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Sieving the Landscape of Gravity Theories

From the Equivalence Principles
to the Near-Planck Regime

Thesis submitted for the degree of
Doctor Philosophiæ

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Abstract

This thesis focusses on three main aspects of the foundations of any theory of gravity where the gravitational field admits a geometric interpretation: (a) the principles of equivalence; (b) their role as selection rules in the landscape of extended theories of gravity; and (c) the possible modifications of the spacetime structure at a "mesoscopic" scale, due to underlying, microscopic-level, quantum-gravitational effects.

The first result of the work is the introduction of a formal definition of the Gravitational Weak Equivalence Principle, which expresses the universality of free fall of test objects with non-negligible self-gravity, in a matter-free environment. This principle extends the Galilean universality of free-fall world-lines for test bodies with negligible self-gravity (Weak Equivalence Principle).

Second, we use the Gravitational Weak Equivalence Principle to build a sieve for some classes of extended theories of gravity, to rule out all models yielding non-universal free-fall motion for self-gravitating test bodies. When applied to metric theories of gravity in four spacetime dimensions, the method singles out General Relativity (both with and without the cosmological constant term), whereas in higher-dimensional scenarios the whole class of Lanczos–Lovelock gravity theories also passes the test.

Finally, we focus on the traditional, manifold-based model of spacetime, and on how it could be modified, at a "mesoscopic" (experimentally attainable) level, by the presence of an underlying, sub-Planckian quantum regime. The possible modifications are examined in terms of their consequences on the hypotheses at the basis of von Ignatowski's derivation of the Lorentz transformations. It results that either such modifications affect sectors already tightly constrained (e.g. violations of the principle of relativity and/or of spatial isotropy), or they demand a radical breakdown of the operative interpretation of the coordinates as readings of clocks and rods.

This thesis is based on the results appeared in the papers:

∗ E. Di Casola, S. Liberati and S. Sonego, "Nonequivalence of equivalence principles", (*Am. J. Phys.* — in press). E-print arXiv:1310.7426 [gr-qc].

∗ E. Di Casola, S. Liberati and S. Sonego, "Weak equivalence principle for self-gravitating bodies: A sieve for purely metric theories of gravity", *Phys. Rev. D* **89** (2014) 084053. E-print arXiv:1401.0030 [gr-qc].

∗ E. Di Casola, S. Liberati and S. Sonego, "Between quantum and classical gravity: Is there a mesoscopic spacetime?", (*Found. Phys.* — submitted). E-print arXiv:1405.5085 [gr-qc].

To my beloved parents, *Euro* & *Velia*.

To my graceful partner, *Alessandra*.

Contents

Preface

All, in the world, exists to lead to, and to end in, a book.

S. Mallarmé, *Poésies*.

This thesis at a glance

To sum up a doctoral dissertation in one motto might be a tough task. Too much content to "squeeze" into a single sentence, too much background information and supplementary details to account for.

This work tries to make an exception, for once. Indeed, the whole point of this document can be condensed in the following clause.

This is the story of a free fall.

There are at least two reasons why the line above offers a comprehensive overview of the meaning of the present work. One is a strictly technical reason; the other has a broader, more "tangential" goal.

The technical aspect highlighted in the motto is the notion of *free fall*, which can be thought of as the main theme for almost two thirds of the thesis. We shall show how a certain version of the classical notion of Galilean free fall can be transformed into a set of selection rules for the vast landscape of extended theories of gravity.

By choosing a simplified — yet, unified — point of view, grounded on the physical assumptions behind Galileo's vision of the free fall, it is possible to sidestep a large class of technical problems and extract a formal sieve, to be used later as a guiding principle when searching for a viable theoretical description of gravitational phenomena.

The upshot of the discussion is that the free-fall motion of small bodies with non-negligible self-gravity exhibits a *universal* character only in the case of *purely metric* theories of gravitation, i.e. theories which encode all the gravitational degrees of freedom in one, and only one, physical field: the metric.

From this result we deduce that General Relativity passes through the sieve (as expected), whereas all theories with additional gravitational degrees of freedom are ruled out. Also, we find that, as soon as the number of dimensions of the underlying spacetime manifold is allowed to grow above four, a plethora of concurring theories, named after Lanczos and Lovelock, now pass the test as well — it is still a small subset of all the theories compatible with other theoretical principles commonly associated with gravitational phenomena, but is large enough to raise new questions.

In this respect, we shall critically review many aspects of both the "golden-age" free fall — whose universality has acquired the name of *Equivalence Principle* —, and of its most recent variations.

The thesis will begin with a bird's eye view of various theories currently challenging General Relativity as the best explanatory framework for gravitational phenomena (Chapter 1). Once the landscape is set, we shall provide (Chapter 2) an examination of Galilean universality of free fall and of the other fundamental statements shaping the traditional models of gravitation theories — collectively called "equivalence principles" —. After that, we shall discuss (Chapter 3) a suitable extension of the free fall motion to self-gravitating systems, denoted as *Gravitational Weak Equivalence Principle*. Its features, implementation, and consequences will be used to build the mentioned sieve for extended theories of gravity, and to later put it to test on various archetypical models.

Finally (Chapter 4), we shall pursue our examination of extensions of the general relativistic framework by investigating the possible fate of Local Lorentz Invariance at scales close to the Planck one. More specifically, we shall discuss the concept of *classical (continuous) spacetime*, and how this notion is supposed to be shaken, changed or even abruptly ruled out by the introduction of quantum effects propagating up to a "mesoscopic", observable scale.

Chapter 5 has the office to deliver the concluding remarks.

Behind the motto

The all-encompassing sentence "This is the story of a free fall" also tells something else. The message does not pertain strictly to Physics or Mathematics (not at first sight, at least); yet, it is of some significance.

The motto restates, decidedly, that *this is a story*. In the Readers' hands rests a tale, like any other true tale, of life and death, love and hate, success and failure, tiny sparkles of inspiration and long ages of *transpiration*.

In the 1940's, Kurt Vonnegut — at the time, a young Anthropology student — suggested an elementary method of graphical representation of all the archetypical plots underlying the stories in literature, mythology, epic poetry, etc.

This thesis, being itself a story, could then be denoted by a variously twisted line laid somewhere between Vonnegut's curves called *Man in Hole*, *Boy Meets Girl*, *Cinderella*, and *Kafka*.[1]

[1] One of Vonnegut's last examples in his list of archetypes was Shakespeare's *Hamlet*; according to him, an unparalleled masterpiece, and an unsung lesson in telling the truth. Simply juxtaposing that eminent play to this thesis would have made a horrible service to both us and the Bard, hence we pityingly omitted the term of comparison. Kafka, on the other hand, deserves an explicit reference here in view of what David Foster Wallace once pointed out about him: that Kafka was among the few who could provide true examples of actual *funniness* (see D. F. Wallace, "Some Remarks on Kafka's Funniness from Which Probably Not Enough Has Been Removed", in *Consider the Lobster. And other essays*, 2005).

The presence of mathematical symbols, bibliographic records, ubiquitous present tense, and logical explanations ought not to fool the Reader, making Him or Her think that this document contains anything different from a bare story. It is true that the narrative techniques adopted herewith try to best fit the sub-genre of scientific literature; also, and most importantly, when reading this thesis, the Reader will not allow for a suspension of His or Her disbelief, as it usually happens with other sorts of narration.

But again, undeniably we daresay, this is a story. We fail at seeing in which sense it could ever be otherwise.

If the story is sound, it will seamlessly lead the Reader until its natural end. If not, it will collapse at some point under the burden of its inconsistencies. Nowhere in the process it will change its innermost nature of a story.

The story of a free fall.

A few acknowledgements

A huge ensemble of people are to be thanked wholeheartedly, for having made this enterprise not only interesting and stimulating, but also, and above all, funny and worthy (almost) every day. Below are gathered a few representative names. Apologies for all possible omissions.

Many many thanks, then, to *Stefano Liberati* ("Il Capo", patient supervisor, inspiring mentor, and good friend), Sebastiano Sonego (goading, enlightening deputy-supervisor), Antonio Romano ("once supervisor, always supervisor").

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If anything good might ever emerge from this work, these are the first persons who must be acknowledged. Needless to say, on the other hand, we are (I am) the only one responsible for any mistake or fallacy within this text and its content.

Trieste, 22$^{\text{nd}}$ October 2014.

Eolo Di Casola

[2]Also, a special thank-you to Prof. Harvey R. Brown and to Prof. Domenico Giulini, the two opponents, for all their stimulating feedbacks and constructive critical remarks.

Notations and Conventions

This system of units [the Planck units, for which $c = G = \hbar = 1$, Ed.] serves to keep both classical relativists and particle physicists happy. This system also serves to keep both classical relativists and particle physicists confused since it is essentially impossible to use dimensional analysis to check results for consistency.

M. Visser, *Lorentzian Wormholes.*

We dedicate here a few paragraphs to present and establish the most common conventions used throughout this work. Other considerations on this crucial theme have been postponed to the footnotes complementing the text.

Physical conventions

Physical constants; Units. In this thesis, each quantity comes with its own physical dimensions in terms of the fundamental units (e.g. velocity as $[l]\,[t]^{-1}$, action as $[m]\,[l]^2\,[t]^{-1}$, momentum density as $[m]\,[l]^{-2}\,[t]^{-1}$, and so forth). MKSA and cgs systems are used in most of the cases.

All the instances of the fundamental constants (c, G, $\hbar$) are written down explicitly to assure consistency of the formulæ and easiness of dimensional check.

The principal physical constants in use are:

c The speed of light in vacuo.

G The gravitational constant (Newton's constant).

$\hbar$ The normalised quantum of action (Planck's constant h over 2π).

Λ The cosmological constant.

Mathematical notation

Indices; tensor notation. We adopt the abstract index notation as presented in Refs. [542] and [332]. Symbols denoting geometric objects ("kernels") are accompanied by a varying number of lowercase, latin, italic, superscript/subscript indices ("dummy", "slot" indices), to keep track of the covariant and contravariant valence of the tensors involved.

On the other hand, lowercase greek, "coordinate" indices span the values $\{1, 2, \ldots, n\}$, with n the number of spacetime dimensions. Coordinates will be generically denoted χ^α (mostly in mathematical context, when dealing with manifolds in general), z^α, or y^α (in more physical contexts).

On Lorentzian manifolds where an indefinite quadratic form (a metric) is defined, non-null coordinates are such that the last index denotes the time coordinate. The notation x^α will be reserved to pseudo-Cartesian rectangular coordinates (x, y, z, ct) in special relativistic context.

Other indices. Uppercase latin indices act as counting indices, their values picked within the set of integers. The same counting indices are used sometimes, with apt placement, when dealing with tetrads, n-beins and objects alike.

Symmetrisation and anti-symmetrisation. Round [respectively, square] parentheses enclosing sequences of indices denote complete symmetrisation [respectively, anti-symmetrisation] in all the enclosed indices, including the normalisation factors; therefore, it is e.g.

$$A_{(ab)} := \frac{1}{2}\left(A_{ab} + A_{ba}\right) \quad , \quad A_{[ab]} := \frac{1}{2}\left(A_{ab} - A_{ba}\right) \ . \tag{1}$$

A single-letter index (subscript or superscript) enclosed in round brackets denotes the non-tensorial character of that index.

Signature, Riemann, etc. The signature convention for the metric is the "mostly plus" one, as in Refs. [353] and [250]: the Lorentzian metric on a given spacetime, when written in locally inertial, pseudo-Cartesian coordinates assumes the form

$$g_{\mu\nu} = \mathrm{diag}\left(+1, +1, \ldots, -1\right) \ . \tag{2}$$

The Riemann curvature tensor ${R_{abc}}^d$ is defined, in an arbitrary coordinate system, as [542]

$${R_{\alpha\beta\gamma}}^\delta := \partial_\beta {\Gamma^\delta}_{\alpha\gamma} - \partial_\alpha {\Gamma^\delta}_{\beta\gamma} + {\Gamma^\lambda}_{\alpha\gamma}{\Gamma^\delta}_{\lambda\beta} - {\Gamma^\lambda}_{\beta\gamma}{\Gamma^\delta}_{\lambda\alpha} \ . \tag{3}$$

The Ricci tensor and the scalar curvature are then given by, respectively

$$R_{ab} := {R_{acb}}^c \ , \tag{4}$$

$$R := g^{ab} R_{ab} \ . \tag{5}$$

Relativistic spacetimes. Following Ref. [250], a relativistic spacetime is defined here as the pair: (i) manifold (with an atlas of differentiable coordinates), and (ii) metric tensor[3]

$$\mathscr{M} \equiv (M, g_{ab}) \ . \tag{6}$$

The topology, dimension and signature of the spacetime are considered as non-dynamical. The spacetime signature is everywhere Lorentzian; that of three-dimensional "physical" space, always Euclidean. When needed, e.g. in the case of affine structures decoupled from the metrical ones, the previous definition of spacetime is extended so as to include also a covariant derivative operator, "D" or "∇", characterising the connexion.

The connexion coefficients compatible with the metric (Levi-Civita connexion coefficients) are written as $\Gamma^a{}_{bc}$ for sake of uniformity with the abstract index notation, although it is intended that the indices there are not of the abstract type — the metric-compatible connexion coefficients are not, in general, tensorial objects — whereas the general affine connexion coefficients are denoted as $\mathit{\Delta}^a{}_{bc}$ (in principle, the indices a, b, c might be abstract indices).

Miscellaneous notations. The calligraphic letter "$\mathscr{B}$" after a bulk term in an action functional denotes apt boundary terms necessary to make the variational problem well-posed, and to extract well-defined field equations.

Script letters such as $\mathcal{S}, \mathcal{E}, \mathcal{G}, \ldots$ found in Chapter 3 refer to the first-order terms in an ϵ-series expansion of the corresponding italic letters $S, E, G, \ldots$.

[3] For the issue of assigning a temporal orientation on a relativistic spacetime see Ref. [470], or [332].

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Chapter 1

Theories of Gravity: A Guided Tour

Little is left to tell.

S. Beckett, *Ohio Impromptu*.

The word *gravitation*, or *gravity*, carries quite a wide range of meanings, each one evoking a cornerstone in the history of physics. At its roots lie concepts familiar to the layperson, such as the planetary motions, the ebb and flow of the tides, and the universal attraction between massive bodies (whereupon, the fall of small objects towards the ground). Nowadays, however, the same name has stretched its semantic boundaries far beyond the limits of the Earth, and of the Solar system, to embrace the entire history of the Cosmos, and the groundbreaking idea that space and time are dynamical entities themselves, rather than immovable, absolute scaffoldings.

Having found a way to lock such a wealth of diverse observations into a single, self-consistent framework remains a grand achievement of theoretical physicists. General Relativity, our current paradigm explaining gravitational phenomena, is perhaps "the most beautiful theory": a crown in the regalia of theoretical physics.

Still, when it comes to gravity, talking about "one" theory, or "the" theory, is a bit of a misnomer. General Relativity is undoubtedly a queen among its peers, but it is just one (admittedly outstanding) model within a dense crowd of other frameworks, each one fighting and racing to dethrone the queen, and become the next ruler.

Nowadays, to go beyond General Relativity is considered a foreseeable step towards the ultimate theory-of-everything, and many proposals attempt to accomplish the mission. The branches of the resulting "family tree" range from tiny variations on the main theme, to radical departures, and one needs a clear understanding of the conceptual and formal aspects of each model to proper assess it and compare it with the dominant scheme.

This Chapter is devoted to offer a wide-angle perspective on the landscape of gravity theories currently challenging General Relativity. We shall keep the presentation as compact as possible, in agreement with the synthetic approach pursued in this work; the plan is to offer a minimal description of some essential elements, highlighting only those aspects which will have an echo in the following. To the supporting bibliography is assigned the risky mission of filling in the numerous blanks.

1.1 Gravitation: the story in a nutshell

A review of the theories of gravitation, however sketchy, cannot dispense with a short presentation of the milestones. The long and complex history of gravity is recollected here through three of its main turning points: the initial sparkle of Newtonian mechanics; the poorly-known, yet crucial contribution by Gunnar Nordström and Adriaan Fokker; the climatic outburst of Einstein's General Relativity.

In this brief sequence, it is tempting to look at the discontinuities, the paradigmatic shifts, the new ideas outnumbering the old intuitions. Rather, we would like to highlight the deep sense of seamless continuity driving the evolution of this branch of theoretical physics; which has been much more effective than any revolutionary afflatus.

Newton recognised the common nature of gravity and of the other mechanical forces, and placed the world on the absolute stage where all the events unfolded, for the pleasure of the audience to see them. Nordström and Fokker extended this notion so as to make it compatible with the relativity of simultaneity. Einstein, finally, framed the missing link, and understood that, in the *grand show* of the Cosmos, the theatre is a mere illusion, and so is the audience: the only surviving truth, is that there is but one, all-embracing, ever-going performance, and we are simply a part of it.[1]

1.1.1 A glance at Newtonian physics

In Newton's theory [376, 353], the mechanical behaviour of point masses is governed by the following, fundamental equation (second law of dynamics), written here in modern vector notation

$$\mathbf{F} = m_{\mathrm{I}}\mathbf{a} \; . \tag{1.1}$$

It is an ordinary differential equation where the force law $\mathbf{F}\left(\mathbf{r},\mathbf{v},t\right)$, a function of position, velocity, and eventually time itself, balances the product of the acceleration $\mathbf{a} = \mathrm{d}^2\mathbf{r}\left(t\right)/\mathrm{d}t^2$, times the *inertial mass*, symbol m_{I}. The latter is an intrinsic property of the particle, and its value can be measured e.g. by means of collision experiments.

All the gravitational phenomena known at Newton's time, i.e. the planetary orbits, the fall of bodies towards the ground, the ebb and flow of tides, can be accounted for by Eq. (1.1), provided that the (always attractive) gravitational

[1]We leave it to the Reader to further decide whether the piece is a Comedy or a Tragedy, for this choice largely exceeds the limits of this metaphor.

force exerted by a point particle "1" on a point particle "2" be of the form

$$\mathbf{F}_{\text{grav}} = -G \frac{m_{\text{A},1} m_{\text{P},2}}{r_{12}^2} \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}} \ , \tag{1.2}$$

with r_{12} the mutual distance of the masses, and $G = 6.67 \cdot 10^{-8}\ \text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}$ a universal coupling constant — Newton's constant — measuring the strength of the interaction. The two quantities m_A and m_P are the *active* and *passive gravitational masses*, respectively — the former refers to the body generating the gravitational attraction, the latter to the one feeling the force —. In view of the action-reaction principle (Newton's third law [94]), they need be proportional with a universal constant; units are chosen then so that such constant reduces to one. Hence it suffices to speak about a *gravitational mass*, m_G, which is in principle different from the inertial one [122, 394].

Gravitational attraction is a conservative force, i.e. the work done by gravity in moving a particle between two points is independent on the path joining them; this feature allows to introduce a scalar potential field $\Psi(\mathbf{r}, t)$, whose gradient provides, point by point and for each moment in time, the value of the gravitational force acting on a test body; in formulæ

$$\mathbf{F}_{\text{grav}} = -m_\text{G} \nabla \Psi(\mathbf{r}, t) \ . \tag{1.3}$$

By using Gauß' flux theorem, it is possible to prove that the configurations of the field Ψ are determined by the distribution of matter — conveniently represented by a mass density function $\rho_\text{G}(\mathbf{r}, t)$ — according to Poisson's equation

$$\Delta \Psi(\mathbf{r}, t) = 4\pi G \rho_\text{G}(\mathbf{r}, t) \ , \tag{1.4}$$

where the Laplacian operator Δ stands for $\sum_i \nabla_i \nabla_i$, $i = 1, 2, 3$. It is worth pointing out that any variation in the distribution of matter generates a variation in the gravitational field which propagates *instantaneously* in space, the information travelling at infinite speed. Newtonian gravity is thus a theory of action-at-a-distance, an aspect that Newton himself accepted quite reluctantly.[2]

Upon substituting Eq. (1.3) in (1.1), and recalling (1.4), Newton's theory of gravity can be expressed in terms of the following system of local equations

$$\begin{cases} \Delta \Psi(\mathbf{r}, t) = 4\pi G \rho_\text{G}(\mathbf{r}, t) \\ m_\text{I} \dfrac{\text{d}^2 \mathbf{r}}{\text{d}t^2} = -m_\text{G} \nabla \Psi(\mathbf{r}, t) \end{cases} . \tag{1.5}$$

This version of Newtonian gravity looks slightly more complicated than the usual one presented in most textbooks. In particular, we keep track of any reference to the inertial or gravitational character of the masses/densities, contrary to the typical cancellation of both m_I and m_G in the last group of equations above.

The possibility of getting rid of any information about the masses involved is not something one can draw from theoretical arguments; rather, it is up to the experiments to show that m_I and m_G are indeed proportional with a universal coefficient, so that units can be chosen so to have the proportionality constant identically equal to one [122, 394].

[2]See e.g. I. Newton, Letter to Bentley, 25 February 1692/93.

1.1.2 Relativistic gravity from Nordström to Fokker

Newton's equations (1.5) are invariant with respect to Galilean transformations; this translates the idea that any mechanical experiment does not provide a different outcome when performed within the class of inertial observers (initial conditions need be transformed accordingly). The same cannot be said when Lorentz transformations are introduced. The problem arises, then, to find a suitable relativistic extension of Newtonian gravity [353].

In the complex path leading to this result [386], two contributions stand out. Nowadays, they are widely overshadowed by the supremacy of General Relativity; it is instructive, however, to briefly recall them here.

The first step is due to Gunnar Nordström, a relativist based in Helsinki; around 1913, he built the first self-consistent, "modern" theory of gravity (if one does not consider some previous preliminary findings by Poincaré, and the prophetic programme laid down by Clifford). After some failed attempts, he managed to incorporate Newton's model into a properly relativistic framework [382, 452, 226].

The starting point is the following action,

$$S_{\mathrm{Nor}} = -\frac{c^3}{16\pi G}\int\Bigg(\eta^{ab}\partial_a\Phi\,\partial_b\Phi + \\ +\sum_J \frac{m_J G}{c^2}\Phi\left(x^\alpha\right)\int \delta^{(4)}\left(x^\alpha - z_J^\alpha\left(\lambda\right)\right)\sqrt{-\eta_{\alpha\beta}\frac{\mathrm{d}z_J^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}z_J^\beta}{\mathrm{d}\lambda}}\mathrm{d}\lambda\Bigg)\,\mathrm{d}^4x\,. \quad (1.6)$$

In the formula above, Φ is a gravitational scalar field defined over Minkowski spacetime, $c = 2.99 \cdot 10^{10}\ \mathrm{cm\,s^{-1}}$ is the invariant value of the speed of light in vacuum, and $z_J^\alpha\left(\lambda\right)$ gives the world-line of the J-th particle particle with rest mass m_J, referred to a general affine parameter λ. Finally, $\delta^{(4)}$ is a Dirac delta distribution pinpointing the particle's world-line.

The action (1.6) yields both the field equation for the gravitational degree of freedom Φ, and those for the world-lines of the particles freely falling in the gravitational field. The former is extracted upon varying the action with respect to Φ, whereas the latter emerge when the variation is performed with respect to the world-line $z_J^\alpha\left(\lambda\right)$; the resulting formulæ read, collectively

$$\begin{cases} \Phi\Box\Phi = -\dfrac{4\pi G}{c^4}T \\ \Phi\dfrac{\mathrm{d}u^a}{\mathrm{d}\tau} = -c^2\eta^{ab}\partial_b\Phi - u^a\dfrac{\mathrm{d}\Phi}{\mathrm{d}\tau} \end{cases}, \quad (1.7)$$

and have to be compared with the system (1.5). The first row contains the field equation for gravity, and there $T := \eta^{ab}T_{ab}$ is the trace of the stress-energy-momentum tensor of the matter,[3] given by the variation

$$T_{ab} := -\frac{2}{\sqrt{-\eta}}\frac{\delta S_{\mathrm{matter}}}{\delta\eta^{ab}}\,, \quad (1.8)$$

with S_{matter} given by the second line in the action (1.6), and $\eta := \det\eta_{\alpha\beta}$. The free-fall equation on the second row has been rewritten in terms of the

[3] Einstein used to call T the *von Laue scalar* [549, 179].

proper time τ, defined by $\mathrm{d}\tau^2 := -\mathrm{d}s^2/c^2$, with $\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$; finally, $u^\alpha := \mathrm{d}x^\alpha/\mathrm{d}\tau$ is the particle's four-velocity.

Eqs. (1.7) are Lorentz-invariant and, above all, nonlinear, as expected from a theory of gravity complying with the mass-energy equivalence (the energy carried by the field Φ ought to gravitate itself, like any other mass).

Nordström's theory is a praiseworthy proposal; sadly enough, it fails completely on the experimental side [558]. With reference to a static, spherically symmetric solution of the field equations (1.7), and to the motion of a test particle with mass m in this environment, the only case in which Nordström's theory fares decently is the prediction of the gravitational redshift factor $\left(1 + mG/c^2r\right)$ in the classical Pound–Rebka experiment. But, unfortunately, the scalar model also suggests an unobserved additional contribution to tidal deformations, in the form of a Coulomb-type interaction $\left(m^2G^2/c^4r^4\right)\operatorname{diag}(-1,1,1)$. On the other hand, the scalar model does not account at all for the unexplained periastron advance of the planet Mercury (even worse, it predicts a periastron *lag* of $-\pi mG/c^2R$, with R the radius of the orbit). Likewise, it is incapable of predicting any bending effect of light rays, in contrast to what was already believed at the time by a naïve application of the Equivalence Principle. The expected time delay (Shapiro delay) of signals in a round trip with a source of gravity in the middle is in slightly better agreement with observation (it gives a value proportional to $2mG/c^3$; not enough to pass the test, though), whereas the results for the radius of a stable circular orbit, and that for the acceleration of a static test particle do not bring any improvement or correction to the values expected from Newtonian theory [558].

Nordström's gravity was quickly dismissed, and as quickly forgotten. Before leaving the stage, however, it triggered interest on one last aspect — one matching an old idea fostered by Poincaré. Which leads us to Adriaan Fokker.

Adriaan Fokker was a post-doc working in Prague with Einstein; in 1914, he proposed a connection between the structure of scalar gravity and the geometry of curved manifolds [178, 386]. An examination of Eq. (1.7) shows in fact that the action can be rearranged so as to give, upon varied with respect to the paths $z^\alpha_J(\lambda)$'s, the equations of the shortest paths (geodesic lines) on a curved spacetime. It suffices to introduce the Lorentzian metric tensor g_{ab} given by

$$g_{ab} := \Phi^2\,\eta_{ab}\ , \tag{1.9}$$

on a generic, four-dimensional differentiable manifold M with the same topology as Minkowski spacetime. Then, the four-velocity U^a gets normalised with respect to g_{ab}, hence defined as $U^\alpha := \mathrm{d}x^\alpha/\mathrm{d}\tilde{\tau}$, with $\left(\mathrm{d}\tilde{\tau}/\mathrm{d}\tau\right)^2 = \Phi^2$. The Euler–Lagrange derivative of the second line of Eq. (1.6) then yields, for the freely falling particles,

$$U^b\nabla_b U^a = 0 = \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tilde{\tau}} + \Gamma^\alpha{}_{\beta\gamma}U^\beta U^\gamma\ , \tag{1.10}$$

with the coefficients $\Gamma^\alpha{}_{\beta\gamma}$'s given by,

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{\Phi}\left(\delta^\alpha_\gamma\partial_\beta\Phi + \delta^\alpha_\beta\partial_\gamma\Phi - \eta_{\beta\gamma}\partial^\alpha\Phi\right)\ . \tag{1.11}$$

The introduction of a non-flat geometry allows one to evaluate the rate of curvature of the spacetime $\mathscr{M} \equiv (M, g_{ab})$. The specific form (1.9) of the metric then yields, for the scalar curvature R, the expression

$$R = -6\frac{\Box\Phi}{\Phi^3} \ , \tag{1.12}$$

and this last equivalence allows to rewrite the first equation in (1.7) as

$$R = \frac{24\pi G}{c^4}\tilde{T} \ , \tag{1.13}$$

where now the stress-energy-momentum tensor is defined as

$$\tilde{T}_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{ab}} \ , \tag{1.14}$$

with $g := \det g_{\alpha\beta} = \Phi^2 \det \eta_{\alpha\beta}$. Consequently, its trace becomes $\tilde{T} = g^{ab}\tilde{T}_{ab}$.

Eq. (1.13) provides a fully geometrical description of gravity. At the same time, it cancels any information about the underlying presence of the flat metric η_{ab}.[4] The curvature-free character of the non-dynamical background is restored by introducing an additional field equation: to this office, one uses the Weyl conformal tensor C_{abcd} [108, 119], i.e. the traceless part of the Riemann curvature tensor, which vanishes whenever the metric equals Minkowski's one up to an overall conformal factor. The complete set of field equations thus reads

$$\begin{cases} R = \dfrac{24\pi G}{c^4}\tilde{T} \\ C_{abcd} = 0 \\ U^b\nabla_b U^a = 0 \end{cases} . \tag{1.15}$$

In this last system, the first two terms are known as the *Einstein–Fokker equations* of scalar gravity [386, 147], whereas the third element completes the set by providing the equations of motion for test particles,

$$\nabla_b U^a = 0 = \Phi^2\frac{\mathrm{d}U^a}{\mathrm{d}\tilde{\tau}} + c^2 g^{ab}\partial_b\Phi + 2U^a\Phi\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde{\tau}} \ . \tag{1.16}$$

It is worth stressing that the trajectories of free particles cannot emerge from the field equations, but must be postulated separately.

1.1.3 General Relativity: gravity, dynamics, and geometry

General Relativity is the currently received framework explaining gravitational phenomena on macroscopic scales. It is a classical (i.e. non-quantum) theory of the gravitational field, whose degrees of freedom are encoded in the ten components of a rank-2, symmetric, covariant tensor field g_{ab} defined on a

[4]This step is more important than what may be judged on merely formal grounds. As pointed out in Ref. [226], within a recollection of the genesis of scalar gravity, getting rid of the Minkowski metric amounts to accepting that the only meaningful notion of distance in space and time is the one given by clocks and rods — which are connected to g_{ab} — rather than that provided by the background geometric scaffolding. Einstein was aware of the problem, and had coined the two expressions "coordinate distances" (*Koordinatenabstand*, from η_{ab}) and "natural distances" (*natürliche Abstände*, from g_{ab}); Eq. (1.13) decidedly supports the latter concept.

manifold M. The model reproduces Newton's scheme in the weak-field, slow-motion limit; it is non-linear, and relativistic; it emerges from a well-posed variational principle, and enforces (local) stress-energy-momentum conservation; passes all the Solar system tests, and accounts for most of the cosmological phenomena [542, 353, 250, 360, 547, 558].

In a groundbreaking conceptual leap, the theory attacks the monolith of "absolutism" in physics at the fundamental level, by promoting a fully relational approach to spacetime [468]. The concept of action-at-a-distance simply disappears. The gravitational field replaces absolute space(time). The mutual interaction between acting and back-reacting fields, rather than the arrangement of fields on a fixed scaffolding, becomes the only way in which natural phenomena can unfold.

The geometric interpretation of the theory still plays a crucial role, in view of the universal coupling of the gravitational field with matter, but now all these geometric quantities must exhibit dynamical character. The tensor g_{ab} yields the metric content of a pseudo-Riemannian structure equipping the manifold, but it evolves in response to the presence of other fields, and reacts upon those fields, in a constant dialogue.

The theory is formulated in terms of an action, the field equations emerging upon setting to zero the first variation of the sum of the gravitational and matter contributions, i.e.

$$\delta S = \delta S_{\mathrm{GR}} + \delta S_{\mathrm{matter}} = 0 \,. \tag{1.17}$$

The part describing the gravitational sector is given by [542],[5]

$$S_{\mathrm{GR}}\left[g^{ab}\right] = \frac{c^4}{16\pi G}\left(\int_\Omega R\sqrt{-g}\,\mathrm{d}^4 y + 2\oint_{\partial\Omega} K\sqrt{h}\,\mathrm{d}^3 y\right) , \tag{1.18}$$

where the inverse metric g^{ab} has been assumed as the independent field (such choice is equivalent to that of picking the twice-covariant form g_{ab}).

The two terms in Eq. (1.18) are the Einstein–Hilbert term (first piece), and the Gibbons–Hawking–York boundary term (second piece) [220, 566]. In the Einstein–Hilbert term, $R = g^{ab}R_{acb}{}^{c}$ is the scalar curvature, obtained by double contraction of the Riemann curvature tensor, and g is the metric determinant; Ω is the coordinate representation of an arbitrary four-dimensional compact volume U on the manifold M, with $\sqrt{-g}\,\mathrm{d}^4 y$ standing for the contracted volume 4-form [542]. In the boundary term on the right, which will be often abbreviated as $\mathscr{B}_{\mathrm{GHY}}$, the normal n^a to the hyper-surface $\partial\Omega$ provides the induced metric h_{ab} via the decomposition $g_{ab} := h_{ab} \pm n_a n_b$, with the sign ambiguity due to the possible timelike/spacelike character of n^a; also, h is the determinant of h_{ab}, whereas K is the trace of the extrinsic curvature, $K := \nabla_a n^a$. The overall multiplying factor $c^4/16\pi G$ is determined by looking at the Newtonian regime of the model [353].

[5]This form of the action for gravity is the one needed to guarantee the well-posedness of the *metric variation* — i.e., the derivation of the field equations from the first variation of the action with respect to the inverse metric g^{ab} (or, which is the same, the metric g_{ab}) —. An alternative route is to vary the action with respect to both the metric *and* the connexion (Palatini variation [353, 411]), considered as separate variables. In this latter case, the presence of the Gibbons–Hawking–York boundary term becomes unnecessary.

The matter sector, on the other hand, is provided by the action

$$S_{\text{matter}}\left[g^{ab},\phi_J\right]=\frac{1}{2}\int\mathscr{L}_{\text{matter}}\left(g^{ab},\phi_J,\partial_c^{(n)}\phi_J,y^\alpha\right)\sqrt{-g}\,\mathrm{d}^4y\ , \tag{1.19}$$

where ϕ_J denotes any degree of freedom other than the metric, for any counting index J, and "$\partial_c^{(n)}$" in front of a field means that the Lagrangian is in general a function of the field and of all its derivatives. The variation of Eq. (1.19) with respect to each ϕ_J gives the matter field equations, whereas the variation with respect to the inverse metric g^{ab} gives the stress-energy-momentum tensor T_{ab}, as in Eq. (1.14) [157, 404, 547],

$$T_{ab}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{ab}}\ . \tag{1.20}$$

In the expression above, T_{ab} is a symmetric, rank-2, covariant tensor; being interpreted as the indicator of the (local) content of matter and energy, it is expected to be covariantly conserved, as it happens in Special Relativity [542, 353]. It is possible to prove that, if the Lagrangian $\mathscr{L}_{\text{matter}}$ does not depend explicitly on the spacetime event (*background independence* [227]), and if the matter field equations $\delta S_{\text{matter}}/\delta\phi_J=0$ hold for each index J, then it is

$$\nabla_a T^{ab}=0\ , \tag{1.21}$$

hence, the (local) conservation of energy and momentum comes for free in General Relativity, provided that the theory is formulated in a background-independent way [424].

The gravitational part of the field equations emerges from the variation of Eq. (1.18) with respect to the inverse metric [157], and gives

$$\frac{c^4}{16\pi G}G_{ab}=\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{GR}}}{\delta g^{ab}}\ , \tag{1.22}$$

where G_{ab} is the Einstein tensor, i.e. the combination $R_{ab}-Rg_{ab}/2$ of the Ricci tensor and scalar curvature (or else, the double-dual of the Riemann tensor [353]). The symmetry properties of the Riemann curvature tensor also enforces the conservation equation (second Bianchi identity [353])

$$\nabla_a G^{ab}=0\ . \tag{1.23}$$

This relation is remarkable, as it allows to conclude that, once the stress-energy-momentum tensor is assumed to be the source of the gravitational field, its covariant conservation emerges independently of the matter field equations. Some key consequences of this result will come into play in Chapter 3.

Upon assembling the field equations, the result reads

$$G_{ab}=\frac{8\pi G}{c^4}T_{ab}\ . \tag{1.24}$$

This is a system of ten second-order, non-linear, hyperbolic partial differential equations for the g_{ab}'s [118], to be compared with Eqs. (1.5) and (1.15). The differential equations are form-invariant under any arbitrary coordinate transformations, i.e. they are *generally invariant*. Given a solution of Eq. (1.24), another one physically equivalent to the first can be obtained via arbitrary coordinate changes. When written explicitly in a particular coordinate chart, the ten field equations can be split into four constraint equations, and six actual evolution equations for the degrees of freedom. Among these, four more equations can be absorbed into apt redefinitions of the coordinates (gauge-fixing [40]); this leaves two actual degrees of freedom for the gravitational field, traditionally associated with the massless particle, or field excitation, mediating the gravitational interaction — the *graviton* [318].

A final aspect worth a mention is that, differently from what happens in Newtonian or scalar gravity, in General Relativity the equations of motion for test particles in a gravitational field (1.10) can be derived from the field equations, and do not need be postulated separately [274].[6] In other words, Eq. (1.24) already yields the geodesic equations (1.10). This can be considered a significant improvement of Einstein's model with respect to the other competing schemes, as it highlights the intrinsic self-consistency and self-completeness of the theory.

1.1.4 Kurt Gödel versus Mach's principle

In 1949, a paper by Kurt Gödel [234] forced general relativists to question severely their theory. Gödel exhibited a new type of cosmological solution of the strongly homogeneous type, with many impressive and troubling features [332, 250, 537]. His cosmos was compatible with the ubiquitous existence of a perfect fluid endowed with negative pressure, animated by uniform, constant rotation. Certainly a quirky spacetime, but not a completely unrealistic one.[7]

Two main issues, however, were particularly disquieting. First, the causal structure of Gödel's universe is completely degenerate: it is possible to trace closed timelike curves on the manifold, and such curves can be made pass through each and any point in view of the strong homogeneity: alarming conclusions can be drawn from that result.[8]

Second, Gödel's universe clashed severely with two formulations of *Mach's principle* [44, 468],[9] which stated that (a) there could be no global rotation of the Universe, and (b) that the local inertial frames were completely determined by the matter content of the Universe.

The violation of (a) was evident, and Gödel himself underlined it in the introduction of his paper, eventually providing a very rough estimate for the rate of constant rotation of his Cosmos, based on data of the average cosmic

[6]The derivation involves the introduction of another hypothesis, the *strengthened dominant energy condition*, which will become of crucial importance in Chapter 3. For a brief mention of the energy conditions, and for some dedicated references, see the footnotes in the next section.

[7]Well, at least if compared to the Lanczos–van Stockum machine [307], or to Taub–N.U.T. universe (aptly defined "a counterexample to almost anything") [354].

[8]It seems that the scenario of a globally non-causal spacetime is considered seriously worrying by general relativists [354, 124]. Indeed, an entire chapter of relativity theory is devoted to find the conditions preventing its emergence (the so-called *energy conditions* [537], constraining the physical plausibility of the tensor T_{ab} on the right side of Eqs. (1.24) so as to avoid closed timelike curves). Not only that: erasing at once all causally pathological solutions of the field equations for gravity is even *postulated* by some, in the form of a *principle of causality*, and encapsulated into General Relativity as a supplementary hypothesis [257].

[9]Ref. [87] lists no less than *eleven* versions of Mach's principle. Ref. [468] stops the counter at eight, underlining the actual absence of "a" single statement, crafted in a precise and unambiguous sense. Of all these versions, some are even *true* in General Relativity, some others are false — among which, those proven invalid by the existence and properties of Gödel's universe — and some even depend largely on the details of the setting when dealing with Newton's bucket experiment.

matter-energy density; that of (b) added up to many other known violations of the statement already known at the time — including e.g. Minkowski spacetime, the Schwarzschild solution, etc. — and showed that, in fact, a single distribution of matter and energy (a pressure-less fluid/dust, or a perfect fluid endowed with negative pressure [13]) resulted into *two physically different* solutions of Einstein's field equations, viz. the static model found by Einstein himself, and Gödel's solution [13]. This implied that the "distant stars" do not determine uniquely, as in Mach's original presentation, the compass of inertia, i.e. the local inertial reference frame, and on top of that the Universe is free to rotate globally.

Gödel's solution was quickly rejected on the basis that its fundamental reference fluid was devoid of any expansion, whence no gravitational redshift could be predicted, and was as quickly forgotten (although a corresponding solution, this time rotating *and* expanding as requested, was found few years later [513]). Still, the resulting debate on the foundations of General Relativity opened new paths, and ultimately ignited the next phase.

Attempts to implement Mach's ideas directly into the framework of gravitational theories resulted in a proposal initially advanced by Jordan, Fierz, and Thiery, and later refined by Brans and Dicke [92]. Their solution was to trade the coupling constant G for a fully dynamical field, acting as a mediator of the interaction between the local frames and the distant stars, to enforce by hand Mach's principle. The age of *extended theories of gravity* had begun.

1.2 Extended theories of gravity: motivations

Brans' and Dicke's model was just the trailblazer of a legion. The landscape of gravitation paradigms, once a thin line of shore peopled only by few inhabitants, soon became a crowded, intricate jungle.

While the critical examination of standard General Relativity went further ahead, discovering potentially serious flaws (singularities [542, 250, 123], breakdown at the quantum level, non-renormalisability [77], and so forth), all sorts of competing proposal revived or flourished, rapidly exhausting the available stock of physical speculations and formal tools. Every alternative proposal aimed at overcoming the problems rooted in Einstein's scheme, predicting new phenomenology, and explaining the ever-increasing amount of observations. Which, in turn, offered many new riddles to be solved.

The goal of the present section is to offer a concise review of the most prominent physical reasons (experimental and theoretical) to broaden the spectrum of gravity theories beyond the limits of classical General Relativity.[10]

1.2.1 Cosmological expansion and large-scale structures

At the time when this work is prepared, the Universe is undergoing a phase of accelerated expansion [11]. Such behaviour is driven by a repulsive force which overcomes the gravitational attraction, and whose effects become non-negligible at the cosmological level, as emerged from many large-scale observations [545].

[10]Purely formal reasons to look for extended theories of gravity are briefly mentioned in §1.3.3, and reappraised in §3.4.3.

The (yet unknown) agent behind this behaviour has been given the evocative name of *dark energy* [417, 419].

At the level of an effective, classical description, the essential features of dark energy can be accounted for by adding a finely-tuned, all-permeating fluid component which interacts only with the gravitational field; its equation of state need be given by $p = -\rho_\Lambda c^2$ [418]. This model is compatible with a minor modification of the Einstein–Hilbert Lagrangian, namely

$$S_{\mathrm{EH}\Lambda} = \frac{c^4}{16\pi G} \int (R - 2\Lambda) \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{GHY}} \,, \tag{1.25}$$

with $\mathscr{B}_{\mathrm{GHY}}$ the Gibbons–Hawking–York boundary term, and Λ a constant term — the *cosmological constant*, of order $10^{-54}\ \mathrm{m}^{-2}$ — whose value is constrained by the observations. The field equations (1.24) are thus upgraded as follows

$$G_{ab} - \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \,. \tag{1.26}$$

The above, seemingly "harmless" addition of the cosmological constant becomes a source of serious issues when put in perspective. If the dark energy budget is expected to have any sort of connection with micro-physics — and indeed it is [109, 407] —, than its value can in principle be extrapolated by quantum arguments. Sadly enough, when the value of the cosmological constant is calculated in this way, the predicted figure Λ_{Q} turns out to be overwhelmingly greater than the measured one [329]. The discrepancy amounts to an embarrassing $120 \div 122$ orders of magnitude, which strongly depletes any credibility of a "fine-tuning argument". As a consequence, many have suggested deeper modifications of General Relativity to avoid the troubling presence of Λ, or to make it meaningful in a quantum context [47, 287, 243, 315, 125, 381, 500, 29].[11]

A different class of problems is related to smaller-scale physical systems, and particularly to galactic and cluster dynamics. In such structures, the dynamical behaviour is usually modelled and predicted using suitably corrected versions of Newton's theory (the latter is assessed as a viable approximation for most practical purposes). Still, a growing wealth of observations, e.g. those of the rotation curves of some galaxies, seem to support the evidence that the only way to fit the current data in terms of Newtonian models is to add a significant amount of "invisible" matter.

This unobservable *dark matter* building up the potential wells where ordinary luminous matters sits in (originating proto-clusters and galactic seeds), has to interact very weakly with the already observed particles, providing essentially a contribution via its gravitational pull [398, 567, 324].

Once again, at the mere level of an effective description, there are powerful tools within General Relativity to account for the presence of dark matter and its effects, as is typically done in computer-aided simulations of large-scale structures formation. In the simplest scenario, one adds another contribution to T_{ab} in Eq. (1.24), and then tunes the equation of state for dark matter to match the data. The clash arises when one finds out that, to fit the wealth of observations, complex feedback mechanisms between visible and dark matter must be plugged in by hand [482, 69]. Such exchanges, however, would imply much

[11] There are, to be fair, also voices less concerned on the topic; see for instance the plea in Ref. [75].

stronger interactions between the luminous and dark sector than expected, to the point that the dark matter particles would become detectable in Earth-based experiments [68, 286, 31, 70].

The search for dark matter candidates is a branch of (experimental and theoretical) particle physics of interest per se. So far, none of the proposed candidates has emerged from direct observations [65, 371]; this, together with the difficulty to fit the available data with the simplest model possible, has triggered speculations in other directions, more oriented towards modifications of the gravitational scheme. The goal is to get rid of the "dark sector" and interpret all the observations strictly in terms of the mainstream model of particle physics — which is excellently tested and constrained — and of possible modifications of the geometric part of Einstein's theory.

In some of the proposed solutions, the changes to General Relativity propagate up to the weak-field, slow-motion regime, thus inducing modifications (at large scales) of the Newtonian model as well [63, 347, 345, 348, 52]. The familiar inverse-square law governing planetary motions gains e.g. Yukawa-like corrections, which could account for the observed behaviour of galaxies and clusters.

1.2.2 Gravity vs the micro-world: dealing with quanta

Our current knowledge of physics up to the Fermi scale can be decidedly considered robust. The picture of the Standard Model of particle physics has been recently enriched [1, 2] by the discovery of a new entity fully compatible with the boson predicted by Brout, Englert and Higgs [184, 258], i.e. the quantised excitation of the field giving mass to the other fundamental particles via a symmetry breaking mechanism [424]. The dominant scheme stably receives new confirmations, both formal and experimental, while the competing models get pushed further and further out of the observable energy window.

With such a stable and successful theory of the micro-world, the next goal is of course to incorporate gravity in the picture. This means that not only quantum physics should be formulated consistently in a curved background — accounting for the interplay between ordinary fields and a non-flat environment — but our knowledge of gravity itself should find its way into the unified landscape of micro-physics, unveiling its hidden, high-energy quantum structure. Sadly enough, these are precisely the points where "Hell breaks loose".

Indeed, every time one tries to accommodate some features of quantum mechanical origin into the traditional framework of gravitation, Einstein's scheme needs radical changes, even if to provide just an effective description. General Relativity, while versatile enough to account for any sort of bizarre spacetime configurations and unlikely physical phenomena, offers a strenuous resistance to new inputs from the micro-world.

The simplest argument in this sense can be stated as follows [77, 103]: to begin with, one can model first the underlying quantum structure of matter encoded into T_{ab}, and see how its presence changes the aspect of Einstein's equations (1.24). Then, one trades the classical stress-energy-momentum tensor T_{ab} for an average $\left\langle \hat{T}_{ab} \right\rangle$ of a corresponding quantum operator $\hat{T}_{ab}$ acting on quantum states $|\zeta\rangle$. Eqs. (1.24) then read [77]

$$G_{ab} = \frac{8\pi G}{c^4} \left\langle \hat{T}_{ab} \right\rangle \ . \tag{1.27}$$

The most relevant feature of this semiclassical, effective description is that, because of the wide range of interactions of the quantum fields (self-interactions, exchanges between fields, interactions with gravity), non-vanishing fluctuations of $\left\langle \hat{T}_{ab} \right\rangle$ exist even when classical matter sources are absent. Even worse, such fluctuations cannot emerge from the variation of any finite action. To get rid of all the infinities in $\left\langle \hat{T}_{ab} \right\rangle$, one has in fact to introduce infinitely many counter-terms in the Lagrangian for gravity [546, 489].

One can then adopt a perturbative scheme, and look for truncated, yet convergent, versions of semiclassical General Relativity [77]. In this case, the loop expansion of the matter and gravitational sector is done in terms of the parameter $\hbar$, i.e. Planck's constant h over 2π, $\hbar = 6.58 \cdot 10^{-16}$ eV s. At the linear level, the divergencies can be removed by introducing the two running coupling constants G_{eff} and Λ_{eff}, and by rewriting the corrections to $\left\langle \hat{T}_{ab} \right\rangle$ as

$$\left\langle \hat{T}_{ab} \right\rangle = \sum_{I=1}^{3} k_I H^I_{ab} \ , \tag{1.28}$$

where the three tensor correction to $\left\langle \hat{T}_{ab} \right\rangle$ are of the general type:

$$H^I_{ab} = H^I_{ab} \left(g_{ab}, R, R_{ab}, {R_{abc}}^d, \Box R, \nabla_a \nabla_b R, \nabla_a \nabla_b R_{cd} \right) \ . \tag{1.29}$$

The semiclassical approach to gravity reveals, however, other layers of challenges: the form of the terms in (1.29), and the presence there of higher derivative corrections, can provide non-unitary evolution of the fields [512], especially in the context of a blind, naïve application of the Feynmann protocol. While some of these issues can be tackled by suitable methods — e.g. the introduction of Faddeev–Popov ghost particles [187] — the solutions are largely unsatisfactory, for they cannot erase all the singularities and infinities emerging at any new stage.

On top of that: in four spacetime dimensions, General Relativity (already without the intervention of the matter sector) is a non-renormalisable theory [546, 489]. Broadly speaking, this means that a perturbative approach attempting to expand the action (1.18) in terms of the parameter $c^4/8\pi G$ generates uncompensated divergencies at any step of the iteration — i.e. for any power of the expansion parameter the integration over momenta becomes divergent —. Curing such divergencies demands the introduction of infinitely many counter-terms, which clashes against the premise is to search for a *convergent*, UV-complete model [169].

All the fundamental issues of General Relativity and frameworks for gravity alike are indeed expected to be solved by a full-fledged theory of *quantum gravity*. This, however, elevates the problem onto a completely different level.

In the annotated reprint of Bryce deWitt's 1978 Cargèse Lectures [151], G. Esposito states that "so far [November 2007, Ed.], no less than 16 major approaches to quantum gravity have been proposed in the literature". The attached list of references ranges from asymptotic safety to twistors, with contributions from string theory, loops/spin-foam, causal dynamical triangulations, canonical and covariant formalism, and so forth; and it could be easily enlarged by adding the most recent achievements in causal sets theory [165, 62], AdS/CFT correspondence [448], group field theory [399, 400, 221].

Since this topic largely exceeds the scope of the thesis, and of this Chapter in particular, we directly address the Reader towards the main references (also, to Chapter 4, for a bird's eye view of few quantum-gravity proposals). That said, we raise the curtain on the landscape of (classical) gravity theories.

1.3 A glance at the landscape of gravity theories

Time to open Pandora's box.[12] and dive into the catalogue.

A word of caution here: any classification is, by definition, inevitably incomplete, provisional, short-sighted, and arbitrary. The catalogue we are about to present makes no exception. Many items are missing for sure, and a thorough search would likely dig out an immense quantity of variations and additions. Hence, we devote one paragraph to explain the sense of this catalogue.

The results presented in this thesis aim at covering a wide range of theories without having to deal with too many details, as an extension of the universality of free fall should be expected to do. This requires, first and foremost, being aware of what the method can be applied to, what could potentially deal with (under suitable reformulation), and what will never be able to address. The catalogue in this section is built with the purpose of highlighting these three fundamental aspects, and to help the Reader orient Himself in the labyrinth of gravity theories. In any case, it ought not to be intended as a comprehensive review.

Before moving on, finally, a due remark on what this catalogue does *not* contain. We have left aside almost all quantum-gravity paradigms, as the focus for the moment is on macroscopic scales, where no micro-structure of spacetime can play a role; still, many *effective* descriptions in terms of higher-curvature corrections, or in higher-dimensional spacetimes (both emerging from fundamental approaches, in some cases), are treated as independent, and included in the catalogue when necessary. Also, all non-geometrical theories of gravity and standpoints alike have been excluded, for the geometrical interpretation of gravitational phenomena is a cornerstone of this work.

That said, a tentative classification of the main lines of research in extended theories of gravity may be sketched as follows:

∗ Theories including additional (dynamical, or non-dynamical) gravitational degrees of freedom besides the metric — the latter remains the only geometric degree of freedom in the scheme —. Examples of this category include scalar-tensor theories, vector-tensor theories, some bimetric models, scalar and stratified scalar schemes, and any admissible combination of these basic ingredients.

∗ Theories presenting higher curvature corrections to the action. These are somewhat "natural" extensions of General Relativity, often emerging from

[12] The outcome of the operation will be better than the one recorded in the antique myth. Well, we hope so. The Reader is free to place a bet on which theory will remain at the bottom of this new box of Pandora's when all the other ones will have fled out.

semiclassical quantum-gravitational standpoints; they, however, generally yield field equations with derivatives of order higher than two. Besides, they are usually presented as if the only gravitational degree of freedom be the metric, but from recent results we know that they indeed contain other dynamical variables related to gravity, which are simply concealed by the particular way their variational principle is formulated.

∗ Schemes requiring some modification/enrichment of the geometric structure, as e.g. torsion, non-metricity, bi-metricity, skew-symmetry, inaffinity. A vast and varied class, in which the addition of other gravitational degrees of freedom not only modifies the action and the field equations, but rather demands a broadening of the geometric notions, to account for the richer phenomenology in the play.

∗ Gravitational models in higher-dimensional environments, however formulated. Another wide-range category, where it is possible to place not only many modifications resulting from unification attempts and/or simultaneous description of gravity and other fields (e.g. the electromagnetic one), but also all the counterparts of General Relativity formulated in spacetime dimensions higher than four.

With this coarse-grained taxonomy in mind, it is now possible to elaborate a bit on each subset, presenting a few specimen per category, and outlining their main properties.

1.3.1 Other gravitational degrees of freedom

The simplest choice one can make is to add a scalar field to standard General Relativity. This is the starting point of all the so-called *scalar-tensor theories* [191, 103, 204], whose self-explaining name immediately evokes the procedure of the extension. A typical scalar-tensor theory is formulated in terms of the following action

$$S_{\rm ST} = \frac{c^4}{16\pi} \int \left[\phi R - \frac{\omega\left(\phi\right)}{\phi} \partial_\alpha \phi\, \partial^\alpha \phi - V\left(\phi\right) \right] \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\rm ST} \;, \tag{1.30}$$

with V, ω general functions of the scalar field. The matter action remains unchanged with respect to the general relativistic case. A comparison of this last formula with Eq. (1.18) shows that, in the scalar-tensor proposal, the gravitational constant G is promoted to a field, with full dynamical character. Scalar-tensor theories are invoked to explain various effects, ranging from primordial inflation to dark matter and dark energy, and are considered the necessary classical-limit counterpart of string-theoretical models [473, 531], since in the latter case the presence of an additional scalar degree of freedom — the *dilaton* [115, 362] — has to be incorporated into the framework together with the graviton.

The first specimen in the scalar-tensor theories sub-class is the already mentioned *Brans–Dicke theory* [92, 64, 156, 189]. It is a scalar-tensor theory with vanishing scalar potential and constant dimensionless parameter ω, whose numerical value must be determined so as to fit observations [316, 38, 7]. The introduction of a scalar degree of freedom has interesting physical consequences, influencing for instance the motion of extended masses and the behaviour of self-gravitating systems [455, 487, 485, 303, 117].

Notice that the term ϕR in Eq. (1.30) is an archetypical example of *non-minimal coupling* between (tensorial) gravity and the scalar field. If one abstains for a moment from the interpretation of ϕ as a gravitational degree of freedom, and considers it a matter field coupled in an unusual way to standard Einsten's theory, then the entire vault of *non-minimally coupled theories* [77, 191, 103] opens up. Motivations to consider non-minimal couplings include: effective description of first-loop corrections in semiclassical quantum gravity and quantum field theories on curved spacetimes [77, 200, 374]; approximations of string theoretical scenarios and grand unification attempts; fixed points in renormalisation group approach [416]; justification of inflationary cosmology [191]; classicalisation of the universe at early stages [103]. Of course, the ϕR term is just one of the many possible choices, the most common being polynomial structures such as

$$\phi^2 R \quad , \quad \left(1+\xi\phi^2+\zeta\phi^4\right) R \quad , \quad \mathrm{e}^{-\alpha\phi}\, R \,\ldots \tag{1.31}$$

A more advanced generalisation of the scalar-tensor scheme is then given by the *Horndeski theory* [264, 302, 74]. This is the most general four-dimensional scalar-tensor field theory compatible with the requirement of providing second-order field equations only, rather than general higher-derivative terms. To write down its specific action, we define first the shorthand notations $\xi := \partial^a\phi\,\partial_a\phi$ and $\delta^{a_1\ldots a_n}_{b_1\ldots b_n} := n!\delta^{[a_1}_{b_1}\delta^{a_2}_{b_2}\ldots\delta^{a_n]}_{b_n}$, and then compose the following combination

$$\begin{aligned} S_{\mathrm{HD}} = \frac{c^4}{16\pi G}\int \Bigg\{ & \delta^{abc}_{def}\left[k_1\nabla^d\nabla_a R_{bc}{}^{ef} - \frac{4}{3}\frac{\partial k_1}{\partial\xi}\nabla^d\nabla_a\phi\nabla^e\nabla_b\phi\nabla^f\nabla_c\phi + \right. \\ & \left. + k_3\nabla_a\phi\nabla^d\phi R_{bc}{}^{ef} - \frac{\partial k_3}{\partial\xi}\nabla_a\phi\nabla^d\phi\nabla^e\nabla_b\phi\nabla^f\nabla_c\phi\right] + \\ & + \delta^{ab}_{cd}\left[\left(F+2W\right)R_{ab}{}^{cd} - 4\frac{\partial F}{\partial\xi}\nabla^c\nabla_a\phi\nabla^d\nabla_b\phi \right. \\ & \left. + 2k_8\nabla_a\phi\nabla^c\phi\nabla^d\nabla_b\phi\right] - 3\left[2\frac{\partial\left(F+2W\right)}{\partial\phi} + \right. \\ & \left. + k_8\xi\right] + k_9\Bigg\}\sqrt{-g}\,\mathrm{d}^4x + \mathscr{B}_{\mathrm{HD}}\ , \end{aligned} \tag{1.32}$$

in which k_1, k_3, k_8, k_9 are four general functions of both ξ and ϕ, whereas $F = F(\phi,\xi)$ is an object constrained by the differential equation $\partial F/\partial\xi = \partial k_1/\partial\phi - k_3 - 2\xi\partial k_3/\partial\xi$, and finally W depends on ϕ alone, hence it can be reabsorbed into a redefinition of F. Horndeski's theory has been advanced to solve the problems of classical instabilities in General Relativity, and to get rid of the ghost fields when trying to accommodate quantum effects in a semiclassical treatment [112]; the theory also reproduces trivially all the other scalar-tensor models with second-order field equations — it suffices to fine-tune the functions and constants — and has a straightforward connection with the Galileon models [144].

Finally, the scalar-tensor paradigm can be further extended to embrace the case when a single scalar field is not enough to account for the effects one wants to explain; this opens the doors to the more general scheme of the *multi-scalar tensor theories* [135, 541]. In such context, either one builds a model containing more than one scalar from scratch, e.g. trying to model at once the early inflationary phase, and the late-time acceleration of the Universe, or one falls back into a multi-scalar-tensor theory by suitably massaging the action and field equation of a higher-curvature model [475, 240, 48].

The scalar field added to the metric is just one possible choice to enrich the set of gravitational degrees of freedom, and the lowest step of a long ladder. Right above it lies the vector term, which gives rise to the sub-class of the *vector-tensor theories* [45, 59, 58, 57]. Let then u^a be the new object joining the metric in the gravitational sector; typically, one picks a unit timelike vector field u^a (a spacelike vector would generate unexpected and unobserved spatial anisotropies, much more difficult to justify on observational grounds), which also becomes immediately a "preferred direction" in spacetime to align the fundamental reference fluid with [208, 571].

The immediate outcome of this choice is the so-called *Einstein–Æther theory* [181, 282, 281, 520], in which the "Æther" part of the name comes from the role of the vector field, which provides a natural "drift" direction. The action reads, in this case,

$$S_{\mathrm{E\AE}} = \frac{c^4}{16\pi G} \int \left(R + P_{mn}{}^{ab} \nabla_a u^m \nabla_b u^n + \lambda \left(g_{ab} u^a u^b + 1 \right) \right) \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{E\AE}} \;, \tag{1.33}$$

where λ is a Lagrange multiplier, and the tensor $P_{mn}{}^{ab}$ is defined by the relation

$$P_{abmn} := C_1 g_{ab} g_{mn} + C_2 g_{am} g_{bn} + C_3 g_{an} g_{bm} + C_4 u_a u_b g_{mn} \;, \tag{1.34}$$

in terms of four coupling constants $C_1, \ldots, C_4$. From its very construction, this theory violates local Lorentz symmetry, for it provides a preferred time direction: this is encoded in the last term in the sum, which constrains the dynamics of the vector field by demanding it to be everywhere normalised to $-c^2$, and timelike. Violations of Lorentz-invariance are severely constrained at particle-physics level [341, 317], whereas less tight limits exist on effects related to strong-field regimes and cosmological scales (e.g. strong self-gravity [202, 199, 180]); also, the model advances a running role for the coupling constant G, and a long list of physical phenomena contributes to narrow the window for the values of the four constants C_I's.

As for another item in this subset, we mention the so-called *Hořava–Lifshitz gravity* theory [263, 262, 501]. Such model is an attempt to restore a power-counting renormalisable theory, starting from earlier results in this direction achieved by Lifshitz for scalar degrees of freedom. Hořava's proposal is formulated on a given spatial foliation of the spacetime manifold — obtained via an Arnowitt–Deser–Misner decomposition of the metric g_{ab} — and the resulting action is given by $(i, j = 1, 2, 3)$

$$S_{\mathrm{HL}} = \frac{c\hbar}{2G} \int \mathrm{d}t \mathrm{d}^3 y \sqrt{-\tilde{g}} N \left(K^{ij} K_{ij} - \lambda K^2 - V\left[\tilde{g}_{ij}, N\right] \right) + \mathscr{B}_{\mathrm{HL}} \;, \tag{1.35}$$

where N is the lapse function, K^{ij} is the spatial extrinsic curvature, $\tilde{g}_{ij}$ the spatial metric on the leaves, λ a dimensionless running coupling constant, and V the potential. Specifically, the renormalisability is obtained if V contains terms with at least sixth order spatial derivatives (but does not include any time derivative, nor depends on the shift function N_i). The particular form of V yields different versions of Hořava–Lifshitz gravity, governing the types of spatial curvature invariants admissible in the action [501, 535]. The resulting model violates Lorentz symmetry, with propagations also at low energies, and in view of such property it has been discovered to contain, as an infrared limit of its projectable version, the Einsten-Æther theory [501, 280] — in this last case, the vector degree of freedom needs be hypersurface-orthogonal to the leaves of the foliation.

The juxtaposition of the three types of gravitational degrees of freedom encountered so far (the scalar field, the unit timelike "æther" vector field, and the usual metric tensor field), makes it is possible to build even further actions for gravity theories. An interesting outcome of such protocol is the *tensor-vector-scalar theory* [54, 53, 51], also known as "TeVeS".[13] Let then ϕ and u^a be the usual scalar and vector field defined above. On top of that, one introduces: the projection tensor $h_{ab} := g_{ab} + u_a u_b$, i.e. the metric on the leaves everywhere perpendicular to the direction of the æther field, u^a; the skew-symmetric tensor $B_{ab} := \partial_a u_b - \partial_b u_a$, of the electromagnetic type; a second scalar field, σ; a dimensionless function f; two dimensionless constants k, K, plus one constant ℓ with dimensions $[l]^1$. Then, by putting all together, the action of TeVeS is given by [54]

$$S_{\text{TeVeS}} = \int \left\{ \frac{c^4}{16\pi G} R + \frac{\sigma^2}{2} h^{ab} \partial_a \phi \partial_b \phi + \frac{G\sigma^4}{4c^4\ell^2} f\left(kG\sigma^2\right) + \right. \\ \left. + \frac{c^4 K}{32\pi G} \left(B^{ab} B_{ab} + 2\frac{\lambda}{K} \left(g_{ab} u^a u^b + 1 \right) \right) \right\} \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\text{TeVeS}} \,. \quad (1.36)$$

TeVeS theory has been considered because it reproduces, in the weak-field limit, the modifications of Newtonian force needed to fit the observed behaviour of galaxies, clusters and other large-scale objects, without any addition of dark components. Still, the model contains instabilities [481], and it is not certain whether it can account for other observed phenomena, such as gravitational lensing [195]. An aspect of this theory worth mentioning is that its scalar field σ is constrained by the field equations only at the kinematical level, yielding the algebraic relation $kG\sigma^2 = F\left(k\ell^2 h^{ab} \partial_a \phi\, \partial_b \phi\right)$, with F an arbitrary function.

Theories with non-dynamical structures

The presence of physical fields devoid of any dynamical nature at the level of the action seems quite hard to justify — especially after the lesson learnt from General Relativity about the essentially dynamical character of Nature — but is nonetheless recorded in many theoretical approaches (quantum field theories on Minkowski spacetime, to name the champion in this context), and even in specific aspects of some otherwise fully dynamical theories (in General Relativity, topology, dimension and signature are not dynamical). The family of extended theories of gravity is packed with models presenting non-dynamical, background scaffoldings, whose ubiquitous coupling with matter and gravity provides an example of *prior geometry* [558].

[13] A different standpoint is offered by the *scalar-vector-tensor theory*, where the same building blocks are rearranged in a different configuration; see e.g. [356].

Nordström's scalar theory of gravity (§1.1.2) is a typical example of this trend: a dynamical quantity — a scalar field, in this case — lives on Minkowski spacetime: the latter is the fixed landscape in which phenomena occur and interactions propagate (along and inside the light-cones). In the same direction goes the proposal by Minkowski himself [349] of a special relativistic theory of gravity grounded on a four-vectorial degree of freedom; such model was proven incorrect by Max Abraham [386], who showed that planetary orbits would have been unstable in the vector framework.[14]

A variation on this theme is the *stratified (multi-) scalar theory*, in all its incarnations [378, 543]. In this case, the single scalar degree of freedom encoding gravitation is traded for at least a couple of similar functions, such that the general metric reads, in pseudo-Cartesian coordinates (x, y, z, t)

$$g_{ab} = f_1 \left(\mathrm{d}_a x \, \mathrm{d}_b x + \mathrm{d}_a y \, \mathrm{d}_b y + \mathrm{d}_a z \, \mathrm{d}_b z\right) - f_2 \, \mathrm{d}_a t \, \mathrm{d}_b t \, , \tag{1.37}$$

with f_1, f_2 arbitrary functions of the scalar field ϕ, and $\mathrm{d}_a x, \mathrm{d}_a y, \mathrm{d}_a z$ the coordinate 1-forms [332]. As a consequence, the spacetime is not conformally flat anymore — although the spatial slices are still conformally flat — and the local Lorentz-invariance is lost.

Nothing prevents one from pushing the idea further, and have for instance a *bimetric theory* of gravity [277, 518, 461, 346, 347, 345]. Then, in addition to g_{ab} (whose dynamics is provided by the Einstein equations), the model requires the introduction of a second, symmetric tensor ζ_{ab}, with (non-) dynamical character.[15] The presence of two metrics makes it possible to decouple the propagation of gravitational interaction, encoded in one element of the pair, from that of all the other interactions. The consequence is that the equation of motion for test particles (geodesic equation) gets modified in the bimetric scheme; similar changes apply to the propagation of light signals [518, 219]. On the bright side, this proposal provides a variational formulation of the stress-energy-momentum tensor for the gravitational field [461].

Finally, a mention to *massive gravity* [533, 568, 107], a classical field theory of a massive spin-2 field living on Minkowski spacetime. Such proposal sees gravity as the outcome of the interaction between the spin-2 graviton and ordinary matter, according the the coupling scheme $h^{ab}T_{ab}$, where h^{ab} encodes the degrees of freedom of the graviton in the Pauli–Fierz action [196]. The adjective "massive" refers to the presence, in the Lagrangian density, of the self-interacting term $h^{ab}h_{ab}$, which yields the mass of the graviton. One issue with massive gravity theory is its prediction of light bending, which accounts for only three quarters of the observed value. To interpret this mismatch, one can notice that the massive term (absent in General Relativity, where the graviton is massless) introduces an additional scalar degree of freedom, which interacts with matter but is transparent to electromagnetic radiation.

1.3.2 Higher curvatures and higher derivatives

This other subset of the family encompasses a wide, yet seemingly more homogeneous, range of theories. The common feature is a modification of the

[14] For a recent reappraisal of the model see also [515].

[15] When the second metric has full dynamical character, it is possible to include the model into the sub-class of theories with extended geometrical structures (§1.3.3), for doubling the metric content allows to double the affine structure as well, and this brings the model into that other sub-group.

Einstein–Hilbert action, and the typical outcome is a system of field equations of order higher than two in the derivatives — few exceptions are known, and duly reported in §1.3.4.

Higher-curvature theories emerge quite frequently in the family of extended theories of gravity; they are often considered as candidates for effective descriptions of quantum corrections to General Relativity (§1.2.2), or as limits of quantum gravity models accounting for the infinite sum of self-interaction of the gravitons. Also, they sometimes succeed in explaining observed phenomena at large and cosmological scales without resorting to the dark sector of matter and energy, and are hence regarded as possible alternative solutions to the problem of the missing mass/energy in the Universe.

The entire sub-class can be compressed into a single formula, which reads

$$S_{\mathrm{HC}} = \frac{c^4}{16\pi G} \int f\left[R_{abc}{}^{d}, g_{ab}\right] \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{HC}} \ , \tag{1.38}$$

where $f\left[R_{abc}{}^{d}, g_{ab}\right]$ is a shorthand notation meaning an analytic function of some scalar invariant built out of the Riemann tensor — or its various contractions/combinations — and the metric tensor. Finally, $\mathscr{B}_{\mathrm{HC}}$ denotes apt boundary terms (if any exist), depending on the choice of the bulk action.

By inspecting Eq. (1.38), an immediate consequence one might draw is that, contrary to the cases treated in the previous section, the theories considered here concern the dynamics of the metric field alone, and no other degree of freedom is involved. Such conclusion, although intuitive, is actually wrong: the large majority of higher-curvature theories are in fact theories with hidden, non-metric degrees of freedom in disguise [475, 240, 48].

The simplest case to be discussed is the so-called $f(R)$ *gravity theory* [101, 140, 505, 126, 67, 259], which requires the introduction, in Eq. (1.38), of a generic function of the scalar curvature R. The model yields fourth-order field equations, which admit a scalar mode propagating together with the spin-2 graviton. The $f(R)$ theory is often invoked when dealing with large-scale physics and cosmological problems, as an alternative to the introduction of the dark sector [106, 373, 499, 380, 510, 440]. In particular, the fact that the model admits Friedmann–Lemaître–Robertson–Walker solutions makes it easy to fine-tune the function f to account for observations and remove singularities — although this cancellation does not work at any level [269]. Also, there is the possibility to reformulate the scheme such that Newton's constant becomes a running coupling constant, curing the issues with Mach's principle.

These last properties of $f(R)$ theory ought to ring a bell in the Reader's mind, for they are precisely the features typical of a scalar-tensor theory; a careful examination of this higher-curvature theory, indeed, shows that the only meaningful formulation of $f(R)$ theory is in terms of a Brans–Dicke scalar-tensor theory with $\omega = -3/2$, as proven in the context of Palatini variation (see also [171]).

A second proposal, one slightly expanding the allowed complexity, is called $f(R_{ab})$ *gravity* theory [89, 314, 17, 506]: it involves the introduction of a general function of the Ricci tensor instead of the curvature scalar. At the lowest level, one finds an action built out of Ricci squared, which turns out to be that of a metric theory with an additional vector degree of freedom [89].

Once again from algebraic manipulations, it is possible to recognise that the scalar curvature R in the Einstein–Hilbert action (1.18) can be rewritten in the following form [404, 402, 366]

$$R = R^{abcd} Q_{abcd} \ , \tag{1.39}$$

where the tensor Q_{abcd}, equipped with the same symmetries as the Riemann curvature tensor, and with identically vanishing covariant divergence, is the combination

$$Q_{abcd} = \frac{1}{2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \ . \tag{1.40}$$

A natural extension of the previous schemes is then given by the $f\left(R_{abc}{}^{d}\right)$ *gravity theory*, where the linear combination in the curvature tensor is traded for a more general smooth function. The most basic version yields polynomials made up from increasing powers of the Kretschmann scalar $R^{abcd} R_{abcd}$, as in the cases of $f(R)$ and $f(R_{ab})$, but of course many more possibilities are permitted.

A nice example in this last sub-class is the *Weyl conformal gravity* theory [146, 555, 91, 336], whose action is given by the quadratic, scalar combination of Weyl tensors

$$S_{\mathrm{CW}} = \frac{c^4}{16\pi G} \int W_{abcd} W^{abcd} \sqrt{-g} \, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{CW}} \ . \tag{1.41}$$

This theory has the relevant advantage of being renormalisable [410, 423, 335]; on the other hand, however, its field equations (Bach equations [237])

$$\nabla_a \nabla_b W^{acbd} - \frac{1}{2} W^{acbd} R_{ab} = 0 \ , \tag{1.42}$$

are of fourth order in the derivatives, which preludes to non-unitary evolution. Weyl gravity admits equivalent reformulations in terms of a metric tensor and a vector degree of freedom, with the latter expressed as the gradient of a scalar field [554].

Going further ahead, if all three objects $R_{abc}{}^{d}, R_{ab}, R$ are present simultaneously and quadratically, the outcome to be found in place of $f\left[R_{abc}{}^{d}, g_{ab}\right]$ is

$$\alpha R^{abcd} R_{abcd} + \beta R^{ab} R_{ab} + \gamma R^2 \ , \tag{1.43}$$

where α, β, γ are dimensionless parameters. This combination, as anticipated, will give rise in general to fourth-order field equations.

There is another reason why the expression (1.43) above deserves a mention: it is possible to show that there exists only one triple of (α, β, γ) such that the resulting field equations are precisely of order two in the derivatives, hence much closer to the General Relativistic ones. This case is the so-called *Gauß–Bonnet gravity theory*, given by the action [133, 150, 526]

$$S_{\mathrm{GB}} = \frac{\alpha c^4}{32\pi G} \int \left(R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right) \sqrt{-g} \, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{GB}} \ . \tag{1.44}$$

Gauß–Bonnet gravity is a very peculiar complement to General Relativity at the level of the action and of the field equations; however, in four spacetime dimensions, δS_{GB} vanishes identically because the integrand is nothing but a topological invariant — the Euler characteristic [370] — whose variation is equal to zero in view of the Gauß–Bonnet theorem [477].

The Gauß–Bonnet term also resurfaces in semiclassical contexts, when one attempts to renormalise gravity at first order in the loop expansion. Indeed, a term proportional to the integrand in (1.44) crops up as a correction to the trace of the averaged quantum operator $\left\langle \hat{T}_{ab} \right\rangle$ and gives non-zero contributions e.g. in a Friedmann–Lemaître–Robertson–Walker universe. This is part of the so-called "trace anomaly" problem [364, 363].

Finally, we notice that, if Gauß–Bonnet gravity is trivial in ordinary four-dimensional spacetimes, the same cannot be said of the further extension given by the $f(\mathscr{L}_{GB})$ *gravity* theory [141, 563, 113], where once again one considers a generic function of the Gauß–Bonnet invariant.

1.3.3 Novel or enriched geometric structures

The common feature of this vast and crowded sub-class of theories is that not only the gravitational degrees of freedom are encoded in additional objects of varying valence besides the metric (in this sense, there is a partial overlap with the content of §1.3.2), but also the new variables are arranged in such a way that they can be ascribed to the onset of richer geometrical structures defined on the base manifolds.

Getting back to Einstein's General Relativity for a moment, the first choice one can make in this sense is to decouple the affine and the metric content of gravity, for in principle they can be interpreted as independent structures [191, 103, 212].[16] The Einstein–Hilbert action for General Relativity becomes then a functional of both the tensor g_{ab} and the connexion coefficients $\Delta^{a}{}_{bc}$, and a rigorous formulation of the variational principle demands that each object be treated separately. This way of obtaining the full set of field equations is known as the *Palatini variation* [411].

At this point, two possibilities arise, related to the dependence of the matter-sector Lagrangian on the variables $g_{ab}, \Delta^{a}{}_{bc}$. If S_{matter} depends only on g_{ab}, and hence the covariant derivatives in S_{matter} are built out of the metric alone, then one selects the *Einstein–Palatini metric-affine theories* of gravity [242, 252, 254, 322, 253, 540, 426, 441, 337, 534, 105, 104]. If, on the other hand, the affine and metric structures are treated separately also in the matter sector, and one allows for dependencies such as $S_{\text{matter}} = S_{\text{matter}}\left[g_{ab}, \Delta^{a}{}_{bc}, \psi^{I}\right]$, then the sub-class is that of the *affine theories* of gravity [296, 197, 291, 121, 480, 339, 88, 308].

It is fair to ask whether the Palatini variation and the purely metric variation coincide for General Relativity and similar metric theories; the answer is positive in the case of Einstein's theory, and in the special group of Lanczos–Lovelock theories (see below), whereas is negative in all the other cases [186, 90]. This misalignment, together with the well-posedness of the variational formulation, can be used as a clue that additional degrees of freedom are hidden within a seemingly purely metric formulation of the action for the gravity theory (see §3.4.3).

[16] This is, in essence, the most compelling *formal* reason to go beyond General Relativity. While such motivation is rather weak in the case of pure Einstein's model — the affine structure emerges uniquely from the metric one —, the same conclusion does not hold for other theories. The force of the formal motivations comes then entirely from an a posteriori argument.

Metric-affine and affine theories are complemented by the further class of *Eddington–Einstein–Schrödinger purely affine* theories of gravity [479, 173, 434, 439, 294]. In this last group, the metric is completely removed from the action, and all one is left with is a general, non-symmetrical connexion [111, 437, 438]. The coefficients ${\mathit{\Delta}^a}_{bc}$, in suitable combinations, replace identically the metric tensor, and even the determinant $\sqrt{-g}$ is traded for a scalar combination of the connexion quantities [434].

As a further example of a metric-affine theory built out of a general connexion, we can cite the *Eddington-inspired Born–Infeld theory* [412, 37, 295]. The model emerges from an old proposal (by Born and Infeld [78]) for non-linear electrodynamics, later applied to explain the behaviour of galaxies and clusters [472, 565]. Indeed, in the Newtonian limit, the theory yields the modified Poisson equation

$$\Delta\phi = 4\pi G\rho + \frac{\kappa}{4}\Delta\rho \; . \tag{1.45}$$

This extension of the weak-field limit equation can be accounted for by picking a gravitational action of the form

$$S_{\mathrm{EiBI}} = \frac{c^4}{16\pi G}\int\left(\sqrt{\left|g_{ab} + \kappa\, R_{(ab)}\right|} - \lambda\sqrt{-g}\right)\mathrm{d}^4y + \mathscr{B}_{\mathrm{EiBI}} \; , \tag{1.46}$$

where $R_{(ab)}$ is the symmetric part of the Ricci tensor constructed out of the general connexion, and λ is related to the standard cosmological constant Λ by the relation $\Lambda = (\lambda - 1)/\kappa$. The matter action, on the other hand, depends on the metric g_{ab} and on the matter fields only [412].

The general affine connexion encountered in the previous items can be used to construct an independent covariant derivative operator "D_a", with connexion coefficients given by the ${\mathit{\Delta}^a}_{bc}$'s; the latter, in turn, can always be decomposed into the sum of a symmetric and a skew-symmetric part, as in

$${\mathit{\Delta}^a}_{bc} = {\mathit{\Delta}^a}_{(bc)} + {\mathit{\Delta}^a}_{[bc]} \; . \tag{1.47}$$

Recalling now that in general $\mathrm{D}_a g_{bc} \neq 0$, as opposite to the metric-compatible Levi-Civita condition $\nabla_a g_{bc} = 0$, one can further massage Eq. (1.47) to get the final structure [528, 477]

$$\begin{aligned}\mathit{\Delta}_{abc} &= \Gamma_{a(bc)} + \frac{1}{2}\left(Q_{a(bc)} - Q_{b(ac)} - Q_{c(ab)}\right) + \frac{1}{2}\left(T_{a[bc]} + T_{bac} + T_{cab}\right)\\ &\equiv \Gamma_{a(bc)} + N_{abc} + C_{abc} \; ,\end{aligned} \tag{1.48}$$

where ${\Gamma^a}_{(bc)}$ are the usual Levi-Civita connexion coefficients, extracted from g_{ab}, whereas $Q_{a(bc)} := \mathrm{D}_a g_{bc}$ is the *non-metricity tensor*, ${T^a}_{[bc]} := 2{\mathit{\Delta}^a}_{[bc]}$ is the *torsion tensor*, the latter composing the *contortion tensor*, C_{abc} [528, 343, 370, 477]. The independent affine connexion can be also substituted in the definition of the Riemann curvature tensor, giving rise once again to an independent object ${\tilde{R}_{abc}}^{\;\;d}$.

Torsion and non-metricity are the major players in this extension of the geometric structure. The role of non-metricity, although vastly overshadowed by that of torsion, has been briefly considered initially by Weyl [554, 98, 460, 276, 442, 476], who first advanced the possibility for Q_{abc} to affect gravitational phenomena; his specific proposal was to set

$$Q_{abc} := k_a g_{bc} \ , \tag{1.49}$$

with k^a a spacetime vector. Weyl's model aimed at enforcing the full scale invariance of physical laws, implemented via a conformal symmetry working for gravity as well, whence the need for a non-vanishing non-metricity tensor. While Einstein quickly proved the model incompatible with the observations (the interaction of the electromagnetic and gravitational field in the interstellar and intergalactic medium should have been discovered in a non-uniform behaviour of charged particles from the sky), it opened a new chapter in geometry and physics, with the introduction of the Weyl structures [476, 432]. The latter have been later reappraised, and used fruitfully both in the context of particle physics and gravitational theories [476, 539, 493, 9, 351, 504, 201].

The introduction of torsion in a gravitational setting allows to properly describe matter with intrinsic angular momentum (spin) [433, 435]: this is the original motivation behind the most eminent model involving torsion, namely the *Einstein–Cartan–Sciama–Kibble theory* [527, 172, 211, 288, 519]. This proposal moves from the consideration that, at the quantum level, the representations of the Poincaré group for stable particles are labelled by mass and spin; also, already in Special Relativity, once spin enters the game, the resulting stress-energy-momentum tensor is not symmetric anymore [251].[17] Still, thanks to a procedure due to Belinfante and Rosenfeld [55], it is possible to find a symmetric, conserved tensor including the usual T_{ab} and the spin tensor $S_{[ab]}{}^{c}$. Such result can be extended to curved spacetimes via the introduction of tensor-valued differential forms [425]. The field equations for Einstein–Cartan–Sciama–Kibble theory can be extracted from the action

$$S_{\mathrm{ECSK}} = \frac{c^4}{16\pi G} \int \mathfrak{e}\, g^{ab} \tilde{R}_{ab} \left[g^{mn}, \alpha T^{k}{}_{mn} \right] \mathrm{d}^4 y + \mathscr{B}_{\mathrm{ECSK}} \ , \tag{1.50}$$

which must be varied with respect to both the metric (or rather, the tetrad field) and the torsion (or the so-called "spin connexion"). In the previous formula, $\tilde{R}_{ab}$ denotes the (non-symmetric) Ricci tensor emerging from the general connexion $\varDelta^{a}{}_{bc}$ (made up of symmetric part and torsional content), while $\mathfrak{e}$ is the determinant of the tetrad $e_I{}^h$ such that $g^{hk} \equiv \eta^{IJ} e_I{}^h e_J{}^k$, and α is a suitable coupling constant. The resulting field equations of the theory can be shown to be

$$\tilde{R}_{ab} - \frac{g_{ab}}{2} \tilde{R} = \frac{8\pi G}{c^4} \tilde{T}_{ab} \tag{1.51}$$

$$T^{a}{}_{[bc]} + \delta^a_b T^{d}{}_{[cd]} - \delta^a_c T^{d}{}_{[bd]} = \frac{8\pi G \alpha}{c^3} S^{a}{}_{[bc]} \tag{1.52}$$

where $\tilde{T}_{ab}$ is the equally non-symmetrical stress-energy-momentum tensor. Besides its stimulating theoretical aspects [536, 375, 451, 492], the model has interesting cosmological and astrophysical consequences [213, 494, 304, 436], and there are research projects looking for apt measurements of the torsion components [210].

[17] At this stage, the spin can be introduced in a general relativistic context without invoking quantum notions, but simplyshaping an apt classical variable; see e.g. [353, 514].

A second noteworthy application of torsion in the context of extended theories of gravity is *Weitzenböck's teleparallel theory* [176, 550, 361, 245, 14, 365, 193]. This model can be considered the antipodal point with respect to the metric paradigm: one postulates the presence of a general linear connexion on a manifold, arranged so that the resulting Riemann curvature tensor vanishes everywhere, but with non-vanishing torsion. If $\{e_I{}^a\}$ is a basis of the tangent space T_pM at a point on a manifold ($I = 1, \ldots, 4$), and f^I are four global functions on M, the Weitzenböckian covariant derivative D_{v^a} along the direction v^a, is given at p by

$$\mathrm{D}_{v^a}\left(f^I e_I{}^b\right) := \left(v^a\left[f^I\right]\right) e_I{}^b\,(p)\ . \tag{1.53}$$

This implies that, in a given coordinate chart y^α on M, the connexion coefficients $\varDelta^\alpha{}_{\beta\gamma}$ can be represented in terms of the matrix functions k^α_I such that $e_I{}^a = k^\alpha_I\left(\partial/\partial y^\alpha\right)^a$, as in

$$\varDelta^\alpha{}_{\beta\gamma} = k^\alpha_I \partial_\beta k^I_\gamma \tag{1.54}$$

This last expression is manifestly non-symmetric, and since $D_{e_I{}^a} e^b_J = 0$, the Riemann curvature of the Weitzenböckian connexion vanishes identically, whereas the torsion is in general non-zero. The latter becomes responsible for gravitational phenomena, and the (flat) metric structure whose metric-compatible curvature tensor vanishes everywhere can be built out of the connexion and a fundamental tetrad (i.e., an inertial reference frame). A possible choice of the action for teleparallel gravity is for instance [14, 194, 15, 16]

$$S_{\mathrm{WT}} = \frac{c^3}{16\pi G}\int T_{abc}\Sigma^{abc}\,\mathfrak{e}\,\mathrm{d}^4y + \mathscr{B}_{\mathrm{WT}}\ , \tag{1.55}$$

where, besides the torsion and the tetrad determinant, one introduces as well the super-potential $\Sigma_{a[bc]}$, defined in terms of the torsion itself and the contortion tensor as $\Sigma_{a[bc]} := C_{abc} - g_{ac}T^d{}_{bd} + g_{ab}T^d{}_{cd}$.

In close analogy with the $f(R)$ theories, there exist various $f(T)$ theories, where T denotes the teleparallel connexion, and f a generic scalar function built out of the possible scalar combinations of the Weitzenböckian "torsional curvature" [369, 60, 290, 102, 284, 458, 390, 313].

Arbitrary connexions endowed with torsion and non-metricity can often be decomposed into apt combination of the metric and of other degrees of freedom; the torsion mentioned above, for instance, can be thought of as the sum of its irreducible components, namely [528, 119, 477, 478]

$$T_{a[bc]} = \frac{1}{3}\left(T_b g_{ac} - T_c g_{ab}\right) - \frac{1}{6}\epsilon_{abcd}S^d + B_{abc}\ , \tag{1.56}$$

with $T_b = T^a{}_{ba}$ the trace vector, $S^a := \epsilon^{abcd}T_{bcd}$ the axial (pseudotrace) vector, and B_{abc} a traceless tensor. Formulæ like this can be plugged in whenever torsion is present, and the resulting field equations can be solved separately for the irreducible components, considered as separate degrees of freedom.

The converse is also true, in a sense: one can build non-standard combinations of the usual metric and other geometric objects, to shape a specific type of non-symmetric connexion. This is the case, for instance, in the theory of *Lyra manifolds* [325, 491, 447, 137]: a Lyra geometry consists of a triple

$\mathscr{L} \equiv (M, g_{ab}, \zeta)$, where a scalar field ζ with the dimension of $[l]^1$ complements a manifold and a metric on it. The (dimensionless) connexion is given by

$$\mathit{\Delta}^a{}_{bc} := \frac{1}{\zeta}\Gamma^a{}_{bc} + \frac{k+1}{\zeta^2} g^{ad} \left(g_{bd}\partial_a\zeta - g_{ab}\partial_d\zeta \right) , \tag{1.57}$$

and gives rise to a related Lyra curvature tensor $\Xi_{abc}{}^d$. The latter is used to define an action for a gravitational theory.

$$S_{\text{Ly}} = \alpha\hbar \int \zeta^4 g^{ab} \Xi_{acb}{}^c \sqrt{-g}\, \mathrm{d}^4y + \mathscr{B}_{\text{Ly}} . \tag{1.58}$$

The main feature of this theory is that it does not contain dimensional couplings, and it is up to the matter sector of the theory to break the scale invariance [293, 490, 486, 569].

Generically non-symmetric structures can be introduced at an even earlier stage; for instance, in the case of *nonsymmetric gravity* [357, 358, 446], the basic ingredient is a non-symmetric "metric" tensor $\hat{g}_{ab}$, with which to form a general connexion $\mathit{\Delta}^a{}_{bc}$, so that the action takes the form

$$S_{\text{NS}} = \frac{c^4}{16\pi G} \int \hat{g}^{ab} \hat{R}_{ab} \sqrt{-\hat{g}}\, \mathrm{d}^4y + \mathscr{B}_{\text{NS}} . \tag{1.59}$$

The non-symmetric Ricci tensor $\tilde{R}_{ab}$ comes from the composition of at least eight separate pieces, arranged so that the resulting equations of motion are at most second order; the model, which is not free from theoretical issues [134, 136], can be sometimes rewritten in terms of a symmetric metric and of a vector degree of freedom of the electromagnetic type [446, 285], which emerges from a Proca-like action.

1.3.4 The higher-dimensional case

Letting the number of dimensions of spacetime grow above the standard number of four — or letting it decrease, for what it matters — allows for the onset of another interesting class of extended theories of gravity: in the lower-dimensional case, the resulting proposals can be used as toy models to test effects in specific regimes, whereas the higher-dimensional landscapes allow for attempted unifications of gravitational physics and other fields, or for the emergence of holographic properties of the actions.

A lower-dimensional manifold can usually host only a drastically simplified gravitational theory, as many geometric objects trivialise and thus reduce the related phenomenology. The exact opposite happens, on the other hand, in higher-dimensional cases. In the elementary 2-dimensional spacetime, for instance, the *Jackiw—Teitelboim gravity* [279] provides the equivalent of a Brans–Dicke theory, with an action given by

$$S_{\text{JT}} = \frac{c^4}{16\pi G} \int \left(\phi R + \frac{1}{2}\partial^a\phi\partial_a\phi + \Lambda \right) \sqrt{-g}\, \mathrm{d}^2y + \mathscr{B}_{\text{JT}} , \tag{1.60}$$

whence the field equations

$$R - \Lambda = \frac{8\pi G}{c^4} T , \tag{1.61}$$

with T the stress-energy-momentum tensor. An extension to three spacetime dimensions is currently used to examine the mechanics of black holes [334].

As for the higher-dimensional case, it is worth stressing that, in any spacetime of integer dimension $n > 4$, it is obviously possible to copycat Einstein's General Relativity by simply rewriting Eq. (1.18) in an arbitrary number n of dimensions; the result reads

$$S_{\mathrm{GR},n} = \frac{c^4}{16\pi G_n}\left(\int_\Omega \overset{(n)}{R}\sqrt{-\overset{(n)}{g}}\,\mathrm{d}^n y + 2\int_{\partial\Omega} \overset{(n-1)}{K}\sqrt{\overset{(n-1)}{h}}\,\mathrm{d}^{(n-1)}y\right) . \tag{1.62}$$

The symbols adopted retain the same geometrical meaning and definitions as in the four-dimensional case, whereas G_n identifies the aptly reformulated value of Newton's constant. This means that, for any integer $k \leq n$, there is an entire ladder of "General Relativit*ies*" given by

$$S_{\mathrm{GR},k,n} = \frac{c^4}{16\pi G_k}\left(\int_{\Omega_k} \overset{(k)}{R}\sqrt{-\overset{(k)}{g}}\,\mathrm{d}^k y + 2\int_{\partial\Omega_{k-1}} \overset{(k-1)}{K}\sqrt{\overset{(k-1)}{h}}\,\mathrm{d}^{(k-1)}y\right) . \tag{1.63}$$

In ordinary four dimensions, then, one can in principle write four-, three-, and two-dimensional Einstein–Hilbert Lagrangians, modulo the caveat remarked at the beginning of this section about the trivialisation of many invariants when the number of dimensions decreases below four.

An example of a higher-dimensional model motivated by the idea of getting rid of dark energy while still explaining the accelerated state of the Universe is the so-called *DGP gravity* [170]. Such proposal advances the existence of a $(4+1)$-dimensional Minkowski spacetime (the *bulk*), in which the ordinary $(3+1)$-dimensional Minkowski spacetime (the *brane*) is embedded. The resulting action principle becomes

$$S_{\mathrm{DGP}} = \left(\frac{c\hbar}{G}\right)^{3/2}\int_{\Omega_5} \overset{(5)}{R}\sqrt{-\overset{(5)}{g}}\,\mathrm{d}^5 z + \frac{c^4}{16\pi G}\int_{\omega_4} R\sqrt{-g}\,\mathrm{d}^4 y + \mathscr{B}_{\mathrm{DGP}} , \tag{1.64}$$

with ω_4 the intersection of the boundary $\partial\Omega$ and the Minkowski brane $\mathscr{M}_4$. One has that the four-dimensional gravity dominates at short range, whereas the five-dimensional effects emerge at long range. This interplay of different regimes introduces corrections to the gravitational potential and possibly explains cosmic acceleration. DGP gravity has been recently challenged by a new wave of cosmological observations, and accounting for the available data may need the re-introduction of the (unwanted) cosmological constant [238, 188, 321].

A similar proposal is provided by the *Randall–Sundrum model* [449], where the universe is thought to be a 5-dimensional bulk environment enclosed by two surfaces (branes) whose position is governed by energy levels. The geometry of the bulk is highly warped, and gravitational interaction can access all the five dimensions; on the other hand, matter fields are confined on a 4-dimensional sub-manifold (specifically, the boundary brane with the lowest energy level). The metric is given by

$$g_{ab} = \frac{1}{ky^2}\left(\mathrm{d}_a y\,\mathrm{d}_b y + \sum_{I=1}^{4}\mathrm{d}_a z^I\,\mathrm{d}_b z^I\right) , \tag{1.65}$$

and the boundaries are set at the value $y = k^{-1}$ and $y = (Wk)^{-1}$ in the fifth dimension, with W the "warp factor" such as Wk is of the order of some TeV's. The brane on which the Standard Model particles reside is the latter.

Higher-dimensional settings are also a necessity within the attempted unification of gravity and other physical fields. Usually, one considers the electromagnetic field; the simplest choice, however, is a scalar field. This is the point of view adopted e.g. in *Kaluza–Klein* theories [289, 525, 359]. In a minimal example of a Kaluza–Klein model, one considers a higher-dimensional spacetime $\mathscr{K} \equiv (M \times K, \hat{g}_{AB})$ for which the manifold is given by the Cartesian product of a standard four-dimensional Lorentzian manifold M, and a k-dimensional Riemannian (i.e. with a definite metric) manifold K, with $k \geq 1$. On the $(4+k)$-dimensional Kaluza–Klein manifold, a metric $\hat{g}_{ab}$ is defined, such that its matrix representation is given in the following block form

$$\hat{g}_{AB} := \begin{pmatrix} g_{ab} & 0 \\ 0 & m_{hk} \end{pmatrix} , \tag{1.66}$$

with $a, b = 1, \ldots, 4$, and $h, k = 5, \ldots, 4+k$.[18] The resulting action for the unified theory reads [33]

$$S_{\text{KK}} = \frac{\hat{c}^4}{16\pi\hat{G}} \int \left(\hat{R} - \hat{\Lambda}\right) \sqrt{\hat{g}}\, \mathrm{d}^{(4+k)} z \;, \tag{1.67}$$

where all the hatted quantities refer to the full Kaluza–Klein manifold. Upon supposing that the supplementary k dimensions wind up at the level of a micorscopic scale ℓ, the above formula boils down to [33]

$$S_{\text{KK}} = \frac{\hat{c}^4 V^{(\ell)}}{16\pi\hat{G}} \int \left[\phi \left(R + {}^{(k)}R + \hat{\Lambda}\right) + \frac{k-1}{k} \frac{\nabla_a \phi \nabla^a \phi}{\phi}\right] \sqrt{-g}\, \mathrm{d}^4 y \;, \tag{1.68}$$

in which $V^{(\ell)}$ is the k-dimensional volume of the compactified submanifold K, $\phi := \sqrt{|\det m_{hk}|}$, and ${}^{(k)}R$ is the scalar curvature of K. It is not difficult to see that Eq. (1.68) corresponds to the action for a four-dimensional Brans–Dicke theory such that $\omega = -(k-1)/k$ [562].

The onset of higher-dimensional landscapes is a common feature of many approaches to quantum gravity and grand unified theories: models based on *string theory* [430, 431, 338] oscillate between 26 and 10-dimensional environments (in most of the cases, compactification is needed at some stage to recover the observed four dimensions); the same holds for refined pictures such as *M-theory* [338], or various *supersymmetric extensions* of standard gravity [372, 297].

We conclude the section (and the Chapter) with what has been called "the most natural extension to General Relativity" [366, 403, 402, 404], because of the many similarities it carries with respect to Einstein's theory: the *Lanczos–Lovelock gravity* theory [323] — or rather, theories.

The main feature of this class is the fact that it starts as a sub-case of the higher curvature schemes, as it presents higher curvature corrections in the action, but in fact the Lagrangian is shaped so as to give only second order field

[18] The original unification of gravity and electromagnetism required a matrix $\hat{g}_{AB}$ with non-diagonal terms as well [525].

equations. A second reason typically offered to claim the "naturalness" of the extension is the capability of the action to be decomposed, as it happens with Einstein's theory, into a term quadratic in the second derivatives of the metric (the *bulk* term), and one which is a total derivative — hence, one leading to a *surface* term. Lanczos–Lovelock theories all share this property, often referred to as *holographic* [403, 415, 298, 564, 405, 414, 408].

The reason why Lanczos–Lovelock theories are included in the present section is the fact that the successive emergence of the elements of the family is dimensional-dependent: the higher the number of spacetime dimensions, the larger the class of admissible Lanczos–Lovelock theories becomes. Also, the formulation of its variational problem shows that the only degrees of freedom are now encoded exclusively into the metric tensor, and no further geometric object is concealed somewhere behind the ostensible structure of the action (see §3.4.4).

The starting point to formulate the model is to consider Lagrangian densities of the form

$$S_{\mathrm{LL}} = \alpha \int Q_{abc}{}^{d} R^{abc}{}_{d} \sqrt{-g}\, \mathrm{d}^n y + \mathscr{B}_{\mathrm{LL}} \ , \tag{1.69}$$

where the tensor $Q_{abc}{}^{d}\left[g_{ab}, R_{abc}{}^{d}\right]$, built out of the metric and the curvature tensor only, has the same symmetries of the Riemann tensor, and has vanishing covariant divergence, $\nabla_d Q_{abc}{}^{d} = 0$. In the simplest case, i.e. when $Q_{abc}{}^{d}$ depends on g_{ab} only (in any number of spacetime dimensions), the only possible choice for it is the form $Q_{abcd} \equiv \frac{1}{2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$ given in Eq. (1.40), and the resulting theory is precisely General Relativity (in dimension n), which results then a proper element of the Lanczos–Lovelock class [404].

When $Q_{abc}{}^{d}$ can also depend linearly on the Riemann tensor, in addition to the previous case, one gets the term

$$Q_{abcd} = R_{abcd} - G_{ac}g_{bd} + G_{bc}g_{ad} + R_{ad}g_{bc} - R_{bd}g_{ac} \ , \tag{1.70}$$

and the full contraction of the previous formula with $R_{abc}{}^{d}$ gives rise to the Gauß–Bonnet term in (1.44). This first non-trivial Lanczos–Lovelock extension of General Relativity becomes a topological invariant in four spacetime dimensions — and thus the action vanishes — but the same does not occur when $n > 4$. This is the anticipated dimensional dependence of the Lanczos–Lovelock models: when n grows, so it does the number of available additional terms in the action, yet the field equations remain constrained to be of second order.

By prolongation of the construction, it is possible to prove that the m-th order Lanczos–Lovelock gravity theory has the form [323, 404] (we put $k = 2m$)

$$\mathscr{L}_{\mathrm{LL},m} = Q_{abc}{}^{d} R^{abc}{}_{d} = \delta^{1357\ldots 2k-1}_{2468\ldots 2k} R^{24}_{13} R^{68}_{57} \ldots R^{2k-2\,2k}_{2k-3\,2k-1} \ , \tag{1.71}$$

which makes use of the completely skew-symmetric, alternating tensor $\delta^{p_1\cdots p_i}_{q_1\cdots q_i}$. The latter is given by the determinant of an $(n \times m)$ matrix, each element in the table being a Kronecker delta, where the first row is given by the sequence $\delta^{p_1}_{q_1}, \ldots, \delta^{p_i}_{q_1}$, and so forth, until the last row, made of $\delta^{p_1}_{q_j}, \ldots, \delta^{p_i}_{q_j}$.

The term on the right in Eq. (1.71) is a homogeneous function of degree m in the curvature tensor, and hence can be expressed as

$$\mathscr{L}_{\mathrm{LL},m} = \frac{1}{m}\left(\frac{\partial \mathscr{L}_{\mathrm{LL},m}}{\partial R_{abc}{}^{d}}\right) R_{abc}{}^{d} = \frac{1}{m} P^{abc}{}_{d} R_{abc}{}^{d} \ , \tag{1.72}$$

where $P^{abc}{}_{d} := mQ^{abc}{}_{d}$, and by definition $P^{abc}{}_{d}$ has as well vanishing covariant divergence, in any of its indices.

Eq. (1.71) and (1.72) show that it is possible to build the most general Lanczos–Lovelock gravity theory by simply building an infinite sum of $\mathscr{L}_{\mathrm{LL},m}$'s, and weighting its terms by apt coupling constants. The symmetry properties of the alternating tensor, however, limit the number of non-trivial terms in the sum; indeed, $\delta^{1357\ldots 2k-1}_{2468\ldots 2k}$ vanishes identically whenever $k > n$, with n the spacetime dimension, and reduces the Lanczos–Lovelock term to a topological invariant whenever $k = n$, in view of the Gauß–Bonnet theorem. This explains why, for $n = 4$, we can in principle build two Lanczos–Lovelock Lagrangians, namely the Einstein–Hilbert one, and the Gauß–Bonnet one, but only the former will give non-trivial contributions to the field equations. But if, say, $n = 5$, the Gauß–Bonnet gravity will provide an actual contribution to the action and to the field equations, and so on for growing n.

Chapter 2

On the Principles of Equivalence

Since others have explained my theory, I can no longer understand it myself.

A. Einstein, in *Einstein and the Poet*.

From the previous Chapter, we inherit the image of a colossal "family tree" of extended theories of gravity: a flourishing and ever-growing organism, where new branches and leaves appear every now and then. To avoid an uncontrolled growth, and the proliferation of ill-formed theories, a Gardener has to take care of the plant, and a Taxonomist has to trace the mutual relationships among the various branches.

To this end, i.e. to prune and inspect the family tree of gravity theories, the Gardener and the Taxonomist will need an appropriate, sharp, and effective instrumentation. This is where the Principles of Equivalence enter the stage.

The Equivalence Principles are founding pillars for any theory of gravity. Broadly speaking, they perform three main offices: establishing some very general prescriptions on the behaviour of physical systems in a gravitational environment; acting as bridges between the world of pure physical intuition, and that of rigorous formalisation; providing neat selection rules to constrain the set of possible frameworks for gravity.

The present Chapter is devoted to a critical discussion of all the most relevant principles of equivalence. In the following pages, the topics are treated via a mixture of a review of some well-established notions, and diffuse original contributions to the debate. The main sources for the material presented herewith are Refs. [388, 558, 557, 155], together with the first sections of [154], and the literature cited therein.

2.1 Introductory remarks

The opening of the Chapter is dedicated to three preliminary, basic steps: framing the general notion of "principle of equivalence", establishing a convention about the word usage, and offering the Reader a critical voice against the notion itself of equivalence principle — with our due, subsequent reply.

In particular, we deem it necessary to detail the different versions of the principle of equivalence used within the context of this thesis, as some forms of the statements are non-standard with respect to the literature on the theme, and some others have been introduced only very recently [155, 154].

2.1.1 Key concepts and milestones

Whenever an experimental campaign finds out that distinct physical quantities or phenomena (in principle, unrelated) are consistently equivalent, or unfold in the same way within the best accuracy attainable, a *Principle of Equivalence* can be formulated. The different conditions examined can thus be traced back to a common origin, or a unified language can be adopted to describe them at once. As a result, different theoretical frameworks can be merged into, or traded for, a single model.

For example, the common features of the free-fall trajectories of test bodies with negligible self gravity, once elevated to the status of an equivalence principle, permit to ascribe the universal character to a geometric property of spacetime, rather than to a property shared by all possible test bodies [244]. Similarly, the numerical equivalence of the gravitational and inertial masses for all test bodies in Newtonian regime allows to unify mechanical and gravitational phenomena under a single theory.

Such general definition of "equivalence principle", however, is quite recent. In its first incarnation [388], the principle of equivalence was just the outcome of Einstein's attempt to widen the validity of his principle of (special) relativity: having shown that, in dynamical terms, any uniform rectilinear motion has the same mechanical content as stasis, his next goal was to prove that also acceleration is relative, and all possible motions, however complicated, are in principle indistinguishable from stasis.

Such a goal was so crucial in the development of Einstein's theory of gravity that he christened the model itself *General* Relativity precisely to overcome the "special" character of inertial frames in Special Relativity. "Principle of Equivalence" was then just another name for a *generalised principle of relativity* [387], this time one holding for all sorts of accelerated motions.

Einstein's proposal, however, was "too good to be true": his pristine identification of the accelerated reference frame in the absence of gravity, with the reference frame at rest in a *uniform* gravitational field had to hold *exactly.* This case, however, simply cannot be, for the latter concept — that of a uniform gravitational field — is chronically ill-defined.[1]

The missed accomplishment of Einstein's original program weakened the theoretical role of the principle of equivalence. Yet, the notion as defined in the opening of this section never really disappeared: it actually kept resurfacing again and again, contributing to the construction of the protocols to test the

[1]This matter is still disputed; for a pair of poles-apart opinions see e.g. [368], and the reply in [155], footnote 30.

theories of gravity beyond General Relativity. The results presented in this work are just the last "rebirth" of the seminal intuition.

2.1.2 A conventional glossary

"Equivalence Principle" is a common expression in gravity theory, found in the early pages of most textbooks. Unfortunately, the term is applied to a wide range of different notions, statements, and ideas, too often without any further specification [524].

Typically, in Newtonian contexts, one speaks of the equivalence principle in reference to the experimental equivalence of the inertial and gravitational masses of point particles. In relativistic contexts, "equivalence principle" is used either as a synonym of universality of free fall for test bodies in a gravitational field, or as the impossibility to distinguish an accelerated reference frame from one freely falling in a given gravitational field.

A further layer of confusion is due to the habit of using the same expression, e.g. "Strong Equivalence Principle", for at least two very different concepts, namely what here is called "Einstein's Equivalence Principle", and the actual "Strong Equivalence Principle" [444].

To prevent the Reader from the onset of painful migraines, we think it is much easier, and more correct, to establish a conventional glossary from afresh, rather than to resort to mutually contradicting sources. The content of the various Equivalence Principles about to be discussed can thus be sketched as follows.

* The equivalence of inertial and gravitational masses (whenever the two concepts make sense) is *Newton's Equivalence Principle.*

* The universality of free fall for non-self-gravitating test bodies is the *Weak Equivalence Principle.*

* The universal behaviour of test bodies (self-gravitating or not) in a gravitational field is the *Gravitational Weak Equivalence Principle.*

* The local equivalence of fundamental, non-gravitational test physics in the presence of a gravitational field, and in an accelerated reference frame in the absence of gravity is *Einstein's Equivalence Principle.*

* The local equivalence of fundamental test physics (gravitational or not) in the presence of an external gravitational field, and in an accelerated reference frame in the absence of gravity is the *Strong Equivalence Principle.*

2.1.3 John Lighton Synge on the Equivalence Principles

Before moving on to review and discuss the various statements, we deem it correct to leave a bit of space to the strongest voice *against* the equivalence principles themselves, namely, that of John Lighton Synge. In the opening pages of his book *Relativity, The General Theory* [517], Synge spends quite disapproving words for the equivalence principles, their role in theory-building, and their actual usefulness.

> "When, in a relativistic discussion, I try to make things clearer by a space-time diagram, the other participants look at it with polite detachment and, after a pause of embarrassment as if some childish indecency had been exhibited, resume the debate in their own terms. Perhaps they speak of the Principle of Equivalence. If so, it is my turn to have a blank mind, for I have never been able to understand this Principle. Does it mean that the signature of the space-time metric is $+2$ (or -2 if you prefer the other convention)? If so, it is important, but hardly a Principle. Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property; it has nothing to do with any observer's world-line. Space-time is either flat or curved [...]. The Principle of Equivalence performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife be now buried with appropriate honours and the facts of absolute space-time faced."

After pondering on Synge's opinion, we concluded that his urgency to "bury the midwife" is slightly excessive. Above all, he seems to grossly underestimate the remarkable achievements emerged after a keen application of the equivalence principles to the landscape of gravity theories.

Some elements of his argument are undeniably correct, and ought to be kept in mind when drafting any theoretical model for gravitational dynamics.[2] Yet, his aversion to the Equivalence Principles ends up being ultimately a short-sighted distaste for the role of physical intuition as the engine of progress within the boundaries of a well-defined theory.

There is more: Synge closes by urging the community to look at (and for) the *facts of spacetime*. The sentence sounds powerful and inspiring, but what does it really mean? Spacetime is a mathematical concept — a *language*, admittedly effective — used to express succinctly a wealth of data and results of experiments. Such language, however, largely depends on the sort of theory of gravity one has in mind, and the latter will inevitably constrain the design of the experiments, and the interpretation of the results.

On top of that, when it comes to General Relativity (or to any background-independent dynamical theory of gravity with a full geometrical interpretation), the passage from the experimental results to the corresponding geometric objects, and vice versa, is often far from obvious, even for very elementary notions, such as "mass", or "energy" of a system. The "facts of spacetime" emerging from neat calculations, in such cases, provide ambiguous answers to elementary physical questions.

To conclude: Synge's idea is praiseworthy, and particularly stimulating to read for anyone working on the principles of equivalence. Sure, the effectiveness of the geometrical language in relativistic contexts is out of the discussion. Yet, we politely suggest less haste in "burying the midwife"; for she might still have something to say about gravity, and the way to build a robust theory of it.

[2]Not the bit concerning the signature, though. Synge could not be aware that models would have been built where signature itself is a dynamical entity [391], or where a Lorentzian spacetime model could emerge from a completely Euclidean underlying background [222].

2.2 The Principles of Equivalence

We can now proceed to examine the principles in more detail, offering an overview of their basic traits. Most of the material contained here is a critical review of notions already widespread in the community, with an apt reformulation of some concepts. The fresh and original content, consisting of an upgraded definition of the Gravitational Weak Equivalence Principle, is outlined and discussed only partially, for a full account of its features will be given in the dedicated Chapter 3.

2.2.1 Newton's Equivalence Principle

Back to the Newtonian formulation of gravity (§1.1.1), we have concluded that the active and passive gravitational masses in Eq. (1.2) need be equal if the action-reaction principle of mechanics holds as well for gravitational phenomena [94]. This allows to consider just one specimen of gravitational mass.

On the other hand, none of the basic laws of dynamics states anything about the relationship between inertial and gravitational masses, which account for completely different physical concepts. One can then design experiments to test the possible discrepancies between the measured values for the two kinds of mass, and see whether they can be found to be e.g. proportional [376, 557].

While the latter possibility is much less probable than any other experimental outcome, it is now known [122, 394, 66, 558, 557] with extremely high accuracy and precision that m_{G} and m_{I} are indeed proportional, and the proportionality coefficient is a universal constant. This allows to drastically simplify Eqs. (1.5), as one can drop all the masses, and get back to the familiar formulation

$$\begin{cases} \Delta\Psi\left(\mathbf{r},t\right) = 4\pi G\rho\left(\mathbf{r},t\right) \\ \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\nabla\Psi\left(\mathbf{r},t\right) \end{cases} . \tag{2.1}$$

Notice that the matter density sourcing the scalar potential in Poisson's equation also lacks any additional subscript; this reflects the fact that, once inertial and gravitational masses are equal in suitable units, then every form of matter gravitates the same way.

It is worth stressing that the experimental equivalence of inertial and gravitational masses can be tested for many theories of gravity, and not only in the Newtonian case; what must be preserved is the Newtonian *regime* in which the tests are performed, for only under those circumstances the concepts of inertial and gravitational masses make sense — they are decidedly Newtonian quantities —. One then has to enforce the weak-field, slow-motion limit to compare m_{G} and m_{I}; possibly, other constraints need be imposed.[3]

The relation $m_{\mathrm{G}} = m_{\mathrm{I}}$ can thus be elevated to the level of a principle; for historical reasons, it is fair to name such principle after Isaac Newton. The

[3]In all those cases in which the Newtonian regime is an approximation holding up to a certain scale λ, the laboratory testing the equivalence must of course be confined to regions with radius smaller than λ. As shown in §1.3, theories exist in which one deals with modified Newtonian dynamics (commonly abbreviated in "MOND" [53]), both at the fundamental level — TeVeS model [54] —, or as the result of an effective description of large-scale phenomena. Whenever MONDian effects are forecasted, the equivalence of inertial and gravitational masses can only be an approximation, valid as long as the scale is such that the modifications remain under the sensitivity threshold.

resulting statement reads [497]

> *Newton's Equivalence Principle* — In the Newtonian limit, the inertial and gravitational masses of a particle are equal.

As a final remark, we point out that this principle is often confused with the universality of free fall (see below). While it is true that the two notions are tightly bound together and largely overlap, the Reader ought to keep them separated [497].

2.2.2 The Weak Equivalence Principle

The universal behaviour of particles in the presence of gravity is a property known and tested since Galileo Galilei's experiments from the leaning tower of Pisa.[4] The precise statement of such property requires some tuning, but the resulting principle is a fundamental pillar of all relevant gravity theories currently available [122, 394, 558, 557, 379].

> *Weak Equivalence Principle* — Test particles with negligible self-gravity behave, in a gravitational field, independently of their properties.

The statement of the principle presented here slightly differs from the formulations typically found in the literature; specifically, much emphasis is put here on the two attributes of the particle, namely its being a *test body*, and its *negligible self-gravity.*

A test body is, by definition, *any physical system which can be acted upon by the surrounding environment, but does not back-react significantly on the environment itself.* In a sense, a test body is a completely "passive" system, whence its use to probe the net effects of the presence of any "active" agent in the environment. The property of being "test" is highly idealised, as any actual system can be considered a test one only approximately [155]. So, for instance, a "test charge" is any system which can feel the presence of a surrounding electromagnetic field, but cannot influence it to the point that the properties of the field change significantly because of the presence of the charge.

The definition of "self-gravity" requires even more care [155]. To grasp the general idea, one can proceed as follows: in the Newtonian regime, introduce the gravitational (self-) energy of any massive system, given by

$$\mathscr{E}_{\mathrm{G}} = \frac{G m_{\mathrm{G}}^2}{r} \ , \tag{2.2}$$

with m_{G} the gravitational mass, and r an aptly defined measure of the size of the system. At the same time, consider the inertial energy of the system, namely

$$\mathscr{E}_{\mathrm{I}} = m_{\mathrm{I}} c^2 \ , \tag{2.3}$$

as emerging from the special relativistic prescription. Provided that Newton's equivalence principle holds, hence $m_{\mathrm{G}} = m_{\mathrm{I}}$, then it is fair to define the ratio [155]

$$\sigma := \frac{\mathscr{E}_{\mathrm{G}}}{\mathscr{E}_{\mathrm{I}}} = \frac{G m^2}{r} \frac{1}{m c^2} = \frac{G m}{r c^2} \ . \tag{2.4}$$

[4] Experiments which may or may not have actually occurred [167].

The scalar σ is a measure of the amount of gravitational binding energy with respect to the overall inertial energy for the body; the lower its value, the less significant the self-gravity of the object. In "ordinary" regimes, and for bodies up to the size and mass of the Sun, σ is very small, whereas it reaches order one for extremely compact objects, e.g. black holes. The far-right term in Eq. (2.4) is also proportional to the ratio between the Schwarzschild radius of a body and its typical size; this justifies the name *compactness* given sometimes to σ.

Finally, the expression "behave independently of their properties" in the statement of the principle simply means that the future histories of the particles will be the same, provided they have the same initial conditions.

The simultaneous presence of the conditions "test body" and "negligible self-gravity" in the statement of the Weak Equivalence Principle is then necessary [155], because the two notions are logically distinct: there might be in principle test bodies with very strong self-gravity (for instance, a micro-black hole in the gravitational field of the Earth), objects with mild self-gravity, but with non-test characters (for instance, the Moon orbiting around the Earth), and finally test bodies with irrelevant self-gravity contribution (e.g., a rock or a pebble freely-falling on the Earth surface, and practically all the systems typically used to test the Weak Equivalence Principle in Earth-based experiments). It is only to this last class of objects that the principle applies, and its conclusion cannot be stretched to cover the other sorts of bodies.

2.2.3 The Gravitational Weak Equivalence Principle

The Gravitational Weak Equivalence Principle builds upon the content of the previous section, removing the constraint of the negligible self-gravity, and introducing the request to work in vacuo. The resulting statement reads [155, 154]:

> *Gravitational Weak Equivalence Principle* — Test particles behave, in a gravitational field and in vacuum, independently of their properties.

This version of the Weak Equivalence Principle dedicated to systems with non-negligible self-gravity has been advanced for the first time in Refs. [155, 154]. Some earlier, slightly different versions of the principle date back to Ref. [558], where the statement is applied to "self-gravitating as well as test bodies".

Since the presence of self-gravity per se does not necessarily spoil the property of being a test body, the condition that the system under consideration have to remain a test one is preserved in our formulation [155]: this assures that it is possible to compare the free-fall trajectories of different bodies, with and without a significant self-gravity, and draw conclusions about a universal behaviour [154].

Another original contribution to the formulation of the principle is the explicit assumption of a *vacuum environment*. In the literature on the topic of self-gravitating systems, such hypothesis is not implemented systematically ab initio, yet typically crops up at later stages to drastically simplify the calculations [350, 427], or to sidestep the complex interpretational issues connected to "dirty" (i.e., matter-imbibed) surroundings of a massive body [538].

It turns out, however, that if one wants the behaviour of free-fall trajectories of self-gravitating systems to be universal, the presence of surroundings devoid of matter is a necessary pre-requisite, rather than a simplifying add-on [154].

The Gravitational Newton's Equivalence Principle

As a side remark, we stress here that the Newtonian counterpart of the Gravitational Weak Equivalence Principle might as well deserve a separate statement, thus yielding a Gravitational Newton's Equivalence Principle, namely [155, 154]:

> *Gravitational Newton's Equivalence Principle* — In the Newtonian limit and in a vacuum environment, the inertial and gravitational masses of a test body with non-negligible self-gravity are equal.

Such further addition might look slightly over-meticulous, but there are good reasons to speak it out explicitly (see §2.4.3).

2.2.4 Einstein's Equivalence Principle

The validity of the Weak, Gravitational Weak, and Newton's Equivalence Principle is restricted to free-fall experiments of both test and non-test bodies, i.e. a subset of all possible mechanical experiments. The fourth statement we introduce covers instead all sorts of non-gravitational physics, provided that the test character of the systems involved remains untouched.

This equivalence principle is named after Einstein, and lies at the roots of description of local physics. The statement reads [155, 154, 558, 557]

> *Einstein's Equivalence Principle* — Non-gravitational, fundamental test physics is not affected, locally and at any point of spacetime, by the presence of a gravitational field.

The key idea behind this principle is the correspondence, and physical equivalence, between local frames in a gravitational field, and arbitrarily accelerated reference frames in the absence of gravity, so that the two can be used interchangeably to describe fundamental, non-gravitational test physics.

Once again, the word "test" means that, from the gravitational point of view, all physical phenomena considered — i.e. thermodynamical, electrodynamical, and so forth — are such that the surrounding environment can be safely considered unaffected, whereas of course it acts upon the particle or the continuum involved. In the same fashion, if the given background is devoid of any gravitational field, the non-gravitational test physics occurring there is intended not to generate a significant one [155].

Einstein's Equivalence Principle assures that non-gravitational test physics is *not* affected, locally, by the presence of a gravitational field. This translates the idea that, in principle, it is always possible to find a sufficiently small region in spacetime (the *local* laboratory) where gravity is absent, and where an observer will record the same results for non-gravitational, fundamental test experiments, as an observer located in another region of spacetime where a gravitational field is present. In most textbooks, this equivalence is presented to occur between an inertial frame in an environment where gravity is identically absent (i.e., Minkowski spacetime), and a non-rotating, uniformly freely-falling small laboratory in a gravitational field (the latter is sometimes considered uniform, but this last condition is too restrictive).

An important comment to be provided at this stage involves the use of the terms "local" and "fundamental" in reference to the test physics considered. "Local" refers to the intuitive idea that, by restricting to a suitably small region in spacetime, effects due to the presence of a gravitational field become progressively negligible. But this is not true in general: the curvature tensor is a local object, whose effects cannot vanish, even in the limit where the size of the laboratory decreases to that of a geometric point. Only when the gravitational field is absent they disappear [394]. In the case e.g. of the motion of a spinning particle, the equations cannot be reduced to those of a free-fall, even in the (ultra-local) point particle limit, and a coupling with the curvature is unavoidable, lest the gravitational field itself be everywhere zero [393, 413, 340, 161].

To be fair, examples like the spinning particle (and many similar cases of point-particle limits) are just *effective* descriptions of compound systems, attempting to frame average behaviours, rather than to pinpoint elementary phenomena.[5] These approximate descriptions of complex bodies are built out of the apt addition, to some elementary equations (for which Einstein's Equivalence Principle *does* hold), of further couplings with the curvature, which are requested to model all the unresolved complications of the physics involved.

Therefore, the word "fundamental" in the statement of the principle is the measure of our ignorance of basic laws of physics, and a warning for the Reader: the most elementary structures indeed comply with Einstein's Equivalence Principle, and the introduction of interactions, higher multipoles and similar non-fundamental quantities spoils the symmetry [393].

Two last remarks: first, since the statement holds "at any point of spacetime", it also contains the notion of *Local Position Invariance*, which is an ingredient traditionally associated with Einstein's Equivalence Principle in the literature on the topic [558, 557]. Also, since the fundamental laws of non-gravitational physics we probe are believed to be Poincaré-invariant in the absence of gravity [341, 317],[6] one deduces that the principle embodies as well the notion of *Local Lorentz Invariance*, i.e. the form of the local equations need be invariant after a change of inertial reference frame made of an arbitrary boost and an arbitrary spatial rotation of the axes [558, 557].

2.2.5 The Strong Equivalence Principle

Einstein's Equivalence Principle above can be further extended by making it encompassing gravitational phenomena as well — the latter are explicitly excluded in the formulation of the principle —. This leads to the so-called Strong Equivalence Principle, namely [155, 558, 557]

[5] A due remark: the "spinning particle" mentioned here is intended in its classical, non-quantum sense — the spin is a purely spatial vector attached to the particle, like the one used in [514] to explain Thomas' precession. The quantum notion of spin is not taken here into account, as it does not comply with the (classical) framework we are building.

[6] The notion itself of "absence of gravity" is fragile, for it is in principle impossible to deprive a (realistic model of the) world of its gravitational interactions, and to be still able to do physics in a meaningful way. At the quantitative level, however, the remarkable agreement of experimental results of, say, particle-physics experiments, with the theoretical calculations done for the systems considered in a Riemann-flat spacetime, seem to strongly suggest that, even though exact Poincaré symmetry might be somehow spoiled by gravity (and of course it is, at least on larger scales), it remains an excellent approximation in the ultra-local limit and for fundamental test physics. On this intriguing topic, see e.g. §9.5.1 of Ref. [96].

Strong equivalence principle — All test fundamental physics (including gravitational phenomena) is not affected, locally, by the presence of a gravitational field.

Having read the above statement, one may easily object that "local gravitational phenomena unaffected by the presence of a gravitational field" sounds suspicious to say the least. Once again, however, the solution is encoded in the restrictive clause of dealing only with *fundamental* and *test* physical conditions, i.e. the phenomena under examination should be such that the background environment be left untouched [155, 154].

In short, the Strong Equivalence Principle establishes the equivalence of local frames in a gravitational field, and local frames however accelerated in absence of gravity, even with respect to fundamental and test gravitational phenomena.

The Strong Equivalence Principle marks a huge leap, for its conclusion is far from being obvious. Gravity is an intrinsically non-linear phenomenon, hence even distant configurations of matter and energy might considerably affect local processes. Adopting this principle seriously constrains the type of theory of gravitation one can build.

The version of this principle one can find in the literature is slightly different, and amounts to the combination of the Gravitational Weak Equivalence Principle, which extends the milder Weak one, plus the local Lorentz Invariance and Local Position Invariance inherited from Einstein's Equivalence Principle [558, 557, 72] — the possibility to adopt the last two invariance conditions also in the context of gravitational physics is guaranteed by the "testness" of the systems considered.

2.3 Equivalence Principles in Practice

To appreciate the crucial role played by the equivalence principles in crafting sound and robust theories of gravity, one has to show them at work. The present section is thus devoted to present, first, the complex web of relations among the various formulations, and, second, the consequences of the principles on the structure of gravity theories. The upshot of the presentation will be the use of the equivalence principles as selection rules to admit or reject entire classes of gravity theories.

2.3.1 The network of relationships

The various equivalence principles presented above are all linked by a tight web of mutual implications, which we outline here to better highlight the similarities, differences, and hierarchical levels among the different statements [155, 154].

To begin with, the Weak Equivalence Principle implies Newton's Equivalence Principle [155]. This can be shown by noticing that, once Newton's Equivalence Principle does not hold anymore, the universality of free fall for test bodies cannot occur, not even in the Newtonian regime.

The converse, however, is not true [497, 396]: the universality of free fall is implied by the equivalence of inertial and gravitational masses only as long as, in the equations of motion, $m_{\rm I}$ and $m_{\rm G}$ appear in the form of their ratio. Since this is the case e.g. in Newtonian mechanics and General Relativity, the

Weak and Newton's Equivalence Principles are often identified. If, however, other sorts of combinations of the two masses are allowed, the universality of free fall is violated even though m_{I} and m_{G} are equal. For instance, in Bohmian mechanics [82] the equations of motion of a test particle in a gravitational field read

$$m_{\mathrm{I}} \frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} = -m_{\mathrm{G}} \boldsymbol{\nabla} \Psi - \boldsymbol{\nabla} Q \ , \tag{2.5}$$

where the quantum potential Q contains the inertial mass, but is not proportional to it; now, notwithstanding the equivalence $m_{\mathrm{I}} = m_{\mathrm{G}}$, the free-fall trajectory of the test particle ends up depending on its mass, which is a severe violation of the Weak Equivalence Principle.

By the same token, we can say that the Gravitational Weak Equivalence Principle implies the Gravitational Newton's Equivalence Principle in an empty background [155]. We also get that, whenever the inertial and gravitational masses of a self-gravitating test bodies are not equal, then also the universality of free fall for self-gravitating test bodies cannot occur. This latter aspect is often probed in experiments, lying at the roots of the so-called Parametrized Post-Newtonian formalism — see below —. Indeed, the non-equivalence of inertial and gravitational mass for two self-gravitating bodies is detected via fractional differences in the mutual accelerations of two systems, and this test is applicable whenever the theory admits a Newtonian regime.

Einstein's Equivalence Principle implies the Weak one. The former guarantees that, locally, the behaviour of a freely-falling (non-spinning, point-wise, non-compound) particle in a gravitational field cannot be distinguished from that of a (non-spinning, point-wise, non-compound) free particle in an environment devoid of gravity. But free particles in absence of gravity behave universally, whence the universality also for the free-fall motion, and the Weak Equivalence Principle.

The same argument allows to state that the Strong Equivalence Principle implies the Gravitational Weak Equivalence Principle, and also Einstein's one, which is contained in the definition of the Strong version.

The implication "Weak Equivalence Principle $\Rightarrow$ Einstein's Equivalence Principle" is still debated; it is believed to be essentially true, to the point that it has been given the status of a *conjecture* — precisely, Schiff's conjecture, from the name of its main standard-bearer [474, 558, 557] — and proofs exist in some restricted cases [319], but a complete argument has not been provided yet (and might even be impossible to exhibit one [395, 377, 129]).

Schiff advances the idea that "test" bodies are in fact compound objects made of elementary building blocks bound together by forces of various nature; this allows to show, for instance, that the previous implication is true if the leading binding interaction is electromagnetic, and the given background gravitational field has spherical symmetry [319]. What is still missing from this proof, for example, is a full argument that, once the binding forces be sensitive to gravity (thus providing a violation of Einstein's Equivalence Principle), also the universality of free fall would be violated, whence the failure of the Weak Equivalence Principle. By negating this implication one would obtain the required statement "Weak Equivalence Principle $\Rightarrow$ Einstein's Equivalence Principle".

A resolution of the conundrum with Schiff's conjecture would also open a path towards the implication "Gravitational Weak Equivalence Principle $\Rightarrow$

Einstein's Equivalence Principle", stemming from the milder connection "Gravitational Weak Equivalence Principle ⇒ Weak Equivalence Principle" [155].

As a final aspect of this tangling network of implications, we conjecture that the content of the Strong Equivalence Principle may be determined entirely by the juxtaposition of the Gravitational Weak Equivalence Principle and of Einstein's Equivalence Principle. This argument would be the natural extension of Schiff's idea to gravitational phenomena and, in particular, to self-gravitational interactions [155].

The simultaneous presence of both the Gravitational Weak Equivalence Principle and Einstein's Equivalence Principle might be necessary because the former cannot describe anything but the free-fall motion of *test objects*, whereas to frame e.g. light gravitational waves another ingredient is required, and that is precisely Einstein's Principle.[7]

In view of these last considerations, we can conclude that the Gravitational Weak Equivalence Principle is a key step towards a better understanding of the universal behaviour of self-gravitating bodies, and the possible cornerstone of a rigorous formulation of the Strong version.

2.3.2 Formal implications of the Equivalence Principles

It is fair to say that, with almost any equivalence principle, comes a leap ahead in the formalisation of gravity theories. We can now see how this process works.

To begin with, consider the content of the Weak Equivalence Principle: the gravitational field, when acting on *any* type of test body with negligible self-gravity, always gives the *same* result (this is precisely the "universality" of the free-fall), and can thus be ascribed to a geometric property of spacetime itself, rather than to a characteristic of the physical systems under consideration. The philosophy behind this logical step is that, any time a universal property emerges for a class of physical objects, one can trade such property for a properly defined geometric structure.[8]

In the specific case of the free-fall world-lines in a gravitational field, this amounts to say that the spacetime manifold M can be given a set of preferred curves, which in turn defines a *path structure*, or rather a *projective structure* on M itself [175, 128, 246]. With the introduction of few further, reasonable hypotheses,[9] the path/projective structure can be massaged into an *affine structure*

[7] In the specific example of the light gravitational wave, this is because the wave can be seen as the propagation of a spin-2 field, to which Einstein's Equivalence Principle does apply [155].

[8] Geometry thus becomes nothing but the physical theory of universal phenomena, or of universally coupling physical agents. In some sense, this is merely the effect of a semantic drift, which drags physical meaning over geometry every time there seems to be no exception to a given rule [155, 244].

[9] Namely, the existence of a conformal structure (of the Lorentzian, or normal-hyperbolic type) on the base manifold. The conformal structure cannot be inferred from the free-fall motions of particles alone, but has to be postulated in this case, being related to the propagation of e.g. light rays, which do not enter the discussion of the systems pertaining to the Weak Equivalence Principle. Also, one has to assume that the path/projective structure and the conformal one are *compatible* — that is, every null auto-parallel curve for the conformal structure must be as well an auto-parallel according to the projective criteria — which in turn equips the manifold M of a so-called *Weyl geometry* structure [175, 442, 460, 476]. The required affine structure on the manifold emerges as soon as one demands the Weyl geometry to be such that any affine auto-parallel curve is itself a projective auto-parallel line, and the nullity of vectors according to the conformal structure is preserved under parallel transport [175].

on spacetime, call the latter "Γ" [497, 175]. Once the affine structure is in place, the preferred trajectories become nothing but the auto-parallel curves of the affine connexion, i.e. the lines whose tangent vector u^a is parallel-transported along itself,

$$u^a \nabla_a u^b = f(\lambda)\, u^b \ , \tag{2.6}$$

with λ a generic parameter along the curve. The association between gravitational field and affine structure becomes perfect when one notices that, in the absence of a gravitational field, the generic free-fall curves reduce to straight lines in Minkowski's spacetime, which themselves are indeed the auto-parallel curves of a flat connexion, and thus express the inertial motions in both Newtonian dynamics and special relativistic mechanics.

With introduction of Einstein's Equivalence Principle, we add the notion that local fundamental test physics complies with the framework of Special Relativity [558, 557]. In a system of inertial, pseudo-Cartesian coordinates $x^\mu := (x, y, z, ct)$, the metric η_{ab} reduces to

$$\eta_{\mu\nu} \equiv \mathrm{diag}\,(1, 1, 1, -1) \ , \tag{2.7}$$

and it is possible to extract from it the only metric-compatible connexion "$\Gamma_{(\eta)}$", i.e. the only connexion such that $\nabla_a^{(\eta)} \eta_{bc} = 0$. $\Gamma_{(\eta)}$ is flat by the definition of η_{ab}, i.e. the associated curvature tensor vanishes identically.

The combination of the Weak and Einstein's Equivalence Principles allows us to move further: the connexions Γ and $\Gamma_{(\eta)}$, which have in principle different origins, must be locally indistinguishable because of the local equivalence of a freely-falling reference frame in a gravitational field, and a non-accelerated one in Minkowski spacetime, whence another daring association: Γ can be thought of as the curved, metric-compatible connexion associated with a generic, non-flat metric g_{ab} defined over the whole spacetime manifold [155, 154].

Actually, the curved metric g_{ab} on M can be introduced at an even earlier stage, without deploying the full content of Einstein's Equivalence Principle, by simply requiring that local chrono-geometric measurements — namely, measurements of spatial and time intervals — be not sensitive to the existence of a surrounding gravitational field [155]. Einstein's principle, however, permits to build upon this conclusion, for it says that the same indifference to the presence of a gravitational field holds, locally, for all fundamental test physics, and this secures the capability to write the various laws of fundamental physics in a curved spacetime as soon as their form is known in Minkowski's one [497, 155, 154].

Notice that, in all this discussion, the notion of "locality" is potentially slippery, and is always intended to be defined once a scale of curvature and a level of accuracy/sensitivity of the measuring apparatus is provided.[10]

There is at this stage another important remark about Einstein's Equivalence Principle to be made. The statement of the principle is about the behaviour of systems in presence and in absence of a gravitational field: this clearly concerns the *solutions* to the equations of motion and/or field equations, rather than the *equations* themselves. Formally, this can be obtained by demanding that

[10] Also, the laws of physics considered must be only those formulated in an entirely local way; whenever non-local or global effects enter the game, as e.g. in quantum field theory on curved spacetime, where there is a dependence on the notion of a global vacuum state, issues of technical and interpretational character arise [496].

the local structure of the Green function associated with a physical law be left untouched when passing from a flat to a curved spacetime environment. This is the condition to be enforced.

It is worth stressing out this aspect because Einstein's Equivalence Principle is often evoked as the rationale behind the "comma-goes-to-semicolon" rule, or "minimal coupling" prescription, namely the idea that the equations of physics in a curved background can be obtained by the corresponding ones in Minkowski spacetime by simply substituting every instance of the metric η_{ab} and of the partial derivative operator ∂_a with the curved metric g_{ab} and the covariant derivative operator ∇_a, respectively. This last protocol, however, crashes even in the simplest case of a scalar field, where minimal coupling allows for the onset of decidedly queer measurable effects [496, 498], which would immediately signal the presence or absence of a gravitational field in a tiny region of spacetime. Rather, the above-mentioned condition on the structure of the Green function sidesteps this issue at once.

The formal consequences of the Gravitational Weak and Strong Equivalence Principle are less immediate to define, whence their smaller diffusion in the community, but particularly the former has a huge impact on the landscape of gravity theories, thanks to its ability to act as a filter, and rule out a wide portion of the models.

The Gravitational Weak Equivalence Principle will be given a full account in Chapter 3; here, we anticipate some key concepts of our main results by saying the following. In the extension of the universality of free fall to self-gravitating systems not affecting appreciably the gravitational environment, one obtains two conditions on the theory as a whole. Of these, one is the explicit requirement that the environment ought to be empty of matter and fields other than the gravitational ones, and the other is that the full information about gravitational phenomena should be entirely and exclusively encoded into the metric field alone, without other gravitational degrees of freedom [154].

As for the Strong Equivalence Principle, it is usually formulated as another "impossibility principle", demanding the presence of the metric field as the sole responsible for gravitational phenomena.

There is, however, at least one recent proposal attempting to give the Strong Equivalence Principle a completely new and different formal content; it has been suggested in Refs. [216, 215], with apparently promising results. For sake of completeness, we leave here a bird's eye view of his achievements and potential pitfalls.

Of gravitons and gluons: a new Strong Equivalence Principle?

Gérard's idea stems from an old analogy between gravity and non-Abelian gauge field theories; namely, that between Yang–Mills theory and General Relativity [216, 215].

To begin with, consider a non-Abelian theory characterised by some Lie algebra with generators q^{h} — the sans-serif superscripts refer to the representation of the algebra, and are summed whenever repeated in the formulæ, regardless of their position —. Suppose then that

$$\mathrm{Tr}\left(q^{\mathsf{h}}q^{\mathsf{k}}\right) = \frac{1}{2}\delta^{\mathsf{hk}} \quad , \quad \left[q^{\mathsf{h}}, q^{\mathsf{k}}\right] = \mathrm{i}\, c^{\mathsf{ihk}} q^{\mathsf{i}} \; , \tag{2.8}$$

with c^{ihk} the structure constants of the theory. A vector gauge potential A_k^{h}, with values in the representation of the algebra, whence the sans-serif index, acts as the Yang–Mills potential in the construction of the skew-symmetric field strength tensor

$$F_{ab}^{\mathsf{h}} := \partial_a A_b^{\mathsf{h}} - \partial_b A_a^{\mathsf{h}} + \kappa c^{\mathsf{hij}} A_a^{\mathsf{i}} A_b^{\mathsf{j}} \; . \tag{2.9}$$

The latter can be also defined in terms of the commutator

$$[\mathrm{D}_a, \mathrm{D}_a] \equiv -\,\mathrm{i}\,\kappa q^{\mathsf{h}} F_{ab}^{\mathsf{h}} \; , \tag{2.10}$$

where in both equations κ is the coupling constant, accounting for the strength of the interaction, whereas the gauge-covariant derivative D_a reads

$$\mathrm{D}_a := \mathbb{I}\partial_a + \mathrm{i}\,\kappa q^{\mathsf{h}} A_a^{\mathsf{h}} \; , \tag{2.11}$$

with $\mathbb{I}$ the identity in the algebra. The field equations for the theory can thus be obtained by simply requiring

$$\mathrm{D}_a F^{\mathsf{h}ab} = \kappa j^{\mathsf{h}b} \; , \tag{2.12}$$

in which one denotes with $j^{\mathsf{h}a}$ the four-vector current sourcing the dynamics of the gauge potential.

Since in such theories the fields carry themselves the charges with which they interact, a non-linear self-coupling is in general expected. This triggers the possibility to describe gravity as well in this unified language. When it comes to gravitational phenomena, the source of self-interaction is mass (and energy), and the roles of the gauge potential and field strength are played by the connexion and curvature, respectively [215]. Starting from the usual definition of the curvature operator[11]

$$R_{abc}{}^{d} := -[\nabla_a, \nabla_b]_c{}^{d} \; , \tag{2.13}$$

it is not difficult to foresee that the new field equations will be [216]

$$\nabla_d R_{abc}{}^{d} = \kappa j_{abc} \; , \tag{2.14}$$

where j_{abc} is the new "current" sourcing the gauge potential, and the gravitational self-interaction is encoded into the "ΓΓΓ" term present on the left of Eq. (2.14).

This proposal drives the attention on the dynamics of the connexion, rather than that of the metric — Eq. (2.14) is second order in the connexion coefficients — and, in the proposal of Refs. [216, 215], Eq. (2.14) is advanced to embody the crucial content of the Strong Equivalence Principle. The non-linearity of gravitational phenomena is nothing but the reflection of the general arrangement of Yang–Mills theories, to the point that, as in the words of Ref. [216], "gravitons gravitate the way gluons glue".

Condition (2.14) is a first example of a compact formalisation of the Strong Equivalence Principle, and reproduces some well-known results in the case of

[11] In this case, the group representation is that of the orthochronous Lorentz group; the connexion coefficients are thought of as Lie algebra-valued 1-forms, and the Riemann tensor is intended as a Lie algebra-valued 2-form. In this sense, the indices in the symbols $\Gamma^a{}_{bc}$ and $R_{abc}{}^d$ have in principle different status, and one can distinguish spacetime indices (referring to the manifold M), and Lorentz indices, defined on the tensor bundle whose base space is M. For an accessible presentation, see [468] and references therein.

General Relativity and scalar-tensor theories. It also presents, however, some delicate issues. For instance, this formulation of the Strong Equivalence Principle does not imply the Weak one, which must be postulated separately at the level of the solutions of the field equations, to fix the value of an otherwise free parameter. This appears odd, for the Strong Equivalence Principle is clearly supposed to include the Weak one as a proper sub-case, hence the claim of Refs. [216, 215] may probably need be slightly readjusted [154].

On top of that, when the protocol of Eq. (2.14) is used as a selection rule to exclude gravity theories other than General Relativity, no general usage prescription is provided in the original sources and, apart from the case of scalar-tensor theories — discussed in some detail, but without any complete derivation of the resulting constraint equations — no hints are present on how to extend the procedure to other models.

2.3.3 From Equivalence Principles to selection rules

We have seen how the Equivalence Principles become effective and powerful "hooks" to better glue mathematical structures to gravitational phenomena. This, however, is only one half of the story. The *constructive* role of the principles highlighted above is perfectly mirrored by their *selective* role. We can now exhibit some examples of the sorts of sieves emerging from a wise application of the principles of equivalence.

Starting from the basic step of the ladder, and adopting only the Weak Equivalence Principle, any theory of gravity providing an affine structure on the spacetime manifold M can be accepted, for the only requirement is that the framework be able to exhibit a class of preferred world-lines on M.

Up to this point, then, the vast majority of the models listed in §1.3 is permitted, for the presence of a privileged class of trajectories is quite a recurring feature [155]. General Relativity is of course inside the set, as any other model admitting a connexion in the base manifold. Interestingly enough, also the metric-affine, affine, and purely affine theories can pass the sieve, and even theories where the metric structure is absent at all, yet an affine connexion is still available, are viable candidates. Finally, a geometrised version of Newtonian theory (*Newton–Cartan theory*, [332, 299, 160]) which includes an affine structure, enters the roster as well.

The introduction of Einstein's Equivalence Principle demands the local existence of a Lorentzian metric on the manifold, and that the affine connexion be the Levi-Civita one. Also, it exacts the physical laws to locally abide by the framework of Special Relativity; this greatly constrains not only the family trees of possible models for gravitational phenomena, but also the entire class of admissible physical theories, whose local equations need be Lorentz-invariant, and whose predictions cannot depend on where and when the experiments are performed. Besides, as stated, the local form of the Green function must be insensitive to the presence of a gravitational field.

Given these premises, a careful application of the above-mentioned "comma-goes-to-semicolon" rule permits to select the so-called *metric theories of gravity*, i.e. all the theories in which the gravitational content of the model is encoded at least into a metric tensor g_{ab} defined over spacetime, which in turn reduces locally to the flat metric η_{ab} [558, 557].

We stress that Einstein's Equivalence Principle demands the presence of *at least* a metric structure incorporating informations about gravitational phenomena, not "at most"; this means that all the theories incorporating other gravitational degrees of freedom, in any form whatsoever, are not ruled out at this stage, provided that they also exhibit a metric tensor, and that they guarantee Local Lorentz Invariance and Local Position Invariance for fundamental non-gravitational test physics.

This last statement allows to clarify a final point: as long as the violations of Local Lorentz Invariance are confined to the gravitational sector, and do not affect the matter sector — which is the one involved in non-gravitational, fundamental test physics — also all the theories commonly denoted as "Lorentz-violating" (Einstein–Æther, or Hořava–Lifshitz) pass the sieve.

Finally, the Gravitational Weak and the Strong Equivalence Principle. Historically, both emerge at a much later stage in the process of theory-building, when the supremacy of General Relativity is already assessed, and essentially focus on the gravitational degrees of freedom, their type and dynamics, and their mutual interactions.

Suppose then that a theory is given, in which one has the metric g_{ab}, at least one scalar field ϕ, and some non-dynamical field A^k (we take momentarily a vector, but any other object would work); the presence of g_{ab} is secured by Einstein's Equivalence Principle, and one might ask what is the effect of the presence of ϕ, A^k. The additional gravitational variables cannot affect local non-gravitational fundamental test physics (the latter "sees" only the metric, and generates a gravitational field which is negligible because of the testness character). At the same time, local gravitational experiments can be influenced by the presence of ϕ, A^k by how the two fields couple to the metric; in particular, the values assumed by the fields ϕ, A^k at the boundaries can affect the outcome of some local gravitational experiment by introducing dependencies on the velocity of the laboratory with respect to the environment, or on the position in space and time.

In their pristine formulation, the Gravitational Weak and Strong Equivalence Principles aim at ruling out the latter effects, and all the theories producing them. This is the reason why, in the statement of the Strong Equivalence Principle found in the literature [558, 557], the universality of free fall for self-gravitating systems — Gravitational Weak form — is accompanied by the requirement of Local Lorentz iInvariance (which prevents velocity-related effects), and Local Position Invariance (which prevents position-related effects).

Now, since General Relativity is described in terms of the metric field only, and the local Minkowskian character of g_{ab} ensures Lorentz and Position invariance, and since practically all the theories considered for a long time [558, 557] all induce some preferred-frame or preferred-position effect, the natural result is that, in four spacetime dimensions, the Strong Equivalence Principle picks Einstein's theory alone, and hence the conjecture holds that the Strong Principle implies only General Relativity.[12]

The point of view we developed in Refs. [155, 154] is slightly different; instead of examining the full content of the Strong Equivalence Principle, we have

[12] And Nordström's gravity, actually [147, 147]. If the latter is not recorded in the conjecture, is because the scalar theory is ruled out at the experimental level — no theoretical, nor formal obstructions have been exhibited so far.

focussed on its sub-set dedicated to the free-fall motion of self-gravitating systems alone, the Gravitational Weak Equivalence Principle. Surprisingly enough, it seems that this proper part of the original statement is powerful enough to recover the whole content of the selection rule implied by the "traditional" Strong form, plus novel conclusions in the case of higher-dimensional environments.

If, then, the typical role ascribed to the Strong Principle can be shifted onto the Gravitational Weak one, the problem arises to understand what further piece of information this principle actually contains [155, 154]. As remarked when recording the statement, the Strong Equivalence Principle deals, at least potentially, with a much ampler class of phenomena than the mere free fall, hence it should be possible in principle to invent and design tests of fundamental physics involving self-gravity (as in the case of gravitational waves, or in mixed scenarios where e.g. electromagnetism and gravity can interact at the fundamental level). The door is open on a new path towards a better understanding of these high-hierarchy principles of equivalence, and their formal and experimental consequences.

We conclude this discussion by adding a last remark. At this stage, one usually introduces two further bottlenecks, by requiring that only "purely dynamical, Lagrangian-based, theories of gravity" be taken into consideration [558].

A *Lagrangian-based theory* is one whose dynamical field equations emerge from a (well-posed) variational principle, as the extrema of the first variation of a given action functional, compatible with the class of assigned boundary conditions. Such theories enjoy relevant properties, the most important being the fact that the conservation equation for the stress-energy-momentum tensor associated to the non-gravitational degrees of freedom, $\nabla^a T_{ab} = 0$, follows from the gravitational field equations if and only if there are *no* non-dynamical variables in the Lagrangian [309].

A *purely dynamical theory* is one in which all the degrees of freedom, including the ones pertaining to gravity, have dynamical status, and no a priori structures act as a background on which the other physical field live, or with which they interact.

While these constraints do not emerge from any Principle of Equivalence, the success of the relational, purely dynamical standpoint in physics is so overwhelming (General Relativity, Standard Model, path-integral formulation of quantum field theory), that non-dynamical, "God-given" scaffoldings have been gradually marginalised, and any true aspect of the world is expected to evolve dynamically. Hence, all theories discussed in the second half of §1.3.1, and presenting some a priori geometric quantity, should be rejected once and for all.

This request is in fact quite restrictive, and the argument might even be turned against itself: all in all, even in the most dynamical of all our models, General Relativity, there are fixed elements (topology, spacetime dimension, signature), and implementing a dynamical character for any of such elements can become extremely troublesome. Models exist, anyway, where some of these remaining background structures are promoted to dynamical fields [421].

2.4 Testing the Equivalence Principles

The validity of a principle relies on its ability to be reaffirmed by any experiment designed to confirm or disprove it. The Equivalence Principle have lined up on this righteous trend since their emergence in the debate on General Relativity and gravity theories, and have offered themselves to any sort of probe. This has guaranteed a wide flow of data in the last century, with an overall confirmation of the principles and their current formulation.

An interesting facet of the history of the experimental path to the principles of equivalence is the fact that, being statements about the most fundamental aspects of nature, and their daring interpretations, they have ignited an extremely rich and diverse pool of possible experiments, ranging from true "classics" (the Michelson–Morley interferometer, the torsion balance), to unexpected detours (old meteorites, signals from distant astrophysical sources, rotating compact objects used as standard clocks).

In the paragraphs below, we have tried to briefly sketch a selection of results concerning the tests of the Equivalence Principles discussed so far: the topic is complex, and largely unnecessary for what follows, so we invite the interested Reader to peruse the vast literature on the topic, starting from the recent review [557], and then diving in the references therein.

2.4.1 Main achievements in testing the principles

As for the Weak Equivalence Principle, most of the tests actually probe the equivalence of inertial and gravitational masses; i.e., they are rather tests for Newton's Equivalence Principle in the context of free-fall experiments for a pair of small, light, uncharged bodies.

A test body is an approximation for a compound system. Suppose then that the gravitational mass m_{G} differs from the inertial one m_{I} because of the details of the interactions occurring within the system; the result reads

$$m_{\mathrm{G}} = m_{\mathrm{I}} + \frac{1}{c^2} \sum_J \xi_J E_J \,, \tag{2.15}$$

where E_J is the internal energy in the compound body due to the J-th interaction, weighted by the coupling constant ξ_J. The fractional difference in the acceleration of two such bodies "1" and "2" would be then given by the so-called *Eötvös ratio*

$$\zeta := \frac{1}{c^2} \sum_J \frac{E_{J,1}/m_{\mathrm{I},1} - E_{J,2}/m_{\mathrm{I},2}}{E_{J,1}/m_{\mathrm{I},1} + E_{J,2}/m_{\mathrm{I},2}} \,. \tag{2.16}$$

Precision experiments on this tone have been carried around for more than a century now [557], mostly involving torsion balances [185]. The settings have been subsequently ameliorated, to account first for the Earth rotational drag, and then for similar effects in satellite probes; the precision reached is currently [8] of the order $\zeta \sim 10^{-13}$, with a forecasted improvement up to $10^{-17} \div 10^{-18}$ for future test both on Earth and in space [509].

Interestingly enough, an Eötvös-like experiment can be designed for peculiar stellar configurations involving quickly-rotating compact objects; in 2014, the discovery of a triple system made up of one pulsar and two companion white dwarfs has attracted much attention for the possibility to make good use of the very different composition of the dwarfs with respect to the pulsar [450]. Other, soon-to-occur space-based experiments involve drag-corrected satellites orbiting around the Sun and carrying differential accelerometers with slightly different internal composition, or sub-orbital rockets [509]. As for the micro-physical test-benches, experiments with anti-hydrogen are currently under consideration [81].

As a side remark, we notice that probing Newton's Equivalence Principle via free-fall experiments, which turn out to be partly also tests for the Weak Equivalence Principle, is not at all the only possible way; the different character of inertial and gravitational masses can be inspected accordingly, e.g. by means of collision experiments to evaluate m_{I}, and by high-precision measurement of weight to get a number for m_{G} [155].

Einstein's Equivalence Principle claims the equivalence of local, fundamental test physics in the presence and in the absence of gravity. In the latter case, the dominating framework for physical laws is Special Relativity, any test of special-relativistic effects (and their violations) becomes an indirect confirmation of the principle itself [557, 224].

What is truly delicate, in all these experiments, is the apt design of a sufficiently "local" and "fundamental" phenomenon, such that the couplings with gravity can be safely neglected, or at least flattened below the sensitivity threshold. Typically, one works with electromagnetic phenomena (recently extended [131, 130, 300] to the whole Standard Model), or with emissions from atoms. In either case, a wealth of secondary effects, due to the non-fundamental nature of the objects involved, must be carefully taken under control.

For example, at low energies, a possible test requires to assume natural units, then measure the value for the speed of light in vacuum c, and look for tiny deviations in the parameter

$$\delta := \left|c^{-2} - 1\right| \ , \tag{2.17}$$

which can be obtained by measuring e.g. anisotropies in the hyperfine transitions of complex nuclei with respect to the corresponding energy levels in standard atomic clocks (the so-called "clock anisotropies"). In such cases, the precision reached so far amounts to $10^{-22} \div 10^{-24}$ — see [443, 306, 120].

Testing Einstein's Equivalence Principle also means testing the invariance of the laws of physics with respect to the position in space and in time of a local laboratory. A useful tool in this sense turned out to be the classical Pound–Rebka experiment on the gravitational redshift, i.e. the difference in wavelength or frequency of two standard clocks placed at different heights in a static gravitational field [558, 557]. The current bounds on the spatial position invariance reach 10^{-5}, from comparison of a Hydrogen maser with a Cesium atomic fountain for over a year's time [46].

As for the local position invariance in time, its violations can propagate to a corresponding variation in the fundamental constants [558, 532], as e.g. the electron-proton mass ratio, the weak interaction constant $\alpha_w := G_f m_p^2 c/\hbar^3$ (with G_f the Fermi constant), and the fine structure constant $\alpha_e := e^2/\hbar c$ (e the elementary electric charge). Then, one follows the evolution of the constants over time, recording the values $\dot{\alpha}/\alpha$, and puts the constraints. The accuracy in this case reaches 10^{-16} for the fine structure constant from ancient meteorite remnants filled with debris of ^{187}Re [397].

Many other options are available [341, 317, 130, 300], based on minimal extensions of the Standard Model (probed in the infrared regime), or on non-minimal such extensions (this time in the ultraviolet regime); see §4.3.1 for more details.

When it comes to tests of the Strong and Gravitational Weak Equivalence Principles, the issue becomes more delicate than ever. The intrinsically non-linear character of gravitational interactions, the universal coupling of gravity to all the other forms of physical agents, and so forth: it all seems to conspire to make any experimental setting an intricate mess, leaving the researcher hopeless.

In an ironic twist, the classical Solar system tests, initially used to reveal possible deviations from the Newtonian predictions, have been progressively refurbished to look for deviations from General Relativity. The main effects investigated, and the best precisions reached, are:

* The deflection of light, as the light rays graze the outer rim of a massive source of a Schwarzschild-like gravitational field, compatible with the general relativistic value within 10^{-4} from data of the Very Long Baseline Interferometers [483].

* The time delay in a round trip about the above-mentioned mass, compatible with the general relativistic value within 10^{-5} from data of the Cassini satellite tracking [71].

* The periastron shift in a quasi-Keplerian orbit. In this case the planet Mercury is still the best source available, and the results are compatible with the general relativistic value within 10^{-7} [484].

* The Nordtvedt effect [384] (a sort of Eötvös-like effect for large, self-gravitating extended masses), for which the Lunar Laser Ranging system is vastly deployed, compatible with the general relativistic value within 10^{-4} — with some assumptions to get consistent results [384, 560, 559].

* The consequences of the existence of preferred frames and preferred locations, affecting spin polarisation and orbital polarisation; compatible with the general relativistic value within $10^{-4} \div 10^{-9}$ [56, 367, 508].

Basically all these results, and a few others involving the sidereal relative change of the gravitational constant and the gravitomagnetic effects, are obtained on the basis of a powerful formalism developed to provide a precise — yet general enough — description of all possible relativistic corrections of the standard Newtonian formalism, including (but not being limited to) those due to General Relativity, a possible "fifth force", and higher-curvature corrections. The so-called *Parametrised Post-Newtonian formalism.*

Which leads us to the next section.

2.4.2 The Parametrized Post-Newtonian formalism

The Parametrised Post-Newtonian formalism is a complex computational algorithm, suitable for testing many types of metric theories of gravity. A fully-detailed review of its features and achievements is available in Refs. [558, 557].

Given the purposes of the present thesis, it suffices to provide a bird's eye view of some key elements.

The cornerstone of the method is made up of two ingredients: the metric of spacetime, whose presence and dynamical character is ensured by Einstein's principle and by the purely dynamical hypothesis, and the matter fields. The latter are supposed to be sensitive only to the metric, and not to any other gravitational degree of freedom — which they can still source in the field equations, but without being acted upon.

Metric and matter become the main reference elements, upon which all the rest of the language is built (including supplementary degrees of freedom accounting for gravity, and non-dynamical geometric quantities). The method focusses on corrections to the Newtonian regime — i.e., in the weak-field, slow-motion limit — whence the "post-Newtonian" expression in the name. Various contributions from the matter terms (density, pressure and anisotropic stress of a continuous medium) and from the motions (velocity of the fluid in a quasi-Lorentzian reference frame, and thus derivative operators in space and time) are recognised to have different orders of "smallness", so that their effect can be represented via powers of a small parameter ϵ in all the expressions.

The formalism then starts with an extremely general form of the metric [384, 383], one accounting for all possible add-ons to the standard Newtonian limit. The particular shape of $g_{\mu\nu}$ is a sum of "metric potentials" built out of the matter parameters ($\rho, p, \mathbf{v}$, etc.), multiplied by the actual parameters of the formalism. In particular, the Parametrized Post-Newtonian method works with ten different parameters, variously interwoven in the expression for the metric, each one accounting for a different effect.

For instance, the form of the spatial part of the metric, written in the nearly global pseudo-Cartesian coordinate system $x^\mu \equiv (x, y, z, ct)$, reads;[13]

$$g_{hk} := \left(1 + 2\gamma \frac{U}{c^2}\right) \delta_{hk} + O\left(\epsilon^2\right) \ , \tag{2.18}$$

where $\epsilon := U/c^2$, and the metric potential U is given by

$$U := \int \frac{\rho\left(\mathbf{x}', t\right)}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3 x' \ . \tag{2.19}$$

The characteristic parameter γ indicates how much spatial curvature is endowed by a unit rest mass (its value reduces to 1 in General Relativity, while remaining unconstrained in other theories). Similar, but much more involved expressions, are available for g_{i4} and g_{44}, as functions of the mentioned ten parameters, and of no less than 19 metric potentials similar to U — see Box 2 in Ref. [557], and [556].

The PPN parameters account for many possible gravitational effects (preferred location effects, preferred frame effects, violation of conservation of total momentum, rate of non-linearity in the laws of gravitational interaction [558]), and are designed so as to reach their standard values when the theory under consideration is General Relativity, in four spacetime dimensions. In this sense, the formalism provides a test for the Strong Equivalence Principle.

[13]This peculiar choice of coordinate, named "PPN gauge", removes at once any residual gauge freedom from the formalism.

To extract the actual values of the coefficients from a given theory of gravity, the procedure requires to identify the gravitational degrees of freedom, assign to them proper boundary/asymptotic values, expand in the post-Newtonian series around those values, substitute in the field equations, solve iteratively for the metric coefficients $g_{\mu\nu}$ up to order $O\left(\epsilon^4\right)$, compare with the expression (2.18) and the object alike, and finally read out the values of the requested parameters [558].

The current bounds in the tests for the Strong Equivalence Principle can be presented as follows: the Eötvös ratio (2.16) applied to macroscopic objects like the Moon [560, 559] is compatible with zero within one part in 10^{-4}, and is expected to vanish if the principle actually holds. In this case, the Lunar Laser Ranging is the state-of-the-art apparatus devoted to this office [557]. Notice, however, that tiny deviations might be "effaced" by the cumulation of a compensating tiny violation of the Weak Equivalence Principle for the microscopic constituents of the extended bodies (tests capable of discriminating and separating the two contributions are currently under development). Also, data extracted from observations of compact objects in highly circular systems [508] support the validity of the principle within 10^{-3} — recall that binary configurations are perfect candidates to confirm or disprove the general relativistic framework [292, 508].

Finally, the preferred-location and preferred-frame effects in the examination of the rate of change $\dot{G}/G$ for Newton's constant support the Strong Principle to a remarkable figure of 10^{-20} [557], with data extracted from Lunar Laser Ranging, binary pulsars, Solar seismology, and Big Bang nucleosynthesis.

2.4.3 Some remarks on the formalism

There are three fundamental elements to be underlined when it comes to a critical reexamination of the PPN formalism, especially in relation to its purpose to test the Strong Equivalence Principle.

First and foremost, a methodological consideration: the protocol lies at the basis of many experiments involving the laser ranging of a self-gravitating body like the Moon. A moment's thought, however, allows to see that what is actually measured in such cases is the equivalence of the inertial and gravitational masses for a self-gravitating body, which is the content of the Gravitational Newton's Equivalence Principle presented in §2.2.3. As Newton's Equivalence Principle for test bodies is linked, but does not coincide necessarily, with the Weak Equivalence Principle, the same difference holds for the Gravitational Weak form and the Gravitational Newton's one.

The fact that inertial and gravitational masses be equal, as seen before, does not prevent the onset of violations of the universality of the free fall, provided that the underlying theory admits equations of motion where not only the ratio $m_{\mathrm{G}}/m_{\mathrm{I}}$, but also other combinations of the two masses are in principle available. Whenever the onset of such other equations of motion is rejected ab initio, the tests for the equivalence of inertial and gravitational mass are also test for the universality of free fall.

It should be pointed out, however, that passing the test for the equivalence $m_{\mathrm{G}} = m_{\mathrm{I}}$ does not prevent completely the emergence of possible violations to the universality of free fall for self gravitating bodies, which is indeed that Gravitational Weak Equivalence Principle we separated from the Strong Equivalence

Principle for the logical reasons exposed in §2.2.3.[14]

Another aspect of some relevance concerns spacetime dimensions. As seen in §1.3.4, there are now sound reasons not to limit one's perspective to strictly four dimensions. Higher dimensional environments have gained popularity in recent years and, thanks to some stimulating proposals (AdS/CFT correspondence, string theory, braneworld scenarios).

The Parametrized Post-Newtonian formalism, in this sense, needs some refurbishing, if it aims at extending its goal to rule out gravity theories on larger manifold settings. As it stands, indeed, the method is clearly tailored on a four dimensional spacetime, where probably all the fields present are defined on the whole structure. Nothing meaningful can be stated in this sense about a model built on a, say, ten-dimensional brane for which six dimensions wind up at microscopic scales.

In the same fashion, the tool is ultimately blunt when it comes to judge e.g. the hierarchy of Lanczos–Lovelock polynomials: those models are precisely purely dynamical and purely metrical theories [404], exactly as those which in principle should pass the test of the Strong Equivalence Principle (and its experimental sieve). It would be interesting to see what the formalism could say about these theories, but the method ought to be retuned to fit in a higher-dimensional setting.

While the Parametrised Post-Newtonian formalism remains a milestone in gravity theories, for its full generality and effectiveness, there might be other paths, and different strategies, to embrace an even larger, or simply a different, subset of the family of extended theories of gravity, and see if, by means of simpler arguments about the behaviour of physical systems, the Strong, or Gravitational Weak Equivalence Principles, may be tested, and used to discriminate among gravitational theories.

We believe to have found a possible way to do so, and we shall discuss it in the next Chapter.

[14]By the same token, also the proposal of Refs [216, 215], in the form implemented in his original sources, ends up testing nothing but the self-gravitating version of Newton's Equivalence Principle, which is a bit less than the expected Strong Equivalence Principle.

Chapter 3

Geodesic Motion and the Gravitational Weak Equivalence Principle

My problem can be formulated as follows: how is it possible to tell a story, in the presence of the whole Universe?

I. Calvino, *Lezioni Americane.*

This is the point of the story where the two worlds collide: the wide landscape of the extended theories of gravity and the sharp pruning tools of the equivalence principles meet half way, their resonance generating a set of equivalence-based selection rules for the family tree.

Our main goal can be summed up as follows: we want to extract, from the Strong Equivalence Principle, a new principle with a selection power comparable with that of the Weak and of Einstein's forms. In Chapter 2, we have advanced that a minimal sub-statement of the Strong form is the Gravitational Weak Equivalence Principle, which encodes most of the currently tested features of the Strong Equivalence Principle, and is generally used to rule out all the theories of gravity but General Relativity. Still, the assessment of the Gravitational Weak Principle remains formally unsatisfactory, and lacks a clear roadmap showing how to impact on the space of theories and sieve it properly. Which leads us naturally to the content of this chapter.

In the following pages we try to settle the issue, and to build a formal structure framing the Gravitational Weak Equivalence Principle. The method proposed here receives inputs from the geometrical interpretation of spacetime (free-fall trajectories as privileged curves on a manifold), from classical results in the dynamics of extended, yet small, physical systems (the Geroch–Jang theorem), and from the variational formulation of gravity theories (well-posedness of variational problems). This, plus a due dose of inevitable limitations, results in a pair of conditions to be satisfied for the geodesic motion to occur also for self-gravitating test bodies. Conditions which can be put to work on the landscape of extended theories of gravity.

What the Reader will find in this chapter is a refined version of the material available in Ref. [154]. The notation has been slightly modified for sake of consistency with the rest of the thesis.

3.1 Self-gravitating bodies

The very first step in our strategy towards a sieve for extended theories of gravity is the examination of the notion of "self-gravitating systems", for on such concept we shall later pivot, to build the expected selection rule.

The difference between a test body with negligible self-gravity, and one whose own gravitational field is non-vanishing, is non-trivial, in view of the peculiar features of gravity (non-linearity, universal coupling, geometric interpretation). This generates a certain range of technical difficulties and physical conundrums when dealing with self-gravitating objects. Some issues can be sidestepped by aptly playing with the relevant scales in the game (§3.1.1); some can be overcome by restricting the observational windows (§3.1.2); some others, finally, turn exactly into the keys to the final solution of the riddle.

In this section we provide the basic setup, and reply to some objections against our proposal for the Gravitational Weak Equivalence Principle.

3.1.1 Apropos of the Gravitational Weak Equivalence Principle

Since the founding pillar of our method is the Gravitational Weak Equivalence Principle, we begin by restating it from §2.2.3.

> *Gravitational Weak Equivalence Principle* — Test particles (both self-gravitating and non-self-gravitating) behave, in a gravitational field and in vacuum, independently of their properties.

This formulation can be considered, basically, qualitative. To build up a formal sieve we need more precision. As recollected in §2.3.2, the Weak Equivalence Principle serves to single out a family of preferred paths in spacetime, which turn out to be the auto-parallel curves of a connexion and, later on, the geodesics of a given (dynamical) metric field. We can now expect that the result of the Gravitational Weak Equivalence Principle be a much similar statement, and thus rephrase the version above with the following [154]:

> *Gravitational Weak Equivalence Principle (geometric version)* — The world lines of small, freely falling test bodies — with, or without non-negligible self-gravity — do not depend on the peculiar physical properties of the bodies themselves.

In reference to this "geometric translation" of the equivalence principle, three aspects need be discussed: the "testness" character of the physical system under consideration; the range in which this statement can be applied meaningfully; the possible physical obstructions preventing its onset.

The hypothesis of working with strictly test bodies cannot be relaxed: since the Gravitational Weak Equivalence Principle extends the Galilean universality of free fall to self-gravitating bodies, for this extension to make sense, the behaviour of the objects must still be *universal*. This means that the environment where they live in must remain unchanged when different systems, with different self-gravitational content, are compared on the basis of their free-fall motion (see [154], and §3.2.3). Which leads back to the very definition of test bodies, as provided in §2.2.2.

At the same time, the "testness" required is a highly idealised scenario: even an extremely tiny object, such as e.g. a micro-black hole orbiting around a massive, extended body like a planet (or a star), will be far from being a test particle when inspected from close enough. The overall motion of the micro-black hole in the gravitational field of the planet might very well approximate that of a test mass in the same environment, but the strong spacetime curvature in the surroundings of the object will greatly distort the nearby zone, to the point that the "test limit" will not be valid anymore.

To overcome this conundrum, we can select a wide enough spacetime region (a world-tube $\mathscr{A}_{\text{body}}$ "sieging" the distribution of self-gravitating matter) such that the effects of the "non-testness" of the system fall below the sensitivity threshold outside $\mathscr{A}_{\text{body}}$, and can thus be neglected for all practical purposes [154].

As soon as the notion of $\mathscr{A}_{\text{body}}$ is introduced, a second issue emerges. In the geometric version of the Gravitational Weak Equivalence Principle, world-lines are required, rather than the world tubes. While that of a world-line is a precise concept from the point of view of its dynamical evolution (one only has to assign an initial position and a four-velocity, together with the equation (1.10) for the geodesics — or more general differential equations for other lines), that of a world-tube is a much more vague concept, and its evolution can only be sketched, particularly if its internal structure is shadowed.

The solution is to restrict the analysis to the right scales: if the system under consideration is small enough, its world-tube $\mathscr{A}_{\text{body}}$ does not differ substantially from a world-line $\mathscr{C}_{\text{body}}$, so that its dynamical evolution becomes better defined, and the issues associated with the extended character can be sidestepped entirely [154].

Switching from world-tubes to world-lines is not a leap one takes with a light heart; if we decided for this simplification of the problem was only because we knew that the essential feature of self-gravitating test bodies — viz., their self-gravity — could be recovered independently of their geometric representation as tubes or lines, and because this sort of approximation is supported by a long series of results in large-scale and cosmological simulations, where immensely vast physical systems, of the size approaching that of an entire galaxy, are seamlessly reduced to "point-wise particles" in almost-free-fall motion [344, 214, 507].[1]

[1] The gist in this last case is the following: at cosmological scales, the trajectories of particles are described in terms of the congruence of timelike curves making up the fundamental reference fluid (and the particles in this case are intended as elementary, infinitesimal, ideal entities). Physical systems constituted by large aggregations of matter are modelled by "bumps" in a density field distribution. In particle-based simulations, these bumps are later traded for point particles endowed

As a second step, we detail on the range of validity of the Gravitational Weak Equivalence Principle. To this end, suppose for a moment that the self-gravitating system under consideration can be described exhaustively in terms of e.g. a point mass equipped with a spin vector, or a quadrupole moment, or any combination of higher multipoles, as done frequently in the study of extended masses [249]. Then, the coupling of such additional geometric structures with the spacetime curvature will in general spoil the "free" character of the free-fall — in the equations of motion there will emerge force terms proportional to the curvature — and in general the behaviour will not be universal anymore. The validity of the Gravitational Weak Equivalence Principle would then disappear.

Such issue, however, only partially overlap with the notion of self-gravity. In fact, even a non-self-gravitating test object can be equipped forcefully with additional multipole structure. This means that the sort of violations of the Gravitational Weak Equivalence Principle expected in this case pertain also to the Weak Equivalence Principle. To give one example, a spinning particle (without any contribution from self-gravity) will be governed by the following, Papapetrou–Mathisson–Dixon equations [353, 413, 340, 161]

$$u^b \nabla_b u^a = -\frac{1}{2m} R_{bcd}{}^a u^d S^{bc} \,, \tag{3.1}$$

where m is the mass, and S_{ab} is the spin tensor.[2] Notice that the driving force on the right-hand side, being dependent on the mass and the spin, makes the behaviour of the spinning particle non-universal. Similar considerations apply to a quadrupole tensor, or to any other multipole structure.

The failure of the Weak, and Gravitational Weak, Equivalence Principle in this case can be attributed entirely to the existence of some sort of spurious "structure" attached to the system under consideration, be it self-gravitating or not. A way to restore the original validity is thus to reject the structure-endowed bodies and focus only on the structure-less ones [154]. While this situation seems much more detached from the actual experimental conditions in which both the principles are tested (see §2.4.2), it is also true that, once the rotational status and internal distribution of the masses of a self-gravitating system has been assessed, multipole corrections can be properly evaluated and excluded from a data-set via appropriate filters.

The contribution given by the proper self-gravitational content deserves instead a separate discussion.

3.1.2 Self-gravity and self-force

Back to our self-gravitating body, there will be a certain amount of gravitational radiation emitted by the system as it moves through spacetime. In principle, nothing prevents the radiation from back-scattering off the surrounding

with a mass proportional to the over-density, and their evolution is studied once the equations of motion and the initial conditions are assigned. In a simulation where enough coarse-graining is present, a single point can be given (very roughly) the mass of a galaxy — which is hardly a small body, let alone a test system — so that its world-line is able to provide indications on the overall motion. Much finer-grained simulations use, as "points", systems whose mass equals millions or billions of Solar masses. Small corrections are then added to account for the effects of the extended and self-gravitating character of the systems before their reduction to a point.

[2]The notion of "spin" used here is strictly classical: the tensor S_{ab}, or the related spatial spin vector S^a, is an object of non-quantum character [514].

gravitational field, and impacting the system itself at a later time. The result is a second contribution, usually extremely tiny, making the motion non-geodesic and acting as a force not much different from the term on the right in Eq. (3.1).

This contribution is known as the *tail term* [427, 570, 350, 445, 248], and it is interesting to sketch the idea behind this further correction to the free-fall motions.[3] One begins with a self-gravitating system and an external background. The former is chosen so as to be as simple as possible, by both enclosing it in an aptly-chosen world-tube, and by reducing its intrinsic complexity (broadly speaking, this amounts to picking a non-rotating, black-hole-like solution for the self-gravitating body [350]); the latter is equally chosen so as to erase unnecessary complexity. The next step is to study, separately, the tidal deformations of the self-gravitating system due to the interactions with the environment, and the tidal deformations of the environment due to the self-gravitating system. To this end, one introduces an "internal zone" where the body dominates and the background is a perturbation, and an "external zone" where the opposite conclusion holds.

In the external zone, the presence of the system is treated using linear perturbations of the background [350]. One introduces the linearised field equations

$$\Box\,\zeta_{ab} + 2R_{a\ b}^{\ c\ d}\,\zeta_{bd} = -\frac{16\pi G}{c^4}T_{ab}\ , \tag{3.2}$$

with T_{ab} the stress-energy-momentum tensor of the system, expressed in this case as a Dirac delta function — the self-gravitating body is approximated by a point particle —, and ζ_{ab} the small perturbation of the background metric, call it $\bar{g}_{ab}$ (the latter provides in turn the d'Alembertian operator, the curvature tensor, etc.). The general solution of Eq. (3.2) is the Green function $G^{ab}{}_{c'd'}\,(z, y')$ in its Hadamard form, a two-point tensor (or bi-tensor) [517, 143, 152] which reads

$$G^{ab}{}_{c'd'}\,(z, y') = \frac{1}{4\pi^2}\left(\frac{U^{ab}{}_{c'd'}\,(z, y')}{\sigma\,(z, y')} + V^{ab}{}_{c'd'}\,(z, y')\log|\sigma| + W^{ab}{}_{c'd'}\,(z, y')\right)\ , \tag{3.3}$$

with σ being Synge's world-function [142, 143, 517], i.e. one half of the squared geodesic distance between the points z and y' — $z = z\,(\tau)$ represents the world-line of the body, whereas y' is a generic point in spacetime.

The two-point functions U and V are singular for $\sigma \to 0$, whereas W is regular; calculating the explicit form of these functions is complex, but such result is not required for the evaluation of the tail term. All one has to notice is that the perturbation ζ_{ab} of the background in the external zone can be expressed as

$$\zeta^{ab}\,(y) = \frac{4m}{r}U^{ab}{}_{c'd'}\,(y, z')\,u^{c'}u^{d'} + \zeta^{ab}_{\text{tail}}\,(y)\ , \tag{3.4}$$

with the "tail term" $\zeta^{ab}_{\text{tail}}\,(y^\alpha)$ given by [350]

$$\zeta^{ab}_{\text{tail}}\,(y) = 4m\int_{-\infty}^{\tau_{\text{ret}}} V^{ab}{}_{c'd'}\,(y, z')\,u^{c'}u^{d'}\,\mathrm{d}\tau'\ . \tag{3.5}$$

[3] The technical details of this treatment of the problem greatly exceed the scope of the thesis. The interested Reader is invited to peruse the literature on the topic of self-force, which is per se a source of stimulating questions and challenges [247]. An excellent starting point is e.g. the short presentation in Ref. [429], and then the thorough review [427]. For an examination of the problem of self-energy in classical Newtonian gravity see Ref. [230].

In the previous formula, $z'^{\beta} = z'^{\beta}(\tau')$ is any point on the world-line representing the system, y^{α} is a generic point in spacetime, $\tau_{\rm ret}$ is the value of the proper time at a "retarded" point (as in the Liénard–Wiechert potentials), $u^{a'}$ is the value of the four-velocity at the same retarded point.

This result can be now incorporated in the treatment of the internal zone, where one sets up the equation of motion of the system influenced by the back-scattering of its own gravitational radiation off the background spacetime; the final outcome of the calculation (which requires some simplifying assumptions and a truncation of the series expansion) reads [350]

$$\ddot{z}^{\alpha}(\tau) = -K^{\alpha\beta\gamma\delta}(z(\tau))\,\nabla_{\delta}\zeta^{\rm tail}_{\beta\gamma}(z(\tau)) \; . \tag{3.6}$$

Here, $z^{\alpha}(\tau)$ — τ the proper time — is the coordinate representation of the world-line of the self-gravitating system; $K^{\alpha\beta\gamma\delta}$ is a tensor given by

$$\begin{aligned} K^{\alpha\beta\gamma\delta} =& \frac{1}{2}\dot{z}^{\alpha}\dot{z}^{\beta}\dot{z}^{\gamma}\dot{z}^{\delta} + g^{\alpha\beta}(z(\tau))\,\dot{z}^{\gamma}\dot{z}^{\delta} - \frac{1}{2}g^{\alpha\delta}(z(\tau))\,\dot{z}^{\beta}\dot{z}^{\gamma} + \\ & - \frac{1}{4}g^{\beta\gamma}(z(\tau))\,\dot{z}^{\alpha}\dot{z}^{\delta} - \frac{1}{4}g^{\alpha\delta}(z(\tau))\,g^{\beta\gamma}(z(\tau)) \end{aligned} \tag{3.7}$$

and $\zeta^{\rm tail}_{\beta\gamma}(z(\tau))$ is the tail term (3.5) itself. The actual determination of the covariant derivative of $\zeta^{\rm tail}_{\alpha\beta}$ present in Eq. (3.6) needs calculations involving the higher derivatives of the world-function and of the Green propagator [517, 350, 152].

Since the tail term clearly depends on the self-gravitational interactions occurring in the system, and renders the motion both non-geodesic — hence, not free — and body-dependent (the mass intervenes in the expression of $\zeta^{\rm tail}_{\alpha\beta}$), it prevents us from implementing the Gravitational Weak Equivalence Principle. The problem is completely general, and affects any theory of gravity; therefore, one can conclude that the free-fall motion of self-gravitating bodies *does not happen at all*, not even in General Relativity [427] (and yet, as seen in §2.3.3, Einstein's theory is supposed to implement the Strong Equivalence Principle, and thus the Gravitational Weak form as well).

The traditional solution is to sidestep the issue completely; this is done by introducing first an intermediate "buffering zone", where the effects of the mutual perturbations in the internal and external zones are both non-negligible; then, one builds up a suitable metric $\tilde{g}_{ab}$ — obtained with the matched asymptotic expansion technique [428, 350] — such that the effects of the tail term are reabsorbed into the dynamics of $\tilde{g}_{ab}$. To this end, one builds the connexion coefficients $\tilde{\Gamma}^{a}{}_{bc}$ corresponding to the metric $\tilde{g}_{ab}$ so as to incorporate, at the leading order in the matched asymptotic expansion, the tail force, and then imposes the further constraint that $\tilde{g}_{ab}$ be a solution of Einstein's field equations [350]. The resulting scheme is that of a full-fledged spacetime $\mathscr{M} \equiv (M, \tilde{g}_{ab})$, where the motion of the self-gravitating system becomes once again geodesic [350, 445].

While this solution is in principle acceptable, it has some drawbacks. As for the tail term (3.5) itself, it requires an integration over the entire history of the body under consideration — the "$-\infty$" in the integration —, which prevents any attempt to compare the behaviour of any two systems with non-negligible self-gravity. Actually, to determine the precise form of $\zeta^{\rm tail}_{\alpha\beta}$, the entire history of the self-gravitating system ought to be under control, and this appears quite an unrealistic possibility.

Second, the introduction of the metric $\tilde{g}_{ab}$ completely decouples the self-gravitating objects from the non-self-gravitating ones: the latter all live on the spacetime generated by the background metric, call it $\bar{g}_{ab}$, for which the self-gravitational phenomena can be neglected, including the tail term. The former, on the other hand, now live on *another* spacetime (that where the metric is $\tilde{g}_{ab}$), which on top of that is body/mass-dependent, therefore the mere experimental comparison of the free-fall motions becomes impossible [154].

To sum up: at the end of this process, the only "free-falling" property still holding will be the existence, for a self-gravitating body, of geodesic motion on *some* spacetime, but certainly not on the *same* spacetime as that of all the other self-gravitating (and non-self-gravitating) systems [154].

To get out of this deadlock, we can proceed as follows: upon noticing that the contribution from the tail term is usually minuscule, we can agree upon considering it as it were below the sensitivity threshold for the sorts of free-fall experiment we are dealing with. Stated otherwise, the Gravitational Weak Equivalence Principle we are formulating is designed so that the tail contribution is neglected ab initio.

This slightly "restrained" version of the Gravitational Weak Equivalence Principle will guarantee that the statement hold true, to begin with, in the case of General Relativity, as it is generally assumed in the literature [558, 557] and confirmed in many high-precision experiments.

3.2 Geodesic motion of small bodies

Since the Gravitational Weak Equivalence Principle aims at achieving the same result as its Weak counterpart, but for the ampler sample of both non-self-gravitating and self-gravitating test bodies, we need to explore in the physical and mathematical laws at the roots of the geodesic trajectories, and find an apt upgrade to the results holding for structureless systems with negligible self-gravity.

3.2.1 The Geroch–Jang–Malament theorem

The main result in the motion of tiny bodies endowed with non-vanishing self-gravity dates back to a theorem first proven by Geroch and Jang as early as 1975 [217]. The proposition has been then ameliorated in 2004 by Ehlers and Geroch [174], and both the versions have been carefully examined in the 2010's by Malament [332], who has settled a few tiny issues and polished the edges.

We begin by establish some warm-up results in the special relativistic case, which will turn out to be useful in a moment [360, 217]. Consider an extended, isolated body represented by a stress-energy-momentum tensor T_{ab}, defined over Minkowski spacetime $\mathscr{M}_{\mathrm{SR}} \equiv \left(\mathbb{R}^4, \eta_{ab}\right)$ and with compact support;[4] its history is then represented by a suitable world-tube $\mathscr{W}$ spanning a region over $\mathscr{M}_{\mathrm{SR}}$. Suppose furthermore that T_{ab} is divergence-free, i.e. that $\nabla_b^{(\eta)} T^{ab} = 0$, with $\nabla_b^{(\eta)}$ the affine connexion associated with the flat metric η_{ab}. One can prove that, for any Killing vector fields ξ^a on $\mathscr{M}_{\mathrm{SR}}$, there exist a vector p^a and a

[4]This last condition characterises the *insular* systems [360].

skew-symmetric tensor $J_{ab} = J_{[ab]}$ such that

$$\int_{\Sigma} T_{ab}\,\xi^b n^a \mathrm{d}\Sigma = J_{ab}\,(\lambda)\,\nabla^a \xi^b - p_a\,(\lambda)\,\xi^a\,. \tag{3.8}$$

The integral is extended to any spacelike three-surface Σ cutting $\mathscr{W}$, and is independent on the choice of Σ in view of the conservation of T_{ab} and Killing's equation $\nabla_{(a}\xi_{b)} = 0$. The two quantities p_a and J_{ab} in Eq. (3.8) depend only on a generic parameter λ (in view of the integration over Σ), and can be interpreted, respectively, as the four-momentum, and the (total) angular momentum about a point, of the system [217]. If one also supposes that the stress-energy-momentum tensor satisfies the dominant energy condition,[5] then the vector p^a emerges as everywhere timelike and future-directed.

Given an arbitrary inertial reference frame in which the isolated system is described, it is possible to find there the world-line of a point sharing the same definition and properties as the Newtonian centre of mass [360]. Such a point is thus called itself *centre of mass*, yet it is a frame-dependent notion, in the sense that in each inertial frame it is possible to define a different such centre. A common feature of all the centres of mass is that they are at rest in the inertial rest frame of the system itself. Of particular significance is then the centre of mass evaluated in the inertial rest frame of the body (proper centre of mass).[6] One can prove [360, 217, 49] that the coordinates $x_0^\alpha = x_0^\alpha\,(\tau)$ of the proper centre of mass are linear functions of the proper time of the point, and that the world-line of the proper centre of mass is a future-pointing, timelike geodesic. Also, the four-momentum and four-velocity of the system are connected by the relation

$$p^\alpha = M u_0^\alpha\,, \tag{3.9}$$

with $M = -p^\alpha p_\alpha / c^2$.

That the world-lines of the centres of mass (and, in particular, that of the proper centre of mass) do not deviate from the "average motion" of the body can be seen by showing that the curve $\mathscr{C}_0$ represented by the $x_0^\alpha\,(\tau)$ remains everywhere inside the convex hull of the body (i.e., the union of all segments of spacelike geodesics with both endpoints in the world-tube $\mathscr{W}$).

Hence, in the absence of gravity, there exists a notion of "almost geodesic motion" of an extended isolated body, which follows from the conservation of the stress-energy-momentum tensor — and the existence of a certain number of Killing symmetries of the background spacetime —. As soon as gravity is "turned on", however, the resulting spacetime does not possess, in general, enough Killing fields to preserve the validity of the special relativistic result. This is the point where the Geroch–Jang Theorem comes handy.

The statement of the Theorem reads

Theorem (Geroch, Jang). *Let $\mathscr{C}$ be a smooth curve in a spacetime $\mathscr{M} \equiv (M, g_{ab})$. Suppose that, given any open subset $\mathscr{U}$ of $\mathscr{M}$ containing $\mathscr{C}$, there exists a smooth symmetric field Θ_{ab} on $\mathscr{M}$ such that: (a) Θ_{ab} satisfies the strengthened*

[5] That is [537], for all points $P \in \mathscr{M}$, and for all unit time-like vectors ξ^a at P, it is $T_{ab}\,\xi^a \xi^b \geq 0$, and, if $T_{ab} \neq 0$, then $T^a{}_b\,\xi^b$ is time-like.

[6] The fact that the proper centre of mass of a body is the centre of mass of the same body evaluated in the body's rest frame is expressed by the relation $p^a J_{ab} = 0$ [360, 217, 49].

dominant energy condition; (b) $\Theta_{ab} \neq 0$ at some point in $\mathscr{U}$; (c) $\Theta_{ab} = 0$ outside of $\mathscr{U}$; (d) $\nabla^b \Theta_{ab} = 0$. Then, $\mathscr{C}$ is a timelike geodesic on $(\mathscr{M}, g_{ab})$.

Before sketching the proof, we notice that the tensor Θ_{ab} evoked in the Theorem shares many properties with the usual stress-energy-momentum tensor of a matter distribution mentioned in the special relativistic argument above. This is not a coincidence: the Geroch–Jang result is designed precisely to suit the needs of describing the almost-geodesic motion of a matter-energy budget in a curved spacetime.

The proof is based on the ultra-local, special relativistic character of any curved spacetime: in a neighbourhood of $\mathscr{C}$, one fixes a flat metric η_{ab} and a corresponding flat derivative operator $\nabla_a^{(\eta)}$ such that the two coincide, on $\mathscr{C}$, with g_{ab} and ∇_a, respectively. The flat structure allows to define the quantities p^a, J_{ab} as in Eq. (3.8). Then one evaluates the difference

$$\nabla_a^{(\eta)} \Theta^{ab} = \left(\nabla_a^{(\eta)} - \nabla_a \right) \Theta^{ab} \, , \tag{3.10}$$

and discovers that such difference can be made arbitrarily small by suitably rescaling the size of the support of Θ_{ab}. Further considerations related to the intersections of the convex hull of the body with the possible slices Σ allow to conclude that $\mathscr{C}$ must be arbitrarily close to some η-geodesic, which is possible only if $\mathscr{C}$ itself is a geodesic with respect to $\nabla_a^{(\eta)}$; but the derivative operators yield $\nabla_a^{(\eta)} = \nabla_a$ on $\mathscr{C}$, therefore $\mathscr{C}$ must be also a geodesic with respect to the full metric g_{ab}, which concludes the proof.

This result, as said, has been reconsidered later on by Ehlers and Geroch [174]. There, it is remarked that the result of the Geroch–Jang Theorem strictly refers to extended bodies *not* equipped with self-gravity — for which Θ_{ab} can actually be identified with the stress-energy-momentum tensor of some matter fields — whereas the (mildly) self-gravitating systems are covered by the upgraded, version of the theorem. In this latter case, the new statement reads

Theorem (Ehlers, Geroch). *Let $\mathscr{C}$ be a smooth curve in a spacetime $\mathscr{M} \equiv (M, g_{ab})$. Consider a close neighbourhood $\mathscr{U}$ of $\mathscr{C}$, and any neighbourhood $\hat{U}$ of g_{ab} in $C^1(\mathscr{U})$. Let there exist, for every such $\mathscr{U}$, if sufficiently small, and every such $\hat{U}$, a Lorentz-signature metric $\tilde{g}_{ab}$ inside $\hat{U}$ whose Einstein tensor $\tilde{G}_{ab}$: i) satisfies the dominant energy condition everywhere in $\mathscr{U}$; ii) is nonzero in some neighbourhood of $\mathscr{C}$; and iii) vanishes on $\partial\mathscr{U}$. Then $\mathscr{C}$ is a g-geodesic.*

Much emphasis is put on the presence of the Einstein tensor instead of a generic, symmetric and divergence-free tensor Θ_{ab}; this substitution allows Ehlers and Geroch to reduce to C^1 the degree of convergence in the space of metrics used in the proof (with Θ_{ab}, one requires in general a C^2-convergence), but later in the paper [174] it is specified that the result holds as well if one considers again the original, generic tensor Θ_{ab}, as nowhere in the proof appear the field equations for the metric g_{ab}. In this sense, the two theorems by Geroch and Jang, and by Ehlers and Geroch, can be used interchangeably; we shall stick to the former for sake of convenience and generality — the presence of the Einstein tensor in the formulation by Ehlers and Geroch can weaken the future use of the statement in the landscape of extended theories of gravity.

The importance of the result by Geroch and Jang for our specific purposes can be understood after highlighting the following elements. To begin with, if the world-tube $\mathscr{U}$ where Θ_{ab} is non-zero can be approximated, for all practical purposes, by a world-line, then the theorem assures that the world-line itself will be a geodesic for the spacetime $\mathscr{M}$. This is desirable, as in §3.1.1 we have established that the test bodies with non-negligible self gravity to which the Gravitational Weak Equivalence Principle applies are described in terms of curves on the spacetime manifold.

Second, in view of the remarks by Ehlers and Geroch, the statement of the theorem is completely general, and does not make any distinction between self-gravitating and non-self gravitating bodies [154]. As long as a Θ_{ab} satisfying properties (a)–(d) of the theorem exists, there will be a world-line within the world-tube which turns out to be a geodesic for the metric g_{ab}.

While this conclusion might seem encouraging, a moment's reflection shows that it is actually worrisome: if two different bodies, a non-self-gravitating test one, and one with non-negligible self gravity, both travel along geodesics, then the Gravitational Weak Equivalence Principle is hardly a principle (its entire content duplicates that of the Weak Equivalence Principle), and most likely not a selection rule, for it singles out all the theories already permitted by the Weak form.

The point here lies in the metric we are dealing with when considering a test body without self-gravity, and one endowed with some self-gravitational content. The problem is the same as the one explored in §3.1.2, only rephrased here in the light of Geroch's and Jang's theorem.

In the former case (non-self-gravitating system), the body does not back-react on the given environment, so it moves along the geodesics of the background metric, call it $\bar{g}_{ab}$ from now on. The spacetime $\bar{\mathscr{M}}$ on which the test particle lives is thus $\bar{\mathscr{M}} = (M, \bar{g}_{ab})$.

In the latter case (self-gravitating body), the self-gravity of the system is actually sourcing the overall gravitational field, for the Θ_{ab} of the body also accounts for its self-gravity, and this contributes to the field equations generating the metric of the compound, non-linearly interacting pair "background plus self-gravitating body". Hence, the particle now lives on the spacetime $\tilde{\mathscr{M}} = (M, \tilde{g}_{ab})$ with $\tilde{g}_{ab} \neq \bar{g}_{ab}$, and it is along the geodesics of this *second* spacetime that it moves, as per the Geroch–Jang theorem.

This, however, looks like an equally worrisome conclusion, for the presence of two different spacetimes makes the possibility of testing the Gravitational Weak Equivalence Principle hopeless: the two gravitational arrangements do not communicate, and the very idea of comparison a the roots of our strategy falls apart.

From this reexamination of the theorem we extract that the test bodies with negligible self gravity will all follow geodesics of the background metric field, hence will determine a subset of preferred trajectories on $\bar{\mathscr{M}}$; in this sense, Geroch's and Jang's theorem provides an independent argument in favour of the Weak Equivalence Principle. Such conclusion, however, does not extend to self-gravitating small bodies, for their world-lines will in general depend on the the bodies themselves, and will unwind on a body-dependent spacetime; which violates the supposed universality of free fall for test particles endowed with non-negligible self-gravitation.

3.2.2 A geodesic for self-gravity

There is a way out. Basically, it amounts to incorporating the intuition by Ehlers and Geroch into that by Geroch and Jang. Suppose that we identify Θ_{ab} in the Geroch–Jang theorem with the stress-energy-momentum tensor T_{ab} of some matter field without self gravity; then, suppose to find *another* symmetric tensor Θ'_{ab}, satisfying conditions (a)–(d) of the theorem, which is still divergence-free with respect to the *same* background metric, i.e.

$$\bar{\nabla}^b \Theta'_{ab} = 0 \ , \tag{3.11}$$

where the symbol $\bar{\nabla}_a$ denotes the covariant derivative built out of the background metric alone (and the indices are raised and lowered with $\bar{g}_{ab}$). If this new tensor accounts also for the self-gravity content of a small body, then, by the statement of Geroch's and Jang's proposition, the world-lines of the physical system represented by Θ'_{ab} will still be the geodesics of the background spacetime — i.e., the lines along which the bodies satisfying the Weak Equivalence Principle move.[7]

Another element to consider is that, if we want to compare the trajectories of the self-gravitating system with those of test particles with negligible self-gravity, the condition of being *test* has to be retained for the self-gravitating systems as well.

Stated otherwise, the self-gravitating system represented by Θ'_{ab} must be such that its self-gravity represents only a *small perturbation* of the overall gravitational field (of which it will be a non-negligible source, however tiny). Once this further assumption is introduced, finding the correct form of Θ'_{ab} reduces to a matter of suitable series expansions in an apt parameter.

Before moving on, a few remarks on some aspects of the Geroch–Jang Theorem which are of great helpfulness when extended theories of gravity come into play. First, the theorem does not say anything about the detailed form of the field equations involved, requiring only the existence of the tensor Θ_{ab} — or of the alternative candidate Θ'_{ab}, as just seen —. At the same time, when self-gravity is "switched off", or when the whole gravitational phenomenon is neglected, Θ_{ab} reduces to the usual stress-energy-momentum tensor T_{ab}, so it appears quite natural to consider any theory of gravity such that its field equations can be cast in the form

$$E_{ab} = T_{ab} \ , \tag{3.12}$$

In this sense, Eq. (3.12) naturally encompasses all metric theories of gravity with full dynamical character, i.e. theories in which the gravitational degrees of freedom are encoded at least in a symmetric, rank-2, covariant tensor g_{ab}. On the left side of Eq. (3.12), there appears the *generalised Einstein tensor* [154], which is itself symmetric and divergence-free with respect to the full metric g_{ab} for consistency with the conservation of T_{ab}. E_{ab} draws its name from the archetypical case of General Relativity, in which it is

$$E_{ab} = \frac{c^4}{8\pi G} G_{ab} \ , \tag{3.13}$$

[7] Θ'_{ab} cannot reduce to the stress-energy-momentum tensor T_{ab} of the system, as the latter does not involve the self-gravity [154].

with G_{ab} the usual Einstein tensor.

Also, the result of the theorem is not influenced by the presence of other gravitational degrees of freedom: as long as the field equations can be reshuffled so as to appear as in Eq. (3.12), nothing in Geroch's and Jang's statement forbids the presence of more dynamical gravitational variables. From the simple scalar field in Brans–Dicke theory in the Jordan representation, to the wildest proliferation of tensors in multi-metric theories, many frameworks outlined in §1.3 can be considered seamlessly.

Finally, the result is dimensional-independent, i.e. it can be exported to any number of spacetime dimensions compatible with the Lorentzian signature of the metric. This as well is an advantage, for it permits us to work with all sorts of lower-dimensional and higher-dimensional schemes, as those presented in §1.3.4.

3.2.3 Limits, boundaries, and constraints

In §3.1 above, we have sketched some issues affecting the overall validity of the Gravitational Weak Equivalence Principle. Here, we discuss a few other technical aspects concerning, and potentially threatening, the construction achieved so far.

In the statement of the Geroch–Jang theorem, the conditions (a)–(d) are sufficient, but not necessary, to assure the existence of the geodesic path for the self-gravitating body. Suppose then that assumption (d) is violated by some tensor Θ'_{ab} for which instead hypotheses (a)–(c) hold; we can prove that the lack of a covariant conservation of Θ'_{ab} implies a non-geodesic motion.

To this end, we adapt a passage from the argument in [445]. Suppose that Θ'_{ab} is a good representative of the stress-energy-momentum of the system (including its self-gravitational content), in the sense that the overall four-momentum of the body can be expressed as the integral

$$p^a = \int_\Sigma {\Theta'^a}_b n^b \sqrt{h}\, \mathrm{d}^3 y \ , \tag{3.14}$$

extended to a spatial slice Σ, with n^a aligned with the "four-velocity" vector u^a providing, point by point of spacetime, the tangent to the world-line representing the body. The force acting on the system is given by

$$f^a = u^b \nabla_b p^a = \int_{\Sigma(\tau)} \pounds_{w^a} \left({\Theta'^a}_b n^b \sqrt{h} \right) \mathrm{d}^3 y \ , \tag{3.15}$$

where the Lie derivative $\pounds_{w^a}$ is taken along the vector w^a generating the passage from one spatial sheet $\Sigma(\tau)$ to another (the lapse function in an Arnowitt–Deser–Misner split). The expression above further reduces to [445]

$$f^a = \int_{\Sigma(\tau)} \nabla_b \Theta'^{ab} w^c n_c \sqrt{h}\, \mathrm{d}^3 y \ . \tag{3.16}$$

If the tensor Θ'_{ab} is not covariantly conserved, nor is trivially proportional to the four-velocity u^a, then a non-zero force emerges along its trajectory, and the latter ceases to be a geodesic, i.e. the required equivalence principle cannot hold anymore.

We can then conclude that the Gravitational Weak Equivalence Principle holds for a self-gravitating test body *if and only if* condition (3.11) is satisfied, by a tensor Θ'_{ab} also abiding by conditions (a)–(c), with Θ'_{ab} such that the difference $\Theta'_{ab} - T_{ab}$ accounts for the self-gravitational content of the system [154].

Another issue concerns hypothesis (c) in the theorem above. Indeed, an isolated body represented by a stress-energy-momentum tensor can certainly be arranged so that T_{ab} has compact support, or is even confined entirely on the world-line of the system; the same cannot be said, however for the tensor Θ'_{ab}; the latter includes the contributions from the gravitational field, which extends all over the manifold M.

A detailed analysis shows [79] that the leading contributions come from the radiative corrections (due to the gravitational self-radiation, as explained in §3.1.2), and from the static gravitational field of the body. If r denotes a radial coordinate on the spacetime, then the radiation terms fall off as r^{-2}, whereas the static field drops as r^{-4}; neither function has compact support. Of the two, however, the radiation corrections can be neglected, in view of the argument provided in §3.1.2.

As for the static field, we remark that the role of hypothesis (c) in the Geroch–Jang theorem is to allow for an integral of a function proportional to Θ_{ab}, performed on a spacelike sub-manifold, to be traded for an integral over a domain coincident only with the volume of the body.

Now, if we call R a variable denoting the average size of the world-tube associated with the self-gravitating system, and M the overall mass enclosed in the tube, then the correction due to the external static gravitational field will be of order GM^2/R. Upon comparing this value with the inertial energy Mc^2, we usually find that only a tiny fraction of the proper energy comes from self-gravity, even in the case of extremely compact objects. All we have to do, then, is to choose R such that the world-tube can still be approximated by a single line, and yet R be significantly larger than the Schwarzschild radius of the system under consideration, so that the fractional energy budget reserved for the higher corrections is negligible. If we manage to set the value of R properly, the integration required by the theorem can be truncated at R, which becomes the boundary of the domain of spacetime outside which the self-gravity can be thought to have compact support for all practical purposes.

Nordström's scalar theory and the Geroch–Jang theorem

We conclude this part with a short remark concerning Nordström's theory (§1.1.2), and models alike. The discussion is quite "raw", and its purpose is just to convey a qualitative idea of the argument.

At first glance, Eq. (3.12) seems to exclude scalar theories from the game, for it strictly requires field equations in a symmetric tensor form; a moment's pondering, however, shows that all is required by the Geroch–Jang theorem is the existence of a certain tensor satisfying specific conditions, regardless of the field equations.

In Nordström's theory such a tensor exists for the matter — it is precisely the tensor whose trace enters the field equations (1.15) — and it satisfies the required conditions, including the conservation equation, which has to be referred to the connexion compatible with the metric $g_{ab} = \Phi^2\, \eta_{ab}$ (problems with condition (c) can be sidestepped via the same, general argument provided above).

This symmetric tensor contributes to the construction, in Nordström's theory, of the analogue of the object Θ'_{ab} of Eq. (3.11), with the inclusion of a (tiny) self-gravitational contribution depending on the gravitational potential Φ. On the other hand, the covariant derivative $\bar{\nabla}_a$ used to take the divergence of Θ'_{ab} must be built out of a background metric $\bar{g}_{ab}$, which in this case will be given by

$$\bar{g}_{ab} = \bar{\Phi}^2\, \eta_{ab}\ , \tag{3.17}$$

with $\bar{\Phi}$ the background gravitational scalar field.

If the protocol just outlined does not clog along its setup, the conclusion will be that the Gravitational Weak Equivalence Principle will be validated in Nordström's theory as well.[8]

3.3 Locking the conditions for geodesic motion

We can now sew together all the ideas gleaned so far. The path is structured in three stages. First, we introduce a perturbative expansion accounting for the idea of the self-gravitating small systems as being still "test" objects with respect to a (dynamical) background; the upshot is the form for the tensor Θ'_{ab} around which the application of the Geroch–Jang theorem pivots. Second, we exhibit a link between Eq. (3.11) and the variational formulation of the general field equations (3.12) for a given theory of gravity, whereupon a formula emerges, unlocking the actual conditions to have Eq. (3.11) satisfied. Finally, we discuss the main aspects, implications, and interpretations of these conditions.

3.3.1 Perturbative expansions

Our proposal is intrinsically perturbative in nature, as it must account for the notion of "testness" of the small, self-gravitating masses. This means that the gravitational degrees of freedom need be split into a background part and a (small) perturbation [154]; to this end, we begin by defining the metric tensor expansion as

$$g_{ab} := \bar{g}_{ab} + \epsilon\, \gamma_{ab}\ , \tag{3.18}$$

where the background is denoted by an over-bar, and ϵ is a bookkeeping parameter embodying the small effect of the perturbation. From now on, any geometric or physical quantity referring to the background will be equally denoted by an over-bar, and the study of only leading-order terms of the perturbation series will demand us to neglect any contribution from ϵ^2 onwards (ϵ^2-terms included).

The decomposition (3.18) propagates up to the field equations, where any

[8] This is comforting; indeed, Nordström's theory is known to abide by the Strong Equivalence Principle, as proven with different methods (PPN expansion [557], and Katz super-potentials [149, 147, 148]). Actually, what both the methods offer is a proof that inertial and gravitational masses are equal in Nordström's scheme for self-gravitating systems as well, but this is a result validating the Gravitational Newton's Equivalence Principle rather than the Gravitational Weak form (let alone the Strong principle, which encompasses an even broader range of physical phenomena).

term can be expanded into a series of ϵ, giving

$$E_{ab} = \bar{E}_{ab} + \epsilon\,\mathcal{E}_{ab} + E^{(2+)}_{ab}\,, \tag{3.19}$$

$$T_{ab} = \bar{T}_{ab} + \epsilon\,\mathcal{T}_{ab} + T^{(2+)}_{ab}\,. \tag{3.20}$$

where the script letters denote linear terms in the series, whereas the superscript "(2+)" denotes all the higher-order terms.

Whenever the perturbation is "switched off", the field equations reduce to the zeroth-order term, which reads

$$\bar{E}_{ab} = \bar{T}_{ab}\ . \tag{3.21}$$

At this point, we make the further assumption $\bar{T}_{ab} = 0$, i.e. we imagine that the self-gravitating system is freely falling in a matter-free background spacetime. Such hypothesis complies with the usual treatment of the problem [350]. This assures that the stress-energy-momentum tensor reduces to the small contribution of the self-gravitating particle alone — call it $T^{(p)}_{ab}$ — so that the general field equations (3.12) become

$$\epsilon\,\mathcal{E}_{ab} = T^{(p)}_{ab} - E^{(2+)}_{ab}\ . \tag{3.22}$$

If we can prove that the background covariant divergence of the tensor $\mathcal{E}_{ab}$ vanishes identically ("background covariant divergence" means the covariant divergence built out of the background metric alone), then the right-hand side of Eq. (3.22) can be assumed as a new stress-energy-momentum tensor, including the self-gravity of the small body, to which the Geroch–Jang theorem applies. The body will thus move along geodesic lines of the background spacetime. In other words, if

$$\bar{\nabla}^b\mathcal{E}_{ab} = 0\ , \tag{3.23}$$

we are allowed to define an effective stress-energy-momentum tensor

$$\Theta'_{ab} := T^{(p)}_{ab} - E^{(2+)}_{ab}\ , \tag{3.24}$$

to which the content of Geroch–Jang Theorem applies[9]. In Θ'_{ab}, indeed, the tensor $T^{(p)}_{ab}$ satisfies all the hypotheses of the proposition by definition, whereas $E^{(2+)}_{ab}$ is such that its addition to the ordinary stress-energy-momentum tensor does not make condition (a) fail, and such that condition (c), as seen in §3.2.3, can be mildly relaxed without spoiling the net result [154].

Therefore, the line $\mathscr{C}$ on which the self-gravitating system moves will be a geodesic of the background metric $\bar{g}_{ab}$. Notice that, since $\bar{g}_{ab}$ is the same metric on whose geodesics the body moves when the self gravity is switched off, it behaves like a non-self-gravitating test body. We have then proper terms for comparison, as both the self-gravitating and non-self-gravitating systems are now moving on the *same* spacetime.

[9] In particular, Θ'_{ab} is symmetric by construction, is non-zero on the curve approximating the world-tube of the self-gravitating system, and falls off sufficiently rapidly outside it (for any given experimental sensitivity λ, one can find a region outside the curve where $|\Theta'_{ab}| < C\lambda^2$, with C a positive constant); also, the strengthened dominant energy condition is satisfied at the leading order because $T^{(p)}_{ab}$ complies with it, and all the contributions in $E^{(2+)}_{ab}$ are at least of order ϵ^2, whereas $T^{(p)}_{ab}$ is of order ϵ.

In view of these conclusions, Eq. (3.23) gains the status of *necessary and sufficient condition* to formalise the content of the Gravitational Weak Equivalence Principle [154].

To sum up, condition (3.23) is the one to be checked in order for a given theory of gravity to satisfy the Gravitational Weak Equivalence Principle. We are now in the position of verifying it for a wide class of theories, i.e. those with field equations in tensor form emerging from a well-posed variational principle — which we need to get Eqs. (3.12) and (3.23).

This method to confirm or disprove the validity of the equivalence principle, however, can be further refined. Given an action for a gravitational theory, we shall show now how to establish a connection between the variational formulation leading to the field equations, and condition (3.23). A connection which makes checking the equivalence principle a matter of inspecting the action itself, without varying it, or perform any ϵ-series expansion.

3.3.2 Variational arguments

Let S be an action for a physical theory involving both gravitational and non-gravitational degrees of freedom, which decouples in the sum

$$S := S_{\text{grav}}\left[g^{ab}, \Pi_J\right] + S_{\text{matter}}\left[g^{ab}, \psi_K\right] \ , \tag{3.25}$$

where the inverse metric g^{ab} is assumed as the independent variable. The term S_{grav} encodes all the gravitational variables, denoted here by the pair $\left(g^{ab}, \Pi_J\right)$, with $\{\Pi_J\}$ a collection of all the other gravitational degrees of freedom besides the metric. Without loss of generality, we can think for the moment that $\{\Pi_J\}$ is made of a single scalar field ϕ [154]. The term S_{matter}, on the other hand, encodes the non-gravitational dynamical variables, represented here by the collection $\{\psi_K\}$ (from now on, a single field ψ will be used). The universal coupling of the gravitational phenomena also demands that S_{matter} depend on g_{ab}, but *not* on ϕ.

The field equations for all the dynamical variables emerge upon varying the action S with respect to all the degrees of freedom, provided that the variational problem be well-posed. To this end, we notice that, in general, Eq. (3.25) is written explicitly as

$$S = \int_\Omega \left(\mathscr{L}_{\text{grav}} + \mathscr{L}_{\text{matter}}\right) \sqrt{-g}\, \mathrm{d}^n y \ , \tag{3.26}$$

where Ω is the coordinate representation of a region in spacetime, and the Lagrangian densities $\mathscr{L}$'s are functions of the fields, and their derivatives of arbitrary order. The variational problem thus reads

$$\delta S = 0 \ , \tag{3.27}$$

Upon switching the variation δ and the integral in (3.26), and extracting the various functional dependencies, it is

$$\delta S = \int_\Omega \left(\frac{\delta S_{\text{grav}}}{\delta g^{ab}} \delta g^{ab} + \frac{\delta S_{\text{grav}}}{\delta \phi} \delta\phi + \frac{\delta S_{\text{matter}}}{\delta g^{ab}} \delta g^{ab} + \frac{\delta S_{\text{matter}}}{\delta \psi_K} \delta\psi_K \right) \mathrm{d}^n y \ , \tag{3.28}$$

provided that all the functional derivatives in round brackets exist — to this end, apt boundary terms might need to be deployed —. The field equations are finally found from Eq. (3.27), upon imposing the condition that the variation of the dynamical fields vanishes on the boundary $\partial\Omega$.

By incorporating the perturbative approach developed in the last section with the variational formalism outlined above, we get that, in general, if the fields admit a decomposition of the type (3.18), then the same will hold for the action. Specifically, the gravitational sector will be written as [154]

$$S_{\text{grav}} = \bar{S}_{\text{grav}} + \epsilon\, \mathcal{S}_{\text{grav}} + S^{(2+)}_{\text{grav}} \ , \tag{3.29}$$

where $\bar{S}_{\text{grav}} = S_{\text{grav}}\left[\bar{g}_{ab}, \bar{\phi}\right]$, i.e. it is the original action evaluated in the background fields only, whereas the linear part $\mathcal{S}$ is given in general by

$$\mathcal{S}_{\text{grav}} := \mathcal{S}_{\text{grav}}\left[\bar{g}_{ab}, \gamma_{ab}, \bar{\phi}, \chi\right] \ , \tag{3.30}$$

where we have set, following (3.18),

$$\phi = \bar{\phi} + \epsilon\, \chi + \phi^{(2+)} \ . \tag{3.31}$$

We now vary the action (3.29) with respect to the dynamical variables — notice that $\bar{g}^{ab}$ and γ^{ab} are independent degrees of freedom —. The variation of Eq. (3.29) with respect to $\bar{g}^{ab}$ gives

$$\frac{\delta \bar{S}_{\text{grav}}}{\delta \bar{g}^{ab}} + \epsilon\, \frac{\delta \mathcal{S}_{\text{grav}}}{\delta \bar{g}^{ab}} + \frac{\delta S^{(2+)}_{\text{grav}}}{\delta \bar{g}^{ab}} = \frac{\delta S_{\text{grav}}}{\delta \bar{g}^{ab}} = \frac{\delta S_{\text{grav}}}{\delta g^{cd}}\, \frac{\partial g^{cd}}{\partial \bar{g}^{ab}} = \frac{\delta S_{\text{grav}}}{\delta g^{ab}} \ . \tag{3.32}$$

Since it is, by definition, $E_{ab} = (2/\sqrt{-g})\, \delta S_{\text{grav}}/\delta g^{ab}$, at first order in ϵ it is also

$$\frac{\delta S_{\text{grav}}}{\delta g^{ab}} = \frac{\sqrt{-g}}{2}\, E_{ab} = \frac{\sqrt{-\bar{g}}}{2}\, \bar{E}_{ab} + \epsilon\, \frac{\sqrt{-\bar{g}}}{2} \left(\frac{\gamma}{2}\, \bar{E}_{ab} + \mathcal{E}_{ab}\right) \ , \tag{3.33}$$

where we have used the expansion (3.19) for the generalised Einstein tensor, and Eq. (B.3) for the expansion of the determinant; also, it is $\gamma := \bar{g}_{ab}\gamma^{ab}$. A comparison of the relation above with Eq. (3.32) allows to separate the two contributions $\bar{E}_{ab}$ and $\mathcal{E}_{ab}$, given by, respectively [154]

$$\bar{E}_{ab} = \frac{2}{\sqrt{-\bar{g}}}\, \frac{\delta \bar{S}_{\text{grav}}}{\delta \bar{g}^{ab}} \ , \tag{3.34}$$

$$\mathcal{E}_{ab} = \frac{2}{\sqrt{-\bar{g}}}\, \frac{\delta \mathcal{S}_{\text{grav}}}{\delta \bar{g}^{ab}} - \frac{\gamma}{2}\, \bar{E}_{ab} \ . \tag{3.35}$$

We can also observe that, since the matter action S_{matter} does not depend on the additional gravitational degrees of freedom, it will be, at the zeroth order,

$$\frac{\delta \bar{S}_{\text{grav}}}{\delta \bar{\phi}} = 0 \ . \tag{3.36}$$

Two other relevant relations we need, emerge from the variation of the first-order action $\mathcal{S}_{\text{grav}}$. We begin by rewriting $\mathcal{S}_{\text{grav}}$ as the derivative of the gravitational action with respect to the small parameter ϵ; we set $\mathrm{d}\mathfrak{V} := \sqrt{-g}\, \mathrm{d}^n y$,[10]

[10]The symbol "$\mathfrak{V}$" is a Gothic — or "Blackletter" — capital "V", a reference to the (hyper-)volume of the spacetime region. The old convention of denoting the tensor densities with Gothic letters is now fading away; once it was common ([360, 479, 477, 478]) to take a general expression such as $\sqrt{-g}A^{abc...l}_{mno...z}$, and denote it with the corresponding Gothic symbol $\mathfrak{A}^{abc...l}_{mno...z}$, but habits are dynamical, and trends rise and fall.

and get

$$
\begin{aligned}
\mathcal{S}_{\rm grav}\left[\bar{g}^{ab},\gamma^{ab},\bar{\phi},\chi\right] &= \left(\frac{\mathrm{d}S_{\rm grav}\left[g^{ab},\phi\right]}{\mathrm{d}\epsilon}\right)_{\epsilon=0} \\
&= \int_\Omega \mathrm{d}^n y \left[\left(\frac{\delta S_{\rm grav}}{\delta g^{ab}}\right)\left(\frac{\mathrm{d}g^{ab}}{\mathrm{d}\epsilon}\right) + \left(\frac{\delta S_{\rm grav}}{\delta \phi}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\epsilon}\right)\right]_{\epsilon=0} \\
&= \int_\Omega \mathrm{d}^n y \left(-\frac{\delta \bar{S}_{\rm grav}}{\delta \bar{g}^{ab}}\,\gamma^{ab} + \frac{\delta \bar{S}_{\rm grav}}{\delta \bar{\phi}}\,\chi\right) \\
&= \int_\Omega \mathrm{d}\bar{\mathfrak{V}} \left(-\frac{1}{2}\,\bar{E}_{ab}\,\gamma^{ab} + \frac{1}{\sqrt{-\bar{g}}}\,\frac{\delta \bar{S}_{\rm grav}}{\delta \bar{\phi}}\,\chi\right) \, . \qquad (3.37)
\end{aligned}
$$

In the above formula, the expression "$\epsilon = 0$" on the second line refers to each term in the sum of the products, hence the fields in the functional derivatives reduce to their background terms, whereas the minus sign in the third and fourth line comes from the usage of γ^{ab}. Upon performing the further variation of Eq. (3.37) with respect to the independent variables γ^{ab} and χ, by comparison one gets the relations [154]

$$
\frac{\delta \mathcal{S}_{\rm grav}}{\delta \gamma^{ab}} = -\frac{\sqrt{-g}}{2}\,\bar{E}_{ab} \, , \qquad (3.38)
$$

$$
\frac{\delta \mathcal{S}_{\rm grav}}{\delta \chi} = \frac{\delta \bar{S}_{\rm grav}}{\delta \bar{\phi}} \, , \qquad (3.39)
$$

which we could have already guessed by noticing that $\epsilon\,\mathcal{S}_{\rm grav}$ corresponds to the first-order variation of the whole gravitational action in terms of the varied fields $\delta g^{ab} = -\epsilon\,\gamma^{ab}$ and $\delta\phi = \epsilon\,\chi$.

The results in Eq. (3.37) are valid as long as the functional derivatives exist, i.e. as long as the variational problem is well-posed. This can be achieved only upon fixing some boundary conditions on the dynamical variables and their derivatives on the boundary $\partial\Omega$, to get rid of spurious instances of terms like $\delta\nabla\Psi$ for some arbitrary field Ψ. Such consideration is relevant for our work, but has general validity: if the variational principle of a physical theory is not well-posed, the theory itself (or, rather, its representation in terms of the given dynamical variables) is intrinsically flawed [171].

All the formal structure for gravity theories built so far is to be background independent [227], i.e. it has to be invariant under arbitrary diffeomorphisms of the coordinates. This means that, if ξ^a is an infinitesimal vector generating the diffeomorphism, order by order in the ϵ-expansion we must have [542]

$$
\delta \bar{S}_{\rm grav} = \delta \mathcal{S}_{\rm grav} = \cdots = 0 \, . \qquad (3.40)
$$

The second condition in the list translates into

$$
\int_\Omega \mathrm{d}^m y \left(\frac{\delta \mathcal{S}_{\rm G}}{\delta \bar{g}^{ab}}\,\delta\bar{g}^{ab} + \frac{\delta \mathcal{S}_{\rm G}}{\delta \gamma^{ab}}\,\delta\gamma^{ab} + \frac{\delta \mathcal{S}_{\rm G}}{\delta \bar{\phi}}\,\delta\bar{\phi} + \frac{\delta \mathcal{S}_{\rm G}}{\delta \chi}\,\delta\chi\right) = 0 \, , \qquad (3.41)
$$

where Eq. (3.39) can be used, together with the zeroth-order condition (3.36), to get rid of the last term in the sum. It is also $\delta\bar{g}^{ab} = \bar{\nabla}^{(b}\xi^{a)}$ and $\delta\bar{\phi} = \xi^a\bar{\nabla}_a\bar{\phi}$, with $\bar{\nabla}_a$ built out of the background metric only. It follows, then,

$$
\int_\Omega \left(-\frac{2\,\bar{\nabla}^b\xi^a}{\sqrt{-\bar{g}}}\,\frac{\delta \mathcal{S}_{\rm G}}{\delta \bar{g}^{ab}} + \frac{\delta\gamma^{ab}}{\sqrt{-\bar{g}}}\,\frac{\delta \mathcal{S}_{\rm G}}{\delta \gamma^{ab}} + \frac{\xi^a\,\bar{\nabla}_a\bar{\phi}}{\sqrt{-\bar{g}}}\,\frac{\delta \mathcal{S}_{\rm G}}{\delta \bar{\phi}}\right) \mathrm{d}\bar{\mathfrak{V}} = 0 \, , \qquad (3.42)
$$

We can integrate by part the first term, erasing a total divergence by demanding that ξ^a vanish on $\partial\Omega$; it results

$$\int_{\mathscr{U}} \mathrm{d}\bar{\mathfrak{V}} \left[\left(\bar{\nabla}^b \left(\frac{2}{\sqrt{-\bar{g}}} \frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \bar{g}^{ab}} \right) + \frac{1}{\sqrt{-\bar{g}}} \frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \bar{\phi}} \bar{\nabla}_a \bar{\phi} \right) \xi^a + \frac{1}{\sqrt{-\bar{g}}} \frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \gamma^{ab}} \delta\gamma^{ab} \right] = 0 \, . \tag{3.43}$$

Under a diffeomorphism, $\delta\gamma^{ab}$ is given by

$$\delta\gamma^{ab} = \xi^c \bar{\nabla}_c \gamma^{ab} - \gamma^{cb} \bar{\nabla}_c \, \xi^a - \gamma^{ac} \bar{\nabla}_c \, \xi^b \, , \tag{3.44}$$

and this can be substituted in Eq. (3.43), yielding the expression

$$\int_{\Omega} \mathrm{d}\bar{\mathfrak{V}} \left(\bar{\nabla}^b \mathcal{E}_{ab} + \frac{\bar{\nabla}_a \bar{\phi}}{\sqrt{-\bar{g}}} \frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \bar{\phi}} - \frac{1}{2} \bar{E}_{bc} \bar{\nabla}_a \gamma^{bc} - \bar{\nabla}_c (\bar{E}_{ab} \, \gamma^{cb}) + \frac{1}{2} \bar{\nabla}^b (\gamma \, \bar{E}_{ab}) \right) \xi^a = 0 \, , \tag{3.45}$$

The diffeomorphism-invariance of the theory demands that this expression vanish for any arbitrary ξ^a, hence we arrive at the final formula [154]

$$\bar{\nabla}^b \mathcal{E}_{ab} = - \frac{1}{\sqrt{-\bar{g}}} \frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \bar{\phi}} \bar{\nabla}_a \bar{\phi} + \frac{1}{2} \bar{E}_{bc} \bar{\nabla}_a \gamma^{bc} + \bar{\nabla}_b (\bar{E}_{ac} \gamma^{bc}) - \frac{1}{2} \bar{E}_{ab} \bar{\nabla}^b \gamma \, , \tag{3.46}$$

where we have used the conservation equation $\bar{\nabla}^a \bar{E}_{ab} = 0$ emerging from the diffeomorphism-invariance of $\bar{S}_{\mathrm{grav}}$, together with the zeroth-order field equations on $\bar{\phi}$, Eq. (3.36).

Eq. (3.46) above is the result we were looking for, i.e. a general condition relating the background covariant divergence of the first-order generalised Einstein tensor, and the gravitational content of an arbitrarily assigned theory of gravity with metric and non-metric gravitational degrees of freedom.

We can now elaborate on the obtained result, and link it with the general picture of a test for the Gravitational Weak Equivalence Principle.

3.3.3 Results, comments, and interpretation

To sum up: the search for a test of the Gravitational Weak Equivalence Principle points at the geodesic character of the spacetime trajectories of small, self-gravitating, yet test bodies on a dynamical background. This specific type of motion is achieved once one finds a suitable tensor, covariantly conserved with respect to the background in which both the self- and non-self-gravitating test bodies move (this comparison is necessary for the result to be physically meaningful). The "testness" of the systems forces this tensor to be the first-order perturbation of the generalised Einstein tensor, i.e. the non-matter contribution to the gravitational field equations of a theory of gravity. Finally, variational arguments allow to find a relationship, given in Eq. (3.46), between $\bar{\nabla}^b \mathcal{E}_{ab}$ and other constituents of the theory examined.

Back to Eq. (3.46), by comparing it with the necessary and sufficient condition for the equivalence principle to hold — Eq. (3.23) — we find that the Gravitational Weak Equivalence Principle is satisfied if and only if [154]

$$\bar{E}_{ab} = 0 \, , \tag{3.47}$$

$$\frac{\delta \mathcal{S}_{\mathrm{G}}}{\delta \bar{\phi}} = 0 \, . \tag{3.48}$$

These are the two necessary and sufficient conditions to implement the geodesic motion of self-gravitating small bodies in a given theory of gravity. Being quite different, the two relations deserve a separate analysis.

Eq. (3.47) demands that, in view of the field equations (3.12), the background surrounding the self-gravitating system be devoid of matter. This is a common assumption when dealing with tests of the Gravitational Weak (and Strong) Equivalence Principle, usually introduced for sake of simplicity and easiness of calculation. In this approach, however, the presence of a matter-empty environment emerges as a fundamental condition to have almost-geodesic motion. That something like this had to crop up can be understood also from the following argument, related to what we have said in §3.1.1.

In the Newtonian regime, consider a self-gravitating body with gravitational potential Ψ, placed in a background with matter distribution represented by the Newtonian density $\bar{\rho}$; the density of the potential energy associated with the combined system is given by $\bar{\rho}\Psi$ — the analogue of "mM/r" for point masses — and the body exerts a gravitational pull on the background. At the same time, because of the action-reaction principle, the background acts on the body by means of the same force, hence the body gets a non-vanishing force contribution and its motion cannot be anymore "free" in any sense [154]. The general relativistic analogue of $\bar{\rho}\Phi$ is the combination $\bar{T}_{ac}\gamma^{cb}$, which is an indicator of some sort of potential; the presence of the covariant derivative is then related in a way to a "force", and this is ultimately the reason behind the last three terms in the sum (3.46).

Eq. (3.48), on the other hand, accounts for a condition on the nature of the gravitational degrees of freedom involved; it states that, in a matter-free environment, the theory will abide by the Gravitational Weak Equivalence Principle if and only if the gravitational degrees of freedom are solely encoded in the metric structure, and no other variables are bound to gravity. Hence, in the sub-class of the purely dynamical theories of gravity, it singles out only the purely metric ones, i.e. the theories of gravity in which g_{ab} alone is in charge of gravitational phenomena.

This result echoes the Strong Equivalence Principle [558, 557], whose main role is generally thought to be the selection of General Relativity only — plus, in four dimensions, Nordström's scalar gravity (but the latter is ruled out at the experimental level).

We have shown, then, that the goal of the Strong Equivalence Principle, i.e. singling out Einstein's theory in the crowd of the extended theories of gravity, can be achieved already at the lower level of the Gravitational Weak Equivalence Principle, which deals with the restricted subset of phenomena involving massive, self-gravitating test bodies, and not the whole category of gravitational physics (which encompasses also e.g. gravitational radiation).

A a final remark: if we suppose that a purely metric theory of gravity is assigned, so that condition (3.48) is satisfied, we may conjecture that the other condition, (3.47), could be sidestepped by deploying an apt gauge transformation, one making the terms involving the γ^{ab}'s disappear. The starting point would be the transformation

$$\gamma^{ab} \mapsto \gamma'^{ab} = \gamma^{ab} + 2\bar{\nabla}^{(a}\zeta^{b)} \ . \tag{3.49}$$

This step, however, would not be effective. The vector ζ^a generating the gauge transformation has four independent degrees of freedom, which are not enough to kill out all the independent variables. On top of that, a change in γ^{ab} would result in a subsequent change of $\mathcal{E}_{ab}$, which in turn would give more terms in the transformed version of Eq. (3.46).

A few remarks on the Yang–Mills-inspired proposal for the Strong Equivalence Principle

The above results allow us to say something more on the "non Abelian Strong Equivalence Principle" of Refs. [216, 215]. There, one adopts the field equations (2.14) in view of the analogy between General Relativity and non-Abelian Yang–Mills field theories. A further simplification proposed (but not thoroughly justified) is to work in vacuo, in the sense that the current j_{abc} sourcing the dynamics of the connexion is set to zero identically. At any rate, it follows that the actual equations embodying the Strong Equivalence Principle become [216]

$$\nabla_d R_{abc}{}^d = 0 \ , \tag{3.50}$$

and must be assumed as the formal translation of the equivalence principle in Strong form. The proposal is then checked by imposing (3.50) as a set of constraints on a metric of the type used in the Parametrised Post-Newtonian formalism and indeed recovers independently the two conditions on the parameters pointing at General Relativity. In [216] it is also advanced that the condition to be checked in the case of e.g. a scalar-tensor theory with additional gravitational variable ϕ is

$$\nabla_d \left(\phi R_{abc}{}^d\right) = 0 \ , \tag{3.51}$$

instead of (3.50).

It has been already pointed out in §2.3.2 that, in this framework, the Weak Equivalence Principle is implemented as a *separate* condition (its role being to set to 1 the value of one post-Newtonian coefficient). Whatever is at stake in this case, then, is neither exactly the Gravitational Weak Equivalence Principle (which includes the Weak form), nor the Strong Equivalence Principle (which builds upon the Weak form). At the very best, we can conclude that condition (3.50) is the *other* element forming the Strong Equivalence Principle together with the separately postulated Weak one.

The situation becomes even less clear when extended theories of gravity are considered. Eq. (3.50) may still be tracked back to a Yang–Mills approach to gravity. What to say, however, about the emergence of Eq. (3.51)? The scalar field and the connexion there appear with different orders of derivations in the same conservation equation, and no Lagrangian leading to the dynamical structure (3.51) is provided. This somewhat weakens the suggested full generality and validity of the condition.

In conclusion, while this alternative formulation of the Strong Equivalence Principle might present interesting aspects in view of its unorthodox point of view, and may have dug out some new hints towards a better understanding of the nature of gravity, it seems that it deserves further study to be brought to its full maturity, so that a complete pattern can emerge — one leading to conditions (3.50) from first principles, possibly incapsulating the add-on for the Weak Equivalence Principle to hold, and providing a full dynamical character to its equation (3.51), or one alike for extended models.

3.4 Sieving the landscape

Now that the sieve has been set up, it is time to let the stream of gravity theories try to pass it, and reject those which do not comply with the requirements.

Indeed, there is something more: the various passages leading to Eq. (3.46) — and conditions (3.47)–(3.48), for the theories abiding by the principle — permit to have the landscape of gravity theories unveil some hidden aspects about the "true" nature of some proposals.

3.4.1 Acid test: General Relativity

To begin with, we check the validity and consistency of our method by applying it to General Relativity. Einstein's theory is of course expected to pass the test, being precisely the framework upon which the Gravitational Weak Equivalence Principle has been originally tailored.

Therefore, we rewrite the action (1.25) (General Relativity, plus cosmological constant and boundary counter-term), and get

$$S_{\mathrm{GR}} = \frac{c^4}{16\pi G_{(n)}} \int_\Omega (R - 2\Lambda)\sqrt{-g}\, \mathrm{d}^n y + \mathscr{B}_{\mathrm{GHY}} \,. \tag{3.52}$$

For later convenience, we have already chosen a generic n-dimensional spacetime (no result will be affected by this). The action is supplemented by the matter-sector action (1.19), which will provide the stress-energy-momentum tensor T_{ab} via Eq. (1.14).

Upon varying Eq. (3.52) with respect to the inverse metric g^{ab}, with boundary conditions $\delta g^{ab} = 0$ on $\partial\Omega$, we are left with what we have called the generalised Einstein tensor; in this specific case it is

$$E_{ab} := \frac{c^4}{8\pi G_{(n)}} \left(G_{ab} + \Lambda g_{ab}\right) \,, \tag{3.53}$$

and it reduces to the ordinary Einstein tensor G_{ab} when the cosmological constant vanishes.

Condition (3.48) holds for General Relativity: this can be seen by noticing that, by construction, the theory is purely metric — in the metric-variation approach —. Then, whenever the model is considered in a matter-vacuum background, $\bar{T}_{ab} = 0$, whence $\bar{E}_{ab} = 0$, and Eq. (3.47) is satisfied identically.

This last statement can be proven independently, via a full calculation of the background-covariant divergence of the first-order generalised Einstein tensor [556], i.e. $\bar{\nabla}^a \mathcal{E}_{ab}$; to this end, we first notice that it is, at first order in the ϵ-expansion,

$$\epsilon\, \mathcal{E}_{ab} = \frac{\epsilon c^4}{8\pi G_{(n)}} \left(\mathcal{G}_{ab} - \Lambda \gamma_{ab}\right) \,, \tag{3.54}$$

with $\epsilon\, \mathcal{G}_{ab}$ the linear term of G_{ab}, and γ_{ab} from decomposition (3.18). In detail, one has

$$\begin{aligned}\mathcal{G}_{ab} =& \frac{1}{2}\Big[\left(\bar{\nabla}^c\bar{\nabla}_a\gamma_{bc} + \bar{\nabla}^c\bar{\nabla}_b\gamma_{ac}\right) - \bar{\nabla}^c\bar{\nabla}_c\gamma_{ab} - \bar{\nabla}_a\bar{\nabla}_b\gamma \\ &- \bar{g}_{ab}\left(\bar{\nabla}^c\bar{\nabla}^d\gamma_{cd} - \bar{\nabla}^c\bar{\nabla}_c\gamma\right) + \bar{g}_{ab}\gamma^{cd}\bar{R}_{cd} - \gamma_{ab}\,\bar{R}\Big] \,,\end{aligned} \tag{3.55}$$

(the full derivation of the above formula is available in Appendix B), and evaluating the background-covariant derivative of Eq. (3.55) is an instructive exercise in differential geometry and index gymnastics. The resulting formula is given in Eq. (B.17).

Upon noticing that, in a vacuum background $\bar{T}_{ab} = 0$, the vacuum field equations become

$$\bar{G}_{ab} = -\Lambda \bar{g}_{ab} \; , \tag{3.56}$$

whence, always in dimension n,

$$\bar{R}_{ab} = \frac{2\Lambda}{n-2} \bar{g}_{ab} \; , \tag{3.57}$$

$$\bar{R} = \frac{2n\Lambda}{n-2} \; . \tag{3.58}$$

The substitution of these two terms in the covariant divergence of Eq. (3.55) leads, after some passages, to the formula

$$\bar{\nabla}^b \mathcal{G}_{ab} = -\Lambda \bar{\nabla}^b \gamma_{ab} \; , \tag{3.59}$$

and this result can be introduced in $\bar{\nabla}^b \mathcal{E}_{ab}$ to find,

$$\bar{\nabla}^b \mathcal{E}_{ab} = 0 \; , \tag{3.60}$$

as expected. General Relativity hence complies with the test of the Gravitational Weak Equivalence Principle [154]. In Einstein's framework, small, self-gravitating bodies without further multipole structure move on geodesics of the background metric (provided that all the caveats listed in §§3.1.2, 3.2.3 are taken into account).

While this result was somehow expected to emerge, a new feature is that the validity of the principle holds even when the cosmological constant is non-zero, and this is a scenario not so often considered in the literature. Indeed, the Parametrised Post-Newtonian formalism described in §2.4.2 for four-dimensional spacetimes traditionally focusses only on the case $\Lambda = 0$.[11]

3.4.2 Other warm-up case studies

For the next step, we move to another classical model where the behaviour of self-gravitating test bodies is well known — in this case, it is known to violate the Gravitational Weak Equivalence Principle, hence also the Strong form —. We refer to the class of scalar-tensor theories (in four spacetime dimensions) [191, 103, 204].

With reference to §1.3.1, the action in this case is given by

$$S_{\mathrm{ST}} = \frac{c^4}{16\pi} \int \left[\phi R - \frac{\omega(\phi)}{\phi} \nabla_a \phi \nabla^a \phi - V(\phi) \right] \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{ST}} \; . \tag{3.61}$$

Since, however, we look for violations of the principle, we can restrict to a slightly simpler sub-case, and consider the Brans–Dicke proposal obtained from

[11] Most likely, because of the extremely tiny value of Λ, but also because the protocol is designed to build Solar-system experiments, where Λ can be safely neglected for all practical purposes — and thus cannot account properly for a more cosmologically-oriented setting.

the general form above by reducing the function $\omega(\phi)$ to a constant, and the potential $V(\phi)$ to zero. We are thus left with

$$S_{\mathrm{BD}} = \frac{c^4}{16\pi} \int \left[\phi R - \frac{\omega}{\phi} \nabla_a \phi \nabla^a \phi \right] \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{BD}} \,. \tag{3.62}$$

The idea behind Brans' and Dicke's model is to promote the gravitational coupling constant G to a field over spacetime, so that the "distant masses" (in the Machian sense) can in a way affect the local inertial frames by inducing a change in the way gravity couples universally to matter and to itself. Variation of Eq. (3.62) with respect to g^{ab} yields the new tensor E_{ab}, namely

$$E_{ab} = \frac{1}{8\pi} \left[\phi\, G_{ab} - \frac{\omega}{\phi} \left(\nabla_a \phi\, \nabla_b \phi - \frac{g_{ab}}{2}\, \nabla^c \phi\, \nabla_c \phi \right) - \nabla_a \nabla_b \phi + g_{ab} \Box \phi \right] , \tag{3.63}$$

which equals the source T_{ab} emerging from the matter sector. In addition to this, one has to vary Eq. (3.62) with respect to the scalar field ϕ, to account for the dynamical behaviour of the long-range scalar field; the result is

$$R - \frac{\omega}{\phi^2}\, \nabla^a \phi\, \nabla_a \phi + \frac{2\omega}{\phi}\, \Box \phi = 0 \,. \tag{3.64}$$

By taking the trace of Eq. (3.63), which equals $T := g^{ab} T_{ab}$ from Eq. (3.12), and substituting this result in the formula above, one gets the other field equation

$$\Box \phi = \frac{8\pi}{3 + 2\omega}\, T \,, \tag{3.65}$$

and the two, (3.64) and (3.65) can be used interchangeably.

By comparing the action (3.62) with condition (3.48), we get that Brans–Dicke theory does not comply with the Gravitational Weak Equivalence Principle, for it is not a purely metric theory of gravity. Even in a matter-vacuum background environment, the non-zero term $\delta \mathcal{S} / \delta \bar{\phi}$ will prevent the onset of the geodesic motion. We have then to expect that Eq. (3.48) has a non-vanishing term on the right-hand side, and this can be seen explicitly by extracting the background-covariant divergence of the tensor $\mathcal{E}_{ab}$, which in this case is given by [154]

$$\begin{aligned}
\mathcal{E}_{ab} = & \frac{1}{8\pi} \Bigg(\bar{\phi}\, \mathcal{G}_{ab} + \chi\, \bar{G}_{ab} - \frac{\omega}{\bar{\phi}} \left(\bar{\nabla}_a \bar{\phi}\, \bar{\nabla}_b \chi + \bar{\nabla}_a \chi\, \bar{\nabla}_b \bar{\phi} \right) + \frac{\omega\, \chi}{\bar{\phi}^2}\, \bar{\nabla}_a \bar{\phi}\, \bar{\nabla}_b \bar{\phi} \\
& - \frac{\omega}{2\bar{\phi}}\, \bar{g}_{ab}\, \gamma^{cd} \bar{\nabla}_c \bar{\phi}\, \bar{\nabla}_d \bar{\phi} + \frac{\omega}{2\bar{\phi}}\, \gamma_{ab}\, \bar{g}^{cd}\, \bar{\nabla}_c \bar{\phi}\, \bar{\nabla}_d \bar{\phi} + \frac{\omega}{\bar{\phi}}\, \bar{g}_{ab}\, \bar{g}^{cd}\, \bar{\nabla}_c \bar{\phi}\, \bar{\nabla}_d \chi \\
& - \frac{\omega\, \chi}{2\bar{\phi}^2}\, \bar{g}_{ab}\, \bar{g}^{cd}\, \bar{\nabla}_c \bar{\phi}\, \bar{\nabla}_d \bar{\phi} - \bar{g}_{ab} \gamma^{cd}\, \bar{\nabla}_c \bar{\nabla}_d \bar{\phi} - \bar{g}_{ab}\, \bar{g}^{cd}\, \Xi^e{}_{cd} \bar{\nabla}_e \bar{\phi} \\
& + \Xi^c{}_{ab} \bar{\nabla}_c \bar{\phi} + \bar{g}_{ab}\, \bar{g}^{cd}\, \bar{\nabla}_c \bar{\nabla}_d \chi + \gamma_{ab}\, \bar{g}^{cd}\, \bar{\nabla}_c \bar{\nabla}_d \bar{\phi} - \bar{\nabla}_a \bar{\nabla}_b \chi \Bigg) \,,
\end{aligned} \tag{3.66}$$

with $\Xi^a{}_{bc}$ defined in (B.5), and $\mathcal{G}_{ab}$ as in (3.55).

This expression looks intimidating, and its manipulation can become unmanageable. Before even starting turning the crank, we notice that there is another way to look at the situation. Consider the background equations in vacuum, namely

$$\bar{E}_{ab} = 0 \ , \tag{3.67}$$

and rewrite them, thanks to (3.63), in the equivalent form

$$\bar{G}_{ab} = \kappa \bar{T}^{(\phi)}_{ab} \ , \tag{3.68}$$

where the new "stress-energy-momentum" contribution associated to the scalar field ϕ is given by

$$\bar{T}^{(\phi)}_{ab} = \frac{\omega}{\bar{\phi}^2}\left(\bar{\nabla}_a\bar{\phi}\,\bar{\nabla}_b\bar{\phi} - \frac{1}{2}\,\bar{g}_{ab}\,\bar{\nabla}^c\bar{\phi}\,\bar{\nabla}_c\bar{\phi}\right) + \frac{\bar{\nabla}_a\bar{\nabla}_b\bar{\phi}}{\bar{\phi}} - \bar{g}_{ab}\,\frac{\bar{\Box}\bar{\phi}}{\bar{\phi}} \ , \tag{3.69}$$

(we suppose of course that $\bar{\phi} \neq 0$ everywhere). We have now a condition in which we are basically dealing with General Relativity — hence, condition (3.48) is satisfied — in the presence of a non-vanishing background matter contribution, which in turn spoils condition (3.47).

We can then see the Brans–Dicke theory either as a framework with an additional scalar degree of freedom, or as a purely metric theory of gravity where one cannot get rid of the zeroth-order stress-energy-momentum tensor. In either case, the Gravitational Weak Equivalence Principle is not satisfied, and from the point of view of our sieve, the model is to be rejected [154].

Passing from the sub-case of Brans–Dicke theory to the entire class of scalar-tensor models represented by the general action (1.30) does not alter the conclusion: the presence of the additional gravitational (scalar) degree of freedom remains, and so it does the possibility to rewrite the background field equations in the form (3.68). The only difference is the level of complication of an expression such as (3.69), or the explicit form of conditions (3.47), (3.48).

With a further step, we can easily rule out the entire family of multi-scalar-tensor theories, for the same reasons expressed above, and all the theories which can be remapped into scalar-tensor theories [154].

Vector-tensor theories, scalar-vector-tensor models, bi-metric frameworks, and so forth: every time gravitational degrees of freedom other than the metric are explicitly encoded in the specific form of the action, we can be sure that the sieve will rule them out. As soon as we demand that the Gravitational Weak Equivalence Principle be enforced, Einstein–Æther theory (1.33) fades out, and so it does Hořava–Lifshits (1.35), the general Horndeski model (1.32), and many non-minimally coupled variations on these themes.

The only abiguity at this stage is represented by those theories which appear, from the formulation of their action, as purely metric, for in this case the mere inspection of the form of S cannot help to fathom the existence of additional gravitational dynamical variables besides those inside g_{ab} [154].

3.4.3 More findings, and "theories in disguise"

Consider a higher-curvature theory like those discussed in §1.3.2 — without loss of generality, we can start with an $f(R)$ model —. The action in the bulk, as emerging from the general prototype (1.38), is

$$S_{f(R)} = \frac{c^4}{16\pi G}\int_\Omega f(R)\,\sqrt{-g}\,\mathrm{d}^4y \ , \tag{3.70}$$

with f a general function, analytic in its argument, and R the curvature scalar of the metric connexion. The resulting field equations read

$$\frac{\mathrm{d}f(R)}{\mathrm{d}R}R_{ab}-\frac{1}{2}\,f(R)\,g_{ab}-\nabla_a\nabla_b\left(\frac{\mathrm{d}f(R)}{\mathrm{d}R}\right)+g_{ab}\,\Box\left(\frac{\mathrm{d}f(R)}{\mathrm{d}R}\right)=\frac{8\pi G}{c^4}T_{ab}\ . \quad (3.71)$$

To obtain the latter, as customary, one introduces a boundary term juxtaposing the action (3.70); for $f(R)$ theories the boundary contribution is given by

$$\mathscr{B}_{f(\phi)}=2\oint_{\partial\Omega}\frac{\mathrm{d}f(R)}{\mathrm{d}R}K\sqrt{h}\,\mathrm{d}^3y\ , \quad (3.72)$$

with the same symbols used in Eq. (1.18) for boundary, induced metric, and trace of the extrinsic curvature.

In Eqs. (3.70) and (3.72), the integrand looks like a function of the metric field alone (and of its derivatives), which would imply that both conditions (3.47) and (3.48) are automatically satisfied in a matter-free environment, provided that the variational principle for the theory be well-defined. We should hence conclude that $f(R)$ theories, and in general all gravity theories with higher-curvature corrections, pass the test for the Gravitational Weak Equivalence Principle as soon as the matter is removed from the physical environment surrounding a self-gravitating test body.

This conclusion, however, is wrong.

In fact, $f(R)$ theories, and all theories alike (with only one exception, discussed in §3.4.4), are indeed frameworks with additional gravitational degrees of freedom besides the metric, and thus cannot pass the test, as condition (3.48) is never satisfied, even in a matter-vacuum environment. The problem in this case is that such dynamical variables are hidden underneath the surface of seemingly purely metric actions, therefore they are hard to spot at first glance.

Luckily enough, the existence of a well-formulated variational principle for a theory of gravity needed to enforce the two conditions is precisely the aspect which allows to dig out the hidden variables and restore the correct answer to the test for the Gravitational Weak Equivalence Principle.

Starting with a semi-heuristic argument, we can observe that, in higher-curvature theories where the integrand in Eq. (1.38) is of the general form

$$f\left(g_{ab},{R_{abc}}^d,R_{ab},R,\ldots,\nabla{R_{abc}}^d,\Box{R_{abc}}^d,\ldots,R\Box R,\ldots\right)\ , \quad (3.73)$$

one usually finds that additional gravitational modes (scalar, vector, tensor, spinor, and so forth) can emerge besides the massless spin-2 graviton. In the specific case of the $f(R)$ theory under discussion, the supplementary mode is a scalar one with non-vanishing mass.

Then the question arises: if $f(R)$ theory is a purely metric one, where does the scalar mode hide, if the only dynamical variables are those within g_{ab}, in principle providing the graviton alone? To answer this question, one usually rewrites the action (3.70) in terms of the metric and an additional, auxiliary scalar field, introduced via a Lagrange-multiplier technique; the new action reads

$$S_{f(\phi)}=\frac{\kappa c^4}{16\pi G}\int_\Omega\left[f(\phi)+\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi}\,(R-\phi)\right]\sqrt{-g}\,\mathrm{d}^4y\ , \quad (3.74)$$

with the same function f as in (3.70). Upon variation with respect to ϕ, from Eq. (3.74) we get the field equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\phi^2}\,(R-\phi) = 0\ , \tag{3.75}$$

and this last one assures that, if $f'' \neq 0$, the new degree of freedom can be identified with the scalar curvature R.

The resulting scheme changes drastically, for we have now a scalar-tensor theory written in terms of the variables g^{ab}, ϕ; in particular, the theory is of the Brans–Dicke type (3.62), with vanishing constant ω, and a potential depending on the specific form of f; the scalar is massive — whereas the graviton remains massless — with the mass related to the second derivative of the potential $V(f)$. Such remapping also allows to identify the boundary term (3.72) with the corresponding one

$$\mathscr{B}_{f(\phi)} = 2\oint_{\partial\Omega} \frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi} K\sqrt{h}\,\mathrm{d}^3 y\ , \tag{3.76}$$

which renders the variational problem well-posed for the scalar-tensor theory as well, in the sense that now all one has to impose is the condition

$$\delta g^{ab} = \delta\phi = 0\ , \tag{3.77}$$

at the boundary $\partial\Omega$.

This last statement, however, ought to ring a bell: if the variational problem for (3.70) has to be well-defined with the boundary terms (3.72), and if the auxiliary variable ϕ coincides with R, then the boundary conditions to impose on an $f(R)$ theory in its "purely metric look" should be

$$\delta g^{ab} = \delta R = 0\ , \tag{3.78}$$

but this is an odd conclusion, as $R \propto \partial_c \partial_d g^{ab}$, hence setting $\delta R = 0$ would require demanding the trivialisation of $\partial_c \partial_d \delta g^{ab}$ on the boundary. While it is true that, in general, $f(R)$ theories have fourth-order field equations (hence, the initial-value formulation requires to define the values of derivatives up to the third order on a given Cauchy surface), still a well-posed variational principle demands to have only the variation of the actual dynamical fields set identically to zero at the boundaries, and not their derivatives.[12]

This last statement leads to the following conclusion: the remapping from higher-curvature gravity theories into purely metric ones with additional gravitational degrees of freedom is not a mere technical tool to simplify the picture or reduce the mathematical efforts required: it is actually a meaningful way to look at higher-curvature corrections, as it is only when the metric and the other variables are decoupled and treated separately, that the variational problem for the actions makes sense [154].

The argument given above for the $f(R)$ theories can be extended to other models; for instance, the class of frameworks where f is a function of R, $\Box R$,

[12] As an aside, it is possible to prove [171, 327] that, if one wants to preserve the equivalence between $f(R)$ theories and scalar-tensor theories *also at the boundary* (as reasonably expected from a faithful remapping), there is no "purely metric" formulation of $f(R)$ theory for which it suffices to set $\delta g^{ab} = 0$ on $\partial\Omega$, even after having rewritten the boundary term (3.72) in a different way.

$\Box^2 R$, and so forth, boils down to a multi-scalar-tensor theory, where one additional scalar degree of freedom can replace a pair of derivatives [475, 240, 48]; therefore, fourth-order gravity is equivalent to metric General Relativity plus one scalar field, sixth-order gravity equals General Relativity plus two scalar fields, eighth-order gravity demands three scalar fields besides the metric, and so forth.

The situation gets slightly more complicated with actions where one considers also the Ricci and Riemann tensors, plus their derivatives. In this case, a help comes from the alternative Palatini formalism [411, 530, 192] widely used in metric-affine, affine, and purely affine theories of gravity.

In the Palatini formalism for the gravitational action, the connexion and the metric are treated as independent variables, and the action is varied with respect to both g_{ab} and a general $\Delta^a{}_{bc}$; it results a pair of sets of equations, namely those for the metric and those for the affine structure. In theories of the type described at the beginning of §1.3.3, the connexion is entirely in charge of the pieces involving the curvature tensor and the Ricci tensor (where the metric is absent), whereas the bits containing R can be reduced to contracted products of the type $g^{ab}R_{ab}$, where once again the dynamical variables get decoupled.

If one supposes that the theory admits more degrees of freedom than those accounted for by the metric alone, decoupling the metric from the curvature is a wise move, as it permits the hidden variables to emerge more easily in terms of the boundary conditions imposed on $\delta g^{ab}, \delta\Delta^a{}_{bc}$ and their derivatives. In the best-case scenario, the two routes — metric and Palatini formalism — turn out to be equivalent in both the space of field equations *and solutions*, which guarantees that $\Delta^a{}_{bc}$ is indeed $\Gamma^a{}_{bc}$, and that the solutions for the metric variation are also solutions for the Palatini one, and vice versa [90].

On the other hand, it may happen that the field equations for the connexion do not boil down to the metric-compatibility condition $\nabla_c g_{ab} = 0$, nor that the spaces of solutions of the two sets coincide (usually, the metric solutions are found to be a subset of all possible solutions for the connexion). This can be taken as an indication that there is a richer structure hidden below the purely metric appearance of the action, and thus the theory under consideration must be reformulated in terms of the metric and other gravitational degrees of freedom [90].

By following this protocol, Vitagliano *et al.* have found [540], for instance, that the dynamics of an $f\big(R, R^{ab}R_{ab}\big)$ theory in Palatini formalism can be identically reformulated in terms of metric General Relativity plus a vector field A_c with a Proca-like action given by

$$S_{\text{Proca}} = -\alpha \int_\Omega \left[\frac{1}{2}F^{ab}F_{ab} + m^2 A^c A_c\right]\sqrt{-g}\,\mathrm{d}^4 y\ , \tag{3.79}$$

with α an apt coupling constant, and $F_{ab} := 2\nabla_{[a}A_{b]}$.

The most important lessons learnt from the case of higher-curvature gravity theories can thus be summed up as follows. First, all these theories cannot pass the test of the Gravitational Weak Equivalence Principle, for the non-metric degrees of freedom hidden inside their structure prevents the onset of free-fall motion for self-gravitating test bodies.

Second, the boundary terms in an action for a dynamical theory crucially contribute to a meaningful, sound, and robust construction of the physical picture, and always ought to be kept under strict control.

In this sense, our construction to test the Gravitational Weak Equivalence Principle can become an independent source of tests of the actual dynamical content of extended theories of gravity [154]. Since the protocol strictly requires a well-defined variational formulation, and the latter demands all the degrees of freedom to be explicitly exhibited (so that one only has to set $\delta g^{ab} = \delta\phi = 0$ at the boundaries), then the search for hidden gravitational variables comes basically for free in the guidelines to our test for the equivalence principle.

3.4.4 An unexpected guest in higher dimensions

Our method tests the Gravitational Weak Equivalence Principle in any spacetime dimension $n \geq 3$.[13] The four-dimensional environment remains, so far, the most interesting to investigate; yet, from the purely "taxonomic" point of view, it is equally interesting to look at higher and lower n's, and explore the effect of the filter in such exotic scenarios, since nothing forbids to do so. In particular, we focus now on the higher-dimensional gravity theories.

In §1.3.4, we have shown that many models fall back in the category of "metric theory plus additional degrees of freedom", hence for them the equivalence principle is destined not to be satisfied, and the test to fail. If we restrict our study to the case in which the additional dimensions do not wind up or compactify, but rather remain "open", then we are left with, among the others, General Relativity, DGP gravity, and the family of dimensional-dependent Lanczos–Lovelock theories.

General Relativity, as seen above, passes the test no matter what the value of n. DGP gravity, if interpreted as a bi-metric model, will be ruled out, for only one of the two metrics — that on the bulk, or that on the brane — can be the "true" one, with the other providing the additional degrees of freedom leading to the violation of condition (3.48).[14]

The scenario is far less explored in the case of Lanczos–Lovelock theories. Lanczos–Lovelock gravity is a higher-curvature theory characterised by the presence of second-order-only field equations; such peculiarity is obtained, as reviewed in §1.3.4, by a carefully chosen (and unique) set of parameters standing in front of the higher-curvature corrections in the action.

One feature of Lanczos–Lovelock Lagrangian densities is that they can be always rewritten as a sum of two pieces: one, called the *bulk term*, is quadratic in the first derivatives of the metric, whereas the other, called the *surface term*, is a total derivative determining a surface term in the action. This result has been found for the first time in General Relativity itself; indeed, the Einstein–Hilbert

[13] The 2-dimensional case has to be excluded for the following reason: the stress-energy-momentum tensor sourcing the field equations for gravity reduces to a scalar function, T, which is not accounted for in the Geroch–Jang Theorem, based instead on a symmetric tensor. One could still build a symmetric T_{ab} from T by putting $T^{(2D)}_{ab} := T g_{ab}$, but then the condition $\nabla^a T_{ab} = 0$ would reduce to $\nabla_b T = 0$, i.e. it would demand T to be a constant, which is too restrictive a condition.

[14] The issue of the boundary terms might also provide some insights on the true nature and number of the gravitational degrees of freedom in DGP model. Also, a natural choice would seem to consider the five-dimensional "bulk" term as the fundamental one, and the four-dimensional action a sort of complement of the boundary term, but the situation is not so clear in this scheme.

Lagrangian density $R\sqrt{-g}$ can be rewritten as

$$R\sqrt{-g} = Q_{abc}{}^{d} R^{abc}{}_{d} \sqrt{-g} \ , \tag{3.80}$$

with the tensor $Q_{abc}{}^{d}$ given by

$$Q_{abc}{}^{d} := \frac{1}{2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right) \ . \tag{3.81}$$

Calculations [404] show that (3.80) splits into the sum

$$Q_{abc}{}^{d} R^{abc}{}_{d} \sqrt{-g} = 2 Q_{a}{}^{bcd} \Gamma^{a}{}_{de} \Gamma^{e}{}_{bc} \sqrt{-g} + 2 \partial_c \left[Q_{a}{}^{bcd} \Gamma^{a}{}_{bd} \sqrt{-g} \right] \ , \tag{3.82}$$

where the bulk and surface term get explicitly separated. The bulk term is nothing but the so-called *gamma-gamma Lagrangian* used by Einstein himself to first formulate General Relativity (see §A.2.3). As for the surface term, it can be rewritten as $2\partial_\alpha \left(H^\alpha \sqrt{-g} \right)$, where H^α is not a four vector, but rather a four-component non-vectorial object.

On top of that, it is possible to prove that the surface term can be determined entirely from the bulk term via the following relation, known as the *holographic property*,

$$\mathfrak{L}_{\text{surf}} = - \frac{1}{(n/2) - 1} \partial_a \left(g_{bc} \frac{\partial \mathfrak{L}_{\text{bulk}}}{\partial \left(\partial_a g_{bc} \right)} \right) \ , \tag{3.83}$$

valid on spacetimes of dimension $n > 2$. Such result is non-trivial, and has gained much attention in the light of a more general renaissance of the concept of holography [403, 405, 366], mostly driven by recent developments in the AdS/CFT correspondence paradigm [448].

The property extends to the entire class of Lanczos–Lovelock theories. The point in this case is the fact that, to prove Eq. (3.82), one simply uses the symmetry and covariant conservation properties of the tensor Q_{abcd}, rather than its precise form. Therefore, whenever the action of a theory of gravity can be recast in form (3.82), with Q_{abcd} sharing the symmetries of the curvature tensor and having $\nabla^a Q_{abcd} = 0$, then the holographic property will emerge again. This similarity between General Relativity and the entire Lanczos–Lovelock class, together with the fact that the resulting field equations for both models only have second-order derivatives, has led some authors to call the Lanczos–Lovelock theories "the most natural extension" of Einstein's scheme [404, 366].

The general Lanczos–Lovelock Lagrangian is a polynomial sum of densities, which we can write as

$$S_{\text{LL}} = \int_\Omega \mathrm{d}^n y \sum_{m \leq n/2} \alpha_{(m)} \, \mathfrak{L}_{\text{LL},m} + \mathscr{B}_{\text{LL}} \ , \tag{3.84}$$

where the sum is constrained by the dimensionality of spacetime, and the Lagrangian density of order m is given by

$$\mathfrak{L}_{\text{LL},m} = \sqrt{-g} \, Q_{abc}{}^{d} R^{abc}{}_{d} = \delta^{1357\ldots 2k-1}_{2468\ldots 2k} R^{24}_{13} R^{68}_{57} \ldots R^{2k-2\, 2k}_{2k-3\, 2k-1} \ . \tag{3.85}$$

The presence of the alternating tensor $\delta^{1357\ldots 2k-1}_{2468\ldots 2k}$ guarantees that, when $k < n$, the Lanczos–Lovelock density of order $k = 2m$ will give a non-trivial contribution; when $k = n$, the resulting term becomes a trivial topological invariant — the Euler characteristic [119, 370] — and its variation will vanish identically as a result of the Gauß–Bonnet theorem [119, 370]. Finally, for $k > n$, the alternating tensor trivialises, and no contribution can emerge.

Relation (3.82), together with the condition $\nabla^a Q_{abcd} = 0$, allows to find regular patterns also in the field equations. Upon variating the action (3.84), a series of manipulations give back the general result [366, 414]

$$E_{ab} = m\, Q_a{}^{cde} R_{bcde} - \frac{1}{2} g_{ab} \mathscr{L}_{\mathrm{LL},m} = \frac{8\pi G}{c^4} T_{ab} \ , \tag{3.86}$$

with T_{ab} emerging from varying the matter action, and $\mathfrak{L}_{\mathrm{LL},m} = \sqrt{-g}\, \mathscr{L}_{\mathrm{LL},m}$. For $m = 1$, the previous formula yields the general relativistic case, because (3.81) implies that $Q_a{}^{cde} R_{bcde} = R_{ab}$, and on the left-hand side one is left with the Einstein tensor.

Given this due introduction to the framework, we can now see what our test of the Gravitational Weak Equivalence Principle can say about Lanczos–Lovelock gravity theories.

In the simplest non-trivial case, i.e. Gauß–Bonnet gravity (which gives non-zero contributions from dimension 5 onwards), the combination (3.84) is given by

$$S_{\mathrm{GB}} = \frac{\alpha c^4}{32\pi G} \int \left(R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right) \sqrt{-g}\, \mathrm{d}^4 y + \mathscr{B}_{\mathrm{GB}} \ , \tag{3.87}$$

to which one usually adds the Einstein–Hilbert term (with, or without, the cosmological constant). The metric variation provides the field equations [404]

$$G_{ab} + \alpha H_{ab} = \frac{8\pi G}{c^4} T_{ab} \ , \tag{3.88}$$

where the symmetric, covariantly conserved tensor H_{ab} is given by

$$H_{ab} = 2 \left[R\, R_{ab} - 2 R_{ac} R^c{}_b - 2 R^{cd} R_{acbd} + R_a{}^{cde} R_{bcde} \right] - \frac{1}{2} g_{ab} \mathscr{L}_{\mathrm{GB}} \ . \tag{3.89}$$

In a vacuum background environment we have $\bar{E}_{ab} = \bar{G}_{ab} + \alpha \bar{H}_{ab} = 0$, and all it remains to check is that the theory itself is a purely metric one. So to do, we can refer to the argument in §3.4.3 on the higher-curvature gravity models and their boundary terms. If such terms are nowhere to be found, the theory will not be purely metric, hence will violate the equivalence principle via a failure of condition (3.48).

It turns out, however, that boundary terms for Gauß–Bonnet theory do exist [404], and they are given by [139]

$$\mathscr{B}_{\mathrm{GB}} = 2 \int_{\partial\Omega} \left[K + 4\alpha \left(J + 2 K_{ab} \tilde{G}^{ab} \right) \right] \sqrt{h} \mathrm{d}^{n-1} y \ , \tag{3.90}$$

with $\tilde{G}^{ab}$ the $(n-1)$-dimensional Einstein tensor built out of the induced metric h_{ab}, and J the trace of the tensor J_{ab} given by

$$J = g^{ab} J_{ab} = g^{ab} \cdot \frac{1}{3} \left(2 K K_{ac} K^c{}_b + K^{cd} K_{cd} K_{ab} - 2 K_{ac} K^{cd} K_{bd} - K^2 K_{ab} \right) \ . \tag{3.91}$$

The boundary term (3.90) allows to avoid the use of additional degrees of freedom, so that the only condition to be set is to have δg^{ab} at the boundaries. Therefore, we can conclude that Gauß–Bonnet gravity is indeed a purely metric theory of gravity,[15] and passes through the sieve, joining General Relativity (the latter, in any number of dimension) in the group of theories implementing the free-fall motion for self-gravitating bodies as well.

The next step is to check whether the result holds for the whole class of Lanczos–Lovelock theories. Once again, for a given number of dimensions n, and in a vacuum background, all the non-trivial terms in the polynomial (3.84) will pass the test if proper boundary terms can be introduced in the action; this is indeed the case, for the general expression of such terms is given by [352, 404]

$$\mathscr{B}_{\rm LL} = \oint_{\partial\Omega} C_p \sqrt{h}\, \mathrm{d}^{n-1}y \; , \tag{3.92}$$

with C_p reading

$$\begin{aligned} C_p = 2p \int_0^1 \mathrm{d}\lambda\, \delta^{h_1 h_2 \ldots h_{2p-1}}_{k_1 k_2 \ldots k_{2p-1}} K^{h_1}_{k_1} \left(\frac{1}{2} R^{h_2 h_3}_{k_2 k_3} - \lambda^2 K^{h_2}_{k_2} K^{h^3}_{k_3} \right) \times \ldots \\ \cdots \times \left(\frac{1}{2} R^{h_{2p-2} h_{2p-1}}_{k_{2p-2} k_{2p-1}} - \lambda^2 K^{h_{2p-2}}_{k_{2p-2}} K^{h_{2p-1}}_{k_{2p-1}} \right) \; . \end{aligned} \tag{3.93}$$

We have thus proven that all the non-trivial Lanczos–Lovelock models for gravity comply with the geodesic motion on a background of a self-gravitating test body; this guarantees that the Gravitational Weak Equivalence Principle is satisfied in this class of extended theories of gravitation [154].

3.5 Wrap-up

The almost-geodesic motion for self-gravitating, extended masses is an experimental fact, verified with remarkable accuracy in the Solar system, and validated (with a lower confidence level) also in large-scale observations and at the cosmological level. The Weak and Einstein's Equivalence Principle cannot say much on this topic, for they pertain either to a different sort of physical system (Weak form, dealing with non-self-gravitating test particles only), or to a different phenomenology (Einstein's form, governing ultra-local, non-gravitational test physics). Since many available models for gravity forecast corrections to the geodesic motion which are not observed, this regularity of Nature might be elevated to the level of another equivalence principle.

The Strong Equivalence Principle is designed precisely to meet this need, but it presents some intrinsic issues: first, it draws conclusions on a wide range of phenomena (e.g. the gravitational radiation) for which no reliable data is currently available; second, its practical implementation (via the Parametrised

[15] This conclusion has been recently challenged (see [100]). The authors rewrite the action and field equations of Gauß–Bonnet–Lanczos–Lovelock theories in terms of the metric tensor, plus a number of differential 3-form fields. While this reformulation of the model is legitimate, it does not disprove our conclusions about the purely metric character of the Lanczos–Lovelock models. The point is that the differential 3-forms introduced in the paper are non-dynamical, for their variation is identically vanishing — see their Eq. (21) — and thus are merely auxiliary fields, encoding no information about the gravitational content of the theory (the latter remains entirely and uniquely confined in the metric degrees of freedom).

Post-Newtonian formalism) basically tests the equivalence of inertial and gravitational masses for extended bodies like the Moon or the Sun, possibly due to a Machian-induced variation of the constant G. This, however, is a test for the Gravitational Newton's Equivalence Principle, rather than of the extension of the Weak form to self-gravitating bodies.

Upon extracting, from the Strong form, a suitable sub-statement, called here "Gravitational Weak Equivalence Principle", we have framed the prolongation of the Galilean universality of free-fall to self-gravitating test bodies. The Gravitational Weak Principle is a proper part of the Strong principle, yet the two do not coincide (unless the gravitational extension of Schiff's conjecture proves to be true).

Once the limits and properties of the equivalence principle we are looking for are finally defined, the goal becomes that of finding an apt formal translation of the statement; this leads to consider the almost-geodesic world-tubes of self-gravitating bodies on a manifold equipped with a metric (at the very least).

The Geroch–Jang theorem points in this sense at a specific tensor, closely resembling the stress-energy-momentum one, whose vanishing covariant divergence locks the geodesic trajectory. Upon finding a tensor of this sort (which embodys the self-gravity of the system considered), conserved with respect to a background, the goal of comparing the motion of test bodies with, and without, negligible self-gravity becomes possible.

Such tensor exists, and emerges from a perturbative expansion of the gravitational and matter fields provided by any given purely dynamical theory of gravity in tensor form. An explicit expression of its divergence can be extracted from variational arguments, provided that the problem is well-posed. This results in the necessary and sufficient conditions (3.47) and (3.48).

The two constraints demand that the motion takes place in a vacuum environment (to prevent the onset of driving forces due to the action-reaction principle [94] applied to the interaction of the self-gravity with the background matter), and that the theory of gravity is purely metric, i.e. that it does not contain gravitational degrees of freedom other than the metric.

With the test ready, we have then applied it on the family tree of the extended gravity theories, to see which one were able to pass through this sieve.

General Relativity passes the test, as expected, and it does so also in the presence of a cosmological term. Besides, our method is supported by the independent results obtained from the Parametrised Post-Newtonian formalism, which, however, does not cover the cosmological extension.

Scalar-tensor theories, both of the Brans–Dicke type, and of the general type, fail the test and hence need be rejected if the Gravitational Weak Equivalence Principle is accepted to hold. This result as well is supported by independent indirect results from the Parametrised Post-Newtonian expansions, which tend to rule out theories where G is a function of space and time, on the basis of an unobserved sidereal variation $\dot{G}/G$ in time (and also in space).

Higher-curvature theories providing higher-derivative corrections also fail the test, for they are actually metric theories with additional gravitational degrees of freedom hidden beneath a seemingly purely metric appearance of the action. This aspect emerges as well from our method, although as a side-result, for our protocol requires well-posedness of the variational formulation of the field equations, and the latter demands all the relevant dynamical fields to be written out explicitly. While theories like $f(R)$ models are not addressed by the post-Newtonian approach, the possibility to remap them consistently into non purely-metric theories of gravity ultimately supports our findings.

Upon relaxing the constraint of working in four spacetime dimensions, two results emerge: first, that any n-dimensional version of General Relativity passes the test, with or without the cosmological constant contribution. Second, that also any dimension-compatible Lanczos–Lovelock action provides a gravity theory passing through the sieve, hence abiding by the Gravitational Weak Equivalence Principle, and implementing the almost-geodesic motion for test bodies with non-negligible self-gravity. This result cannot be confirmed by means of post-Newtonian results, as they are tailored on a strictly four-dimensional experimental setting; at any rate, independent arguments based on a completely different approach — Katz super-potentials, see [149, 148] — seem to agree with our conclusion, at least for what concerns the Gravitational Newton's Equivalence Principle.

Chapter 4

"Mesoscopic" Effects of Quantum Spacetime

> We shall treat the steel in the armour plate as it were a perfect fluid.
>
> Lavrentiev and Shabat, *Complex Variables*.

So far, we have looked at the landscape of gravity theories from a point of view centred on macroscopic objects and scales; yet, many recent proposals for alternative gravitational frameworks arise as attempts to construct an ultraviolet completion of the gravitational sector around the Planck scale. To better pinpoint the viability and mutual relationships of such models, the macroscopic regime is not the only zone where to look at.

A possible alternative might be to zoom in over the picture of spacetime up to the Planck scale, or rather, slightly above that threshold, and devote some pondering to the innermost nature of the notion of spacetime itself. We are led to look at "spacetime" in the sense of an "umbrella term", concealing a complex, perhaps discontinuous microstructure that, once examined, might provide useful insights and constraints on the type of gravity theories to be expected on larger scales.

Spacetime, then. Is it ultimately an entity, as believed by Newton, or a relation, as assumed by Aristotle? The most effective models currently at hand seems to favour the latter position, space and time being the names we give to specific relations between the gravitational field and the matter making up our clocks and rods.

The idea of spacetime as a manifold equipped with a metric structure is a great tool to describe macroscopic phenomena where quantum effects are systematically washed out. Yet, such model is likely believed to fail whenever the interplay of microscopic fluctuations and strong gravity starts to dominate [12, 330] (even though the validity of the smooth spacetime picture is prolonged at any scale, even the smallest ones, where instead it is reasonable to expect the emergence of novel phenomenology). The onset of a different regime is conjectured to occur at extremely high energies (of the order e.g. of the Planck scale,[1] around $E_{\wp} \sim 12.4 \times 10^{27}$ eV); this contributes to explain why no positive detections of any effect have emerged so far in accelerator-based experiments [267, 34, 50], whose best performance peak around 10^{12} eV.

More extreme conditions can be found in astrophysical and cosmological contexts; the energies and densities supposed to have been present during the earliest phases of the Universe's life nurture the hope to detect some relic traces of the "Planck era" imprinted in fossil radiation [10], or else in violations of fundamental CPT symmetry affecting dispersion relations [342, 223]. Compact objects also offer energy thresholds much higher than those attainable on Earth, with effects magnified by the redshift over cosmological distances, and are hence good candidates where to test the combined effects of quantum mechanics and general relativity [420, 454, 471, 521]. Yet, "clean" sources offering sharply identifiable features are hard to find.[2]

In the following pages, we recap the main points of a critical analysis of the notion of spacetime motivated by the quest for a "mesoscopic" regime of physical phenomena [153] in which the quantum effects propagate up to a scale compatible with our observational window (or at least approach it), and intertwine with classical structure. The kind of argument we pursue needs be general, for we want it to be compatible with as many quantum gravity proposals as possible; also, it is constrained by the large wealth of at our disposal, to check for violations and corrections to the ordinary laws of physics.

The results, finally, fit in a much broader, and more ambitious, research programme. Understanding something more about the structure of spacetime on small and "semi-small" scales could indeed shine a light also on the types of gravity theories one could expect to emerge below a certain threshold, and this further sews together the conclusions presented here with the findings of the previous Chapters.

4.1 Space and Time

Like many other fundamental concepts, "space" and "time" seem easy to define at first glance. The closer we get to the two ideas, however, the more complex it becomes any attempt to say something meaningful and precise about them.

The nature and status of space and time have been debated since the earliest days of philosophical thinking, and are still under scrutiny [43]. Within this ceaseless re-discussion of the foundations, we are mainly interested in three aspects.

[1] Any physical quantity referring to the Planck scale will be denoted here by the subscript "$\wp$", as in $\ell_{\wp}$ for a length, $m_{\wp}$ for a mass, $E_{\wp}$ for an energy, and so forth.

[2] One can study e.g. high-energy emissions from Gamma-Ray Bursts as possible carriers of information about quantum gravity effects — the latter manifesting themselves mainly through violations of the principle of relativity and, hence, of Lorentz symmetry; see [30, 4, 3, 203] —. An examination of the time delay of photons with energies in the Gamma-Ray range (MeV–GeV scale) with respect to photons in the hard X-rays (KeV scale), coming from a sample of five Gamma-Ray Bursts, has been recently performed by Castignani et al. [110]. The Reader is warmly invited to peruse the references and the included bibliography.

* The classical, formal definition of spacetime, i.e. the model used in General Relativity (and most gravitation theories) and also, with minor modifications, in Special Relativity — and in geometrised Newtonian physics.

* The notion of spacetime as an emergent concept, arising from quantum gravity proposals, i.e. spacetime as a non-fundamental entity, derived from the interplay and/or evolution of quantum variables.

* The operational construction of spacetime, i.e. the scheme in which space, time, and spacetime (with their formal properties) are built out of the readings of clocks and rods.

To examine the properties of a "mesoscopic" regime, we need to understand how the latter can be made fit in one of these frameworks, as a consequence of which modifications in the construction protocol [153]. Below, we initially focus on the classical and quantum descriptions of spacetime.

4.1.1 The classical spacetime

A spacetime $\mathscr{M}$ is, classically, a smooth, n-dimensional manifold endowed with a pseudo-Riemannian metric structure and a connexion [542, 332, 470, 250], which can be denoted by the symbol

$$\mathscr{M} \equiv (M, g_{ab}, \mathrm{D}) \ . \tag{4.1}$$

The smooth manifold M is a topological space X equipped with a coordinate atlas.[3] On $\mathscr{M}$ are then defined: the metric structure g_{ab}, represented by a smooth, rank-2, symmetric, covariant tensor field of Lorentzian signature; and the affine structure, given in general by the operator D, defined as[4]

$$\mathrm{D} : \mathfrak{X}(M) \times T^r_s M \mapsto T^r_s M \ , \tag{4.2}$$

with D_{V^a} a derivation in the algebra of tensors for all vectors V^a.

In General Relativity (and in all metric theories of gravity), D is the Levi-Civita affine structure ∇, and its properties can be entirely determined from those of g_{ab}, whence the typical omission of the symbol "D" from Eq. (4.1). In particular, the metric compatibility condition $\nabla_a g_{bc} = 0$ and the symmetry property $\Delta^a{}_{bc} = \Delta^a{}_{(bc)}$ for all indices, allows one to identically determine the connexion coefficients $\Gamma^a{}_{bc}$ as linear combinations of the first derivatives of the metric.[5]

[3] I.e. a family of pairs $\{(U_I, \chi_I)\}$, the *charts*, in which each U_I is an open set, and the union of the U_I covers M. The symbols χ_I denotes instead a C^∞-homeomorphism, for all running indices I, from U_I onto an open subset of $\mathbb{R}^n$ (in this sense, each χ_I is represented by n smooth functions χ^α_I, the *coordinate functions*, $\alpha = 1, 2, \ldots, n$). Also, given U_I, U_J such that their intersection is non-empty, the composite map $\chi_{IJ} := \chi_I \chi_J^{-1}$ is infinitely differentiable.

[4] The symbol $\mathfrak{X}(M)$ denotes the (algebra of the) vector fields defined on the manifold; $T^r_s M$ is used for the tensor fields of rank (r, s) on M; see for instance [119, 370, 523, 551].

[5] General Relativity, being a theory formulated in terms of geometric objects defined on a manifold, enjoys another property: any diffeomorphism ψ acting on M and push-forwarding the structures $g_{ab}, \mathrm{D}, \varUpsilon$ (with $\varUpsilon$ denoting all the other fields defined on M), gives an $\mathscr{M}'$ which is physically indistinguishable from $\mathscr{M}$. This is the so-called *Leibnitz equivalence*, and can be interpreted as a type of gauge freedom of spacetime theories [385, 552].

The special relativistic, ultra-local (i.e. point-wise) approximation of the spacetime $\mathscr{M}$ is Minkowski spacetime [470, 225], viz. the one where the curvature vanishes everywhere, hence $\mathscr{M}$ becomes

$$\mathscr{M}_{\mathrm{SR}} \equiv (\mathbb{R}^n, \eta_{ab}) \ , \tag{4.3}$$

with η_{ab} the flat Minkowski metric. In a system of pseudo-Cartesian, global coordinates x^α (see below, §4.2.2), η_{ab} reduces to a diagonal matrix with entries diag $(1, 1, 1, -1)$. Minkowski spacetime is itself a solution of Einstein's field equations for gravity (in the absence of the cosmological constant), rather than a separate entity defined per se, out of the gravitational framework.

The geometric description of spacetime allows to introduce a well-defined notion of *observer*. An observer O is a smooth curve $\mathscr{C}_O$ on $\mathscr{M}$ such that its tangent vector is everywhere timelike and future-pointing. The notion of *locally inertial reference frame* for the observer O is then obtained by erecting, at each point of $\mathscr{C}_O$, a quadruplet ${e_\alpha}^a$ of orthonormal axes (α is here the running index denoting the specific axis of the frame), in the sense that

$$g_{ab}\, {e_\alpha}^a {e_\beta}^b = \eta_{\alpha\beta} \ , \tag{4.4}$$

with $\eta_{\alpha\beta}$ the set of scalars arranged in the matrix diag $(1, 1, 1, -1)$. The axes mirror the physical structure of an ultra-local laboratory where the effects of gravity and curvature can be safely neglected. The four vectors ${e_\alpha}^a$ form the observer's *tetrad*, or *vierbein*, and, in reference to the coordinate functions y^α, can be thought of as vector-valued 1-forms ${e_\alpha}^a \mathrm{d}y^\alpha$.

Definition (4.1) can be slightly modified to encompass an even wider semantic range for the term "spacetime". For example, classical Newtonian mechanics can be described in an entirely geometrised environment, the *Newton–Cartan* spacetime [332, 299, 160], and in that case the analogue of $\mathscr{M}$ is represented by the structure

$$\mathscr{N} := \left(M, \tau_a, \gamma^{ab}, \nabla\right) \tag{4.5}$$

where M is a manifold, τ_a is a smooth 1-form, γ^{ab} is a C^∞, symmetric tensor field of signature $(1, 1, \ldots, 0)$ — the last two items are such that $\tau_b \gamma^{ab} = 0$ —, and ∇ is a derivative operator compatible with both τ_a and γ^{ab}, i.e. $\nabla_a \tau_b = \nabla_a \gamma^{bc} = 0$.

The covector τ_a is the "temporal metric" providing the measurements of time intervals, and this induces a proper temporal orientation for all possible directions ζ^a via the sign of the contraction $\tau_a \zeta^a$.

While all these definitions work in any number of dimensions greater or equal than two, observations, experiments, and theoretical arguments point towards the number *four* as the one best representing the dimensionality of the spacetime we have probed so far [522].

As a final remark, we observe that, from the physicist's point of view, there is another difference when it comes to the interpretation of the formal structures associated to the notion of spacetime. Broadly speaking, in a given $\mathscr{M} \equiv (M, g_{ab})$, one can separate a set of "pre-physical" quantities, such as dimension, topology, and differentiable structure, from the group of dynamical objects, such as the metric (or the tetrads), the curvature, all matter fields, and so forth. The former items are usually encoded into a suitable smooth manifold, acting as a background stage "on" which the fields live, whereas the latter terms are given a proper dynamical evolution via field equations.

Such separation is helpful, yet ought to be taken with a grain of salt: the topological arrangement of spacetime can greatly influence the physical properties therein [421], and in a sense so it does the number of dimensions, although neither concept has a dynamical character — at least in General Relativity.

4.1.2 The quantum world(s)

The smooth model of spacetime is an effective paradigm over many orders of magnitude, ranging from particle-physics levels to cosmological scales. What happens, however, when one decreases the size further (or increases the energy), is not clear yet.

Around the Planck scale, i.e. around 1.6×10^{-33} cm in length ($\ell_{\wp}$, i.e. $\sqrt{\hbar G/c^3}$),[6] or 12.4×10^{27} eV in energy ($E_{\wp}$, viz., $\sqrt{\hbar c^5/G}$), it is widely believed that quantum fluctuations and gravitational phenomena merge inextricably, with bizarre effects [409, 158]. The Compton wavelength of a "Planck particle" coincides with its Schwarzschild radius, and the particle can become energetic enough to probe the levels which would make it become a black hole itself. Any hope to measure space and time, or to confirm the smoothness of the spacetime manifold, would be spoiled by the mixed presence of quantum uncertainty and strong gravitational pulls.

Such conclusions make it reasonable to advance that, at Planck scales, our current notion of spacetime should radically change, requiring severe modifications of Eq. (4.1) [36, 331, 162, 389]. A semi-conservative standpoint tackles the problem by simply looking for a suitable discretisation of the gravitational field g_{ab}, as in quantum field theory (this time, however, the background is itself dynamical), with all the related paraphernalia of quantised eigenvalues, commutation relations, S-matrix expansion etc. Such approach naturally leads to an interpretation in terms of "quantum geometry", with the discrete eigenvalues of quantum operators mirroring some sort of "chunked" geometric objects recovering the pseudo-Riemannian spacetime in apt limits.

Among the many available proposals [401], we focus here on three main points of view, namely Causal Dynamical Triangulations, Loop Quantum Gravity, and Causal Sets Theory, as they can be seen as different paradigms for the possible emergence of the classical spacetime as we know it.

Spacetime from quantum geometry: Causal Dynamical Triangulations

Quantising the gravitational field is an ambitious, yet-uncompleted programme [151], which fights against formal subtleties, technical issues, and interpretational problems. To solve the riddle, one direction which has been explored consists in building a non-perturbative quantum field theory of gravity [20], provided that in this case there is no fixed background geometry where the fields live "on" (as is Minkowski spacetime for quantum electrodynamics, or quantum chromodynamics).

[6] The concept of "Planck length" is not Lorentz-invariant, hence $\ell_{\wp}$ is in a way ill-defined. The issue can be settled by introducing first a Lorentz-invariant notion, such as that of four-volume, and then extracting from it a length scale. This is the way the future instances of the term "Planck length" are intended in this work.

The paradigm is known as *Causal Dynamical Triangulations* [320, 19], and it starts from the gravitational path integral, i.e. a sum over all possible geometries with pseudo-Riemannian signature compatible with given boundary constraints [23],

$$\mathscr{Z}\left(G_{(n)}, \Lambda\right) := \int_{g_{ab}\in\{\mathtt{g}\}} \mathrm{e}^{\mathrm{i}\, S_{\mathrm{EH}}[g]}\, \mathfrak{D}g_{ab}\ , \tag{4.6}$$

where $G_{(n)}$ stands for Newton's constant in any dimension n, $\{\mathtt{g}\}$ denotes the class of Lorentzian metrics complying with the assigned boundary conditions, S_{EH} is the Einstein–Hilbert Lagrangian (3.52) equipped as well with the cosmological constant term Λ, and $\mathfrak{D}g_{ab}$ is a proper measure over the space of attainable geometries.

To evaluate $\mathscr{Z}$, one first performs a Wick-rotation of expression (4.6) — i.e. introduces the map $t \mapsto \mathrm{i}\,\tau$ — so that the resulting path integral is momentarily Euclidean. Then, the class $\{\mathtt{g}\}$ is regularised to a sub-class $\{\mathtt{g}'\}$ by considering all its possible representations in terms of conjoined, piecewise-flat manifolds, in a procedure borrowing from Regge's "skeleton calculus" [353]. The underlying idea somehow mirrors an exhaustion process in the integration of curved surfaces by means of flat facets approximating the actual surface. Notice that the "discretisation" of geometries is expected to occur at sub-Planckian level, so it is destined to never be observable [22]. As a further, fundamental hypothesis, one requires that the geometries under consideration be causally well-behaved [18], in the sense that spaces with branching causal structure, or singular light-cone structure, are rejected ab initio and confined to a set of measure zero. A notion of evolution of spatial leaves in a privileged time variable thus naturally emerges.

The piece-wise flat approximating structure (for which the simplest choice is to pick 4-simplexes [370]) is later refined by imposing that the edges of the simplicial complexes all have the same length, a, which acts as an ultraviolet cut-off. A similar regulator N is applied to the Euclidean volume element. The integration (4.6) is then performed directly on the regularised domain $\{\mathtt{g}'\}$, sidestepping the need for gauge freedom appearing in the continuum treatment. Divergencies of the path integral (4.6) due to the exponential growth of configurations can be tamed by introducing a bare cosmological constant, say $\aleph$,[7] whose run counter-balances the over-population of possible geometries.

The continuum dynamics of the metric field — and, hence, the familiar spacetime of the type (4.1) — is recovered when the two limits $a \to 0$ and $N \to \infty$ are taken, which amounts to having the microscopic building blocks disappear, shrunken to an infinitesimal size [320]. The problem arises, however, as to whether the continuum limit emerging in this way is truly a macroscopically extended, four-dimensional, pseudo-Riemannian spacetime as in Eq. (4.1). The answer is, in general, "no": even with the causal clause ruling out many pathological geometries, the theory admits a certain proliferation of "baby universes" in the spatial directions, with a final picture somewhat differing from the expected smooth manifold.

Even so, causal dynamical triangulations theory has achieved interesting results in reconciling the quantisation of the gravitational field and the macroscopic manifold model. Numerical Monte-Carlo simulations [320, 21, 22] per-

[7]Grateful acknowledgements $\mathscr{A}$rletta $\mathscr{N}$owodworska for kindly suggesting the *aleph* glyph "א" as a choice for the bare cosmological constant.

formed in four spacetime dimensions have recovered a few known geometries of cosmological interest, e.g. de Sitter or Friedmann–Lemaître–Robertson–Walker spacetimes.[8]

More specifically [21, 22], the picture of spacetime emerging from the Causal Dynamical Triangulations can be represented in terms of a bidimensional phase diagram where the axes are labelled respectively with the inverse bare cosmological constant $\aleph^{-1}$, and an asymmetry parameter δ encoding the dependence of the action on the lattice spacing a ($\delta = 0$ implies $a = 1$, and the larger δ, the finer the lattice spacing). In the plot, three distinct phases emerge, termed A, B, and C — it reminds a bit the diagram for the aggregation states of water, with a triple point separating the liquid/solid/gaseous states —. Of these phases, only C exhibits a well-behaved, spatially extended universe, correctly evolving dynamically in the time steps (the simulations point at a de Sitter geometry with constant scalar curvature R). Phase A, on the other hand, shows a proliferation of distinct, almost disconnected mini-universes, linked by tiny spacelike tubes, with size along the time direction close to that of the elementary simplex; this region is characterised by a series of merging and splitting events as the simulation progresses, and thus the geometry there is considered as "oscillating". Phase B, finally, shows a situation where only one spatial slice has a size large enough to overcome the cut-off threshold, and hence "time" is completely absent there, with the resulting, single universe spatially extended, yet "frozen" in the evolution parameter; in this sense, there is no classical geometry at all, nor traces of possible classicalisation in phase B.

The phase diagram outlined here has wide similarities with the one exhibited by the Hořava–Lifshitz model of classical gravity (see §1.3.1 and references therein), to the point that it has been suggested [21, 22] that Hořava gravity might be the actual classical-limit theory of Causal Dynamical Triangulations, instead of General Relativity in the ADM decomposition. This conjecture marries the non-perturbative quantisation program pursued by the causal triangulations, with the power-counting renormalisability of classical gravity guaranteed by the Lifshitzian model, with the two approaches supporting each other.

Spacetime from quantum operators: Loop Quantum Gravity

The grail of the quantum counterpart of the gravitational field (both at the kinematical and dynamical level) is pursued as well by the proposal known as *Loop Quantum Gravity* [468, 467, 463, 465]. In this case, attention is drawn onto the relational character of the metric structure, and on the consequences of the canonical quantisation; the emergence of a spacetime in the form (4.1), although clearly a goal of the paradigm, is a somewhat secondary task.

The idea, once again, is to construct a non-perturbative quantum theory of gravity from the ground up, and this requires a Hilbert space equipped with a Poisson algebra of operators, a set of space states, and the transition amplitudes yielding the dynamical content [463]. The Hilbert space is a certain sum over a set of abstract graphs (modulo an equivalence relation singling out redundant copies); it has combinatorial and separable character. The quantum operators are associated to quantities with the dimensions of an an area and a volume,

[8]Besides, an outcome of this programme is the finding that, at the Planckian level around 10^{-35} m, the spacetime structure undergoes dynamical dimensional reduction and effectively approximates a 2-dimensional fractal object [320].

whose discrete eigenvalues read, respectively [464],

$$A_\Sigma := \frac{8\pi \Im G\hbar}{c^3} \sum_{l\in\Sigma} \sqrt{j_l\,(j_l+1)} \quad , \quad V_R = \alpha\Im \left(\frac{G\hbar}{c^3}\right)^{3/2} \sum_{n\in R} V_n\,, \tag{4.7}$$

with Σ a collection of links of the graph, j_l a half-integer, R a region of spacetime, and V_n related to the "gravitational field operators" L^i_l, $i = 1, 2, 3$, the latter interpreted as the flux of a spatial vector triad across a surface pierced by the link l. Notice the presence, in both A_Σ and V_n, of the Planck length $\ell_\wp = \sqrt{G\hbar/c^3}$, respectively squared and elevated to the third power. While $\ell_\wp$ is not a Lorentz-invariant concept, here it emerges via the discrete eigenvalues associated to quantum operators with a geometric interpretation, thus recovering a sort of compatibility with the required Lorentz symmetry of the "loopy" approach [466].[9]

As it happens in Causal Dynamical Triangulations, the model assumes a foliation, although a locally Lorentz-invariant one [466], and the dynamics is that of spatial leaves evolving in a time coordinate. Finally, $\Im$ is a non-zero real number, the Immirzi parameter [273, 272, 42, 159], and α is a real number depending on the valence of the nodes considered. The space states of the theory, called spin network states, are a basis in the Hilbert space, formed by eigenvectors of the area and volume operators; they are characterised by three elements, namely a graph Γ, and the quantum numbers j_l, v_n, with v_n as emerging from the diagonalisation of V_n along an orthonormal basis of triads.

This abstract, group-based construction is reconciled with classical spacetime in the following sense: a spin network state is the representation of a "granular" space (dynamically evolving in time) in which each node of the graph Γ is a "seed", or "grain" of space, with volume given by v_n. Given any two such infinitesimal chunks of space, they are adjacent if they are connected by a link l; the latter pierces the "surface" between the two, which carries a quantum of area given by $A_l = \sqrt{j_l\,(j_l+1)}8\pi\Im\hbar G/c^3$. Hence, the Hilbert space given by the graph can be thought of as describing quantum space at a given moment in time, or, rather, as a "boundary state" providing the quantum space enclosing a finite region of a four-dimensional spacetime [463].

A slightly different explanation of the emergence of spacetime in Loop Quantum Gravity goes as follows [270]: the quantum superposition of spin network states (the latter represented by labelled graphs) gives the physical three-dimensional space, which is itself a dynamical entity, obeying the Wheeler–DeWitt equation. The spin representations at the vertices of the graphs give a measure of the "size" of the "atoms of space", whereas the representations on the edges provide the corresponding data for the "surface" of the "areas of the facets" connecting two adjacent chunks of space. The dynamical evolution of the spin network states, after a suitable combination of limiting and approximation processes have been put to use, gives back the usual spacetime of General Relativity, although described in terms of an (arbitrary) Arnowitt–Deser–Misner decomposition.

Notice at this point that nowhere in the model is explicitly set that the structures evolving in time are three-dimensional geometries; this conclusion

[9] There is, however, a residual possibility of violating the Lorentz symmetry also in the context of Loop Quantum Gravity; for a recent account see [206], and for an earlier analysis extended to the whole program of canonical quantum gravity see [207].

emerges instead from the fact that the relevant group is SU (2), and this — via Penrose's spin-geometry theorem [422] — ensures that the spin network states determine a three-dimensional geometry. By further massaging the equations, it is possible to prove that, in the given Hilbert space, one has an over-complete basis of wave packets which can be interpreted as classical geometries with evolving intrinsic and extrinsic curvatures.

Loop Quantum Gravity has been linked to many other formulations of quantum gravity, e.g. discrete General Relativity on a lattice with a boundary, the Ashtekar formulation of Einstein's theory in terms of tetrads and spin connexion [231, 468]. Also Causal Dynamical Triangulations can be harmonised with the "loopy" scenario; to bridge the gap, one has to assign a symplectic structure to the quantised simplicial complexes described above. The unit normal vectors to the facets of the causally evolving polyhedra can then be promoted to quantum operators, and this glues together the two pictures [467, 463].

If the kinematical picture appears quite settled, at least at the purely formal level, dynamics still posits serious issues in terms of actually calculating the observables of interest, and interpreting the available results. The dynamical content is expected to emerge from the transition amplitudes associated to boundary states, expressed as linear functionals on the Hilbert space [355, 76, 183, 182]. Such amplitudes should yield the probability of passing from one boundary state to another, i.e. the notion of a dynamical process. It seems possible to reconstruct the structure of classical General Relativity from the case of a simple vertex amplitude, via a process called *evaluation* of the $SL(2, \mathbb{C})$ spin network. Yet the programme is vastly under construction, and requires the accomplishment of many intermediate goals (Hamiltonian constraint, physical interpretation [462, 116], etc.).

Spacetime from causal ordering: the theory of Causal Sets

We conclude the section with a glance at the proposal of *Causal Sets Theory* [85, 165, 84, 256]. In this case, one starts from a purely abstract environment — the Causal Set — equipped with apt properties, and then tries to recover the usual spacetime structure via a coarse-grained, statistical procedure.[10]

A causal set $\mathcal{C}$ is a set of abstract elements ("points", whence the discreteness germane to the model) endowed with a relation of partial ordering satisfying the properties of reflexivity, anti-symmetry, and transitivity. Also, one assumes that $\mathcal{C}$ is locally finite, in the sense that each Alexandrov neigbourhood has finite cardinality [84, 165]. The relationship between pairs of points in $\mathcal{C}$ can be thought of as a relation between pairs of causally connected events on a spacetime of the type (4.1); in this sense, when thought of as a scheme to represent classical spacetimes, the theory introduces a very high level of non-locality, in the sense that two connected points on a causal set can correspond to locations on an $\mathscr{M}$ incredibly far from each other, and yet in causal contact [84, 456, 233].[11]

[10] Cau-Sets theory fits in the category of the "emergent gravity" models (where the geometrodynamical structure is the result of an apt limit); its fundamentally discrete standpoint has connections with some of the fully "quantum" proposals for gravity reviewed above. Still, Causal Sets theory has not been harmonised yet with quantum mechanics, nor with quantum field theory, and remains to date a reformulation of classical spacetime.

[11] Therefore, no comparison with other discrete models such as General Relativity on a lattice, or Causal Dynamical Triangulations is actually fair: the discreteness in these last models can be tracked back to the existence of a lattice spacing, which dictates both the type of non-continuity

The Causal Sets approach is based on the mathematical result [333] that a conformal isomorphism is the only one-to-one map between two spacetimes (both distinguishing past from future) which preserves the causal structure of the two metrics. From the point of view of causal relations, then all possible "causally reasonable" solutions of the field equations for gravity can be divided into equivalence classes. The conformal factor left unspecified by the isomorphism can then be determined by measurements of spacetime volume, hence from pure number counting. These results have been distilled into the motto of the framework, which reads

"Geometry equals Order plus Number."

The causal sets are, in fact, abstract entities; the problem arises, then, as to whether they actually represent a spacetime of the type (4.1). As yet, a general protocol to have a smooth spacetime (rather, a class of conformally isomorphic spacetimes) emerge from a causal set is not available. The problem has been then turned around, and formulated as the search of classes of causal sets approximating given pseudo-Riemannian manifolds, to determine the geometric properties of the latter from the characteristics of the former. There exists a well-defined notion of "faithful" approximation of a spacetime by a causal set [255]: one embeds a $\mathcal{C}$ in a given $\mathscr{M}$ and checks that: the partial ordering on the set mirrors causal relations on the spacetime; the distribution of points in $\mathcal{C}$ brought on $\mathscr{M}$, the *sprinkling*, is uniform (this is obtained by using a random Poisson distribution, which ensures that no preferred directions in spacetime can emerge after the spacetime is locally Lorentz-boosted [83, 166]); the smallest length scale present in $\mathscr{M}$ is no larger than the embedding scale — which is usually taken of the order of the Planck scale.

The goal of the model is to build up a path-integral formulation for Causal Sets theory as in Eq. (4.6), recovering the general relativistic case when the embedding scale goes to zero, i.e. in the continuum limit [163, 256]. The domain of integration should be given in this case by all possible causal sets compatible with a faithful embedding, or by all causal sets once the "pathological" ones are confined into a subset of measure zero.

Causal sets theory is a radical departure from the usual approach to spacetime, and is still an ongoing programme, with many technical and fundamental aspects waiting to be settled; still, it has offered so far promising results and insights [86, 457, 61, 62, 93]. For instance, when studying the dynamics of a scalar field on a given $\mathcal{C}$ faithful to a spacetime $\mathscr{M}$, one has to construct the discrete counterpart of the d'Alembertian operator "$\Box$", named "B" [232, 164]; the latter, however, yields a stochastic character, and thus makes sense only on a statistical average of realisations (i.e., sprinklings) of the causal set on the spacetime. To tame the exploding fluctuations of B at a point, which prevent its comparison with the corresponding continuous d'Alembertian, a non-locality scale is introduced as a cut-off; such scale ℓ_{nonloc} is different, and greater, than the embedding one, and introduces a new layer of phenomenology, where the dynamics of the field gets corrections even though the microscopic structure and the continuum limits are left unaltered.

and non-locality (via chains of progressively less close neighbourhoods).

4.1.3 The "mesoscopic" regime

The sub-Planckian world most likely demands the sacrifice of the large-scale model, but the way this transition from one regime to the other ought to occur is far from obvious.

If we assume for a moment that the change is not a drastic, abrupt "switch" occurring at Planck scale, then it is fair to expect that there will be at least a phase in which the pseudo-Riemannian arrangement is only *slightly* modified, with marginal — yet, measurable — corrections from the underlying quantum structure.

The scenario we must face becomes therefore the following. The pseudo-Riemannian model of spacetime works fine up to the new, "mesoscopic" scale ℓ_{meso}, where corrections and modifications of quantum nature cannot be neglected anymore, and gets definitely broken at the Planck scale $\ell_{\wp}$, where the strong couplings of gravity and quanta spoils the very notion of space and time [153].

This phenomenon is known and expected already in some quantum gravity proposals. The non-locality scale ℓ_{nonloc} demanded by the discrete d'Alembertian operator B in causal sets theory [232, 164] is precisely the type of mesoscopic scale one might look for to observe the emergence of tiny corrections to the ordinary laws of physics, or to the geometric picture. Let us suppose then for a moment that the transition at the Planck scale is not abrupt, and let us ask ourselves: what happens around ℓ_{meso}?

Seeing the problem from a different perspective, we can ask: what sorts of modifications can be imposed, in full generality, upon the pseudo-Riemannian manifold structure (or upon the laws of physics), to make it account for the onset of new, quantum-driven phenomenology?

An approach to the mesoscopic regime: Relative Locality

A recent proposal partly addressing the problem of the mesoscopic regime and of its consequences is known under the name Relative Locality [28, 27, 25]. The leading principles of the theory move from the observation that the measurements of fundamental, non-gravitational test physics we usually perform are, in most cases, point-wise coincidences of events in which the outcome is a measure of energy and direction. Besides, even measurements of length and duration (which themselves can be realised as point coincidences) are performed almost ubiquitously in a very limited range of energies, and in principle might acquire additional contributions as the energy ramps up [203].

In a way, the attention is thus shifted, from the usual local spacetime manifold with coordinates $x^\alpha \equiv (x, y, z, ct)$, to the momentum-based quadruplet $p_\alpha \equiv (p_1, p_2, p_3, E/c)$, which is then assumed as the fundamental entity. Physics is thus supposed to unfold, even at the classical level, on the momentum space $\mathscr{P}$, with the x^α's becoming themselves functions of the p_α's [27].

Relative Locality demands a full inversion of the logic behind Eq. (4.1): the base space $\mathscr{P}$ is represented by a manifold whose coordinates are the components of the four-momentum p_α, and the co-local inertial reference frame representing the observer is thus identified with the cotangent space $T_0^*\mathscr{P}$ in the origin of $\mathscr{P}$. The physical features of such modified version of the phase space is encoded in its property of being, in general, non-associative and non-commutative.

One can assign a metric and an affine structure on $\mathscr{P}$: the metric structure contains information about the dispersion relations of particles, and is in general represented by a rank-2, symmetric, contravariant tensor k^{ab} only at the leading order in a series expansion in terms of a scale, ℓ, which can be considered either as the Planck one, or a larger, mesoscopic one.[12] It results

$$\mathrm{d}s_p^2 := k^{ab}p_a p_b + \frac{1}{\ell}\Upsilon^{abc}p_a p_b p_c + \dots \tag{4.8}$$

In a similar fashion, the connexion is assumed to be generically non-symmetric (in some versions of the theory, the tensor algebra on $T^r_s\mathscr{P}$ is not only non-commutative [301], but even non-associative [27]), and at the leading order gets contributions from the Levi-Civita part ${\Gamma_a}^{bc}$, from the torsion ${T_a}^{bc}$, and from the non-metricity Q^{abc}, with further corrections related to ℓ as in Eq. (4.8).

In principle, the scale ℓ can be identified with the Planck one. If, however, one believes that the Planck regime demands a breakdown even of the Relative Locality framework — after all, no "atomicity" is expected to emerge in this scheme —, nothing forbids from advancing that the corrections shown above could be charged upon a larger scale ℓ_{meso}, sufficiently far from the Planckian threshold to consider the smooth manifold approximation reliable. As for the observable effects: the non-metricity provides a delay in the arrival time on Earth of two signals started simultaneously at the source point and carried by photons with different energies, whereas torsion accounts for a sort of birefringence effect in which the mentioned two photons, started along parallel directions, are detected along two directions with a non-vanishing angle [203].

Many conclusions emerging from the relative locality paradigm are counter-intuitive, and some results are still debated [26, 266, 265]; the knowledge of the effects on macroscopic objects, for instance, are far from being settled. On top of that, an exhaustive mathematical formalisation and physical interpretation for the model is still missing; the peculiar nature of the "momentum space" $\mathscr{P}$ and its characteristics still need be fully addressed.[13]

Gravity, geometry, fields, and relativity: our program

After this preparatory review, we can now set the stage for our argument. A point to stress is that what we want to frame is a statement working at the fundamental level — in the sense that it touches only founding hypotheses and pillars of the structure — and being as general as possible, so that the conclusion holds independently of the specific quantum gravity model, or effective description, adopted to accommodate potential effects of ℓ_{meso}

First, a remark on the "orders of magnitude" in the game. The expected mesoscopic scale needs be much greater than the Planck one. At the same time, ℓ_{meso} must be much smaller than any possible curvature radius associated with the gravitational fields generated by macroscopic objects, as the pseudo-Riemannian model is confirmed with high accuracy in that regime.

[12] Notice that, being the momentum space the original starting point to build up the geometric and tensor structures, all the indices need be reversed with respect to the usual placement.

[13] To give some ideas: the peculiar topology of $\mathscr{P}$, which seems to have a privileged point — the origin —; the ultimate fate of the symplectic structure of the phase space, which has been so far left aside; the reasons behind the selection of a non-associative structure over an associative one, and vice versa; the description of the observers (the tetrads exist on $\mathscr{P}$, or on $T^*_0\mathscr{P}$?); and so forth.

We conclude that, whatever it be, the mesoscopic regime must emerge in regions where, essentially, the spacetime manifold under consideration is the Minkowskian one for all practical purposes, and where the laws of physics are described by the framework of Special Relativity. Then, two possibilities arise: that the quantum-related modifications enter the game as changes in the behaviour of physical systems living on $\mathscr{M}_{\text{SR}}$; or, that it is the Minkowskian spacetime itself which has to be revised, and modified accordingly [153].

In what follows, we shall adopt the latter point of view. This leads to the idea of examining the pillars of Special Relativity (which lies at the basis of Minkowski spacetime), and trying to relax its founding axioms, to accommodate the possible effects of the propagation of Planckian physics to the mesoscopic regime. The possibility that the changes occur at the level of physical apparatuses and physical laws, on the other hand, is briefly explored at the very end of the Chapter.

4.2 Space and Time. Again

We have mentioned that Minkowski spacetime is a solution of the field equations for gravity in General Relativity (the cosmological constant must be set to zero identically), and so we could in principle start from the general, formal definition of spacetime given so far. Special Relativity, however, can be formulated in a different, yet equivalent, fashion, one more suitable to study its founding pillars from the point of view of someone looking for tiny, quantum-driven deviations from the ordinary structure. We adopt this second approach here, as it allows us to better underline the details of the problem, and of a possible solution.

We shall thus adopt an operational standpoint, as presented e.g. in [96, 453], and forget for a moment all the formal apparatus presented in §4.1, as we want to construct a model of space and time from the outset, and explore the resulting features.

The only two primitive notions we take for granted are those of *event* and *observer*. An event is any physical phenomenon occurring in a sufficiently small region of space, and lasting shortly enough, to be approximated by a "point-wise" happening. An observer is e.g. a small computer supposed capable of measuring intervals of proper time, sending and receiving signals, and time-stamping the events it, indeed, observes.

4.2.1 The operationalist standpoint

Talking about space and time, in many practical situations, means talking simply about duration and distance, and about clocks and rods (see the remarks in e.g. [96, 552, 553]).

It is because of Newton's approach to time (which he pictured as an ever-present, immaterial flow, constantly streaming from future to past, like an endless river) that we still believe that clocks run after some extra-sensorial, metaphysical entity, rather than chasing one another [468, 43]. A moment's reflection, however, allows us to see that, once two devices measuring time are assigned, they can be compared with each other, and this erases any link with the absolute

Newtonian abstract time. A similar argument based on the spatial contiguity of adjacent objects makes the idea of absolute space superfluous.

Many subtle points in the debate on the ontological status of space and time can be jumped over by adopting the "operationalist" point of view, and trading the abstract concepts of space and time (whatever the two words mean) for the notions of measurements of durations and distances [96].[14] The result is an "paradigmatic overturn", where e.g. time is what a clock measures, rather than the opposite.

The point to be kept in mind is that all the properties of the measured entities are in fact properties of the measuring apparatuses: the geometric features of distances, i.e. of "space", are nothing but reflections of the physical characteristics of the rulers determining those distances, and the same holds for "time" and the clocks.

A word of caution here. One has to be aware that the operational approach will never be "fundamental" in the sense of "pertaining to fundamental constituents", for a ruler is an extremely complex object where atomic, nuclear, and electromagnetic interactions occur. This implies that conclusions and results about nature supposed to have general validity might contain, well hidden inside, dynamical relationships germane to the specific measuring devices.

On the other hand, the operational method has the advantage that the interpretation comes for free, since the most fundamental quantities — as space and time — are precisely those which are measured, and the link between theory and experiment is therefore straightforward.

4.2.2 Observers; time and space

Let O be an Observer.[15] The universe of natural phenomena, which exists regardless the presence of the observer, can be described as set $\mathcal{M}$ (the *spacetime*) of all events an observer can possibly label. Notice that $\mathcal{M}$, differently from $\mathscr{M}$ of Eq. (4.1), is just a collection of physical events, and all its formal properties have yet to be defined.[16]

In fact, there are infinite observers: it suffices to postulate that any single event occurs in the presence of one (and only one) observer. In this sense, the intuitive notion of "points of space" is traded for the operational one of "infinitely many observers". Each observer O records the events by labelling them uniquely with the time at which they occur, where "time" here means the outcome of a reading of the observer's own clock (the proper time).

[14] Throughout this Chapter, we shall frequently refer to the monograph [96], where a neat and precise presentation of the operational construction of spacetime can be found. The Reader, however, ought to be aware that the author of [96] by no means advocates or supports unconditionally the operational approach, stressing instead the fully dynamical character of the measuring apparatuses used to determine the length and duration of spacetime intervals.

[15] Hat tip to Sebastiano Sonego, and to his unpublished *Notes on Classical Mechanics*.

[16] The classical spacetime $\mathscr{M}$ of Eq.(4.1) is not a collection of events, for events have physical meaning, whereas the coordinates on $\mathscr{M}$ have none. In the language of classical spacetime, the universe of the events we are building here is the *space of point-coincidences*, obtained as follows: rewrite all the physical fields defined on M in terms of a finite number m of scalars ψ_J, counting them with a running index J. Consider then the map Ψ defined by the ordered m-tuple $\Psi := (\psi_1, \ldots, \psi_m)$, with domain in M and values in $\mathbb{R}^m$. The space of point-coincidences $\mathcal{M}$ is then the sub-domain of $\mathbb{R}^m$ spanned by $\Psi(M)$, and is in general a manifold — yet, it is different from $\mathscr{M}$ [553].

One usually requires that the clock — any clock, for any observer — be arranged so that the time variable t it provides allows for the simplest description possible of the elementary laws of physics. In the words of John Archibald Wheeler, "time is defined so that motion looks simple" [353]. In a similar fashion, the intuitive notion of "space" (to be defined in a moment) is crafted so that the local geometric environment looks as simple as possible, i.e. isotropic and homogeneous.

Both the simplicity assumptions mentioned above about temporal and spatial variables concern essentially the form of the dynamical equations adopted for the non-gravitational interactions. The actual physical characteristics of space and time emerge then at the experimental level, confirming or disproving the adequacy of the simplifications advanced in first place, in a continuous feedback mechanism.

To relate events and measurements occurring in the presence of different observers, the first step is to set up a protocol to synchronise the clocks [453]. In this sense, we have first to assume that such synchronisation is possible — which is not granted; see e.g. [470] —. The details of the particular procedure are of no interest here; what matters is that any procedure be able to let the observers deploy a function t, given by

$$t : \mathcal{M} \mapsto \mathbb{R} \ , \tag{4.9}$$

such that the restriction of t to the history of any observer gives the proper time read off the clock by the observer Himself. The passage in Eq. (4.9) is legitimate only if we are postulating that the universe of the events form a *proper-time-synchronisable spacetime* [470]. The function t can be used to determine the time interval ("duration") between any pair of events, simply by subtraction.

Two events P, Q are hence simultaneous if and only if $t_P = t_Q$. It is then possible to define the *space of simultaneity*, Σ_{t_Q}, as the set of all the events simultaneous with respect to a given one Q occurring at a time t_Q.[17] In formulæ

$$\Sigma_{t_Q} := \{P \in \mathscr{U} \mid t_P = t_Q\} \ . \tag{4.10}$$

It results that the only meaningful definition of "space" in the operational approach is one determined by the observers, and no concept of space independent of time can exist in physics. The notion of "point of space" is essentially equal to that of "observer".

The observers can use the rulers to evaluate the physical properties of the spatial environment, and describe it accordingly.[18] The rulers, which have the role of units of length, are chosen such that the measured distance $d\,(P, Q)$ between pairs of points does not depend on time — i.e., the space is, in a sense, infinitely "rigid".

It follows that an effective formal description for the space in the proximity of any observer is provided by a three-dimensional topological space, equipped with an affine and an Euclidean metric structure. This reflects the experimental

[17]For an examination of the general issue of simultaneity in relativistic theories see e.g. [229].

[18]A possible substitute for the set of rulers is a "standard signal" with known velocity, which is a natural extension of the former notion, and the most suitable one for large-scale measurements, where the rulers would fail at remaining undeformed.

result that the local space is, with excellent approximation, homogenous and isotropic.[19]

One can thus erect, at any point of space, a positively-oriented Cartesian triad of axes $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ aligned along three non-coplanar directions of the rulers; on the axes, the observer in the origin can read the values of three coordinates, (x, y, z), respectively (the units of length on the three axes can be made to coincide without loss of generality) [459].

The notions of observer, time coordinate (reading on a clock), and spatial triads of axes (giving the spatial coordinates on each space of simultaneity) can be fused into the one of *reference frame*. A reference frame K is an ordered set $(O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}, t)$ where O can be identified with any point of space and any moment in some proper time. The three spatial axes are usually chosen to be orthogonal with respect to the Euclidean metric on Σ_t for all t's, even though this is not a compulsory step.

4.2.3 Reference frames; relative motion

The reference frame K defined above is not unique: two such reference frames differing from a spatial rotation of the axes, or by a uniform translation in space and/or in time, are physically equivalent, where "physically equivalent" has here an operational characterisation as well.

The kind of idea one ought to have in mind is the following. Suppose to have a catalogue of possible physical experiments (kinematical, dynamical, electromagnetic, thermodynamical, etc.); the protocol to build and perform each experiment is clearly and unambiguously formulated, item by item. Assume then to have two reference frames, where the same experiment from the catalogue is performed. The two frames are then said to be *physically equivalent* if the results of the two experiments are the same.

This allows to conclude that there is indeed an equivalence class of reference frames $\{K\}$, whose physically equivalent elements are related by spatial rotations and spacetime translations.

We can sum up the result in the form of a Postulate, which also contains the first seed of the "principle of relativity", i.e. the physical equivalence within a given class of reference frames. It reads [153]

Postulate "$\mathcal{A}$". *There exist reference frames constituted by observers, clocks, rulers (or units for length), and synchronisation procedures, such that the distance between two arbitrary observers — or points — does not depend on time, and such that the resulting local spatial geometry is Euclidean. Two reference frames related by a rotation of the spatial axes, or by a spatial/temporal translation, are physically equivalent.*

Among all reference frames, those in which the motion of an isolated particle is both rectilinear and uniform are called *inertial*.

[19] Although seemingly general, this construction indeed pertains to quite a restricted scale, where the role of the gravitational field can be safely neglected. An observer might decide to replicate this construction, identifying however the spatial points with e.g. galaxies. His conclusions about the geometry of space would be then much different, with the emergence of a "local" environment changing over time.

Consider now a reference frame K; let K' be another reference frame, in rectilinear, uniform motion with respect to K, with velocity $\boldsymbol{v}$. One can compare the outcome of kinematical and mechanical experiments in both reference frames.

One finds that, upon performing experiments in both K and K', the same results emerge, provided that the initial and boundary conditions are transformed appropriately. Notice that the kind of experiments mentioned here do not involve only measurements of durations or distances (geometrical or kinematical tests), but involve various dynamical phenomena, all confirming the physical equivalence of the two inertial frames. Such conclusion allows to state that the class of reference frames in Postulate $\mathcal{A}$ also includes all the reference frames of the type K'.

This conclusion is logically separate from the ones concerning the frames who differ from spatial rotations and spacetime translations, and can be given the status of a separate Postulate, namely [153]

Postulate "$\mathcal{B}$". *It is inertial any reference frame K' moving with translatory, uniform, rectilinear motion with respect to an inertial reference frame K.*

4.2.4 Hypotheses behind Lorentz transformations

Consider now two arbitrary inertial reference frames, K and K'; call $\boldsymbol{v}$ the velocity of the second frame as measured in the first one.

In general, since the fundamental quantities to be attached to any event are its coordinates x, y, z, t, we can expect that physical laws will be generally expressed as relationships of the form $\psi = \psi(x, y, z, t)$, where ψ is some observable. Notice that the expression $\psi(x, y, z, t)$ does not represent the value assumed by ψ at a certain point in space and moment in time; rather, it gives the coincidence of the outcome of measuring ψ, and reading off the values of the time and the distances on the observer's clock and rulers.

It is then fair to ask what is the most general transformation of the coordinates x, y, z, t between inertial frames which complies with postulates $\mathcal{A}$–$\mathcal{B}$. I.e., we look for the specific form of the maps

$$x'^{\alpha} = f^{\alpha}(x, y, z, t; \boldsymbol{v}) \ , \tag{4.11}$$

for $\alpha = 1, 2, 3, 4$.[20] Indeed, there are various ways to get the result [96, 198, 360], and the one relying on the smallest set of assumptions is the derivation from first principles due to von Ignatowski [271, 495, 96, 317]. In von Ignatowski's construction, the transformations can be derived from the following set of hypotheses:

1. Spatial and temporal homogeneity — viz., the equivalence of all positions in space, and moments in time.

2. Spatial isotropy — i.e., the equivalence of all possible directions in space.

3. Principle of relativity — i.e., absence of a preferred frame.

[20] We are using sets of mutually orthogonal spatial axes in both the inertial reference frames, even though the triples $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ are in general not aligned with each other.

4. Pre-causality — viz., irreversibility of the time ordering of two events in the passage between one frame and another.

The hypothesis of homogeneity in space and time assures that the transformations (4.11) be linear in their arguments, for the time and spatial intervals $\Delta x', \Delta y', \Delta z', \Delta t'$ measured in K' can only depend on the corresponding intervals measured in K, and not on the coordinates. Such result heavily relies on the operational interpretation of the coordinates.

Spatial isotropy is used, in combination with the principle of relativity, to prove the principle of *reciprocity* [73], i.e. the fact that the velocity $\boldsymbol{v}'$ of K in the frame K' is $-\boldsymbol{v}$. On the purely computational side, spatial isotropy is also deployed to simplify the mutual orientation of the two frames, and work with only two variables, namely t and one spatial coordinate, say x.

The relativity principle demands that the maps f^{α}'s in Eq. (4.11) form a group. Any violation of such rule would imply the kinematical non-equivalence of inertial frames.[21]

Finally, the pre-causality hypothesis further restricts the shape of the transformations, by demanding that the temporal order of events be conserved when passing from one inertial reference frame to another — viz., $\partial t'/\partial t > 0$.

By massaging the equations, and restoring an arbitrary mutual orientation of the axes for K and K', the coordinate transformations boil down to [360, 35, 96, 228]

$$\begin{cases} x'^i = R^i_j x^j - v^i \left(\dfrac{\gamma - 1}{v^2} x^k v_k - \gamma t \right) \\ t' = \gamma \left(t - \dfrac{x^k v_k}{C^2} \right) \end{cases} , \tag{4.12}$$

where $i, j, k \in \{1, 2, 3\}$, $\gamma := \left(1 - v^2/C^2\right)^{-1/2}$, R^i_j denotes the entries of an orthogonal (proper rotation) matrix with constant and velocity-independent coefficients, and C stands for an *invariant velocity* [99], i.e. one whose value remains the same in any inertial reference frame (the numerical value of C, however, is unspecified, and can be any real value, even infinite).

Eq. (4.12) yields the sought-for coordinate transformations between inertial frames, viz. the *Lorentz transformations*. These need be complemented by the translations in both space and time, which are a direct consequence of the local homogeneity of both space and time. As long as gravitational phenomena and spacetime curvature are not involved, uniform translations are admissible spacetime transformations leaving the fundamental, non-gravitational test physics untouched, and must hence be included. The resulting symmetry group is the Poincaré group [495].

Experimentally, one finds that C equals, with high accuracy, the value of the speed of light in vacuo, c, hence light signals travel at an invariant speed — and can thus be used e.g. in one of the many allowed synchronisation procedures for the clocks —. The limit $C \to \infty$ gives the Galilean transformation of classical,

[21] In fact, the transformations between inertial frames at the kinematical level are verified or disproved by measurements performed by clocks and rods — we are still playing the game according to the operationalist's rules — and those measurements involve for sure the behaviour of complex, dynamical objects. This means that the kinematical equivalence or non-equivalence of inertial reference frames is tested already at the dynamical level.

pre-relativistic mechanics.[22]

Before moving on, a final remark. From the argument above, we see that any statement or quantity calling itself Lorentz-invariant or Lorentz-symmetric must comply with *all four* hypotheses 1.–4. One often finds sources in which "Lorentz symmetry" simply refers to the principle of relativity, without any reference (or, with implicit reference) to the other conditions [469, 466, 83]. While it is true that the relativity principle is a necessary condition to be enforced by a Lorentz-invariant setting, it is not the only one.

4.3 Mesoscopic effects & Lorentzian structure

We can now look for possible "mesoscopic" effects of a quantum gravitational infrastructure hidden below the veil of our observable universe. Our idea can be stated as follows.

If the quantum seeds expected to shatter the fabric of classical spacetime affect so deeply the basic axioms of pre-general relativistic physics, then their percolation to a mesoscopic, observable level cannot but be translated into a modification in the founding pillars and axioms of classical spacetime itself. Therefore, we can review what happens if we start shaking, one at a time, the set of postulates contained in the previous section.

We do not look for specific new effects or novel phenomenology: our research is a matter of principle(s). What we want to identify is the precise step, in the logical path to Lorentz transformations, where the traditional protocol jams, and a window opens to the onset of quantum-driven corrections.

4.3.1 Tinkering with the pillars

The analysis we plan to pursue moves from the set of postulates $\mathcal{A}$–$\mathcal{B}$ and hypotheses 1.–4. Since the former are much more general, we proceed in reverse order, and work first with the latter, studying the effects of relaxing the hypotheses.

In the four assumptions underlying Lorentz transformations, pre-causality appears to be the most robust: relaxing it would seriously prevent any actual physical investigation. Therefore, we leave it as it stands.

The principle of relativity is a perfect candidate to be abandoned, probably the easiest to drop out of the series, and the most commonly attacked in the literature [341, 317, 39, 502, 503, 548]. A common trend is to start with a Lorentz-symmetric theory, for instance the Standard Model of particle physics,

[22] To be precise, the Galilean *regime* of the Lorentz transformations actually requires *four* conditions to be realised [96, 261]. One is the slow-motion condition $v \ll c$ (which has a more physical significance than the limit $C \to \infty$, as the latter indeed clashes against the finite value measured for the speed of light in vacuo); then, the condition $\mathrm{d}x/\mathrm{d}ct \ll 1$, i.e., that only large time intervals are involved with respect to the spatial ones. Also, one has to require that the spatial gradients overwhelm the temporal derivatives, $\nabla^i \gg \left(v^i/c^2\right) \partial/\partial t$ for all $i = 1, 2, 3$, and, finally, that bodies all move at non-relativistic speeds, such that the velocity composition law reduces to $\dot{x}'^i \sim \dot{x}^i - v^i$. If one only sticks to the first two conditions [312], and reverses the second, i.e. considers the slow-motion regime of large *spatial* intervals, another, viable group structure emerges, namely Lévy-Leblond's *Carroll transformations*, named after the author of *Alice in Wonderland* and *Through the Looking-Glass* [312, 168]. Carroll's group, although of somehow littler interest, has recently provided nice contributions in relativistic electrodynamics [168], and remains an exquisite tiny deviation of special relativistic physics.

write down all the operators compatible with the remaining symmetries but the equivalence of inertial reference frames, and evaluate the resulting phenomenology [317]. The expected effects can be probed using both particle-inspired tests, and astrophysical sources acting as very-high-energy laboratories (supernovæ, blazars, pulsars, active galactic nuclei and gamma ray bursts).

A violation of the frame-independence of physical laws generates, for instance, shifts in the thresholds for elementary particles reactions — pion productions from protons — and even permits reactions otherwise forbidden (vacuum Cherenkov effect [32], photon decay [283]). Anomalous decays such as helicity flip or photon splitting [260] become possible. Since the violation also affects the possible maximum velocity of matter and light, it can be found in the peak of emission from supernova remnants, or Gamma-Ray Bursts [326, 511]. Other effects encompass vacuum birefringence [6] and time delays on long-baseline distances (whence the tests on far astrophysical sources).

To date, quite tight experimental constraints have been cast upon possible violations of the relativity principle, from both the particle-physics side, and the astrophysical direction. In a rotational-invariant minimal extension of the Standard Model, such violations are compatible with the null hypothesis, even though the outcome vastly depends on the sought-for order of the violation, and on the specific sector of the Standard Model where they are investigated [317]. The strengths of the constraints peak at best somewhere between 10^{-6} (from observations of protons in cosmic ray for an order-4 violation, in a neutrino-flavour independent scenario) and 10^{-20} (for order-2 violations, again in protons from cosmic rays) [317]. Results of order 10^{-16} can emerge from observations of photons from Gamma-Ray Bursts, or positron-electron pairs (both for order-3 violations in a neutrino-flavour independent setting) [317].

The relaxation of the principle of relativity has also been examined, from a group-theory perspective, within the programme of Very Special Relativity theory [127]. There, the key role is played by specific sub-symmetries of the Poincaré and Lorentz groups, with the onset of a preferred, fixed "æther" direction, the *spurion*.[23] Also in this case, the possible violations are tightly constrained by experiments and observations [138].

A third hypothesis to play with is spatial isotropy, whence the emergence of a preferred direction in space, rather than one in spacetime. While the catalogue of available proposals in this sense is certainly less ample than that embracing violations of the relativity principle, some conclusions can be drawn [495, 305, 239]. By limiting the analysis of an isotropy breakdown at the kinematical level [495], one finds out that

* Anisotropic kinematics is consistent and theoretically admissible. It can be made emerge from a slight modification of the general proof to find the Lorentz transformations.

* It is fully compatible with the absence of a preferred frame in spacetime, thus abides by the principle of relativity. This is reflected by the fact that the anisotropic transformations still form a group.

* In a geometric interpretation based on $\mathscr{M}_{\mathrm{SR}}$ of Eq. (4.3), it is compatible

[23]For a geometric reappraisal of Very Special Relativity, see e.g. Ref. [218].

with the trade of the pseudo-Riemannian structure for a pseudo-Finslerian one [41, 114, 495].[24]

The analyses show that part of the problem with anisotropic kinematics can be traced back to the protocol for clock synchronisation (and can thus be erased by an apt choice of the method), hence has a purely conventional character, whereas there is a residual real anisotropy which cannot be gauged away, and results in measurable effects.[25]

Finally, we can try to relax homogeneity [153]. It turns out, however, that this is a much more delicate pillar to play with, for homogeneity intervenes at two different stages in the proof of the transformations (4.12). At the fundamental level, and even before the beginning of the proof itself, it enforces the operational meaning of the coordinates x^{α} in any inertial reference frame; this assures that, in both frames K and K' considered above, the difference $\Delta t := t_Q - t_P$ [respectively, $\Delta t' := t'_Q - t'_P$] between the time coordinates of two events P, Q will be interpreted as the (time) interval between the events, as recorded by a clock attached to the origin of the reference frame; likewise, the difference $\Delta x := x_Q - x_P$ [resp., $\Delta x' := x'_Q - x'_P$] will be interpreted as the distance, at a given time in K [resp., in K'], between the two given points.

Subsequently, homogeneity constraints the maps f^{α}'s to be linear in their arguments. To see this, consider two events $P \equiv (t, x)$ and $Q \equiv (t + \Delta t, x)$ as measured at a constant spatial position x in K (without loss of generality, we can use here only the pair of coordinates x, t); the duration $\Delta t = (t + \Delta t) - t$ between the two is mapped, via Eq. (4.11), into

$$f\left(x, t_A + \Delta t; v\right) - f\left(x, t_A; v\right) = F\left(t, \Delta t, x; v\right) \ , \tag{4.13}$$

with F a general function. At the same time, the transformed difference on the left will still be a time interval $\Delta t'$ as measured in K' — because of the operational definition of the coordinates in the class of inertial frames, holding in view of Postulates $\mathcal{A}, \mathcal{B}$ — and hence $\Delta t'$ cannot depend on where and when the interval is measured, again by the homogeneity assumption. Therefore, F can only be a function of T and v, and dimensional considerations require the function F to give

$$\Delta t' = F\left(v\right) \Delta t \ . \tag{4.14}$$

It follows, finally

$$t' = F\left(v\right) t + H\left(x; v\right) \tag{4.15}$$

with H an arbitrary function — which, however, can be proven to be itself homogeneous in the x-variable by an analogue argument for the the spatial measurements. This concludes the sketch of the proof.

Now, if homogeneity goes missing, the mentioned differences $\Delta t, \Delta x$ loose any relationship with actual durations and distances. Even worse, if Postulate $\mathcal{B}$ holds, i.e. if the principle of relativity is adopted, the same loss of meaning occurring in a reference K holds for any other inertial observer, which means that

[24]I.e., it yields a structure $\mathscr{M}_{\text{Finsler}} \equiv (M, k_{ab})$ in which the metric $k_{ab}\left(x^{\alpha}, v^{a}\right)$ depends not only on the coordinates, but also on the velocities. In other words, k_{ab} is a rank-2, covariant, symmetric tensor defined on the tangent bundle TM, rather than on M alone.

[25]This can be easily seen in a $(1+1)$-setting, where anisotropy in space is reduced to the non-equivalence of the positive and negative directions along the spatial line [495].

the pseudo-Cartesian coordinates in any inertial reference frame are deprived of their operational interpretation.

This fact, however, clashes with assumption $\mathcal{A}$, i.e., it conflicts with one of the pillars of local physics established, for the class of inertial observers long before the Lorentz transformations were considered [153]. Stated otherwise: relaxing the homogeneity hypothesis affects the innermost structure of classical spacetime, and demands radical departures from the early roadmap used to construct the notion of spacetime itself.

4.3.2 A "no-go argument"

What can be noticed by going upstream through the set of hypotheses behind Lorentz transformations, is that the closer we get to the very roots of the construction of spacetime, the more "rigid" the structure becomes. This is in a way expected, for the operationalist's standpoint precisely begins with supposedly robust foundations, and then applies successive layers of looser and looser ends.

At the same time, the rigidity of the homogeneity assumption prevents the onset not only of any departure of the spacetime structure from the Minkowskian one, but also any minimal variation e.g. of the linearity of the coordinate transformations. Where, then, is any room left to accommodate the existence of a mesoscopic regime? Our answer goes as follows [153]:

If spacetime obeys the complete set of axioms and hypotheses $\mathcal{A}$–$\mathcal{B}$ and 1. to 4., then it must have a full, ordinary Minkowskian structure, and calling the regime "mesoscopic" makes little sense, for no detectable differences emerge at the scale $\ell_{\rm meso}$, provided that the structure of the field equations and of the equations of motion do not get any change when approaching the Planck scale..

If one is willing to relax some of the assumptions, the two hypotheses most reasonable to be changed in the context of coordinate transformations are the absence of a preferred frame, and spatial isotropy. In either case, tight constraints exist on the observable effects.

Very few possibilities remain, then, and all point at a change in postulate $\mathcal{A}$. This, however, implies severe modifications to the innermost character of the reference frames, rather than "mild" changes such as those coming from e.g. allowing for spatial anisotropies [153].

Our "no-go" argument can also be rephrased in terms of physical laws, and their symmetries [153].

By a rigorous application of the full set of statements $\mathcal{A}$–$\mathcal{B}$, plus 1. to 4., all the resulting fundamental laws of physics are expected to be strictly Poincaré-invariant, at any scale up to $\ell_\wp$, and no deviation from the spacetime structure of standard Special Relativity is forecasted.

When a modification of some of the postulates is allowed around $\ell_{\rm meso}$, the fundamental laws of physics will become no longer Poincaré-invariant. In this context, whenever spatial isotropy and the absence of a preferred frame are kept holding, the sorts of expected modifications might be tightly constrained, even though not necessarily unphysical [153].

Still, we have to stress that the request of exact invariance under Poincaré maps does not prevent the onset of new phenomenology around the scale $\ell_{\rm meso}$, for such invariance is only part of the postulates and axioms governing our description of the local environment.

4.3.3 Results, and some speculations

To conclude: if the spacetime is fully Minkowskian at mesoscopic scales as well, than any onset of new phenomenology is postponed down to the microscopic, Planck scale only, and its emergence must be abrupt, as it happens in a second-order phase transition. In the same fashion, one can say that also the emergence of Poincaré-violating physical laws can only occur from $\ell_\wp$ downwards.

If, on the other hand, the modifications proposed pertain to the content of postulate $\mathcal{A}$, the current interpretation of spacetime gets deeply revolutionised, and in principle new phenomenology might emerge, via many mechanisms. Yet, calling this regime "mesoscopic" would once again be a misnomer, for large-scale effects would be generically expected to manifest.

Finally, imposing exact Poincaré invariance for the local laws of physics constrains the form of the maps (4.11), but does not rule out a priori the possible emergence of new, Poincaré-symmetric terms (suppressed by powers of the scale ℓ_{meso}) accounting for new phenomenology.

The intrinsic rigidity of the Minkowskian structure can still be reconciled with the onset of a mesoscopic scale if the latter affects the structure of the physical laws and their field equations — yet, in a Lorentz-invariant way —. In this sense, non-locality may play an important role.

A word of caution here: in the context of this analysis, the expression *non locality* has quite a specific connotation, different from the "mainstream" definition related e.g. to Bell's theorem, the Einstein–Podolski–Rosen paradox, and similar statements. We refer here mainly to the non-local contributions forecasted in some quantum-gravity scenarios, such as the "de-coherence scale" needed in Causal Sets to tame the divergencies of the discrete D'Alembertian operator B.[26]

Similar types of non-locality effects — arising in a sense at the semiclassical level — are a built-in feature of many physical pictures, from non-commutative quantum field theory to extensions of the Standard Model of particles, spanning also string theory and Loop Quantum Gravity [205], and this may suggest that the intrinsic fuzziness of the microscopic sub-Planckian regime may propagate up to an observable scale precisely through non-local corrections.

A sort of non-locality is also present in the relative locality paradigm, where the locality condition is indeed "relative" in the sense that only quantities and measurements performed in the close proximity of an observer ("co-local") are truly well-defined; anything occurring far from an assigned origin of a reference frame gets "smeared" or "blurred" in a fuzzy region where no point-wise coincidences of events are anymore distinguishable — the size of the region turns out to be proportional to a corresponding volume in the phase space [27, 203].

These conclusions hint at the possibility that the onset of non-locality be a foreseeable consequence, at the mesoscopic level, of the sub-Planckian breakdown of the classical spacetime model, notwithstanding the fact that none of

[26] Kudos to D. T. Benincasa and A. Belenchia for some enlightening explanations on this topic.

such frameworks strictly violates the relativity principle, nor the spatial isotropy, nor in a sense spacetime homogeneity (the coarse-grained limit fully recovers the smooth structure $\mathscr{M}$). What could then be expected is the emergence of non-local, yet possibly Lorentz-invariant contributions to e.g. the dynamical evolution of the physical fields, or their interactions. The already vast catalogue of effects, ranging from modified dispersion relations to superluminal propagation of modes, might thus be enriched by the emergence of non-locality correction e.g. to the propagator of the fields, suppressed by apt scales restoring the already-tested large-scale regime.

A different direction where to look at concerns instead the physical meaning of coordinates. Back to our first operational definitions, consider two clocks based on some exponential decay (of the same substance, and in the best possible equivalence as per construction and calibration), both measuring durations, and, hence, time. Suppose now to increase the energy of just one of the two clocks, and follow the evolution of the phenomenon used to define the timepiece as the energy ramps up.[27] It is not obvious that the readings of the two clocks should entail the same results in terms of resolution, when the energy regime in which they are operated varies significantly. Nor is assured that any relationship or regularity could be found, between the resolutions at different energy levels.

But then the question arises [95]: which of the two apparatuses is measuring the "right" proper time? In the operational approach, the answer is "both", for no specification about the energetic environment of the clock was made when the object was selected to account for the measurements of durations. At the same time, now we have two different time variables, uncorrelated, and impossible to synchronise.[28]

Another aspect involves the breakdown of the operational interpretation of coordinates: suppose indeed that the x^α's do not account anymore for the measurements of lengths and time intervals. Then, in the absence of independent detections, all notions such as velocity, acceleration, momentum etc. become at once ill-defined, whereupon any hope to recover even the simplest kinematical laws collapses.

At the same time, if an independent definition of e.g. velocity (or momentum) is permitted, then the space of degrees of freedom gets inevitably enlarged, with the v^α's now becoming separate variables with a physical significance — the p_α's are another legitimate choice —. In this last scenario, a possible consequence is the need for a suitably extended geometric structure on which to describe the unfolding of physical phenomena; the relative locality proposal briefly outlined above suggests to look at a space parametrised by pairs (x^α, p_β), but alternatives exist [24] in which the manifold covered by coordinates $\left(x^\alpha, v^\beta\right)$ is the aptest direction where to look at.

These "skeletons of examples" tell us two things: first, as already remarked, that a hidden danger of the operational approach is to adopt complex objects,

[27] Notice that such proposal does not affect the functioning itself of the clock; rather, it is the *resolution* of the timepiece which is at stake in this experiment.

[28] A similar problem arises when two identical clocks are placed in a gravitational field: one is kept at a fixed height on the surface of Earth, while the other is abandoned in free fall. Which one is measuring the "true" time? The answer in this case is "neither", as time is just a manifestation of the gravitational field, and it is only the dynamics of the latter which actually matters [468].

with yet-unspecified dynamics, to represent the fundamental kinematical variables; second, that the importance of the notion of coordinates might really be over-estimated, even in a special-relativistic context, and that in principle we ought not to be afraid of the possibility to adopt a different language, where only dynamical physical fields exist, and their relational structure gives meaningful observables.

Acknowledging that coordinates are indeed fields, or at least a complex outcome of the interaction of many physical fields, would render much less traumatic the relaxing of the homogeneity hypothesis, for the latter would become just a manifestation of some local, and large-scale, configuration of fields whose dynamics at small scales — mesoscopic or trans-Planckian — remains mostly unknown. Stated otherwise, the rigidity of the Minkowskian framework would turn out as the outcome of a particular realisation of a field configuration in a precise window of energies and other observables, rather than an immovable, infinitely extendible property constraining the entire realm of physical laws.

Chapter 5

Upshot / Outlook

> Though the truth may vary, this ship will carry our bodies safe to shore.
>
> Of Monsters and Men, *Little Talks*.

The "story of a free fall" comes to an end.

As the farewell approaches, we dedicate a few sentences to sum up the achieved results, at the same time encapsulating our findings in in a broader perspective. After that, we expand a bit on some possible directions for future explorations.

5.1 A bird's eye view at the achievements

This thesis has focussed on the foundations of gravity theories from two, almost poles-apart standpoints: the "macroscopic" one (equivalence principles, almost-geodesic motion of self-gravitating test bodies, and related formal selection rules), and the "microscopic" one (near-Planckian regimes, and axiomatisation of local spacetime).

The main findings tend to support the received paradigm of General Relativity (and its special relativistic, ultra-local limit); still, other conclusions have emerged, consensed as follows.

5.1.1 Equivalence principles, and conjectures

The Equivalence Principles lying at the foundations of gravity theories have proven to be effective and sharp tools to establish viable models for gravitational phenomena, notwithstanding the inevitable limits of their formulation — and implementation in actual experimental settings.

Sometimes, such principles are regarded as outdated traces of a long-gone past, when the theory of gravity (and the theory of the theories thereof) had just entered its early childhood. The end of their "career", however, might still be far ahead. In fact, these statements still work effectively nowadays, and provide formal descriptions of significant portions of the phenomenology, or sieves to rule out branches of the "family tree".

The importance of their selective role has even increased: the remarkable growth of the landscape of competing theoretical frameworks requires a finer taxonomy, and the Equivalence Principles come handy when the interwoven families and sub-groups of theories are to be distinguished.

We have provided an exploration of main aspects — and a few subtleties — of these statements, putting the network of mutual relationships under the lens. The puzzle, however, is still missing a closing keystone. The Strong Equivalence Principle lacks a proper formal counterpart, and in some sense even a concrete example of its possible experimental validation, besides those already covered by the Gravitational Weak and the Gravitational Newton's principles.

5.1.2 Gravitational Weak Equivalence Principle, and its tests

The version of the Gravitational Weak Equivalence Principle presented in this thesis has been designed to be an extension of the Galilean free fall for bodies exhibiting some self-gravitational content. In its formulation, the key ingredients of the geodesic motion become the cornerstones of the resulting selection rule, whereas some details of the physical systems under consideration are framed via suitable approximations — and hence neglected — according to a simplifying standpoint.

The sieve thus obtained rules out whichever theory admits non-metric, dynamical gravitational degrees of freedom, and/or any situation in which the background stress-energy-momentum distribution does not vanish identically. While the latter condition can be imposed as an additional hypothesis for many theories of gravity (it is "environmental"), the former requirement actively filters the landscape of frameworks.

To spot the presence of non-metric gravitational degrees of freedom, one can look directly at the action of a theory, provided that the latter is written in such a way that the variational problem for the action itself is well-posed. This last condition makes the actual dynamical variables emerge, as it is on them that one imposes the boundary conditions to ensure the well-posedness of the variational problem, later extracting the field equations.

In the sample of theories examined (purely dynamical, Lagrangian-based, metric schemes of gravity), only General Relativity — also in the presence of a cosmological constant — and Lanczos–Lovelock theories pass through the sieve, as they are the only purely metric theories complying with the other requirements.[1] In this sense, our findings confirm the results usually attributed to the Strong Equivalence Principle (which aims at singling out General Relativity only, among the experimentally-verified theories in four spacetime dimensions),

[1]A word at this point on Nordström's gravity. Although ruled out by experiments and observations, the model remains theoretically viable. And in fact it passes the PPN-based tests for the Strong Equivalence Principle; its agreement with the Gravitational Weak and Gravitational Newton's principle is also supported by independent arguments based on Katz super-pontentials. While the type of sieve developed in this work cannot be applied to Nordström's gravity (the field equations need be in tensor form), the underlying necessary and sufficient condition for the Gravitational Weak Equivalence Principle to hold, Eq. (3.11), might be true for this scalar theory. A full-fledged examination of the case is under development.

with the emergence of the higher-dimensional Lanczos–Lovelock models as further candidates validating the statement.

In its present form, the selection rule based on the Gravitational Weak Equivalence Principle covers a significant portion of the family tree of extended theories of gravity, but attempts are ongoing, to expand the formulation even more, making it embrace larger areas of the landscape.

5.1.3 Classical spacetime structure, and beyond

Zooming from the macroscopic picture into the quantum region, we have also explored how the different notions of classical spacetime react to the presence of a "mesoscopic" regime disclosing the critical threshold at the Planck scale.

When one adopts the smooth manifold paradigm, the very existence of the mesoscopic scale is excluded ab initio. The validity of the continuous geometry is prolonged at any level and scale. When the operative standpoint is assumed, instead, it is the founding postulates at the very base of the construction which limit the possible onset of a mesoscopic regime.

In particular, the homogeneity of space and time permitting an operative interpretation of the coordinates prevents the emergence of near-Planckian modifications. Which might as well be an indication that the operative interpretation itself is concealing some hidden assumptions, constraining the structure beyond the pristine intentions.

Inspired by the proof from first principles established by von Ignatowski, we have thus explored the consequences of a breakdown of the hypotheses behind the Lorentz transformations, obtaining that either what can be relaxed is already tightly constrained by experiments and observations (violations of spatial isotropy, or of the principle of relativity), or the remaining option seems to be to decouple the coordinates adopted in an inertial reference frame and the outcomes of temporal and distance measurements.

This last conclusion admits two interpretations. The "geometric" one, which charges the onset of the mesoscopic regime onto some universal property of Nature, can suggest e.g. the adoption of richer structures (non-commutative phase space, non-associative velocity space) where to accommodate the breakdown of the operative interpretation of the coordinates.

On the other hand, a more "physical" point of view (according to which the regime is due to non-universal properties) may explain a possible novel "meso-scale" phenomenology in terms of modifications in the behaviour of dynamical fields, perhaps suppressed by apt scales — these fields, then, might still be defined on some background manifold, remaining unaltered as the dynamics change —. An example of this second approach is e.g. the non-locality scale in the propagator emerging from the Causal Sets Theory approach to the emergence of gravity.[2]

Both the answers above may appear radical; to date, however, they seem to be an acceptable reply to an equally radical attitude reaffirming the invariable continuation of the "scale-invariant" paradigm in spite of well-known technical and interpretational issues.

[2]Acknowledgments to S. Liberati for suggesting this scenario.

5.2 Some hints and proposals for future work

Because the job is never really done, and any accomplishment is but the forking point where even more questions arise, and challenges emerge.

5.2.1 Foundations of the Equivalence Principles

In the debate on the Equivalence Principles, two big questions remain unanswered: one is the relationship between the Weak Equivalence Principle and Einstein's one, which has gained the short-hand name of "Schiff's Conjecture"; the other is the extension of the above relationship to the Gravitational Weak form of the principle.

The best results in proving Schiff's conjecture are still restricted to very limited and selected sub-cases, where symmetries and simplifications of different nature allow to sidestep some of the technical and conceptual difficulties involved. A general proof of the conjecture, or a decided counter-example, might be a valuable contribution, and a significant milestone towards an understanding of the delicate interplay between ultra-local fundamental test physics, and gravitational phenomena.

Deeply bound to this problem is the other issue of the true nature (and formulation) of the Strong Equivalence Principle. Which is made of the Gravitational Weak part — on which we have focussed — plus "something else" which, however, is seldom (if not ever) considered in the experimental settings or in the protocols used to design the experiments.

The PPN formalism in fact tests the free-fall of extended, self-gravitating bodies like the Moon orbiting around the Earth (suitable fine tunings can accommodate some selected features of binary systems in a regime where stronger gravity is at work), and this phenomenology is entirely covered by the Gravitational Weak form. What else, then, can the Strong Equivalence Principle help discriminate?

An answer might be: the unfolding of gravitational phenomena other than the free fall, in theories of gravity beyond General Relativity. In this sense, a better understanding of the physics of e.g. gravitational waves (and, hopefully, a direct detection) could open new paths towards the ultimate solution of the riddle.

The Strong Equivalence Principle might also have something to say on the true nature of gravity beyond the linear regime, and the way various types of non-linearities can be discriminated. In this sense, the Lanczos–Lovelock theories might play a significant role, with theoretical guidelines adapted to the higher-dimensional spacetimes required for the Lovelock Lagrangians to be non-trivial.

The proposal of Refs. [216, 215] is another source of unanswered questions. The deep link between General Relativity and non-Abelian Yang–Mills theories probably deserves a "second chance". It is true that the analogy requires embarking oneself in the analysis of gauge formulations of gravity (a branch of the "family tree" not explored in detail here), but the preliminary results obtained so far look somewhat promising, and invite to better understand the role of the dynamics of the connexion and the curvature in the description of gravitational phenomena.

5.2.2 A larger arena...

The conditions proposed in this work for the Gravitational Weak Equivalence Principle have been crafted for metric theories of gravity, in agreement with a long-standing tradition. The landscape, however, is larger, and more complex frameworks await their "custom" version of the principle.

Consider for instance a purely affine theory of gravity (§1.3.3), i.e. one where the action is built out entirely of a skew-symmetric connexion, which gives even the analogue of the metric determinant $\sqrt{-g}$ needed to integrate on a manifold.[3] Or a metric-affine and/or affine theory; what matters is the premise that the affine structure be an independent dynamical field.

Such models might be of some significance because the autoparallel worldlines and the geodesic curves form, in general, two separate classes of one-dimensional submanifolds on M. Some questions then arise. For instance: how to build a proper implementation (if any) of the Gravitational Weak Equivalence Principle for these theories, and which derivative operator to place in Eq. (3.11)? Also: if the Gravitational Weak Equivalence Principle is implemented independently of the Einstein Principle,[4] how many different Gravitational Weak Principle*s* can emerge, and what is their mutual relation? And, again: is there a way to relate this decoupling of affine and metric structure to a formal definition of the Strong Equivalence Principle?

Finding an extension of the Gravitational Weak principle to a purely affine setting (and a metric-affine, and an affine one) might teach some lessons, both at the level of enlarging the knowledge base of the nature of the equivalence principles, and by establishing a new formulation of the test to check the free fall of self-gravitating bodies.

5.2.3 ...For an even finer sieve

In a general, n-dimensional spacetime, two groups of theories pass through the sieve constructed here: Einstein's General Relativity (for any n), and all the dimensionally-compatible Lanczos–Lovelock theories.

In principle, this might be considered a fair performance for a test filtering all metric theories of gravity. One, however, could want to explain what further conditions are needed if the goal becomes to select *only* General Relativity in the bundle of metric theories, as the Strong Equivalence Principle is conjectured to do.

The different Lanczos–Lovelock models, although structurally quite similar to the Einstenian framework, are not precisely identical to it, and one can present some physically relevant differences. To name two, and by limiting ourselves to the lowest order of the Lovelock actions: the propagation of the wave fronts and the critical collapse in Gauß–Bonnet gravity [118, 561, 236, 328].

In the former case, one finds [118] that the wave fronts (defined as the analytical discontinuities in the highest derivatives of the dynamical variables) are not anymore tangent to the light-cones of the causal structure governed by g_{ab},

[3]In this sense, it would be interesting to see whether such models are able to recover the local Minkowskian structure of spacetime, as an acid test of their physical viability.

[4]The Gravitational Weak Equivalence Principle, being the extension to self-gravitating bodies of the Galilean free fall (Weak Equivalence Principle) is independent of the existence and consequences of Einstein's Equivalence Principle, even though the former is usually implemented only *after* the metric structure has been introduced, as a consequence of the latter principle.

nor even are tangent to a second-order cone. On top of that, if a coupling between the homogeneous polynomials in the Riemann tensor and the Einstein tensor is permitted, the propagation of the wave fronts is not tangent anymore, in general, to a convex cone.

The reason behind such conclusion seems to be a technical one, related to the properties of the normal form of the field equations for the metric coefficients [118, 561]. It may admit, however, a possible physical interpretation in terms of the graviton propagator, its higher Feynmann graphs, and ultimately in terms of the way the intrinsic non-linearity of the gravitational phenomena is described in the Gauß–Bonnet theory.

As for the critical collapse, such phenomenon is prevented in the context of Gauß–Bonnet gravity by the presence of a second fundamental scale, given by the coupling constant α in Eq. (3.88), which acts as a regulator and tames the critical behaviour occurring instead in General Relativity. The argument can be rephrased in terms of the relative presence of two scales, namely G and α in the Einstein–Gauß–Bonnet gravity model [236, 328] (the role of a *third* scale, e.g. the cosmological constant Λ, might be explored as well).

The upshot of this minimal review is that the family of Lanczos–Lovelock gravity theories is not as close to General Relativity as one might think by looking at some literature on the topic. Therefore, it may be possible to find further criteria, with physical significance — and sufficient generality —, to highlight in a compact form all the differences in the hierarchy of models, and trace them back to some structural, fundamental aspect.

A possible answer could be to look once again at the structure of the action, and notice that, as the number of couplings among the $\Gamma^{\alpha}{}_{\beta\gamma}$'s in the polynomials ramps up, the degree of non-linearity of the gravity theory increases. General Relativity, being the simplest specimen, offers hence the *minimal non-linearity* within the Lanczos–Lovelock class. Such minimal non-linearity is also related to the absence of supplementary scales, the latter emerging inevitably as the rank of the Lovelock scalar density progresses.

A formal argument supporting such statement is not available yet, but it might be a programme where to invest some effort, considering not only the "taxonomic" relevance of an answer to the question, but also the more general consequences of a better understanding of non-linear phenomena.

5.2.4 Spacetime/Quantum structure

The operative approach is a useful tool to sketch the description of physical phenomena, and sidesteps at once the problem of interpreting the experimental results. Yet, it hides a delicate balance of unspoken details and hidden assumptions [98], whose presence ends up constraining the framework to the point that the Minkowskian outcome (or some of its mild extensions) becomes almost inevitable.

Above all, it is a standpoint rooted in the "human" perspective on the world, and it seems hard to adapt it meaningfully to the scales close to the Planckian sill. The notion of e.g. observer given in this work (a tiny computer made of a clock, a memory to time-stamp the events, and some transmitter-receiver to communicate with other observers) is admittedly more versatile than that of "human/sentient being seeing things",[5] yet it is possibly of little use at the scales where the "mesoscopic" regime might emerge.

Also, the simple thought experiment concerning "elementary" clocks at different energies shows that, even without abandoning "human" scales, there might appear tiny deviations from the expected laws of physics — maybe in a Poincaré-invariant way.

What could then happen at the "mesoscopic" scale? Non-locality might play a significant role — there are reasonable suspicions that it is already relevant, at macroscopic scales and in a different meaning, as Bell's theorem seems to ultimately suggest —. Or, different formal structures might be needed to account for the universal deformations of fields and dynamical variables. One way or another, the infinite prolongation of the continuous paradigm seems destined to break, with the underlying quantum fabric of spacetime finally becoming manifest at some point.

Such speculations are perhaps not robust enough to support the forecast of a "mesoscopic regime" of spacetime, but we deem nonetheless that they could trigger new questions, and promote a more critical approach to the foundations of classical spacetime.

Curtain

This was the story of a free fall.

To all those who were there, and supported, helped, gave advise, provided laughs and desserts, or just shared the tiniest bit of themselves as the world-lines unwound: thank you. Thank you very much. It has been a pleasure, and an honour.

Falling is hard, much harder than expected.

Freely falling is exactly as hard. But makes one free.

Take care of yourselves.

Farewell,

— Eolo

[5]Acknowledgments to S. Sonego for proposing this modification.

Appendices
& Bibliography

Appendix A

Variational Principles and Boundary Terms

> Our stability is but balance, and wisdom lies in masterful administration of the unforeseen.
>
> R. Bridges, *The Testament of Beauty*.

The role played in this thesis by the notion of a "well-defined variational formulation" for a dynamical theory suggested to include a few more details on the topic. In the following, the basic definitions and results are complemented by a short discussion of the Einstein–Hilbert action for General Relativity.

A.1 Action functionals and field equations

Let $\phi^I(\chi^\alpha)$ be a collection of fields — i.e., of functions of the coordinates χ^α on a manifold M, numbered by a running index I. We introduce the compact notation $\left\{\phi^I\right\} := \left(\phi^I, \partial_\alpha\phi^I, \partial_\alpha\partial_\beta\phi^I, \dots\right)$ to denote the fields and all their derivatives up to arbitrary order. One can then consider the *Lagrangian function*

$$\mathfrak{L}\left(\left\{\phi^I\right\}, \chi^\alpha\right) := \sqrt{-g}\,\mathscr{L}\left(\left\{\phi^I\right\}, \chi^\alpha\right) \ , \tag{A.1}$$

where $\mathscr{L}$ an ordinary scalar function, and we have used the classical "Gothic" notation for tensor densities [360, 479, 477, 478], such that, for a general tensor, it is

$$\mathfrak{A}^{abc\dots}_{def\dots} := \sqrt{-g} A^{abc\dots}_{def\dots} \tag{A.2}$$

The quantity in (A.1) can be used to build up an *action functional* (a function of the field configurations), given by

$$S\left[\phi^I, g^{ab}\right] := \kappa \int_\Omega \mathfrak{L}\left(\left\{\phi^I\right\}, \chi^\alpha\right) \mathrm{d}^n\chi \tag{A.3}$$

with Ω the coordinate representation of a region over M, g^{ab} the inverse metric defined on the manifold, and κ a dimensionful constant accounting for the correct dimensional arrangement of the right-hand side.

Action functionals, fields, and Lagrangian densities are tools in widespread use in theoretical physics, for they often represent dynamical models of actual systems [5, 235, 459, 424]. The behaviour of such systems can be determined from the *field equations*, providing the extremal configurations of the action functional itself. To this end, it is necessary to introduce the notion of differentiability of a functional

For sake of simplicity (but without any loss of generality), consider the case in which $I = 1$, and drop the counting index. If ϕ is a smooth function, then for all $\delta\phi$ such that $\phi + \delta\phi$ is still a smooth function, the functional S is said to be *differentiable* if

$$S\left[\phi + \delta\phi\right] = S\left[\phi\right] + \delta S\left[\phi, \delta\phi\right] + \Upsilon\left[\phi, \delta\phi\right] \ , \tag{A.4}$$

where $\delta S\left[\phi, \delta\phi\right]$ is *linear* in $\delta\phi$, which means that, for a fixed ϕ, it is (c is any number)

$$\delta S\left[\phi, \delta\phi_1 + \delta\phi_2\right] = \delta S\left[\phi, \delta\phi_1\right] + \delta S\left[\phi, \delta\phi_2\right] \ , \tag{A.5}$$

$$\delta S\left[\phi, c\delta\phi\right] = c\delta S\left[\phi, \delta\phi\right] \ . \tag{A.6}$$

As for the term Υ, it must be of order ϵ^2, in the sense that, if $|\delta\phi| < \epsilon$ and $|\nabla_a \delta\phi| < \epsilon$, then it is

$$\left|\Upsilon\left[\phi, \delta\phi\right]\right| < C\epsilon^2 \ , \tag{A.7}$$

with C a positive real number. The object $\delta S\left[\phi, \delta\phi\right]$, when it exists, is called the *variation* of the action functional, and is thus the linear part in $\delta\phi$ of the difference $S\left[\phi + \delta\phi\right] - S\left[\phi\right]$.

The existence of the variation δS, and the possibility to have the decomposition (A.4), are in general not automatically guaranteed; when this occurs, one says that the variational problem for the action functional is *well-posed*. The well-posedness of the variational problem usually requires the introduction of apt supplementary conditions of regularity for the field ϕ at the boundary of the region where the integration is performed, the *boundary conditions*, which ensure that δS exists, it is linear in $\delta\phi$, and that Υ provides a sufficiently negligible contribution with respect to δS.

For a given differentiable action functional, any configuration $\bar{\phi}$ for which the variation vanishes, $\delta S\left(\bar{\phi}, \delta\phi\right) = 0$, for all field variations $\delta\phi$, is called an *extremal*. Whenever it is possible to write

$$\delta S\left[\phi, \delta\phi\right] = \int \frac{\delta S}{\delta\phi} \delta\phi \, \mathrm{d}^n\chi \tag{A.8}$$

for some function ϕ, the term $\delta S/\delta\phi$ is the *functional derivative* of the action with respect to ϕ.

The *field equations*, i.e. the equations governing the evolution (dynamical or kinematical) of the configuration ϕ, emerge then as the outcome of extremising the action functionals, that is, implementing the condition

$$\delta S = \delta \int_\Omega \mathfrak{L}\left(\{\phi\}, \chi^\alpha\right) \mathrm{d}^n\chi = 0 \tag{A.9}$$

which reduces to setting to zero the *Euler–Lagrange derivative* of the Lagrangian scalar density, viz.

$$\Delta_\phi \mathfrak{L}\left([\phi]\,,\chi^\alpha\right) = 0\ , \tag{A.10}$$

for any possible variation $\delta\phi$. The symbol Δ_ϕ stands for

$$\Delta_\phi := \frac{\partial}{\partial\phi} - \partial_\alpha \frac{\partial}{\partial\left(\partial_\alpha\phi\right)} + \partial_\alpha\partial_\beta \frac{{}^s\partial}{\partial\left(\partial_\alpha\partial_\beta\phi\right)} - \ldots\ , \tag{A.11}$$

where the symmetric derivative with respect to the derivatives of the fields in the formula above equals

$$\frac{{}^s\partial}{\partial\left(\partial_\alpha\partial_\beta\ldots\partial_\lambda\phi\right)}\partial_\gamma\partial_\iota\ldots\partial_\zeta\Xi = \delta^\alpha{}_{(\gamma}\ldots\delta^\lambda{}_{\zeta)}\ , \tag{A.12}$$

The well-posedness of the variational problem, as said, is related to the boundary conditions for the dynamical fields. In a pseudo-Riemannian setting, namely when Ω is the coordinate representation of a region defined over a spacetime $\mathscr{M}$, the boundary $\partial\Omega$ can be generally separated into a timelike part (the *spatial boundary*), and a pair of spacelike hypersurfaces (the *endpoints*).

A.2 The Einstein–Hilbert action

We can now apply the considerations of the previous section to the specific case of General Relativity. The matter is discussed along three main directions: the pure Einstein–Hilbert Lagrangian, without any additional boundary term; the same action, but equipped with the Gibbons–Hawking–York boundary term; the less-known, yet instructive, gamma-gamma Lagrangian.

Before proceeding with the argument, a remark concerning the physical role of the boundary conditions in this particular case [171]. Broadly speaking, the choice of the spatial boundary conditions mirrors the choice of a defined "vacuum" of the theory under examination (e.g. the asymptotically flat vacuum, with respect to which many solutions of the field equations are allowed), whereas conditions on the endpoints, or initial data, assign a particular state in the vacuum (in this way one can discriminate e.g. the Kerr solution from the Schwarzschild, or Minkowski one, within the class of aymptotically flat spacetimes).

A.2.1 Standard, "naïve" formulation

We start with the pure Einstein–Hilbert Lagrangian, viz.

$$S_{\mathrm{EH}} = \frac{c^4}{16\pi G}\int_\Omega R\sqrt{-g}\,\mathrm{d}^4y = \frac{c^4}{16\pi G}\int_\Omega g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4y\ . \tag{A.13}$$

The field equations emerge upon setting $\delta S_{\mathrm{EH}} = 0$, and varying with respect to the inverse metric g^{ab}; in the language of Eq. (A.10), this means

$$\Delta_{g^{hk}}\left(R\sqrt{-g}\right) = \Delta_{g^{hk}}\left(g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}\right) = 0\ . \tag{A.14}$$

The various terms can be found through standard calculations [542, 353, 370]. Two pieces need be evaluated, namely $\Delta_{g^{hk}}\sqrt{-g}$, and $\Delta_{g^{hk}}R_{\alpha\beta}$. The first is given by

$$\Delta_{g^{hk}}\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}\ , \tag{A.15}$$

and, when grouped together with the term $R_{ab}\delta g^{ab}$, gives the contribution (the overall coupling constant can be neglected)

$$\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\delta g^{\alpha\beta} = G_{\alpha\beta}\delta g^{\alpha\beta} \ . \tag{A.16}$$

Forgetting for a second the term $g^{\alpha\beta}\Delta_{g^{hk}}R_{\alpha\beta}$, and focussing on the relation above, we have that the field equations for the metric will emerge upon setting to zero the variation of Eq. (A.13) for all possible $\delta g^{\alpha\beta}$'s in Ω. A glance at Eq. (A.16) shows that this is the case if

$$G_{\alpha\beta} = 0 \ , \tag{A.17}$$

i.e., the Einstein field equations in vacuo.

There is a problem, however: the term $g^{\alpha\beta}\Delta_{g^{hk}}R_{\alpha\beta}$ still has to be evaluated. How does it come that we already have the field equations with a piece of the variation missing?

Let us complete the calculation. The variation of the Ricci tensor gives, after some rearrangements,

$$\Delta_{g^{hk}}R_{\alpha\beta} = \nabla_\gamma \delta\Gamma^\gamma{}_{\alpha\beta} - \nabla_\alpha \delta\Gamma^\gamma{}_{\gamma\beta} \ . \tag{A.18}$$

Further manipulations make it possible to show that $g^{\alpha\beta}\Delta_{g^{hk}}R_{\alpha\beta}$ amounts to

$$g^{\alpha\beta}\Delta_{g^{hk}}R_{\alpha\beta} = \left(\delta^\alpha_\gamma \nabla_\delta g^{\delta\beta} - \nabla_\gamma g^{\alpha\beta}\right)\delta\Gamma^\gamma{}_{\alpha\beta} + \nabla_\gamma\left(g^{\alpha\beta}\delta\Gamma^\gamma{}_{\alpha\beta} - g^{\gamma\beta}\delta\Gamma^\alpha{}_{\alpha\beta}\right) \ . \tag{A.19}$$

Upon substituting this last expression in Eq. (A.13), and using Gauß' theorem, we have

$$\int_\Omega g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4y = \int_\Omega \left(\delta^\alpha_\gamma \nabla_\delta g^{\delta\beta} - \nabla_\gamma g^{\alpha\beta}\right)\delta\Gamma^\gamma{}_{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4y + \\ + \oint_{\partial\Omega}\left(g^{\alpha\beta}\delta\Gamma^\gamma{}_{\alpha\beta} - g^{\gamma\beta}\delta\Gamma^\alpha{}_{\alpha\beta}\right)n_\gamma \mathrm{d}\Sigma \tag{A.20}$$

where $n_\gamma\mathrm{d}\Sigma$ denotes the oriented 3-volume element on $\partial\Omega$, defined by the normal vector n^γ. In the formula above, the "bulk" four-dimensional term vanishes in view of the metric-compatibility condition $\nabla_\alpha g^{\beta\gamma} = 0$, and only the surface term survives.

Now we can select the boundary conditions. It is reasonable to expect that the metric field have fixed value at the boundary, which implies a vanishing variation,

$$\delta g^{\alpha\beta} = 0 \ , \tag{A.21}$$

identically on $\partial\Omega$. This first fixing, however, does not seem to help at this stage, for the terms in the surface integral in Eq. (A.20) depend on the derivatives of the variation of the metric at the boundary, since $\delta\Gamma^\alpha{}_{\beta\gamma} \sim \partial_\alpha \delta g_{\beta\gamma}$. One might then decide to put, blindly

$$\partial_\gamma \delta g^{\alpha\beta} = 0 \ , \tag{A.22}$$

and get rid of the boundary terms. Yet, this would be too restrictive a constraint on the metric field [171]. Indeed, suppose to accept condition (A.22), as if not only the value of the field, but also of its first derivative were constant on the boundary. This amounts to restricting the class of spacetime geometries over which one is extremising the action, i.e. reducing the possible "competing paths" solving the field equations. But then, nothing prevents to require that also the second derivatives have vanishing variations on the boundary. And the third derivatives; and the fourth, and so on. In this way, we can constrain the entire Taylor series representing the metric on the boundary, to the point that there will be only one geometry compatible with the given boundary conditions, and the extremisation of the action; this, however, would be hardly a protocol to find a solution of the field equations: the space of possible configurations among which to choose via the dynamical equations has become too cramped.

The point is that the class of competing field configurations must be kept as large as possible, hence the smallest number of constraints on the fields, the more effective the variational method [171]. Therefore, condition (A.22) must be rejected, whereas only condition (A.21) must be enforced to derive the form of the field equations. This implies that there must be other ways to get rid of the boundary contribution (A.20) in presence of the constraint $\delta g^{\alpha\beta} = 0$ on $\partial\Omega$ alone.

A.2.2 The Gibbons–Hawking–York counter-term

Let us get back to the non-vanishing boundary term in Eq. (A.20), namely

$$\left(g^{\alpha\beta}\delta\Gamma^{\gamma}{}_{\alpha\beta} - g^{\gamma\beta}\delta\Gamma^{\alpha}{}_{\alpha\beta}\right) n_{\gamma} \ . \tag{A.23}$$

This boils down to the expression

$$g^{\alpha\beta} n^{\gamma} \left(\partial_{\beta}\delta g_{\alpha\gamma} + \partial_{\gamma}\delta g_{\alpha\beta}\right) \ , \tag{A.24}$$

and the latter can be further simplified by decomposing the metric into the orthogonal and parallel parts with respect to the hyper-surface $\partial\Omega$; one has, indeed

$$g_{\alpha\beta} = h_{\alpha\beta} \pm n_{\alpha} n_{\beta} \ , \tag{A.25}$$

with n^{α} the normal to $\partial\Omega$, and $h_{\alpha\beta}$ a symmetric tensor normal to n^{α}, the induced (transverse) metric. The ambiguity of the sign in the formula above is due to the possible choice of n^{α} as timelike or spacelike. Without loss of generality, we can suppose that n^{α} is timelike, and pick the minus sign (so the three-surface where $h_{\alpha\beta}$ is defined is spacelike, and $h_{\alpha\beta}$ is a positve-definite metric tensor on $\partial\Omega$).

We can now use the condition $\delta g^{\alpha\beta} = 0$ at the boundary to derive that, in analogous fashion, the variations $\delta h^{\alpha\beta}$ and δn^{α} will all vanish on the boundary. Not only that: since the metric is constant on $\partial\Omega$, so is the tangential derivative of its variation on the boundary, i.e.

$$h^{\alpha\beta}\partial_{\alpha}\delta g_{\beta\gamma} = 0 \ . \tag{A.26}$$

This allows to conclude that the surface term in Eq. (A.20) amounts to

$$\sqrt{-g} g^{\alpha\beta}\delta R_{\alpha\beta}\big|_{\partial\Omega} = \sqrt{|h|}\, h^{\alpha\beta} n^{\gamma} \partial_{\gamma}\delta g_{\alpha\beta} \ , \tag{A.27}$$

and also that any function of the normal vector n^{α}, the induced metric $h_{\alpha\beta}$, and of the tangential derivative $h^{\alpha\beta}\partial_{\beta}$ will have a vanishing variation on the boundary $\partial\Omega$.

We have now to get rid of the term on the right-hand side of Eq. (A.27); to this end, we begin by introducing the trace of the extrinsic curvature K of the hyper-surface $\partial\Omega$, which is given by

$$K := \nabla_\alpha n^\alpha = g^{\alpha\beta}\nabla_\alpha n_\beta = h^{\alpha\beta}\left(\partial_\alpha n_\beta + \Gamma^\gamma{}_{\alpha\beta} n_\gamma\right) \ , \tag{A.28}$$

and we can get rid of the term $n^\alpha n^\beta \nabla_\alpha n_\beta$ because n^α is perpendicular to its covariant derivative.

Consider now the variation δK, multiply it by 2, and impose the usual condition $\delta g^{\alpha\beta} = 0$ on the boundary; the result reads

$$2\delta K = 2\delta\left(h^{\alpha\beta}\left(\partial_\alpha n_\beta - \Gamma^\gamma{}_{\alpha\beta} n_\gamma\right)\right) = -2h^{\alpha\beta} n_\gamma \delta\Gamma^\gamma{}_{\alpha\beta} = h^{\alpha\beta} n^\gamma \partial_\gamma \delta g_{\alpha\beta} \ , \tag{A.29}$$

i.e., precisely the term we are asked to rule out to avoid imposing the unnatural condition $\delta\partial_\gamma g_{\alpha\beta} = 0$. We can thus conclude that the term to be added to the Einstein–Hilbert Lagrangian is given by the surface contribution

$$\mathscr{B}_{\mathrm{GHY}} = 2\oint_{\partial\Omega} K\sqrt{h}\,\mathrm{d}^3 y \ , \tag{A.30}$$

which is indeed the Gibbons–Hawking–York term [220, 566].

Only with the addition of this complement to the Einstein–Hilbert action, the metric variation for the gravitational Lagrangian becomes well-posed in a true sense, as it was to prove.

A.2.3 The gamma-gamma Lagrangian

A well-posed metric variation for General Relativity can be formulated as well in the absence of the Gibbons–Hawking–York fixing term, provided that the Einstein–Hilbert action is rearranged appropriately.

The idea is to subtract total derivatives from the Lagrangian (A.13), and deploy the minimal boundary conditions (A.21). This tweak was used by Einstein himself [177] long before the remarks by Gibbons, Hawking, and York, to exhibit an alternative proposal for his gravitational action.

Einstein suggests to write the Lagrangian function for General Relativity in the following form, usually known as the *gamma-gamma Lagrangian*,

$$\mathfrak{L}_{\Gamma\Gamma} = \sqrt{-g}\, g^{\alpha\beta}\left(\Gamma^\gamma{}_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma} - \Gamma^\gamma{}_{\gamma\delta}\Gamma^\delta{}_{\alpha\beta}\right) \ . \tag{A.31}$$

This scalar density contains only first-order derivatives of the metric, hence does not require any further fixing of the derivatives at the boundary. One can prove that (A.31) differs from the Einstein–Hilbert Lagrangian by a pure divergence term; namely

$$\mathfrak{L}_{\Gamma\Gamma} = \sqrt{-g}\left(R - \nabla_\alpha B^\alpha\right) \ , \tag{A.32}$$

with the object B^α — not a vector — given by

$$B^\alpha = g^{\beta\gamma}\Gamma^\alpha{}_{\beta\gamma} - g^{\alpha\beta}\Gamma^\gamma{}_{\beta\gamma} \ . \tag{A.33}$$

Eq. (A.32) shows that the resulting “bulk” field equations for gravity are the same, no matter if one starts with (A.13) (plus the Gibbons–Hawking–York boundary term) or with (A.31).

The gamma-gamma Lagrangian and the Einstein–Hilbert one — the latter being equipped with the Gibbons–Hawking–York supplementary term — can be obtained from one another by introducing a boundary term of the form $f\left(g_{\alpha\beta}, n^{\alpha}, h^{\alpha\beta}\partial_{\beta}\right)$, which vanishes identically on $\partial\Omega$ as stated in the previous section; specifically, it is

$$\int_{\Omega} \mathfrak{L}_{\Gamma\Gamma} \mathrm{d}^4 y = \int_{\Omega} R\sqrt{-g}\,\mathrm{d}^4 y - \oint_{\partial\Omega} B^{\alpha} n_{\alpha} \sqrt{|h|} \mathrm{d}^3 y \,, \tag{A.34}$$

and, upon massaging B^{α} from (A.33), one finds

$$B^{\alpha} n_{\alpha} = -2K + 2h^{\alpha\beta}\partial_{\beta} n_{\alpha} - n^{\alpha} h^{\beta\gamma} \partial_{\beta} g_{\alpha\gamma} \,, \tag{A.35}$$

so that the specific form of the function $f\left(g_{\alpha\beta}, n^{\alpha}, h^{\alpha\beta}\partial_{\beta}\right)$ reads

$$f = 2h^{\alpha\beta}\partial_{\beta} n_{\alpha} - n^{\alpha} h^{\beta\gamma} \partial_{\beta} g_{\gamma\alpha} \,. \tag{A.36}$$

By recalling that the condition $\delta g^{\alpha\beta} = 0$ on the boundary makes it vanish any function of the tangential derivatives, normal vector, and metric itself, then the complete equivalence of the two formulations of the variational problem is proven.

Appendix B

First-order perturbations

...ma l'Anselmo, previdente, fin le brache
avea d'acciar.

G. Visconti Venosta, *Anselmo the Vailant.*

In this Appendix, we collect a number of useful expressions for the differences, to the first order in ϵ, between the geometric objects built out of two metrics g_{ab} and $\bar{g}_{ab}$ connected via the relation (3.18). Thus, all the equations presented here hold only up to order ϵ.

B.1 General-use formulæ

First of all, we note that Eq. (3.18) implies [556]

$$g^{ab} = \bar{g}^{ab} - \epsilon\,\gamma^{ab}\,, \tag{B.1}$$

where $\bar{g}^{ab}$ is the inverse of $\bar{g}_{ab}$, and $\gamma^{ab} := \bar{g}^{ac}\,\bar{g}^{bd}\,\gamma_{cd}$.

To find the relation between the determinants of the metric coefficients, let us first expand g around the unperturbed metric $\bar{g}_{ab}$:

$$g = \bar{g} + \epsilon\,\frac{\partial g}{\partial g_{ab}}\,\gamma_{ab}\ , \tag{B.2}$$

where the partial derivatives are evaluated at $g_{ab} = \bar{g}_{ab}$. Using the property $\partial g/\partial g_{ab} = g\,g^{ab}$, and defining $\gamma := \bar{g}^{ab}\,\gamma_{ab}$, we find the simple relation

$$g = \bar{g}\,(1 + \epsilon\,\gamma)\ . \tag{B.3}$$

The Christoffel symbols $\Gamma^a{}_{bc}$ and $\bar{\Gamma}^a{}_{bc}$ of the metrics g_{ab} and $\bar{g}_{ab}$, respectively, are related as

$$\Gamma^a{}_{bc} = \bar{\Gamma}^a{}_{bc} + \epsilon\,\Xi^a{}_{bc}\ , \tag{B.4}$$

where $\Xi^a{}_{bc} := \bar{g}^{ad}\,\Xi_{dbc}$, and the tensor Ξ_{abc} is

$$\Xi_{abc} = \frac{1}{2}\left(\bar{\nabla}_b\gamma_{ca} + \bar{\nabla}_c\gamma_{ba} - \bar{\nabla}_a\gamma_{bc}\right). \tag{B.5}$$

This result can be easily obtained using the expression

$$\begin{aligned}\nabla_a g_{bc} &= \bar{\nabla}_a g_{bc} - \epsilon\,\Xi^d{}_{ab}\,g_{dc} - \epsilon\,\Xi^d{}_{ac}\,g_{bd}\\ &= \bar{\nabla}_a\bar{g}_{bc} + \epsilon\,\bar{\nabla}_a\gamma_{bc} - \epsilon\,\Xi_{cab} - \epsilon\,\Xi_{bac}\,,\end{aligned} \tag{B.6}$$

which follows from Eq. (B.4). Since the covariant derivatives ∇_a and $\bar{\nabla}_a$ are associated with g_{ab} and $\bar{g}_{ab}$, respectively, the compatibility condition for the Riemannian connection gives $\nabla_a g_{bc} = \bar{\nabla}_a\bar{g}_{bc} = 0$. Thus,

$$\Xi_{cab} + \Xi_{bac} = \bar{\nabla}_a\gamma_{bc}\,. \tag{B.7}$$

Equation (B.5) is then obtained following the same steps by which one finds the usual expression for the Christoffel symbols in terms of partial derivatives of the metric.

The first-order difference $\epsilon\,\mathcal{R}_{abc}{}^d = R_{abc}{}^d - \bar{R}_{abc}{}^d$ between the Riemann curvature tensors follows from Eq. (B.4), and one has

$$\mathcal{R}_{abc}{}^d = \bar{\nabla}_b\Xi^d{}_{ac} - \bar{\nabla}_a\Xi^d{}_{bc}\,. \tag{B.8}$$

This implies, for the difference $\epsilon\,\mathcal{R}_{ab} = R_{ab} - \bar{R}_{ab}$ between the Ricci tensors,

$$\mathcal{R}_{ab} = \mathcal{R}_{acb}{}^c = \bar{\nabla}_c\Xi^c{}_{ab} - \bar{\nabla}_a\Xi^c{}_{cb}\,; \tag{B.9}$$

and, for the difference $\epsilon\,\mathcal{R} = R - \bar{R}$ between the curvature scalars $R = g^{ab}R_{ab}$ and $\bar{R} = \bar{g}^{ab}\bar{R}_{ab}$,

$$\mathcal{R} = \bar{g}^{ab}\,\bar{\nabla}_c\Xi^c{}_{ab} - \bar{g}^{ab}\,\bar{\nabla}_a\Xi^c{}_{cb} - \gamma^{ab}\bar{R}_{ab}\,, \tag{B.10}$$

where Eqs. (B.1) and (B.9) have been used.

Finally, for the difference $\epsilon\,\mathcal{G}_{ab} = G_{ab} - \bar{G}_{ab}$ between the Einstein tensors we find, defining $\Xi^{ab}{}_b := \bar{g}^{bc}\,\Xi^a{}_{bc}$:

$$\begin{aligned}\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\,\bar{g}_{ab}\,\mathcal{R} - \frac{1}{2}\,\bar{R}\,\gamma_{ab} &= \bar{\nabla}_c\Xi^c{}_{ab} - \bar{\nabla}_a\Xi^c{}_{cb}\\ &- \frac{\bar{g}_{ab}}{2}\,\bar{\nabla}_c\Xi^{cd}{}_d + \frac{\bar{g}_{ab}}{2}\bar{\nabla}^c\Xi^e{}_{ec} + \frac{\bar{g}_{ab}}{2}\gamma^{cd}\bar{R}_{cd} - \frac{\gamma_{ab}}{2}\bar{R}\,.\end{aligned} \tag{B.11}$$

B.2 Divergence of the first-order Einstein tensor

The quantity $\bar{\nabla}^b\mathcal{G}_{ab}$ intervenes frequently in the calculation of $\bar{\nabla}^b\mathcal{E}_{ab}$ — noticeably, in §§ 3.4.1 , 3.4.2 —, so we evaluate it here in full generality. We begin by substituting the expression (B.5) for $\Xi^a{}_{bc}$ into Eq. (B.11), to obtain

$$\begin{aligned}\mathcal{G}_{ab} =& \frac{1}{2}\Big[\left(\bar{\nabla}^c\bar{\nabla}_a\gamma_{bc} + \bar{\nabla}^c\bar{\nabla}_b\gamma_{ac}\right) - \bar{\nabla}^c\bar{\nabla}_c\gamma_{ab} - \bar{\nabla}_a\bar{\nabla}_b\gamma\\ &- \bar{g}_{ab}\left(\bar{\nabla}^c\bar{\nabla}^d\gamma_{cd} - \bar{\nabla}^c\bar{\nabla}_c\gamma\right) + \bar{g}_{ab}\gamma^{cd}\bar{R}_{cd} - \gamma_{ab}\,\bar{R}\Big].\end{aligned} \tag{B.12}$$

In a flat background spacetime (a situation common, for instance, in the study of gravitational radiation [542]), it is a straightforward exercise to show that $\partial^b \mathcal{G}_{ab} = 0$, the key point in the proof being a heavy use of the commutative property for partial derivatives. In the case of a non-flat background, on the other hand, switching covariant derivative operators $\bar{\nabla}_a$ generates instances of the Riemann and Ricci tensors. The three terms in $\bar{\nabla}^b \mathcal{G}_{ab}$ where this happens can be written, rearranging the indices and using the property $\bar{\nabla}_a \bar{g}_{cd} = 0$, as:

$$\begin{aligned} \bar{\nabla}_b \bar{\nabla}_c \bar{\nabla}_a \gamma^{bc} - \bar{\nabla}_a \bar{\nabla}_b \bar{\nabla}_c \gamma^{bc} = {} & \bar{R}_{abcd}\, \bar{\nabla}^d \gamma^{bc} \\ & + 2\bar{R}_{ab} \bar{\nabla}_c \gamma^{bc} + 2\bar{\nabla}_b \bar{R}_{ac} \gamma^{bc} - \bar{\nabla}_a \bar{R}_{bc} \gamma^{bc}\,; \end{aligned} \tag{B.13}$$

$$\bar{\nabla}_b \bar{\nabla}_c \bar{\nabla}^b {\gamma_a}^c - \bar{\nabla}_c \bar{\nabla}_b \bar{\nabla}^b {\gamma_a}^c = -\bar{R}_{bcda}\, \bar{\nabla}^b \gamma^{cd}\,; \tag{B.14}$$

$$\bar{\nabla}_a \bar{\nabla}_b \bar{\nabla}^b \gamma - \bar{\nabla}_b \bar{\nabla}_a \bar{\nabla}^b \gamma = -\bar{R}_{ab} \bar{\nabla}^b \gamma\;; \tag{B.15}$$

where in Eq. (B.13) we have used the identity, holding in general [542, 250],

$$\nabla_a {R_{bcd}}^a = \nabla_c R_{bd} - \nabla_b R_{cd} \tag{B.16}$$

but applied here to the background quantities. Using these expressions, we find at the end

$$\begin{aligned} \bar{\nabla}^b \mathcal{G}_{ab} = & \frac{1}{2} \big(2\bar{R}_{ab} \bar{\nabla}_c \gamma^{bc} + 2\gamma^{bc} \bar{\nabla}_b \bar{R}_{ac} - \bar{R}_{ab} \bar{\nabla}^b \gamma \\ & + \bar{R}_{bc} \bar{\nabla}_a \gamma^{bc} - \bar{R} \bar{\nabla}^b \gamma_{ab} - \gamma_{ab} \bar{\nabla}^b \bar{R} \big)\;, \end{aligned} \tag{B.17}$$

which is the requested result.

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Terminat hora diem, terminat auctor opus. Adhuc.