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Geometry and analysis of control-affine systems: motion planning, heat and Schrödinger evolution

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To Meg.

Knowledge and ability were tools,
not things to show off.
– Haruki Murakami, 1Q84

CONTENTS

1 INTRODUCTION

Dynamics, from how a car moves up to the evolution of a quantum particle, are modelled in general by differential equations. Control theory deals with dynamics where it is possible to act on some part of the equation by means of controls, e.g., how to park a car or how to steer a quantum particle to a desired state.

More precisely, a control system on a smooth manifold M is an ordinary differential equation in the form

$$\dot{q}(t) = f(q(t), u(t)), \tag{1.0.1}$$

where $u : [0, T] \to U$ is an integrable and bounded function – called control – taking values in some set $U \subset \mathbb{R}^m$, and $f : M \times U \to TM$ is a continuous function such that $f(\cdot, u)$ is a smooth vector field for each $u \in U$. Thus, fixing a control and an initial point q_0, system (1.0.1) has a unique maximal solution $q_u(t)$. Every curve $\gamma : [0, T] \to M$ that can be written as solution of system (1.0.1) for some control u and with starting point $\gamma(0)$, is said to be *admissible*.

With a control system it is possible to associate an *optimal control problem*. Namely, one considers a *cost* $J : (u, T) \mapsto [0, +\infty)$, where $T > 0$ and $u \in L^1([0, T], \mathbb{R}^m)$. Then, given two points q_0, $q_1 \in M$ one is interested in minimizing the functional J among all admissible controls $u \in L^1([0, T], \mathbb{R}^m)$, $T > 0$, for which the corresponding solution of (1.0.1) with initial condition $q_u(0) = q_0$ is such that $q_u(T) = q_1$. This is written as

$$\begin{cases} \dot{q}(t) = f(q(t), u(t)), \\ q(0) = q_1, \qquad q(T) = q_1, \\ J(u, T) \longrightarrow \min. \end{cases} \tag{1.0.2}$$

The final time T can either be fixed, or free to be selected in a certain interval of time.

From the optimal control problem associated with a cost J, one defines the value function $V : M \times M \to [0, +\infty]$. This is a function that associates to every pair of points q_0, $q_1 \in M$ the infimum of the cost of controls admissible for the corresponding optimal control problem (1.0.2). If there are no such controls, then $V(q_0, q_1) = +\infty$.

The aim of this thesis is to study two different problems arising from control theory, regarding control-affine systems with unbounded controls, i.e., with $U = \mathbb{R}^m$. A control-affine system on a smooth manifold M is a control system in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)), \tag{1.0.3}$$

where, $u : [0, T] \to \mathbb{R}^m$ is an integrable control function and $\{f_0, f_1, \dots, f_m\}$ is a family of smooth vector fields. The vector fields $f_1, \dots, f_m$ are called control vector fields, while f_0 is called the *drift*. For most of the dissertation we will consider as a cost $\mathcal{J} : (u, T) \mapsto [0, +\infty)$ the L^1-norm of u. Namely we will be interested in the optimal control problem

$$\begin{cases} \dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)), \\ q(0) = q_0, \qquad q(T) = q_1, \\ \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{i=1}^{m} u_i(t)^2}\, dt \longrightarrow \min. \end{cases} \tag{1.0.4}$$

From a mathematical point of view, these systems describe the underlying geometry of hypoelliptic operators, as we will see later. In applications, they appear in the study of many mechanical systems, from the already mentioned car parking problem up to most kind of robot motion planning, and recently in research fields such as mathematical models of human behavior, quantum control or motion of self-propulsed micro-organism (see [ADL08, BDJ+08, BCG02a]). A suggestive application of these systems and of hypoelliptic diffusions, in the particular case where $f_0 \equiv 0$, appeared in the field of cognitive neuroscience to model the functional architecture of the area V1 of the primary visual cortex, as proposed by Petitot, Citti, and Sarti [PT99, Pet09, CS06].

We will focus on the following two general problems for these kind of systems.

1. *Complexity of non-admissible trajectories.* A common issue in control theory, used for example in robot motion planning, is to steer the system along a given curve Γ. Since, in general, Γ is not admissible, i.e., it is not a solution of system (1.0.3), the best one can do is to steer the system along an approximating trajectory. The first part of the thesis is dedicated to quantify the cost of this approximation – called complexity – depending on the relation between Γ and (1.0.3). As a preliminary step, it is necessary to study the value function associated with the optimal control problem (1.0.4), estimating its behavior along the curve Γ. This research appears in [Pra14, JP].

2. *Singular diffusions.* In the second part of the thesis we will focus on a class of two dimensional driftless control systems in the form (1.0.3), to which it is possible to associate intrinsically a Laplace-Beltrami operator. Due to the control vector fields becoming collinear on a curve $\mathcal{Z}$, this operator will present some singularities. This research appears in [BP, BPS]. In [BP] our interest lies on how $\mathcal{Z}$ affects the diffusion dynamics. In particular, we will try to understand if solutions to the heat and Schrödinger equations associated with this Laplace-Beltrami operator are able to cross $\mathcal{Z}$, and whether some heat is absorbed in the process or not. On the other hand, in [BPS] we are interested in how the presence of the singularity affects the spectral properties of the operator, in particular under a magnetic Aharonov–Bohm-type perturbation. Recent results on this topic, that have been part of the research developed during the PhD, but are not presented here, are contained in the work in progress [PP].

The mathematical motivation of the problems considered in this thesis lies in sub-Riemannian geometry. Thus, next section will be devoted to a short introduction to this topic. Our contributions will then be described in Sections 1.2, 1.3.2, and 1.3.3, while Section 1.4 is devoted to expose some open problems and future lines of research.

Other material that is related to these topics and that has been part of the research developed during the PhD, but is not presented here, is contained in the following papers and preprints.

P1. U. Boscain, J.-P. Gauthier, D. Prandi and A. Remizov, *Image reconstruction via non-isotropic diffusion in Dubins/Reed-Shepp-like control systems*, to appear on Proceedings of the 51th IEEE Conference on Decision and Control, December 2014.

P2. R. Chertovskih, J.-P. Gauthier, D. Prandi and A. Remizov, *Image reconstruction and hypoelliptic diffusion: new ideas – new results.*, in preparation.

P3. D. Prandi, Sobolev and BV integral differential quotients, in preparation.

1.1 SUB-RIEMANNIAN GEOMETRY

Sub-Riemannian geometry can be thought of as a generalization of Riemannian geometry, where the dynamics is subject to non-holonomic constraints. Classically (see, e.g., [Mon02]), a sub-Riemannian structure on M is defined by a smooth vector distribution $\Delta \subset TM$ – i.e., a sub-bundle of TM – of constant rank k and a Riemannian metric $\mathbf{g}$ defined on Δ. From this structure, one derives the so-called *Carnot-Carathéodory distance* d_{SR} on M: The length of any absolutely continuous path tangent to the distribution – called *horizontal* – is defined through the Riemannian metric, and the distance $d_{SR}(q_0, q_1)$ is then defined as the infimum of the length of all horizontal paths joining q_0 to q_1. If no such path exists, $d_{SR}(q_0, q_1) = +\infty$.

Locally, it is always possible to find an orthonormal frame $\{f_1, \dots, f_m\}$ for Δ. This allows to identify horizontal trajectories with admissible trajectories of the *non-holonomic control system*

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)). \tag{1.1.1}$$

The problem of finding the shortest curve joining two fixed points q_0, $q_1 \in M$ is then naturally formulated as the optimal control problem

$$\begin{cases} \dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)), \\ q(0) = q_1, \qquad q(T) = q_1, \\ \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{i=1}^{m} u_i(t)^2}\, dt \longrightarrow \min. \end{cases} \tag{1.1.2}$$

With this point of view, the Carnot-Carathéodory distance is the value function associated with (1.1.2).

This framework is however more general than classical sub-Riemannian geometry. Indeed, choosing $f_1, \dots, f_m$ to be possibly non-linearly independent, this optimal control formulation allows to define sub-Riemannian structures endowed with a *rank-varying distribution* $\Delta(q) = \operatorname{span}\{f_1(q), \dots, f_m(q)\}$. Namely, it is possible to define a Riemannian norm on $\Delta(q)$ as

$$\|v\|_q = \min\left\{ |u| \mid v = \sum_{i=1}^{m} u_i f_i(q) \right\}, \quad \text{for any } v \in \Delta(q),$$

from which the metric $\mathbf{g}_q$ follows by polarization. Through this metric we obtain the Carnot-Carathéodory distance, coinciding with the value function of the optimal control problem associated with the non-holonomic control system, as in the classical case. Since it well known that every distribution can be globally represented as the linear span of a finite family of (possibly not linearly independent) vector fields (see [Suso8, ABB12a, DLPR12]), it is always possible to represent globally a sub-Riemannian structure as a non-holonomic system.

Although it is outside the scope of the following discussion, we remark that this control theoretical setting can be stated in purely geometrical terms, as done in [ABB12a].

1.1.1 Metric properties

Once the Carnot-Carathéodory distance is defined, the first natural question is: is it finite? This amounts to ask if every pair of points of M is joined by an horizontal curve. This property, in the control theoretic language, is known as *controllability* or *accessibility*.

A partial answer (for analytic corank-one distributions) can be found in Carathéodory paper [Car09] on formalization of classical thermodynamics, where the role of horizontal curves is roughly taken by adiabatic processes[1]. However it is not until the 30's, that Rashevsky [Ras38] and Chow [Cho39] independently extendend Carathéodory result to a general criterion for smooth distributions. The key assumption of this theorem is the *Hörmander condition* (or *Lie bracket-generating condition*) for Δ, i.e., that the Lie algebra generated by the horizontal vector fields spans at any point the whole tangent space.

Theorem 1.1.1 (Chow-Rashevsky Theorem)**.** *Let M be a connected sub-Riemannian manifold, such that Δ satisfies the Hörmander condition. Then, the Carnot-Carathéodory distance is finite, continuous, and induces the manifold topology.*

Heuristically, the Chow-Rashevsky theorem is a consequence of the fact that, in coordinate representation,

$$e^{-tg} \circ e^{-tf} \circ e^{tg} \circ e^{tf}(q) = q + t^2[f,g](q) + o(t^2). \tag{1.1.3}$$

Iterating this procedure shows that, if the Lie bracket-generating condition is satisfied, it is possible to move in every direction and hence to connect every couple of points on M.

Let us remark that the converse is not true without assuming M and Δ to be analytic (see [Nag66]). From now on, we will always assume the Lie bracket-generating condition to be satisfied.

Although finite, the Carnot-Carathéodory distance presents a quite different behavior than the Riemannian one. It is a basic fact of Riemannian geometry that small balls around a fixed point are, when looked in coordinates, roughly Euclidean. This isotropic behavior is essentially due to the fact that geodesics tangent vectors are parametrized on the Euclidean sphere in the tangent space. In sub-Riemannian geometry this is no more true, and as a consequence the Carnot-Carthéodory distance is highly anisotropic. Indeed, in order to move in directions that are not contained in the distribution, it is necessary to construct curves like (1.1.3). This suggest that the number of brackets we have to build to attain a certain direction is directly related to the cost of moving in that direction.

1 Indeed, the proof of this fact relies on the theory of Carnot cycles. This is why the sub-Riemannian distance is known as "Carnot-Carathéodory" distance.

In order to exploit this fact, it is necessary to choose an appropriate coordinate system. Let $\Delta^1 = \Delta$ and define recursively $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta]$, for every $s \in \mathbb{N}$. By the Hörmander condition, the evaluations of the sets Δ^s at q form a flag of subspaces of $T_q M$,

$$\Delta^1(q) \subset \Delta^2(q) \subset \ldots \subset \Delta^r(q) = T_q M. \tag{1.1.4}$$

The integer $r = r(q)$, which is the minimum number of brackets required to recover the whole $T_q M$ is called *degree of non-holonomy* (or *step*) of Δ at q. Finally, let $w_1 \leqslant \ldots \leqslant w_n$ be the *weights* associated with the flag, defined by $w_i = s$ if $\dim \Delta^{s-1}(q) < i \leqslant \dim \Delta^s(q)$, setting $\dim \Delta^0(q) = 0$. A system of coordinates $z = (z_1, \ldots, z_n)$ at q is *privileged* whenever the non-holonomic order of z_i is exactly w_i – i.e., if $f_{i_1} \cdots f_{i_{w_i}} z_i = 0$ for any $\{i_1, \ldots, i_{w_i}\} \subset \{1, \ldots, m\}$ but $f_{i_1} \cdots f_{i_{w_i}} f_{i_{w_i+1}} z_i \neq 0$ for some $\{i_1, \ldots, i_{w_i}, i_{w_i+1}\} \subset \{1, \ldots, m\}$. In particular, any system of privileged coordinates at q induces a splitting of the tangent space as a direct sum,

$$T_q M = \Delta^1(q) \oplus \Delta^2(q)/\Delta^1(q) \oplus \ldots \oplus \Delta^r(q)/\Delta^{r-1}(q),$$

where each $\Delta^s(q)/\Delta^{s-1}(q)$ is spanned by $\partial_{z_i}|_q$ with $w_i = s$.

Starting from the 80's, various authors exploited privileged coordinates to obtain the following result, showing the strong anisotropy of the Carnot-Carathéodory distance. For early versions see [NSW85, Ger90, Gro96], while a general and detailed proof can be found in [Bel96].

Theorem 1.1.2 (Ball-box Theorem). *Let $z = (z_1, \ldots, z_n)$ be a system of privileged coordinates at $q \in M$. Then, there exist $C, \varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, it holds*

$$\mathrm{Box}\left(\frac{1}{C}\varepsilon\right) \subset B_{SR}(q, \varepsilon) \subset \mathrm{Box}(C\varepsilon).$$

Here, we let $B_{SR}(q, \varepsilon)$ be both the sub-Riemannian ball of radius $\varepsilon > 0$ centered in q and its coordinate representation $z(B_{SR}(q, \varepsilon))$. Moreover, we let

$$\mathrm{Box}(\varepsilon) = \{x \in \mathbb{R}^n \mid |x_i| \leqslant \varepsilon^{w_i}\}. \tag{1.1.5}$$

An immediate consequence of this theorem is the Hölder equivalence of the Carnot-Carathéodory distance and the Euclidean one. Namely, in any coordinate system centered at q and for q' sufficiently close to q, it holds

$$|q' - q| \lesssim d_{SR}(q, q') \lesssim |q' - q|^{\frac{1}{r}}. \tag{1.1.6}$$

Here we used "$\lesssim$" to denote an inequality up to a multiplicative constant.

As a consequence of the anisotropy of the distance, the Hausdorff dimension of a sub-Riemannian manifold is different from its topological dimension. A point q is said to be *regular* if $\dim \Delta^s$ is constant near q for any $1 \leqslant s \leqslant r$. If every point is regular, then the sub-Riemannian manifold is said to be *equiregular*. This allows to prove the following celebrated theorem [Mit85].

Theorem 1.1.3 (Mitchell's measure theorem). *The Hausdorff dimension $\dim_q^{\mathcal{H}} M$ of a sub-Riemannian manifold at a regular point q is given by*

$$\dim_q^{\mathcal{H}} M = \sum_{s=1}^{r} s(\dim \Delta^s(q) - \dim \Delta^{s-1}(q)).$$

In particular, if $\dim \Delta(q) < \dim M$, then $\dim M < \dim_q^{\mathcal{H}} M$. Moreover, the $(\dim_q^{\mathcal{H}} M)$-dimensional Hausdorff measure is absolutely continuous with respect to any smooth volume, near q.

The theorem has been proved only at regular points since near these points the Ball-Box Theorem holds with uniform constants. See [GJ13] for some more general results in this direction.

1.1.2 The sub-Laplacian

A differential operator P is hypoelliptic if for any $a : U \subset M \to \mathbb{R}$ it holds that $Pa \in C^\infty(U)$ implies $a \in C^\infty(U)$. The deep connection between second-order hypoelliptic operators and sub-Riemannian geometry became evident after the celebrated work [Hör67]. In this paper, Hörmander proved that the Lie bracket-generating condition is sufficient for the hypoellipticity of a second order differential operators with local expression

$$\mathcal{L} = \sum_{i=1}^{m} f_i^2 + \text{“first-order terms”},$$

where the f_i's are first-order differential operators. Then, interpreting $\{f_1, \dots, f_m\}$ as a family of vector fields, it is possible to define a sub-Riemannian structure on M.

The operator $\mathcal{L} = \sum_{i=1}^m f_i^2$ is commonly called the *sub-Laplacian* on M associated with the frame $\{f_1, \dots, f_m\}$. From a sub-Laplacian it is possible to recover the Carnot-Carathéodory metric d_{SR} defined by the frame. In fact, letting the sub-Riemannian gradient $\nabla_H u = \sum_{i=1}^m f_i u$, it holds that

$$d_{SR}(q_0, q_1) = \sup \left\{ u(x) - u(y) \mid u \in C_c^\infty(M) \text{ and } |\nabla_H u|^2 \leqslant 1 \text{ a.e.} \right\},$$

where $|\nabla_H u|^2 = \sum_{i=1}^m (f_i u)^2$. This has allowed to find many estimates on the fundamental solution of $\mathcal{L}$ in terms of the associated Carnot-Carathéodory distance (see, e.g., [FS74, RS76]), and was at the origin of the renovated interested in sub-Riemannian geometry in the 70's [Gav77].

However, the correspondence between hypoelliptic operators and sub-Riemannian manifolds is not one-to-one. Indeed, it is easy to check that the sub-Riemannian gradient does not depend on the family of vector fields $\{f_1, \dots, f_m\}$, but is intrinsically defined by the sub-Riemannian structure[2]. On the other hand, the sub-Laplacian $\tilde{\mathcal{L}}$ associated with a different family of vector fields $\{g_1, \dots, g_m\}$, generating the same distribution, differs from $\mathcal{L}$ by a first-order differential operator. Thus, the same sub-Riemannian structure is associated with different sub-Laplacians.

Since we are interested in having a diffusion operator intrinsically associated with the sub-Riemannian structure, we have to resolve this ambiguity. The same problem arises in Riemannian geometry, when defining the Laplace-Beltrami operator, and it is resolved through the Green identity. We will proceed in the same way. Namely, instead of defining the sub-Laplacian through a local frame of the distribution, we consider a global smooth volume form $d\mu$, and let the sub-Laplacian $\mathcal{L}$ to be the only operator satisfying the Green identity:

$$-\int_M f(\mathcal{L} g)\, d\mu = \int_M \mathbf{g}(\nabla_H f, \nabla_H g)\, d\mu, \qquad \text{for any } f, g \in C_c^\infty(M). \tag{1.1.7}$$

Hence, in order to have an intrinsically-defined sub-Laplacian, one needs the volume $d\mu$ to be intrinsically defined by the geometric structure of the manifold.

2 Indeed, the sub-Riemannian gradient of u is the only vector field such that $\mathbf{g}_q(\nabla_H u(q), \nu) = du(\nu)$, for any $q \in M$ and $\nu \in \Delta(q)$.

In the Riemannian case this problem is readily settled. Indeed, on any Riemannian manifold there are three common ways to define an intrinsic volume: The Riemannian metric defines the Riemannian volume, with coordinate expression $dV = \sqrt{\mathbf{g}}dx_1 \wedge \cdots \wedge dx_2$, while the Riemannian distance allows to define the n-dimensional Hausdorff and spherical Hausdorff volumes. Since these three volumes are proportional up to a constant (see, e.g., [Fed69]), they are equivalent for the definition of the Laplace-Beltrami operator through (1.1.7).

In the sub-Riemannian setting, through the sub-Riemannian structure it is possible to define an intrinsic measure – called Popp measure – that plays the role of the Riemannian volume, and which is smooth if M is equiregular. In the equiregular case, by Theorem 1.1.3, we have at our disposal also the $(\dim^{\mathcal{H}} M)$-dimensional Hausdorff measure and spherical Hausdorff measure, which are commensurable one with respect to the other (see, e.g., [Fed69]) and are absolutely continuous with respect to the Popp measure. Recent results [ABB12b], have however proved that the density of the Hausdorff measures with respect to the Popp measure is not, in general, smooth. Thus these measures define different intrinsic sub-Laplacians. When the manifold is not equiregular, moreover, these sub-Laplacians can present terms that diverge near singular points, as we will discuss in the next section.

This said, considering sub-Riemannian manifolds endowed with additional structure can resolve this ambiguity. For example, for left-invariant sub-Riemannian structures, i.e., Lie groups equipped with a left-invariant distribution and metric, both the Popp and the Hausdorff measures are left-invariant and hence Haar measures. The uniqueness up to a constant of Haar measures, allows then to define the sub-Laplacian through (1.1.7), as studied in [ABGR09].

1.2 COMPLEXITY OF CONTROL-AFFINE MOTION PLANNING

The concept of complexity was first developed for the non-holonomic motion planning problem in robotics. Given a control system on a manifold M, the motion planning problem consists in finding an admissible trajectory connecting two points, usually under further requirements, such as obstacle avoidance. If a cost function is given, it makes sense to try to find the trajectory costing the least. This study is critical for applications. As examples we cite: mechanical systems with controls on the acceleration (see e.g., [BL05], [BLS10]) where the drift is the velocity, or quantum control (see e.g., [D'A08], [BM06]), where the drift is the free Hamiltonian.

Different approaches are possible to solve this problem (see [LSL98]). Here we focus on those based on the following scheme:

1. find any (usually non-admissible) curve or path solving the problem,
2. approximate it with admissible trajectories.

The first step is independent of the control system, since it depends only on the topology of the manifold and of the obstacles, and it is already well understood (see [SS83]). In the first part of this thesis, we are interested in the second step, which depends only on the local nature of the control system near the path. Our goal is to understand how to measure the complexity of the approximation task for control-affine systems. Namely, we are interested in systems in the form

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)), \tag{1.2.1}$$

By complexity we mean a function of the non-admissible curve $\Gamma \subset M$ (or path $\gamma : [0, T] \to M$), and of the precision of the approximation, quantifying the difficulty of the latter by means of the cost function.

The following two sections will be dedicated to some generalities on systems of the form (1.2.1) and to the preliminary results contained in [Pra14], respectively. These results are essential for the study of the complexity carried out in Section 1.2.3, where we will present the results contained in [JP].

1.2.1 Control-affine systems

It has been known since the the 70's that under the *strong Hörmander condition*, i.e., if $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and with unbounded controls, systems in the form (1.2.1) are controllable. Such a result is proved for example in [BL75], considering (1.0.3) as a perturbation of a non-holonomic control system. From now on, we will always assume the strong Hörmander condition to be satisfied. Such assumption is generically satisfied, e.g., by finite-dimensional quantum control systems with two controls, as the ones studied in [BC04, BCG02a, DD01].

Although out of the scope of the present work, we have to mention that from as early as the 60's the problem of controllability of such systems under the Hörmander condition – i.e., that the Lie algebra generated by the drift *and* the control vector fields spans the whole tangent space at any point – has been subject to a lot of attention, see for example [Kal60, Her64, BL75, Sus82]. In particular, the main focus has been the so-called *small time local controllability* around an equilibrium point, i.e., if given an equilibrium point $q \in M$ and any time $T > 0$ the end-points of admissible trajectories defined on $[0, T]$ and starting from q cover a neighborhood of q. This problem is important, for example, in the context of quasi-static motions for robots with controls on the acceleration. For a review on results obtained in this direction see e.g., [Kaw90].

System (1.2.1) can be seen, from a geometrical point of view, as a generalization of sub-Riemannian geometry, where the distribution $\Delta(q)$ is replaced by the affine distribution $f_0(q) + \Delta(q)$. Thus, in addition to the L^1 cost $\mathcal{J}$ considered in (1.0.4), it makes sense to study also the cost

$$\mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{i=1}^m u_i(t)^2}\, dt,$$

that measures the "Riemannian" length of admissible curves. We then fix a time $\mathcal{T} > 0$ and consider the two value functions $V^{\mathcal{J}}(q_0, q_1)$ and $V^{\mathcal{I}}(q_0, q_1)$ as the infima of the costs $\mathcal{J}$ and $\mathcal{I}$, respectively, over all controls steering system (1.0.3) from q_0 to q_1 in time $T \leqslant \mathcal{T}$. Contrarily to what happens in sub-Riemmanian geometry with the Carnot-Carathéodory distance, these value functions are not symmetric, and hence do not induce a metric space structure on M. In fact, system (1.2.1) is not reversible – i.e., changing orientation to an admissible trajectory does not yield an admissible trajectory.

The reason for introducing a maximal time of definition for the controls – not needed in the sub-Riemannian context – is that, by taking $\mathcal{T}$ sufficiently small, it is possible to prevent any exploitation of the geometry of the orbits of the drift (that could be, for example, closed). Let us also remark that, since the controls can be defined on arbitrarily small times, it is possible to approximate admissible trajectories via trajectories for the sub-Riemannian associated system (i.e., the one obtained by posing $f_0 \equiv 0$ in (1.0.3)) rescaled on small intervals.

1.2.2 Hölder continuity of the value function

The work [Pra14], is dedicated to generalize the Chow-Rashewsky theorem and the Ball-box theorem to system (1.0.3) with the cost $\mathcal{J}$. This is a technical but essential result for the understanding of complexities, as we will see in the following section. Indeed, the first result we obtain is a global continuity result for the value function.

Theorem 1.2.1. *For any $0 < \mathcal{T} \leqslant +\infty$, the function $V^{\mathcal{J}} : M \times M \to [0, +\infty)$ is continuous. Moreover, letting d_{SR} be the sub-Riemannian distance induced by $\{f_1, \dots, f_m\}$, it holds*

$$V^{\mathcal{J}}(q, q') \leqslant \min_{0<t\leqslant\mathcal{T}} d_{SR}(e^{tf_0}q, q'), \qquad \textit{for any } q,\ q' \in M.$$

Letting $\mathcal{R}_{f_0}(q, \varepsilon)$ be the reachable set from q with cost $\mathcal{J}$ less than ε, Theorem 1.2.1 shows that

$$\bigcup_{0<t\leqslant\mathcal{T}} B_{SR}(e^{tf_0}q, \varepsilon) \subset \mathcal{R}_{f_0}(q, \varepsilon). \tag{1.2.2}$$

Thus, the cost to steer the sub-Riemannian system from one point to another is always larger or equal than the cost to steer the control-affine system between the same points. Moreover, the fact that in coordinates it holds

$$e^{tf_0 - \varepsilon f_1} \circ e^{tf_0 + \varepsilon f_1}(q) = 2tf_0(q) + t\varepsilon[f_0, f_1](q) + o(\varepsilon t),$$

suggests that exploiting the drift it is actually possible to move more easily in some directions. Indeed, we will prove that this is the case, but only on very special directions realized as brackets of the drift with the control vector fields. Although this will not suffice to improve (1.2.2), we will be able to obtain a ball-box-like estimation of the reachable set from the outside.

Assume that the drift is regular, in the sense that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, where Δ^s is defined through the vector fields $\{f_1, \dots, f_m\}$ as in the sub-Riemannian case. In particular, this allows to build systems of privileged coordinates rectifying f_0. Let $\{\partial_{z_i}\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$ such that ∂_{z_ℓ} be the coordinate expression of f_0, and consider the following sets:

$$\Xi(\eta) = \bigcup_{0\leqslant\xi\leqslant\mathcal{T}} \left(\xi\partial_{z_\ell} + \mathrm{Box}(\eta)\right)$$

$$\Pi(\eta) = \bigcup_{0\leqslant\xi\leqslant\mathcal{T}} \{z \in \mathbb{R}^n \colon |z_\ell - \xi| \leqslant \eta^s,\ |z_i| \leqslant \eta^{w_i} + \eta\xi^{\frac{w_i}{s}} \text{ for } w_i \leqslant s,\ i \neq k,$$
$$\text{and } |z_i| \leqslant \eta(\eta + \xi^{\frac{1}{s}})^{w_i - 1} \text{ for } w_i > s\},$$

In particular, observe that $\Pi(\eta)$ is contained in $\mathrm{Box}(\eta)$, defined in (1.1.5), and that $\Pi(\eta) \cap \{z_\ell < 0\} = \mathrm{Box}(\eta) \cap \{z_\ell < 0\}$. We then get the following generalization of the Ball-Box theorem

Theorem 1.2.2. *Let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at q for $\{f_1, \dots, f_m\}$, rectifying f_0 as the* k*-th coordinate vector field ∂_{z_ℓ}, for some $1 \leqslant \ell \leqslant n$. Then, there exist $C, \varepsilon_0, T_0 > 0$ such that, if the maximal time of definition of the controls satisfies $\mathcal{T} < T_0$, it holds*

$$\Xi\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_{f_0}(q,\varepsilon) \subset \Pi(C\varepsilon), \qquad \text{for } \varepsilon < \varepsilon_0. \tag{1.2.3}$$

Here, with abuse of notation, we denoted by $\mathcal{R}_{f_0}(q,\varepsilon)$ the coordinate representation of the reachable set.

This theorem represent the key step for generalizing the estimates on the complexity of curves from sub-Riemannian control systems to control-affine systems.

Finally, as in the sub-Riemannian case, as a consequence of Theorem 1.2.2 we get the following local Hölder equivalence between the value function and the Euclidean distance.

Theorem 1.2.3. *Let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at q for $\{f_1, \dots, f_m\}$, rectifying f_0 as the* k*-th coordinate vector field ∂_{z_ℓ}, for some $1 \leqslant \ell \leqslant n$. Then, there exist $T_0, \varepsilon_0 > 0$ such that, if the maximal time of definition of the controls satisfies $\mathcal{T} < T_0$ and $V^{\mathcal{J}}(q,q') \leqslant \varepsilon_0$, it holds*

$$\operatorname{dist}\left(z(q'), z(e^{[0,\mathcal{T}]f_0}q)\right) \lesssim V^{\mathcal{J}}(q,q') \lesssim \operatorname{dist}\left(z(q'), z(e^{[0,\mathcal{T}]f_0}q)\right)^{\frac{1}{r}}.$$

Here for any $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $\operatorname{dist}(x,A) = \inf_{y\in A}|x-y|$ denotes the Euclidean distance between them and r is the degree of non-holonomy of the sub-Riemannian control system defined by $\{f_1, \dots, f_m\}$.

In this result, instead of the Euclidean distance from the origin that appeared in (1.1.6), we have the distance from the integral curve of the drift. This is due to the fact that moving in this direction has null cost.

It is worth to mention that these results regarding control-affine systems are obtained by reducing them, as in [AL10], to time-dependent control systems in the form

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t)\, f_i^t(q(t)), \qquad \text{a.e. } t \in [0,T], \tag{1.2.4}$$

where $f_i^t = (e^{-tf_0})_* f_i$ is the pull-back of f_i through the flow of the drift. On these systems, that are linear in the control, we are able to define a good notion of approximation of the control vector fields. Namely, we will define a generalization of the nilpotent approximation, used in the sub-Riemannian context, taking into account the fact that in system (1.2.4), exploiting the time, we can generate the direction of the brackets between f_0 and the f_js. This approximation and an iterated integral method yield fine estimates on the reachable set.

1.2.3 Complexity and motion planning

The core of the first part of the thesis is [JP], in collaboration with F. Jean. Here, we focus on extending the concept of complexity, already introduced in the sub-Riemannian setting by Gromov [Gro96, p. 278] and Jean [Jea01a], to the control-affine case, and to give weak estimates of these quantities.

Heuristically, the complexity of a curve Γ (or path $\gamma : [0,T] \to M$) at precision ε is defined as the ratio

$$\frac{\text{“cost” to track } \Gamma \text{ at precision } \varepsilon}{\text{“cost” of an elementary } \varepsilon\text{-piece}}. \tag{1.2.5}$$

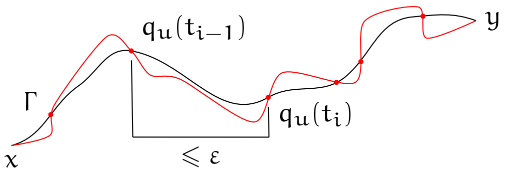

Figure 1: Interpolation by cost complexity

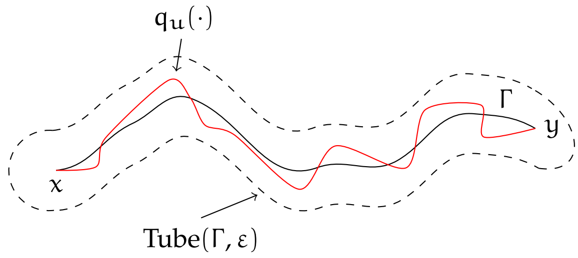

Figure 2: Tubular approximation complexity

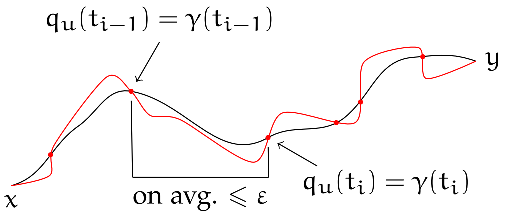

Figure 3: Time interpolation complexity

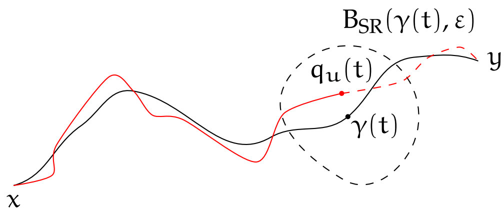

Figure 4: Neighboring approximation complexity

In order to obtain a precise definition of complexity, we need to give a meaning to the notions appearing above. Namely, we have to specify what do we mean by "cost"[3], tracking at precision ε, and elementary ε-piece. Indeed, these choices will depend on the type of motion planning problem at hand.

First of all, we classify motion planning problems as *time-critical* or *static*, depending on wether the constraints depend on time or not. The typical example of static motion planning problem is the obstacle avoidance problem with fixed obstacles. On the other hand, the same problem where the position of the obstacles depends on time, or the rendez-vous problems, are examples of time-critical motion planning problems.

For static motion planning problems, the solution of the first step of the motion planning scheme (introduced at the beginning of the paper) is usually given as a curve, i.e., a dimension 1 connected submanifolds of $\Gamma \subset M$ diffeomorphic to a closed interval. On the other hand, in time-critical problems we have to keep track of the time. Thus, for this type of problems, the solution of the first step is a path, i.e., a smooth injective function $\gamma : [0, T] \to M$. As a consequence, when computing the complexity of paths we will require the approximating trajectories to respect also the parametrization, and not only the geometry, of the path. While in the sub-Riemannian case, due to the time rescaling properties of the control system, these concepts coincide, this is not the case for control-affine systems.

In this thesis, we consider four distinct notions of complexity, two for curves (static problems) and two for paths (time-critical problems). In both cases, one of the two will be based on the interpolation of the given curve or path, while the other will consider trajectories that stays

3 The cost appearing in (1.2.5) is not necessarily related with the cost function ($\mathcal{J}$ or $\mathcal{I}$) taken into account. This is the reason for the quotation marks.

near the curve or path. Thus, for this complexity, we will need to fix a metric. In this work we will consider only the sub-Riemannian metric of the associated sub-Riemannian control system (obtained by putting $f_0 \equiv 0$ in (1.2.1)).

We remark that the two complexity for curves are the same as the sub-Riemannian ones already introduced in [Gro96, Jea01a]. This is true also for what we call the neighboring approximation complexity of a path, since in the sub-Riemannian case it coincides with the tubular approximation complexity. On the other hand, what we call the interpolation by time complexity never appeared in the literature, to our knowledge. Here, we define them for the cost $\mathcal{J}$, but the same definitions holds for $\mathcal{I}$.

Fix a curve Γ. Then, denoting by q_u the trajectory associated with a control u and with starting point $q_u(0) = x$, we define for any $\varepsilon > 0$ the following complexities.

- *Interpolation by cost complexity:* (see Figure 1) For $\varepsilon > 0$, let an *ε-cost interpolation* of Γ to be any control $u \in \mathcal{U}$ such that there exist $0 = t_0 < t_1 < \ldots < t_N = T \leqslant \mathcal{T}$ for which the trajectory q_u with initial condition $q_u(0) = x$ satisfies $q_u(T) = y$, $q_u(t_i) \in \Gamma$ and $\mathcal{J}(u|_{[t_{i-1},t_i)}, t_i - t_{i-1}) \leqslant \varepsilon$, for any $i = 1, \ldots, N$. Then, let

  $$\Sigma^{\mathcal{J}}_{\text{int}}(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \mathcal{J}(u, T) \mid q_u \text{ is an } \varepsilon\text{-cost interpolation of } \Gamma \right\}.$$

  This function measures the number of pieces of cost ε necessary to interpolate Γ. Namely, following a trajectory given by a control admissible for $\Sigma^{\mathcal{J}}_{\text{int}}(\Gamma, \varepsilon)$, at any given moment it is possible to go back to Γ with a cost less than ε.

- *Tubular approximation complexity:* (see Figure 2) Let $\text{Tube}(\Gamma, \varepsilon)$ to be the tubular neighborhood of radius ε around the curve Γ w.r.t. the small sub-Riemannian system associated with (1.2.1) (obtained by putting $f_0 \equiv 0$), and define

  $$\Sigma^{\mathcal{J}}_{\text{app}}(\Gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \mathcal{J}(u, T) \;\middle|\; \begin{array}{l} 0 < T \leqslant \mathcal{T}, \\ q_u(0) = x,\ q_u(T) = y, \\ q_u\big([0, T]\big) \subset \text{Tube}(\Gamma, \varepsilon) \end{array} \right\}$$

  This complexity measures the number of pieces of cost ε necessary to go from x to y staying inside the sub-Riemannian tube $\text{Tube}(\Gamma, \varepsilon)$. Such property is especially useful for motion planning with obstacle avoidance. In fact, if the sub-Riemannian distance of Γ from the obstacles is at least $\varepsilon_0 > 0$, then trajectories obtained from controls admissible for $\Sigma^{\mathcal{J}}_{\text{app}}(\Gamma, \varepsilon)$, $\varepsilon < \varepsilon_0$, will avoid such obstacles.

We then define the following complexities for a path $\gamma : [0, T] \to M$ at precision $\varepsilon > 0$.

- *Interpolation by time complexity:* (see Figure 3) Let a δ-time interpolation of γ to be any control $u \in L^1([0, T], \mathbb{R}^m)$ such that its trajectory $q_u : [0, T] \to M$ with $q_u(0) = \gamma(0)$ is such that $q_u(T) = \gamma(T)$ and that, for any interval $[t_0, t_1] \subset [0, T]$ of length $t_1 - t_0 \leqslant \delta$, there exists $t \in [t_0, t_1]$ with $q_u(t) = \gamma(t)$. Then, fix a $\delta_0 > 0$ and let

  $$\sigma_{\text{int}}(\gamma, \varepsilon) = \inf \left\{ \frac{T}{\delta} \;\middle|\; \begin{array}{c} \delta \in (0, \delta_0) \text{ and exists } u \in L^1([0, T], \mathbb{R}^m), \\ \delta\text{-time interpolation of } \gamma, \text{ s.t. } \delta\, J(u, T) \leqslant \varepsilon \end{array} \right\}.$$

  Controls admissible for this complexity will define trajectories such that the minimal average cost between any two consecutive times such that $\gamma(t) = q_u(t)$ is less than ε. It is thus well suited for time-critical applications where one is interested in minimizing the time between the interpolation points - e.g. motion planning in rendez-vous problem.

- *Neighboring approximation complexity:* (see Figure 4) Let $B_{SR}(p,\varepsilon)$ denote the ball of radius ε centered at $p \in M$ w.r.t. the small sub-Riemannian system associated with the control-affine system, and define

$$\sigma_{app}^{\mathcal{J}}(\gamma,\varepsilon) = \frac{1}{\varepsilon}\inf\left\{ \mathcal{J}(u,T) \;\middle|\; \begin{array}{l} q_u(0)=x,\ q_u(T)=y, \\ q_u(t)\in B_{SR}(\gamma(t),\varepsilon),\ \forall t \end{array} \right\}.$$

This complexity measures the number of pieces of cost ε necessary to go from x to y following a trajectory that at each instant $t \in [0,T]$ remains inside the sub-Riemannian ball $B_{SR}(\gamma(t),\varepsilon)$. Such complexity can be applied to motion planning in rendez-vous problems where it is sufficient to attain the rendez-vous only approximately.

We remark that for the interpolation by time complexity the "cost" in (1.2.5) is the time, while for all the other complexities it is the cost function associated with the system. For the motivation of the bound on δ in the definition of the interpolation by time complexity, see Remark 4.3.3. Finally, whenever a metric is required, we use the sub-Riemannian one. Although such metric is natural for control-affine systems satisfying the Hörmander condition, nothing prevents from defining complexities based on different metrics.

Two functions $f(\varepsilon)$ and $g(\varepsilon)$, tending to ∞ when $\varepsilon \downarrow 0$ are *weakly equivalent* (denoted by $f(\varepsilon) \asymp g(\varepsilon)$) if both $f(\varepsilon)/g(\varepsilon)$ and $g(\varepsilon)/f(\varepsilon)$, are bounded when $\varepsilon \downarrow 0$. When $f(\varepsilon)/g(\varepsilon)$ (resp. $g(\varepsilon)/f(\varepsilon)$) is bounded, we will write $f(\varepsilon) \preccurlyeq g(\varepsilon)$ (resp. $f(\varepsilon) \succcurlyeq g(\varepsilon)$). In the sub-Riemannian context, the complexities are always measured with respect to the L^1cost of the control, $\mathcal{J}$. Then, for any curve $\Gamma \subset M$ and path $\gamma : [0,T] \to M$ such that $\gamma([0,T]) = \Gamma$ it holds $\Sigma_{int}^{\mathcal{J}}(\Gamma,\varepsilon) \asymp \Sigma_{app}^{\mathcal{J}}(\Gamma,\varepsilon) \asymp \sigma_{app}^{\mathcal{J}}(\gamma,\varepsilon)$.

Let us remark that in the sub-Riemannian setting the asymptotic behavior of $\sigma(\Gamma,\varepsilon)$ as $\varepsilon \downarrow 0$ is strictly related with the Hausdorff dimension $\dim^{\mathcal{H}} \Gamma$. A complete characterization of weak asymptotic equivalence of metric complexities of a path is obtained in [Jea03]. We state here this result in the special case where M is an equiregular sub-Riemannian manifold.

Theorem 1.2.4. *Let M be an equiregular sub-Riemannian manifold and let $\Gamma \subset M$ be a curve. Then, if there exists $k \in \mathbb{N}$ such that $T_q\Gamma \subset \Delta^k(q) \setminus \Delta^{k-1}(q)$ for any $q \in \Gamma$, it holds*

$$\Sigma_{int}(\Gamma,\varepsilon) \asymp \Sigma_{app}(\Gamma,\varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

In particular, this implies that

$$\dim^{\mathcal{H}} \Gamma = k.$$

Here, similarly to what happened in Theorem 1.1.3, the equiregularity is needed in order to have a uniform Ball-Box theorem near Γ. Indeed, to get the general result of [Jea03], it is necessary to use a finer form of the Ball-Box theorem that holds uniformly around singular points, proved in [Jea01b].

We mention also that for a restricted set of sub-Riemannian systems, i.e., one-step bracket generating or with two controls and dimension not larger than 6, strong asymptotic estimates and explicit asymptotic optimal syntheses are obtained in a series of papers by Gauthier, Zakalyukin and others (e.g., see[RMGMP04, GZ05, GZ06] and [BG13] for a review).

Our first result is the following. It completes the description of the sub-Riemannian weak asymptotic estimates started in Theorem 1.2.4, describing the case of the interpolation by time complexity.

Theorem 1.2.5. *Assume that* $\{f_1, \dots, f_m\}$ *defines an equiregular sub-Riemannian structure and let* $\gamma : [0,T] \to M$ *be a path. Then, if there exists* $k \in \mathbb{N}$ *such that* $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ *for any* $t \in [0,T]$, *it holds*

$$\sigma_{int}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

Here the complexity is measured w.r.t. the cost $\mathcal{J}(u,T) = \|u\|_{L^1([0,T],\mathbb{R}^m)}$.

Since in the sub-Riemannian context one is only interested in the cost $\mathcal{J}$, Theorems 1.2.4 and 1.2.5 completely characterize the weak asymptotic equivalences of complexities of equiregular sub-Riemannian manifolds.

Then, exploiting the results of the previous section, we are able to prove the following theorem for the genuinely control-affine case, in the same spirit as Theorem 1.2.4 and 1.2.5.

Theorem 1.2.6. *Assume that the sub-Riemannian structure defined by* $\{f_1, \dots, f_m\}$ *is equiregular, and that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ *for some* $s \geqslant 2$. *Then, for any curve* $\Gamma \subset M$, *whenever the maximal time of definition of the controls* $\mathcal{T}$ *is sufficiently small, it holds*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{J}}_{app}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{app}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^\kappa}.$$

Here $\kappa = \max\{k \colon T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$, *for any* p *in an open subset of* $\Gamma\}$.

Moreover, for any path $\gamma : [0,T] \to M$ *such that* $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}$ *for any* $t \in [0,T]$, *it holds*

$$\sigma^{\mathcal{J}}_{int}(\gamma, \delta) \asymp \sigma^{\mathcal{I}}_{int}(\gamma, \delta) \asymp \sigma^{\mathcal{J}}_{app}(\gamma, \varepsilon) \asymp \sigma^{\mathcal{I}}_{app}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa, s\}}}.$$

Here $\kappa = \max\{k \colon \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ *for any* t *in an open subset of* $[0,T]\}$.

This theorem shows that, asymptotically, the complexity of curves is uninfluenced by the drift, and only depends on the underlying sub-Riemannian system, while the one of paths depends also on how "bad" the drift is with respect to this system. We remark also that for the path complexities it is not necessary to have an a priori bound on $\mathcal{T}$.

We conclude this section by considering the problem of *long time local controllability* (henceforth simply *LTLC*), i.e., the problem of staying near some point for a long period of time $T > 0$. This is essentially a stabilization problem around a non-equilibrium point.

Since the system (1.2.1) satisfies the strong Hörmander condition, using unbounded controls it is always possible to satisfy some form of LTLC. Hence, we try to quantify the minimal cost needed, by posing the following. (To lighten the notation, we consider only the cost $\mathcal{J}$.) Let $T > 0$, $q_0 \in M$, and $\gamma_{q_0} : [0,T] \to M$, $\gamma_{q_0}(\cdot) \equiv q_0$.

- *LTLC complexity by time:*

  $$\mathrm{LTLC}_{\mathtt{time}}(q_0, T, \delta) = \sigma^{\mathcal{J}}_{\mathrm{int}}(\gamma_{q_0}, \delta).$$

  Here, we require trajectories defined by admissible controls to pass through q_0 at intervals of time of length at most δ.

- *LTLC complexity by cost:*

  $$\mathrm{LTLC}_{\mathtt{cost}}(q_0, T, \varepsilon) = \sigma^{\mathcal{J}}_{\mathrm{app}}(\gamma_{q_0}, \varepsilon).$$

  Admissible controls for this complexity, will always be contained in the sub-Riemannian ball of radius ε centered at q_0.

Clearly, if $f_0(q_0) = 0$, then $\mathrm{LTLC}_{time}(q_0, T, \delta) = \mathrm{LTLC}_{cost}(q_0, T, \varepsilon) = 0$, for any $\varepsilon, \delta, T > 0$. Although γ_{q_0} is not a path by our definition, since it is not injective and $\dot{\gamma}_{q_0} \equiv 0$, the arguments of Theorem 1.2.6 can be applied also to this case. Hence, we get the following asymptotic estimate for the LTLC complexities.

Corollary 1.2.7. *Assume that the sub-Riemannian structure defined by $\{f_1, \dots, f_m\}$ is equiregular, and that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ for some $s \geqslant 2$. Then, for any $q_0 \in M$ and $T > 0$ it holds*

$$\mathrm{LTLC}_{time}(q_0, T, \varepsilon) \asymp \mathrm{LTLC}_{cost}(q_0, T, \varepsilon) \asymp \frac{1}{\varepsilon^s}.$$

1.3 DIFFUSIONS ON SINGULAR MANIFOLDS

The second part of the thesis is devoted to the study of diffusions on some family of sub-Riemannian structures and their generalizations. This interest is motivated by the fact that these structures allow for an intrinsic Laplace-Beltrami operator to be defined without the ambiguities due to choice of the measure that arise in general sub-Riemannian structures (see Section 1.1.2).

1.3.1 The Laplace-Beltrami operator in almost-Riemannian geometry

A *2-dimensional almost-Riemannian structures* (abbreviated to 2-ARS) is a rank-varying sub-Riemannian structure on a 2-dimensional manifold M that can be defined locally by a pair of smooth vector fields satisfying the Lie bracket-generating condition. The name almost-Riemannian is due to the fact that these manifolds can be regarded as equipped with a generalized Riemannian metric $\mathbf{g}$, whose eigenvalues are allowed to diverge approaching the singular set $\mathcal{Z}$ where the two vector fields become collinear. Such structures were introduced in the context of hypoelliptic operators [Gru70, FL82], then appeared in problems of population transfer in quantum systems [BCG⁺02b, BC04, BCC05], and have applications to orbital transfer in space mechanics [BC08, BCST09].

Almost-Riemannian manifolds present very interesting phenomena. For instance, geodesics can pass through the singular set with no singularities, even if all Riemannian quantities (e.g., the metric, the Riemannian area, the curvature) diverge while approaching $\mathcal{Z}$ (see Figures 7 and 8 in Section 1.3.3 for examples of this type of geodesics in the Grushin cylinder). Moreover, the presence of a singular set allows the conjugate locus to be nonempty even if the Gaussian curvature, where it is defined, is always negative (see [ABS08]). See also [ABS08, ABC⁺10, BCG13, BCGS13] for Gauss–Bonnet-type formulas, a classification of 2-ARS from the point of view of Lipschitz equivalence and normal forms for generic 2-ARS.

Given a 2-ARS on M it is always possible to write, in coordinates (x, y), its local orthonormal frame as

$$Y_1(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Y_2(x,y) = \begin{pmatrix} 0 \\ f(x,y) \end{pmatrix},$$

for some smooth function f. With this choice, the singular set $\mathcal{Z}$ is the zero-level set of f. Outside $\mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{f(x,y)^2} \end{pmatrix},$$

$$d\omega = \frac{1}{|f(x,y)|} dx\, dy,$$

$$K(x,y) = \frac{f(x,y)\partial_x^2 f(x,y) - 2\partial_x f(x,y)^2}{f(x,y)^2}.$$

Since almost-Riemannian structures are not equiregular, both the Popp measure $d\mathcal{P}$ and the 2-dimensional Hausdorff measures diverge on $\mathcal{Z}$. On the other hand, on $M \setminus \mathcal{Z}$ the Popp measure coincides with the Riemannian volume and is thus proportional to the 2-dimensional Hausdorff measures. This allows to define an intrinsic sub-Laplacian $\mathcal{L}$ through formula (1.1.7) applied to smooth functions compactly supported on $M \setminus \mathcal{Z}$. Namely, we get

$$\mathcal{L}\,\varphi = \partial_x^2 \varphi + f^2 \partial_y^2 \varphi - \frac{\partial_x f}{f} \partial_x \varphi + f(\partial_y f)\partial_y \varphi.$$

Due to the explosion of the metric and of the volume when approaching the singularity, this operator is singular on $\mathcal{Z}$. Since $\mathcal{L}$ is actually the Laplace-Beltrami operator of the Riemannian manifold $M \setminus \mathcal{Z}$, it is called the Laplace-Beltrami operator associated with the 2-ARS.

Let us remark that this Laplace-Beltrami operator does not coincide with the hypoelliptic operator classically associated with the 2-ARS. Indeed, on trivializable structures over $\mathbb{R}^2$ – i.e., structures on $\mathbb{R}^2$ admitting a global orthonormal frame – the latter corresponds to the "sum of squares" sub-Laplacian L. This sub-Laplacian can be defined through (1.1.7) using the 2-dimensional Lebesgue measure, and thus it is not intrinsic. Moreover, since the Lebesgue measure is locally finite on $\mathbb{R}^2$, the operator L is not singular on $\mathcal{Z}$.

In [BL] the following has been proved for a class of 2-ARS, with strong evidence suggesting that the same is true in general.

Theorem 1.3.1. *Let M be a 2-dimensional compact orientable manifold endowed with a 2-ARS. Assume moreover that $\mathcal{Z}$ is an embedded one-dimensional sub-manifold of M and that $\Delta + [\Delta, \Delta] = TM$. Then the Laplace-Beltrami operator $\mathcal{L}$ is essentially self-adjoint on $L^2(M, d\mathcal{P})$ and has discrete spectrum.*

The proof proceeds in two steps. First, the statement is proved for the Laplace-Beltrami operator associated with a compactified version of the Grushin plane ([Gru70, FL82]), and then this result is extended to a general compact 2-ARS, through perturbation theory [Kat95].

The Grushin plane is the 2-ARS defined globally by the couple of vector fields

$$X_1(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2(x,y) = \begin{pmatrix} 0 \\ x \end{pmatrix}, \qquad x, y \in \mathbb{R}^2, \tag{1.3.1}$$

and, thanks to the normal forms obtained in [BCG13], it is a good model for general 2-ARS satisfying $\Delta + [\Delta, \Delta] = TM$. For this structure, the Laplace-Beltrami operator $\mathcal{L}$ and the "sum of squares" sub-Laplacian L are, respectively,

$$\mathcal{L} = \partial_x^2 - \frac{1}{x}\partial_y + x^2 \partial_y^2 \quad \text{and} \quad L = X_1^2 + X_2^2 = \partial_x^2 + x^2 \partial_y^2.$$

Theorem 1.3.1 has a number of implications. First, it shows that the singularity splits the manifold in two connected components that a quantum particle or the heat cannot cross: The explosion of the area naturally acts as a barrier which prevents the crossing of $\mathcal{Z}$ by the particles. Moreover, since the Carnot-Caratheodory distance between points on different sides of the singularity is finite, there is no hope to get estimates for the fundamental solution of $\mathcal{L}$ in terms of this distance. However, such estimates have been found by [Léa87] for L, showing that this operator and $\mathcal{L}$ have quite different properties. The first one is not intrinsic, but however keeps track of intrinsic quantities such as the Carnot-Carthéodory distance. In particular, the corresponding heat flow crosses the set $\mathcal{Z}$, contrarily to what happens for $\mathcal{L}$.

The statement on the compactness of the spectrum of $\mathcal{L}$ contained in Theorem 1.3.1 is, up to our knowledge, the only result regarding spectral properties of Laplace-Beltrami operators associated with almost-Riemannian structures. We remark that, differently from the Riemannian setting, for genuinely almost-Riemannian manifolds this result is not trivial since the considered structures have always infinite volume.

1.3.2 The Laplace-Beltrami operator on conic and anti-conic surfaces

Our work [BP], in collaboration with U. Boscain, is devoted to extend Theorem 1.3.1 to more general singular surfaces, as well as to understand whether the effect of the singularity on the heat diffusion is repulsive or absorbing.

Namely, we consider a family of Riemannian manifolds depending on a parameter $\alpha \in \mathbb{R}$, defined on the disconnected cylinder $M = (\mathbb{R}\setminus\{0\}) \times \mathbb{S}^1$, and whose metric has orthonormal basis

$$X_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2(x,\theta) = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \qquad x \in \mathbb{R},\ \theta \in \mathbb{S}^1. \tag{1.3.2}$$

In other words, we will consider the Riemannian metric $\mathbf{g} = dx^2 + |x|^\alpha d\theta^2$. Notice that for $\alpha = 1$ this reduces, up to a sign difference in X_2, to the Grushin structure (1.3.1) defined on the cylinder.

Through a standard procedure, it is possible to extend this metric to $M_{\text{cylinder}} = \mathbb{R} \times \mathbb{S}^1$ when $\alpha \geqslant 0$, and to $M_{\text{cone}} = M_{\text{cylinder}} / \sim$ when $\alpha < 0$. Here, $(x_1, \theta_1) \sim (x_2, \theta_2)$ if and only if $x_1 = x_2 = 0$. We will let M_α be this extended metric space. Notice that in the cases $\alpha = 1, 2, 3, \dots$, M_α is an almost Riemannian structure in the sense of Section 1.3.1, while in the cases $\alpha = -1, -2, -3, \dots$ it corresponds to a singular Riemannian manifold with a semi-definite metric.

One of the main features of these metrics is the fact that, except in the case $\alpha = 0$, the corresponding Riemannian volumes have a singularity at $\mathcal{Z}$,

$$d\mu = \sqrt{\det g}\, dx\, d\theta = |x|^{-\alpha} dx\, d\theta.$$

Due to this fact, the corresponding Laplace-Beltrami operators contain some diverging first order terms,

$$\mathcal{L} = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{2} \partial_j \left(\sqrt{\det g}\, g^{jk} \partial_k \right) = \partial_x^2 + |x|^{2\alpha} \partial_\theta^2 u - \frac{\alpha}{x} \partial_x. \tag{1.3.3}$$

Here, we proceed as in Section 1.3.1 and initially define $\mathcal{L}$ as an operator on $C_c^\infty(M_\alpha \setminus \mathcal{Z}) = C_c^\infty(M)$.

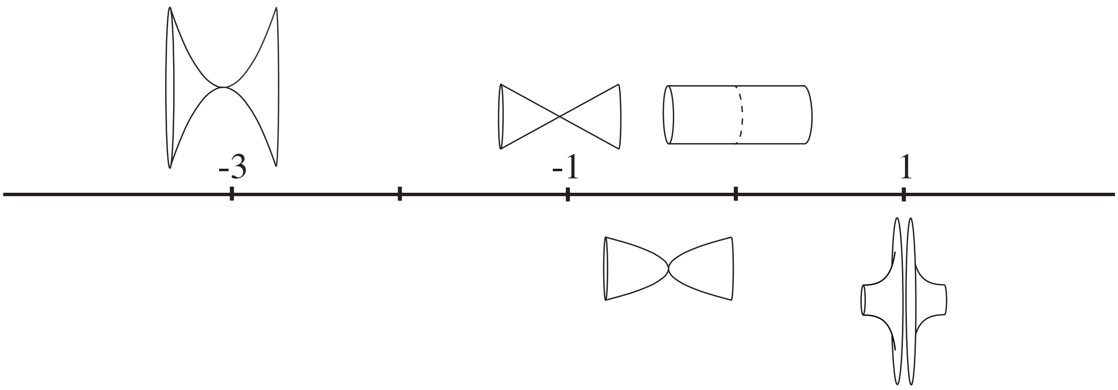

Figure 5: Geometric interpretation of M_α. The figures above the line are actually isometric to M_α, while for the ones below the isometry is singular in $\mathcal{Z}$.

We have the following geometric interpretation of M_α (see Figure 5). For $\alpha = 0$, this metric is that of a cylinder. For $\alpha = -1$, it is the metric of a flat cone in polar coordinates. For $\alpha < -1$, it is isometric to a surface of revolution $\mathcal{S} = \{(t, r(t)\cos\vartheta, r(t)\sin\vartheta) \mid t > 0, \vartheta \in \mathbb{S}^1\} \subset \mathbb{R}^3$ with profile $r(t) = |t|^{-\alpha} + O(t^{-2\alpha})$ as $|t|$ goes to zero. For $\alpha > -1$ ($\alpha \neq 0$) it can be thought as a surface of revolution having a profile of the type $r(t) \sim |t|^{-\alpha}$ as $t \to 0$, but this is only formal, since the embedding in $\mathbb{R}^3$ is deeply singular at $t = 0$. As already mentioned, the case $\alpha = 1$ corresponds to the Grushin metric defined in (1.3.1), considered on the cylinder.

In [BP] we considered the following problems about M_α.

(Q1) Do the heat and free quantum particles flow through the singularity? In other words, we are interested to the following: consider the heat or the Schrödinger equation

$$\partial_t \psi = \mathcal{L}\,\psi, \tag{1.3.4}$$

$$i\partial_t \psi = -\mathcal{L}\,\psi, \tag{1.3.5}$$

where $\mathcal{L}$ is given by (1.3.3). Take an initial condition supported at time $t = 0$ in $M^- = \{x \in M \mid x < 0\}$. Is it possible that at time $t > 0$ the corresponding solution has some support in $M^+ = \{x \in M \mid x > 0\}$? [4]

(Q2) Does equation (1.3.4) conserve the total heat (i.e. the L^1 norm of ψ)? This is known to be equivalent to the stochastic completeness of M_α – i.e., the fact that the stochastic process, defined by the diffusion $\mathcal{L}$, almost surely has infinite lifespan. In particular, we are interested in understanding if the heat is absorbed by the singularity $\mathcal{Z}$.

The same question for the Schrödinger equation has a trivial answer, since the total probability (i.e., the L^2 norm) is always conserved under the Schrödinger evolution, by Stone's theorem.

In order for this two questions to have a meaning it is necessary to interpret $\mathcal{L}$ as a self-adjoint operator acting on $L^2(M, d\mu)$. However, since $\mathcal{L}$ is defined only on $C_c^\infty(M)$, it cannot be self-adjoint and hence one has to apply the theory of self-adjoint extensions. As a comparison, in order to have a well defined evolution for the equation $\partial_t\psi = \partial_x^2\psi$ on the half-line

4 Notice that this is a necessary condition to have some positive controllability results by means of controls defined only on one side of the singularity, in the spirit of [BCG].

$[0,+\infty)$, it is necessary to pose appropriate boundary conditions at 0: Dirichlet, Neumann, or a combination of the two. These conditions, indeed, guarantee that ∂_x^2 is essentially self-adjoint on $L^2([0,+\infty))$.

Passage through the singularity

The rotational symmetry of M_α suggests to proceed by a Fourier decomposition of $\mathcal{L}$ in the θ variable. Thus, we decompose the space $L^2(M, d\mu) = \bigoplus_{k=0}^{\infty} H_k$, where $H_k \cong L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$, and the corresponding operators on each H_k will be

$$\widehat{\mathcal{L}}_k = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}k^2. \tag{1.3.6}$$

It is a standard fact that $\mathcal{L}$ is essentially self-adjoint on $L^2(M, d\mu)$ if all of its Fourier components $\widehat{\mathcal{L}}_k$ are essentially self-adjoint on $L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$, while the contrary is not true.

As remarked at the end of Section 1.3.1, if the Laplace-Beltrami operator is essentially self-adjoint – i.e., if it admits only one self-adjoint extension that is the Friederichs extension $\mathcal{L}_F$ – then (Q1) has a negative answer. Indeed, by definition, $\mathcal{L}_F$ acts separately on the two sides of the singularity hence inducing two independent dynamics. The following theorem – that extends Theorem 1.3.1 – classifies the essential self-adjointness of $\mathcal{L}$ and of its Fourier components.

Theorem 1.3.2. *Consider M_α for $\alpha \in \mathbb{R}$ and the corresponding Laplace-Beltrami operator $\mathcal{L}$ as an unbounded operator on $L^2(M, d\mu)$. Then the following holds.*

- *If $\alpha \leqslant -3$ then $\mathcal{L}$ is essentially self-adjoint;*
- *if $\alpha \in (-3,-1]$, only the first Fourier component $\widehat{\mathcal{L}}_0$ is not essentially self-adjoint;*
- *if $\alpha \in (-1,1)$, all the Fourier components of $\mathcal{L}$ are not essentially self-adjoint;*
- *if $\alpha \geqslant 1$ then $\mathcal{L}$ is essentially self-adjoint.*

As a corollary of this theorem, we get the following answer to (Q1).

$\alpha \leqslant -3$	Nothing can flow through $\mathcal{Z}$
$-3 < \alpha \leqslant -1$	Only the average over $\mathbb{S}^1$ of the function can flow through $\mathcal{Z}$
$-1 < \alpha < 1$	It is possible to have full communication between the two sides
$1 \leqslant \alpha$	Nothing can flow through $\mathcal{Z}$

More precisely, when $-3 < \alpha \leqslant -1$ there exists a self-adjoint extension of $\mathcal{L}$, called the *bridging extension* and denoted by $\mathcal{L}_B$, such that the heat and Schrödinger flows allow the passage of only the first Fourier component through the singularity. On the other hand, when $-1 < \alpha < 1$, there exists a self-adjoint extension of $\mathcal{L}$, still called the bridging extension, such that (Q1) has a positive answer, i.e., all the Fourier components can flow through the singularity.

Remark 1.3.3. Notice that in the case $\alpha \in (-3, 0)$, since the singularity reduces to a single point, one would expect to be able to "transmit" through $\mathcal{Z}$ only a function independent of θ

(i.e. only the average over $\mathbb{S}$). Theorem 1.3.2 shows that this is the case for $\alpha \in (-3,-1]$, but not for $\alpha \in (-1,0)$. Looking at M_α, $\alpha \in (-1,0)$, as a surface embedded in $\mathbb{R}^3$ the possibility of transmitting Fourier components other than $k = 0$, is due to the deep singularity of the embedding. In this case we say that the contact between M^+ and M^- is *non-apophantic*.

Stochastic completeness

It is a well known result that each non-positive self-adjoint operator A on a Hilbert space $\mathcal{H}$ defines a strongly continuous contraction semigroup, denoted by $\{e^{tA}\}_{t\geqslant 0}$. If $\mathcal{H} = L^2(M, d\mu)$ and it holds $0 \leqslant e^{tA}\psi \leqslant 1$ $d\mu$-a.e. whenever $\psi \in L^2(M, d\mu)$, $0 \leqslant \psi \leqslant 1$ $d\mu$-a.e., the semigroup $\{e^{tA}\}_{t\geqslant 0}$ and the operator A are called *Markovian*.

When $\{e^{tA}\}_{t\geqslant 0}$ is the evolution semigroup of the heat equation, the Markov property can be seen as a physical admissibility condition. Namely, it assures that when starting from an initial datum ψ representing a temperature distribution (i.e., a positive and bounded function) the solution $e^{tA}\psi$ remains a temperature distribution at each time, and, moreover, that the heat does not concentrate. Hence in the following we will focus only on the Markovian self-adjoint extensions of $\mathcal{L}$.

The interest for Markovian operators lies also in the fact that, under an additional assumption which is always satisfied in the cases we consider, Markovian operators are generators of Markov processes $\{X_t\}_{t\geqslant 0}$ (roughly speaking, stochastic processes which are independent of the past).

Since essentially bounded functions are approximable with functions in $L^2(M, d\mu)$, the Markov property allows to extend the definition of e^{tA} from $L^2(M, d\mu)$ to $L^\infty(M, d\mu)$. Let 1 be the constant function $1(x,\theta) \equiv 1$. Then (Q2) is equivalent to the following property.

Definition 1.3.4. A Markovian operator A is called *stochastically complete* (or conservative) if $e^{tA}1 = 1$, for any $t > 0$. It is called *explosive* if it is not stochastically complete.

It is well known that this property is equivalent to the fact that the Markov process $\{X_t\}_{t\geqslant 0}$, with generator A, has almost surely infinite lifespan.

We will consider also the following stronger property of $\{X_t\}_{t\geqslant 0}$.

Definition 1.3.5. A Markovian operator is called *recurrent* if the associated Markov process $\{X_t\}_{t\geqslant 0}$ satisfies, for any set Ω of positive measure and any point x,

$$\mathbb{P}_x\{\text{there exists a sequence } t_n \to +\infty \text{ such that } X_{t_n} \in \Omega\} = 1.$$

Here $\mathbb{P}_x$ denotes the probability measure in the space of paths emanating from a point x associated with $\{X_t\}_{t\geqslant 0}$.

We are particularly interested in distinguishing how the stochastic completeness and the recurrence are influenced by the singularity $\mathcal{Z}$ or by the behavior at ∞. Thus we will consider the manifolds with borders $M_0 = M \cap ([-1,1] \times \mathbb{S}^1)$ and $M_\infty = M \setminus [-1,1] \times \mathbb{S}^1$, with Neumann boundary conditions. Indeed, with these boundary conditions, when the Markov process $\{X_t\}_{t\geqslant 0}$ hits the boundary it is reflected, and hence the eventual lack of recurrence or stochastic completeness on M_0 (resp. on M_∞) is due to the singularity $\mathcal{Z}$ (resp. to the behavior at ∞). If a Markovian operator A on M is recurrent (resp. stochastically complete) when restricted on M_0 we will call it recurrent (resp. stochastically complete) at 0. Similarly, when the same happens on M_∞, we will call it recurrent (resp. stochastically complete) at ∞.

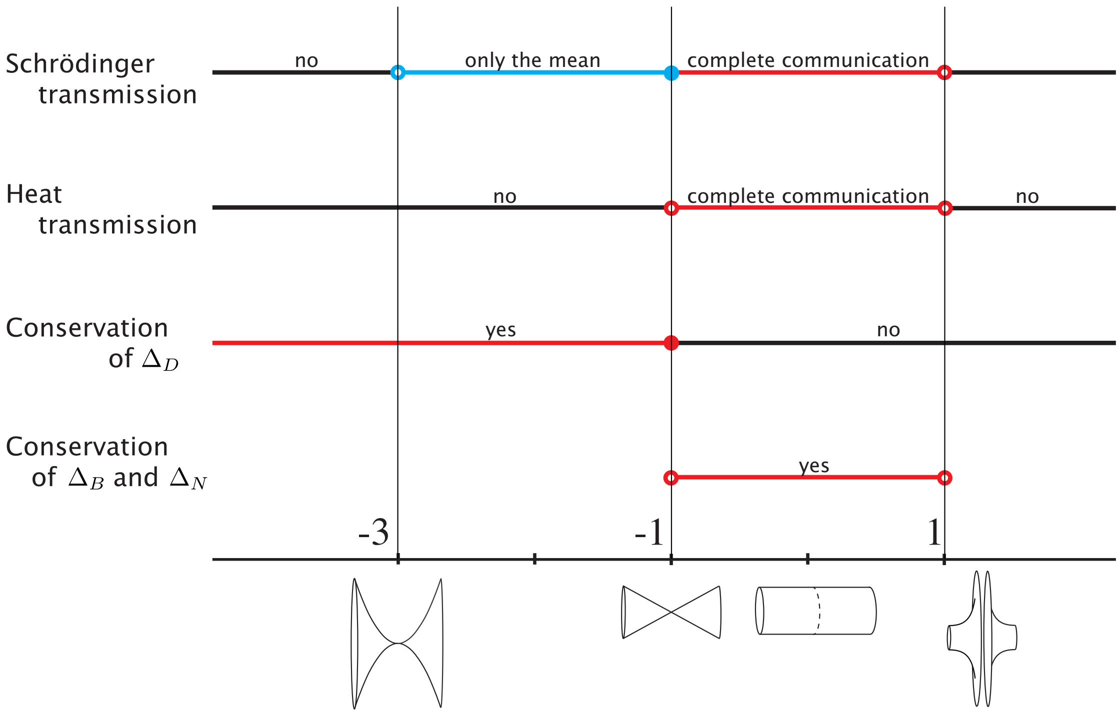

Figure 6: A summary of the results obtained in [BP].

In this context, it makes sense to give special consideration to three specific self-adjoint extensions of $\mathcal{L}$, corresponding to different conditions at $\mathcal{Z}$. Namely, we will consider the already mentioned Friedrichs extension $\mathcal{L}_F$, that corresponds to an absorbing condition, the Neumann extension $\mathcal{L}_N$, that corresponds to a reflecting condition, and the bridging extension $\mathcal{L}_B$, that corresponds to a free flow through $\mathcal{Z}$ and is Markovian only for $\alpha \in (-1, 1)$. Observe that $\mathcal{L}_F$ and $\mathcal{L}_N$ are always self-adjoint Markovian extensions, although it may happen that $\mathcal{L}_F = \mathcal{L}_N$. In this case $\mathcal{L}_F$ is the only Markovian extension, and the operator $\mathcal{L}$ is called Markov unique. This happens, for example, when $\mathcal{L}$ is essentially self-adjoint.

The following result will answer to (Q2).

Theorem 1.3.6. *Consider M_α, for $\alpha \in \mathbb{R}$, and the corresponding Laplace-Beltrami operator $\mathcal{L}$ as an unbounded operator on $L^2(M, d\mu)$. Then it holds the following.*

- *If $\alpha < -1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is stochastically complete at 0 and recurrent at ∞;*
- *if $\alpha = -1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is recurrent both at 0 and at ∞;*
- *if $\alpha \in (-1, 1)$, then $\mathcal{L}$ is not Markov unique and, moreover,*
  - *any Markovian extension of $\mathcal{L}$ is recurrent at ∞,*
  - *$\mathcal{L}_F$ is explosive at 0, while both $\mathcal{L}_B$ and $\mathcal{L}_N$ are recurrent at 0,*
- *if $\alpha \geqslant 1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is explosive at 0 and recurrent at ∞;*

In particular, Theorem 1.3.6 implies that for $\alpha \in (-3, -1]$ no mixing behavior defines a Markov process. On the other hand, for $\alpha \in (-1, 1)$ we can have a plethora of such processes.

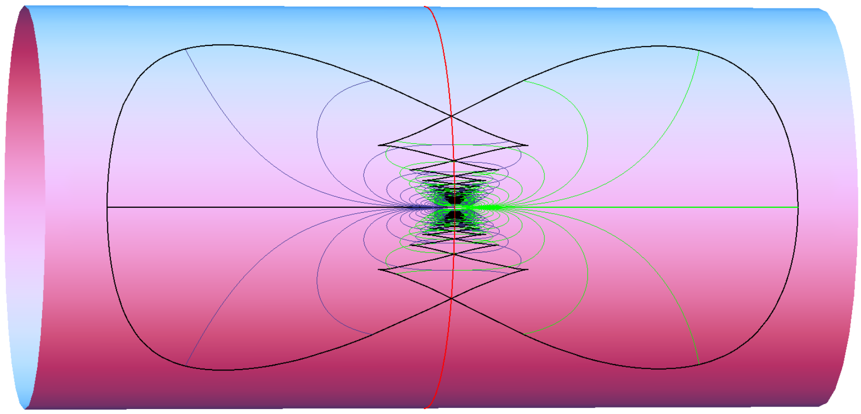

Figure 7: The geodesics for the Grushin cylinder starting from the singular set (red circle), for $t \in [0, 1.7]$. The black (self-intersecting) curve line is the wave front (i.e., the end point of all geodesics at time 1.7). For the explicit expression of these geodesics see for instance [BL].

Classifying all possible Markov processes in this interval of parameters is the aim of [PP], in collaboration with A. Posilicano.

Since the singularity $\mathcal{Z}$ is at finite distance from any point of M_α, one can interpret a Markov process that is explosive at 0 as if $\mathcal{Z}$ were absorbing the heat. Thus, as a corollary of Theorem 1.3.6, we get the following answer to (Q2).

$\alpha \leqslant -1$	The heat is absorbed by $\mathcal{Z}$
$-1 < \alpha < 1$	The Friedrichs extension is absorbed by $\mathcal{Z}$, while the Neumann and the bridging extensions are not.
$1 \leqslant \alpha$	The heat is absorbed by $\mathcal{Z}$

In Figure 6, we plotted a summary of the results we obtained.

1.3.3 Spectral analysis of the Grushin cylinder and sphere

In the last part of the thesis, following the work [BPS], in collaboration with U. Boscain and M. Seri, we study the spectral properties of the Laplace-Beltrami operator $\mathcal{L}$ in two relevant almost-Riemannian structures with infinite volume, one of which is non-compact: the Grushin structures on the cylinder and on the sphere.

Note that in the following we use the convention $0 \notin \mathbb{N}$. When needed, we denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We will also denote the ceiling and floor functions with $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$, respectively.

Almost-Riemannian structures under consideration

In the following we introduce the two main structures studied in this section and we describe some of their properties.

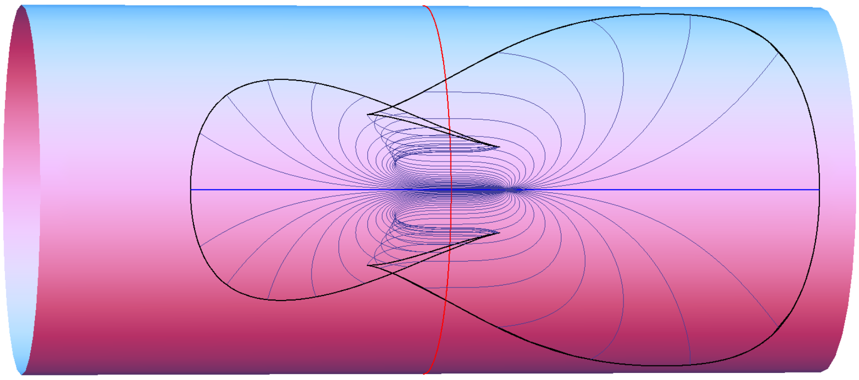

Figure 8: The geodesics for the Grushin cylinder starting from the the point $(0.3, 0)$, for $t \in [0, 1.7]$. Notice that they cross the singular set (red circle) with no singularities. For the explicit expression of these geodesics see for instance [BL].

Definition 1.3.7. The Grushin almost-Riemannian structure on the cylinder is the structure on $M = \mathbb{R} \times \mathbb{S}^1$ whose generating frame is

$$X_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2(x,\theta) = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

For this structure the singular set is $\mathcal{Z} = \{0\} \times \mathbb{S}^1$. On $M \setminus \mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}, \qquad d\omega = \frac{1}{|x|} dx\, dy, \qquad K(x,y) = -\frac{2}{x^2}.$$

Notice that $d\omega$ is not integrable on any open set intersecting the θ axis.

The associated Laplace-Beltrami operator is

$$\mathcal{L} u = \partial_x^2 u - \frac{1}{x} \partial_x u + x^2 \partial_\theta^2 u. \tag{1.3.7}$$

By Theorem 1.3.1 this operator, with domain $C_c^\infty(M \setminus \mathcal{Z})$, is essentially self-adjoint on $L^2(M, d\omega)$. Hence, it separates in the direct sum of its restrictions to $M_\pm = \mathbb{R}_\pm \times \mathbb{S}^1$. Therefore, w.l.o.g. we will focus on $\mathcal{L}$ on M_+. Notice that the Grushin metric restricted to M_+ is not geodesically complete, since geodesics can exit M_+ in finite time. The same happens for M_-.

Definition 1.3.8. The Grushin almost-Riemannian structure on the sphere is the structure on $M = \mathbb{S}^2$ whose generating frame is, in polar coordinates (x, ϕ),

$$Y_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Y_2(x,\theta) = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}.$$

Indeed, the Grushin almost-Riemannian structure on the sphere $\mathbb{S}^2$ is the trivializable almost-Riemannian structure obtained by taking as generating frame two unitary rotations along two orthogonal axis. More precisely, let $\mathbb{S}^2 = \{y_1^2 + y_2^2 + y_3^2 = 1\}$ and

$$X_1 = \begin{pmatrix} 0 \\ -y_3 \\ y_2 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} -y_3 \\ 0 \\ y_1 \end{pmatrix}.$$

Then, passing in spherical coordinates $(y_1, y_2, y_3) = (\cos x \cos\phi, \cos x \sin\phi, \sin x)$ and opportunely rotating X_1, X_2, we recover

$$\begin{cases} Y_1(q) = \cos(\phi - \pi/2) X_1(q) - \sin(\phi - \pi/2) X_2(q), \\ Y_2(q) = \sin(\phi - \pi/2) X_1(q) + \cos(\phi - \pi/2) X_2(q). \end{cases}$$

For this structure the singular set is $\mathcal{Z} = \{0\} \times \mathbb{S}^1$, while the singularities for $x = \pm\pi/2$ are apparent and due to the choice of the coordinates. On $\mathbb{S}^2 \setminus \mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\tan(x)^2} \end{pmatrix}, \qquad d\omega = \frac{1}{|\tan(x)|}\, dx\, d\phi, \qquad K(x,y) = -\frac{2}{\sin(x)^2}.$$

Notice that $d\omega$ is not integrable on $\mathbb{S}^2$.

The associated Laplace-Beltrami operator on $L^2(\mathbb{S}^2 \setminus \mathcal{Z})$ is

$$\mathcal{L}\, u = \partial_x^2 u - \frac{1}{\sin(x)\cos(x)}\, \partial_x u + (\tan x)^2 \partial_\phi^2 u. \tag{1.3.8}$$

This operator, with domain $C_c^\infty(\mathbb{S}^2 \setminus \mathcal{Z})$, is essentially self-adjoint in $L^2(\mathbb{S}^2, d\omega)$ and its spectrum is purely discrete, by Theorem 1.3.1. Similarly to the cylinder case, this operator separates in the direct sum of its restrictions to the north and south hemispheres $S_\pm$, cutted at the equatorial singularity. Thus, we will restrict to consider $\mathcal{L}$ on the north hemisphere S_+.

This almost-Riemannian metric has been first defined in [BCG+02c].

Spectral analysis of the Laplace-Beltrami operators

Very little is known regarding how the spectral properties of the sub-Laplacian interwines with the underlying sub-Riemannian geometry. For compact Riemannian manifolds, a first result in this direction is the Weyl law (see, e.g., [Cha84]), that describes the growth ratio of the counting function of the eigenvalues of the Laplace-Beltrami operator

$$N(E) := \#\{\lambda \in \sigma_d(-\mathcal{L}) \mid \lambda \leqslant E\}. \tag{1.3.9}$$

Namely, $N(E)$ can be expressed asymptotically as a function of the dimension d, the volume of the Euclidean ball ω_d, and the volume of the manifold $\mathrm{vol}(M)$,

$$N(E) \sim \frac{\omega_d}{(2\pi)^d} \mathrm{vol}(M)\, E^{d/2}. \tag{1.3.10}$$

Similar results have been obtained also for some families of non-compact Riemannian manifolds eventually with boundary. See, e.g., [Ivr80, Nor72, Vas92, Par95, Sha02, Bor07, GM08].

In the context of Riemannian manifolds, Weyl asymptotics of the form (1.3.10) can be obtained through pseudo-differential calculus techniques. Some promising results in this direction have been obtained in step 2 equiregular sub-Riemannian structures (see [Pon08]), where notably the power exponent is $\dim^{\mathcal{H}} M/2$. However, these techniques rely heavily on the Heisenberg group structure of the tangent cone, and cannot be applied on the non-equiregular almost-Riemannian structures.

We will now focus on the two structures introduced in the previous section. For these, we will present an explicit description of the spectrum, the eigenfunctions and their properties.

This allows us to compute the Weyl law for $-\mathcal{L}$, obtaining as leading order $N(E) \sim E\log(E)$, which is fairly unusual for Laplace-Beltrami operators on 2-dimensional Riemannian manifolds.

In the following theorem we describe explicitly the spectrum of the Laplace-Beltrami operator on the Grushin cyilinder.

Theorem 1.3.9 (Grushin cylinder case). *The operator $-\mathcal{L}$ on $L^2(M_+)$, defined in (1.3.7), has absolutely continuous spectrum $\sigma(-\mathcal{L}) = [0, \infty)$ with embedded discrete spectrum*

$$\sigma_d(-\mathcal{L}) = \{\lambda_{n,k} = 4|k|n \mid n \in \mathbb{N},\ k \in \mathbb{Z} \setminus \{0\}\}.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x,\theta) = e^{ik\theta}\frac{1}{x}W_{n,\frac{1}{2}}(|k|x^2),$$

where $W_{\nu,\mu}$ is the Whittaker W-function of parameters ν and μ.

Through the above explicit description, it is then possible to calculate the Weyl law for the Grushin cylinder.

Corollary 1.3.10 (Grushin cylinder case). *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{2}\log(E) + (\gamma - 2\log(2))\frac{E}{2} + O(1),$$

where γ is the Euler-Mascheroni constant.

Similar results can be obtained also for the Laplace-Beltrami operator of the Grushin sphere.

Theorem 1.3.11 (Grushin sphere case). *The operator $-\mathcal{L}$ on $L^2(S_+)$, defined in (1.3.8), has purely discrete spectrum*

$$\sigma(-\mathcal{L}) := \{\lambda_{n,k} := 4n(n + |k|) \mid n \in \mathbb{N},\ k \in \mathbb{Z}\}.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x,\phi) = e^{ik(\phi+\frac{\pi}{2})}\cos(x)^k F\left(-(n+1),\ n+k+1;\ 1+k;\ \cos(x)^2\right)$$

where $F(a,b;c;x)$ is the Gauss Hypergeometric function with parameters a, b, c.

Corollary 1.3.12 (Grushin sphere case). *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{4}\log(E) + \left(\gamma - \log(2) - \frac{1}{2}\right)\frac{E}{2} + O(\sqrt{E}),$$

where γ is the Euler-Mascheroni constant.

Spectra of the Aharonov-Bohm perturbed Laplace-Beltrami operator

A magnetic field B on a Riemannian manifold is an exact 2-form taking purely imaginary values. Classically the action of B is given by the Lorentz force Φ, which is a $(1,1)$ tensor field given by

$$g(\Phi(X), Y) = B(X, Y) \qquad X, Y \in \mathrm{Vec}(M).$$

The trajectories of charged particles under the magnetic field B are then the solution of the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = \Phi(\dot\gamma)$. In particular, if $B = 0$ the Lorentz force is null $\Phi \equiv 0$.

In quantum mechanic the situation is different. Given a magnetic field B there exists a 1-form A, taking purely imaginary values and called magnetic vector potential, such that $B = dA$. The magnetic Laplace-Beltrami operator is then defined via the following "magnetic" Green identity (compare it with (1.1.7)),

$$-\int_M f(\mathcal{L}_A\, g)\, d\mu = \int_M \mathbf{g}((\nabla_H + A)f, (\nabla_H + A)g)\, d\mu, \qquad \text{for any } f, g \in C_c^\infty(M).$$

The evolution of a charged particle with wave function ψ in the magnetic field is then given by the Schrödinger equation

$$i\frac{d}{dt}\psi = -\mathcal{L}_A\,\psi.$$

When M is simply connected (i.e., if $H^1_{dR}(M) = 0$) for any two magnetic vector potentials A and A', it holds that $A - A'$ is exact. Thus the two magnetic Laplace-Beltrami operators $\mathcal{L}_A$ and $\mathcal{L}_{A'}$ are unitarily equivalent, by gauge invariance, and the evolution of charged particles depends only on the magnetic field, as in the classical case.

When M is non-simply connected (i.e., if $H^1_{dR}(M) \neq 0$) this is no more true, as $A - A'$ needs only to be closed. This is known as the *Aharonov-Bohm effect* [AT98, dOP08]. Two choices of magnetic potential may lead to in-equivalent magnetic Laplace-Beltrami operator. In $\mathbb{R}^2$ with a bounded obstacle, this phenomenon can be seen through a difference of wave phase arising from two non-homotopic paths that circumvent the obstacle, and has been experimentally observed [TOM+86].

If in the Euclidean case the effect of the hidden magnetic fields is surprising but somewhat simple, the same cannot be said for what concerns asymptotically hyperbolic manifolds with cusps. In such cases, as proved in [GM08], a change in vector potentials can drastically modify the spectral properties of the operator, e.g. by destroying the absolutely continuous component of the spectrum. This phenomenon can be useful for counting eigenvalues embedded in the absolutely continuous spectrum in non-separable problems [GM08, MT08].

In this section, we investigate the Aharonov-Bohm effect on the spectrum of the Laplace-Beltrami operators of the Grushin cylinder and sphere. To this purpose, we introduce a magnetic vector potential for the zero magnetic field whose flux is non zero and we show that the aforementioned drastic effect on the spectrum is present. Additionally, we show that the degeneracy of the eigenvalues is extremely sensitive to the vector potential.

To mimic the Ahronov-Bohm effect for the Laplace-Beltrami operator in the Grushin cylinder we consider the connection (e.g. a one-form) $\omega^b = -ib\, d\theta$, $b \in \mathbb{R}$. The associated magnetic Laplace-Beltrami operator on the Grushin cylinder is then

$$\mathcal{L}^b = \partial_x^2 - \frac{1}{x}\partial_x + |x|^2(\partial_\theta^2 - 2ib\,\partial_\theta - b^2).$$

As expected, for $b = 0$ this coincides with $\mathcal{L}$.

As first results, we obtain the following explicit description of the spectrum of the operator $\mathcal{L}^b$, which will allow to compute the corresponding Weyl law.

Theorem 1.3.13 (Grushin cylinder case)**.** *The operator* $-\mathcal{L}^b$ *on* $L^2(M_+)$ *has a non-empty discrete spectral component*

$$\sigma_d(-\mathcal{L}^b) = \left\{\lambda^b_{n,k} := 4n|k-b| \mid n \in \mathbb{N},\ k \in \mathbb{Z}\setminus\{b\}\right\}. \tag{1.3.11}$$

When $b \in \mathbb{Z}$ the operator has in addition absolutely continuous spectrum $[0, +\infty)$. When $b \notin \mathbb{Z}$ the spectrum has no absolutely continuous part. In any case, the eigenfunctions are

$$\psi^b_{n,k}(x, \theta) = e^{ik\theta} \frac{1}{x} W_{n,\frac{1}{2}}(|k - b|x^2).$$

Corollary 1.3.14. *If $b \in \mathbb{Z}$, the Weyl law is the one of Corollary 1.3.10. If $b \notin \mathbb{Z}$, let $\kappa \in \mathbb{Z}$ be the closest integer to b. Then, the Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{2}\log(E) + \frac{E}{2}\left(\frac{1}{2|\kappa - b|} + \gamma - 2\log(2) - \frac{\psi(1 - |\kappa - b|) + \psi(1 + |\kappa - b|)}{2}\right) + O(1),$$

where γ is the Euler-Mascheroni constant and $\psi(x)$ is the digamma function. Here, the $O(1)$ is uniformly bounded with respect to b.

Remark 1.3.15. Notice that $N(E)$ diverges for $b \to \kappa$ since, in this limit, part of the discrete spectrum degenerates and gives rise to an absolutely continuous one.

For this operator, we can also explicitly describe the degeneracy of the spectrum, depending on the value of b.

Theorem 1.3.16 (Degeneracy of the spectrum in the Grushin cylinder case)**.** *Let $d(n)$ denote the number of divisors of n. Then,*

- *If $b \in \mathbb{R} \setminus \mathbb{Q}$, the spectrum is simple.*
- *If $b \in \mathbb{Q}$, the discrete spectrum is degenerate in the following sense: each eigenvalue λ has multiplicity bounded from above by $2d(\lambda/4)$.*
- *If $b \in \mathbb{Z}$, the eigenvalues achieve the maximal degeneracy and the multiplicity is exactly*

$$\begin{cases} 2d(\lambda/4), & \text{if } \lambda/4 \text{ is odd,} \\ 2d(\lambda/4) - 2, & \text{if } \lambda/4 \text{ is even,} \end{cases}$$

(in particular it is bounded below by 2).

Remark 1.3.17. A direct consequence of the previous theorem is that the maximal multiplicity of the eigenvalues has very slow growth. In fact, it is well known [Apo76] that as $n \to \infty$

$$d(n) = o(n^\epsilon), \qquad \text{for any } \epsilon > 0.$$

Finally, we give deeper information on the decompactification of the spectrum in the limit $b \to k$.

Corollary 1.3.18 (Decompactification of the spectrum on the Grushin cylinder)**.** *Fix $k \in \mathbb{Z}$. Then, for every $n \in \mathbb{N}$, the spacing between the eigenvalues*

$$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| \to 0 \text{ as } b \to k.$$

Moreover, for any fixed interval $I = [x_1, x_2] \subset [0, \infty)$ and any $N \in \mathbb{N}$

$$\#\{n \in \mathbb{N} \mid \lambda^b_{n,k} \in I\} \geqslant N \text{ as } b \to k.$$

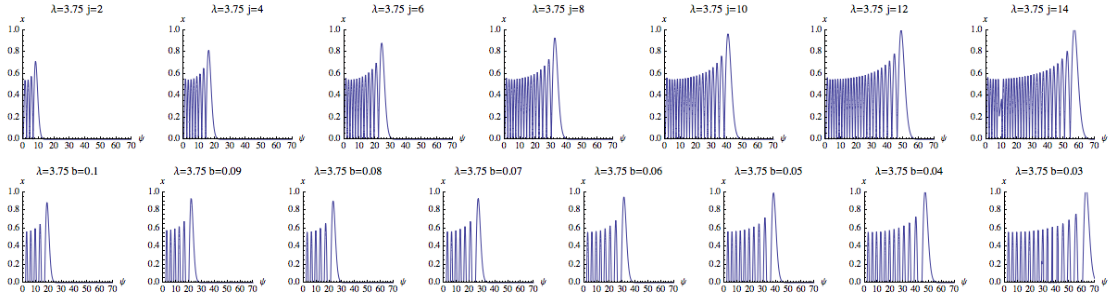

Figure 9: The first row shows the spreading of the projection onto $\theta = 0$ of $\psi^{b_j}_{n_j,0}(x)$ as j increases for $\lambda = 3.75$. The second row shows the spreading of the projection onto $\theta = 0$ of $\psi^{b}_{n(b),0}(x)$ as $b \to 0$ for $\lambda = 3.75$. See Theorem 1.3.19 and Remark 1.3.20

In the following theorem, we show that the Ahronov-Bohm perturbation strongly affects the structure of a subset of the eigenfunctions and makes it degenerate into the set of generalized eigenfunctions. Up to our knowledge this is the first result of this kind.

Theorem 1.3.19 (Degeneration of the eigenfunctions on the Grushin cylinder)**.** *Fix* $k \in \mathbb{Z}$. *Then for any* $\lambda \in \mathbb{Q}$, $\lambda > 0$, *there exist a sequence of pairs* $(b_j, n_j) \in \left(k - \frac{1}{2}, k + \frac{1}{2}\right) \times \mathbb{N}$, *with* $b_j \to k$ *and* $n_j \to \infty$, *such that*

$$\psi^{b_j}_{n_j,k}(x,\theta) \to e^{ik\theta}\frac{\sqrt{\lambda}}{2}J_1(\sqrt{\lambda}x) \tag{1.3.12}$$

uniformly on compact sets, where $J_\nu(z)$ *is the Bessel function of the first kind of order* ν. *The limit function on the r.h.s. is the generalized eigenfunction of* $\mathcal{L}^b$ *with generalized eigenvalue* λ *(see Remark 6.1.1).*

Remark 1.3.20. Theorem 1.3.19 can be rewritten as follows. For every $\lambda > 0$, let

$$n(b) := 2\left\lceil \frac{\lambda}{8|b-k|} \right\rceil. \tag{1.3.13}$$

Then

$$\lim_{b\to k}\psi^{b}_{n(b),k}(x,\theta) = e^{ik\theta}\frac{\sqrt{\lambda}}{2}J_1(\sqrt{\lambda}x)$$

uniformly on compact sets. The proof is similar to the one of Theorem 1.3.19 with n_j replaced by $n(b)$.

Remark 1.3.21. Figure 1.3 shows the collapse of the eigenfunctions to the generalized eigenfunctions, while Figure 1.4 shows the collapse of the eigenvalues to the continuous spectrum.

We now shift our attention to the Grushin sphere. Here, we consider the magnetic Laplace-Beltrami operator induced by the magnetic vector potential $\omega_b = -ib\, d\phi$ on the north emisphere of $\mathbb{S}^2$ with removed north pole $\mathbb{S}^\circ_+$ and Dirichlet boundary conditions. See Section 6.2.3 for more details. The corresponding operator is

$$\mathcal{L}^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x + \tan(x)^2\left(\partial_\phi^2 - 2ib\partial_\phi - b^2\right). \tag{1.3.14}$$

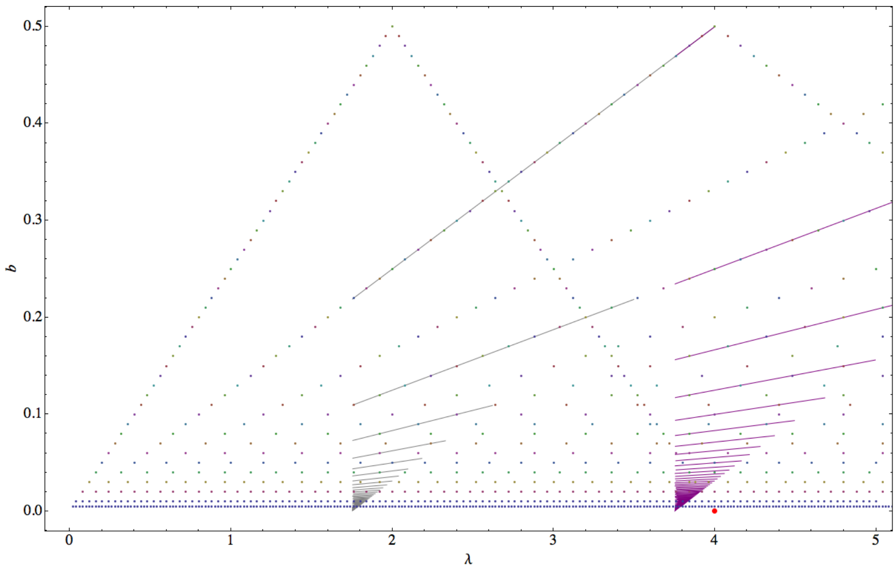

Figure 10: The dots correspond to the eigenvalues up to energy 5 for some values of b as it gets closer to $\kappa = 0$. The thick red dot represents the only embedded eigenvalue $\lambda = 4$ of the operator with $b = 0$ up to energy 5. The grey line is the the curve $\lambda^b_{n(b),\kappa}$ (see Remark 1.3.20) converging to 1.75 as $b \to \kappa$. The purple one is the curve $\lambda^b_{n(b),\kappa}$ converging to 3.75.

We then have the description of the spectrum and the corresponding Weyl law, depending on b.

Theorem 1.3.22 (Grushin sphere case)**.** *The operator $-\mathcal{L}^b$ defined in (1.3.14) and acting on $L^2(S^\circ_+)$, has purely discrete spectrum*

$$\sigma(-\mathcal{L}^b) = \{\lambda_{n,k} = 4n(n + |k - b|) \mid n \in \mathbb{N}, k \in \mathbb{Z}\}.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x, \phi) = e^{ik\phi} e^{i(k-b)\frac{\pi}{2}} \cos(x)^{k-b}\, F\left(-(n+1), n+k-b+1; 1+k-b; \cos(x)^2\right).$$

As a consequence of the explicit description of the spectrum, we obtain the following.

Corollary 1.3.23. *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{4}\log(E) + \left(\gamma - \log(2) - \frac{1}{2}\right) E + O(\sqrt{E}),$$

where γ is the Euler-Mascheroni constant, and the big O is uniformly bounded w.r.t. b.

Notice that the first two orders of the asymptotic expansion of $N(E)$ are the same with or without the Aharnov-Bohm perturbation. Indeed, the dependence on b is hidden in the remainder term.

The degeneracy of the spectrum for the Laplace-Beltrami operator on Grushin sphere is less explicit than the one in Theorem 1.3.18. Indeed, we can only obtain the following result.

Corollary 1.3.24 (Degeneracy of the spectrum in the Grushin sphere case)**.** *If* $b \in \mathbb{R} \setminus \mathbb{Q}$ *the spectrum is simple, if* $b \in \mathbb{Q}$ *the spectrum is finitely degenerate.*

A brief but more detailed discussion of the topic can be found in Section 6.2.

More general results on conic and anti-conic type surfaces

To conclude this last part of the thesis, we consider the Aharonov-Bohm perturbation on the conic and anti-conic structures introduced in Section 1.3.2. Here, the Aharonov-Bohm effect affects not only the spectrum but also the self-adjointness properties of the operator.

As before, we turn it on by considering the connection $\omega_b = -ib\,d\theta$, where $b \in \mathbb{R}$. The corresponding Laplace-Beltrami operator is

$$\mathcal{L}_b = \partial_x^2 + |x|^{2\alpha}\partial_\theta^2 + |x|^{2\alpha}\left(\partial_\theta^2 - 2ib\partial_\theta - b^2\right).$$

Through the same Fourier decomposition as in (1.3.2), we get that $L^2(M, d\mu) = \bigoplus_{k=0}^{\infty} H_k$, where $H_k \cong L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, and the corresponding operators on each H_k is

$$\hat{\mathcal{L}}^b_{\alpha,k} = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}(b-k)^2. \tag{1.3.15}$$

The same argument of Theorem 1.3.2 applied to (1.3.15) yields the following.

Theorem 1.3.25. *If* $b \notin \mathbb{Z}$*, the operator* $\mathcal{L}^b_\alpha$ *with domain* $C_c^\infty(M)$ *is essentially self-adjoint in* $L^2(M, d\omega)$ *if* $|\alpha| \geqslant 1$*, and Theorem 1.3.2 still applies for* $|\alpha| < 1$*.*

On the other hand, if $b \in \mathbb{Z}$*, Theorem 1.3.2 holds with the following change: if* $-3 < \alpha \leqslant -1$ *for every* $k \neq b$ *the operator* $\hat{\mathcal{L}}^b_{,k}$ *is essentially self-adjoint, while* $\hat{\mathcal{L}}^b_{,b}$ *is not.*

The Aharonov-Bohm effect on the spectrum extends to this more general setting as follows.

Theorem 1.3.26. *For* $\alpha > 0$*, the operator* $-\mathcal{L}^b_\alpha$ *on* $L^2(M, d\omega_\alpha)$ *has a non-empty discrete spectral component* $\sigma_d(-\mathcal{L}^b_\alpha) \subset [0, +\infty)$*.*

When $b \in \mathbb{Z}$ *the operator has absolutely continuous spectrum* $[0, +\infty)$ *with embedded discrete spectrum. When* $b \notin \mathbb{Z}$ *the spectrum has no absolutely continuous part.*

Proof. For $b \neq k$, the spectrum of the operators $\hat{\mathcal{L}}^b_{\alpha,k}$ (or of any of their self-adjoint extensions) is purely discrete (see e.g. [Tit62, Chapter 5]). For $b = k$, on the other hand, the essential spectrum of $\hat{\mathcal{L}}^b_{\alpha,k}$ is non-empty and in particular it contains the half line $[0, +\infty)$ (see e.g. [Wei87, Theorem 15.3]). □

The previous theorems suggest that, for $b = 0$ and $\alpha > 0$, the 0-th Fourier component, is the only responsible for the continuous spectrum. The Aharonov-Bohm perturbation, when $b \in \mathbb{Z}$, shift this role to the b-th Fourier component. When $b \notin \mathbb{Z}$ no Fourier component produces a continuous spectrum. This is a well-known phenomena in the case of asymptotically hyperbolic manifolds with finite volume [GM08], but completely new in this setting.

Further study of the cases $\alpha < 0$ is outside the scope of this thesis. As a side remark, note that the case $\alpha = -1$ considered on $\mathbb{R}_+ \times \mathbb{S}^1$ coincides with the standard Aharonov-Bohm Laplacian in polar coordinates. Moreover, in the case $\alpha = -1/2$, $-\mathcal{L}^b_\alpha$ has discrete spectrum accumulating at 0 and absolutely continuous spectrum in $[0, +\infty)$. When $b \notin \mathbb{Z}$ an additional family of eigenvalues accumulating at 0 appears.

1.4 PERSPECTIVES AND OPEN PROBLEMS

The results exposed in this thesis are part of ongoing work. Here, we list some of the natural extensions of this work.

1.4.1 Complexity of non-admissible trajectories

1. Currently, we are working on improving two aspects of Theorem 1.2.6. First, we are considering the case where $f_0(\gamma(t)) = \dot\gamma(t) \mod \Delta^{s-h}(\gamma(t))$ for some $1 \leqslant h \leqslant s$ and for any $t \in [0, T]$. We have strong evidence suggesting that this yields smaller complexities. Secondly, we are trying to weaken the assumption $f_0 \in \Delta^s \setminus^{s-1}$ to $f_0(q) \notin \Delta(q)$ for any $q \in M$.

2. In mechanical systems, where one controls the acceleration and the drift is the velocity, one is usually interested in quasi-static motion planning, i.e., moving along trajectories near the zero-level set of the drift. In order to develop a complete theory of control-affine complexities for the costs $\mathcal{J}$ and $\mathcal{I}$, it is then necessary to extend our results to curves or paths contained in the zero-level set of f_0.

3. We focused on costs based on the L^1-norm. While in the sub-Riemannian case, thanks to the rescaling properties of non-holonomic control systems, this is essentially equivalent to minimize the L^p-norm, such a statement is no more true for control-affine system. Thus, we intend to study what happens for this kind of costs. This problem is not just a mathematical curiosity, but is critical for a fruitful application of these results to quantum control, where the cost is usually the L^2-norm.

4. In this thesis we obtained only weak asymptotic estimates for the complexities. It is then natural to look for strong asymptotic estimates and asymptotic optimal syntheses in the spirit of [RMGMP04, GZ05, GZ06] . The techniques employed by Gauthier and Zakalyukin should indeed admit a natural generalization to the control-affine case.

1.4.2 Singular diffusions

1. As already mentioned, we are currently collaborating with A. Posilicano on [PP] which is a direct continuation of the results exposed in Section 1.3.2. Our aim is to completely classify all the Markovian self-adjoint extensions of $\mathcal{L}$ in the case $\alpha \in (-1, 1)$, where the deficiency indexes of the Laplace-Beltrami operator are infinite. This would allow for a better understanding of which kind of transmissions are possible in this context. The main motivation for this classification, however, is the interest these metrics have for $\alpha \in (0, 1)$ in the control of partial differential equations, see e.g., [Mor13].

2. It would be nice to understand the scattering properties of these operators (taking the bridging extension as a reference), and to derive the associated transmission and reflection coefficients. This would give informations on how much of a wave packet would be reflected or transmitted, when hitting the singularity.

3. We give results on the behavior of Markov processes by working solely on their generators. It is natural to try to understand if it is possible to obtain the same results from

a purely probabilistic point of view, for example by defining such Markov processes as limits of random walks.

4. The spectral properties derived in Section 1.3.3 for two specific almost-Riemannian structures suggest that in general the first order of the Weyl law should be of the form $E \log E$, at least for generic structures. At the moment we are working with M. Seri in this direction, trying to adapt pseudo-laplacian techniques [CdV82, CdV83] to the almost-Riemannian case.

5. As already mentioned, some work has been done in order to develop a pseudo-differential calculus on step 2 sub-Riemannian manifolds, through the group structure of the tangent cone. It would be very interesting to develop a general pseudo-differential calculus for sub-Riemannian manifolds, that exploits the natural Hamiltonian formulation on the cotangent bundle. This is a work in progress with Y. Chitour and M. Seri.

Part I

Complexity of control-affine motion planning

2 PRELIMINARIES OF SUB-RIEMANNIAN GEOMETRY

In this chapter we present some preliminaries in sub-Riemannian geometry. Recall from Section 1.1, and in particular from (1.1.2), that a sub-Riemannian structure is determined by a control-affine system without drift of the form

$$\dot{q} = \sum_{i=1}^{m} u_i f_i(q), \qquad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{SR}$$

where $\{f_1, \dots, f_m\}$ are smooth (not necessarily linearly independent) vector fields that we always assume to satisfy the following.

(SR1) The family of smooth vector fields $\{f_1, \dots, f_m\}$ satisfies the *Hörmander condition*, i.e., its iterated Lie brackets generate the whole tangent space at any point.

The chapter is divided in two sections. In Section 2.1 we discuss more in detail some of the sub-Riemannian notions already presented in the introduction, as priviliged coordinates and the ball-box theorem. Then, in Section 2.2 we present some properties of families of coordinates depending continuously on the points of some curve or path, needed in Chapter 4.

2.1 PRIVILEGED COORDINATES AND NILPOTENT APPROXIMATION

In this section we discuss more in detail some classical notions and results of sub-Riemannian geometry, already mentioned in the introduction. In particular, we focus on the equivalent, in the sub-Riemannian context, of the linearization of a vector field. This classical procedure, called *nilpotent approximation*, is possible only in carefully chosen sets of coordinates, called *privileged coordinates*.

Let us recall some notation. Let $\Delta^1 = \Delta := \{f_1, \dots, f_m\}$ and define recursively $\Delta^{s+1} = \Delta^s + [\Delta^s, \Delta]$, for every $s \in \mathbb{N}$. Since by (SR1) the family $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, the values of the sets Δ^s at q form a flag of subspaces of T_qM,

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^r(q) = T_qM. \tag{2.1.1}$$

The integer $r = r(q)$, which is the minimum number of brackets required to recover the whole T_qM, is called *degree of non-holonomy* (or *step*) of the family $\{f_1, \dots, f_m\}$ at q. Set $n_s(q) = \dim \Delta^s(q)$. The integer list $(n_1(q), \dots, n_r(q))$ is called the *growth vector* at q. From now on we fix $q \in M$, and denote by r and $(n_1, \dots, n_r)$ its degree of non-holonomy and its growth vector, respectively. Finally, let $w_1 \leqslant \dots \leqslant w_n$ be the *weights* associated with the flag, defined by $w_i = s$ if $n_{s-1} < i \leqslant n_s$, setting $n_0 = 0$.

For any smooth vector field f, we denote its action, as a derivation on smooth functions, by $f : a \in C^\infty(M) \mapsto fa \in C^\infty(M)$. For any smooth function a and every vector field f with $f \not\equiv 0$ near q, their *(non-holonomic) order* at q is

$$\begin{aligned} \mathrm{ord}_q(a) &= \min\{s \in \mathbb{N} \colon \exists i_1, \dots, i_s \in \{1, \dots, m\} \text{ s.t. } (f_{i_1} \dots f_{i_s}\, a)(q) \neq 0\}, \\ \mathrm{ord}_q(f) &= \max\{\sigma \in \mathbb{Z} \colon \ \mathrm{ord}_q(fa) \geqslant \sigma + \mathrm{ord}_q(a) \text{ for any } a \in C^\infty(M)\}. \end{aligned}$$

In particular, it can be proved that $\mathrm{ord}_q(a) \geqslant s$ if and only if $a(q') = O(d_{SR}(q', q))^s$.

The following proposition clarifies the relationship between non-holonomic orders and the flag (2.1.1).

Proposition 2.1.1. *Let (SR1) be satisfied, i.e. assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and let $q \in M$ and $s \in \mathbb{N}$. Then, for any smooth vector field f it holds that $\mathrm{ord}_{q'} f = -s$ for any q' near q if and only if $f(q') \in \Delta^s(q') \setminus \Delta^{s-1}(q')$ for any q' near q.*

Remark 2.1.2. In the previous proposition, the fact that the assumptions hold in a neighborhood of q is essential. Indeed, although it is true that $\mathrm{ord}_q f \geqslant -s$ implies $f(q) \in \Delta^s(q)$, when the growth vector is not constant around q the contrary is false. To see this, it suffices to consider the sub-Riemannian control system on $\mathbb{R}^2$ with (privileged) coordinates (x, y), defined by the vector fields ∂_x and $x^2\partial_y$. Outside $\{x = 0\}$, the non-holonomic degree of these vector fields is -1, while on $\{x = 0\}$ we need two brackets to generate the y direction and hence the non-holonomic degree of ∂_y is -3. Then, the vector field $f(x, y) = \partial_x + x\partial_y$ is such that $f(0, 0) \in \Delta^1(0, 0)$ but $\mathrm{ord}_{(0,0)} f = -2$.

Definition 2.1.3. A *system of privileged coordinates* at q for $\{f_1, \dots, f_m\}$ is a system of local coordinates $z = (z_1, \dots, z_n)$ centered at q and such that $\mathrm{ord}_q(z_i) = w_i$, $1 \leqslant i \leqslant n$.

The family of smooth vector fields $\{g_1, \dots, g_n\}$ is an adapted basis at $q \in M$ if $\mathrm{span}\{g_1(q), \dots, g_{n_s}(q)\} = \Delta^s(q)$ for any $1 \leqslant s \leqslant r$. (Equivalently: if $\mathrm{span}\{g_1(q), \dots, g_n(q)\} = T_qM$ and $g_i \in \Delta^{w_i}(q)$ for any $1 \leqslant i \leqslant n$.) By continuity, if $\{g_1, \dots, g_n\}$ is an adapted basis at q, it is a basis of $T_{q'}M$ for any q' near q, which in general will not be adapted.

Through an adapted basis at q, it is always possible to define a system of privileged coordinates at q. Namely, for any permutation $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$, the inverse $z = (z_1, \dots, z_n)$ of the local diffeomorphism

$$\phi : (z_1, \dots, z_n) \mapsto e^{z_{i_n} g_{i_n}} \circ \dots \circ e^{z_{i_1} g_{i_1}}(q),$$

is a system of privileged coordinates at q, called *canonical coordinates of the second kind*. In particular, in this system g_{i_n} is rectified, i.e., $z_* g_{i_n} \equiv \partial_{z_{i_n}}$, where z_* is the push-forward operator on vector fields associated with the coordinates, defined as $z_* f = dz \circ f \circ z^{-1}$. We immediately obtain the following.

Proposition 2.1.4. *Let (SR1) be satisfied, i.e. assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and let $q \in M$ and f be a smooth vector field such that $f(q) \neq 0$ and $f(q) \in \Delta^s(q) \setminus \Delta^{s-1}(q)$ for some $s \in \mathbb{N}$. Then, there exists a system of privileged coordinates $z = (z_1, \dots, z_n)$ at q rectifying f, i.e., such that $z_* f \equiv \partial_k$ for some $1 \leqslant k \leqslant n$.*

Consider any system of privileged coordinates $z = (z_1, \dots, z_n)$. We now show that it allows to compute the order of functions or vector fields in a purely algebraic way. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we define the weighted degree of the monomial $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ as $w(\alpha) =$

$w_1\alpha_1 + \cdots + w_n\alpha_n$ and the weighted degree of the monomial vector field $z^\alpha \partial_{z_j}$ as $w(\alpha) - w_j$. Then given $a \in C^\infty(M)$ and a smooth vector field f with Taylor expansions

$$a(z) \sim \sum_\alpha a_\alpha z^\alpha \quad \text{and} \quad f(z) \sim \sum_{\alpha,j} f_{\alpha,j} z^\alpha \partial_{z_j},$$

their orders at q can be computed as

$$\operatorname{ord}_q(a) = \min\{w(\alpha) \colon\ a_\alpha \neq 0\} \quad \text{and} \quad \operatorname{ord}_q(f) = \min\{w(\alpha) - w_j \colon\ f_{\alpha,j} \neq 0\}.$$

A function or a vector field is said to be *homogeneous* if all the nonzero terms of its Taylor expansion have the same weighted degree.

We recall that, for any $a, b \in C^\infty(M)$ and any smooth vector fields f, g, the order satisfies the following properties

$$\begin{aligned} \operatorname{ord}_q(a+b) &= \min\{\operatorname{ord}_q(a), \operatorname{ord}_q(b)\}, & \operatorname{ord}_q(ab) &= \operatorname{ord}_q(a) + \operatorname{ord}_q(b), \\ \operatorname{ord}_q(f+g) &= \min\{\operatorname{ord}_q(f), \operatorname{ord}_q(g)\}, & \operatorname{ord}_q([f,g]) &\geqslant \operatorname{ord}_q(f) + \operatorname{ord}_q(g). \end{aligned} \tag{2.1.2}$$

Consider the control vector fields f_i, $1 \leqslant i \leqslant m$. By the definition of order, it follows that $\operatorname{ord}_q(f_i) \geqslant -1$. Then we can express f_i in coordinates as

$$z_* f_i = \sum_{j=1}^n (h_{ij} + r_{ij}) \partial_{z_j},$$

where h_{ij} are homogeneous polynomials of weighted degree $w_j - 1$ and r_{ij} are functions of order larger than or equal to w_j.

Definition 2.1.5. The nilpotent approximation at q of f_i, $1 \leqslant i \leqslant m$, associated with the privileged coordinates z is the vector field with coordinate representation

$$z_* \widehat{f}_i = \sum_{j=1}^n h_{ij}\, \partial_{z_j}.$$

The nilpotentized sub-Riemannian control system is then defined as

$$\dot q = \sum_{j=1}^m u_j(t) \widehat{f}_j(q). \tag{NSR}$$

The family of vector fields $\{\widehat{f}_1, \dots, \widehat{f}_m\}$ is bracket-generating and nilpotent of step r (i.e., every iterated bracket $[f_{i_1}, [\dots, [f_{i_{k-1}}, f_{i_k}]]]$ of length larger than r is zero).

The main property of the nilpotent approximation is the following (see for example [Bel96, Proposition 7.29]).

Proposition 2.1.6. *Let (SR1) be satisfied, i.e. assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at $q \in M$ for $\{f_1, \dots, f_m\}$. For $T > 0$ and $u \in L^1([0,T]; \mathbb{R}^m)$, with $|u| \equiv 1$, let $\gamma(\cdot)$ and $\hat\gamma(\cdot)$ be the trajectories associated with u in (SR) and (NSR), respectively, and such that $\gamma(0) = \hat\gamma(0) = q$. Then, there exist $C, T_0 > 0$, independent of u, such that, for any $t < T_0$, it holds*

$$|z_i(\gamma(t)) - z_i(\hat\gamma(t))| \leqslant C t^{w_i + 1}, \qquad i = 1, \dots, n.$$

We recall, finally, the celebrated Ball-Box Theorem, that gives a rough description of the shape of small sub-Riemannian balls.

Theorem 2.1.7 (Ball-Box Theorem). *Let (SR1) be satisfied, i.e. assume that* $\{f_1, \dots, f_m\}$ *satisfies the Hörmander condition, and let* $z = (z_1, \dots, z_n)$ *be a system of privileged coordinates at* $q \in M$ *for* $\{f_1, \dots, f_m\}$*. Then there exist* $C, \varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$*, it holds*

$$\mathrm{Box}\left(\frac{1}{C}\varepsilon\right) \subset B_{SR}(q, \varepsilon) \subset \mathrm{Box}\,(C\varepsilon)\,.$$

Here, $B_{SR}(q, \varepsilon)$ *is identified with its coordinate representation* $z(B_{SR}(q, \varepsilon))$ *and, for any* $\eta > 0$*, we let*

$$\mathrm{Box}\,(\eta) = \{z \in \mathbb{R}^n : \; |z_i| \leqslant \eta^{w_i}\}, \tag{2.1.3}$$

Remark 2.1.8. The constants C and ε_0 in the above theorem depend on q and are not uniform, in general. However, this is clearly true on some compact set $N \subset M$ if there exists for each $q \in N$ a system of privileged coordinates z^q such that $q \mapsto z^q$ depends continuously on q. Observe also that this is always true except when q is a singular point for the sub-Riemannian structure, i.e., if the growth vector is not constant near q.

We now state an uniform version of the Ball-Box theorem along integral curves of vector fields, which we will need in Section 3.2.2. This is a particularization of a much more general result contained in [Jea01b].

Proposition 2.1.9. *Let (SR1) be satisfied, i.e. assume that* $\{f_1, \dots, f_m\}$ *satisfies the Hörmander condition, and let* $z = (z_1, \dots, z_n)$ *be a system of privileged coordinates at* $q \in M$ *for* $\{f_1, \dots, f_m\}$*. Let* f *be a smooth vector field such that* q *be regular on its integral curve. Namely, there exists* t_0 *such that* $\dim \Delta^s(e^t(q))$, $s \in \mathbb{N}$, *is constant for* $t < t_0$*. Then, there exist* $C, \varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *and* $t < t_0$*, it holds*

$$z(e^{tf_0}(q)) + \mathrm{Box}\left(\frac{1}{C}\varepsilon\right) \subset B_{SR}(e^{tf_0}(q), \varepsilon) \subset z(e^{tf_0}(q)) + \mathrm{Box}\,(C\varepsilon)\,.$$

As a corollary of the Ball-Box Theorem, we get the following result on the regularity of the distance.

Corollary 2.1.10. *Let (SR1) be satisfied, i.e. assume that* $\{f_1, \dots, f_m\}$ *satisfies the Hörmander condition, and let* $z = (z_1, \dots, z_n)$ *be a system of privileged coordinates at* $q \in M$ *for* $\{f_1, \dots, f_m\}$*. Then there exists* $C, \varepsilon > 0$ *such that*

$$\frac{1}{C}|z(q')| \leqslant d_{SR}(q, q') \leqslant C|z(q')|^{1/r}, \qquad q' \in B_{SR}(q, \varepsilon).$$

2.2 CONTINUOUS FAMILIES OF COORDINATES

In this section we consider properties of families of coordinates depending continuously on points of some curve or path.

From the definition of privileged coordinates, we immediately get the following.

Proposition 2.2.1. *Let $\gamma : [0,T] \to M$ be a path. Let $t > 0$ and let z be a system of privileged coordinates at $\gamma(t)$ for $\{f_1, \dots, f_m\}$. Then, there exists $C > 0$ such that*

$$|z_j(\gamma(t+\xi))| \leqslant C|\xi| \quad \text{for any } j = 1, \dots, n \text{ and any } t+\xi \in [0,T]. \tag{2.2.1}$$

Moreover, if for $k \in \mathbb{N}$ it holds that $\dot\gamma(t) \notin \Delta^{k-1}(\gamma(t))$, then there exist $C_1, C_2, \xi_0 > 0$ and a coordinate z_α, of weight $\geqslant k$, such that for any $t \in [0,T]$ and any $|\xi| \leqslant \xi_0$ with $t+\xi \in [0,T]$ it holds

$$C_1 \xi \leqslant z_\alpha(\gamma(t+\xi)) \leqslant C_2 \xi. \tag{2.2.2}$$

Finally, if $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$, the coordinate z_α can be chosen to be of weight k.

Proof. By the smoothness of γ, there exists a constant $C > 0$ such that $|(z_j)_*\dot\gamma(t+\xi)| \leqslant C$ for any $j = 1, \dots, n$ and any $t+\xi \in [0,T]$. Thus, we obtain

$$|z_j(\gamma(t+\xi))| \leqslant \left| \int_t^{t+\xi} |(z_j)_*\dot\gamma(t+\eta)| \, d\eta \right| \leqslant C\,|\xi|.$$

Let us prove (2.2.2). Let $\{f_1, \dots, f_n\}$ be an adapted basis associated with the system of coordinates z. In particular it holds that $z_* f_i(\gamma(t)) = \partial_{z_i}$. Moreover, let $k' \geqslant k$ be such that $\dot\gamma(t) \in \Delta^{k'}(\gamma(t)) \setminus \Delta^{k'-1}(\gamma(t))$ and write $\dot\gamma(t) = \sum_{w_i \leqslant k'} a_i(t) f_i(\gamma(t))$ for some $a_i \in C^\infty([0,T])$. Hence

$$z_*\dot\gamma(t) = \sum_{w_i \leqslant k'} a_i(t)\, z_* f_i(\gamma(t)) = \sum_{w_i \leqslant k'} a_i(t)\, \partial_{z_i}.$$

Since there exists i with $w_i = k'$ such that $a_i(t) \neq 0$, this implies that $(z_i)_*\dot\gamma(t) \neq 0$. Since $k' \geqslant k$, we have then proved (2.2.1). □

As a consequence of Remark 2.1.8, in order to apply the estimates of Theorem 2.1.7 uniformly on γ it suffices to consider a continuous family of coordinates $\{z^t\}_{t \in [0,T]}$ such that each z^t is privileged at $\gamma(t)$ for $\{f_1, \dots, f_m\}$. We will call such a family a *continuous coordinate family for* γ.

Let us recall that, fixed any basis $\{f_1, \dots, f_n\}$ adapted to the flag in a neighborhood of $\gamma([0,T])$, letting z^t be the inverse of the diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_1 f_1} \circ \dots \circ e^{z_n f_n}(\gamma(t)), \tag{2.2.3}$$

defines a continuous coordinate family for γ.

The following proposition precises Proposition 2.2.1.

Proposition 2.2.2. *Let $\gamma : [0,T] \to M$ be a path and let $k \in \mathbb{N}$ such that $\dot\gamma(s) \in \Delta^k(\gamma(s))$ for any $t \in [0,T]$. Then, for any continuous coordinate family $\{z^t\}_{t \in [0,T]}$ for γ there exists constants $C, \xi_0 > 0$ such that for any $t \in [0,T]$ and $0 \leqslant \xi \leqslant \xi_0$ it holds*

$$|z_j^t(\gamma(t+\xi))| \leqslant C\xi \quad \text{if } w_j \leqslant k \quad \text{and} \quad |z_j^t(\gamma(t+\xi))| \leqslant C\xi^{\frac{w_j}{k}} \text{ if } w_j > k. \tag{2.2.4}$$

Proof. Fix $t \in [0,T]$ and let $\{f_1, \dots, f_n\}$ be an adapted basis associated with the privileged coordinate system z^t. To lighten the notation, we do not explicitly write the dependence on time of such basis. Writing $z_*^t f_i(z) = \sum_{j=1}^n f_i^j(z)\partial_{z_j^t}$, it holds that f_i^j is of weighted order $\geqslant w_j - w_i$, and hence there exists a constant $C > 0$ such that

$$|f_i^j(z)| \leqslant C\|z\|^{(w_j - w_i)^+}. \tag{2.2.5}$$

Here $\|z\|$ is the pseudo-norm $|z_1|^{\frac{1}{w_1}} + \cdots + |z_n|^{\frac{1}{w_n}}$ and $h^+ = \max\{0, h\}$ for any $h \in \mathbb{R}$. Due to the compactness of $[0, T]$, the constant C can be choosen to be uniform w.r.t. the time.

Since $\dot\gamma(\xi) \in \Delta^k(\gamma(\xi))$ for $\xi > 0$, there exist functions $a_i \in C^\infty([0, T])$ such that

$$\dot\gamma(\xi) = \sum_{w_i \leqslant k} a_i(\xi) f_i(\gamma(\xi)) \qquad \text{for any } \xi \in [0, T]. \tag{2.2.6}$$

Observe that, for any $t \in [0, T]$, it holds that

$$\frac{1}{\xi}\int_t^{t+\xi} |a_i(\eta)|\, d\eta = |a_i(t)| + O(\xi) \quad \text{as } \xi \downarrow 0, \tag{2.2.7}$$

where $O(\xi)$ is uniform w.r.t. t. In particular, for any ξ sufficiently small, this integral is bounded.

By (2.2.6), for any $t \in [0, T]$ we get

$$z_j^t(\gamma(t+\xi)) = \sum_{w_i \leqslant k} \int_t^{t+\xi} a_i(\eta) f_i^j(z^t(\gamma(\eta)))\, d\eta, \qquad \text{for any } t+\xi \in [0, T] \tag{2.2.8}$$

Then, applying (2.2.5) we obtain

$$\begin{aligned} \max_{\rho\in[0,\xi]} |z_j^t(\gamma(t+\rho))| &\leqslant \sum_{w_i \leqslant k} \int_t^{t+\xi} |a_i(\eta)|\,|f_i^j(z^t(\gamma(\eta)))|\, d\eta \\ &\leqslant C\left(\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho))\|\right)^{(w_j - k)^+} \sum_{w_i \leqslant k} \int_t^{t+\xi} |a_i(\eta)|\, d\eta. \end{aligned} \tag{2.2.9}$$

Up to enlarging the constant C, this and (2.2.7) yield

$$\begin{aligned} \frac{\max_{\rho\in[0,\xi]} |z_j^t(\gamma(t+\rho^k))|}{\xi^{w_j}} &\leqslant C\left(\frac{\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|}{\xi}\right)^{(w_j-k)^+} \sum_{w_i \leqslant k} \frac{1}{\xi^k}\int_t^{t+\xi^k} |a_i(\eta)|\, d\eta \\ &\leqslant C\left(\frac{\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|}{\xi}\right)^{(w_j-k)^+}. \end{aligned} \tag{2.2.10}$$

Clearly, if $\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|/\xi \leqslant C$ uniformly in t, inequality (2.2.10) proves (2.2.4). Then, let us assume by contradiction that $\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|/\xi$ is unbounded as $\xi \downarrow 0$. For any ξ let $\bar\xi \in [0, \xi]$ to be such that $\|z^t(\gamma(t+\bar\xi^k))\| = \max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|$. Then, there exists a sequence $\xi_\nu \to +\infty$ such that

$$b_\nu = \frac{|z_j^t(\gamma(t+\bar\xi_\nu^k))|}{\xi_\nu^{w_j}} \longrightarrow +\infty \quad \text{and} \quad \frac{1}{n}\frac{\|z^t(\gamma(t+\bar\xi_\nu^k))\|}{\xi_\nu} \leqslant b_\nu^{\frac{1}{w_j}} \leqslant \frac{\|z^t(\gamma(t+\bar\xi_\nu^k))\|}{\xi_\nu}.$$

Moreover, by (2.2.10), it has to hold that $w_j > k$. Then, again by (2.2.10), follows that

$$b_\nu \leqslant Cn\, b_\nu^{1-\frac{k}{w_j}} \longrightarrow 0 \quad \text{as } \nu \to +\infty.$$

This contradicts the fact that $b_\nu \to +\infty$, and proves that there exists $\xi_0 > 0$, a priori depending on t, such that $\|z^t(\gamma(t+\bar\xi^k))\|/\xi \leqslant C$ for any $\xi < \xi_0$. Since $[0, T]$ is compact, both constants ξ_0, C are uniform for $t \in [0, T]$, thus completing the proof of (2.2.4) and of the proposition. □

3 HÖLDER CONTINUITY OF THE VALUE FUNCTION

In this chapter we prove the results contained in [Pra14] and already mentioned in Section 1.2.2, regarding the value function control-affine systems satisfying the strong Hörmander condition w.r.t. the cost

$$\mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{i=1}^{m} u_i(t)^2}\, dt. \tag{3.0.11}$$

Let us remark that in this chapter we will also discuss some results where the maximal time of definition of the controls is infinite, i.e., $\mathcal{T} = +\infty$.

The chapter is divided in two sections. In Section 3.1 we consider control systems in the form (1.2.4), and prove the continuity of the value function for general time-dependent vector fields, under the following assumptions.

(T1) The map $t \mapsto f_i^t$ is smooth for $1 \leqslant i \leqslant m$ and $t \in I$.

(T2) The family of smooth vector fields $\{f_1^t, \dots, f_m^t\}$ satisfies the strong Hörmander condition, i.e., for any t the family $\{f_1^t, \dots, f_m^t\}$ satisfies the Hörmander condition.

Then, in Theorem 3.1.9, restricting to the case where the time dependency is explicitly given as $f_i^t = (e^{-tf_0})_* f_i$, we establish some estimates on the reachable sets, in the same spirit as the Ball-Box theorem.

Finally, in Section 3.2 we consider control-affine systems satisfying the following assumptions.

(D1) The family $\{f_0, f_1, \dots, f_m\}$ satisfies the strong Hörmander condition, i.e., the family $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition and thus defines a sub-Riemannian control system.

(D2) The point q is regular for the integral curve of the drift, i.e., is such that $\dim \Delta^s(e^{tf_0}(q))$, $s \in \mathbb{N}$, is constant for small t.

(D3) The point q is regular w.r.t. the drift, in the sense that there exists $s \in \mathbb{N}$ such that $f_0(q') \in \Delta^s(q') \setminus \Delta^{s-1}(q')$, for any q' near q.

Here, after proving the relation between control-affine systems and time-dependent systems, we prove the continuity of the value function. Then, in Lemma 3.2.6, exploiting the affine nature of the control system, we give an upper bound on the time needed to join two points q and q' as a function of $V_T(q, q')$. From this fact and the estimates of Section 3.1 we finally obtain Theorems 1.2.3 and 1.2.2.

3.1 TIME-DEPENDENT SYSTEMS

Consider the following time-dependent non-holonomic control system

$$\dot{q} = \sum_{i=1}^{m} u_i f_i^t(q), \qquad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \quad t \in I, \tag{TD}$$

where $I = [0, b)$ for some $b \leqslant +\infty$ and $\{f_1^t, \dots, f_m^t\}$ is a family of non-autonomous smooth vector fields depending smoothly on time and satisfying (T1) and (T2). We let $f_u^t = \sum_{i=1}^m u_i f_i^t$.

As we will see later on in Section 3.2, when considering families of time-dependent vector fields of the form $f_i^t = (e^{-tf_0})_* f_i$ assumption (T2) will follow from the strong Hörmander condition for the affine control system with drift f_0 and control vector fields $\{f_1, \dots, f_m\}$ (i.e., assumption (D1)).

Since we will discuss multiple control systems at the same time, to better distinguish them, in the following we will call the horizontal curves of a sub-Riemannian system (SR)-admissible. Analogously, we define (TD)-admissible curves as absolutely continuous curves $\gamma : [0, T] \subset I \to M$ such that $\dot{\gamma}(t) = f^t_{u(t)}(\gamma(t))$ for a.e. $t \in [0, T]$ and for some control $u \in L^1([0, T], \mathbb{R}^m)$. Observe, however, that contrary to what happens in the sub-Riemannian case, the (TD)-admissibility property is not invariant under time reparametrization, e.g., a time reversal. Thus, we define the cost (and not the length) of γ to be

$$J(\gamma) = \min \|u\|_{L^1([0,T],\mathbb{R}^m)},$$

where the minimum is taken over all controls u such that γ is associated with u and is attained due to convexity. The *value function* induced by the time-dependent system is then defined as

$$\rho(q, q') = \inf\{J(\gamma) \colon \ \gamma \text{ is (TD)-admissible and } \gamma : q \rightsquigarrow q'\}.$$

Clearly, the value function is non-negative. It is not a metric since, in general, it fails both to be symmetric and to satisfy the triangular inequality. Moreover, as the following example shows, ρ could be degenerate. Namely, it could happen that $q \neq q'$ but $\rho(q, q') = 0$.

Example 3.1.1. Let $M = \mathbb{R}$, with coordinate x and consider the vector field $f^t = (1-t)^{-2}\partial_x$ defined on $[0, 1)$. For any $x_0 \in \mathbb{R}$, $x_0 \neq 0$, and for any sequence $t_n \uparrow 1$, let $u_n \in L^1([0, t_n])$ be defined as $u_n \equiv (1 - t_n)x_0$. By definition, each u_n steers the system from 0 to x_0. Hence,

$$\rho_1(0, x_0) \leqslant \inf_{n \in \mathbb{N}} \|u_n\|_{L^1([0,t_n])} = \inf_{n \in \mathbb{N}} \int_0^{t_n} (1 - t_n)x_0 \, dt = x_0 \inf_{n \in \mathbb{N}} t_n(1 - t_n) = 0.$$

This proves that, for any $x_0 \in \mathbb{R}$, $\rho_1(0, x_0) = 0$.

For $T > 0$, $q \in M$ and $\varepsilon > 0$, we denote the reachable set from q with cost less than ε by

$$\mathcal{R}(q, \varepsilon) = \{q' \in M \colon \ \rho(q, q') < \varepsilon\}.$$

We will also consider the reachable set from q in time less than $T > 0$ and cost less than ε, and denote it by $\mathcal{R}_T(q, \varepsilon)$. Clearly $\mathcal{R}_T(q, \varepsilon) \subset \mathcal{R}(q, \varepsilon)$.

In general, the existence of minimizers for the optimal control problem associated with (TD) is not guaranteed. We conclude this section with an example of this fact.

Example 3.1.2. Let $M = \mathbb{R}$, with coordinate x, and consider the vector field $f^t = e^{-t}\partial_x$ for $t \in [0, 1)$. Fix $x_0 \in \mathbb{R}$, $x_0 \neq 0$. Observe that, for any $T > 0$ and any control $u \in L^1([0, T])$ steering the system from 0 to x_0 , it holds

$$|x_0| = \left| \int_0^T u(t)e^{-t}\, dt \right| \leqslant \int_0^T |u(t)|e^{-t}\, dt < \|u\|_{L^1([0,T])}. \tag{3.1.1}$$

This implies $\rho(0, x_0) \geqslant |x_0|$. Let now $u_n \in L^1([0, 1/n])$ be defined as $u_n(t) = x_0 n e^t$. Clearly u_n steers the system from 0 to x_0. Moreover,

$$\rho(0, x_0) \leqslant \inf_{n\in\mathbb{N}} \|u_n\|_{L^1([0,1/n])} = |x_0| \inf_{n\in\mathbb{N}} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = |x_0|.$$

This proves that $\rho(0, x_0) = |x_0|$. Hence, the non-existence of minimizers follows from (3.1.1).

3.1.1 Finiteness and continuity of the value function

In this section, we extend the Chow–Rashevsky Theorem to time-dependent non-holonomic systems under the strong Hörmander condition. Namely, we will prove the following.

Theorem 3.1.3. *Assume that* $\{f_1^t, \dots, f_m^t\}_{t\in I}$ *satisfies (T1) and (T2). Then, the function* $\rho : M \times M \to [0, +\infty)$ *is continuous. Moreover, for any* $t_0 \in I$ *and any* $q, q' \in M$*, letting* d_{SR} *be the sub-Riemannian distance induced by* $\{f_1^{t_0}, \dots, f_m^{t_0}\}$*, it holds* $\rho(q, q') \leqslant d_{SR}(q, q')$.

Let us introduce some notation. Following [ABB12a], the flows between times $s, t \in \mathbb{R}$ of an autonomous vector field f and of a non-autonomous vector field $\tau \mapsto f^\tau$ will be denoted by, respectively,

$$e^{(t-s)f} : M \to M \quad \text{and} \quad \overrightarrow{\exp} \int_s^t f^\tau\, d\tau : M \to M.$$

Fix $q \in M$ and assume, for the moment, that $t_0 = 0$. Let $\ell \in \mathbb{N}$ and $\mathcal{F} = (i_1, \dots, i_\ell) \in \{1, \dots, m\}^\ell$. For any $\mathcal{T} \in I$, $\mathcal{T} > 0$, we define the *switching end-point map* at time $\mathcal{T}$ and associated with $\mathcal{F}$ to be the function $E_{\mathcal{T},\mathcal{F}} : \mathbb{R}^\ell \to M$ defined as

$$\begin{aligned} E_{\mathcal{T},\mathcal{F}}(\xi) &= \overrightarrow{\exp} \int_{\frac{\ell-1}{\ell}\mathcal{T}}^{\mathcal{T}} \frac{\ell}{\mathcal{T}} \xi_\ell\, f_{i_\ell}^\tau\, d\tau \circ \cdots \circ \overrightarrow{\exp} \int_0^{\frac{\mathcal{T}}{\ell}} \frac{\ell}{\mathcal{T}} \xi_1\, f_{i_1}^\tau\, d\tau\, (q) \\ &= \overrightarrow{\exp} \int_{\frac{\ell-1}{\ell}}^1 \ell\xi_\ell\, f_{i_\ell}^{\tau\mathcal{T}}\, d\tau \circ \cdots \circ \overrightarrow{\exp} \int_0^{\frac{1}{\ell}} \ell\xi_1\, f_{i_1}^{\tau\mathcal{T}}\, d\tau\, (q). \end{aligned} \tag{3.1.2}$$

Here, we applied a standard change of variables formula for non-autonomous flows. Let then

$$g_{\mathcal{T},\mathcal{F}}^\tau = \begin{cases} \ell\xi_1\, f_{i_1}^{\tau\mathcal{T}} & \text{if } 0 \leqslant \tau < 1/\ell, \\ \ell\xi_2\, f_{i_2}^{(\tau-1/\ell)\mathcal{T}} & \text{if } 1/\ell \leqslant \tau < 2/\ell, \\ \vdots & \\ \ell\xi_\ell\, f_{i_\ell}^{(\tau-(\ell-1)/\ell)\mathcal{T}} & \text{if } (\ell-1)/\ell \leqslant \tau < 1, \end{cases} \tag{3.1.3}$$

so that we can write

$$E_{\mathcal{T},\mathcal{F}}(\xi) = \overrightarrow{\exp}\int_0^1 g^{\tau}_{\mathcal{T},\mathcal{F}}(\xi)\, d\tau\,(q).$$

Clearly, $t \mapsto \overrightarrow{\exp}\int_0^t g^{\tau}_{\mathcal{T},\mathcal{F}}(\xi)\, d\tau\,(q)$, $t \in [0,1]$, is a (TD)-admissible trajectory. Thus, $E_{\mathcal{T},\mathcal{F}}(\xi)$, $T > 0$, is the end-point of a piecewise smooth (TD)-admissible curve of cost $\sum_i |\xi_i|$.

We recall that, by the series expansion of $\overrightarrow{\exp}$ (see [ABB12a]), for any non-autonomous smooth vector field f^{τ}, it holds $\overrightarrow{\exp}\int_0^t f^{\tau}\, d\tau\,(q) = e^{t\, f^0}(q) + O(t^2)$. Thus, we can define

$$E_{0,\mathcal{F}}(\xi) = \lim_{\mathcal{T}\downarrow 0} E_{\mathcal{T},\mathcal{F}}(\xi) = e^{\xi_\ell f^0_\ell} \circ \ldots \circ e^{\xi_1 f^0_1}(q) = \overrightarrow{\exp}\int_0^1 g^{\tau}_{0,\mathcal{F}}(\xi)\, d\tau\,(q),$$

where, $g^{\tau}_{0,\mathcal{F}}(\xi)$ is defined in (3.1.3). Then $t \mapsto \overrightarrow{\exp}\int_0^t g^{\tau}_{0,\mathcal{F}}(\xi)\, d\tau\,(q)$, $t \in [0,1]$, is an (SR)-admissible curve for the sub-Riemannian structure defined by $\{f^0_1, \ldots, f^0_m\}$ and $E_{0,\mathcal{F}}(\xi)$ is the end-point of a piecewise smooth trajectory in (SR).

After [Sus76], we say that a point $q' \in M$ is (TD)-*reachable* from q at time $t_0 = 0$, if there exist $\ell \in \mathbb{N}$, $\mathcal{F} \in \{1, \ldots, m\}^{\ell}$, $\mathcal{T} > 0$ and $\xi \in \mathbb{R}^{\ell}$, such that $E_{\mathcal{T},\mathcal{F}}(\xi) = q'$. In this case it is clear that $\rho(q, q') \leqslant \sum_i |\xi_i|$. Moreover, if $\xi' \mapsto E_{\mathcal{T},\mathcal{F}}(\xi')$ has rank n at ξ, the point q' is said to be (TD)-*normally reachable* at time $t_0 = 0$. Finally, the point q' is said to be (SR)-*reachable* or (SR)-*normally reachable* for the vector fields $\{f^0_1, \ldots, f^0_m\}$, if these properties holds for $\mathcal{T} = 0$.

In the case $t_0 > 0$, taking $\mathcal{T} > 0$ such that $\mathcal{T} + t_0 \in I$ and changing the interval of integration in (3.1.2) from $[0, \mathcal{T}]$ to $[t_0, t_0 + \mathcal{T}]$, it is clear how to define (TD)-*reachable* and (TD)-*normally reachable* points from q at time t_0, and (SR)-*reachable* and (SR)-*normally reachable* points for the vector fields $\{f^{t_0}_1, \ldots, f^{t_0}_m\}$.

The proof of the following lemma is an adaptation of [Sus76, Lemma 3.1].

Lemma 3.1.4. *Let $q' \in M$ be (SR)-normally reachable for the vector fields $\{f^{t_0}_1, \ldots, f^{t_0}_m\}$ from q, by some $\ell \in \mathbb{N}$, $\xi \in \mathbb{R}^{\ell}$ and $\mathcal{F} \in \{1, \ldots, m\}^{\ell}$. Then, there exist $\varepsilon_0, \mathcal{T}_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, the point q' is (TD)-normally reachable at time t_0, by the same ℓ and $\mathcal{F}$, and some $\xi' \in \mathbb{R}^{\ell}$, with $\sum_j |\xi_j - \xi'_j| \leqslant \varepsilon$, and any $\mathcal{T} < \mathcal{T}_0$.*

Proof. Without loss of generality, we assume $t_0 = 0$.

Let $U \subset \mathbb{R}^{\ell}$ be a neighborhood of ξ such that $E_{0,\mathcal{F}}$ has still rank n when restricted to it. Then, there exists $B = \{x\colon\ \sum_j |x_j - \xi_j| \leqslant \varepsilon\} \subset U$ such that $E_{0,\mathcal{F}}$ maps diffeomorphically a neighborhood of B in U onto a neighborhood of q. It follows, from standard properties of differential equations, that, for $\mathcal{T} > 0$ sufficiently small, the map $E_{\mathcal{T},\mathcal{F}}$ is well defined on B and that $E_{\mathcal{T},\mathcal{F}} \longrightarrow E_{0,\mathcal{F}}$ as $\mathcal{T} \downarrow 0$ in the C^1-topology over B. Thus, there exists $\mathcal{T}_1 > 0$ such that, for $\mathcal{T} < \mathcal{T}_1$, $E_{\mathcal{T},\mathcal{F}}$ has rank n at every point of B.

Since the map $E_{0,\mathcal{F}}$ is an homeomorphism from B onto a neighborhood of q, and $E_{\mathcal{T},\mathcal{F}} \longrightarrow E_{0,\mathcal{F}}$ uniformly as $\mathcal{T} \downarrow 0$, it follows that there exists a fixed neighborhood V of q and $\mathcal{T}_2 > 0$ such that $V \subset E_{\mathcal{T},\mathcal{F}}(B)$, for any $\mathcal{T} < \mathcal{T}_2$. Then, for any $\mathcal{T} < \min\{\mathcal{T}_1, \mathcal{T}_2\}$, there exists $\xi' \in B$ such that the point $q' = E_{\mathcal{T},\mathcal{F}}(\xi')$ is (TD)-normally reachable. □

We will use the following consequence of Lemma 3.1.4. We remark that the result holds even if $\{f^t_1, \ldots, f^t_m\}_{t \in I}$ satisfies the Hörmander condition only at the time $t_0 \in I$.

Lemma 3.1.5. *Let d_{SR} be the sub-Riemannian distance induced by $\{f^{t_0}_1, \ldots, f^{t_0}_m\}$, then for any $t_1 \in I$, such that $t_1 - t_0 > 0$ is sufficiently small, and for any $q, q' \in M$ it holds that*

$$\inf\{J(\gamma)\colon\ \gamma : [t_0, t_1] \to M \text{ is (TD)-admissible}, \gamma(t_0) = q \text{ and } \gamma(t_1) = q'\} \leqslant d_{SR}(q, q').$$

In particular, $\rho(q,q') \leqslant d_{SR}(q,q')$.

Proof. Fix $\varepsilon > 0$. By Chow's theorem it is clear that q' is (SR)-reacheable from q. Moreover, since there exist (SR)-normally reachable points from q' arbitrarily close to q' (see e.g., [ABB12a, Lemma 3.21]), follows that q' is always (SR)-normally reacheable from q by ξ such that $\sum_j |\xi_j| \leqslant d_{SR}(q,q') + \varepsilon/2$. Hence, by Lemma 3.1.4, if ε and $\eta > 0$ are sufficiently small, we have that q' is (TD)-normally reachable from q at time t_0 by ξ' such that $\sum_j |\xi_j'| \leqslant d_{SR}(q,q') + \varepsilon$ and $T < t_1$. This clearly implies that

$$\inf\{J(\gamma)\colon \gamma \text{ is (TD)-admissible}, \gamma(t_0) = q \text{ and } \gamma(t_1) = q'\} \leqslant d_{SR}(q,q') + \varepsilon.$$

Finally, the lemma follows letting $\varepsilon \downarrow 0$. □

We now prove the main theorem of the section.

Proof of Theorem 3.1.3. By Lemma 3.1.5, we only need to prove the continuity of ρ. We will prove only the lower semicontinuity, since the upper semicontinuity follows by similar arguments.

We start by proving the lower semicontinuity of $\rho(q,\cdot)$ at q'. Consider a sequence $q_k \to q'$ and let $u_k \in L^1([0,T_k],\mathbb{R}^m)$ be controls such that each one steers system (TD) from q to q_k and $\liminf_k V_\infty(q,q_k) = \liminf_k \|u_k\|_{L^1}$. Then, by Lemma 3.1.5, for any $\varepsilon > 0$ there exists a sequence of $\tilde T_k > 0$ and a sequence of controls $v_k \in L^1([T_k,\tilde T_k],\mathbb{R}^m)$ all steering system (TD) from q_k to q' and such that $\|v_k\|_{L^1([T_k,\tilde T_k],\mathbb{R}^m)} \leqslant d_{SR}(q_k,q') + \varepsilon$. Since $d_{SR}(q_k,q') \to 0$, this implies that

$$\rho(q,q') \leqslant \lim_{n\to\infty}\left(\|u_k\|_{L^1([0,T_k],\mathbb{R}^m)} + \|v_k\|_{L^1([T_k,\tilde T_k],\mathbb{R}^m])}\right) = \liminf_n \rho(q,q_k) + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ proves that $\rho(q,\cdot)$ is lower semicontinuous at q'.

In order to prove the lower semicontinuity of $\rho(\cdot,q')$ at q, let us define

$$\varphi_\varepsilon(p) = \inf\{J(\gamma)\colon \gamma : [\varepsilon,T] \subset I \to M \text{ is (TD)-admissible and } \gamma : p \rightsquigarrow q'\}.$$

We claim that for any $p \in M$ it holds that $\varphi_\varepsilon(p) \longrightarrow \rho(p,q')$ as $\varepsilon \downarrow 0$. Since it is clear that $\varphi_\varepsilon(\cdot) \geqslant \rho(\cdot,q')$, it suffices to prove that

$$\lim_{\varepsilon\downarrow 0} \varphi_\varepsilon(p) \leqslant \rho(p,q') \qquad \text{for any } p \in M. \tag{3.1.4}$$

To this aim, fix $p \in M$ and $\eta > 0$ and let $\gamma : [0,T] \to M$ be such that $\gamma : p \rightsquigarrow q'$ and that $J(\gamma) \leqslant \rho(p,q') + \eta$. It is clear that $\gamma(2\varepsilon) \to p$ as $\varepsilon \downarrow 0$, and hence that $\rho(p,\gamma(2\varepsilon)) \to 0$ as $\varepsilon \downarrow 0$, by the first part of the proof. Thus, for any $\varepsilon > 0$ sufficiently small, there exists a (TD)-admissible curve $\gamma_\varepsilon : [\varepsilon, 2\varepsilon] \to M$ such that $\gamma_\varepsilon : p \rightsquigarrow \gamma(2\varepsilon)$ and $J(\gamma_\varepsilon) \leqslant \rho(p,\gamma(2\varepsilon)) + \eta$. By concatenating γ_ε with $\gamma|_{[2\varepsilon,T]}$, we get that

$$\varphi_\varepsilon(p) \leqslant J(\gamma_\varepsilon) + J(\gamma) \leqslant \rho(p,\gamma(2\varepsilon)) + \rho(p,q') + 2\eta.$$

Letting $\varepsilon \downarrow 0$ and then $\eta \downarrow 0$, this proves (3.1.4) and thus the claim.

Let now $q_k \to q$ and fix $\eta > 0$. By Lemma 3.1.5 this implies that $\rho(q_k,q) \to 0$ and that for any $\varepsilon > 0$ sufficiently small, there exists a (TD)-admissible curve $\gamma_\varepsilon : [0,\varepsilon] \to M$ such that $\gamma_\varepsilon : q_k \rightsquigarrow q$ and $J(\gamma_\varepsilon) \leqslant \rho(q_k,q) + \eta$. Hence

$$\rho(q_k,q') \leqslant c(\gamma_\varepsilon) + \varphi_\varepsilon(q) \leqslant \rho(q_k,q) + \varphi_\varepsilon(q) + \eta.$$

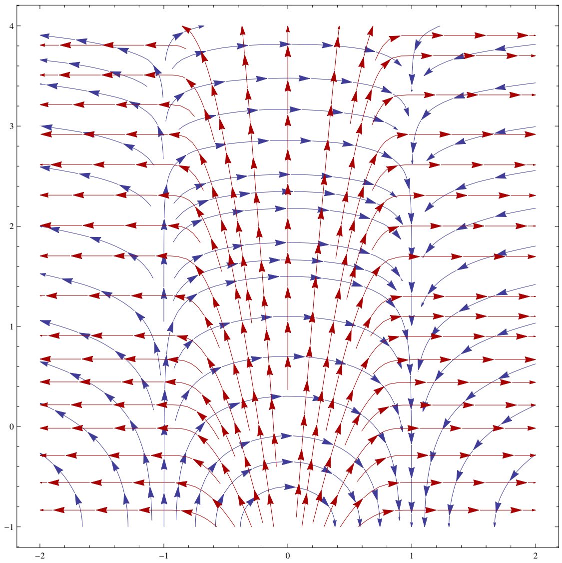

Figure 11: The two vector fields of Example 3.1.7 with $h(x) = c\, e^{-\frac{1}{1-x^2}}$ for $x \in [-1, 1]$.

By the previous claim, letting $\varepsilon, \eta \downarrow 0$, this implies that $\rho(q_k, q') \leqslant \rho(q_k, q) + \rho(q, q')$. Since $\rho(q_k, q) \to 0$, taking the liminf as $k \to +\infty$, this proves the lower semicontinuity of $\rho(\cdot, q')$ at q, completing the proof. □

Remark 3.1.6. From the proof of Theorem 3.1.3, it follows that hypothesis (T2) is not sharp. Indeed, the following would suffice to prove the theorem.

(T2′) The family of smooth vector fields $\{f_1^t, \dots, f_m^t\}_{t \in I}$ satisfies the strong Hörmander condition at $t = 0$ and in an open neighborhood of $\sup I$.

We will conclude this section by showing that, in our framework, it is essential to assume the Hörmander condition on both ends of I and hence that assumption (T2′) is minimal. Although outside the scope of the present work, we remark that stronger assumptions on the regularity of the vector fields, i.e., that they are uniformly Lipschitz, would allow to prove Theorem 3.1.3 assuming only that $\{f_1^t, \dots, f_m^t\}_{t \in I}$ satisfies the Hörmander condition at one time $t_0 \in I$.

The following example proves that if the family $\{f_1^t, \dots, f_m^t\}_{t \in I}$ satisfies the Hörmander condition only near $t = 0$, then the value function is in general not continuous. Through a slight modification, the same argument can also be used to prove that the same holds if the Hörmander condition is satisfied only at a neighborhood of $\sup I$ or of any $t_0 \in I$.

Example 3.1.7. Let $M = (-2, 2) \times (-1, +\infty)$, with coordinates (x, y), and consider the vector fields

$$f(x, y) = \frac{((y+1)(1-x^2), -x)}{\sqrt{(y+1)^2(1-x^2)^2 + x^2}}, \qquad g(x, y) = \frac{(x, h(x)(y+2))}{\sqrt{x^2 + h(x)^2(y+2)^2}},$$

where $h : [-2, 2] \to \mathbb{R}$ is a smooth cutoff function such that $\operatorname{supp} h \subset [-1, 1]$, $h \geqslant 0$ and $h(0) = 1$ (see Figure 11). Fix $0 < \varepsilon < 1$, $C \geqslant 16$ and let ϕ, $\psi : [0, 1] \to \mathbb{R}$ be two smooth functions such that

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant \varepsilon, \\ 0 & \text{if } 2\varepsilon \leqslant t \leqslant 1, \end{cases} \qquad \psi(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 2\varepsilon, \\ C & \text{if } 3\varepsilon \leqslant t \leqslant 1, \end{cases}$$

and such that ϕ is nonincreasing while ψ is nondecreasing. Finally, consider the time-dependent system on M specified by the vector fields $f^t(x, y) = \phi(t)f(x, y)$ and $g^t(x, y) = \psi(t)g(x, y)$, $t \in [0, 1]$. We will show that $\{f^t, g^t\}$ satisfies the Hörmander condition for $t \in [0, \varepsilon]$, but that the value function associated with the family $\{f^t, g^t\}_{t\in[0,1]}$ is not lower semicontinuous.

We start by showing that $f(p)$ and $g(p)$ are transversal for any $p = (x, y) \in M$, thus proving the Hörmander condition for $\{f^t, g^t\}$, $t \in [0, \varepsilon]$. If $x \in (-2, -1] \cup [1, 2)$, then, by definition of h, $g(p) = (1, 0)$ is clearly transversal to $f(p)$. On the other hand, if $x \in (-1, 1) \setminus \{0\}$ and $g(p)$ is parallel to $f(p)$, a simple computation shows that $h(x) < 0$, which is a contradiction. Finally, for $x = 0$, it is clear that $g(p) = (0, y + 2)$ and $f(p) = (y + 1, 0)$ are never parallel. We remark that this implies also that the value function ρ_ε, induced by controls defined on $[0, \varepsilon]$, is a distance equivalent to the Euclidean one. In particular, $|p_1 - p_2| \leqslant 2\rho_\varepsilon(p_1, p_2)$ for any $p_1, p_2 \in M$.

Fix now $q' = (1, 0)$. The set of points from which q' is reachable using only f is exactly $\mathcal{O}_{q'} = \{(1, y) : y > -1\}$. Let then $q_0 \in (-1, 0) \times \{0\}$ be such that $\rho_\varepsilon(q_0, (-1, 0)) \leqslant \frac{1}{4} \min_{p\in\mathcal{O}_{q'}} \rho_\varepsilon(q_0, p)$. In order to show that $\rho_1(q_0, \cdot)$ is not lower semicontinuous at q', consider any sequence $\{q_n\}_{n\in\mathbb{N}} \subset (1/2, 1) \times \{0\}$ such that $q_n \longrightarrow q'$. By continuity of ρ_ε and the fact that $-q_n \longrightarrow (-1, 0)$, we can always assume that, up to subsequences, $\rho_\varepsilon(q_0, -q_n) \leqslant \frac{1}{2} \min_{p\in\mathcal{O}_{q'}} \rho_\varepsilon(q_0, p)$.

Since $g^t \equiv 0$ for $t \geqslant 2\varepsilon$, if $u \in L^1([0, 1], \mathbb{R}^2)$ is a control steering the system from q_0 to q', the control $u|_{[0,2\varepsilon]}$ steers the system from q_0 to some $p \in \mathcal{O}_{q'}$. Exploiting the fact that $\rho_{2\varepsilon} \geqslant \rho_\varepsilon$ by monotonicity of ψ, this implies that

$$\rho_1(q_0, q') \geqslant \min_{p\in\mathcal{O}_{q'}} \rho_\varepsilon(q_0, p) \geqslant 2\rho_\varepsilon(q_0, -q_n). \tag{3.1.5}$$

Let now $u \in L^1([0, 1], \mathbb{R}^m)$ be the control constructed as follows. From time 0 to ε, $u|_{[0,\varepsilon]}$ is the minimizer of ρ_ε steering the system from q_0 to $-q_n$. Then, $u|_{(\varepsilon,3\varepsilon)} \equiv 0$ and, after this, the control acts only on f^t for time $t \in [3\varepsilon, 1]$, steering the system from $-q_n$ to q_n. Hence,

$$|q_n - (-q_n)| = \left| \int_{3\varepsilon}^{1} u(t) f^t(x(t), y(t))\, dt \right| = C \int_{3\varepsilon}^{1} |u(t)|\, dt. \tag{3.1.6}$$

Since $|q_n - (-q_n)| < 2$, $C \geqslant 16/|q_0 - q'| \geqslant 8/\rho_\varepsilon(q_0, q')$, and by (3.1.6), it holds that

$$\rho_1(q_0, q_n) \leqslant \int_0^1 |u(t)|\, dt = \rho_\varepsilon(q_0, -q_n) + \frac{1}{C}|q_n - (-q_n)| \leqslant \frac{3}{4}\rho_1(q_0, q').$$

Taking the $\liminf$ as $n \to \infty$ shows that $\rho_1(q_0, \cdot)$ is not l.s.c. at q'.

3.1.2 Estimates on reachable sets

In this section, we concentrate on a particular class of time-dependent systems. Namely, let $\{f_1, \dots, f_m\}$ be a family of smooth vector fields, f_0 be a smooth vector field, and consider the time-dependent system

$$\dot{q} = \sum_{i=1}^{m} u_i f_i^t, \qquad f_i^t = (e^{-tf_0})_* f_i(q), \quad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m. \tag{3.1.7}$$

Here, $(e^{-tf_0})_*$ is the push-forward operator associated with the flow of f_0. Throughout this section we will assume the following.

(T3) The family of smooth vector fields $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition.

Observe, in particular, that from (T3) it follows immediately that the family $\{(e^{-tf_0})_* f_1, \dots, (e^{-tf_0})_* f_m\}$ satisfies (T2), i.e., the strong Hörmander condition for time-dependent systems. Moreover, (T1) is an immediate consequence of the definition of the f_i^t's.

As we will see in the next section, this class of systems arises naturally when dealing with control systems that are affine with respect to the control.

Before proceeding to estimate the shape of the reachable sets, we need to define a suitable approximation of system (3.1.7). Namely, fix a system of privileged coordinates (in the sub-Riemannian sense) at q for $\{f_1, \dots, f_m\}$. Assume that $f_0(q) \neq 0$, and let $s \in \{1, \dots, r\}$ be such that $\operatorname{ord}_q f_0 = -s$. In particular, by Remark 2.1.2, this implies that $f_0(q) \in \Delta^s(q) \setminus \Delta^{s-1}(q)$. In this case, there exist, in coordinates, an homogeneous vector field f_0^{-s}, of weighted degree $-s$, and a vector field $f_0^{>-s}$, of weighted degree $\geqslant -s+1$, such that $f_0^{-s} \not\equiv 0$ near $z(q) = 0$ and

$$z_* f_0 = f_0^{-s} + f_0^{>-s}. \tag{3.1.8}$$

For any smooth vector field f, let $(\operatorname{ad}^1 f_0) f = [f_0, f]$ and $(\operatorname{ad}^\ell f_0) f = [f_0, (\operatorname{ad}^{\ell-1} f_0) f]$, for any $\ell \in \mathbb{N}$. We recall (see for example [Her91]) that the following Taylor expansion holds

$$(e^{-tf_0})_* f \sim \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} (\operatorname{ad}^\ell f_0) f. \tag{3.1.9}$$

Since $\operatorname{ord}_q(f_j) \geqslant -1$, by (2.1.2) we have that $\operatorname{ord}_q((\operatorname{ad}^\ell f_0) f_j) \geqslant -\ell s - 1$. Then, by decomposition (3.1.8), for any $\ell \geqslant 0$, there exists, in coordinates, an homogeneous vector field F_j^ℓ of weighted degree $-\ell s$, and a remainder r^ℓ of order $\geqslant -\ell s - 1$, such that

$$z_* \left[(\operatorname{ad}^\ell f_0) f_j \right] = F_j^\ell + r^\ell. \tag{3.1.10}$$

Definition 3.1.8. The *homogeneous series approximation* at q of f_j^t, $1 \leqslant j \leqslant m$, associated with the privileged coordinates z, is the vector field with coordinate representation

$$\widehat{f}_j^t = \sum_{\ell=0}^{\rho} \frac{t^\ell}{\ell!} F_j^\ell, \tag{3.1.11}$$

where $\rho = \lfloor r-1/s \rfloor$ and r is the non-holonomic degree of $\{f_1, \dots, f_m\}$ at q. The approximated time-dependent control system is then defined as

$$\dot{q} = \sum_{j=1}^{m} u_j(t) \widehat{f}_j^t(q). \tag{ATD}$$

If a system, in some system of privileged coordinates, coincides with its homogeneous series approximation , we will say that it is series homogeneous.

The homogeneous series approximation encodes the idea that the time t is of weight $s = -\operatorname{ord}_q(f_0)$. This is a consequence of the fact that, due to the expansion (3.1.9), t allows to build brackets of f_0 with the f_js. In this sense, the homogeneous series approximation is a generalization of the nilpotent approximation.

We are now ready to state the main theorem of this section.

Theorem 3.1.9. *Let (T3) be satisfied, i.e., assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at $q \in M$ for $\{f_1, \dots, f_m\}$. Then there exist $C, T, \varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and any $q' \in \mathcal{R}_T(q, \varepsilon)$, setting $s \in \mathbb{N}$ to be such that $\operatorname{ord}_q f_0 = -s$ it holds*

$$|z_i(q')| \leqslant C\left(\varepsilon^{w_i} + \varepsilon T^{\frac{w_i}{s}}\right) \quad \text{if } w_i \leqslant s, \tag{3.1.12}$$

$$|z_i(q')| \leqslant C\varepsilon\left(\varepsilon + T^{\frac{1}{s}}\right)^{w_i - 1} \quad \text{if } w_i > s. \tag{3.1.13}$$

Moreover, if the system is series homogeneous, then it holds the stronger estimate

$$|z_i(q')| \leqslant C\varepsilon^{w_i} \quad \text{if } w_i \leqslant s. \tag{3.1.14}$$

To prove this theorem we need the following proposition, estimating the difference between (3.1.7) and (ATD).

Proposition 3.1.10. *Let (T3) be satisfied, i.e., assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition, and let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at $q \in M$ for $\{f_1, \dots, f_m\}$. For $T > 0$ and $u \in L^1([0, T]; \mathbb{R}^m)$, let $\gamma(\cdot)$ and $\hat{\gamma}(\cdot)$ be the trajectories associated with u in (3.1.7) and (ATD), respectively, and such that $\gamma(0) = \hat{\gamma}(0) = q$. Then there exist $C, \varepsilon_0, T_0 > 0$, independent of u, such that, if $t < T_0$ and $\int_0^t |u|\, ds = \varepsilon < \varepsilon_0$, and setting $s \in \mathbb{N}$ to be such that $\operatorname{ord}_q f_0 = -s$ it holds*

$$|z_i(\gamma(t)) - z_i(\hat{\gamma}(t))| \leqslant C\varepsilon\left(\varepsilon + t^{\frac{1}{s}}\right)^{w_i}, \qquad i = 1, \dots, n. \tag{3.1.15}$$

Remark 3.1.11. This proposition generalizes Proposition 2.1.6. In fact, in the sub-Riemannian case, since $f_0 \equiv 0$, any curve γ associated with $u \in L^1([0, t], \mathbb{R}^m)$, $t > 0$, is associated also to $u_\tau(\cdot) = \frac{\tau}{t} u(\frac{\tau}{t}\cdot)$, for any $\tau > 0$. Thus, since $\int_0^\tau |u_\tau|\, ds = \int_0^t |u|\, ds = \varepsilon$, (3.1.15) reduces to

$$|z_i(\gamma(t)) - z_i(\hat{\gamma}(t))| \leqslant \inf_{\tau > 0} C(\varepsilon^{w_i + 1} + \tau\varepsilon^r) = C\varepsilon^{w_i + 1}.$$

Finally, assuming that u satisfies the hypotheses of Proposition 2.1.6, i.e., that $|u| = 1$, we get $t = \varepsilon$.

Proof. Let $z(\gamma(\cdot)) = x(\cdot)$, $z(\hat{\gamma}(\cdot)) = y(\cdot)$, and $\|z\| = \sum_{\ell=1}^n |z_\ell|^{1/w_i}$. We mimic the proof of Proposition 7.29 in [Bel96]. The first step is to prove that there exists a constant $C > 0$ such that $\|x(t)\|, \|y(t)\| \leqslant C\varepsilon$ for t and $\varepsilon = \int_0^t |u|\, ds$ small enough. We prove this for $\|x(t)\|$, the same argument works also for $\|y(t)\|$.

In z coordinates, the equation of the control system (3.1.7) is,

$$\dot{x}_i(t) = \sum_{j=1}^m u_j(t)(z_i)_* f_j^t(\gamma(t)), \qquad i = 1, \dots, n.$$

Due to the fact that $z_*f_j^t = z_*f_j + \mathcal{O}(t)$ uniformly in a neighborhood of q, that $\mathrm{ord}_q(z_i) = w_i$ and that $\mathrm{ord}_q(f_j) \geqslant -1$, we have that there exist T_0 and $C > 0$ such that $|(z_i)_*f_j^t(q)| \leqslant \frac{C}{2}|(z_i)_*f_j(q)| \leqslant C\|x(t)\|^{w_i-1}$, for any $t < T_0$. Thus we get

$$|\dot{x}_i(t)| \leqslant C\sum_{j=1}^{m}|u_j(t)|\|x_i(t)\|^{w_j-1}. \tag{3.1.16}$$

As in the proof for the sub-Riemannian case, choosing N sufficiently large, so that all N/w_i are even integers, and setting $|||z||| = (\sum_{\ell=1}^{n}|z_\ell|^{N/w_i})^{\frac{1}{N}}$ we get a norm equivalent to $\|z\|$. Deriving with respect to time and using (3.1.16) we get $\frac{d}{dt}|||x(t)||| \leqslant C\sum_{j=1}^{n}|u_j(t)|$. Finally, by integration, equivalence of the norms, and the fact that $x(0) = z(q) = 0$, we conclude that $\|x(t)\| \leqslant C\varepsilon$.

Now we move to proving (3.1.15). By construction of (ATD) and the Taylor expansion of f_j^t, for any $\ell \leqslant \rho = \lfloor r-1/s \rfloor$, there exist homogeneous polynomials h_{ji}^ℓ of order $w_i - \ell s - 1$ and remainders r_{ji}^ℓ of order larger than or equal to $w_i - \ell s$, such that we can write

$$(z_i)_*f_j^t = \sum_{\ell=0}^{\rho}\frac{t^\ell}{\ell!}(h_{ji}^\ell + r_{ji}^\ell) + O(t^{\rho+1}),$$
$$(z_i)_*\widehat{f}_j^t = \sum_{\ell=0}^{\rho}\frac{t^\ell}{\ell!}h_{ji}^\ell.$$

Here, the O is intended as $t \downarrow 0$ and is uniform in a compact neighborhood of the origin. Then,

$$\begin{aligned}\dot{x}_i(t)-\dot{y}_i(t) &= \sum_{j=1}^{m}u_j(t)\Big(\sum_{\ell=0}^{\rho}\frac{t^\ell}{\ell!}\big(h_{ji}^\ell(x) - h_{ji}^\ell(y) + r_{ji}^\ell(x)\big) + O(t^{\rho+1})\Big)\\ &= \sum_{j=1}^{m}u_j(t)\Big(\sum_{\ell=0}^{\rho}\frac{t^\ell}{\ell!}\Big(\sum_{w_k<w_i-\ell s}\big(x_k(t) - y_k(t)\big)Q_{jik}^\ell(x,y) + r_{ji}^\ell(x)\Big) + O(t^{\rho+1})\Big),\end{aligned}$$

where Q_{jik}^ℓ are homogeneous polynomial in x and y, of order $w_i - w_k - \ell s - 1$. We observe that, if $w_i - w_k - \ell s - 1 < 0$, then $Q_{jik}^\ell \equiv 0$. Thus, for sufficiently small $\|x\|$ and $\|y\|$, we have

$$|Q_{jik}^\ell(x,y)| \leqslant C\big(\|x\|^{(w_i-w_k-\ell s-1)^+} + \|y\|^{(w_i-w_k-\ell s-1)^+}\big), \quad |r_{ji}^\ell(x)| \leqslant C\|x\|^{(w_i-\ell s)^+}.$$

Here, we let $(\xi)^+ = \max\{\xi, 0\}$, for any $\xi \in \mathbb{R}$. Using the inequalities of the first step, taking $t < T$ sufficiently small, and enlarging the constant C, we get

$$\begin{aligned}|\dot{x}_i(t) - \dot{y}_i(t)| &\leqslant C|u(t)|\Big(\sum_{\ell=0}^{\rho}\frac{t^\ell}{\ell!}\Big(\sum_{w_k<w_i-\ell s}\big|x_k(t) - y_k(t)\big|\varepsilon^{w_i-w_k-\ell s-1} + \varepsilon^{(w_i-\ell s)^+}\Big) + t^{\rho+1}\Big)\\ &\leqslant C|u(t)|\Big(\sum_{\ell=0}^{\rho}t^\ell\Big(\sum_{w_h<w_i}\big|x_h(t) - y_h(t)\big|\varepsilon^{w_i-w_h-1} + \varepsilon^{(w_i-\ell s)^+}\Big) + t^{\rho+1}\Big).\end{aligned}$$

In the last inequality we applied the change of variable $w_k \mapsto w_h - \ell s$ in each of the sums.

We can integrate the system by induction, since it is in triangular form. For $w_i = 1$, since $(w_i - \ell s)^+ = 0$ for any $\ell \geqslant 1$, the inequality reduces to

$$|\dot{x}_i(t) - \dot{y}_i(t)| \leqslant C|u(t)| \left(\sum_{\ell=0}^{\rho} t^\ell \varepsilon^{(w_i - \ell s)^+} + t^{\rho+1} \right) \leqslant C|u(t)|(\varepsilon^{w_i} + t).$$

Here we enlarged the constant C. Thus, integrating the previous inequality, we get $|x_i(t) - y_i(t)| \leqslant C\varepsilon(\varepsilon^{w_i} + t) \leqslant C\varepsilon(\varepsilon + t^{\frac{1}{s}})^{w_i}$.

Let, then, $w_i > 1$ and assume that $|x_h(t) - y_h(t)| \leqslant C\varepsilon(\varepsilon + t^{\frac{1}{s}})^{w_h}$ for any $w_h < w_i$. To complete the proof it suffices to show that $|\dot{x}_i(t) - \dot{y}_i(t)| \leqslant C|u(t)|(\varepsilon + t^{\frac{1}{s}})^{w_i}$, since (3.1.15) will follow, as above, by integration. Thus, we have, enlarging again the constant C and taking t sufficiently small,

$$\begin{aligned} |\dot{x}_i(t) - \dot{y}_i(t)| &\leqslant C|u(t)| \left(\sum_{\ell=0}^{\rho} t^\ell \left(\sum_{w_h < w_i} \left(\varepsilon + t^{\frac{1}{s}}\right)^{w_h} \varepsilon^{w_i - w_h} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\rho+1} \right) \\ &\leqslant C|u(t)| \left(\sum_{\ell=0}^{\rho} t^\ell \left(\sum_{w_h < w_i} t^{\frac{w_h}{s}} \varepsilon^{w_i - w_h} + \varepsilon^{(w_i - \ell s)^+} \right) + t^{\rho+1} \right). \end{aligned} \tag{3.1.17}$$

If $t \leqslant \varepsilon^s$, from (3.1.17) it is clear that $|\dot{x}_i(t) - \dot{y}_i(t)| \leqslant C|u(t)|\varepsilon^{w_i}$. Here we used the fact that $\rho + 1 \geqslant w_i/s$. On the other hand, if $\varepsilon < t^{1/s}$, it holds

$$|\dot{x}_i(t) - \dot{y}_i(t)| \leqslant C|u(t)| \left(\sum_{\ell=0}^{\rho} \left(\sum_{w_h < w_i} t^{\frac{w_h}{s} + \ell + \frac{w_i - w_h}{s}} + t^{\ell + \frac{w_i - \ell s}{s}} \right) + t^{\rho+1} \right) \leqslant C|u(t)| t^{\frac{w_i}{s}}.$$

Putting together these two estimates, we get that $|\dot{x}_i(t) - \dot{y}_i(t)| \leqslant C|u(t)|(\varepsilon^{w_i} + t^{\frac{w_i}{s}}) \leqslant C|u(t)|(\varepsilon + t^{\frac{1}{s}})^{w_i}$, completing the proof of the proposition. □

Proof of Theorem 3.1.9. We start by claiming that (3.1.14) implies (3.1.12). In fact, if $\gamma : q \rightsquigarrow q'$ is the trajectory associated in (3.1.7) to a control $u \in L^1([0, T], \mathbb{R}^m)$, and $\hat{\gamma}$ is the trajectory associated with the same control in the homogeneous series approximation (ATD), with $\hat{\gamma}(0) = q$, it holds

$$|z_i(q')| \leqslant |z_i(\hat{\gamma}(T))| + |z_i(\hat{\gamma}(T)) - z_i(\gamma(T))|.$$

Thus, by Proposition 3.1.10, the claim is proved.

Hence, from now on we assume our system to be in the form (ATD). Let us define, for $1 \leqslant j \leqslant n$ and $0 \leqslant \alpha \leqslant r$, the vector fields φ_j^α as

$$\varphi_j^\alpha = \sum_{\ell=0}^{\alpha} \frac{t^\ell}{\ell!} F_j^\ell,$$

where F_j^ℓ are defined in (3.1.10). We do not explicitly denote the dependence on time, to lighten the notation. Observe that, if $\alpha = \rho$, then, by (3.1.11), $\varphi_j^\alpha = \widehat{f}_j^t$.

We claim that, letting $x^{(\alpha)}(\cdot)$ be the trajectory associated with a control $u \in L^1([0, T], \mathbb{R}^m)$ in system (TD) with $\{\varphi_1^\alpha, \dots, \varphi_m^\alpha\}$ as vector fields, then, for some constant $C > 0$ and any $i \in \{1, \dots, n\}$ and $\alpha \geqslant 1$, it holds

$$|x_i^{(\alpha)}(T) - x_i^{(\alpha-1)}(T)| \leqslant \begin{cases} 0 & \text{if } w_i \leqslant \alpha s, \\ C\varepsilon(\varepsilon + T^{\frac{1}{s}})^{w_i - 1} & \text{if } w_i > \alpha s. \end{cases} \tag{3.1.18}$$

In fact, due to the homogeneity of the F_j^ℓ, proceeding as in the proof of Proposition 3.1.10, we get that for $w_i \leqslant \alpha s$ it holds

$$|\dot{x}_i^{(\alpha)}(t) - \dot{x}_i^{(\alpha-1)}(t)| \leqslant C|u(t)| \sum_{\ell=0}^{\alpha-1} t^\ell \sum_{w_h < w_i} |x_h^{(\alpha)}(t) - x_h^{(\alpha-1)}(t)| \varepsilon^{w_i - w_h - 1}.$$

By induction on $1 \leqslant w_i \leqslant \alpha s$, this proves the first part of the claim. On the other hand, if $w_i > \alpha s$, it holds

$$|\dot{x}_i^{(\alpha)}(t) - \dot{x}_i^{(\alpha-1)}(t)| \leqslant C|u(t)| \bigg(\sum_{\ell=0}^{\alpha-1} t^\ell \sum_{w_h < w_i} |x_h^{(\alpha)}(t) - x_h^{(\alpha-1)}(t)| \varepsilon^{w_i - w_h - 1} + t^\alpha \varepsilon^{w_i - \alpha s - 1} \bigg).$$

Then, again by induction over w_i, we get that $|x_i^{(\alpha)}(T) - x_i^{(\alpha-1)}(T)| \leqslant C T^\alpha \varepsilon^{w_i - \alpha s}$. Finally, the claim follows considering the cases $T \leqslant \varepsilon^s$ and $T > \varepsilon^s$.

Due to the fact that $\varphi_j^0 = \widehat{f}_j$, by Theorem 2.1.7 it holds $|x_i^{(0)}(T)| \leqslant C\varepsilon^{w_i}$. Thus, applying (3.1.18) and enlarging the constant C, we get

$$|z_i(q')| = |x_i^{(r)}(T)| \leqslant \sum_{\ell=1}^{r} \left| x_i^{(\ell)}(T) - x_i^{(\ell-1)}(T) \right| + \left| x_i^{(0)}(T) \right| \leqslant \begin{cases} C\varepsilon^{w_i} & \text{if } w_i \leqslant s, \\ C\varepsilon(\varepsilon + T^{\frac{1}{s}})^{w_i - 1} & \text{if } w_i > s. \end{cases}$$

This proves (3.1.13) and (3.1.14), completing the proof of the theorem. □

We end this section by showing that the estimate (3.1.13) is sharp, at least in some directions. Indeed, for a system which is series homogeneous at q in some privileged coordinates z, and satisfies the hypotheses of Theorem 3.1.9, it holds that $z_*((\text{ad}^k f_0) f_j)$ is an homogeneous vector field of weighted degree $-sk-1$. Thus, since $\varepsilon t^k \leqslant \varepsilon(\varepsilon + t^{\frac{1}{s}})^{sk}$, the following proposition shows that (3.1.13) is sharp in this direction. The proof is an adaption of an argument from [Cor07].

Proposition 3.1.12. *Let (T3) be satisfied, i.e., assume that $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition. Let, moreover $q \in M$, $i \in \{1, \dots, m\}$ and $k \geqslant 0$. Then, for any coordinate system y at q, there exist $T, \varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and $t < T$ there exists a (TD)-admissible curve $\gamma : [0,t] \to M$, with $J(\gamma) \leqslant \varepsilon$, and such that*

$$y(\gamma(t)) = \varepsilon t^k dy\big((ad^k f_0) f_j(q)\big) + O(\varepsilon t^{k+1}) \quad \textit{as } \varepsilon t \to 0.$$

Proof. Let $t, \eta > 0$ be fixed, and define $u \in L^1([0,T], \mathbb{R}^m)$ as $u_i(\tau) \equiv \eta$, $u_j(\tau) \equiv 0$ for $j \neq i$, $\tau \in [0,t]$. Then, fix any $\Phi \in C^k([0,1])$ such that $\Phi^{(i)}(0) = \Phi^{(i)}(1) = 0$, for $0 \leqslant i < k$. Thus, by integrating by parts and the fact that $\frac{d}{dt}(e^{-tf_0})_* g = (e^{-tf_0})_* (\text{ad}(f_0) g)$, we get

$$\int_0^t \Phi^{(k)}(\tau/t)(e^{-\tau f_0})_* f_i(q)\, d\tau = t^k \int_0^t \Phi(\tau/t)(e^{-\tau f_0})_* \left((\text{ad}^k f_0) f_i \right)(q)\, d\tau,$$

for any t and q. This implies that the flows generated by $\Phi^{(k)}(\tau/t)(e^{-\tau f_0})_* f_i$ and $t^k\Phi(\tau/t)(e^{-\tau f_0})_*\left((\mathrm{ad}^k f_0)f_i\right)$ coincide. Using the series expansions of the chronological exponential and $(e^{-tf_0})_*$, see [ABB12a, Section 2.4], it holds

$$\begin{aligned}\overrightarrow{\exp}\int_0^t \sum_{j=1}^m \Phi^{(k)}(\tau/t)u_j(\tau)(e^{-\tau f_0})_* f_j \, d\tau &= \overrightarrow{\exp}\int_0^t \eta\Phi^{(k)}(\tau/t)(e^{-\tau f_0})_* f_i \, d\tau \\ &= \overrightarrow{\exp}\int_0^t \eta t^k\, \Phi(\tau/t)(e^{-\tau f_0})_* \left((\mathrm{ad}^k f_0)f_i\right) \, d\tau \\ &= \overrightarrow{\exp}\int_0^1 \eta t^{k+1}\, \Phi(s)(e^{-tsf_0})_* \left((\mathrm{ad}^k f_0)f_i\right) \, ds \\ &= \overrightarrow{\exp}\int_0^1 \eta t^{k+1}\, \Phi(s) \left((\mathrm{ad}^k f_0)f_i + O(t)\right) \, ds \\ &= \mathrm{Id} + \eta t^{k+1}(\mathrm{ad}^k f_0)f_i + O(\eta t^{k+2})\end{aligned}$$

Finally, considering any coordinate system and letting $\varepsilon = \eta t$, this completes the proof. □

3.2 CONTROL-AFFINE SYSTEMS

In this section we apply the results of Section 3.1 to control-affine systems. Let $\{f_1, \dots, f_m\}$ be a family of vector fields, f_0 be a smooth vector field, called the *drift*, and consider the control-affine system

$$\dot q = f_0(q) + \sum_{i=1}^m u_i\, f_i(q), \qquad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m. \tag{D}$$

Throughout this section we will assume the following.

(D1) The family of smooth vector fields $\{f_0, f_1, \dots, f_m\}$ satisfies the *strong Hörmander condition*, i.e., the family $\{f_1, \dots, f_m\}$ satisfies the Hörmander condition.

An absolutely continuous curve $\gamma : [0, T] \to M$ is (D)-admissible if $\dot\gamma(t) = f_0(\gamma(t)) + f_{u(t)}(\gamma(t))$ for some control $u \in L^1([0,T], \mathbb{R}^m)$. Its cost is defined as

$$J(\gamma) = \min \mathcal{J}(u, T),$$

where the minimum is taken over all controls u such that γ is associated with u. The two value functions we are interested in are

$$\begin{aligned} V_T(q, q') &= \inf\{J(\gamma)\colon\ \gamma : [0, T'] \to M \text{ is (D)-admissible}, \gamma : q \rightsquigarrow q', T' \leqslant T\}, \\ V_\infty(q, q') &= \inf\{J(\gamma)\colon\ \gamma \text{ (D)-admissible and } \gamma : q \rightsquigarrow q'\}. \end{aligned}$$

Here, we omit any reference to the cost under consideration, since we will only consider (3.0.11). It is clear that $V_T(q, q') \searrow V_\infty(q, q')$ as $T \to +\infty$, for any $q, q' \in M$. Moreover, we observe that, $V_T(q, e^{tf_0}q) = 0$ for any $0 \leqslant t \leqslant T$. Finally, the reachable sets with respect to these value functions, from any $q \in M$ and for $\varepsilon, T > 0$, are

$$\mathcal{R}^{f_0}_T(q, \varepsilon) = \{q' \in M\colon\ V_T(q, q') < \varepsilon\}, \quad \mathcal{R}^{f_0}_\infty(q, \varepsilon) = \{q' \in M\colon\ V_\infty(q, q') < \varepsilon\}.$$

3.2.1 Connection with time-dependent systems

Applying the variations formula (see [ABB12a]), system (D) can be written as a composition of a time-dependent system in the form (3.1.7) and of a translation along the drift. Namely, for any $u \in L^1([0,T],\mathbb{R}^m)$, it holds

$$\overrightarrow{\exp}\int_0^T \left(f_0 + \sum_{i=1}^m u_i(t)\, f_i\right) dt = e^{Tf_0} \circ \overrightarrow{\exp}\int_0^T \sum_{i=1}^m u_i(t)\, (e^{-tf_0})_* f_i \, dt. \tag{3.2.1}$$

We call *time-dependent system associated with* (D) the following control system,

$$\dot{q} = \sum_{i=1}^m u_i\, (e^{-tf_0})_* f_i(q), \qquad q \in M, \quad u = (u_1, \dots, u_m) \in \mathbb{R}^m. \tag{3.2.2}$$

Observe that, since diffeomorphisms preserve linear independence, the strong Hörmander condition for (D), implies that $\{(e^{-tf_0})_* f_1, \dots, (e^{-tf_0})_* f_m\}_{t\in[0,+\infty)}$ satisfies (T3), i.e., the strong Hörmander condition for time-dependent systems.

Observe that by the same argument as Lemma 3.1.5 we can prove

Lemma 3.2.1. *For any $\eta > 0$ sufficiently small and for any $q_0, q_1 \in M$, it holds*

$$\inf\{\mathcal{J}(u,\eta) \mid \text{ if } q_u(0) = q_0 \text{ then } q_u(\eta) = q_1\} \leqslant d_{SR}(q_0, q_1).$$

Exploiting these facts, we can prove the following.

Proposition 3.2.2. *Let (D1) be satisfied, i.e., assume that $\{f_0, f_1, \dots, f_m\}$ satisfies the strong Hörmander condition. Then, for any $T > 0$, the functions $V_T, V_\infty : M \times M \to [0, +\infty)$ are continuous. Moreover, letting d_{SR} be the sub-Riemannian distance induced by $\{f_1, \dots, f_m\}$, for any $q, q' \in M$ it holds*

$$V_T(q,q') \leqslant \min_{0\leqslant t\leqslant T} d_{SR}(e^{tf_0}q, q'), \qquad V_\infty(q,q') \leqslant \min_{t\geqslant 0} d_{SR}(e^{tf_0}q, q').$$

Proof. The continuity of the two functions, and the fact that $V_T(q,q'), V_\infty(q,q') \leqslant d_{SR}(q,q')$, for any $q, q' \in M$, follows from the same arguments used in Theorem 3.1.3, adapting Lemmata 3.1.4 and 3.1.5 to the system (D). In particular, one has to consider $(\mathcal{T},\mathcal{F},\xi) \mapsto e^{\mathcal{T}f_0} \circ E_{\mathcal{T},\mathcal{F}}(\xi)$ instead of $(\mathcal{T},\mathcal{F},\xi) \mapsto E_{\mathcal{T},\mathcal{F}}(\xi)$.

To prove the second part of the statement, we let, for any $t \in [0,T)$,

$$\varphi_t(p) = \inf\{J(\gamma) \colon \ \gamma : [t, T'] \to M \text{ is (D)-admissible}, \gamma : p \rightsquigarrow q', \ T' \leqslant T\}.$$

By Lemma 3.2.1 we immediately have $\varphi_t(p) \leqslant d_{SR}(p, q')$. Moreover, we observe that $V_T(q, e^{tf_0}q) = 0$ for any $0 \leqslant t < T$, and hence that for any such t it holds

$$V_T(q,q') \leqslant \varphi_t(e^{tf_0}q) \leqslant d_{SR}(e^{tf_0}q, q').$$

Taking the minimum for $0 \leqslant t < T$, proves the inequality regarding V_T. To complete the proof it suffices to observe that $V_\infty(q,q') \leqslant V_T(q,q')$ for any $T > 0$. □

We point out that in system (D), as in time-dependent systems, the existence of minimizers is not assured. Moreover, this lack of minimizers is possible even if they exist for the associated time-dependent system, as the following example points out.

Example 3.2.3. Consider the following vector fields on $\mathbb{R}^3$, with coordinates (x, y, z),

$$f_1(x, y, z) = \partial_x, \qquad f_2(x, y, z) = \partial_y + x\partial_z.$$

Since $[f_1, f_2] = \partial_z$, $\{f_1, f_2\}$ is a bracket-generating family of vector fields. The sub-Riemannian control system associated with $\{f_1, f_2\}$ on $\mathbb{R}^3$ corresponds to the Heisenberg group.

Let now $f_0 = \partial_z$ be the drift. Since $[f_1, \partial_z] = [f_2, \partial_z] = 0$ it holds that $(e^{-tf_0})_* f_1 = f_1$ and $(e^{-tf_0})_* f_2 = f_2$. Hence, the associated time-dependent system is actually not time-dependent. Thus, by (3.2.1), a curve $\gamma : [0, T] \to \mathbb{R}^3$ is (SR)-admissible for $\{f_1, f_2\}$ if and only if $\tilde{\gamma}(\cdot) = e^{\cdot f_0} \circ \gamma(\cdot)$ is (D)-admissible. As a consequence of this, for any $q \in \mathbb{R}^3$ and any $\varepsilon > 0$,

$$\mathcal{R}^{f_0}_\infty(q, \varepsilon) = \bigcup_{t \geqslant 0} e^{tf_0} \circ B_{SR}(q, \varepsilon). \tag{3.2.3}$$

As already pointed out, minimizers for the sub-Riemannian system exist between any pair of points in $B_{SR}(q, \varepsilon)$, if ε is sufficiently small. Let us show that, for any point in $\mathcal{R}^{f_0}_\infty(q, \varepsilon)$ with positive z coordinate, we have an explicit minimizer, while for the others there exists no minimizer. Without loss of generality we can consider $q = 0$. Then, since $e^{t' f_0}(x', y', z') = (x', y', z' + t')$, every point $(x, y, z) \in \mathcal{R}^{f_0}_\infty(0, \varepsilon)$ with $z > 0$, can be reached optimally considering the sub-Riemannian minimizing curve between the origin and $(x, y, 0)$ rescaled on time z.

If, instead, $z \leqslant 0$, we cannot construct any sub-Riemannian trajectory from 0 to $(x, y, z - t)$, $t > 0$, with cost $\leqslant d_{SR}(0, (x, y, z))$. This is due to the fact that the minimizing trajectories in Heisenberg group are the lifts of arcs on the plane (x, y), spanning area equal to the z coordinates, and that $|z - t| = -z + t > |z|$. Since, by Proposition 3.2.2, $V_\infty(0, (x, y, z)) \leqslant d_{SR}(0, (x, y, z))$, this implies that there exists no minimizer for $V_\infty(q, (x, y, z))$.

3.2.2 Estimates on reachable sets

In this section we apply Theorem 3.1.9, in order to prove Theorem 1.2.3. Fixed a point $q \in M$, we will need the following assumptions, already stated at the beginning of the chapter.

(D2) the point q is *regular for the integral curve of the drift*, i.e., is such that $\dim \Delta^s(e^{tf_0}(q))$, $s \in \mathbb{N}$, is constant for small t;

(D3) the point q is *regular w.r.t. the drift*, in the sense that there exists $s \in \mathbb{N}$ such that $f_0(q') \in \Delta^s(q') \setminus \Delta^{s-1}(q')$, for any q' near q.

By Proposition 2.1.1, (D3) is equivalent to $\operatorname{ord}_{q'} f_0 = -s$ for any q' near q. We remark also that, by (D3) and Proposition 2.1.4, it is always possible to find a system of privileged coordinates at q satisfying the assumptions of Theorem 1.2.3.

Set $s \in \mathbb{N}$ to be such that $\operatorname{ord}_q f_0 = -s$ and define the following sets, for parameters $\eta > 0$ and $T > 0$. We remark that $\mathrm{Box}\,(\eta)$ is defined as in (2.1.3) and that $\{\partial_{z_i}\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$.

$$\begin{aligned}
\Xi_T(\eta) &= \bigcup_{0\leqslant\xi\leqslant T}\Big(\xi\partial_{z_k} + \mathrm{Box}\,(\eta)\Big),\\
\Pi_T(\eta) &= \mathrm{Box}\,(\eta) \cup \bigcup_{0<\xi\leqslant T}\{z\in\mathbb{R}^n\colon\ 0\leqslant z_k-\xi\leqslant\eta^s,\ |z_i|\leqslant\eta^{w_i}+\eta\xi^{\frac{w_i}{s}} \text{ for } w_i\leqslant s, i\neq k,\\
&\qquad\qquad \text{and } |z_i|\leqslant\eta(\eta+\xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\},\\
\widehat{\Pi}_T(\eta) &= \mathrm{Box}\,(\eta) \cup \bigcup_{0<\xi\leqslant T}\{z\in\mathbb{R}^n\colon\ 0\leqslant z_k-\xi\leqslant\eta^s,\ |z_i|\leqslant\eta^{w_i} \text{ for } w_i\leqslant s, i\neq k,\\
&\qquad\qquad \text{and } |z_i|\leqslant\eta(\eta+\xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i > s\}.
\end{aligned}$$

As for Corollary 2.1.10 in the sub-Riemannian case, Theorem 1.2.3 is a direct consequence of some local estimates on the shape of the accessible sets, contained in the following. We cannot expect anything global, since in general the sets $\mathcal{R}^{f_0}_\infty(q,\varepsilon)$ are noncompact.

Theorem 3.2.4. *Let (D1) be satisfied, i.e., assume that $\{f_0, f_1, \dots, f_m\}$ satisfies the strong Hörmander condition, and let $q \in M$ be a point satisfying (D2) and (D3). Assume, moreover, that $z = (z_1, \dots, z_n)$ is a system of privileged coordinates at q for $\{f_1, \dots, f_m\}$, such that $z_* f_0 = \partial_{z_k}$, for some $1 \leqslant k \leqslant n$. Then, there exist $C, \varepsilon_0, T_0 > 0$ such that, setting $s \in \mathbb{N}$ to be such that $f_0(q) \in \Delta^s(q) \setminus \Delta^{s-1}(q)$,*

$$\Xi_T\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}^{f_0}_T(q,\varepsilon) \subset \Pi_T(C\varepsilon), \qquad \textit{for } \varepsilon < \varepsilon_0 \textit{ and } T < T_0. \tag{3.2.4}$$

Here, with abuse of notation, we denoted by $\mathcal{R}^{f_0}_T(q,\varepsilon)$ the coordinate representation of the reachable set. In particular,

$$\mathrm{Box}\left(\frac{1}{C}\varepsilon\right) \cap \{z_k \leqslant 0\} \subset \mathcal{R}^{f_0}_T(q,\varepsilon) \cap \{z_k \leqslant 0\} \subset \mathrm{Box}\,(C\varepsilon) \cap \{z_k \leqslant 0\}.$$

Moreover, if the system is nilpotent, it holds

$$\Xi_T\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}^{f_0}_T(q,\varepsilon) \subset \widehat{\Pi}_T(C\varepsilon), \qquad \textit{for } \varepsilon < \varepsilon_0 \textit{ and } T < T_0. \tag{3.2.5}$$

Example 3.2.5. Consider the sub-Riemannian control system of Example 3.2.3, which is nilpotent and corresponds to the Heisenberg group, and endow it with the drift $f_0(x, y, z) = \partial_z + g(x, y, z)$, where $g(\cdot) \in \mathrm{span}\{f_1(\cdot), f_2(\cdot)\}$. Then, by Proposition 2.1.4, for any g we can find a system of privileged coordinates $z = (z_1, z_2, z_3)$ at $(0, 0, 0)$ such that $z_* f_0 = \partial_{z_3}$. Hence, since $(0, 0, 0)$ satisfies (D2) and (D3), by Theorem 3.2.4, for sufficiently small ε and T, it holds

$$\Xi_T\left(C^{-1}\varepsilon\right) \subset \mathcal{R}^{f_0}_T((0,0,0),\varepsilon) \subset \widehat{\Pi}_T(C\varepsilon),$$

where (see Figure 12),

$$\begin{aligned}
\Xi_T\left(C^{-1}\varepsilon\right) &= \left[-C^{-1}\varepsilon, C^{-1}\varepsilon\right] \times \left[-C^{-1}\varepsilon, C^{-1}\varepsilon\right] \times \left[-\left(T + C^{-1}\varepsilon^2\right), T + C^{-1}\varepsilon^2\right],\\
\widehat{\Pi}_T(C\varepsilon) &= \mathrm{Box}\,(C\varepsilon) \cup \bigcup_{0\leqslant\xi\leqslant T}\left(\left[-\left(C\varepsilon + C\varepsilon\xi^{\frac{1}{2}}\right), C\varepsilon + C\varepsilon\xi^{\frac{1}{2}}\right] \times [0, C\varepsilon^2]\right).
\end{aligned}$$

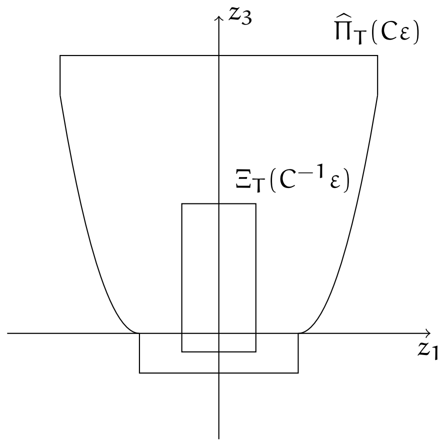

Figure 12: The section at $z_2 = 0$ of the boxes of Example 3.2.5.

In particular, since the coordinates (x, y, z) are priviliged, by (3.2.3) this is true although not at all sharp, when $g \equiv 0$.

In order to prove Theorem 3.2.4, we need the following lemma.

Lemma 3.2.6. *Let (D1) be satisfied, i.e., assume that $\{f_0, f_1, \dots, f_m\}$ satisfies the strong Hörmander condition. Let $z = (z_1, \dots, z_n)$ be a system of privileged coordinates at $q \in M$ and set $s \in \mathbb{N}$ to be such that $\operatorname{ord}_q f_0 = -s$. Then there exist $C, \varepsilon_0, T_0 > 0$ such that, for any $q' \in \mathcal{R}_T^{f_0}(q, \varepsilon_0)$ for $\varepsilon < \varepsilon_0$ and $T < T_0$, and such that*

(i) for any $t < \varepsilon_0$ it holds that $\operatorname{ord}_q f_0 = -s$, where $q'(t) = e^{-tf_0}(q')$,

(ii) $dz_k\big(f_0(z(q'))\big) \neq 0$, for some k with $w_k = -s$,

it holds that, if $u \in L^1([0,T], \mathbb{R}^m)$ is a control steering the system (D) from q to q', with $\|u\|_1 = \varepsilon$, then

$$T \leqslant C\big(\varepsilon^s + \max\{z_k(q'), 0\}\big).$$

Proof. For any $\eta > 0$, let γ be the trajectory associated with $u \in L^1([0,T], \mathbb{R}^m)$ in the system (D), and satisfying $\gamma : q \rightsquigarrow q'$. Let $\tilde{\gamma}$ be the trajectory associated with u and starting from q, in the time-dependent system (3.2.2). Thus $\gamma(t) = e^{tf_0} \circ \tilde{\gamma}(t)$ and $\rho(q, \tilde{\gamma}(T)) \leqslant \varepsilon$.

Recall that, for any vector field g and point $p \in M$, it holds that $z_k(e^{Tg}(p)) - z_k(p) = \int_0^T dz_k\big(g(e^{tg}(p))\big)$. Thus, by the mean value theorem, there exists $\tau \in [0,T]$ such that

$$z_k(q') = z_k(\gamma(T)) = T\, dz_k\big(f_0(e^{\tau f_0}(\tilde{\gamma}(T)))\big) + z_k(\tilde{\gamma}(T)). \tag{3.2.6}$$

Since $e^{\tau f_0}(\tilde{\gamma}(T)) = e^{-(T-\tau)f_0}(q')$, by hypothesis (ii) and the smoothness of f_0, there exist $T_0, C_1 > 0$, independent of γ, such that $dz_k\big(f_0(e^{\tau f_0}(\tilde{\gamma}(T)))\big) \geqslant C_1$ for $T < T_0$. Hence, by Theorem 3.1.9 (since $w_k = s$), there exist $C_2, \bar{\varepsilon} > 0$ such that, if $\varepsilon < \bar{\varepsilon}$ and $T < T_0$,

$$T \leqslant \frac{|z_k(\tilde{\gamma}(T))| + \max\{z_k(q'), 0\}}{C_1} \leqslant \frac{C_2\big(\varepsilon^s + T\varepsilon\big) + \max\{z_k(q'), 0\}}{C_1}.$$

Since the constants are independent of γ, taking $C = C_2/C_1$ and $\varepsilon_0 \leqslant \min\{T_0, \bar{\varepsilon}, (C-1)/C^2\}$ completes the proof. □

Proof of Theorem 1.2.2. The first inclusion in (3.2.4) follows from Theorem 3.2.2, Proposition 2.1.9, and the fact that $z_i(e^{tf_0}(q)) = 0$, if $i \neq k$, and $z_k(e^{tf_0}(q)) = t$. In fact, combining them, we have that, for any $\varepsilon < \varepsilon_0$ and any $T > 0$,

$$\Xi_T\left(\frac{1}{C}\varepsilon\right) \subset \bigcup_{0\leqslant t\leqslant T} B_{SR}(e^{tf_0}q, \varepsilon) \subset \mathcal{R}_T^{f_0}(q, \varepsilon).$$

To prove the second inclusion, we let $q' \in \mathcal{R}_T^{f_0}(q, \varepsilon)$. Fix any $\eta > 0$ and consider a control $u \in L^1([0,\tau], \mathbb{R}^m)$, $\tau \leqslant T$, such that its associated trajectory γ, in the system (D), satisfies $\gamma : q \rightsquigarrow q'$ and $c_{f_0}(\gamma) \leqslant \varepsilon + \eta$. We distinguish two cases. First we assume that $z_k(q') \leqslant 0$. In this case, by Lemma 3.2.6 it follows there exists $C, \varepsilon_0, T_0 > 0$ such that if $\tau < T_0$ and $\varepsilon < \varepsilon_0$, then $\tau \leqslant C\varepsilon^s$. Moreover (3.2.1) implies that $e^{-\tau f_0}(q') \in \mathcal{R}_T(q, \varepsilon)$. Then, enlarging the constant C, Theorem 3.1.9 yields

$$\begin{aligned}
|z_i(q')| &= \left|z_i\left(e^{-\tau f_0}(q')\right)\right| \leqslant C\left(\varepsilon^{w_i} + \varepsilon\tau^{\frac{w_i}{s}}\right) \leqslant C\varepsilon^{w_i}, \quad \text{if } w_i \leqslant s \text{ and } i \neq k,\\
|z_k(q')| &\leqslant \tau + \left|z_k\left(e^{-\tau f_0}(q')\right)\right| \leqslant \tau + C\left(\varepsilon + \tau^{\frac{1}{s}}\right)^s \leqslant C\varepsilon^s,\\
|z_i(q')| &= \left|z_i\left(e^{-\tau f_0}(q')\right)\right| \leqslant C\varepsilon\left(\varepsilon + \tau^{\frac{1}{s}}\right)^{w_i - 1} \leqslant C\varepsilon^{w_i}, \quad \text{if } w_i > s.
\end{aligned}$$

Here, we used the fact that, for any $p \in M$, from $z_* f_0 = \partial_{z_k}$, it holds $z_i(p) = z_i\left(e^{-Tf_0}(p)\right)$ and $|dz_k(f_0(p))| \equiv 1$. Thus, if $T \leqslant T_0$, it holds $q' \subset \mathrm{Box}\,(C\varepsilon) \subset \Pi\,(C\varepsilon)$.

On the other hand, if $z_k(q') > 0$, Lemma 3.2.6 yields that $\tau \leqslant C\left(\varepsilon^s + z_k(q')\right)$. Then, applying again Theorem 3.1.9, we get

$$\begin{aligned}
|z_i(q')| &= \left|z_i\left(e^{-\tau f_0}(q')\right)\right| \leqslant C\left(\varepsilon^{w_i} + \varepsilon\tau^{\frac{w_i}{s}}\right), \quad \text{if } w_i \leqslant s \text{ and } i \neq k,\\
z_k(q') &\leqslant \tau + \left|z_k\left(e^{-\tau f_0}(q')\right)\right| \leqslant \tau + C\left(\varepsilon + \tau^{\frac{1}{s}}\right)^s \leqslant \tau + C\varepsilon^s,\\
|z_i(q')| &= \left|z_i\left(e^{-\tau f_0}(q')\right)\right| \leqslant C\varepsilon\left(\varepsilon + \tau^{\frac{1}{s}}\right)^{w_i - 1}, \quad \text{if } w_i > s.
\end{aligned}$$

Letting $\tau = \xi$, this proves that $q' \subset \Pi\,(C\varepsilon)$, completing the proof of (3.2.4).

To prove (3.2.5) it suffices to use the same argument as above, applying the result on nilpotent systems in Theorem 3.1.9. □

Remark 3.2.7. Theorem 1.2.2 suggests that the behavior of system (D), when moving in the direction $-f_0$, is essentially sub-Riemannian. However, although this is true locally in time, it is false in general. For example, consider the Euclidean plane endowed with a rotational drift, i.e., such that $\{e^{tf_0}(q)\}_{t\in(0,+\infty)}$ is diffeomorphic to $\mathbb{S}^1$ for any $q \neq 0$. Then, $V_\infty(q, e^{-tf_0}(q)) = 0$ for any $t > 0$ and thus we can move in the direction $-f_0$ for free.

Remark 3.2.8. By the arguments used in the proof of [BS90, Theorem 2.2] applied to our setting, it is possible to recover the first inclusion of Theorem 1.2.2. There, a different notion of approximation of control-affine systems is used. Namely, the authors do not consider the associated time-dependent system, preferring to define $\widehat{f}_i$ as a vector field such that $\mathrm{ord}_q(f_i - \widehat{f}_i) \geqslant 0$. In the literature this is sometimes called a first-order approximation. However, such notion does not take into account the role played by the time in control-affine systems, and thus it does not seem well suited to derive the estimates necessary to obtain the second inclusion. We thank the anonymous referee of [Pra14] for bringing this fact to our attention.

Proof of Theorem 1.2.3. Since every norm on $\mathbb{R}^n$ is equivalent, $\mathrm{dist}(z(q'), [0,T]\partial_{z_k})$ is equivalent to

$$a(q') = \sum_{\substack{1\leqslant i\leqslant n\\ i\neq k}} |z_i(q')| + \min_{t\in[0,T]} |z_k(q') - t|.$$

Thus, to complete the proof it suffices to prove that it holds $C^{-1}a(q') \leqslant V_T(q,q') \leqslant Ca(q')^{1/r}$.

By Theorem 3.1.9, $\Xi_T(C^{-1}\varepsilon) \subset \mathcal{R}_T^{f_0}(q,\varepsilon) \subset \Pi_T(C\varepsilon)$ for any $\varepsilon < \varepsilon_0$. The first inclusion is equivalent to the fact that, for every $\varepsilon < \varepsilon_0$ such that $Ca(q') \leqslant \varepsilon^r$, one has $V_T(q,q') \leqslant \varepsilon$. From this follows that $V_T(q,q') \leqslant C^{1/r}a(q')^{1/r}$. The same reasoning applied to the other inclusion proves that

$$|z_i(q')| \leqslant C(V_T(q,q')^{w_i} + V_T(q,q')T^{\frac{w_i}{s}}) \text{ if } w_i \leqslant s,\ i \neq k,$$
$$\min_{t\in[0,T]} |z_k(q') - t| \leqslant CV_T(q,q')^s,$$
$$|z_i(q')| \leqslant C(V_T(q,q')^{w_i} + V_T(q,q')T^{\frac{w_i-1}{s}}) \text{ if } w_i > s.$$

Clearly, this implies that $a(q') \leqslant CV_T(q,q')$, for some larger constant, completing the proof of the theorem. □

4 COMPLEXITY AND MOTION PLANNING

In this chapter we present the contents of the work [JP] regarding asymptotic estimates of the four complexities for control-affine systems introduced in Section 1.2.3. In particular, we prove Theorems 1.2.5 and 1.2.6.

Let us consider the control affine system

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t) f_i(q(t)), \qquad \text{a.e. } t \in [0, T], \tag{4.0.7}$$

where $u : [0, T] \to \mathbb{R}^m$ is an integrable control function and $f_0, f_1, \ldots, f_m$ are (not necessarily linearly independent) smooth vector fields. When posing $f_0 = 0$ in (4.0.7) we obtain the (small) sub-Riemannian control system associated with (4.0.7), i.e., the driftless control system in the form

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t) f_i(q(t)), \qquad \text{a.e. } t \in [0, T], \tag{4.0.8}$$

Our working assumptions will be the following.

(C1) The family $\{f_0, f_1, \ldots, f_m\}$ satisfies the strong Hörmander condition, i.e., the family $\{f_1, \ldots, f_m\}$ satisfies the Hörmander condition and thus defines a sub-Riemannian control system.

(C2) The distribution Δ defined by $\{f_1, \ldots, f_m\}$ is equiregular, i.e., the quantities $\dim \Delta^s(q)$ are independent of q.

(C3) There exists $s \geqslant 2$ such that $f_0(\cdot) \subset \Delta^s(\cdot) \setminus \Delta^{s-1}(\cdot)$.

Observe that these assumptions implies the assumptions (D1)–(D3) made in the previous chapter.

Our first result is the following, already stated in the Introduction as Theorem 1.2.5, that completes the description of the sub-Riemannian weak asymptotic estimates started in Theorem 1.2.4, describing the case of the interpolation by time complexity. It is proved in Section 4.5.

Theorem 4.0.9. *Assume that $\{f_1, \ldots, f_m\}$ defines an equiregular sub-Riemannian structure and let $\gamma : [0, T] \to M$ be a path. Then, if there exists $k \in \mathbb{N}$ such that $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ for any $t \in [0, T]$, it holds*

$$\sigma_{int}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

Here the complexity is measured w.r.t. the cost $\mathcal{J}(u, T) = \|u\|_{L^1([0,T],\mathbb{R}^m)}$.

The main result of this chapter is then following theorem, already stated in the Introduction as Theorem 1.2.6.

Theorem 4.0.10. *Assume that the sub-Riemannian structure defined by* $\{f_1, \dots, f_m\}$ *is equiregular, and that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ *for some* $s \geqslant 2$*. Then, for any curve* $\Gamma \subset M$*, whenever the maximal time of definition of the controls* $\mathcal{T}$ *is sufficiently small, it holds*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{J}}_{app}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{app}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\kappa}}.$$

Here $\kappa = \max\{k\colon\ T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$*, for any* p *in an open subset of* $\Gamma\}$*.*

Moreover, for any path $\gamma : [0, T] \to M$ *such that* $f_0(\gamma(t)) \neq \dot{\gamma}(t) \mod \Delta^{s-1}$ *for any* $t \in [0, T]$*, it holds*

$$\sigma^{\mathcal{J}}_{int}(\gamma, \delta) \asymp \sigma^{\mathcal{I}}_{int}(\gamma, \delta) \asymp \delta^{\frac{1}{\max\{\kappa, s\}}}, \qquad \sigma^{\mathcal{J}}_{app}(\gamma, \varepsilon) \asymp \sigma^{\mathcal{I}}_{app}(\gamma, \varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa, s\}}}.$$

Here $\kappa = \max\{k\colon\ \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ *for any* t *in an open subset of* $[0, T]\}$*.*

The chapter is divided in 5 sections. In Section 4.1 we present some technical results regarding families of coordinates depending continuously on the base point and adapted to the drift. These results will be essential in the sequel. Section 4.2 collects some useful properties of the costs $\mathcal{J}$ and $\mathcal{I}$, proved mainly in [Pra14], while Section 4.3 is devoted to relate the complexities of the control-affine system with those of the associated sub-Riemannian systems, and to prove Theorem 4.0.9. In this section we also prove Proposition 4.3.5, that gives a first result in the direction of Theorem 4.0.10 showing when the sub-Riemannian and control-affine complexities coincide. Finally, the proof of the main result is contained in Sections 4.4 and 4.5, for curves and paths respectively.

4.1 COORDINATE SYSTEMS ADAPTED TO THE DRIFT

We now focus on coordinate systems adapted to the drift. In particular, if for some $s \in \mathbb{N}$ it holds that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, it makes sense to consider the following definition.

Definition 4.1.1. A *privileged coordinate system adapted to* f_0 *at* q is a system of privileged coordinates z at q for $\{f_1, \dots, f_m\}$ such that there exists a coordinate z_ℓ such that $z_* f_0 \equiv \partial_{z_\ell}$.

Observe that completing f_0 to an adapted basis $\{f_1, \dots, f_0, \dots, f_n\}$ allows us to consider the coordinate system adapted to f_0 at q, given by the inverse of the diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_\ell f_0} \circ \dots \circ e^{z_n f_n}(q). \tag{4.1.1}$$

The following definition combines continuous coordinate families for a path $\gamma : [0, T] \to M$, defined in Section 2.2, with coordinate systems adapted to a drift.

Definition 4.1.2. A *continuous coordinate family for* γ *adapted to* f_0 is a continuous coordinate family $\{z^t\}_{t\in[0,T]}$ for γ, such that each z^t is a privileged coordinate system adapted to f_0 at $\gamma(t)$.

Such coordinates systems can be built as per (4.1.1), letting the point q vary on the curve.

Recall that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ for some s, and consider a path $\gamma : [0, T] \to M$ such that $\dot{\gamma}(t) \in \Delta^s(\gamma(t))$ and that $f_0(\gamma(t)) \neq \dot{\gamma}(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in [0, T]$. In this case, there exists $f_\alpha \subset \Delta^s \setminus \Delta^{s-1}$ and two functions $\varphi_\ell, \varphi_\alpha \in C^\infty([0, T])$, $\varphi_\alpha \geqslant 0$, such that

$$\dot{\gamma}(t) \mod \Delta^{s-1}(\gamma(t)) = \varphi_\ell(t) f_0(\gamma(t)) + \varphi_\alpha(t) f_\alpha(\gamma(t)).$$

Moreover, by the assumption $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$, if $\varphi_\ell(t) = 1$ then $\varphi_\alpha(t) > 0$. Then, using f_α as an element of the adapted basis used to define a continuous coordinate family for γ adapted to f_0, it holds $(z_i^t)_*\dot\gamma(t) = \varphi_i(t)$ for $i = \alpha, \ell$ and any $t \in [0,T]$. The following lemma will be essential to study this case.

Lemma 4.1.3. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\gamma : [0,T] \to M$ be a path such that $\dot\gamma(t) \in \Delta^s(\gamma(t))$ and such that $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in [0,T]$. Consider the continuous coordinate family $\{z^t\}_{t\in[0,T]}$ for γ adapted to f_0 defined above. Then, there exist constants $\xi_0, \rho, m > 0$ and a coordinate $\alpha \neq \ell$ of weight s such that for any $t \in [0,T]$ and $0 \leqslant \xi \leqslant \xi_0$, it holds*

$$(z_\ell^t)_*\dot\gamma(t+\xi) \leqslant 1-\rho \quad \text{if } t \in E_1 = \{\varphi_\ell < 1-2\rho\}, \tag{4.1.2}$$
$$(z_\alpha^t)_*\dot\gamma(t+\xi) \geqslant m \quad \text{if } t \in E_2 = \{1-2\rho \leqslant \varphi_\ell \leqslant 1+2\rho\}, \tag{4.1.3}$$
$$(z_\ell^t)_*\dot\gamma(t+\xi) \geqslant 1+\rho \quad \text{if } t \in E_3 = \{\varphi_\ell > 1+2\rho\}. \tag{4.1.4}$$

In particular, it holds that $E_1 \cup E_2 \cup E_3 = [0,T]$.

Proof. Since $\varphi_\alpha > 0$ on $\varphi_\ell^{-1}(1)$, by continuity of φ_ℓ and φ_α there exists $\rho > 0$ such that $\varphi_\alpha > 0$ on $\varphi_\ell^{-1}([1-2\rho, 1+2\rho])$. Since $E_2 = \varphi_\ell^{-1}([1-2\rho, 1+2\rho])$ is closed, letting $2m = \min_{E_2}\varphi_\alpha > 0$ property (4.1.3) follows by the uniform continuity of $(t,\xi) \mapsto (z_\alpha^t)_*\dot\gamma(t+\xi)$ on $E_2 \times [0,\xi_0]$, for sufficiently small ξ_0. Finally, the uniform continuity of $(t,\xi) \mapsto (z_\ell^t)_*\dot\gamma(t+\xi)$ over $\overline{E_1} \times [0,\xi_0]$ and $\overline{E_3} \times [0,\xi_0]$ yields (4.1.2) and (4.1.4). □

We end this section by observing that when the path is well-behaved with respect to the sub-Riemannian structure, it is possible to construct a very special continuous coordinate family, rectifying both γ and f_0 at the same time.

Proposition 4.1.4. *Let $\gamma : [0,T] \to M$ be a path and $k \in \mathbb{N}$ be such that $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ for any $t \in [0,T]$, there exists a continuous coordinate family $\{z^t\}_{[0,T]}$ for γ adapted such that*

1. *there exists a coordinate z_α of weight k such that $z_*^t\dot\gamma \equiv \partial_{z_\alpha}$;*

2. *for any ξ, $t \in [0,T]$ it holds that $z_\alpha^t = z_\alpha^{t-\xi} + \xi$ and $z_i^t = z_i^\xi$ if $i \neq \alpha$.*

Moreover, if there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ and such that $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in [0,T]$ whenever $s = k$, such family can be chosen adapted to f_0.

Proof. By the assumptions on $\dot\gamma$, it is possible to choose $f_\alpha \subset \Delta^k \setminus \Delta^{k-1}$ such that $\dot\gamma(t) = f_\alpha(\gamma(t))$. Let then $\{f_1, \dots, f_n\}$ be the adapted basis obtained by completing f_α and f_0. Finally, to complete the proof it is enough to consider the family of coordinates given by the inverse of the diffeomorphisms

$$(z_1, \dots, z_n) \mapsto e^{z_\ell f_0} \circ \dots \circ e^{z_\alpha f_\alpha}(\gamma(t)).$$

□

4.2 COST FUNCTIONS

In this section we discuss some properties of the cost functions under consideration, namely

$$\mathcal{J}(u,T)=\int_0^T\sqrt{\sum_{i=0}^m u_i(t)^2}\,dt \qquad \text{and} \qquad \mathcal{I}(u,T)=\int_0^T\sqrt{1+\sum_{i=0}^m u_i(t)^2}\,dt,$$

and of the associated value functions. These are respectively denoted by $V^{\mathcal{J}}(\cdot,\cdot)$ and $V^{\mathcal{I}}(\cdot,\cdot)$. For $\mathcal{J}$ such function is defined by

$$V^{\mathcal{J}}(q,q')=\inf\left\{\mathcal{J}(u,T)|\,T>0,\ q_u(0)=q,\ q_u(T)=q'\right\}. \tag{4.2.1}$$

The definition of $V^{\mathcal{I}}$ is analogous.

Most of the properties discussed in this section will be derived from the results obtained in the previous chapter. We remark, however, that in the notation of the previous chapter $V^{\mathcal{J}}=V_{\mathcal{T}}$, i.e., here we are always assuming a maximal bound on the time of definition of the controls.

It is quite easy to extend Proposition 3.2.2 to $\mathcal{I}$, thus obtaining the following.

Theorem 4.2.1. *For any $\mathcal{T}>0$, the functions $V^{\mathcal{J}}$ and $V^{\mathcal{I}}$ are continuous from $M\times M\to[0,+\infty)$ (in particular they are finite). Moreover, for any $q,q'\in M$ it holds*

$$V^{\mathcal{J}}(q,q')\leqslant\min_{0\leqslant t\leqslant\mathcal{T}} d_{SR}(e^{tf_0}q,q'),$$
$$V^{\mathcal{I}}(q,q')\leqslant\min_{0\leqslant t\leqslant\mathcal{T}}\left(t+d_{SR}(e^{tf_0}q,q')\right).$$

Here e^{tf_0} denotes the flow of f_0 at time t and d_{SR} denotes the Carnot-Carathéodory distance w.r.t. the system (4.0.8), obtained from (4.0.7) by putting $f_0=0$.

We notice also that, since Example 3.2.3 is easily extendable to $\mathcal{I}$, it follows that, for neither $\mathcal{J}$ nor $\mathcal{I}$, the existence of minimizers is assured. However, the following proposition assures that a minimizer for $\mathcal{J}$ and $\mathcal{I}$ always exists when moving in the drift direction.

Proposition 4.2.2. *Assume that there exists $s\in\mathbb{N}$ such that $f_0\subset\Delta^s\setminus\Delta^{s-1}$. For any $0<t<\mathcal{T}$, the unique minimizer between any $q_0\in M$ and $e^{tf_0}q_0$ for the cost $\mathcal{J}$ is the null control on $[0,t]$. Moreover, if $f_0\notin\Delta(q_0)$, i.e. $s\geqslant 2$, and the maximal time of definition of the controls $\mathcal{T}$ is sufficiently small, the same is true for $\mathcal{I}$.*

Proof. Since, for $t\in[0,\mathcal{T}]$, we have that $V^{\mathcal{J}}(q,e^{tf_0}q)=0$, the first statement is trivial.

To prove the second part of the statement we proceed by contradiction. Namely, we assume that there exists a sequence $\mathcal{T}_n\longrightarrow 0$ such that for any $n\in\mathbb{N}$ there exists a control $\nu_n\in L^1([0,t_n],\mathbb{R}^m)\subset\mathcal{U}_{\mathcal{T}_n}$, $\nu_n\not\equiv 0$, steering the system from q_0 to $e^{\mathcal{T}_n f_0}(q_0)$ and such that

$$t_n+\|\nu_n\|_{L^1([0,t_n],\mathbb{R}^m)}=\mathcal{I}(\nu_n,t_n)\leqslant\mathcal{I}(0,\mathcal{T}_n)=\mathcal{T}_n. \tag{4.2.2}$$

Let $z=(z_1,\dots,z_n)$ be a privileged coordinate system adapted to f_0 at q, as per Definition 4.1.1. Thus, by Theorem 1.2.2, it holds

$$|z_\ell(e^{\mathcal{T}_n f_0}(q_0))|\leqslant t_n+C\|\nu_n\|^2_{L^1([0,t_n],\mathbb{R}^m)}. \tag{4.2.3}$$

Since $z_\ell(e^{\mathcal{T}_n f_0}(q_0)) = \mathcal{T}_n$, putting together (4.2.2) and (4.2.3) yields $\|v_n\|_{L^1([0,t_n],\mathbb{R}^m)} \leqslant C\|v_n\|^2_{L^1([0,t_n],\mathbb{R}^m)}$ for any $n \in \mathbb{N}$. Since by the continuity of $V^{\mathcal{J}}$ we have that $\|v_n\|_{L^1([0,t_n],\mathbb{R}^m)} \to 0$, this is a contradiction. □

We remark that, in the case of $\mathcal{I}$, the assumption $f_0 \notin \Delta(q_0)$ of Proposition 4.2.2 is essential. In particular, in the following example we show that when $f_0 \subset \Delta$ even if a minimizer between q_0 and $e^{tf_0}(q_0)$ exists, it could not coincide with an integral curve of the drift.

Example 4.2.3. Consider the control-affine system on $\mathbb{R}^2$,

$$\frac{d}{dt}x = f_0(x) + u_1 f_0(x) + u_2 f(x), \tag{4.2.4}$$

where $f_0 = (1,0)$ and $f = (\phi_1, \phi_2)$ for some $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathbb{R}$, with $\phi_2 \neq 0$ and $\partial_x(\phi_1/\phi_2)|_{(0,0)} \neq 0$. Since f_0 and f are always linearly independent, the underlying small sub-Riemannian system is indeed Riemannian with metric

$$g = \begin{pmatrix} 1 & -\phi_1/\phi_2 \\ -\phi_1/\phi_2 & \frac{1-\phi_1^2}{\phi_2^2} \end{pmatrix}.$$

Let us now prove that the curve $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (tT, 0)$ is not a minimizer of the Riemannian distance between $(0,0)$ and $(T,0)$. In particular, it is enough to prove that γ is not a geodesic for small $T > 0$. For γ the geodesic equation writes

$$\begin{cases} t^2\Gamma^1_{11}(\gamma(t)) = 0, \\ t^2\Gamma^2_{11}(\gamma(t)) = 0, \end{cases} \quad \text{for any } t \in [0,1] \qquad \Longleftrightarrow \qquad \Gamma^1_{11}(\cdot,0) = \Gamma^2_{11}(\cdot,0) = 0 \text{ near } 0.$$

Here, $\Gamma^i_{k\ell}$ are the Christoffel numbers of the second kind associated with g. A simple computation shows that

$$\Gamma^1_{11} = \frac{\phi_1}{\phi_2}\partial_{x_1}\left(\frac{\phi_1}{\phi_2}\right), \quad \Gamma^2_{11} = \partial_{x_1}\left(\frac{\phi_1}{\phi_2}\right).$$

Thus, if $\partial_{x_1}(\phi_1/\phi_2)|_{(0,0)} \neq 0$, then $\Gamma^2_{11}(0,0) \neq 0$, showing that γ is not a geodesic.

We now show that this fact implies that for any minimizing sequence $u_n = (u_n^1, u_n^2) \in L^1([0,t_n], \mathbb{R}^2$ for $V^{\mathcal{J}}$ between $(0,0)$ and $e^{Tf_0}((0,0)) = (T,0)$, such that $J(u_{n+1}, t_{n+1}) \leqslant J(u_n, t_n)$, then $u_n^2 \neq 0$ for sufficiently big n. To this aim, fix any $t_n \to 0$, let $u_n(s) = u(s/t_n)$ and $q_n(\cdot)$ be the trajectory associated with u_n in system (4.2.4). Moreover, let $v = (v_1, 0) \in L^1([0,S], \mathbb{R}^2)$ be the minimizer of $\mathcal{I}$ between $(0,0)$ and $(T,0)$ in the system $\dot{x}_1 = 1 + v_1$. Since the trajectory of v is exactly γ, by rescaling it holds $\mathrm{length}(\gamma) = \mathcal{I}(v,S)$. Then, by standard results in the theory of ordinary differential equations, it follows that $q_n(t_n) \to (T,0)$ and the fact that γ is not a Riemannian minimizing curve implies that

$$\|u_n\|_{L^1} = \|u\|_{L^1} < \mathrm{length}(\gamma) = \mathcal{I}(v,S).$$

Hence, for sufficiently big n it holds that $\mathcal{I}(u_n, t_n) < \mathcal{I}(v,S)$, proving the claim.

As a consequence of Proposition 4.2.2, we get the following property for the complexities with respect to the costs $\mathcal{J}$ and $\mathcal{I}$. It generalizes to the control-affine setting the trivial minimality of the sub-Riemannian complexity on the path $\Gamma = \{q\}$.

Corollary 4.2.4. *Assume that there exists $s \geqslant 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $x \in M$ and $y = e^{Tf_0}x$, for some $0 < T < \mathcal{T}$. Then, for any $\varepsilon > 0$, the minimum over all curves $\Gamma \subset M$ (resp. paths $\gamma : [0,T] \to M$) connecting x and y of $\Sigma^{\mathcal{J}}_{int}(\cdot, \varepsilon)$ and $\Sigma^{\mathcal{J}}_{app}(\cdot, \varepsilon)$ (resp. $\sigma^{\mathcal{J}}_{int}(\cdot, \delta)$ and $\sigma^{\mathcal{J}}_{app}(\cdot, \varepsilon)$) is attained at $\Gamma = \{e^{tf_0}\}_{t\in[0,T]}$ (resp. at $\gamma(t) = e^{tf_0}x$). Moreover, the same is true for the cost $\mathcal{I}$, whenever $\mathcal{T}$ is sufficiently small.*

When we consider two points on different integral curves of the drift, it turns out that the two costs $\mathcal{J}$ and $\mathcal{I}$ are indeed equivalent, as proved in the following.

Proposition 4.2.5. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $q, q' \in M$ be such that there exists a set of privileged coordinates adapted to f_0 at q. Then, there exists $C, \varepsilon_0, \mathcal{T} > 0$ such that, for any $u \in \mathcal{U}$ such that, for some $T < \mathcal{T}$, $q_u(T) = q'$ and $\mathcal{J}(u, T) < \varepsilon_0$, it holds*

$$\mathcal{J}(u,T) \leqslant \mathcal{I}(u,T) \leqslant C\mathcal{J}(u,T).$$

The proof of this fact relies on the following particular case of Lemma 3.2.6.

Lemma 4.2.6. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $q \in M$ and let $z = (z_1, \ldots, z_n)$ be a system of privileged coordinate system adapted to f_0 at q. Then, there exist $C, \varepsilon_0, \mathcal{T} > 0$ such that, for any $u \in \mathcal{U}$, with $\mathcal{J}(u,T) < \varepsilon_0$ for some $T < \mathcal{T}$, it holds*

$$T \leqslant C\big(\mathcal{J}(u,T)^s + z_\ell(q_u(T))^+\big).$$

Here, we let $\xi^+ = \max\{\xi, 0\}$.

This Lemma is crucial, since it allows to bound the time of definition of any control through its cost. We now prove Proposition 4.2.5.

Proof of Proposition 4.2.5. The first inequality is trivial. The second one follows by applying Lemma 4.2.6, and computing

$$\mathcal{I}(u,T) \leqslant T + \mathcal{J}(u,T) \leqslant (C\varepsilon_0^{s-1} + 1)\mathcal{J}(u,T).$$

□

4.3 FIRST RESULTS ON COMPLEXITIES

In this section we collect some first results regarding the various complexities we defined.

Firstly, we prove a result on the behavior of complexities. For all the complexities under consideration, except the interpolation by time complexity, such result will hold with respect to a generic cost function $J : \mathcal{U} \to [0, +\infty)$, satisfying some weak hypotheses.

Proposition 4.3.1. *Assume that for any $q_1 \in M$ and any $q_2 \notin \{e^{tf_0}q_1\}_{t\in[0,\mathcal{T}]}$, it holds $V^J(q_1, q_2) > 0$. Then, the following holds.*

i. For any curve $\Gamma \subset M$ it holds the following.

a) If the maximal time of definition of the controls, $\mathcal{T}$, is sufficiently small, then $\lim_{\varepsilon\downarrow 0} \Sigma_{int}(\Gamma, \varepsilon) = \lim_{\varepsilon\downarrow 0} \Sigma_{app}(\Gamma, \varepsilon) = +\infty$.

b) If Γ is an admissible curve for (4.0.7), then $\varepsilon\Sigma_{int}(\Gamma, \varepsilon)$ and $\varepsilon\Sigma_{app}(\Gamma, \varepsilon)$ are bounded from above, for any $\varepsilon > 0$.

ii. For any path $\gamma : [0,T] \to M$ it holds the following.

a) If γ is not a solution of (4.0.7), $\lim_{\varepsilon\downarrow 0} \sigma_{app}(\gamma,\varepsilon) = +\infty$.

b) If the cost is either $\mathcal{J}$ or $\mathcal{I}$, $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, $\dot\gamma(t) \subset \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ and $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in [0,T]$, then $\lim_{\varepsilon\downarrow 0} \sigma_{int}(\gamma,\varepsilon) = +\infty$ whenever $\delta_0 < \eta$.

c) If γ is an admissible curve for (4.0.7), then $\varepsilon\sigma_{int}(\gamma,\varepsilon)$ and $\varepsilon\sigma_{app}(\gamma,\varepsilon)$ are bounded by above, for any $\delta, \varepsilon > 0$.

Proof. The last statement for curves and paths follows simply by considering the control whose trajectory is the curve or the path itself, which is always admissible regardless of ε.

We now prove the first statement for the interpolation by cost complexity of a curve Γ. The same reasonings will hold for Σ_{app} and σ_{app}. Let x, y be the two endpoints of Γ and assume $\mathcal{T}$ to be sufficiently small so that $V^J(x,y) > 0$. Then, the first statement follows from

$$\lim_{\varepsilon\downarrow 0} \Sigma_{int}(\Gamma,\varepsilon) \geqslant V(x,y) \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} = +\infty.$$

Consider now the interpolation by time complexity and proceed by contradiction. Namely, let us assume that there exists a constant $C > 0$ such that $\sigma_{int}(\gamma,\varepsilon) \leqslant C$ for any $\varepsilon > 0$. Then, by definition of σ_{int}, this implies that for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in [CT/2, \delta_0)$ and a δ_ε-time interpolation $u_\varepsilon \in L^1([0,T],\mathbb{R}^m)$ such that $\delta_\varepsilon J(u_\varepsilon, T) \leqslant \varepsilon$.

Firstly, observe that by Lemma 4.2.6 and the assumptions on f_0 and $\dot\gamma$, we obtain that there exist $\eta > 0$ and an interval $I \subset [0,T]$, with $|I| > \eta$, such that

$$V\big(\gamma(t_1^h), \gamma(t_2^h)\big) \longrightarrow 0 \text{ as } h \downarrow 0 \implies t_2^h - t_1^h \longrightarrow 0 \text{ as } h \downarrow 0, \tag{4.3.1}$$

whenever $t_1^h \in I$ and $t_2^h > t_1^h$ for any h in a right neighborhood of zero.

For any ε, let $0 = t_0^\varepsilon < t_1^\varepsilon < \ldots < t_{N_\varepsilon}^\varepsilon = T$ be a partition of $[0,T]$ such that $q_{u_\varepsilon}(t_i^\varepsilon) = \gamma(t_i^\varepsilon)$ for any $i \in \{0, N_\varepsilon\}$ and $t_i^\varepsilon - t_{i-1}^\varepsilon \leqslant \delta_\varepsilon$. It is clear that, up to removing some t_i^ε's, we can assume that $t_i^\varepsilon - t_{i-1}^\varepsilon \geqslant \delta_\varepsilon/2 \geqslant CT/4$. Let us fix, $\tau_1^\varepsilon = t_{i_\varepsilon}^\varepsilon \in I$ for some index i_ε and $\tau_2^\varepsilon = t_{i_\varepsilon+1}^\varepsilon$. Such τ_1^ε always exists, since $|I| > \delta_0$. Since, by the definition of u_ε and the choice of the cost, follows that $V^J(\gamma(\tau_1^\varepsilon), \gamma(\tau_2^\varepsilon)) \to 0$ as $\varepsilon \downarrow 0$ we obtain a contradiction. In fact, this implies that

$$0 = \lim_{\varepsilon\downarrow 0} \left(\tau_2^\varepsilon - \tau_1^\varepsilon\right) \geqslant \frac{CT}{4} > 0.$$

□

Remark 4.3.2. Result ii.b, regarding the interpolation by time complexity, holds for any cost satisfying the assumptions of Proposition 4.3.1, such that for any path γ it holds (4.3.1), and that, for any $u \in L^1([0,T],\mathbb{R}^m)$, there exists a constant such that, if $t_1, t_2 \in [0,T]$, $t_1 < t_2$, then

$$J(u|_{[t_1,t_2]}(\cdot + t_1), t_2 - t_1) \leqslant CJ(u,T).$$

Remark 4.3.3. The bound on δ_0 in Proposition 4.3.1 is essential. For example, consider the cost $\mathcal{J}(u,T) = \|u\|_{L^1([0,T],\mathbb{R}^m)}$, and a curve such that, for some $N \in \mathbb{N}$, it holds $\gamma(jT/N) = e^{j(T/N)f_0}(\gamma(0))$ for any $j = 1, \ldots, T/N$ (see, e.g., Figure 13). In this case, the null control is a (T/N)-time interpolation of γ, with $\mathcal{J}(0,T) = 0$. In particular, if $\delta_0 > T/N$, it holds $\sigma_{int}(\gamma,\varepsilon) \leqslant N$.

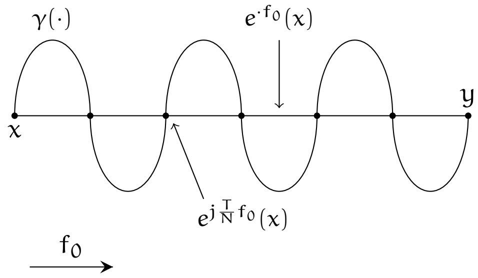

Figure 13: An example of a curve satisfying Remark 4.3.3, with a rectified drift.

In the following, we will denote with an apex "SR-s" – e.g. $\Sigma_{\text{int}}^{\text{SR-s}}$ – the complexities associated with the small sub-Riemannian system (4.0.8) defined at p. 61, and with an apex "SR-b", e.g. $\Sigma_{\text{int}}^{\text{SR-b}}$, the ones associated with the big sub-Riemannian system

$$\dot{q}(t) = \sum_{i=1}^{m} u_i(t)\, f_i(q(t)), \qquad \text{a.e. } t \in [0,T], \tag{4.3.2}$$

We immediately get the following.

Proposition 4.3.4. *Let $\Gamma \subset M$ be a curve and $\gamma : [0,T] \to M$ be a path.*

i. Any complexity relative to the cost $\mathcal{J}$ is smaller than the same complexity relative to $\mathcal{I}$. Namely, for any $\varepsilon, \delta > 0$, it holds

$$\Sigma_{int}^{\mathcal{J}}(\Gamma,\varepsilon) \leqslant \Sigma_{int}^{\mathcal{I}}(\Gamma,\varepsilon), \quad \Sigma_{app}^{\mathcal{J}}(\Gamma,\varepsilon) \leqslant \Sigma_{app}^{\mathcal{I}}(\Gamma,\varepsilon),$$
$$\sigma_{int}^{\mathcal{J}}(\gamma,\varepsilon) \leqslant \sigma_{int}^{\mathcal{I}}(\gamma,\varepsilon), \quad \sigma_{app}^{\mathcal{J}}(\gamma,\varepsilon) \leqslant \sigma_{app}^{\mathcal{I}}(\gamma,\varepsilon).$$

ii. For any cost, the neighboring approximation complexity of some path is always bigger than the tubular approximation complexity of its support. Namely, for any $\gamma : [0,T] \to M$ and any $\varepsilon > 0$, it holds

$$\Sigma_{app}^{\mathcal{J}}(\gamma([0,T]),\varepsilon) \leqslant \sigma_{app}^{\mathcal{J}}(\gamma,\varepsilon), \qquad \Sigma_{app}^{\mathcal{I}}(\gamma([0,T]),\varepsilon) \leqslant \sigma_{app}^{\mathcal{I}}(\gamma,\varepsilon)$$

iii. Any complexity relative to the cost $\mathcal{I}$ is bigger than the same complexity computed for the system (4.3.2). Namely, for any $\varepsilon, \delta > 0$, it holds

$$\Sigma_{int}^{SR\text{-}b}(\Gamma,\varepsilon) \leqslant \Sigma_{int}^{\mathcal{I}}(\Gamma,\varepsilon), \quad \Sigma_{app}^{SR\text{-}b}(\Gamma,\varepsilon) \leqslant \Sigma_{app}^{\mathcal{I}}(\Gamma,\varepsilon),$$
$$\sigma_{int}^{SR\text{-}b}(\gamma,\varepsilon) \leqslant \sigma_{int}^{\mathcal{I}}(\gamma,\varepsilon), \quad \sigma_{app}^{SR\text{-}b}(\gamma,\varepsilon) \leqslant \sigma_{app}^{\mathcal{I}}(\gamma,\varepsilon).$$

iv. In the case of curves, the complexities relative to the cost $\mathcal{I}$ are always smaller than the same complexities computed for the system (4.0.8). Namely, for any $\varepsilon > 0$ it holds

$$\Sigma_{int}^{\mathcal{I}}(\Gamma,\varepsilon) \leqslant \Sigma_{int}^{SR\text{-}s}(\Gamma,\varepsilon), \quad \Sigma_{app}^{\mathcal{I}}(\Gamma,\varepsilon) \leqslant \Sigma_{app}^{SR\text{-}s}(\Gamma,\varepsilon).$$

Proof. The inequality in (ii) is immediate, since any control admissible for the $\sigma_{app}(\gamma, \varepsilon)$ is also admissible for $\Sigma_{app}(\gamma([0, T]), \varepsilon)$.

On the other hand, the inequalities in (iii) between the complexities in (4.3.2) and the ones in (4.0.7), with cost $\mathcal{I}$, is a consequence of the fact that, for every control $u \in \mathcal{U}$, the trajectory q_u is admissible for (4.3.2) and associated with the control $u_0 = (1, u) : [0, T] \to \mathbb{R}^{m+1}$ with $\|u_0\|_{L^1([0,T],\mathbb{R}^{m+1})} = \mathcal{I}(u, T)$. The inequalities in (i) between the complexities in (4.0.7) with respect to the different costs follows from the fact that $\mathcal{J} \leqslant \mathcal{I}$.

Finally, to complete the proof of the proposition, observe that, by Theorem 1.2.1, it holds that

$$V^{\mathcal{I}}(q, q') \leqslant d_{SR}(q, q'), \qquad \text{for any } q,\ q' \in M.$$

This shows, in particular, that every ε-cost interpolation for (4.0.8), is an ε-cost interpolation for (4.0.7), proving the statement regarding the cost interpolation complexity in (iv). The part concerning the tubular approximation follows in the same way. □

We conclude this section by proving an asymptotic equivalence for the complexities of a control-affine system in a very special case. In particular, we will prove that if we cannot generate the direction of Γ with an iterated bracket of f_0 and some $f_1, \ldots, f_m$, then the curve complexities for the systems (4.0.7), (4.0.8) and (4.3.2) behaves in the same way.

Let $\mathcal{L}_{f_0}$ be the ideal of the Lie algebra $\mathrm{Lie}(f_0, f_1, \ldots, f_m)$ generated by the adjoint endomorphism $\mathrm{ad}(f_0) : f \mapsto \mathrm{ad}(f_0)f = [f_0, f]$, $f \in \mathrm{Vec}(M)$. Then the following holds.

Proposition 4.3.5. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, and let $\Gamma \subset M$ be a curve such that there exists $k \in \mathbb{N}$ for which $T\Gamma \subset \Delta^k \setminus \Delta^{k-1}$. Assume, moreover, that for any $q \in \Gamma$ it holds that $T_q\Gamma \not\subset \mathcal{L}_{f_0}(q)$. Then, for sufficiently small $\mathcal{T}$,*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{int}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{J}}_{app}(\Gamma, \varepsilon) \asymp \Sigma^{\mathcal{I}}_{app}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}. \tag{4.3.3}$$

Proof. By the fact that $T_q\Gamma \not\subset \mathcal{L}_{f_0}(q)$, follows that $T_q\Gamma \subset \mathrm{Lie}^k_q(f_0, f_1, \ldots, f_m) \setminus \mathrm{Lie}^{k-1}_q(f_0, f_1, \ldots, f_m)$. Thus, approximating Γ in the big or in the small sub-Riemannian system is equivalent, and by Theorem 1.2.4 follows

$$\Sigma^{\text{SR-s}}_{\text{int}}(\Gamma, \varepsilon) \asymp \Sigma^{\text{SR-b}}_{\text{int}}(\Gamma, \varepsilon) \asymp \Sigma^{\text{SR-s}}_{\text{app}}(\Gamma, \varepsilon) \asymp \Sigma^{\text{SR-b}}_{\text{app}}(\Gamma, \varepsilon) \asymp \frac{1}{\varepsilon^k}.$$

The statement then follows by applying Proposition 4.3.4. □

Remark 4.3.6. Observe that if $f_0 \in \Delta$ in a neighborhood U of Γ, it holds that $\mathrm{Lie}^k_q(f_0, f_1, \ldots, f_m) = \Delta^k(q)$ for any $q \in U$. Then, by the same argument as above, we get that (4.3.3) holds. This shows that, where $f_0 \subset \Delta$, the asymptotic behavior of complexities of curves is the same as in the sub-Riemannian case.

4.4 COMPLEXITY OF CURVES

This section is devoted to prove the statement on curves of Theorem 4.0.10. Namely, we will prove the following.

Theorem 4.4.1. *Assume that there exists* $s \geqslant 2$ *such that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$*. Let* $\Gamma \subset M$ *be a curve and define* $\kappa = \max\{k\colon\ T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$ *for some* $p \in \Gamma\}$*. Then, if the maximal time of definition of the controls* $\mathcal{T}$ *is small enough,*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{I}}_{int}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{J}}_{app}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{I}}_{app}(\Gamma,\varepsilon) \asymp \frac{1}{\varepsilon^{\kappa}},$$

Due to the fact that the value functions associated with the costs $\mathcal{J}$ and $\mathcal{I}$ are always smaller than the sub-Riemannian distance associated with system (4.0.8), the $\preccurlyeq$ immediately follows from the results in [Jea03].

Proposition 4.4.2. *Let* $\Gamma \subset M$ *be a curve such that there exists* $k \in \mathbb{N}$ *for which* $T\Gamma \subset \Delta^k$*. Then,*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma,\varepsilon) \preccurlyeq \Sigma^{\mathcal{I}}_{int}(\Gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{k}}, \qquad \Sigma^{\mathcal{J}}_{app}(\Gamma,\varepsilon) \preccurlyeq \Sigma^{\mathcal{I}}_{app}(\Gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{k}}.$$

Proof. By (i) in Proposition 4.3.4, follows that we only have to prove the upper bound for the complexities relative to the cost $\mathcal{I}$. Moreover, by the same proposition and [Jea03, Theorem 3.14], follows immediately that $\Sigma^{\mathcal{I}}_{int}(\Gamma,\varepsilon)$ and $\Sigma^{\mathcal{I}}_{app}(\Gamma,\varepsilon) \preccurlyeq \varepsilon^{-k}$, completing the proof of the proposition. □

In order to prove $\succcurlyeq$, we will need to exploit a sub-additivity property of the complexities. In order to have this property, it is necessary to exclude certain bad behaving points, called cusps. Near these points, the value function behaves like the Euclidean distance does near algebraic cusps (e.g., $(0,0)$ for the curve $y = \sqrt{|x|}$ in $\mathbb{R}^2$). In the sub-Riemannian context, they have been introduced in [Jea03].

Definition 4.4.3. The point $q \in \Gamma$ is a *cusp* for the cost J if it is not an endpoint of Γ and if, for every $c, \eta > 0$, there exist two points $q_1, q_2 \in \Gamma$ such that q lies between q_1 and q_2, with q_1 before q and q_2 after q w.r.t. the orientation of Γ (in particular $q \neq q_1, q_2$), $V^J(q_1,q_2) \leqslant \eta$ and $V^J(q,q_2) \geqslant c\ V(q_1,q_2)$.

In [Jea03] is proved that no curve has cusps in an equiregular sub-Riemannian stucture. As the following example shows, the equiregularity alone is not enough for control-affine systems.

Example 4.4.4. Consider the following vector fields on $\mathbb{R}^3$, with coordinates (x,y,z),

$$f_1(x,y,z) = \partial_x, \qquad f_2(x,y,z) = \partial_y + x\partial_z.$$

Since $[f_1,f_2] = \partial_z$, $\{f_1,f_2\}$ is a bracket-generating family of vector fields. The sub-Riemannian control system associated with $\{f_1,f_2\}$ on $\mathbb{R}^3$ corresponds to the Heisenberg group.

Let now $f_0 = \partial_z \subset \Delta^2 \setminus \Delta$ be the drift, and let us consider the curve $\Gamma = \{(t^2,0,t) \mid t \in (-\eta,\eta)\}$. Let $q = (0,0,0)$. Since $T_q\Gamma \notin \Delta(q)$, by smoothness of Γ and Δ, for η sufficiently small $T\Gamma \subset \Delta^2 \setminus \Delta$. We now show that the point q is indeed a cusp for the cost $\mathcal{J}$. In fact, for any $\xi > 0$ such that $2\xi < \mathcal{T}$, it holds that the null control defined over time $[0,2\xi]$ steers the control affine system from $q_1 = (\xi^2,0,-\xi) \in \Gamma$ to $q_2 = (\xi^2,0,\xi) \in \Gamma$. Hence, by Proposition 4.2.2, $V^{\mathcal{J}}(q_1,q_2) = 0$. Moreover, since q and q_2 are not on the same integral curve of the drift, $V^{\mathcal{J}}(q,q_2) > 0 = V^{\mathcal{J}}(q_1,q_2)$. This proves that q is a cusp for $\mathcal{J}$.

The following proposition shows that cusps appear only where the drift becomes tangent to the curve at isolated points, as in the above example.

Proposition 4.4.5. *Assume that there exists* $s \geqslant 2$ *such that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. *Let* $\Gamma \subset M$ *be a curve such that* $T\Gamma \subset \Delta^k \setminus \Delta^{k-1}$. *Moreover, if* $s = k$, *let* Γ *be such that either* $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$ *for any* $p \in \Gamma$ *or* $f_0|_\Gamma \subset T\Gamma \oplus \Delta^{s-1}$. *Then* Γ *has no cusps for the cost* $V^{\mathcal{J}}$.

Proof. If $f_0|_\Gamma \subset T_p\Gamma \oplus \Delta^{s-1}(p)$, the statement is a consequence of Proposition 4.2.2. Hence, we assume that $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$ for any $p \in \Gamma$. Let $\gamma : [0, \mathfrak{T}] \to M$ be a path parametrizing Γ and consider the continuous coordinate family $\{z^t\}_{t\in[0,\mathfrak{T}]}$ adapted to f_0 given by Proposition 4.1.4. In particular, it holds that $z^t_*\dot\gamma(\cdot) \equiv \partial_{z_\alpha}$ for some coordinate z_α of weight k and for any $t \in [0, \mathfrak{T}]$. We now fix any $t_0 \in (0, \mathfrak{T})$ and prove that $\gamma(t_0)$ is not a cusp. In fact, letting $\eta > 0$ be sufficiently small, by Theorem 1.2.2 and the fact that $z^t_\ell(\gamma(\cdot)) \equiv 0$ we get

$$\begin{aligned} V^{\mathcal{J}}(\gamma(t_0), \gamma(t_0+\eta)) &\leqslant C \sum_{j=1}^n |z_j^{t_0}(\gamma(t_0+\eta))|^{\frac{1}{w_j}} = C|z_\alpha^{t_0}\gamma(t_0+\eta)|^{\frac{1}{k}} \\ &= 2C|z_\alpha^{t_0-\eta}(\gamma(t_0+\eta))|^{\frac{1}{k}} \leqslant CV(\gamma(t_0-\eta), \gamma(t_0+\eta)). \end{aligned}$$

Letting $t_1 = t_0 - \eta$ and $t_2 = t_0 + \eta$, this proves that $V^{\mathcal{J}}(\gamma(t_0), \gamma(t_2)) \leqslant V^{\mathcal{J}}(\gamma(t_1), \gamma(t_2))$. By definition, this implies that $\gamma(t_0)$ is not a cusp, completing the proof of the proposition. □

Finally, we can prove the sub-additivity of the curve complexities.

Proposition 4.4.6. *Let* $\Gamma' \subset \Gamma \subset M$ *be two curves. Then, if the endpoints of* Γ' *are not cusps for the cost* $\mathcal{J}$, *there exists a constant* $C > 0$ *such that for sufficiently small* $\mathfrak{T}$ *it holds*

$$\Sigma^{\mathcal{J}}_{int}(\Gamma', \varepsilon) \preccurlyeq \Sigma^{\mathcal{J}}_{int}(\Gamma, \varepsilon), \qquad \Sigma^{\mathcal{J}}_{app}(\Gamma', \varepsilon) \preccurlyeq \Sigma^{\mathcal{J}}_{app}(\Gamma, \varepsilon).$$

Proof. Cost interpolation complexity. Let $u \in L^1([0,T], \mathbb{R}^m)$ be a control admissible for $\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma, \varepsilon)$, and let $0 = t_1 < \ldots < t_N = T$ be such that $\|u\|_{L^1([t_{i-1},t_i])} \leqslant \varepsilon$. Recall that by Theorem 1.2.1, $V^{\mathcal{J}}$ is a continuous function. Since for small $\mathfrak{T} > 0$, for any $\varepsilon > 0$ and for any $q_0 \in M$ the reachable set $\mathcal{R}_{\mathfrak{T}}(q, \varepsilon)$ is bounded, it holds that $\mathcal{R}_{\mathfrak{T}}(q, \varepsilon) \searrow \{e^{tf_0}(q_0) \mid t \in [0, \mathfrak{T}]\}$ as $\varepsilon \downarrow 0$, in the sense of pointwise convergence of characteristic functions. From this follows that, for ε and $\mathfrak{T}$ sufficiently small, there exist $i_1 \neq i_2$ such that $q_u(t_i) \in \Gamma'$ for any $i \in \{i_1, \ldots, i_2\}$ and $q_u(t_i) \notin \Gamma'$ for any $i \notin \{i_1, \ldots, i_2\}$. Since x' and y' are not cusps, there exists $c > 0$ such that, letting x' and y' be the endpoints of Γ', it holds $V^{\mathcal{J}}(x', q_u(t_{i_1})) \leqslant c\, V^{\mathcal{J}}(q_u(t_{i_1-1}, q_u(t_{i_1})) \leqslant \varepsilon$ and $V^{\mathcal{J}}(q_u(t_{i_2}), y') \leqslant V^{\mathcal{J}}(q_u(t_{i_2}), q_u(t_{i_2+1})) \leqslant c\varepsilon$. Thus, there exists a constant $C > 0$ such that

$$\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma', \varepsilon) \leqslant \frac{\mathcal{J}(u|_{[t_{i_1},t_{i_2}]})}{\varepsilon} + 2c \leqslant C\frac{\mathcal{J}(u|_{[t_{i_1-1},t_{i_2+1}]})}{\varepsilon} \leqslant C\frac{\mathcal{J}(u)}{\varepsilon}.$$

Taking the infimum over all controls u, admissible for $\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma, \varepsilon)$ completes the proof.

Tubular approximation complexity. Let $u \in L^1([0,T], \mathbb{R}^m)$ be a control admissible for $\Sigma^{\mathcal{J}}_{\mathrm{app}}(\Gamma, \varepsilon)$. Then, letting q_u be its trajectory such that $q_u(0) = x$, there exists two times t_1 and t_2 such that $q_u(t_1) \in B_{SR}(x', C\varepsilon)$ and $q_u(t_2) \in B_{SR}(y', C\varepsilon)$. Then, since $V^{\mathcal{J}} \leqslant d_{SR}$ by Theorem 1.2.1, the same argument as above applies. □

Thanks to the sub-additivity, we can prove the $\succcurlyeq$ part of Theorem 4.4.1 in the case where the curve is always tangent to the same stratum $\Delta^k \setminus \Delta^{k-1}$.

Proposition 4.4.7. *Assume, that there exists* $s \in \mathbb{N}$ *such that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. *Let* $\Gamma \subset M$ *be a curve such that there exists* $k \in \mathbb{N}$ *for which* $T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$ *for any* $p \in \Gamma$. *Then, for sufficiently small time* $\mathfrak{T}$, *it holds*

$$\Sigma_{int}^{\mathfrak{T}}(\Gamma, \varepsilon) \succcurlyeq \Sigma_{int}^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}, \qquad \Sigma_{app}^{\mathfrak{T}}(\Gamma, \varepsilon) \succcurlyeq \Sigma_{app}^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}.$$

Proof. By Proposition 4.3.4, $\Sigma_{\text{int}}^{\mathfrak{T}}(\Gamma, \varepsilon) \succcurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma, \varepsilon)$ and $\Sigma_{\text{app}}^{\mathfrak{T}}(\Gamma, \varepsilon) \succcurlyeq \Sigma_{\text{app}}^{\mathcal{J}}(\Gamma, \varepsilon)$. We will only prove that $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma, \varepsilon) \succcurlyeq \varepsilon^{-k}$, since the same arguments apply to $\Sigma_{\text{app}}^{\mathcal{J}}(\Gamma, \varepsilon)$.

Let $\gamma : [0, \mathfrak{T}] \to M$ be a path parametrizing Γ. We will distinguish three cases.

CASE 1 $f_0(p) \notin \Delta^{s-1}(p) \oplus T_p\Gamma$ **FOR ANY** $p \in \Gamma$ Fix $\eta > 0$ and consider a control $u \in L^1([0, T], \mathbb{R}^m)$, admissible for $\Sigma_{\text{int}}(\Gamma, \varepsilon)$ such that

$$\frac{\|u\|_{L^1}}{\varepsilon} \leqslant \Sigma_{\text{int}}(\Gamma, \varepsilon) + \eta. \tag{4.4.1}$$

Let $u_i = u|_{[t_{i-1}, t_i]}$, $i = 1, \dots, N = \left\lceil \frac{\|u\|_{L^1}}{\varepsilon} \right\rceil$ to be such that $\|u_i\|_{L^1} = \varepsilon$ for any $1 \leqslant i < N$, $\|u_N\|_{L^1} \leqslant \varepsilon$. Moreover, let s_i be the times such that $\gamma(s_i) = q_u(t_i)$.

By (4.4.1), it holds $N \leqslant \lceil \Sigma_{\text{int}}(\Gamma, \varepsilon) + \eta + 1 \rceil$. However, we can assume w.l.o.g. that $N \leqslant \lceil \Sigma_{\text{int}}(\Gamma, \varepsilon) + \eta \rceil$. In fact, $N > \lceil \Sigma_{\text{int}}(\Gamma, \varepsilon) + \eta \rceil$ only if $\|u_N\| < \varepsilon$. In this case we can simply restrict ourselves to compute $\Sigma_{\text{int}}(\tilde{\Gamma}, \varepsilon)$ where $\tilde{\Gamma}$ is the segment of Γ comprised between x and $q_u(t_{N-1})$. Indeed, by Propositions 4.4.5 and 4.4.6, it follows that $\Sigma_{\text{int}}(\tilde{\Gamma}, \varepsilon) \preccurlyeq \Sigma_{\text{int}}(\Gamma, \varepsilon)$.

We now assume that ε and $\mathfrak{T}$ are sufficiently small, in order to satisfy the hypotheses of Theorem 1.2.2 at any point of Γ. Moreover, let $\{z^t\}_{t \in [0, \mathfrak{T}]}$ be the continuous coordinate family for Γ adapted to f_0 given by Proposition 4.1.4. Then, it holds

$$\mathfrak{T} = \sum_{i=1}^{N} (s_i - s_{i-1}) = \sum_{i=1}^{N} |z_\alpha^{s_i - 1}(\gamma(s_i))| = \sum_{i=1}^{N} |z_\alpha^{s_i - 1}(q_u(t_i))| \leqslant C(\Sigma_{\text{int}}(\Gamma, \varepsilon) + \eta)\varepsilon^k. \tag{4.4.2}$$

Here, in the last inequality we applied Theorem 1.2.2 and the fact that $z_\ell^{s_i - 1}(q_u(t_i)) = 0$ by Proposition 4.1.4. Finally, letting $\eta \downarrow 0$ in (4.4.2), we get that for any ε sufficiently small it holds $\Sigma_{\text{int}}(\Gamma, \varepsilon) \geqslant C\mathfrak{T}\,\varepsilon^{-k}$. This completes the proof in this case.

CASE 2 $s = k$ **AND** $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$ **FOR ANY** $p \in \Gamma$ Let $\{z^t\}_{t \in [0, \mathfrak{T}]}$ be a continuous coordinate family for γ adapted to f_0. In this case, since $(z_\ell^t)_* f_0 = 1$, it holds that $(z_\ell^t)_* \dot{\gamma}(\cdot) \neq 0$. Hence, there exist $C_1, C_2 > 0$ such that for any $t, \xi \in [0, T]$

$$C_1(t - \xi) \leqslant z_\ell^t(\gamma(\xi)) \leqslant C_2(t - \xi), \quad \text{if } (z_\ell^t)_* \dot{\gamma}(\cdot) > 0; \tag{4.4.3}$$
$$C_1(t - \xi) \leqslant -z_\ell^t(\gamma(\xi)) \leqslant C_2(t - \xi), \quad \text{if } (z_\ell^t)_* \dot{\gamma}(\cdot) < 0. \tag{4.4.4}$$

If (4.4.4) holds, then we can proceed as in Case 1 with $\alpha = \ell$. In fact, $|z_\ell^{s_{i-1}}(q_u(t_i)| \leqslant C\varepsilon^s$ by Theorem 1.2.2. On the other hand, if (4.4.3) holds, by applying Theorem 1.2.2 we get

$$\mathfrak{T} = \sum_{i=1}^N (s_i - s_{i-1}) \leqslant \frac{1}{C_1} \sum_{i=1}^N |z_\ell^{s_{i-1}}(\gamma(s_i))| = \frac{1}{C_1} \sum_{i=1}^N |z_\ell^{s_{i-1}}(q_u(t_i))|$$
$$\leqslant \frac{1}{C_1} \sum_{i=1}^N (C\varepsilon^s + t_i - t_{i-1}) \leqslant C\big(\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon) + \eta\big)\varepsilon^s + T.$$

By taking $\mathcal{T}$ sufficiently small, it holds $T \leqslant \mathcal{T} < \mathfrak{T}$. Then, letting $\eta \downarrow 0$ this proves that $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon) \geqslant ((T-\mathcal{T})/C)\varepsilon^{-s} \succcurlyeq \varepsilon^{-s}$. This completes the proof of this case.

CASE 3 $s = k$ AND $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$ FOR SOME $p \in \Gamma$ In this case, there exists an open interval $(t_1, t_2) \subset [0, \mathfrak{T}]$ such that $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in (t_1, t_2)$. Thus, $\Gamma' = \gamma((t_1,t_2))$, satisfies the assumption of Case 1 and hence $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \succcurlyeq \varepsilon^{-k}$. Moreover, by Proposition 4.4.5, we can assume that $\gamma(t_1)$ and $\gamma(t_2)$ are not cusps. Then, by Proposition 4.4.6 we get

$$\frac{1}{\varepsilon^k} \preccurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \preccurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon),$$

completing the proof of the proposition.

□

Finally, we are in a condition to prove the main theorem of this section.

Proof of Theorem 4.4.1. Since it is clear that $T\Gamma \subset \Delta^\kappa$, the upper bound follows by Proposition 4.4.2. Moreover, by Proposition 4.3.4 it suffices to prove that $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon)$ and $\Sigma_{\text{app}}^{\mathcal{J}}(\Gamma,\varepsilon) \succcurlyeq \varepsilon^{-\kappa}$. Since the arguments are analogous, we only prove this for $\Sigma_{\text{int}}^{\mathcal{J}}$.

By smoothness of Γ, the set $A = \{p \in \Gamma \mid T_p\Gamma \in \Delta^\kappa(p) \setminus \Delta^{\kappa-1}(p)\}$ has non-empty interior. Let then $\Gamma' \subset A$ be a non-trivial curve such that either $f_0(p) \notin T_p\Gamma' \oplus \Delta^{s-1}(p)$ for any $p \in \Gamma'$ or that $f_0|_{\Gamma'} \subset T\Gamma' \oplus \Delta^{s-1}$. Then, since by Proposition 4.4.5 we can choose Γ' such that it does not contain any cusps, applying Proposition 4.4.6 yields that $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \preccurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon)$. Finally, the result follows from the fact that, by Proposition 4.4.7, it holds $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \succcurlyeq \varepsilon^{-\kappa}$. □

4.5 COMPLEXITY OF PATHS

In this section we will prove the statement on paths of Theorems 4.0.9 and 4.0.10.

Recall the definition of δ-time interpolation given in Section 1.2.3, and define the following function of a path $\gamma : [0,T] \to M$ and a time-step $\delta > 0$

$$\omega(\gamma,\delta) = \delta \inf \big\{ J(u,T) | \, u \text{ is a } \delta\text{-time interpolation of } \gamma \big\}.$$

Controls admissible for the above infimum define trajectories touching γ at intervals of time of length at most δ. Then, function $\omega(\gamma, \delta)$ measures the minimal average cost on each of these intervals. It is possible to express the interpolation by time complexity through ω. Namely,

$$\sigma_{\text{int}}(\gamma, \varepsilon) = \inf_{\delta \leqslant \delta_0} \left\{ \frac{T}{\delta} \,\middle|\, \omega(\gamma, \delta) \leqslant \varepsilon \right\} = \sup_{\delta \leqslant \delta_0} \left\{ \frac{T}{\delta} \,\middle|\, \omega(\gamma, \delta') \geqslant \varepsilon \text{ for any } \delta' \geqslant \delta \right\}. \quad (4.5.1)$$

From (4.5.1) follows immediately that, for any $k \in \mathbb{N}$,

$$\sigma_{\text{int}}(\gamma, \varepsilon) \preccurlyeq \varepsilon^{-k} \iff \omega(\gamma, \delta) \preccurlyeq \delta^{\frac{1}{k}} \quad \text{and} \quad \sigma_{\text{int}}(\gamma, \varepsilon) \succcurlyeq \varepsilon^{-k} \iff \omega(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{k}}. \quad (4.5.2)$$

Exploiting this fact, we are able to prove Theorem 4.0.9.

Proof of Theorem 4.0.9. Let $\{z^t\}_{t \in [0,T]}$ to be the continuous family of coordinates for γ given by Proposition 4.1.4. We start by proving that $\omega(\gamma, \delta) \preccurlyeq \delta^{\frac{1}{k}}$ which, by (4.5.2), will imply $\sigma^{\text{SR-s}}_{\text{int}}(\gamma, \varepsilon) \preccurlyeq \varepsilon^{-k}$. Fix any partition $0 = t_0 < t_1 < \ldots < t_N = T$ such that $\delta/2 \leqslant t_i - t_{i-1} \leqslant \delta$. If δ is sufficiently small, from Theorem 2.1.7 follows that there exists a constant $C > 0$ such that for any $i = 0, \ldots, N$ in the coordinate system z^{t_i} it holds that $\text{Box}(\gamma(t_i), C\delta^{\frac{1}{k}}) \subset B_{SR}(\gamma(t_i), \delta^{\frac{1}{k}})$. Hence, since $z_\alpha^{t_{i-1}}(\gamma(t_i)) = t_i - t_{i-1}$, that $z_j^{t_{i-1}}(\gamma(t_i)) = 0$ for any $j \neq \alpha$, and that $N \leqslant \lceil 2T/\delta \rceil \leqslant CT/\delta$, we get

$$\omega(\gamma, \delta) \leqslant \delta \sum_{i=1}^{N} d_{SR}(\gamma(t_{i-1}), \gamma(t_i)) \leqslant C\delta \sum_{i=1}^{N} \sum_{j=1}^{n} |z_j^{t_{i-1}}(\gamma(t_i))|^{\frac{1}{w_j}} = C\delta \sum_{i=1}^{N} (t_i - t_{i-1})^{\frac{1}{k}} \leqslant CT\delta^{\frac{1}{k}}.$$

This proves completes the proof of the first part of the Theorem.

Conversely, to prove that $\sigma_{\text{int}}(\gamma, \varepsilon) \preccurlyeq \varepsilon^{-k}$ we need to show that $\omega(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{k}}$. To this aim, let $\eta > 0$ and $u \in L^1$ be a control admissible for $\omega(\gamma, \delta)$ such that

$$\|u\|_{L^1([t_{i-1}, t_i])} \leqslant \frac{\omega(\gamma, \delta)}{\delta} + \eta.$$

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be times such that $q_u(t_i) = \gamma(t_i)$, $i = 0, \ldots, N$, $0 < t_i - t_{i-1} \leqslant \delta$. Moreover, let $u_i \in L^1([t_{i-1}, t_i])$ be the restriction of u between t_{i-1} and t_i. Observe that, up to removing some t_i's, we can assume that $t_i - t_{i-1} \in \left(\frac{\delta}{2}, \frac{3}{2}\delta\right]$. This implies that $\lceil 2T/(3\delta) \rceil \leqslant N \leqslant \lceil 2T/\delta \rceil$.

To complete the proof it suffices to show that $\|u_i\|_{L^1([t_{i-1}, t_i])} \geqslant C\delta^{\frac{1}{k}}$. In fact, for any $\eta > 0$, this yields

$$\frac{\omega(\gamma, \delta)}{\delta} \geqslant \|u\|_{L^1([0,T], \mathbb{R}^m)} - \eta = \sum_{i=1}^{N} \|u_i\|_{L^1([t_{i-1}, t_i])} - \eta \geqslant C \sum_{i=1}^{N} \delta^{\frac{1}{k}} - \eta \geqslant C \frac{2T}{3\delta} \delta^{\frac{1}{k}} - \eta.$$

Letting $\eta \downarrow 0$, this will prove that $\omega(\gamma, \delta) \succcurlyeq \delta^{\frac{1}{k}}$, completing the proof.

Observe that, by Theorem 2.1.7, for any $i = 1, \ldots, N$ in the coordinate system $z^{t_{i-1}}$ it holds $B_{SR}(\gamma(t_i), \|u_i\|_{L^1([t_{i-1}, t_i])}) \subset \text{Box}\left(\gamma(t_i), C\|u_i\|_{L^1([t_{i-1}, t_i])}\right)$. Since $z_\alpha^{t_{i-1}}(t_i) = t_i - t_{i-1}$, this implies that

$$\frac{\delta}{2} \leqslant t_i - t_{i-1} = |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leqslant C\, \|u_i\|^k_{L^1([t_{i-1}, t_i])},$$

proving the claim and the theorem. □

The rest of the section will be devoted to the proof of the statement on paths of Theorem 4.0.10. Namely, we will prove the following.

Theorem 4.5.1. *Assume that there exists* $s \geqslant 2$ *such that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. *let* $\gamma : [0,T] \to M$ *be a path such that* $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ *for any* $t \in [0,T]$ *and define* $\kappa = \max\{k\colon \gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ *for any* t *in an open subset of* $[0,T]\}$. *Then, it holds*

$$\sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \asymp \sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \asymp \sigma^{\mathcal{J}}_{app}(\gamma,\varepsilon) \asymp \sigma^{\mathcal{J}}_{app}(\gamma,\varepsilon) \asymp \frac{1}{\varepsilon^{\max\{\kappa,s\}}},$$

where the asymptotic equivalences regarding the interpolation by time complexity are true only when δ_0, *i.e., the maximal time-step in* $\sigma_{int}(\gamma,\varepsilon)$, *is sufficiently small.*

Differently to what happened for curves, the $\preccurlyeq$ part does not immediately follow from the estimates of sub-Riemannian complexities, but requires additional care. It is contained in the following proposition.

Proposition 4.5.2. *Assume that there exists* $s \in \mathbb{N}$ *such that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. *Let* $\gamma : [0,T] \to M$ *be a path such that* $\dot\gamma(t) \in \Delta^k(\gamma(t))$. *Then, it holds*

$$\sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}, \qquad \sigma^{\mathcal{J}}_{app}(\Gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{J}}_{app}(\Gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}. \tag{4.5.3}$$

Proof. By (i) in Proposition 4.3.4, follows that we only have to prove the upper bound for the complexities relative to the cost $\mathcal{J}$. We will start by proving (4.5.3) for $\sigma^{\mathcal{J}}_{\text{int}}$. In particular, by (4.5.2) it will suffices to prove $\omega^{\mathcal{J}}(\gamma,\delta) \preccurlyeq \delta^{\frac{1}{k}}$

Let $\{z^t\}_{t\in[0,T]}$ be a continuous coordinate family for γ adapted to f_0. Let $\tilde\gamma_t(\xi) = e^{-(\xi-t)f_0}(\gamma(\xi))$. Then, since $z^t_* f_0 = \partial_{z_\ell}$, it holds

$$z^t_\ell(\tilde\gamma_t(\xi)) = z^t_\ell(\gamma(\xi)) - (\xi - t), \qquad z^t_i(\tilde\gamma_t(\xi)) = z^t_i(\gamma(\xi)) \quad \text{for any } i \neq \ell. \tag{4.5.4}$$

Fix $\xi > 0$ sufficiently small for Proposition 2.2.2 to hold and choose a partition $0 < t_1 < \ldots < t_N = T$ such that $\delta/2 \leqslant t_i - t_{i-1} \leqslant \delta$. In particular, $N \leqslant \lceil 2T/\delta \rceil$. We then select a control $u \in L^1([0,T],\mathbb{R}^m)$ such that its trajectory q_u in (4.0.7), with $q_u(0) = x$, satisfies $q_u(t_i) = \gamma(t_i)$ for any $i = 1,\ldots,N$ as follows. For each i, we choose $u_i \in L^1([t_{i-1},t_i],\mathbb{R}^m)$ steering system (TD) from $\gamma(t_{i-1}) = \tilde\gamma_{t_{i-1}}(t_{i-1})$ to $\tilde\gamma_{t_{i-1}}(t_i)$. Then, by (3.2.1) and the definition of $\tilde\gamma_{t_{i-1}}$, the control u_i steers system (4.0.7) from $\gamma(t_{i-1})$ to $\gamma(t_i)$.

Since by [Pra14, Theorem 8] it holds $V^{\mathcal{J}}_{TD} \leqslant d_{SR}$, by (4.5.4), Proposition 2.2.2 and Theorem 2.1.7, if δ is sufficiently small we can choose u_i such that there exists $C > 0$ for which

$$\begin{aligned}\mathcal{J}(u_i, t_i - t_{i-1}) &\leqslant C \sum_{j=1}^{n} |z_j^{t_{i-1}}(\tilde\gamma_{t_{i-1}}(t_i))|^{\frac{1}{w_j}} \leqslant C \sum_{j=1}^{n} |z_j^{t_{i-1}}(\gamma(t_i))|^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} \\ &\leqslant C \left(\sum_{w_j \leqslant k} \delta^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} + \sum_{w_j > k} \delta^{\frac{1}{k}} \right) \leqslant C \delta^{\frac{1}{\max\{k,s\}}}.\end{aligned} \tag{4.5.5}$$

Hence, we obtain that

$$\mathcal{J}(u,T) \leqslant N\,\mathcal{J}(u_i, t_i - t_{i-1}) \leqslant 3C\frac{T}{\delta}\delta^{\frac{1}{\max\{k,s\}}}. \tag{4.5.6}$$

Since the control u is admissible for $\omega^{\mathcal{J}}(\gamma,\delta)$, this implies that $\omega^{\mathcal{J}}(\gamma,\delta) \preccurlyeq \delta^{\frac{1}{\max\{k,s\}}}$. This proves the first part of the theorem.

To complete the proof for $\sigma_{app}(\gamma,\varepsilon)$, let $\delta = \varepsilon^{\max\{k,s\}}$. Then, by Theorems 2.1.7 and 1.2.2, there exists a constant $C > 0$ such that $\mathcal{R}^{f_0}_{\delta}(\gamma(t),\varepsilon) \subset B_{SR}(\gamma(t), C\varepsilon)$ for any $t \in [0,T]$. In particular, $d_{SR}(\gamma(t_i), q_u(t)) \leqslant C\varepsilon$ for any $t \in [t_{i-1}, t_i]$. Moreover, again by Theorem 2.1.7, Proposition 2.2.2, and the fact that $\dot\gamma(\cdot) \in \Delta^k(\gamma(\cdot))$, this choice of δ implies also that $d_{SR}(\gamma(t_{i-1}),\gamma(t)) \leqslant C\varepsilon$ for any $t \in [t_{i-1}, t_i]$. Hence, for any $t \in [t_{i-1}, t_i]$, we get

$$d_{SR}(\gamma(t), q_u(t)) \leqslant d_{SR}(\gamma(t_{i-1}), q_u(t)) + d_{SR}(\gamma(t_{i-1}),\gamma(t)) \leqslant 2C\varepsilon.$$

Thus, u is admissible for $\sigma^{\mathcal{J}}_{app}(\gamma, C\varepsilon)$. Finally, from (4.5.6) we get that $\sigma^{\mathcal{J}}_{app}(\gamma, C\varepsilon) \leqslant \varepsilon^{-1}\mathcal{J}(u,T) \leqslant 3CT\varepsilon^{-\max\{k,s\}}$, proving that $\sigma_{app}(\gamma,\varepsilon) \preccurlyeq \varepsilon^{-\max\{k,s\}}$. This completes the proof. □

Now, we prove the $\succcurlyeq$ part of the statement, in the case where $\dot\gamma$ is always contained in the same stratum $\Delta^k \setminus \Delta^{k-1}$.

Proposition 4.5.3. *Assume that there exists $s \geqslant 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\gamma : [0,T] \to M$ be a path, such that $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ for any $t \in [0,T]$. Moreover, if $s = k$, assume that $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}$ for any $t \in [0,T]$. Then, it holds*

$$\sigma^{\mathcal{I}}_{int}(\gamma,\varepsilon) \succcurlyeq \sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}, \qquad \sigma^{\mathcal{I}}_{app}(\gamma,\varepsilon) \succcurlyeq \sigma^{\mathcal{J}}_{app}(\gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}.$$

Proof. By Proposition 4.3.4, $\sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{I}}_{int}(\gamma,\varepsilon)$ and $\sigma^{\mathcal{J}}_{app}(\gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{I}}_{app}(\gamma,\varepsilon)$. Hence, to complete the proof it suffices to prove the asymptotic lower bound for $\sigma^{\mathcal{J}}_{int}(\gamma,\varepsilon)$ and $\sigma^{\mathcal{J}}_{app}(\gamma,\varepsilon)$. In the following, to lighten the notation, we write σ_{int} and σ_{app} instead of $\sigma^{\mathcal{J}}_{int}$ and $\sigma^{\mathcal{J}}_{app}$.

Interpolation by time complexity. By (4.5.2), it suffices to prove that $\omega(\gamma,\delta) \succcurlyeq \delta^{\frac{1}{\max\{k,s\}}}$. Let $\eta > 0$ and $u \in L^1([0,T],\mathbb{R}^m)$ be a control admissible for $\omega(\gamma,\delta)$ such that

$$\mathcal{J}(u,T) = \|u\|_{L^1([0,T],\mathbb{R}^m)} \leqslant \frac{\omega(\gamma,\delta)}{\delta} + \eta. \tag{4.5.7}$$

Let $N = \lceil T/\delta \rceil$ and $0 = t_0 < t_1 < \ldots < t_N = T$ be times such that $q_u(t_i) = \gamma(t_i)$, $i = 0,\ldots,N$, and $0 < t_i - t_{i-1} \leqslant \delta$. Observe that, up to removing some t_i's, we can always assume $\delta/2 \leqslant t_i - t_{i-1} \leqslant (3/2)\delta$ and $N \geqslant \lceil (2T)/(3\delta) \rceil$. Moreover, let $u_i = u|_{[t_{i-1},t_i]}$. Proceding as in the proof of Theorem 4.0.9, p. 74, we get that in order to show that $\omega(\gamma,\delta) \succcurlyeq \delta^{\frac{1}{\max\{k,s\}}}$ it suffices to prove

$$\|u_i\|_{L^1([t_{i-1},t_i])} \geqslant C\delta^{\frac{1}{\max\{s,k\}}}, \qquad i = 1,\ldots,N. \tag{4.5.8}$$

We distinguish three cases.

CASE 1 $k > s$ Let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 4.1.4. Then, since $z^t_\ell(\gamma(\cdot)) = 0$ and $z^t_\alpha(\gamma(\xi)) = \xi - t$, by Theorem 1.2.2 it holds

$$\frac{\delta}{2} \leqslant (t_i - t_{i-1}) = |z^{t_{i-1}}_\alpha(\gamma(t_i))| \leqslant C\|u_i\|^k_{L^1([t_{i-1},t_i])}.$$

This proves (4.5.8).

CASE 2 $k < s$ Also in this case, let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 4.1.4. Then, by Lemma 4.2.6 we get

$$\frac{\delta}{2} \leqslant t_i - t_{i-1} \leqslant C\|u_i\|^s_{L^1([t_{i-1},t_i])},$$

which immediately proves (4.5.8).

CASE 3 $k = s$ Let $\{z^t\}_{t\in[0,T]}$ be a continuous coordinate family for γ adapted to f_0. By the mean value theorem there exists $\xi \in [t_{i-1}, t_i]$ such that

$$z_\ell^{t_{i-1}}(\gamma(t_i)) = \int_{t_{i-1}}^{t_i} (z_\ell^t)_*\dot\gamma(t)\,dt = \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i - t_{i-1}). \tag{4.5.9}$$

Consider the partition $\{E_1, E_2, E_3\}$ of $[0, T]$ given by Lemma 4.1.3 and let $\delta \leqslant \delta_0$. Then, depending to which E_j belongs t_{i-1}, we proceed differently.

(a) $t_{i-1} \in E_1$: By Lemma 4.2.6 and (4.5.9) we get

$$t_i - t_{i-1} \leqslant C\|u_i\|^s_{L^1([t_{i-1},t_i])} + z_\ell^{t_{i-1}}(\gamma(t_i))^+ = C\|u_i\|^s_{L^1([t_{i-1},t_i])} + \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i - t_{i-1}).$$

Then, by (4.1.2) of Lemma 4.1.3, we get

$$\|u_i\|_{L^1([t_{i-1},t_i])} \geqslant \left(\frac{1-(z_\ell^{t_{i-1}})_*\dot\gamma(\xi)}{C}\right)^{\frac{1}{s}} (t_i - t_{i-1})^{\frac{1}{s}} \geqslant \left(\frac{\rho}{C}\right)^{\frac{1}{s}} \delta^{\frac{1}{s}}.$$

This proves (4.5.8).

(b) $t_{i-1} \in E_2$: By (4.1.3) of Lemma 4.1.3, (4.5.9) and Theorem 1.2.2 we get

$$m(t_i - t_{i-1}) \leqslant |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leqslant C\left(\|u_i\|^s_{L^1([t_{i-1},t_i])} + \|u_i\|_{L^1([t_{i-1},t_i])}|z_\ell^{t_{i-1}}(\gamma(t_i))|\right).$$

Reasoning as in (4.5.5) yields that we can assume $\|u_i\|_{L^1([t_{i-1},t_i])} \leqslant C\delta^{\frac{1}{s}}$. Then, by (4.5.9) and letting $\delta \leqslant (m/(2+4\rho))^s$, we get

$$\|u_i\|_{L^1([t_{i-1},t_i])} \geqslant (m - \delta^{\frac{1}{s}}(1+2\rho))^{\frac{1}{s}}(t_i - t_{i-1})^{\frac{1}{s}} \geqslant \left(\frac{m}{2}\right)^{\frac{1}{s}} \delta^{\frac{1}{s}},$$

proving (4.5.8).

(c) $t_{i-1} \in E_3$: By Theorem 1.2.2 it follows that

$$|z_\ell^{t_{i-1}}(\gamma(t_i))| \leqslant C\|u_i\|^s_{L^1([t_{i-1},t_i])} + (t_i - t_{i-1}). \tag{4.5.10}$$

Then, by (4.5.9) and (4.5.10) we obtain

$$\|u_i\|_{L^1([t_{i-1},t_i])} \geqslant \left(\frac{(z_\ell^{t_{i-1}})_*\dot\gamma(\xi) - 1}{C}\right)^{\frac{1}{s}} (t_i - t_{i-1})^{\frac{1}{s}} \geqslant \left(\frac{\rho}{C}\right)^{\frac{1}{s}} \delta^{\frac{1}{s}}.$$

The last inequality follows from (4.1.4) of Lemma 4.1.3. This proves (4.5.8).

Neighboring approximation complexity. Fix $\eta > 0$ and let $u \in L^1([0,T],\mathbb{R}^m)$ be admissible for $\sigma_{app}(\gamma,\varepsilon)$ and such that $\|u\|_{L^1([0,T],\mathbb{R}^m)} \leqslant \sigma_{app}(\gamma,\varepsilon)+\eta$. Let $q_u : [0,T] \to M$ be the trajectory of u with $q_u(0) = \gamma(0)$. Let then $N = \lceil \sigma_{app}(\gamma,\varepsilon)+\eta \rceil$ and $0 = t_0 < t_1 < \ldots < t_N = T$ be such that $\|u\|_{L^1([t_{i-1},t_i])} \leqslant \varepsilon$ for any $i = 1,\ldots,N$. By Lemma 3.2.1 and the fact that $q_u(t) \in B_{SR}(\gamma(t),\varepsilon)$ for any $t \in [0,T]$, we can build a new control, still denoted by u, such that $q_u(t_i) = \gamma(t_i)$, $i = 1,\ldots,N$, and $\|u\|_{L^1([t_{i-1},t_i])} \leqslant 3\varepsilon$.

Fixed a $\delta_0 > 0$, w.l.o.g. we can assume that $t_i - t_{i-1} \leqslant \delta_0$. In fact, we can split each interval $[t_{i-1},t_i]$ not satisfying this property as $t_{i-1} = \xi_1 < \ldots < \xi_M = t_i$, with $\xi_\nu - \xi_{\nu-1} \leqslant \delta_0$. Then, as above, it is possible to modify the control u so that $q_u(\xi_\nu) = \gamma(\xi_\nu)$ for any $\nu = 1,\ldots,M$. Since $M \leqslant \lceil T/\delta_0 \rceil$ and $q_u(\cdot) \in B_{SR}(\gamma(\cdot),\varepsilon)$, we have $\|u\|_{L^1([\xi_i,\xi_{i-1}])} \leqslant 5\varepsilon$ and the new total number of intervals is $\leqslant (1+\lceil T/\delta_0 \rceil)\lceil \sigma_{app}(\gamma,\varepsilon)+\eta \rceil \leqslant C(\sigma_{app}(\gamma,\varepsilon)+\eta)$.

We claim that to prove $\sigma_{app}(\gamma,\varepsilon) \succcurlyeq \varepsilon^{-\max\{s,k\}}$, it suffices to show that there exists a constant $C > 0$, independent of u, such that

$$t_i - t_{i-1} \leqslant C\varepsilon^{\max\{s,k\}}, \qquad \text{for any } i = 1,\ldots,N. \tag{4.5.11}$$

In fact, since $N \leqslant C(\sigma_{app}(\gamma,\varepsilon)+\eta)$, this will imply that

$$T = \sum_{i=1}^{N} t_i - t_{i-1} \leqslant C(\sigma_{app}(\gamma,\varepsilon)+\eta)\varepsilon^{\max\{s,k\}}.$$

Letting $\eta \downarrow 0$, we get that $\sigma_{app}(\gamma,\varepsilon) \succcurlyeq \varepsilon^{-\max\{s,k\}}$, proving the claim.

We now let δ_0 sufficiently small in order to apply Lemma 4.1.3, Theorem 1.2.2, and Lemma 4.2.6. As before, we distinguish three cases.

CASE 1 $k > s$ Let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 4.1.4. By Theorem 1.2.2, using the fact that $\gamma(t_i) = q_u(t_i)$ for $i = 1,\ldots,N$, we get

$$(t_i - t_{i-1}) = |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leqslant C\varepsilon^k. \tag{4.5.12}$$

This proves (4.5.11).

CASE 2 $k < s$ Again, let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 4.1.4. As for the interpolation by time complexity, by Lemma 4.2.6 and the fact that $q_u(t_i) = \gamma(t_i)$, we get

$$(t_i - t_{i-1}) \leqslant C\varepsilon^s,$$

thus proving (4.5.11).

CASE 3 $k = s$ Let $\{z^t\}_{t\in[0,T]}$ to be a continuous coordinate family for γ adapted to f_0. Consider the partition $\{E_1,E_2,E_3\}$ of $[0,T]$ given by Lemma 4.1.3 and recall (4.5.9). We distinguish three cases.

(a) $t_{i-1} \in E_1$: By Lemma 4.2.6 and (4.5.9) we get

$$t_i - t_{i-1} \leqslant C\varepsilon^s + z_\ell^{t_{i-1}}(\gamma(t_i)) = 2C\varepsilon^s + \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i - t_{i-1}).$$

By (4.1.2) of Lemma 4.1.3, this implies

$$t_i - t_{i-1} \leqslant \left(\frac{2C}{1 - (z_\ell^{t_{i-1}})_* \dot\gamma(\xi)} \right) \varepsilon^s \leqslant \frac{2C}{\rho} \varepsilon^s.$$

Hence, (4.5.11) is proved.

(b) $t_{i-1} \in E_2$: By (4.1.3) of Lemma 4.1.3, (4.5.9) and Theorem 1.2.2 we get

$$m(t_i - t_{i-1}) \leqslant |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leqslant C\left(\varepsilon^s + \varepsilon|z_\ell^{t_{i-1}}(\gamma(t_i))|\right) \leqslant C\left(\varepsilon^s + \varepsilon^{s+1} + \varepsilon(t_i - t_{i-1})\right).$$

This, by taking ε sufficiently small and enlarging C, implies (4.5.11).

(c) $t_{i-1} \in E_3$: By Theorem 1.2.2 it follows that

$$|z_\ell^{t_{i-1}}(\gamma(t_i))| \leqslant C\varepsilon^s + (t_i - t_{i-1}). \tag{4.5.13}$$

Then, by (4.5.9) and (4.5.13) we obtain

$$t_i - t_{i-1} \leqslant \frac{C}{(z_\ell^{t_{i-1}})_* \dot\gamma(\xi) - 1} \varepsilon^s \leqslant \frac{C}{\rho} \varepsilon^s.$$

The last inequality follows from (4.1.4) of Lemma 4.1.3, and proves (4.5.11).

□

As for the case of curves, in order to extend Proposition 4.5.3 to paths not always tangent to the same strata, we will need the following sub-additivity property. Let us remark that due to the definition of the path complexities, we do not need to make any assumption regarding cusps.

Proposition 4.5.4. *Let $\gamma : [0, T] \to M$ be a path and let $t_1, t_2 \subset [0, T]$.*

i. If there exists $k \in \mathbb{N}$ such that $\sigma_{int}^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \varepsilon) \succcurlyeq \varepsilon^{-k}$, then $\sigma_{int}^{\mathcal{J}}(\gamma, \varepsilon) \succcurlyeq \varepsilon^{-k}$,

ii. $\sigma_{app}^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \varepsilon) \preccurlyeq \sigma_{app}^{\mathcal{J}}(\gamma, \varepsilon)$.

Proof. Time interpolation complexity. By (4.5.2), it suffices to prove that $\omega^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \delta) \preccurlyeq \omega^{\mathcal{J}}(\gamma, \delta)$. Let $u \in L^1([0, T], \mathbb{R}^m)$ be a control admissible for $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma, \varepsilon)$, and let $0 = \xi_1 < \ldots < \xi_N = T$ be the times where $q_u(\xi_i) = \gamma(\xi_i)$. Let $i_1 \neq i_2$ such that $t_1 \leqslant \xi_i \leqslant t_2$ for any $i \in \{i_1, \ldots, i_2\}$. Observe that, by Theorems 2.1.7 and 1.2.1, we have $V^{\mathcal{J}}(\gamma(t_1), \gamma(\xi_{i_1})) \leqslant d_{SR}(\gamma(t_1), \gamma(\xi_{i_1})) \leqslant C\delta^{\frac{1}{r}}$ and $V^{\mathcal{J}}(\gamma(\xi_{i_2}), \gamma(t_2)) \leqslant d_{SR}(\gamma(\xi_{i_2}), \gamma(t_2)) \leqslant C\delta^{\frac{1}{r}}$, where δ is sufficiently small, C is independent of δ, and r is the nonholonomic degree of the distribution. Thus, assuming w.l.o.g. $C \geqslant 1$,

$$\omega^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \delta) \leqslant \delta\mathcal{J}(u|_{[t_{i_1},t_{i_2}]}) + 2C\delta^{1+\frac{1}{r}} \leqslant C\delta\mathcal{J}(u) + C\delta^{1+\frac{1}{r}}.$$

Taking the infimum over all controls u admissible for $\omega^{\mathcal{J}}(\gamma, \delta)$, and recalling that, by Proposition 4.5.2, it holds $\omega^{\mathcal{J}}(\gamma, \delta) \preccurlyeq \delta^{\frac{1}{r}}$, completes the proof.

Neighboring approximation complexity. In this case, the proof is identical to the one of Proposition 4.4.6 for the tubular approximation complexity. The sole difference is that here, by definition of $\sigma_{\text{app}}^{\mathcal{J}}$, we do not need to assume the absence of cusps. □

We can now complete the proof of Theorem 4.0.10, by proving Theorem 4.5.1.

Proof. The proof is analogous to the one of Theorem 4.4.1, using Propositions 4.5.2, 4.5.3 and 4.5.4. □

Part II

Diffusions on singular manifolds

5 THE LAPLACE-BELTRAMI OPERATOR ON CONIC AND ANTI-CONIC SURFACES

In this chapter we consider the Riemannian metric on $M = \big(\mathbb{R}\setminus\{0\}\big)\times\mathbb{S}$ whose orthonormal basis has the form:

$$X_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2(x,\theta) = \begin{pmatrix} 0 \\ |x|^{\alpha} \end{pmatrix}, \qquad x\in\mathbb{R},\ \theta\in\mathbb{S}^1. \tag{5.0.14}$$

Here $x \in \mathbb{R}$, $\theta \in \mathbb{S}$ and $\alpha \in \mathbb{R}$ is a parameter. In other words we are interested in the Riemannian manifold (M, g), where

$$g = dx^2 + |x|^{-2\alpha} d\theta^2, \text{ i.e., in matrix notation } g = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-2\alpha} \end{pmatrix}. \tag{5.0.15}$$

One of the main features of these metrics is the fact that, except in the case $\alpha = 0$, the corresponding Riemannian volumes have a singularity at $\mathcal{Z} = \{x = 0\}$,

$$d\mu = \sqrt{\det g}\, dx\, d\theta = |x|^{-\alpha} dx\, d\theta.$$

Due to this fact, the corresponding Laplace-Beltrami operators contain some diverging first order terms,

$$\mathcal{L} = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{2} \partial_j \left(\sqrt{\det g}\, g^{jk} \partial_k\right) = \partial_x^2 + |x|^{2\alpha}\partial_\theta^2 u - \frac{\alpha}{x}\partial_x \tag{5.0.16}$$

As already anticipated in the introduction, the purpose of this chapter is to answer the following two questions.

(Q1) Do the heat and free quantum particles flow through the singularity? In other words, we are interested to the following: consider the heat or the Schrödinger equation

$$\partial_t \psi = \mathcal{L}\,\psi, \tag{5.0.17}$$

$$i\partial_t \psi = -\mathcal{L}\,\psi, \tag{5.0.18}$$

where $\mathcal{L}$ is given by (5.0.16). Take an initial condition supported at time $t = 0$ in $M^- = \{x \in M \mid x < 0\}$. Is it possible that at time $t > 0$ the corresponding solution has some support in $M^+ = \{x \in M \mid x > 0\}$?

(Q2) Does equation (5.0.17) conserve the total heat (i.e. the L^1 norm of ψ)? This is known to be equivalent to the stochastic completeness of M_α – i.e., the fact that the stochastic process, defined by the diffusion $\mathcal{L}$, almost surely has infinite lifespan. In particular, we are interested in understanding if the heat is absorbed by the singularity $\mathcal{Z}$.

The same question for the Schrödinger equation has a trivial answer, since the total probability (i.e., the L^2 norm) is always conserved by Stone's theorem.

The answer to (Q1) is related to the self-adjointness of $\mathcal{L}$, and is contained in Theorem 1.3.2, whose statement will be recalled in Theorem 5.2.3. On the other hand, the answer to (Q2) is related to the Markovianity and the stochastic completeness of $\mathcal{L}$, which is strictly related to the theory of Dirichlet forms. The answer to this question is contained in Theorem 1.3.6, whose statement is recalled in Theorem 5.4.17.

The chapter is divided into 4 sections. The first section is devoted to the geometric interpretation of the above defined structures, both from a topological and from a metric point of view. Then, in Section 5.2, after some preliminaries regarding self-adjointness, we analyze in detail the Fourier components of the Laplace-Beltrami operator on M_α, proving Theorems 5.2.3 and 5.2.10. We conclude this section with a description of the maximal domain of the Laplace-Beltrami operator in terms of the Sobolev spaces on M_α, contained in Proposition 5.2.13. In this section we only concern ourselves with real self-adjoint extensions of $\mathcal{L}$. The discussion regarding the complex self-adjoint extension of $\widehat{\mathcal{L}}_0$ is the subject of Section 5.3. Finally, in Section 5.4, we introduce and discuss the concepts of Markovianity, stochastic completeness and recurrence through the potential theory of Dirichlet forms. After this, we study the Markov uniqueness of $\mathcal{L}|_{C_c^\infty(M)}$ and characterize the domains of the Friedrichs, Neumann and bridging extensions (Propositions 5.4.11 and 5.4.12). Then, we define stochastic completeness and recurrence at 0 and at ∞, and, in Proposition 5.4.15, we discuss how these concepts behave if the $k = 0$ Fourier component of the self-adjoint extension is itself self-adjoint. In particular, we show that the Markovianity of such an operator A implies the Markovianity of its first Fourier component $\widehat{A}_0$, and that the stochastic completeness (resp. recurrence) at 0 (resp. at ∞) of A and $\widehat{A}_0$ are equivalent. Then, in Proposition 5.4.14 we prove that stochastic completeness or recurrence are equivalent to stochastically completeness or recurrence both at 0 and at ∞. Finally, we prove Theorem 5.4.17.

5.1 GEOMETRIC INTERPRETATION

In this section, we discuss how the topology and the metric properties of M_α depend on α.

5.1.1 Topology of M_α

Define $M_{\text{cylinder}} = \mathbb{R} \times \mathbb{S}$ and $M_{\text{cone}} = M_{\text{cylinder}} / \sim$, where $(x_1, \theta_1) \sim (x_2, \theta_2)$ if and only if $x_1 = x_2 = 0$. In the following we are going to suitably extend the metric structure to M_{cylinder} through (5.0.14) when $\alpha \geqslant 0$, and to M_{cone} through (5.0.15) when $\alpha < 0$.

Recall that, on a general two dimensional Riemannian manifold for which there exists a global orthonormal frame, the distance between two points can be defined equivalently as

$$d(q_1, q_2) = \inf \Big\{ \int_0^1 \sqrt{u_1(t)^2 + u_2(t)^2}\, dt \mid \gamma : [0,1] \to M \text{ Lipschitz}, \gamma(0) = q_1, \\ \gamma(1) = q_2 \text{ and } u_1, u_2 \text{ are defined by } \dot\gamma(t) = u_1(t) X_1(\gamma(t)) + u_2(t) X_2(\gamma(t)) \Big\}, \tag{5.1.1}$$

$$d(q_1,q_2)=\inf\left\{\int_0^1\sqrt{g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\,dt \mid \gamma:[0,1]\to M \text{ Lipschitz },\right.$$
$$\left.\gamma(0)=q_1,\ \gamma(1)=q_2\right\},\quad (5.1.2)$$

where $\{X_1,X_2\}$ is the global orthonormal frame for (M,g).

Case $\alpha\geqslant 0$

Similarly to what is usually done in sub-Riemannian geometry (see Section 1.1), when $\alpha\geqslant 0$, formula (5.1.1) can be used to define a distance on M_{cylinder} where X_1 and X_2 are given by formula (5.0.14). We have the following.

Lemma 5.1.1. *For any* $\alpha\geqslant 0$*, formula* (5.1.1) *endows* M_{cylinder} *with a metric space structure, which is compatible with its original topology.*

Proof. By (5.1.1), it is clear that $d: M_{\text{cylinder}}\times M_{\text{cylinder}}\to[0,+\infty)$ is symmetric, satisfies the triangular inequality and $d(q,q)=0$ for any $q\in M_{\text{cylinder}}$. Observe that the topology on M_{cylinder} is induced by the distance $d_{\text{cylinder}}((x_1,\theta_1),(x_2,\theta_2))=|x_1-x_2|+|\theta_1-\theta_2|$. Here and henceforth, for any $\theta_1,\ \theta_2\in\mathbb{S}$ when writing $\theta_1-\theta_2$ we mean the non-negative number $\theta_1-\theta_2 \mod 2\pi$. In order to complete the proof it suffices to show that for any $\{q_n\}_{n\in\mathbb{N}}\subset M_{\text{cylinder}}$ and $\bar q\in M_{\text{cylinder}}$ it holds

$$d(q_n,\bar q)\longrightarrow 0 \text{ if and only if } d_{\text{cylinder}}(q_n,\bar q)\longrightarrow 0. \quad (5.1.3)$$

In fact, this clearly implies that if $d(q_1,q_2)=0$ then $q_1=q_2$, proving that d is a distance, and moreover proves that d and d_{cylinder} induce the same topology.

Assume that $d(q_n,\bar q)\to 0$ for some $\{q_n\}_{n\in\mathbb{N}}\subset M_{\text{cylinder}}$ and $\bar q=(\bar x,\bar\theta)\in M_{\text{cylinder}}$. In this case, for any $n\in\mathbb{N}$ there exists a control $u_n:[0,1]\to\mathbb{R}^2$ such that $\|u_n\|_{L^1([0,1],\mathbb{R}^2)}\to 0$ and that the associated trajectory $\gamma_n(\cdot)=(x_n(\cdot),\theta_n(\cdot))$ satisfies $\gamma_n(0)=q_n$ and $\gamma_n(1)=\bar q$. This implies that, for any $t\in[0,1]$

$$|x_n(t)-\bar x|\leqslant\int_0^t|u_1(t)|\,dt\leqslant\|u_n\|_{L^1([0,1],\mathbb{R}^2)}\longrightarrow 0.$$

Hence, $x_n(t)\longrightarrow\bar x$. In particular, this implies that $|x_n(t)|\leqslant\|u_n\|_{L^1([0,1],\mathbb{R}^2)}+|\bar x|$ for any $t\in[0,1]$, and hence

$$\begin{aligned}|\theta_n(0)-\bar\theta|\leqslant\int_0^1|u_2(t)||x_n(t)|^\alpha\,dt&\leqslant\left(\|u_n\|_{L^1([0,1],\mathbb{R}^2)}+|\bar x|\right)^\alpha\int_0^1|u_2(t)|\,dt\\&\leqslant\|u_n\|_{L^1([0,1],\mathbb{R}^2)}(\|u_n\|_{L^1([0,1],\mathbb{R}^2)}+|\bar x|)^\alpha\longrightarrow 0.\end{aligned}$$

Here, when taking the limit, we exploited the fact that $\alpha\geqslant 0$. Thus also $\theta_n(0)\longrightarrow\bar\theta$, and hence $q_n=(x_n(0),\theta_n(0))\longrightarrow(\bar x,\bar\theta)=\bar q$ w.r.t. d_{cylinder}.

In order to complete the proof of (5.1.3), we now assume that for some $q_n=(x_n,\theta_n)$ and $\bar q=(\bar x,\bar\theta)$ it holds $d_{\text{cylinder}}(q_n,\bar q)\longrightarrow 0$ and claim that $d(q_n,\bar q)\longrightarrow 0$. We start by considering the case $\bar q\notin\mathcal{Z}$, and w.l.o.g. we assume $\bar q\in M^+$. Since M^+ is open with respect to d_{cylinder}, we may assume that $q_n\in M^+$. Consider now the controls

$$u_n(t)=\begin{cases}2\,(\bar x-x_n)\,(1,0) & \text{if } 0\leqslant t\leqslant\frac12,\\ 2\,(\bar\theta-\theta_n)|\bar x|^{-\alpha}\,(0,1) & \text{if } \frac12<t\leqslant 1,\end{cases}$$

A simple computation shows that each u_n steers the system from q_n to $\bar{q}$. The claim then follows from

$$d(q_n, \bar{q}) \leqslant \|u_n\|_{L^1([0,1],\mathbb{R}^2)} \leqslant |\bar{x} - x_n| + |\bar{\theta} - \theta_n||\bar{x}|^{-\alpha} \leqslant (1 + |\bar{x}|^{-\alpha})\, d_{\text{cylinder}}(q_n, \bar{q}) \longrightarrow 0.$$

Let now $\bar{q} \in \mathcal{Z}$ and observe that w.l.o.g. we can assume $q_n \notin \mathcal{Z}$ for any $n \in \mathbb{N}$. In fact, if this is not the case it suffices to consider $\tilde{q}_n = (x_n + 1/n, \theta_n) \notin \mathcal{Z}$, observe that $d(q_n, \tilde{q}_n) \to 0$ and apply the triangular inequality. Then, we consider the following controls, steering the system from q_n to $\bar{q}$,

$$v_n(t) = \begin{cases} 3\left((\bar{\theta} - \theta_n)^{1/2\alpha} - x_n\right)(1,0) & \text{if } 0 \leqslant t \leqslant \frac{1}{3}, \\ 3(\bar{\theta} - \theta_n)^{1/2}(0,1) & \text{if } \frac{1}{3} < t \leqslant \frac{2}{3}, \\ -3(\bar{\theta} - \theta_n)^{1/2\alpha}(1,0) & \text{if } \frac{2}{3} < t \leqslant 1. \end{cases}$$

Since $\bar{x} = 0$ and $\alpha \geqslant 0$, we have

$$d(q_n, \bar{q}) \leqslant \|v_n\|_{L^1([0,1],\mathbb{R}^2)} \leqslant |(\theta_n - \bar{\theta})^{1/2\alpha} - x_n| + |\bar{\theta} - \theta_n|^{1/2} + |\theta_n - \bar{\theta}|^{1/2\alpha} \longrightarrow 0.$$

This proves (5.1.3) and hence the lemma. □

Case $\alpha \geqslant 0$

In this case X_1 and X_2 are not well defined in $x = 0$. However, to extend the metric structure, one can use formula (5.1.2), where g is given by (5.0.15). Notice that this metric identifies points on $\{x = 0\}$, in the sense that they are at zero distance. Hence, formula (5.1.2) gives a structure of a well-defined metric space not to M_{cylinder} but to M_{cone}. Indeed, we have the following (for the proof see Appendix 5.1.1).

Lemma 5.1.2. *For $\alpha < 0$, formula (5.1.2) endows M_{cone} with a metric space structure, which is compatible with its original topology.*

Proof. By (5.1.2), it is clear that $d : M_{\text{cone}} \times M_{\text{cone}} \to [0, +\infty)$ is symmetric, satisfies the triangular inequality and $d(q, q) = 0$ for any $q \in M_{\text{cone}}$.

Observe that the topology on M_{cone} is induced by the following metric

$$d_{\text{cone}}((x_1, \theta_1), (x_2, \theta_2)) = \begin{cases} |x_1 - x_2| + |\theta_1 - \theta_2| & \text{if } x_1 x_2 > 0, \\ |x_1 - x_2| & \text{if } x_1 = 0 \text{ or } x_2 = 0, \\ |x_1 - x_2| + |\theta_1| + |\theta_2| & \text{if } x_1 x_2 < 0. \end{cases}$$

By symmetry, to show the equivalence of the topologies induced by d and by d_{cone}, it is enough to show that the two distances are equivalent on $[0, +\infty) \times \mathbb{S}$. Moreover, since by definition of g it is clear that $d(q_1, q_2) = 0$ for any $q_1, q_2 \in \mathcal{Z}$ and that d is equivalent to the Euclidean metric on $(0, +\infty) \times \mathbb{S}$, we only have to show that for any $\{q_n\} \subset (0, +\infty) \times \mathbb{S}$, $q_n = (x_n, \theta_n)$, and $\bar{q} = (0, \bar{\theta}) \in \mathcal{Z}$, it holds that

$$d(q_n, \bar{q}) \longrightarrow 0 \text{ if and only if } d_{\text{cone}}(q_n, \bar{q}) \longrightarrow 0. \tag{5.1.4}$$

We start by assuming that $d(q_n, \bar{q}) \longrightarrow 0$. Then, there exists $\gamma_n : [0,1] \to M$ such that $\gamma_n(0) = q_n$ and $\gamma_n(1) = \bar{q}$ and $\int_0^1 \sqrt{g(\dot{\gamma}_n(t), \dot{\gamma}_n(t))}\, dt \longrightarrow 0$. This implies that

$$|x_n| \leqslant \int_0^1 \sqrt{g(\dot{\gamma}_n(t), \dot{\gamma}_n(t))}\, dt \longrightarrow 0,$$

and thus that $x_n \longrightarrow 0$. This suffices to prove that $d_{\text{cone}}(q_n, \bar{q}) \longrightarrow 0$.

On the other hand, if $d_{\text{cone}}(q_n, \bar{q}) \longrightarrow 0$, it suffices to consider the curves

$$\gamma_n(t) = \begin{cases} ((1-2t)x_n, \theta_n) & \text{if } 0 \leqslant t < \frac{1}{2}, \\ (0, \theta_n + (2t-1)(\bar{\theta} - \theta_n)) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly γ_n is Lipschitz and $\gamma_n(0) = q_n$ and $\gamma_n(1) = \bar{q}$. Finally, since $g|_{\mathcal{Z}} = 0$, the proof is completed by

$$d(q_n, \bar{q}) \leqslant \int_0^1 \sqrt{g_{\gamma_n(t)}(\dot{\gamma}_n(t), \dot{\gamma}_n(t))}\, dt = \int_0^{\frac{1}{2}} \sqrt{g_{\gamma_n(t)}((-2x_n, 0), (-2x_n, 0))}\, dt = x_n \longrightarrow 0.$$

□

5.1.2 Surfaces of revolution

For $\alpha \in \mathbb{R}$, we call M_α the generalized Riemannian manifold obtained in the previous section. Namely,

- $\alpha \geqslant 0$: $M_\alpha = M_{\text{cylinder}}$ and metric structure induced by (5.0.14);
- $\alpha < 0$: $M_\alpha = M_{\text{cone}}$ and metric structure induced by (5.0.15).

The corresponding metric space is called (M_α, d). Moreover, with abuse of notation, we call $\mathcal{Z}$ the singular set, i.e.,

$$\mathcal{Z} = \begin{cases} \{0\} \times \mathbb{S}, & \alpha \geqslant 0, \\ \{0\} \times \mathbb{S}/\sim & \alpha < 0. \end{cases}$$

The singularity splits the manifold M_α in two sides $M^+ = (0, +\infty) \times \mathbb{S}$ and $M^- = (-\infty, 0) \times \mathbb{S}$.

Notice that in the cases $\alpha = 1, 2, 3, \ldots$, M_α is an almost Riemannian structure in the sense of Section 1.3.1, while in the cases $\alpha = -1, -2, -3, \ldots$ it corresponds to a singular Riemannian manifold with a semi-definite metric.

Remark 5.1.3. The curvature of M_α is given by $K_\alpha(x) = -\alpha(1+\alpha)x^{-2}$. Notice that M_α and M_β with $\beta = -(\alpha+1)$ have the same curvature for any $\alpha \in \mathbb{R}$. For instance, the cylinder with Grushin metric has the same curvature as the cone corresponding to $\alpha = -2$, but they are not isometric even locally (see [BCG13]).

This section is devoted to precise the following geometric interpretation of M_α (see Figure 14). For $\alpha = 0$, this metric is that of a cylinder. For $\alpha = -1$, it is the metric of a flat cone in polar coordinates. For $\alpha < -1$, it is isometric to a surface of revolution $\mathcal{S} = \{(t, r(t)\cos\vartheta, r(t)\sin\vartheta) \mid t > 0, \vartheta \in \mathbb{S}\} \subset \mathbb{R}^3$ with profile $r(t) \sim |t|^{-\alpha}$ as $|t|$ goes to zero. For $\alpha > -1$ ($\alpha \neq 0$) it can be thought as a surface of revolution having a profile of the type $r(t) \sim |t|^{-\alpha}$ as $t \to 0$, but this is only formal, since the embedding in $\mathbb{R}^3$ is deeply singular at $t = 0$. The case $\alpha = 1$ corresponds to the Grushin metric on the cylinder.

Let us first recall the definition of isometric (generalized) Riemannian manifolds.

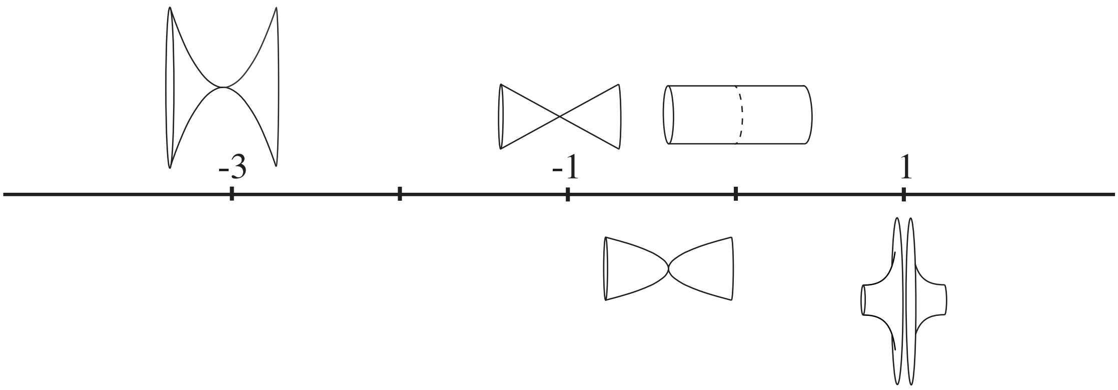

Figure 14: Geometric interpretation of M_α. The figures above the line are actually isometric to M_α, while for the ones below the isometry is singular in $\mathcal{Z}$.

Definition 5.1.4. Given two manifolds M and N, endowed with two (possibly semi-definite) metrics g^M and g^N, we say that M is C^1-*isometric* to N if and only if there exists a C^1-diffeomorphism $\Phi : M \to N$ such that $\Phi^* g_N = g_M$. Here Φ^* is the pullback of Φ. Recall that, in matrix notation, for any $q \in M$ it holds

$$(\Phi^* g^N)_q(\xi, \eta) = (J_\Phi)^T \, g^M_{\Phi(q)} J_\Phi(\xi, \eta). \tag{5.1.5}$$

Here J_Φ is the Jacobian matrix of Φ.

We have the following.

Proposition 5.1.5. *If $\alpha < -1$ the manifold M_α is C^1-isometric to a surface of revolution $\mathcal{S} = \{(t, r(t)\cos\vartheta, r(t)\sin\vartheta) \mid t \in \mathbb{R}, \vartheta \in \mathbb{S}\} \subset \mathbb{R}^3$ with profile $r(t) = |t|^{-\alpha} + O(t^{-2(\alpha+1)})$ as $|t| \downarrow 0$ (see figure 15), endowed with the metric induced by the embedding in $\mathbb{R}^3$.*

If $\alpha = -1$, M_α is globally C^1-isometric to the surface of revolution with profile $r(t) = t$, endowed with the metric induced by the embedding in $\mathbb{R}^3$.

Proof. For any $r \in C^1(\mathbb{R})$, consider the surface of revolution $\mathcal{S} = \{(t, r(t)\cos\vartheta, r(t)\sin\vartheta) \mid t > 0, \vartheta \in \mathbb{S}\} \subset \mathbb{R}^3$. By standard formulae of calculus, we can calculate the corresponding (continuous) semi-definite Riemannian metric on $\mathcal{S}$ in coordinates $(t, \vartheta) \in \mathbb{R} \times \mathbb{S}$ to be

$$g_{\mathcal{S}}(t, \vartheta) = \begin{pmatrix} 1 + r'(t)^2 & 0 \\ 0 & r^2(t) \end{pmatrix}.$$

Let now $\alpha < -1$ and consider the C^1 diffeomorphism $\Phi : (x, \theta) \in \mathbb{R} \times \mathbb{S} \mapsto (t(x), \vartheta(\theta)) \in \mathcal{S}$ defined as the inverse of

$$\Phi^{-1}(t, \vartheta) = \begin{pmatrix} x(t) \\ \theta(\vartheta) \end{pmatrix} = \begin{pmatrix} \int_0^t \sqrt{1 + r'(s)^2} ds \\ \vartheta \end{pmatrix}. \tag{5.1.6}$$

Observe that Φ is well defined due to the fact that r' is bounded near 0. Since $\partial_t(\Phi^{-1}) = \partial_t x(t) = \sqrt{1 + r'(t)^2}$, by (5.1.5) the metric is transformed in

$$\Phi^* g_{\mathcal{S}}(x, \theta) = \left(J_\Phi^{-1}\right)^T g_{\mathcal{S}}(\Phi(x, \theta)) J_\Phi^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r\big(\Phi(x, \theta)\big)^2 \end{pmatrix}.$$

We now claim that there exists a function $r \in C^1(\mathbb{R})$ such that $r(t(x)) = |x|^{-\alpha}$ near $\{x = 0\}$. Moreover, this function has expression

$$r(t) = \begin{cases} t^{-\alpha} + O(t^{-2(\alpha+1)}), & \text{if } t \geqslant 0, \\ -(-t)^{-\alpha} + O(t^{-2(\alpha+1)}) & \text{if } t < 0. \end{cases} \tag{5.1.7}$$

Notice that, this function generates the same surface of revolution as $r(t) = |t|^{-\alpha} + O(t^{-2(\alpha+1)})$, but is of class C^1 in 0 while the latter is not.

The fact that $r(t(x)) = |x|^{-\alpha}$ is equivalent to $r(t) = |x(t)|^{-\alpha}$, i.e.,

$$r(t) = \left(\int_0^t \sqrt{1 + r'(s)^2}\, ds \right)^{-\alpha}. \tag{5.1.8}$$

This integral equation has a unique solution. Indeed, after algebraic manipulation and a differentiation, it is equivalent to the Cauchy problem

$$\begin{cases} r'(t) = \sqrt{\dfrac{1}{\alpha^{-2} r(t)^{-2(1+1/\alpha)} - 1}}, \\ r(0) = 0. \end{cases} \tag{5.1.9}$$

It is easy to check that, thanks to the assumption $\alpha < -1$, the r.h.s. of the ODE is Hölder continuous of parameter $1 + 1/\alpha$ at 0 (but not Lipschitz). This guarantees the existence of a solution, but not its unicity. Indeed, this equation admits two kinds of solutions, either $r_1(t) \equiv 0$ or $r_2(t) \not\equiv 0$, where the transition between $r_1(t)$ and $r_2(t - t_0)$ can happen at any $t_0 \geqslant 0$. However, the only admissible solution of (5.1.8) is r_2, as can be directly checked.

We now prove the representation (5.1.7). Assume w.l.o.g. that t, and hence $x(t)$, be positive. Due to the Hölder continuity of the r.h.s. of the ODE in (5.1.9), we get that $|r'(t)| \leqslant Ct^{1+1/\alpha}$. Hence,

$$|x'(t) - x'(0)| = |\sqrt{1 + r'(t)^2} - 1| \leqslant |r'(t)| \leqslant Ct^{1+1/\alpha}.$$

Here, we used the 1/2-Hölder property of the square root. Finally, a simple computation shows that $|x(t) - tx'(0)| = O(t^{2+1/\alpha})$, which yields

$$r(t) = (x(t))^{-\alpha} = \left(t + O(t^{2+1/\alpha}) \right)^{-\alpha} = t^{-\alpha} + O(t^{-(2+1/\alpha)(\alpha+1)}) = t^{-\alpha} + O(t^{-2(\alpha+1)}).$$

Here, in the last step we used the fact that $-(2 + 1/\alpha)(\alpha + 1) < -2(\alpha + 1)$. This proves the claim and thus the first part of the statement.

Let now $\alpha = -1$. In this case, by letting $r(t) = t$, the metric on the surface of revolution is

$$g_{\mathcal{S}}(t, \vartheta) = \begin{pmatrix} 2 & 0 \\ 0 & t^2 \end{pmatrix}.$$

Consider the diffeomorphism $\Psi : (x, \theta) \in \mathbb{R} \times \mathbb{S} \mapsto (t, \vartheta) \in \mathcal{S}$ defined as

$$\Psi(x, \theta) = \sqrt{2} \begin{pmatrix} x \\ \theta \end{pmatrix}. \tag{5.1.10}$$

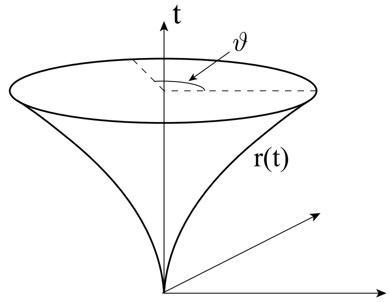

Figure 15: The surface of revolution of Proposition 5.1.5 with $\alpha = -2$, i.e. $r(t) = t^2$.

Then the statement follows from the following computation,

$$\Phi^* g_{\mathcal{S}}(x,\theta) = \left(J_\Psi^{-1}\right)^T g_{\mathcal{S}}(\Psi(x,\theta))J_\Psi^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & r\big(\Psi(x,\theta)\big)^2/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}.$$

□

Remark 5.1.6. If $\alpha > -1$ we cannot have a result like the above, since the change of variables (5.1.6) is no more regular. In fact, the function $r(t) = t^{-\alpha}$ has an unbounded first derivative near 0.

5.2 SELF-ADJOINT EXTENSIONS

In this section we prove Theorem 5.2.3, characterizing the self-adjointness of $\mathcal{L}$ on $L^2(M, d\mu)$, and present some results on its possible self-adjoint extensions. In particular, we will show when it is possible for a free particle to cross the singularity, thus answering (Q1).

5.2.1 Preliminaries

Let $\mathcal{H}$ be an Hilbert space with scalar product $(\cdot,\cdot)_\mathcal{H}$ and norm $\|\cdot\|_\mathcal{H} = \sqrt{(\cdot,\cdot)_\mathcal{H}}$. Given an operator A on $\mathcal{H}$ we will denote its domain by $D(A)$ and its adjoint by A^*. Namely, if A is densely defined, $D(A^*)$ is the set of $\varphi \in \mathcal{H}$ such that there exists $\eta \in \mathcal{H}$ with $(A\psi,\varphi)_\mathcal{H} = (\psi,\eta)_\mathcal{H}$, for all $\psi \in D(A)$. For each such φ, we define $A^*\varphi = \eta$.

An operator A is symmetric if

$$(A\psi,\varphi)_\mathcal{H} = (\psi, A\varphi)_\mathcal{H}, \qquad \text{for all } \psi \in D(A).$$

A densely defined operator A is *self-adjoint* if and only if it is symmetric and $D(A) = D(A^*)$, and is *non-positive* if and only if $(A\psi,\psi) \leqslant 0$ for any $\psi \in D(A)$.

Given a strongly continuous group $\{T_t\}_{t\in\mathbb{R}}$ (resp. semigroup $\{T_t\}_{t\geqslant 0}$), its generator A is defined as

$$Au = \lim_{t\to 0} \frac{T_t u - u}{t}, \quad D(A) = \{u \in \mathcal{H} \mid Au \text{ exists as a strong limit}\}.$$

When a group (resp. semigroup) has generator A, we will write it as $\{e^{tA}\}_{t\in\mathbb{R}}$ (resp. $\{e^{tA}\}_{t\geqslant 0}$). Then, by definition, $u(t) = e^{tA}u_0$ is the solution of the functional equation

$$\begin{cases} \partial_t u(t) = Au(t) \\ u(0) = u_0 \in \mathcal{H}. \end{cases}$$

Recall the following classical result.

Theorem 5.2.1. *Let $\mathcal{H}$ be an Hilbert space, then*

1. (Stone's theorem)*The map $A \mapsto \{e^{itA}\}_{t\in\mathbb{R}}$ induces a one-to-one correspondence*

$$A \text{ self-adjoint operator } \iff \{e^{itA}\}_{t\in\mathbb{R}} \text{ strongly continuous unitary group;}$$

2. *The map $A \mapsto \{e^{tA}\}_{t\geqslant 0}$ induces a one-to-one correspondence*

$$A \text{ non-positive self-adjoint operator } \iff \{e^{tA}\}_{t\geqslant 0} \text{ strongly continuous semigroup;}$$

For any Riemannian manifold $\mathcal{M}$ with measure dV, via the Green identity follows that $\mathcal{L}\,|_{C_c^\infty(\mathcal{M})}$ is symmetric. However, from the same formula, follows that

$$D(\mathcal{L}\,|_{C_c^\infty(\mathcal{M})}{}^*) = \{u \in L^2(\mathcal{M}, dV) \mid \mathcal{L}\,u \in L^2(\mathcal{M}, dV)\} \supsetneq C_c^\infty(\mathcal{M}),$$

where $\mathcal{L}\,u$ is intended in the sense of distributions. Hence, $\mathcal{L}$ is not self-adjoint on $C_c^\infty(\mathcal{M})$.

Since, by Theorem 5.2.1, in order to have a well defined solution of the Schrödinger equation the Laplace-Beltrami operator has to be self-adjoint, we have to extend its domain in order to satisfy this property. For the heat equation, on the other hand, we will need also to worry about the fact that it stays non-positive while doing so. We will tackle this problem in the next section, where we will require the stronger property of being Markovian (i.e., that the evolution preserves both the non-negativity and the boundedness).

Mathematically speaking, given two operators A, B, we say that B is an extension of A (and we will write $A \subset B$) if $D(A) \subset D(B)$ and $A\psi = B\psi$ for any $\psi \in D(A)$. The simplest extension one can build starting from A is the closure $\bar{A}$. Namely, $D(\bar{A})$ is the closure of $D(A)$ with respect to the graph norm $\|\cdot\|_A = \|A\cdot\|_{\mathcal{H}} + \|\cdot\|_{\mathcal{H}}$, and $\bar{A}\psi = \lim_{n\to+\infty} A\psi_n$ where $\{\psi_n\}_{n\in\mathbb{N}} \subset D(A)$ is such that $\psi_n \to \psi$ in $\mathcal{H}$. Since if A is symmetric $A \subset \bar{A} \subset A^*$, any self-adjoint extension B of A will be such that $\bar{A} \subset B \subset A^*$. For this reason, we let $D_{min}(A) = D(\bar{A})$ and $D_{max}(A) = D(A^*)$. Moreover, from this fact follows that any self-adjoint extension B will be defined as $B\psi = A^*\psi$ for $\psi \in D(B)$, so we are only concerned in specifying the domain of B. The simplest case is the following.

Definition 5.2.2. The densely defined operator A is *essentially self-adjoint* if its closure $\bar{A}$ is self-adjoint.

It is a well known fact, dating as far back as the series of papers [Gaf54, Gaf55], that the Laplace-Beltrami operator is essentially self-adjoint on any complete Riemannian manifold. On the other hand, it is clear that if the manifold is incomplete this is no more the case, in general (see [Mas05, Gri09]). It suffices, for example, to consider the case of an open set $\Omega \subset \mathbb{R}^n$, where to have the self-adjointness of the Laplacian, we have to pose boundary conditions (Dirichlet, Neumann or a mixture of the two). In our case, Theorem 5.2.3 will give an answer to the problem of whether $\mathcal{L}\,|_{C_c^\infty(M)}$ is essentially self-adjoint or not.

The problem of determining the self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$ on $L^2(M, d\mu)$ has been widely studied in different fields. A lot of work has been done in the case $\alpha = -1$, in the setting of Riemannian manifolds with conical singularities (see e.g., [Che80, Moo96]), and the same methods have been applied in the more general context of metric cusps or horns (see e.g., [Che79, Brü96]) that covers the case $\alpha < -1$. See also [LP98]. Concerning $\alpha > -1$, on the other hand, the literature regarding Δ is more thin (see e.g., [Mor]).

In the following we will consider only the real self-adjoint extensions, i.e., all the function spaces taken into consideration are composed of real-valued functions. The complex case will be discussed in Section 5.3.

Applying the definition of minimal and maximal domain we immediately obtain that

$$D_{\min}(\mathcal{L}|_{C_c^\infty(M)}) = \text{closure of } C_c^\infty(M) \text{ with respect to the norm}$$
$$\|\Delta\cdot\|_{L^2(M_\alpha,d\mu)} + \|\cdot\|_{L^2(M_\alpha,d\mu)}\},$$
$$D_{\max}(\mathcal{L}|_{C_c^\infty(M)}) = \{u \in L^2(M_\alpha, d\mu) : \Delta u \in L^2(M_\alpha, d\mu) \text{ in the sense of distributions}\}.$$

Recall that the Riemannian gradient is given by $\nabla u(x,\theta) = (\partial_x u(x,\theta), |x|^{2\alpha}\partial_\theta u(x,\theta))$. Following [Gri09], we let the Sobolev spaces on the Riemannian manifold M endowed with measure $d\mu$ be

$$H^1(M, d\mu) = \{u \in L^2(M, d\mu) : |\nabla u| \in L^2(M, d\mu)\}, \quad H_0^1(M, d\mu) = \text{closure of } C_c^\infty(M) \text{ in } H^1(M, d\mu),$$
$$H^2(M, d\mu) = \{u \in H^1(M, d\mu) : \Delta u \in L^2(M, d\mu)\}, \quad H_0^2(M, d\mu) = H^2(M, d\mu) \cap H_0^1(M, d\mu).$$

We define the Sobolev spaces $H^1(M_\alpha, d\mu)$ and $H^2(M_\alpha, d\mu)$ in the same way. We remark that with these conventions $H_0^2(M, d\mu)$ is in general bigger than the closure of $C_c^\infty(M)$ in $H^2(M, d\mu)$. Moreover, it may happen that $H^1(M, d\mu) = H_0^1(M, d\mu)$. Indeed this property will play an important role in the next section. In Proposition 5.2.13, is contained a description of $D_{\max}(\mathcal{L}|_{C_c^\infty(M)})$ in terms of these Sobolev spaces.

Although in general the structure of the self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$ can be very complicated, the Friedrichs (or Dirichlet) extension $\mathcal{L}_F$, is always well defined and self-adjoint. Namely,

$$D(\mathcal{L}_F) = H_0^2(M, d\mu).$$

Observe that, since $L^2(M, d\mu) = L^2(M^+, d\mu) \oplus L^2(M^-, d\mu)$ and $H_0^1(M, d\mu) = H_0^1(M^+, d\mu) \oplus H_0^1(M^-, d\mu)$, it follows that

$$D(\mathcal{L}_F) = \{u \in H_0^1(M^+, d\mu) \mid \Delta u \in L^2(M^+, d\mu)\} \oplus \{u \in H_0^1(M^-, d\mu) \mid \Delta u \in L^2(M^-, d\mu)\}.$$

This implies that $\mathcal{L}_F$ actually defines two separate dynamics on M^+ and on M^- and, hence, there is no hope for an initial datum concentrated in M^+ to pass to M^-, and vice versa. Thus, if $\mathcal{L}|_{C_c^\infty(M)}$ is essentially self-adjoint (i.e., the only self-adjoint extension is $\mathcal{L}_F$) the question (Q1) has a negative answer.

5.2.2 Fourier decomposition and self-adjoint extensions of Sturm-Liouville operators

The rotational symmetry of M_α suggests to proceed by a Fourier decomposition in the θ variable, through the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}} \subset L^2(\mathbb{S})$. Thus, we decompose the space

$L^2(M, d\mu) = \bigoplus_{k=0}^{\infty} H_k$, $H_k \cong L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, and the corresponding operators on each H_k will be

$$\widehat{\mathcal{L}}_k = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}k^2. \tag{5.2.1}$$

The aim of this section is to prove Theorem 1.3.2, that we recall here.

Theorem 5.2.3. *Consider M_α for $\alpha \in \mathbb{R}$ and the corresponding Laplace-Beltrami operator $\mathcal{L}$ as an unbounded operator on $L^2(M, d\mu)$. Then the following holds.*

- *If $\alpha \leqslant -3$ then $\mathcal{L}$ is essentially self-adjoint;*
- *if $\alpha \in (-3, -1]$, only the first Fourier component $\widehat{\mathcal{L}}_0$ is not essentially self-adjoint;*
- *if $\alpha \in (-1, 1)$, all the Fourier components of $\mathcal{L}$ are not essentially self-adjoint;*
- *if $\alpha \geqslant 1$ then $\mathcal{L}$ is essentially self-adjoint.*

There exist various theories allowing to classify the self-adjoint extensions of symmetric operators. We will use some tools from the Neumann theory (see [RS75]) and, when dealing with one-dimensional problems, from the Sturm-Liouville theory. Let $\mathcal{H}$ be a complex Hilbert space and i be the imaginary unit. The *deficiency indexes* of A are then defined as

$$n_+(A) = \dim\ker(A + i), \qquad n_-(A) = \dim\ker(A - i).$$

Then A admits self-adjoint extensions if and only if $n_+(A) = n_-(A)$, and they are in one to one correspondence with the set of partial isometries between $\ker(A - i)$ and $\ker(A + i)$. Obviously, A is essentially self-adjoint if and only if $n_+(A) = n_-(A) = 0$.

Following [Zet05], we say that a self-adjoint extension B of A in $\mathcal{H}$ is a *real self-adjoint extension* if $u \in D(B)$ implies that $\overline{u} \in D(B)$ and $B(\overline{u}) = \overline{Bu}$. When $\mathcal{H} = L^2(M, d\mu)$, i.e. the real Hilbert space of square-integrable real-valued functions on M, the self-adjoint extensions of A in $L^2(M, d\mu)$ are the restrictions to this space of the real self-adjoint extensions of A in $L^2_{\mathbb{C}}(M, d\mu)$, i.e. the complex Hilbert space of square-integrable complex-valued functions. This proves that A is essentially self-adjoint in $L^2(M, d\mu)$ if and only if it is essentially self-adjoint in $L^2_{\mathbb{C}}(M, d\mu)$. Hence, when speaking of the deficiency indexes of an operator acting on $L^2(M, d\mu)$, we will implicitly compute them on $L^2_{\mathbb{C}}(M, d\mu)$.

We start by proving the following general proposition that will allow us to study only the Fourier components of $\mathcal{L}|_{C_c^\infty(M)}$, in order to understand its essential self-adjointness.

Proposition 5.2.4. *Let A_k be symmetric on $D(A_k) \subset H_k$, for any $k \in \mathbb{Z}$ and let $D(A)$ be the set of vectors in $\mathcal{H} = \bigoplus_{k\in\mathbb{Z}} H_k$ of the form $\psi = (\psi_1, \psi_2, \dots)$, where $\psi_k \in D(A_k)$ and all but finitely many of them are zero. Then $A = \sum_{k\in\mathbb{Z}} A_k$ is symmetric on $D(A)$, $n_+(A) = \sum_{k\in\mathbb{Z}} n_+(A_k)$ and $n_-(A) = \sum_{k\in\mathbb{Z}} n_-(A_k)$.*

Proof. Let $\psi = (\psi_1, \psi_2, \dots) \in D(A)$. Then, by symmetry of the A_k's and the fact that only finitely many ψ_k are nonzero, it holds

$$(Au, v)_{\mathcal{H}} = \sum_{k\in\mathbb{Z}} (A_k u_k, v_k)_{H_k} = \sum_{k\in\mathbb{Z}} (u_k, A_k v_k)_{H_k} = (u, Av)_{\mathcal{H}}.$$

This proves the symmetry of A.

Observe now that $\psi = (\psi_1, \psi_2, \dots) \in \ker(A \pm i)$ if and only if $0 = A\psi \pm i = (A_1\psi_1 \pm i, A_2\psi_2 \pm i, \dots)$. This clearly implies that $\dim\ker(A \pm i) = \sum_{k\in\mathbb{Z}} \dim\ker(A_k \pm i)$, completing the proof. □

Observe that, for any $k \in \mathbb{Z}$, the Fourier component $\widehat{\mathcal{L}}_k$, defined in (5.2.1), is a second order differential operator of one variable. Thus, it can be studied through the Sturm-Liouville theory (see [Zet05, EGNT13]). Let $J = (a_1, b_1) \cup (a_2, b_2)$, $-\infty \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant +\infty$, and for $1/p$, q, $w \in L^1_{loc}(J)$ consider the Sturm-Liouville operator on $L^2(J, w(x)dx)$ defined by

$$Au = \frac{1}{w}\Big(-\partial_x(p\,\partial_x u) + qu \Big). \tag{5.2.2}$$

Letting $J = \mathbb{R} \setminus \{0\}$, $w(x) = |x|^{-\alpha}$, $p(x) = -|x|^{-\alpha}$, and $q(x) = -k^2|x|^{\alpha}$, we recover $\widehat{\mathcal{L}}_k$.

For a Sturm-Liouville operator the maximal domain can be explicitly characterized as

$$D_{max}(A) = \{u : J \to \mathbb{R} \mid u,\ p\,\partial_x u \text{ are absolutely continuous on } J, \text{ and } u,\ Au \in L^2(J, w(x)dx)\}. \tag{5.2.3}$$

In (5.2.5), at the end of the section, we will give a precise characterization of the minimal domain.

Definition 5.2.5. The endpoint (finite or infinite) a_1, is limit-circle if all solutions of the equation $Au = 0$ are in $L^2((a_1, d), w(x)dx)$ for some (and hence any) $d \in (a_1, b_1)$. Otherwise a_1 is limit-point.

Analogous definitions can be given for b_1, a_2 and b_2.

Let us define the Lagrange parenthesis of $u, v : J \to \mathbb{R}$ associated to (5.2.2) as the bilinear antisymmetric form

$$[u, v] = u\,p\,\partial_x v - v\,p\,\partial_x u.$$

By [Zet05, (10.4.41)] or [EGNT13, Lemma 3.2], we have that $[u, v](d)$ exists and is finite for any $u, v \in D_{max}(\widehat{\mathcal{L}}_k)$ and any endpoint d of J.

Definition 5.2.6. The Sturm-Liouville operator (5.2.2) is *regular* at the endpoint a_1 if for some (and hence any) $d \in (a_1, b_1)$, it holds

$$\frac{1}{p},\ q,\ w \in L^1((a_1, d)).$$

A similar definition holds for b_1, a_2, b_2.

In particular, for any $k \in \mathbb{Z}$, the operator $\widehat{\mathcal{L}}_k$ is never regular at the endpoints $+\infty$ and $-\infty$, and is regular at 0^+ and 0^- if and only if $\alpha \in (-1, 1)$.

We will need the following theorem, that we state only for real extensions and in the cases we will use.

Theorem 5.2.7 (Theorem 13.3.1 in [Zet05]). *Let A be the Sturm-Liouville operator on $L^2(J, w(x)dx)$ defined in (5.2.2). Then*

$$n_+(A) = n_-(A) = \#\{\text{limit-circle endpoints of } J\}.$$

Assume now that $n_+(A) = n_-(A) = 2$, and let a and b be the two limit-circle endpoints of J. Moreover, let $\phi_1, \phi_2 \in D_{max}(A)$ be linearly independent modulo $D_{min}(A)$ and normalized by $[\phi_1, \phi_2](a) = [\phi_1, \phi_2](b) = 1$. Then, B is a self-adjoint extension of A over $L^2(J, w(x)dx)$ if and only if $Bu = A^*u$, for any $u \in D(B)$, and one of the following holds

1. Disjoint dynamics: *there exists* $c_+, c_- \in (-\infty, +\infty]$ *such that* $u \in D(B)$ *if and only if*

$$[u, \phi_1](0^+) = c_+[u, \phi_2](a) \quad \textit{and} \quad [u, \phi_1](0^-) = d_+[u, \phi_2](b).$$

2. Mixed dynamics: *there exist* $K \in SL_2(\mathbb{R})$ *such that* $u \in D(B)$ *if and only if*

$$U(bu) = K\,U(a), \qquad \textit{for } U(x) = \begin{pmatrix} [u, \phi_1](x) \\ [u, \phi_2](x) \end{pmatrix}.$$

Remark 5.2.8. Let ϕ_1^a and ϕ_2^a be, respectively, the functions ϕ_1 and ϕ_2 of the above theorem, multiplied by a cutoff function $\eta : \bar{J} \to [0,1]$ supported in a (right or left) neighborhood of a in J and such that $\eta(a) = 1$ and $\eta'(a) = 0$. Let ϕ_1^b and ϕ_2^b be defined analogously. Then, from (5.2.5), follows that we can write

$$D_{\max}(A) = D_{\min}(A) + \operatorname{span}\{\phi_1^a, \phi_1^b, \phi_2^a, \phi_2^b\}. \tag{5.2.4}$$

The following lemma classifies the end-points of $\mathbb{R} \setminus \{0\}$ with respect to the Fourier components of $\mathcal{L}|_{C_c^\infty(M)}$.

Lemma 5.2.9. *Consider the Sturm-Liouville operator* $\widehat{\Delta}_k$ *on* $\mathbb{R} \setminus \{0\}$*. Then, for any* $k \in \mathbb{Z}$ *the endpoints* $+\infty$ *and* $-\infty$ *are limit-point. On the other hand, regarding* 0^+ *and* 0^- *the following holds.*

1. *If* $\alpha \leqslant -3$ *or if* $\alpha \geqslant 1$*, then they are limit-point for any* $k \in \mathbb{Z}$*;*
2. *if* $-3 < \alpha \leqslant -1$*, then they are limit-circle if* $k = 0$ *and limit-point otherwise;*
3. *if* $-1 < \alpha < 1$*, then they are limit-circle for any* $k \in \mathbb{Z}$*.*

Before the proof, let us remark that, since $[u, v](d) = 0$ for any limit-point end-point d, by the Patching Lemma (see [Zet05, Lemma 10.4.1]) and [Zet05, Lemma 13.3.1], Lemma 5.2.9 gives the following characterization of the minimal domain of $\widehat{\Delta}_k$,

$$D_{\min}(\widehat{\Delta}_k) = \left\{ u \in D_{\max}(\widehat{\Delta}_k) \mid [u, v](0^+) = [u, v](0^-) = 0 \text{ for all } v \in D_{\max}(\widehat{\Delta}_k) \right\}. \tag{5.2.5}$$

Proof of Lemma 5.2.9. By symmetry with respect to the origin of $\widehat{\Delta}_k$, it suffices to check only 0^+ and $+\infty$.

Let $k = 0$, then for $\alpha \neq -1$ the equation $\widehat{\Delta}_0 u = u'' - (\alpha/x)u' = 0$ has solutions $u_1(x) = 1$ and $u_2(x) = x^{1+\alpha}$. Clearly, u_1 and u_2 are both in $L^2((0,1), |x|^{-\alpha}dx)$, i.e., 0^+ is limit-circle, if and only if $\alpha \in (-3, 1)$. On the other hand, u_1 and u_2 are never in $L^2((1, +\infty), |x|^{-\alpha}dx)$ simultaneously, and hence $+\infty$ is always limit-point. If $\alpha = -1$, the statement follows by the same argument applied to the solutions $u_1(x) = 1$ and $u_2(x) = \log(x)$.

Let now $k \neq 0$ and $\alpha \neq -1$. Then $\widehat{\Delta}_k u = u'' - (\alpha/x)u' - x^{2\alpha}k^2 u = 0$, $x > 0$, has solutions $u_1(x) = \exp\left(\frac{kx^{1+\alpha}}{1+\alpha}\right)$ and $u_2(x) = \exp\left(-\frac{kx^{1+\alpha}}{1+\alpha}\right)$. If $\alpha > -1$, both u_1 and u_2 are bounded and nonzero near $x = 0$, and either u_1 or u_2 has exponential growth as $x \to +\infty$. Hence, in this case, $u_1, u_2 \in L^2((0,1), |x|^{-\alpha})$ if and only if $\alpha < 1$, while $+\infty$ is always limit-point. On the other hand, if $\alpha < -1$, u_1 and u_2 are bounded away from zero as $x \to +\infty$ and one of them has exponential growth at $x = 0$. Since the measure $|x|^{-\alpha}dx$ blows up at infinity, this implies that both 0^+ and $+\infty$ are limit-point. Finally, the same holds for $\alpha = -1$, considering the solutions $u_1(x) = x^k$ and $u_2(x) = x^{-k}$. □

We are now able to classify the essential self-adjointness of the operator $\mathcal{L}|_{C_c^\infty(M)}$.

Proof of Theorem 5.2.3. Let $D \subset C_c^\infty(M)$ be the set of $C_c^\infty(M)$ functions which are finite linear combinations of products $u(x)v(\theta)$. Since $L^2(M, d\mu) = L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx) \otimes L^2(\mathbb{S}, d\theta)$, the set D is dense in $L^2(M, d\mu)$ and hence, by Proposition 5.2.4 the operator $\Delta|_D$ is essentially self adjoint if and only if so are all $\widehat{\Delta}_k|_{D\cap H_k}$. Since $n_\pm(\Delta|_D) = n_\pm(\mathcal{L}|_{C_c^\infty(M)})$, this is equivalent to $\mathcal{L}|_{C_c^\infty(M)}$ being essentially self-adjoint.

To conclude, recall that by Theorem 5.2.7 the operator $\widehat{\Delta}_k$ is not essentially self-adjoint on $L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$ if and only if it is in the limit-circle case at at least one of the four endpoints $-\infty$, 0^-, 0^+ and $+\infty$. Hence applying Lemma 5.2.9 is enough to complete the proof. □

5.2.3 The first Fourier component $\widehat{\mathcal{L}}_0$

In order to have a more precise answer to (Q1) for $\alpha \in (-3, 1)$, we now describe the self-adjoint extension realising the maximal communication between the two sides. We will call such extension the *bridging extension*.

We start with the case $\alpha \in (-3, -1]$ since here it suffices to study the equation on the first Fourier component. Indeed, by Theorem 5.2.3, in this case any self-adjoint extension A of $\mathcal{L}|_{C_c^\infty(M)}$ can be decomposed as

$$A = \widehat{A}_0 \oplus \left(\bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \widehat{\mathcal{L}}_k \right). \tag{5.2.6}$$

Here, $\widehat{A}_0$ is a self-adjoint extension of $\widehat{\mathcal{L}}_0$ and, with abuse of notation, we denoted the only self-adjoint extension of $\widehat{\mathcal{L}}_k$ by $\widehat{\mathcal{L}}_k$ as well.

We will thus focus on the first Fourier component $\widehat{\mathcal{L}}_0|_{C_c^\infty(\mathbb{R}\setminus\{0\})}$ on $L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, when $\alpha \in (-3, 1)$, and describe its real self-adjoint extensions. For a description of the complex self-adjoint extensions of $\widehat{\mathcal{L}}_0|_{C_c^\infty(\mathbb{R}\setminus\{0\})}$, we refer to Theorem 5.3.3. We remark that this operator is regular at the origin, in the sense of Sturm-Liouville problems (see Definition 5.2.6), if and only if $\alpha > -1$. Hence, for $\alpha \leqslant -1$, the boundary conditions will be asymptotic, and not punctual.

Let ϕ_D^+ and ϕ_N^+ be two smooth functions on $\mathbb{R}\setminus\{0\}$, supported in $[0, 2)$, and such that, for any $x \in [0, 1]$ it holds

$$\phi_D^+(x) = 1, \qquad \phi_N^+(x) = \begin{cases} (1+\alpha)^{-1}\, x^{1+\alpha} & \text{if } \alpha \neq -1, \\ \log(x) & \text{if } \alpha = -1. \end{cases} \tag{5.2.7}$$

Let also $\phi_D^-(x) = \phi_D^+(-x)$ and $\phi_N^+(x) = \phi_N^-(-x)$. Finally, recall that, on $\mathbb{R}\setminus\{0\}$ endowed with the Euclidean structure, the Sobolev space $H^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$ is the space of functions $u \in L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$ such that $|\partial_x u|, |\partial_x^2 u| \in L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$. Then, the following holds.

Theorem 5.2.10. *Let $D_{\min}(\widehat{\mathcal{L}}_0)$ and $D_{\max}(\widehat{\mathcal{L}}_0)$ be the minimal and maximal domains of $\widehat{\mathcal{L}}_0|_{C_c^\infty(\mathbb{R}\setminus\{0\})}$ on $L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, for $\alpha \in (-3, 1)$. Then,*

$$D_{\min}(\widehat{\mathcal{L}}_0) = \text{closure of } C_c^\infty(\mathbb{R}\setminus\{0\}) \text{ in } H^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$$

$$D_{\max}(\widehat{\mathcal{L}}_0) = \{u = u_0 + u_D^+\phi_D^+ + u_N^+\phi_N^+ + u_D^-\phi_D^- + u_N^-\phi_N^- \colon\ u_0 \in D_{\min}(\widehat{\mathcal{L}}_0) \text{ and } u_D^\pm, u_N^\pm \in \mathbb{R}\},$$

Moreover, A is a self-adjoint extension of $\widehat{\mathcal{L}}_0$ if and only if $Au = (\widehat{\mathcal{L}}_0)^* u$, for any $u \in D(A)$, and one of the following holds *(i)* Disjoint dynamics: *there exist* $c_+, c_- \in (-\infty, +\infty]$ *such that*

$$D(A) = \{u \in D_{\max}(\widehat{\mathcal{L}}_0) \colon\ u_N^+ = c_+ u_D^+ \text{ and } u_N^- = c_- u_D^+\}.$$

(ii) Mixed dynamics: *there exist* $K \in SL_2(\mathbb{R})$ *such that*

$$D(A) = \{u \in D_{\max}(\widehat{\mathcal{L}}_0) \colon\ (u_D^-, u_N^-)^T = K\,(u_D^+, u_N^+)^T\}.$$

Finally, the Friedrichs extension $(\widehat{\mathcal{L}}_0)_F$ *is the one corresponding to the disjoint dynamics with* $c_+ = c_- = 0$ *if* $\alpha \leqslant -1$ *and with* $c_+ = c_- = +\infty$ *if* $\alpha > -1$.

From the above theorem (see Remark 5.2.12) it follows that $u_N^\pm = \lim_{x\to 0^\pm} |x|^{-\alpha}\,\partial_x u(x)$ and, if $-1 < \alpha < 1$, that $u_D^\pm = u(0^\pm)$. Moreover, the last statement implies that

$$D((\widehat{\mathcal{L}}_0)_F) = \begin{cases} \{u \in D_{\max}(\widehat{\mathcal{L}}_0) \colon\ u_N^+ = u_N^- = 0\} & \text{if } \alpha \leqslant -1, \\ \{u \in D_{\max}(\widehat{\mathcal{L}}_0) \colon\ u(0^+) = u(0^-) = 0\} & \text{if } \alpha > -1. \end{cases}$$

In particular, if $\alpha \leqslant -1$ the Friedrichs extension does not impose zero boundary conditions.

Clearly, the disjoint dynamics extensions will give an evolution for which (Q1) has negative answer. On the other hand, the mixed dynamics extensions will permit information transfer between the two halves of the space. Since by Theorem 5.2.3, to classify the self-adjoint extensions for $\alpha \in (-3, -1]$ it is enough to study $\widehat{\mathcal{L}}_0$, this analysis completely classifies the self-adjoint extensions in this case. On the other hand, since for $\alpha \in (-1, 1)$ all the Fourier components are not essentially self-adjoint, a complete classification requires more sophisticated techniques. We will, in turn, study some selected extensions.

Remark 5.2.11. The mixed dynamics extension with $K = \mathrm{Id}$ is the *bridging extension* of the first Fourier component, which we will denote by $(\widehat{\mathcal{L}}_0)_B$. If $\alpha \in (-3, -1]$, the bridging extension $\mathcal{L}_B$ of $\mathcal{L}|_{C_c^\infty(M)}$ is then defined by (5.2.6) with $A_0 = (\widehat{\mathcal{L}}_0)_B$. The bridging extension for $\alpha \in (-1, 1)$ is described in the following section.

Proof of Theorem 5.2.10. We start by proving the statement on $D_{\min}(\widehat{\mathcal{L}}_0)$. The operator $\widehat{\mathcal{L}}_0$ is transformed by the unitary map $U_0 : L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx) \to L^2(\mathbb{R}\setminus\{0\})$, $U_0 v(x) = |x|^{-\alpha/2} v(x)$, in

$$\mathbf{\Delta}_0 = \partial_x^2 - \frac{\alpha}{2}\left(\frac{\alpha}{2} + 1\right)\frac{1}{x^2}.$$

By [AA12] and [Zet05, Lemma 13.3.1], it holds that $D_{\min}(\mathbf{\Delta}_0)$ is the closure of $C_c^\infty(\mathbb{R}\setminus\{0\})$ in the norm of $H^2(\mathbb{R}\setminus\{0\}, dx)$, i.e.,

$$\|u\|_{H^2(\mathbb{R}\setminus\{0\},dx)} = \|u\|_{L^2(\mathbb{R}\setminus\{0\},dx)} + \|\partial_x u\|_{L^2(\mathbb{R}\setminus\{0\},dx)} + \|\partial_x^2 u\|_{L^2(\mathbb{R}\setminus\{0\},dx)}.$$

From this follows that $D_{\min}(\widehat{\mathcal{L}}_0) = U_0^{-1} D_{\min}(\mathbf{\Delta}_0)$ is given by the closure of $C_c^\infty(\mathbb{R}\setminus\{0\})$ in $W = U_0^{-1} H^2(\mathbb{R}\setminus\{0\}, dx)$, w.r.t. the induced norm

$$\begin{aligned} \|v\|_W &= \|U_0 v\|_{H^2(\mathbb{R}\setminus\{0\},dx)} \\ &= \|v\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \left\||x|^{\alpha/2}\partial_x(|x|^{-\alpha/2}v)\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \||x|^{\alpha/2}\partial_x^2(|x|^{-\alpha/2}v)\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} \end{aligned}$$

(5.2.8)

To prove the statement, it suffices to show that on $C_c^\infty(\mathbb{R}\setminus\{0\})$ the induced norm (5.2.8) is equivalent to the norm of $H^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, which is

$$\|v\|_{H^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} = \|v\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \left\|\partial_x v\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \|\widehat{\mathcal{L}}_0 v\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)}. \tag{5.2.9}$$

To this aim, observe that

$$\partial_x v(x) = |x|^{\alpha/2}\partial_x(|x|^{-\alpha/2}v) + \frac{\alpha}{2}\frac{v}{x}, \qquad \widehat{\mathcal{L}}_0 v = |x|^{\alpha/2}\partial_x^2\left(|x|^{-\alpha/2}v\right) + \frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)\frac{v}{x^2}. \tag{5.2.10}$$

Moreover, by a cutoff argument, it is clear that we can prove the bound separately for v supported near the origin and away from it.

Let $v \in C_c^\infty(\mathbb{R}\setminus\{0\})$ be supported in $(-1,0)\cup(0,1)$. By (5.2.10) and the fact that if $|x| \leqslant 1$ then $|x|^{-1}, |x|^{-2} \geqslant 1$, it follows immediately that $\|v\|_{H^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} \leqslant \|v\|_W$. In order to prove the opposite inequality, observe that $x^{-2} \geqslant x^{-1}$ and $v \in H_0^2((0,1),dx) \oplus H_0^2((-1,0),dx)$. Thus, by [AA12, (3.5)] we obtain

$$\begin{aligned}\left\|vx^{-1}\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \left\|vx^{-2}\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} &\leqslant 2\|vx^{-2}\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} \\ &= 2\|vx^{-2-\alpha/2}\|_{L^2((0,1))} \leqslant 2C\|\partial_x(vx^{-\alpha/2})\|_{H^2((0,1))} = 2C\|v\|_W.\end{aligned} \tag{5.2.11}$$

Finally, let $v \in C_c^\infty(\mathbb{R}\setminus\{0\})$ be supported in $(1,+\infty)$ (the same argument will work also between $(-\infty,-1)$). In this case, $x^{-2} < x^{-1} < 1$. Thus, by (5.2.10), (5.2.8), (5.2.9) and the triangular inequality, we get that for any $v \in C_c^\infty(\mathbb{R}\setminus\{0\})$ it holds

$$\begin{aligned}&\left|\|v\|_W - \|v\|_{H^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)}\right| \\ &\quad\leqslant C\left(\left\|vx^{-1}\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} + \left\|vx^{-2}\right\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)}\right) \leqslant 2C\|v\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)},\end{aligned}$$

for some constant $C > 0$. Since $\|v\|_W$ and $\|v\|_{H^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)} \geqslant \|v\|_{L^2(\mathbb{R}\setminus\{0\},|x|^{-\alpha}dx)}$, this completes the proof of the first part of the theorem.

We now proceed to the classification of the self-adjoint extensions of $\widehat{\mathcal{L}}_0$. For this purpose, recall the definition of $\phi_D^\pm$ and $\phi_N^\pm$ given in (5.2.7) and let

$$\phi_N(x) = \phi_N^+(x) + \phi_N^-(x), \qquad \phi_D(x) = \phi_D^+(x) + \phi_D^-(x).$$

Observe that $\phi_D \in L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$ and that $\widehat{\mathcal{L}}_0\phi_D(x) = 0$ for any $x \notin (-2,-1)\cup(1,2)$. Since the function is smooth, this implies that $\phi_D \in D_{\max}(\widehat{\mathcal{L}}_0)$. The same holds for ϕ_N. Moreover, a simple computation shows that $[\phi_D^+, \phi_N^+](0^+) = [\phi_D^+, \phi_N^+](0^-) = 1$, and hence ϕ_N and ϕ_D satisfy the hypotheses of Theorem 5.2.7. In particular, by Remark 5.2.8, this implies that

$$D_{\max}(\widehat{\mathcal{L}}_0) = D_{\min}(\widehat{\mathcal{L}}_0) + \operatorname{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}.$$

We claim that for any $u = u_0 + u_D^+\phi_D^+ + u_N^+\phi_N^+ + u_D^-\phi_D^- + u_N^-\phi_N^- \in D_{\max}$ it holds

$$[u,\phi_N](0^+) = u_D^+, \qquad [u,\phi_D](0^+) = u_N^+, \qquad [u,\phi_N](0^-) = u_D^-, \qquad [u,\phi_N](0^-) = u_N^-.$$

(5.2.12)

This, by Theorem 5.2.7 will complete the classification of the self-adjoint extensions. Observe that, (5.2.5) and the bilinearity of the Lagrange parentheses imply that $[u_0, \phi_N](0^\pm) = [u_0, \phi_D](0^\pm) = 0$. The claim then follows from the fact that

$$\begin{aligned}
&[\phi_D^+, \phi_N](0^+) = [\phi_N^+, \phi_D](0^+) = [\phi_D^-, \phi_N](0^-) = [\phi_N^-, \phi_D](0^-) = 1,\\
&[\phi_D^-, \phi_N](0^+) = [\phi_N^-, \phi_D](0^+) = [\phi_D^+, \phi_N](0^-) = [\phi_N^+, \phi_D](0^-) = 0.
\end{aligned}$$

To complete the proof, it remains only to identify the Friedrichs extension $(\widehat{\mathcal{L}}_0)_F$. Recall that such extension is always defined, and has domain

$$D((\widehat{\mathcal{L}}_0)_F) = \{u \in H_0^1(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx) \mid \widehat{\mathcal{L}}_0 u \in L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)\}.$$

Since if $\alpha \leqslant -1$, $\phi_N \notin H^1(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, it is clear that the Friedrichs extension corresponds to the case where $u_N^+ = u_N^- = 0$, i.e., to $c_+ = c_- = 0$. On the other hand, if $\alpha > -1$, since all the end-points are regular, by [EGNT13, Corollary 10.20] holds that the Friedrichs extension corresponds to the case where $u(0^\pm) = u_D^\pm = 0$, i.e., to $c_+ = c_- = +\infty$. □

Remark 5.2.12. If $u \in D_{\max}(\widehat{\mathcal{L}}_0)$, it holds

$$u_D^+ = [u, \phi_N](0^+) = \lim_{x\downarrow 0}\big(u(x) - x\,\partial_x u(x)\big) \qquad \text{and} \qquad u_N^+ = [u, \phi_D](0^+) = \lim_{x\downarrow 0} x^{-\alpha}\,\partial_x u(x).$$

This implies, in particular, that if $\alpha > -1$ then $u_D^+ = u(0^+)$. Indeed this holds if and only if the end-point 0^+ is regular in the sense of Sturm-Liouville operators, see Definition 5.2.6. Clearly the same computations hold at 0^-.

We conclude this section with a description of the maximal domain, in the case $\alpha \in (-1, 1)$.

Proposition 5.2.13. *For any $\alpha \in \mathbb{R}$, it holds that*

$$D_{\max}(\mathcal{L}|_{C_c^\infty(M)}) = \begin{cases} H^2(M, d\mu) = H_0^2(M, d\mu) & \text{if } \alpha \leqslant -3 \text{ or } \alpha \geqslant 1,\\ H^2(M, d\mu) \oplus \operatorname{span}\{\phi_N^+, \phi_N^-\} & \text{if } -3 < \alpha \leqslant -1,\\ H^2(M, d\mu) \supsetneqq H_0^2(M, d\mu) & \text{if } -1 < \alpha < 1.\end{cases}$$

Here we let, with abuse of notation, $\phi_N^\pm(x, y) = \phi_N^\pm(x)$.

Proof. Recall that, by definition, $H^2(M, d\mu) \subset D_{\max}(\mathcal{L}|_{C_c^\infty(M)})$. Moreover, if $\alpha \leqslant -3$ or if $\alpha \geqslant 1$, by Theorem 5.2.3 it holds $D_{\max}(\mathcal{L}|_{C_c^\infty(M)}) = D(\mathcal{L}_F) = H_0^2(M, d\mu) \subset H^2(M, d\mu)$. This proves the first statement.

On the other hand, by Remark 5.2.8, if $\alpha \in (-3, -1]$, since $\widehat{\mathcal{L}}_k$ is essentially self-adjoint for any $k \neq 0$ we can decompose the maximal domain as

$$D_{\max}(\mathcal{L}|_{C_c^\infty(M)}) = D_{\max}(\widehat{\mathcal{L}}_0) \oplus \left(\bigoplus_{k\in\mathbb{Z}\setminus\{0\}} D(\widehat{\mathcal{L}}_k)\right)$$

Moreover, letting π_0 be the projection on the $k = 0$ Fourier component and defining $(\pi_0^{-1}u_0)(x, \theta) = u_0(x)$ for any $u_0 \in L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, the previous decomposition and the fact that $D_{\min}(\mathcal{L}|_{C_c^\infty(M)}) \subset H^2(M, d\mu) \subset D_{\max}(\mathcal{L}|_{C_c^\infty(M)})$ implies that

$$\begin{aligned}
D_{\max}(\mathcal{L}|_{C_c^\infty(M)}) &= \{u = u_0 + \pi_0^{-1}\tilde{u} \mid u_0 \in D_{\min}(\mathcal{L}|_{C_c^\infty(M)}),\ \tilde{u} \in \operatorname{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}\}\\
&= H^2(M, d\mu) + \operatorname{span}\{\phi_D^+, \phi_N^+, \phi_D^-, \phi_N^-\}.
\end{aligned}$$

Here, in the last equality, we let $\phi_D(x, y) = \phi_D(x)$ and $\phi_N(x, y) = \phi_N(x)$. A simple computation shows that $\phi_D \in H^1(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ and $\phi_N \notin H^1(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$. Since $\widehat{\mathcal{L}}_0 \phi_D = 0$, it follows that $\phi_D \in H^2(M, d\mu)$, while $\phi_N \notin H^2(M, d\mu)$. This implies the statement.

To complete the proof it suffices to prove that if $\alpha \in (-1, 1)$ it holds $D_{max}(\mathcal{L}|_{C_c^\infty(M)}) \subset H^2(M, d\mu)$. In fact, the inequality $H^2(M, d\mu) \neq H_0^2(M, d\mu)$ will then follow from the fact that $\mathcal{L}_F$ is not the only self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$. By Parseval identity, $\phi, \Delta\phi \in L^2(M, d\mu)$ if and only $\phi_k, \widehat{\mathcal{L}}_k \phi_k \in L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ for any $k \in \mathbb{Z}$ and thus the statement is equivalent to $D_{max}(\widehat{\mathcal{L}}_k) \subset H^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ for any $k \in \mathbb{Z}$. Let $u \in D_{max}(\widehat{\mathcal{L}}_k)$. Since $\lim_{x \to 0^\pm} x^{-\alpha} \partial_x u(x) = [u, \phi_D](0^\pm)$, this limit exists and is finite. Moreover, since $\pm\infty$ are limit-point, it holds $\lim_{x \to \pm\infty} x^{-\alpha} \partial_x u(x) = [u, \phi_D](\pm\infty) = 0$. Hence, $x^{-\alpha} \partial_x u$ is square integrable near 0 and at infinity, and from the characterization (5.2.3) follows that $\widehat{\mathcal{L}}_k u \in L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$. This proves that $u \in H^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha} dx)$ and thus the proposition. □

5.3 COMPLEX SELF-ADJOINT EXTENSIONS

The natural functional setting for the Schrödinger equation on M_α is the space of square integrable complex-valued function $L^2_{\mathbb{C}}(M, d\mu)$. It is easy to see that the self-adjoint extension of A over $L^2(M, d\mu)$ studied in the previous section are exactly the restrictions to this space of the real self-adjoint extension of A over $L^2_{\mathbb{C}}(M, d\mu)$.

All the theory of Section 5.2 extends to the complex case, in particular, we have the following generalization of Theorem 5.2.7.

Theorem 5.3.1 (Theorem 13.3.1 in [Zet05]). *Let A be the Sturm-Liouville operator on $L^2_{\mathbb{C}}(J, w(x)dx)$ defined in (5.2.2). Then*

$$n_+(A) = n_-(A) = \#\{\text{limit-circle endpoints of } J\}.$$

Assume now that $n_+(A) = n_-(A) = 2$, and let a and b be the two limit-circle endpoints of J. Moreover, let $\phi_1, \phi_2 \in D_{max}(A)$ be linearly independent modulo $D_{min}(A)$ and normalized by $[\phi_1, \phi_2](a) = [\phi_1, \phi_2](b) = 1$. Then, B is a self-adjoint extension of A over $L^2_{\mathbb{C}}(J, w(x)dx)$ if and only if $Bu = A^*u$, for any $u \in D(B)$, and one of the following holds

1. Disjoint dynamics: *there exists $c_+, c_- \in (-\infty, +\infty]$ such that $u \in D(B)$ if and only if*

$$[u, \phi_1](0^+) = c_+[u, \phi_2](0^+) \quad \text{and} \quad [u, \phi_1](0^-) = d_+[u, \phi_2](0^-).$$

2. Mixed dynamics: *there exist $K \in SL_2(\mathbb{R})$ and $\gamma \in (-\pi, \pi]$ such that $u \in D(B)$ if and only if*

$$U(0^-) = e^{i\gamma} K\, U(0^+), \qquad \text{for } U(x) = \begin{pmatrix} [u, \phi_1](x) \\ [u, \phi_2](x) \end{pmatrix}.$$

Finally, B is a real self-adjoint extension if and only if it satisfies (1) the disjoint dynamic or (2) the mixed dynamic with $\gamma = 0$.

As a consequence of Theorem 5.3.1, we get a complete description of the essential self-adjointness of $\mathcal{L}|_{C_c^\infty(M)}$ over $L^2_{\mathbb{C}}(M, d\mu)$, extending Theorem 5.2.3, and of the complex self-adjoint extensions of $\widehat{\mathcal{L}}_0$, extending Theorem 5.2.10.

Theorem 5.3.2. *Consider M_α for $\alpha \in \mathbb{R}$ and the corresponding Laplace-Beltrami operator $\mathcal{L}|_{C_c^\infty(M)}$ as an unbounded operator on $L^2_\mathbb{C}(M, d\mu)$. Then it holds the following.*

(i) If $\alpha \leqslant -3$ then $\mathcal{L}|_{C_c^\infty(M)}$ is essentially self-adjoint;

(ii) if $\alpha \in (-3, -1]$, only the first Fourier component $\widehat{\Delta}_0$ is not essentially self-adjoint;

(iii) if $\alpha \in (-1, 1)$, all the Fourier components of $\mathcal{L}|_{C_c^\infty(M)}$ are not essentially self-adjoint;

(iv) if $\alpha \geqslant 1$ then $\mathcal{L}|_{C_c^\infty(M)}$ is essentially self-adjoint.

Theorem 5.3.3. *Let $D_{\min}(\widehat{\mathcal{L}}_0)$ and $D_{\max}(\widehat{\mathcal{L}}_0)$ be the minimal and maximal domains of $\widehat{\mathcal{L}}_0|_{C_c^\infty(\mathbb{R}\setminus\{0\})}$ on $L^2_\mathbb{C}(\mathbb{R}\setminus\{0\}, |x|^{-\alpha})$, for $\alpha \in (-3, 1)$. Then,*

$$D_{\min}(\widehat{\mathcal{L}}_0) = \text{closure of } C_c^\infty(\mathbb{R}\setminus\{0\}) \text{ in } H^2_\mathbb{C}(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$$
$$D_{\max}(\widehat{\mathcal{L}}_0) = \{u = u_0 + u_D^+\phi_D^+ + u_N^+\phi_N^+ + u_D^-\phi_D^- + u_N^-\phi_N^- : \ u_0 \in D_{\min}(\widehat{\mathcal{L}}_0) \text{ and } u_D^\pm, u_N^\pm \in \mathbb{C}\},$$

Moreover, A is a self-adjoint extension of $\widehat{\mathcal{L}}_0$ if and only if $Au = (\widehat{\mathcal{L}}_0)^*u$, for any $u \in D(A)$, and one of the following holds

(i) Disjoint dynamics: *there exist $c_+, c_- \in (-\infty, +\infty]$ such that*

$$D(A) = \left\{u \in D_{\max}(\widehat{\mathcal{L}}_0) : \ u_N^+ = c_+ u_D^+ \text{ and } u_N^- = c_- u_D^+\right\}.$$

(ii) Mixed dynamics: *there exist $K \in SL_2(\mathbb{R})$ and $\gamma \in (-\pi, \pi]$ such that*

$$D(A) = \left\{u \in D_{\max}(\widehat{\mathcal{L}}_0) : \ (u_D^-, u_N^-) = e^{i\gamma} K (u_D^+, u_N^+)^T\right\}.$$

Finally, the Friedrichs extension $(\widehat{\mathcal{L}}_0)_F$ is the one corresponding to the disjoint dynamics with $c_+ = c_- = 0$ if $\alpha \leqslant -1$ and with $c_+ = c_- = +\infty$ if $\alpha > -1$.

5.4 BILINEAR FORMS

In this section we prove Theorem 1.3.6, that answers to (Q2).

5.4.1 Preliminaries

This introductory section is based on [FOT11]. Let $\mathcal{H}$ be an Hilbert space with scalar product $(\cdot,\cdot)_\mathcal{H}$. A non-negative symmetric bilinear form densely defined on $\mathcal{H}$, henceforth called only a *symmetric form on* $\mathcal{H}$, is a map $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ such that $D(\mathcal{E})$ is dense in $\mathcal{H}$ and $\mathcal{E}$ is bilinear, symmetric, and non-negative (i.e., $\mathcal{E}(u,u) \geqslant 0$ for any $u \in D(\mathcal{E})$). A symmetric form is *closed* if $D(\mathcal{E})$ is a complete Hilbert space with respect to the scalar product

$$(u,v)_\mathcal{E} = (u,v)_\mathcal{H} + \mathcal{E}(u,v), \qquad u, v \in D(\mathcal{E}). \tag{5.4.1}$$

To any densely defined non-positive definite self-adjoint operator A it is possible to associate a symmetric form $\mathcal{E}_A$ such that

$$\mathcal{E}_A(u,v) = (-Au, v)$$
$$D(A) = \{u \in D(\mathcal{E}_A) : \ \exists v \in \mathcal{H} \text{ s.t. } \mathcal{E}(u,\phi) = (v,\phi) \text{ for all } \phi \in D(\mathcal{E}_A)\}.$$

Indeed, we have the following.

Theorem 5.4.1 ([Kat95, FOT11]). *Let $\mathcal{H}$ be an Hilbert space, then the map $A \mapsto \mathcal{E}_A$ induces a one to one correspondence*

$$A \text{ non-positive definite self-adjoint operator } \iff \mathcal{E}_A \text{ closed symmetric form.}$$

In particular, the inverse correspondence can be characterized by $D(A) \subset D(\mathcal{E}_A)$ and $\mathcal{E}_A(u,v) = (-Au, v)$ for all $u \in D(A)$, $v \in D(\mathcal{E}_A)$.

Consider now a σ-finite measure space $(X, \mathcal{F}, m)$.

Definition 5.4.2. A symmetric form $\mathcal{E}$ on $L^2(X,m)$ is *Markovian* if for any $\varepsilon > 0$ there exists $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $-\varepsilon \leqslant \psi_\varepsilon \leqslant 1+\varepsilon$, $\psi_\varepsilon(t) = t$ if $t \in [0,1]$, $0 \leqslant \psi_\varepsilon(t) - \psi_\varepsilon(s) \leqslant t - s$ whenever $s < t$ and

$$u \in D(\mathcal{E}) \implies \psi_\varepsilon(u) \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(\psi_\varepsilon(u), \psi_\varepsilon(u)) \leqslant \mathcal{E}(u,u).$$

A closed Markovian symmetric form is a *Dirichlet form*.

A semigroup $\{T_t\}_{t\geqslant 0}$ on $L^2(X,m)$ is *Markovian* if

$$u \in L^2(X,m) \text{ s.t. } 0 \leqslant u \leqslant 1 \quad m-\text{a.e.} \implies 0 \leqslant T_t u \leqslant 1 \quad m-\text{a.e. for any } t > 0.$$

A non-positive self-adjoint operator is *Markovian* if it generates a Markovian semigroup.

When the form is closed, the Markov property can be simplified, as per the following Theorem. For any $u : X \to \mathbb{R}$ let $u_\sharp = \min\{1, \max\{u, 0\}\}$.

Theorem 5.4.3 (Theorem 1.4.1 of [FOT11]). *The closed symmetric form $\mathcal{E}$ is Markovian if and only if*

$$u \in D(\mathcal{E}) \implies u_\sharp \in D(\mathcal{E}) \text{ and } \mathcal{E}(u_\sharp, u_\sharp) \leqslant \mathcal{E}(u,u).$$

Since any function of $L^\infty(X,m)$ is approximable by functions in $L^2(X,m)$, the Markov property allows to extend the definition of $\{T_t\}_{t\geqslant 0}$ to $L^\infty(X,m)$, and moreover implies that it is a contraction semigroup on this space. When $\{T_t\}_{t\geqslant 0}$ is the evolution semigroup of the heat equation, the Markov property can be seen as a physical admissibility condition. Namely, it assures that when starting from an initial datum u representing a temperature distribution (i.e., a positive and bounded function) the solution $T_t u$ remains a temperature distribution at each time, and, moreover, that the heat does not concentrate.

The following theorem extends the one-to-one correspondence given in Theorems 5.2.1 and 5.4.1 to the Markovian setting.

Theorem 5.4.4 ([FOT11]). *Let A be a non-positive self-adjoint operator on $L^2(X,m)$. The following are equivalents*

1. *A is a Markovian operator;*
2. *$\mathcal{E}_A$ is a Dirichlet form;*
3. *$\{e^{tA}\}_{t\geqslant 0}$ is a Markovian semigroup.*

Given a non-positive symmetric operator A we can always define the (non-closed) symmetric form

$$\mathcal{E}(u,v) = (-Au, v), \quad D(\mathcal{E}) = D(A).$$

The Friedrichs extension A_F of A is then defined as the self-adjoint operator associated via Theorem 5.4.1 to the closure $\mathcal{E}_0$ of this form. Namely, $D(\mathcal{E}_0)$ is the closure of $D(A)$ with respect to the scalar product (5.4.1), and $\mathcal{E}_0(u,v) = \lim_{n\to+\infty} \mathcal{E}(u_n, v_n)$ for $u_n \to u$ and $v_n \to v$ w.r.t. $(\cdot,\cdot)_{\mathcal{E}}$. It is a well-known fact that the Friedrichs extension of a Markovian operator is always a Dirichlet form (see, e.g., [FOT11, Theorem 3.1.1]).

A Dirichlet form $\mathcal{E}$ is said to be *regular* on X if $D(\mathcal{E}) \cap C_c(X)$ is both dense in $D(\mathcal{E})$ w.r.t. the scalar product (5.4.1) and dense in $C_c(X)$ w.r.t. the $L^\infty(X)$ norm. To any regular Dirichlet form $\mathcal{E}_A$ it is possible to associate a Markov process $\{X_t\}_{t\geqslant 0}$ which is generated by A (indeed they are in one-to-one correspondence to a particular class of Markov processes, the so-called Hunt processes, see [FOT11] for the details).

If its associated Dirichlet form is regular, by Definitions 1.3.4 and 1.3.5, a Markovian operator is said *stochastically complete* if its associated Markov process has almost surely infinite lifespan, and *recurrent* if it intersects any subset of X with positive measure an infinite number of times. If it is not stochastically complete, an operator is called *explosive*. Observe that recurrence is a stronger property than stochastic completeness. Since we will only consider regular Dirichlet forms, we refer to [FOT11] for a definition of recurrence valid for general Dirichlet forms.

We will need the following characterizations.

Theorem 5.4.5 (Theorem 1.6.6 in [FOT11])**.** *A Dirichlet form $\mathcal{E}$ is stochastically complete if and only if there exists a sequence $\{u_n\} \subset D(\mathcal{E})$ satisfying*

$$0 \leqslant u_n \leqslant 1, \quad \lim_{n\to+\infty} u_n = 1 \quad m-\textit{a.e.},$$

such that

$$\mathcal{E}(u_n, v) \to 0 \quad \textit{for any } v \in D(\mathcal{E}) \cap L^1(X, m).$$

We let the *extended domain* $D(\mathcal{E})_e$ of a Dirichlet form $\mathcal{E}$ to be the family of functions $u \in L^\infty(X, m)$ such that there exists $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E})$, Cauchy sequence w.r.t. the scalar product (5.4.1), such that $u_n \longrightarrow u$ m-a.e. . The Dirichlet form $\mathcal{E}$ can be extended to $D(\mathcal{E})_e$ as a non-negative definite symmetric bilinear form, by $\mathcal{E}(u,u) = \lim_{n\to+\infty} \mathcal{E}(u_n, u_n)$.

Theorem 5.4.6 (Theorems 1.6.3 and 1.6.5 in [FOT11])**.** *Let $\mathcal{E}$ be a Dirichlet form. The following are equivalent.*

1. *$\mathcal{E}$ is recurrent;*
2. *there exists a sequence $\{u_n\} \subset D(\mathcal{E})$ satisfying*

$$0 \leqslant u_n \leqslant 1, \quad \lim_{n\to+\infty} u_n = 1 \quad m-\textit{a.e.},$$

such that

$$\mathcal{E}(u_n, v) \to 0 \quad \textit{for any } v \in D(\mathcal{E})_e.$$

3. $1 \in D(\mathcal{E})_e$, *i.e., there exists a sequence* $\{u_n\} \subset D(\mathcal{E})$ *such that* $\lim_{n\to+\infty} u_n = 1 \quad m-$*a.e. and* $\mathcal{E}(u_n, u_n) \to 0$.

Remark 5.4.7. As a consequence of this two theorems we have that if $m(X) < +\infty$, stochastic completeness and recurrence are equivalent.

We conclude this preliminary part, by introducing a notion of restriction of closed forms associated to self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$.

Definition 5.4.8. Given a self-adjoint extension A of $\mathcal{L}|_{C_c^\infty(M)}$ and an open set $U \subset M$, we let the *Neumann restriction* $\mathcal{E}_A|_U$ of $\mathcal{E}_A$ to be the form associated with the self-adjoint operator $A|_U$ on $L^2(U, d\mu)$, obtained by putting Neumann boundary conditions on ∂_U.

In particular, by Theorem 5.4.1 and an integration by parts, it follows that $D(\mathcal{E}_A|_U) = \{u|_U \mid u \in D(\mathcal{E}_A)\}$.

5.4.2 Markovian extensions of $\mathcal{L}|_{C_c^\infty(M)}$

The bilinear form associated with $\mathcal{L}|_{C_c^\infty(M)}$ is

$$\mathcal{E}(u,v) = \int_{M_\alpha} g(\nabla u, \nabla v)\, d\mu = \int_{M_\alpha} \left(\partial_x u\, \partial_x v + |x|^{2\alpha} \partial_\theta u\, \partial_\theta v\right) d\mu, \quad D(\mathcal{E}) = C_c^\infty(M).$$

By [FOT11, Example 1.2.1], $\mathcal{E}$ is a Markovian form. The Friederichs extension is then associated with the form

$$\mathcal{E}_F(u,v) = \int_M \left(\partial_x u\, \partial_x v + |x|^{2\alpha} \partial_\theta u\, \partial_\theta v\right) d\mu, \qquad D(\mathcal{E}_F) = H_0^1(M, d\mu),$$

where the derivatives are taken in the sense of Schwartz distributions. By its very definition, and the fact that $D(\mathcal{E}_F) \cap C_c^\infty(M) = C_c^\infty(M)$, follows that $\mathcal{E}_F$ is always a regular Dirichlet form on M (equivalently, on M^+ or on M^-). Its associated Markov process is absorbed by the singularity.

The following Lemma will be crucial to study the properties of the Friederichs extension. Let $M_0 = (-1,1) \times \mathbb{S}$, $M_\infty = (1,+\infty) \times \mathbb{S}$ and recall the notion of Neumann restriction given in Definition 5.4.8.

Lemma 5.4.9. *If* $\alpha \leqslant -1$, *it holds that* $1 \in D(\mathcal{E}_F|_{M_0})$. *Moreover,* $1 \notin D(\mathcal{E}_F|_{M_0})_e$ *if* $\alpha > -1$ *and* $1 \in D(\mathcal{E}_F|_{M_\infty})_e$ *if and only if* $\alpha \geqslant -1$.

Proof. To ease the notation, we let $\widehat{\mathcal{E}}_k$ to be the Dirichlet form associated to the Friederichs extension of $\widehat{\mathcal{L}}_k$. In particular, for $k = 0$,

$$\widehat{\mathcal{E}}_0(u,v) = \int_{\mathbb{R}\setminus\{0\}} \partial_x u\, \partial_x v\, |x|^{-\alpha} dx, \qquad D(\widehat{\mathcal{E}}_0) = H_0^1(\mathbb{R}\setminus\{0\}, |x|^{-\alpha} dx).$$

Let $\pi_k : L^2(M, d\mu) \to H_k = L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha} dx)$ be the projection on the k-th Fourier component. Then, from the rotational invariance of $D(\mathcal{E}_F)$ follows that

$$D(\mathcal{E}_F) = \bigoplus_{k\in\mathbb{Z}} D(\widehat{\mathcal{E}}_k), \qquad \mathcal{E}_F(u,v) = \sum_{k\in\mathbb{Z}} \widehat{\mathcal{E}}_k(\pi_k u, \pi_k v).$$

In particular, since $\pi_0 1 = 1$ and $\pi_k 1 = 0$ for $k \neq 0$, follows that $1 \in D(\mathcal{E}_F|_{M_0})$ (resp. $1 \in D(\mathcal{E}_F|_{M_\infty})_e$) if and only if $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ (resp. $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$). Here, with abuse of notation, we denoted as 1 both the functions $1 : M \to \{1\}$ and $1 : \mathbb{R} \to \{1\}$. Thus, to complete the proof of the lemma, it suffices to prove that $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ if $\alpha \leqslant -1$, that $1 \notin D(\widehat{\mathcal{E}}_0|_{(0,1)})_e$ if $\alpha \geqslant -1$ and that $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$ if and only if $\alpha \geqslant -1$.

For any $0 < r < R < +\infty$, let $f^\alpha_{r,R}$ be the only solution to the Cauchy problem

$$\begin{cases} \widehat{\mathcal{L}}_0 f = 0, \\ f(r) = 1, \qquad f(R) = 0. \end{cases}$$

Namely,

$$f^\alpha_{r,R}(x) = \begin{cases} \dfrac{R^{1+\alpha} - x^{1+\alpha}}{R^{1+\alpha} - r^{1+\alpha}} & \text{if } \alpha \neq -1, \\[2ex] \dfrac{\log\left(\frac{R}{x}\right)}{\log\left(\frac{R}{r}\right)} & \text{if } \alpha = -1. \end{cases}$$

Then, the 0-equilibrium potential (see [FOT11] and Remark 5.4.10) of $[0,r]$ in $[0,R]$, is given by

$$u_{r,R}(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant r, \\ f^\alpha_{r,R}(x) & \text{if } r < x \leqslant R, \\ 0 & \text{if } x > R. \end{cases} \tag{5.4.2}$$

It is a well-known fact that $u_{r,R}$ is the minimizer for the capacity of $[0,r]$ in $[0,R)$. Namely, for any locally Lipschitz function v with compact support contained in $[0,R]$ and such that $v(x) = 1$ for any $0 < x < r$, it holds

$$\int_0^{+\infty} |\partial_x u_{r,R}|^2 x^{-\alpha}\, dx \leqslant \int_0^{+\infty} |\partial_x v|^2 x^{-\alpha}\, dx \tag{5.4.3}$$

Since it is compactly supported on $[0,+\infty)$ and locally Lipschitz, it follows that $u_{r,R} \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})$ and $1 - u_{r,R} \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ for any $0 < r < R < +\infty$.

Consider now $\alpha \geqslant -1$, and let us prove that $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$. To this aim, it suffices to show that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset D(\widehat{\mathcal{E}}_0|_{(1,+\infty)}) = \{u|_{(1,+\infty)} \mid u \in H^1((0,+\infty), x^{-\alpha} dx)\}$ such that $u_n \longrightarrow 1$ a.e. and $\widehat{\mathcal{E}}_0|_{(1,+\infty)}$. Let

$$u_n = \begin{cases} u_{n,2n} & \text{if } \alpha \neq -1, \\ u_{n,n^2} & \text{if } \alpha = -1. \end{cases}$$

It is clear that $u_n \longrightarrow 1$ a.e., moreover, a simple computation shows that

$$\widehat{\mathcal{E}}_0|_{(1,+\infty)}(u_n, u_n) = \int_1^{+\infty} |\partial_x u_n|^2\, x^{-\alpha}\, dx = \begin{cases} \frac{1+\alpha}{2^{1+\alpha}-1}\, n^{-(1+\alpha)} & \text{if } \alpha \neq -1, \\ \frac{1}{\log(n)} & \text{if } \alpha = -1. \end{cases}$$

Hence $\widehat{\mathcal{E}}_0|_{(1,+\infty)} \longrightarrow 0$ if $\alpha \geqslant -1$, proving that $1 \in D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$.

We now prove that $1 \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$ if $\alpha \leqslant -1$. Consider the following sequence in $H^1((0,1), x^{-\alpha}dx)$,

$$u_n = \begin{cases} u_{1/2n,1/n} & \text{if } \alpha \neq -1, \\ u_{1/n^2,1/n} & \text{if } \alpha = -1. \end{cases}$$

A direct computation of $\int_0^1 |\partial_x u_n|^2 x^{-\alpha} dx$, the fact that $\operatorname{supp} u_n \subset [0, 1/n]$ and $0 \leqslant u_n \leqslant 1$, prove that $u_n \longrightarrow 0$ in $H^1((0,1), x^{-\alpha}dx)$. Since $1 - u_n \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$, which is closed, this proves that $1 - u_n \longrightarrow 1$ in $D(\widehat{\mathcal{E}}_0|_{(0,1)})$, and hence the claim.

To complete the proof, it remains to show that $1 \notin D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})_e$ if $\alpha < -1$. The same argument can be then used to prove that $1 \notin D(\widehat{\mathcal{E}}_0|_{(0,1)})_e$ if $\alpha > -1$. We proceed by contradiction, assuming that there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset D(\widehat{\mathcal{E}}_0|_{(1,+\infty)})$ such that $v_n \longrightarrow 1$ a.e. and $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \longrightarrow 0$. Since the form $\widehat{\mathcal{E}}_0|_{(1,+\infty)}$ is regular on $[1, +\infty)$, we can take $v_n \in C_c^\infty([1, +\infty))$. Moreover, we can assume that $v_n(1) = 1$ for any $n \in \mathbb{N}$. In fact, if this is not the case, it suffices to consider the sequence $\tilde{v}_n(x) = v_n(x)/v_n(1)$. Let $R_n > 0$ be such that $\bigcup_{m\leqslant n} \operatorname{supp} v_m \subset [1, R_n]$. Moreover, extend v_n to 1 on $(0,1)$, so that $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) = \int_0^{+\infty} |\partial_x v_n|^2 x^{-\alpha} dx$. Since the same holds for u_{1,R_n}, by (5.4.3), the fact that $R_n \longrightarrow +\infty$ and $\alpha < -1$, we get

$$\lim_{n\to+\infty} \widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \geqslant \lim_{n\to+\infty} \widehat{\mathcal{E}}_0|_{(1,+\infty)}(u_{1,R_n}, u_{1,R_n}) = \lim_{n\to+\infty} \frac{1+\alpha}{R_n^{1+\alpha} - 1} = -(1+\alpha) > 0.$$

This contradicts the fact that $\widehat{\mathcal{E}}_0|_{(1,+\infty)}(v_n, v_n) \longrightarrow 0$, completing the proof. □

Remark 5.4.10. The 0-equilibrium potential defined in (5.4.2) admits a probabilistic interpretation. Namely, it is the probability that the Markov process associated with $\widehat{\mathcal{L}}_0$ and starting from x, exits the first time from the interval $\{r < x < R\}$ through the inner boundary $\{x = r\}$.

It is possible to define a semi-order on the set of the Markovian extensions of $\mathcal{L}|_{C_c^\infty(M)}$ as follows. Given two Markovian extensions A and B, we say that $A \subset B$ if $D(\mathcal{E}_A) \subset D(\mathcal{E}_B)$ and $\mathcal{E}_A(u,u) \geqslant \mathcal{E}_B(u,u)$ for any $u \in D(\mathcal{E}_A)$. With respect to this semi-order, the Friederichs extension is the minimal Markovian extension. Let $\mathcal{L}_N$ be the maximal Markovian extension (see [FOT11]). This extension is associated with the Dirichlet form $\mathcal{E}^+$ defined by

$$\mathcal{E}^+(u,v) = \int_M \left(\partial_x u \, \partial_x v + |x|^{2\alpha} \partial_\theta u \, \partial_\theta v\right) d\mu,$$
$$D(\mathcal{E}^+) = \{u \in L^2(M, d\mu) \mid \mathcal{E}^+(u,u) < +\infty\} = H^1(M, d\mu),$$

where the derivatives are taken in the sense of Schwartz distributions. We remark that $\mathcal{E}^+$ is a regular Dirichlet form on $\overline{M^+} = M_\alpha \setminus M^-$ and $\overline{M^-} = M_\alpha \setminus M^+$ (see, e.g., [FOT11, Lemma 3.3.3]). Its associated Markov process is reflected by the singularity.

When $\mathcal{L}|_{C_c^\infty(M)}$ has only one Markovian extension, i.e., whenever $\mathcal{L}_F = \mathcal{L}_N$, we say that it is *Markov unique*. Clearly, if $\mathcal{L}|_{C_c^\infty(M)}$ is essentially self-adjoint, it is also Markov unique. The next proposition shows that Markov uniqueness is a strictly stronger property than essential self-adjointness.

Proposition 5.4.11. *The operator $\mathcal{L}|_{C_c^\infty(M)}$ is Markov unique if and only if $\alpha \notin (-1, 1)$.*

Proof. As observed above, the statement is an immediate consequence of Theorem 5.2.3 for $\alpha \leqslant -3$ and $\alpha \geqslant 1$. If $\alpha \in (-3,-1]$, since by Theorem 5.2.3 all $\widehat{\mathcal{L}}_k$ for $k \neq 0$ are essentially self-adjoint, it holds that $\mathcal{L}_N = \widehat{A}_0 \oplus (\bigoplus_{k\in\mathbb{N}} \widehat{\mathcal{L}}_k)$ for some self-adjoint extension $\widehat{A}_0$ of $\widehat{\mathcal{L}}_0$. Recall the definition of $\phi_D^\pm$ and $\phi_N^\pm$ given in (5.2.7) and with abuse of notation let $\phi_D^\pm(x,\theta) = \phi_D^\pm(x)$ and $\phi_N^\pm(x,\theta) = \phi_N^\pm(x)$. Since $\mathcal{E}^+(\phi_N^\pm,\phi_N^\pm) = +\infty$ if and only if $\alpha \leqslant -1$, we get that $\phi_N^+,\phi_N^- \notin D(\mathcal{E}^+) \supset D(\mathcal{L}_N)$ if $\alpha \leqslant -1$. Hence, by Theorem 5.2.10, it holds that $\widehat{A}_0 = (\widehat{\mathcal{L}}_0)_F$ and hence that $\mathcal{L}_N = \mathcal{L}_F$.

On the other hand, if $\alpha \in (-1,1)$, the result follows from Lemma 5.4.9. In fact, it implies that $\phi_D \notin H_0^1(M,d\mu) = D(\mathcal{E}_F)$ but, since $\mathcal{E}^+(\phi_D,\phi_D) < +\infty$, we have that $\phi_D \in D(\mathcal{E}^+)$. This proves that $\mathcal{L}_F \subsetneq \mathcal{L}_N$. ☐

By the previous result, when $\alpha \in (-1,1)$ it makes sense to consider the bridging extension, associated to the operator $\mathcal{L}_B$ and the form $\mathcal{E}_B$, defined by

$$\mathcal{E}_B(u,v) = \int_{M_\alpha} \left(\partial_x u\, \partial_x v + |x|^{2\alpha} \partial_\theta u\, \partial_\theta v\right) d\mu,$$
$$D(\mathcal{E}_B) = \{u \in H^1(M,d\mu) \mid u(0^+,\theta) = u(0^-,\theta) \text{ for a.e. } \theta \in \mathbb{S}\}.$$

From Theorem 5.4.3 and the fact that $\mathcal{E}_B = \mathcal{E}^+|_{D(\mathcal{E}_B)}$ follows immediately that $\mathcal{E}_B$ is a Dirichlet form, and hence $\mathcal{L}_F \subset \mathcal{L}_B \subset \mathcal{L}_N$. Moreover, due to the regularity of $\mathcal{E}^+$ and the symmetry of the boundary conditions appearing in $D(\mathcal{E}_B)$, follows that $\mathcal{E}_B$ is regular on the whole M_α. Its associated Markov process can cross, with continuous trajectories, the singularity.

We conclude this section by specifying the domains of the Markovian self-adjoint extensions associated with $\mathcal{E}_F$, $\mathcal{E}^+$ and, when it is defined, $\mathcal{E}_B$.

Proposition 5.4.12. *It holds that* $D(\mathcal{L}_F) = H_0^2(M,d\mu)$, *while*

$$D(\mathcal{L}_N) = \{u \in H^1(M,d\mu) \mid (\Delta u, v) = (\nabla u, \nabla v) \textit{ for any } v \in H^1(M,d\mu)\}.$$

Moreover, if $\alpha \in (-1,1)$, *the domain of* $\mathcal{L}_B$ *is*

$$D(\mathcal{L}_B) = \{H^2(M_\alpha,d\mu) \mid u(0^+,\cdot) = u(0^-,\cdot),\ \lim_{x\to 0^+} |x|^{-\alpha}\partial_x u(x,\cdot) = \lim_{x\to 0^-} |x|^{-\alpha}\partial_x u(x,\cdot) \textit{ for a.e. } \theta \in \mathbb{S}\}.$$

Proof. In view of Theorem 5.4.1, to prove that A is the operator associated with $\mathcal{E}_A$ it suffices to prove that $D(A) \subset D(\mathcal{E}_A)$ and that $\mathcal{E}_A(u,v) = (-Au,v)$ for any $u \in D(A)$ and $v \in D(\mathcal{E}_A)$. The requirement on the domain is satisfied by definition in all three cases. We proceed to prove the second fact.

Friedrichs extension. By integration by parts it follows that $\mathcal{E}_F(u,v) = (-\mathcal{L}_F u,v)$ for any $u,v \in C_c^\infty(M)$, and this equality can be extended to $u \in H_0^2(M,d\mu) = D(\mathcal{L}_F)$ and $v \in H_0^1(M,d\mu) = D(\mathcal{E}_F)$.

Neumann extension. The property that $\mathcal{E}^+(u,v) = (-\mathcal{L}_N u,v)$ for any $u \in D(\mathcal{L}_N)$ and $v \in D(\mathcal{E}^+)$ is contained in the definition.

Bridging extension. By an integration by parts, it follows that

$$\int_{M_\alpha} \left(\partial_x u\, \partial_x v + x^{2\alpha} \partial_\theta u\, \partial_\theta v\right) d\mu = (-\mathcal{L}_B u, v) - \int_{\mathbb{S}} v|x|^{-\alpha}\partial_x u\Big|_{x=0^-}^{0^+} d\theta = (-\mathcal{L}_B u, v).$$

☐

5.4.3 Stochastic completeness and recurrence on M_α

We are interested in localizing the properties of stochastic completeness and recurrence of a Markovian self-adjoint extension A of $\mathcal{L}|_{C_c^\infty(M)}$. Due to the already mentioned repulsing properties of Neumann boundary conditions, the natural way to operate is to consider the Neumann restriction introduced in Definition 5.4.8.

Observe that, if $U \subset M$ is an open set such that $\bar{U} \cap (\{-\infty, 0, +\infty\} \times \mathbb{S}) = \varnothing$, then the Neumann restriction $\mathcal{E}_A|_U$ is always recurrent on U. In fact, in this case, there exist two constants $0 < C_1 < C_2$ such that $C_1 \,dx\,d\theta \leqslant d\mu \leqslant C_2 \,dx\,d\theta$ on U and clearly $1 \in D(\mathcal{E}_A|_U) = H^1(U, dx\,d\theta)$, that by Theorem 5.4.6 implies the recurrence. For this reason, we will concentrate only on the properties "at 0" or "at ∞".

Definition 5.4.13. Given a Markovian extension A of $\mathcal{L}|_{C_c^\infty(M)}$, we say that it is *stochastically complete at* 0 (resp. *recurrent at* 0) if its Neumann restriction to $M_0 = (-1, 1) \times \mathbb{S}$, is stochastically complete (resp. recurrent). We say that A is exploding at 0 if it is not stochastically complete at 0. Considering $M_\infty = (1, \infty) \times \mathbb{S}$, we define stochastic completeness, recurrence and explosiveness at ∞ in the same way.

In order to justify this approach, we will need the following.

Proposition 5.4.14. *A Markovian extension A of $\mathcal{L}|_{C_c^\infty(M)}$ is stochastically complete (resp. recurrent) if and only if it is stochastically complete (resp. recurrent) both at 0 and at ∞.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A)$ such that $u_n \to 1$ a.e. and $\mathcal{E}_A(u_n, u_n) \to 0$. Since $D(\mathcal{E}_A|_{M_0}) = \{u|_{M_0} \mid u \in D(\mathcal{E}_A)\}$ and $D(\mathcal{E}_A|_{M_\infty}) = \{u|_{M_\infty} \mid u \in D(\mathcal{E}_A)\}$ follows that $\{u_n|_{M_0}\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A|_{M_0})$ and $\{u_n|_{M_\infty}\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A|_{M_\infty})$. Moreover, it is clear that $u_n|_{M_0}, u_n|_{M_\infty} \to 1$ a.e. and $\mathcal{E}_A|_{M_0}(u_n|_{M_0}, u_n|_{M_0})$, $\mathcal{E}_A|_{M_\infty}(u_n|_{M_\infty}, u_n|_{M_\infty}) \to 0$. By Theorem 5.4.6, this proves that if $\mathcal{E}_A$ is recurrent it is recurrent also at 0 and ∞.

On the other hand, if $A|_{M_0}$ and $A|_{M_\infty}$ are recurrent, we can always choose the sequences $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A|_{M_0})$ and $\{v_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A|_{M_\infty})$ approximating 1 such that they equal 1 in a neighborhood N of $\partial_{M_0} = \partial_{M_\infty} = (\{1\} \times \mathbb{S}) \cup (\{-1\} \times \mathbb{S})$. In fact the constant function satisfies the Neumann boundary conditions we posed on $\partial M_0 = \partial M_\infty$ for the operators associated with $\mathcal{E}_A|_{M_0}$ and $\mathcal{E}_A|_{M_\infty}$. Hence, by gluing u_n and v_n we get a sequence of functions in $D(\mathcal{E}_A)$ approximating 1. The same argument gives also the equivalence of the stochastic completeness, exploiting the characterization given in Theorem 5.4.5. $\square$

Before proceeding with the classification of the stochastic completeness and recurrence of $\mathcal{L}_F$, $\mathcal{L}_N$ and $\mathcal{L}_B$, we need the following result. For an operator acting on $L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$, the definition of stochastic completeness and recurrence at 0 or at ∞ is given substituting M_0 and M_∞ in Definition 5.4.13 with $(-1, 1)$ and $(1, +\infty)$.

Proposition 5.4.15. *Let A be a Markovian self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$ and assume it decomposes as $A = \widehat{A}_0 \oplus \tilde{A}$, where $\widehat{A}_0$ is a self-adjoint operator on H_0 and $\tilde{A}$ is a self-adjoint operator on $\bigoplus_{k\neq 0} H_k$. Then, $\widehat{A}_0$ is a Markovian self-adjoint extension of $\widehat{\Delta}_0$. Moreover, A is stochastically complete (resp. recurrent) at 0 or at ∞ if and only if so is $\widehat{A}_0$.*

Proof. Let $\pi_k : L^2(M, d\mu) \to H_k = L^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$ be the projection on the k-th Fourier component. In particular, recall that $\pi_0 u = (2\pi)^{-1}\int_0^{2\pi} u(x,\theta)\,d\theta$. Let $u \in D(\widehat{A}_0) \subset L^2(\mathbb{R}, |x|^{-\alpha}dx)$ be such that $0 \leqslant u \leqslant 1$. Hence, posing $\tilde{u}(x,\theta) = u(x)$, due to the splitting of A follows that $\tilde{u} \in D(A)$ and by the markovianity follows that $0 \leqslant A\tilde{u} \leqslant 1$. The first part of the statement is then proved by observing that, since $\pi_0 \tilde{u} = u$ and $\pi_k \tilde{u} = 0$ for $k \neq 0$, we have $A\tilde{u}(x,\theta) = \widehat{A}_0 u(x)$ for any $(x,\theta) \in M$.

We prove the second part of the statement only at 0, since the arguments to treat the at ∞ case are analogous. First of all, we show that the stochastic completeness of A and $\widehat{A}_0$ at 0 are equivalent. If $1 : M_0 \to \mathbb{R}$ is the constant function, it holds that $\pi_0 1 = 1 : (-1,1) \to \mathbb{R}$. Moreover, due to the splitting of A, we have that $e^{tA} = e^{t\widehat{A}_0} \oplus e^{t\tilde{A}}$ Hence, it follows that $e^{tA}1 = e^{t\widehat{A}_0}1$. This, by Definition 1.3.4, proves the claim.

To prove the equivalence of the recurrences at 0, we start by observing that $D(\mathcal{E}_A) = D(\mathcal{E}_{\widehat{A}_0}) \oplus D(\mathcal{E}_{\tilde{A}})$ and that

$$\mathcal{E}_A(u,v) = \mathcal{E}_{\widehat{A}_0}(\pi_0 u, \pi_0 v) + \mathcal{E}_{\tilde{A}}(\oplus_{k\neq 0}\pi_k u, \oplus_{k \neq 0}\pi_k v), \qquad \text{for any } u,v \in D(\mathcal{E}_A) \quad (5.4.4)$$

In particular, since $\pi_0 1 = 1$ this implies that $\mathcal{E}_A|_{M_0}(1,1) = \mathcal{E}_{\widehat{A}_0}|_{(-1,1)}(1,1)$. By Theorem 5.4.6, this proves that if $\widehat{A}_0$ is recurrent at 0, so is A. Assume now that $A|_{M_0}$ is recurrent. By Theorem 5.4.6 there exists $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_A|_{M_0})$ such that $0 \leqslant u_n \leqslant 1$ a.e., $u_n \longrightarrow 1$ a.e. and $\mathcal{E}_A|_{M_0}(u_n, v) \to 0$ for any v in the extended domain $D(\mathcal{E}_A|_{M_0})_e$. By dominated convergence, it follows that $\pi_0 u_n = (2\pi)^{-1}\int_0^{2\pi} u_n(\cdot,\theta)\, d\theta \to 1$ for a.e. $x \in (-1,1)$. For any $v \in D(\mathcal{E}_{\widehat{A}_0}|_{(-1,1)})_e$, let $\tilde{v}(x,\theta) = v(x)$. It is easy to see that $\tilde{v} \in D(\mathcal{E}_A|_{M_0})_e$ Then, by applying (5.4.4) we get

$$\mathcal{E}_{\widehat{A}_0}|_{(-1,1)}(\pi_0 u_n, v) = \mathcal{E}_A|_{M_0}(u_n,\tilde{v}) \longrightarrow 0, \qquad \text{for any } v \in D(\mathcal{E}_{\widehat{A}_0}|_{(-1,1)})_e.$$

Since $0 \leqslant \pi_0 u_n \leqslant 1$, this proves that $\widehat{A}_0|_{(-1,1)}$ is recurrent □

The following proposition answers the problem of stochastic completeness or recurrence of the Friedrichs extension.

Proposition 5.4.16. *Let Δ_F be the Friedrichs extension of $\mathcal{L}|_{C_c^\infty(M)}$. Then, the following holds*

	at 0	*at* ∞
$\alpha < -1$	*recurrent*	*stochastically complete*
$\alpha = -1$	*recurrent*	*recurrent*
$\alpha > -1$	*explosive*	*recurrent*

In particular, $\mathcal{L}_F$ is stochastically complete for $\alpha < -1$, recurrent for $\alpha = -1$ and explosive for $\alpha > -1$.

Proof. The part regarding the recurrence is a consequence of Lemma 5.4.9 and Theorem 5.4.6, while the last statement is a consequence of Proposition 5.4.14. Thus, to complete the proof it suffices to prove that $\mathcal{L}_F$ is stochastically complete at $+\infty$ if $\alpha < -1$ and not stochastically complete at 0 if $\alpha > -1$.

By Proposition 5.4.15 and the fact that $\mathcal{L}_F = \oplus_{k\in\mathbb{Z}}(\widehat{\mathcal{L}}_k)_F$, we actually need to prove this fact only for $(\widehat{\mathcal{L}}_0)_F$. Moreover, since the Friederichs extension decouples the dynamics on the two sides of the singularity, we can work only on $(0,+\infty)$ instead that on $\mathbb{R}\setminus\{0\}$. As in Lemma 5.4.9, we let $\widehat{\mathcal{E}}_0$ to be the Dirichlet form associated to the Friederichs extension of $\widehat{\mathcal{L}}_0$.

We start by proving the explosion for $\alpha > -1$ on $(0,1)$. Let us proceed by contradiction and assume that $(\widehat{\mathcal{L}}_0)_F$ is stochastically complete on $(0,1)$. By Theorem 5.4.5, there exists $u_n \in D(\widehat{\mathcal{E}}_0|_{(0,1)})$, $0 \leqslant u_n \leqslant 1$, $u_n \longrightarrow 1$ a.e. and such that $\widehat{\mathcal{E}}_0|_{(0,1)}(u_n, v) \longrightarrow 0$ for any $v \in$

$D(\widehat{\mathcal{E}}_0|_{(0,1)}) \cap L^1((0,1), x^{-\alpha}dx)$. Since $\widehat{\mathcal{E}}_0|_{(0,1)}$ is regular on $(0,1]$, we can choose the sequence such that $u_n \in C_c^\infty((0,1])$. In particular $u_n(0) = \lim_{x\downarrow 0} u_n(x) = 0$ for any n. Let us define, for any $0 < R \leqslant 1$,

$$v_R(x) = \lim_{r\downarrow 0} (1 - u_{r,R}(x)) = \begin{cases} x^{1+\alpha}/R^{1+\alpha} & \text{if } 0 \leqslant x < R, \\ 1 & \text{if } 0 \leqslant x \geqslant R, \end{cases}$$

where $u_{r,R}$ is defined in (5.4.2). Observe that, by the probabilistic interpretation of $u_{r,R}$ given in Remark 5.4.10, follows that $v_R(x)$ is the probability that the Markov process associated with $(\widehat{\mathcal{L}}_0)_F$ and starting from x exits the interval $(0,R)$ before being absorbed by the singularity at 0. A simple computation shows that $v_R \in D(\widehat{\mathcal{E}}_0|_{(0,1)}) \cap L^1((0,1), x^{-\alpha}dx)$. Thus, by definition of $\{u_n\}_{n\in\mathbb{N}}$ and a direct computation we get

$$0 = \lim_{n\to+\infty} \widehat{\mathcal{E}}_0|_{(0,1)}(u_n, v_R) = \frac{1+\alpha}{R^{1+\alpha}} \lim_{n\to+\infty} \int_0^R \partial_x u_n \, dx = \frac{1+\alpha}{R^{1+\alpha}} \lim_{n\to+\infty} u_n(R).$$

Hence, $u_n(R) \longrightarrow 0$ for any $0 < R < 1$, contradicting the fact that $u_n \longrightarrow 1$ a.e..

To complete the proof, we need to show that if $\alpha < -1$, $(\widehat{\mathcal{L}}_0)_F$ is stochastically complete on $(1,+\infty)$. Since on $(1,+\infty)$ the metric is regular, we can complete it to a C^∞ Riemannian metric on the whole interval $(0,+\infty)$. Then, the result follows by applying the characterization of stochastic completeness on model manifolds contained in [Gri09] and Theorem 5.4.14. □

We are now in a position to completely answer to (Q2).

Theorem 5.4.17. *Consider M_α, for $\alpha \in \mathbb{R}$, and the corresponding Laplace-Beltrami operator $\mathcal{L}$ as an unbounded operator on $L^2(M, d\mu)$. Then it holds the following.*

- *If $\alpha < -1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is stochastically complete at 0 and recurrent at ∞;*
- *if $\alpha = -1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is recurrent both at 0 and at ∞;*
- *if $\alpha \in (-1,1)$, then $\mathcal{L}$ is not Markov unique and, moreover,*
  - *any Markovian extension of $\mathcal{L}$ is recurrent at ∞,*
  - *$\mathcal{L}_F$ is explosive at 0, while both $\mathcal{L}_B$ and $\mathcal{L}_N$ are recurrent at 0,*
- *if $\alpha \geqslant 1$ then $\mathcal{L}$ is Markov unique, and $\mathcal{L}_F$ is explosive at 0 and recurrent at ∞;*

Proof. By Propositions 5.4.11 and 5.4.16, we are left only to prove statement (iii)-(a) and the second part of (iii)-(b), i.e., the stochastic completeness of $\mathcal{L}_N$ and $\mathcal{L}_B$ at 0 when $\alpha \in (-1,1)$.

Statement (iii)-(a) follows from [FOT11, Theorem 1.6.4], since for $\alpha \in (-1,1)$ the Friederichs extension (which is the minimal extension of $\mathcal{L}|_{C_c^\infty(M)}$) is recurrent at ∞. To complete the proof it suffices to observe that, for these values of α, it holds that $1 \in H^1(M_0, d\mu) = D(\mathcal{E}^+|_{M_0})$ and clearly $\mathcal{E}^+|_{M_0}(1,1) = 0$. By Theorem 5.4.6, this implies the recurrence of $\mathcal{E}^+$ at 0. The recurrence of $\mathcal{E}_B$ at 0 follows analogously, observing that 1 is also continuous on $\mathcal{Z}$ and hence it belongs to $D(\mathcal{E}_B|_{M_0})$ □

6 SPECTRAL ANALYSIS OF THE GRUSHIN CYLINDER AND SPHERE

In this chapter, we study spectral properties of the Laplace-Beltrami operator on two relevant almost-Riemannian manifolds, namely the Grushin structures on the cylinder and on the sphere, proving the theorems stated in Section 1.3.3. As proved in [BL] and recalled in Theorem 1.3.2, in these structures the singular set acts as a barrier for the evolution of the heat and of a quantum particle, although geodesics can cross it.

6.1 SPECTRAL ANALYSIS AND THE AHRONOV-BOHM EFFECT FOR THE GRUSHIN CYLINDER

In this section we study the spectrum of the Grushin cylinder, with or without an Aharonov-Bohm magnetic field.

6.1.1 Grushin metric and associated Laplace-Beltrami operator

Recall from Section 1.3.3 that the Grushin almost-Riemannian structure on the cylinder defines on $M_+ \cup M_- = (\mathbb{R}\setminus\{0\}) \times \mathbb{S}^1$ the metric $g = dx^2 + x^{-2}d\theta^2$, the volume $dV = \sqrt{|g|}\,dx\,d\theta = \frac{1}{|x|}\,dx\,d\theta$ and the Laplace-Beltrami operator

$$\mathcal{L}\,u = \partial_x^2 u - \frac{1}{x}\partial_x u + x^2\partial_\theta^2 u.$$

As already mentioned, this operator with domain $C_c^\infty((\mathbb{R}\setminus\{0\}) \times \mathbb{S}^1)$ is essentially self-adjoint in $L^2(M, d\omega)$ and hence the evolutions on the two sides of the singularity are decoupled. Thus, we will henceforth consider the self-adjoint operator $\mathcal{L}$ acting only on M_+. Namely, the domain $D(\mathcal{L})$ will be the closure w.r.t. the graph norm of $C_c^\infty(M_+)$.

Recall the Fourier decomposition of $L^2(M_+, dV)$ w.r.t. the varaible θ introduced in Section 1.3.3,

$$L^2(M_+, dV) = \bigoplus_{k\in\mathbb{Z}} H_k, \qquad \text{where } H_k \simeq L^2\left(\mathbb{R}_+, \frac{1}{x}dx\right).$$

The operator $\mathcal{L}$ decomposes as $\mathcal{L} = \bigoplus_{k\in\mathbb{Z}} \widehat{\mathcal{L}}_k$, where

$$\widehat{\mathcal{L}}_k = \partial_x^2 - \frac{1}{x}\partial_x - k^2x^2.$$

Since $\mathcal{L}$ is essentially self-adjoint, each $\widehat{\mathcal{L}}_k$ is a self-adjoint operator on the closure w.r.t. the graph norm of $C_c^\infty(\mathbb{R}_+)$.

Consider the unitary transformation $U : L^2\left(\mathbb{R}_+, \frac{1}{x}dx\right) \to L^2(\mathbb{R}_+, dx)$ defined by $Uv(x) := \sqrt{x}v(x)$. The operator $\widehat{\mathcal{L}}_k$ is then transformed to

$$L_k := U\widehat{\mathcal{L}}_k U^{-1} = \partial_x^2 - \frac{3}{4}\frac{1}{x^2} - k^2x^2, \quad D(L_k) = UD(\widehat{\mathcal{L}}_k). \tag{6.1.1}$$

6.1.2 Spectral properties of the Laplace-Beltrami operator

The following relations for the spectrum of direct sums $A = \oplus_{k\in\mathbb{Z}} A_k$ are well known (see e.g., [RS80]):

$$\sigma_p(A) = \bigcup_{k\in\mathbb{Z}} \sigma_p(A_k) \tag{6.1.2}$$

$$\sigma_c(A) = \left(\left(\bigcup_{k\in\mathbb{Z}} \sigma_p(A_k)\right)^c \cap \left(\bigcup_{k\in\mathbb{Z}} \sigma_r(A_k)\right)^c \cap \left(\bigcup_{k\in\mathbb{Z}} \sigma_p(A_k)\right)\right)$$
$$\cup \left\{\lambda \in \bigcap_{k\in\mathbb{Z}} \rho(A_k) \mid \sup_k \|R_\lambda(A_k)\| = +\infty\right\}. \tag{6.1.3}$$

Since the spectrum is invariant under unitary transformations, from this it follows that we can reduce the study of the spectrum of $\mathcal{L}$ to that of the operators L_k. Exploiting this reduction we can easily prove Theorem 1.3.9.

Proof of Theorem 1.3.9. The operator $-L_0$ is the Schrödinger operator on the real line with a Calogero potential of strength $3/4$. It is well-known that this operator has continuous spectrum $[0, +\infty)$, see e.g., [RS80, Sec. VIII.10].

Let now $k \neq 0$ and let us compute the solutions of the eigenvalue problem

$$(L_k - \lambda)u = 0 \iff (\widehat{\mathcal{L}}_k - \lambda)U^{-1}u = 0. \tag{6.1.4}$$

Through the change of variables $|k|x^2 \mapsto z$ and multiplying by $4(k^2)z$, we obtain

$$\partial_z^2 v(z) + \left(-\frac{1}{4} + \frac{\lambda}{4z|k|}\right) = 0.$$

This is the well-known Whittaker equation, whose solutions are the Whittaker functions $M_{\frac{\lambda}{4|k|},\frac{1}{2}}(z)$ and $W_{\frac{\lambda}{4|k|},\frac{1}{2}}(z)$. The solutions of the eigenvalue problem (6.1.4) are then

$$u_1(x) = \frac{1}{\sqrt{x}} M_{\frac{\lambda}{4|k|},\frac{1}{2}}(|k|x^2), \qquad u_2(x) = \frac{1}{\sqrt{x}} W_{\frac{\lambda}{4|k|},\frac{1}{2}}(|k|x^2).$$

Through the asymptotic expansions of $M_{\nu,\mu}$ and $W_{\nu,\mu}$ (see e.g., [BE56]) one easily sees that u_1 is never square-integrable near infinity. On the other hand, $u_2 \in L^2(\mathbb{R}_+)$ if and only if there exists a non-negative integer ℓ such that $-\ell = \frac{1}{2} - \nu + \mu = \frac{1}{2} - \frac{\lambda}{4|k|} + \frac{1}{2}$. Namely, for any $k \in \mathbb{N}$ there exists a sequence $\{\lambda_{n,k} = 4|k|n\}_{n\in\mathbb{N}}$ of eigenvalues with (non-normalized) eigenfunction $x \mapsto \psi_{n,k}(x) = W_{n,\frac{1}{2}}(|k|x^2)/\sqrt{x}$.

Let $k = 0$. Then, the operator L_0 given by (6.1.1) can be interpreted as a Laplace operator with a relatively infinitesimally-bounded perturbation. It is a well known result [RS80] that its spectrum is purely absolutely continuous and equal to $[0, \infty)$.

Finally, the statement follows from the definition of U^{-1} and the relations (6.1.2) and (6.1.3). □

Remark 6.1.1. Observe that (6.1.1) can be explicitly solved, it's solutions being of the form

$$\begin{cases} c_1 x^{3/2} + \frac{c_2}{\sqrt{x}} & \text{for } \lambda = 0, \\ c_1 \sqrt{x}\, J_1(\sqrt{\lambda} x) + c_2 \sqrt{x}\, Y_1(\sqrt{\lambda} x) & \text{for } \lambda > 0, \end{cases} \tag{6.1.5}$$

where J_1 and Y_1 are the Bessel functions of order 1. In particular, for $\lambda \geqslant 0$ one has the explicit form of the generalized eigenfunctions of the absolutely continuous spectrum of L_0.

With the eigenvalue counting function $N(E)$ defined as in (1.3.9), we are able to prove Corollary 1.3.10.

Proof of Corollary 1.3.10. Obviously, by Theorem 1.3.9, the following holds,

$$\#\{\lambda \in \sigma_p(-\mathcal{L}) \mid \lambda \leqslant E\} = \#\{(n,k) \in \mathbb{N} \times \mathbb{Z} \setminus \{0\} \mid 4n|k| \leqslant E\}. \tag{6.1.6}$$

For fixed $k \in \mathbb{Z} \setminus \{0\}$, this implies that the couples (n,k) admissible in the above, are those such that $n \leqslant E/(4|k|)$. Moreover, it is clear that for any $|k| > E/4$ there exist no couple (n,k) is admissible. These facts and (6.1.6) yield the estimation

$$N(E) = \sum_{0<|k|\leqslant \frac{E}{4}} \frac{E}{4|k|} = \frac{E}{2} \sum_{\ell=1}^{\lfloor E/4 \rfloor} \frac{1}{\ell}.$$

The well-known asymptotic formula (see e.g., [CG96])

$$\sum_{m=1}^{n} \frac{1}{m} = \log(n) + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \tag{6.1.7}$$

where γ is the Euler-Mascheroni constant, then implies the following asymptotic estimate as $E \to +\infty$,

$$\begin{aligned} N(E) &= \frac{E}{2}\left(\log\left(\frac{E}{4} + \gamma + O\left(\frac{1}{E}\right)\right)\right) \\ &= \frac{E}{2}\log(E) + (\gamma - 2\log(2))\,\frac{E}{2} + O(1). \end{aligned}$$

□

6.1.3 Aharonov-Bohm effect

Now we look at the Aharonov-Bohm effect on the Grushin cylinder. As already introduced in Section 1.3.3, the magnetic Laplace-Beltrami operator on M_+ with vector potential $\omega^b = -ib\, d\theta$, $b \in \mathbb{R}$, is

$$\mathcal{L}^b = \partial_x^2 - \frac{1}{x}\partial_x + x^2\partial_\theta^2 - 2ibx^2\partial_\theta - b^2x^2.$$

After the transformation U, introduced in Section 6.1.1, we obtain the following operator acting on $L^2(M_+, dx\,d\theta)$,

$$L_b = U\mathcal{L}^b U^{-1} = \partial_x^2 - \frac{3}{4}\frac{1}{x^2} + x^2\left(\partial_\theta - ib\right)^2.$$

Through a straightforeward extension of the proof of Theorem 1.3.9, we immediately get Theorem 1.3.13. For $b \in \mathbb{Z}$ it is evident that the role of L_0 in the proof of Theorem 1.3.9 is now taken by L_b.

Proof of Corollary 1.3.14. W.l.o.g. we restrict ourselves to $b \in (-1/2, 1/2)$, therefore $\kappa = 0$. Clearly, if $b = 0$ the statement reduces to the one of Corollary 1.3.10. Thus we can assume $b \neq 0$.

Replacing k with $|k-b|$ in the proof of Corollary 1.3.10 we observe that for $k = 0$ the additional term $E/4|b|$ appears in the count. Thus, we can rewrite the counting function as

$$N(E) = \frac{E}{4}\sum_{k=1}^{\lfloor E/4\rfloor}\frac{1}{k+b} + \frac{E}{4|b|} + \frac{E}{4}\sum_{k=1}^{\lfloor E/4\rfloor}\frac{1}{k-b}$$

We now apply the following identity (see e.g., [OLBC10])

$$\sum_{k=1}^{n}\frac{1}{k+x} = \psi(n+x+1) - \psi(1+x),$$

and the asymptotic estimate as $x \to \infty$

$$\psi(x+1) = \log(x) + \gamma + \frac{1}{2x} + O\left(\frac{1}{x^2}\right),$$

where $\psi(x)$ is the digamma function and γ is the Euler-Mascheroni constant. By a straightforward computation we obtain

$$\begin{aligned} N(E) &= \frac{E}{4}\left(\psi\left(\lfloor E/4\rfloor + b + 1\right) - \psi(1+b)\right) + \frac{E}{4|b|} + \frac{E}{4}\left(\psi\left(\lfloor E/4\rfloor - b + 1\right) - \psi(1-b)\right) \\ &= \frac{E}{2}\log(E) + \frac{E}{2}\left(\frac{1}{2|b|} + \gamma - 2\log(2) - \frac{\psi(1-b)+\psi(1+b)}{2}\right) + O(1). \end{aligned}$$

The general result then follows by shifting the above computation with $b \mapsto |\kappa - b|$. □

We can now precisely determine the degeneracy of the eigenvalues, depending on the value of b.

Proof of Theorem 1.3.16. The proof is divided in three cases.

Case 1, $b \in \mathbb{R}\setminus\mathbb{Q}$: This immediately implies that $|k-b| \in \mathbb{R}\setminus\mathbb{Q}$. It is then straightforward to show that there exist no $(n', k') \neq (n, k)$ such that $\lambda^b_{n',k'} = \lambda^b_{n,k}$.

Case 2, $b \in \mathbb{Q}$: Let us write $b = p/q$ with $p, q \in \mathbb{Z}$ such that $(p, q) = 1$. Fix (n, k) and $(n', k') \neq (n, k)$ such that $\lambda^b_{n,k} = \lambda^b_{n',k'}$. Then,

$$4n'|qk' - p| = q\lambda^b_{n,k}. \tag{6.1.8}$$

W.l.o.g. assume that $qk' > p$. Then, since $4n'|qk'-p|$ cannot divide q because $(q,p)=1$, we have that it must divide $\lambda_{n,k}$.

From $q \neq 1$, $\{|qk'-p| \mid k' \in \mathbb{Z}\} \subseteq (q\mathbb{Z}-p) \subsetneq \mathbb{Z}$, we obtain that the number of couples (n',k') such that $4n'|k'-b| = \lambda^b_{n,k}$ is bounded above by $2d(\lambda^b_{n,k}/4)$, where $d(n)$ denotes the number of divisors of n. In fact, if $|k'-b| = d_1$ for some $d_1 \in \mathbb{Q}$ divides $\lambda^b_{n,k}/4$, then $n' = \lambda^b_{n,k}/(4d_1)$. Observe that, due to the presence of a non integer b in the term $|k-b|$, not all the possible divisors can be considered. However, if a $k' > b$ can be taken, then there exists a $k'' < b$ that will give an additional couple (k'',n').

Case 3, $b \in \mathbb{Z}$: In this case, equation (6.1.8) reduces to $4n'|k'-b| = \lambda_{n,k}$. Then, for any (n,k) with $k \neq b$, a simple computation shows that

$$\lambda^b_{k,n+b} = \lambda^b_{n,k+b} = \lambda^b_{n,-k+b} = \lambda^b_{k,-n+b}.$$

However if $n|k|$ is even, the combination $n = k = \lambda^K_{n,k+K}/8$ is repeated twice. Therefore the degeneracy is given by

$$\begin{cases} 2d(\lambda/4), & \text{if } \lambda/4 \text{ is odd,} \\ 2d(\lambda/4)-2, & \text{if } \lambda/4 \text{ is even.} \end{cases}$$

Finally, this degeneracy cannot be achieved for $b \in \mathbb{Q}\setminus\mathbb{Z}$. In fact, it would require $\mathbb{Z} \ni k' = (qn+p)/q$ which is impossible for $(q,p)=1$. □

Corollary 1.3.14 suggests that in the limit $b \to k$, the number of eigenvalues in a finite interval explodes. Corollary 1.3.18 makes this statement more precise, namely

- for any fixed $k \in \mathbb{Z}$ and for any $n \in \mathbb{N}$, the spacing between the eigenvalues

  $$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| \to 0 \text{ as } b \to k;$$

- for any fixed interval $I = [x_1,x_2] \subset [0,\infty)$ and any $N \in \mathbb{N}$

  $$\#\{n \in \mathbb{N} \mid \lambda^b_{n,k} \in I\} \geqslant N \text{ as } b \to k.$$

Proof of Corollary 1.3.18 (Corollary of Theorem 1.3.13). Observe that

$$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| = 4|k-b|. \tag{6.1.9}$$

Taking the limit for $k \to b$ in the above yields immediately the first statement.

To prove the second statement, assume w.l.o.g. $k \geqslant 0$ and define

$$L(b) := \left\lceil \frac{x_1}{4|k-b|} \right\rceil, \quad R(b) := \left\lfloor \frac{x_2}{4|k-b|} \right\rfloor.$$

Then $\lambda^b_{L(b),k} \geqslant x_1$ and $\lambda^b_{R(b),k} \leqslant x_2$. If now

$$|k-b| \leqslant \frac{x_2-x_1}{4(N+1)},$$

by (6.1.9) we obtain that

$$\#\{\lambda^b_{i,k} \mid L(b) \leqslant i \leqslant R(b)\} \geqslant N.$$

This completes the proof of the second statement and hence of the corollary. □

This limiting process affects also the eigenfunctions. Theorem 1.3.19 describes how the spectrum of the k-th Fourier components decompactifies in the limit $b \to k$ and produces the absolutely continuous part of the spectrum.

Proof of Theorem 1.3.19. Recall that $\psi^b_{n,k}(x,\theta) = e^{ik\theta} W_{n,\frac{1}{2}}(|k-b|x^2)/x$. Since w.l.o.g. we can assume $k = 0$, to complete the proof it suffices to show that

$$W_{n_j,\frac{1}{2}}(|b_j|x^2) \to \frac{\sqrt{\lambda}\,x}{2} J_1(\sqrt{\lambda}x). \tag{6.1.10}$$

Let us recall the following classical results (see resp. [MOS66] and [BE56]).

$$W_{n,1/2}(z) = (-1)^{n-1} z e^{-\frac{1}{2}z} L^1_{n-1}(z),$$
$$\lim_{n\to\infty} n^{-\alpha} L^1_n(x/n) = x^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{x}).$$

Here L^α_n is the generalized Laguerre polynomial of degree n with parameter α and the limit is in the sense of uniform convergence on compact sets.

Define

$$n_j := 2j \text{ and } b_j := \frac{\lambda}{4(n_j+1)},$$

so that $\lambda^{b_j}_{n_j+1,0} = \lambda$ for all $j > 0$. Then

$$\lim_{j\to\infty} W_{n_j,\frac{1}{2}}(|b_j|x^2) = \lim_{j\to\infty} \frac{\lambda n_j x^2}{4(n_j+1)} \exp\left(-\frac{1}{2n_j}\frac{\lambda n_j x^2}{4(n_j+1)}\right) n_j^{-1} L^1_{n_j}\left(\frac{1}{n_j}\frac{\lambda n_j x^2}{4(n_j+1)}\right)$$
$$= \frac{\sqrt{\lambda}\,x}{2} J_1(\sqrt{\lambda}x).$$

This completes the proof of (6.1.10). □

Remark 6.1.2. By changing the parity of the sequence n_j used in the previous proof, we could change the sign of the limit in (1.3.12).

6.2 SPECTRAL ANALYSIS AND AHRONOV-BOHM EFFECT FOR THE GRUSHIN SPHERE

In this section we consider the Grushin almost-Riemannian metric on the sphere introduced in Section 1.3.3.

6.2.1 Grushin metric and associated Laplace-Beltrami operator

The Grushin almost-Riemannian structure defines the metric $g = dx^2 + \tan(x)^{-2} d\theta^2$ on $S_+ \cup S_-$, the sphere S^2 without the equatorial line. The natural volume form defined by this metric is $dV = \sqrt{|g|}\,dx\,d\theta = |\tan(x)|^{-1}\,dx\,d\theta$ and the associated Laplace-Beltrami operator is

$$\mathcal{L}u = \partial_x^2 u - \frac{1}{\sin(x)\cos(x)}\partial_x u + \tan^2(x)\partial_\phi^2 u$$

As shown in [BL], the operator $\mathcal{L}$ with domain $C_c^\infty(S_+ \cup S_-)$ is essentially self-adjoint in $L^2(\mathbb{S}^2, d\omega)$ and hence the evolutions on the two sides of the singularity are decoupled. Moreover it has purely discrete spectrum. In the following we will consider the self-adjoint operator $\mathcal{L}$ restricted to $L^2(S_+, dV)$. Namely, the domain $D(\mathcal{L})$ will be the closure w.r.t. the graph norm of $C_c^\infty(S_+)$.

As in the previous section we can separate the space using the orthoromal eigenbase of $\mathbb{S}^1$ getting

$$L^2(S_+, d\omega) = \bigoplus_{k=-\infty}^{\infty} H_k^{S_+}, \quad H_k^{S_+} \simeq L^2([0, \pi/2), \tan(x)dx). \tag{6.2.1}$$

On each $H_k^{S_+}$ the operator separates as

$$\tilde{\mathcal{L}}_k := \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x - \tan^2(x)k^2.$$

6.2.2 Spectral properties of the Laplace-Beltrami operator

Exploiting decomposition (6.2.1), we can describe the spectrum of $\mathcal{L}$.

Proof of Theorem 1.3.11. We look for solutions $\phi \in H_k^{S_+}$ of the eigenvalue equation

$$-\tilde{\mathcal{L}}_k\phi(x) = \lambda\phi(x).$$

Since k appears in $\tilde{\mathcal{L}}_k$ only squared, the eigenvalues are simmetric with respect to $k = 0$. To simplify the notation, in the following we will assume $k \geqslant 0$, but the same considerations hold for $k < 0$ substituting $|k|$ to k.

With the change of variables $z = \cos(x)^2$ and writing $\phi(x) = (-z)^{\frac{k}{2}}\varphi(z)$, the eigenvalue equation becomes

$$4(-z)^{\frac{k}{2}}\left(z(1-z)\partial_z^2\varphi(z) + (1+k)(1-z)\partial_z\varphi(z) + \frac{\lambda}{4}\varphi(z)\right) = 0.$$

The equation in bracket is a particular example of the well-known Euler's hypergeometric equations. Two linearly independent solutions can be found in terms of Gauss Hypergeometric Functions $F(a, b; c; z)$ (see [BE56, Vol. 1, Ch. 2]) as follows:

$$\phi_1(x) = i^{-k}\cos(x)^{-k}F\left(-\frac{k}{2} - \frac{\sqrt{\lambda + k^2}}{2}, -\frac{k}{2} + \frac{\sqrt{\lambda + k^2}}{2}; 1-k; \cos(x)^2\right),$$

$$\phi_2(x) = i^{k}\cos(x)^{k}F\left(\frac{k}{2} - \frac{\sqrt{\lambda + k^2}}{2}, \frac{k}{2} + \frac{\sqrt{\lambda + k^2}}{2}; 1+k; \cos(x)^2\right).$$

Notice here that in the case $\frac{k}{2} \pm \frac{\sqrt{\lambda+k^2}}{2}, k-1 \in \mathbb{N}_0$ the first solution is not defined, in fact we are in the so called *degenerate case* and the only regular solution is ϕ_2. Therefore we do not need to introduce the other corresponding linearly independent solution.

A solution is an eigenfunction of the Laplace-Beltrami operator if it is in $H_k^{S_+}$. For this to be true, the solutions has to be square-integrable with respect to the measure $d\omega := \tan(x)^{-1}dx$

near 0. This is equivalent to be $O(\sin(x))$ for $x \to 0$ and, in particular, it is requires that the solution be zero at $x = 0$.

Let us recall that

$$\phi_1(0) = i^{-k} \frac{\Gamma(1-k)}{\Gamma\left(-\frac{k}{2} - \frac{\sqrt{k^2+\lambda}}{2} + 1\right)\Gamma\left(-\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} + 1\right)}, \tag{6.2.2}$$

$$\phi_2(0) = i^{k} \frac{\Gamma(k+1)}{\Gamma\left(\frac{k}{2} - \frac{\sqrt{k^2+\lambda}}{2} + 1\right)\Gamma\left(\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} + 1\right)}. \tag{6.2.3}$$

Moreover, observing that

$$\pm\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} \geqslant 0 \text{ for all } k \in \mathbb{N}_0 \text{ and } \lambda \in \mathbb{R}^0_+,$$

it is immediate to obtain $\phi_1(0) \neq 0$ if $k \geqslant 1$. By the previous considerations, this implies that $\phi_1 \notin H_k^{S+}$ for $k \geqslant 1$. Since $k = 0$ corresponds to the degenerate case, where the two solutions coincide, in the following we can restrict ourselves to consider only ϕ_2.

By (6.2.3), in order for $\phi_2(0) = 0$ to hold there has to exist $n \in \mathbb{N}_0$ such that λ satisfies

$$\frac{k}{2} + 1 - \frac{\sqrt{k^2+\lambda}}{2} = -n.$$

Solving the above for λ, yields the following expression for the candidate eigenvalue

$$\lambda = \lambda^+_{k,n} := 4(1+n)(1+n+k).$$

In order to prove that the candidate eigenvalues $\lambda^+_{k,n}$ are indeed eigenvalues, we check the order of convergence of the solutions. For this purpose we use the well-known identity [OLBC10, 15.2(ii)] for $a = -m \in \mathbb{Z}_- \cup \{0\}$, $b > 0$ and $c \notin \mathbb{Z}_- \cup \{0\}$

$$F(-m, b; c; z) = \sum_{\ell=0}^{m} (-1)^m \binom{m}{\ell} \frac{(b)_\ell}{(c)_\ell} x^\ell. \tag{6.2.4}$$

Plugging the values of the parameters for ϕ_2 in the above, and setting $\lambda = \lambda^+_{k,n}$, we obtain

$$F\left(-(n+1), n+k+1; k+1; \cos(x)^2\right) = \sum_{\ell=0}^{n+1} (-1)^{n+1} \binom{n+1}{\ell} \frac{(n+k+1)_\ell}{(k+1)_\ell} \cos(x)^{2\ell}.$$

This shows that ϕ_2 and his derivative have the correct behaviour in 0 and are regular at $\pi/2$, completing the proof.

Finally, in order to obtain the expression of the eigenvalues and eigenfunctions given in the statement, it suffices to replace $n+1$ with n in the definition of $\lambda^b_{n,k}$. The theorem then follows by the symmetry w.r.t. $k = 0$ of the problem. □

We are now in a position to derive the Weyl law for the Laplace-Beltrami operator of the Grushin sphere.

Proof of Corollary 1.3.12. By the simmetry of the eigenvalue problem w.r.t. $k = 0$, it follows that $N(E) = 2\#\{k \in \mathbb{N}, n \in \mathbb{N} \mid \lambda_{n,k} \leqslant E\} + \#\{n \in \mathbb{N} \mid \lambda_{n,0} \leqslant E\}$. Let $N_0(E)$ be the counting function for this second sum. It is easy to see that $\lambda(0, n) \leqslant E$ for $n \in \left[0, \lfloor\sqrt{E}/2\rfloor\right]$. Therefore $N_0(E) = O(\sqrt{E})$.

Let $N_+(E)$ be the counting function for positive values of k. A simple computation shows that $\lambda(k, n) \leqslant E$ if and only if

$$0 < k \leqslant \left\lfloor \frac{E - 4n^2}{4n} \right\rfloor.$$

Additionally, notice that if $n > \lfloor\sqrt{E}/2\rfloor =: \eta_1(E)$, then $\frac{E-4n^2}{4n} < 0$.

Let

$$K(n) := \frac{E - 4n^2}{4n}. \tag{6.2.5}$$

Then we have

$$\lfloor K(n) \rfloor \leqslant \# \left\{ k \in \left[0, \frac{E - 4n^2}{4n}\right] \cap \mathbb{N} \right\} \leqslant \lceil K(n) \rceil,$$

and consequently

$$\sum_{n=1}^{\eta_1(E)} \lfloor K(n) \rfloor \leqslant N_+(E) \leqslant \sum_{n=1}^{\eta_1(E)} \lceil K(n) \rceil.$$

Due to the asymptotic estimate (6.1.7), we immediately get the following asymptotic estimate as $E \to +\infty$

$$\begin{aligned} N(E) = 2N_+(E) + N_0(E) &= 2 \sum_{n=1}^{\eta_1(E)} K(n) + O(\sqrt{E}) \\ &= \frac{E}{2} \sum_{n=1}^{\eta_1(E)} \frac{1}{n} - 2 \sum_{n=1}^{\eta_1(E)} n + O(\sqrt{E}) \\ &= \frac{E}{2} \left(\log(\sqrt{E}/2) + \gamma \right) - \frac{E}{4} + O(\sqrt{E}) \\ &= \frac{E}{4} \log(E) + \left(\gamma - \log(2) - \frac{1}{2} \right) \frac{E}{2} + O(\sqrt{E}). \end{aligned}$$

This completes the proof. □

It follows from Theorem 1.3.11 that for $k \neq 0$ the operator $\tilde{\mathcal{L}}_k$ acting on H_k presents an infinite amount of eigenvalues accumulating at infinity that can be explicitly described by

$$\sigma_d(H_k) := \{\lambda_{n,|k|} = 4(1+n)(1+n+|k|) \mid n \in \mathbb{N}, k \in \mathbb{Z}\}.$$

Due to the symmetry w.r.t. k of $\lambda_{n,|k|}$, all the eigenvalues are at least double degenerate. Moreover, this degeneracy must be finite. In fact it is enough to observe that each operator has a ground state of energy greater than

$$\lambda_{1,|k|} = 4(|k| + 1),$$

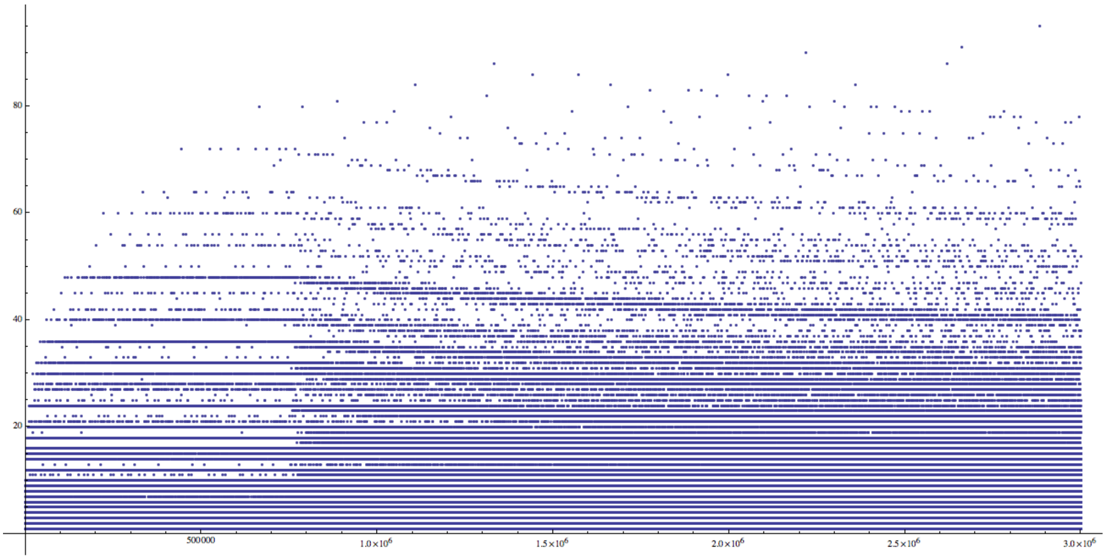

Figure 16: Degeneracy (halved) of the first 4.893.535 eigenvalues (namely $\lambda_{n,k} < 3 \cdot 10^6$). Observe that only one of those eigenvalues attains the higher multiplicity of 110, far below than our upper bound.

and that the function $n \mapsto \lambda_{n,|k|}$ is increasing in $n \in \mathbb{N}$.

The degeneracy of an eigenvalue λ can be easily bounded above by $(\lambda - 4)/2$, but this is far from being optimal. In fact, the computation of the first five million eigenvalues (see Figure 16) suggests the growth of the degeneracy to be irregular and slow as in the case of the Grushin cylinder (see Theorem 1.3.16). We remark that plotting more eigenvalues yielded qualitatively the same plot.

Unfortunately, it is not possible to obtain a more precise description of the degeneracy with the simple techniques employed in Theorem 1.3.16. Indeed, in this case the problem reduces to counting the number of solutions of a non-linear Diophantine equation, which is well-known to be an hard problem.

6.2.3 Aharonov-Bohm effect

We now consider the Aharonov-Bohm on the Grushin sphere. Since $\mathbb{S}^2$ is simply connected, any closed form is exact and hence we cannot hope to obtain an Aharonov-Bohm effect without artificially poking a hole in the manifold. This is the same phenomena as in the original Aharonov-Bohm effect [AT98, dOP08].

In this section we will thus consider the magnetic Laplace-Beltrami operator induced by the magnetic vector potential $\omega^b = -ib\, d\phi$, $b \in \mathbb{R}$, on the north hemisphere of $\mathbb{S}^2$ with the north pole removed, denoted by $\mathbb{S}^\circ_+$. Note that on $\mathbb{S}^\circ_+$ the corresponding magnetic field is 0. The resulting operator is

$$\mathcal{L}^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x + \tan(x)^2 \left(\partial_\phi^2 - 2ib\partial_\phi - b^2\right). \tag{6.2.6}$$

That decomposes in the Fourier components

$$\tilde{\mathcal{L}}_k^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x - \tan(x)^2(k-b)^2$$

It is important to remark that, due to the forceful removal of the origin (the north pole), this magnetic Laplace-Beltrami operator is not essentially self-adjoint on $C_c^\infty(S_+^\circ)$. As is customary for the standard Aharonov-Bohm effect on $\mathbb{R}^2$, we will consider the self-adjoint extension obtained by posing Dirichlet boundary conditions on the origin. Namely, we will take as domain the closure of $C_c^\infty(S_+^\circ)$ w.r.t. the Sobolev norm $W_0^{1,2}(S_+^\circ)$.

We immediately get Theorem 1.3.22 by the same arguments of Theorem 1.3.11, replacing k with $k-b$. We then easily obtain Corollary 1.3.23.

Proof of Corollary 1.3.23. Without loss of generality, we can assume $b \in (-1/2, 1/2)$, i.e., $\kappa = 0$. If $b = 0$ then the statement reduces to the one of Corollary 1.3.12, so let us assume, by the simmetry of the eigenvalue expression, that $b < 0$.

We then proceed similarly to the proof of Corollary 1.3.12, splitting the counting function in two components $N_-(E)$ and $N_+(E)$, depending on wether k is smaller or bigger than b. The estimates on $N_+(E)$ and $N_-(E)$ are then obtained as in Corollary 1.3.12, paying attention to the presence of b. Indeed, in the notation of the proof of that corollary, in both cases we obtain

$$K(n) = \frac{E - 4n^2}{4n} + |b|.$$

Since the sum has to be computed for $n \leqslant \eta_1(E)$, given by

$$\eta_1(E) = \left\lfloor \frac{|b| + \sqrt{E + |b|^2}}{2} \right\rfloor,$$

it is easy to see that b only appears (linearly) in the $O(\sqrt{E})$ term. Since $|b| \leqslant 1/2$ this completes the proof. □

As already anticipated in the previous section, the degeneracy of the spectrum for the Grushin sphere seems to be of similar nature as for the Grushin cylinder, at least from a numerical point of view, but having a precise control on it is much more involved and probably not possible at present. However, one can still prove that the degeneracy is very unstable with respect to the parameter b and, in particular, that the spectrum is simple for $b \in \mathbb{R} \setminus \mathbb{Q}$ and finitely degenerate for $b \in \mathbb{Q}$. This is summarised in Corollary 1.3.23 and it follows from an argument very close to the one in the proof of Corollary 1.3.16.

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