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Contents

Introduction

The present thesis is devoted to the study of a particular decomposition technique for optimal transportation problems with convex cost. The aim is to find a partition of the space in indecomposable subsets that ensures the existence of a map.

First of all let us introduce the problem. Consider two positive functions f and g in $\mathbb{R}^d$ such that their integrals evaluated on the whole space coincide. f will be the initial distribution of mass and g the final. The "optimal transportation problem" consists in finding the "best way" to move the mass from the initial to the final place.

This problem was introduced in 1781 by Gaspard Monge. He considered a function $t : \mathbb{R}^d \to \mathbb{R}^d$ for which the local balance of mass holds:

$$\int_{t^{-1}(E)} f(x)dx = \int_E g(y)dy, \quad \text{for any } E \subset \mathbb{R}^d \text{ Borel.} \tag{1}$$

Such a function is called *transport map.*

Monge asked to find the best transport map where "best" means solving the following problem (*Monge Problem*):

$$\inf\Big\{ \int_{\mathbb{R}^d} |x - t(x)| f(x)dx : t \text{ is a transport}\Big\}. \tag{2}$$

From this idea many developments followed. Two of them are particularly worthy to be mentioned. The first is due to L. Kantorovich: in 1942 he applied the machinery of linear programming to the Monge problem to solve economics problems. The second is due to Y. Brenier: in 1987 he used optimal transportation to prove a new projection theorem on the set of measure preserving maps. His aim was the application to fluid mechanics. Optimal transportation theory has been used for a wide number of application later: nonlinear partial differential equations (PDEs), calculus of variations, probability, economics, statistical mechanics, and many other fields. Nevertheless the core of this problem is far from being trivial.

As we said above, this thesis is devoted to approach the Monge problem by means of a decomposition argument. This task has been completed successfully for a strictly convex norm cost by L. Caravenna in [12] and for the case of a general norm cost by S. Bianchini and S. Daneri in [8]. In this thesis we develop their arguments in order to face lower semicontinuous and convex cost functions.

Since Monge wrote his *"Mémoire sur la théorie des déblais et des remblais"* where he introduced the problem, the original statement has been generalized in many directions. For instance, the euclidean norm $|\cdot|$ has been replaced by a general cost c and f, g by two positive probability measures μ, ν for which a local balance condition is required:

$$\nu(E) = \mu(t^{-1}(E)), \quad \text{for any } E \subset \mathbb{R}^d \text{ measurable.}$$

As we mentioned before, the solution to the Monge problem is far from being trivial even in Euclidean spaces. Consider for instance as μ a Dirac delta and as ν the sum of two Dirac deltas. It is immediate to see that in this case the Monge problem does not admit solutions. Moreorer, looking at (2) it is evident that the constraint on the transport map t is non linear. This leads to a lot of difficulties in finding solutions.

To avoid this problem, in 1942 L.V. Kantorovich suggested a notion of weak solution to the Monge problem. His idea consists in looking for *transference plans* instead of transport maps (*Monge-Kantorovich Problem*). A *transference plan* is a probability measure π in $\mathbb{R}^d \times \mathbb{R}^d$ such that its first marginal is equal to μ $\big($i.e. $\mu(A) = \pi(A \times \mathbb{R}^d)$ for every A Borel set$\big)$ and its second marginal is equal to ν $\big($i.e. $\nu(B) = \pi(\mathbb{R}^d \times B)$ for every B Borel set$\big)$. Denoting by $\Pi(\mu,\nu)$ the class of plans, the problem becomes the following:

$$\min \bigg\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathsf{c}(x,y)\pi(\mathrm{d}x\mathrm{d}x') : \pi \in \Pi(\mu,\nu) \bigg\}. \tag{3}$$

Contrary to the original one, the dependence of the new problem from π is linear. Moreover, since the constraint $\pi \in \Pi(\mu,\nu)$ is convex, weak-* topology can be used to provide existence of solutions in the case c is lower semicontinuous (see [21, 22]). Notice that when a plan is concentrated on a graph we recover the Monge problem and when t is a solution to the Monge problem $(Id \times t)_\sharp \mu$ is an admissible plan. In this sense the problem proposed by Kantorovich is a weak formulation of the original problem.

A key point in the history of the optimal transportation is the *duality formulation* suggested by Kantorovich. This formulation is linked to the linear programming and it starts from an easy idea: consider two functions $\phi \in L^1(\mu)$ and $\psi \in L^1(\nu)$ such that $\psi(y) - \phi(x) \leq \mathsf{c}(x,y)$ for π-almost every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, where $\pi \in \Pi(\mu,\nu)$ is optimal for the problem (3). The idea consists in maximizing $\psi(y) - \phi(x)$ instead of minimising $\mathsf{c}(x,y)$. This change of point of view is very important because under mild assumption the following proposition holds:

(0.1) **PROPOSITION:** *Let* c *be a lower semi-continuous nonnegative function on*

$\mathbb{R}^d \times \mathbb{R}^d$, *and let* μ, ν *bet two Borel Probability measures on* $\mathbb{R}^d$. *Then*

$$\inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathtt{c}(x,y) d\pi = \sup_{\psi(y)-\phi(x) \leq c(x,y)} \int_{\mathbb{R}^d} \psi(y) d\nu - \int_{\mathbb{R}^d} \phi(x) d\mu.$$

1. The problem

After this brief introduction to the problem we are ready to state the problem. We consider a non negative convex l.s.c. function $\mathtt{c} : \mathbb{R}^d \to \mathbb{R}$ with superlinear growth and we are interested in solving the following optimal transportation problem: given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, find a minimizer π of the problem

$$\inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathtt{c}(x' - x)\, \pi(\mathrm{d}x\mathrm{d}x'), \quad \pi \in \Pi(\mu, \nu) \right\}, \tag{4}$$

where $\Pi(\mu, \nu)$ is the set of transference plans $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ, ν respectively. W.l.o.g. we can assume that the above minimum (the transference cost $\mathcal{C}(\mu, \nu)$) is not ∞.

It is well known that in this setting the Monge-Kantorovich problem (4) has a solution (*optimal transference plan*) and a standard question is whether the Monge Problem admits a solution (*optimal transport map*).

In this thesis we prove a decomposition result from which one deduces the existence of an optimal transport map. The result is actually stronger, showing that for any fixed optimal plan $\bar{\pi}$ it is possible to give a partition of the space $\mathbb{R}^d$ into sets $S^h_{\mathfrak{a}}$ which are essentially indecomposable (a precise definition will be given in the following): it is standard from this property of the partition to deduce the existence of an optimal map.

In the case of norm cost, there is a large literature on the existence of optimal maps: see for example [1, 12, 11, 15, 16, 23]. The original Sudakov strategy has been finally implemented in the norm case in [8]. In the case of convex cost, an attempt to use a similar approach of decomposing the transport problems can be found in [14].

In order to state the main result, in addition to the standard family of transference plans $\Pi(\mu, \nu)$ we introduce the notion of *transference plan subjected to a partition*: given $\underline{\pi} \in \Pi(\mu, \nu)$ and a partition $\{S_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})\}_{\mathfrak{a} \in \mathfrak{A}}$ of $\mathbb{R}^d$, with $\mathtt{f} : \mathbb{R}^d \to \mathfrak{A}$ Borel, let $\underline{\pi}_{\mathfrak{a}}$ be the conditional probabilities of the disintegration of $\underline{\pi}$ w.r.t. $\{S_{\mathfrak{a}} \times \mathbb{R}^d\}_{\mathfrak{a}}$,

$$\underline{\pi} = \int \underline{\pi}_{\mathfrak{a}} m(d\mathfrak{a}), \qquad m := \mathtt{f}_\sharp \mu.$$

Define the family of probabilities $\underline{\nu}_{\mathfrak{a}}$ as the second marginal of $\underline{\pi}_a$ (the first being the conditional probability of μ when disintegrated on $\{S_{\mathfrak{a}}\}_{\mathfrak{a}}$). Then set

$$\Pi(\mu, \{\underline{\nu}_{\mathfrak{a}}\}) := \left\{\pi : \pi = \int \pi_{\mathfrak{a}} m(d\mathfrak{a}) \text{ with } \pi_a \in \Pi(\mu_a, \underline{\nu}_a)\right\}.$$

Clearly this is a nonempty subset of $\Pi(\mu, \nu)$.

A second definition is the notion of *cyclically connected sets*. We recall that given a cost $\mathtt{c} : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ and a set $\Gamma \subset \{\mathtt{c} < \infty\}$, the set $S \in \mathbb{R}^d$ is $\mathtt{c}$*-cyclically connected* if for every couple of point $x, x' \in S$ there are a family $(x_i, y_i) \in \Gamma$, $i = 0, \dots, N-1$, such that

$$\mathtt{c}(x_{i+1 \mod N}, y_i) < \infty \qquad \text{and} \qquad x_0 = x,\ x' = x_j \text{ for some } j \in \{0, \dots, N-1\}.$$

When the cost $\mathtt{c}$ is clear from the setting, we will only say *cyclically connected*.

We will need to define the disintegration of the Lebesgue measure on a partition. The formula of the disintegration of a σ-finite measure ϖ w.r.t. a partition $\{S_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})\}_{\mathfrak{a}}$ is intended in the following sense: fix a strictly positive function f such that $\varpi' := f\varpi$ is a probability and write

$$\varpi = f^{-1}\varpi' = \int \big(f^{-1}\varpi'_{\mathfrak{a}}\big)\sigma(d\mathfrak{a}), \qquad \sigma = \mathtt{f}_\sharp \varpi'.$$

It clearly depends on the choice of f, but not the property of being absolutely continuous as stated below.

We say that a set $S \subset \mathbb{R}^d$ is *locally affine* if it is open in its affine span $\operatorname{aff} S$. If $\{S_{\mathfrak{a}}\}_{\mathfrak{a}}$ is a partition into disjoint locally affine sets, we say that the disintegration is *Lebesgue regular* (or for shortness *regular*) if the disintegration of $\mathcal{L}^d$ w.r.t. the partition satisfies

$$\mathcal{L}^d \llcorner_{\cup_{\mathfrak{a}} S_a} = \int_{\mathfrak{A}} \xi_{\mathfrak{a}} \eta(d\mathfrak{a}), \qquad \xi_{\mathfrak{a}} \ll \mathcal{H}^h \llcorner_{S_{\mathfrak{a}}},\ h = \dim \operatorname{aff} S_{\mathfrak{a}}.$$

At this point we are able to state the main result.

(1.1) THEOREM: *Let $\underline{\pi} \in \Pi(\mu, \nu)$ be an optimal transference plan, with $\mu \ll \mathcal{L}^d$. Then there exists a family of sets $\{S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}\}_{\substack{h=0,\dots,d \\ \mathfrak{a} \in \mathfrak{A}^h}}$ in $\mathbb{R}^d$ such that the following holds:*

1. *$S^h_{\mathfrak{a}}$ is a locally affine set of dimension h;*
2. *$O^h_{\mathfrak{a}}$ is a h-dimensional convex set contained in an affine subspace parallel to* $\operatorname{aff} S^h_{\mathfrak{a}}$ *and given by the projection on $\mathbb{R}^d$ of a proper extremal face of* $\operatorname{epi} \mathtt{c}$*;*
3. *$\mathcal{L}^d(\mathbb{R}^d \setminus \cup_{h,\mathfrak{a}} S^h_{\mathfrak{a}}) = 0$;*
4. *the partition is Lebesgue regular;*
5. *if $\pi \in \Pi(\mu, \{\underline{\nu}^h_{\mathfrak{a}}\})$ then optimality in (4) is equivalent to*

$$\sum_h \int \left[\int \mathbb{1}_{O^h_{\mathfrak{a}}}(x' - x)\pi^h_{\mathfrak{a}}(dxdx')\right] m^h(d\mathfrak{a}) < \infty, \tag{5}$$

where $\pi = \sum_h \int_{\mathfrak{A}^h} \pi^h_{\mathfrak{a}} m^h(d\mathfrak{a})$ is the disintegration of π w.r.t. the partition $\{S^h_{\mathfrak{a}} \times \mathbb{R}^d\}_{h,\mathfrak{a}}$;

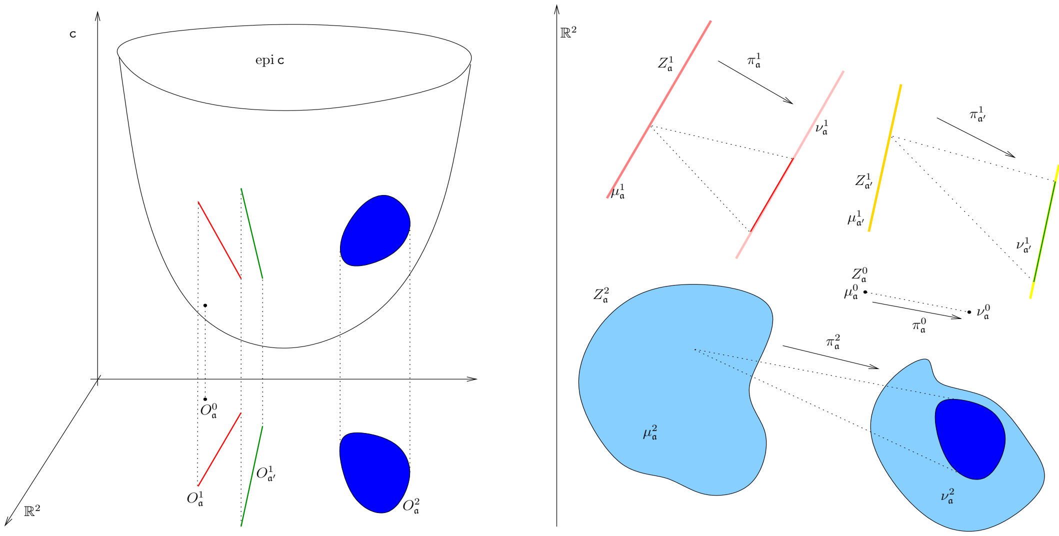


Figure 1: Theorem 1.1 in the plane case, $h = 0, 1, 2$: the projection $O^h_{\mathfrak{a}}$ of the extremal face of $\mathrm{epi}\,\mathsf{c}$ provides the convex cost $\mathbb{1}_{O^h_{\mathfrak{a}}}$ used in the decomposed transportation problems $\pi^h_{\mathfrak{a}} \in \mathcal{P}(Z^h_{\mathfrak{a}} \times \mathbb{R}^d)$.

6. *for every carriage Γ of $\pi \in \Pi(\mu, \{\underline{\nu}^h_{\mathfrak{a}}\})$ there exists a μ-negligible set N such that each $S^h_{\mathfrak{a}} \setminus N$ is $\mathbb{1}_{O^h_{\mathfrak{a}}}$-cyclically connected.*

Using the fact that $\mathsf{c}\llcorner_{O^h_{\mathfrak{a}}}$ is linear, a simple computation allows to write

$$\begin{aligned}
\int \mathsf{c}(x'-x)\pi(dxdx') &= \sum_h \int_{\mathfrak{A}^h} \mathsf{c}(x'-x)\pi^h_{\mathfrak{a}} m^h(d\mathfrak{a}) \\
&= \sum_h \int_{\mathfrak{A}^h} \mathsf{c}(x'-x)\mathbb{1}_{O^h_{\mathfrak{a}}}(x'-x)\pi^h_{\mathfrak{a}} m^h(d\mathfrak{a}) \qquad (6)\\
&= \sum_h \int \bigg[a^h_{\mathfrak{a}} + b^h_{\mathfrak{a}} \cdot \bigg(\int x' \underline{\nu}^h_{\mathfrak{a}} - \int x \mu^h_{\mathfrak{a}} \bigg) \bigg] m^h(d\mathfrak{a}),
\end{aligned}$$

where $a^h_{\mathfrak{a}} + b^h_{\mathfrak{a}} \cdot x$ is a support plane of the face $O^h_{\mathfrak{a}}$.

Following the analysis of [8], the decomposition $\{S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ will be called *Lyapunov decomposition* subjected to the plan $\underline{\pi}$. Note that the indecomposability of $S^h_{\mathfrak{a}}$ yields a *uniqueness* of the decomposition in the following sense: if $\{S^k_{\mathfrak{b}}, O^k_{\mathfrak{b}}\}_{k,\mathfrak{b}}$ is another partition, then by (5) one obtains that $O^k_{\mathfrak{b}} \subset O^h_{\mathfrak{a}}$ (or $O^h_{\mathfrak{a}} \subset O^k_{\mathfrak{b}}$) on $S^k_{\mathfrak{b}} \cap S^h_{\mathfrak{a}}$ (up to μ-negligible sets), and then Point (6) of the above theorem gives that $S^k_{\mathfrak{b}} \subset S^h_{\mathfrak{a}}$ (or $S^h_{\mathfrak{a}} \subset S^k_{\mathfrak{b}}$). But then the indecomposability condition for $\{S^h_a, O^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ (or $\{S^k_{\mathfrak{b}}, O^k_{\mathfrak{b}}\}_{k,\mathfrak{b}}$) is violated.

We remark again that the indecomposability is valid only in the convex set $\Pi(\mu, \{\underline{\nu}^h_{\mathfrak{a}}\}) \subset \Pi(\mu,\nu)$, in general by changing the plan $\underline{\pi}$ one obtains another decomposition. In the case $\nu \ll \mathcal{L}^d$, this decomposition is independent on $\underline{\pi}$: this is proved at the end of Section I.9, Theorem I.9.2.

2. Preliminary results

In order to illustrate the main result, we present some special cases. A common starting point is the existence of a couple of potentials φ, ψ (see [24, Theorem 1.3]) such that

$$\psi(x') - \varphi(x) \leq \mathtt{c}(x'-x) \quad \text{for all } x, x' \in \mathbb{R}^d$$

and

$$\psi(x') - \varphi(x) = \mathtt{c}(x'-x) \quad \text{for } \pi\text{-a.e. } (x,x') \in \mathbb{R}^d \times \mathbb{R}^d, \tag{7}$$

where π is an arbitrary optimal transference plan. Since μ, ν have compact support and $\mathtt{c}$ is locally Lipschitz, then φ, ψ can be taken Lipschitz, in particular $\mathcal{L}^d$-a.e. differentiable. By (7) and the assumption $\mu \ll \mathcal{L}^d$ one obtains that for π-a.e. (x,x') the gradient $\nabla\varphi$ satisfies the inclusion

$$\nabla\varphi(x) \in \partial^- \mathtt{c}(x'-x), \tag{8}$$

being $\partial^- \mathtt{c}$ the subdifferential of the convex function $\mathtt{c}$.

Assume now $\mathtt{c}$ strictly convex. Being the proper extremal faces of epi $\mathtt{c}$ only points, the statement of Theorem (1.1) gives that the decomposition is trivially $\big\{\{x\}, O_x\big\}_x$, where O_x is some vector in $\mathbb{R}^d$. In this case for all $p = \nabla\varphi(x)$ there exists a unique $q = x' - x$ such that (8) holds. Then one obtains that $O_x = \{q\}$.
The second case is when $\mathtt{c}$ is a convex norm: in this case the sets $O^h_{\mathfrak{a}}$ become cones $C^h_{\mathfrak{a}}$. This case has been studied in [8]: in the next section we will describe this result more deeply, because our approach is heavily based on their result.
The cases of convex costs with convex constraints or of the form $h(\|x'-x\|)$, with $h : \mathbb{R}^+ \to \mathbb{R}^+$ strictly increasing and $\|\cdot\|$ an arbitrary norm in $\mathbb{R}^2$ are studied in [14].

As an application of these reasonings, we show how (8) can be used in order to construct of an optimal map, i.e. a solution of the Monge transportation problem with convex cost (see [13]): indeed, one just minimize among $\pi \in \Pi(\mu, \{\underline{\nu}^h_{\mathfrak{a}}\})$ the secondary cost $|\cdot|^2/2$ ($|\cdot|$ being the standard Euclidean norm), and by the cyclically connectedness of $S^h_{\mathfrak{a}}$ one obtains potentials $\{\varphi^h_{\mathfrak{a}}, \phi^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$. Since μ, ν have compact support, then again these potentials are $\mu^h_{\mathfrak{a}}$-a.e. differentiable, and a simple computation shows that $x'-x$ is the unique minimizer of

$$\frac{|p|^2}{2} - \nabla\phi(x)\cdot p + \mathbb{1}_{O^h_{\mathfrak{a}}}(p).$$

The fact that this construction is Borel regular w.r.t. $h, \mathfrak{a}$ is standard, and follows by the regularity properties of the map $h, \mathfrak{a} \to S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}$ in appropriate Polish spaces, see the definitions at the beginning of Section I.3.

(2.1) COROLLARY: *There exists an optimal map* $\mathtt{T} : \mathbb{R}^d \to \mathbb{R}^d$ *such that* $(\mathbb{I}, \mathtt{T})_\sharp\mu$ *is an optimal transference plan belonging to* $\Pi(\mu, \{\underline{\nu}^h_{\mathfrak{a}}\})$.

Note that by varying $\underline{\pi}$ and the secondary cost one obtains infinitely many different optimal maps.

(2.2) REMARK: *In the proof we will only consider the case of μ, ν compactly supported. This assumption avoids some technicalities, and it is fairly easy to recover the general case.*

Indeed, let $K_n \nearrow \mathbb{R}^d$ be a countable family of compact sets and consider $\pi_n := \pi \llcorner_{K_n \times K_n}$. Assume that Theorem 1.1 is proved for all π_n: let $(S^{h,n}_{\mathfrak{a}}, O^{h,n}_{\mathfrak{a}})$ be the corresponding decomposition. Up to reindexing and regrouping the sets, one can take $S^{h,n}_{\mathfrak{a}} \nearrow S^h_{\mathfrak{a}}$, and since $\dim O^{h,n}_{\mathfrak{a}}$ is increasing with n, then $O^{h,n}_{\mathfrak{a}} = O^h_{\mathfrak{a}}$ for n sufficiently large. Hence $\{S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ is the desired decomposition.

3. Description of the approach

The main idea of the proof is to recast the problem in $\mathbb{R}^{d+1}$ with a 1-homogeneous cost $\bar{c}$ and use the strategy developed in [8].

Define

$$\bar{\mu} := (1, \mathbb{I})_\sharp \mu, \qquad \bar{\nu} := (0, \mathbb{I})_\sharp \nu,$$

and the cost

$$\bar{\mathtt{c}}(t,x) := \begin{cases} t\,\mathtt{c}\big(-\frac{x}{t}\big) & t > 0, \\ \mathbb{1}_{(0,0)} & t = 0, \\ +\infty & \text{otherwise}, \end{cases} \tag{9}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. It is clear that the minimisation problem (4) is equivalent to

$$\int_{(\mathbb{R}^+ \times \mathbb{R}^d) \times (\mathbb{R}^+ \times \mathbb{R}^d)} \bar{\mathtt{c}}(t - t', x - x')\, \bar{\pi}(\mathrm{d}t \mathrm{d}x \mathrm{d}t' \mathrm{d}x'), \qquad \bar{\pi} \in \Pi(\bar{\mu}, \bar{\nu}). \tag{10}$$

In particular, every optimal plan π for the problem (4) selects an optimal $\bar{\pi} := \big((1,\mathbb{I}) \times (0,\mathbb{I})\big)_\sharp \pi$ for the problem (10) and viceversa.

The potentials $\bar{\phi}$, $\bar{\psi}$ for (10) can be constructed by the Lax formula from the potentials ϕ, ψ of the problem (4):

$$\begin{aligned} \bar{\phi}(t,x) &:= \min_{x' \in \mathbb{R}^d} \big\{ -\psi(x') + \bar{\mathtt{c}}(t, x - x') \big\}, \qquad t \geq 0 \\ \bar{\psi}(t,x) &:= \max_{x' \in \mathbb{R}^d} \big\{ -\phi(x') - \bar{\mathtt{c}}(1 - t, x' - x) \big\}, \qquad t \leq 1. \end{aligned} \tag{11}$$

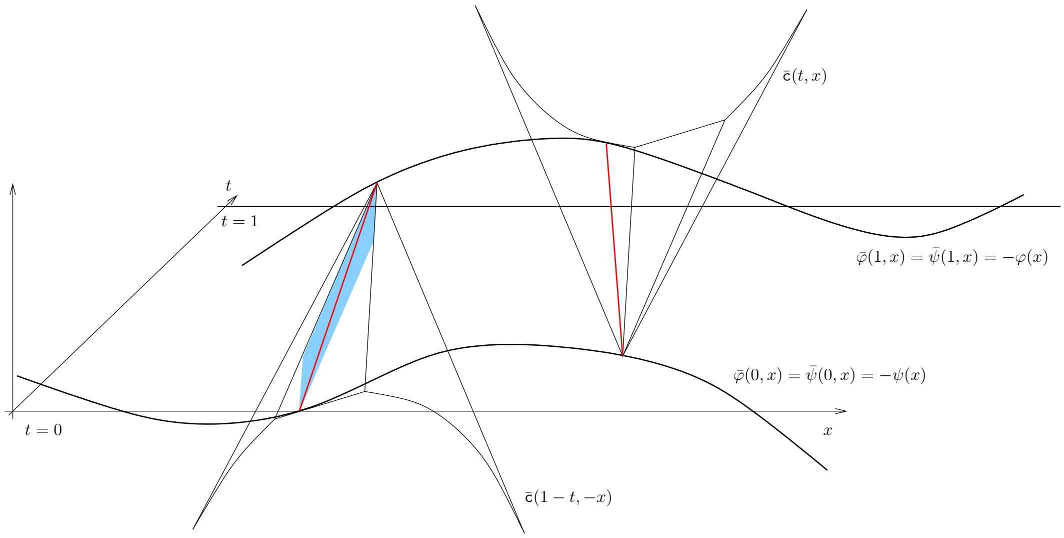


Figure 2: The formulation in $\mathbb{R}^{d+1}$ as a HJ equation: in general in the common region $0 \leq t \leq 1$ it holds $\bar{\psi} \leq \bar{\phi}$, but in the (red) optimal rays and the depicted region the equality holds.

It clearly holds

$$\bar{\phi}(0,x) = \bar{\psi}(0,x) = -\psi(x) \qquad \text{and} \qquad \bar{\phi}(1,x) = \bar{\psi}(1,x) = -\phi(x),$$

so that the function $\bar{\phi}, \bar{\psi}$ are at $t = 0,1$ conjugate forward/backward solutions of the Hamilton-Jacoby equation

$$\partial_t u + H(\nabla u) = 0, \tag{12}$$

with Hamiltonian $H = (\mathsf{c})^*$, the Legendre transform of c. (This is actually the reason for the choice of the minus sign in the definition of (9).)

By standard properties of solutions to (12) one has

$$\bar{\phi}(t,x) - \bar{\phi}(t',x') \leq \bar{\mathsf{c}}(t-t', x-x'), \qquad \text{for every } t \geq t' \geq 0, \ x,x' \in \mathbb{R}^d,$$

and for all $\bar{\pi}$ optimal

$$\bar{\phi}(z) - \bar{\phi}(z') = \bar{\mathsf{c}}(z-z'), \qquad \text{for } \bar{\pi}\text{-a.e. } z = (t,x), z' = (t',x') \in [0,+\infty) \times \mathbb{R}^d.$$

Being $\bar{\phi}$ a potential, it holds:

$$\bar{\phi}(t,x) - \bar{\phi}(t',x') \leq \bar{\mathsf{c}}(t-t', x-x'), \qquad \text{for every } t > t' \in \mathbb{R}^+ \text{ and } x,x' \in \mathbb{R}^d.$$

and for all $\bar{\pi}$ optimal

$$\bar{\phi}(z) - \bar{\phi}(z') = \bar{\mathsf{c}}(z-z'), \qquad \bar{\pi}\text{-a.e. } z = (t,x), z' = (t',x') \in \mathbb{R}^d.$$

Being $\bar{c}$ a 1-homogeneous cost, one can use the same approach of [17] in order to obtain a first *directed locally affine partition* $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$, where $Z^h_{\mathfrak{a}}$ is a relatively open (in its affine span) set of affine dimension $h+1$, $h \in \{0,\dots,d\}$, and $C^h_{\mathfrak{a}}$ is a convex extremal face of $\bar{c}$ (a *cone* due to 1-homogeneity) obtained by

$$C^h_{\mathfrak{a}} = \mathbb{R}^+ \cdot \partial^+ \bar{\phi}(z), \qquad z = (t,x) \in Z^h_{\mathfrak{a}}.$$

The definition of $\partial^+ \bar{\phi}$ is the standard formula

$$\partial^+ \bar{\phi}(z) := \big\{ z' \in [0,+\infty) \times \mathbb{R}^d : \bar{\phi}(z') - \bar{\phi}(z) = \bar{c}(z'-z) \big\}.$$

By the results of [8], this first decomposition satisfies already many properties stated in Theorem 1.1:

1. $Z^h_{\mathfrak{a}}$ is locally affine of dimension $h+1$;
2. $C^h_{\mathfrak{a}}$ is an extremal cone of $\bar{c}$ of dimension $h+1$ parallel to $Z^h_{\mathfrak{a}}$;
3. $\bar{\mu}(\cup_{h,a} Z^h_{\mathfrak{a}}) = 1$;
4. $\bar{\pi}' \in \Pi(\bar{\mu}, \bar{\nu})$ is optimal iff

$$\sum_{h=0}^d \int \bigg[\int \mathbb{1}_{C^h_{\mathfrak{a}}}(z'-z) (\bar{\pi}')^h_{\mathfrak{a}} \bigg] m(d\mathfrak{a}) < \infty,$$

begin $\bar{\pi}' = \sum_h \int (\bar{\pi}')^h_{\mathfrak{a}} m(d\mathfrak{a})$ the disintegration of $\bar{\pi}'$ w.r.t. $\{Z^h_{\mathfrak{a}}, \times \mathbb{R}^d\}_{h,\mathfrak{a}}$.

We note here that this decomposition is *independent* on $\bar{\pi}$, because it is only based on the potentials $\bar{\phi}$, $\bar{\psi}$. Observe that the choice of the signs in (11) yields that z and z' are exchanged w.r.t. x, x' in (5).

A family of sets $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ satisfying the first two points above (plus some regularity properties) will be called *directed locally affine partition*; the precise definition can be found in Definition I.3.1.

While the indecomposability stated in Point (6) is know to be not true also in the norm cost case, the main problem we face here is that the regularity of the partition is stated in terms of the Lebesgue measure $\mathcal{L}^{d+1}$, and this has no direct implication on the structure of the disintegration of $\bar{\mu}$, begin the latter supported on $\{t = 1\}$.
The first new result is thus the fact that, due to the transversality of the cones $C^h_{\mathfrak{a}}$ w.r.t. the plane $\{t=1\}$, $\cup_{h,\mathfrak{a}} Z^h_{\mathfrak{a}} \cap \{t=1\}$ is $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$−conegligible and the disintegration of $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$ w.r.t. $Z^h_{\mathfrak{a}}$ is *regular* for all $\bar{t} > 0$, i.e.

$$\mathcal{H}^d \llcorner_{\{t=\bar{t}\}} = \sum_{h=0}^d \int \xi^h_{\mathfrak{a}} \eta^h(d\mathfrak{a}), \qquad \xi^h \ll \mathcal{H}^h \llcorner_{Z^h_{\mathfrak{a}} \cap \{t=\bar{t}\}}.$$

Note that since $C^h_{\mathfrak{a}}$ is transversal to $\{t = \bar{t}\}$ by the definition of $\bar{\mathtt{c}}$, then $Z^h_{\mathfrak{a}} \cap \{t=\bar{t}\}$ has affine dimension h (and this is actually the reason for the notation). We thus obtain the first result of the thesis, which is a decomposition into a directed locally affine partition which on one hand is *independent* on the optimal transference plan, on the other hand its elements are not indecomposable in the sense of Point (6) of Theorem 1.1.

(3.1) THEOREM: *There exists a directed locally affine partition* $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ *such that*

1. $\mathcal{H}^d(\{t=1\} \setminus \cup_{h,\mathfrak{a}} Z^h_{\mathfrak{a}}) = 0$;
2. *the disintegration of* $\mathcal{H}^d \llcorner_{\{t=1\}}$ *w.r.t. the partition* $\{Z^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ *is regular;*
3. $\bar{\pi}$ *is an optimal plan iff*

$$\sum_h \int \mathbb{1}_{C^h_{\mathfrak{a}}}(z-z')\pi^h_{\mathfrak{a}}(dzdz')m^h(d\mathfrak{a}) < \infty,$$

where $\pi = \sum_h \int \pi^h_{\mathfrak{a}} m^h(d\mathfrak{a})$ *is the disintegration of* π *w.r.t. the partition* $\{Z^h_{\mathfrak{a}} \times \mathbb{R}^{d+1}\}_{h,\mathfrak{a}}$.

Now the technique developed in [8] can be applied to each set $Z^h_{\mathfrak{a}}$ with the cost $C^h_{\mathfrak{a}}$ and marginals $\bar{\mu}^h_{\mathfrak{a}}$ and $\bar{\nu}^h_{\mathfrak{a}}$. As it is shown in [8], the next steps *depend* on the marginal $\bar{\nu}^h_{\mathfrak{a}}$, so that one need to fix a transference plan $\underline{\pi}$ in the theorem. In general this first decomposition is not $\mathbb{1}_{C^h_{\mathfrak{a}}}$-cyclically connected in the sense of Point (6), so that further partitioning has to be performed.

For simplicity we fix here the indexes $h, \mathfrak{a}$, while in general in order to obtain a Borel construction one has to consider also the dependence $h, \mathfrak{a} \mapsto Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}$.

In each $Z^h_{\mathfrak{a}}$ the problem thus becomes a transportation problem with marginals $\mu^h_{\mathfrak{a}}$, $\bar{\nu}^h_{\mathfrak{a}}$ and cost $\mathbb{1}_{C^h_{\mathfrak{a}}}$. The analysis of [8] yields a decomposition of Z into locally affine sets $Z^{h'}_{\beta}$ of affine dimension $h'+1$, together with extremal cones $C^{h'}_{\beta}$ such that $\{Z^{h'}_{\beta}, C^{h'}_{\beta}\}_{h',\beta}$ is a locally affine directed partition.
The main problem is that the *regularity* of the partition refers to the Lebesgue measure in $\mathbb{R}^{h+1}$, while we need to disintegrate $\mu^h_{\mathfrak{a}} \ll \mathcal{H}^h \llcorner_{\{t=1\}}$. The novelty is thus that we use the transversality of the cones C w.r.t. the planes $\{t=\bar{t}\}$ is order to deduce the regularity of the partition.
A similar approach is used also in the decomposition with the potentials above.

Refined partition with cone costs

To avoid heavy notations, in this section we set $\breve{Z} = Z^h_{\mathfrak{a}}$, $\breve{C} = C^h_{\mathfrak{a}}$ and with a slight abuse of notation $\breve{\mu} = \bar{\mu}^h_{\mathfrak{a}}$, $\breve{\nu} = \underline{\nu}^h_{\mathfrak{a}}$.

Fix a carriage

$$\breve{\Gamma} \subset \{w - w' \in \breve{C}\} \cap \big(\{t=1\} \times \{t=0\}\big)$$

of a transport plan $\breve{\pi} \in \Pi(\breve{\mu}, \breve{\nu})$ of $\mathbb{1}_{\breve{C}}$-finite cost, and let $\mathtt{w}_n$ be countably many points such that

$$\{\mathtt{w}_n\}_n \subset \mathtt{p}_1\Gamma \subset \mathrm{clos}\{\mathtt{w}_n\}_n,$$

where $\mathtt{p}_i$ denotes the projection on the i-th component of $(w,w') \in \mathbb{R}^h \times \mathbb{R}^h$, $i=1,2$.

For each n define the set H_n of points which can be reached from $\mathtt{w}_n$ with an axial path of finite cost,

$$H_n := \Big\{ w : \exists I \in \mathbb{N}, \big\{(w_i, w'_i)\big\}_{i=1}^I \subset \breve{\Gamma} \ \big(w_1 = \mathtt{w}_n \ \wedge \ w_{i+1} - w'_i \in \breve{C}\big) \Big\},$$

and let the function θ' be given by

$$\theta'(w) := \sum_n 3^{-n} \chi_{H_n}(w).$$

Notice that θ' depends on the set Γ and the family $\{\mathtt{w}_n\}_n$.

The fact that $C \cap \{t = \bar{t}\}$ is a compact convex set of linear dimension h allows to deduce that the sets H_n are of finite perimeter, more precisely the topological boundary $\partial H_n \cap \{t = \bar{t}\}$ is $\mathcal{H}^{h-1}$-locally finite, and that θ' is SBV in $\mathbb{R}^+ \times \mathbb{R}^h$.

The first novelty of this thesis is to observe that we can replace θ' with two functions which make explicit use of the transversality of C: define indeed

$$\theta(w) := \sup \Big\{ \theta'(w'), w' \in \mathtt{p}_2\Gamma \cap \{w - C\} \Big\} \tag{13}$$

and let ϑ be the u.s.c. envelope of θ. It is fairly easy to verify that $\theta'(w) = \theta'(w') = \theta(w) = \theta(w')$ for $(w, w') \in \check{\Gamma}$ (Lemma I.6.4), and moreover (13) can be seen as a Lax formula for the HJ equation with Lagrangian $\mathbb{1}_C$.
Again simple computations imply that θ is SBV, and moreover being each level set a union of cones it follows that $\partial\{\theta \geq \vartheta\} \cap \{t = \bar{t}\}$ is of locally finite $\mathcal{H}^{h-1}$-measure. Hence in each slice $\{t = \bar{t}\}$, $\vartheta > \theta$ only in $\mathcal{H}^h$-negligible set, and for ϑ the Lax formula becomes

$$\vartheta(w) := \max \Big\{ \vartheta(w'), w' \in \mathtt{p}_2\Gamma \cap \{w - C\} \Big\}.$$

We now start the analysis of the decomposition induced by the level sets of θ or ϑ. The analysis of [8] yields that up to a negligible set N there exists a locally affine partition $\{Z^{h'}_\beta, C^{h'}_\beta\}_{h',\beta}$: the main point in the proof is to show that the set of the so-called *residual points* are $\mathcal{H}^h$-negligible is each plane $\{t = \bar{t}\}$ and that the disintegration is $\mathcal{H}^h \llcorner_{\{t=1\}}$-regular. Since the three functions differ only on a $\check{\mu}$-negligible set, we use θ to construct the partition and ϑ for the estimate of the residual set and the disintegration: the reason is that if $(w, w') \in \check{\Gamma}$ then $\theta(w) = \theta(w')$, relation which is in general false for ϑ (however they clearly differ on a $\check{\pi}$-negligible set, because $\check{\mu} \ll \mathcal{H}^h \llcorner_{\{t=1\}}$).

The strategy we use can be summarized as follows: first prove regularity results for ϑ and then deduce the same properties for θ up to a $\mathcal{H}^h_{\{t=\bar{t}\}}$-negligible set. We show how this reasoning works in order to prove that optimal rays of θ can be prolonged for $t > 1$: for $\mathcal{H}^h \llcorner_{\{t=1\}}$-a.e. w there exist $\varepsilon > 0$ and $w'' \in w + \check{C} \cap \{t = 1 + \varepsilon\}$ such that $\theta(w'') = \theta(w)$. This property is known in the case of HJ equations, see for example the analysis in [9] (or the reasoning in Section I.4.1).

The advantage of having a Lax formula for ϑ is that for *every* point $w \in \mathbb{R}^+ \times \mathbb{R}^h$ there exists at least one optimal ray connecting w to $t = 0$: the proof follows closely the analysis for the HJ case. Moreover the non-degeneracy of the cone C implies that it is possible to make (several) selections of the initial point $\mathbb{R}^+ \times \mathbb{R}^d \ni w \mapsto w'(w) \in \{t = 0\}$ in such a way along the optimal ray $[\![w, w'(w)]\!]$ the following *area estimate* holds:

$$\mathcal{H}^h(A_t) \geq \left(\frac{t}{\underline{t}}\right)^h \mathcal{H}^h(A_{\underline{t}}), \qquad A_t = \left\{ \left(1 - \frac{t}{\underline{t}}\right) w + \frac{t}{\underline{t}} w'(w), w \in A_{\underline{t}} \right\},$$

where $A_{\underline{t}} \subset \{t = \underline{t}\}$ (see [9, 17] for an overview of this estimate). In particular by letting $\underline{t} \searrow \bar{t}$ one can deduce that $\mathcal{H}^h \llcorner_{\{t=\bar{t}\}}$-a.e. point w belongs to a ray starting in $\{t > \bar{t}\}$. Since θ differs from ϑ in a $\mathcal{H}^h_{\{t=\bar{t}\}}$-negligible set, one deduce that the same property holds also for optimal rays of θ.

The property that the optimal rays can be prolonged is the key point in order to show that the residual set N is $\mathcal{H}^h \llcorner_{\{t=\bar{t}\}}$-negligible for all $\bar{t} > 0$ and that the disintegration is regular.

The technique to obtain the indecomposability of Point (6) is now completely similar to the approach in [8]. For every $\breve{\Gamma}$, $\{\mathtt{w}_n\}_n$ one construct the function $\theta_{\Gamma,\mathtt{w}_n}$ and the equivalence relation

$$E_{\Gamma,\mathtt{w}_n} := \big\{\theta_{\Gamma,\mathtt{w}_n}(w) = \theta_{\Gamma,\mathtt{w}_n}(w')\big\},$$

then prove that there is a minimal equivalence relation $\bar{E}$ given again by some function $\bar{\theta}$, and deduce from the minimality that the sets of *positive* $\breve{\mu}$-measure are not further decomposable. Since $\breve{\mu} \ll \mathcal{H}^h \llcorner_{\{t=1\}}$, one can prove that Point (6) of Theorem 1.1 holds.

We thus obtain the following theorem.

(3.2) THEOREM: *Given a directed locally affine partition* $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ *and a transference plan* $\underline{\pi} \in \Pi(\bar{\mu}, \bar{\nu})$ *such that*

$$\underline{\pi} = \sum_h \int \underline{\pi}^h_{\mathfrak{a}} m^h(d\mathfrak{a}), \qquad \int \mathbb{1}_{C^h_{\mathfrak{a}}}(z - z')\underline{\pi}^h_{\mathfrak{a}}(dzdz') < \infty, \tag{14}$$

then there exists a directed locally affine partition $\{Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}, C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}\}_{h,\mathfrak{a},\ell,\mathfrak{b}}$ *such that*

1. $Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}} \subset Z^h_{\mathfrak{a}}$ *has affine dimension* $\ell + 1$ *and* $C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}$ *is an* $(\ell+1)$*-dimensional extremal cone of* $C^h_{\mathfrak{a}}$*; moreover* $\mathrm{aff}\, Z^h_a = \mathrm{aff}(z + C^h_{\mathfrak{a}})$ *for all* $z \in Z^h_{\mathfrak{a}}$*;*
2. $\mathcal{H}^d(\{t = 1\} \setminus \cup_{h,\mathfrak{a},\ell,\mathfrak{b}} Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}) = 0$*;*
3. *the disintegration of* $\mathcal{H}^d \llcorner_{\{t=1\}}$ *w.r.t. the partition* $\{Z^\ell_{\mathfrak{c}}\}_{\ell,\mathfrak{c}}$*,* $\mathfrak{c} = (\mathfrak{a}, \mathfrak{b})$*, is regular, i.e.*

$$\mathcal{H}^d \llcorner_{\{t=1\}} = \sum_\ell \int \xi^\ell_{\mathfrak{c}} \eta^\ell(d\mathfrak{c}), \qquad \xi^\ell_{\mathfrak{c}} \ll \mathcal{H}^\ell \llcorner_{Z^\ell_{\mathfrak{c}} \cap \{t=1\}};$$

4. *if* $\bar{\pi} \in \Pi(\bar{\mu}, \{\underline{\nu}^h_{\mathfrak{a}}\})$ *with* $\underline{\nu}^h_{\mathfrak{a}} = (\mathtt{p}_2)_\sharp \underline{\pi}^h_{\mathfrak{a}}$*, then* $\bar{\pi}$ *satisfies* (14) *iff*

$$\bar{\pi} = \sum_\ell \int \bar{\pi}^\ell_{\mathfrak{c}} m^\ell(d\mathfrak{c}), \qquad \int \mathbb{1}_{C^\ell_{\mathfrak{c}}}(z - z')\bar{\pi}^\ell_{\mathfrak{c}} < \infty;$$

5. *if* $\ell = h$*, then for every carriage* Γ *of any* $\bar{\pi} \in \Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$ *there exists a* $\bar{\mu}$*-negligible set* N *such that each* $Z^{h,h}_{\mathfrak{a},\mathfrak{b}} \setminus N$ *is* $\mathbb{1}_{C^{h,h}_{\mathfrak{a},\mathfrak{b}}}$*-cyclically connected.*

The proof of Theorem 1.1 is now accomplished by repeating the reasoning at most d times as follows.

First one uses the decomposition of Theorem 3.1 to get a first directed locally affine partition.

Then starting with the sets of maximal dimension d, one uses Theorem 3.2 in order to obtain (countably many) indecomposable sets of affine dimension $d+1$ as in Point (5) of Theorem 3.2. The remaining sets forms a directed locally affine partition with sets of affine dimension $h \leq d$. Note that if $C^h_{\mathfrak{a}}$ is an extremal face of $\bar{\mathtt{c}}$ and $C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}$ is an extremal fact of $C^h_{\mathfrak{a}}$, then clearly $C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}$ is an extremal face of $\bar{\mathtt{c}}$.
Applying Theorem 3.2 to this remaining locally affine partition, one obtains indecomposable sets of dimension $h+1$ and a new locally affine partition made of sets with affine dimension $\leq h$, and so on.

The last step is to project the final locally affine partition $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ of $\mathbb{R}^+ \times \mathbb{R}^d$ made of indecomposable sets (in the sense of Point (5) of Theorem 3.2) in the original setting $\mathbb{R}^d$. By the definition of $\bar{\mathtt{c}}$ it follows that $c(-x) = \bar{\mathtt{c}}(1,x)$, so that any extremal cone $C^h_{\mathfrak{a}}$ of $\bar{\mathtt{c}}$ corresponds to the extremal face $O^h_{\mathfrak{a}} = -C^h_{\mathfrak{a}} \cap \{t=1\}$ of c. Thus the family

$$S^h_{\mathfrak{a}} := Z^h_{\mathfrak{a}} \cap \{t=1\}, \qquad O^h_{\mathfrak{a}} := -C^h_{\mathfrak{a}} \cap \{t=1\}$$

satisfies the statement, because $\bar{\mu}(\{t=1\}) = \bar{\nu}(\{t=0\}) = 1$.

(3.3) REMARK: *As a concluding remark, we observe that similar techniques work also without the assumption of superlinear growth and allowing c to take infinite values. Indeed, first of all one decomposes the space $\mathbb{R}^d$ into indecomposable sets S_γ w.r.t. the convex cost*

$$C := \operatorname{clos}\{\mathtt{c} < \infty\},$$

using the analysis on the cone cost case. Notice that since w.l.o.g. C has dimension d, this partition is countable.

Next in each of these sets one studies the transportation problem with cost c. Using the fact that these sets are essentially cyclically connected for all carriages Γ, then one deduces that there exist potentials ϕ_β, ψ_β, and then the proof outlined above can start.

The fact that the intersection of C (or of the cones $C^h_{\mathfrak{a}}$) is not compact in $\{t=\bar{t}\}$ can be replaced by the compactness of the support of μ, ν, while the regularity of the functions θ', θ and ϑ depends only on the fact that $C \cap \{t=\bar{t}\}$ is a convex closed set of dimension d (or h for $C^h_{\mathfrak{a}}$).

4. A case study: the bidimensional case

In the last part of this thesis we consider a special case in order to better explain our procedure. The bidimensional case of the problem (4) (i.e. when $d = 2$) is the first case where the function θ is required in the reduction argument. Similarly to the general case we consider the embedding in $[0, +\infty) \times \mathbb{R}^2$, the 1-homogenous convex cost $\bar{\mathtt{c}}$, and the potentials $\bar{\phi}$ and $\bar{\psi}$.

A first directed locally affine partition $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h \in \{0,1,2\}, \mathfrak{a}}$ is given by the existence of the potentials. We recall that this partition is made of $(h + 1)$-dimensional subspaces of $[0, +\infty) \times \mathbb{R}^2$ which their intersection with $\{t = 1\}$ is a h-dimensional convex set corresponding to a face of epi $\bar{\mathtt{c}}$.

Then we investigate each subset of the partition according to their dimension: we fix h and we show how to face each subset.

When $h = 0$ the subsets $\{Z^0_{\mathfrak{a}}, C^0_{\mathfrak{a}}\}_{\mathfrak{a}}$ give naturally a map and $\xi^a_{\mathfrak{a}}$ are Dirac deltas in the disintegration

$$\mathcal{H}^2 \llcorner_{\{t=\bar{t}\} \cap \bigcup_{\mathfrak{a}} Z^0_{\mathfrak{a}}} = \int \xi^0_{\mathfrak{a}} \eta^0(d\mathfrak{a}), \qquad \text{where } \bar{t} > 0.$$

The case $h = 1$ can be refined directly: with an explicit computation we decompose each subset in the sum of 1-dimensional subsets in $[0, +\infty) \times \mathbb{R}^2$ and indecomposable subsets as in point 5 of Theorem 3.2. It remains to prove the regularity of the disintegration: the proof relies in the following proposition:

(4.1) PROPOSITION: *Let $\bar{t} > 0$, U be a relative open subset of $\{t = \bar{t}\}$, and $\{Z^1_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathfrak{a}} \cap U$ a family of segments such that for $\mathcal{H}^2$-almost every $z \in U$ there is $\mathfrak{a}$ such that $z \in Z^1_{\mathfrak{a}}$ and*

$$\textit{for every } \mathfrak{a}, \mathfrak{a}' \in \mathfrak{a}, \quad \mathrm{int}_{\mathrm{rel}} Z^1_{\mathfrak{a}} \cap Z^1_{\mathfrak{a}'} \neq \emptyset \implies \mathfrak{a} = \mathfrak{a}'.$$

Then,

$$\mathcal{H}^2 \llcorner_{\{t=\bar{t}\} \cap \bigcup_{\mathfrak{a}} Z^1_{\mathfrak{a}}} = \int \xi^1_{\mathfrak{a}} \eta^1(d\mathfrak{a}), \qquad \textit{and } \xi^1_{\mathfrak{a}} \ll \mathcal{H}^1.$$

To conclude we treat the case $h = 2$. Contrary to the previous case it is not possible to refine subsets directly and therefore we apply the analysis made with θ in the general case. Notice that the disintegration in this case is trivial.

By the above procedure we refine the partition given by the potentials as in Theorem 3.2 and therefore we can conclude as in Theorem 1.1. This means that also in this case we provide a solution to the Monge problem.

5. Structure of the thesis

The thesis is divided into two main parts: in the first we analyze the general case and in the second the case study.

5.1. THE GENERAL CASE

In detail the first part is organized as follows.

The problem is defined in Section I.1 and the embedded problem in the Subsection I.1.1. In Subsection I.1.2 are introduced the potentials for this problem.

In Section I.2 we introduce some notations and tools we use in the next sections. Apart from standard spaces, we recall some definitions regarding multifunctions and linear/affine subspaces, adapted to our setting. Finally some basic notions on optimal transportation are presented. In particular we define the cyclically monotonicity and introduce the linear preorder.

In Section I.3 we state the fundamental definition of *directed locally affine partition* $\mathbf{D} = \{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$: this definition is the natural adaptation of the same definition in [8], with minor variation due to the presence of the preferential direction t. Proposition I.3.3 shows how to decompose $\mathbf{D}$ into a countable disjoint union of directed locally affine partitions $\mathbf{D}(h,n)$ such that the sets $Z^h_{\mathfrak{a}}$ in $D(h,n)$ have fixed affine dimension, are almost parallel to a given h-dimensional plane V^h_n, the projections of the $Z^h_{\mathfrak{a}}$ on V^h_n contains a given h-dimensional cube, and the projection of $C^h_{\mathfrak{A}}$ on V^h_n is close a given cone C^h_n. The sets $\mathbf{D}(h,k)$ are called *sheaf sets* (Definition I.3.4).

As we said in the introduction, the line of the proof is to refine a directed locally affine partition in order to obtain either indecomposable sets or diminish their dimension of at least 1: in Section I.4 we show how the potentials $\bar{\phi}$, $\bar{\psi}$ can be used to construct a directed locally affine partition. The approach is to associate forward and backward optimal rays to each point in $\mathbb{R}^+ \times \mathbb{R}^d$, and then define the *forward/backward regular transport set*: the precise definition is given in Definition I.4.6, we just want to observe that these points are in some sense *generic*. After proving some regularity properties, Theorems (I.4.14), (I.4.15) and Proposition I.4.16 shows how to construct a directed locally affine partition $\mathbf{D}_{\bar{\phi}} = \{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$, formula (I.35).
The second part of the section shows that the partition induced by $Z^h_{\mathfrak{a}}$ covers all $\{t=1\}$ up to a $\mathcal{H}^d$-negligible set and that the disintegration of $\mathcal{H}^d$ w.r.t. $Z^h_{\mathfrak{a}}$ is regular. Here we need to refine the approach of [8], which gives only the regularity of the disintegration for $\mathcal{L}^{d+1}\llcorner_{t>0}$. Proposition I.4.21 shows that in $\mathcal{H}^d\llcorner_{\{t=1\}}$ is regular and Proposition I.4.22 completes the analysis proving that the conditional probabilities of the disintegration of $\mathcal{H}^d\llcorner_{\{t=1\}}$ are a.c. with respect to $\mathcal{H}^h\llcorner_{Z^h_{\mathfrak{a}}\cap\{t=1\}}$.

The next four sections describe the iterative step: given a directed locally affine partition

$\mathbf{D}$ such that the disintegration of $\mathcal{H}^d \llcorner_{\{t=1\}}$ is regular, obtain a refined locally affine partition $\mathbf{D}'$, again with a regular disintegration, but such that the sets of maximal dimension are indecomposable in the sense of Point (6) of Theorem 1.1.

First of all in Section I.5 we define the notion of optimal transportation problems in a sheaf set $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{\mathfrak{a}}$, with h-fixed: the key point is that the transport can occur only along the directions in the cone $C^h_{\mathfrak{a}}$, see (I.38). For the directed locally affine partition obtained from $\bar{\phi}$, this property is equivalent to the optimality of the transport plan. We report a simple example which shows why from this point onward we need to fix a transference plan, Example I.5.1.

The fact that the elements of a sheaf set are almost parallel to a given plane makes natural to map them into *fibration*, which essentially a sheaf set whose elements $Z^h_{\mathfrak{a}}$ are parallel. This is done in Section I.5.1, and Proposition I.5.4 shown the equivalence of the transference problems.

The proof outlined in Section 3 is developed starting from Section I.6. For any fixed carriage $\Gamma \subset \{t = 1\} \times \{t = 0\}$ we construct in Section I.6.1 first the family of sets H_n, and then the partition functions θ', θ: the properties we needs (mainly the regularity of the level sets) are proved in Section I.6.1. In Section I.6.1 we show how by varying Γ we obtain a family of equivalence relations (whose elements are the level sets of θ) closed under countable intersections.

The next section (Section I.6.2) uses the techniques developed in [6] in order to get a minimal equivalence relation: the conclusion is that there exists a function $\bar{\theta}$, constructed with a particular carriage Γ, which is finer that all other partitions, up to a $\bar{\mu}$-negligible set. The final example (Example I.6.12) address a technical point: it shows that differently from [8] it is not possible to identify the sets of cyclically connected points with the Lebesgue points of the equivalence classes.

Section I.7 strictly follows the approach of [8] in order to obtain from the fibration a refined locally affine partition. Roughly speaking the construction is very similar to the construction with the potential $\bar{\phi}$: one defined the optimal directions and the regular points. After listing the necessary regularity properties of the objects introduced at the beginning of this section, in Section I.7.2 we give the analog partition function of the potential case and obtain the refined locally affine partition $\tilde{\mathbf{D}}' = \{Z^\ell_{\mathfrak{a},\mathfrak{b}}, C^\ell_{\mathfrak{a},\mathfrak{b}}\}_{\ell,\mathfrak{a},\mathfrak{b}}$.

Section I.8 addresses the regularity problem of the disintegration. As said in the introduction, the main idea is to replace $\bar{\theta}$ with its u.s.c. envelope $\bar{\vartheta}$, which has the property that its optimal rays reach $t = 0$ for *all* point in $\mathbb{R}^+ \times \mathbb{R}^d$. A slight variation of the approach used with the potential $\bar{\phi}$ gives that $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$-a.e. point is regular (Proposition I.8.5) for the directed locally affine partition given by $\bar{\vartheta}$. Using the fact that $\bar{\theta} = \bar{\vartheta}$ $\mathcal{H}^h \llcorner_{\{t=\bar{t}\}}$-a.e., one obtains the regularity of $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$-a.e. point for the directed locally affine partition induced by $\bar{\theta}$ (Corollary I.8.6). The area estimate for optimal rays of $\bar{\vartheta}$ (Lemma I.8.3) allows with an easy argument to prove the regularity of the disintegration, Proposition I.8.7.

The final section (Section I.9 explains how the steps outlined in the last four sections can be used in order to obtain the proof of Theorem 1.1.

5.2. A CASE STUDY

The second part is devoted to a case study.

In Section II.1 we state the problem and we find a first directed locally affine partition using the potential. Then we prove the area estimate for the optimal ray for the potential. Successively we apply the area estimate to prove the regularity of this partition and in particular that the residual set is negligible.

An analysis of the first directed locally affine partition with respect to the dimension of the subsets is given in Section II.2. In this section we explain the particular techniques used according to the dimension. In particular Subsection II.2.1 is devoted to comment the case $h = 0$. Subsection II.2.2 is an explicit computation of the case $h = 1$: here it is shown directly how to refine the problem. Subsection II.2.3 introduce the analysis of the case $h = 2$.

This analysis is developed in Section II.3 where we define θ and use it to refine a $Z^2_{\mathfrak{a}}$ in the union of irreducible and lower dimension subsets.

In the last Section II.4 there is a report of the results obtained and the conclusion of the argument.

5.3. APPENDIX

Finally in Appendix A we recall the result of [7] concerning linear preorders and the existence of minimal equivalence relations and their application to optimal transference problems.

The general case

I

I.1. Setting

The main topic of this Thesis is the Monge problem in $\mathbb{R}^d$. We consider a non negative, lower semicontinuous, and convex real valued cost function $c : \mathbb{R}^d \to \mathbb{R}$ and the following problem:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} c(x-x')\,\pi(\mathrm{d}x\mathrm{d}x'), \quad \pi \in \Pi(\mu,\nu). \tag{I.1}$$

Moreover, we assume c with superlinear growth and locally bounded. In particular c is locally Lipschitz and by super linear growth the proper faces of the epigraph of c are bounded.

In this setting it can be proved the existence of a couple of potentials φ and ψ (see [24, Theorem 1.3]) such that:

$$\psi(x') - \varphi(x) \leq c(x-x') \quad \text{for all } x, x' \in \mathbb{R}^d$$

and

$$\psi(x') - \varphi(x) = c(x-x') \quad \text{for } \pi\text{-a.e. } x, x' \in \mathbb{R}^d \times \mathbb{R}^d.$$

I.1.1. EMBEDDING

In the following, we develop the strategy of [8]. To this purpose, we have to highlight an affine structure that it is not evident in the problem I.1 but it become clear if we recast the problem in $[0,+\infty)\times\mathbb{R}^d$.

$$\bar\mu := (1,\mathbb{I})_\sharp\mu, \quad \bar\nu := (0,\mathbb{I})_\sharp\nu,$$

and the cost

$$\bar{\mathsf{c}}(t,x) = \begin{cases} t\,c\big(-\frac{x}{t}\big) & \text{if t} > 0, \\ \mathbb{1}_{(0,0)} & \text{if t} = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$.

It is straightforward that the minimisation problem (I.1) is equivalent to

$$\int_{\big(\mathbb{R}^+\times\mathbb{R}^d\big)\times\big(\mathbb{R}^+\times\mathbb{R}^d\big)} \bar{\mathsf{c}}(t-t',x-x')\,\bar\pi(\mathrm{d}t\mathrm{d}x\mathrm{d}t'\mathrm{d}x'), \quad \bar\pi\in\Pi(\bar\mu,\bar\nu). \tag{I.2}$$

In particular, every optimal plan π for the problem (I.1) selects an optimal $\bar\pi := \big((1,\mathbb{I})\times(0,\mathbb{I})\big)_\sharp\pi$ for the problem (I.2).

For simplicity, we assume the supports of $\bar{\mu}$ and $\bar{\nu}$ are compact sets as stated in Remark 2.2. The general case can be obtained by considering a countable union of disjoint compact sets on which these measures are concentrated.

I.1.2. POTENTIALS

A couple of potentials for (I.2) can be constructed by the Lax formula from the potentials of the problem (I.1):

$$\bar{\phi}(t,x) = \min_{x' \in \mathbb{R}^d} \big\{ -\psi(x') + \bar{\mathsf{c}}(t, x - x') \big\}, \quad t \geq 0 \tag{I.3}$$

and

$$\bar{\psi}(t,x) = \max_{x' \in \mathbb{R}^d} \big\{ -\phi(x') - \bar{\mathsf{c}}(1-t, x' - x) \big\}, \quad t \leq 1.$$

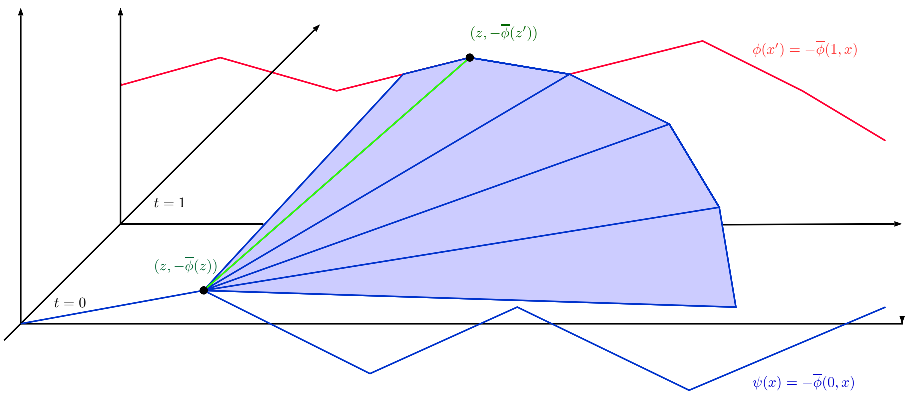


Figure I.1: Equation I.3. The potential $\bar{\phi}$ is constructed from the potentials ϕ and ψ.

Notice that the existence of max and min in our setting is standard since we assumed growth estimate on c and ϕ and ψ can be taken to be globally Lipschitz. Moreover:

$$\bar{\phi}(0,x') = -\psi(x') \quad \text{and} \quad \bar{\phi}(1,x) = -\varphi(x).$$

(I.1.1) REMARK: *Being $\bar{\phi}$ a potential, it holds:*

$$\bar{\phi}(t,x) - \bar{\phi}(t',x') \leq \bar{\mathsf{c}}(t-t', x-x'), \qquad \textit{for every } t > t' \in \mathbb{R}^+ \textit{ and } x, x' \in \mathbb{R}^d.$$

and

$$\bar{\phi}(z) - \bar{\phi}(z') = \bar{\mathsf{c}}(z - z'), \qquad \textit{for } \bar{\pi} \textit{ optimal and } \overline{\pi}\textit{-a.e. } z = (t,x), z' = (t',x') \in [0,+\infty) \times \mathbb{R}^d.$$

I.2. General notations and definitions

As standard notation, we will write $\mathbb{N}$ for the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Q}$ for the rational numbers, $\mathbb{R}$ for the real numbers. The set of positive rational and real numbers will be denoted by $\mathbb{Q}^+$ and $\mathbb{R}^+$ respectively. To avoid the analysis of different cases when parameters are in $\mathbb{R}$ or $\mathbb{N}$, we set $\mathbb{R}^0 := \mathbb{N}$. The first infinite ordinal number will be denoted by ω, and the first uncountable ordinal number is denoted by Ω.

The d-dimensional real vector space will be denoted by $\mathbb{R}^d$. The euclidian norm in $\mathbb{R}^d$ will be denoted by $|\cdot|$. For every $k \leq d$, the open unit ball in $[0,+\infty)\times\mathbb{R}^h$ with center z and radius r will be denoted with $B(z,r)$ and for every $x \in \mathbb{R}^h$, $\bar{t} \geq 0$, $B^h(\bar{t},x,r) := B(\bar{t},x,r) \cap \{t = \bar{t}\}$.

Moreover, for every $a, b \in [0,+\infty) \times \mathbb{R}^d$ define the close segment, the open segment, and the section at $t = \bar{t}$ respectively as :

$$[\![a,b]\!] := \{\lambda a + (1-\lambda)b : \lambda \in [0,1]\}, \quad]\!]a,b[\![:= \{\lambda a + (1-\lambda)b : \lambda \in]0,1[\}, \quad [\![a,b]\!](\bar{t}) := [\![a,b]\!] \cap \{t = \bar{t}\}.$$

The closure of a set A in a topological space X will be written $\operatorname{clos} A$, and its interior by $\operatorname{int} A$. If $A \subset Y \subset X$, then the relative interior of A in Y is $\operatorname{int}_{\mathrm{rel}} A$: in general the space Y will be clear from the context. The topological boundary of a set A will be denoted by ∂A, and the relative boundary is $\partial_{\mathrm{rel}} A$. The space Y will be clear from the context.

If A, A' are subset of a real vector space, we will write

$$A + A' := \{z + z', z \in A, z' \in A'\}.$$

If $T \subset \mathbb{R}$, then we will write

$$TA := \{tz, t \in T, z \in A\}.$$

The convex envelope of a set $A \subset [0,+\infty) \times \mathbb{R}^d$ will be denoted by $\operatorname{conv} A$. If $A \subset [0,+\infty) \times \mathbb{R}^d$, its convex direction envelope is defined as

$$\operatorname{conv}_{\mathrm{d}} A := \{t = 1\} \cap (\mathbb{R}^+ \cdot \operatorname{conv} A).$$

If $x \in \prod_i X_i$, where $\prod_i X_i$ is the product space of the spaces X_i, we will denote the projection on the $\bar{i}$-component as $\mathtt{p}_{\bar{i}} x$ or $\mathtt{p}_{x_{\bar{i}}} x$: in general no ambiguity will occur. Similarly we will denote the projection of a set $A \subset \prod_i X_i$ as $\mathtt{p}_{\bar{i}} A$, $\mathtt{p}_{x_{\bar{i}}} A$. In particular for every $\bar{t} \geq 0$ and $x \in \mathbb{R}^d$, $\mathtt{p}_t(\bar{t}, x) := x$.

I.2.1. Functions and multifunctions

A multifunction $\mathbf{f}$ will be considered as a subset of $X \times Y$, and we will write

$$\mathbf{f}(x) = \{y \in Y : (x,y) \in \mathbf{f}\}.$$

The inverse will be denoted by

$$\mathtt{f}^{-1} = \big\{(y,x) : (x,y) \in \mathtt{f}\big\}.$$

With the same spirit, we will not distinguish between a function $\mathtt{f}$ and its graph graph $\mathtt{f}$, in particular we say that the function $\mathtt{f}$ is *σ-continuous* if graph $\mathtt{f}$ is σ-compact. Note that we do not require that its domain is the entire space.

If $\mathtt{f}$, $\mathtt{g}$ are two functions, their composition will be denoted by $\mathtt{g} \circ \mathtt{f}$.

The epigraph of a function $\mathtt{f} : X \to \mathbb{R}$ will be denoted by

$$\operatorname{epi} \mathtt{f} := \big\{(x,t) : \mathtt{f}(x) \leq t\big\}.$$

The identity map will be written as $\mathbb{I}$, the characteristic function of a set A will be denoted by

$$\chi_A(x) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A, \end{cases}$$

and the indicator function of a set A is defined by

$$\mathbb{1}_A(x) := \begin{cases} 0 & x \in A, \\ \infty & x \notin A. \end{cases}$$

I.2.2. AFFINE SUBSPACES AND CONES

We now introduce some spaces needed in the next sections: we will consider these spaces with the topology given by the Hausdorff distance $\mathtt{d}_{\mathrm{H}}$ of their elements in every closed ball $\operatorname{clos} B(0,r)$ of $\mathbb{R}^d$, i.e.

$$\mathtt{d}(A, A') := \sum_n 2^{-n} \mathtt{d}_{\mathrm{H}}\big(A \cap B(0,n), A' \cap B(0,n)\big).$$

for two generic elements A, A'.

We will denote points in $[0,+\infty) \times \mathbb{R}^d$ as $z = (t,x)$.

For $h, h', d \in \mathbb{N}_0$, $h' \leq h \leq d$, define $\mathcal{G}(h, [0,+\infty) \times \mathbb{R}^d)$ to be the set of $(h+1)$-dimensional subspaces of $[0,+\infty) \times \mathbb{R}^d$ such that their slice at $t = 1$ is a h-dimensional subspace of $\{t = 1\}$, and let $\mathcal{A}(h, [0,+\infty) \times \mathbb{R}^d)$ be the set of $(h+1)$-dimensional affine subspaces of $[0,+\infty) \times \mathbb{R}^d$ such that their slice at $t = 1$ is a h-dimensional affine subspace of $\{t = 1\}$. If $V \in \mathcal{A}(h, [0,+\infty) \times \mathbb{R}^d)$, we define $\mathcal{A}(h', V) \subset \mathcal{A}(h', [0,+\infty) \times \mathbb{R}^d)$ as the $(h'+1)$-dimensional affine subspaces of V such that their slice at time $t = 1$ is a h'-dimensional affine subset.

We define the projection on $A \in \mathcal{A}(h, [0,+\infty) \times \mathbb{R}^d)$ with $\bar{t}$ fixed as $\mathtt{p}^t_A$:

$$\mathtt{p}^t_A(\bar{t}, x) = \big(\bar{t}, \mathtt{p}_{A \cap \{t = \bar{t}\}} x\big).$$

If $A \subset [0,+\infty) \times \mathbb{R}^d$, then define its affine span as

$$\operatorname{aff} A := \bigg\{ \sum_i t_i z_i, i \in \mathbb{N}, t_i \in \mathbb{R}, z_i \in A, \sum_i t_i = 1 \bigg\}.$$

The *linear dimension* of the set $\mathrm{aff}\, A \subset [0,+\infty) \times \mathbb{R}^d$ is denoted by $\dim A$. The orthogonal space to $\mathrm{span}\, A := \mathrm{aff}(A \cup \{0\})$ will be denoted by $A^\perp$. For brevity, in the following the dimension of $\mathrm{aff} A \cap \{t = \bar{t}\}$ will be called *dimension at time* $\bar{t}$ (or if there is no ambiguity *time fixed dimension*) and denoted by $\dim_{\bar{t}} A$.

Let $\mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d)$ be the set of closed convex non degenerate cones in $[0,+\infty) \times \mathbb{R}^d$ with vertex in $(0,0)$ and dimension $h+1$: non degenerate means that their linear dimension is $h+1$ and their intersection with $\{t = 1\}$ is a compact convex set of dimension h. Note that if $C \in \mathcal{C}(h, [0,\infty)\times\mathbb{R}^d)$, then $\mathrm{aff}\, C \in \mathcal{A}(h, [0,\infty)\times\mathbb{R}^d)$ and conversely if $\mathrm{aff}\, C \in \mathcal{A}(h, [0,\infty)\times\mathbb{R}^d)$ and $C \cap \{t = 1\}$ is bounded then $C \in \mathcal{C}(h, [0,\infty) \times \mathbb{R}^d)$.

Set also for $C \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d)$

$$\mathcal{D}C := C \cap \{t = 1\},$$

and

$$\begin{aligned}\mathcal{DC}(h, [0,+\infty) \times \mathbb{R}^d) &:= \big\{\mathcal{D}C : C \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d)\big\} \\ &= \big\{K \subset \{t = 1\} : K \text{ is convex and compact}\big\}.\end{aligned}$$

The latter set is the set of *directions* of the cones $C \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d)$. We will also write for $V \in \mathcal{G}(h, [0,+\infty) \times \mathbb{R}^d)$

$$\mathcal{C}(h', V) := \big\{C \in \mathcal{C}(h', [0,+\infty) \times \mathbb{R}^d) : \mathrm{aff}\, C \subset V\big\}, \quad \mathcal{DC}(h, V) := \big\{\mathcal{D}C : C \in \mathcal{C}(h, V)\big\}.$$

Define $\mathcal{K}(h)$ as the set of all h-dimensional compact and convex subset of $\{t = 1\}$. If $K \in \mathcal{K}(h)$, set the *open set*

$$\mathring{K}(r) := \big(K + B^{d+1}(0, r)\big) \cap \mathrm{aff}\, K. \tag{I.4}$$

Define $K(r)$:= $\mathrm{clos}\, K(r) \in \mathcal{K}(h)$. Notice that $K = \cap_n \mathring{K}(2^{-n})$.

For $r < 0$ we also define the open set

$$\mathring{K}(-r) := \Big\{z \in \{t = 1\} : \exists \epsilon > 0 \ \big(B^{d+1}(z, r + \epsilon) \cap \mathrm{aff}\, K \subset K\big)\Big\}, \tag{I.5}$$

so that $\mathrm{int}_{\mathrm{rel}} K = \cup_n \mathring{K}(-2^{-n})$: as before $K(-r) := \mathrm{clos}\, \mathring{K}(-r) \in \mathcal{K}(h, [0,+\infty) \times \mathbb{R}^d)$ for $0 < -r \ll 1$.

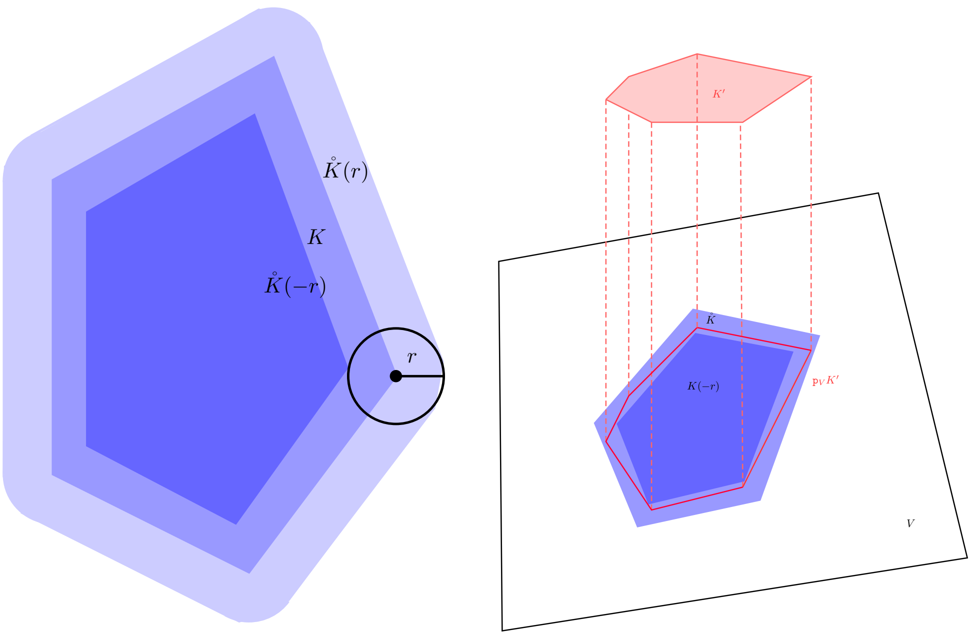
If V is a h-dimensional subspace of $\{t = 1\}$, $K \in \mathcal{K}(h)$ such that $K \subset V$ and given two real numbers $r, \lambda > 0$, consider the subsets $L_{\mathrm{d}}(h, K, r, \lambda)$ of $\mathcal{K}$ defined by

$$\begin{aligned}L_{\mathrm{d}}(h, K, r, \lambda) := \Big\{K' \in \mathcal{K}(h) : \ &(i) \ K(-r) \subset \mathtt{p}_V \mathring{K}', \\ &(ii) \ \mathtt{p}_V K' \subset \mathring{K}, \\ &(iii) \ \mathtt{d}_{\mathrm{H}}(\mathtt{p}_V K', K') < \lambda\Big\}.\end{aligned} \tag{I.6}$$

The subscript d refers to the fact that we are working in $\{t = 1\} \times \mathbb{R}^d$.

Recall that according to the definition of $C \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d)$, $C \cap \{t = 1\}$ is compact. Define

$$L(h, C, r, \lambda) := \Big\{C \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d) : C \cap \{t = 1\} \in L_{\mathrm{d}}(h, K, r, \lambda)\Big\}.$$



(a) The sets $\mathring{K}(-r), \mathring{K}(r)$ for a given $K \in \mathcal{K}(h)$.

(b) The set $L_{\mathrm{d}}(h, K, r, \lambda)$.

Figure I.2: The sets defined in (I.4), (I.5) and (I.6).

It is fairly easy to see that for all $r, \lambda > 0$ the family

$$\mathfrak{L}(h, r, \lambda) := \left\{ L(h, C, r', \lambda'), C \in \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d), 0 < r' < r, 0 < \lambda' < \lambda \right\} \tag{I.7}$$

generates a prebase of neighborhoods of $\mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$. In particular, being the latter separable, we can find countably many sets $L(h, C_n, r_n, \lambda_n)$, $n \in \mathbb{N}$, covering $\mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$, and such that $\big(C_n \cap \{t = 1\}\big)(-r_n) \in \mathcal{K}(h)$.

Let $C \in \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$ and $r > 0$. For simplicity, we define

$$\mathring{C}(r) := \{0\} \cup \mathbb{R}^+ \cdot \Big(\big(\mathcal{D}C + B^{d+1}(0, r)\big) \cap \operatorname{aff} \mathcal{D}C \Big),$$

$$\mathring{C}(-r) := \{0\} \cup \mathbb{R}^+ \cdot \Big\{ z \in \{t = 1\} : \exists \epsilon > 0 \ \big(B^{d+1}(z, r + \epsilon) \cap \operatorname{aff} \mathcal{D}C \subset \mathcal{D}C \big) \Big\},$$

$$C(r) := \operatorname{clos} C(r) \quad \text{and} \quad C(-r) := \operatorname{clos} C(-r).$$

Notice that (I.7) can be rewritten using these new definitions.

I.2.3. PARTITIONS

We say that a subset $Z \subset [0, +\infty) \times \mathbb{R}^d$ is *locally affine* if there exists $h \in \{0, \ldots, d\}$ and $V \in \mathcal{A}(h, [0, +\infty) \times \mathbb{R}^d)$ such that $Z \subset V$ and Z is relatively open in V, i.e. $\operatorname{int}_{\mathrm{rel}} Z \neq \emptyset$.

Notice that we are not considering here 0-dimensional sets (points), because we will not use them in the following.

A *partition* in $[0,+\infty) \times \mathbb{R}^d$ is a family $\mathcal{Z} = \{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ of disjoint subsets of $[0,+\infty) \times \mathbb{R}^d$. We do not require that $\mathcal{Z}$ is a *covering* of $[0,+\infty) \times \mathbb{R}^d$, i.e. $\cup_\mathfrak{a} Z_\mathfrak{a} = [0,+\infty) \times \mathbb{R}^d$.

A *locally affine partition* $\mathcal{Z} = \{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ is a partition such that each $Z_\mathfrak{a}$ is locally affine. We will often write

$$\mathcal{Z} = \bigcup_{k=0}^{d} \mathcal{Z}^h, \quad \mathcal{Z}^h = \big\{ Z_\mathfrak{a}, \mathfrak{a} \in \mathfrak{A} : \dim Z_\mathfrak{a} = h+1 \big\},$$

and to specify the dimension of $Z_\mathfrak{a}$ we will add the superscript $(\dim Z_\mathfrak{a} - 1)$: thus, the sets in $\mathcal{Z}^h$ are written as $Z^h_\mathfrak{a}$, and $\mathfrak{a}$ varies in some set of indexes $\mathfrak{A}^{d-h}$ (the reason of this notation will be clear in the following. In particular $\mathfrak{A}^{d-h} \subseteq \mathbb{R}^{d-h}$).

I.2.4. MEASURES, DISINTEGRATION, AND TRANSFERENCE PLANS

We will denote the Lebesgue measure of $[0,+\infty) \times \mathbb{R}^d$ as $\mathcal{L}^{d+1}$, and the k-dimensional Hausdorff measure on an affine k-dimensional subspace V as $\mathcal{H}^h \llcorner_V$. In general, the restriction of a function/measure to a set $A \in [0,+\infty) \times \mathbb{R}^d$ will be denoted by the symbol $\llcorner_A$ following the function/measure.

The product of two locally finite Borel measures ϖ_0, ϖ_1 will be denoted by $\varpi_0 \otimes \varpi_1$.

The Lebesgue points $\mathrm{Leb}(A)$ of a set $A \subset [0,+\infty) \times \mathbb{R}^d$ are the points $z \in A$ such that

$$\lim_{r\to 0} \frac{\mathcal{L}^{d+1}(A \cap B(z,r))}{\mathcal{L}^{d+1}(B(z,r))} = 1.$$

If ϖ is a locally bounded Borel measure on $[0,+\infty) \times \mathbb{R}^d$, we will write $\varpi \ll \mathcal{L}^{d+1}$ if ϖ is a.c. w.r.t. $\mathcal{L}^{d+1}$, and we say that z is a *Lebesgue point of* $\varpi \ll \mathcal{L}^{d+1}$ if

$$\mathtt{f}(z) > 0 \quad \wedge \quad \lim_{r\to 0} \frac{1}{\mathcal{L}^{d+1}(B(z,r))} \int_{B(z,r)} \big|\mathtt{f}(z') - \mathtt{f}(z)\big| \mathcal{L}^{d+1}(dz') = 0,$$

where we denote by $\mathtt{f}$ the Radon-Nikodym derivative of ϖ w.r.t. $\mathcal{L}^{d+1}$, i.e. $\varpi = \mathtt{f}\mathcal{L}^{d+1}$. We will denote this set by $\mathrm{Leb}\,\varpi$.

For a generic *Polish space* X (i.e., a separable and complete metric space), the Borel sets and the set of Borel probability measures will be respectively denoted by $\mathcal{B}(X)$ and $\mathcal{P}(X)$. The *Souslin sets* Σ^1_1 of a Polish space X are the projections on X of the Borel sets of $X \times X$. The σ-algebra generated by the Souslin sets will be denoted by Θ.

Two Radon measures ϖ_0, ϖ_1 on X are *equivalent* if for all $B \in \mathcal{B}(X)$

$$\varpi_0(B) = 0 \quad \Longleftrightarrow \quad \varpi_1(B) = 0,$$

and we denote this property by $\varpi_0 \simeq \varpi_1$.

If ϖ is a measure on a measurable space X and $\mathtt{f} : X \to Y$ is an ϖ-measurable map, then the *push-forward of* ϖ by $\mathtt{f}$ is the measure $\mathtt{f}_\# \varpi$ on Y defined by

$$\mathtt{f}_\# \varpi(B) = \varpi(\mathtt{f}^{-1}(B)), \quad \text{for all } B \text{ in the } \sigma\text{-algebra of } Y.$$

Finally we briefly recall the concept of disintegration of a measure over a partition.

(I.2.1) DEFINITION: A *partition* in $\mathbb{R}^d$ is a family $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ of disjoint subsets of $\mathbb{R}^d$. We say that $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ is a *Borel partition* if $\mathfrak{A}$ is a Polish space, $\underset{\mathfrak{a}\in\mathfrak{A}}{\cup} Z_\mathfrak{a}$ is Borel and the *quotient map* $\mathtt{h} : \underset{\mathfrak{a}\in\mathfrak{A}}{\cup} Z_\mathfrak{a} \to \mathfrak{A}$, $\mathtt{h} : z \mapsto \mathtt{h}(z) = \mathfrak{a}$ such that $z \in Z_\mathfrak{a}$, is Borel-measurable. We say that $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ is σ-*compact* if $\mathfrak{A} \subset \mathbb{R}^k$ for some $k \in \mathbb{N}$, $\underset{\mathfrak{a}\in\mathfrak{A}}{\cup} Z_\mathfrak{a}$ is σ-compact and $\mathtt{h}$ is σ-continuous.

The sets in the σ-algebra $\{\mathtt{h}^{-1}(F) : F \in \mathcal{B}(\mathfrak{A})\}$ are also called in the literature *saturated sets*. Notice that we do not require $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ to be a covering of $\mathbb{R}^d$.

(I.2.2) DEFINITION: Given a Borel partition in $\mathbb{R}^d$ into sets $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ with quotient map $\mathtt{h} : \underset{\mathfrak{a}\in\mathfrak{A}}{\cup} Z_\mathfrak{a} \to \mathfrak{A}$ and a probability measure $\varpi \in \mathcal{P}(\mathbb{R}^d)$ s.t. $\varpi\big(\underset{\mathfrak{a}\in\mathfrak{A}}{\cup} Z_\mathfrak{a}\big) = 1$, a *disintegration* of ϖ w.r.t. $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ is a family of probability measures $\{\varpi_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}} \subset \mathcal{P}(\mathbb{R}^d)$ such that

$$\mathfrak{A} \ni \mathfrak{a} \mapsto \varpi_\mathfrak{a}(B) \quad \text{is an } \mathtt{h}_\#\varpi\text{-measurable map } \forall\, B \in \mathcal{B}(\mathbb{R}^d), \tag{I.8}$$

$$\varpi\big(B \cap \mathtt{h}^{-1}(F)\big) = \int_F \varpi_\mathfrak{a}(B)\, d\mathtt{h}_\#\varpi(\mathfrak{a}), \quad \forall\, B \in \mathcal{B}(\mathbb{R}^d),\, F \in \mathcal{B}(\mathfrak{A}). \tag{I.9}$$

As proven in Appendix A of [6] (for a more comprehensive analysis see [19]), we have the following theorem.

(I.2.3) THEOREM: *Under the assumptions of Definition I.2.2, the disintegration* $\{\varpi_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ *is* unique *and* strongly consistent*, namely*

$$\text{if } \mathfrak{a} \mapsto \varpi^1_\mathfrak{a},\ \mathfrak{a} \mapsto \varpi^2_\mathfrak{a} \text{ satisfy (I.8)-(I.9)} \quad \Longrightarrow \quad \varpi^1_\mathfrak{a} = \varpi^2_\mathfrak{a} \text{ for } \mathtt{h}_\#\varpi\text{-a.e. } \mathfrak{a} \in \mathfrak{A};$$

$$\varpi_\mathfrak{a}(Z_\mathfrak{a}) = 1 \quad \text{for } \mathtt{h}_\#\varpi\text{-a.e. } \mathfrak{a} \in \mathfrak{A}.$$

The measures $\{\varpi_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ are also called *conditional probabilities*.

To denote the (strongly consistent) disintegration $\{\varpi_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ of a probability measure $\varpi \in \mathcal{P}(\mathbb{R}^d)$ on a Borel partition $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ we will often use the formal notation

$$\varpi = \int_\mathfrak{A} \varpi_\mathfrak{a}\, dm(\mathfrak{a}), \quad \varpi_\mathfrak{a}(Z_\mathfrak{a}) = 1, \tag{I.10}$$

with $m = \mathtt{h}_\#\varpi$, $\mathtt{h}$ being the quotient map.

Since the conditional probabilities $\varpi_\mathfrak{a}$ are defined m-a.e., many properties (such as $\varpi_\mathfrak{a}(Z_\mathfrak{a}) = 1$) should be considered as valid only for m-a.e. $\mathfrak{a} \in \mathfrak{A}$: for shortness, we will often consider the $\varpi_\mathfrak{a}$ redefined on m-negligible sets in order to have statements valid $\forall \mathfrak{a} \in \mathfrak{A}$.

We also point out the fact that, according to Definition I.2.2, in order that a disintegration of ϖ over a partition can be defined, ϖ has to be concentrated on the union of the sets of the partition (which do not necessarily cover the whole $\mathbb{R}^d$). In general, if we remove this assumption, since the formulas (I.8)-(I.9) make sense nonetheless for $B \subset \bigcup_{\mathfrak{a}\in\mathfrak{A}} Z_\mathfrak{a}$, by means of formula (I.10) we "reconstruct" only $\varpi \llcorner_{\bigcup_{\mathfrak{a}\in\mathfrak{A}} Z_\mathfrak{a}}$.

Let $m' \in \mathcal{P}(\mathfrak{A})$, $\{\varpi'_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}} \subset \mathcal{P}(\mathbb{R}^d)$ such that

$$\mathfrak{A} \ni \mathfrak{a} \mapsto \varpi'_\mathfrak{a}(B) \quad \text{is } m'\text{-measurable}, \forall\, B \in \mathcal{B}(\mathbb{R}^d).$$

Then, one can define the probability measure ϖ' on $\mathbb{R}^d$ by

$$\varpi'(B) = \int_\mathfrak{A} \varpi'_\mathfrak{a}(B)\, dm'(\mathfrak{a}), \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^d). \tag{I.11}$$

The measure defined in (I.11) will be denoted as

$$\varpi' = \int_\mathfrak{A} \varpi'_\mathfrak{a}\, dm'.$$

Notice that, despite the notation is the same as in (I.10), the family $\{\varpi'_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ in the above definition is not necessarily a disintegration of ϖ', both because the measure m' is not necessarily a quotient measure of a Borel partition and because the measures $\varpi'_\mathfrak{a}$ are not necessarily concentrated on the sets of a partition. In the rest of the thesis, such an ambiguity will not occur, since we will always point out whether a measurable family of probability measures is generated by a disintegration or not.

(I.2.4) **REMARK:** *If instead of $\varpi \in \mathcal{P}(\mathbb{R}^d)$ we consider the Lebesgue measure $\mathcal{L}^d$ (more generally, a Radon measure) a disintegration $\{\upsilon_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ is to be considered in the following sense. First choose a partition $\{A_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^d$ into sets with unit Lebesgue measure, then let*

$$\mathcal{L}^d \llcorner_{A_i} = \int \upsilon_{\mathfrak{a},i} d\eta_i(\mathfrak{a}), \quad \eta_i := \mathtt{h}_\# \mathcal{L}^d \llcorner_{A_i},$$

be the standard disintegration of the probability measure $\mathcal{L}^d \llcorner_{A_i}$, and finally

$$\upsilon_\mathfrak{a} := \sum_i 2^i \upsilon_{\mathfrak{a},i}, \quad \eta := \sum_i 2^{-i} \eta_i.$$

Clearly, in this definition the "conditional probabilities" $\upsilon_\mathfrak{a}$ and the "image measure" η depend on the choice of the sets $\{A_i\}_{i\in\mathbb{N}}$.

I.2.5. OPTIMAL TRANSPORTATION PROBLEMS

For a generic Polish space X, measures $\mu, \nu \in \mathcal{P}(X)$ and Borel *cost function* $\mathtt{c} : X \times X \to [0, \infty]$, we will consider the sets of probability measures

$$\Pi(\mu, \nu) := \Big\{ \pi \in \mathcal{P}(X \times X) : (\mathtt{p}_1)_\# \pi = \mu, (\mathtt{p}_2)_\# \pi = \nu \Big\}, \tag{I.12}$$

$$\Pi^f_{\mathtt{c}}(\mu,\nu) := \Big\{\pi \in \Pi(\mu,\nu) : \int_{X\times X} \mathtt{c}\, d\pi < +\infty\Big\}, \tag{I.13}$$

$$\Pi^{\mathrm{opt}}_{\mathtt{c}}(\mu,\nu) := \Big\{\pi \in \Pi(\mu,\nu) : \int_{X\times X} \mathtt{c}\, d\pi = \inf_{\pi'\in\Pi(\mu,\nu)} \int_{X\times X} \mathtt{c}\, d\pi'\Big\}. \tag{I.14}$$

The elements of the set defined in (I.12) are called *transference* or *transport plans* between μ and ν, those in (I.13) *transference* or *transport plans with finite cost* and the set defined in (I.14) is the set of *optimal plans*. The quantity

$$\mathtt{C}(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{X\times X} \mathtt{c}\, d\pi$$

is the *transportation cost*.

In the following we will always consider costs and measures s.t. $\mathtt{C}(\mu,\nu) < +\infty$, thus $\Pi^f_{\mathtt{c}}(\mu,\nu) \neq \emptyset$.

(I.2.5) REMARK: *The Monge-Kantorovich problem can be rephrased in this context as* $\Pi^{\mathrm{opt}}_{\mathtt{c}}(\mu,\nu) \neq \emptyset$.

We recall (see e.g. [6, 20]) that any optimal plan $\pi \in \Pi^{\mathrm{opt}}_{\mathtt{c}}(\mu,\nu)$ is $\mathtt{c}$-*cyclically monotone*, i.e. there exists a σ-compact *carriage* $\Gamma \subset X \times X$ such that $\pi(\Gamma) = 1$ and for all $I \in \mathbb{N}$, $\{(x_i, y_i)\}_{i=1}^I \subset \Gamma$,

$$\sum_{i=1}^{I} \mathtt{c}(x_i, y_i) \leq \sum_{i=1}^{I} \mathtt{c}(x_{i+1}, y_i),$$

where we set $x_{I+1} := x_1$. Any such Γ is called $\mathtt{c}$-*cyclically monotone carriage*. However, in order to deduce the optimality of a transference plan the $\mathtt{c}$-cyclical monotonicity condition itself is not sufficient and one has to impose additional conditions. Most of the conditions in the literature exploit the dual formulation of Monge-Kantorovich problem (see [25]), namely

$$\mathtt{C}(\mu,\nu) = \sup_{\substack{\phi,\,\psi:X\to[-\infty,+\infty) \\ \phi\ \mu\text{-meas. and } \psi\ \nu\text{-meas.}}} \Big\{ \int \phi(x)\, d\mu(x) + \int \psi(y)\, d\nu(y) :\ \phi(x) + \psi(y) \leq \mathtt{c}(x,y)\Big\}.$$

For example (see Lemma 5.3 of [6]) if there exists a pair of functions

$$\phi,\, \psi : X \to [-\infty, +\infty), \qquad \phi\ \mu\text{-measurable and } \psi\ \nu\text{-measurable}, \tag{I.15}$$

$$\psi(y) - \phi(x) \leq \mathtt{c}(x,y), \quad \forall\, x, y \in X,$$

$$\psi(y) - \phi(x) = \mathtt{c}(x,y), \quad \pi\text{-a.e. for some } \pi \in \Pi(\mu,\nu), \tag{I.16}$$

then ϕ, ψ are optimizers for the dual problem and $\pi \in \Pi^{\mathrm{opt}}_{\mathtt{c}}(\mu,\nu)$. Conditions on the cost guaranteeing the existence of such potentials (and indeed of more regular ones) are e.g. the following ones:

1. $\mathtt{c}$ is l.s.c. and satisfies $\mathtt{c}(x,y) \leq \mathtt{f}(x) + \mathtt{g}(y)$ for some $\mathtt{f} \in L^1(\mu)$, $\mathtt{g} \in L^1(\nu)$ ([22]);

2. c is real-valued and satisfies the following assumption ([1])

$$\nu\Big(\Big\{y : \int \mathsf{c}(x,y)\, d\mu(x) < +\infty\Big\}\Big) > 0, \quad \mu\Big(\Big\{x : \int \mathsf{c}(x,y)\, d\nu(y) < +\infty\Big\}\Big) > 0;$$

3. $\{\mathsf{c} < +\infty\}$ is an open set O minus a $\mu \otimes \nu$-negligible set N ([3]).

The weakest sufficient condition for optimality, which does not rely on the existence of global potentials and implies the results recalled above, has been given in [6].

I.2.6. LINEAR PREORDERS, UNIQUENESS AND OPTIMALITY

Let $\mathsf{c} : X \times X \to [0, +\infty]$ be a Borel cost function on a Polish space X such that $\mathsf{c}(x,x) = 0$ for all $x \in X$, let $\mu, \nu \in \mathcal{P}(X)$ be such that $\Pi^f_{\mathsf{c}}(\mu,\nu) \neq \emptyset$ and let $\Gamma \subset X \times X$ be a c-cyclically monotone carriage of some $\pi \in \Pi^f_{\mathsf{c}}(\mu,\nu)$ satisfying w.l.o.g. $\{(x,x) : x \in X\} \subset \Gamma$. A standard formula for constructing a pair of optimal potentials is the following: for fixed $(x_0, y_0) \in \Gamma$ and $(x,y) \in \Gamma$, define

$$\begin{aligned} \phi(x) &:= \inf\left\{ \sum_{i=0}^{I} \mathsf{c}(x_{i+1}, y_i) - \mathsf{c}(x_i, y_i) : (x_i, y_i) \in \Gamma, I \in \mathbb{N}, x_{I+1} = x \right\}, \qquad \text{(I.17)} \\ \psi(y) &:= \mathsf{c}(x,y) + \phi(x). \end{aligned}$$

If one of the assumptions (1)-(3) holds, then this ϕ, ψ satisfy (I.15)-(I.16). However, for general Borel costs c, the assumptions (1)-(3) are not satisfied. In particular, for any choice of (x_0, y_0), there may be a set of positive μ-measure on which ϕ is not well defined (namely, the infimum in (I.17) is taken over an empty set) or takes the value $-\infty$ (see the examples in [6]).

To explain why this can happen and briefly recall the strategy adopted in [6] to overcome this problem in a more general setting, we need the following definition.

(I.2.6) DEFINITION: An *axial path with base points* $\{(x_i, y_i)\}_{i=1}^I \subset \Gamma$, $I \in \mathbb{N}$, starting at $x = x_1$ and ending at x' is the sequence of points

$$(x, y_1) = (x_1, y_1), (x_2, y_1), \ldots, (x_i, y_{i-1}), (x_i, y_i), (x_{i+1}, y_i), \ldots, (x_I, y_I), (x', y_I).$$

We will say that the axial path *goes from* x to x': note that $x \in \mathsf{p}_1\Gamma$. A *closed axial path* or *cycle* is an axial path with base points in Γ such that $x = x'$. A (Γ, c)-*axial path* is an axial path with base points in Γ whose points are contained in $\{\mathsf{c} < \infty\}$ and a (Γ, c)-*cycle* is a closed (Γ, c)-axial path.

Notice that, in order that (I.17) is well defined, for μ-a.e. point $x \in \mathsf{p}_1\Gamma$ there must be a (Γ, c)-axial path going from x_0 to x. Moreover, being Γ c-cyclically monotone, ϕ is surely finite valued in the case in which for μ-a.e. point $x \in \mathsf{p}_1\Gamma$ there exists also a (Γ, c)-axial path going from x to x_0 (and thus to a.a. any other point in Γ). In particular, x and x_0 are connected by a (Γ, c)-cycle.

The first idea in [6] is then to partition X into the equivalence classes $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ induced by the $(\Gamma, \mathtt{c})$-cycle equivalence relation and disintegrate μ, ν over $\{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$ and π over $\{Z_\mathfrak{a} \times Z_\mathfrak{b}\}_{\mathfrak{a},\mathfrak{b}\in\mathfrak{A}}$.

Since $\mathtt{c}(x, x) = 0\ \forall x \in X$ and $\Gamma \supset \operatorname{graph}\mathbb{I}$, then $(x, y) \in \Gamma$ implies that x and y belong to the same $(\Gamma, \mathtt{c})$-cycle (consider the path (x, y), (y, y), (y, y), (x, y)) and in particular that

$$\pi\left(\bigcup_{\mathfrak{a}\in\mathfrak{A}} Z_\mathfrak{a} \times Z_\mathfrak{a}\right) = 1. \tag{I.18}$$

If the disintegration is strongly consistent (see Theorem I.2.3), we get

$$\begin{aligned} \mu &= \int \mu_\mathfrak{a}\, dm(\mathfrak{a}), \quad \mu_\mathfrak{a}(Z_\mathfrak{a}) = 1, \\ \nu &= \int \nu_\mathfrak{a}\, dm(\mathfrak{a}), \quad \nu_\mathfrak{a}(Z_\mathfrak{a}) = 1, \\ \pi &= \int \pi_{\mathfrak{a}\mathfrak{a}}\, d(\mathbb{I} \times \mathbb{I})_\# m(\mathfrak{a}), \quad \pi_\mathfrak{a}(Z_\mathfrak{a} \times Z_\mathfrak{a}) = 1, \end{aligned} \tag{I.19}$$

where $m = \mathtt{h}_\#\mu = \mathtt{h}_\#\nu$ because there exists at least a plan in $\Pi^f_\mathtt{c}(\mu, \nu)$ –in this case π– such that (I.18) is satisfied.

Notice that the fact that π is concentrated on the diagonal equivalence classes $\{Z_\mathfrak{a} \times Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathfrak{A}}$, i.e. formula (I.18), is equivalent to say that the quotient measure $(\mathtt{h} \times \mathtt{h})_\#\pi$ satisfies

$$(\mathtt{h} \times \mathtt{h})_\#\pi = (\mathbb{I} \times \mathbb{I})_\# m,$$

i.e. it is concentrated on the diagonal of $\mathfrak{A} \times \mathfrak{A}$ (see (I.19)).

Now, as a consequence of the fact that $\mu_\mathfrak{a}$-a.a. points in $Z_\mathfrak{a}$ can be connected to $\mu_\mathfrak{a}$-a.a. other points in $Z_\mathfrak{a}$ by a $(\Gamma \cap Z_\mathfrak{a} \times Z_\mathfrak{a}, \mathtt{c})$-cycle and $\exists\, \pi_{\mathfrak{a}\mathfrak{a}} \in \Pi^f_\mathtt{c}(\mu_\mathfrak{a}, \nu_\mathfrak{a})$ $\mathtt{c}$-cyclically monotone which is concentrated on $\Gamma \cap Z_\mathfrak{a} \times Z_\mathfrak{a}$, using (I.17) we are able to construct optimal potentials $\phi_\mathfrak{a}$, $\psi_\mathfrak{a} : Z_\mathfrak{a} \to [-\infty, +\infty)$ for the transportation problem in $\Pi(\mu_\mathfrak{a}, \nu_a)$ and conclude that

$$\pi_{\mathfrak{a}\mathfrak{a}} \in \Pi^{\mathrm{opt}}_\mathtt{c}(\mu_\mathfrak{a}, \nu_\mathfrak{a}), \qquad \text{for } m\text{-a.e. } \mathfrak{a}.$$

Let us then consider another $\pi' \in \Pi^f_\mathtt{c}(\mu, \nu)$. After the disintegration w.r.t. $\{Z_\mathfrak{a} \times Z_\mathfrak{b}\}_{\mathfrak{a},\mathfrak{b}\in\mathfrak{A}}$ we get

$$\pi' = \int \pi'_{\mathfrak{a}\mathfrak{b}} dm'(\mathfrak{a}, \mathfrak{b}), \quad \pi'_{\mathfrak{a}\mathfrak{b}}(Z_\mathfrak{a} \times Z_\mathfrak{b}) = 1,$$

with

$$m' \in \Pi^f_{(\mathtt{h}\times\mathtt{h})_\#\mathtt{c}}(m, m), \quad \text{where} \quad (\mathtt{h} \times \mathtt{h})_\#\mathtt{c}(\mathfrak{a}, \mathfrak{b}) = \inf_{Z_\mathfrak{a}\times Z_\mathfrak{b}} \mathtt{c}(x, y).$$

Hence one has the following theorem, which gives a sufficient condition for optimality based on behavior of optimal transport plans w.r.t. disintegration on $(\Gamma, \mathtt{c})$-cycle equivalence relations.

(I.2.7) THEOREM: *Let Γ be a c-cyclically monotone carriage of a transference plan $\pi \in \Pi^f_{\mathsf{c}}(\mu,\nu)$. If the partition $\{Z_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$ w.r.t. the (Γ,c)-cycle equivalence relation satisfies*

$$\text{the disintegration on } \{Z_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}} \text{ is strongly consistent,} \tag{I.20}$$

$$\pi'\left(\bigcup_{\mathfrak{a}} Z_{\mathfrak{a}} \times Z_{\mathfrak{a}}\right) = 1, \quad \forall\, \pi' \in \Pi^f_{\mathsf{c}}(\mu,\nu), \tag{I.21}$$

then π is an optimal transference plan.

Indeed, if (I.20) and (I.21) are satisfied, then $\pi' = \int \pi'_{\mathfrak{a}\mathfrak{a}}\, d(\mathbb{I}\times\mathbb{I})_\# m(\mathfrak{a})$ with $\pi'_{\mathfrak{a}\mathfrak{a}} \in \Pi^f_{\mathsf{c}}(\mu_{\mathfrak{a}},\nu_{\mathfrak{a}})$ and one obtains the conclusion by integrating w.r.t. m the optimality of the conditional plans $\pi_{\mathfrak{a}\mathfrak{a}}$, namely

$$\int \mathsf{c}(x,y)\, d\pi_{\mathfrak{a}\mathfrak{a}}(x,y) \le \int \mathsf{c}(x,y)\, d\pi'_{\mathfrak{a}\mathfrak{a}}(x,y).$$

The second crucial point in [6] is then to find weak sufficient conditions such that the assumptions of Theorem I.2.7 are satisfied.

Before introducing them, we show how the request that the sets of a Borel partition satisfying (I.21) coincide with the equivalence classes of the (Γ,c)-cycle relation can be weakened, yet yielding the possibility of constructing optimal potentials on each class –and then, as a corollary, to prove the optimality of a c-cyclically monotone plan π. First, we need the following

(I.2.8) DEFINITION: A set $E \subset \mathsf{p}_1\Gamma$ is (Γ,c)-*cyclically connected* if $\forall\, x,y \in E$ there exists a (Γ,c)-cycle connecting x to y.

According to the above definition, the equivalence classes of $\preccurlyeq_{(\Gamma,\mathsf{c})}$ are maximal (Γ,c)-cyclically connected sets, namely (Γ,c)-cyclically connected sets which are maximal w.r.t. set inclusion.

Then notice that, given a Borel partition $\{Z'_{\mathfrak{b}}\}_{\mathfrak{b}\in\mathfrak{B}} \subset \mathbb{R}^d$ such that

$$\pi\left(\bigcup_{\mathfrak{b}} Z'_{\mathfrak{b}} \times Z'_{\mathfrak{b}}\right) = 1, \quad \forall\, \pi \in \Pi^f_{\mathsf{c}}(\mu,\nu)$$

and whose sets are (Γ,c)-cyclically connected but not necessarily maximal, then it is still possible to define on each of them a pair of optimal potentials and prove the optimality of π such that $\pi(\Gamma) = 1$.

Moreover, one can weaken this condition by removing a μ-negligible set in the following way. Let $\mu = \int \mu'_{\mathfrak{b}}\, dm'(\mathfrak{b})$, $\mu'_{\mathfrak{b}}(Z_{\mathfrak{b}}) = 1$.

(I.2.9) DEFINITION: The partition $\{Z'_{\mathfrak{b}}\}_{\mathfrak{b}\in\mathfrak{B}}$ is (μ,Γ,c)-*cyclically connected* if $\exists\, F \subset X$ μ-conegligible s.t. $Z'_{\mathfrak{b}} \cap F$ is (Γ,c)-cyclically connected $\forall\, \mathfrak{b} \in \mathfrak{B}$. Equivalently, $\exists$ an m'-conegligible set $\mathfrak{B}' \subset \mathfrak{B}$ s.t. $\forall\, \mathfrak{b}' \in \mathfrak{B}'$ $\exists\, N'_{\mathfrak{b}} \subset Z'_{\mathfrak{b}}$, with‘ $\mu'_{\mathfrak{b}}(N'_{\mathfrak{b}}) = 0$, s.t. $Z'_{\mathfrak{b}} \setminus N'_{\mathfrak{b}}$ is (Γ,c)-cyclically connected.

When the $(\mu, \Gamma, \mathtt{c})$-cyclically connectedness property holds for all $\mathtt{c}$-cyclically monotone carriages of all transport plans of finite cost –hence it is possible to construct optimal potentials starting from any $\mathtt{c}$-cyclically monotone Γ– we have the following

(I.2.10) **DEFINITION:** We say that $\{Z'_{\mathfrak{b}}\}$ is $\Pi^f_{\mathtt{c}}(\mu, \nu)$-*cyclically connected* if it is $(\mu, \Gamma, \mathtt{c})$-cyclically connected $\forall\, \Gamma$ $\mathtt{c}$-cyclically monotone s.t. $\pi(\Gamma) = 1$ for some $\pi \in \Pi^f_{\mathtt{c}}(\mu, \nu)$.

Notice that the μ-conegligible set F in the definition of $(\mu, \Gamma, \mathtt{c})$-cyclically connected partition depends on the set Γ.

In this thesis, in particular for the proof of Theorems 3.2 and to prove the existence of an optimal map, the importance of $\Pi^f_{\mathtt{c}}(\mu, \nu)$-cyclically connected partitions is given by the following proposition.

(I.2.11) **PROPOSITION:** *Let $\{Z'_{\mathfrak{b}}\}_{\mathfrak{b} \in \mathfrak{B}}$ be a $\Pi^f_{\mathtt{c}}(\mu, \nu)$-cyclically connected Borel partition satisfying*

$$\pi\Big(\bigcup_{\mathfrak{b}} Z'_{\mathfrak{b}} \times Z'_{\mathfrak{b}}\Big) = 1, \quad \forall\, \pi \in \Pi^f_{\mathtt{c}}(\mu, \nu) \tag{I.22}$$

for a cost function of the form

$$\mathtt{c}(x, y) = \mathbb{1}_M(x, y), \qquad M \supset \big\{(x, x) : x \in X\big\}. \tag{I.23}$$

Let $\mathtt{c}_{\mathtt{m}} : X \times X \to [0, +\infty]$ be any secondary cost *of the form*

$$\mathtt{c}_{\mathtt{m}}(x, y) = \begin{cases} \mathtt{m}(x, y) & \mathtt{c}(x, y) < +\infty, \\ +\infty & \textit{otherwise}, \end{cases}$$

where $\mathtt{m}$ is l.s.c. and there exist $\mathtt{f} \in L^1(\mu)$, $\mathtt{g} \in L^1(\nu)$ s.t. $\mathtt{m}(x, y) \leq \mathtt{f}(x) + \mathtt{g}(y)$. Then, any $\mathtt{c}_{\mathtt{m}}$-cyclically monotone plan $\pi_{\mathtt{m}} \in \Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu, \nu)$ is optimal for $\mathtt{c}_{\mathtt{m}}$. More precisely, for any $\mathtt{c}_{\mathtt{m}}$-cyclically monotone set $\Gamma_{\mathtt{m}}$ with $\pi_{\mathtt{m}}(\Gamma_{\mathtt{m}}) = 1$, there exist Borel functions $\phi^{\mathtt{m}}$, $\psi^{\mathtt{m}}$ such that the restrictions

$$\phi^{\mathtt{m}}_{\mathfrak{b}} := \phi^{\mathtt{m}} \llcorner_{Z'_{\mathfrak{b}}}, \qquad \psi^{\mathtt{m}}_{\mathfrak{b}} := \psi^{\mathtt{m}} \llcorner_{Z'_{\mathfrak{b}}} \tag{I.24}$$

are Borel optimal potentials for $\Pi^{\mathrm{opt}}_{\mathtt{c}_{\mathtt{m}}}(\mu'_{\mathfrak{b}}, \nu'_{\mathfrak{b}})$, for all $\mathfrak{b}$ in an m'-conegligible set $\mathfrak{B}' \subset \mathfrak{B}$.

PROOF. Notice that $\Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu, \nu) \subset \Pi^f_{\mathtt{c}}(\mu, \nu)$. Let $\Gamma_{\mathtt{m}} \subset \bigcup_{\mathfrak{b}} Z'_{\mathfrak{b}} \times Z'_{\mathfrak{b}}$ be a $\mathtt{c}_{\mathtt{m}}$-cyclically monotone carriage for $\pi_{\mathtt{m}} \in \Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu, \nu)$. Then, there exists a conegligible set $F \subset X$ such that $Z'_{\mathfrak{b}} \cap F$ is $(\Gamma_{\mathtt{m}}, \mathtt{c})$-cyclically connected for all $\mathfrak{b} \in \mathfrak{B}$. Hence, formula (I.17), together with the validity of the Point (1) at page 10, yields potentials $\phi^{\mathtt{m}}_{\mathfrak{b}}$, $\psi^{\mathtt{m}}_{\mathfrak{b}}$ for the transport problem in $\Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu_{\mathfrak{b}}, \nu_{\mathfrak{b}})$ with cost $\mathtt{c}_{\mathtt{m}}$. In particular, the conditional probability $\pi_{\mathtt{m}, \mathfrak{b}\mathfrak{b}}$ is optimal in $\Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu_{\mathfrak{b}}, \nu_{\mathfrak{b}})$, and thus by (I.22) it follows as in Theorem I.2.7 that $\pi_{\mathtt{m}}$ is optimal in $\Pi^f_{\mathtt{c}_{\mathtt{m}}}(\mu, \nu)$.

The fact that one can find Borel functions $\phi^{\mathtt{m}}$, $\psi^{\mathtt{m}}$ such that (I.24) holds is an application of standard selection principles, and it can be found in [6]. ■

In order to state the main result of [6] which is at the core of their sufficient condition concerning optimality, we need the concept of (linear) preorder.

(I.2.12) DEFINITION: A *preorder* on X is a set $A \subset X \times X$ s.t.

$$\begin{aligned} &(x,x) \in A, \quad \forall x \in X \\ &(x,y) \in A \quad \wedge \quad (y,z) \in A \quad \Longrightarrow \quad (x,z) \in A. \end{aligned}$$

A preorder $A \subset X \times X$ is *linear* if

$$X \times X = A \cup A^{-1}.$$

The statement $(x,y) \in A$ will also be denoted by $x \preccurlyeq_A y$ and A is also called the *graph of the (linear) preorder* $\preccurlyeq_A$. Any preorder $\preccurlyeq_A$ induces the equivalence relation $\simeq_A$ on X

$$x \simeq_A y \qquad \Longleftrightarrow \qquad x \preccurlyeq_A y \quad \text{and} \quad y \preccurlyeq_A x.$$

We also denote the graph of the equivalence relation $\simeq_A$ by

$$A \cap A^{-1} \qquad \text{or} \qquad \preccurlyeq_A \cap (\preccurlyeq_A)^{-1}.$$

Going back to our problem, one can see that the $(\Gamma, \mathtt{c})$-axial relation gives a Borel preorder on X, namely

$$x \preccurlyeq_{(\Gamma,\mathtt{c})} y \quad \text{ if there exists a } (\Gamma, \mathtt{c})\text{-axial path going from } y \text{ to } x.$$

The reason for introducing (linear) preorders in this context is given by the following theorem [6].

(I.2.13) THEOREM: *Let $A \subset X \times X$ be a Borel graph of a linear preorder on X with equivalence classes $\{Z^A_{\mathfrak{c}}\}_{\mathfrak{c}\in\mathfrak{C}}$ satisfying*

$$\{\mathtt{c} < +\infty\} \subset A, \tag{I.25}$$

$$\preccurlyeq_{(\Gamma,\mathtt{c})} \subset A, \text{ for some } \mathtt{c}\text{-cyclically monotone set } \Gamma \text{ s.t. } \pi(\Gamma) = 1,\ \pi \in \Pi^f_{\mathtt{c}}(\mu,\nu). \tag{I.26}$$

Then, the disintegration w.r.t. the partition $\{Z^A_{\mathfrak{c}}\}_{\mathfrak{c}\in\mathfrak{C}}$ is strongly consistent and

$$\pi'\left(\bigcup_{\mathfrak{c}} Z^A_{\mathfrak{c}} \times Z^A_{\mathfrak{c}}\right) = 1, \qquad \forall\, \pi' \in \Pi^f_{\mathtt{c}}(\mu,\nu). \tag{I.27}$$

For future convenience we give the following definition.

(I.2.14) DEFINITION: A preorder $\preccurlyeq_A$ on X is $\mathtt{c}$-*compatible* if (I.25) holds.

(I.2.15) REMARK: *Let A be a $\mathtt{c}$-compatible linear preorder. Whenever a carriage Γ satisfies* (I.26) *the $\preccurlyeq_{(\Gamma,\mathtt{c})}$-equivalence classes are contained in the equivalence classes of $\simeq_A$ and then, as noticed before, since $\Gamma \supset \text{graph}\,\mathbb{I}$ and $\mathtt{c}(x,x) = 0$ for all x,*

$$\Gamma \subset \bigcup_{\mathfrak{c}} Z^A_{\mathfrak{c}} \times Z^A_{\mathfrak{c}}, \quad \pi\left(\bigcup_{\mathfrak{c}} Z^A_{\mathfrak{c}} \times Z^A_{\mathfrak{c}}\right) = 1.$$

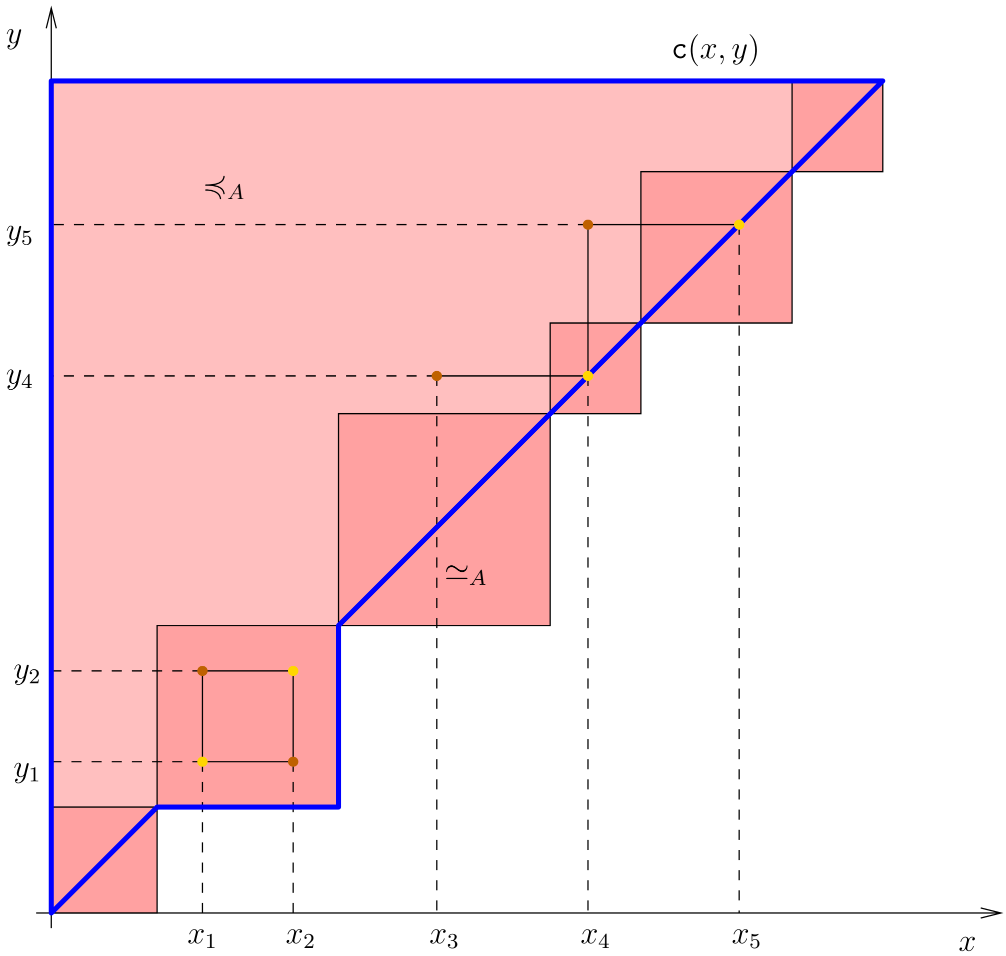


Figure I.3: The graph of the cost c is given by the indicator function of the region inside the blue curve. The graph of a c-compatible linear preorder $\preccurlyeq_A$ is given by the union of the pink and of the red region. The red region corresponds to the graph of the induced equivalence relation $\simeq_A$. We draw also an axial path connecting x_5 to x_3 with base points (x_5, y_5), (x_4, y_4), and a (Γ, c)-cycle connecting (x_1, y_1) to (x_2, y_2).

Viceversa, if $\pi'\big(\cup_{\mathtt{c}} Z^A_{\mathtt{c}} \times Z^A_{\mathtt{c}}\big) = 1$ *for some* $\pi' \in \Pi^f_{\mathtt{c}}(\mu,\nu)$ *and* $\pi'(\Gamma') = 1$*, then by the* $\mathtt{c}$*-compatibility of* A

$$\preccurlyeq_{(\Gamma' \cap \cup_{\mathtt{c}} Z^A_{\mathtt{c}} \times Z^A_{\mathtt{c}}, \mathtt{c})} \subset A$$

and then also its equivalence classes are contained in the equivalence classes of $\simeq_A$*. In particular,* (I.26) *could also be rewritten as* $\pi\big(\cup_{\mathtt{c}} Z^A_{\mathtt{c}} \times Z^A_{\mathtt{c}}\big) = 1$*.*

We point out that, while a $\mathtt{c}$*-compatible linear preorder satisfying* (I.26) *for some* Γ *can always be constructed using the axiom of choice,* (I.27) *may not hold if the linear preorder is not Borel (see [6]): hence, the main assumption of the theorem is the Borel regularity. Finally, notice that the partition into equivalence classes of* $\preccurlyeq_{(\Gamma' \cap \cup_{\mathtt{c}} Z^A_{\mathtt{c}} \times Z^A_{\mathtt{c}}, \mathtt{c})}$ *with* Γ' *as above is* $(\mu, \Gamma', \mathtt{c})$*-cyclically connected in the sense of Definition I.2.9.*

To conclude this section we give a last definition:

(I.2.16) DEFINITION: If $\preccurlyeq_A$ is $\mathtt{c}$-compatible and (I.26) holds for every $\pi \in \Pi^f_{\mathtt{c}}(\mu,\nu)$, then A is called $(\mathtt{c},\mu,\nu)$*-compatible*.

Hence, Theorem I.2.13 can also be restated saying that whenever A is a Borel $\mathtt{c}$-compatible linear preorder satisfying (I.26) for some Γ of finite cost, then it is $(\mathtt{c},\mu,\nu)$-compatible.

According to the terminology used in [6], $(\mathtt{c},\mu,\nu)$-compatibility can also be restated saying that the diagonal in the quotient space

$$(\mathbb{I} \times \mathbb{I}) \circ \mathtt{h} \circ \mathtt{p}_1(A)$$

is a *set of uniqueness* for $\Pi^f_{(\mathtt{h}\times\mathtt{h})_\#\mathtt{c}}(m,m)$, where $\mathtt{h}$ is the quotient map associated to the partition $\simeq_A$: this means that there exists a unique transference plan in $\Pi^f_{(\mathtt{h}\times\mathtt{h})_\#\mathtt{c}}(m,m)$, namely $(\mathbb{I} \times \mathbb{I})_\# m$.

I.3. Directed locally affine partitions

The key element in our proof is the definition of locally affine partition: this definition is not exactly the one given given in [8] because we require that if the cone has linear dimension $h+1$, then its intersection with $t=1$ is a compact convex set of linear dimension h.

(I.3.1) DEFINITION: A *directed locally affine partition* in $[0,+\infty) \times \mathbb{R}^d$ is a partition into locally affine sets $\{Z^h_{\mathfrak{a}}\}_{\substack{h=0,\dots,d \\ \mathfrak{a}\in\mathfrak{A}^{d-h}}}$, $Z^h_{\mathfrak{a}} \subset [0,+\infty) \times \mathbb{R}^d$ and $\mathfrak{A}^{d-h} \subset \mathbb{R}^{d-h}$, together with a map

$$\mathbf{d} : \bigcup_{h=0}^{d} \{h\} \times \mathfrak{A}^{d-h} \to \bigcup_{h=0}^{d} \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$$

satisfying the following properties:

1. the set

$$\mathbf{D} = \Big\{ \big(h, \mathfrak{a}, z, \mathbf{d}(h, \mathfrak{a})\big) : k \in \{0, \dots, d\}, \mathfrak{a} \in \mathfrak{A}^{d-h}, z \in Z^h_{\mathfrak{a}} \Big\},$$

   is σ-compact in $\cup_h \big(\{h\} \times \mathbb{R}^{d-h} \times ([0, +\infty) \times \mathbb{R}^d) \times \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)\big)$, i.e. there exists a family of compact sets

$$K_n \subset \bigcup_h \big(\{h\} \times \mathbb{R}^{d-h} \times ([0, +\infty) \times \mathbb{R}^d)\big)$$

   such that $Z^h_{\mathfrak{a}} \cap \mathtt{p}_z K_n$ is compact and

$$\mathtt{p}_{h,\mathfrak{a}} K_n \ni (h, \mathfrak{a}) \mapsto (Z^h_{\mathfrak{a}} \cap \mathtt{p}_z K_n, \mathbf{d}(h, \mathfrak{a}))$$

   is continuous w.r.t. the Hausdorff topology;

2. denoting $C^h_{\mathfrak{a}} := \mathbf{d}(h, \mathfrak{a})$, then

$$\forall z \in Z^h_{\mathfrak{a}} \ \Big(\mathrm{aff}\, Z^h_{\mathfrak{a}} = \mathrm{aff}(z + C^h_{\mathfrak{a}})\Big);$$

3. the plane aff $Z^h_{\mathfrak{a}}$ satisfies aff $Z^h_{\mathfrak{a}} \in \mathcal{A}(h, [0, +\infty) \times \mathbb{R}^d)$.

(I.3.2) **REMARK:** *Using the fact that $C^h_{\mathfrak{a}}$ is not degenerate, one sees immediately that Point 3 is unnecessary.*

The map $\mathbf{d}$ will be called *direction map* of the partition, or *direction vector field* for $h = 0$. Sometimes in the following we will write

$$\mathbf{d}(z) = \mathbf{d}(h, \mathfrak{a}) \quad \text{for} \quad z \in Z^h_{\mathfrak{a}},$$

being $Z^h_{\mathfrak{a}}$ a partition, or we will use also the notation $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$. For shortness we will write

$$\mathbf{Z}^h := \mathtt{p}_z \mathbf{D}(h) = \bigcup_{\mathfrak{a} \in \mathfrak{A}_h} Z^h_{\mathfrak{a}}, \qquad \mathbf{Z} := \mathtt{p}_z \mathbf{D} = \bigcup_{h=0}^{d} \mathbf{Z}^h = \bigcup_{h=0}^{d} \bigcup_{\mathfrak{a} \in \mathfrak{A}_h} Z^h_{\mathfrak{a}}. \tag{I.28}$$

For each $C^h_n \in \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$, (the index n is because of the proposition below), consider a family $\mathrm{e}^h(n)$ of vectors $\{\mathrm{e}^h_i(n), i = 0, \dots, h\}$ in $\mathbb{R}^d$ such that

$$\mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d) \ni C(\{\mathrm{e}^h(n)\}) := \bigg\{ \sum_{i=0}^{h} t_i (1, \mathrm{e}^h_i(n)), t_i \in [0, \infty) \bigg\} \subset \mathring{C}^h_n(-r_n). \tag{I.29}$$

Define also

$$U(\{\mathrm{e}^h(n)\}) := \{t = 0\} \times \mathrm{conv}\, \mathrm{e}^h(n). \tag{I.30}$$

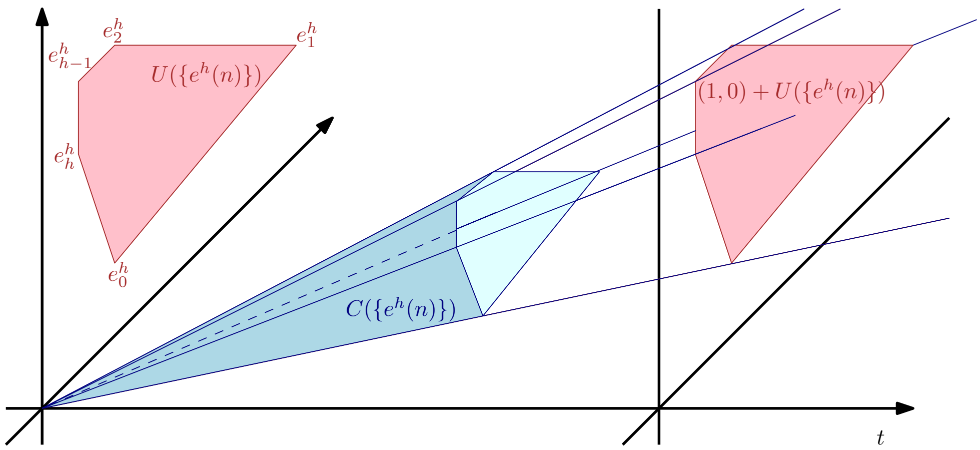


Figure I.4: Definition of $C(\{\mathrm{e}^h(n)\})$ and $U(\{\mathrm{e}^h(n)\})$, formulas (I.29) and (I.30).

Note that

$$\mathrm{aff}\,\big\{(0,0),(1,\mathrm{e}^h_0(n)),\dots,(1,\mathrm{e}^h_h(n))\big\}\in\mathcal{A}(h,[0,\infty)\times\mathbb{R}^d),$$

so that $C(\{\mathrm{e}^h(n)\})\in\mathcal{C}(h,[0,+\infty)\times\mathbb{R}^d)$.

The following proposition is the adaptation of Proposition 3.15 of [8] to the present situation.

(I.3.3) PROPOSITION: *There exists a countable covering of* $\mathbf{D}$ *into disjoint* σ*-compact sets* $\mathbf{D}(h,n)$*,* $h=0,\dots,d$ *and* $n\in\mathbb{N}$*, with the following properties: there exist*

- *vectors* $\{\mathrm{e}^h_i(n)\}_{i=0}^h\subset\mathbb{R}^d$*, with linear span*

$$V^h_n=\mathrm{span}\big\{(1,\mathrm{e}^h_0(n)),\dots,(1,\mathrm{e}^h_h(n))\big\}\in\mathcal{A}(h,[0,+\infty)\times\mathbb{R}^d),$$

- *a cone* $C^h_n\in\mathcal{C}(h,V^h_n)$*,*
- *a given point* $z^h_n\in V^h_n$*,*
- *constants* $r^h_n,\lambda^h_n\in(0,\infty)$*,*

such that, setting

$$\mathfrak{A}^h_n:=\mathtt{p}_{\mathfrak{a}}\mathbf{D}(h,n),\qquad C^h_{\mathfrak{a}}=\mathtt{p}_{\mathcal{C}(h,[0,\infty]\times\mathbb{R}^d)}\mathbf{D}(h,n)(\mathfrak{a}),$$

it holds:

1. $\mathtt{p}_{\{0,\dots,d\}}\mathbf{D}(h,n)=\{h\}$ *for all* $n\in\mathbb{N}$*, i.e. the intersections of the elements* $Z^h_{\mathfrak{a}}$*,* $C^h_{\mathfrak{a}}$ *with* $\{t=1\}$ *have linear dimension* h*, for* $\mathfrak{a}\in\mathfrak{A}^h_n$*;*
2. *the cone generated by* $\{\mathrm{e}^h_i(n)\}$ *is not degenerate and strictly contained in* C^h_n*,*

$$C(\{\mathrm{e}^h_i(n)\})\in\mathcal{C}(h,V^h_n),\qquad C(\{\mathrm{e}^h_i(n)\})\subset\mathring{C}^h_n(-r^h_n);$$

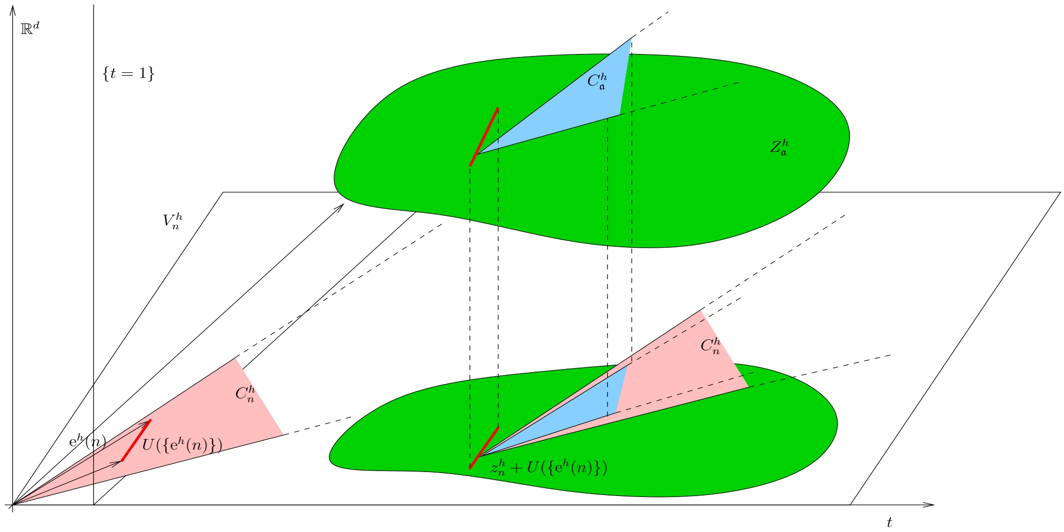


Figure I.5: The decomposition presented in Proposition I.3.3.

3. *the cones* $C^h_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}^{d-h}_n$, *have a uniform opening,*

$$C^h_n(-r^h_n) \subset \mathtt{p}^{\bar{t}}_{V^h_n} \mathring{C}^h_{\mathfrak{a}};$$

4. *the projections of cones* $C^h_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}^{d-h}_n$, *are strictly contained in* C^h_n,

$$\mathtt{p}^{\bar{t}}_{V^h_n} C^h_{\mathfrak{a}} \subset \mathring{C}^h_n;$$

5. *the projection at constant* t *on* V^h_n *is not degenerate: there is a constant* $\kappa > 0$ *such that*

$$\left|\mathtt{p}^t_{V^h_n}(z - z')\right| \geq \kappa |z - z'| \quad \text{for all } z, z' \in C^h_{\mathfrak{a}} \cap \{t = \bar{t}\},\ \mathfrak{a} \in \mathfrak{A}^h_n, \bar{t} \geq 0;$$

6. *the projection at constant* t *of* $Z^h_{\mathfrak{a}}$ *on* V^h_n *contains a given cube,*

$$z^h_n + U(\{\mathrm{e}^h_i(n)\}) \subset \mathtt{p}^{\bar{t}}_{V^h_n} Z^h_{\mathfrak{a}}.$$

Note that clearly the $Z^h_{\mathfrak{a}}$ are transversal to $\{t = \text{constant}\}$.

PROOF. The only difference w.r.t. the analysis done in [8] is the fact that we are using projections with t constant, instead of projecting on V^h_n. However the assumption of Point 3 of Definition I.3.1 gives that the projection of $Z^h_{\mathfrak{a}}$, $C^h_{\mathfrak{a}}$ at t fixed is a set of linear dimension h, and thus we can take as a base for the partitions sets of the form (I.29), (I.30). ∎

Following the same convention of (I.28), we will use the notation $\mathbf{Z}^h_n := \mathtt{p}_z \mathbf{D}(h, n)$.

By the above proposition and the transversality to $\{t = \bar{t}\}$, the sets $\mathfrak{A}^h_n$ can be now chosen to be

$$\mathfrak{A}^h_n := \mathbf{Z}^h_n \cap (\mathtt{p}^t_{V_h})^{-1}(z_n), \quad \mathfrak{A}^h := \bigcup_{n \in \mathbb{N}} \mathfrak{A}^h_n. \tag{I.31}$$

(I.3.4) **DEFINITION:** We will call a directed locally affine partition $\mathbf{D}(h,n)$ a *h-dimensional directed sheaf set with base directions C^h_n, $C^h_n(-r_n)$ and base rectangle $z_n + \lambda_n U(\mathrm{e}^h_i)$* if it satisfies the properties listed in Proposition I.3.3 for some $\{\mathrm{e}^h_i(n)\}_{i=0}^h \subset \mathbb{R}^d$, $V^h_n = \mathrm{span}\{(1,\mathrm{e}^h_0),(1,\mathrm{e}^h_i),(1,\mathrm{e}^h_n)\}$, $C^h_n \in \mathcal{C}(h, V^h_n)$, $z_n \in \mathbb{R}^{d+1}$, $r_n, \lambda_n \in (0,\infty)$.

(I.3.5) **REMARK:** *In the following we are only interested in the sets $Z^h_{\mathfrak{a}}$ such that $Z^h_{\mathfrak{a}} \cap \{t=1\} \neq \emptyset$. Thus, the definition of $\mathbf{D}$ could be restricted to these sets, and the quotient space $\mathfrak{A}^{d-h}$ can be taken to be a subset of an affine subspace $\{t=1\} \times \mathbb{R}^{d-h}$.*

I.4. Construction of the first directed locally affine partition

In this section we show how to use the potential $\bar\phi$ to find a directed locally affine partition in the sense of the previous section. The approach follows closely [17]: the main variations are in proving regularity, Sections I.4.1 and I.4.2.

(I.4.1) **DEFINITION:** We define the *sub-differential* of $\bar\phi$ at z as

$$\partial^- \bar\phi(z) := \big\{ z' \in [0,+\infty) \times \mathbb{R}^d : \bar\phi(z) - \bar\phi(z') = \bar{\mathtt{c}}(z - z') \big\},$$

and the *super-differential* of $\bar\phi$ at z as

$$\partial^+ \bar\phi(z) := \big\{ z' \in [0,+\infty) \times \mathbb{R}^d : \bar\phi(z') - \bar\phi(z) = \bar{\mathtt{c}}(z' - z) \big\}.$$

(I.4.2) **DEFINITION:** We say that a segment $[\![z,z']\!]$ is an *optimal ray* for $\bar\phi$ if

$$\bar\phi(z') - \bar\phi(z) = \bar{\mathtt{c}}(z' - z).$$

We say that a segment $[\![z,z']\!]$ is a *maximal optimal ray* if it is maximal with respect to set inclusion.

(I.4.3) **DEFINITION:** The *backward direction multifunction* is given by

$$\mathcal{D}^- \bar\phi(z) = \left\{ \frac{z - z'}{\mathtt{p}_t(z - z')} : z' \in \partial^- \bar\phi(z) \setminus \{z\} \right\},$$

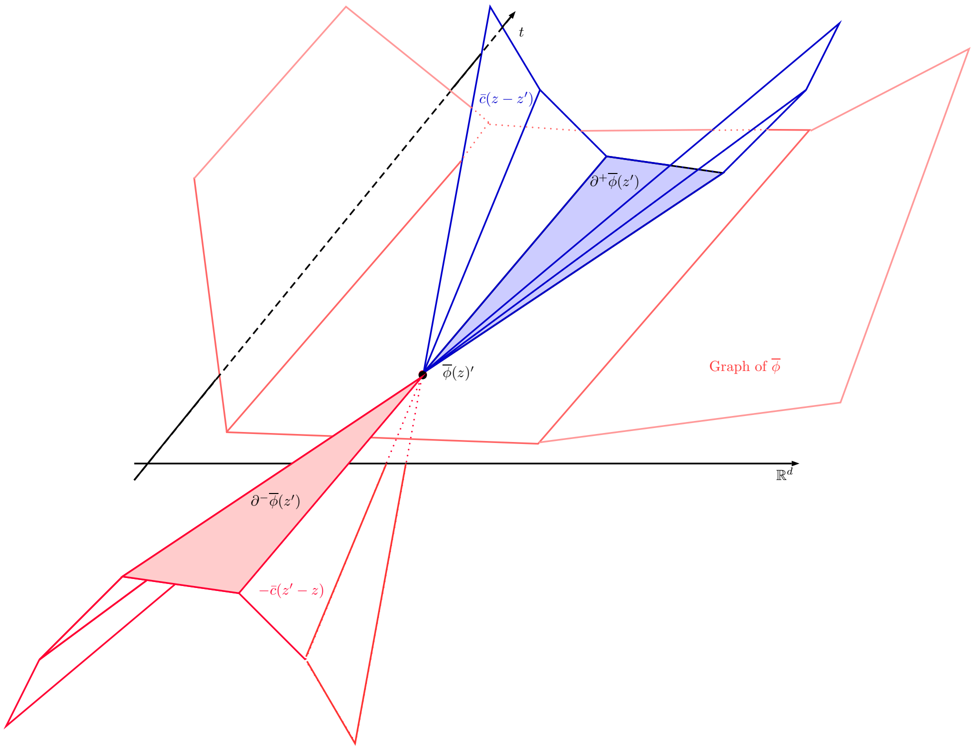


Figure I.6: The sets $\partial^-\bar{\phi}(z)$, $\partial^+\bar{\phi}(z)$ of Definition I.4.1 are obtained intersecting epi $\bar{\mathsf{c}}$, $-$epi $\bar{\mathsf{c}}$ with graph $\bar{\phi}$, respectively.

and *forward direction multifunction* is given by

$$\mathcal{D}^+\bar{\phi}(z) = \left\{ \frac{z'-z}{\mathtt{p}_t(z'-z)} : z' \in \partial^+\bar{\phi}(z) \setminus \{z\} \right\}.$$

(I.4.4) DEFINITION: The *convex cone generated* by $\mathcal{D}^-\bar{\phi}$ (resp. by $\mathcal{D}^+\bar{\phi}$) is the cone

$$F^-_{\bar{\phi}}(z) = \mathbb{R}^+ \cdot \mathrm{conv}\, \mathcal{D}^-\bar{\phi}(z) \qquad \big(\text{resp. } F^+_{\bar{\phi}}(z) = \mathbb{R}^+ \cdot \mathrm{conv}\, \mathcal{D}^+\bar{\phi}(z)\big).$$

(I.4.5) DEFINITION: The *backward transport set* is defined respectively by

$$T^-_{\bar{\phi}} := \big\{ z : \partial^-\bar{\phi}(z) \neq \{z\} \big\},$$

the *forward transport set* by

$$T^+_{\bar{\phi}} := \big\{ z : \partial^+\bar{\phi}(z) \neq \{z\} \big\},$$

and the *transport set* by

$$T_{\bar{\phi}} = T^-_{\bar{\phi}} \cap T^+_{\bar{\phi}}.$$

(I.4.6) DEFINITION: The *h-dimensional backward/forward regular transport sets* are defined for $h = 0, \dots, d$ respectively as

$$R^{-,h}_{\bar{\phi}} := \left\{ z \in T^-_{\bar{\phi}} : \begin{array}{ll} (i) & \mathcal{D}^-\bar{\phi}(z) = \mathrm{conv}\, \mathcal{D}^-\bar{\phi}(z) \\ (ii) & \dim(\mathrm{conv}\, \mathcal{D}^-\bar{\phi}(z)) = h \\ (iii) & \exists z' \in T^-_{\bar{\phi}} \cap (z + \mathrm{int}_{\mathrm{rel}} F^-_{\bar{\phi}}(z)) \\ & \text{ such that } \bar{\phi}(z) = \bar{\phi}(z') + \bar{\mathtt{c}}(z'-z) \text{ and } (i), (ii) \text{ hold for } z' \end{array} \right\},$$

and

$$R^{+,h}_{\bar{\phi}} := \left\{ z \in T^+_{\bar{\phi}} : \begin{array}{ll} (i) & \mathcal{D}^+\bar{\phi}(z) = \mathrm{conv}\, \mathcal{D}^+\bar{\phi}(z) \\ (ii) & \dim(\mathrm{conv}\, \mathcal{D}^+\bar{\phi}(z)) = h \\ (iii) & \exists z' \in T^+_{\bar{\phi}} \cap (z - \mathrm{int}_{\mathrm{rel}} F^+_{\bar{\phi}}(z)) \\ & \text{ such that } \bar{\phi}(z') = \bar{\phi}(z) + \bar{\mathtt{c}}(z-z') \text{ and } (i), (ii) \text{ hold for } z' \end{array} \right\}.$$

Define the *backward (resp. forward) transport regular set* as

$$R^-_{\bar{\phi}} := \bigcup_{h=0}^{d} R^{-,h}_{\bar{\phi}} \qquad \left(\text{resp. } R^+_{\bar{\phi}} := \bigcup_{h=0}^{d} R^{+,h}_{\bar{\phi}}\right),$$

and the *regular transport set* as

$$R_{\bar{\phi}} := R^+_{\bar{\phi}} \cap R^-_{\bar{\phi}}.$$

Finally define the *residual set* N by

$$N_{\bar{\phi}} := T_{\bar{\phi}} \setminus R_{\bar{\phi}}.$$

(I.4.7) PROPOSITION: *The set $\partial^\pm\bar{\phi}$, $T^\pm_{\bar{\phi}}$, $\mathcal{D}^\pm\bar{\phi}$, $F^\pm_{\bar{\phi}}$, $R^{\pm,h}_{\bar{\phi}}$, $R^\pm_{\bar{\phi}}$, $R_{\bar{\phi}}$ are $\sigma-$compact.*

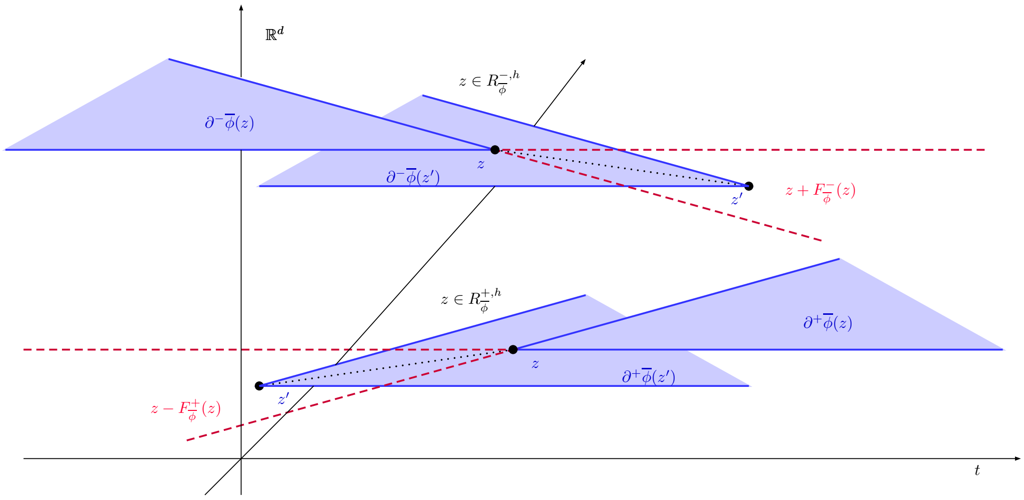


Figure I.7: The sets $R^-_{\bar{\phi}}$ and $R^+_{\bar{\phi}}$ of Definition I.4.6.

PROOF. $\partial^\pm \bar{\phi}$. The map

$$(z, z') \mapsto \Phi(z, z') := \bar{\phi}(z') - \bar{\phi}(z) - \bar{\mathtt{c}}(z' - z)$$

is continuous. Therefore, $\partial^\pm \bar{\phi} = \Phi^{-1}(0)$ is σ-compact.

$T^\pm_{\bar{\phi}}$. The set $T^-_{\bar{\phi}}$ is the projection of the σ-compact set

$$\bigcup_n \Big\{ \partial^- \bar{\phi} \cap \big\{ (z, z') : |z - z'| \geq 2^{-n} \big\} \Big\},$$

and hence σ-compact. The same reasoning can be used for T^+.

$\mathcal{D}^\pm \bar{\phi}$. Since

$$\{(z, z') : t(z) > t(z')\} \ni (z, z') \mapsto \frac{z - z'}{t(z) - t(z')} \in \{t = 1\}$$

is continuous, it follows that $\mathcal{D}^- \bar{\phi}$ is σ-compact, being the image of a σ-compact set by a continuous function. The same reasoning holds for $\mathcal{D}^+ \bar{\phi}$.

A similar analysis can be carried out for the σ-compactness of $F^\pm_{\bar{\phi}}$.

$R^{\pm,h}_{\bar{\phi}}$. Since the maps $A \mapsto \operatorname{conv} A$ is continuous with respect to the Hausdorff topology, and the dimension of a convex set is a lower semicontinuous map, the only point to prove is that the set

$$\Big\{ (z, z', C) \in [0, +\infty) \times \mathbb{R}^h \times [0, +\infty) \times \mathbb{R}^h \times \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^h) : z' \in z - \mathrm{int}_{\mathrm{rel}} C \Big\}$$

is σ-compact. This follows by taking considering the cones $C(-r)$ and writing the previous set as the countable union of σ-compact sets as follows

$$\bigcup_n \Big\{ (z, z', C) \in [0, +\infty) \times \mathbb{R}^h \times [0, +\infty) \times \mathbb{R}^h \times \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^h) : z' \in z + C(-2^{-n}) \setminus B(0, 2^{-n}) \Big\}.$$

Hence the set

$$\Big\{(z,z',C)\ :\ \begin{array}{rl} (i) & z,z'\in T^-_{\bar\phi} \\ (ii) & C=F^-_{\bar\phi}(z) \\ (iii) & z'\in z+\mathrm{int}_{\mathrm{rel}}C \\ (iv) & \dim(\mathrm{conv}\mathcal{D}^-\bar\phi(z))=\dim(\mathrm{conv}\mathcal{D}^-\bar\phi(z'))=h \\ (v) & \mathcal{D}^-\bar\phi(z)=\mathrm{conv}\,\mathcal{D}^-\bar\phi(z),\mathcal{D}^-\bar\phi(z')=\mathrm{conv}\,\mathcal{D}^-\bar\phi(z')\end{array}\Big\}$$

is σ-compact, and thus $R^{-,h}$ is σ-compact, too. The proof for $R^+_{\bar\phi}$ is analogous, and hence the regularity for $R_{\bar\phi}$ follows. ■

(I.4.8) **PROPOSITION:** *Let $z,z',z''\in[0,+\infty)\times\mathbb{R}^d$, then the following statements hold:*

1. *$z'\in\partial^-\bar\phi(z)$ and $z\in\partial^-\bar\phi(z'')$ imply $z'\in\partial^-\bar\phi(z'')$;*
2. *$z''\in\partial^+\bar\phi(z)$ and $z\in\partial^+\bar\phi(z')$ imply $z''\in\partial^+\bar\phi(z')$.*

PROOF. It easily follows from Definition I.4.1. ■ Moreover, it is easy to prove that:

$$z'\in\partial^\pm\bar\phi(z)\quad\Longrightarrow\quad\partial^\pm\bar\phi(z')\subset\partial^\pm\bar\phi(z).$$

(I.4.9) **DEFINITION:** Let z and z' such that $\bar\phi(z')-\bar\phi(z)=\mathsf{c}(z'-z)$ and define

$$\mathcal{Q}_{\bar\phi}(z,z'):=\mathrm{p}_{[0,+\infty)\times\mathbb{R}^d}\big\{\big((z,\bar\phi(z))+\mathrm{epi}\,\mathsf{c}\big)\cap\big((z',\bar\phi(z'))-\mathrm{epi}\,\mathsf{c}\big)\big\}. \tag{I.32}$$

(I.4.10) **LEMMA:** *It holds,*

$$\mathcal{Q}_{\bar\phi}(z,z')\subseteq\partial^-\bar\phi(z')\cap\partial^+\bar\phi(z).$$

Moreover

$$\mathbb{R}^+\big(\mathcal{Q}_{\bar\phi}(z,z')-z\big)=\mathbb{R}^+\big(z'-\mathcal{Q}_{\bar\phi}(z,z')\big)=F(z,z'). \tag{I.33}$$

where $F(z,z')$ is the projection of the minimal extremal face of epi c *containing of $(z'-z,\bar\phi(z')-\bar\phi(z))$.*

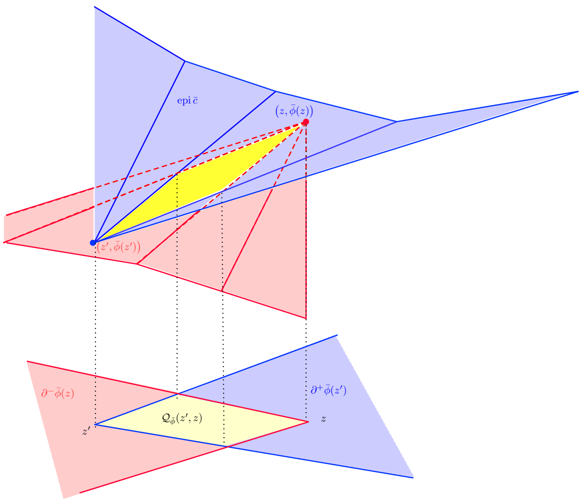
PROOF. Let $(\bar z,\bar r)\in\big((z,\bar\phi(z))+\mathrm{epi}\,\mathsf{c}\big)\cap\big((z',\bar\phi(z'))-\mathrm{epi}\,\mathsf{c}\big)$: by definition,

$$\bar r-\bar\phi(z)\geq\mathsf{c}(\bar z-z)\qquad\text{and}\qquad\bar\phi(z')-\bar r\geq\mathsf{c}(z'-\bar z).$$

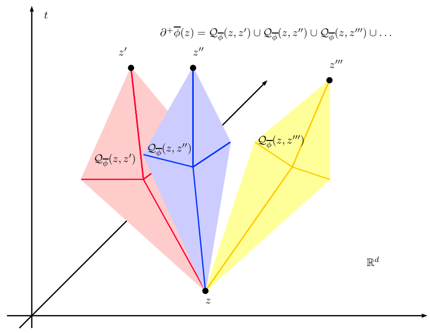
Hence, from $\bar\phi(z')-\bar\phi(\bar z)\leq\mathsf{c}(z'-\bar z)$,

$$\begin{aligned}\bar\phi(\bar z)-\bar\phi(z)&\geq\bar\phi(\bar z)-\bar r+\mathsf{c}(\bar z-z)\\&\geq\bar\phi(\bar z)-\bar\phi(z')+\mathsf{c}(z'-\bar z)+\mathsf{c}(\bar z-z)\geq\mathsf{c}(\bar z-z).\end{aligned}$$

Then $\bar z\in\partial^+\bar\phi(z)$ and similarly one can prove $\bar z\in\partial^-\bar\phi(z')$.



(a) The set $\mathcal{Q}(z, z')$ and Lemma I.4.10.



(b) The representation formula (I.34) for $\partial^+ \bar{\phi}(z)$.

Figure I.8

The second part of the statement is an elementary property of convex sets: if K is a compact convex set and $0 \in K$, then

$$K \cap \operatorname{span}(K \cap (-K))$$

is the extremal face of K containing 0 in its relative interior. Since for us K is a cone, the particular form (I.33) follows. ■

In particular, one deduces immediately that $\partial^\pm \bar{\phi}$ is the union of sets of the form (I.32), Figure I.8:

$$\partial^- \bar{\phi}(z) = \bigcup_{z' \in \partial^- \bar{\phi}(z)} \mathcal{Q}_{\bar{\phi}}(z', z), \qquad \partial^+ \bar{\phi}(z) = \bigcup_{z' \in \partial^+ \bar{\phi}(z)} \mathcal{Q}_{\bar{\phi}}(z, z'). \tag{I.34}$$

(I.4.11) **PROPOSITION:** *Let F be the projection on $[0, +\infty) \times \mathbb{R}^d$ of an extremal face of* epi $\bar{\mathsf{c}}$*. The following holds:*

1. $F \cap \{t = 1\} \subseteq \mathcal{D}^- \bar{\phi}(z) \iff \exists \delta > 0$ *such that* $B(z, \delta) \cap (z - F) \subseteq \partial^- \bar{\phi}(z)$.
2. *If* $F \cap \{t = 1\} \subseteq \mathcal{D}^- \bar{\phi}(z)$ *is maximal w.r.t. set inclusion, then*

$$\forall z' \in B(z, \delta) \cap (z - \operatorname{int}_{\mathrm{rel}} F) \ \big(\mathcal{D}^- \bar{\phi}(z') = F \cap \{t = 1\}\big),$$

   with $\delta > 0$ given by the previous point.
3. *The following conditions are equivalent:*
   (a) $\mathcal{D}^- \bar{\phi}(z) = F^-_{\bar{\phi}}(z) \cap \{t = 1\}$;

(b) the family of cones

$$\left\{\mathbb{R}^+ \cdot \left(z - \mathcal{Q}_{\bar\phi}(z', z)\right), z' \in \partial^- \bar\phi(z)\right\}$$

has a unique maximal element w.r.t. set inclusion, which coincides with $F^-_{\bar\phi}(z)$*;*

(c) $\partial^- \bar\phi(z) \cap \mathrm{int}_{\mathrm{rel}}(z - F^-_{\bar\phi}(z)) \neq \emptyset$*;*

(d) $\mathcal{D}^- \bar\phi(z) = \mathrm{conv}\, \mathcal{D}^- \bar\phi(z)$*.*

We recall that $F^-_{\bar\phi}$ is defined in Definition I.4.4.

PROOF. *Point* (1). Only the first implication has to be proved. The assumption implies that there exists a point

$$z' \in \left(z - \mathrm{int}_{\mathrm{rel}} F\right) \cap \partial^- \bar\phi(z)$$

and thus $\partial^- \bar\phi(z)$ contains $\mathcal{Q}_{\bar\phi}(z', z)$ by Lemma I.4.10. It is fairly easy to see that this yields the conclusion, because there exists $\delta > 0$ such that

$$B(z, \delta) \cap \left(z - F\right) \subseteq \mathcal{Q}_{\bar\phi}(z', z).$$

Point (2). The transitivity property of Lemma I.4.8 implies one inclusion. The opposite one follows because $\bar z$ is an inner point of $\mathcal{Q}_{\bar\phi}(z', z)$.

Point (3). (3b) implies (3a): by Lemma I.4.10 it follows that the set $\mathcal{D}^- \bar\phi(z)$ can be decomposed as the union of extremal faces with inner directions: since the dimension of extremal faces must increase by one at each strict inclusion, every increasing sequence of extremal faces has a maximum. If the maximal face $F^{\max}$ is unique, we apply Lemma I.4.10 to a point $\bar z$ in an inner direction, obtaining that $F^{\max} = F^+_{\bar\phi}(z)$.

(3a) implies (3d) and (3d) implies (3c): these implications follow immediately from the definition of $\mathcal{D}^- \bar\phi$.

(3c) implies (3b): if there is a direction in the interior of an extremal face, than by Lemma I.4.10 we conclude that the whole face is contained in $\mathcal{D}^- \bar\phi(z)$. ∎

A completely similar proposition can be proved for $\partial^+ \bar\phi$: we state it without proof.

(I.4.12) **PROPOSITION:** *Let F be the projection on $[0, +\infty) \times \mathbb{R}^d$ of an extremal face of* epi $\bar{\mathtt{c}}$*. The following holds:*

1. $F \cap \{t = 1\} \subseteq \mathcal{D}^+ \bar\phi(z) \iff \exists \delta > 0$ *such that* $B(z, \delta) \cap (z + F) \subseteq \partial^+ \bar\phi(z)$*.*
2. *If* $F \cap \{t = 1\} \subseteq \mathcal{D}^+ \bar\phi(z)$ *is maximal w.r.t. set inclusion, then*

$$\forall z' \in B(z, \delta) \cap (z + \mathrm{int}_{\mathrm{rel}} F) \ \left(\mathcal{D}^+ \bar\phi(z') = F \cap \{t = 1\}\right),$$

with $\delta > 0$ given by the previous point.

3. *The following conditions are equivalent:*

(a) $\mathcal{D}^+ \bar\phi(z) = F^+_{\bar\phi}(z) \cap \{t = 1\}$*;*

(b) the family of cones

$$\left\{\mathbb{R}^+ \cdot \left(z + \mathcal{Q}_{\bar{\phi}}(z', z)\right), z' \in \partial^+\bar{\phi}(z)\right\}$$

has a unique maximal element by set inclusion, which coincides with $F^+_{\bar{\phi}}(z')$*;*

(c) $\partial^+\bar{\phi}(z) \cap \mathrm{int}_{\mathrm{rel}}(z + F^+_{\bar{\phi}}(z)) \neq \emptyset$*;*

(d) $\mathcal{D}^+\bar{\phi}(z) = \mathrm{conv}\mathcal{D}^+\bar{\phi}(z)$.

As a consequence of Point (3) of the previous propositions, we will call sometimes $F^-_{\bar{\phi}}(z)$, $F^+_{\bar{\phi}}(z)$ the *maximal backward/forward extremal face.*

Now we construct a map which gives a directed affine partition in $[0, +\infty) \times \mathbb{R}^d$ up to a residual set. Define first

$$\begin{array}{cccc} \mathtt{v}^-_{\bar{\phi}} & : & R^-_{\bar{\phi}} & \to & \cup_{h=0}^d \mathcal{A}(h, [0, +\infty) \times \mathbb{R}^d) \\ & & z & \mapsto & \mathtt{v}^-_{\bar{\phi}}(z) := \mathrm{aff}\, \partial^-\bar{\phi}(z) \end{array}$$

(I.4.13) LEMMA: *The map* $\mathtt{v}^-_{\bar{\phi}}$ *is* σ*-continuous.*

PROOF. Since $\partial^-\bar{\phi}(z)$ is σ-continuous by Proposition I.4.7 and the map $A \mapsto \mathrm{aff}\, A$ is σ-continuous in the Hausdorff topology, the conclusion follows. ■

Notice that we are assuming the convention $\mathbb{R}^0 = \mathbb{N}$.

(I.4.14) THEOREM: *The map* $\mathtt{v}^-_{\bar{\phi}}$ *induces a partition*

$$\bigcup_{h=0}^d \left\{ Z^{h,-}_{\mathfrak{a}} \subset [0, +\infty) \times \mathbb{R}^d, \mathfrak{a} \in \mathbb{R}^{d-h} \right\}$$

on $R^-_{\bar{\phi}}$ *such that the following holds:*

1. *the sets* $Z^{h,-}_{\mathfrak{a}}$ *are locally affine;*
2. *there exists a projection* $F^{h,-}_{\mathfrak{a}}$ *of an extremal face* $F^{h,-}_{\mathfrak{a}}$ *with dimension* $h+1$ *of the cone* $\mathrm{epi}\, \bar{\mathtt{c}}$ *such that*

$$\forall z \in Z^{h,-}_{\mathfrak{a}}, \quad \mathrm{aff}\, Z^{h,-}_{\mathfrak{a}} = \mathrm{aff}(z - F^{h,-}_{\mathfrak{a}}) \quad \textit{and} \quad \mathcal{D}^-\bar{\phi}(z) = (F^{h,-}_{\mathfrak{a}}) \cap \{t = 1\};$$

3. *for all* $z \in T^-$ *there exists* $r > 0$, $F^{h,-}_{\mathfrak{a}}$ *such that*

$$B(z, r) \cap (z - \mathrm{int}_{\mathrm{rel}} F^{h,-}_{\mathfrak{a}}) \subseteq Z^{h,-}_{\mathfrak{a}}.$$

The choice of $\mathfrak{a}$ is in the spirit of Proposition I.31.

PROOF. Being a map, $\mathtt{v}^-_{\bar{\phi}}$ induced clearly a partition $\{Z^{h,-}_{\mathfrak{a}}, h = 0, \ldots, d, \mathfrak{a} \in \mathbb{R}^{d-h}\}$.

Point (1). Let $z \in Z^{h,-}_{\mathfrak{a}}$. By assumption, $z \in R^-_{\bar{\phi}}$ (or more precisely $z \in R^{h,-}_{\bar{\phi}}$ for some h), so that by Point (i) of Definition I.4.6 of $R^{-,h}_{\bar{\phi}}$ there exists z' such that

$$z' \in z - \mathrm{int}_{\mathrm{rel}} \partial^-\bar{\phi}(z).$$

In the same way, by Point (iii) of Definition I.4.6 of $R^{-,h}_{\bar\phi}$ there exists z'' such that

$$z'' \in z + \mathrm{int}_{\mathrm{rel}} \partial^- \bar\phi(z).$$

By Lemma I.4.10 we conclude that z is contained in the interior of $\mathcal{Q}_{\bar\phi}(z', z'')$, and this is a relatively open subset of $Z^{h,-}_{\mathfrak{a}}$, being of dimension

$$\dim \partial^- \bar\phi(z) = h + 1.$$

Point (2). Since $z \in R^{-,h}(\mathfrak{a})$, then the maximal backward extremal face $F^{h,-}_{\mathfrak{a}}$ is given by $F^-_{\bar\phi}(z)$. Using the fact that z is contained in a relatively open set of $Z^{h,+}_{\mathfrak{a}}$, the statements are a consequence of Proposition I.4.11.

Point (3). If $z \in T^-_{\bar\phi}$, then $\partial^- \bar\phi(z) \neq \emptyset$. We can thus take a maximal cone of the family

$$\big\{\mathbb{R}^+ \cdot \mathcal{Q}_{\bar\phi}(z, z'), z' \in \partial^- \bar\phi(z)\big\},$$

and the point $z' \in \partial^- \bar\phi(z)$ such that $\mathcal{Q}_{\bar\phi}(z, z')$ is maximal with respect to the set inclusion: it is thus fairly simple to verify that

$$\mathrm{int}_{\mathrm{rel}} \mathcal{Q}_{\bar\phi}(z, z') \subset Z^{h,-}_{\mathfrak{a}}$$

for some $h \in \{0, \dots, d\}$, $\mathfrak{a} \in \mathbb{R}^{d-h}$. Hence, if $F^{h,+}_{\mathfrak{a}}$ is a projection on $[0, +\infty) \times \mathbb{R}^d$ of an extremal face of a cone for $z \in \mathrm{int}_{\mathrm{rel}} \mathcal{Q}_{\bar\phi}(z, z')$, then from (I.33) the conclusion follows. ■

A completely similar statement holds for R^+, by considering of σ-continuous map

$$\begin{array}{rccc} \mathtt{v}^+_{\bar\phi} & : & R^+_{\bar\phi} & \rightarrow & \cup_{h=0}^d \mathcal{A}(h, [0, +\infty) \times \mathbb{R}^d) \\ & & z & \mapsto & \mathtt{v}^+_{\bar\phi}(z) := \mathrm{aff}\, \partial^+ \bar\phi(z) \end{array}$$

(I.4.15) THEOREM: *The map $\mathtt{v}^+_{\bar\phi}$ induces a partition*

$$\bigcup_{h'=0}^{d} \Big\{ Z^{h',+}_{\mathfrak{a}'} \subset [0, +\infty) \times \mathbb{R}^d, \mathfrak{a}' \in \mathbb{R}^{d-h'} \Big\}$$

on $R^+_{\bar\phi}$ such that the following holds:

1. *the sets $Z^{h',+}_{\mathfrak{a}'}$ are locally affine;*
2. *there exists a projection $F^{h',+}_{\mathfrak{a}'}$ of an extremal face with dimension $h' + 1$ of the cone* epi $\bar{\mathtt{c}}$ *such that*

$$\forall z \in Z^{h',+}_{\mathfrak{a}'}, \quad \mathrm{aff}\, Z^{h',+}_{\mathfrak{a}'} = \mathrm{aff}(z + F^{h',+}_{\mathfrak{a}'}) \quad \textit{and} \quad \mathcal{D}^- \bar\phi(z) = F^{h',+}_{\mathfrak{a}'} \cap \{t = 1\};$$

3. *for all $z \in T^+$ there exists $r > 0$, $F^{h',+}_{\mathfrak{a}'}$ such that*

$$B(z, r) \cap (z + \mathrm{int}_{\mathrm{rel}} F^{h',+}_{\mathfrak{a}'}) \subseteq Z^{h',+}_{\mathfrak{a}'}.$$

In general $h \neq h'$, but on $R_{\bar\phi}$ the two dimensions (and hence the affine spaces $\mathrm{aff}\, \partial^{\pm}\bar\phi(z)$) coincide.

(I.4.16) PROPOSITION: *If $z \in R_{\bar\phi}$ then*

$$\mathtt{v}^-_{\bar\phi}(z) = \mathtt{v}^+_{\bar\phi}(z).$$

PROOF. By the definition of $R_{\bar\phi}$, it follows that $h = h'$ because we have inner directions both forward and backward, and since each z is in the relatively open set

$$\mathrm{int}_{\mathrm{rel}}\big(Z^{h,-}_{\mathfrak{a}} \cap Z^{h',+}_{\mathfrak{a}'}\big),$$

then $\mathrm{aff}\, \partial^- \bar\phi(z) = \mathrm{aff}\, \partial^+ \bar\phi(z)$, i.e. $\mathtt{v}^-_{\bar\phi}(z) = \mathtt{v}^+_{\bar\phi}(z)$. ■

Define thus on $R_{\bar\phi}$

$$\mathtt{v}_{\bar\phi} := \mathtt{v}^-_{\bar\phi} \llcorner_R = \mathtt{v}^+_{\bar\phi} \llcorner_R,$$

and let

$$\Big\{ Z^h_{\mathfrak{a}}, \mathfrak{a} \in \mathbb{R}^{d-h} \Big\}$$

be the partition induced by $\mathtt{v}_{\bar\phi}$: since $R_{\bar\phi} = \cup_h (R^{-,h}_{\bar\phi} \cap R^{+,h}_{\bar\phi})$, it follows that

$$Z^h_{\mathfrak{a}} = Z^{h,-}_{\mathfrak{a}} \cap Z^{h,+}_{\mathfrak{a}},$$

once the parametrization of $\mathcal{A}(h, \mathrm{aff}\, Z^h_{\mathfrak{a}})$ is chosen in a compatible way. We can then introduce the extremal cones of $\mathrm{epi}\, \bar\phi$

$$C^h_a := \mathrm{epi}\, \bar\phi \cap \big(\mathtt{v}_{\bar\phi}(z) - z\big) = F^{h,+}_a = F^{h,-}_a.$$

Finally, define the set

$$\mathbf{D}_{\bar\phi} \subset \bigcup_{h=0,\dots,d} \Big(\{h\} \times \mathbb{R}^{d-h} \times \mathbb{R}^h \times \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^d) \Big)$$

by

$$\mathbf{D}_{\bar\phi} := \Big\{ \big(h, \mathfrak{a}, z, C\big) : C = C^h_{\mathfrak{a}}, z \in Z^h_{\mathfrak{a}} \Big\}. \tag{I.35}$$

(I.4.17) LEMMA: *The set $\mathbf{D}_{\bar\phi}$ is σ-compact.*

PROOF. Since $\mathtt{v}_{\bar\phi}$ is σ-continuous, the conclusion follows. ■

The next two sections will prove that this partition satisfies the condition of Theorem 3.1.

I.4.1. BACKWARD AND FORWARD REGULARITY

The first point we need to prove is that $\mathcal{H}^d$-almost every point in $\{t = 1\}$ is regular, i.e. it belongs to $R_{\bar\phi}$.

We recall below the result obtained in [8, 17], rewritten in our settings.

(I.4.18) PROPOSITION: *$\mathcal{L}^{d+1}$-almost every point in $[0,+\infty)\times\mathbb{R}^d$ is regular.*

Next we introduce a key tool for proving the regularity: the area estimate.

(I.4.19) LEMMA: *Let $\bar{t} > s > \varepsilon > 0$, and consider a Borel and bounded subset $S \subset \{t = \bar{t}\}$ made of backward regular points. Then for every $(\bar{t}, x) \in S$ there exists a point $\sigma_s(\bar{t}, x) \in \mathrm{int}_{\mathrm{rel}}\big(\partial^-\bar{\phi}(\bar{t}, x) \cap \{t = s\}\big)$ such that*

$$\mathcal{H}^d(\sigma_s(S)) \geq \left(\frac{s-\varepsilon}{\bar{t}-\varepsilon}\right)^d \mathcal{H}^d(S). \tag{I.36}$$

PROOF. First of all we recall that from (11) every point has always an optimal ray reaching $\{t = 0\}$. Using the assumption that the points in S are backward regular and the transitivity property stated in Proposition I.4.8, it follows that

$$\dim \partial^-\bar{\phi}(z) \cap \{t = \varepsilon\} = h, \qquad z \in S \cap R^{-,h}_{\bar{\phi}}.$$

In particular, it contains a given cone $z - K$ made of inner rays of $\partial^-\bar{\phi}(z)$.

Using the fact that $\mathcal{C}(h, [0,+\infty)\times\mathbb{R}^d)$ is separable and a decomposition analogous to the one of Proposition I.3.3, we can assume that there is a fixed h-dimensional cone K' such that

$$K' \subset \mathtt{p}^{\bar{t}}_{\mathrm{span}\,K'}\big((\partial^-\bar{\phi}(z) - z) \cap \{t = \varepsilon\}\big).$$

Hence we can slice the sets $\partial^-\bar{\phi}(S)$ by a family of parallel planes in $\mathcal{A}(d-h, [0,+\infty)\times\mathbb{R}^d)$ whose intersection with (a suitable translate of) K' is an inner direction of K'.

In this way, we find a $(d-h)$-dimensional problem one each affine plane A such that for every $(\bar{t}, x) \in S \cap A$ there exists a unique point in $\mathrm{int}_{\mathrm{rel}}\,\partial^-\bar{\phi}(\bar{t}, x) \cap \{t = \varepsilon\} \cap A$. We can now follow the strategy adopted in [12, Lemma 2.13] and obtain the area formula. ■

(I.4.20) REMARK: *We underline that the dimension of $\partial^-\bar{\phi}(z)$ is constant along the inner ray selected in the proof of the previous lemma. A similar property holds along inner rays of $\partial^+\bar{\phi}(z)$, $z \in R^+_{\bar{\phi}}$.*

We can now prove the regularity of $\mathcal{H}^d \llcorner_{\{t=1\}}$-a.e. point.

(I.4.21) PROPOSITION: *$\mathcal{H}^d$−almost every point in $\{t = 1\}$ is regular for $\bar{\phi}$.*

PROOF. By Proposition I.4.18 and Fubini theorem there is $\varepsilon > 0$ arbitrary small such that $\mathcal{H}^d$-a.e. point z of $\{t = 1 \pm \varepsilon\}$ is a regular point for $\bar{\phi}$.

Let $\varepsilon' > 0$ be fixed according to Lemma I.4.19. The area estimate I.36 gives that the measure of points in $\{t = 1 - \varepsilon\}$ which belong to an inner ray of a backward regular point in $\{t = 1 + \varepsilon\}$ is larger than

$$\left(\frac{1-\varepsilon-\varepsilon'}{1+\varepsilon-\varepsilon'}\right)^d \mathcal{H}^d(S).$$

By assumption these points are also regular (and thus forward regular).

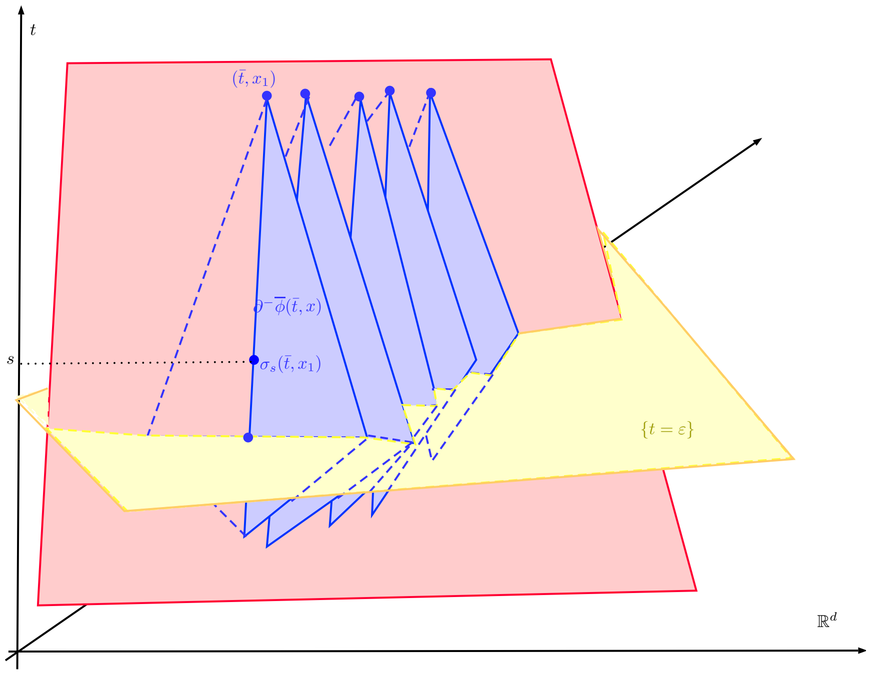


Figure I.9: The strategy to prove Lemma I.4.19: the pink plane is the transversal plane where $\partial^- \bar{\phi}(z)$ has a unique inner ray.

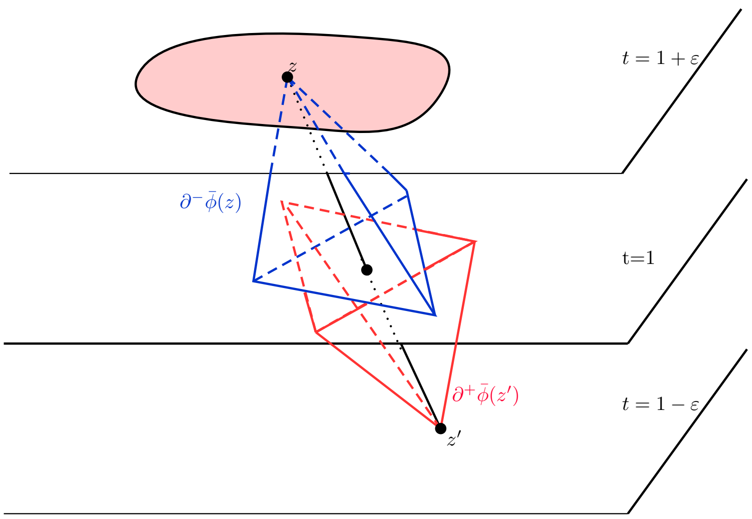


Figure I.10: If z, z' are regular points, then also the inner ray $[\![z, z']\!]$ is made of regular points (proof of Proposition I.4.21).

Observe that an inner optimal ray starting from a backward regular point and arriving in a regular point is made of regular points, see Figure I.10. Therefore, by the arbitrariness of ε and ε' we conclude the proof. ∎

Hence Point 1 of Theorem 3.1 is proved.

I.4.2. REGULARITY OF THE DISINTEGRATION

By [8, (3) of Theorem 1.1] we know that

$$\mathcal{L}^{d+1} \llcorner_{\bigcup_{\mathfrak{a}} Z^h_{\mathfrak{a}}} = \int f(\mathfrak{a}, z) \mathcal{H}^{h+1} \llcorner_{Z^h_{\mathfrak{a}}} (dz) \eta^h(d\mathfrak{a}).$$

so that by Fubini Theorem

$$\mathcal{H}^d \llcorner_{\{t=1+\varepsilon\} \cap \bigcup_{\mathfrak{a}} Z^h_{\mathfrak{a}}} = \int f(\mathfrak{a}, x) \mathcal{H}^h \llcorner_{\{t=1+\varepsilon\} \cap Z^h_{\mathfrak{a}}} (dx) \eta^h(d\mathfrak{a}) \qquad \text{for a.e. } \varepsilon > 0.$$

Recalling the decomposition of Lemma I.4.19, we fix the set of indexes

$$\mathfrak{A}^h_{\varepsilon,K} := \Big\{ \mathfrak{a} \in \mathfrak{A}^h : Z^h_{\mathfrak{a}} \cap \{t = 1 + \varepsilon\} \neq \emptyset \text{ and } K \subset \mathtt{p}_{\mathrm{aff}\,K} C^h_{\mathfrak{a}} \Big\},$$

with $K \in \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^d)$ given.

An easy argument based on the push forward of $\mathcal{H}^d$ along the rays selected in the proof of Lemma I.4.19 (see for example [8, Section 5]) gives that there is

$$c(\mathfrak{a}, x) \in \big((1 - \varepsilon/2)^d, 2^d \big)$$

such that

$$\mathcal{H}^d \llcorner_{\{t=1\}\cap\bigcup_{\mathfrak{a}} Z^h_{\mathfrak{a}}} = \int_{\mathfrak{A}^h_{\varepsilon,K}} c(\mathfrak{a},x) f(\mathfrak{a},x) \mathcal{H}^h \llcorner_{\{t=1+\varepsilon\}\cap Z^h_{\mathfrak{a}}}(dx)\eta^h(d\mathfrak{a}).$$

The lower estimate of c is given immediately by Lemma I.8.3 for $\bar{t} = 1+\varepsilon/2$, $\varepsilon' = \varepsilon/2$ and $s = 1$.

The upper estimate follows by inverting the roles of $\bar{t} = 1+\varepsilon$ and $s = 1$: in this case the ray starts in $Z^h_{\mathfrak{a}} \cap \{t = 1\}$ and ends in $Z^h_{\mathfrak{a}} \cap \{t = 1+\varepsilon\}$, and we are estimating the area between $t = 1$ and $t = 1+\varepsilon/2$. Using the same rays of Lemma I.4.19 in the backward direction and applying (I.36), one obtains the second bound.

Notice now that in the partition of the proof of Lemma I.4.19 the inner rays are parallel inside the elements of the partition: once the cone K and the transversal planes V_K are chosen, in each element $Z^h_{\mathfrak{a}}$ the rays $Z^h_{\mathfrak{a}} \cap V_K$ are parallel, so that the map along

$$\begin{array}{cccc} \mathtt{t}_{V_K} & : & \bigcup_{\mathfrak{A}^h_{\varepsilon,K}} Z^h_{\mathfrak{a}} \cap \{t = 1+\varepsilon/2\} & \to & \bigcup_{\mathfrak{A}^h_{\varepsilon,K}} Z^h_{\mathfrak{a}} \cap \{t=1\} \\ & & Z^h_{\mathfrak{a}} \ni x & \mapsto & \mathtt{t}_{V_k}(x) := (x+V_K)\cap Z^h_{\mathfrak{a}} \cap \{t=1\} \end{array} \tag{I.37}$$

is just a translation (see Figure I.11). We thus deduce that

$$(\mathtt{t}_{V_k})_\sharp \mathcal{H}^h \llcorner_{Z^h_{\mathfrak{a}}\cap\{t=1+\varepsilon/2\}} = \mathcal{H}^h \llcorner_{\mathtt{t}_{V_K}(Z^h_{\mathfrak{a}}\cap\{t=1+\varepsilon/2\}},$$

and that $c(\mathfrak{a},x) = c(\mathfrak{a})$.

Define

$$f'(\mathfrak{a}, \mathtt{t}_{V_K}(x)) := c(\mathfrak{a}) f(\mathfrak{a},x),$$

so that we can write

$$\mathcal{H}^d \llcorner_{\{t=1\}\cap\bigcup_{\mathfrak{A}^h_{\varepsilon,K}} Z^h_{\mathfrak{a}}} = \int_{\mathfrak{A}^h_{\varepsilon,K}} f'(\mathfrak{a},x) \mathcal{H}^h \llcorner_{\{t=1\}\cap Z^h_{\mathfrak{a}}}(dx)\eta^h(d\mathfrak{a}).$$

By the uniqueness of the disintegration, the previous formula gives the regularity of the disintegration of $\mathcal{H}^d \llcorner_{\mathtt{t}_{V_K}(\bigcup_{A^h_{\varepsilon,K}} Z^h_{\mathfrak{a}}\cap\{t=1\})}$. By varying K and ϵ and using the fact that $Z^h_{\mathfrak{a}}$ are transversal to $\{t=1\}$ and relatively open, we obtain the following proposition:

(I.4.22) PROPOSITION: *The disintegration*

$$\mathcal{H}^d \llcorner_{\cup_{h,\mathfrak{a}} Z^h_{\mathfrak{a}}\cap\{t=1\}} = \sum_h \int v^h_{\mathfrak{a}} \eta^h(d\mathfrak{a})$$

w.r.t. the partition $\{Z^{\mathfrak{a}}_h \cap \{t=1\}\}_{h,\mathfrak{a}}$ *is regular:*

$$v^h_{\mathfrak{a}} \ll \mathcal{H}^h \llcorner_{Z^h_{\mathfrak{a}}}.$$

This concludes the proof of Point (2) of Theorem 3.1. The last point of Theorem 3.1 is an immediate consequence of the fact that $\bar{\phi}$ is a potential, and thus the mass is moving only along optimal rays $\operatorname{graph}\bar{\phi} \cap (z - \operatorname{epi}\bar{\mathtt{c}})$, and for all regular points z

$$\mathtt{p}_{[0,+\infty)\times\mathbb{R}^d}\big(\operatorname{graph}\bar{\phi}\cap(z-\operatorname{epi}\bar{\mathtt{c}})\big) \subset z - C^h_{\mathfrak{a}}.$$

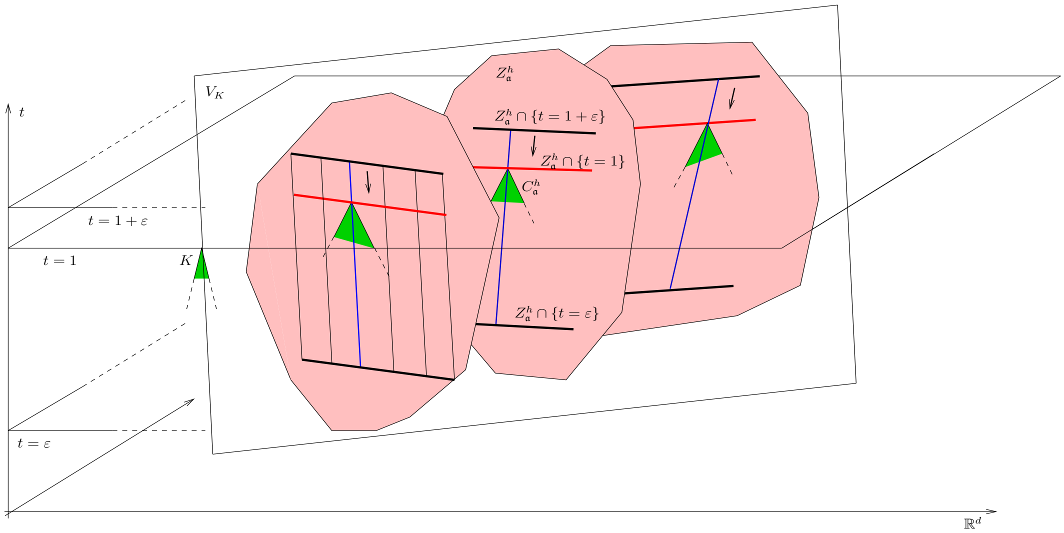


Figure I.11: The parallel translation of (I.37) along the direction $C^h_{\mathfrak{a}} \cap V_K$.

(I.4.23) **REMARK:** *The fact that $\eta^h \simeq \mathcal{H}^{d-h} \llcorner_{\mathfrak{A}^h}$, with $\mathfrak{A}^h$ chosen as in Remark I.3.5, is again a simple consequence of the estimate on the push-forward along optimal rays and Fubini Theorem. This result is exactly the same as the one stated in [8, Theorem 5.18]: we refer to that paper for the proof, because the form of the image measure is not essential in the construction and can be seen as an additional regularity of the partition.*

I.5. Optimal transport and disintegration of measures on directed locally affine partitions

In this section and the following three ones we show how to refine a directed locally affine partition $\mathbf{D}$ either to lower the dimension of the sets or to obtain indecomposable sets. This procedure will then be applied at most d-times in order to obtain the proof of Theorem 1.1.

Following the structure of the first directed locally affine partition $\mathbf{D}_{\bar{\phi}}$ constructed in the previous section, we will consider the following three measures:

1. the measure $\mathcal{H}^d \llcorner_{\{t=1\}}$, with $\mathcal{H}^d(\{t = 1\} \setminus \cup_{h,\mathfrak{a}} Z^h_{\mathfrak{a}}) = 0$;

2. the probability measure $\bar{\mu} := \delta_{\{t=1\}} \times \mu$, such that $\mu \ll \mathcal{L}^d$, and thus in particular $\bar{\mu} \ll \mathcal{H}^d \llcorner_{\{t=1\}}$;

3. a probability measure $\bar{\nu}$ supported on $\{t = 0\}$.

On $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ we can define the natural transference cost

$$\mathtt{c}_{\mathbf{Z}}(z, z') := \begin{cases} 0 & z \in Z^h_{\mathfrak{a}}, z - z' \in C^h_{\mathfrak{a}}, \\ \infty & \text{otherwise.} \end{cases} \tag{I.38}$$

Since

$$\{\mathtt{c}_{\mathbf{Z}} < \infty\} = \big\{(z, z') : z \in \mathbf{Z}, z - z' \in \mathbf{d}(z)\big\},$$

i.e. it coincides with the projection $(\mathtt{p}_z, (\mathtt{p}_{\mathbb{R}^{d+1}} \circ \mathtt{p}_C))\mathbf{D}$ of $\mathbf{D}$, then it is σ-continuous.

From Point (3) of Theorem 3.1, it follows for $\mathbf{D}$ each optimal transference plan $\bar{\pi}$ has finite transference cost w.r.t. $\mathtt{c}_{\mathbf{Z}}$, so that the set $\Pi^f_{\mathtt{c}_{\mathbf{Z}}}(\bar{\mu}, \bar{\nu})$ is not empty. From the observation (see Example I.5.1 below) that in general the construction *depends* on the selected transference plan $\underline{\bar{\pi}}$ through the marginals $\{\underline{\bar{\nu}}^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$, we will consider transference plans $\bar{\pi} \in \Pi(\bar{\nu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ such that

$$\int \mathtt{c}_{\mathbf{Z}} \bar{\pi} < \infty.$$

i.e. $\bar{\pi} \in \Pi^f_{\mathtt{c}_{\mathbf{Z}}}(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$.

Consider the disintegrations on the partition $\{Z^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$: if $z \mapsto (h(z), \mathfrak{a}(z))$ is the σ-continuous function whose graph is the projection $\mathtt{p}_{h,\mathfrak{a},z}\mathbf{D}$, then

$$\bar{\mu} = \sum_{h=0}^{d} \int_{\mathfrak{A}^h} \bar{\mu}^h_{\mathfrak{a}} \xi^h(d\mathfrak{a}), \qquad \xi^h := \mathfrak{a}_\sharp \bar{\mu} \llcorner_{\mathbf{Z}^h}.$$

In the same way we can disintegrate $\bar{\pi} \in \Pi(\bar{\nu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ w.r.t. the partition $\{Z^h_{\mathfrak{a}} \times \mathbb{R}^{d+1}\}_{h,\mathfrak{a}}$,

$$\bar{\pi} = \sum_{h=0}^{d} \int_{\mathfrak{A}^h} \bar{\pi}^h_{\mathfrak{a}} \xi^h(d\mathfrak{a}), \qquad \bar{\mu}^h_{\mathfrak{a}} = (\mathtt{p}_1)_\sharp \bar{\pi}^h_{\mathfrak{a}}.$$

Write also

$$\bar{\nu} = \sum_{h=0}^{d} \int_{\mathfrak{A}^h} \underline{\bar{\nu}}^h_{\mathfrak{a}} \xi^h(d\mathfrak{a}), \qquad \underline{\bar{\nu}}^h_{\mathfrak{a}} = (\mathtt{p}_2)_\sharp \bar{\pi}^h_{\mathfrak{a}},$$

even if the above formula does not correspond to a real disintegration.

In the following example we show why in general the partition depends on the plan $\underline{\bar{\pi}}$.

(I.5.1) EXAMPLE: *For $d = 2$ let*

$$\bar{\mu} = \frac{1}{8}\mathcal{H}^2 \llcorner_A, \qquad A = \big\{(1, x) : \big|x - (\pm 2, 0)\big| \leq 1\big\},$$

$$\bar{\nu} = \frac{1}{8}\big(2 - ||x| - 2|\big)\mathcal{H}^1 \llcorner_B, \qquad B = \big\{(0, x) : x \in \{0\} \times [-4, 4]\big\}.$$

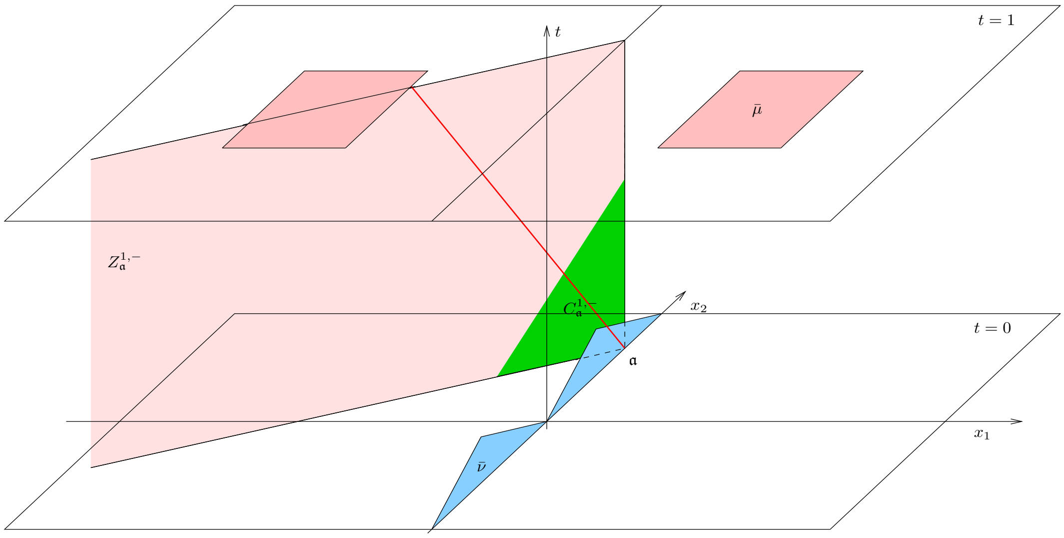


Figure I.12: The transport problem studied in Example I.5.1.

and let the transportation cost $\bar{\mathtt c}$ *be*

$$\bar{\mathtt c}(t,x) = \begin{cases} |x|_\infty & t > 0, \\ \mathbb{1}_{\{0\}}(x) & t = 0, \\ +\infty & t < 0, \end{cases} \qquad |(x_1,x_2)|_\infty = \max\{|x_1|,|x_2|\}.$$

An pair of optimal plans $\bar\pi^\pm$ *are given by*

$$\bar\pi^\pm = (\mathbb{I},\mathtt{T}^\pm)_\sharp\bar\mu, \qquad \textit{where} \quad \mathtt{T}^\pm(x) := \big(0,0,(x_2 \pm x_1)\big),\ x = (x_1,x_2) \in \mathbb{R}^2,$$

and, taking as a potential $\bar\phi(t,x) = |x_1|$*, the decomposition obtained by the first step can be easily checked to be*

$$\begin{aligned} Z^2_{\mathfrak a_1} &= \big\{(t,x), x_1 < 0\big\}, \qquad & C^2_{\mathfrak a_1} &= \big\{(t,x), |x_2| \leq -x_1\big\}, \\ Z^2_{\mathfrak a_1} &= \big\{(t,x), x_1 > 0\big\}, \qquad & C^2_{\mathfrak a_2} &= \big\{(t,x), |x_2| \leq +x_1\big\}. \end{aligned}$$

Being the second marginals of the disintegration of $\bar\pi^\pm = \bar\pi^{2,\pm}_{a_1} + \bar\pi^{2,\pm}_{a_2}$ *w.r.t. the partition* $\{Z^h_{\mathfrak a_i}\}_{i=1,2}$ *given by*

$$\bar\nu^{2,\pm}_{\mathfrak a_i} = (\mathtt p_2)_\sharp\bar\pi^{2,\pm}_{\mathfrak a_i} = \frac{1}{4}\big(2 - ||x| - 2|\big)\mathcal{H}^1 \llcorner_{B^\pm_i}, \qquad B^\pm_i = \big\{(1,0,x_2) : \pm(-1)^i x_2 \in [0,4]\big\},$$

the sets $\Pi(\bar\mu,\{\bar\nu^{h,-}_{\mathfrak a_i}\})$, $\Pi(\bar\mu,\{\bar\nu^{h,+}_{\mathfrak a_i}\})$ *are different.*

If we further proceed with the decomposition, we will obtain that the indecomposable partition corresponding to $\bar\pi^\pm$ *is*

$$\begin{aligned} Z^{1,\pm}_{\mathfrak a} &= \big\{(t,x), x_2 = \mathfrak a \mp x_1, \pm(\operatorname{sign}\mathfrak a)x_1 > 0\big\}, \\ C^{1,\pm}_{\mathfrak a} &= \big\{(t,x), x_2 = \mp x_1, \pm(\operatorname{sign}\mathfrak a)x_1 \geq 0\big\}. \end{aligned}$$

The parameterization is such that

$$Z^{1,\pm}_{\mathfrak{a}} \cap \{x_1 = 0\} = \{(0, \mathfrak{a})\}, \qquad \mathfrak{a} \in \mathbb{R}.$$

We conclude with the observation that if instead we consider the transference plan $\bar{\pi} = (\bar{\pi}^+ + \bar{\pi}^-)/2$, *then the first decomposition is already indecomposable in the space* $\Pi(\bar{\mu}, \{\bar{\nu}^2_{\mathfrak{a}_i}\})$, *because* $\bar{\nu}^2_{\mathfrak{a}_i} = \bar{\nu}$.

Consider now the disintegration of the Hausdorff measure $\mathcal{H}^d \llcorner_{\{t=1\}}$ w.r.t. the partition $\{Z^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$:

$$\mathcal{H}^d \llcorner_{\{t=1\}} = \sum_{h=0}^{d} \int_{\mathfrak{A}^h} \upsilon^h_{\mathfrak{a}} \eta^h(d\mathfrak{a}), \qquad \eta^h := \mathfrak{a}_\sharp \mathcal{H}^d \llcorner_{\mathbf{Z}^h \cap \{t=1\}}.$$

Following Point (2) of Theorem 3.1 on the regularity of the disintegration and taking into account the choice of the variable $\mathfrak{a}$ considered in Remark I.3.5, we will recursively assume that the following

1. the measures $\upsilon^h_{\mathfrak{a}}$ are equivalent to $\mathcal{H}^h \llcorner_{Z^h_{\mathfrak{a}} \cap \{t=1\}}$,
2. the measures η^h are equivalent to $\mathcal{H}^{d-h} \llcorner_{\mathfrak{A}^h}$ (see Remark I.4.23).

We will write

$$\bar{\mu}^h_{\mathfrak{a}} = \mathtt{f}_{\mathfrak{a}} \mathcal{H}^h, \quad \mathtt{f}_{\mathfrak{a}} \text{ Borel.}$$

I.5.1. MAPPING A SHEAF SET TO A FIBRATION

Consider one of the h-dimensional directed sheaf sets $\mathbf{D}(h,n)$, $h = 0, \dots, d$, constructed in Proposition I.3.3, and define the map

$$\begin{array}{rccc} \mathtt{r} & : & \mathbf{D}(h,n) & \to & \mathbb{R}^{d-h} \times \big([0,+\infty) \times \mathbb{R}^h\big) \times \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h) \\ & & (\mathfrak{a}, z, C^h_{\mathfrak{a}}) & \mapsto & \mathtt{r}(\mathfrak{a}, z, C^h_{\mathfrak{a}}) := \big(\mathfrak{a}, \mathtt{p}^t_{\operatorname{aff} C^h_n} z, \mathtt{p}^t_{\operatorname{aff} C^h_n} C^h_{\mathfrak{a}}\big) \end{array} \tag{I.39}$$

Remember that C^h_n is the reference cone for each cone in $\mathbf{D}(h,n)$ and $\operatorname{aff} C^h_n$ is the reference plane for each $Z^h_{\mathfrak{a}}$ in this sheaf.

Being the projection of a σ-compact set, $\mathtt{r}$ is σ-continuous. Clearly, since z determines $\mathfrak{a}$ and $\mathfrak{a}$ determines $C^h_{\mathfrak{a}}$, also the maps

$$\begin{array}{rccc} \tilde{\mathtt{r}} & : & \mathbf{Z}^h_n & \to & \mathbb{R}^{d-h} \times \big([0,+\infty) \times \mathbb{R}^h\big) \times \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h) \\ & & z & \mapsto & \tilde{\mathtt{r}}(z) := \big(\mathfrak{a}(z), \mathtt{p}^t_{\operatorname{aff} C^h_n} z, \mathtt{p}^t_{\operatorname{aff} C^h_n} C^h_{\mathfrak{a}(z)}\big) \end{array}$$

$$\begin{array}{rccc} \hat{\mathtt{r}} & : & \mathfrak{A}^h_n & \to & \mathbb{R}^{d-h} \times \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h) \\ & & \mathfrak{a} & \mapsto & \hat{\mathtt{r}}(z) := \big(\mathfrak{a}, \mathtt{p}^t_{\operatorname{aff} C^h_n} C^h_{\mathfrak{a}}\big) \end{array}$$

are σ-continuous. We will use the notation

$$w \in [0,+\infty) \times \mathbb{R}^h, \quad \tilde{Z}^h_{\mathfrak{a}} := \big(\mathtt{i}_h \circ \mathtt{p}^t_{\operatorname{aff} C^h_n}\big) Z^h_{\mathfrak{a}} \quad \text{and} \quad \tilde{C}^h_{\mathfrak{a}} := \big(\mathtt{i}_h \circ \mathtt{p}^t_{\operatorname{aff} C^h_n}\big) C^h_{\mathfrak{a}},$$

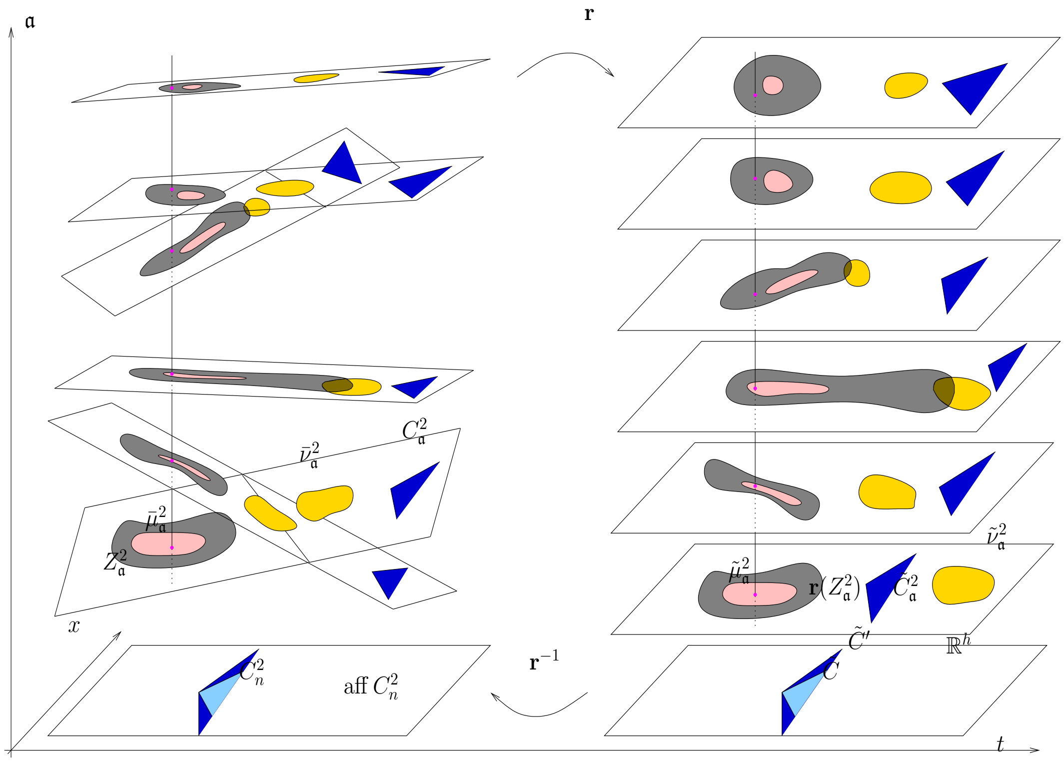


Figure I.13: The map $\mathtt{r}$ defined in I.39.

where $\mathtt{i}_h : V^h_n = \text{aff}\, C^h_n \to \mathbb{R}^h$ is the identification map. Moreover set $\tilde{\mathbf{Z}}^h_n := \cup_{\mathfrak{a}}\{\mathfrak{a}\} \times \tilde{Z}^h_{\mathfrak{a}}$.

From Points (3) and (4) of Proposition I.3.3 we deduce the following result.

(I.5.2) LEMMA: *There exists two cones C_n, $C_n(-r_n)$ in $\mathcal{C}(h, [0, +\infty) \times \mathbb{R}^h)$ such that*

$$\forall \mathfrak{a} \in \mathfrak{A}^h_n \ \Big(C_n(-r_n) \subset \mathring{\tilde{C}}{}^h_{\mathfrak{a}} \ \wedge \ \tilde{C}^h_{\mathfrak{a}} \subset \mathring{C}_n\Big).$$

PROOF. Take $C_n := (\mathtt{i}_h \circ \mathtt{p}^t_{\text{aff}\, C^h_n})C^h_n$. ■

(I.5.3) DEFINITION: A σ-compact subset $\tilde{\mathbf{D}}$ of $\mathbb{R}^{d-h} \times ([0, +\infty) \times \mathbb{R}^h) \times \mathcal{C}(h, [0, \infty) \times \mathbb{R}^h)$ such that

- the cone $\tilde{\mathbf{D}}(\mathfrak{a}, w)$ is independent of w,
- there are two non degenerate cones $\tilde{C}, \tilde{C}' \in \mathcal{C}(h, [0, +\infty) \times \mathbb{R}^h)$, $\tilde{C} \subset \tilde{C}'$, (replacing of $C_n(-r_n)$, C_n) satisfying Lemma I.5.2,

will be called an *h-dimensional directed fibration.*

In particular we have that $\mathbf{r}(\mathbf{D}(h, n))$ is an h-dimensional directed fibration.

The fact that we are considering transference problems in $\Pi(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ allows to rewrite them in the coordinates $(\mathfrak{a}, w) \in \mathbb{R}^{d-h} \times \big([0,+\infty) \times \mathbb{R}^h\big)$. Indeed, consider the multifunction $\check{\mathbf{r}}$ whose inverse is the map

$$\begin{array}{rccc} \check{\mathtt{r}}^{-1} & : & \mathfrak{A}^h_n \times \big([0,+\infty)\times\mathbb{R}^h\big) & \to & [0,\infty)\times\mathbb{R}^d \\ & & (\mathfrak{a}, w) & \mapsto & \check{\mathtt{r}}^{-1}(\mathfrak{a}, w) := \mathrm{aff}\, Z^h_{\mathfrak{a}} \cap (\mathtt{i}_k \circ \mathtt{p}^t_{\mathrm{aff}\, C^h_n})^{-1}(w) \end{array} \tag{I.40}$$

and define the transport cost

$$\tilde{\mathtt{c}}^h_n\big(\mathfrak{a}, w, \mathfrak{a}', w'\big) := \begin{cases} 0 & \mathfrak{a} = \mathfrak{a}', w - w' \in \tilde{C}^h_{\mathfrak{a}}, \\ \infty & \text{otherwise.} \end{cases} \tag{I.41}$$

It is clear that

$$\mathtt{c}_{\mathbf{Z}}\big(\check{\mathtt{r}}^{-1}(\mathfrak{a}, w), \check{\mathtt{r}}^{-1}(\mathfrak{a}', w')\big) = \tilde{\mathtt{c}}^h_n\big(\mathfrak{a}, w, \mathfrak{a}, w'\big).$$

Define the measures $\tilde{\mu}, \tilde{\nu} \in \mathcal{P}(\mathbb{R}^{d+1})$ by

$$\tilde{\mu} := \int_{\mathfrak{A}^h_n} \tilde{\mu}_{\mathfrak{a}} \xi^h(d\mathfrak{a}), \quad \tilde{\mu}_{\mathfrak{a}} := (\mathtt{i}_h \circ \mathtt{p}^t_{\mathrm{aff}\, C^h_n})_\sharp \underline{\bar{\mu}}^h_{\mathfrak{a}},$$

$$\tilde{\nu} := \int_{\mathfrak{A}^h_n} \tilde{\nu}_{\mathfrak{a}} \xi^h(d\mathfrak{a}), \quad \tilde{\nu}_{\mathfrak{a}} := (\mathtt{i}_h \circ \mathtt{p}^t_{\mathrm{aff}\, C^h_n})_\sharp \bar{\nu}^h_{\mathfrak{a}}.$$

Since the marginals of the conditional probabilities $\bar{\pi}^h_{\mathfrak{a}}$ are fixed for all $\bar{\pi} \in \Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$, then it is fairly easy to deduce the next proposition.

(I.5.4) PROPOSITION: *It holds*

$$\Pi^f_{\mathtt{c}_{\mathbf{Z}}}(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\}) = \big(\check{\mathbf{r}}^{-1} \otimes \check{\mathbf{r}}^{-1}\big)_\sharp \Pi^f_{\tilde{\mathtt{c}}^h_n}(\tilde{\mu}, \tilde{\nu}).$$

Moreover $\tilde{\mu}(\{t = 1\}) = 1$, $\tilde{\nu}(\{t = 0\}) = 1$.

By Point (5) of Proposition I.3.3 one deduces that $(\mathtt{p}_{\mathrm{aff}\, C^h_n})_\sharp \upsilon^h_{\mathfrak{a}}$ are equivalent to $\mathcal{H}^h \llcorner_{\tilde{Z}^h_{\mathfrak{a}} \cap \{t=1\}}$, being $\upsilon^h_{\mathfrak{a}}$ the conditional probabilities of the disintegration of $\mathcal{H}^d \llcorner_{\{t=1\}}$, so that in particular the measure $\check{\mathbf{r}}_\sharp \mathcal{H}^d \llcorner_{\mathbf{Z}^h_a \cap \{t=1\}}$ is equivalent to $\mathcal{H}^d \llcorner_{\tilde{\mathbf{Z}}^h_n \cap \{t=1\}}$. We have used the fact that $\check{\mathbf{r}}$ is single valued on $\mathbf{Z}^h_n$.

I.6. Analysis of the cyclical monotone relation on a fibration

In this section we study the cyclical monotone relation generated by transference plans with finite cost on a fibration $\tilde{\mathbf{D}}$.

We recall that a directed fibration is a σ-compact subset of

$$\Big\{(\mathfrak{a}, w, C) \in \mathbb{R}^{d-h} \times \big([0,+\infty) \times \mathbb{R}^h\big) \times \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h)\Big\}$$

with the properties that $\mathtt{p}_{\mathfrak{a},C}\tilde{\mathbf{D}}$ is the graph of a σ-compact map $\mathfrak{a} \mapsto C_{\mathfrak{a}} \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h)$ and there are two cones $\tilde{C}, \tilde{C}' \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h)$ such that

$$\forall \mathfrak{a} \in \mathtt{p}_{\mathfrak{a}}\tilde{\mathbf{D}} \ \Big(\tilde{C} \subset \mathring{\tilde{C}}_{\mathfrak{a}} \ \wedge \ \tilde{C}_{\mathfrak{a}} \subset \mathring{\tilde{C}}'\Big). \tag{I.42}$$

We will use the notation $\tilde{\mathfrak{A}} := \mathtt{p}_{\mathfrak{a}}\tilde{\mathbf{D}} \subset \mathbb{R}^{d-h}$, $\tilde{Z}_{\mathfrak{a}} := \mathtt{p}_w\tilde{\mathbf{D}}(\mathfrak{a})$, $\tilde{\mathbf{Z}} := \mathtt{p}_{\mathfrak{a},w}\tilde{\mathbf{D}}$: essentially the notation is the same for $\mathbf{D}(h,n)$, only neglecting the index h and n.

The properties (I.42) of the two cones $\tilde{C}, \tilde{C}' \in \mathcal{C}(h, [0,+\infty) \times \mathbb{R}^h)$ allows us to choose coordinates $w = (t,x) \in [0,+\infty) \times \mathbb{R}^h$ such that

$$\tilde{C} = \operatorname{epi} \mathtt{co}, \qquad \tilde{C}' = \operatorname{epi} \mathtt{co}'$$

for two 1-Lipschitz 1-homogeneous convex functions $\mathtt{co}, \mathtt{co}' : \mathbb{R}^h \to [0,+\infty)$ such that $\mathtt{co}'(x) < \mathtt{co}(x)$ for all $x \neq 0$. In the same way, let $\mathtt{co}_{\mathfrak{a}} : \mathbb{R}^h \to [0,+\infty)$ be 1-Lipschitz 1-homogeneous convex functions such that

$$\tilde{C}_{\mathfrak{a}} = \operatorname{epi} \mathtt{co}_{\mathfrak{a}}.$$

Clearly from (I.42) for $x \neq 0$ it holds $\mathtt{co}'(x) < \mathtt{co}_{\mathfrak{a}}(x) < \mathtt{co}(x)$. Moreover from the assumption that $\tilde{C}' \cap \{t = 1\}$ is bounded, we have that $\mathtt{co}'(x) > 0$ for $x \neq 0$.

Define the transference cost $\tilde{\mathtt{c}}$ as in (I.41)

$$\tilde{\mathtt{c}}\big(\mathfrak{a}, w, \mathfrak{a}', w'\big) := \begin{cases} 0 & \mathfrak{a} = \mathfrak{a}', w - w' \in \tilde{C}_{\mathfrak{a}}, \\ \infty & \text{otherwise.} \end{cases}$$

Since $\tilde{\mathtt{c}}(\mathfrak{a}, w, \mathfrak{a}', w') < \infty$ implies $\mathfrak{a} = \mathfrak{a}'$, we will often write

$$\tilde{\mathtt{c}}_{\mathfrak{a}}(w, w') := \tilde{\mathtt{c}}\big(\mathfrak{a}, w, \mathfrak{a}, w'\big).$$

From the straightforward geometric property of a convex cone C

$$w \in C \quad \Longrightarrow \quad w + C \subset C, \tag{I.43}$$

one deduces that

$$\tilde{\mathtt{c}}(\mathfrak{a}, w, \mathfrak{a}', w'), \tilde{\mathtt{c}}(\mathfrak{a}', w', \mathfrak{a}'', w'') < \infty \quad \Longrightarrow \quad \tilde{\mathtt{c}}(\mathfrak{a}, w, \mathfrak{a}'', w'') < \infty.$$

Note that in particular $\mathfrak{a} = \mathfrak{a}' = \mathfrak{a}''$.

Consider two probability measures $\tilde{\mu}$, $\tilde{\nu}$ in $\tilde{\mathfrak{A}} \times ([0, +\infty) \times \mathbb{R}^h)$ such that their disintegrations

$$\tilde{\mu} = \int_{\tilde{\mathfrak{A}}} \tilde{\mu}_{\mathfrak{a}} \tilde{\xi}(d\mathfrak{a}),\ \tilde{\xi} := (\mathtt{p}_{\tilde{\mathfrak{A}}})_\sharp \tilde{\mu}, \qquad \tilde{\nu} = \int_{\tilde{\mathfrak{A}}} \tilde{\nu}_{\mathfrak{a}} \tilde{\xi}'(d\mathfrak{a}),\ \tilde{\xi}' = (\mathtt{p}_{\tilde{\mathfrak{A}}})_\sharp \tilde{\nu}, \tag{I.44}$$

satisfy

$$\tilde{\xi} = \tilde{\xi}' \qquad \text{and} \qquad \tilde{\mu}_{\mathfrak{a}}(\tilde{Z}_{\mathfrak{a}} \cap \{t = 1\}) = \tilde{\nu}_{\mathfrak{a}}(\tilde{Z}^h_{\mathfrak{a}} \cap \{t = 0\}) = 1.$$

It is fairly easy to see that if $\tilde{\pi} \in \Pi^f(\mu, \nu)$, then

$$\tilde{\pi} = \int_{\tilde{\mathfrak{A}}} \tilde{\pi}_{\mathfrak{a}} \tilde{\xi}(d\mathfrak{a}) \qquad \text{with} \qquad \tilde{\pi}_{\mathfrak{a}} \in \Pi^f(\tilde{\mu}_{\mathfrak{a}}, \tilde{\nu}_{\mathfrak{a}}), \tag{I.45}$$

and conversely if $\mathfrak{a} \mapsto \tilde{\pi}_{\mathfrak{a}} \in \Pi^f(\tilde{\mu}_{\mathfrak{a}}, \tilde{\nu}_{\mathfrak{a}})$ is an $\tilde{\xi}$-measurable function, then the transference plan given by the integration in (I.45) is in $\Pi^f(\tilde{\mu}, \tilde{\nu})$.

We denote by $\mathit{\Gamma}(\tilde{\pi})$ the family of σ-compact carriages $\tilde{\Gamma}$ of $\tilde{\pi} \in \Pi^f(\tilde{\mu}, \tilde{\nu})$,

$$\mathit{\Gamma}(\tilde{\pi}) := \Big\{ \tilde{\Gamma} \subset \{\tilde{\mathtt{c}} < \infty\} \cap \{t = 1\} \times \{t = 0\} : \tilde{\pi}(\tilde{\Gamma}) = 1 \Big\},$$

and set

$$\mathit{\Gamma} := \bigcup_{\tilde{\pi} \in \Pi^f(\tilde{\mu}, \tilde{\nu})} \mathit{\Gamma}(\tilde{\pi}).$$

The section of a set $\tilde{\Gamma} \in \mathit{\Gamma}$ at $(\mathfrak{a}, \mathfrak{a})$ will be denoted by $\tilde{\Gamma}(\mathfrak{a}) \subset \{t = 1\} \times \{t = 0\}$.

I.6.1. A LINEAR PREORDER ON $\tilde{\mathfrak{A}} \times [0, +\infty) \times \mathbb{R}^h$

Let $\tilde{\Gamma} \in \mathit{\Gamma}$. The following lemma is taken from [8, Lemma 7.3]: we omit the proof because it is completely similar.

(I.6.1) LEMMA: *There exist a $\tilde{\xi}$-conegligible set $\tilde{\mathfrak{A}}' \subset \mathbb{R}^{d-h}$ and a countable family of σ-continuous functions $\mathtt{w}_n : \tilde{\mathfrak{A}}' \to \{t = 1\} \times \mathbb{R}^h$, $n \in \mathbb{N}$, such that*

$$\forall n \in \mathbb{N}, \mathfrak{a} \in \tilde{\mathfrak{A}}' \ \Big(\mathtt{w}_n(\mathfrak{a}) \subset \mathtt{p}_1 \tilde{\Gamma}(\mathfrak{a}) \subset \mathrm{clos}\, \{\mathtt{w}_n(\mathfrak{a})\}_{n \in \mathbb{N}} \Big).$$

Define the set $H_n \subset \tilde{\mathfrak{A}} \times \mathbb{R}^h$ by

$$H_n := \bigg\{ (\mathfrak{a}, w) : \exists I \in \mathbb{N}, \big\{ (w_i, w'_i) \big\}_{i=1}^I \subset \tilde{\Gamma}(\mathfrak{a}) \ \Big(w_1 = \mathtt{w}_n(\mathfrak{a}) \wedge \tilde{\mathtt{c}}(\mathfrak{a}, w_{i+1}, \mathfrak{a}, w'_i), \tilde{\mathtt{c}}(\mathfrak{a}, w, \mathfrak{a}, w'_I) < \infty \Big) \bigg\}. \tag{I.46}$$

This set represents the points which can be reached from $\mathtt{w}_n(\mathfrak{a})$ by means of axial path of finite costs (see Definition I.2.6 and Figure I.14).

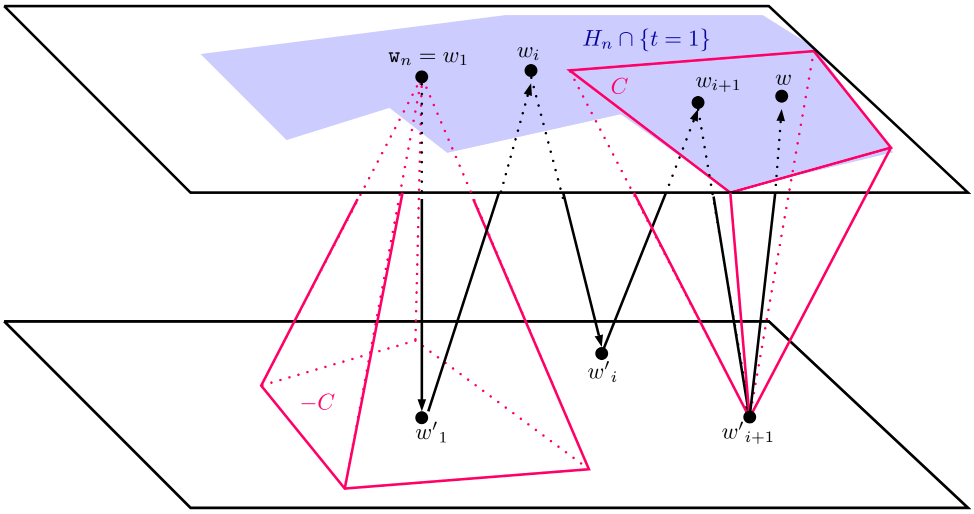


Figure I.14: Construction of the set H_n, formula I.46.

(I.6.2) **PROPOSITION:** *The set H_n is σ-compact in $\mathbb{R}^{d-h} \times ([0, +\infty) \times \mathbb{R}^h)$, and moreover, defining the Borel set $\tilde{\mathfrak{A}}^\dagger_n := \{\mathfrak{a} : H_n(\mathfrak{a}) \neq \emptyset\}$, then there exists a Borel function $\mathtt{h}_n : \tilde{\mathfrak{A}}^\dagger \times \mathbb{R}^h \to [0, +\infty)$ such that for all $x, x' \in \mathbb{R}^h$*

$$\mathtt{h}_n(\mathfrak{a}, x') \leq \mathtt{h}_n(\mathfrak{a}, x) + \mathtt{co}_\mathfrak{a}(x' - x)$$

and

$$\big\{(\mathfrak{a}, t, x) : t > \mathtt{h}_n(\mathfrak{a}, x)\big\} \subset H_n \subset \big\{(\mathfrak{a}, t, x) : t \geq \mathtt{h}_n(\mathfrak{a}, x)\big\}.$$

The above statement is the analog of Proposition 7.4 of [8], and we omit the proof. The function $\mathtt{h}_n$ is given explicitly by

$$\begin{aligned} \mathtt{h}_n(x) &= \inf\big\{\mathtt{co}_\mathfrak{a}(x-y), y \in \mathtt{i}_d(H_n \cap \{t = 0\})\big\} \\ &= \min\Big\{\mathtt{co}_\mathfrak{a}(x-y), y \in \operatorname{clos}\big(\mathtt{i}_d(H_n \cap \{t = 0\})\big)\Big\}. \end{aligned}$$

The separability of $\mathbb{R}^d$ and the non degeneracy of the cone $\tilde{C}_\mathfrak{a}$ yields the next lemma.

(I.6.3) **LEMMA:** *There exist countably many cones $\{w'_i + \tilde{C}_\mathfrak{a}\}_{i\in\mathbb{N}}$, $\{w'_i\}_{i\in\mathbb{N}} \subset \mathtt{i}_h(\mathtt{p}_2\tilde{\Gamma}(\mathfrak{a}) \cap H_n(\mathfrak{a}))$, such that*

$$\mathtt{i}_h\big(\mathring{H}_n(\mathfrak{a})\big) = \bigcup_{i\in\mathbb{N}} w'_i + \mathring{\tilde{C}}_\mathfrak{a}.$$

Moreover, the set $\partial(\mathtt{i}_h(H_n(\mathfrak{a}))) \cap \{t = \bar{t}\}$ is $(h-1)$-rectifiable for all $\bar{t} > 0$.

The estimate given in the proof below is well known, we give it for completeness.

PROOF. We need to prove only the second part. Let $K = \tilde{C}_\mathfrak{a} \cap \{t = \bar{t}\}$, and consider in $\mathbb{R}^h$

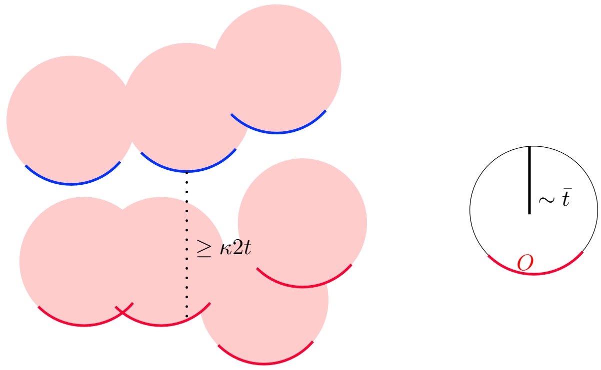


Figure I.15: We can consider a straight line in $\{t = \bar{t}\}$ traversal to O. The distance between two points of $(\partial H_n) \cap B^h(z, r)$ on this line and belonging to some translations of O is of the order of $\bar{t}$. (Lemma I.6.3).

a set H of the form

$$H = \bigcup_{i \in \mathbb{N}} w'_i + \mathring{K}.$$

If a point w belongs to ∂H, then it belongs to the boundary of $w' + K$ for a suitable w'.

Being $K = C_{\mathfrak{a}} \cap \{t = 1\}$ a compact convex set, the set ∂K can be divided into finitely many L-Lipschitz graphs O_i, $i = 1, \dots, I$. By restricting their domains, for all $\bar{\imath}$ fixed we can assume that if two points w_j, $j = 1, 2$, are such that

$$w_1 \in (w'_1 + O_{\bar{\imath}}) \setminus (w'_2 + K) \qquad \text{and} \qquad w_2 \in (w'_2 + O_{\bar{\imath}}) \setminus (w'_1 + K),$$

then either they belong to a common $2L$-Lipschitz graph or their distance is of order $\operatorname{diam} K \approx \bar{t}$ (see Figure I.15).

The previous assumption on the sets O_i implies that the points in $\partial H_n \cap B(0, R)$ of the form $w' + O_{\bar{\imath}}$, with $\bar{\imath}$ fixed, can be arranged into at most $\frac{R}{\bar{t}}$ $2L$-Lipschitz graphs: hence we can estimate

$$\mathcal{H}^{h-1}\big(\partial H \cap B(0, R)\big) \approx \max\left\{\frac{R}{\bar{t}}, 1\right\} \cdot R^{h-1} \approx \frac{R^h}{\bar{t}} + R^{h-1}.$$

For $R \ll \bar{t}$ we made use of the observation that there can be only 1 Lipschitz graph inside the $B(0, R)$. ■

Construction of the linear preorder $\preccurlyeq_{\mathtt{W}}$

Denote with $\mathtt{W} = \{\mathtt{w}_n\}_{n \in \mathbb{N}}$ the countable family of functions constructed in Lemma I.6.1.

Define first the function

$$\begin{array}{rccl} \theta'_{\mathtt{W},\tilde\Gamma} : & \tilde{\mathfrak{A}}' \times ([0,+\infty)\times\mathbb{R}^h) & \to & \tilde{\mathfrak{A}}' \times [0,1] \\ & (\mathfrak{a},w) & \mapsto & \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w) := \Big(\mathfrak{a}, \max\Big\{0, \sum_n 2\cdot 3^{-n}\chi_{H_n}(\mathfrak{a},w)\Big\}\Big) \end{array} \tag{I.47}$$

It is fairly easy to show that $\theta'_{\mathtt{W},\tilde\Gamma}$ is Borel. The dependence on $\tilde\Gamma$ occurs because the family $\mathtt{W}$ is chosen once $\tilde\Gamma$ has been selected.

Since we are interested only in the values of the functions on $\mathtt{p}_1\tilde\Gamma$ and the measure $\tilde\mu$ is a.c., then once the function $\theta'_{\mathtt{W},\tilde\Gamma}$ has been computed we define a new function $\theta_{\mathtt{W},\tilde\Gamma}$ by

$$\big\{w : \theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w) \geq \underline{\theta}\big\} = \bigcup_{\substack{w'\in\mathtt{p}_2\tilde\Gamma(\mathfrak{a}) \\ \theta'_{\mathtt{W},\Gamma}(w')\geq\underline{\theta}}} w' + \tilde C_{\mathfrak{a}}, \qquad \underline{\theta}\in[0,1]. \tag{I.48}$$

Being $\mathtt{p}_2\tilde\Gamma(\mathfrak{a})$ σ-compact and $\mathfrak{a}\mapsto\tilde C_a$ σ-compact, it is standard to prove that $\theta_{\mathtt{W},\tilde\Gamma}$ is Borel if $\theta'_{\mathtt{W},\tilde\Gamma}$ is.

The main reason for the introduction of the function θ will be clear in Section I.8: indeed, θ and its upper semicontinuous envelope ϑ satisfy a Lax representation formula similar to the Lax formula for HJ equation (Remark I.6.6), so that the techniques used in order to prove regularity of the disintegration for $\bar\phi$ (Sections I.4.1 and I.4.2) can be adapted to this context.

(I.6.4) LEMMA: *The functions $\theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}), \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a})$ are locally SBV on every section $\{t=\bar t\}$, and*

$$(w,w')\in\tilde\Gamma(\mathfrak{a}) \quad\Longrightarrow\quad \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w) = \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w') = \theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w) = \theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w').$$

Hence the function $\theta_{\mathtt{W},\tilde\Gamma}$ has the same values of $\theta'_{\mathtt{W},\tilde\Gamma}$ on $\mathtt{p}_1\tilde\Gamma(\mathfrak{a})\cup\mathtt{p}_2\tilde\Gamma(\mathfrak{a})$.

PROOF. From the definition of $H_n(\mathfrak{a})$, formula (I.46), it is fairly easy to see that

$$(w,w')\in\tilde\Gamma\ \Big(w\in H_n(\mathfrak{a}) \ \Leftrightarrow\ w'\in H_n(\mathfrak{a})\Big),$$

hence $\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w) = \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},w')$. Moreover

$$\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a},[0,+\infty)\times\mathbb{R}^h) \subset \bigg\{\sum_n s_n 3^{-n}, s_n\in\{0,2\}\bigg\},$$

so that its range in $\mathcal{L}^1$-negligible (it is a subset of the ternary Cantor set). By (I.46), the sets $H_n(\mathfrak{a})\cap\{t=\bar t\}$ is the union of compact convex sets containing a ball of radius $\mathcal{O}(\bar t)$, and then by Lemma I.6.3 it is of locally finite perimeter: more precisely, in each ball in $\mathbb{R}^h$ of radius r its perimeter is $\mathcal{O}(r^h/\bar t + r^{h-1})$.

Being $\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a})\llcorner_{\{t=\bar t\}}$ given by the sum of the functions of $3^{-n}\chi_{H_n\cap\{t=\bar t\}}$ with (relative) perimeter $\approx 3^{-n}(r^h/\bar t + r^{h-1})$ in each ball of $\mathbb{R}^h$ of radius r, it follows that $\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a})\llcorner_{\{t=\bar t\}}$ is locally BV with

$$\text{Tot.Var.}(\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a})\llcorner_{\{t=\bar t\}}, B(x,r)) \approx \frac{r^h}{\bar t} + r^{h-1}.$$

Being a countable sum of rectifiable sets, it is locally SBV in each plane $\{t = \bar{t}\}$: more precisely, only the jump part of $D_w\theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ is non zero.

The same analysis can be repeated for $\theta_{\mathtt{W},\tilde{\Gamma}}$, using the definition (I.48). This concludes the proof of the regularity.

The fact that $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w) = \theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w')$ if $(w, w') \in \tilde{\Gamma}(\mathfrak{a})$ is a fairly easy consequence of $\theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w) = \theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w')$ and the definition of θ. Indeed, it is clear that $\theta \geq \theta'$; on the other hand, if $w'_i \in \mathtt{p}_2\tilde{\Gamma}(\mathfrak{a})$ is a maximizing sequence for $w \in \mathtt{p}_1\tilde{\Gamma}(\mathfrak{a})$, then the definition of θ' gives

$$\theta'(w) \geq \theta'(w'_i),$$

and then $\theta'(w) = \theta'(w') = \theta(w)$. Since $\theta(w') \leq \theta(w)$ by (I.48), the conclusion follows. ∎

(I.6.5) LEMMA: $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ *and* $\theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ *are SBV in* $[0, +\infty) \times \mathbb{R}^h$.

PROOF. Being every sub levels of $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ and $\theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ the sum of cones $w' + \tilde{C}_{\mathfrak{a}}$, the boundary of level sets is locally Lipschitz and the thesis follows. ∎

To estimate the regularity of the disintegration of the locally affine partition generated by θ (Section I.7), we define the function $\vartheta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ as the upper semicontinuous envelope of $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$:

$$\big\{\vartheta(\mathfrak{a}) \geq \underline{\theta}\big\} = \mathrm{clos}\big\{\theta(\mathfrak{a}) \geq \underline{\theta}\big\}.$$

Being the topological boundaries of level sets of $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ rectifiable, the $\mathcal{H}^h \llcorner_{\{t=\bar{t}\}}$-measure of the points where $\vartheta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ and $\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a})$ are different is 0.

(I.6.6) REMARK: *We observe here the relation with the Lax formula for Hamilton-Jacoby equation (with inverted time). In fact, if we define the Lagrangian*

$$L_{\mathfrak{a}}(w) = \mathbb{1}_{\tilde{C}_{\mathfrak{a}}}(w),$$

then formula (I.48) *can be rewritten as*

$$\theta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w) = \sup\Big\{\theta'_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w') - L_{\mathfrak{a}}(w - w'), w' \in \{t = 0\}\Big\}.$$

Moreover, the definition of $\vartheta_{\mathtt{W},\tilde{\Gamma}}$ *yields that*

$$\vartheta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w) = \max\Big\{\vartheta_{\mathtt{W},\tilde{\Gamma}}(\mathfrak{a}, w') - L_{\mathfrak{a}}(w - w'), w' \in \{t = 0\}\Big\}.$$

Being the maximum reached in some point, it follows that $\vartheta_{\mathtt{W},\tilde{\Gamma}}$ *in some sense replaces the potential* $\bar{\phi}$*. The advantages of using* $\theta_{\mathtt{W},\tilde{\Gamma}}$ *instead of* $\vartheta_{\mathtt{W},\tilde{\Gamma}}$ *will be clear in the following sections.*

We remark here only that the disintegration of the Lebesgue measure $\mathcal{H}^d \llcorner_{\{t=1\}}$ *on the sub levels of* θ *or of* ϑ *is equivalent, as observed above.*

The space $\mathbb{R}^{d-h} \times [0,1]$ is naturally linearly ordered by the lexicographic ordering $\trianglelefteq$: set for $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{d-h})$, $s \in [0,1]$,

$$(\mathfrak{a}, s) \lhd (\mathfrak{a}', s') \iff \Big[\exists i \in \{1,\dots,d-h\} \, \Big(\forall j < i \, \big(\mathfrak{a}_j = \mathfrak{a}'_j\big) \,\wedge\, \mathfrak{a}_i < \mathfrak{a}'_i\Big)\Big] \\ \vee \quad \big[\mathfrak{a}_i = \mathfrak{a}'_i \,\wedge\, s < s'\big]. \tag{I.49}$$

The pull-back of $\trianglelefteq$ by $\theta_{\mathtt{W},\tilde\Gamma}$ is the linear preorder $\preccurlyeq_{\mathtt{W},\tilde\Gamma}$ defined by

$$\preccurlyeq_{\mathtt{W},\tilde\Gamma} := \big(\theta_{\mathtt{W},\tilde\Gamma} \otimes \theta_{\mathtt{W},\tilde\Gamma}\big)^{-1}(\trianglelefteq^{-1}),$$

and the corresponding equivalence relation on $\tilde{\mathfrak{A}}' \times \mathbb{R}^h$ is

$$E_{\mathtt{W},\tilde\Gamma} := \preccurlyeq_{\mathtt{W},\tilde\Gamma} \cap \preccurlyeq^{-1}_{\mathtt{W},\tilde\Gamma} = \Big\{(w,w') : \theta_{\mathtt{W},\tilde\Gamma}(w) = \theta_{\mathtt{W},\tilde\Gamma}(w')\Big\}.$$

By construction $(\mathfrak{a}, w) \sim_{E_{\mathtt{W},\tilde\Gamma}} (\mathfrak{a}', w')$ implies that $\mathfrak{a} = \mathfrak{a}'$. By convention we will also set

$$E_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}) = \Big\{(w,w') : \theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}, w) = \theta_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}, w')\Big\}.$$

(I.6.7) LEMMA: *Assume that $(\mathfrak{a}, w), (\mathfrak{a}'', w'') \in \mathtt{p}_1\tilde\Gamma$ can be connected by a closed axial path of finite cost. Then $(\mathfrak{a}, w) \sim_{E_{\mathtt{W},\tilde\Gamma}} (\mathfrak{a}'', w'')$.*

PROOF. Clearly $\mathfrak{a} = \mathfrak{a}''$, and thus the condition can be stated as follows: there exist $I \in \mathbb{N}$, $(w_i, w'_i) \in \tilde\Gamma(\mathfrak{a})$, $i = 1, \dots, I$, such that $\tilde{\mathtt{c}}_{\mathfrak{a}}(w_{i+1}, w'_i) < \infty$, $i = 1, \dots, I$ with $w_{I+1} = w_1$, and moreover $w = w_{i_1}$, $w'' = w_{i_2}$ for some $i_1, i_2 \in I$. This implies that

$$\forall n \in \mathbb{N} \, \big(w \in H_n(\mathfrak{a}) \;\Leftrightarrow\; w'' \in H_n(\mathfrak{a})\big),$$

which proves that $\theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}, w) = \theta'_{\mathtt{W},\tilde\Gamma}(\mathfrak{a}'', w'')$. From Lemma I.6.4 the conclusion follows. ∎

A consequence of Lemma I.6.4 is thus that $\tilde\Gamma \subset E_{\mathtt{W},\tilde\Gamma}$. If Γ' is another carriage contained in $\{\tilde{\mathtt{c}} < \infty\}$, then

$$(w,w') \in \Gamma'(\mathfrak{a}) \quad\Longrightarrow\quad w \preccurlyeq_{\mathtt{W},\tilde\Gamma} w',$$

because

$$\theta_{\mathtt{W},\Gamma'}(w' + \tilde C_{\mathfrak{a}}) \geq \theta_{\mathtt{W},\Gamma'}(w') \tag{I.50}$$

by construction. In particular from Theorem A.2.2 we deduce the following proposition.

(I.6.8) PROPOSITION: *If $\tilde\pi' \in \Pi^f(\tilde\mu, \tilde\nu)$, then $\tilde\pi'$ is concentrated on $E_{\mathtt{W},\tilde\Gamma}$.*

Construction of a σ-closed family of equivalence relations

The linear preorder $\preccurlyeq_{\mathtt{W},\tilde\Gamma}$ depends on the family $\mathtt{W}$ of functions we choose and on the carriage $\tilde\Gamma$: by varying the $\tilde{\mathtt{c}}$-cyclically monotone carriage $\tilde\Gamma \in \mathit{\Gamma}$ and the family $\mathtt{W}$ dense in $\tilde\Gamma$ and we obtain in general different preorders.

We can easily compose the linear preorders $\preccurlyeq_{\mathtt{W}_\beta,\tilde\Gamma_\beta}$, $\beta < \alpha$ countable ordinal number, by using the lexicographic preorder on $[0,1]^\alpha$: in fact, define the function (recall the notation $(\mathfrak{a}, s) \in \mathbb{R}^{d-h} \times [0,1]$)

$$\begin{array}{rccc} \theta_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}} : & \mathbb{R}^{d-h} \times [0,+\infty) \times \mathbb{R}^h & \to & \mathbb{R}^{d-h} \times [0,1]^\alpha \\ & (\mathfrak{a}, w) & \mapsto & \theta_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}(\mathfrak{a}, w) := \big(\mathfrak{a}, \{\mathtt{p}_s \theta_{\mathtt{W}_\beta,\tilde\Gamma_\beta}(\mathfrak{a}, w)\}_{\beta<\alpha}\big) \end{array} \tag{I.51}$$

As in the previous section $\theta_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}$ is Borel, and the function should be considered defined in the domain $\cap_\beta \tilde{\mathfrak{A}}'_\beta$, where $\tilde{\mathfrak{A}}'_\beta$ is the domain of the family of functions $\mathtt{W}_\beta$.

If $\trianglelefteq$ is the lexicographic preorder in $\mathbb{R}^{d-h} \times [0,1]^\alpha$ as in (I.49), then set

$$\preccurlyeq_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}} := \big(\theta_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}} \otimes \theta_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}\big)^{-1}(\trianglelefteq), \quad E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}} := \preccurlyeq_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}} \cap \preccurlyeq^{-1}_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}.$$

Clearly $\tilde\pi(E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}) = 1$, since $\tilde\pi(E_{\mathtt{W}_\beta,\tilde\Gamma_\beta}) = 1$ for all $\beta < \alpha$. To be an equivalence relation on $\mathbb{R}^{d-h} \times [0,+\infty) \times \mathbb{R}^h$, we can assume that $\mathbb{I} \subset E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}$.

The next lemma is a simple consequence of the fact that a countable union of countable sets is countable. Its proof can be found in [8, Proposition 7.5].

(I.6.9) LEMMA: *The family of equivalence relations*

$$\tilde{\mathcal{E}} := \Big\{E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}, \mathtt{W}_\beta = \big\{\mathtt{w}_{n,\beta}\big\}_{n\in\mathbb{N}}, \alpha \in \Omega\Big\}$$

is closed under countable intersection. Moreover, for ever $E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}$ there exists $\bar{\tilde\Gamma} \in \mathit{\Gamma}$ and $\bar{\mathtt{W}}$ such that

$$E_{\bar{\mathtt{W}},\bar{\tilde\Gamma}} \subset E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}.$$

I.6.2. PROPERTIES OF THE MINIMAL EQUIVALENCE RELATION

Let $\bar E_{\{\mathtt{W}_\beta,\tilde\Gamma_\beta\}_{\beta<\alpha}}$ be the minimal equivalence relation chosen as in Lemma I.6.9 after a minimal equivalence relation of Theorem A.3.1 in Appendix A.3 has been selected.

Let $\bar\theta' : \mathbb{R}^{d-h} \times ([0,+\infty) \times \mathbb{R}^h) \to \mathbb{R}^{d-h} \times [0,1]$ be the function obtained through (I.47) with the set $\bar{\tilde\Gamma}$ and the family of functions $\bar{\mathtt{W}}$, and let $\bar\theta$ be the corresponding function given by (I.48). For shortness in the following we will use only the notation $\bar E$, $\bar\theta$ and $\bar\preccurlyeq$, and the convention is that $\bar\theta$ is defined on a σ-compact set $\tilde{\mathfrak{A}}' \times ([0,\infty) \times \mathbb{R}^h)$ as in the discussion following (I.51).

Let $\tilde\Gamma \in \mathit{\Gamma}$ be a σ-compact cyclically monotone set, and let $\theta_{\mathtt{W},\tilde\Gamma} : \mathbb{R}^{d-h} \times ([0,+\infty) \times \mathbb{R}^h) \to \mathbb{R}^{d-h} \times [0,1]^{\mathbb{N}}$ be constructed as in Section I.6.1.

By Corollary A.3.2, it follows that there exists a $\tilde\mu$-conegligible σ-compact set $\tilde B \subset \mathbb{R}^{d-h} \times ([0,+\infty) \times \mathbb{R}^h)$ and a Borel function $\mathtt{s} : \mathbb{R}^{d-h} \times [0,1] \to \mathbb{R}^{d-h} \times [0,1]$ such that $\theta_{\mathtt{W}} = \mathtt{s} \circ \bar\theta$ on $\tilde B$: since

$$\mathtt{p}_{\mathfrak{a}} \bar\theta = \mathtt{p}_{\mathfrak{a}} \theta_{\mathtt{W},\tilde\Gamma} = \mathbb{I},$$

it follows that we can write $\mathtt{s}(\mathfrak{a}, s) = (\mathfrak{a}, \mathtt{s}(\mathfrak{a}, s))$, with a slight abuse of notation. The set $\tilde{B}$ depends on $\theta_{\mathtt{W},\tilde{\Gamma}}$.

Applying this result to the equivalence classes of positive $\tilde{\mu}_{\mathfrak{a}}$-measure, where $\tilde{\mu}_{\mathfrak{a}}$ are the conditional probabilities given by (I.44), we obtain the following proposition.

(I.6.10) PROPOSITION: *There exists a set $\tilde{\mathfrak{A}}'' \subset \tilde{\mathfrak{A}}'$ of full $\tilde{\xi}$-measure such that*

$$\forall \mathfrak{a} \in \tilde{\mathfrak{A}}'', \underline{\theta} \in [0,1] \ \biggl(\tilde{\mu}_{\mathfrak{a}}(\bar{\theta}^{-1}(\underline{\theta})) > 0 \quad \Longrightarrow \quad \exists \underline{\theta}' \in [0,1] \ \Bigl(\tilde{\mu}_{\mathfrak{a}}\bigl(\bar{\theta}^{-1}(\underline{\theta}) \setminus \theta^{-1}_{\mathtt{W},\tilde{\Gamma}}(\underline{\theta}')\bigr) = 0 \Bigr) \biggr).$$

PROOF. Since the equivalence classes under consideration have positive $\tilde{\mu}_{\mathfrak{a}}$-measure, the $\tilde{\mu}$-negligible set $(\tilde{\mathfrak{A}}' \times [0,+\infty) \times \mathbb{R}^h) \setminus \tilde{B}$ satisfies

$$\tilde{\xi}\Bigl(\Bigl\{ \mathfrak{a} : \exists \underline{\theta} \ \bigl(\tilde{\mu}_{\mathfrak{a}}(\bar{\theta}^{-1}(\underline{t}) \setminus \tilde{B}) > 0 \bigr) \Bigr\} \Bigr) = 0.$$

In the remaining $\tilde{\xi}$-conegligible subset $\tilde{\mathfrak{A}}''$ of $\tilde{\mathfrak{A}}'$ the value $\underline{\theta}' = \mathtt{s}(\underline{\theta})$ satisfies the statement. ■

The essential cyclical connectedness of $\bar{\theta}^{-1}(\underline{\theta})$ now follows from the following lemma, valid for a generic $\theta_{\mathtt{W},\tilde{\Gamma}}$. This lemma justify the choice of the density properties of the functions $\mathtt{w}_n$, Lemma I.6.1.

(I.6.11) LEMMA: *If $\tilde{\mu}_a(\theta^{-1}_{\mathtt{W},\tilde{\Gamma}}(\underline{\theta})) > 0$, $\mathfrak{a} \in \tilde{\mathfrak{A}}'$, then it is $\mathcal{H}^h_{\{t=1\}}$-essentially $\tilde{\Gamma}$-cyclically connected.*

PROOF. Fix $\mathfrak{a} \in \tilde{\mathfrak{A}}'$ and assume the opposite. Then there are two sets A_1, A_2 in $\mathtt{p}_1(\tilde{\Gamma}(\mathfrak{a})) \cap \theta^{-1}_{\mathtt{W},\tilde{\Gamma}}(\underline{\theta})$ of positive $\tilde{\mu}_{\mathfrak{a}}$-measure such that each point of A_1 cannot reach any point of A_2.

If $(\bar{w}, \bar{w}') \in \tilde{\Gamma}(\mathfrak{a}) \cup A_1 \times \{t = 0\}$ is such that $\bar{w}$ is a $\mathcal{H}^h_{\{t=1\}}$-Lebesgue points of A_1, then using the non degeneracy of $\tilde{C}_{\mathfrak{a}}$ and the density of $\mathtt{W} = \{\mathtt{w}_n\}_n$, we obtain that there exists a $\mathtt{w}_{\bar{n}}(\mathfrak{a}) \in A_1 \cap (\bar{w}' + \tilde{C}_{\mathfrak{a}})$ with $H_{\bar{n}} \cap A_1 \neq \emptyset$.

By the assumption that $\theta_{\mathtt{W},\tilde{\Gamma}}$ is constant, we deduce that $A_2 \in H_{\bar{n}}$, so that there is an axial path connecting $\bar{w}$ to A_2. ■

The next example shows that, differently from [8, Theorem 7.2], the Lebesgue points of $\{\bar{\theta}_{\mathfrak{a}} = t\}$ are not necessarily cyclically connected.

(I.6.12) EXAMPLE: *Consider the sets in $\mathbb{R}^2$*

$$A_0 := \bigl\{ x_1 \geq 0, x_2 = 0 \bigr\}, \quad A_1 := A_0 + B(0,1),$$

and the map

$$\mathtt{T} : \mathbb{R}^2 \setminus A_0 \to \mathbb{R}^2 \setminus A_1, \qquad \mathtt{T}(x) = x + \biggl(1 - \frac{1}{2}\mathrm{dist}(x, A_0) \biggr)^+ \frac{x - \mathrm{dist}(x, A_0)}{|x - \mathrm{dist}(x, A_0)|}.$$

It is immediate to see that $\mathtt{T}$ is optimal for the cost $\mathtt{c}(x - x') = \mathbb{1}_{|x| \leq 1}(x)$ and the measures

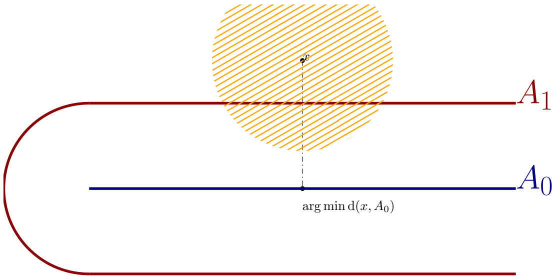


Figure I.16: Example I.6.12.

μ, $\mathtt{T}_\sharp\mu$ for all $\mu \in \mathcal{P}(A_1 \setminus A_0)$, and the sets and functions

$$\tilde{\Gamma} = \mathit{graph}(\mathtt{T}), \quad \mathtt{W} \cap \setminus A_0 = \emptyset \qquad\qquad \theta_{\mathtt{W},\tilde{\Gamma}}(x) = \begin{cases} 2/3 & x \notin A_0, \\ 0 & x \in A_0, \end{cases}$$

$$\bar{\tilde{\Gamma}} = \tilde{\Gamma} \cup \Big\{(x, x') : x \in A_0, x' \in (x + B(0,1))\Big\}, \quad \mathrm{clos}(\bar{\mathtt{W}} \cap A_0) = A_0,$$

$$\bar{\theta}_{\bar{\mathtt{W}},\bar{\tilde{\Gamma}}}(x) = \tfrac{2}{3},$$

satisfy Proposition I.6.10, the equivalence class for $\bar{\theta}$ being $\mathbb{R}^2$ but for θ is $\mathbb{R}^2 \setminus A_0$.

I.7. Decomposition of a fibration into a directed locally affine partition

In this section we use the function $\bar{\theta}$ constructed in the previous section to obtain a partition of subsets of $\mathbb{R}^{d-h} \times [0, +\infty) \times \mathbb{R}^h$ which is locally affine and satisfies some regularity properties: these properties are needed to prove the disintegration theorem of the next section.

The decomposition presented in this section can be performed for using any σ-continuous function with the property (I.50). In particular, in Section I.8 we will use the function $\bar{\vartheta}$.

Using Lusin Theorem (134Yd of [18]) we can assume that $\bar{\theta}$ is σ-continuous up to a $(\tilde{\mu} + \mathcal{H}^d \llcorner_{\{t=1\}})$-negligible set.

I.7.1. DEFINITIONS OF TRANSPORT SETS AND RELATED SETS

We define the following sets: they correspond to the sets used in [8], Section 4, adapted to the space $\mathbb{R}^{d-h} \times [0,+\infty) \times \mathbb{R}^h$, and are the analog of the sets used in Section I.4 for the potential $\bar\phi$ and the cost $\bar{\mathtt{c}}$, replaced by $\bar\theta$ and $\tilde{\mathtt{c}}_{\mathfrak{a}}$.

Sub/super differential of $\bar\theta$ we define the *cone sub/super-differential* of $\bar\theta$ at $(\mathfrak{a},w)$ as

$$\partial^-\bar\theta(\mathfrak{a}) := \Big\{(w,w') : \tilde{\mathtt{c}}_{\mathfrak{a}}(w,w') < +\infty, \bar\theta(\mathfrak{a},w) = \bar\theta(\mathfrak{a},w')\Big\}, \tag{I.52}$$

$$\partial^+\bar\theta(\mathfrak{a}) := \Big\{(w,w') : \tilde{\mathtt{c}}_{\mathfrak{a}}(w',w) < +\infty, \bar\theta(\mathfrak{a},w) = \bar\theta(\mathfrak{a},w')\Big\}.$$

Note that $\partial^-\bar\theta = (\partial^+\bar\theta)^{-1}$.

Optimal Ray the *optimal rays* are the segments whose end points $(\mathfrak{a},w),(\mathfrak{a},w')$ satisfy

$$\bar\theta(\mathfrak{a},w) = \bar\theta(\mathfrak{a}',w') \qquad \text{and} \qquad w \in w' + \tilde C_{\mathfrak{a}}.$$

Backward/forward transport set the *forward/backward transport set* are defined by

$$T^-_{\bar\theta}(\mathfrak{a}) := \big\{w : \partial^-\bar\theta(\mathfrak{a},w) \neq \{w\}\big\} = \mathtt{p}_1\big(\partial^-\bar\theta(\mathfrak{a}) \setminus \mathbb{I}\big),$$
$$T^+_{\bar\theta}(\mathfrak{a}) := \big\{w : \partial^+\bar\theta(\mathfrak{a},w) \neq \{w\}\big\} = \mathtt{p}_1\big(\partial^+\bar\theta(\mathfrak{a}) \setminus \mathbb{I}\big).$$

Set of fixed points the *set of fixed points* is given by

$$F_{\bar\theta}(\mathfrak{a}) := \mathbb{R}^h \setminus \big(T^-_{\bar\theta}(\mathfrak{a}) \cup T^+_{\bar\theta}(\mathfrak{a})\big).$$

Backward/forward direction multifunction The *backward/forward direction multifunction* is given by

$$\mathcal{D}^-\bar\theta(\mathfrak{a}) := \left\{\left(w, \frac{w-w'}{\mathtt{p}_t(w-w')}\right), w=(t,x) \in T^-_{\bar\theta}(\mathfrak{a}), w'=(t',x') \in \partial^-\bar\theta(\mathfrak{a},w) \setminus \{w\}\right\},$$
$$\mathcal{D}^+\bar\theta(\mathfrak{a}) := \left\{\left(w, \frac{w'-w}{\mathtt{p}_t(w'-w)}\right), w=(t,x) \in T^+_{\bar\theta}(\mathfrak{a}), w'=(t',x') \in \partial^+\bar\theta(\mathfrak{a},w) \setminus \{w\}\right\},$$

normalized such that $\mathtt{p}_t\mathcal{D}^\pm\bar\theta(\mathfrak{a},w) = 1$.

Convex cone generated by $\mathcal{D}^\pm\bar\theta$ define

$$C^-_{\bar\theta}(\mathfrak{a},w) := \mathbb{R}^+ \cdot \operatorname{conv}\mathcal{D}^-\bar\theta(\mathfrak{a},w), \qquad C^+_{\bar\theta}(\mathfrak{a},w) := \mathbb{R}^+ \cdot \operatorname{conv}\mathcal{D}^+\bar\theta(\mathfrak{a},w).$$

Backward/forward regular transport set the *ℓ-dimensional backward/forward regular transport sets* are defined for $\ell = 0,\dots,h$ respectively as

$$\begin{aligned} R^{-,\ell}_{\bar\theta}(\mathfrak{a}) := \Big\{ w \in T^-(\mathfrak{a}) :\ & (i)\ \mathcal{D}^-\bar\theta(\mathfrak{a},w) = \operatorname{conv}\mathcal{D}^-\bar\theta(\mathfrak{a},w) \\ & (ii)\ \dim\big(\operatorname{conv}\mathcal{D}^-\bar\theta(\mathfrak{a},w)\big) = \ell \\ & (iii)\ \exists w' \in T^-_{\bar\theta}(\mathfrak{a}) \cap \big(w + \operatorname{int}_{\mathrm{rel}} C^-_{\bar\theta}(\mathfrak{a},w)\big)\ \big((i),(ii) \text{ hold for } w'\big)\Big\}, \end{aligned}$$

$$R^{+,\ell}_{\bar\theta}(\mathfrak{a}) := \Big\{ w \in T^+(\mathfrak{a}) : \ (i)\ \mathcal{D}^+\bar\theta(\mathfrak{a},w) = \operatorname{conv}\mathcal{D}^+\bar\theta(\mathfrak{a},w) \\ (ii)\ \dim\big(\operatorname{conv}\mathcal{D}^+\bar\theta(\mathfrak{a},w)\big) = \ell \\ (iii)\ \exists w' \in T^+_{\bar\theta}(\mathfrak{a}) \cap \big(w - \operatorname{int}_{\mathrm{rel}} C^+_{\bar\theta}(\mathfrak{a},w)\big)\ \big((i),(ii) \text{ hold for } z'\big) \Big\}.$$

The *backward/forward regular transport sets* and the *regular transport set* are defined respectively by

$$R^-_{\bar\theta}(\mathfrak{a}) := \bigcup_{\ell=0}^{h} R^{-,\ell}_{\bar\theta}(\mathfrak{a}), \quad R^+_{\bar\theta}(\mathfrak{a}) := \bigcup_{\ell=0}^{h} R^{-,\ell}_{\bar\theta} \qquad \text{and} \qquad R_{\bar\theta}(\mathfrak{a}) := R^-_{\bar\theta}(\mathfrak{a}) \cap R^+_{\bar\theta}(\mathfrak{a}).$$

Finally define the *residual set* $N_{\bar\theta}$ by

$$N_{\bar\theta}(\mathfrak{a}) := T_{\bar\theta}(\mathfrak{a}) \setminus R_{\bar\theta}(\mathfrak{a}).$$

The next statements are completely analog to [8, Section 4], and we will omit the proof.

(I.7.1) PROPOSITION: *The set $\partial^\pm\bar\theta$, $T^\pm_{\bar\theta}$, $F_{\bar\theta}$, $\mathcal{D}^\pm\bar\theta$, $C^\pm_{\bar\theta}$, $R^{\pm,\ell}_{\bar\theta}$, $R^\pm_{\bar\theta}$, $R_{\bar\theta}$ are σ-compact.*

The next lemma follows easily from (I.50): in the language of [8], we can say that the level sets of $\bar\theta(\mathfrak{a})$ are complete $\tilde{\mathtt{c}}_{\mathfrak{a}}$-Lipschitz graphs ([8, Definition 4.1]).

(I.7.2) LEMMA: *If $(w,w') \in \partial^+\bar\theta(\mathfrak{a})$, then*

$$Q_{\bar\theta,\mathfrak{a}}(w,w') := (w + \tilde C_{\mathfrak{a}}) \cap (w' - \tilde C_{\mathfrak{a}}) \subset \partial^+\bar\theta(\mathfrak{a},w). \tag{I.53}$$

Moreover,

$$\mathbb{R}^+ \cdot (Q_{\bar\theta,\mathfrak{a}}(w,w') - w) = \mathbb{R}^+ \cdot (w' - Q_{\bar\theta,\mathfrak{a}}(w,w')) =: O_{\mathfrak{a}}(w,w')$$

where $O_{\mathfrak{a}}(w,w')$ is the minimal extremal face of $\tilde C_{\mathfrak{a}}$ containing $w'-w$.

In particular, one deduces that as in the potential case

$$\partial^-\bar\theta(\mathfrak{a},w) = \bigcup_{w' \in \partial^-\bar\theta(\mathfrak{a},w)} Q_{\bar\theta,\mathfrak{a}}(w',w), \qquad \partial^+\bar\theta(\mathfrak{a},w) = \bigcup_{w' \in \partial^+\bar\theta(\mathfrak{a},w)} Q_{\bar\theta,\mathfrak{a}}(w,w').$$

Moreover, by (I.43),

$$w' \in \partial^\pm\bar\theta(\mathfrak{a},w) \quad \Longrightarrow \quad \partial^\pm\bar\theta(\mathfrak{a},w') \subset \partial^\pm\bar\theta(\mathfrak{a},w). \tag{I.54}$$

Using the same ideas of the proof of Proposition I.4.11, we obtain an analogous proposition for $\mathcal{D}^-\bar\theta$.

(I.7.3) PROPOSITION: *Let $O_{\mathfrak{a}} \subset \tilde C_{\mathfrak{a}}$ be an extremal face. Then the following holds.*

1. $O_{\mathfrak{a}} \cap \{t=1\} \subset \mathcal{D}^-\bar\theta(\mathfrak{a},w) \quad \Longleftrightarrow \quad \exists \delta > 0 \ \Big(B(w,\delta) \cap \big(w - O_{\mathfrak{a}}\big) \subset \partial^-\bar\theta(\mathfrak{a},w)\Big).$

2. If $O_{\mathfrak{a}} \cap \{t = 1\} \subset \mathcal{D}^-\bar{\theta}(\mathfrak{a}, w)$ *is maximal w.r.t. set inclusion, then*

$$\forall w' \in B^h(w, \delta) \cap \big(w - \mathrm{int}_{\mathrm{rel}} O_{\mathfrak{a}}\big) \ \Big(\mathcal{D}^-\bar{\theta}(\mathfrak{a}, w') = O_{\mathfrak{a}} \cap \{t = 1\}\Big),$$

with $\delta > 0$ *given by the previous point.*

3. The following conditions are equivalent:

(a) $\mathcal{D}^-\bar{\theta}(\mathfrak{a}, w) = C^-_{\bar{\theta}}(\mathfrak{a}, w) \cap \{t = 1\}$*;*

(b) the family of cones

$$\big\{\mathbb{R}^+ \cdot Q_{\mathfrak{a}}(w', w), w' \in \partial^-\bar{\theta}(\mathfrak{a}, w)\big\}$$

has a unique maximum by set inclusion, which coincides with $C^-_{\bar{\theta}}(\mathfrak{a}, w)$*;*

(c) $\partial^-\bar{\theta}(\mathfrak{a}, w) \cap (z - \mathrm{int}_{\mathrm{rel}} C^-_{\bar{\theta}}(\mathfrak{a}, w)) \neq \emptyset$*;*

(d) $\mathcal{D}^-\bar{\theta}(\mathfrak{a}, w) = \mathrm{conv}\mathcal{D}^-\bar{\theta}(\mathfrak{a}, w)$.

A completely similar proposition can be proved for $\mathcal{D}^+\bar{\theta}$.

(I.7.4) PROPOSITION: *Let* $O_{\mathfrak{a}} \subset \tilde{C}_{\mathfrak{a}}$ *be an extremal face. Then the following holds.*

1. $O_{\mathfrak{a}} \cap \{t = 1\} \subset \mathcal{D}^+\bar{\theta}(\mathfrak{a}, w) \iff \exists \delta > 0 \ \Big(B(w, \delta) \cap \big(w + O_{\mathfrak{a}}\big) \subset \partial^+\bar{\theta}(\mathfrak{a}, w)\Big)$.

2. If $O_{\mathfrak{a}} \cap \{t = 1\} \subset \mathcal{D}^+\bar{\theta}(\mathfrak{a}, w)$ *is maximal w.r.t. set inclusion, then*

$$\forall w' \in B(w, \delta) \cap \big(w + \mathrm{int}_{\mathrm{rel}} O_{\mathfrak{a}}\big) \ \Big(\mathcal{D}^+\bar{\theta}(\mathfrak{a}, w') = O_{\mathfrak{a}} \cap \{t = 1\}\Big),$$

with $\delta > 0$ *given by the previous point.*

3. The following conditions are equivalent:

(a) $\mathcal{D}^+\bar{\theta}(\mathfrak{a}, w) = C^+_{\bar{\theta}}(\mathfrak{a}, w) \cap \{t = 1\}$*;*

(b) the family of cones

$$\big\{\mathbb{R}^+ \cdot Q_{\mathfrak{a}}(w, w'), w' \in \partial^+\bar{\theta}(\mathfrak{a}, w)\big\}$$

has a unique maximum by set inclusion, which coincides with $C^+_{\bar{\theta}}(\mathfrak{a}, w)$*;*

(c) $\partial^+\bar{\theta}(\mathfrak{a}, w) \cap \mathrm{int}_{\mathrm{rel}}(z + C^+_{\bar{\theta}}(\mathfrak{a}, w)) \neq \emptyset$*;*

(d) $\mathcal{D}^+\bar{\theta}(\mathfrak{a}, w) = \mathrm{conv}\mathcal{D}^+\bar{\theta}(\mathfrak{a}, w)$.

As a consequence of Point (3) of the previous propositions, we will call sometimes $C^-_{\bar{\theta}}(\mathfrak{a}, w)$, $C^+_{\bar{\theta}}(\mathfrak{a}, w)$ the *maximal backward/forward extremal face.*

(I.7.5) REMARK: *In Section I.8 we will need to compute the same objects for the function* $\bar{\vartheta}$*. The definitions are exactly the same, as well as the statements of Propositions I.7.1, I.7.3, I.7.4 and Lemma I.7.2, just replacing the function* $\bar{\theta}$ *with* $\bar{\vartheta}$*. We thus will consider the sets*

$$\partial^\pm\bar{\vartheta}, \quad T^\pm_{\bar{\vartheta}}, \quad F_{\bar{\vartheta}}, \quad \mathcal{D}^\pm\bar{\vartheta}, \quad C^\pm_{\bar{\vartheta}}, \quad R^{\pm,\ell}_{\bar{\vartheta}}, \quad R^\pm_{\bar{\vartheta}}, \quad R_{\bar{\vartheta}}, \quad N_{\bar{\vartheta}},$$

and for the exact definition we refer to the analog for $\bar{\theta}$*.*

I.7.2. PARTITION OF THE TRANSPORT SET

In this section we construct a map which give a directed locally affine partition in $\mathbb{R}^{d-h} \times [0, +\infty) \times \mathbb{R}^h$: more precisely, up to a residual set, we will find a directed locally affine partition on each fiber $\{\mathfrak{a}\} \times [0, +\infty) \times \mathbb{R}^h$, and the dependence of this partition from the parameter $\mathfrak{a}$ is σ-continuous.

Define the map

$$\begin{array}{rcccl} \mathtt{v}^-_{\bar\theta} & : & R^-_{\bar\theta} & \to & \mathbb{R}^{d-h} \times \cup_{\ell=0}^h \mathcal{A}(\ell, \mathbb{R}^h) \\ & & (\mathfrak{a}, w) & \mapsto & \mathtt{v}^+_{\bar\theta}(\mathfrak{a}, w) := \big(\mathfrak{a}, \operatorname{aff} \partial^- \bar\theta(\mathfrak{a}, w)\big) \end{array}$$

(I.7.6) LEMMA: *The map $\mathtt{v}^-_{\bar\theta}$ is σ-continuous.*

PROOF. Since $\partial^-\bar\theta(\mathfrak{a}, w)$ is σ-continuous by Proposition I.7.1 and the map $A \mapsto \operatorname{aff} A$ is σ-continuous in the Hausdorff topology, the conclusion follows. ■

We recall the convention $\mathbb{R}^0 = \mathbb{N}$.

(I.7.7) THEOREM: *The map $\mathtt{v}^-_{\bar\theta}$ induces a partition*

$$\bigcup_{\ell'=0}^h \Big\{ Z^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}, \mathfrak{a} \in \mathbb{R}^{d-h}, \mathfrak{b}' \in \mathbb{R}^{h-\ell'} \Big\}$$

on $R^-_{\bar\theta}$ such that the following holds:

1. *each set $Z^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}$ is locally affine;*
2. *there exists an extremal face $O^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}$ with dimension ℓ' of the cone $\tilde C_{\mathfrak{a}}$ such that*
$$\forall w \in Z^{\ell',-}_{\mathfrak{a},\mathfrak{b}'} \ \Big(\operatorname{aff} Z^{\ell',-}_{\mathfrak{a},\mathfrak{b}'} = \operatorname{aff}(w + O^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}) \quad \wedge \quad \mathcal{D}^-\bar\theta(\mathfrak{a}, w) = O^{\ell',-}_{\mathfrak{a},\mathfrak{b}'} \cap \{t = 1\} \Big);$$
3. *for all $w \in T^-_{\bar\theta}(\mathfrak{a})$ there exists $r > 0$, $O^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}$ such that*
$$B(w, r) \cap \big(w - \operatorname{int}_{\mathrm{rel}} O^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}\big) \subset Z^{\ell',-}_{\mathfrak{a},\mathfrak{b}'}.$$

The choice $\mathfrak{b} \in \mathbb{R}^{h-\ell}$ is in the spirit of Proposition I.3.3.

PROOF. For the proof see [8, Theorem 4.18]. ■

A completely similar statement holds for $R^+_{\bar\theta}$, by means of σ-continuous map

$$\begin{array}{rcccl} \mathtt{v}^+_{\bar\theta} & : & R^+_{\bar\theta} & \to & \mathbb{R}^{d-h} \times \cup_{\ell=0}^h \mathcal{A}(\ell, [0, +\infty) \times \mathbb{R}^h) \\ & & (\mathfrak{a}, w) & \mapsto & \mathtt{v}^+_{\bar\theta}(\mathfrak{a}, w) := \big(\mathfrak{a}, \operatorname{aff} \partial^+ \bar\theta(\mathfrak{a}, w)\big) \end{array}$$

(I.7.8) THEOREM: *The map $\mathtt{v}^+_{\bar\theta}$ induces a partition*

$$\bigcup_{\ell=0}^h \Big\{ Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}} \subset [0, +\infty) \times \mathbb{R}^h, \mathfrak{a} \in \mathbb{R}^{d-h}, \mathfrak{b} \in \mathbb{R}^{h-\ell} \Big\}$$

on $R^+_{\bar\theta}$ such that the following holds:

1. *each set* $Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}}$ *is locally affine;*
2. *there exists an extremal face* $O^{\ell,+}_{\mathfrak{a},\mathfrak{b}}$ *with dimension* ℓ *of the cone* $\tilde{C}_{\mathfrak{a}}$ *such that*

$$\forall w \in Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}} \ \Big(\operatorname{aff} Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}} = \operatorname{aff}(z + O^{\ell,+}_{\mathfrak{a},\mathfrak{b}}) \quad \wedge \quad \mathcal{D}^+\bar{\theta}(\mathfrak{a}, w) = O^{\ell,+}_{\mathfrak{a},\mathfrak{b}} \cap \{t = 1\}\Big);$$

3. *for all* $w \in T^+_{\bar{\theta}}(\mathfrak{a})$ *there exists* $r > 0$, $O^{\ell,+}_{\mathfrak{a},\mathfrak{b}}$ *such that*

$$B(w, r) \cap \big(w + \operatorname{int}_{\mathrm{rel}} O^{\ell,+}_{\mathfrak{a},\mathfrak{b}}\big) \subset Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}}.$$

In general $\ell \neq \ell'$, but on $R_{\bar{\theta}}$ the two dimensions (and hence the affine spaces $\mathtt{p}_2\, \mathtt{v}^{\pm}_{\bar{\theta}}$) coincide.

(I.7.9) PROPOSITION: *On the set* $R_{\bar{\theta}}$ *one has*

$$\mathtt{v}^-_{\bar{\theta}}(\mathfrak{a}, w) = \mathtt{v}^+_{\bar{\theta}}(\mathfrak{a}, w).$$

PROOF. For the proof see [8, Corollary 4.19]. ■ Define thus on $R_{\bar{\theta}}$

$$\mathtt{v}_{\bar{\theta}} := \mathtt{v}^+_{\bar{\theta}} \llcorner_{R_{\bar{\theta}}} = \mathtt{v}^-_{\bar{\theta}} \llcorner_{R_{\bar{\theta}}},$$

and let

$$\Big\{ Z^\ell_{\mathfrak{a},\mathfrak{b}}, (\mathfrak{a}, \mathfrak{b}) \in \mathbb{R}^{d-h} \times \mathbb{R}^{h-\ell} \Big\}$$

be the partition induced by $\mathtt{v}_{\bar{\theta}}$: since $R_{\bar{\theta}} = \cup_\ell R^{-,\ell}_{\bar{\theta}} \cap R^{+,\ell}_{\bar{\theta}}$, it follows that

$$Z^\ell_{\mathfrak{a},\mathfrak{b}} = Z^{\ell,-}_{\mathfrak{a},\mathfrak{b}} \cap Z^{\ell,+}_{\mathfrak{a},\mathfrak{b}},$$

once the parametrization of $\mathcal{A}(\ell', \operatorname{aff} Z^\ell_{\mathfrak{a}})$ is fixed accordingly.

Finally, define the set $\tilde{\mathbf{D}}'$ by

$$\begin{aligned} \tilde{\mathbf{D}}' &:= \Big\{ (\ell, \mathfrak{a}, \mathfrak{b}, w, C) : C = \mathtt{p}_2 \mathtt{v}_{\bar{\theta}}(\mathfrak{a}, w) \cap \tilde{C}_{\mathfrak{a}}, w \in Z^\ell_{\mathfrak{a},\mathfrak{b}} \Big\} \\ &\subset \bigcup_0^h \big\{ \{\ell\} \times \mathbb{R}^{d-h} \times \mathbb{R}^{h-\ell} \times ([0, \infty) \times \mathbb{R}^\ell) \times \mathcal{C}(\ell, \mathbb{R}^h) \big\}. \end{aligned}$$

(I.7.10) LEMMA: *The set* $\tilde{\mathbf{D}}'$ *is* σ*-compact.*

PROOF. Since $\mathtt{v}_{\bar{\theta}}$, $\mathfrak{a} \mapsto \tilde{C}_{\mathfrak{a}}$ are σ-continuous, the conclusion follows. ■

We thus conclude that $\tilde{\mathbf{D}}'$ corresponds the following directed locally affine partition of $\mathbb{R}^d$:

$$\hat{\mathbf{D}}' := \Big\{ (\ell, \mathfrak{c}, z, C), \mathfrak{c} = (\mathfrak{a}, \mathfrak{b}), z = (\mathfrak{a}, w) : (\ell, \mathfrak{a}, \mathfrak{b}, w, C) \in \tilde{\mathbf{D}}' \Big\}. \tag{I.55}$$

We will use the notation $\mathfrak{c} = (\mathfrak{a}, \mathfrak{b})$ and

$$Z^\ell_{\mathfrak{c}} := Z^\ell_{\mathfrak{a},\mathfrak{b}} = \mathtt{p}_z \tilde{\mathbf{D}}'(\ell, \mathfrak{c}), \quad C^\ell_{\mathfrak{c}} := O^\ell_{\mathfrak{a},\mathfrak{b}} = \mathtt{p}_C \tilde{\mathbf{D}}'(\ell, \mathfrak{c}), \quad \tilde{\mathbf{Z}}'^\ell := \mathtt{p}_{\mathfrak{a},w} \tilde{\mathbf{D}}'(\ell),$$

where $O^\ell_{\mathfrak{a},\mathfrak{b}}$ is the extremal fact of $\tilde{C}_{\mathfrak{a}}$ corresponding to the space $\operatorname{span} \mathcal{D}\bar{\theta}(\mathfrak{a}, w)$.

I.8. Disintegration on directed locally affine partitions

In this section we show how to use the function $\bar{\vartheta}$ in order to prove that the directed locally affine partition $\tilde{\mathbf{D}}$ is regular w.r.t. the measure $\mathcal{H}^d \llcorner_{\{t=1\}}$. This is the main difference w.r.t. the analysis of [8], where the regularity is proved w.r.t. the measure $\mathcal{L}^{d+1}$.

Let thus $\bar{\vartheta}$ be the upper semi continuous envelope of $\bar{\theta}$.

(I.8.1) **LEMMA:** *For all $s \geq 0$, the following holds: if $t \geq s$, then*

$$\bar{\vartheta}(\mathfrak{a}, t, x) = \max \Big\{ \bar{\vartheta}(\mathfrak{a}, s, y) - \mathbb{1}_{\tilde{C}_\mathfrak{a}}(t - s, x - y), y \in \mathbb{R}^h \Big\}.$$

PROOF. Recalling that $\bar{\theta}$ is defined by

$$\bar{\theta}(\mathfrak{a}, t, x) = \sup \Big\{ \bar{\theta}(\mathfrak{a}, s, y) - \mathbb{1}_{\tilde{C}_\mathfrak{a}}(t - s, x - y) : y \in \mathbb{R}^h \Big\},$$

the proof follows immediately by considering a sequence of maximizers y_n for $\bar{\theta}(\mathfrak{a}, t, x)$. ■

Since $\bar{\vartheta}$ satisfies

$$\bar{\vartheta}(\mathfrak{a}, w + \tilde{C}_\mathfrak{a}) \geq \bar{\vartheta}(\mathfrak{a}, w) \tag{I.56}$$

i.e. it is a *complete $\tilde{c}$-Lipschitz foliation* according to [8], the same completeness property (I.53) holds: if $(w, w') \in \partial^+ \bar{\vartheta}(\mathfrak{a})$, then

$$Q_{\bar{\vartheta}, \mathfrak{a}}(w, w') := (w + \tilde{C}_\mathfrak{a}) \cap (w' - \tilde{C}_\mathfrak{a}) \subset \partial^+ \bar{\vartheta}(\mathfrak{a}, w).$$

Recalling for the notations Remark I.7.5, a first connection between $\bar{\theta}$ and $\bar{\vartheta}$ is shown in the following lemma.

(I.8.2) **LEMMA:** *If $\bar{\vartheta}(\mathfrak{a}, t, x) = \bar{\theta}(\mathfrak{a}, t, x)$, then $\partial^- \bar{\theta}(\mathfrak{a}, t, x) \subset \partial^- \bar{\vartheta}(\mathfrak{a}, t, x)$.*

PROOF. Let $(s, y) \in \partial^- \bar{\theta}(\mathfrak{a}, t, x)$, so that $\bar{\theta}(\mathfrak{a}, s, y) = \bar{\theta}(\mathfrak{a}, t, x)$. The inclusion $\partial^- \bar{\theta}(\mathfrak{a}, t, x) \subset \partial^- \bar{\vartheta}(\mathfrak{a}, t, x)$ follows from the estimate:

$$\bar{\theta}(\mathfrak{a}, s, y) \leq \bar{\vartheta}(\mathfrak{a}, s, y) \leq \bar{\vartheta}(\mathfrak{a}, t, x) = \bar{\theta}(\mathfrak{a}, t, x).$$

This concludes the proof. ■

I.8.1. REGULARITY OF THE PARTITION $\tilde{\mathbf{D}}'$

The proof to show the regularity of $\mathcal{H}^d \llcorner_{\{t=1\}}$-a.e. point for $\bar{\vartheta}$ is very similar to the analysis done in Section I.4.1: the two proofs differ because we have now to consider a family of HJ equations (one for each $\mathfrak{a} \in \tilde{\mathfrak{A}}$), and that the Lagrangian is the indicator function of a cone $\tilde{C}_{\mathfrak{a}}$.

Once we have the regularity for $\bar{\vartheta}$, we use the fact that $\bar{\theta}(\bar{t}) = \bar{\vartheta}(\bar{t})$ for $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$-a.e. point in order to deduce that the same regularity holds for $\bar{\theta}$.

Consider a Borel bounded set $S \subset \{t = \bar{t}\}$ made of backward regular points for $\bar{\vartheta}$. Since by the definition of $\bar{\vartheta}$ each point has an optimal ray reaching $t = 0$, for all $s > 0$ we can find *inner* optimal rays, i.e. with directions belonging to the interior of $C^-_{\bar{\vartheta}}$.

(I.8.3) LEMMA: *Let $\bar{t} > s > \varepsilon > 0$. Then for every $(\bar{t}, x) \in S$ there exists a point $\sigma_s(\bar{t}, x) \in \mathrm{int}_{\mathrm{rel}}\big(\partial^- \bar{\vartheta}(\mathfrak{a}, \bar{t}, x) \cap \{t = s\}\big)$ such that*

$$\mathcal{H}^h(\sigma_s(S)) \geq \left(\frac{s - \varepsilon}{\bar{t} - \varepsilon}\right)^h \mathcal{H}^h(S).$$

PROOF. For each fixed $\mathfrak{a}$ the proof is the same as the one of $\bar{\phi}$, just replacing it with $\bar{\vartheta}$. In particular we obtain that for each fixed $\varepsilon > 0$ every point $z \in S$ has a cone of optimal backward directions K_z such that

$$K_z \in \mathcal{C}(\ell, [0, +\infty) \times \mathbb{R}^h) \qquad \text{and} \qquad (z - K_z) \cap \{t = \varepsilon\} \subset \partial^- \bar{\vartheta}(z) \cap \{t = \varepsilon\},$$

where $\ell = \dim \mathcal{D}^- \bar{\vartheta}(z)$.

As in the proof of Lemma I.4.19, we can thus partition the set S according to the requirement that the projection of K on a $(\ell + 1)$-dimensional reference plane V' contains a reference cone $K' \in \mathcal{C}(\ell, [0, +\infty) \times \mathbb{R}^d)$.

Slicing the problem on $(d - \ell)$-dimensional planes V'' transversal to V', it follows that

$$\sigma_s(z) := \bar{\vartheta}(z) \cap V'' \cap \{t = s\}$$

is singleton for all $z \in S$. We can then use the same approach used in [8, Section 8].

Consider the two measures $\hat{\mu} := \mathcal{H}^d \llcorner_S$ and its image measure $\hat{\nu} := (\sigma_s)_\sharp \hat{\mu}$. By (I.56) and Proposition I.6.8 applied to $\bar{\vartheta}$, every transport $\hat{\pi} \in \Pi^f_{\tilde{\mathsf{c}}}(\hat{\mu}, \hat{\nu})$ with finite cost w.r.t. $\tilde{\mathsf{c}}$ occurs on the level sets of $\bar{\vartheta}$: in particular, in each plane V'' there exists a unique transference plan with finite cost.

We can then use [8, Lemma 8.4] in order to obtain a family of cone vector fields converging to σ_s for $\mathcal{H}^d$-a.e. point. Being the area estimate

$$\mathcal{H}^h(\sigma^n_s(S)) \geq \left(\frac{s - \varepsilon}{\bar{t} - \varepsilon}\right)^h \mathcal{H}^h(S)$$

u.s.c. w.r.t. pointwise convergence $\sigma^n_s(z) \to \sigma_s(z)$ [8, Lemma 5.6], we obtain the statement. ■

We can now repeat the same proof of Proposition I.4.21 in order to obtain the regularity of $\mathcal{H}_{\{t=\bar{t}\}}$-a.e. point. The only variation w.r.t. the proof of Proposition I.4.21 is that we have to use the regularity of the disintegration of $\mathcal{L}^{d+1}$ on the directed locally affine partition

$\tilde{\mathbf{D}}_{\bar{\vartheta}} = \{Z^\ell_{\mathfrak{a},\mathfrak{b}}(\bar{\vartheta}), C^\ell_{\mathfrak{a},\mathfrak{b}}(\bar{\vartheta})\}_{\ell,\mathfrak{a},\mathfrak{b}}$ induced by $\bar{\vartheta}$ through the map

$$\begin{array}{rccl} \mathtt{v}_{\bar{\vartheta}} & : & R_{\bar{\vartheta}} & \rightarrow & \mathbb{R}^{d-h} \times \cup_{\ell=0}^h \mathcal{A}(\ell, \mathbb{R}^h) \\ & & (\mathfrak{a}, w) & \mapsto & \mathtt{v}_{\bar{\vartheta}}(\mathfrak{a}, w) := \big(\mathfrak{a}, \mathrm{aff}\, \partial^- \bar{\vartheta}(\mathfrak{a}, w)\big) \end{array}$$

(The fact that $\mathtt{v}_{\bar{\vartheta}}$ induces a locally affine directed partition is the same statement of Theorem I.4.14 or Theorem I.4.15, see Remark I.7.5.)
The regularity of $\tilde{\mathbf{D}}_{\vartheta}$ is one of the fundamental results of [8], Theorem 8.1 and Corollary 8.2:

(I.8.4) THEOREM: *If $\{Z^\ell_{\mathfrak{a},\mathfrak{b}}(\bar{\vartheta}), C^\ell_{\mathfrak{a},\mathfrak{b}}(\bar{\vartheta})\}_{\ell,\mathfrak{a},b}$ is the locally affine partition induced by the function $\bar{\vartheta}$, then $\mathcal{L}^{d+1}$-a.e. point is regular and the disintegration of $\mathcal{L}^{d+1}$ w.r.t. $\{Z^\ell_{\mathfrak{a},\mathfrak{b}}(\bar{\vartheta})\}_{\ell,\mathfrak{a},\mathfrak{b}}$ is regular.*
A similar statement holds for the directed locally affine partition $\tilde{\mathbf{D}}' = \{Z^\ell_{\mathfrak{a},\mathfrak{b}}, C^\ell_{\mathfrak{a},\mathfrak{b}}\}_{\ell,\mathfrak{a},\mathfrak{b}}$ obtained through the function $\bar{\theta}$.

Once we are given that $\mathcal{L}^{d+1}$-a.e. point is regular for $\bar{\vartheta}$, replacing the area estimate Lemma I.4.19 with Lemma I.8.3 in the proof of Proposition I.4.21 yields the following result.

(I.8.5) PROPOSITION: *For all $\bar{t} > 0$, the set of regular points for $\bar{\vartheta}$ in $\{t = \bar{t}\}$ is $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$-conegligible.*

We now transfer the regularity w.r.t. $\bar{\vartheta}$ to the regularity w.r.t. $\bar{\theta}$. The sketch of the proof is as follows: since by Fubini theorem, for $\mathcal{H}^1$-a.e. $\bar{t}$ it holds that $\mathcal{H}^d \llcorner_{\{t=\bar{t}\}}$-a.e. point is regular for $\bar{\theta}$, and the same occurs for $\bar{\vartheta}$, we can use the fact that $\bar{\theta}(t,x) = \bar{\vartheta}(t,x)$ for $\mathcal{H} \llcorner_{\{t=\bar{t}\}}$-a.e. x and every $\bar{t} > 0$ in order to obtain that the points $z, \sigma_s(z)$ used in Lemma I.8.3 are regular points for $\bar{\theta}$. The key observation is that the inner rays for $\bar{\vartheta}$ will be also inner rays for $\bar{\theta}$.

(I.8.6) PROPOSITION: *If $\bar{t} > 0$ then $\mathcal{H}^h \llcorner_{\{t=\bar{t}\}}$-a.e. point is regular for $\bar{\theta}$.*

PROOF. By Theorem I.8.4 we can fix $\varepsilon' > 0$ such that $\mathcal{H}^d_{t=\bar{t}\pm\varepsilon'}$-a.e. point is regular for both $\bar{\theta}$ and $\bar{\vartheta}$. By the area estimate of Lemma I.8.3, we can also assume that

$$S \subset R_{\bar{\theta}} \cap R_{\bar{\vartheta}} \cap \{t = \bar{t} + \epsilon\} \cap \{\bar{\theta} = \bar{\vartheta}\}$$

and for all $z \in S$

$$\sigma_{\bar{t}-\varepsilon'}(z) \in R_{\bar{\theta}} \cap R_{\bar{\vartheta}} \cap \{t = \bar{t} - \varepsilon'\} \cap \{\bar{\theta} = \bar{\vartheta}\}.$$

In particular we deduce that

$$\sigma_{\bar{t}-\varepsilon'}(z) \in \partial^- \bar{\theta}(z). \tag{I.57}$$

If $z - \sigma_s(z)$ belongs to an inner direction of

$$C_{\bar{\theta}}(z) = C^+_{\bar{\theta}}(z) = C^-_{\bar{\theta}}(z),$$

then the same observation at the end of the proof of Proposition I.4.21 will give immediately the statement: the arbitrariness of ε' is used as in the proof of the proposition in order to obtain that $\mathcal{H}^d \llcorner_{\{t=1\}}$-a.e. point is regular.

We thus left with proving this last property of $\sigma_s(z)$, i.e. $z - \sigma_s(z) \in C_{\bar\theta}$.

Being $\sigma_s(z) - z$ an inner ray of $-C_{\bar\vartheta}(z)$ and $C_{\bar\theta}(z)$, $C_{\bar\vartheta}$ extremal cones of $\tilde C_{\mathtt{p}_\mathfrak{a} z}$, it follows that if $C_{\bar\theta}(z) \subsetneq C_{\bar\vartheta}$, by the extremality property then for $s < \bar t + \varepsilon'$

$$(z - \sigma_s(z)) \cap C_{\bar\theta}(z) = \emptyset,$$

contradicting (I.57). ■

The proof of the regularity of the disintegration of $\mathcal{H}^d \llcorner_{\{t=1\}}$ w.r.t. the partition $\{Z^\ell_{\mathfrak{a},\mathfrak{b}}\}_{\ell,\mathfrak{a},\mathfrak{b}}$ follows the same line of Section I.4.2: just replace the use of Lemma I.4.19 with Lemma I.8.3.

The statement is analogous.

(I.8.7) PROPOSITION: *The disintegration*

$$\mathcal{H}^d \llcorner_{\cup_{\ell,\mathfrak{a},\mathfrak{b}} Z^\ell_{\mathfrak{a},\mathfrak{b}} \cap \{t=1\}} = \sum_\ell \int v^\ell_{\mathfrak{a},\mathfrak{b}} \eta^\ell(d\mathfrak{a} d\mathfrak{b})$$

w.r.t. the partition $\{Z^h_{\mathfrak{a},\mathfrak{b}} \cap \{t=1\}\}_{h,\mathfrak{a},\mathfrak{b}}$ *is regular:*

$$v^h_{\mathfrak{a},\mathfrak{b}} \ll \mathcal{H}^\ell \llcorner_{Z^\ell_{\mathfrak{a},\mathfrak{b}}}.$$

I.8.2. PROOF OF THEOREM 3.2

Let $\hat{\mathbf{D}}' = \{Z^\ell_\mathfrak{c}, C^\ell_\mathfrak{c}\}_{\ell,\mathfrak{c}}$ be the directed locally affine given by (I.55). By Corollary I.8.6 we have that the complement of $\cup_{\ell,\mathfrak{c}} Z^\ell_\mathfrak{c}$ is $\mathcal{H}^d \llcorner_{\{t=1\}}$-negligible; Proposition I.4.22 yields that the disintegration is regular.

Consider now the map $\check{\mathtt{r}}$ defined in (I.40): $\check{\mathtt{r}}$ is invertible on $\hat{\mathbf{D}}'$ and, as observed at the end of Section I.5.1, the two measures $\check{\mathtt{r}}^{-1}\mathcal{H}^\ell \llcorner_{Z^\ell_\mathfrak{c}}$ and $\mathcal{H}^\ell \llcorner_{\check{\mathtt{r}}^{-1}(Z^\ell_\mathfrak{c})}$ are equivalent. Hence the first three points of Theorem 3.2 follows: namely

- the sets
$$Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}} := \check{\mathtt{r}}^{-1}(Z^\ell_\mathfrak{c}) \subset Z^h_\mathfrak{a}$$
has affine dimension $\ell + 1$ and
$$C^{h,\ell}_{\mathfrak{a},\mathfrak{b}} := \check{\mathtt{r}}^{-1}(C^\ell_\mathfrak{c})$$
is an $(\ell+1)$-dimensional extremal cone of $C^h_\mathfrak{a}$, $\ell \le h$ and $\mathfrak{c} = (\mathfrak{a}, \mathfrak{b})$;
- $\bar\mu(\cup_{h,\ell,\mathfrak{a},\mathfrak{b}} Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}) = 1$;
- the disintegration of $\mathcal{H}^d \llcorner_{\{t=1\}}$ w.r.t. the partition $\{Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}\}_{h,\ell,\mathfrak{a},\mathfrak{b}}$ is regular;

Since the preorder $\bar\preccurlyeq$ induced by $\bar\theta$ is Borel and $\bar\pi'(\bar\preccurlyeq) = 1$, then by Theorem A.2.2 the transference plan is concentrated on the diagonal $\bar E = \bar\preccurlyeq \cap \bar\preccurlyeq^{-1}$. Hence by construction the transference of mass occurs along the equivalence classes: these directions are exactly the optimal rays defined by (I.52). On the regular set by definition these directions are the extremal cone $C^\ell_\mathfrak{c}$ in $Z^\ell_\mathfrak{c}$:

- if $\bar{\pi} \in \Pi(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ with $\underline{\bar{\nu}}^h_{\mathfrak{a}} = \mathtt{p}_2 \underline{\bar{\pi}}^h_{\mathfrak{a}}$, then $\bar{\pi}$ satisfies (14) iff

$$\bar{\pi} = \sum_{h,\ell} \int \bar{\pi}^{h,\ell}_{\mathfrak{a},\mathfrak{b}} m^{h,\ell}(d\mathfrak{a} d\mathfrak{b}), \qquad \int \mathbb{1}_{C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}}(x - x') \bar{\pi}^{h,\ell}_{\mathfrak{a},\mathfrak{b}} < \infty;$$

The indecomposability of the sets $Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}$ with $\ell = h$ is a consequence of Proposition I.6.10 and Lemma I.6.11, stating that the function θ constructed by a given Γ is constant on $Z^{h,h}_{\mathfrak{a},\mathfrak{b}}$ and the sets $Z^{h,h}_{\mathfrak{a},\mathfrak{b}}$ are indecomposable. This proves Point (5) of Theorem 3.2:

- if $\ell = h$, then for every carriage Γ of $\bar{\pi} \in \Pi(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ there exists a $\bar{\mu}$-negligible set N such that each $Z^{h,h}_{\mathfrak{a},\mathfrak{b}} \setminus N$ is $\mathbb{1}_{C^{h,h}_{\mathfrak{a},\mathfrak{b}}}$-cyclically connected.

In the next section we will use this theorem in order to prove Theorem 1.1.

I.9. Proof of Theorem 1.1

By Theorem 3.1 we have a first directed locally affine decomposition $\mathbf{D}_{\bar{\phi}}$, and by Theorem 3.2 a method of refining a given locally affine partition in order to obtain indecomposable sets or lower the dimension of the sets by at least 1. It is thus clear that after at most d steps we obtain a locally affine decomposition $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}$ with the properties stated in Point (5) of Theorem 3.2.

(I.9.1) THEOREM: *Given a transference plan $\underline{\bar{\pi}} \in \Pi(\bar{\mu}, \bar{\nu})$ optimal w.r.t. the cost $\tilde{\mathtt{c}}$, then there is a directed locally affine partition $\bar{\mathbf{D}} = \{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ such that*

1. *$Z^h_{\mathfrak{a}}$ has affine dimension $h+1$ and $C^h_{\mathfrak{a}}$ is an $(h+1)$-dimensional proper extremal cone of* $\operatorname{epi} \bar{\mathtt{c}}$*; moreover* $\operatorname{aff} Z^h_{\mathfrak{a}} = \operatorname{aff}(z + C^h_{\mathfrak{a}})$ *for all $z \in Z^h_{\mathfrak{a}}$;*
2. *$\mathcal{H}^d(\{t = 1\} \setminus \cup_{h,\mathfrak{a}} Z^h_{\mathfrak{a}}) = 0$;*
3. *the disintegration of $\mathcal{H}^d \llcorner_{\{t=1\}}$ w.r.t. the partition $\{Z^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ is regular, i.e.*

$$\mathcal{H}^d \llcorner_{\{t=1\}} = \sum_h \int \xi^h_{\mathfrak{a}} \eta^h(d\mathfrak{a}), \qquad \xi^h_{\mathfrak{a}} \ll \mathcal{H}^h \llcorner_{Z^h_{\mathfrak{a}} \cap \{t=1\}};$$

4. *if $\bar{\pi} \in \Pi(\bar{\mu}, \{\underline{\bar{\nu}}^h_{\mathfrak{a}}\})$ with $\underline{\bar{\nu}}^h_{\mathfrak{a}} = \mathtt{p}_2 \underline{\bar{\pi}}^h_{\mathfrak{a}}$, then $\bar{\pi}$ is optimal iff*

$$\bar{\pi} = \sum_h \int \bar{\pi}^h_{\mathfrak{a}} m^h(d\mathfrak{a}), \qquad \int \mathbb{1}_{C^h_{\mathfrak{a}}}(z - z') \bar{\pi}^h_{\mathfrak{a}} < \infty;$$

5. *for every carriage* Γ *of any* $\bar{\pi} \in \Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$ *there exists a* $\bar{\mu}$*-negligible set* N *such that each* $Z^h_{\mathfrak{a},\mathfrak{b}} \setminus N$ *is* $\mathbb{1}_{C^h_{\mathfrak{a}}}$*-cyclically connected.*

The last step is to project back the decomposition for $\{t = 1\} \times \mathbb{R}^d$ to $\mathbb{R}^d$, and cut the cones $C^h_{\mathfrak{a}}$ at $\{t = 1\}$.

- Take $S^h_{\mathfrak{a}} := Z^h_{\mathfrak{a}} \cap \{t = 1\}$ and $O^h_{\mathfrak{a}} = -C^h_{\mathfrak{a}} \cap \{t = 1\}$: the minus sign is because of the definition of $\bar{\mathtt{c}}$ in formula (9). By the transversality of $C^h_{\mathfrak{a}}$ if follows that $\dim O^h_{\mathfrak{a}} = \dim O^h_{\mathfrak{a}} = h$. Since $Z^h_{\mathfrak{a}}$ is parallel to $C^h_{\mathfrak{a}}$, then $O^h_{\mathfrak{a}}$ is parallel to $S^h_{\mathfrak{a}}$. Being $C^h_{\mathfrak{a}}$ an extremal cone of $\bar{\mathtt{c}}$ and the latter 1-homogeneous, it follows that $O^h_{\mathfrak{a}}$ is an extremal face of $\mathtt{c}$.
- The fact that the partition cover $\mathcal{L}^d$-a.e. point and that the disintegration is regular are straightforward.
- Being $\bar{\mu}(\{t = 1\}) = \bar{\nu}(\{t = 0\}) = 1$, then it is clear that we can assume that every carriage Γ is a subset of $\{t = 1\} \times \{t = 0\}$. This implies that when computing the cyclical indecomposability we use only vectors in $O^h_{\mathfrak{a}}$, and thus the last point of Theorem 1.1 follows from Point (5) of Theorem I.9.1.

This concludes the proof of Theorem 1.1.

I.9.1. THE CASE OF $\nu \ll \mathcal{L}^d$

In general the end points of optimal rays are in $Z^h_{\mathfrak{a}} + C^h_{\mathfrak{a}} \cap \{t = 1\}$, which in general is larger that $Z^h_{\mathfrak{a}}$. As an example, one can consider the case $\nu = \delta_{x=0}$, and verify that $Z^h_a = C^h_{\mathfrak{a}} \setminus \{0\}$. However in the case $\nu \ll \mathcal{L}^d$, the partition is independent of π, i.e. following [8] we call it *universal*.

The key observation is that we can replace the roles of the measures $\bar{\mu}$, $\bar{\nu}$, obtaining then a decomposition $\{W^{h'}_{\mathfrak{a}'}, C^{h'}_{\mathfrak{a}'}\}_{h',\mathfrak{a}'}$ for $\{t = 0\}$. Now recall that along optimal rays $\bar{\theta}$ is constant: being inner ray of the cones $C^h_{\mathfrak{a}}$, $C^{h'}_{\mathfrak{a}'}$, then it follows that $C^h_{\mathfrak{a}} = C^{h'}_{\mathfrak{a}'}$, and from this it is fairly easy to see that $Z^h_{\mathfrak{a}} = W^{h'}_{\mathfrak{a}'}$. In particular, for each optimal transference plan $\bar{\pi}$ it follows that its second marginals are given by the disintegration of $\bar{\nu}$ on $Z^h_{\mathfrak{a}}$, i.e. they are independent of $\bar{\pi}$.

Translating this decomposition into the original setting, we can thus strengthen Theorem 1.1 as follows.

(I.9.2) THEOREM: *Let* $\mu, \nu \ll \mathcal{L}^d$*. Then there exists a family of sets* $\{S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}\}_{\substack{h=0,\dots,d \\ \mathfrak{a}\in\mathfrak{A}^h}}$ *in* $\mathbb{R}^d$ *such that the following holds:*

1. $S^h_{\mathfrak{a}}$ *is a locally affine set of dimension* h*;*
2. $O^h_{\mathfrak{a}}$ *is a* h*-dimensional convex set contained in an affine subspace parallel to* $\operatorname{aff} S^h_{\mathfrak{a}}$ *and given by the projection on* $\mathbb{R}^d$ *of a proper extremal face of* $\operatorname{epi} \mathtt{c}$*;*

3. $\mathcal{L}^d(\mathbb{R}^d \setminus \cup_{h,\mathfrak{a}} S^h_{\mathfrak{a}}) = 0$;

4. *the partition is Lebesgue regular;*

5. *if* $\pi \in \Pi(\mu, \nu)$ *then optimality in* (4) *is equivalent to*

$$\sum_h \int \bigg[\int \mathbb{1}_{O^h_{\mathfrak{a}}}(x' - x) \pi^h_{\mathfrak{a}}(dxdx') \bigg] m^h(d\mathfrak{a}) < \infty,$$

   where $\pi = \sum_h \int_{\mathfrak{A}^h} \pi^h_{\mathfrak{a}} m^h(d\mathfrak{a})$ *is the disintegration of* π *w.r.t. the partition* $\{S^h_{\mathfrak{a}} \times \mathbb{R}^d\}_{h,\mathfrak{a}}$;

6. *for every carriage* Γ *of* $\pi \in \Pi(\mu, \nu)$ *there exists a* μ*-negligible set* N *such that each* $S^h_{\mathfrak{a}} \setminus N$ *is* $\mathbb{1}_{O^h_{\mathfrak{a}}}$*-cyclically connected.*

A study case

II

II.1. Setting

As case study we consider the following Monge problem in $\mathbb{R}^2$:

$$\min\Big\{\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathsf{c}(x-x')\,\pi(\mathrm{d}x\mathrm{d}x'):\quad \pi\in\Pi(\mu,\nu)\Big\}, \tag{II.1}$$

where μ and $\nu \ll \mathcal{L}^2$ are two positive probabilities measure on $\mathbb{R}^2$, $\Pi(\mu,\nu)$ the set of probabilities measures on $\mathbb{R}^2\times\mathbb{R}^2$ such that the first marginal is μ, the second ν, and c is a non negative, lower semicontinuous, and convex real valued cost with the same growth estimate of the general case.

As before, we consider the embedding in $[0,+\infty)\times\mathbb{R}^2$ and the relative problem associated to (II.1):

$$\int_{\big(\mathbb{R}^+\times\mathbb{R}^2\big)\times\big(\mathbb{R}^+\times\mathbb{R}^2\big)} \bar{\mathsf{c}}(t-t',x-x')\,\overline{\pi}(\mathrm{d}t\mathrm{d}x\mathrm{d}t'\mathrm{d}x'),\quad \bar{\pi}\in\Pi(\bar{\mu},\bar{\nu}). \tag{II.2}$$

Then we consider the associated potentials:

$$\bar{\phi}(t,x)=\min_{x'\in\mathbb{R}^2}\big\{-\psi(x')+\bar{\mathsf{c}}(t,x-x')\big\},\quad t\geq 0,$$

and

$$\bar{\psi}(t,x)=\max_{x'\in\mathbb{R}^2}\big\{-\phi(x')-\bar{\mathsf{c}}(1-t,x'-x)\big\},\quad t\leq 1.$$

II.1.1. CONSTRUCTION OF THE FIRST DIRECTED LOCALLY AFFINE PARTITION

In the following we adopt the same notation and the results obtained in Section I.4.

By means of the potentials $\bar{\phi}$ we find a directed locally affine partition $\mathbf{D}_{\bar{\phi}}\subset\bigcup_{h\in\{0,1,2\}}\Big(\{h\}\times\mathbb{R}^{2-h}\times([0,+\infty)\times\mathbb{R}^2)\times\mathcal{C}(h,[0,+\infty)\times\mathbb{R}^2)\Big)$,

$$\mathbf{D}_{\bar{\phi}}:=\Big\{\big(h,\mathfrak{a},z,C\big):C=F^h_{\mathfrak{a}},z\in Z^h_{\mathfrak{a}}\Big\}.$$

II.1.2. AREA ESTIMATE

The next step is to prove that $\mathbf{D}_{\bar{\phi}}$ is regular enough to be refined.

A powerful tool to prove this kind of regularity is the area estimate that we introduce in this section.

Let $0 < t_1 < t_2$ be two different real numbers and S a subset of $\{t = t_2\}$ such that $\mathcal{H}^2(S) < +\infty$. Let us assume that for every $z \in S$ there exists an optimal ray that reaches t_1. This is not restrictive by formula (I.3)

(II.1.1) DEFINITION: Take $\bar{t} \in]t_1, t_2[$ a real number. With the previous notation we define the following sets:

$$\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t}) := \left\{ z \in S : \partial^- \bar{\phi}(z) \cap \{t = \bar{t}\} \text{ does not contain any segment } \right\},$$

$$\Sigma^-_{\bar{\phi},1}(S, t_1, t_2, \bar{t}) := \left\{ z \in S : \partial^- \bar{\phi}(z) \cap \{t = \bar{t}\} \text{ contains a segment but no two dimensional convex } \right\},$$

$$\Sigma^-_{\bar{\phi},2}(S, t_1, t_2, \bar{t}) := \left\{ z \in S : \partial^- \bar{\phi}(z) \cap \{t = \bar{t}\} \text{ contains at least a two dimensional convex } \right\},$$

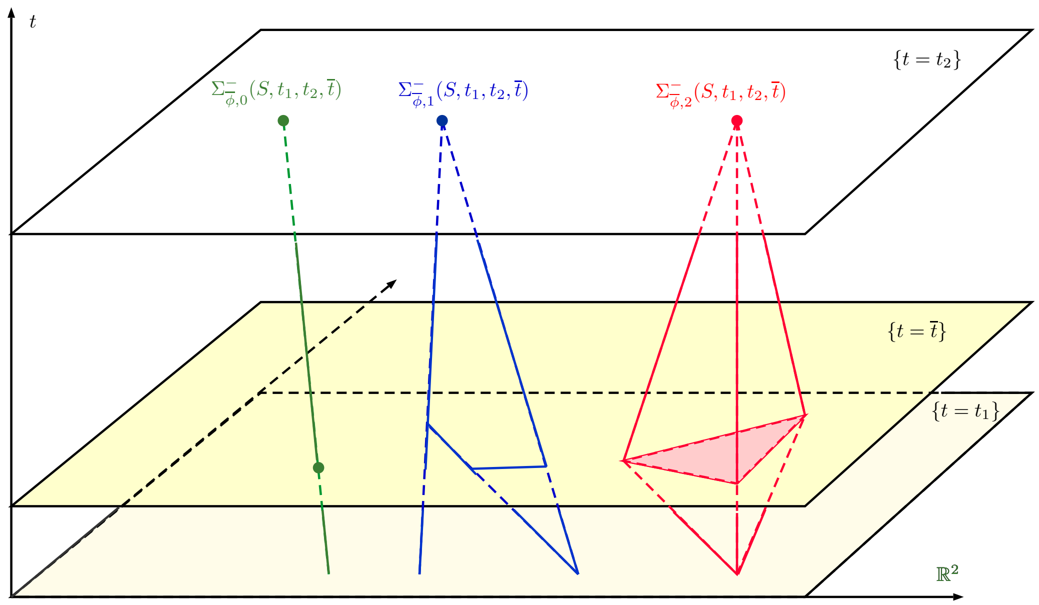


Figure II.1: Definition II.1.1.

First of all we prove that the subset of $\Sigma^-_{\bar{\phi},0}(S, t_2, t_2, \bar{t})$ made of points z such that there are $z_1, z_2 \in \partial^- \bar{\phi}(z)$ and $\text{dist}_H([\partial \bar{\mathtt{c}}(\cdot - z_1)](z), [\partial \bar{\mathtt{c}}(\cdot - z_2)](z)) > 0$ is rectifiable.

(II.1.2) DEFINITION: Let $S \subset \mathbb{R}^d$ be an $\mathcal{H}^k-$measurable set. We say that S is $k-$*countably rectifiable* if there exist countable many Lipschitz functions $f_i : \mathbb{R}^k \mapsto \mathbb{R}^d$ such that $S \subset \cup f_i(\mathbb{R}^d)$.

(II.1.3) THEOREM: *Let $S \subseteq \mathbb{R}^d$ and assume that for any $z \in S$ there exists $\rho_z > 0$, $m_{\cdot}z > 0$ and a $k-$plane $L(z) \subset \mathbb{R}^d$ such that*

$$S \cap B_{\rho_z}(z) \subset z + \{z \in \mathbb{R}^d : |\mathrm{p}_{L(z)^\perp} z| \leq m_z |\mathrm{p}_{L(z)} z|\},$$

where p_L is the orthogonal projection onto L, $\mathrm{p}_{L^\perp}$ onto the orthogonal of L. Then S is contained in the union of countably many Lipschitz $k-$graph whose Lipschitz constants do not exceeded $2\sup_z m_z$.

As in [4, Section 6] we can argue as following. For every $K \subset \mathbb{R}^3$ compact set such that $0 \notin K$ we can define

$$G^+(K) = \{w \in \mathbb{R}^3 : w \cdot k > 0, \text{ for every } k \in K\} \quad \text{and} \quad G(K) = G^+(K) \cap G^+(-K).$$

In the following we will indicate the usual notion of sub-differential with the notation ∂.

(II.1.4) LEMMA: *Let $z \in D_t$, $z_1, z_2 \in \partial^- \bar{\phi}(z)$, and assume that $\mathrm{dist}_H([\partial \bar{\mathsf{c}}(\cdot - z_1)](z), [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)) > 0$.*
Let $(z^k)_{k\in\mathbb{N}}$ a sequence of points in D_t converging to z such that there exists $z_1^k \in \partial^- \bar{\phi}(z^k)$ with $z_1^k \to z_1$.
Then, if $[0, +\infty) \times \mathbb{R}^2$ is the derived set of $\left\{ \frac{z^k - z}{|z^k - z|} \right\}_{k\in\mathbb{N}}$,

$$D \cap G^+\big([\partial \bar{\mathsf{c}}(\cdot - z_1)](z) - [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)\big) = \emptyset.$$

PROOF. Up to a subsequence $\big(\frac{z^k - z}{|z^k - z|}\big)_{k\in\mathbb{N}}$ converges to some l.
Observe that there exists $\alpha_1^k \in [\partial \bar{\mathsf{c}}(\cdot - z_1^k)](\overline{z})$ and $\alpha_2 \in [\partial \bar{\mathsf{c}}(\cdot - z_2)](\overline{z})$ such that

$$\bar{\mathsf{c}}(z^k - z_1^k) = \bar{\mathsf{c}}(z - z_1^k) + \alpha_1^k \cdot (z^k - z) - o(|z^k - z)|)$$

and

$$\bar{\mathsf{c}}(z^k - z_2) = \bar{\mathsf{c}}(z - z_2) + \alpha_2 \cdot (z^k - z) - o(|z^k - z)|).$$

Notice that for every subsequence α_1^k converging to some α_1, necessarily α_1 belongs to $[\partial^- \bar{\mathsf{c}}(\cdot - z_1)](z)$.
Therefore,

$$\begin{aligned} \bar{\phi}(z) + \alpha_2 \cdot (z^k - z) + o(|z^k - z|) &= \bar{\phi}(z) - \bar{\mathsf{c}}(z - z_2) + \bar{\mathsf{c}}(z^k - z_2) \\ &= \bar{\phi}(z_2) + \bar{\mathsf{c}}(z^k - z_2) \\ &\geq \bar{\phi}(z^k) \\ &= \bar{\phi}(z_1^k) + \bar{\mathsf{c}}(z^k - z_1^k) \\ &= \bar{\phi}(z_1^k) + \bar{\mathsf{c}}(z - z_1^k) + \alpha_1^k \cdot (z^k - z) + o(|z^k - z|) \\ &\geq \bar{\phi}(z) + \alpha_1^k \cdot (z^k - z) + o(|z^k - z|). \end{aligned}$$

Therefore

$$(\alpha_2 - \alpha_1^k) \cdot (z^k - z) \geq o(|z^k - z|).$$

Passing to the limit we obtain that:

$$(\alpha_2 - \alpha_1) \cdot l \geq 0.$$

■

(II.1.5) PROPOSITION: *Fix $t > 0$. Then, the set*

$$J_t = \left\{ z \in D_t : \exists z_1, z_2 \in T^-_{\bar{\phi}} \cap \partial^- \bar{\phi}(z), \mathrm{dist}_H([\partial \bar{\mathsf{c}}(\cdot - z_1)](z), [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)) > 0 \right\}.$$

is countably $n-1$ rectifiable.

PROOF. It is not restrictive to suppose that there is $\rho > 0$ and $d_1, d_2 \in \{t = 1\}$ such that for every $z \in J_t$ there are z_1 and z_2 in $\partial^- \bar{\phi}(z)$ such that $\frac{z - z_1}{\mathsf{p}_t(z - z_1)} \in B_1(d_1, \rho)$, $\frac{z - z_2}{\mathsf{p}_t(z - z_2)} \in B_1(d_2, \rho)$ and,

$$\mathrm{dist}_H([\partial \bar{\mathsf{c}}(\cdot - z_1)](z), [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)) > 0.$$

If there is a sequence z^i converging to z such that there are two sequence z_1^i and z_2^i converging respectively to z_1 and z_2 then this sequence must not be in

$$G([\partial \bar{\mathsf{c}}(\cdot - z_1)](z) - [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)),$$

according to Lemma II.1.4. In order to get the thesis it is sufficient to applies the Theorem (II.1.3) ■

(II.1.6) COROLLARY: *Fix $t > 0$. Then, the set*

$$J_{\bar{t}} = \left\{ z \in \{t = \bar{t}\} : \exists z_1, z_2 \in T^+_{\bar{\phi}} \cap \partial^+ \bar{\phi}(z), \mathrm{dist}_H([\partial \bar{\mathsf{c}}(\cdot - z_1)](z), [\partial \bar{\mathsf{c}}(\cdot - z_2)](z)) > 0 \right\}.$$

is countably $n-1$ rectifiable.

(II.1.7) REMARK: *If we worked with a two dimensional slice of $[0, +\infty) \times \mathbb{R}^2$ the proof would be easier. For instance, consider a section of $[0, +\infty) \times \mathbb{R}^2$ made with a plane orthogonal to $\{t = 1\}$. Without loss of generality we can identify this section with $[0, +\infty) \times \mathbb{R}$. Take $0 < t_1 < t_2$ and let us consider a bounded set S in the real line $\{t_2\} \times \mathbb{R}$.*
Let us suppose that for every z in S there is a point z' on $\{t_1\} \times \mathbb{R}$ such that $z' \in \partial^- \bar{\phi}(z)$. Then, for every $n \in \mathbb{N}$ and $\bar{t} \in]t_1, t_2[$ the set

$$\left\{ z \in \Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t}) : \text{ there are } z_1, z_2 \in \partial^- \bar{\phi}(z) \cap (\{\bar{t}\} \times \mathbb{R}) \text{ such that } |z_1 - z_2| > \frac{1}{n+1} \right\}$$

has only a finite number of points. This is a simple consequence of the boundedness of R_c. Then, by a covering argument one can prove that

$$\left\{ z \in \Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t}) : \text{ there are } z_1, z_2 \in \partial^- \bar{\phi}(z) \cap (\{\bar{t}\} \times \mathbb{R}) \text{ such that} |z_1 - z_2| > 0 \right\}$$

is negligible.

(II.1.8) **REMARK:** *Let $z \in \{t = t_2\}$ and $z' \in \{t_1\} \times \mathbb{R}$ such that*

$$\bar{\phi}(z) - \bar{\phi}(z') = \bar{\mathsf{c}}(z - z').$$

Let us indicate with F the projection on $[0, +\infty) \times \mathbb{R}^2$ of a fixed bi-dimensional or three-dimensional face of epi$\bar{\mathsf{c}}$. *Then, if $[\![z', z]\!] \cap (z - \mathrm{int}_{\mathrm{rel}} F)$ is not empty, then z is not in $\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})$*

PROOF. It is a consequence of Lemma I.4.10. ■

Now we are ready to introduce and prove the formula of the area estimate. The following lemma is just technical result that will be applied later in the proof of the formula.

(II.1.9) **LEMMA:** *Let C be a cone in $\mathcal{C}(1, [0, +\infty) \times \mathbb{R}^2)$. Let f be a Borel function that associates to $z \in S$ the segment $\{t = \bar{t}\} \cap (z - C)$. Then, for every $\varepsilon > 0$ there exists a finite family of hyperplanes $\{\alpha_i\}_i$ such that the measure of the set*

$$\{z \in S : \text{ there exists } j \text{ such that } f(z) \cap (z + \alpha_j) \text{ is an inner relative single point of } f(z)\}$$

is $\mathcal{H}^2(S)$.

PROOF. We can assume that f associates to each point of S a segment which length is l. Let $J > 1$ a natural number and consider the vectors

$$e_j := \left(\bar{t}, \cos\left(\frac{j}{J}\pi\right), \sin\left(\frac{j}{J}\pi\right)\right), \quad j \in \{0, \dots, J-1\}.$$

Consider now the set:

$$S'_j := \left\{ w \in A : \left| \frac{a-b}{\|a-b\|} \cdot e_j \right| \geq l \cos\left(\frac{\pi}{2J}\right) \text{ where } a \text{ and } b \text{ are two different points belonging to } f(w) \right\}$$

Defining $\{S_j\}_{j \in \{0, \dots, J-1\}}$ as the sets:

$$S_0 := S'_0,$$

$$S_j := S'_j \setminus \bigcup_{0}^{j-1} S'_i, \quad j \in \{1, \dots, J-1\},$$

we obtain a partition made of disjoint sets.
In the plane $([0, +\infty) \times \mathbb{R}^2) \cap \{t = \bar{t}\}$ for every j let us consider $e_j^\perp$ a normal vector orthogonal to e_j.
There exists a natural number K such that

$$S'_{j,k} := \left\{ z \in S_j : \left(z + \left(\bar{t} - t_2, k \cos\left(\frac{j}{J}\pi\right), k \sin\left(\frac{j}{J}\pi\right)\right) + \mathbb{R} e_j^\perp\right) \cap \mathrm{int}_{\mathrm{rel}} f(z) \neq \emptyset \right\}, \quad k \in \{-K, \dots, K\}$$

made a partition of S_j. As before we can construct from these sets a new disjoint partition $\{S_{j,k}\}_k$ of S. This conclude the argument.

■

Recall that S is a subset of $\{t = t_2\}$. The proof of the area estimate is divided into three streps. First we divide the points of S according to the dimension of the sub differential and then we prove the formula in each case. Therefore we conclude the argument gluing the three cases together.

(II.1.10) LEMMA: *For any $\bar{t} \in]t_1, t_2[$ there exists a set $S' \subseteq \{t = \bar{t}\}$ such that:*

1. *there exists a surjective function σ from $\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})$ to S',*
2. *for every $z \in \Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})$, $\sigma(z) \in \partial^- \bar{\phi}(z)$*
3. *the following formula holds:*

$$\mathcal{H}^2(S') \geq \left(\frac{\bar{t} - t_1}{t_2 - t_1}\right)^2 \mathcal{H}^2(\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})).$$

PROOF.

Remarks 2.2 allow us to assume that we are going to consider optimal rays in a bounded cone.

Remark II.1.8 ensures us that we can avoid to consider directions that are in the relative inner part of a face of the cost. Proposition II.1.5 let us to consider rays that are on the same face. This means that we can have two possible cases: fix a point z in $\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})$, $[\partial^- \bar{\phi}(z)] \cap \{t = \bar{t}\}$ could be made of one point or of two points in the relative boundary of the same bi-dimensional face.

In the first case one can argue as following. By the inner regularity of $\mathcal{H}^2$, for every $\varepsilon_1 > 0$ there exists K_1 compact such that

$$\mathcal{H}^2(\Sigma^-_{\bar{\phi},0}(S, t_1, t_2, \bar{t})) < \mathcal{H}^2(K_1) + \varepsilon_1.$$

Fix a dense sequence $\{z^i\}_{i=1}^{\infty}$ on $[\overline{S} - C] \cap \{t = t_1\}$ and consider

$$\overline{\sigma}_I(z) \in \arg\max\{\bar{\phi}(z) - \bar{\mathsf{c}}(z^i - z) : i \in \{1, \dots, I\}\}$$

We can assume $\overline{\sigma}_I(z)$ is equal to $z^{\bar{i}}$ where $\bar{i}$ is the first $i \in \{1, \dots, I\}$ that achieve the maximum.

$\overline{\sigma}_I$ converges pointwise to a function $\overline{\sigma}$ such that

$$\bar{\phi}(z) = \bar{\phi}(\overline{\sigma}(z)) + \bar{\mathsf{c}}(\overline{z} - \sigma(z)).$$

Let us define $\sigma(z) := [\![\overline{\sigma}(z), z]\!](\bar{t})$.
By Egoroff Theorem, for every $\varepsilon_2 > 0$ there is a compact K_2 such that $\overline{\sigma}_I$ converges uniformly to the function $\overline{\sigma}$ and

$$\mathcal{H}^2(K_1) < \mathcal{H}^2(K_2) + \varepsilon_2.$$

Lusin Theorem ensure us that we can assume the functions continuous.
Uniformly convergences of σ_I to σ ensure us that

$$\sigma_I(K_2) \xrightarrow{H} \sigma(K_2).$$

Therefore, by the upper semicontinuity of the Husdorff distance we have

$$\mathcal{H}^2(K_2) = \limsup_I \left(\frac{\bar{t}-t_1}{t_2-t_1}\right)^2 \mathcal{H}^2(\sigma_I(K_2)) \leq \mathcal{H}^2(\sigma(K_2)) \leq \mathcal{H}^2(\sigma(S)).$$

In the latter case we can associate to each point of S a segment on $\{t=\bar{t}\}$ that has as extreme points the intersection of the backward rays and $\{t=\bar{t}\}$. Then, one can partition S according to the Lemma (II.1.9) and using the same notation of this Lemma, obtain the partition $\{A_i\}_{i\in I}$. Without loss of generality we can assume that for each $z' \in A_i$

$$z + \left(\bar{t}-t_2, \cos\left(\frac{j}{J}\pi\right), \sin\left(\frac{j}{J}\pi\right)\right) \cap \mathrm{int}_{\mathrm{rel}} f(z) \neq \emptyset.$$

This means that in A_j we can obtain a selection on the rays considering the new cost:

$$\tilde{\bar{\mathsf{c}}}(z-z') := \bar{\mathsf{c}}(z-z') + (\mathrm{p}_{\{t=\bar{t}\}}(z-z')\cdot e_j)^+,$$

This cost produces new potential $\tilde{\bar{\phi}}$ that coincides with $\bar{\phi}$ on one of the two optimal rays. This means we find a measurable selection on the rays. Reproducing the previous computation we can obtain the estimate also in this case. ■

(II.1.11) LEMMA: *For any $\bar{t} \in]t_1,t_2[$ there exists a set $S' \subseteq \{t=\bar{t}\}$ such that:*

1. *there exists a surjective function σ from $\Sigma^-_{\bar{\phi},1}(S,t_1,t_2,\bar{t})$ to S',*
2. *for every $z' \in \Sigma^-_{\bar{\phi},1}(S,t_1,t_2,\bar{t})$ there are two different point $z_1,z_2 \in [\partial^-\bar{\phi}(z)] \cap \{t=\bar{t}\}$ such that*
$$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}[\![z_1,z_2]\!],$$
3. *the following formula holds:*
$$\mathcal{H}^2(S') \geq \left(\frac{\bar{t}-t_1}{t_2-t_1}\right)^2 \mathcal{H}^2(\Sigma^-_{\bar{\phi},1}(S,t_1,t_2,\bar{t})).$$

PROOF. Partition $\Sigma^-_2(S,t_1,t_2,\bar{t})$ according to Lemma II.1.9. Notice that elements belonging to different sets of the partition can not cross in a point with time greater than t_1, otherwise they will increase the dimension of $\partial^-\bar{\phi}$ evaluated in the two points. In each slice provided by the LemmaII.1.9 we can prove the thesis reminding Remark II.1.7 and applying the same technique used for $\Sigma^-_{\bar{\phi},1}$. ■

(II.1.12) LEMMA: *For any $\bar{t} \in]t_1,t_2[$ there exists a set $S' \subseteq \{t=\bar{t}\}$ such that:*

1. *there exists a surjective function σ from $\Sigma^-_{\bar{\phi},2}(S,t_1,t_2,\bar{t})$ to S',*
2. *for every $z \in \Sigma^-_{\bar{\phi},2}(S,t_1,t_2,\bar{t})$ there are three different and non align point $z_1,z_2,$ and $z_3 \in [\partial^-\bar{\phi}(z)] \cap \{t=\bar{t}\}$ such that*
$$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}\mathrm{conv}\{z_1,z_2,z_3\},$$

3. *the following formula holds:*

$$\mathcal{H}^2(S') \geq \left(\frac{\bar{t}-t_1}{t_2-t_1}\right)^2 \mathcal{H}^2(\Sigma^-_{\bar{\phi},2}(S,t_1,t_2,\bar{t})).$$

PROOF. Since the epigraph of c admit at most an countably infinite number of three dimensional faces, $\bar{\mathtt{c}}$ admit at most a countably infinite number of four dimensional faces. We will indicate these faces as $\{F_i\}_{i=1}^{\infty}$ and define S_i as the points of $\Sigma^-_{\bar{\phi},2}(S,t_1,t_2,\bar{t})$ for which there exists $\delta > 0$ such that

$$B(x,\delta) \cap (x - F_i) \subseteq \partial^- \bar{\phi}(x).$$

Notice that for every S_i we can assume there is ${S'}_i$ contained in $D_{\bar{t}}$ such that

1. there exist a surjective function σ from S_i to ${S'}_i$,
2. $\sigma(z)$ belongs to the internal part of $\partial^-\bar{\phi}(z)$,
3. $|S_i| = |{S'}_i|$.

In order to obtain the thesis we have to prove that $\{S_i\}_{i-1}^{\infty}$ and $\{{S'}_i\}_{i-1}^{\infty}$ are two families made of pairwise disjoint sets. Proposition II.1.5 and Corollary II.1.6 ensure us that this is the case. ■

(II.1.13) COROLLARY: *For any $\bar{t} \in]t_1, t_2[$ there exists a set $S' \subseteq \{t = \bar{t}\}$ such that:*

1. *there exists a surjective function σ from S to S',*
2. (a) *for every $z \in \Sigma^-_{\bar{\phi},0}(S,t_1,t_2,\bar{t})$, $\sigma(z) \in \partial^-\bar{\phi}(z)$,*
   (b) *for every $z \in \Sigma^-_{\bar{\phi},1}(S,t_1,t_2,\bar{t})$ there are two different point $z_1, z_2 \in [\partial^-\bar{\phi}(z)] \cap \{t = \bar{t}\}$ such that*
   $$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}[\![z_1, z_2]\!],$$
   (c) *for every $z \in \Sigma^-_{\bar{\phi},2}(S,t_1,t_2,\bar{t})$ there are three different and non align point $z_1, z_2,$ and $z_3 \in [\partial^-\bar{\phi}(z)] \cap \{t = \bar{t}\}$ such that*
   $$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}\mathrm{conv}\{z_1, z_2, z_3\},$$
3. *the following formula holds:*

$$\mathcal{H}^2(S') \geq \left(\frac{\bar{t}-t_1}{t_2-t_1}\right)^2 \mathcal{H}^2(S). \tag{II.3}$$

PROOF. Consequence of previous Lemmas. ■

When S is a subset of $\{t = t_1\}$ and for every $z \in S$ there exists an optimal ray that reaches t_2 we can state the following definition and prove very similar results:

(II.1.14) **DEFINITION:** Take $t_1 < \bar{t} < t_2$ a real number. With the previous notation we define the following sets:

$$\Sigma^+_{\bar{\phi},0}(S,t_1,t_2,\bar{t}) := \big\{z \in S : \partial^+\bar{\phi}(z) \cap \{t = \bar{t}\} \text{ does not contain any segment } \big\},$$

$$\Sigma^+_{\bar{\phi},1}(S,t_1,t_2,\bar{t}) := \big\{z \in S : \partial^+\bar{\phi}(z) \cap \{t = \bar{t}\} \text{ contains a segment but no two dimensional convex } \big\},$$

$$\Sigma^+_{\bar{\phi},2}(S,t_1,t_2,\bar{t}) := \big\{z \in S : \partial^+\bar{\phi}(z) \cap \{t = \bar{t}\} \text{ contains at least a two dimensional convex } \big\},$$

(II.1.15) **PROPOSITION:** *For any $\bar{t} \in]t_1, t_2[$ there exists a set $S' \subseteq \{t = \bar{t}\}$ such that:*

1. *there exists a surjective function σ from S to S',*
2. (a) *for every $z \in \Sigma^+_{\bar{\phi},0}(S,t_1,t_2,\bar{t})$, $\sigma(z) \in \partial^+\bar{\phi}(z)$,*
   (b) *for every $z \in \Sigma^+_{\bar{\phi},1}(S,t_1,t_2,\bar{t})$ there are two different point $z_1, z_2 \in [\partial^+\bar{\phi}(z)] \cap \{t = \bar{t}\}$ such that*
   $$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}[\![z_1, z_2]\!],$$
   (c) *for every $z \in \Sigma^+_{\bar{\phi},2}(S,t_1,t_2,\bar{t})$ there are three different and non align point $z_1, z_2,$ and $z_3 \in [\partial^+\bar{\phi}(z)] \cap \{t = \bar{t}\}$ such that*
   $$\sigma(z) \in \mathrm{int}_{\mathrm{rel}}\mathrm{conv}\{z_1, z_2, z_3\},$$
3. *the following formula holds:*

$$\mathcal{H}^2(S') \geq \left(\frac{t_2 - \bar{t}}{t_2 - t_1}\right)^2 \mathcal{H}^2(S). \tag{II.4}$$

II.1.3. NEGLIGIBILITY OF NON REGULAR POINTS

Once we have obtained the area estimates (II.3) and (II.4), we apply these formulas to prove the regularity of the partition. In particular we prove that almost every point in $\{t = 1\}$ is regular.

(II.1.16) **REMARK:** *Let us consider $z' \in \partial^-\bar{\phi}(z)$ such that $]\!]z', z[\![$ is contained in an optimal ray in the relative inner part of $\partial^-\bar{\phi}(z)$. The dimension of $\partial^-\bar{\phi}$ is constant along the open segment $]\!]z', z[\![$. Moreover, the respective property holds for $\partial^+\bar{\phi}$.*

(II.1.17) **LEMMA:** *$\mathcal{H}^2$-almost every point of $\{t = 1\}$ is backward regular.*

PROOF. Let $\tau, \varepsilon \in]0, 1[$ and fix a compact subset $K \subseteq \{t = 1 + \varepsilon\}$. Since the property of $\bar{\phi}$ ensure us that for every $w \in K$ there is an optimal ray for $\bar{\phi}$ that reach D_0, there is an inner optimal ray reaching $t = 1 - \tau$ as in Corollary II.1.13, and then we can apply the area estimate of the corollary. It is easy to see that the relative internal part of optimal rays

involved in the area estimate is made of backward regular points.
Therefore the measure of regular point in $\{t = 1\}$ is greater or equal than

$$\frac{\tau}{\varepsilon + \tau}\mathcal{H}^2(K).$$

By the arbitrariness of ε, the measure of backward regular points is arbitrary near to the measure of K. ■

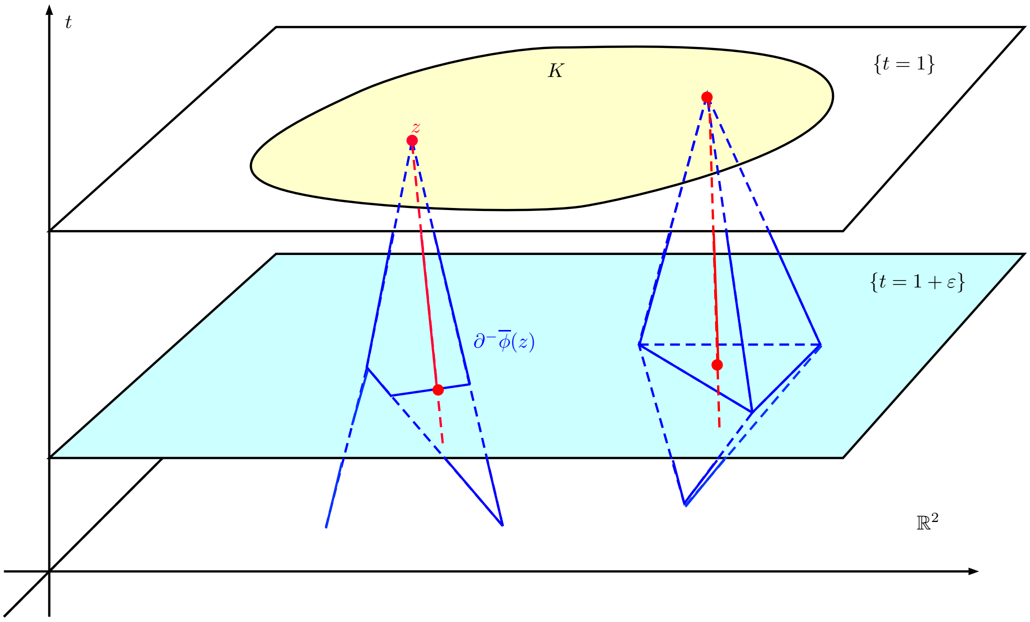


Figure II.2: The red line is the regular ray starting from z and used in the area estimate of Corollary II.1.13.

(II.1.18) LEMMA: *$\mathcal{H}^2$-almost every point of $\{t = 1\}$ is forward regular.*

PROOF. Let $\varepsilon_1, \varepsilon_2 > 0$. Similarly to the previous Lemma we can fix a compact $K \subseteq \{t = 1+\varepsilon_2\}$ and by means of the area estimate we can find a sub set $K' \subseteq \{t = 1-\varepsilon_1\}$ of measure arbitrary close to the measure of K. By the arbitrariness of ε_1 and ε_2 we can assume that fixing a subset $S \in \{t = 1 - \varepsilon_1\}$ up to an arbitrary small part of points every point reach $\{t = 1 + \varepsilon_2\}$. To conclude the proof we want to repeat the same proof as before introducing the dual potential

$$\bar{\phi}^*(z) := \sup_{z' \in \{t=1+\varepsilon_2\}} \{\bar{\phi}(z') - \bar{c}(z - z')\}.$$

It lacks to prove that the optimal rays for $\bar{\phi}^*$ coincide with the ones of $\bar{\phi}$. Indeed, notice that the area estimate ensures us that almost every point on $\{t = 1 - \varepsilon_1\}$ could be connected through an optimal ray for $\bar{\phi}$ to a point on $\{t = 1 + \varepsilon_2\}$ and these optimal rays are optimal also for $\bar{\phi}^*$. Since an analogous of Lemma I.4.10 holds for $\bar{\phi}^*$ and taken two points on the previous optimal rays $\mathcal{Q}^*_{\bar{\phi}}$ coincides with $\mathcal{Q}^*_{\bar{\phi}}$, the thesis follows. ■

(II.1.19) COROLLARY: *$\mathcal{H}^2$-almost every point of $\{t = 1\}$ is regular.*

PROOF. The proof is straightforward consequence of previous lemmas. Notice that a point selected by the area estimate has to have the same forward and backward differential dimension. ■

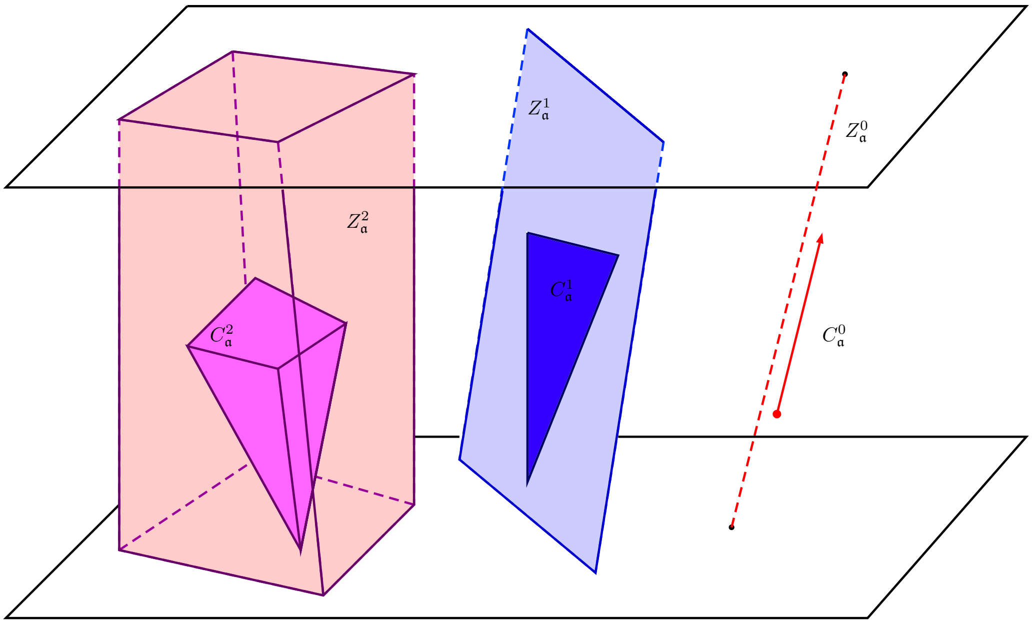


Figure II.3: $Z^h_{\mathfrak{a}}$ and F^h_a

II.2. Analysis on the first directed locally affine partition

The previous section provides a first directed locally affine partition

$$\mathbf{D}_{\bar{\phi}} = \Big\{(k, \mathfrak{a}, z, C) : C = F^h_{\mathfrak{a}}, z \in Z^h_{\mathfrak{a}}\Big\}.$$

In this section we show how to refine the partition and reduce our analysis to irreducible sets. To this purpose we group the sets of the partition according to their dimension and then we study them case by case.

II.2.1. $h = 0$ CASE

First of all we have to consider the case $h = 0$ that correspond to the case where the subset $Z^0_{\mathfrak{a}}$ is a single line which intersection with $\{t = 0\}$ and $\{t = 1\}$ are two single points. Moreover, $F^0_{\mathfrak{a}}$ is a single vector.

In this case is is naturally to consider the map that associate to $Z^0_{\mathfrak{a}} \cap \{t = 1\}$, $Z^0_{\mathfrak{a}} \cap \{t = 1\}$. Notice that the disintegration of $\delta_1 \times \mathcal{H}^2$ on these subsets is made of deltas up to a negligible set and this conclude the case.

II.2.2. $h = 1$ CASE

In the case $h = 0$ the map is found and it is unique, that is not always the case. In higher dimension we choose to consider a secondary cost. This technique depends on the existence of a couple of potentials for the secondary cost and a priori they could not exists (see[13, Example 3.14]). This is the reason why we need a deeper analysis in order to apply Proposition I.2.11 in this context.

The two dimensional case it is not trivial because the map is not evident as in the previous case and the regularity of disintegration has to be proved. For the first problem we study the cyclically connectedness of the sets and for the latter we use the fact that non crossing segments on a plane can not be disposed too bad.

First we show directly how we can reduce our argument to cyclically connected $Z^1_{\mathfrak{a}}$ sets and then we prove the regularity of the disintegration in this case.

Construction of a sheaf

Define $\mathcal{S}$ as the subset of all closed segment contained in $\{t=1\}$. For every $L \in \mathcal{S}$ and $r>0$ set

$$\mathring{L}(-r) := \Big\{ z \in \{t=1\} : \big(B(z,r) \cap \operatorname{aff} L\big) \subseteq L \Big\} \quad \text{and} \quad \mathring{L} = \operatorname{int}_{\mathrm{rel}} L.$$

Let $e_V \in \{1\} \times \mathbb{S}^1$, $r>0$, and $\lambda \in]0,1[$. Consider $V = \operatorname{span}\{e_V\}$ and $L \in \mathcal{S}$ such that $L \subseteq V$ and define

$$\begin{aligned}\mathcal{S}(L,r,\lambda) := \Big\{ L' \in \mathcal{S} :\ & (i)\ L(-r) \subset \mathtt{p}_V \mathring{L}', \\ & (ii)\ \mathtt{p}_V L' \subset \mathring{L}, \\ & (iii)\ |(z-z')\cdot e_V| > \lambda |z-z'| \text{ with } z \neq z' \in L \}.\end{aligned}$$

It is fairly easy to see that for all r, λ the family

$$\mathfrak{S}(L,r,\lambda) := \Big\{ \mathcal{S}(L,r',\lambda'), 0<r'<r, 0<\lambda'<\lambda \Big\}$$

generates a prebase of neighborhoods of $\mathcal{S}$. In particular, we can find a countably many sets $\mathfrak{S}(L_n,r_n,\lambda_n)$, $n \in \mathbb{N}$, covering $\mathcal{S}$ being the latter separable.

Moreover, define

$$\mathbf{D}_{\bar{\phi}}(1,n) := \{(1,\mathfrak{a},z,C) \in \mathbf{D}_{\bar{\phi}} : C \cap \{t=1\} \in \mathfrak{S}(L_n,r_n,\lambda_n) \text{ and } z \in Z^h_a)\}$$

Fix $n \in \mathbb{N}$ and define the map

$$\begin{array}{rccl} \mathtt{r} & : & \mathbf{D}_{\bar{\phi}}(1,n) & \rightarrow & \mathbb{R} \times \big([0,+\infty) \times \mathbb{R}\big) \times \mathcal{C}(1,[0,+\infty) \times \mathbb{R}^h) \\ & & (\mathfrak{a},z,F^1_{\mathfrak{a}}) & \mapsto & \mathtt{r}(\mathfrak{a},z,C^k_{\mathfrak{a}}) := \big(\mathfrak{a}, \mathtt{p}_{\operatorname{aff} F^1_n} z, \mathtt{p}_{\operatorname{aff} F^1_n} F^1_{\mathfrak{a}}\big) \end{array}$$

Being the projection of a σ-compact set, $\mathtt{r}$ is σ-continuous. Clearly, since z determines $\mathfrak{a}$ and $\mathfrak{a}$ determines $F^1_{\mathfrak{a}}$, also the maps

$$\begin{array}{rccl} \tilde{\mathtt{r}} & : & \mathbf{Z}^1_n & \rightarrow & \mathbb{R} \times \big([0,+\infty) \times \mathbb{R}\big) \times \mathcal{C}(1,[0,+\infty) \times \mathbb{R}) \\ & & z & \mapsto & \tilde{\mathtt{r}}(z) := \big(\mathfrak{a}(z), \mathtt{p}_{\operatorname{aff} F^1_n} z, \mathtt{p}_{\operatorname{aff} F^1_n} F^1_{\mathfrak{a}(z)}\big) \end{array}$$

$$\begin{array}{rccl} \hat{\mathtt{r}} & : & \mathfrak{A}^k_n & \rightarrow & \mathbb{R} \times \mathcal{C}(1,[0,+\infty) \times \mathbb{R}) \\ & & \mathfrak{a} & \mapsto & \hat{\mathtt{r}}(z) := \big(\mathfrak{a}, \mathtt{p}_{\operatorname{aff} F^1_n} C^1_{\mathfrak{a}}\big) \end{array}$$

are σ-continuous. We will use the notation $w \in [0,+\infty) \times \mathbb{R}$, $\tilde{Z}^1_{\mathfrak{a}} := (\mathtt{i}_1 \circ \mathtt{p}_{\operatorname{aff} C^1_n}) Z^1_{\mathfrak{a}}$ and $\tilde{F}^1_{\mathfrak{a}} := (\mathtt{i}_1 \circ \mathtt{p}_{\operatorname{aff} F^1_n}) F^1_{\mathfrak{a}}$, where $\mathtt{i}_1 : V^k_n \to \mathbb{R}^k$ is the identification map. Moreover set $\tilde{\mathbf{Z}}^1_n := \cup_{\mathfrak{a}} \big\{\{\mathfrak{a}\} \times \tilde{Z}^1_{\mathfrak{a}}\big\}$.

In particular we have that $\mathbf{r}(\mathbf{D}(k,n))$ is a k-dimensional directed fibration. Notice that $\tilde{C}$ is bounded, because the $C^h_{\mathfrak{a}}$ are.

The fact that we are considering transference problems in $\Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$ allows to rewrite them in the coordinates $(\mathfrak{a}, w) \in \mathbb{R}^{d-k} \times \big([0,+\infty) \times \mathbb{R}^h\big)$. Indeed, consider the multifunction $\check{\mathtt{r}}$ whose inverse is the map

$$\begin{array}{rccl} \check{\mathtt{r}}^{-1} & : & \mathfrak{A}^h_n \times \big([0,+\infty) \times \mathbb{R}^h\big) & \rightarrow & [0,\infty) \times \mathbb{R}^d \\ & & (\mathfrak{a},w) & \mapsto & \check{\mathtt{r}}^{-1}(\mathfrak{a},w) := \operatorname{aff} Z^h_{\mathfrak{a}} \cap (\mathtt{i}_k \circ \mathtt{p}_{\operatorname{aff} C^k_n})^{-1}(w) \end{array}$$

and define the transport cost

$$\tilde{\mathtt{c}}_n^k\big(\mathfrak{a}, w, \mathfrak{a}', w'\big) := \begin{cases} 0 & \mathfrak{a} = \mathfrak{a}', w - w' \in \tilde{C}_{\mathfrak{a}}^k, \\ \infty & \text{otherwise.} \end{cases}$$

It is clear that

$$\mathtt{c}\big(\check{\mathtt{r}}^{-1}(\mathfrak{a}, w), \check{\mathtt{r}}^{-1}(\mathfrak{a}', w')\big) = \tilde{\mathtt{c}}_n^h\big(\mathfrak{a}, w, \mathfrak{a}, w'\big).$$

In order to precede our analysis in the case $h = 1$, we fix $\mathfrak{a} \in \mathfrak{A}$ and consider a couple $(F_{\mathfrak{a}}^1, Z_{\mathfrak{a}}^1) \in D_{\bar{\phi}}$. In the following we prove that we can decompose $Z_{\mathfrak{a}}^1$ in the union of subsets Γ-cyclically connected and subsets belonging to the previous case $h = 0$.

(II.2.1) LEMMA: *Let (w_1, w'_1) and (w_2, w'_2) be two couples in Γ made of points of $Z_{\mathfrak{a}}^1$.*

If $]\!]w_1, w'_1[\![\cap]\!]w_2, w'_2[\![\neq \emptyset$, then w_1 and w_2 are Γ-cyclically connected.

PROOF. It is an easy consequence of the definition of Γ-cyclically connected set. ■

(II.2.2) COROLLARY: *If $\mathcal{H}^1$−a.e. point of $\mathtt{p}_1\Gamma \cap Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ have just one optimal ray, they are not Γ-cyclically connected.*

PROOF. Consider points with more than one optimal rays. These points can be partitionate according to the direction of these two rays, without loss of generality we can assume that for every $\varepsilon > 0$ there are two point $d_1, d_2 \in \{t = 1\}$ such that for almost every $w \in Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ there are $w_1, w_2 \in \Gamma(w)$ such that

$$\max\{|w - w_1 - d_1|, |w - w_1 - d_1|\} < \varepsilon.$$

Moreover, we can assume $|d_1 - d_2| > 3\rho$.

By Lemma II.2.1, it is easy to see that two different points in $Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ with more than one ray can not be closer than ρ without being Γ-cyclically connected. ■

(II.2.3) LEMMA: *Assume that $\mathcal{H}^1$−a.e. point in $\mathtt{p}_1\Gamma \cap Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ has an relative inner optimal ray. Then the Lebesgue points of $Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ are Γ-cyclically connected.*

PROOF. Let us suppose $w \in Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ and $w' \partial^- \bar{\phi}(w) \cap \mathtt{p}_2\Gamma$ such that $]\!]w, w'[\![$ is a relative internal optimal ray. We claim there is $\rho_w > 0$ such that w is Γ-cyclically connected with every point in $B^2(w, \rho_w)$. By the σ-continuity of Γ there is $\rho > 0$ such that for almost every $u \in B^2(w, \rho)$, $\Gamma(u) \cap (w - F_{\mathfrak{a}}^1) \neq \emptyset$. On the other hand it is not restrictive to assume $\big(B^2(w, \rho) \cap \{t = 1\}\big) \subset (w' + \mathrm{int}_{\mathrm{rel}} C_{\mathfrak{a}}^1)$ and therefore $w' \in u - F_{\mathfrak{a}}^1$.

By the compactness of $Z_{\mathfrak{a}}^1 \cap \{t = 1\}$ we obtain the thesis of the Lemma. ■

It remains to prove that conditional probabilities are absolutely continuous:

(II.2.4) LEMMA: *Let $\big\{[\![a_\alpha, b_\alpha]\!] : \alpha \in A\big\}$ be a family of segments in $\mathbb{R}^2$ and $d \in \mathcal{S}^1$ be a direction such that:*

1. $(b_\alpha - a_\alpha) \cdot d \le \frac{1}{2}|b_\alpha - a_\alpha|$, *for every* $\alpha \in A$.

2. *there is* $h > 0$ *such that* $a_\alpha \cdot d \le -h$ *and* $b_\alpha \cdot d \ge h$, *for every* $\alpha \in A$.

3. $[\![a_\alpha, b_\alpha]\!] \cap]\!]a_\beta, b_\beta[\![= \emptyset$, *for every distinct* $\alpha, \beta \in A$

Define $S := \{x \in \mathbb{R}^2 : |x \cdot d| \le h\}$ *and* $\delta(x) := \frac{b_\alpha - x}{|b_\alpha - x|}$ *for* $\alpha \in A$ *such that* $x \in]\!]a_\alpha, b_\alpha[\![$. *Then,* δ *is locally Lipschitz on* S.

PROOF. Let $\varepsilon > 0$ and consider $S_\varepsilon := \{x \in \mathbb{R}^2 : |x \cdot d| \le h - \varepsilon\}$.

For every $x_1, x_2 \in \mathbb{R}^2$ there are $y_1, y_2, \in \mathbb{R}^2$ such that

$$x_i + \delta(x_i) = y_i, \qquad \text{for } i = 1, 2.$$

$$|\delta(x_1) - \delta(x_2)| \le |x_1 - x_2| + |y_1 - y_2|$$

Since two segment can not cross in S_ε it means that there is a constant k depending only on h and ε such that:

$$|y_1 - y_2| \le k|x_1 - x_2|.$$

This means that in S_ε, δ is Lipschitz. ∎

Previous Lemma let us to apply Theorem 9.4 of [1] and prove the following Proposition:

(II.2.5) PROPOSITION: *Let* U *be a relative open subset of* $\{t = 1\}$ *and* $\{Z^1_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathfrak{a}}$ *a family of segments in* $\{t = 1\} \cap U$ *such that for* $\mathcal{H}^2$*-almost every* $z \in U$ *there is* $\mathfrak{a}$ *such that* $z \in Z^1_{\mathfrak{a}}$ *and*

$$\text{for every } \mathfrak{a}, \mathfrak{a}' \in \mathfrak{a}, \quad \mathrm{int}_{\mathrm{rel}} Z^1_{\mathfrak{a}} \cap Z^1_{\mathfrak{a}'} \neq \emptyset \implies \mathfrak{a} = \mathfrak{a}'.$$

Then, $\mathcal{H}^2_{\llcorner U} = \int_{\mathfrak{a}} \int_{Z^1_{\mathfrak{a}}} \eta_{\mathfrak{a}} \xi(\mathrm{d}\mathfrak{a})$ *and for every* $\mathfrak{a} \in \mathfrak{a}$, $\eta_{\mathfrak{a}} \ll \mathcal{H}^1$.

II.2.3. $h = 2$ CASE

Now we deal with the case $h = 2$. In this case the conclusion it is not trivial as in the case $h = 0$ and it cannot be done directly as in the case $h = 1$. Our strategy is to reduce again our argument the Proposition I.2.11.

Therefore, the key point of this argument will be the analysis of the cyclical monotone relation generated by transference plans with finite cost. For this purpose and in order to simplify the notation we will fix a three dimensional subspace of $\mathbf{D}_{\bar{\phi}}$, Z. Since the problem admits only a countably infinite number of three dimensional subsets, it is not restrictive.

II.3. Reduction of $Z^2_{\mathtt{a}}$

In this section we apply the theory developed in Section I.6 to refine the sets $Z^2_{\mathtt{a}}$ in the union of irreducible subsets and lower dimension subsets. For the sake of completeness we repeat the statement without proofs.

By the regularity of $\mathbf{D}_{\bar{\phi}}$ we have proved, we can fix an arbitrary point z in the internal part of Z and define $C := \mathbb{R}^+ \cdot \mathcal{D}^+\bar{\phi}(z)$.

Define the transference cost $\tilde{\mathtt{c}}$ as

$$\tilde{\mathtt{c}}(z, z') := \begin{cases} 0 & \text{if } z - z' \in C, \\ \infty & \text{otherwise.} \end{cases}$$

From the straightforward geometric property of a convex cone F

$$z \in C \quad \Rightarrow \quad z + C \subset C,$$

one deduces that

$$\tilde{\mathtt{c}}(z, z'), \tilde{\mathtt{c}}(z', z'') < \infty \quad \Longrightarrow \quad \tilde{\mathtt{c}}(z, z'') < \infty.$$

Consider two probability measures μ and ν in $\mathbb{R}^2$ and their relative embedding $\bar{\mu} = \delta_1 \otimes \mu$ and $\bar{\nu} = \delta_0 \otimes \nu$ in D.

We denote by $\mathit{\Gamma}(\bar{\pi})$ be the family of σ-compact carriages Γ of $\bar{\pi} \in \Pi^f(\bar{\mu}, \bar{\nu})$,

$$\mathit{\Gamma}(\bar{\pi}) := \Big\{ \Gamma \subset \{\bar{c} < \infty\} \cap \big(\{t = 1\} \times \{t = 0\}\big) : \bar{\pi}(\Gamma) = 1 \Big\},$$

and set

$$\mathit{\Gamma} := \bigcup_{\bar{\pi} \in \Pi^f(\bar{\mu}, \bar{\nu})} \mathit{\Gamma}(\bar{\pi}).$$

II.3.1. A linear preorder on Z

Let $\Gamma \in \bar{\mathit{\Gamma}}$. The following lemma is taken from [8, Lemma 7.3]: we omit the proof because it is just an easier version.

(II.3.1) LEMMA: *There exists a sequence $\{\mathtt{w}_n\}_{n\in\mathbb{N}}$ in Z such that*

$$\forall n \in \mathbb{N}, \, \mathtt{w}_n \in \mathtt{p}_1\Gamma \quad \textit{and} \quad \operatorname{clos}\{\mathtt{w}_n\}_{n\in\mathbb{N}} \supset \mathtt{p}_1\Gamma.$$

Where $\mathtt{p}_1\bar{\Gamma}$ is the projection of Γ on $\{t = 1\}$.

(II.3.2) DEFINITION:

$$H_n := \left\{ w \in Z : \text{there are } N \in \mathbb{N} \text{ and } \big\{(w_i, w'_i)\big\}_{i=0}^n \subseteq \Gamma \text{ such that } \begin{array}{cc} (a) & w_0 = \mathtt{w}_n \\ (b) & w_{i+1} \in w'_i + F \\ (c) & w \in w'_N + F \end{array} \right\}$$

This set represents the points which can be reached from $\mathtt{w}_n$ by means of axial path of finite costs.

(II.3.3) PROPOSITION: *The set H_n is σ-compact in $[0,+\infty)\times\mathbb{R}^2$, and moreover, there exists a Borel function $\mathtt{h}_n:\mathbb{R}^2\to[0,+\infty)$ such that*

$$\forall x,x'\in\mathbb{R}^2\ \big(\mathtt{h}_n(x')\le\mathtt{h}_n(x)+\mathtt{co}(x'-x)\big)$$

$$\big\{(t,x):t>\mathtt{h}_n(x)\big\}\subset H_n\subset\big\{(t,x):t\ge\mathtt{h}_n(x)\big\}.$$

The above statement is the analog of Proposition 7.4 of [8], and we omit the proof.

The function $\mathtt{h}_n$ is given explicitly by

$$\mathtt{h}_n(x)=\inf\big\{\mathtt{co}_F(x-x'),x'\in\mathtt{i}(H_n\cap\{t=0\})\big\}=\min\Big\{\mathtt{co}_F(x-x'),x'\in\mathrm{clos}\,\big(\mathtt{i}(H_n\cap\{t=0\})\big)\Big\}.$$

The separability of $\mathbb{R}^2$ and the non degeneracy of the cone F yields the next lemma.

(II.3.4) LEMMA: *There exists countably many cones $\{w_i'+F\}_{i\in\mathbb{N}}$, $\{w_i'\}_{i\in\mathbb{N}}\subset\mathtt{p}_2\Gamma\cap H_n$, such that*

$$\mathrm{int}H_n=\bigcup_{i\in\mathbb{N}}w_i'+\mathrm{int}F,$$

and the set $\partial H_n\cap\{t=\bar t\}$ is $(k-1)$-rectifiable for all $\bar t>0$.

Construction of the linear preorder $\preccurlyeq_{\mathtt{W}}$

Denote with $\mathtt{W}=\{\mathtt{w}_n\}_{n\in\mathbb{N}}$ the sequence constructed in the previous section.

Define first the function

$$\begin{array}{rccc}\theta'_{\mathtt{W},\Gamma} & : & D & \to & [0,1]\\ & & w & \mapsto & \theta'_{\mathtt{W},\Gamma}(w):=\max\Big\{0,\sum_n 2\cdot 3^{-(n+1)}\chi_{H_n}(w)\Big\}\end{array}$$

It is fairly easy to show that $\theta'_{\mathtt{W},\Gamma}$ is Borel. The dependence on Γ occurs because the set $\mathtt{W}$ is chosen once Γ has been selected.

Since we are interested only in the values of the functions on $\mathtt{p}_1\Gamma$, and the measure μ is a.c., then once the function $\theta'_{\mathtt{W},\Gamma}$ has been computed we define a new function $\theta_{\mathtt{W},\Gamma}$ by

$$\big\{w:\theta_{\mathtt{W},\Gamma}(\mathfrak{a},w)\ge\tau\big\}=\bigcup_{\substack{w'\in\mathtt{p}_2\Gamma\\ \theta'_{\mathtt{W},\Gamma}(w')\ge\tau}}w'+F,\qquad\tau\in[0,1].$$

Since $\mathtt{p}_2\Gamma$ is σ-compact, it is standard to prove that $\theta_{\mathtt{W},\Gamma}$ is Borel if $\theta'_{\mathtt{W},\Gamma}$ is.

(II.3.5) LEMMA: *The functions $\theta'_{\mathtt{W},\Gamma}$, $\theta_{\mathtt{W},\Gamma}$ are locally SBV on every section $\{t=\bar t\}$, and*

$$(w,w')\in\Gamma\quad\Longrightarrow\quad\theta'_{\mathtt{W},\Gamma}(w)=\theta'_{\mathtt{W},\Gamma}(w')=\theta_{\mathtt{W},\Gamma}(w)=\theta_{\mathtt{W},\Gamma}(w').$$

Hence the function $\theta_{\mathtt{W},\Gamma}$ has the same values of $\theta'_{\mathtt{W},\Gamma}$ on $\mathtt{p}_1\Gamma \cup \mathtt{p}_2\Gamma$.
A completely similar argument shows that $\theta'_{\mathtt{W},\Gamma}$, $\theta_{\mathtt{W},\Gamma}$ are SBV also in $[0,+\infty) \times \mathbb{R}^2$.
Define the function $\vartheta_{\mathtt{W},\Gamma}$ as the upper semicontinuous envelope of $\theta_{\mathtt{W},\Gamma}$.

(II.3.6) LEMMA: *For all $\bar t > 0$ it holds*

$$\mathcal{H}^2\big(\{x : \vartheta_{\mathtt{W},\Gamma}(\bar t, x) > \theta_{\mathtt{W},\Gamma}(\bar t, x)\}\big) = 0.$$

An analogous computation shows that θ coincides with its l.s.c.-envelope up to a $\mathcal{H}^2$-negligible set in each set $\{t = \bar t\}$, $\bar t > 0$.

(II.3.7) COROLLARY: *For every $\bar t > 0$ and $\tau \in [0,1]$ the set*

$$\partial\big\{\theta_{\mathtt{W},\Gamma} > \tau\big\} \cap \{t = \bar t\} \qquad \Big(\partial\big\{\theta_{\mathtt{W},\Gamma} \geq \tau\big\} \cap \{t = \bar t\}\Big)$$

is rectifiable, with total variation uniformly locally bounded by $r^2/\bar t + r$ in each ball of radius r.

In particular we deduce that for the level sets of $\theta_{\mathtt{W},\Gamma}$ of positive $\mathcal{H}^2$-measure satisfies

$$\partial\big[\{\theta_{\mathtt{W},\Gamma} = \tau\big\} \cap \{t = \bar t\} \cap B(x,r)\big] \approx \frac{r^2}{\bar t} + r.$$

(II.3.8) REMARK: *We observe here the relation with the Lax formula for Hamilton-Jacoby equation (with inverted time). In fact, if we define the Lagrangian*

$$L(w) = \mathbb{1}_F(w),$$

then formula (I.48) *can be rewritten as*

$$\theta_{\mathtt{W},\Gamma}(w) = \sup\Big\{\theta'_{\mathtt{W},\Gamma}(w') - L(w - w'), w' \in \{t = 0\}\Big\}.$$

Moreover, from the definition of $\vartheta_{\mathtt{W},\Gamma}$ yields that

$$\vartheta_{\mathtt{W},\Gamma}(w) = \max\Big\{\vartheta_{\mathtt{W},\Gamma}(w') - L(w - w'), w' \in \{t = 0\}\Big\}.$$

It thus follows that $\vartheta_{\mathtt{W},\Gamma}$ in some sense replaces the potentials ϕ, ψ for a transport problem. The advantages of using $\vartheta_{\mathtt{W},\Gamma}$ instead of $\theta_{\mathtt{W},\Gamma}$ will be clear in the following sections.

We notice here only that the disintegration of the Lebesgue measure $\mathcal{L}^3 \llcorner_{\{t=1\}}$ w.r.t. the equivalence classes of θ or of ϑ are equivalent, because the two functions differ on a negligible set.

The pull-back of $\leq$ by $\theta_{\mathtt{W},\Gamma}$ is the linear preorder $\preccurlyeq_{\mathtt{W},\Gamma}$ defined by

$$\preccurlyeq_{\mathtt{W},\Gamma} := \big(\theta_{\mathtt{W},\Gamma} \otimes \theta_{\mathtt{W},\Gamma}\big)^{-1}(\leq^{-1}),$$

and the equivalence relation on $[0,+\infty) \times \mathbb{R}^2$

$$E_{\mathtt{W},\Gamma} := \preccurlyeq_{\mathtt{W},\Gamma} \cap \preccurlyeq_{\mathtt{W},\Gamma}^{-1} = \Big\{(w,w') : \theta_{\mathtt{W},\Gamma}(w) = \theta_{\mathtt{W},\Gamma}(w')\Big\}.$$

(II.3.9) LEMMA: *Assume that $w, w'' \in \mathtt{p}_1\Gamma$ can be connected by a closed axial path of finite cost. Then, $w \sim_{E_{\mathtt{W},\Gamma}} w''$.*

A consequence of Lemma II.3.5 is thus that $\Gamma \subset E_{\mathtt{W},\Gamma}$. If Γ' is another carriage contained in $\{\tilde{\mathtt{c}} < \infty\}$, then

$$(w, w') \in \Gamma' \quad \Longrightarrow \quad w \preccurlyeq_{\mathtt{W},\Gamma} w',$$

because

$$\theta'_{\mathtt{W},\Gamma}(w' + F) \geq \theta'_{\mathtt{W},\Gamma}(w')$$

by construction. In particular from Theorem (A.2.2) we deduce the following Proposition.

(II.3.10) PROPOSITION: *If $\bar{\pi} \in \Pi^f(\bar{\mu}, \bar{\nu})$, then $\bar{\pi}$ is concentrated on $E_{\mathtt{W},\Gamma}$.*

Construction of a σ-closed family of equivalence relations

The linear preorder $\preccurlyeq_{\mathtt{W},\Gamma}$ depends on the set $\mathtt{W}$: by changing the c_F-cyclically monotone carriage $\Gamma \in \mathit{\Gamma}$ and the family $\mathtt{W}$ dense in Γ, we obtain in general different preorders.

We can easily compose linear preorders $\preccurlyeq_{\mathtt{W}_\beta,\Gamma_\beta}$, $\beta < \alpha$ countable ordinal number, by using the lexicographic preorder on $[0,1]^\alpha$: in fact, define the function

$$\begin{array}{rccl} \theta_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}} & : & D & \to & [0,1]^\alpha \\ & & w & \mapsto & \theta_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}(w) := \{\mathtt{p}_s\theta_{\mathtt{W}_\beta,\Gamma_\beta}(w)\}_{\beta<\alpha} \end{array}$$

As in the previous section $\theta_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}$ is Borel.

If $\trianglelefteq$ is the lexicographic preorder in $[0,1]^\alpha$, then set

$$\preccurlyeq_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}} := \big(\theta_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}} \otimes \theta_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}\big)^{-1}(\trianglelefteq), \quad E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}} := \preccurlyeq_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}} \cap \preccurlyeq^{-1}_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}.$$

Clearly $\bar{\pi}(E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}) = 1$, since $\bar{\pi}(E_{\mathtt{W}_\beta,\Gamma_\beta}) = 1$ for all $\beta < \alpha$. To be an equivalence relation on D, we can assume that $\mathbb{I} \subset E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}$.

The next lemma is a simple consequence of the fact that a countable union of countable sets is countable. It proof can be found in [8], Proposition 7.5.

(II.3.11) LEMMA: *The family of equivalence relations*

$$\mathcal{E} := \Big\{E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}, \mathtt{W}_\beta = \big\{\mathtt{w}_{n,\beta}\big\}_{n\in\mathbb{N}}, \alpha \in \Omega\Big\}$$

is closed under countable intersection. Moreover, for ever $E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}$ there exists $\bar{\Gamma} \in \mathit{\Gamma}$ and $\bar{\mathtt{W}}$ such that

$$E_{\bar{\mathtt{W}},\bar{\Gamma}} \subset E_{\{\mathtt{W}_\beta,\Gamma_\beta\}_{\beta<\alpha}}.$$

Let $\bar{E}_{\{\mathtt{W}_\beta,\tilde{\Gamma}_\beta\}_{\beta<\alpha}}$ be the minimal equivalence relation chosen as in Lemma II.3.11 after a minimal equivalence relation of Theorem A.3.1 in Appendix A.3 has been selected. Let $\bar{\theta}' : \mathbb{R}^{2-h} \times ([0,+\infty) \times \mathbb{R}^h) \to \mathbb{R}^{2-h} \times [0,1]$ be the function obtained through (I.47) with the set $\bar{\tilde{\Gamma}}$ and the family of functions $\bar{\mathtt{W}}$, and let $\bar{\theta}$ be the function given by (I.48).

Let $\Gamma \in \mathit{\Gamma}$ be a σ-compact cyclically monotone set, and let $\bar{\theta}_{\mathtt{W},\Gamma} : [0,\infty) \times \mathbb{R}^2 \to [0,1]^{\mathbb{N}}$ be constructed as before.

By Corollary A.3.2, it follows that there exists a $\bar{\mu}$-conegligible σ-compact set $B \subset D$ and a Borel function $\mathtt{s} : [0,1] \to [0,1]$ such that $\bar{\theta}_{\mathtt{W}} = \mathtt{s} \circ \bar{\theta}$ on B. The set B depends on $\bar{\theta}_{\mathtt{W},\Gamma}$.

Applying this result to the equivalence classes of positive $\bar{\mu}$-measure, we obtain the following proposition.

(II.3.12) PROPOSITION: *For all $\tau \in [0,1]$, it holds:*

$$\bar{\mu}\big((\bar{\theta}')^{-1}(\tau)\big) > 0 \ \Rightarrow\ \exists \tau' \in [0,1], \quad \bar{\mu}\big((\bar{\theta}')^{-1}(\tau) \setminus \bar{\theta}^{-1}_{\mathtt{W},\Gamma}(\tau')\big) = 0.$$

Notice that by Corollary II.3.7 it follows that the set

$$\mathrm{int}\left(\bar{\theta}^{-1}(\tau) \cap \bar{\theta}^{-1}_{\mathtt{W},\Gamma}(\tau')\right)$$

has topological boundary rectifiable.

II.3.3. AREA ESTIMATE

We have proved that the sub/super differential of ϑ enjoys of the same property of the one of ϕ. In particular every point Since ϑ has the same property of ϕ, we can repeat the same analysis made in subsection II.1.2.

Let $0 < t_1 < t_2$ be two different real numbers and S a subset of $\{t = t_2\}$ such that $\mathcal{H}^2(S) < +\infty$. Let us assume that for every $w \in S$ there exists an optimal ray that reaches t_1.

(II.3.13) DEFINITION: Take $t_1 < \bar{t} < t_2$ a real number. With the previous notation we define the following sets:

$$\Sigma^-_{\vartheta,0}(S,t_1,t_2,\bar{t}) := \{w \in S : \partial^-\vartheta(w) \cap \{t = \bar{t}\} \text{ does not contain any segment }\},$$

$$\Sigma^-_{\vartheta,1}(S,t_1,t_2,\bar{t}) := \{w \in S : \partial^-\vartheta(w) \cap \{t = \bar{t}\} \text{ contains a segment but any two dimensional convex }\},$$

$$\Sigma^-_{\vartheta,2}(S,t_1,t_2,\bar{t}) := \{w \in S : \partial^-\vartheta(w) \cap \{t = \bar{t}\} \text{ contains at least a two dimensional convex }\},$$

(II.3.14) DEFINITION: Let $w \in [0,+\infty) \times \mathbb{R}^2$ and $w' \in \partial^-\vartheta(w) \setminus \{w\}$. We can define the *time-fixed normal* to $w - F$

$$N_{w,F}(w') := \Big\{n \in \{t = t(w')\} : n \cdot \big(w' - e\big) \le 0 \text{ for every } e \in (w - F) \cap \{t = t(w')\}\Big\}.$$

(II.3.15) LEMMA: *Fix $w \in [0,+\infty) \times \mathbb{R}^2$. Let assume $\bar{t} \in]0, t(w)[$, $h > 0$, and assume there are two points w_1 and w_2 in $\partial^- \vartheta(w) \cap (w - \partial F) \cap \{t = \bar{t}\}$ such that*

$$\inf\{|n_1 - n_2| : n_1 \in N_{w,F}(w_1) \text{ and } n_2 \in N_{w,F}(w_2)\ \} > h.$$

Then, there is $\delta > 0$, $h > 0$, and a cone $C \subseteq \{t = t(w)\}$ such that if

1. $w' \in B^2(w, \delta) \cap (w + \mathrm{int}_{\mathrm{rel}} C)$,
2. *there exists* $w'_2 \in \partial^- \vartheta(w') \cap (w - F) \cap \{t = \bar{t}\}$,
3. $\sup\{|n - n'| : n \in N_{w,F}(w_2)$ *and* $n' \in N_{w',F}(w'_2)\} < \frac{h}{3}$,

then $w \in \Sigma^-_{\vartheta,2}(w, \bar{t}, t(w), s)$ for every $s \in]\bar{t}, t(w)[$.

PROOF. Let us assume there is a point $\{w'_2\}$ as in the hypothesis. It is easy to prove that there are $\bar{n}_1 \in N_{w,F}(w_1)$ and $\bar{n}_2 \in N_{w,F}(w_2)$ such that

$$|\bar{n}_1 - \bar{n}_2| = \inf\{|n_1 - n_2| : n_1 \in N_{w,F}(w_1) \text{ and } n_2 \in N_{w,F}(w_2)\ \}.$$

Without loss of generality in the following we can assume that

$$|\bar{n}_1 - \bar{n}_2| \leq \inf\{|n_1 - n_2| : n_1 \in N_{w,F}(w_1) \text{ and } n_2 \in N_{w,F}(w'_2)\ \}.$$

Since (II.3.15) there is $\tilde{w} \in (w - \partial C) \cap \{t = \tau\}$ and $\tilde{n} \in N_{w,F}(\tilde{w})$ such that

$$\mathrm{dist}\Big(\tilde{w}, [\![w_1, w_2]\!]\Big) > 0 \quad \text{and} \quad \min\{|\tilde{n} - n_1|, |\tilde{n} - n_2|\} > 0.$$

Consider C as the cone of $\{t = 0\}$ generated by the convex combination of the projections on $\{t = 0\}$ of $\tilde{w} - w_2$ and $w_1 - w_2$. ■

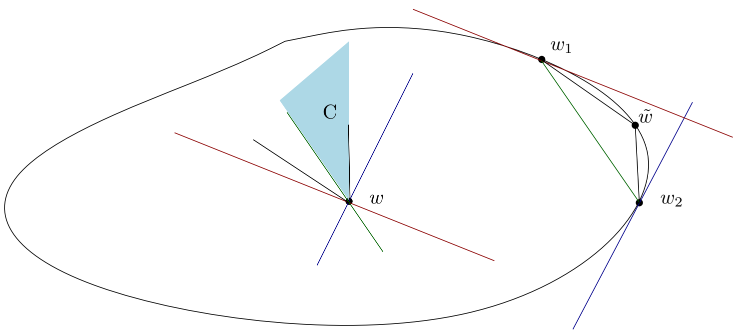


Figure II.4: Proposition II.3.15

(II.3.16) COROLLARY: *The subset of $\Sigma^-_{\vartheta,0}(S, t_1, t_2, \bar{t}) \cup \Sigma^-_{\vartheta,1}(S, t_1, t_2, \bar{t})$ made of w such that*

there are $w_1, w_1 \in \partial^- \vartheta(w) \cap \{t = t_1\}$ *and* $\mathrm{dist}_H(N_{C,w}(w_1), \mathrm{dist}_H(N_{C,w}(w_2))) > 0$,

is $\mathcal{H}^1$-rectifiable.

PROOF. Similarly to Proposition II.1.5 we would like to apply Theorem II.1.3. First of all we need to partitionate the space according to the distance between the normal in w_1 and the normal in w_2. Precisely, we can assume that the reference distances w_1 and w_2 are greater than h and every optimal ray has direction belonging to a ball centred in $w_1 - w$ or $w_2 - w$ and with time-fixed normal near to $N_{w,F}(w_1)$ or $N_{w,F}(w_2)$ less than $\frac{h}{3}$ in the mean of Lemma II.3.15.
Once we have partitionate the space we can apply Lemma II.3.15 and obtain the thesis. ■

(II.3.17) COROLLARY: *For any $\bar{t} \in]t_1, t_2[$ there exists a set $S' \subseteq \{t = \bar{t}\}$ such that:*

1. *there exists a surjective function σ from S to S',*
2. (a) *for every $w \in \Sigma^-_{\vartheta,0}(S, t_1, t_2, \bar{t})$, $\sigma(w) \in \partial^-\vartheta(w)$,*
   (b) *for every $w \in \Sigma^-_{\vartheta,1}(S, t_1, t_2, \bar{t})$ there are two different point $w_1, w_2 \in \partial^-\vartheta(w) \cap \{t = \bar{t}\}$ such that*
   $$\sigma(w) \in \mathrm{int}_{\mathrm{rel}}[\![w_1, w_2]\!],$$
   (c) *for every $w \in \Sigma^-_{\vartheta,2}(S, t_1, t_2, \bar{t})$ there are three different and non align point $w_1, w_2,$ and $u_3 \in \partial^-\vartheta(w) \cap \{t = \bar{t}\}$ such that*
   $$\sigma(w) \in \mathrm{int}_{\mathrm{rel}}\mathrm{conv}\{w_1, w_2, u_3\},$$
3. *the following formula holds:*
   $$\mathcal{H}^2(S') \geq \left(\frac{\bar{t} - t_1}{t_2 - t_1}\right)^2 \mathcal{H}^2(S).$$

PROOF. The proof is the same of Corollary II.1.13 of the Subsection II.1.2 except for the sequence used to prove the area estimate.

Since ϕ is Lipscitz, a dense sequence is used to approximate the potential and find the formula. In this setting this is not possible. Therefore, to overcome this difficulty we apply a very similar analysis applied in [8].

We can find a cone approximation considering the map T that associate to every point in S the point in S' founded by the scheme of the Corollary II.1.13. In this way, the map T is well defined and we can push the mass the measure $\mathcal{H}^2_{\llcorner S}$ to $T_\sharp\mathcal{H}^2_{\llcorner S}$ supported in $\{t = 0\}$. As in [8] we can find a cone approximation and prove the estimate. ■

When S is a sub set of $\{t = t_1\}$ and for every $w \in S$ there exists an optimal ray that reaches t_2 we can state the following definition and prove very similar results:

(II.3.18) DEFINITION: Take $t_1 < \bar{t} < t_2$ a real number. With the previous notation we define the following sets:

$$\Sigma^+_{\vartheta,0}(S, t_1, t_2, \bar{t}) := \{w \in S : \partial^+\vartheta(w) \cap \{t = \bar{t}\} \text{ does not contain any segment }\},$$

$\Sigma^+_{\vartheta,1}(S,t_1,t_2,\bar{t}) := \{w \in S : \partial^+\vartheta(w) \cap \{t=\bar{t}\}$ contains a segment but any two dimensional convex $\}$,

$\Sigma^+_{\vartheta,2}(S,t_1,t_2,\bar{t}) := \{w \in S : \partial^+\vartheta(w) \cap \{t=\bar{t}\}$ contains at least a two dimensional convex $\}$,

(II.3.19) PROPOSITION: *For any $\bar{t} \in]t_1,t_2[$ there exists a set $S' \subseteq D_{\bar{t}}$ such that:*

1. *there exists a surjective function σ from S to S',*
2. (a) *for every $w \in \Sigma^+_{\vartheta,0}(S,t_1,t_2,\bar{t})$, $\sigma(w) \in \partial^+\vartheta(w)$,*

   (b) *for every $w \in \Sigma^+_{\vartheta,1}(S,t_1,t_2,\bar{t})$ there are two different point $w_1, w_2 \in [\partial^+\vartheta(w)]_{\bar{t}}$ such that*
   $$\sigma(w) \in \mathrm{int_{rel}}[\![w_1,w_2]\!],$$

   (c) *for every $w \in \Sigma^+_{\vartheta,3}(S,t_1,t_2,\bar{t})$ there are three different and non align point $w_1, w_2,$ and $u_3 \in [\partial^+\vartheta(w)]_{\bar{t}}$ such that*
   $$\sigma(w) \in \mathrm{int_{rel}}\mathrm{conv}\{w_1,w_2,u_3\},$$
3. *the following formula holds:*
   $$\mathcal{H}^2(S') \geq \left(\frac{t_2-\bar{t}}{t_2-t_1}\right)^2 \mathcal{H}^2(S).$$

II.3.4. NEGLIGIBILITY OF NON REGULAR POINTS

(II.3.20) LEMMA: *If $\bar{\vartheta}(t,x) = \bar{\theta}(t,x)$, then $\partial^-\bar{\theta}(t,x) \subset \partial^-\bar{\vartheta}(t,x)$ for $t > 0$.*

PROOF. Let $(s,y) \in \partial^-\bar{\theta}(t,x)$. The inclusion $\partial^-\bar{\theta}(t,x) \subset \partial^-\bar{\vartheta}(t,x)$ follows from the estimate:
$$\bar{\theta}(s,y) \leq \bar{\vartheta}(s,y) \leq \bar{\vartheta}(t,x) = \bar{\theta}(t,x).$$
Being $\bar{\theta}(s,y) = \bar{\theta}(t,x)$ we conclude. ■

(II.3.21) PROPOSITION: *$\mathcal{H}^2$−almost every point of $\{t=1\}$ is regular for θ.*

PROOF. Since $\partial^\pm\vartheta$ have the transitive property stated in Proposition I.54 as $\partial^\pm\phi$ have and ϑ reach $\{t=0\}$ as ϕ does, the proof is the same if ϑ coincided with θ.

Therefore notice for instance in the backward regularity that since the area estimate selects an inner direction in a maximal dimension component, $\mathcal{Q}_\theta(w,w') \subseteq \mathcal{Q}_\vartheta(w,w')$and that the dimension of $\partial^-\vartheta(w)$ coincides with the dimension of $\partial^-\theta(w)$, where w is the point of the optimal ray selected at $t = 1+\varepsilon$ and w' at $t=\tau>0$. ■

II.4. Translation of the result in the original problem

In this section we state the main Theorem of this work.

The first directed locally affine partition $\mathbf{D}_{\bar{\phi}}$ of Section II.1 has been refined following the scheme of Section II.2.

In particular in Subsection II.2.1 we noticed $Z^0_{\mathfrak{a}}$ are irreducible and give naturally a map.

In Subsection II.2.2 we showed how to refine directly each subset in the case $h = 1$.

The remaining case is described in Subsection II.2.3 and analysed in the Section II.3. At the end of this procedure we obtain a refinement $\{Z^{h,\ell'}_{\mathfrak{a},\mathfrak{b}}, C^{h,\ell'}_{\mathfrak{a},\mathfrak{b}'}\}_{h,\mathfrak{a},\ell',\mathfrak{b}'}$ of $\mathbf{D}_{\bar{\phi}}$ where the subsets $\{Z^{h,2}_{\mathfrak{a},\mathfrak{b}'}$ are cyclically connected. Applying the same argument of Subsection II.2.2 to each $Z^{h,1}_{\mathfrak{a},\mathfrak{b}'}$ of this new partition we obtain the following theorem:

(II.4.1) THEOREM: *Given a directed locally affine partition $\{Z^h_{\mathfrak{a}}, C^h_{\mathfrak{a}}\}_{h,\mathfrak{a}}$ and a transference plan $\bar{\pi} \in \Pi(\bar{\mu}, \bar{\nu})$ such that*

$$\bar{\pi} = \sum_h \int \bar{\pi}^h_{\mathfrak{a}} m^h(d\mathfrak{a}), \qquad \int \mathbb{1}_{C^h_{\mathfrak{a}}}(x - x')\bar{\pi}^h_{\mathfrak{a}} < \infty,$$

then there exists a directed locally affine partition $\{Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}, C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}\}_{h,\mathfrak{a},\ell,\mathfrak{b}}$ such that

1. *$Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}} \subset Z^h_{\mathfrak{a}}$ has affine dimension $\ell + 1$ and $C^{h,\ell}_{\mathfrak{a},\mathfrak{b}}$ is an $(\ell + 1)$-dimensional extremal cone of $C^h_{\mathfrak{a}}$;*

2. *$\bar{\mu}(\cup_{h,\mathfrak{a},\ell,\mathfrak{b}} Z^{h,\ell}_{\mathfrak{a},\mathfrak{b}}) = 1$;*

3. *the disintegration of $\mathcal{H}^2 \llcorner_{\{t=1\}}$ w.r.t. the partition $\{Z^\ell_{\mathfrak{c}}\}_{\ell,\mathfrak{c}}$ is regular, i.e.*

$$\mathcal{H}^2 \llcorner_{\{t=1\}} = \sum_\ell \int \xi^\ell_{\mathfrak{c}} \eta^\ell(d\mathfrak{c}), \qquad \xi^\ell_{\mathfrak{c}} \ll \mathcal{H}^\ell \llcorner_{Z^\ell_{\mathfrak{c}} \cap \{t=1\}};$$

4. *if $\bar{\pi} \in \Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$ with $\bar{\nu}^h_{\mathfrak{a}} = \mathtt{p}_2 \bar{\pi}^h_{\mathfrak{a}}$, then $\bar{\pi}$ satisfies (14) iff*

$$\bar{\pi} = \sum_\ell \int \bar{\pi}^\ell_{\mathfrak{c}} m^\ell(d\mathfrak{c}), \qquad \int \mathbb{1}_{C^\ell_{\mathfrak{c}}}(x - x')\bar{\pi}^\ell_{\mathfrak{c}} < \infty;$$

5. *if $\ell = h$, then for every carriage Γ of $\bar{\pi} \in \Pi(\bar{\mu}, \{\bar{\nu}^h_{\mathfrak{a}}\})$ there exists a $\bar{\mu}$-negligible set N such that each $Z^{h,h}_{\mathfrak{a},\mathfrak{b}} \setminus N$ is $\mathbb{1}_{C^{h,h}_{\mathfrak{a},\mathfrak{b}}}$-cyclically connected.*

Being the problem (4) recasted in $[0, +\infty) \times \mathbb{R}^2$ as (10), we have proved also the following Theorem:

(II.4.2) THEOREM: *Let $\pi \in \Pi(\mu, \nu)$ be an optimal transference plan, with $\mu \ll \mathcal{L}^2$. Then there exists a family of sets $\{S^h_{\mathfrak{a}}, O^h_{\mathfrak{a}}\}_{\substack{h=0,\dots,d \\ \mathfrak{a}\in\mathfrak{A}^h}}$ in $\mathbb{R}^2$ such that the following holds:*

1. $S^h_{\mathfrak{a}}$ *is a locally affine set of dimension* h*;*
2. $O^h_{\mathfrak{a}}$ *is a* h*-dimensional convex set contained in an affine subspace parallel to* $\operatorname{aff} S^h_{\mathfrak{a}}$ *and given by the projection on* $\mathbb{R}^2$ *of a proper extremal face of* $\operatorname{epi} \mathtt{c}$*;*
3. $\mathcal{L}^2(\mathbb{R}^2 \setminus \cup_{h,\mathfrak{a}} S^h_{\mathfrak{a}}) = 0$*;*
4. *the partition is Lebesgue regular;*
5. *if* $\pi \in \Pi(\mu, \{\nu^h_{\mathfrak{a}}\})$ *then optimality in* (4) *is equivalent to*

$$\sum_h \int \bigg[\int \mathbb{1}_{O^h_{\mathfrak{a}}}(x' - x) \pi^h_{\mathfrak{a}}(dxdx') \bigg] m^h(d\mathfrak{a}) < \infty,$$

   where $\pi = \sum_h \int_{\mathfrak{A}^h} \pi^h_{\mathfrak{a}} m^h(d\mathfrak{a})$ *is the disintegration of* π *w.r.t. the partition* $\{S^h_{\mathfrak{a}} \times \mathbb{R}^2\}_{h,\mathfrak{a}}$*;*
6. *for every carriage* Γ *of* $\pi \in \Pi(\mu, \{\nu^h_{\mathfrak{a}}\})$ *there exists a* μ*-negligible set* N *such that each* $S^h_{\mathfrak{a}} \setminus N$ *is* $\mathbb{1}_{O^h_{\mathfrak{a}}}$*-cyclically connected.*

Notice this is the Theorem II.4.1 rephrased in the original setting.
By Theorem II.4.2 and Proposition I.2.11 we prove the following Theorem:

(II.4.3) THEOREM: *Let* $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $\mu \ll \mathcal{L}^d$*. Then, there exists an optimal transport map* $\mathtt{T}$ *for the Monge problem* (II.1).

Appendix - Equivalence relations, disintegration and uniqueness

A

The following theorems have been proved in Section 4 of [6]. For a more comprehensive analysis, see [19].

A.1. Disintegration of measures

Let E be an equivalence relation on X, and let $\mathtt{h} : X \mapsto X/E$ be the quotient map. The set $\mathfrak{A} := X/E$ can be equipped with the σ-algebra

$$\mathtt{A} := \Big\{ A \subset \mathfrak{A} : \mathtt{h}^{-1}(A) \in \mathcal{B}(X) \Big\}.$$

Let $\mu \in \mathcal{P}(X)$, and define $\xi := \mathtt{h}_\sharp \mu$.

A *disintegration of* μ *consistent with* E is a map $\mathfrak{A} \ni \mathfrak{a} \mapsto \mu_\mathfrak{a} \in \mathcal{P}(X)$ such that

1. for all $B \in \mathcal{B}(X)$ the function $\mathfrak{a} \mapsto \mu_\mathfrak{a}(B)$ is ξ-measurable,
2. for all $B \in \mathcal{B}(X)$, $A \in \mathtt{A}$

$$\mu(B \cap \mathtt{h}^{-1}(A)) = \int_A \mu_\mathfrak{a}(B) \xi(d\mathfrak{a}).$$

The disintegration is *unique* if the *conditional probabilities* $\mu_\mathfrak{a}$ are uniquely defined ξ-a.e.. It is *strongly consistent* if $\mu_\mathfrak{a}(E_\mathfrak{a}) = 1$.

(A.1.1) THEOREM: *Under the previous assumptions, there exists a unique consistent disintegration.*

If the image space is a Polish space and $\mathtt{h}$ *is Borel, then the disintegration is strongly consistent.*

A.2. Linear preorders and uniqueness of transference plans

We now recall some results about uniqueness of transference plans. Let $A \subset X \times X$ be a Borel set such that

1. $A \cup A^{-1} = X$, where
$$A^{-1} = \big\{(x, x') : (x', x) \in A\big\};$$
2. $(x, x'), (x', x'') \in A \;\Rightarrow\; (x, x'') \in A$.

We will say that A is (the graph of) a *preorder* if Condition (2) holds, and a *linear preorder* if all points are comparable (Condition 1). It is easy to see that

$$E := A \cap A^{-1}$$

is an equivalence relation. Let $\mathtt{h} : X \mapsto X/E$ be a quotient map.

(A.2.1) THEOREM: *If $\mu \in \mathcal{P}(X)$, then the disintegration of μ w.r.t. E is strongly consistent:*
$$\mu = \int \mu_{\mathfrak{a}} \xi(d\mathfrak{a}), \quad \xi := \mathtt{h}_\sharp \mu, \; \mu_{\mathfrak{a}}(E_{\mathfrak{a}}) = 1.$$

Let $\bar\pi \in \mathcal{P}(X \times X)$ such that $\bar\pi(E) = 1$, and let $\bar\mu := (\mathtt{p}_1)_\sharp \bar\pi$, $\bar\nu := (\mathtt{p}_2)_\sharp \bar\pi$ be its marginals. Consider the disintegration

$$\bar\pi = \int \bar\pi_{\mathfrak{a}} \bar\xi(d\mathfrak{a}), \quad \bar\xi = (\mathtt{h} \circ \mathtt{p}_1)_\sharp \bar\pi.$$

Let $\bar\mu_{\mathfrak{a}}$, $\bar\nu_{\mathfrak{a}}$ be the conditional probabilities of $\bar\mu$, $\bar\nu$ w.r.t. E:

$$\bar\mu = \int \bar\mu_{\mathfrak{a}} \bar\xi(d\mathfrak{a}) = \int (\mathtt{p}_1)_\sharp \bar\pi_{\mathfrak{a}} \bar\xi(d\mathfrak{a}), \quad \bar\nu = \int \bar\nu_{\mathfrak{a}} \bar\xi(d\mathfrak{a}) = \int (\mathtt{p}_2)_\sharp \bar\pi_{\mathfrak{a}} \bar\xi(d\mathfrak{a}),$$

(A.2.2) THEOREM: *If $\pi \in \Pi(\bar\mu, \bar\nu)$ satisfies*

$$\int \mathbb{1}_A \pi < +\infty,$$

then $\pi(E) = 1$, and moreover the disintegration of π on E satisfies

$$\pi = \int \pi_{\mathfrak{a}} \bar\xi(d\mathfrak{a}), \quad \pi_{\mathfrak{a}} \in \Pi(\bar\mu_{\mathfrak{a}}, \bar\nu_{\mathfrak{a}}).$$

A.3. Minimality of equivalence relations

Consider a family of equivalence relations on X,

$$\mathcal{E} = \Big\{ E_{\mathfrak{e}} \subset X \times X, \mathfrak{e} \in \mathfrak{E} \Big\}.$$

Assume that $\mathcal{E}$ is closed under countable intersection

$$\{E_{\mathfrak{e}_i}\}_{i\in\mathbb{N}} \subset \mathcal{E} \quad \Rightarrow \quad \bigcap_{i\in\mathbb{N}} E_{\mathfrak{e}_i} \in \mathcal{E},$$

and let $\mu \in \mathcal{P}(X)$.

By Theorem A.1.1, we can construct the family of disintegrations

$$\mu = \int_{\mathfrak{A}_{\mathfrak{e}}} \mu_{\mathfrak{a}} \xi_{\mathfrak{e}}(d\mathfrak{a}), \quad \mathfrak{e} \in \mathfrak{E}.$$

(A.3.1) THEOREM: *There exists $E_{\bar{\mathfrak{e}}} \in \mathfrak{E}$ such that for all $E_{\mathfrak{e}}$, $\mathfrak{e} \in \mathcal{E}$, the following holds:*

1. *if $\mathtt{A}_{\mathfrak{e}}$, $\mathtt{A}_{\bar{\mathfrak{e}}}$ are the σ-subalgebras of the Borel sets of X made of the saturated sets for $E_{\mathfrak{e}}$, $E_{\bar{\mathfrak{e}}}$ respectively, then for all $A \in \mathtt{A}_{\mathfrak{e}}$ there is $A' \in \mathtt{A}_{\bar{\mathfrak{e}}}$ s.t. $\mu(A \triangle A') = 0$;*
2. *if $\xi_{\mathfrak{e}}$, $\xi_{\bar{\mathfrak{e}}}$ are the restrictions of μ to $\mathtt{A}_{\mathfrak{e}}$, $\mathtt{A}_{\bar{\mathfrak{e}}}$ respectively, then $\mathtt{A}_{\mathfrak{e}}$ can be embedded (as measure algebra) in $\mathtt{A}_{\bar{\mathfrak{e}}}$ by Point (1): let*

   $$\xi_{\bar{\mathfrak{e}}} = \int \xi_{\bar{\mathfrak{e}},\mathfrak{a}} \xi_{\mathfrak{e}}(d\mathfrak{a})$$

   be the disintegration of $\xi_{\bar{\mathfrak{e}}}$ consistent with the equivalence classes of $\mathtt{A}_{\mathfrak{e}}$ in $\mathtt{A}_{\bar{\mathfrak{e}}}$.
3. *If*

   $$\mu = \int \mu_{\mathfrak{e},\mathfrak{a}} \xi_{\mathfrak{e}}(d\mathfrak{a}), \quad \mu = \int \mu_{\bar{\mathfrak{e}},\beta} \xi_{\bar{\mathfrak{e}}}(d\beta)$$

   are the disintegration consistent with $E_{\mathfrak{e}}$, $E_{\bar{\mathfrak{e}}}$ respectively, then

   $$\mu_{\mathfrak{e},\mathfrak{a}} = \int \mu_{\bar{\mathfrak{e}},\mathfrak{b}} \xi_{\bar{\mathfrak{e}},\mathfrak{a}}(d\mathfrak{b}).$$

   for $\xi_{\mathfrak{e}}$-.a.e. $\mathfrak{a}$.

In particular, assume that each $E_{\mathfrak{e}}$ is given by

$$E_{\mathfrak{e}} = \{\theta_{\mathfrak{e}}(x) = \theta_{\mathfrak{e}}(x')\}, \quad \theta_{\mathfrak{e}} : X \to X', \ X' \text{ Polish}, \ \theta_{\mathfrak{e}} \text{ Borel}.$$

(A.3.2) COROLLARY: *There exists a μ-conegligible set $F \subset X$ such that $\theta_{\mathfrak{e}}$ is constant on $F \cap \theta_{\bar{\mathfrak{e}}}^{-1}(x')$, for all $x' \in X'$.*

PROOF. Consider the function $\vartheta := (\theta_{\mathfrak{c}}, \theta_{\bar{\mathfrak{c}}})$: by the minimality of $\theta_{\bar{\mathfrak{c}}}$, it follows that

$$\xi_{\bar{\mathfrak{c}}} = \int \xi_{\bar{\mathfrak{c}},(x',x'')} \xi_\vartheta(dx'dx''), \quad \xi_\vartheta := \vartheta_\sharp \mu.$$

Since $(\mathtt{p}_2)_\sharp \xi_\vartheta = \xi_{\bar{\mathfrak{c}}}$, then also

$$\xi_\vartheta = \int \xi_{\vartheta,x'} \xi_{\bar{\mathfrak{c}}}(dx'),$$

and thus

$$\xi_{\bar{\mathfrak{c}}} = \int \bigg[\int \xi_{\bar{\mathfrak{c}},(x',x'')} \xi_{\vartheta,x''}(dx'dx'') \bigg] \xi_{\bar{\mathfrak{c}}}(dx').$$

This implies that $\xi_{\bar{\mathfrak{c}}}$-a.e. x'

$$\int \xi_{\bar{\mathfrak{c}},(x',x'')} \xi_{\vartheta,x'}(dtds) = \delta_{x'},$$

or equivalently that

$$\xi_{\vartheta,x'''} = \delta_{\mathtt{x}(x'''),\mathtt{x}''(x''')}, \quad \xi_{\bar{\mathfrak{c}},(\mathtt{x}'(x'''),\mathtt{x}''(x'''))} = \delta_{x'''}.$$

Hence ξ_ϑ is concentrated on a graph: by choosing $x' = x'''$, there exists $\mathtt{s} = \mathtt{s}(x')$ Borel such that $\xi_\vartheta = (\mathbb{I}, \mathtt{s})_\sharp \xi_{\bar{\mathfrak{c}}}$. This is equivalent to say that there exists a μ-conegligible set F such that $\theta_{\mathfrak{c}} = \mathtt{s} \circ \theta_{\bar{\mathfrak{c}}}$ on F. ■

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