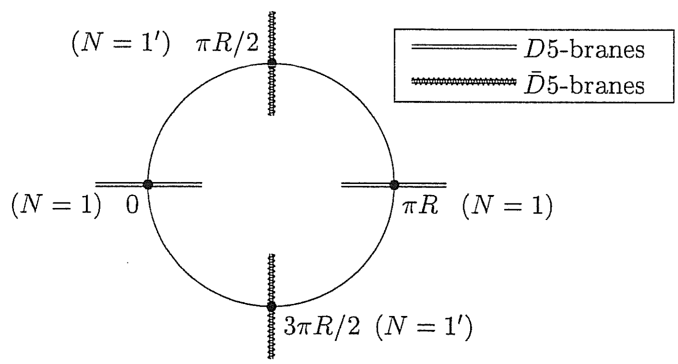

Figure 2.5: *Brane positions along the SS direction for the $\mathbf{Z}_6'\times\mathbf{Z}_2'$ model. The different supersymmetries left unbroken at the massless level in the 55 and $\overline{55}$ sectors are also indicated.*

Open string spectrum

We now turn to the determination of a solution for the Chan–Paton matrices satisfying the above conditions for the global cancellation of massless tadpoles, eqs. (2.35)–(2.40). For simplicity, we consider the case of maximal unbroken gauge symmetry where all $D5$ and $\bar{D}5$ branes are located respectively at $P_{11\bullet}$ and $P_{11'\bullet}$ (see Fig. 2.4

and its caption). The $\mathbf{Z}_2'$ projection then requires that an equal number of image branes be located respectively at $P_{12\bullet}$ and $P_{12'\bullet}$. We do not consider the case in which branes and antibranes coincide also along the SS direction, since this configuration is unstable even classically, because of the presence of open string tachyons. On the other hand, fixing the branes at antipodal points along the SS direction allows a metastable configuration without open string tachyons for sufficiently large SS radius.

The untwisted tadpoles imply $n_9 = n_5 = n_{\bar{5}} = 32$, whereas a definite solution of the twisted tadpoles is given by

$$\gamma_9 = \gamma_5 = \gamma_{\bar{5}}\delta_{\bar{5}} = \begin{pmatrix} \gamma_{16} & 0 \\ 0 & -\gamma_{16} \end{pmatrix} ; \tag{2.43}$$

$$\delta_9 = \begin{pmatrix} I_{16} & 0 \\ 0 & I_{16} \end{pmatrix} , \quad \delta_5 = \delta_{\bar{5}} = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix} , \tag{2.44}$$

where ($\phi = \exp(i\pi/6)$):

$$\gamma_{16} = \mathrm{diag}\{\phi I_2, \phi^5 I_2, \phi^3 I_4, \bar{\phi} I_2, \bar{\phi}^5 I_2, \bar{\phi}^3 I_4\} . \tag{2.45}$$

It is easy to verify that with such a choice all massless tadpoles cancel (although (2.41) and (2.42) do not vanish). Notice that the above choice for $\gamma_{9,5}$ coincides with that of [14]. The structure of the twist matrices given in (2.43) and (2.44) reflects our choice for brane positions; in particular, the matrix δ implements the translation β in the Chan–Paton degrees of freedom. Hence, as far as the massless spectrum is concerned, we can effectively restrict our attention to the 16 branes and antibranes at P_{111} and $P_{11'1}$ respectively, and work with 16×16 Chan–Paton matrices.

The massless open string spectrum can now be easily derived, and is summarized in Table 2.2. In the 99 sector, the bosonic spectrum is unaffected by the $\mathbf{Z}_2'$ element and therefore coincides with that of the $\mathbf{Z}_6'$ orbifold[3]; all fermions (both gauginos and charginos) are instead massive. The 55 and $\overline{55}$ sectors are supersymmetric at the massless level, but with respect to different supersymmetries: $N = 1$ and $N = 1'$. The 55 and $\overline{55}$ gauge groups G_5 and $G_{\bar{5}}$ are reduced by the non-trivial action of the translation in these sectors, and the corresponding states are in conjugate representations. A similar reasoning also applies for the 95 and $9\bar{5}$ sectors. Finally, the $5\bar{5}$ sector does not contain massless states, thanks to the separations between $D5$-branes and $\bar{D}5$-branes. There are massive scalars and fermions in the bifundamental of $G_5 \times G_{\bar{5}}$, and charged would-be tachyons of mass $\alpha' m^2 = -1/2 + R^2/(16\alpha')$.

Notice that the above solution of the tadpole cancellation conditions is not unique. In fact, another interesting and more symmetric solution is obtained by choosing δ_9 of the same form as $\delta_5 = \delta_{\bar{5}}$ in (2.44). This solution is not maximal in the sense that the resulting G_9 gauge group is reduced and equal to $G_9 = G_5 \times G_{\bar{5}} = U(4)^2 \times U(2)^4$.

[3]These are as in [75], but differ slightly from [14] and [74].

However, it has the nice feature that now also the tadpoles (2.41) and (2.42) do vanish. Clearly, there exist other solutions, which we do not report here. Note for instance that a non-vanishing twist matrix δ in the 9 sector can be considered as a $\mathbf{Z}_2$ Wilson line along the SS radius. Since δ implements a SS gauge symmetry breaking, this reflects the close interplay between Wilson line symmetry breaking [97] and SS gauge symmetry breaking.

Let us now comment on the brane content of this orbifold. From the tadpoles, we learn that there is no local $\mathbf{Z}_6$ and $\mathbf{Z}_2$ twisted RR charge at all in the model ($\mathbf{Tr}\gamma = \mathbf{Tr}\gamma^3 = 0$), but there is a $\mathbf{Z}_3$-charge, since $\mathbf{Tr}\gamma^2 \neq 0$, that globally cancels between $D9$ and $O9$-planes, and $D5$, $\bar{D}5$, $O5$ and $\bar{O}5$-planes. On a $\mathbf{Z}_6$ orbifold, a regular D-brane must have 5 images. Since we start with 32 branes, it is clear that the branes in this model cannot all be regular. In fact, the presence of a non-vanishing $\mathbf{Z}_3$ RR (and NSNS) charge suggests that fractional $D5$ and $\bar{D}5$ branes are present at $\mathbf{Z}_3$ fixed planes[4] of the orbifold. The configuration is then the following. We have 2 regular $D5$ and $\bar{D}5$ branes (and 5 images for each) and 2 fractional $\mathbf{Z}_3$ $D5$ and $\bar{D}5$ branes (and one $\mathbf{Z}_2$ image for each). In our maximal configuration, they are all located at $P_{11\bullet}$ ($D5$) and $P_{11'\bullet}$ ($\bar{D}5$). Clearly, there are the additional $\mathbf{Z}_2'$ images located at $P_{12\bullet}$ and $P_{12'\bullet}$. Regular branes can move around freely, whereas fractional branes are stuck at the fixed points. However, one can still shift a fractional brane from one fixed point to another, suggesting that this freedom represents the T-dual of discrete Wilson lines in orbifolds. Notice that also $D9$-branes have $\mathbf{Z}_3$ RR charge. Although it is not appropriate to speak about fractional $D9$-branes, this kind of object represents the T-dual version of the usual lower-dimensional fractional branes. In some sense, they are stuck in the gauge bundle, and do not admit continuous Wilson lines, but only discrete ones.

The $\mathbf{Z}_2'$ twist acts trivially in the gauge-bundle of the $D9$-branes, whereas in the 5 sector it is T-dual to a discrete Wilson line given by the matrix δ. More precisely, the breaking of the gauge group in the 5 and $\overline{5}$ sector is the T-dual version of a Wilson line symmetry breaking. The additional $(-)^F$ action is on the other hand responsible for the $D5 \to \bar{D}5$ flip for half of the branes.

2.4.2 A $\mathbf{Z}_3 \times \mathbf{Z}_3'$ model

It has been shown in [22] and reviewed in the previous sections that SS symmetry breaking can be obtained also in $\mathbf{Z}_3$ models through a suitable freely acting and SUSY-breaking $\mathbf{Z}_3'$ projection. In this section, we will construct a new $\mathbf{Z}_3 \times \mathbf{Z}_3'$ model, based on this structure, that will prove to be much simpler than the $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model.

[4]In our case, the $\mathbf{Z}_3$ fixed plane is at the origin. However, seen as $D7$-branes wrapped on vanishing two-cycles [98], these branes wrap only the $\mathbf{Z}_3$ vanishing cycles.

The $\mathbf{Z}_3 \times \mathbf{Z}_3'$ orbifold group is defined in the following way [22]. The $\mathbf{Z}_3$ generator θ acts as a SUSY-preserving rotation with twist $v_i^\theta = 1/3(1,1,0)$, while the $\mathbf{Z}_3'$ generator β acts as a SUSY-breaking rotation with $v_i^\beta = 1/3(0,0,2)$ and an order-three diagonal translation δ in T_1^2.

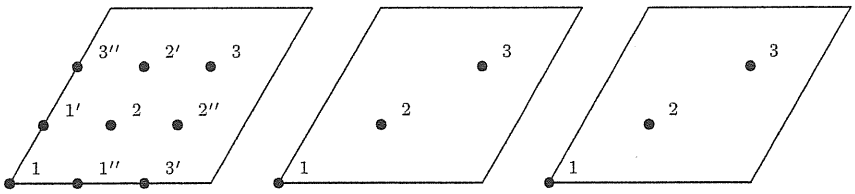

Figure 2.6: *The fixed-point structure in the $\mathbf{Z}_3 \times \mathbf{Z}_3'$ model. We label the 9 θ fixed planes with $P_{ab\bullet}$, the 27 $\theta\beta$ fixed points with $P_{a'bc}$, the 27 $\theta\beta^2$ fixed points with $P_{a''bc}$, and the 3 β fixed planes with $P_{\bullet\bullet c}$.*

Sector	$\mathbf{Z}_6' \times \mathbf{Z}_2'$	$\mathbf{Z}_3 \times \mathbf{Z}_3'$
Untwisted	1 graviton, 5 scalars	1 graviton, 11 scalars, 1+1 spinors
θ twisted	6 chiral multiplets	6 hypermultiplets
θ^2 twisted	18 scalars	–
θ^3 twisted	6 chiral multiplets	–
$\theta\beta$ twisted	6 chiral multiplets	9 chiral multiplets
$\theta^3\beta$ twisted	6 chiral multiplets	–
$\theta\beta^2$ twisted	–	9 chiral multiplets

Table 2.1: *Massless closed string spectrum for $\mathbf{Z}_6' \times \mathbf{Z}_2'$ and $\mathbf{Z}_3 \times \mathbf{Z}_3'$ models. We used θ as the generator of $\mathbf{Z}_6'$ ($\mathbf{Z}_3$) and β as the generator of $\mathbf{Z}_2'$ ($\mathbf{Z}_3'$). Hypermultiplets are multiplets of N=1 SUSY in 6D, while chiral multiplets are multiplets of N=1 SUSY in 4D. The SUSY generators are different in the different sectors, as explained in the text. The two spinors in the untwisted sector of $\mathbf{Z}_3 \times \mathbf{Z}_3'$ have opposite chirality.*

Closed string spectrum

It is convenient to consider first the massless closed string spectrum in the parent Type IIB orbifold, before the Ω projection. In this case, we get an untwisted sector, and both SUSY-preserving and SUSY-breaking twisted sectors.

The untwisted sector contains the 4D space-time part of the NSNS spectrum, i.e. the graviton, the axion and the dilaton; furthermore, there are 10 scalars arising from fields with internal indices in the NSNS sector, 12 scalars from the RR sector and 2 spinors for each chirality from the NSR+RNS sectors.

The twists θ and θ^2 act only on two of the three tori, and their action preserves $N=2$ SUSY in 6D. More precisely, they preserve the supercharges $Q_2^{L,R}$ and $Q_3^{L,R}$ in the 4D notation of [22]. Each twisted sector contains a 6D $N=2$ tensor multiplet. The states are located at the $\mathbf{Z}_3$ fixed points, and are $\mathbf{Z}_3$-invariant, while $\mathbf{Z}_3'$ acts exchanging states from one fixed point to the other, so that in the first torus the three $\mathbf{Z}_3$ fixed points are identified.

The twists $\theta\beta$ and $(\theta\beta)^2$ act instead on all the compact space, preserving two supercharges, $Q_4^{L,R}$. The twisted spectrum contains a 4D $N=2$ hypermultiplet. In these sectors, the elements θ and $\theta\beta^2$ act by exchanging states from one fixed point to the other in the first torus, so that, as before, there is only one physical fixed point in this torus. The $\theta\beta^2$ and $(\theta\beta^2)^2$ twisted sectors can be treated similarly, the only difference being the position of the fixed points and the unbroken supercharges $Q_1^{L,R}$.

The twists β and β^2 are SUSY-breaking, and the corresponding twisted sectors yield each a real would-be tachyon of mass $m^2 = -4/3 + 2T_2/(3\sqrt{3}\alpha')$ (where T_2 is the imaginary part of the Kähler structure of the SS torus) and 16 massive RR 16 scalars. These states are β-invariant and located at β fixed points, and again the remaining elements only switch fixed points.

It is now easy to understand the effect of the Ω projection. In the untwisted sector, Ω removes the axion, half of the NSNS and RR scalars, and half of the fermions. In the twisted sectors, Ω relates Q^L to Q^R and projects away half of the supersymmetries, so that the surviving states fill supermultiplets of $N=1$ SUSY in 4D or 6D. Furthermore, Ω relates the twist $\theta\beta^i$ to $(\theta\beta^i)^2$, and only half of the corresponding states survive the projection. The spectrum is therefore reduced to 2 hypermultiplets of 6D $N=1$ SUSY from θ twists, for each θ fixed point; 1 chiral multiplet of 4D $N=1$ SUSY from $\theta\beta$ twists, for each $\theta\beta$ fixed point, and the same for $\theta\beta^2$ twists; 1 real would-be tachyon and 16 massive scalars from β twists. The massless closed string spectrum is summarized in Table 2.1.

Tadpole cancellation

The computation of the partition functions on the A, M and K amplitudes and the extraction of the tadpoles is again standard. The only novelty occurs in the untwisted sector, with β^n $(n = 1, 2)$ inserted in the trace. In these sectors, the

Model	$\mathbf{Z}'_6 \times \mathbf{Z}'_2$	$\mathbf{Z}_3 \times \mathbf{Z}'_3$
G_9 : $G_5 = G_{\bar{5}}$:	$U(4)^2 \times U(8)$ $U(2)^2 \times U(4)$	$SO(8) \times U(8) \times U(4)$ –
99 scalars	$(\mathbf{4},\mathbf{4},\mathbf{1}),\ (\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{1}),\ (\mathbf{1},\mathbf{1},\mathbf{28}),$ $(\mathbf{1},\mathbf{1},\overline{\mathbf{28}}),\ (\mathbf{6},\mathbf{1},\mathbf{1}),\ (\mathbf{1},\mathbf{4},\bar{\mathbf{8}}),$ $(\bar{\mathbf{4}},\mathbf{1},\mathbf{8}),\ (\mathbf{1},\bar{\mathbf{6}},\mathbf{1}),\ (\bar{\mathbf{4}},\mathbf{4},\mathbf{1}),$ $(\mathbf{4},\mathbf{1},\mathbf{8}),\ (\mathbf{1},\bar{\mathbf{4}},\bar{\mathbf{8}})$	$2(\mathbf{8},\mathbf{8},\mathbf{1}),\ 2(\mathbf{1},\overline{\mathbf{28}},\mathbf{1}),$ $(\mathbf{8},\mathbf{1},\mathbf{4}),\ (\mathbf{1},\mathbf{1},\bar{\mathbf{6}})$
99 fermions	–	$2(\mathbf{8},\mathbf{1},\mathbf{4}),\ 2(\mathbf{1},\mathbf{1},\bar{\mathbf{6}}),$ $(\mathbf{1},\bar{\mathbf{8}},\mathbf{4}),\ (\mathbf{1},\bar{\mathbf{8}},\bar{\mathbf{4}})$
55 chiral mult.	$(\mathbf{2},\mathbf{2},\mathbf{1}),\ (\bar{\mathbf{2}},\bar{\mathbf{2}},\mathbf{1}),\ (\mathbf{1},\mathbf{1},\mathbf{6}),$ $(\mathbf{1},\mathbf{1},\bar{\mathbf{6}}),\ (\mathbf{1}_{\mathbf{A}},\mathbf{1},\mathbf{1}),\ (\mathbf{1},\mathbf{2},\bar{\mathbf{4}}),$ $(\bar{\mathbf{2}},\mathbf{1},\mathbf{4}),\ (\mathbf{1},\bar{\mathbf{1}}_{\mathbf{A}},\mathbf{1}),\ (\bar{\mathbf{2}},\mathbf{2},\mathbf{1}),$ $(\mathbf{2},\mathbf{1},\mathbf{4}),\ (\mathbf{1},\bar{\mathbf{2}},\bar{\mathbf{4}})$	–
95 chiral mult.	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1};\bar{\mathbf{2}},\mathbf{1},\mathbf{1}),\ (\mathbf{1},\mathbf{4},\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{1}),$ $(\mathbf{4},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\bar{\mathbf{4}}),\ (\mathbf{1},\mathbf{1},\bar{\mathbf{8}};\mathbf{2},\mathbf{1},\mathbf{1}),$ $(\mathbf{1},\bar{\mathbf{4}},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{4}),\ (\mathbf{1},\mathbf{1},\mathbf{8};\mathbf{1},\bar{\mathbf{2}},\mathbf{1})$	–

Table 2.2: *Massless open string spectrum for $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ and $\mathbf{Z}_3 \times \mathbf{Z}'_3$ models. In the 55 sector, chiral multiplets in the representation of G_5 are reported. The matter content of the $\overline{55}$ sector is the same as in the* 55 *sector, but in conjugate representations of $G_{\bar{5}} = G_5$. In the 95 sector, chiral multiplets are present in representations of $G_9 \times G_5$. Again, the matter content in the $9\bar{5}$ sector is obtained from that in the 95 sector by conjugation.*

oscillator contribution to the partition function is given by

$$\Theta_n(\tau) = \sum_{a,b=0}^{1/2} \eta_{ab} \frac{\theta\left[{a \atop b}\right]^3(\tau)}{\eta^9(\tau)} \frac{(-2\sin 2\pi n/3)\theta\left[{a \atop b+2n/3}\right](\tau)}{\theta\left[{1/2 \atop 1/2+2n/3}\right](\tau)} , \tag{2.46}$$

and the corresponding partition function on each surface reads:

$$\begin{aligned} Z_A\left[{1 \atop \beta^n}\right] &= \frac{v_4}{2NN'} \int_0^\infty \frac{dt}{64t^3} \sum_m e^{2i\pi n(\delta\cdot m_1)}\, \Lambda_1[m]\, \Lambda_2[m] \Theta_n(it)\, (\mathbf{Tr}\,\delta_n)^2 , \\ Z_M\left[{1 \atop \beta^n}\right] &= -\frac{v_4}{8NN'} \int_0^\infty \frac{dt}{4t^3} \sum_m e^{2i\pi n(\delta\cdot m_1)}\, \Lambda_1[m]\, \Lambda_2[m] \Theta_n\,(it - 1/2)\ \mathbf{Tr}\,\delta_{2n} , \\ Z_K\left[{1 \atop \beta^n}\right] &= \frac{v_4}{2NN'} \int_0^\infty \frac{dt}{4t^3} \sum_m e^{2i\pi n(\delta\cdot m_1)}\, \Lambda_1[m/\sqrt{2}]\, \Lambda_2[m/\sqrt{2}] \Theta_{2n}(2it) , \end{aligned} \tag{2.47}$$

where NN' is the total order of the group (i.e. 9 in our case) and $\Lambda_i[m]$ is the 2D lattice of the i-th torus as defined in (A.3).

The tadpoles for massless closed string modes are easily computed. We skip the explicit form of the usual 10D tadpole, arising in all orientifold models, that fixes to 32 the number of $D9$-branes and requires $\gamma_\Omega^t = \gamma_\Omega$. All other tadpoles are associated to twisted states occurring only at fixed points or fixed planes. We list them here using the already introduced notation. We denote the twist matrices associated to the $\mathbf{Z}_3$ and $\mathbf{Z}_3'$ actions by γ and δ, and we assume $\gamma^3 = \eta_\gamma I$, $\delta^3 = \eta_\delta I$, where $\eta_\gamma, \eta_\delta = \pm 1$. The tadpoles are at the 9 θ fixed planes, the 27 $\theta\beta$ fixed points and the 27 $\theta\beta^2$ fixed points; they are given by $(1_{NSNS} - 1_{RR})v_4/72 \int dl$ times:

$$
\begin{aligned}
\theta &: \frac{v_3}{3} \sum_{a,b} \left[(8 - \eta_\gamma \,\mathbf{Tr}\, \gamma)^2 + (8 - \mathbf{Tr}\, \gamma^2)^2 \right], \\
\theta\beta &: \frac{1}{3\sqrt{3}} \sum_{a',b,c} \left[(4 + \eta_\gamma \eta_\delta \,\mathbf{Tr}\, \gamma\delta)^2 + (4 + \mathbf{Tr}\, \gamma^2\delta^2)^2 \right], \\
\theta\beta^2 &: \frac{1}{3\sqrt{3}} \sum_{a'',b,c} \left[(4 + \eta_\gamma \,\mathbf{Tr}\, \gamma\delta^2)^2 + (4 + \eta_\delta \,\mathbf{Tr}\, \gamma^2\delta)^2 \right]. \qquad (2.48)
\end{aligned}
$$

We wrote explicitly the contributions from the θ^k and θ^{N-k} sectors, arising from the same physical closed string state.

By taking the transverse channel expressions of the amplitudes (2.47) through S and P modular transformations, the additional tadpoles for the non-SUSY β twisted sectors arising at the 3 fixed planes $P_{\bullet\bullet c}$ can be derived. They yield the following result for the massive would-be tachyonic NSNS states:

$$
\begin{aligned}
\frac{v_1 v_2}{4\sqrt{3}}\, q^{-\frac{1}{3}} \sum_c \sum_{m=-\infty}^{\infty} \Bigg\{ & \hat{\Lambda}_1\left(2m + \frac{1}{3}\right) (\mathbf{Tr}\, \delta)^2 + \hat{\Lambda}_1\left(2m - \frac{1}{3}\right) (\mathbf{Tr}\, \delta^2)^2 + \qquad (2.49) \\
& \hat{\Lambda}_1\left(2m - \frac{2}{3}\right) (16 - \mathbf{Tr}\, \delta^2)^2 + \hat{\Lambda}_1\left(2m + \frac{2}{3}\right) (16 - \eta_\delta \,\mathbf{Tr}\, \delta)^2 \Bigg\},
\end{aligned}
$$

where we have retained the lattice sum along the SS directions. These tadpoles are associated to massive states for sufficiently large radii along the SS torus, and are therefore irrelevant in that case. They imply that would-be tachyons and massive RR 7-forms are exchanged between $D9$-branes and/or $O9$-planes. Contrarily to the $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model, there is no choice for the twist matrix δ that makes eq. (2.49) vanish.

Open string spectrum

In the following, we take $\eta_\gamma = \eta_\delta = 1$, because all the other choices lead to equivalent theories. It is then easy to see that the twisted tadpoles (2.48) are canceled by choosing ($\phi = \exp 2i\pi/3$):

$$
\begin{aligned}
\gamma &= \mathrm{diag}(I_{16}, \phi\, I_8, \phi^{-1}\, I_8); \\
\delta &= \mathrm{diag}(\phi\, I_4, \phi^{-1}\, I_4, I_{24}). \qquad (2.50)
\end{aligned}
$$

Notice that the order of the entry in (2.50) is crucial to cancel the tadpoles; the above choice is such that $\gamma\delta^2 = \gamma_\theta$, where γ_θ is the twist matrix of the 4D $N=1$ $\mathbf{Z}_3$ model constructed in [14].

The massless open string spectrum is easily determined. The maximal gauge group is $SO(8) \times U(8) \times U(4)$. The $U(8) \times U(4)$ factor comes from the $U(12)$ gauge factor of the 4D $N=1$ $\mathbf{Z}_3$ model, which is further broken by the $\mathbf{Z}'_3$ projection. As in the previous model, this can be interpreted as a Wilson line symmetry breaking. In this perspective, $\delta = I$ and γ as above, and the tadpoles in (2.48) are canceled thanks to a (discrete) Wilson line W equal to δ along the first torus in (2.50). Notice that all the gauginos are massive. The spectrum of charged massless states is easily obtained and reported in Table 2.2.

Chapter 3

Anomalies in orbifold models

Quantum corrections play an important rôle in quantum field theory. As it is well known they can modify a theory and, in certain cases, spoil properties of crucial importance for the consistence of the theory itself, such as unitarity.

Given a theory with a classical lagrangian symmetric under the action of a certain symmetry group, quantum corrections can spoil the invariance, making the symmetry anomalous. A classical example of this is the axial symmetry breaking in an effective theory describing the interaction of a pseudoscalar (π^0) with the ordinary quantum electrodynamic. Even though both axial and vector current are classically conserved the axial current is anomalous at the quantum level and this ensures the possibility for the pseudoscalar to decay into two fotons [99, 100].

The failure of a global symmetry is in general a property that modifies the phenomenology of a theory without any other effects on the consistency of the theory itself, if instead a gauge symmetry is anomalous then the theory becomes inconsistent. Gauge invariance is crucial for a consistent interpretation of a quantum theory with gauge fields. In particular it ensures that the unphysical degrees of freedom of the gauge field decouple from the physical spectrum, guaranteeing that the S-matrix governing the interactions of the physical particles is unitary. In this sense a gauge anomaly can be computed directly by the inspection of the Feynman graphs having as external legs a longitudinally polarized and transversally polarized gauge bosons. If such an amplitude is non-zero then it can be shown that a gauge anomaly is present and, on the other hand, such an amplitude is exactly a failure of unitarity for the theory.

A quantum theory of gravity is a special realization of a gauge theory, where the gauge group is the Lorentz group. In this sense also quantum gravity can be affected by gauge anomalies, in this case representing a breakdown of general covariance. Since the current related to this symmetry is the energy-momentum tensor such an anomaly corresponds to a failure of the conservation of it.

In [101] Alvarez-Gaumé and Witten have shown an effective way to compute gravitational anomalies showing that it can be done in strict analogy with the computation undertaken to obtain gauge anomalies in ordinary Yang-Mills theories.

Anomalies can arise also in string theory and they represented the most pressing problem affecting it until the discovery of the so-called Green-Schwarz (GS) mechanism [34] in 1986. In particular while closed strings are unaffected by anomalies, in type I and heterotic theory it happens that the spectrum is potentially anomalous. The anomalies in these last cases are canceled through the GS mechanism, that is crucial also in string model building since its generalizations are responsible for anomaly cancellation in all the tadpole-free models built from type I string theory [78].

To give a brief sketch of the mechanism we introduce the anomaly computation for the ten dimensional type I string theory. From a diagrammatic point of view it is given by the open one-loop amplitudes with the insertion of six vertex operators. Open string loop-amplitudes involve not only the usual annulus amplitude, with two edges, but also the MS amplitudes, or unoriented amplitudes, with only one edge, so that we can distinguish between (I) annulus amplitudes with all the vertexes placed on the same edge (oriented planar graphs), (II) annulus amplitude with both the edges supporting some vertex (oriented non-planar graphs) and (III) unoriented amplitudes, as summarized in fig. (3.1). The oriented planar diagram contains a trace on the gauge degrees of freedom with all the vertexes inserted and an "empty" trace containing the identity. The trace is performed in the fundamental representation and so the empty trace contributes with an n in the case of $SO(n)$ gauge group. The unoriented diagram is related to the topology of Möbius strip, (fig.(3.1)), that has only one edge and so there is only one trace, still containing all the vertexes. The other graphs, related as the first one to the annulus amplitude, have always two nonempty traces. The open/closed string duality allows us to consider all these amplitudes as a tree level exchange of closed strings between charged objects (O-planes and D-branes), exactly as in the case of the computations of tadpoles worked out previously, the only difference being the presence of the vertexes. From this point of view the anomaly due to these loop diagrams can be canceled if in the low energy theory some closed string excitation is coupled in an anomalous way to the O-planes/D-branes, in such a way that the open loop amplitude is canceled by suitable closed tree-level amplitudes. Closed tree-level amplitudes cannot cancel all the open string loop-amplitude, in particular the unoriented and the planar graph are an exchange between a D-brane and a D-brane/O-plane, with the final state with 0 particles and the initial one with six particles, and would mean an inconsistency of the theory, so that it is necessary that the sum of these two amplitudes is zero. All the other amplitudes can instead be canceled. The cancellation between the unoriented and the planar graph is ensured by tadpole cancellation condition, with the request that, in type I string case, the gauge group is $SO(32)$.

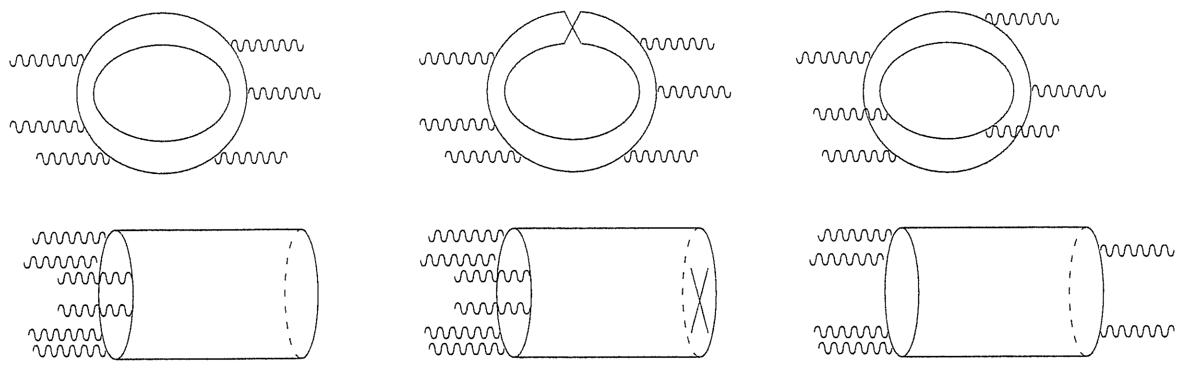

Figure 3.1: *On the left side the contribution to the ten dimensional anomaly due to the planar orientable diagram. In the upper picture the contribution is seen as an open string loop amplitude, while in the lower picture as a closed string exchange between a configuration with six external entering particles and the vacuum (brane). In the center the contribution due to the non-orientable diagram, seen as open and closed string amplitude. This two amplitudes are proportional to the square root of the amplitude computed in fig. 2.1 so that the total contribution is zero if the theory is tadpole-free. On the right side one example of contribution due to the non-planar graph (clearly it is not the only one) both in the open channel (upper picture) and in the closed (lower picture).*

This is a general result: as shown in [78] tadpole cancellation is a sufficient condition ensuring that all the loop amplitudes that cannot be canceled by closed tree-level amplitudes are automatically zero. It is also shown that the condition is not in general a necessary condition.

In the rest of the chapter we show how the anomalies can be computed and which are the correct anomalous couplings ensuring that the various models are anomaly free. In a modern notation the tree-level anomalous amplitudes that guarantee the cancellation are known as *"inflow of anomaly"*.

3.1 Anomalies and inflow in string models

The characterization of anomalies can be performed in a systematic way knowing the strict relation with the topology of the manifold (see for example [96, 102, 103]). It is possible to show that given a theory defined on a D dimensional manifold X the gauge (gravitational) anomaly of a chiral spinor propagating along a submanifold M of X and interacting with a gauge field is related to the index of the spinor complex, namely

$$A = \lim_{L \to \infty} \operatorname{Tr}\left[\Gamma^{D+1} e^{-(i\not{D}_M/L)^2}\right], \qquad (3.1)$$

where the trace is performed over the eigenstates of $\not{D}_M$, the Dirac operator on the submanifold M and Γ^{D+1} is the chiral matrix over X. The index theorem allows us to write the anomaly polynomial using topological quantities related to the shape of the manifold, before of giving the final form is convenient to introduce some notation. Given a gauge theory with gauge (Lie) group G we have the usual field strength as a two-form F that can be decomposed as $\sum_a F^a T^a$ where the F^a's are two-forms acting as scalars on the gauge degrees of freedom and the T^a's are the elements of the Lie algebra of G. We can define, in general, polynomials where a generic term of degree one is given by F, a term of degree two by $F \wedge F$ and so on. In this sense we can also think a power expansion in F and define

$$e^{\frac{iF}{2\pi}} \equiv 1 + \frac{i}{2\pi}F - \frac{1}{2 \cdot (2\pi)^2} F \wedge F + \dots \,. \tag{3.2}$$

Moreover we can introduce, over each term of the polynomial, a trace over a given representation of the gauge group

$$\mathrm{Tr}_\theta \left[F \wedge F \right] = \sum_{a,b} F^a \wedge F^b \mathrm{Tr}_\theta \left[T^a T^b \right]. \tag{3.3}$$

In this sense it is easy to define the Chern class in the representation θ of a given field strength F as

$$\mathrm{ch}_\theta(F) = \mathrm{Tr}_\theta \left[e^{\frac{iF}{2\pi}} \right]. \tag{3.4}$$

The result is a sum containing a 0-form, a 2-form etc. It is possible to introduce similar polynomials also for the Riemann tensor R, that can be seen as a 2-form with values in the space of the real $D \times D$ matrices, where D is the dimension of the manifold under consideration. In this sense it is possible to introduce R as a real matrix of 2-forms and it is always possible to bring it to the skew-symmetric form by means of an orthogonal transformation

$$R = \begin{pmatrix} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & 0 & \dots & 0 \\ 0 & \begin{matrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{matrix} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \begin{matrix} 0 & \lambda_{D/2} \\ -\lambda_{D/2} & 0 \end{matrix} \end{pmatrix}. \tag{3.5}$$

Given this form is possible to introduce the Roof genus, Hirzebruch polynomials and Euler class respectively as:

$$\hat{A}(R) = \prod_i \frac{\lambda_i}{\sinh \lambda_i}, \quad \hat{L}(R) = \prod_i \frac{\lambda_i}{\tanh \lambda_i}, \quad e(R) = \prod_i \frac{\lambda_i}{2\pi}, \tag{3.6}$$

where the λ's are 2-forms and the various functions are always a formal resummation of the related Taylor expansion. In a more friendly notation it is possible to see that all these objects are, as in the case of the Chern class, polynomials in R, as shown in [76].

We introduced a manifold X and splitted it into the sum of the submanifold M where the chiral spinor propagates plus, roughly speaking, a "complementar" space (this naive decomposition is formally done passing to the tangent/normal bundle). This decomposition is reflected also in the curvature and, in particular, having the Riemann tensor in the skew-symmetric form it is possible to decompose it in the part related to M, called with abuse of notation R, plus the complementar part called R'. Clearly this decomposition is reflected in the various polynomials defined before. With all these objects introduced it is possible to write the anomaly as

$$A = \int_M \mathrm{ch}(F) \wedge \frac{\hat{A}(R)}{\hat{A}(R')} \wedge e(R'). \tag{3.7}$$

Eq. (3.7) is a formal notation meaning that all the various polynomials must be expanded and then, of the full sum of forms, must be retained only the form of degree equal to the dimension of M. In this sense the integral is well defined.

There is an anomaly contribution also form self-dual or anti-self-dual forms present in the theory. As in the case of fermions the contribution can be obtained from an index theorem and so related to topological properties of the space. The final form of the contribution is

$$A = -\frac{1}{8}\int_M \frac{\hat{L}(R)}{\hat{L}(R')} \wedge e(R'). \tag{3.8}$$

We define the anomaly polynomial for a chiral spinor and a self dual tensor as

$$I_{1/2} = \mathrm{ch}(F) \wedge \frac{\hat{A}(R)}{\hat{A}(R')} \wedge e(R'), \tag{3.9}$$

$$I_A = -\frac{1}{8}\frac{\hat{L}(R)}{\hat{L}(R')} \wedge e(R'). \tag{3.10}$$

These anomalies can be related to the loop amplitudes described before, and are canceled by closed tree level amplitudes where closed RR states are exchanged between the various extended objects.

To compute the cancellation it is necessary to introduce the couplings between D-branes and O-planes to closed string modes in the most general way. For this purpose we recall that the Wess-Zumino coupling in the effective world volume of a Dp-brane is (see for example [76] and references therein)

$$S = \mu_p \int C \wedge e^{\mathcal{F}} \wedge \sqrt{\frac{\hat{A}(R)}{\hat{A}(R')}} \tag{3.11}$$

where μ_p is the charge of the brane, C is the formal sum over all the RR form potentials and $\mathcal{F} = 2\pi\alpha' F - B$ is the difference between the gauge curvature F and the NSNS potential B.

We also recall that the coupling for an Op-plane is [76]

$$S = \mu_p \int C \wedge \sqrt{\frac{\hat{L}(R)}{\hat{L}(R')}}. \tag{3.12}$$

These anomalous couplings modifies the usual equation of motion and Bianchi identity for the RR field strength H, usually bound to be $H = dC$. On the other hand, H must be invariant under a gauge variation since it is a physical quantity, so that new anomalous transformations for C are imposed, depending exactly on the new anomalous couplings. Since the potential C is present in the action only through the terms (3.11) and (3.12) the new anomalous variation is revealed exactly by them. This means that given the anomalous coupling and δC the inflow of anomaly is computed exactly from the anomalous variation of the anomalous couplings.

Since each space-time defect (D-brane or O-plane) contributes independently to the the new C transformation it follows that δC is a sum over the various defects of the various anomalous gauge variation. On the other side the full action itself is already a sum of anomalous coupling terms, so that the anomaly δS is a double sum over the space-time defects, and one can recognize the D-brane - D-brane, D-brane - O-plane, O-plane - O-plane contributions independently to be equal to:

$$\begin{aligned} I_{BB} &= -\frac{1}{4}\mathrm{ch}_{n\otimes\bar{n}}(F) \wedge \frac{\hat{A}(R)}{\hat{A}(R')} \wedge e(R') = -\frac{1}{4} I_{1/2}^{n\otimes\bar{n}}(F,R,R'), \qquad (3.13) \\ I_{BO} &= \tfrac{1}{8}\,\mathrm{ch}_{n\oplus\bar{n}}(2F) \wedge \frac{\hat{A}(R)}{\hat{A}(R')} \wedge e(R') = \frac{1}{8} I_{1/2}^{n\oplus\bar{n}}(2F,R,R'), \\ I_{OO} &= -\frac{1}{16}\frac{\hat{L}(R)}{\hat{L}(R')} \wedge e(R') = -\frac{1}{2} I_A(R,R'). \end{aligned}$$

We can now see how the cancellation works in the most simple case of type I string theory. The anomaly polynomial is computed given the spectrum. Type I theory contains a a gravitational multiplet and a vector multiplet. The full anomaly, so, is exactly

$$I_{grav} = -\frac{1}{2} I_{1/2}(R) + \frac{1}{2} I_{3/2}(R), \tag{3.14}$$

$$I_{vect} = \frac{1}{2} I_{1/2}(F,R). \tag{3.15}$$

In the previous equation the $I_{3/2}(R)$ anomaly is due to the presence of a spin 3/2 particle in the gravitational multiplet. Its contribution is computed knowing that

$$I_{1/2}(R) - I_{3/2}(R) - I_A(R) = 0. \tag{3.16}$$

The vector contribution contains a Chern class of F computed over the antisymmetric $n(n-1)/2$ representation of the $SO(n)$ group of string theory. Since

$$\mathrm{ch}_{n(n-1)/2}(F) = \frac{1}{2}(\mathrm{ch}_{n\otimes\bar{n}}(F) - \mathrm{ch}_n(2F)) \tag{3.17}$$

the anomaly polynomial can be written as

$$I = \frac{1}{2}I_A(R) + \frac{1}{4}I_{1/2}^{n\otimes\bar{n}}(F,R) - \frac{1}{4}I_{1/2}^{n\oplus\bar{n}}(F,R) \tag{3.18}$$

and cancellation against the anomaly inflow (3.13) computed previously is straightforward

$$I + I_{BB} + I_{BO} + I_{OB} + I_{OO} = 0. \tag{3.19}$$

Anomaly cancellation in orbifold models

In the previous section we introduced a general way to compute the anomaly polynomial of a theory given the low energy spectrum and how this anomaly is canceled through a non trivial anomalous coupling between the closed RR form and D-branes (O-planes). The cancellation is valid also in the case of orbifold models, but, since the spectrum is modified by the new topological properties of the background, also the anomaly polynomial is different and so the anomalous couplings. It is possible to compute the anomaly polynomial exactly as in the previous case, knowing the contribution of a chiral spinor, Rarita-Schwinger field and antisymmetric tensor and given the spectrum of the model. On the other hand another technique can be followed, taking in account for a moment the anomaly polynomial as computed directly from the evaluation of the loop amplitudes. Calling I_{Ann}, I_{MS} and I_{KB} the annulus, Möbius strip and Klein bottle contribution we obtain, in the most trivial case

$$I_{Ann} = \frac{1}{4}\mathrm{Tr}_R\left[(-1)^F e^{-tH(R,F)}\right] \tag{3.20}$$

$$I_{MS} = \frac{1}{4}\mathrm{Tr}_R\left[\Omega(-1)^F e^{-tH(R,F)}\right] \tag{3.21}$$

$$I_{KB} = \frac{1}{8}\mathrm{Tr}_{RR}\left[\Omega(-1)^{F+\tilde{F}} e^{-tH(R,F)}\right] \tag{3.22}$$

where $H(F,R)$ contains also the vertex operators of the six external legs and the traces are performed only on the Ramond excitations with the insertion of the target space fermion number projector $(-1)^F$. We are not going to compute directly this amplitude, we introduce it because in the case of an orbifold with orbifold group of order N generated by g it is modified in a very simple way, since the only effect is the insertion of a projector on g-invariant states in the annulus and MS traces and a projection and a sum over the twisted states in the KB amplitude

$$I_{Ann} = \frac{1}{4N} \sum_{k=0}^{N-1} \mathrm{Tr}_R \left[g^k (-1)^F e^{-tH(R,F)} \right] , \tag{3.23}$$

$$I_{MS} = \frac{1}{4N} \sum_{k=0}^{N-1} \mathrm{Tr}_R \left[g^k \Omega (-1)^F e^{-tH(R,F)} \right] , \tag{3.24}$$

$$I_{KB} = \frac{1}{8N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \mathrm{Tr}^{(m)}_{RR} \left[g^k \Omega (-1)^{F+\bar{F}} e^{-tH(R,F)} \right] . \tag{3.25}$$

The presence of the twist operator modifies the final formula only by means of an overall counting of the fixed points, through the action on the zero modes, and with the insertion of a matrix γ_k, the embedding of the action of g^k on the Chan-Paton degrees of freedom, in the Chern class

$$\mathrm{ch}(F) \to \mathrm{ch}(\gamma_k F) = \mathrm{tr} \left[\gamma_k e^{iF/2\pi} \right] . \tag{3.26}$$

The details of these derivation can be found in [76], we collect here a summary of the final formula valid for K3 orbifolds

$$I^{99,55}_{BB}(R, F_{9,5}) = \frac{1}{4N} \sum_{k=1}^{N-1} \left(2 \sin \frac{\pi k}{N} \right)^2 \mathrm{ch}^2(\gamma_k \, F_{9,5}) \, I_{1/2}(R) , \tag{3.27}$$

$$I^{95}_{BB}(R, F_9, F_5) = -\frac{1}{2N} \sum_{k=0}^{N-1} \mathrm{ch}(\gamma_k \, F_9) \, \mathrm{ch}(\gamma_k \, F_5) \, I_{1/2}(R) , \tag{3.28}$$

$$I^{9}_{BO}(R, F_9) = -\frac{1}{4N} \sum_{k=1}^{N-1} \left(2 \sin \frac{\pi k}{N} \right)^2 \mathrm{ch}(\gamma_{2k} \, 2F_9) \, I_{1/2}(R) , \tag{3.29}$$

$$I^{5}_{BO}(R, F_5) = \frac{1}{4N} \sum_{k=0}^{N-1} \left(2 \cos \frac{\pi k}{N} \right)^2 \mathrm{ch}(\gamma_{2k} \, 2F_5) \, I_{1/2}(R) , \tag{3.30}$$

$$I_{OO}(R) = -\frac{1}{2N} \sum_{k=0}^{N-1} \left[\left(2 \sin \frac{2\pi k}{N} \right)^2 - N'_k \right] I_A(R) , \tag{3.31}$$

where N is the order of the orbifold group while N'_k, different from zero only for even-N, is the number of fixed points of $g^{N/2}$ that are fixed also under the action of g^k.

The novelties with respect to the trivial case can be understood easily from a more physical point of view. In presence of an orbifold some of the states are bound to live on the fixed points, and there is a copy of each spectrum for each fixed point. The factor counting the number of fixed points, so, is simply the way in which this degeneracy is taken into account. The presence of the γ matrices, instead, is simply the remnant that in an orbifold the gauge group is not the most general $SO(32)$ but is a subgroup of it fixed by the orbifold action, so exactly by the projection through the γ matrices.

3.2 Localized anomalies

Taking into account a field (string) theory defined on a space-time with the four standard extended dimensions and extra (compact) ones the analysis of anomalies becomes more subtle. On one hand from a phenomenological point of view we are interested in the low energy physics, and so all the relevant "physical" quantities are integrated over the extra dimensions. This, in some sense, means that we are interested only in the effects due to the low energy states in the Kaluza-Klein expansion, since the effects due to the massive excitations are anyway suppressed by their mass. On the other hand there are effects that, even though are due to heavy states, can be dangerous for the model itself, and there can be quantities of interest that are zero if integrated over the extra dimensions while are not zero as a function of the full space-time. Anomalies show a good example of such quantities. As an example we will review the model studied in [36], consisting in a field theory defined on a space time with an extra compact dimension, taken to be a circle modded out by two reflections $S^1/\mathbf{Z}_2 \times \mathbf{Z}_2'$. The theory contains a 5D massless fermion of unit charge coupled to a $U(1)$ gauge field.

The reduction to four dimensions is straightforward and, since there is no massless 4D chiral fermions, is naively anomaly-free. This means that the triangle diagram with all massless external legs is zero. A more accurate analysis shows, instead, that the interaction between massless and massive KK states can produce anomalous triangle diagrams, with at least one external leg massive, and this is mapped in a 5D anomaly that is zero if integrated over the extra dimension.

The presence of this anomaly is an effective breakdown both for the full 5D theory and for the reduced 4D theory, since even though one is interested in the interactions between the lowest mass KK modes, it is possible to glue together anomalous diagrams in such a way that all the external legs are massless while the massive states, necessary to have anomalous triangle diagrams, propagates as internal lines (see for example fig. 3.2).

3.3 Localized anomalies in orbifold models

In a string model the Green-Schwarz mechanism ensures that all the anomalies are canceled provided that tadpole cancellation conditions are implemented, but is anyway of great interest to understand whether an anomaly is not present or it is present but is canceled by the mechanism. The interest is due to the fact that, as we will show, the mechanism acting on 4D anomalous $U(1)$ gauge symmetries ensures the anomaly cancellation but giving a mass to the gauge boson. Moreover, understanding how the

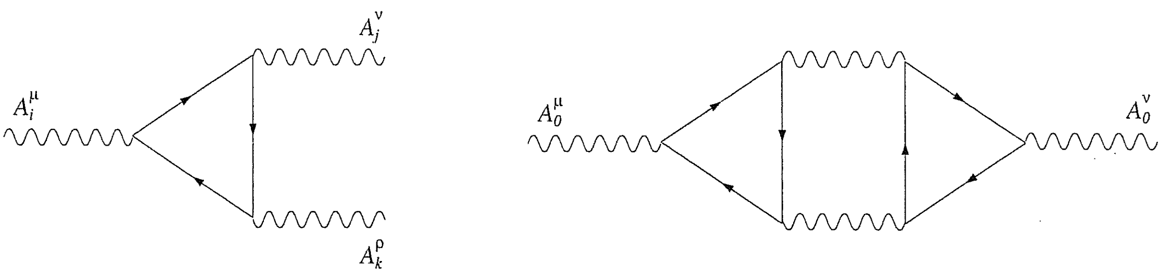

Figure 3.2: *The anomalous diagram for the theory of the example given in [36]. On the left the usual anomaly triangle for a $U(1)$ gauge theory in 4D. In the case of interest the theory is a reduction from a 5D theory and so the gauge fields entering as external legs have an extra KK index (i, j, k). Due to the fact that the 4D theory has an anomaly-free spectrum the triangle amplitude is non zero only if the sum $i+j+k$ is odd, so that at least one of the external legs must be a massive particle. This is a breakdown also for the 4D physics since it is possible to glue two triangles as in the right picture in such a way that all the external legs are massless.*

cancellation is obtained in string theory gives an hint about the way in which it is possible to add counterterms to a model built from a "bottom-up" approach to take care of the presence of anomalies. We will study examples of this for models with only four extended dimensions, with a detailed analysis for the particular models discussed in the previous chapter.

As said all the anomalies are canceled by the Green-Schwarz mechanism, but an important novelty occurs for globally vanishing anomalies, that correspond to an anomaly polynomial I that vanishes when integrated over the orbifold: $\int I = 0$. In this case, the GS mechanism can be mediated not only by RR axions (or their dual 2-forms), as for globally non-vanishing anomalies, but also by KK modes of RR 4-forms. The occurrence of one or the other mechanism depends on the way the anomaly is factorized in terms of forms $X_n(F, R)$ of definite even degree n, constructed out of the gauge and gravitational curvature 2-forms F and R. If it has the form $I \sim X_2 X_4$, the GS mechanism will be mediated by twisted RR axions, arising at the fixed points (or fixed-planes) where the anomaly is distributed. If it has instead the form $I \sim X_6$, the relevant fields are twisted RR 4-forms, arising at fixed planes that contain all the fixed points where the anomaly is distributed. Notice that localized irreducible anomalies are always of the second type, whereas mixed $U(1)$ anomalies can be of both types. As we shall now illustrate with simple and general examples, the fate of the symmetry suffering from a globally vanishing anomaly is radically different in the two alternative mechanisms.

Consider first anomalies of the type $I \sim X_2 X_4$. In this case, the relevant GS mechanism can be easily understood by distinguishing anomalous couplings localized at different points in the internal space. The qualitative novelty can be illustrated by focusing on the case of a $U(1)$ gauge anomaly distributed at two distinct fixed points $z = z_{1,2}$, corresponding to a term of the type $I = X_2X_4|_{z_1} - X_2X_4|_{z_2}$ in the anomaly polynomial. This anomaly is canceled through a GS mechanism mediated by two axions, $C_0^{1,2}$, living at the two fixed points $z_{1,2}$[1]. The action is:

$$S_{GS} = \int d^4x \left[\frac{1}{2}|dC_0^1 + X_1|^2 + C_0^1 X_4\right]_{z_1} + \int d^4x \left[\frac{1}{2}|dC_0^2 + X_1|^2 - C_0^2 X_4\right]_{z_2} \quad (3.32)$$

where the 1-form X_1 denotes the $U(1)$ gauge field associated to the curvature 2-form X_2, such that $X_2 = dX_1$. The modified kinetic terms in (3.32) require that $\delta C_0^{1,2} = -X_0(z_{1,2})$ under a gauge transformation[2] with parameter $X_0(x,z)$, under which $\delta X_1 = dX_0$. The variation of the X_4 terms in (3.32) then provides the required inflow of anomaly that restores gauge invariance. The form of the action (3.32) is fixed by the requirement of having *full* gauge invariance, and implies that the $U(1)$ field becomes massive, independently of whether the anomaly vanishes or not globally. Indeed, one can choose a gauge in which $C_0^1 = -C_0^2$, where the kinetic terms in (3.32) are diagonalized and mass terms for the 4D gauge field are generated. This fact has not been appreciated so far in the literature, where only integrated anomalies were studied.

Consider next the case of anomalies of the type $I \sim X_6$. A globally non-vanishing anomaly of this kind, associated to a global tadpole for a RR 4-form, would lead to an inconsistency, because it cannot be canceled by a standard GS mechanism; indeed, the latter should be mediated by a RR 4-form, that in 4D is a non-propagating field, whose dual in 4D would be a manifestly non-physical and meaningless (-2)-form [12]. Instead, if the anomaly is globally vanishing, and therefore associated to a local tadpole for a RR 4-form, the situation is different. The crucial observation is that this type of anomaly always appears in conjunction with twisted RR states living on fixed planes rather than fixed points in the internal space. Such states propagate in 6D rather than 4D, and this opens up new possibilities, since a 4-form is now a physical propagating field and can mediate a GS mechanism. Moreover a 4-form in 6D is dual to a 0-form, and not to a meaningless (-2)-form. However, internal derivatives will play a role and the corresponding states will thus be massive KK modes from the 4D point of view. The situation is most conveniently illustrated with a simple example consisting of an irreducible term in the anomaly polynomial of the form $I = X_6|_{z_1} - X_6|_{z_2}$, where the points z_1 and z_2 differ only in the fixed-plane direction. The relevant 6D action for the RR 4-form C_4 responsible for the inflow is:

$$S_{GS} = \int d^6x \, \frac{1}{2}|dC_4 + X_5|^2 + \int d^4x \, C_4\Big|_{z_1} - \int d^4x \, C_4\Big|_{z_2} , \quad (3.33)$$

[1]The same basic mechanism works for anomalies localized on fixed-planes, that will thus be canceled by RR axions propagating in 4 or 6 dimensions.

[2]In our set-up the gauge fields that can have anomalies localized at distinct fixed points are in general linear combinations of fields coming from $D9$, $D5$ and $\bar{D}5$-branes.

where X_5 is the Chern–Simons 5-form associated to X_6, such that $X_6 = dX_5$. The kinetic term in (3.33) requires that $\delta C_4 = -X_4$ under a 10D gauge transformation, where X_4 is defined as usual from the gauge variation of X_5: $\delta X_5 = dX_4$. The variation of the second and third terms in (3.33) then provides the required inflow of anomaly[3]. Contrarily to the previous case, no $U(1)$ gauge factor is broken by (3.33). Since C_4 enters in (3.33) only through massive KK states, it is interesting to understand its effect in the 4D low-energy effective field theory. In order to do that, notice that the 2-form $\delta(z-z_1) - \delta(z-z_2)$ can be written locally as $d\eta(z)$ for some 1-form $\eta(z)$. Equation (3.33) can then be interpreted as a 6D action with Lagrangian $L_{GS} = \frac{1}{2}|dC_4 + X_5|^2 - \eta dC_4$. We can now integrate out the massive modes of C_4 and evaluate their action on-shell. This is easily done by substituting back into the Lagrangian the equations of motion for C_4, that imply $dC_4 + X_5 = {}^*\eta$ (where * denotes the 6D Hodge operator); it yields $L_{GS}^{\rm eff} = -\frac{1}{2}|\eta|^2 + \eta X_5$. Finally, we obtain the local 6D Chern–Simons term

$$S_{GS}^{\rm eff} = \int d^6x\, \eta X_5 \,. \tag{3.34}$$

Note that this gives, at it should, the same gauge variation as the original action, since $\delta(\eta X_5) = \eta dX_4$, which gives $-d\eta X_4 = -(\delta(z-z_1) - \delta(z-z_2))X_4$ after integration by parts. Moreover, the discontinuous coefficient η is achieved exactly as proposed in [81], the only difference being that the involved 4-form is a dynamical field in the full 10D theory, which behaves like an auxiliary field only in the 4D effective theory. Importantly, the results of [81] ensure that the term (3.34) is compatible with local supersymmetry at the fixed points.

Summarizing, it is clear that there is an important qualitative difference between anomalies that vanish globally and other that do not. From a purely 4D effective field theory point of view, the condition $\int I = 0$ on the anomaly polynomial I guarantees that the corresponding anomaly can be canceled through the addition of a local Chern–Simons counterterm with a discontinuous coefficient[4]. In open string models, however, anomalies with $I \sim X_2 X_4$ always lead to a spontaneous symmetry breaking, and only those with $I \sim X_6$ are canceled through a local counterterm. It would be interesting to understand whether there is some deeper physical principle determining this distinction, besides factorization properties.

All the above considerations apply qualitatively to any orientifold model. For 6D SUSY models, for instance, part of the GS mechanism is mediated by untwisted

[3] In short, localized irreducible 6-form terms in the 4D anomaly polynomial look like reducible terms in a 6D anomaly polynomial, given by the product of the 6-form term and a field-independent δ-function 2-form.

[4] For example, an orbifold field theory that is globally free of anomalies can be regulated in a gauge-invariant way by adding heavy Pauli–Villars fields with mass terms that also have a discontinuous coefficient; the appropriate Chern–Simons term is then automatically generated when integrating out the regulator [83, 104].

RR forms, and these can play the same role as 6D twisted sectors in 4D models. In particular, we have verified local anomaly cancellation in the SUSY 6D $\mathbf{Z}_2$ model of [9, 13]. In the case of maximal unbroken gauge group with all $D5$-branes at a same fixed point, irreducible $\mathrm{Tr}\, F_9^4$ and $\mathrm{Tr}\, R^4$ terms in the anomaly polynomial do not vanish locally and are indeed canceled by a local GS mechanism similar to that described in (3.33), but mediated by untwisted RR 6-forms propagating in 10D. Again, from a 6D effective theory point of view, these amount to a local Chern–Simons term.

Let us now be more concrete and apply the general arguments outlined above to the $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ and $\mathbf{Z}_3 \times \mathbf{Z}'_3$ models. We will begin with the $\mathbf{Z}_3 \times \mathbf{Z}'_3$ model, which does not have irreducible anomalies at all, and then analyse the more complicated $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ model, where some are present. Fortunately, the techniques of [75] can be easily generalized to study the local structure of anomalies. It is convenient to define δ_{abc} as a 6D Dirac δ-function in the internal orbifold, localized at the fixed point with positions labeled a, b and c in the three T^2's respectively, as reported in Figs. 2.4 and 2.6. We also define $\delta_{ab\bullet}$ as a 4D δ-function in the internal space, localized at the fixed planes with positions a and b in the first two T^2's, and similarly for $\delta_{a\bullet c}$ and $\delta_{\bullet bc}$. Moreover, we will denote by F_i^α the field strength of the i-th factor of the gauge group, ordered as in Table 2.2 ($i = 1, 2, 3$ in all cases), in the α (9, 5 or $\bar{5}$) D-brane sector, and with "tr" the traces in fundamental representations of the gauge groups.

3.3.1 Local anomalies in the $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ and $\mathbf{Z}_3 \times \mathbf{Z}'_3$ models

In this subsection we discuss the computation of anomalies for the $\mathbf{Z}'_6 \times \mathbf{Z}'_2$ and $\mathbf{Z}_3 \times \mathbf{Z}'_3$ models, and the deduction of anomalous couplings by factorization. We proceed along the lines of the previous sections. The polynomial are written as function of the full 10D space, so that all the anomalies are taken in account. Since we are interested only in anomalies that are localized in the extra dimensions we take in account only loop amplitude with the insertion of the various orbifold operators, moreover, due to the localization on the fixed points, the various term will be proportional to Dirac delta functions.

It is useful to define the following field-dependent topological charges for D-branes and fixed points:

$$X^\alpha(F_\alpha, R) = \mathrm{Tr}\,[\Gamma_X^\alpha\, e^{iF_\alpha}]\, \sqrt{A(R)}\;, \tag{3.35}$$

$$Y^\alpha(F_\alpha, R) = \mathrm{Tr}\,[\Gamma_Y^\alpha\, e^{iF_\alpha}]\, \sqrt{A(R)}\;, \tag{3.36}$$

$$Z(R) = \sqrt{L(R/4)}\;. \tag{3.37}$$

The labels X and Y distinguish between the two different sectors contributing to the anomaly in each of the models under analysis. These charges must be intended as sums of components with growing degree n, which we shall denote by X^α_n, Y^α_n and Z_n.

The factorization of the anomaly polynomial will be performed along the lines of [75] and the results will be given in the compact differential form notation where $C^{\pm}_{abc}$ denotes the formal sum/difference of a RR axion (0-form) χ_{abc} and its 4D dual 2-form b_{abc} arising at a generic fixed point P_{abc}: $C^{\pm}_{abc} = \chi_{abc} \pm b_{abc}$. The inflows mediated by these fields can then be schematically written as $\langle C^{\pm}_{abc} C^{\pm}_{abc} \rangle = \pm 1$. A similar notation is adopted also for twisted states associated to a fixed plane, say $P_{ab\bullet}$, which consist of an axion $\chi_{ab\bullet}$ and its 6D dual 4-form $c_{ab\bullet}$, and a self-dual 2-form $b_{ab\bullet}$; we define in this case $D_{ab\bullet} = \chi_{ab\bullet} + b_{ab\bullet} + c_{ab\bullet}$. Since these fields live in 6D, $D5$-branes or fixed-points and $D9$-branes or fixed-planes couple to different 4D components $D^{9,5}_{ab\bullet}$. In particular, the 6D 2- and 4-form fields $b_{ab\bullet}$ and $c_{ab\bullet}$ give rise in 4D to 2- and 4-forms $b_{ab\bullet}$ and $c_{ab\bullet}$ when no index is in the fixed-plane direction, but also to 0- and 2-forms $\tilde{\chi}_{ab\bullet}$ and $\tilde{b}_{ab\bullet}$ when 2 indices are in the fixed-plane direction. In this notation, $D^9_{ab\bullet} = \chi_{ab\bullet} + \tilde{\chi}_{ab\bullet} + \tilde{b}_{ab\bullet}$ and $D^5_{ab\bullet} = \chi_{ab\bullet} + b_{ab\bullet} + c_{ab\bullet}$. Since $\chi_{ab\bullet}$ and $\tilde{b}_{ab\bullet}$, as well as $\tilde{\chi}_{ab\bullet}$ and $b_{ab\bullet}$, are dual from the 4D point of view, whereas $\chi_{ab\bullet}$ and $c_{ab\bullet}$ are dual from the 6D point of view, the only non-vanishing inflows mediated by these fields can be formally summarized in $\langle D^9_{ab\bullet} D^5_{ab\bullet} \rangle = 1$. This setting allows us to understand the form of the anomalous couplings in sectors with fixed planes, including those left unexplained in [75].

$\mathbf{Z}_3 \times \mathbf{Z}'_3$ model

In the $\mathbf{Z}_3 \times \mathbf{Z}'_3$ model, X refers to the $\theta\beta$ twisted sector, whereas Y refers to the $\theta\beta^2$ twisted sector, so that $\Gamma_X = \gamma\,\delta$ and $\Gamma_Y = \gamma\,\delta^2$. The anomaly polynomial is easily computed and is given by $I = I^{99}_A + I^9_M$, where

$$I^{99}_A = \frac{\mu^2}{2} \sum_{a,b,c} \left(\delta_{a'bc} X^9\, X^9 - \delta_{a''bc} Y^9\, Y^9 \right) , \tag{3.38}$$

$$I^9_M = 2\,\mu^2 \sum_{a,b,c} \left(\delta_{a'bc} X^9\, Z - \delta_{a''bc} Y^9\, Z \right) , \tag{3.39}$$

are the contributions from the annulus and Möbius strip surfaces respectively, and $\mu = 3^{-7/4}$.

The sum of the two contributions is

$$I = \mu^2 \sum_{a,b,c} \left[\delta_{a'bc} X^9_2 \left(X^9_4 + 4 Z_4 \right) - \delta_{a''bc} Y^9_2 \left(Y^9_4 + 4 Z_4 \right) \right] . \tag{3.40}$$

where the explicit form for the quantities X^α_n, Y^α_n and Z_n are ($m_i = (1,1,0)$, $n_i = (1,-1,0)$, $s_i = (1,-1,-1)$):

$$X^9_2 = -\sqrt{3}\, m_i\, \mathrm{tr} F^\alpha_i \ , \quad Y^9_2 = -\sqrt{3}\, n_i\, \mathrm{tr} F^\alpha_i \ , \tag{3.41}$$

$$X^9_4 = Y^9_4 = -\frac{1}{2} \left[s_i\, \mathrm{tr} F^{\alpha\,2}_i + \frac{1}{12}\, \mathrm{tr} R^2 \right] , \quad Z_4 = -\frac{1}{192}\, \mathrm{tr} R^2 \ . \tag{3.42}$$

There are two anomalous combinations of $U(1)$ factors, X_1^9 and Y_1^9, defined by $X_2^9 = dX_1^9$, $Y_2^9 = dY_1^9$. These have opposite anomalies at the two types of fixed points. This means that the combination $X_1^9 - Y_1^9$ has true 4D anomalies, whereas $X_1^9 + Y_1^9$ suffers only from a globally vanishing anomaly of the type corresponding to (3.32). It can easily be verified that the integrated anomaly coincides with the contribution of the massless chiral fermions in the representations reported in Table 2.2.

The anomaly polynomial can be easily factorized, and yields the following anomalous couplings:

$$S_{D9} = \mu \int \sum_{a,b,c} \left(\delta_{a'bc} C^-_{a'bc} X^9 + \delta_{a''bc} C^+_{a''bc} Y^9 \right) , \tag{3.43}$$

$$S_F = 4\,\mu \int \sum_{a,b,c} \left(\delta_{a'bc} C^-_{a'bc} Z + \delta_{a''bc} C^+_{a''bc} Z \right) . \tag{3.44}$$

that is equivalent to

$$\begin{aligned} \mathcal{L} = \mu \sum_{a,b,c} \delta_{a'bc} \Big[- d\chi_{a'bc} \cdot X_1^9 - \chi_{a'bc} \big(X_4^9 + 4\,Z_4 \big) \Big] \\ + \mu \sum_{a,b,c} \delta_{a''bc} \Big[- d\chi_{a''bc} \cdot Y_1^9 + \chi_{a''bc} \big(Y_4^9 + 4\,Z_4 \big) \Big] , \end{aligned} \tag{3.45}$$

The first term in each row corresponds to the cross-term in a mixed kinetic term of the form (3.32) for the two axions.

$\mathbf{Z}_6' \times \mathbf{Z}_2'$ model

In the $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model, X refers to the θ and $\theta\beta$ sectors, whereas Y refers to the θ^2 and $\theta^2\beta$ sectors; Γ_X is defined as γ_9 in the 9 sector and γ_{16} in the 5 and $\bar{5}$ sectors, and Γ_Y as γ_9^2 in the 9 sector and γ_{16}^2 in the 5 and $\bar{5}$ sectors. The anomaly is given by $I = \sum_{\alpha\beta} I_A^{\alpha\beta} + \sum_\alpha I_M^\alpha$ in terms of the contributions from each sector of the annulus and Möbius strip, which are given by

$$I_A^{\alpha\beta} = -\frac{\mu^2}{2} \sum_{b=1}^{n^{\alpha\beta}} \sum_{c=1}^{3} \rho^{\alpha\beta} \Big[\delta_{1bc} \left(X^\alpha X^\beta + \epsilon^{\alpha\beta}\, Y^\alpha Y^\beta \right) - \delta_{1b'c} \left(X^{\bar{\alpha}} X^{\bar{\beta}} + \epsilon^{\bar{\alpha}\bar{\beta}}\, Y^{\bar{\alpha}} Y^{\bar{\beta}} \right) \Big] \tag{3.46}$$

$$I_M^\alpha = 4\,\mu^2 \sum_{b=1}^{n^{\alpha\alpha}} \sum_{c=1}^{3} \rho^{\alpha\alpha} \Big[\delta_{1bc}\, Y^\alpha Z - \delta_{1b'c}\, Y^{\bar{\alpha}} Z \Big] , \tag{3.47}$$

where $\mu = 12^{-3/4}$; for $\alpha\beta = 99, 55, 95, 59$, the coefficient $\rho^{\alpha\beta}$ is equal to $1, 4, 2, 2$, $\epsilon^{\alpha\beta}$ is $0, 0, 1, 1$, and $n^{\alpha\beta}$ is $4, 2, 2, 2$.

The complete anomaly polynomial, so, is

$$
\begin{aligned}
I = \mu^2 \sum_{c=1}^{3} \Big\{ & 2 \sum_{b=1}^{2} \delta_{1bc} \Big[X_2^5 \left(-2X_4^5 + X_4^9\right) + X_2^9 X_4^5 + Y_2^5 \left(Y_4^9 + 8Z_4\right) + Y_2^9 Y_4^5 - 4Y_6^9 \Big] \\
& -2 \sum_{b=1}^{2} \delta_{1b'c} \Big[X_2^{\bar 5} \left(-2X_4^{\bar 5} + X_4^9\right) + X_2^9 X_4^{\bar 5} + Y_2^{\bar 5} \left(Y_4^9 + 8Z_4\right) + Y_2^9 Y_4^{\bar 5} - 4Y_6^9 \Big] \\
& + \sum_{b=1}^{4} \Big(\delta_{1bc} - \delta_{1b'c} \Big) \Big[-X_2^9 X_4^9 + 4Y_2^9 Z_4 + 4Y_6^9 \Big] \Big\} ,
\end{aligned}
\tag{3.48}
$$

in terms of the components of the charges, which read in this case ($p_i = (1,1,2)$, $q_i = (1,1,-2)$, $r_i = (1,-1,0)$):

$$X_2^\alpha = -\, p_i \,\mathrm{tr} F_i^\alpha \,, \quad Y_2^\alpha = -\sqrt{3}\, r_i \,\mathrm{tr} F_i^\alpha \,, \tag{3.49}$$

$$X_4^\alpha = -\frac{\sqrt{3}}{2} r_i \,\mathrm{tr} F_i^{\alpha\,2} \,, \quad Y_4^\alpha = -\frac{1}{2}\left[q_i \,\mathrm{tr} F_i^{\alpha\,2} + \frac{\alpha - 1}{48} \mathrm{tr} R^2 \right] , \quad Z_4 = -\frac{1}{192} \mathrm{tr} R^2 \tag{3.50}$$

$$Y_6^\alpha = -\frac{r_i}{2\sqrt{3}} \left[\mathrm{tr} F_i^{\alpha\,3} + \frac{1}{36} \mathrm{tr} F_i \,\mathrm{tr} R^2 \right] . \tag{3.51}$$

When integrated over the internal space, eq. (3.48) is in agreement[5] with the contribution of the massless chiral fermions in the representations listed in Table 2.2. There are however additional anomalies (of all types, including irreducible terms) that do not involve the gauge fields associated to the $D5$ or $\bar{D}5$ branes, which are distributed with opposite signs at different fixed points and are therefore not detectable in the 4D effective theory. These anomalies are generated by KK modes of charged fields in the 99 sector. In total, there are 4 truly anomalous $U(1)$'s, two $U(1)$'s that have only localized anomalies, and localized irreducible anomalies.

Written in the form of equation (3.46), the anomaly can be easily factorized, and we find the following anomalous couplings:

$$S_{D9} = \mu \sum_{c=1}^{3} \int \Big[\sum_{b=1}^{4} \Big(\delta_{1bc}\, C_{1bc}^{+} X^9 + \delta_{1b'c}\, C_{1b'c}^{-} X^9 \Big) - \sum_{a=1}^{3} \Big(\delta_{a\bullet c}\, D_{a\bullet c}\, Y^9 \Big) \Big] , \tag{3.52}$$

$$S_{D5} = -2\,\mu \sum_{c=1}^{3} \int \sum_{b=1}^{2} \Big[\delta_{1bc} \Big(C_{1bc}^{+} X^5 - D_{1\bullet c}\, Y^5 \Big) + \delta_{1b'c} \Big(C_{1b'c}^{-} X^{\bar 5} + D_{1\bullet c}\, Y^{\bar 5} \Big) \Big] , \tag{3.53}$$

$$S_F = 4\,\mu \sum_{c=1}^{3} \int \Big[\sum_{b=1}^{4} \Big(\delta_{1bc}\, D_{1\bullet c}\, Z - \delta_{1b'c}\, D_{1\bullet c}\, Z \Big) - 2 \sum_{a=1}^{3} \Big(\delta_{a\bullet c}\, D_{a\bullet c}\, Z \Big) \Big] . \tag{3.54}$$

It can also be written as couplings for the two kinds of twisted axions χ and $\tilde{\chi}$ and the twisted 4-form c. Defining for convenience the combination of Kronecker δ-functions

[5] In particular, it reproduces the results of [74] for gauge anomalies, apart from irrelevant chirality conventions.

$\delta_b = \delta_{b,1} + \delta_{b,2}$, we find:

$$\begin{aligned}
\mathcal{L} = \mu \sum_{c=1}^{3} \sum_{b=1}^{4} \delta_{1bc} \Big[& - d\chi_{1bc} \cdot (X_1^9 - 2\,\delta_b\, X_1^5) + \chi_{1bc}(X_4^9 - 2\,\delta_b\, X_4^5) \\
& - d\tilde{\chi}_{1\bullet c} \cdot (2\,\delta_b\, Y_1^5) + \chi_{1\bullet c}(4\, Z_4 + 2\,\delta_b\, Y_4^5) + c_{1\bullet c}(4 - 8\,\delta_b) \Big] \\
+ \mu \sum_{c=1}^{3} \sum_{b=1}^{4} \delta_{1b'c} \Big[& - d\chi_{1b'c}(X_1^9 - 2\,\delta_{b'}\, X_1^{\bar{5}}) - \chi_{1b'c}(X_4^9 - 2\,\delta_{b'}\, X_4^{\bar{5}}) \\
& + d\tilde{\chi}_{1\bullet c} \cdot (2\,\delta_{b'}\, Y_1^{\bar{5}}) - \chi_{1\bullet c}(4\, Z_4 + 2\,\delta_{b'}\, Y_4^{\bar{5}}) - c_{1\bullet c}(4 - 8\,\delta_{b'}) \Big] \\
+ \mu \sum_{a=1}^{3} \sum_{c=1}^{3} \delta_{a\bullet c} \Big[& d\chi_{a\bullet c} \cdot Y_1^9 - \tilde{\chi}_{a\bullet c}(Y_4^9 + 8\, Z_4) - dc_{a\bullet c} \cdot Y_5^9 \Big] \,. \qquad (3.55)
\end{aligned}$$

The terms relevant to the cancellation of localized irreducible anomalies are the last terms of each square bracket. The other terms, instead, are relevant to the cancellation of reducible $U(1)$ anomalies.

Chapter 4

Stability in orbifold models

As explained in the introduction quantum corrections inducing a potential for the compactification moduli are of crucial importance in the study of freely acting orbifolds. In chapter 1 we showed how a freely acting orbifold model contains a would be tachyon, of mass $m \sim 1 - aR^2$, with $a > 0$ being a proper model-dependent constant and R being the SS radius. The presence of a would-be tachion, unavoidable, gives a constraint on the moduli space to be fulfilled. Furthermore the SUSY breaking scale is proportional to $1/R^2$, so that the moduli space is constrained also from a phenomenological point of view. Due to these constraints it is clear that the study of the loop-induced potential is relevant, at least in the restricted case of the SS radius potential.

In this chapter we introduce how to work out loop computations in order to compute the vacuum energy of a model, that is exactly the potential for all the parameters present. The study is completely general, since it is clear that in principle all non-SUSY models can receive loop corrections to the potential.

Special attention is devoted to the case of closed loop-amplitude, where, due to the $Sl(2, \mathbf{Z})$ invariance of the torus, the computation implies an integral over the torus fundamental region. We show how to unfold this region to a more suitable one in order to perform computations in an exact analytic way. The problem of the unfolding of the fundamental region, studied in [91], is presented in the most general case in Appendix B.

We take into account the potential in some simple toy model, essentially with the SS radius as only one modulus, in order to extract the main model-independent behavior. We obtain that no stabilization is present if the orbifold group is odd-order. Then we study the chiral N=1 $\mathbf{Z}_6' \times \mathbf{Z}_2$ model of chapter 1, showing that a stabilization is possible but also that it is necessary to include, in the study, all the moduli in order to solve a well-defined minimum problem.

4.1 General form of the vacuum energy

The vacuum energy at one-loop level in any quantum field theory depends simply on the mass spectrum of the theory and its degeneracy. In D space-time dimensions, in a Schwinger proper time parametrization, it can be written as

$$E_D = -V_{D-1} \int_0^\infty \frac{dt}{2t} \sum_i d_i (-)^{2J_i} \int \frac{d^D k}{(2\pi)^D} e^{-t(k^2+m_i^2)}, \tag{4.1}$$

with J_i and m_i the spin and mass of the state i, d_i its degeneracy, k its euclidean momentum and V_{D-1} the volume of the $D-1$ spatial dimensions. More succinctly,

$$E_D = -\int_0^\infty \frac{dt}{2t} \operatorname{Tr}_{\mathcal{H}} e^{-tH}, \tag{4.2}$$

where $\mathcal{H}$ is the full Hilbert space of our system (including V_{D-1}) and $H = k^2 + m^2$. We are interested in the explicit form of (4.1) for orbifold and orientifold-derived theories, with the usual tower of massive string states, where supersymmetry is broken by a SS mechanism. We focus our attention to the case in which the twisted periodicity conditions defining the SS breaking are taken only along one direction, a circle of radius R, henceforth denoted SS direction. As we will see shortly, similar considerations apply also when the SS direction is an $S^1/\mathbf{Z}_2$ orbifold, such as in the chiral 4D model considered in section 5. More general configurations with two (or more) SS directions will not be considered here.

It is convenient to treat separately the contribution to the vacuum energy of states propagating or not along the SS direction. We denote the symmetry breaking in the two cases respectively as longitudinal and transverse SS breaking.

4.1.1 Longitudinal SS breaking

This is the situation that will concern us mostly, since it applies to all closed strings and to open strings on $D9$-branes. In both cases, the vacuum energy can be written as in (4.1). For open strings this is clear, t being the modulus of the annulus and d_i the string degeneracy of the state. For closed strings the situation is more complicated because the $PSL(2,\mathbb{Z})$ modular invariance of the torus restricts the integration over the modulus τ to the fundamental domain. Nevertheless, generalizing standard techniques (see *e.g.* [89, 90] and Appendix B) one can unfold the fundamental domain to the strip and rewrite the whole closed string contribution, including the Klein bottle term, in the form (4.1), where d_i is now the string degeneracy of the state, with the level-matching conditions imposed by means of the τ_1 integration. Quite interestingly, in this way only untwisted closed string states will explicitly appear in the computation. Similarly, winding modes along the SS direction will not be present, so that as far as the R dependence is concerned, the whole amplitude looks effectively like that of a *purely* quantum field theory with an infinite number of states.

It is useful to distinguish closed and open string contributions to the vacuum energy. Recall that in string theory $H^{(closed)} = L_0 + \bar{L}_0 = \alpha' p^2/2 + (N + \bar{N})/\alpha'$ and $H^{(open)} = L_0 = \alpha' p^2 + N/\alpha'$. Therefore, rescaling $t \to \pi\alpha' t$ and $t \to 2\pi\alpha' t$ respectively in the two cases, one gets

$$\begin{aligned} \rho_D^{(C)} &= -\int_0^\infty \frac{dt}{2\, t^{\frac{D+2}{2}}} \sum_i d_i^{(C)} (-)^{2J_i}\, e^{-\pi t \alpha' m_i^2}, \\ \rho_D^{(O)} &= -2^{-D/2} \int_0^\infty \frac{dt}{2\, t^{\frac{D+2}{2}}} \sum_i d_i^{(O)} (-)^{2J_i}\, e^{-2\pi t \alpha' m_i^2}, \end{aligned} \tag{4.3}$$

where (C) and (O) stand respectively for closed and open, and we have defined the energy densities

$$\rho_D \equiv \frac{E_D (4\pi^2 \alpha')^{\frac{D}{2}}}{V_{D-1}}, \tag{4.4}$$

with V_{D-1} the $(D-1)$-dimensional spatial volume of the non-compact dimensions. The generic mass of a given closed and open string Kaluza-Klein (KK) state at level n along the SS direction is respectively

$$\begin{aligned} m_{i,n}^2 &= \frac{2(N+\bar{N})}{\alpha'} + \ldots + \frac{(n+q)^2}{R^2} \equiv m_i^2 + \frac{(n+q)^2}{R^2}, \\ m_{i,n}^2 &= \frac{N}{\alpha'} + \ldots + \frac{[n+q(G)]^2}{R^2} \equiv m_i^2 + \frac{[n+q(G)]^2}{R^2}, \end{aligned} \tag{4.5}$$

where q is the twisted charge given by the SS breaking, N and $\bar{N}$ are the string oscillator numbers and the dots stand for the KK and winding mode contributions along the other compact directions. The index i in (4.3) includes thus a sum over N, $\bar{N}$, KK and winding modes over all the compact directions (but the SS direction) of states of given charge q and then a sum over all possible twists q. Since the SS breaking can (and must) be implemented in the gauge sector as well, for open strings q depends also on the gauge degrees of freedom, $q = q(G)$. The m_i^2 mass terms are typically functions of the geometric moduli of the compactification, except R, but for simplicity of notation this dependence will be left implicit in the following. Eq.(4.3) can be rewritten as

$$\begin{aligned} \rho_D^{(C)} &= -\sum_{n,i} \sum_{F=0,1} \int_0^\infty \frac{dt}{2\, t^{\frac{D+2}{2}}} d_i(q,F)\, e^{i\pi F}\, e^{-\pi t \alpha' \left(m_i^2 + \frac{(n+q)^2}{R^2}\right)}, \\ \rho_D^{(O)} &= -2^{-D/2} \sum_{n,i,G} \sum_{F=0,1} \int_0^\infty \frac{dt}{2\, t^{\frac{D+2}{2}}} d_i(q,F)\, e^{i\pi F} e^{-2\pi t \alpha' \left(m_i^2 + \frac{[n+q(G)]^2}{R^2}\right)}, \end{aligned} \tag{4.6}$$

where F is the space-time fermion number operator, G denotes a sum over the gauge indices and $d_i(q,F)$ are the string degeneracy factors, in general depending on q and F, that include also the degeneracy arising from the expansion of the modular functions. By a Poisson resummation on the index n and some algebra, it is not difficult to explicitly compute $\rho_D^{(C)}$ and $\rho_D^{(O)}$. It is convenient to separate the $m_i^2 = 0$ contribution, denoted by $\rho_{D,0}^{(C,O)}$, from the remaining ones. One gets for both closed and open strings:

$$\rho_{D,0}^{(C,O)} = -\left(\frac{\sqrt{\alpha'}}{R}\right)^D \frac{\Gamma(\frac{D+1}{2})}{\pi^{\frac{D+1}{2}}} \sum_{q;F=0,1} e^{i\pi F} d_0(q,F) \frac{\mathrm{Li}_{D+1}(e^{2i\pi q}) + \mathrm{Li}_{D+1}(e^{-2i\pi q})}{2}, \quad (4.7)$$

where

$$\mathrm{Li}_p(z) = \sum_{w=1}^{\infty} \frac{z^w}{w^p} \quad (4.8)$$

are the polylogarithm functions, the sum over the gauge degrees of freedom for open strings is implicit, and we denoted by d_0 the degeneracy of states with $m_i^2 = 0$. As far as the $m_i^2 \neq 0$ contributions are concerned, we get

$$\rho_{D,i}^{(C,O)} = -2\left(\frac{\sqrt{\alpha'}}{R}\right)^{\frac{D-1}{2}} (\alpha' m_i^2)^{\frac{D+1}{4}} \sum_{q;F=0,1} e^{i\pi F} d_i(q,F) \sum_{w=1}^{\infty} \frac{\cos(2i\pi q w)}{w^{\frac{D+1}{2}}} K_{\frac{D+1}{2}}(2\pi R w m_i), \quad (4.9)$$

where K_n are the modified Bessel functions and again the sum over gauge indices has been omitted. The full vacuum energy ρ_D is obtained by summing the total closed and open string energy density contributions, $\rho_D = \rho_D^{(C)} + \rho_D^{(O)}$, where

$$\rho_D^{(C,O)} = \rho_{D,0}^{(C,O)} + \sum_{i\neq 0} \rho_{D,i}^{(C,O)} \quad (4.10)$$

and $i \neq 0$ indicates that states with $m_i^2 = 0$ should not be included in the sum. Both $\rho_D^{(C)}$ and $\rho_D^{(O)}$ are finite, as expected by the non-local nature of the SS breaking. Potential divergences should be local in space-time, but locally SUSY is preserved and hence no divergences at all are present. Indeed, in both eqs.(4.7) and (4.9) there would be a potential R-independent UV divergence arising from the $w = 0$ term, where the index w, entering in (4.8), is obtained by a Poisson resummation on the index n of eq.(4.6). This term vanishes because at each mass level i the total number of bosons, summed over all possible twists q, equal the total number of fermions:

$$\sum_q d_i(q,0) = \sum_q d_i(q,1) \quad \forall i\,. \quad (4.11)$$

String models with SS SUSY breaking generically have winding modes that become tachyonic below certain values of R, where the vacuum energy diverges. Though the m_i^2 mass terms defined in (4.5) are always positive, this divergence appears in (4.9) from the sum over all massive string states. As is well known, the degeneracy of massive string states, for large masses, has a leading exponential behavior $d_i \sim \exp(2\pi c\sqrt{\alpha'} m)$, with c a given constant. On the other hand, for large values of its argument, the modified Bessel function $K_n(z)$ admits an asymptotic expansion whose leading term is $\sim \exp(-z)$. Hence, we see that the infinite sum over i in (4.9) converges only for $R > R_T = \sqrt{\alpha'}c$. When the SS twist $q = F$, one can easily recognize eqs.(4.7) and (4.9) to be closely related to the free energy of string/field theory-derived models, with $1/T = 2\pi R$ (see *e.g.* [105]) and $T_H = 1/(2\pi R_T)$ being the Hagedorn temperature.

The general form of the vacuum energy for $m_i R >> 1$ is easily extracted. We see from (4.7) that the $m_i^2 = 0$ contributions is power-like in R, whereas for large R (4.9) is exponentially suppressed in R. More precisely we get

$$\rho_D \sim \frac{1}{R^D}\sum_q \left[d_0(q,1) - d_0(q,0)\right] C_D(q) + O\left(\frac{e^{-m_i R}}{R^{\frac{D}{2}}}\right), \tag{4.12}$$

where $d_0(q,1)$ and $d_0(q,0)$ are the total (closed + open) number of fermionic and bosonic massless states in $D+1$ dimensions (before the SS compactification) with charge q and $C_D(q)$ are certain functions easily obtained from (4.7). For $\mathbf{Z}_2$ or $\mathbf{Z}_3$ twists, in which we get two independent twists $q = 0, 1/2$ or $q = 0, 1/3$, the constraint (4.11) implies that for large R the vacuum energy is dominated by the difference between the total number of fermionic and bosonic D-dimensional, rather than $D+1$-dimensional, massless states $d_0(0,1)$ and $d_0(0,0)$ [19];

$$\rho_D \sim \frac{d_0(0,1) - d_0(0,0)}{R^D} + O\left(\frac{e^{-m_i R}}{R^{\frac{D}{2}}}\right). \tag{4.13}$$

In these cases, an exponentially small one-loop cosmological constant requires $d_0(0,0) = d_0(0,1)$ [19, 23]. However, for more general twists, we notice that the leading power-like behavior can be vanishing in a non-trivial way, thanks to a compensation between bosonic and fermionic contributions with different twists, even if $d_0(0,0) \neq d_0(0,1)$. It would be quite interesting to fully exploit this observation and see if there exist string models with a spectrum satisfying this property.

All the above considerations are easily generalized to the case in which the SS direction is an $S^1/\mathbf{Z}_2$ orbifold. Bulk states propagating along the orbifold are now classified according to their $\mathbf{Z}_2$ parities. The massive spectra of $\mathbf{Z}_2$-even and $\mathbf{Z}_2$-odd states differ only by the presence or not of a zero mode along the orbifold. Both contributions can be summed together in the form (4.6), where the KK level runs over all integers. Possible left-over $n = 0$ terms in the process of recombining $\mathbf{Z}_2$-even and $\mathbf{Z}_2$-odd contributions must vanish, since SUSY is broken by the compactification and they do not depend on R. Equation (4.6) and all the analysis that follows is then still valid for the $S^1/\mathbf{Z}_2$ orbifold. The remaining compact directions can be instead arbitrary, as far as SUSY is broken only by the twist on R. Their structure will affect the explicit form of m_i^2 as well as the degeneracy factors $d_i(q,F)$.

4.1.2 Transverse SS breaking

States that do not propagate along the SS direction do not have a KK decomposition along that direction and hence their contribution to the vacuum energy requires a separate analysis. In string theory, states of this kind can arise either as twisted closed strings located at fixed points orthogonal to the SS direction or as open strings on D-branes transverse to the SS direction. In our class of models, the first kind of states appears always with unbroken tree-level SUSY and hence will never contribute to the one-loop vacuum energy. The same applies also to open strings on D-branes transverse to the SS direction, that present unbroken SUSY at the classical level. The only exception arises for open strings stretched between $D5$-branes/$O5$-planes and $\bar{D}5$-branes/$\bar{O}5$-planes, where SUSY is broken at tree level, for the 4D model discussed in section 5. In this case, the one-loop open string amplitude is more conveniently expressed as a tree-level exchange of closed string states, propagating from one object to the other. This contribution can be thus summarized as follows:

$$\rho_D \sim -V_n \sum_i \sum_{\hat{w}} Q^{(i)} \bar{Q}^{(i)} \mathcal{G}_d^{(i)}[\Delta x(\hat{w})]\,, \tag{4.14}$$

where $\mathcal{G}_d^{(i)}$ is the d-dimensional propagator of a particle of mass m_i and spin J_i and V_n is the volume of the compact longitudinal directions along which the states can propagate. $\Delta x(\hat{w})$ is the transverse D-brane/O-plane-$\bar{D}$-brane/$\bar{O}$-plane distance modulo windings, since a given closed string state can wind w times along all transverse directions before ending on a D-brane/O-plane, and $Q^{(i)} = Q_D^{(i)}/Q_O^{(i)}$ is the D-brane/O-plane charge for each state i. The same applies to the anti-brane and anti-plane charges $\bar{O}^{(i)}$. As in last subsection, the sum over i runs over the string and winding modes along the longitudinal directions, whereas the sum over $\hat{w}$ runs over all the possible windings along the transverse d dimensions. One should recall that the subscripts d and n in (4.14) represent the number of space-time directions in which closed strings propagate and thus it can be different for untwisted and various twisted closed string states. As in the last subsection, the massless $i = 0$ contributions in (4.14) are power-like in R, whereas massive ones give an infinite sum over modified Bessel functions K_n and thus are exponentially suppresed in R, for large R. The sign of the vacuum energy contribution is now given by the brane/plane charge and is always negative. This is intuitively clear, since objects with opposite charges feel an attractive potential between each other. The divergence due to the open string tachyon arising below a given radius is determined by looking at the asymptotic form of the modified Bessel functions and at their degeneracies for large masses.

4.2 A nine-dimensional model

We compute in this section the energy density ρ_9 of the simple 9D model introduced in chapter 2, with Type I string theory compactified on $S^1/(\Omega \times \mathbf{Z}_2)$, where $\mathbf{Z}_2$ is generated by g, the product of a πR translation σ along the circle and $(-)^F$, with F the space-time fermion number operator; $g = \sigma(-)^F$. Such a computation has already been done in [30]; we review it in the following as a simple example of the kind of computation we are going to perform in the next sections. We are interested in the radius dependence of $\rho_9 = \rho_9(R)$; for simplicity we do not include continuous Wilson lines but discuss the dependence of $\rho_9(R)$ on the twist matrices γ_g that embed the $\mathbf{Z}_2$ group in the Chan-Paton degrees of freedom. They can alternatively be considered as $\mathbf{Z}_2$ discrete Wilson lines.

The only massless tadpoles are those for the dilaton, graviton (NSNS) and for the untwisted 10-form (RR), whose cancellation requires the presence of 32 $D9$-branes. All neutral fermions are anti-periodic along the circle and thus massive, with a mass $\sim 1/R$. In the twisted sector we get a tower of real would-be tachyons starting from $\alpha' m^2 = R^2/(4\alpha') - 2$. The twist matrix γ_g is arbitrary and must only satisfy the group algebra $\gamma_g^2 = \pm I$. The gauge group is $SO(n) \times SO(32-n)$, with massless fermions in the bifundamentals $(\mathbf{n}, \mathbf{32}-\mathbf{n})$, for $\gamma_g = \mathrm{diag}(I_n, -I_{32-n})$. If γ_g is taken traceless, the twisted tadpoles associated to the above would-be tachyons cancel as well. From this perspective, such a choice is different from the others. If γ_g is chosen antisymmetric, one gets a $U(16)$ group, or subgroups thereof, with massless fermions in antisymmetric representations[1]. We focus in the following on symmetric twist matrices for definiteness, but since the energy density $\rho_9(R)$ depends on γ_g only through its trace, the analysis applies equally well for other more general choices.

The full energy density of the model is obtained by summing the closed string contribution $\rho_9^{(C)}$ (torus+Klein bottle) and the open one, $\rho_9^{(O)}$ (annulus+Möbius strip).

4.2.1 Closed string contribution

Since the Klein bottle amplitude vanishes identically (the Ω-projection acts in a supersymmetric manner on the closed string spectrum), the whole contribution is given by the torus amplitude:

$$\rho_9^{(C)} = -\int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2^{11/2}} \frac{1}{2|\eta|^{24}} \sum_{n,w\in\mathbb{Z}} \left\{ |\theta_2^4|^2 \Lambda_{n,w} (-)^n + |\theta_4^4|^2 \Lambda_{n,w+\frac{1}{2}} + |\theta_3^4|^2 \Lambda_{n,w+\frac{1}{2}} (-)^n \right\}, \tag{4.15}$$

[1] In this model, local tadpole cancellation for both untwisted NSNS and RR tadpoles cannot be achieved. By choosing respectively γ_g symmetric or antisymmetric one can cancel respectively the massive NSNS or RR tadpoles.

where

$$\Lambda_{n,w} = q^{\frac{\alpha'}{4}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right)^2} \bar{q}^{\frac{\alpha'}{4}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right)^2} , \tag{4.16}$$

and $\theta_i = \theta_i(0|\tau)$ are the usual theta functions. In eq.(4.15) and throughout all the chapter, we often omit the modular dependence of θ_i on τ and always leave implicit its vanishing argument $z = 0$. Using the unfolding technique (see Appendix B), (4.15) can be written as

$$\rho_9^{(C)} = -\frac{R}{\sqrt{\alpha'}} \int_0^\infty \frac{d\tau_2}{2\tau_2^6} \sum_{N\in\mathbb{N}} d_N^2\, e^{-4\pi\tau_2 N} \sum_{p=\in\mathbb{Z}} \left[\frac{1-(-)^p}{2}\right] e^{-p^2 \frac{\pi R^2}{4\tau_2\alpha'}} , \tag{4.17}$$

where

$$\frac{1}{2}\int_{-1/2}^{1/2} d\tau_1 \frac{|\theta_2^4|^2}{|\eta|^{24}} \equiv \sum_{N\in\mathbb{N}} d_N^2\, e^{-4\pi\tau_2 N} . \tag{4.18}$$

Eq.(4.17) can also be written by using a Poisson resummation as

$$\rho_9^{(C)} = -\int_0^\infty \frac{d\tau_2}{2\,\tau_2^{11/2}} \sum_{N\in\mathbb{N}} d_N^2 e^{-4\pi\tau_2 N} \sum_{n\in\mathbb{Z}} \left[e^{-\pi\tau_2 \frac{\alpha' n^2}{(R/2)^2}} - e^{-\pi\tau_2 \frac{\alpha'(n+1/2)^2}{(R/2)^2}} \right] . \tag{4.19}$$

Eq.(4.19) is precisely of the general form (4.6) with $D = 9$, $q = F/2$, $i = N$, $\alpha' m_N^2 = 4N$, $d_N(0,0) = d_N(1/2,1) = d_N^2$, $d_N(0,1) = d_N(1/2,0) = 0$, and $R/2 \to R$. The rescaling of the radius is standard and is due to the identification $x \sim x + \pi R$ induced by the freely acting action. We thus notice, as mentioned in the last section, that the full string theory contribution to the cosmological constant (all KK, winding, untwisted and twisted string states) is automatically encoded in a field theory contribution with only untwisted states, whose KK modes are shifted by the SS mechanism (bosons/fermions periodic/anti-periodic along the SS circle R) and a reduced massive string spectrum, where only the diagonal $N = \bar{N}$ states contribute [93]. The integration on τ_2 in (4.19) can thus be read off directly from (4.7) for $N = 0$ and (4.9) for $N \neq 0$.

4.2.2 Open string contribution

The annulus and Möbius strip contributions, in the absence of Wilson lines, are

$$\rho_9^A = -(\mathrm{Tr}\,\gamma_g)^2 \int_0^\infty \frac{dt}{4(2t)^{11/2}} \frac{\theta_2^4}{\eta^{12}}(it) \sum_{m\in\mathbb{Z}} (-)^m e^{-2\pi t m^2 \frac{\alpha'}{R^2}} ,$$

$$\rho_9^M = +32 \int_0^\infty \frac{dt}{4(2t)^{11/2}} \frac{\theta_2^4}{\eta^{12}}(it - \frac{1}{2}) \sum_{m\in\mathbb{Z}} (-)^m e^{-2\pi t m^2 \frac{\alpha'}{R^2}} . \tag{4.20}$$

The integrations of (4.20) are easily performed and one gets

$$\rho_{9,0}^A + \rho_{9,0}^M = \left[32 - (\mathrm{Tr}\,\gamma_g)^2\right] \left(\frac{\sqrt{\alpha'}}{R}\right)^9 2^{\frac{15}{2}}\, d_0\, \frac{\Gamma(5)}{\pi^5}\, \xi(10)(2-2^{-9}) , \tag{4.21}$$

$$\rho_{9,N}^A + \rho_{9,N}^M = \left[32(-)^N - (\mathrm{Tr}\,\gamma_g)^2\right] \left(\frac{\sqrt{\alpha'}}{R}\right)^4 2^{\frac{7}{2}}\, d_N\, N^{\frac{5}{2}} \sum_{n=1}^\infty \frac{1-(-)^n}{n^5} K_5\left[\pi R n \sqrt{\frac{N}{\alpha'}}\right] ,$$

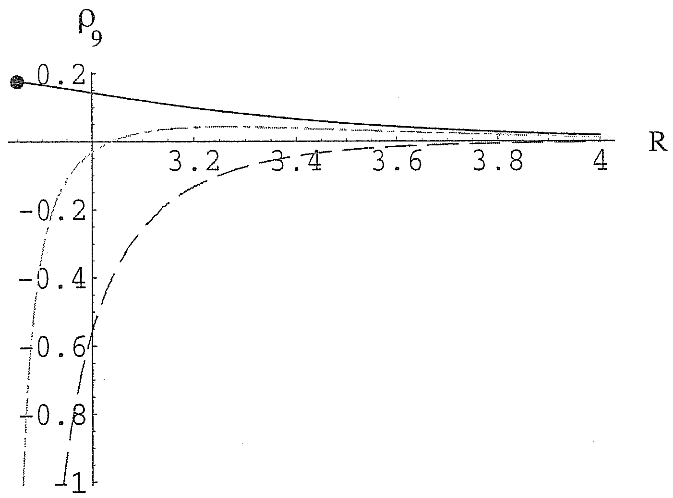

Figure 4.1: Behavior of the vacuum energy density for different choices of the twist matrix γ_g. The upper line refers to the $SO(16) \times SO(16)$ group, the intermediate line to $SO(17) \times SO(15)$ and the lowest one to $SO(18) \times SO(14)$. R is in units of α'. The bullet in the solid line represents the point where a tachyon appears.

with d_N as in (4.18). It is not difficult to realize that these expressions are of the general form (4.6) with $D = 9$, $i = N$, $\alpha' m_N^2 = N$ and $q_{ij}(G) = F/2 + \hat{q}_i + \hat{q}_j$, where i, j run over the fundamental representation of $G = SO(32)$. The parameters $\hat{q}_i = 0, 1/2$ are related to the eigenvalues of γ_g by

$$\gamma_g = \mathrm{diag}\left(e^{2i\pi\hat{q}_1}, \ldots, e^{2i\pi\hat{q}_{32}}\right). \tag{4.22}$$

As mentioned, the resulting 9D gauge group is $SO(n) \times SO(32-n)$ if $\gamma_g = \mathrm{diag}(I_n, -I_{32-n})$, with bosons and fermions in antisymmetric, symmetric and bifundamental representations, depending on the KK and string mass level N. Notice that the leading open string contribution to the energy density is positive whenever $14 \leq n \leq 18$, with a maximum for $n = 16$.

The full energy density $\rho_9 = \rho_9^{(C)} + \rho_9^{(O)} = \rho_9^{(C)} + \rho_9^A + \rho_9^M$ can then be numerically evaluated as a function of R by truncating the infinite sums of modified Bessel functions.

The R-dependence of ρ_9 crucially depends on γ_g. For $n \leq 14$ or $n \geq 18$, $\rho_9 < 0$ monotonically and leads the system to a tachyonic instability, like in the early work of [18]. For $15 \leq n \leq 17$, ρ_9 can be positive and a *maximum* close to R_T is obtained for $n = 15$ or $n = 17$.

In figure 4.1 we show these different behaviors plotting $\rho_9(R)$ for $n = 16$ (blue/solid line), $n = 15 \sim n = 17$ (green/dotted-dashed line) and $n = 14 \sim n = 18$ (red/dashed line).

4.3 Strings on twisted ALE spaces

An interesting class of models whose energy density can be studied are orbifold or orientifold models on Asymptotically Locally Euclidean (ALE) spaces, non-trivially fibered along an S^1, of the form $(\mathbb{C}\times S^1)/\mathbf{Z}_N$, or their compact versions $(T^2\times S^1)/\mathbf{Z}_N$. The $\mathbf{Z}_N$ generator is a product of a $4\pi/N$ rotation along $\mathbb{C}$ and of a translation of $2\pi R/N$ along the circle. The rotation is taken to be of angle $4\pi/N$ so that $g^N=1$ on spinors. A non-trivial fibration is necessary to implement the Scherk-Schwarz supersymmetry breaking and to lift the mass of the would-be tachyons. Upon compactification on S^1, such spaces give rise to a Melvin background [37] (see [106] and [39, 107] for strings and D-branes on Melvin backgrounds).

Consider then Type IIB string theory on $(\mathbb{C}\times S^1)/\mathbf{Z}_N$ or $(T^2\times S^1)/\mathbf{Z}_N$ (as we will see, our results are the same for the non-compact or compact version). In order to keep our analysis as simple as possible, we focus on N odd. Other values of N would require the introduction of $O7$-planes and $D7$-branes, in addition to $D9$-branes and the $O9$-plane, when considering orientifolds. Moreover, for N odd, would-be tachyons appear localized in space-time and an understanding of the possible tachyonic instabilities can be obtained [85, 86, 87, 88].

Uncharged fermions in all these models are massive for $N\neq 3$, with a mass $\sim 1/R$. In each k-twisted sector we get a tower of complex would-be tachyons starting from $\alpha' m^2 = -4\frac{k}{N} + \left(\frac{k}{N}\right)^2 \frac{R^2}{\alpha'}$.

4.3.1 Orbifold models

The only relevant one-loop world-sheet surface in this case is the torus. Its contribution to the vacuum energy density $\rho_7^{N,T}(R)$ can be written, using the unfolding technique discussed in Appendix B, as an integral over the strip of the untwisted sector only

$$\rho_7^{N,T} = -\frac{1}{N}\frac{R}{\sqrt{\alpha'}}\int_0^\infty \frac{d\tau_2}{2\tau_2^5}\sum_{k=1}^{N-1}\sum_{M\in\mathbb{N}}\left[d_M(B)^k - d_M(F)^k\right]e^{-4\pi\tau_2 M}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left(n+\frac{k}{N}\right)^2}, \tag{4.23}$$

where we have defined for later convenience the coefficients $d_M(B,F)^k$ as

$$\sum_{M\in\mathbb{N}}\left[d_M(B)^k - d_M(F)^k\right]e^{-4\pi\tau_2 M} = \int\frac{d\tau_1}{4}\left|2\sin(\frac{2\pi k}{N})\frac{\sum_{\alpha,\beta}\eta_{\alpha\beta}\theta\left[{\alpha\atop\beta}\right]^3\theta\left[{\alpha\atop \beta+\frac{2k}{N}}\right]}{\eta^9\,\theta\left[{\frac{1}{2}\atop\frac{1}{2}+\frac{2k}{N}}\right]}\right|^2. \tag{4.24}$$

If, similarly to the 9D case of last section, the torus contribution to the vacuum energy is exactly encoded in a field theory-like contribution with only KK and untwisted states, (4.23) should admit a rewriting such as (4.6), where the twist q will be

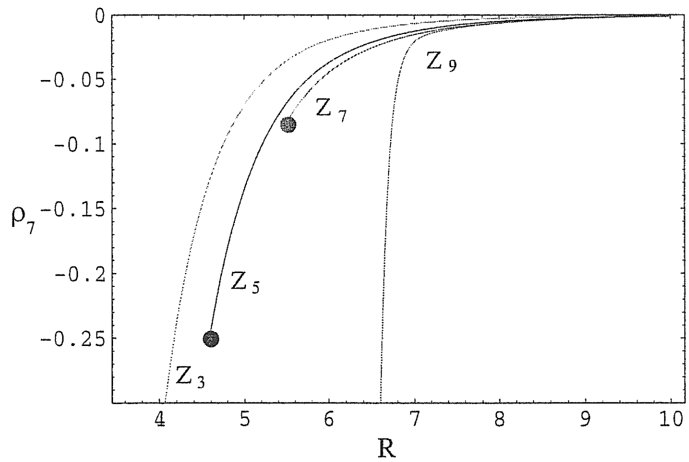

Figure 4.2: R-dependence of the torus contribution to the cosmological constant for $\mathbf{Z}_N$ orbifolds. R is the radius of the SS circle, in units of α'. The bullets represent the points where tachyons appear.

given by the Lorentz $SO(2)$ charge of the states along the twisted directions, with the correct degeneracies. It is useful to compute the latter in some detail for the massless states of IIB string theory.

The above orbifold breaks the Lorentz group $SO(10) \to SO(8)\times SO(2)$. A generic field Φ will have the following periodicity conditions along S^1;[2]

$$\Phi(y + 2\pi R) = e^{2i\pi\hat{q}\alpha}\Phi(y), \tag{4.25}$$

where $\hat{q}$ is its charge under the $SO(2) \simeq U(1)$ internal Lorentz group and $\alpha = 2/N$ is the twist induced by the $\mathbf{Z}_N$ action. Upon reduction on S^1, a field with charge $\hat{q}$ will have a tower of KK states with masses $M_n^{(q)} = (n+q)/R$, where $q = \hat{q}\alpha$. Massless states are not present whenever $q \notin \mathbb{Z}$.

The $U(1)$ charges of the massless states of IIB string theory are easily obtained. The Ramond-Ramond (RR) four, two and zero-forms have the following $SO(8)\times U(1)$ decomposition:

$$\begin{aligned} \mathbf{35} &= \mathbf{15}_0 \oplus \mathbf{10}_1 \oplus \mathbf{10}_{-1}, \\ \mathbf{28} &= \mathbf{15}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1} \oplus \mathbf{1}_0, \\ \mathbf{1} &= \mathbf{1}_0, \end{aligned} \tag{4.26}$$

where the subscript denotes the $U(1)$ Lorentz charge $\hat{q}$. One gets for the Neveu-Schwarz/Neveu-Schwarz (NSNS) graviton, B-field and dilaton:

$$\begin{aligned} \mathbf{35} &= \mathbf{21}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1} \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}, \\ \mathbf{28} &= \mathbf{15}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1} \oplus \mathbf{1}_0, \\ \mathbf{1} &= \mathbf{1}_0. \end{aligned} \tag{4.27}$$

[2]Notice that due to the freely acting action, the radius R entering in (4.25) is N times smaller than the R appearing in eq.(4.23).

For generic $N \neq 2, 4$ we thus get 70 bosonic massless states. For fermions one has two copies of:

$$\begin{aligned} \mathbf{56} &= \mathbf{24}_{1/2} \oplus \mathbf{24}_{-1/2} \oplus \mathbf{4}_{3/2} \oplus \mathbf{4}_{-3/2}, \\ \mathbf{8} &= \mathbf{4}_{1/2} \oplus \mathbf{4}_{-1/2}. \end{aligned} \tag{4.28}$$

Notice that all fermions are always massive except for the case $N = 3$, where we get 16 massless states from the decomposition of the two gravitinos.

We are now ready to explicitly show how these massless states and the corresponding twists arise from (4.23). We focus on the $N = 3$ case, but the analysis can be generalized to the other cases. Reintroducing also the vanishing $k = 0$ term in (4.23) we have ($\tau_2 \to t$ and $R \to NR$):

$$\begin{aligned} \rho_7^T = -\frac{R}{\sqrt{\alpha'}} \int_0^\infty \frac{dt}{2t^5} \sum_{M\in\mathbb{N}} \sum_{n\in\mathbb{Z}} e^{-4\pi t M} \Big\{ & \big[d_M^0(B) - d_M^0(F)\big] e^{-(3n)^2 \frac{\pi R^2}{\alpha' t}} + \\ & \big[d_M^1(B) - d_M^1(F)\big] e^{-(3n+1)^2 \frac{\pi R^2}{\alpha' t}} + \\ & \big[d_M^2(B) - d_M^2(F)\big] e^{-(3n+2)^2 \frac{\pi R^2}{\alpha' t}} \Big\}. \end{aligned} \tag{4.29}$$

Notice that the above sums can be rewritten as follows:

$$\sum_{n\in\mathbb{Z}} e^{-(3n)^2 \frac{\pi R^2}{\alpha' t}} = \sum_{n\in\mathbb{Z}} \frac{1 + e^{\frac{2i\pi n}{3}} + e^{\frac{-2i\pi n}{3}}}{3} e^{-n^2 \frac{\pi R^2}{\alpha' t}}, \tag{4.30}$$

$$\sum_{n\in\mathbb{Z}} \left[e^{-(3n+1)^2 \frac{\pi R^2}{\alpha' t}} + e^{-(3n+2)^2 \frac{\pi R^2}{\alpha' t}} \right] = \sum_{n\in\mathbb{Z}} \frac{2 - e^{\frac{2i\pi n}{3}} - e^{\frac{-2i\pi n}{3}}}{3} e^{-n^2 \frac{\pi R^2}{\alpha' t}}.$$

Since $d_M(B,F)^1 = d_M(B,F)^2 \ \forall M$, eq.(4.23) can be rewritten, by performing a Poisson resummation in n, as

$$\begin{aligned} \rho_7^T = -\int_0^\infty \frac{dt}{2t^{\frac{9}{2}}} \sum_{n\in\mathbb{Z}} \sum_{M\in\mathbb{N}} e^{-4\pi t M} \Big\{ & \Big[d_M(0,0) - d_M(0,1) \Big] e^{-\frac{\pi t \alpha' n^2}{R^2}} \\ & + \Big[d_M(\tfrac{1}{3},0) - d_M(\tfrac{1}{3},1) \Big] e^{-\frac{\pi t \alpha' (n+1/3)^2}{R^2}} \Big\}, \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} d_M(0,0/1) &= \frac{1}{3} \big[d_M(B/F)^0 + 2 d_M(B/F)^1 \big], \\ d_M(\tfrac{1}{3},0/1) &= \frac{2}{3} \big[d_M(B/F)^0 - d_M(B/F)^1 \big] \end{aligned} \tag{4.32}$$

are the total number of bosonic and fermionic states at level M with twist 0 and 1/3 respectively. One can easily check that for $M = 0$ the above coefficients precisely coincide with the field theory results above. Indeed, from (4.24) one finds $d_0(B)^0 =$

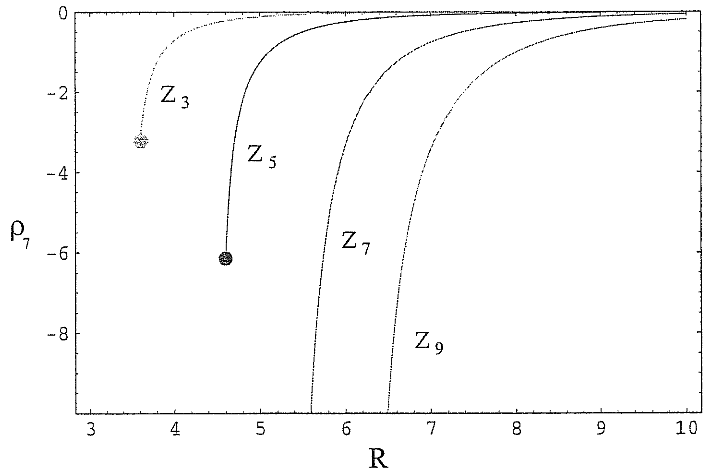

Figure 4.3: R-dependence of the cosmological constant for $\mathbf{Z}_N$ orientifolds. R is the radius of the SS circle, in units of α'. The bullets represent the points where tachyons appear.

$d_0(F)^0 = 128$, $d_0(B)^1 = d_0(B)^2 = 41$, $d_0(F)^1 = d_0(F)^2 = -40$ and hence $d_0(0,0) = 70$, $d_0(0,1) = 16$, $d_0(1/3,0) = 58$, $d_0(1/3,1) = 112$, as expected. Eq.(4.31) is hence precisely of the form (4.6) with $D = 7$, $i = M$, $\alpha' m_M^2 = 4M$ and $q = 0, 1/3$.

Interestingly, even for $\mathbf{Z}_3$ (and most likely for all $\mathbf{Z}_N$, at least with N odd) the *full* string theory computation is encoded in a field theory computation where only untwisted states with the uncompactified level matching conditions $M = \bar{M}$ enter.

The form of $\rho_7^N(R)$ for various $\mathbf{Z}_N$ orbifold models is reported in figure 4.2. As anticipated in the introduction, we find that $\rho_7^N(R) > \rho_7^M(R)$ for $N < M$ and for all the values of R for which both energy densities are well defined.

4.3.2 Orientifold models

Orientifold models are obtained as usual by modding out the above orbifolds by the world-sheet parity operator Ω. As in the 9D model discussed in last section, massless tadpoles are cancelled by introducing 32 $D9$-branes. We do not include continuous Wilson lines but consider the dependence of $\rho_7^N(R)$ on the twist matrices γ_g, whose only constraint comes from the group algebra: $\gamma_g^N = \pm I$. The resulting gauge group depends on the precise form of γ_g, although the energy density is sensitive only on its trace. In addition to the torus amplitude (4.23), we have now to consider also the Klein bottle, annulus and Möbius strip surfaces.

The Klein bottle contribution, $\rho_7^{N,K}$, is easily obtained and has a form similar to (4.23). Defining

$$\sum_{M\in\mathbb{N}}[D_M(B)^{2k} - D_M(F)^{2k}]e^{-4\pi M t} = -2\sin(\frac{4\pi k}{N})\frac{\sum_{\alpha,\beta}\eta_{\alpha\beta}\,\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^3\theta\begin{bmatrix}\alpha\\ \beta+\frac{4k}{N}\end{bmatrix}}{2\,\eta^9\,\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+\frac{4k}{N}\end{bmatrix}}(2it)\,, \quad (4.33)$$

$\rho_7^{N,K}$ can be in fact nicely combined with $\rho_7^{N,T}$ above, so that the full closed string contribution reads exactly as (4.23) with $d_M(B,F)^k \to [d_M(B,F)^k + D_M(B,F)^k]/2$. As expected, the degeneracies of the massless states again agree with the field theory expectations. This can be easily checked by noting that from (4.33) one gets $D_0(B)^2 = D_0(B)^4 = 9$, $D_0(F)^2 = D_0(F)^4 = 0$ and that the RR four and zero-form, the NSNS B field and half of the fermions are now projected out, leaving the decomposition of the remaining states as above. The open string contribution $\rho_7^{N,(O)}$ is easily computed:

$$\rho_7^{N,(O)} = -2^{-7/2} \int_0^\infty \frac{dt}{2t^{9/2}} \sum_{M\in\mathbb{N},\, n\in\mathbb{Z}, k=1}^{N-1} \left[(\mathrm{Tr}\gamma_k)^2 - (-1)^M \mathrm{Tr}\gamma_{2k}\right] e^{2\pi i n \frac{k}{N}} e^{-2\pi t\alpha' \frac{n^2}{R^2}} D_M^k e^{-2\pi t M} \tag{4.34}$$

where the coefficients D_N^k are the same as those defined in (4.33) and $\gamma_k = (\gamma_g)^k$. Eq.(4.34) is not manifestly in the form (4.6) because the embedding of the SS breaking in the gauge sector through the twist matrices γ_g requires a little bit of algebra. However, we do not need to work out these details, since the vacuum energy depends only on the trace of γ_g. The integration in t is by now standard and leads to a power-like behavior in R for the $M = 0$ terms and a sum over modified Bessel functions for $M \neq 0$.

The form of $\rho_7^N(R)$ for various $\mathbf{Z}_N$ orientifold models is reported in figure 4.3 for a proper choice of γ_g that maximizes $\rho_7^N(R)$. As can be seen from the figure, the qualitative structure of ρ_7^N is not modified by the orientifold projection. In particular, we still find that $\rho_7^N(R) > \rho_7^M(R)$ for $N < M$. For different choices of γ_g, ρ_7^N has always the same form as in figure 4.3, but the above ordering of ρ_7^N can be lost.

The vacuum energy density $\rho_7^N(R)$ receives a non-vanishing contribution only for states that do not propagate along the two twisted directions for both orbifold and orientifold models. This implies that the above computation apply equally well for non-compact $(\mathbb{C} \times S^1)/\mathbf{Z}_N$ or compact $(T^2 \times S^1)/\mathbf{Z}_N$ models. The energy density is located at the tip of the cone or at the fixed points of T^2, in the two cases. This is expected since these are precisely the loci where would-be tachyons are localized.

It is interesting to notice that if we plot ρ_7^N as a function of R/N, the effective radius of the SS direction, we get again the behavior as in figures 4.2 and 4.3, but now $\rho_7^N(R/N) < \rho_7^M(R/M)$ for $N < M$, $\forall N, M$ and R. It would be interesting to have some dynamical understanding of this ordering of the energy densities for this class of models.

The $\mathbf{Z}_2$ orbifold and orientifold models can also be considered. In this case space-time is flat with fermions antiperiodic along R and thus we recover exactly the previous 9D model of section 2 or its compactified version on a T^2 torus.

4.4 A four dimensional model

In this section we study the vacuum energy density of the 4D IIB orientifold model compactified on $T^6/(\mathbf{Z}_6' \times \mathbf{Z}_2')$ described in the chapter 1. The complex structure of the first and second torus is fixed by the $\mathbf{Z}_6'$ action, whereas the third torus can be taken rectangular, since the action is a $\mathbf{Z}_2$ reflection. We define $P_{1,2}$ such that the volume of the i-th torus equals to $4\pi^2 P_i^2/[3(1+\delta_{i,1})]$, and R and S to be the radii of the two circles of T_3^2.

We compute the cosmological constant ρ_4 as a function of the SS radius and the other moduli. Contrary to the previous models, ρ_4 gets a non-vanishing contribution both from states with longitudinal and transverse SS SUSY breaking. For simplicity we do not include continuous Wilson lines.

4.4.1 Longitudinal SS breaking contribution

There are two main longitudinal contributions coming from the torus amplitude and the open sector of $D9$-branes.

Closed string contribution. Due to the unfolding technique, the longitudinal closed string contribution to the energy density, denoted by $\rho_4^{(C,l)}$, can be written as

$$\rho_4^{(C,l)} = -\int_0^\infty \frac{dt}{2t^3} \sum_i \sum_{j=U,T} d_i^{(C,j)} e^{-\pi t\alpha' m_i^{2(C,j)}} \sum_{n\in\mathbb{Z}} [e^{-\pi t\alpha' \frac{(2n)^2}{R^2}} - e^{-\pi t\alpha' \frac{(2n+1)^2}{R^2}}]\,, \qquad (4.35)$$

where U and T stand for the untwisted and θ^2/θ^4-twisted string sectors—the only sectors with a non-vanishing contribution—and the index i includes the sum over the string, KK and winding modes over the non-SS directions with the level matching conditions imposed by means of the τ_1 integration[3].

The masses of the i-th level for the untwisted ($j=U$) and twisted ($j=T$) sectors are given by the following expression:

$$m_i^{2(C,j)} = \frac{2(N+\bar{N})}{\alpha'} + \frac{\alpha' n_1^2}{S^2} + \frac{w_1^2 S^2}{\alpha'} + \delta_{j,U} \sum_{a=1}^{2} \left[\frac{2\alpha' N_a^2}{\sqrt{3}P_a^2} + \frac{\sqrt{3} W_a^2 P_a^2}{2\alpha'} \right], \qquad (4.36)$$

where we have defined the combinations of KK and winding modes N_a^2 and W_a^2 as

$$\begin{aligned} N_a^2 &= n_{2a}^2 + n_{2a+1}^2 + n_{2a}n_{2a+1}\,, \\ W_a^2 &= w_{2a}^2 + w_{2a+1}^2 - w_{2a}w_{2a+1}\,. \end{aligned} \qquad (4.37)$$

[3]This equals $\bar{N} = N + \sum_{a=1}^{5} m_a n_a$ in the untwisted sector and $N = \bar{N} + w_1 n_1$ in the θ^2/θ^4 twisted ones.

The degeneracy of the i-th state is given by $d_i^{(C,j)} = d_N^j d_{\bar{N}}^j/24$, with

$$\sum_{N,\bar{N}\in\mathbb{N}} d_N^j\, d_{\bar{N}}^j q^N \bar{q}^{\bar{N}} = \begin{cases} |\theta_2^4|^2|\eta|^{-24}, & j = U\,, \\ 9|\theta_2^2|^2|\eta|^{-12}\left|\theta\begin{bmatrix}1/2\\1/3\end{bmatrix}\theta\begin{bmatrix}1/2\\5/6\end{bmatrix}^{-1}\right|^4, & j = T\,. \end{cases} \tag{4.38}$$

Once again, eq.(4.35) is of the same form as (4.3).

Open string contribution. The longitudinal open string contribution to the energy density, denoted by $\rho_4^{(O,l)}$, is

$$\rho_4^{(O,l)} = -\int_0^\infty \frac{dt}{4(2t)^3}\sum_i\sum_{j=U,T} d_i^{(O,j)} e^{-2\pi t\alpha' m_i^{2(O,j)}} \sum_{n\in\mathbb{Z}}[e^{-\pi t\alpha'\frac{(2n)^2}{R^2}} - e^{-\pi t\alpha'\frac{(2n+1)^2}{R^2}}]\,, \tag{4.39}$$

where

$$m_i^{2(O,j)} = \frac{N}{\alpha'} + \frac{\alpha' n_1^2}{S^2} + \delta_{j,U}\sum_{a=1}^{2}\frac{2\alpha' N_a^2}{\sqrt{3}P_a^2}, \tag{4.40}$$

with N_a^2 as in (4.37) and:

$$d_i^{(O,U)} = \left[(\mathrm{Tr}\,\gamma_\beta)^2 - 32(-1)^N\right]\frac{d_N^U}{6}\,, \tag{4.41}$$

$$d_i^{(O,T)} = \left[\left(\mathrm{Tr}\,\gamma_\theta^2\gamma_\beta\right)^2 - \left(\mathrm{Tr}\,\gamma_\theta^4\right)(-1)^N\right]\frac{d_N^T}{6}\,, \tag{4.42}$$

with γ_θ and γ_β the twist matrices associated respectively to the SUSY and non-SUSY twists θ and β, and $d_N^{U/T}$ defined as in (4.38).

The untwisted contribution is essentially the same as the open contribution (4.20) of the 9D model, the only difference being an overall factor and the Kaluza-Klein lattices from the compactified directions. As in the 9D case, γ_β is unconstrained (provided that $\gamma_\beta^2 = \pm I$). The value of $\mathrm{Tr}\,\gamma_\theta^4$ is fixed to be 8 by tadpole cancellation.

4.4.2 Transverse SS breaking contribution

This contribution arises only when considering $D5$, $\bar{D}5$-branes, as well as $O5$ and $\bar{O}5$-planes. As shown in section 2, these terms can be written as a sum of propagators of closed string states with mass m_i, propagating along the number of compact dimensions that are orthogonal to the brane.

Tadpole cancellations are automatically encoded when we add up the Klein bottle, annulus and Möbius strip contributions in the closed string channel. Notice that the cancellation is obtained for all the KK modes along the SS direction due to the choice of the brane positions, ensuring the local tadpole cancellation in that dimension.

In terms of closed string states, the full amplitude is a sum of two contributions, one coming from the untwisted and one from the θ^2/θ^4 twisted string sector.

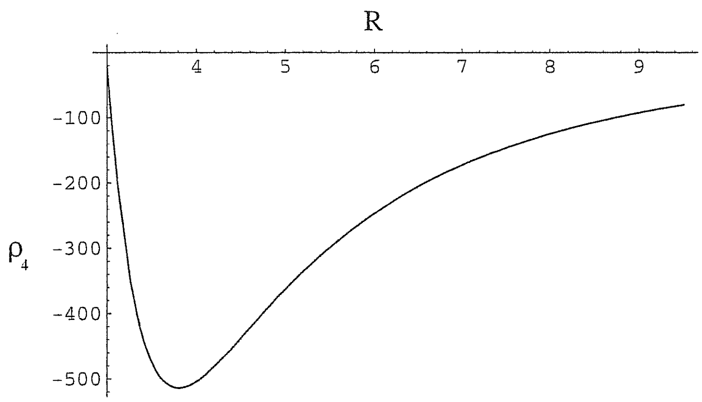

Figure 4.4: R-dependence of the cosmological constant for the 4-dimensional model. R refers to the radius of the Scherk-Schwarz dimension and is in unit of α'. The other moduli are fixed.

Untwisted string contribution This is given by the sum of the relevant Klein bottle, annulus and Möbius strip amplitudes. Denoting by $\rho_4^{(U,t)}$ the contribution to the vacuum energy of this sector, we get:

$$\rho_4^{(U,t)} = -\frac{8P_2^2\pi^2}{3\alpha'} \sum_{i,\hat{w}} d_i^U \left\{ 8G_4^{(i)} \left[\Delta x(\hat{w})^{(O,U)}\right] + \left[1 - 2(-1)^N\right] G_4^{(i)} \left[\Delta x(\hat{w})^{(C,U)}\right] \right\} \tag{4.43}$$

where $G_d^{(i)}(x)$ is the d-dimensional propagator of a scalar particle of mass ($a = O/C$):

$$m_i^{2(a,\,U)} = \frac{4N}{\alpha'} + (1 + 3\delta_{a,C}) \frac{2P_2^2 W_2^2}{\sqrt{3}}\,, \tag{4.44}$$

and

$$\left(\Delta x(\hat{w})^{(a,U)}\right)^2 = \frac{\pi^2 R^2}{\alpha'} \left(w + \frac{1}{2}\right)^2 + (1 + 3\delta_{a,O})\pi^2 \left[\frac{S^2}{\alpha'} n^2 + \frac{2P_1^2 W_1^2}{\sqrt{3}}\right]. \tag{4.45}$$

In eq.(4.43), the three terms in curly brackets are given by the annulus, Klein bottle and Möbius strip surfaces, respectively. Notice that the two labels O and C are needed because the winding and KK modes exchanged between two D-branes (annulus contribution) are not equal to those exchanged between O-planes (Klein bottle) or an O-plane and a D-brane (Möbius strip). More precisely, O-planes couple only to even winding modes along the longitudinal directions (second torus) and this explains the factor $(1 + 3\delta_{a,C})$ in (4.44). Similarly, with our choice of $D/\bar{D}5$-brane positions, closed strings exchanged between two branes have an integer winding mode along the first torus and the S direction, whereas half-windings appear between O-planes. This explains the factor $(1 + 3\delta_{a,O})$ in (4.45). The degeneracy is given by $d_i^U = d_N^U$, with d_N^U defined in (4.38).

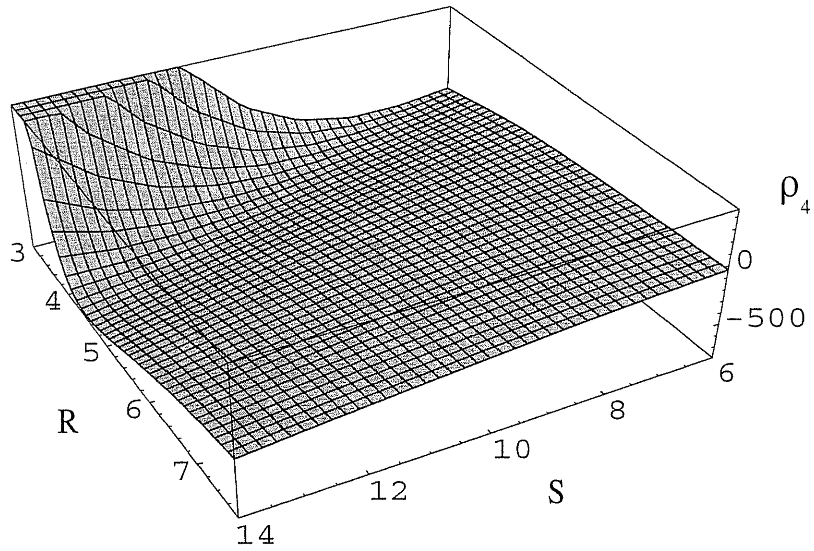

Figure 4.5: Dependence of the cosmological constant for the 4-d model on the moduli, expressed in units of α'. R is the Scherk-Schwarz radius and S is the radius of the other dimensions.

Twisted string contribution As before, this is given by the sum of the Klein bottle, annulus and Möbius strip amplitudes. It is given by

$$\rho_4^{(T,t)} = -\frac{4\pi}{3}\sum_{i,\hat{w}} d_i^T \left\{2G_2^{(i)}\left[\Delta x(\hat{w})^{(O,T)}\right] + \left[1-2(-1)^N\right]G_2^{(i)}\left[\Delta x(\hat{w})^{(C,T)}\right]\right\}, \quad (4.46)$$

where the index i runs over N, the string oscillator number, and hence

$$m_i^{2(a,T)} = \frac{4N}{\alpha'}, \quad (4.47)$$

independent of a, whereas w includes the winding modes in the second torus:

$$\left(\Delta x(\hat{w})^{(a,T)}\right)^2 = \frac{\pi^2 R^2}{\alpha'}\left(w+\frac{1}{2}\right)^2 + (1+3\delta_{a,O})\frac{\pi^2 S^2}{\alpha'}w'^2\,. \quad (4.48)$$

The degeneracy is given by

$$\sum_i d_i^T\, e^{-4\pi i l} = 3\,\frac{\theta_2^2}{\eta^6}\,\frac{\theta\begin{bmatrix}1/6\\0\end{bmatrix}}{\theta\begin{bmatrix}1/6\\1/2\end{bmatrix}}\,\frac{\theta\begin{bmatrix}1/3\\0\end{bmatrix}}{\theta\begin{bmatrix}1/3\\1/2\end{bmatrix}}\,. \quad (4.49)$$

All the considerations performed after (4.45) apply also here, in order to understand the form of (4.46) and (4.48).

It is interesting to note that the transverse contribution is always negative, due to the fact that the Klein bottle and annulus amplitudes are negative and dominant over the positive Möbius strip amplitude.

4.4.3 Behavior of the cosmological constant

The full 4D cosmological constant is given by adding together all the contributions, coming from both the longitudinal and the transverse SS breaking sectors:

$$\rho_4 = \rho_4^{(C,l)} + \rho_4^{(O,l)} + \rho_4^{(U,t)} + \rho_4^{(T,t)} . \tag{4.50}$$

For generic values of $\mathrm{Tr}\,\gamma_\beta \neq 0$, the longitudinal and transverse contributions, both negative, result in a monotonic $\rho_4 < 0$, leading the system towards the tachyonic instability. The same instability affects the IIB orbifold model before the Ω-projection, as can be seen by studying the torus contribution. Things are more interesting when $\mathrm{Tr}\,\gamma_\beta = 0 = \mathrm{Tr}\,\gamma_\theta^2\gamma_\beta$. In this case, the $D9$ annulus contribution vanishes and the longitudinal contribution to ρ_4 is greater than zero and can thus compensate the always negative transverse contribution. This feature is characteristic of a longitudinal breaking when the SUSY-breaking operator is of order even. In presence of operators of order odd, as in section 4, it is not possible to take such a choice for the twist matrices. For this reason, although we have not performed a detailed analysis, we expect that 4D models with odd SS SUSY twists, such as the model constructed in [72], will unavoidably have $\rho_4 < 0\ \forall R$, and end up in the tachyonic regime. In the following we thus focus to the case in which both γ_β and $\gamma_\theta^2\gamma_\beta$ are taken to be traceless.

In Fig. 4.4 we present the behavior of the one-loop cosmological constant as a function of the SS radius where the other radii has been fixed to $S = P_1 = P_2 \sim 10$ in units of α'. We found a minimum for ρ_4 which is essentially due to a compensation between the longitudinal (positive) and transverse (negative) monotonic contributions.

In order to get a better understanding of the fate of this minimum as the other moduli are varied, we study the behavior of ρ_4 as a function of the two parameters R and $S = P_1 = P_2$. The numeric result (figure 4.5) shows that the structure of the minimum in R is still preserved as long as $S > 8$ but $\rho_4(R, S)$ drives both moduli to larger values and hence to a decompactification limit.

Other cases similar to the latest one have been considered by studying ρ_4 as a function of two moduli, one of them being the SS circle radius R and the other one the value of one or more of the extra moduli. In all the studies the scenario with a minimum in the R direction is preserved, whereas the runaway behavior depends on how the extra moduli are fixed. As an example, fixing all the moduli in the first and second torus, $\rho_4(R, S)$ still develops a minimum in R but the behavior along S drives the system to smaller radii and thus towards the tachyonic instability.

Chapter 5

String vacua with torsion from freely-acting orbifolds

The possibility offered by string quantization on a background space with non-trivial torsion has been introduced in the heterotic context by Strominger [43], the main motivation being an enlargement of the class of known string backgrounds.

The starting point is a background spacetime allowing a warp factor and, in general, non trivial torsion and vacuum for the gauge field. We follow [43] (see also [66] for a recent review) and take a metric ansatz of the type

$$ds^2 = g^0_{MN} dx^M dx^N = e^{2\Delta(y)}(\hat{g}_{\mu\nu} dx^\mu dx^\nu + \hat{g}_{mn} dy^m dy^n), \tag{5.1}$$

where as usual greek indexes refers to the 4D spacetime while latin uppercase indexes to the whole space and latin lowercase indexes to the 6D internal space. Furthermore we introduce the rescaled metric $g_{MN} = e^{-2\phi} g^0_{MN}$, where ϕ is the dilaton. The $N = 1$ SUSY variation in the case of heterotic string theory can be written as

$$\begin{aligned} \delta\psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MNP} \Gamma^{NP} \epsilon \\ \delta\chi &= -\frac{1}{4} F_{MN} \Gamma^{MN} \epsilon \\ \delta\gamma &= \nabla\!\!\!/ \phi + \frac{1}{24} H_{MNP} \Gamma^{MNP} \epsilon \end{aligned} \tag{5.2}$$

where ψ is the gravitino, χ is the gluino, γ is the dilatino and $\Gamma^{M_1 M_2 \ldots M_n}$ is the usual total antisymmetrized product of n γ matrices. The covariant derivative is computed with respect to the rescaled metric g_{MN}.

In the case of trivial warp and backgrounds the conditions implied by equations (5.2) reduce to $\nabla_M \epsilon = 0$. This well-known condition is extremely strong because it implies the integrability condition $[\nabla_M, \nabla_N]\epsilon = 0$, equivalent to

$$R_{MNPQ} \Gamma^{PQ} \epsilon = 0, \tag{5.3}$$

with R_{MNPQ} the usual Riemann tensor. The implications of this equation are reviewed, for example, in [102]. In particular, choosing as a background for the 4D spacetime a maximally symmetric space (de Sitter, anti de-Sitter, Minkowski) the condition is satisfied only in the Minkowski case. The conditions on the internal space are instead broader. Essentially we are looking for a covariantly constant spinor, or, in other words, for a spinor that is mapped on itself after parallel transport along any contractible closed curve. The symmetry group that is responsible for the transformations of a spinor under such a transport is the so-called Holonomy group, that is in general a subgroup of the $SO(6)$ group of rotations. If the holonomy group is the full $SO(6)$ then no spinor is left invariant, since no spinor is a singlet of $SO(6)$. We need that the holonomy group is reduced to some G admitting invariant spinors. This is the case of $SU(3) \in SO(6)$, who admits exactly one the covariantly constant spinor.

Moreover, given a manifold with a Riemannian metric g_{mn} and a covariantly constant spinor ϵ we can build the form $K_{ij} = \bar{\epsilon}\Gamma_{ij}\epsilon$, the tensor $J^i_j = g^{ik}K_{kj}$ and the form $w_{ijk} = \epsilon^T\Gamma_{ijk}\epsilon$. It can be shown that J is a complex structure for the metric so that locally the manifold is complex. It can also be shown that the manifold is also a Kähler manifold, i.e. its metric can be deduced from a scalar potential.

On the other hand the requirement (5.3) implies that $\Gamma^k R_{ik}\epsilon = 0$, where R_{ik} is the Ricci tensor. This implies that the Ricci tensor is zero and the manifold is Calabi-Yau.

These well-known deductions are crucially modified if the warp $\Delta(y)$ and a non-trivial H are allowed. Assuming that the dilaton, warp and torsion depend only on the internal coordinates and that the components of H tangent to the space-time vanish, then the equation for the dilatino is splitted in a 4D space-time equation and an equation for the internal coordinates.

The 4D equation can be written in terms of the the warp-rescaled metric $\hat{g}_{\mu\nu}$ as

$$\hat{\nabla}_\mu\epsilon + \frac{1}{2}\hat{\Gamma}^n_\mu\hat{\nabla}_n(\Delta(y) - \phi)\epsilon = 0. \tag{5.4}$$

As in the previous simpler case an integrability equation is implied. Imposing that the 4D space-time is maximally symmetric we deduce that $\hat{g}_{\mu\nu}$ is the usual Minkowski metric while the warp factor is equal to the dilaton.

To study the geometric implications offered by the gravitino equation related to the internal coordinates it is useful to introduce a tensor J from ϵ exactly as in the previous case.

The gravitino equation is read, in this way, as a condition on this "would be" complex structure, imposing in particular that it is covariantly constant with respect to a covariant derivative defined from the usual connection plus a torsion term taken to be equal to H

$$\nabla_m J^p_n - H^p_{sm}J^s_n - H^s{}_{mn}J^p_s = 0. \tag{5.5}$$

The previous case is clearly obtained imposing $H = 0$. Also the dilatino equation can be seen as a constraint, imposing that the Nihenuis tensor computed for J

$$N^p_{mn} = J^q_m(\partial_q J^p_n - \partial_n J^p_q) - J^q_n(\partial_q J^p_m - \partial_m J^p_q), \tag{5.6}$$

is zero. These conditions ensure that the manifold is complex with complex structure J.

Furthermore these two conditions are solved if the complex structure is related to the torsion. This relation can be written, introducing complex coordinates, as

$$H = \frac{i}{2}(\bar{\partial} - \partial)J \tag{5.7}$$

where now J is considered a (1,1) form, ∂ is the holomorphic part of the usual exterior derivative and $\bar{\partial}$ the antiholomorphic one.

The equation for the dilatino instead is a relation between the dilaton and H. It can be cast in the following form

$$d^\dagger J = 8i(\partial - \bar{\partial})\phi, \tag{5.8}$$

where $d^\dagger$ is defined as the adjoint of the exterior derivative d. It is interesting to note that the last formula measure exactly the non-Kählerity of the new manifold, while the Kähler case is reached if the dilaton, i.e. the warp factor, is a constant.

Also the gluino equation can be recast in a complex coordinates formalism, becoming a condition on the complex form of F:

$$J^{a\bar{b}} F_{a\bar{b}} = 0, \tag{5.9}$$

$$F_{\bar{a}\bar{b}} = F_{ab} = 0. \tag{5.10}$$

We have shown the main properties of an heterotic string background with non-trivial warp and torsion, obtaining the conditions that the various background fields must satisfy. In what follow we show an explicit example of such a background, obtained starting from a freely acting orbifold model through a suitable net of dualities.

We start from a simple toy model, with all SUSY broken through the SS mechanism. We show how a reparametrization maps it on a modified Melvin background that is T-dual to a background with non trivial vacuum for the B-field. We establish the power and the limits of the dualities, then we undertake the study of a SUSY model to give the explicit example of a non-Kahler background geometry for heterotic string theory preserving N=2 SUSY.

5.1 A simple toy model

It is well-known the duality between a SS compactification/freely acting orbifold and the so called "flux-tube" of [40], a generalization of Melvin backgrounds [37]. To review it take into account the simple freely-acting orbifold introduced in chapter 2, with the orbifold group $\mathbf{Z}_2$ acting as a reflection in a 2D flat space (T^2 or $\mathbb{C}$) parametrized by z and as a translation in a compact direction x, namely

$$g\ : z \to -z, \ \ g : x \to x + \pi R,$$

where $x \in [0, 2\pi R]$ and the eventual periodicity of z is defined as

$$z \sim z + S(m\tau + n),$$

m and n being two integer numbers, τ the torus modular parameter and S a radial parameter. For simplicity we take $\tau = i$, so that the volume of the torus is S^2. With these choices the metric is diagonal and multiple of the identity

$$ds^2 = dx^2 + dzd\bar{z}.$$

It is possible to redefine the parametrization as

$$z' = e^{ix/R}z, \ \ x' = x.$$

In the new variables the metric is no more trivial

$$ds^2 = dx'^2 + (dz' - i\frac{z'}{R}dx)\,(d\bar{z}' + i\frac{\bar{z}'}{R}dx),$$

and while the x' periodicity is the same as before the new z' one is different

$$z' \sim z' + (im + n)e^{ix/R}. \tag{5.11}$$

Passing to orthogonal coordinates $x' = x_1$, $z' = x_2 + ix_3$, we have

$$ds^2 = \left(1 + \frac{x_2^2 + x_3^2}{R^2}\right) dx_1^2 + dx_2^2 + dx_3^2 + \frac{2dx_1}{R}(x_3 dx_2 - x_2 dx_3). \tag{5.12}$$

The new metric is clearly invariant under a translation along x_1, so that the x_1 periodicity is easily proved. It is not difficult to prove that also the periodicity along x_2 and x_3 is still respected, but expressed in the new terms of (5.11).

The presence of non-trivial off-diagonal terms in the metric makes this background interesting from the point of view of T-duality, since the the off-diagonal terms are mapped to a non-trivial B-field. This has been introduced in [41, 42]. Let us briefly review the T-duality along the lines of [110] to understand its power and limits.

Considering the non-linear sigma model associated by string theory to this background we have

$$S_O = \frac{1}{4\pi\alpha'}\int d^2\sigma\left[\sqrt{\gamma}(\gamma^{\mu\nu}g_{ab} + \epsilon^{\mu\nu}h_{ab})(\partial_\mu x^a)(\partial_\nu x^b) + \alpha'\sqrt{\gamma}R^{(2)}\phi(x)\right], \tag{5.13}$$

where $\gamma^{\mu\nu}$ is the inverse of the world sheet metric, g_{ab} the target-space metric (5.12), h_{ab} is the vacuum expectation value for the antisymmetric B-field, zero in our case,

$R^{(2)}$ is the curvature scalar associated to $\gamma_{\mu\nu}$ and $\phi(x)$ is the dilaton field in the conventions of [95][1].

T-duality along a direction x_i is allowed if translation along x_i leave S invariant, in other words if g_{ab} and h_{ab} are invariant under a translation along x_i.

In our case x_1 is exactly of this kind, in other words it is a Killing vector for our metric so the T-duality along it is allowed. The T-dual model is reached following the world-sheet procedure described in [109, 110]. There it is shown how a new action can be written by introducing a lagrange multiplier $\hat{x}_1$

$$S_F = \frac{1}{4\pi\alpha'}\int d^2\sigma\Big\{\sqrt{\gamma}\gamma^{\mu\nu}\left[g_{11}V_\mu V_\nu + 2g_{1i}V_\mu(\partial_\nu x^i) + g_{ij}(\partial_\mu x^i)(\partial_\nu x^j)\right] + \epsilon^{\mu\nu}\left[h_{1i}V_\mu(\partial_\nu x^i) + h_{ij}(\partial_\mu x^i)(\partial_\nu x^j)\right] + \epsilon^{\mu\nu}\hat{x}_1(\partial_\mu V_\nu) + \alpha'\sqrt{\gamma}R^{(2)}\phi(x)\Big\}, \quad (5.14)$$

where i and j are summed only on 2 and 3. This action is equivalent to the first one: by integrating the Lagrange multiplier the condition $\epsilon^{\mu\nu}(\partial_\mu V_\nu)$ is obtained, implying that V_μ is irrotational and can be written as a differential of a scalar field to be identified with x_1. Replacing V_μ with $\partial_\mu x_1$ one obtains S_O. The T-duality is realized if instead the equation of motion for V are solved and V is replaced with its solution. A new action S_D is obtained of the form (5.13), but with new metric and h:

$$\hat{g}_{11} = \frac{1}{g_{11}}, \quad \hat{g}_{1i} = \frac{h_{1i}}{g_{11}}, \quad \hat{g}_{ij} = g_{ij} - \frac{g_{1i}g_{1j} - h_{1i}h_{1j}}{g_{11}}; \quad (5.15)$$

$$\hat{h}_{1i} = \frac{g_{1i}}{g_{11}}, \quad \hat{h}_{ij} = h_{ij} + \frac{g_{1i}h_{1j} - g_{1j}h_{1i}}{g_{11}}. \quad (5.16)$$

The new action is classically equivalent to the original one, from a quantum point of view this is also true provided that the dilaton is suitably shifted [110], for a review see [111, 112]

$$\hat{\phi} = \phi - \frac{1}{2}\log[g_{11}]. \quad (5.17)$$

The described duality is essentially equivalent to the procedure described by [113, 114], where the symmetry induced by the Killing vector is gauged in order to obtain the action (5.14).

In the case of our interest the duality and the results are clear if the x_2 and x_3 dimensions are non-compact. In the compact case while the original action is manifestly invariant under a translation along a lattice element the final one does not show this invariance. In particular already S_F is not invariant, and an obstruction to the T-duality seems to be present. From a geometric point of view this is related to the fact that while x_1 is a good Killing vector, generating a symmetry group locally leaving invariant the system, the fact that the new periodicity (5.11) is not

[1] Note that this dilaton field differs from that introduced to study the SUSY equation of motion. In particular the new dilaton is 4 times the old one

x_1-invariant means that globally the symmetry is not an invariance for our system, and so the duality cannot be realized in this simple way[2].

This does not conclude the discussion in the compact case, since, as shown by Hassan in [115], it is possible to generalize the duality also to geometries dependent on the coordinate with respect to which duality is performed. It is possible that, as SUSY in the case studied therein, also the periodicity can be realized through a non-trivial non-local fibration of the torus along the new SS variable $\hat{x}_1$ (see also [116]).

5.2 An heterotic model

Let us consider the heterotic string ($SO(32)$ or $E_8 \times E_8$) on $R^4 \times S^1 \times (T^4 \times S^1)/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts as $z_k \to \exp(2i\pi v_k) z_k$ on the two complex coordinates $z_{1,2}$ of $T^4 = T^2 \times T^2$, where $v_k = (1/2, -1/2)$, and at the same time as an half-shift on the circle: $x \to x + \pi R$, where R is the radius of the circle. For simplicity, we take the two tori to be rectangular, with complex structures $\tau_n = i$, $n = 1, 2$.

This model is a simple supersymmetric freely-acting compact orbifold model. It corresponds to a Scherk-Schwarz compactification, where fields are twisted according to their $SO(4)$ Lorentz quantum numbers along the direction of T^4. In 4D notation, the model has $N = 2$ supersymmetry. Modular invariance imposes that a rotation has to be implemented also in the internal lattice sector. In a fermionic world-sheet formulation in terms of 16 complex fields λ^A, $\mathbf{Z}_2$ acts as $\lambda^A \to \exp(2i\pi V_A)\lambda^A$, $A = 1, \ldots, 16$. Modular invariance imposes that $v_k^2 - V_A^2 = n$, with n any integer number. For simplicity, we focus in the following on a standard embedding, in which $V_A = (1/2, -1/2, 0, \ldots, 0)$.

The associated 1+1 dimensional σ-model is clearly an exact free super conformal field theory (SCFT) (in the RNS formalism), with the above identification for the fields. In an $N = 1/2$ superspace language [117], the relevant SCFT associated to the 5 directions $(z_1^{(0)}, z_2^{(0)}, x)$ and the $\lambda^{A(0)}$'s fields is the following:

$$\mathcal{L} = -i \int d\theta \left[DX\bar{\partial}X + \frac{1}{2} \sum_{k=1,2} (DZ_k^{(0)} \bar{\partial} \bar{Z}_k^{(0)} + D\bar{Z}_k^{(0)} \bar{\partial} Z_k^{(0)}) - i \sum_{A=1}^{16} \Lambda_A^{\star(0)} D\Lambda^{A(0)} \right], \quad (5.18)$$

where $X, Z_k^{(0)}$ are superfields of the form $\Phi^k = x^k + \theta\psi^k$, with ψ^k left-moving world-sheet fermions, and $\Lambda^{A(0)}$ superfields $\Lambda^{A(0)} = \lambda^{A(0)} + \theta F^{A(0)}$, where $\lambda^{A(0)}$ are right-moving world-sheet fermions and $F^{A(0)}$ auxiliary fields. The $N = 1/2$ covariant derivative is defined as

$$D = \frac{\partial}{\partial\theta} + i\theta\partial\,, \quad (5.19)$$

[2] This seems to have been overlooked in [55] where, however, these vacua have been neglected for other reasons.

in complex coordinates, where $\partial = \partial/\partial z = \partial_\tau + i\partial_\sigma$. The $\mathbf{Z}_2$ action implies that $(Z_k^{(0)}, \Lambda_{1,2}^{(0)}, X) \sim (-Z_k^{(0)}, -\Lambda_{1,2}^{(0)}, X + \pi R)$. As in the previous section we define world-sheet superfields

$$
\begin{aligned}
Z_{1,2} &= Z_{1,2}^{(0)} \exp(-iX/R), \quad \bar{Z}_{1,2} = \bar{Z}_{1,2}^{(0)} \exp(iX/R)\,, \\
\Lambda_{1,2} &= \Lambda_{1,2}^{(0)} \exp(-iX/R), \quad \Lambda^*_{1,2} = \Lambda_{1,2}^{*(0)} \exp(iX/R)\,,
\end{aligned} \tag{5.20}
$$

so that all fields become single-valued along the SS direction x. In this way, integrating over the Grassmanian variable θ and solving for the auxiliary fields F^*_A and F^A one finds:

$$
\mathcal{L} = g_{\mu\nu}\partial x^\mu \bar{\partial} x^\nu + i g_{\mu\nu}\psi^\mu \left[\bar{\partial}\psi^\nu + \Gamma^\nu_{\rho\sigma}\bar{\partial}x^\rho\psi^\sigma\right] + \lambda^*_A\left[\partial\lambda^A + iA^A_x\partial x^x\lambda^A\right], \tag{5.21}
$$

where $x^\mu = (x, z^1, \bar{z}^1, z^2, \bar{z}^2)$, $A^A_x = 1/R(i, -i, 0, \ldots, 0)$ is a discrete $\mathbf{Z}_2$ Wilson line[3], and $g_{\mu\nu}$ is the metric:

$$
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{|z_1|^2 + |z_2|^2}{R^2}\right)dx^2 + \frac{1}{2}dz_k d\bar{z}_k + \frac{i}{2R}dx(z_k d\bar{z}_k - \bar{z}_k dz_k)\,. \tag{5.22}
$$

As told the metric (5.22) is a generalization of the 4D Melvin background metric for a compact space, that is known to correspond to a SS dimensional reduction on a circle of a higher dimensional flat space [37]. As before the periodicity conditions for the z^k's are x-dependent: if $z^{k(0)} \sim z^{k(0)} + 1 \sim z^{k(0)} + i$, the new coordinates satisfy the periodicity conditions

$$
z^k \sim z^k + e^{-ix/R} \sim z^k + ie^{-ix/R}\,. \tag{5.23}
$$

The T^4 torus is now non-trivially fibered along the SS direction x. From the construction above, it is clear that such backgrounds are exact classical solutions to all orders in α', being related by a simple rescaling to a free SCFT. The latter theory has actually $N = (4, 0)$ world-sheet SUSY and implies that the background manifold is an hyper-Kähler manifold, in particular a complex manifold. Since the complex structure J turns out to play an important role in the following, we explicitly derive its form for the metric (5.22). The complex structure J associated to the original freely-acting orbifold $S^1 \times (T^4 \times S^1)/\mathbf{Z}_2$, being globally well-defined despite the $\mathbf{Z}_2$ shift, can be taken to be the trivial one, as in flat space. After the change of coordinates (5.20), J takes the form

$$
J^\mu{}_\nu = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
x_4/R & -x_3/R & 0 & 1 & 0 & 0 \\
-x_3/R & -x_4/R & -1 & 0 & 0 & 0 \\
-x_6/R & x_5/R & 0 & 0 & 0 & 1 \\
x_5/R & x_6/R & 0 & 0 & -1 & 0
\end{pmatrix}, \tag{5.24}
$$

[3]The factor of i arises because in our conventions the generators of the gauge group are chosen to be anti-hermitian. This is a non-trivial Wilson line because, due to the shift, the effective radius of the SS direction is $R/2$, rather than R.

where the first two entry in J are taken respectively along the directions of the two circles where the $\mathbf{Z}_2$ does not (x_1 direction) and does act ($x_2 \equiv x$ direction), and we have introduced real coordinates defined as $z_k = x_{2k+1}+ix_{2k+2}$. It is straightforward to verify that J is actually a complex structure, with $J^2 = -I$ and vanishing associated Nijenhuis tensor.

5.2.1 Massless spectrum

The massless spectrum of this model is easily obtained, being closely related to that of the heterotic string on the well-known $T^4/\mathbf{Z}_2 \times T^2$ orbifold. In terms of 4D $N = 2$ SUSY multiplets, we get one gravitational multiplet, 3 $U(1)$ vector multiplets and 4 hypermultiplets from the gravitational sector, *i.e.* from the decomposition of the 10D metric, antisymmetric tensor field and dilaton. The gauge sector is also straightforward. Focusing on the $SO(32)$ case, we have one vectormultiplet in the adjoint of the $SO(28) \times SO(4)$ group, that is the unbroken gauge group in 4D, and one hypermultiplet in the bifundamental $(\mathbf{28}, \mathbf{4})$.

Notice that this spectrum is essentially a truncation of that of the $SO(32)$ heterotic string on $T^4/\mathbf{Z}_2 \times T^2$.[4] In the $T^4/\mathbf{Z}_2 \times T^2$ case, we would have obtained all the states as before, but in addition other states arising from twisted sectors. More precisely, 16 neutral and 16 charged hypermultiplets (one for each of the 16 fixed points of $T^4/\mathbf{Z}_2$), the latter in the $(\mathbf{28}, 4)$ representation of $SO(28) \times SO(4)$, with $\mathbf{4}$ the spinor representation of $SO(4)$. In presence of the $\mathbf{Z}_2$ shift, the twisted vacuum state carries a non-trivial winding number and is thus massive. We see, then, that compared to the $T^4/\mathbf{Z}_2 \times T^2$ orbifold case (no shift), many moduli (geometrical and not) have been lifted, precisely like in presence of fluxes.

5.2.2 The T-dual model

T-duality can be performed exactly as in the simple toy model discussed before. In order to avoid the problem with periodicity described, we take the non-compact limit of the above string vacua, where the model looks like $R^4 \times S^1 \times (C^2 \times T^2)/\mathbf{Z}_2$. In this case, $\partial/\partial x$ defines an isometry and we can perform a T-duality transformation. The T-duality rules for generic non-flat backgrounds are known [110][5]. In the heterotic case we are considering, they have been derived by [118] at leading order in a derivative expansion from a low-energy effective action point of view. They recover the usual T-duality rules of the flat space reviewed, for example, in [111, 112]. As far as we know, so far there is no a satisfactory world-sheet derivation of such rules[6]. For our

[4]This is true only at the massless level.

[5]In general such T-duality rules get higher-order α' corrections.

[6]Refs.[119] and [120] discuss a σ-model approach to T-duality in heterotic theories, but they do not seem to recover the usual T-duality rules for simple toroidal compactifications, missing some

purposes, it will be useful to consider directly the T-dual version of the complex structure J. Following [121], simple T-duality rules for the inverse metric g^{-1}, the complex structure J and the gauge connection A can be written in terms of a matrix $Q^{\mu}{}_{\nu}$ (see [121] for details),

$$Q_{\mu\nu} = (I - S)_{\mu\nu} + S_{\mu}^{\ \theta} K_{\theta\nu}, \tag{5.25}$$

where I is the identity matrix, $S_{\mu\nu}$ is a the diagonal matrix $\delta_{i\mu}\delta_{i\nu}$ i is the direction along which the T-duality is performed, δ_{ij} is the usual Kronecker symbol and

$$K_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu} + \frac{1}{2}\sum_a A^a_\mu A^a_\nu. \tag{5.26}$$

Defining with abuse of notation the new objects with the same names of the old ones,

$$\begin{aligned} J &\to Q\,J\,Q^{-1}\,, \\ g &\to Q\,g\,Q^{T}\,, \\ A^A &\to A^A\,Q^{-1}\,. \end{aligned} \tag{5.27}$$

One obtains the new metric

$$ds^2 = dx_1^2 + \frac{R^2(f-\eta\alpha')}{f^2}dx_2^2 + \frac{2R}{f^2}dx_2(x_4dx_3 - x_4dx_3 + x_5dx_6 - x_6dx_5) + \tag{5.28}$$

$$\sum_{i=3}^{6} dx_i^2 - \frac{f+\eta\alpha'}{f^2}\left[(x_4dx_3 - x_3dx_4)^2 + (x_5dx_6 - x_6dx_5)^2 - \right. \tag{5.29}$$

$$\left.(x_4x_6dx_3dx_5 - x_4x_5dx_3dx_6 - x_3x_6dx_4dx_5 + x_3x_5dx_4dx_6)\right] \tag{5.30}$$

where $\eta = 1$ and f is

$$f = R^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + \eta\alpha'. \tag{5.31}$$

The B-field takes the form

$$B = \frac{R}{f}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & x_3 & x_6 & -x_5 \\ 0 & -x_4 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 & 0 & 0 \\ 0 & -x_5 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.32}$$

The Wilson Lines are mapped to a non trivial background for the gauge field

$$A_1 = -A_2 = \frac{1}{f}\begin{pmatrix} 0 \\ 1 \\ -2\pi v x_4 \\ 2\pi v x_3 \\ -2\pi v x_6 \\ 2\pi v x_5 \end{pmatrix}. \tag{5.33}$$

corrections due to Wilson lines.

The complex structure matrix is

$$J = \frac{1}{f} \begin{pmatrix} 0 & R^2 & -x_4 R & x_3 R & x_6 R & -x_5 R \\ -f(1+\eta\alpha'/R^2) & 0 & x_3 f/R & x_4 f/R & -x_5 f/R & -x_6 f/R \\ x_4 f/R & -x_3 R & x_3 x_4 & f - x_3^2 & -x_3 x_6 & x_3 x_5 \\ -x_3 f/R & -x_4 R & -f + x_4^2 & -x_3 x_4 & -x_4 x_6 & x_4 x_5 \\ -x_6 f/R & x_5 R & -x_4 x_5 & x_3 x_5 & x_5 x_6 & f - x_5^2 \\ x_5 f/R & x_6 R & -x_4 x_6 & x_3 x_4 & -f + x_6^2 & -x_5 x_6 \end{pmatrix} . \tag{5.34}$$

The form of H is easily given in components, using the notation of [96]:

$$\begin{aligned} H_{1ab} &= 0, & H_{\hat{a}\hat{b}\hat{c}} &= \frac{2\alpha'}{f^2} \sum_{d=3}^{6} \epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} \, x_{\hat{d}}, \\ H_{234} &= -\frac{2R}{f^2}(R^2 + x_5^2 + x_6^2), & H_{235} = -H_{246} &= \frac{2R}{f^2}(x_4 x_5 + x_3 x_6), \\ H_{256} &= -\frac{2R}{f^2}(R^2 + x_3^2 + x_4^2), & H_{236} = H_{245} &= \frac{2R}{f^2}(x_4 x_6 - x_3 x_5), \end{aligned} \tag{5.35}$$

where $\hat{a}$ is an index running only in $\{3,4,5,6\}$ and $\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is the usual totally antisymmetric tensor of the 4D subspace $x_3 \dots x_6$, $\epsilon_{3456} = 1$.

The description here is valid also in the type IIB string case, provided that one puts the Wilson line parameter to zero. This is done putting, in the previous formulas $\eta = 0$.

The dual dilaton is as usual

$$\Phi \to \Phi - \frac{1}{4} \log \left[\frac{\mathrm{Det}\, g}{\mathrm{Det}\, \tilde{g}} \right] . \tag{5.36}$$

The background defined by (5.28) corresponds to an heterotic SUSY vacuum on a non-Kähler manifold with non-trivial torsion H.

As shown it must satisfy the equations of motion and the relation between the torsion and the complex structure written in the new conventions of [96]

$$\begin{aligned} d^\dagger \hat{J} &= 2i(\partial - \bar{\partial})\Phi \,, \\ \hat{J}^{a\bar{b}} F_{a\bar{b}} &= 0 \,, \\ F_{ab} = F_{\bar{a}\bar{b}} &= 0 \,, \end{aligned} \tag{5.37}$$

$$H = i(\bar{\partial} - \partial)\hat{J} \,, \tag{5.38}$$

Moreover the torsion H should satisfy also the corrected Bianchi identity, written with the conventions of [96] as

$$dH = \frac{\alpha'}{4}\left(\mathrm{tr}\, R^2 - \mathrm{tr} F^2\right) . \tag{5.39}$$

Notice that the equations of motion (5.37), as well as the definition (5.38) for H or the Bianchi identity (5.39) are local expressions valid for any six-dimensional compactification manifold, and hence should be satisfied also in the non-compact limit we are considering. Starting from the explicit form of (5.28), it is straightforward, although laborious, to verify that the equations of motion (5.37) are, in fact, exactly verified. Although in the original model $F = 0$, the T-dual field strength is non vanishing and thus the last two equations in (5.37) are satisfied in a non-trivial way. Notice that going to complex coordinates is not necessary to check (5.37) or the Bianchi identity (5.39). For instance, in real coordinates (5.37) read as follow (omitting the tilde for the T-dual fields):

$$\begin{aligned} \partial_\mu J^\mu{}_\nu &= -2J_\nu{}^\mu \partial_\mu \Phi\,, \\ J^{\mu\nu} F_{\mu\nu} &= 0\,, \\ J^\mu{}_\rho J^\nu{}_\sigma F_{\mu\nu} &= F_{\rho\sigma}\,. \end{aligned} \tag{5.40}$$

On the other hand, the torsion H, as defined in (5.38), does not satisfy the Bianchi identity (5.39), but only its two-derivative version

$$dH = -\frac{\alpha'}{4}\mathrm{tr}F^2\,. \tag{5.41}$$

The reason for this discrepancy can be traced back to the T-duality transformation rules we have used. The latter, as we said, have been derived from a low-energy effective action approach, in a derivative expansion. As such, the term $\mathrm{tr}\,R^2$, being a 4-derivative term, is of higher-order with respect to dH or $\mathrm{tr}\,F^2$ in (5.39). Since higher-order corrections to the T-duality rules are in general expected for non-trivial backgrounds, we see that no inconsistency arises. On the contrary, it is somehow unexpected that the equations of motion (5.37) are satisfied by our background exactly, and not only at leading order in α'. Although we do not have a full understanding of this result, we believe it has to do with the higher degree of symmetry we have, in particular to the fact that our background is actually $N = 2$ space-time supersymmetric, with an associated $N = (4, 0)$ SCFT.

5.2.3 Geometric description

Supersymmetric string vacua with non-vanishing fluxes are best classified by the group structures (or G-structures), rather than the holonomy, of the compactification manifold. Roughly speaking, a d-dimensional manifold admits a group structure $G \subseteq SO(d)$ if all tensors (and spinors) can be decomposed globally into representations of G. Classifications of a large class of supersymmetric string and M-theory vacua in terms of G-structures has been derived in [70]. In a 4D heterotic context, it has been shown in [66] how $SU(3)$-structures are particularly useful in classifying vacua with torsion. The latter can be decomposed into 5 classes, denoted $\mathcal{W}_i$, $i = 1, \ldots, 5$, according to their different representations under $SU(3)$. The equations of motion (5.37) and the relation (5.38) between the complex structure J and the torsion H can be rephrased as a constraint on the possible torsion classes of H. One finds that [66] (see also [108]) $\mathcal{W}_1$, $\mathcal{W}_2$ and the combination $2\mathcal{W}_4 + \mathcal{W}_5$ must vanish, with $\mathcal{W}_4$ and $\mathcal{W}_5$ real and exact. All the above considerations must hold also in the non-compact limit and thus apply to our T-dual heterotic configuration. In what follows, along the lines of [66], we compute $\mathcal{W}_i$ corresponding to our particular string vacuum and show that it is actually of the most general form, where all three classes $\mathcal{W}_3$, $\mathcal{W}_4$ and $\mathcal{W}_5$ are non-vanishing[7].

We introduce a basis of vielbeins e^i

$$
\begin{aligned}
e^1_\mu &= (1,\, 0,\, 0,\, 0,\, 0,\, 0)\,, \\
e^2_\mu &= \frac{R}{f}\,(0,\, R,\, -x_4,\, x_3,\, x_6,\, -x_5)\,, \\
e^3_\mu &= \frac{1}{f}\left(0,\, Rx_4,\, f - x_4^2,\, x_3x_4,\, x_4x_6,\, -x_4x_5\right), \\
e^4_\mu &= \frac{1}{f}\left(0,\, -Rx_3,\, x_3x_4,\, f - x_3^2,\, -x_3x_6,\, x_3x_5\right), \\
e^5_\mu &= \frac{1}{f}\left(0,\, -Rx_6,\, x_4x_6,\, -x_3x_6,\, f - x_6^2,\, x_5x_6\right), \\
e^6_\mu &= \frac{1}{f}\left(0,\, Rx_5,\, -x_4x_5,\, x_3x_5,\, x_5x_6,\, f - x_5^2\right),
\end{aligned}
\tag{5.42}
$$

so that the complex structure J reads

$$
J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6\,. \tag{5.43}
$$

It is useful to define a (3,0)-form (with respect to the above defined complex structure) Ψ:

$$
\Psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)\,. \tag{5.44}
$$

The requirement that $\mathcal{W}_1 = \mathcal{W}_2 = 0$ imply respectively that $dJ = d\Psi = 0$, as can be easily verified. This is a simple consistency check, since $\mathcal{W}_1 = \mathcal{W}_2 = 0$ is a necessary and sufficient condition for the manifold to be complex, a condition that we have already explicitly checked. The torsion class $\mathcal{W}_4$ can be directly derived from the dilaton Φ:

$$
\mathcal{W}_4 = d\Phi = -\frac{1}{f^2}\Big(\sum_{i=3}^{6} x_i dx_i\Big)\,. \tag{5.45}
$$

[7]Since our background has $N = 2$, rather than $N = 1$, SUSY, a more refined classification in terms of $SU(2)$-structures should be possible. We did not find an easy way to do that, and hence we restrict our attention to $SU(3)$-structures.

On the other hand, $\mathcal{W}_5$ can be computed starting from the real part of Ψ (see [66] for details) and satisfies the relation $2\mathcal{W}_4 + \mathcal{W}_5 = 0$. Finally, $\mathcal{W}_3$ is non-vanishing and it is obtained by taking the (2,1)-form from $\mathcal{W} = dJ - J \wedge \mathcal{W}_4$, that is

$$\begin{aligned}
\mathcal{W} = -\frac{1}{f^2}\Bigg\{ & R\, dx_1 \wedge \Bigg[R \sum_{i=3}^{6} x_i \, dx_2 \wedge dx_i + (2f - x_3^2 - x_4^2) dx_3 \wedge dx_4 + \\
& (2f - x_5^2 - x_6^2) dx_5 \wedge dx_6 - (x_4 x_5 + x_3 x_6)(dx_3 \wedge dx_5 - dx_4 \wedge dx_6) + \\
& (x_3 x_5 - x_4 x_6)(dx_3 \wedge dx_6 + dx_4 \wedge dx_5) \Bigg] + \\
& 2R dx_2 \wedge (x_3 dx_3 + x_4 dx_4) \wedge (x_5 dx_5 + x_6 dx_6) + \\
& (f - 2(x_3^2 + x_4^2)) dx_3 \wedge dx_4 \wedge (x_5 dx_5 + x_6 dx_6) + \\
& (f - 2(x_5^2 + x_6^2))(x_3 dx_3 + x_4 dx_4) \wedge dx_5 \wedge dx_6 \Bigg\}.
\end{aligned} \tag{5.46}$$

Chapter 6

Conclusions

This thesis has been devoted to the study of string models obtained by quantizing string theory over freely-acting orbifolds. It is based mainly on [72, 84, 91, 94].

We reviewed briefly the definition of an orbifold and how it is possible to compactify open string theory on it. We have shown how freely acting orbifold can be used to obtain a complete SUSY breaking through the so called SS SUSY breaking mechanism obtaining stable anomaly free models. Special attention has been devoted to two chiral 4D open string model. In this setting, we derived the known $\mathbf{Z}_6' \times \mathbf{Z}_2'$ model and constructed a new and very simple $\mathbf{Z}_3 \times \mathbf{Z}_3'$ model. Both are classically stable, since all massless NSNS and RR tadpoles vanish. The compactification backgrounds are non-SUSY deformations of usual Calabi–Yau orbifolds. In the $\mathbf{Z}_3 \times \mathbf{Z}_3'$ model, the deformation is induced by the $\mathbf{Z}_3'$ element, which is a diagonal translation in a torus together with a non-SUSY rotation along another torus. This deformation is very similar to the one that gives rise to Melvin space-time backgrounds, where a generic rotation along a non-compact plane is performed together with a $2\pi R$ translation along a circle [38][1].

A detailed study of anomaly cancellation in orbifold models has been introduced. We reviewed the well-known GS mechanism in its simple type I string realization and its generalization to orbifold models. Then we reviewed the relevance of the analysis of the localized anomalies in models with extra (compact) dimensions and a detailed study of local anomaly cancellation in the two SS models introduced has been performed. All pure gauge and mixed gauge-gravitational anomalies cancel, thanks to a generalized GS mechanism that involves also twisted RR 4-forms, necessary to cancel localized irreducible 6-form terms in the anomaly polynomial, which vanish only globally. The 4D remnant of this mechanism is a local Chern–Simons term. The local (and global) cancellation of reducible anomalies is instead ensured by twisted RR axions. In the latter case, even $U(1)$ gauge fields affected by anomalies that vanish only globally in 4D are spontaneously broken by the GS mechanism. Also

[1]See [39] for a discussion of D-branes on Melvin backgrounds.

closed string models are free from irreducible anomalies as shown in [122], whereas reducible ones are canceled.

We showed how one-loop instabilities can affect freely acting orbifolds with complete SUSY breaking and, more in general, non-SUSY string models. We showed how to compute the instabilities and we obtained that typically in these models the SS direction tends to shrink and to reach the tachyon instability, as in the early work of [18]. Only for orientifolds with a $\mathbf{Z}_2$ SS twist and with a proper choice of Chan-Paton twist matrices this situation is modified. In this case the tachyonic instability is avoided, but the radius increases with a runaway behavior toward the decompactification limit in which SUSY is restored. When more geometric moduli are involved, such as in the 4D model of section 5, the situation is more interesting but less clear.

Our results are preliminary in various respects. First of all the issue of moduli stabilization should be considered for all moduli at the same time, both geometrical and not. This approach is clearly tremendously hard unless some other arguments, like that in [32], can predict the behavior of the vacuum energy density as moduli are varied. In addition, all our results hold at one-loop level only and thus can be spoiled by higher loop corrections. It has been shown in [123], for instance, that higher loop corrections to the vacuum energy density may lead, for large R, to $\log R$ dependencies, in addition to the usual $1/R^n$ terms. Nevertheless, we think that our results indicate that the issue of moduli stabilization is somehow more interesting for models admitting a non-SUSY $\mathbf{Z}_2$ action and for which the corresponding $\rho(R) > 0$ for large R. Interestingly, models with a $\mathbf{Z}_2$ non-SUSY action and hence anti-periodic fermions in the bulk are affected by possible semi-classical instabilities at strong coupling, where space-time is eaten by a bubble of nothing [124].

We studied simple 7D model with SUSY completely broken through a SS mechanism involving a $\mathbf{Z}_N$ freely acting orbifold. In the context of closed string tachyon condensation, it would be interesting to see if the jump in the vacuum energy ρ_7 when the $\mathbf{Z}_N \to \mathbf{Z}_{N'}$ $(N' < N)$ transition takes place is exactly accounted for the tachyon condensation. In this case, assuming that no energy in form of radiation is released in the transition (like in the above case of the semi-classical false vacuum decay) and knowing that the final stage is flat space-time vacuum, one could have some hint on the value of the potential of some twisted closed string tachyon.

In order to compute the closed string loop-amplitudes we introduced and discussed a technique to unfold the torus fundamental region and simplify the form of the various integrals. The introduced technique is completely general and it turns out to be very useful when considering one-loop closed string amplitudes. It has been applied in more simple cases to compute, for example, threshold corrections to gauge couplings in heterotic string theory [90, 125, 126], or mass corrections to closed string states.

Finally we explored the duality between freely acting orbifolds and quantization of string theory on backgrounds with torsion. We introduced the subject by reviewing [43], where it is shown how the presence of a warp factor in the metric and a non-trivial torsion are responsible for a modification of the usual conditions to have $N = 1$ SUSY in an heterotic string model. We explained the features of the new background internal geometry, that is no-more a Kähler space.

We introduced the net of dualities connecting freely acting orbifold models with torsionfull backgrounds. We explicitly show in a toy model how the duality works, observing that an obstruction to the T-duality we need is present if the dimensions z's along which the SS operators acts as a rotation are compact. The obstruction is related to the fact that while all the fields are independent of the SS direction x and T-duality along it is in principle admissible, the periodicity for the z's is x-dependent and this seems to spoil the possibility of performing a T-duality. We discarded for the moment this case, leaving for future work the analysis of the T-duality in that case along the lines of [115]. In the non-compact case we studied a $N = 2$ heterotic model showing explicitly the duality SS/fluxes and giving an explicit example of geometric background with non-trivial torsion. We analyze the background according to the classification given in [66], showing that it represents a good example of a non-Kähler manifold with non-vanishing group structure classes $\mathcal{W}_3$, $\mathcal{W}_4$ and $\mathcal{W}_5$.

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Appendix A

Lattice sums

In this appendix we study the lattice sums related to open string loop amplitudes with the insertion of a SS operator g acting as a translation. The closed case (torus amplitude) has already been analyzed, see for instance [21, 22] or Appendix C.

We denote the 2D lattice sum over the i-th torus by:

$$\Lambda_i(\tau) = \sum_{n,m} \Lambda_i[m,n](\tau) = \sum_{m,n} q^{\frac{1}{2}|P_L^{(i)}|^2}\, \bar{q}^{\frac{1}{2}|P_R^{(i)}|^2}, \qquad \text{(A.1)}$$

where $q = \exp[2i\pi\tau]$ and the lattice momenta are given by

$$\begin{aligned} P_L^{(i)} &= \frac{1}{\sqrt{2\,\mathrm{Im}\,T_i\,\mathrm{Im}\,U_i}}\Big[-m_1\,U_i + m_2 + T_i\left(n_1 + n_2 U_i\right)\Big]\,, \\ P_R^{(i)} &= \frac{1}{\sqrt{2\,\mathrm{Im}\,T_i\,\mathrm{Im}\,U_i}}\Big[-m_1\,U_i + m_2 + \bar{T}_i\left(n_1 + n_2 U_i\right)\Big]\,, \end{aligned} \qquad \text{(A.2)}$$

in terms of the standard dimensionless moduli T_i and U_i, parametrizing respectively the Kähler and complex structure of the torus. We also define:

$$\Lambda_i[m] \equiv \Lambda_i[m,0](it)\;, \quad \Lambda_i[w] \equiv \Lambda_i[0,w](it)\;, \qquad \text{(A.3)}$$

and denote respectively by $\hat{\Lambda}_i[m]$ and $\hat{\Lambda}_i[w]$ the corresponding Poisson resummed lattice sums, where the dependence on the transformed modular parameter l is understood.

Annulus It is convenient to define $\Lambda[N,D\,|\,g]$ as the annulus lattice sum for Neumann (N) and Dirichlet (D) boundary conditions (b.c.) with the insertion of the operator g. The only non-trivial case to be considered is when g equals the identity or a translation δ. The relevant Poisson resummed lattice sums are found to be (omitting the index i in $\hat{\Lambda}$):

$$\Lambda[N\,|\,I\,] = \sum_m \hat{\Lambda}[m]\, W_m^{(i)} (W_m^{(j)})^{-1}\,, \tag{A.4}$$

$$\Lambda[D\,|\,I\,] = \sum_w \hat{\Lambda}[w]\, W_w^{(i)}\, (W_w^{(j)})^{-1}\,, \tag{A.5}$$

$$\Lambda[N\,|\,\delta\,] = \sum_m \hat{\Lambda}[m+\delta]\, W_m^{(i)} (W_m^{(j)})^{-1}\,, \tag{A.6}$$

$$\Lambda[D\,|\,\delta\,] = 0\,, \tag{A.7}$$

where $W_w^{(i)}$ encodes the position X_i of the i-th brane along the corresponding torus and $W_m^{(i)}$ is a generic Wilson line along the torus, parametrized by the θ_i phase factors:

$$W_w^{(i)} = \exp[iw\cdot X_i/R]\,, \quad W_m^{(i)} = \exp[im\cdot\theta_i]\,. \tag{A.8}$$

The sum (A.7) vanishes because a translation has no fixed points and hence the operator δ is not diagonal on the states. The action of the translation in (A.6) produces a phase in the KK modes that, in the Poisson resummed lattice sums, gives a shift on m. Notice that D-branes couple to all KK and winding modes.

Möbius strip In this case, the N b.c. give lattice sums similar to those in the annulus, since Ω does not act on KK modes. For D b.c., the non-trivial cases are obtained when $g = R$ and $g = R\delta$, where R and δ are respectively a rotation and a translation of order 2 on the torus (actually only on a circle). Indicating with $\Lambda[N,D\,|\,\Omega g]$ the Möbius strip lattice sum contribution, we therefore get:

$$\Lambda[N\,|\,\Omega I\,] = \sum_m \hat{\Lambda}[2m]\, W_{2m}^{(i)} \tag{A.9}$$

$$\Lambda[N\,|\,\Omega\delta\,] = \sum_m \hat{\Lambda}[2m+2\delta]\, W_{2m}^{(i)}\,, \tag{A.10}$$

$$\Lambda[D\,|\,\Omega R\,] = \sum_w \hat{\Lambda}[2w]\, W_{2w}^{(i)}\,, \tag{A.11}$$

$$\Lambda[D\,|\,\Omega R\delta\,] = \sum_w e^{2i\pi\delta\cdot w}\,\hat{\Lambda}[2w]\, W_{2w}^{(i)}\,. \tag{A.12}$$

The fact that only even KK and winding mode appear in the above equations implies that O-planes couple only to even KK momenta and winding modes. Notice, moreover, that eq. (A.11) represents the interaction of a $D5$- or $\bar{D}5$-brane with $O5$-planes in the R fixed points, i.e. $y = 0$ and $y = \pi R$ along the SS direction, whereas eq. (A.12) represents the interaction of a $D5$- or $\bar{D}5$-brane with the $O5$-planes (actually $\bar{O}5$-planes due to the $(-)^F$ action that comes together with δ) located at the $R\delta$ fixed points, i.e. $y = \pi R/2$ and $y = 3\pi R/2$ along the SS direction. Similarly, eqs. (A.9) and (A.10) represent respectively the $D9$ (or $\bar{D}9$) interactions with $O9$ and $\bar{O}9$-planes.

Klein bottle Define $\Lambda_i[h \,|\, \Omega\, g]$ as the Klein bottle lattice sum in the h twisted sector with the insertion of the operator g in the trace. Since lattice sums can only appear for the usual untwisted sector or for sectors twisted by a translation of order 2, $h = I, \delta$, where δ is the translation. On the other hand, non-trivial lattice contributions are obtained when g is a generic translation, as well as a $\mathbf{Z}_2$ reflection R (aside the identity). As in the analogue annulus case, the insertion of a translation gives rise to KK-dependent phases $\exp(2i\pi\delta \cdot m)$, whereas the δ twisted sector presents half-integer winding modes for Λ_i. Therefore, the relevant Poisson resummed lattice sums are given by:

$$\begin{aligned}
&\hat{\Lambda}[I \,|\, \Omega] = \sum_m \hat{\Lambda}[2m] \,, \\
&\hat{\Lambda}[I \,|\, \Omega\, \delta] = \sum_m \hat{\Lambda}\left[2m + 2\delta\right] \,, \\
&\hat{\Lambda}[I \,|\, \Omega\, R] = \hat{\Lambda}[I \,|\, \Omega\, R\, \delta] = \sum_w \hat{\Lambda}[2w] \,, \\
&\hat{\Lambda}[\delta \,|\, \Omega] = \hat{\Lambda}[\delta \,|\, \Omega\, \delta] = 0 \,, \\
&\hat{\Lambda}[\delta \,|\, \Omega\, R] = \hat{\Lambda}[\delta \,|\, \Omega\, R\, \delta] = \sum_w e^{2i\pi\delta\cdot w} \hat{\Lambda}[2w] \,.
\end{aligned} \tag{A.13}$$

Notice that (A.13) confirms that O-planes couple only to even KK momenta or even winding modes, differently from D-branes.

Appendix B

The unfolding technique

The main aim of this appendix is to describe how the known lattice-reduction techniques [89, 90, 93, 125, 127] can be modified and applied to amplitudes related to $\mathbf{Z}_n$ orbifold to unfold the fundamental region $\mathcal{F}$ and, at the same time, to simplify the structure of the integrand itself. This last task is pursued since, when considering a theory quantized on an orbifold, the typical one-loop amplitude is given by the sum of a large finite number of terms. The technique here described allows also a great reduction of these terms.

The technique, initially studied for $\mathbf{Z}_n$ orbifolds, is presented in a fully general way and can be applied to every modular invariant amplitude integrated over the fundamental region $\mathcal{F}$.

We start reviewing the known lattice reduction technique to unfold the fundamental region $\mathcal{F}$ to the strip. Then we present the way in which we want to modify it when the integrand is a finite sum of terms. We start from the generic amplitude

$$A = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} I(\tau),$$

where τ is a complex variable, $\mathcal{F}$ is the fundamental region and and $I(\tau)$ is a modular invariant function, i.e. a function that is invariant under the transformations

$$\tau \to \frac{a_1\tau + a_2}{a_3\tau + a_4}, \quad a_i \in \mathbb{Z}, \quad a_1a_4 - a_2a_3 = 1.$$

These transformations can be represented via the group $PSl(2, \mathbb{Z})$ of the matrices

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad a_i \in \mathbb{Z}, \quad a_1a_4 - a_2a_3 = 1,$$

generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

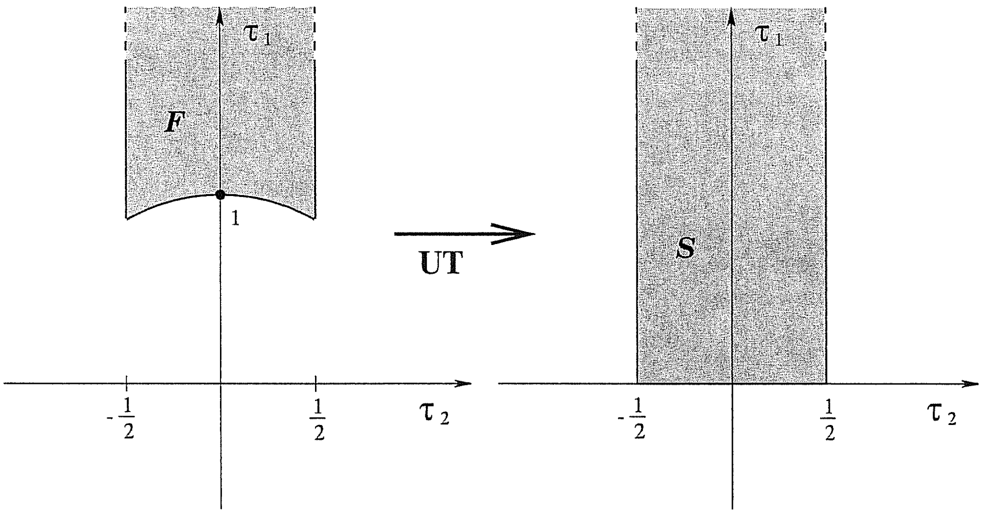

Figure B.1: The fundamental region $\mathcal{F}$, on the left and the strip S, on the right. The unfolding technique maps integrals over $\mathcal{F}$ into integrals over S.

modded out a $\mathbf{Z}_2$ group acting as $a_i \to -a_i$. The fundamental region is taken exactly to gauge away the modular invariance in a one-loop string amplitude, where the integrating region was, originally, the full plane $\mathbb{C}$:

$$\mathcal{F} = \frac{\mathbb{C}}{PSl(2,\mathbb{Z})}.$$

The integration is difficult due to the form of $\mathcal{F}$ but we can use modular invariance to unfold this region to a more suitable one. In general this is done [89] considering some special properties of the complex lattice

$$\Lambda(\tau) = \sum_{m,n\in\mathbb{Z}} e^{-\frac{|m+n\tau|^2}{\tau_2}}. \tag{B.1}$$

For each m and n in the sum we can compute the maximum common divisor (MCD) $p = (m, n)$ and write (B.1) as

$$\Lambda(\tau) = \sum_{p\in\mathbb{Z}} \sum_{\substack{c\in\mathbb{N},\\ d\in\mathbb{Z}\,|\,(c,d)=1}} e^{-\frac{p^2\cdot|c+d\tau|^2}{\tau_2}};$$

now, as shown in [89], given any c and d such that $(c, d) = 1$, there exists a class of transformations

$$\begin{pmatrix} a_1 & a_2 \\ c & d \end{pmatrix}$$

in $PSl(2,\mathbb{Z})$ mapping

$$e^{-\frac{p^2\cdot|c+d\tau|^2}{\tau_2}} \to e^{-\frac{p^2}{\tau_2}}.$$

These transformations are related each other by a T transformation so that it is possible to choose only a couple of a_i such that the related transformation maps from $\mathcal{F}$ to S. So, given any c, d in the sum, it is possible to choose one and only one transformation in $PSl(2,\mathbb{Z})/T$ mapping from $\mathcal{F}$ to S, and, in particular, the lattice can be written as the orbit under $PSl(2,\mathbb{Z})/T$ of the reduced lattice

$$\hat{\Lambda}(\tau) = \sum_{m\in\mathbb{Z}} e^{-\frac{p^2}{\tau_2}}.$$

Noting that the original lattice was modular invariant and so was the integration measure and $I(\tau)$, the general amplitude A can be written as

$$A = \sum_{g\in PSl(2,\mathbb{Z})/T} \int_{\mathcal{F}} g\left\{\frac{d^2\tau}{\tau_2^2} I(\tau) \times \frac{\hat{\Lambda}(\tau)}{\Lambda(\tau)}\right\} = \sum_{g\in PSl(2,\mathbb{Z})/T} \int_{g\{\mathcal{F}\}} \frac{d^2\tau}{\tau_2^2} I(\tau) \times \frac{\hat{\Lambda}(\tau)}{\Lambda(\tau)}, \quad \text{(B.2)}$$

where the right side of (B.2) is obtained simply by a change of variables. Now we can remember the definition of $\mathcal{F}$ and note that

$$\sum_{g\in PSl(2,\mathbb{Z})/T} g\{\mathcal{F}\} = \sum_{g\in PSl(2,\mathbb{Z})/T} g\left\{\frac{\mathbb{C}}{PSl(2,\mathbb{Z})}\right\} = \frac{\mathbb{C}}{T} = S$$

where S, the orbit of $\mathcal{F}$ under the action of $PSl(2,\mathbb{Z})/T$, is simply $\mathbb{C}/T$, i.e. the strip $\tau_1 \in [-1/2, 1/2]$, $\tau_2 > 0$, described in the right side of fig. B.1. We obtain

$$A = \int_S \frac{d^2\tau}{\tau_2^2} I(\tau) \times \frac{\hat{\Lambda}(\tau)}{\Lambda(\tau)}. \quad \text{(B.3)}$$

This kind of "unfolding technique" (UT) is completely general and can be applied independently of the form of $I(\tau)$, that is leaved unchanged. In some sense it closes the problem of unfolding the fundamental region $\mathcal{F}$. It is also useful to note that the lattice at the denominator can be written in a simple way through the elliptic theta functions

$$\Lambda(\tau) = \sqrt{\tau_2}\, |\theta_2(2\tau)|^2 .$$

As introduced this does not conclude our purposes: it is interesting to use modular invariance of $I(\tau)$ in a more subtle way to simplify the integration. In general $I(\tau)$ is a sum of terms that are not modular invariant but that are in a finite dimensional representation, or multiplet, of the modular group

$$I(\tau) = \sum_{i=1}^{N} I_i(\tau).$$

This happens, for example, when considering amplitudes arising from orbifold models. If N is large calculations can be really cumbersome but it is possible to perform a particular UT to simplify it in a large class of cases. Essentially we split the UT in two steps. In the first step we study the properties of the multiplet, and try to reduce it as the orbit of a suitable element $I_0(\tau)$ under the action of a subset G of the modular group,

$$A' = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{i=1}^{N} I_i(\tau) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{g\in G} g\left\{I_0(\tau)\right\}.$$

Then, noting that $I_0(\tau)$ is left invariant by a subgroup Γ of the modular group, we look for a suitable lattice that is invariant under Γ and that can be reduced through Γ exactly as $\Lambda(\tau)$ is reduced by $PSl(2,\mathbb{Z})/T$ to $\hat{\Lambda}(\tau)$

$$1 = \frac{\Lambda'(\tau)}{\Lambda'(\tau)} = \sum_{\gamma\in\Gamma} \gamma \left\{ \frac{\hat{\Lambda}'(\tau)}{\Lambda'(\tau)} \right\}.$$

So it is possible to write the integral as

$$A' = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{g\in G} g\left\{I_0(\tau)\right\} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{g\in G,\,\gamma\in\Gamma} g\circ\gamma \left\{ I_0(\tau) \times \frac{\hat{\Lambda}'(\tau)}{\Lambda'(\tau)} \right\}.$$

It is possible to choose Γ in such a way that

$$\sum_{g\in G} g\{\Gamma\} = \frac{PSl(2,\mathbb{Z})}{T}$$

so that

$$A' = \int_{S} \frac{d^2\tau}{\tau_2^2} I_0(\tau) \times \frac{\hat{\Lambda}'(\tau)}{\Lambda'(\tau)}. \tag{B.4}$$

In the next section we describe these two steps for a large class of multiplets, giving a systematic and general way to unfold this kind of integrals.

B.1 Finite irreducible representation of the modular group and the unfolding technique

In this section we explain how the ideas presented previously can be applied to various finite irreducible representations of the modular group. We study the general properties of the multiplets, we show the connection between the classification of the infinite dimensional subgroup of the modular group and the possible multiplet. We describe how the unfolding can be worked out for multiplets related to a large class of subgroups and we see how this analysis is independent of the details of the multiplets.

A finite-dimensional representation is completely defined when we give the net of identifications of its elements under the action of the modular group. As in the example given in the previous section, given any multiplet $\{I_i(\tau),\ i = 1, \ldots, N\}$ there is an N dimensional subset G of $PSl(2,\mathbb{Z})$, $G = \{g_i,\ i = 1, \ldots, N\}$ such that $I_i(\tau) = g_i I_0(\tau)$. Since we are interested only in the modular properties of the multiplet all the information we need are contained in G. We have to note also that, given a multiplet, the classification made with G is not one to one: given G mapping I_0 in all the multiplet in general it does not map also I_1 in the multiplet, so that each multiplet is related to more than one G. Since we are mainly interested in multiplets where at least one element is T invariant we call G the set of elements that maps the generic T-invariant term in all the others[1]. Furthermore given G is defined uniquely also the subgroup Γ such that given any $\gamma \in \Gamma$, $\gamma I_0(\tau) = I_0(\tau)$ and $\sum_{g\in G} g\{\Gamma\} = PSl(2,\,\mathbb{Z})$. We classify each multiplet using this last subgroup Γ.

We begin with the class of multiplets relative to $\Gamma_0^1[n]$, the subgroup of $PSl(2,\mathbb{Z})$ of the matrices of the form:

$$\begin{pmatrix} na+1 & b \\ nc & nd+1 \end{pmatrix} \sim \begin{pmatrix} na-1 & b \\ nc & nd-1 \end{pmatrix}$$

where the identification is due to the equivalence between a matrix A and $-A$. We present firstly the example $n = 5$, then we generalize the result to any prime number n and then to any $n \neq 4$. The $n = 4$ case is treated separately.

As said we are interested only in the modular properties of the elements and we can describe completely them, at least at this stage, giving their transformations under T and S. This means that the net of identifications is important, while the "names" we use for the elements is not. Nevertheless to simplify the notation we introduce here the names used to identify the various terms arising from the computation of the vacuum energy for a $\mathbf{Z}_n$ orbifold. We define

$$\left[\frac{i}{N}, \frac{j}{N}\right](\tau) = Tr\Big|_i \left[\theta^j q^{L_0} \bar{q}^{\bar{L}_0}\right], \quad q = e^{2\pi i\tau}$$

where θ is the $\mathbf{Z}_n$ generator, L_0 and $\bar{L}_0$ are the Virasoro generators and the trace is performed over the light-cone degrees of freedom of the θ^i twisted sector of closed type IIB string theory, i.e. the sector where the world-sheet fields are identified as

$$\Phi(s + 2\pi, t) = \theta^i \Phi(s, t).$$

The trace contains also the usual GSO projection and a sum over spin structures.

In the $n = 5$ case we refer to the twelve dimensional multiplet described by the

[1] As we will show this G is unique also when there are more than one T-invariant terms.

graph

$$
\begin{array}{ccccccc}
\left[\frac{0}{5},\frac{1}{5}\right](\tau) & \longleftrightarrow & \left[\frac{1}{5},\frac{0}{5}\right](\tau) & & \left[\frac{2}{5},\frac{0}{5}\right](\tau) & \longleftrightarrow & \left[\frac{0}{5},\frac{2}{5}\right](\tau) \\
 & & \downarrow & & \downarrow & & \\
 & & \left[\frac{1}{5},\frac{1}{5}\right](\tau) & & \left[\frac{2}{5},\frac{2}{5}\right](\tau) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \left[\frac{1}{5},\frac{2}{5}\right](\tau) & \longleftrightarrow & \left[\frac{2}{5},\frac{4}{5}\right](\tau) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \left[\frac{1}{5},\frac{3}{5}\right](\tau) & \longleftrightarrow & \left[\frac{2}{5},\frac{1}{5}\right](\tau) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \left[\frac{1}{5},\frac{4}{5}\right](\tau) & & \left[\frac{2}{5},\frac{3}{5}\right](\tau) & &
\end{array}
$$

where the vertical and horizontal arrows refers to T and S transformations respectively, S and T being the previously described generators of the modular group. In this and in all the nets we usually omit the vertical lines connecting $\left[\frac{i}{n},\frac{n-i}{n}\right](\tau)$ to $\left[\frac{i}{n},\frac{0}{n}\right](\tau)$ and the horizontal line between $\left[\frac{i}{n},\frac{i}{n}\right](\tau)$ and $\left[\frac{i}{n},\frac{n-i}{n}\right](\tau)$.

As it is clear there are two special T invariant terms that are also invariant under the action of $\Gamma_0^1[5]$, while the other terms are invariant under a subgroup that comes directly from $\Gamma_0^1[5]$. As an example $\left[\frac{1}{5},\frac{0}{5}\right](\tau)$ and $\left[\frac{2}{5},\frac{0}{5}\right](\tau)$ are invariant under the action of $S\Gamma_0^1[5]S$. Since there is a complete symmetry in the system, the unfolding can be performed equivalently from one or the other of the two T invariant terms and as we will see the symmetry between the two sectors is present also in the final result.

We have described the multiplet and concluded the first step: we take $I_0(\tau)$ to be $\left[\frac{0}{5},\frac{1}{5}\right](\tau)$. Now we can introduce the lattice

$$\Lambda_1{}^{(5)}(\tau)=\sum_{m,n\in\mathbb{Z}} e^{-\frac{|5m+1+5n\tau|^2}{25\tau_2}}=\sum_{m,n\in\mathbb{Z}} e^{-\frac{|5m+4+5n\tau|^2}{25\tau_2}}. \tag{B.5}$$

It is invariant under $\Gamma_0^1[5]$ and we can also note that the $p=(5m+1,5n)$ can be 1, 2, 3, 4 *mod* 5, so that we can write (B.5) as

$$\Lambda_1{}^{(5)}(\tau)=\sum_{\substack{m,\,c,\,d\,\in\,\mathbb{Z}\\(5c+1,5d)=1}} e^{-(5m+1)^2\frac{|5c+1+5d\tau|^2}{25\tau_2}}+\sum_{\substack{m,\,c,\,d\,\in\,\mathbb{Z}\\(5c+3,5d)=1}} e^{-(5m+2)^2\frac{|5c+3+5d\tau|^2}{25\tau_2}}.$$

Now we can consider the orbit of $Exp(\tau_2^{-1})$ under the action of $\Gamma_0^1[5]/T$ and obtain

$$\sum_{\gamma\in\Gamma_0^1[5]/T}\gamma\left\{e^{-\frac{1}{25\tau_2}}\right\}=\sum_{\substack{c,\,d\,\in\,\mathbb{Z}\\(5c+1,5d)=1}} e^{-\frac{|5c+1+5d\tau|^2}{25\tau_2}}.$$

We also note that under a ST^2ST^3S transformation the subgroup $\Gamma_0^1[5]/T$ is shifted in the subset of all the transformations of the form

$$\begin{pmatrix} na+3 & b \\ nc & nd+2 \end{pmatrix} \sim \begin{pmatrix} na-3 & b \\ nc & nd-2 \end{pmatrix}.$$

We call this subset $\Gamma_0^2[5]$, and it is simple to verify that the orbit of $Exp(\tau_2^{-1})$ under the action of $\Gamma_0^2[5]$ is

$$\sum_{\gamma\in\Gamma_0^2[5]/T} \gamma\left\{e^{-\frac{1}{25\tau_2}}\right\} = \sum_{\substack{c,\, d\in\mathbb{Z} \\ (5c+2,5d)=1}} e^{-\frac{|5c+2+5d\tau|^2}{25\tau_2}}.$$

Now it is clear that the lattice $\Lambda_1{}^{(5)}(\tau)$ can be written as

$$\Lambda_1{}^{(5)}(\tau) = \sum_{\gamma\in\Gamma_0^1[5]/T} \gamma\left\{\hat{\Lambda}_1^{(5)}(\tau) + ST^2ST^3S\left\{\hat{\Lambda}_2^{(5)}(\tau)\right\}\right\}$$

where

$$\hat{\Lambda}_i^{(n)}(\tau) = \sum_{m\in\mathbb{Z}} e^{-\frac{(nm+i)^2}{n^2\tau_2}}.$$

This allows us to write

$$\begin{aligned} A_5 &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{i,j} \left[\frac{i}{5}, \frac{j}{5}\right](\tau) = \sum_{g\in G} \int_{\mathcal{F}} g\left\{\frac{d^2\tau}{\tau_2^2}\left[\frac{0}{5}, \frac{1}{5}\right](\tau)\right\} = \\ & \sum_{\substack{g\,\in\, G, \\ \gamma\,\in\,\Gamma_0^1[5]/T}} \int_{\mathcal{F}} g\circ\gamma\left\{\frac{d^2\tau}{\tau_2^2}\frac{\left[\frac{0}{5}, \frac{1}{5}\right](\tau)}{\Lambda_1{}^{(5)}(\tau)} \times \left[\hat{\Lambda}_1^{(5)}(\tau) + ST^2ST^3S\left\{\hat{\Lambda}_2^{(5)}(\tau)\right\}\right]\right\}. \end{aligned}$$

Now we note that ST^2ST^3S maps exactly $\left[\frac{0}{5}, \frac{1}{5}\right](\tau)$ in $\left[\frac{0}{5}, \frac{2}{5}\right](\tau)$ and $\Lambda_1{}^{(5)}(\tau)$ in $\Lambda_2{}^{(5)}(\tau)$, where $\Lambda_i{}^{(5)}(\tau)$ is a simple extension of the previous definition:

$$\Lambda_i{}^{(5)}(\tau) = \sum_{m,n\in\mathbb{Z}} e^{-\frac{|5m+i+5n\tau|^2}{25\tau_2}} = \sum_{m,n\in\mathbb{Z}} e^{-\frac{|5m-i+5n\tau|^2}{25\tau_2}},$$

and, furthermore, we note that $\Gamma_0^1(5)$ commutes with ST^2ST^3S and $G\circ ST^2ST^3S \sim G$, so that,

$$A_5 = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{i,j} \left[\frac{i}{5}, \frac{j}{5}\right](\tau) = \int_{S} \frac{d^2\tau}{\tau_2^2} \sum_{i=1}^{2} \frac{\left[\frac{0}{5}, \frac{i}{5}\right](\tau)}{\Lambda_i{}^{(5)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(5m+i)^2}{25\tau_2}}. \qquad \text{(B.6)}$$

As anticipated the final form is symmetric in the T-invariant terms.

The symmetry of the result, that contains all the T invariant terms, is encoded in the fact that the invariance group is not the full subgroup $\Gamma_0[5]$ of the matrices

$$\begin{pmatrix} a & b \\ 5c & d \end{pmatrix}$$

as one can expect generalizing the results of [90], but a subgroup of it, that can be mapped in the full $\Gamma_0[5]$ by the set of transformations $\{I, ST^2ST^3S\}$, where ST^2ST^3S maps exactly one T-invariant term in the other.

The extension to a general $\Gamma_0[n]$ for a generic prime number n is straightforward. The multiplet contains $(n^2-1)/2$ terms, joined in $(n-1)/2$ sectors of the form

$$\begin{array}{ccc} \left[\frac{0}{n}, \frac{1}{n}\right](\tau) & \xrightarrow{S} & \left[\frac{1}{n}, \frac{0}{n}\right](\tau) \\ & & \downarrow \\ & & \cdots \\ & & \downarrow \\ & & \left[\frac{1}{n}, \frac{n-1}{n}\right](\tau) \end{array} \tag{B.7}$$

with $(n+1)$ elements each.

There are $(n-1)/2$ T invariant elements, and we can guess these elements are invariant under the action of $\Gamma_0^1[n]$, while the other transformations in $\Gamma_0[n] \setminus \Gamma_0^1[n]$ exchange the various T invariant terms. So the first step of the unfolding is completed taking one of the T invariant terms as fundamental element and $\Gamma_0^1[n]$ as unfolding group. The second step is a generalization of the calculations worked out in the $n=5$ case, we have only to use $\Lambda_1^{(n)}$ and the final result is

$$A_n = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{i,j} \left[\frac{i}{n}, \frac{j}{n}\right](\tau) = \int_{S} \frac{d^2\tau}{\tau_2^2} \sum_{i=1}^{(n-1)/2} \frac{\left[\frac{0}{n}, \frac{i}{n}\right](\tau)}{\Lambda_i^{(n)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(nm+i)^2}{n^2\tau_2}}. \tag{B.8}$$

Now we can study the case of multiplets relative to arbitrary n's. In this case the multiplet contains terms that are invariant under $\Gamma_0^1[n]$, i.e. there are parts of the multiplet of the form described in the graph (B.7), but in general there are also different building blocks. This is not a problem and at this stage the full form of the multiplet is not indispensable. As a general recipe we can say that, given a certain multiplet, one sees if there exists a T invariant term, then one studies the part of the multiplet related to this element via S and T^aS transformations and finds the order of the invariance subgroup. Then the unfolding is completed by considering one of the invariant terms as generator of the full multiplet and the subgroup $\Gamma_0^1[n]$ as base for the second step. The presence of further T-invariant terms is encoded in $\Gamma_0^1[n]$, exactly as explained in the $\Gamma_0^1[5]$ case. Formula (B.8) can be applied with attention to the fact that if n is not a prime number then the index i does not run over $\{1, 2, \ldots, (n-1)/2\}$ but only on the numbers p previously described. For example in the $n=9$ case, of interest, for example, for the 36-dimensional irreducible multiplet arising when computing the free energy in a $\mathbf{Z}_9$ model, the index i takes values 1, 2, 4. The only note is about the identification of the T-invariant terms of the multiplet analyzed with the terms $\left[\frac{0}{n}, \frac{i}{n}\right](\tau)$ used in (B.8). This is done by choosing one T-invariant term and calling it $\left[\frac{0}{n}, \frac{1}{n}\right](\tau)$, then one can derive the net of identifications for the original multiplet and for $\left[\frac{0}{n}, \frac{1}{n}\right](\tau)$ knowing that

$$\begin{array}{ccc}
\left[\frac{h}{N}, \frac{k}{N}\right](\tau) & \xrightarrow{S} & \left[\frac{k}{N}, \frac{-h}{N}\right](\tau) \\
T \big\downarrow & & \\
\left[\frac{h}{N}, \frac{h+k}{N}\right](\tau) & &
\end{array}$$

and $\left[\frac{h}{N}, \frac{k}{N}\right](\tau) \sim \left[\frac{-h}{N}, \frac{-k}{N}\right](\tau) \sim \left[\frac{nN-h}{N}, \frac{pN-k}{N}\right](\tau)$, $\forall\ n, p \in \mathbb{N}$. At this point the identification is easily done by comparing the two nets.

As an example we present the twelve-dimensional multiplet arising from a $\mathbf{Z}_6$ orbifold. The full multiplet contains 36 elements, but is not irreducible, it is given by the sum of the usual trivial one-dimensional multiplet, a three-dimensional multiplet related to $\Gamma_0^1[2]$, two equivalent four-dimensional multiplets related to $\Gamma_0^1[3]$ and two equivalent twelve-dimensional multiplets related to $\Gamma_0^1[6]$: $\mathbf{36} = \mathbf{1} \oplus \mathbf{3} \oplus 2 \times \mathbf{4} \oplus 2 \times \mathbf{12}$. This latter multiplet has the form:

$$\begin{array}{ccccccc}
\left[\frac{0}{6}, \frac{1}{6}\right](\tau) & \longleftrightarrow & \left[\frac{1}{6}, \frac{0}{6}\right](\tau) & & & & \\
 & & \big\downarrow & & & & \\
 & & \left[\frac{1}{6}, \frac{1}{6}\right](\tau) & & & & \\
 & & \big\downarrow & & & & \\
 & & \left[\frac{1}{6}, \frac{2}{6}\right](\tau) & & \longleftrightarrow & & \left[\frac{2}{6}, \frac{5}{6}\right](\tau) \\
 & & \big\downarrow & & & & \big\downarrow \\
 & & \left[\frac{1}{6}, \frac{3}{6}\right](\tau) & \longleftrightarrow & \left[\frac{3}{6}, \frac{5}{6}\right](\tau) & & \left[\frac{2}{6}, \frac{1}{6}\right](\tau) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & \left[\frac{1}{6}, \frac{4}{6}\right](\tau) & & \left[\frac{3}{6}, \frac{2}{6}\right](\tau) & \longleftrightarrow & \left[\frac{2}{6}, \frac{3}{6}\right](\tau)
\end{array}$$

where an horizontal line links $\left[\frac{1}{6}, \frac{4}{6}\right](\tau)$ to $\left[\frac{2}{6}, \frac{1}{6}\right](\tau)$. The multiplet contains the only T invariant term $\left[\frac{0}{6}, \frac{1}{6}\right](\tau)$, as we expect from the fact that 1 is the only integer satisfying the condition of being at the same time prime with 6 and less than 3.

The integral is:

$$A_6 = \int_S \frac{d^2\tau}{\tau_2^2} \frac{\left[\frac{0}{6}, \frac{1}{6}\right](\tau)}{\sum_{m,n \in \mathbb{Z}} e^{-\frac{|6m+1+6n\tau|^2}{36\tau_2}}} \times \sum_{m \in \mathbb{Z}} e^{-\frac{(6m+1)^2}{36\tau_2}}.$$

This concludes the analysis of the multiplets with T-invariant terms, that can be summarized as follows. For a given $\Gamma_0^1[n]$ multiplet the unfolded integral equals

$$A_n = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{i,j} \left[\frac{i}{n}, \frac{j}{n}\right](\tau) = \int_S \frac{d^2\tau}{\tau_2^2} \sum_{i=1} \frac{\left[\frac{0}{n}, \frac{i}{n}\right](\tau)}{\Lambda_i^{(n)}(\tau)} \times \hat{\Lambda}_i^{(n)}(\tau) \qquad \text{(B.9)}$$

where i is in the set of numbers such that $(i, n) = 1$ and $i < (n-1)/2$.

This is not the only one class of multiplets. From $\mathbf{Z}_n \times \mathbf{Z}_n$ orbifolds one can see that there are multiplets where all the term are equivalent and where the invariance subgroup is not based on T and ST^nS but on T^n and ST^nS, the relevant subgroup being $\tilde{\Gamma}[n]$ made of matrices of the form:

$$\begin{pmatrix} na+1 & nb \\ nc & nd+1 \end{pmatrix}.$$

Due to the fact that is more difficult to take in account $\tilde{\Gamma}[n]/T$ [2], or, equivalently, that all the terms in the multiplet are equivalent we deduce that the UT useful in this case is the general formula (B.3) described in the second section.

B.1.1 An exception: $\Gamma_0[4]$

The $\Gamma_0[4]$ case represents an exception to the recipe previously given. It is the only exception suggested by orbifold models, so, if we conjecture the one to one correspondence multiplet/invariance group and note that from orbifold models we have regular multiplets for all $\Gamma_0[n]$ $n \neq 4$, than we can guess that all the $n \neq 4$ cases are regular.

The $\Gamma_0[4]$ case is exceptional because the multiplet

$$\begin{array}{ccccc}
\left[\frac{0}{4}, \frac{1}{4}\right](\tau) & \longleftrightarrow & \left[\frac{1}{4}, \frac{0}{4}\right](\tau) & & \\
& & \downarrow & & \\
& & \left[\frac{1}{4}, \frac{1}{4}\right](\tau) & & \\
& & \downarrow & & \\
& & \left[\frac{1}{4}, \frac{2}{4}\right](\tau) & \longleftrightarrow & \left[\frac{2}{4}, \frac{1}{4}\right](\tau) \\
& & \downarrow & & \\
& & \left[\frac{1}{4}, \frac{3}{4}\right](\tau) & &
\end{array}$$

contains two T-invariant terms even though only 1 respects the conditions $(i, 4) = 1$, $i < 2$. The unfolding is written in the same way as before, but now starting from one term we do not obtain the second one. This is due to the fact that the contributions obtained starting from one or the other term are equal, as we will see in the example in the next section. We can conclude that, even though the multiplet is exceptional, the unfolding is not and formula (B.9) can be applied as in the other cases.

B.1.2 Two examples

The $n = 2$ and $n = 3$ cases have already been taken into consideration in [90, 93] and in [90] respectively. These are two examples of how the technique can be applied efficiently in various cases.

[2] This is due to the fact that $T \circ \tilde{\Gamma} \notin \tilde{\Gamma}$

In particular in [90] the authors computed the threshold corrections to gauge couplings in an orbifold compactification of heterotic string. Following [125] they started from a one-loop amplitude, clearly modular invariant, that, due to the presence of twisted sectors and projections, is made of a finite sum of elements. In particular they considered the case of the three and four-dimensional multiplet

$$\begin{array}{ccc} \left[\frac{0}{2},\frac{1}{2}\right](\tau) & \longleftrightarrow & \left[\frac{1}{2},\frac{0}{2}\right](\tau) \\ & & \downarrow \\ & & \left[\frac{1}{2},\frac{1}{2}\right](\tau) \end{array}$$

and

$$\begin{array}{ccc} \left[\frac{0}{3},\frac{1}{3}\right](\tau) & \longleftrightarrow & \left[\frac{1}{3},\frac{0}{3}\right](\tau) \\ & & \downarrow \\ & & \left[\frac{1}{3},\frac{1}{3}\right](\tau) \\ & & \downarrow \\ & & \left[\frac{1}{3},\frac{2}{3}\right](\tau) \end{array}$$

The first step of the unfolding was performed as described here, while the second step was treated in a modified version to better fit the form of the terms they took into account.

In [93], instead, the vacuum energy for an orbifold of type IIB string theory was computed. In particular the authors performed the unfolding in the $n = 2$ case in the same way here presented. The result is essentially that given in (B.15). In the next section we show how the computation can be extended to a generic orbifold through the UT shown.

B.2 The vacuum energy for a class of seven dimensional orbifold models

In this section we compute the one-loop vacuum energy for the class of models of chapter 4, where type IIB string theory is compactified over $\mathbb{R}^7 \times (\mathbb{C} \times S/\mathbf{Z}_N)$. The group $\mathbf{Z}_N$ acts as an order N rotation on $\mathbb{C}$ and an order N translation around the compact circle S, taken to be of radius R. We consider firstly the odd-N case, then we extend the result to the even-N one.

As seen in Appendix C, independently of the details of the action of $\mathbf{Z}_N$, the full amplitude contains N^2 elements named $\left[\frac{i}{N},\frac{j}{N}\right](\tau)$, usually in a reducible multiplet:

$$Z^{(9)} = \frac{1}{N}\int_{\mathcal{F}}\frac{d^2\tau}{2\tau_2^2}\sum_{i,j=0}^{N}\left[\frac{i}{N},\frac{j}{N}\right](\tau)$$

where the $\left[\frac{i}{N}, \frac{j}{N}\right](\tau)$ are defined in the previous section and in Appendix C.

The reduction of the multiplet is quite easy. First of all the term $\left[\frac{0}{N}, \frac{0}{N}\right](\tau)$ is a singlet, and is SUSY, so its contribution is zero. Then, due to the periodicity properties, the (N^2-1)-dimensional multiplet obtained is reduced into two equivalent $(N^2-1)/2$-dimensional multiplets. We consider one of them. It contains the $(N-1)/2$ T-invariant terms $\{\left[\frac{0}{N}, \frac{1}{N}\right](\tau), \ldots, \left[\frac{0}{N}, \frac{(N-1)/2}{N}\right](\tau)\}$. These terms are the key ones, being, with the exception of the $\mathbf{Z}_4$ orbifold, the only T-invariant terms in all the multiplet. Now the reduction is done by considering the net for each of these multiplets, or better, some relevant features of the multiplet itself. We consider the $\mathbf{Z}_9$ example first, then we see how the final form can be extracted in the general case without any further analysis. In the $\mathbf{Z}_9$ case we have the term $\left[\frac{0}{9}, \frac{1}{9}\right](\tau)$ that is left invariant by the action of $\Gamma_0^1[9]$. The full multiplet, so, contains also two other T-invariant terms, because the number of p such that $(p,9)=1$ and $p<9/2$ is three: $p=1,2,4$. Clearly these terms are $\left[\frac{0}{9}, \frac{2}{9}\right](\tau)$ and $\left[\frac{0}{9}, \frac{4}{9}\right](\tau)$, while the term $\left[\frac{0}{9}, \frac{3}{9}\right](\tau)$ is in an independent irreducible multiplet, based on $\Gamma_0^1[3]$. The reduction is concluded, now we can apply directly formula (B.9), since the identification between the elements of our multiplets and those used in the previous section is trivial.

$$Z^{(9)} = 2\int_S \frac{d^2\tau}{\tau_2^2} \sum_{i=1,2,4} \frac{\left[\frac{0}{9}, \frac{i}{9}\right](\tau)}{\Lambda_i{}^{(9)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(9m+i)^2}{9^2\tau_2}} +$$
$$2\int_S \frac{d^2\tau}{\tau_2^2} \frac{\left[\frac{0}{9}, \frac{3}{9}\right](\tau)}{\Lambda_1{}^{(3)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(3m+1)^2}{3^2\tau_2}} \tag{B.10}$$

where the overall factor of 2 takes in account the terms $\left[\frac{0}{9}, \frac{p}{9}\right](\tau)$ with $p>4$. Since

$$\sum_{m\in\mathbb{Z}} e^{-\frac{(9m+3)^2}{9^2\tau_2}} \equiv \sum_{m\in\mathbb{Z}} e^{-\frac{(3m+1)^2}{3^2\tau_2}},$$

$\Lambda_1^{(3)}(\tau) \equiv \Lambda_3^{(9)}(\tau)$ and $\hat{\Lambda}_1^{(3)}(\tau) \equiv \hat{\Lambda}_3^{(9)}(\tau)$ and so there is a more interesting version of (B.10):

$$Z^{(9)} = \int_S \frac{d^2\tau}{\tau_2^2} \sum_{i=0}^{8} \frac{\left[\frac{0}{9}, \frac{i}{9}\right](\tau)}{\Lambda_i{}^{(9)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(9m+i)^2}{9^2\tau_2}}. \tag{B.11}$$

It is easy to understand that formula (B.11) can be extended to every other case, independently of the form of the initial multiplet.

$$Z^{(N)} = \int_S \frac{d^2\tau}{\tau_2^2} \sum_{i=0}^{N} \frac{\left[\frac{0}{N}, \frac{i}{N}\right](\tau)}{\Lambda_i{}^{(N)}(\tau)} \times \sum_{m\in\mathbb{Z}} e^{-\frac{(Nm+i)^2}{N^2\tau_2}}. \tag{B.12}$$

We note that while the so-called "first step" is completely general, the "second step" requires that we take a choice and pick a special object $\Lambda_i^{(n)}$. This does not spoil the fact that the procedure can be applied to a generic multiplet, independently of its origin. Nevertheless it is interesting to note that strictly dependently on the details of the form of the various terms one can pick a different and more suitable object to perform the second step. In our case we can specialize (B.12) noting that, due to the translation action, a lattice term

$$\Lambda_i^{(N)}(R,\tau) = \sum_{m,n\in\mathbb{Z}} e^{-\frac{R^2|Nm+1+Nn\tau|^2}{N^2\tau_2}}$$

is present in $\left[\frac{0}{N},\,\frac{i}{N}\right](\tau)$. We can use this lattice to perform the second step, in such a way that the denominator is canceled by the lattice $\Lambda_i^{(N)}(R,\tau)$, obtaining the final result, valid for any odd N,

$$Z^{(N)} = \frac{v_7}{N}\frac{R}{\sqrt{\alpha'}}\int_0^\infty \frac{d\tau_2}{2\tau_2^5}\sum_{k=1}^{N-1}\sum_{M\in\mathbb{N}}\left[d_M(B)^k - d_M(F)^k\right]e^{-4\pi\tau_2 M}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left(n+\frac{k}{N}\right)^2}, \tag{B.13}$$

where we have defined the coefficients $d_M(B,F)^k$ as

$$\sum_{M\in\mathbb{N}}\left[d_M(B)^k - d_M(F)^k\right]e^{-4\pi\tau_2 M} = \int\frac{d\tau_1}{4}\left|2\sin\left(\frac{2\pi k}{N}\right)\frac{\sum_{\alpha,\beta}\eta_{\alpha\beta}\theta\left[{\alpha \atop \beta}\right]^3\theta\left[{\alpha \atop \beta+\frac{2k}{N}}\right]^3}{\eta^9\,\theta\left[{\frac{1}{2} \atop \frac{1}{2}+\frac{2k}{N}}\right]}\right|^2. \tag{B.14}$$

The even-N case can be easily deduced. The multiplet is reduced in the same way, the only exception being that the term $\left[\frac{0}{N},\,\frac{N/2}{N}\right](\tau)$ is special, because $\theta^{N/2}$ acts only as $(-1)^F$ and so while the fermionic part is as in (B.12) the bosonic is different and, in particular, there is an extra factor from the momentum in the $\mathbb{C}$ dimensions. So, to conclude, the general amplitude is described in (B.13) with k taking values $\{1,\ldots,(N-2)/2,(N+2)/2,\ldots,N-1\}$ and an extra term from the term $k=N/2$ of the form

$$\Delta = \frac{v_7 v_2}{N}\frac{R}{\sqrt{\alpha'}}\int_0^\infty\frac{d\tau_2}{2\tau_2^6}\sum_{M\in\mathbb{N}}\left[d_M(B)^{N/2} - d_M(F)^{N/2}\right]e^{-4\pi\tau_2 M}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left(n+\frac{1}{2}\right)^2}, \tag{B.15}$$

where

$$\sum_{M\in\mathbb{N}}\left[d_M(B)^{N/2} - d_M(F)^{N/2}\right]e^{-4\pi\tau_2 M} = \int d\tau_1\left|\frac{\theta\left[{1/2 \atop 0}\right]^4}{\eta^{12}}\right|^2 \tag{B.16}$$

In (B.15) there is an extra volume v_2 taking in account the fact that these states propagates also in the $\mathbb{C}$ dimensions.

The $\mathbf{Z}_4$ case is not special, and the formula described above is still valid. The only remark we can make is the fact that starting from the two different elements we obtain the same result. This is true because the Scherk-Schwarz present in $\left[\frac{0}{4},\,\frac{1}{4}\right](\tau)$ case is that used usually, $\Lambda_1^{(4)}(R,\tau)$, while $\left[\frac{2}{4},\,\frac{1}{4}\right](\tau)$ there is a lattice that is exactly $ST^2S\{\Lambda_1^{(4)}(R,\tau)\}$ so that the unfolding for $\left[\frac{2}{4},\,\frac{1}{4}\right](\tau)$ proceed as in $\left[\frac{0}{4},\,\frac{1}{4}\right](\tau)$ but with an extra ST^2S that transforms $\left[\frac{2}{4},\,\frac{1}{4}\right](\tau)$ in $\left[\frac{0}{4},\,\frac{1}{4}\right](\tau)$.

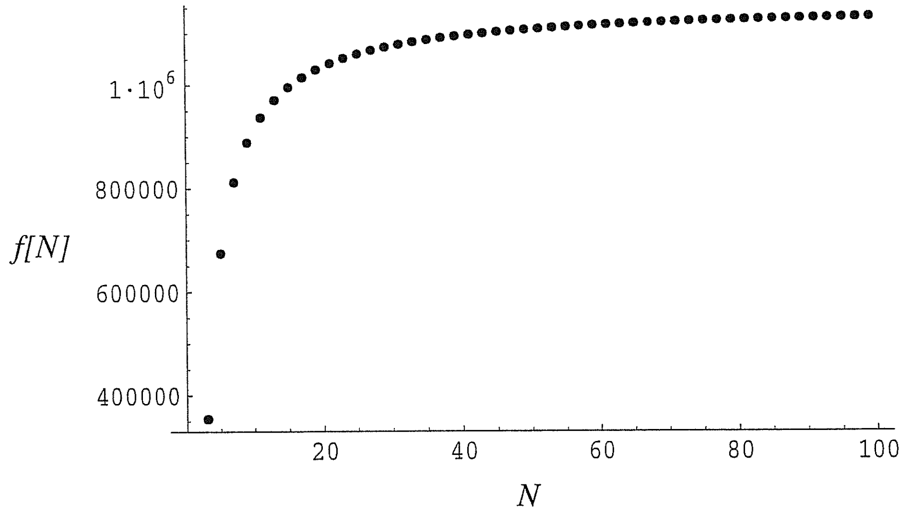

Figure B.2: The value of the $f[N]$ function in different cases. As clear $f[N]$ is a monotonic function with asymptotic value $f[\infty] \sim 1.16 * 10^6$.

It is easy to compute the leading term for $Z^{(N)}$, given by the $M = 0$ contribution. It is given by

$$Z^{(N)} \sim v_7 \left(\frac{\alpha'}{R^2}\right)^{\frac{7}{2}} \frac{3}{\pi^4} f[N] + \frac{1+(-1)^N}{2} \frac{v_7 v_2}{N} \left(\frac{\alpha'}{R^2}\right)^{\frac{9}{2}} \frac{c}{\pi^5}$$

where the function $f[N]$ contains all the N-dependence in the odd-N case and $c \sim 3 \cdot 2^{22}$

$$f[N] \sim \frac{2^9}{N} \sum_{k=1}^{(N-1)/2} \frac{sin\left(\frac{\pi k}{N}\right)^8}{\left(\frac{k}{N}\right)^8}.$$

The behavior of $f[N]$ is summarized in fig. B.2, it is essentially monotonic and in the limit $N \to \infty$ it approaches the constant value

$$Lim_{N\to\infty} f[N] = f[\infty] = 2^9 \int_0^1 dx \left(\frac{sin(\pi x)}{x}\right)^8 \sim 1.16 * 10^6.$$

The monotonic behavior ensures that the one parameter set of functions $Z^{(N)}$ has, at least at the leading order, a monotonic behavior in N in the odd-N case, as found in the lowest N case in chapter 4. In the even-N case the extra factor from the term $k = N/2$ changes the behavior, that depends also on the ratio between the volume v_2 and R^2.

Appendix C

One loop closed string amplitudes

In this appendix we describe the one loop closed string amplitudes used to compute the partition functions of chapter 4. The main task is to obtain the form of the $\left[\frac{i}{N}, \frac{j}{N}\right](\tau)$ objects used in Appendix B an their modular properties, in particular we give a recipe to study the net of identifications between the $\left[\frac{i}{N}, \frac{j}{N}\right](\tau)$ under a modular transformation.

We take the orbifold of the form $M = \mathbb{C}^n \times T^{3-n}/\mathbf{Z}_N$, where $\mathbf{Z}_N$ is generated by an order N rotation in the various $\mathbb{C}$ or T manifold, the rotation angle being $2\pi \times \vec{v}$. The vector v is of the form

$$\vec{v} = \frac{1}{N}(a, b, c)$$

with a, b, c natural numbers. World-sheet supersymmetry impose that the rotation is applied also on world sheet fermions. In order to have that the action of the rotation is of order N also on the fermionic degrees of freedom we require that $a + b + c$ is an even number. We do not impose any other constraint since we are interested also in non-SUSY orbifolds. We take n to be generic, since the difference between $\mathbb{C}$ and T^2 is revealed only by the zero modes of string theory.

The partition function is given by

$$Z = \frac{1}{N} \int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2^2} \sum_{i,j=0}^{N} \left[\frac{i}{N}, \frac{j}{N}\right](\tau)$$

where

$$\left[\frac{i}{N}, \frac{j}{N}\right](\tau) = Tr\Big|_i \left[\theta^j q^{L_0} \bar{q}^{\bar{L}_0}\right], \quad q = e^{2\pi i \tau}$$

where θ is the $\mathbf{Z}_N$ generator and the trace is performed over the light-cone degrees of freedom of the θ^i twisted sector.

We can also join the described rotation with a translation in some compact direction where the rotation acts in a suitable way. It is not necessary to impose any requirements about the relation between rotation and translation at this stage. We describe a completely general case giving the building blocks for $\left[\frac{i}{N}, \frac{j}{N}\right](\tau)$, without any constraint. Clearly, when putting together the terms to build the partition function, the requirements must be fulfilled, as explained in chapter 2.

Since in the theory the fermionic modes are decoupled from the bosonic ones and, furthermore, the bosonic zero modes are decoupled from the other bosonic modes, $\left[\frac{i}{N}, \frac{j}{N}\right](\tau)$ factorize in the product of three terms: a term from bosonic non-zero modes, one from fermionic modes and one from momentum, that is the bosonic zero mode. This last term contains also a factor arising from the number of fixed points of the operators θ^i and θ^j, as described in the appendix of [72].

The fermionic modes give the contribution

$$\left[\frac{h}{N}, \frac{k}{N}\right]_F(\tau) = \left|\sum_{\alpha,\beta\in\{0,1/2\}} \frac{(-1)^{2\alpha+2\beta+4\alpha\beta}}{2} \prod_{d=0}^{3} \eta^{-1}(\tau)\, e^{-2\pi h\beta v_d}\, \theta\begin{bmatrix}\alpha + hv_d\\ \beta + kv_d\end{bmatrix}(\tau)\right|^2 \tag{C.1}$$

where v_0 is taken to be zero while v_1, v_2 and v_3 are the components of the above described $\vec{v}$. The θ functions are the well-known modular functions whose definition and modular properties are described, for example, in the appendix of [14].

The sum over the spin structure can be simplified using the Jacobi identity described in the appendix A of [112], using it (C.1) is written as

$$\left[\frac{h}{N}, \frac{k}{N}\right]_F(\tau) = \left|\prod_{d=0}^{3} \eta^{-1}(\tau)\, \theta\begin{bmatrix}1/2 + hv'_d\\ 1/2 + kv'_d\end{bmatrix}(\tau)\right|^2 \tag{C.2}$$

where

$$v'_0 = \frac{1}{2}(v_1 + v_2 + v_3), \quad v'_1 = \tfrac{1}{2}(-v_1 + v_2 + v_3)$$
$$v'_2 = \frac{1}{2}(v_1 - v_2 + v_3), \quad v'_3 = \tfrac{1}{2}(v_1 + v_2 - v_3).$$

From the modular properties of the θ functions it is easy to see that T maps $\left[\frac{i}{N}, \frac{j}{N}\right]_F(\tau)$ in $\left[\frac{i}{N}, \frac{i+j}{N}\right]_F(\tau)$, while S maps $\left[\frac{i}{N}, \frac{j}{N}\right]_F(\tau)$ in $\left[\frac{j}{N}, \frac{-i}{N}\right]_F(\tau)$. It is also interesting to note that, due to the periodicity properties, $\left[\frac{i}{N}, \frac{j}{N}\right]_F(\tau) \sim \left[\frac{-i}{N}, \frac{-j}{N}\right]_F(\tau)$ and $\left[\frac{i}{N}, \frac{j}{N}\right]_F(\tau) \sim \left[\frac{aN+i}{N}, \frac{bN+j}{N}\right]_F(\tau)$ for all a, $b \in \mathbb{N}$.

The term from the bosonic non-zero modes can be split in the product of four terms related to the four v_i. Each term is of the form

$$\left[\frac{h}{N}, \frac{k}{N}\right]_B^{(d)}(\tau) = |\eta(\tau)|^{-4}$$

if hv_d, $kv_d \in \mathbb{N}$ or of the form

$$\left[\frac{h}{N}, \frac{k}{N}\right]_B^{(d)}(\tau) = \left|\eta(\tau)\, \theta^{-1}\begin{bmatrix}\frac{1}{2} + hv_d\\ \frac{1}{2} + kv_d\end{bmatrix}(\tau)\right|^2$$

in the other cases.

The term from the bosonic zero modes, like the previous one, can be split in the product of four terms, each of them having a different form depending on the action of a translation/rotation. Each term equals one if a non-trivial rotation is present, if no rotation is present it equals

$$\left[\frac{h}{N},\,\frac{k}{N}\right]^{(d)}_{0B}(\tau) = \frac{\sqrt{G}}{\alpha'\,\tau_2}\sum_{\vec{m},\vec{n}} e^{-\frac{\pi}{\alpha'\tau_2}[(m+kw_d)+(n+hw_d)\tau]_i(G+B)_{ij}[(m+kw_d)+(n+hw_d)\bar{\tau}]_j} \tag{C.3}$$

if the submanifold is compact and the $\mathbf{Z}_N$ acts also as a translation of length $\hat{w}_d$ in this direction. The two dimensional lattice is written in the most general case, with G and B respectively the metric and antisymmetric field. Also the translation is generic, we only ask that it is of order N, without any other constraint on the form of the two-dimensional vector w_d. The formula (C.3) is extended in the non-compact case and resummed to $V/4\pi^2\alpha'\tau_2$, V being, as usual, the infinite volume of the two-dimensional non-compact manifold itself. Clearly in presence of a non compact manifold it is meaningless to introduce an order N translation.

Furthermore there is an overall factor $\left[\frac{h}{N},\,\frac{k}{N}\right]_{\mathcal{N}}$ counting the number of fixed points.

The full $\left[\frac{h}{N},\,\frac{k}{N}\right](\tau)$ is given by

$$\left[\frac{h}{N},\,\frac{k}{N}\right](\tau) = \left[\frac{h}{N},\,\frac{k}{N}\right]_{\mathcal{N}}\left[\frac{h}{N},\,\frac{k}{N}\right]_F(\tau)\prod_{d=0}^{4}\left[\frac{h}{N},\,\frac{k}{N}\right]^{(d)}_B(\tau)\left[\frac{h}{N},\,\frac{k}{N}\right]^{(d)}_{0B}(\tau) \tag{C.4}$$

and it has the same modular and periodicity properties of $\left[\frac{h}{N},\,\frac{k}{N}\right]_F(\tau)$, so that

$$\begin{array}{ccc} \left[\frac{h}{N},\,\frac{k}{N}\right](\tau) & \xrightarrow{\;S\;} & \left[\frac{k}{N},\,\frac{-h}{N}\right](\tau) \\ T\big\downarrow & & \\ \left[\frac{h}{N},\,\frac{h+k}{N}\right](\tau) & & \end{array} \tag{C.5}$$

and $\left[\frac{h}{N},\,\frac{k}{N}\right](\tau) \sim \left[\frac{-h}{N},\,\frac{-k}{N}\right](\tau) \sim \left[\frac{nN-h}{N},\,\frac{pN-k}{N}\right](\tau)$, $\forall\; n,\,p \in \mathbb{N}$.

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