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A variational approach to statics and dynamics of elasto-plastic systems

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a Francesco

Contents

Introduction

The present thesis is the result of the study of several variational problems arising from mechanical systems of elastic and elasto-plastic bodies. The thesis is divided in two main parts. In the first part, concerning the *macroscopic* theory, we focus on the dynamics of these systems. Our analysis leads us to solve systems of nonlinear partial differential equations. When the inertia and the viscosity of the bodies are taken into account, such systems of PDEs contain a hyperbolic equation for the displacement coupled with some flow rules, which govern the evolution of internal variables used to describe the nonlinear phenomena occuring in the considered systems. In our specific cases, these are the plastic response of the bodies to the stress and the deterioration of the adhesive which keeps two bodies glued together. The quasistatic limit of the dynamics evolution is studied when the inertia and the viscosity of the bodies are neglected.

In the second part of the thesis we study plastic bodies from a *mesoscopic* point of view. This approach involves the concept of dislocations, that we attack by mean of tools of geometric measure theory, in particular the theory of currents and of Cartesian maps. In this setting minimization problems are considered and their solution requires also to solve systems of partial differential equations involving elliptic equations.

The thesis consists of two chapters. In the first one we consider evolution problems in two mechanical systems: respectively, an elasto-plastic body and the system of two elastic bodies that are glued by an adhesive on a interface. The former is an elastic body where nonconservative deformations might take place. In particular, the stress satisfies a constitutive equation which does not involve the whole deformation gradient. A part of the deformation gradient (the plastic strain) does not contribute to internal forces, but still evolves according to a flow rule which depends on the stress. Instead, the system of glued bodies is the union of two perfectly elastic bodies, where the movements and the high stress provoke the destruction of molecular links of the adhesive, deteriorating the glue and weakening its effect. This phenomenon is much studied in literature and is referred to as delamination process; it takes place untill the glue is completely ineffective, causing the rupture of the connecting surface and allowing the two bodies to separate.

In both devices we consider the inertia and the viscosity of the bodies. We prove some original existence results for the evolution of the displacement of the systems, and we give an energetic formulation of the solution, that is, we prove that a displacement u solves the considered system of PDEs if and only if it satisfies an energy balance and some "dynamic equilibrium" conditions. Once we have obtained the existence results, we analyse the behaviour of the solutions when the density and the viscosity of the materials are neglected. This analysis is equivalent to study the asymptotic limit of the solutions when the data of the problem become slower and slower. In literature, sometimes such analysis is referred to as slow-time limit. In the plasticity contest, we prove that the solutions approximate a quasistatic evolution in perfect plasticity. The quasistatic evolution is here not expressed explicitly as a solution of a system of PDEs, but it is convenient to consider a weak formulation, by requiring that it satisfies a stability condition and an energy balance. This is the classical formulation given by Suquet in [73]. One of the main difficulty is that the solutions of the quasistatic evolution do not belong to Sobolev spaces, but it is necessary to introduce the space of functions with bounded deformation and to deal with space of measures. In the case of delamination, the solutions tend to the solution of an energetic problem that is not exactly a quasistatic evolution. Indeed we can prove only an energy inequality, and we also show that the strict inequality takes place in some cases. In some specific cases a more detailed description of the quasistatic limit is also given.

All the results described in this part are essentially contained in the two papers [19] and [67].

The second chapter contains results obtained in a two-years long study about the structure of dislocations in single crystals and about the nature of the deformation gradients around the dislocations. In the presence of dislocations the deformation gradient is well defined only in a local sense. Indeed it is a matrix-valued field which is not curl-free, its curl concentrating upon some Lipschitz curves inside the crystal called dislocations. As a consequence the displacement cannot be defined univocally, but it can be reconstructed only in simple connected domains not intersecting the dislocation lines. From a physical point of view, dislocations are responsible of the dissipative phenomena in the bulk, and macroscopically, their presence gives rise to plastic response of the material.

A relevant part of the discussion is dedicated to construct a mathematical model for dislocations by mean of integral currents. The equilibria of a single crystal with dislocations is obtained by minimizing suitable energies depending on the strain and on the dislocations. The description of dislocations by integral currents is useful to give a precise mathematical formulation of these minimum problems under very general assumptions on the geometry of dislocations and on the regularity properties of the strain. In order to solve this variational problems we make use of some tools of geometric measure theory, as Cartesian maps and the concept of currents carried by the graphs of Sobolev maps. We prove the existence of minimizers in many cases, under different hypotheses on the energy and on the class of admissible deformations. Some of these results require a finer description of the behaviour of the strain in a neighborhood of the dislocation line. This is obtained by recostructing the deformation in a specific space of functions which take values in the three dimensional torus. As a result, it is possible to explicitly compute the boundary of the graphs of such maps, allowing us to use well-known convergence results on the sequence of graphs. These provide the existence of minimizers. From the existence of minimizers we are able to compute the variation of the energy, and then to obtain an explicit formula of the so-called Peach-Köeler force, which is a force acting between dislocations.

Many of the results described in this part are contained in the three papers [68], [69], and [70].

Chapter 1

Dynamic evolution problems in visco-elasto-plasticity and delamination

Preamble

The quasistatic evolution of rate independent systems has been often obtained as the limit case of a viscosity driven evolution (see [73], [47], [24], [21], [77], [48], [37], [38], [43], [44], [49], [50], [64], [65], [46]). In the present chapter we present a case study on the approximation of a quasistatic evolution in linearly elastic perfect plasticity (see [73] and [20]) and of a quasistatic evolution in delamination (see [39] and [66]) by dynamic evolutions. The corresponding approximation in a finite dimensional setting has been presented in [3]. Similar approximations obtained by dynamic processes can be found in literature (for approximation of quasistatic evolutions of similar mechanical problems see, for instance, [7] for the perfect plasticity, and [64] for the delamination). The results we present in the first three sections are contained in the paper [19], written in collaboration with Gianni Dal Maso, while Sections 1.4 and 1.5 contain the results of the paper [67].

The visco-elasto-plastic model. In Section 1.2 we consider a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime. This model couples dynamic visco-elasticity with Perzyna visco-plasticity (see [34], [75], and [45]). The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. The linearized strain Eu, defined as the symmetric part of the gradient of the displacement u, is decomposed as $Eu = e + p$, where e is the elastic component and p is the plastic one. The part of the strain that contributes to the stress σ is only the elastic part. The constitutive law for the stress is

$$\sigma = A_0 e + \mu A_1 \dot{e}, \tag{1.0.1}$$

which is the sum of an elastic part $A_0 e$ and a viscous part $\mu A_1 \dot{e}$, where A_0 is the elasticity tensor, A_1 is the viscosity tensor, $\dot{e}$ is the derivative of e with respect to time, and $\mu > 0$ is a parameter connected with the viscosity of the material. In our model, we assume, as usual, that A_0 is symmetric and positive definite, while we only assume that A_1 is symmetric and positive semidefinite, so that we are allowed to consider also $A_1 = 0$, which corresponds to the case where the elastic viscosity is neglected.

The balance of momentum gives the equation

$$\rho\ddot{u} - \mathrm{div}\sigma = f, \tag{1.0.2}$$

where f is the volume force, and $\rho > 0$ is the mass density. As in Perzyna visco-plasticity, the evolution of the plastic part is governed by the flow rule

$$\mu\dot{p} = \sigma_D - \pi_K\sigma_D,$$

where σ_D is the deviatoric part of σ and π_K is the projection onto a prescribed convex set K in the space of deviatoric symmetric matrices, which can be interpreted as the domain of visco-elasticity. Indeed, if σ_D belongs to K during the evolution, then there is no production of plastic strain, so that, if $p = 0$ at the initial time, then $p = 0$ for every time and the solution is purely visco-elastic. This can be interpreted as follows: when the strain is small enough, then the body behaves as perfectly elastic, while only high strain (and then stress) is needed to produce irreversible deformations.

The complete system of equations is

$$Eu = e + p, \tag{1.0.3a}$$
$$\sigma = A_0e + \mu A_1\dot{e}_{A_1}, \tag{1.0.3b}$$
$$\rho\ddot{u} - \mathrm{div}\sigma = f, \tag{1.0.3c}$$
$$\mu\dot{p} = \sigma_D - \pi_K\sigma_D, \tag{1.0.3d}$$

where e_{A_1} denotes the projection of e into the image of A_1. This system is supplemented by a Dirichlet boundary conditions w, a Neumann boundary condition, and by initial conditions. Other dynamic models of elasto-plasticity with viscosity have been considered in [6] and [7]. The main difference with respect to our model is that they couple visco-elasticity with perfect plasticity, while we couple visco-elasticity with visco-plasticity.

In Section 1.2 we prove two results of existence and uniqueness of a solution to (1.0.3) with initial and boundary conditions (Theorem 1.2.1 and Theorem 1.2.9). In the first existence result we assume that the visco-elastic tensor A_1 is only positive semidefinite. This general assumptions has as a consequence the lack of some a-priori estimates on the norm of the elastic part e of the solution. In order to obtain good estimates, we need to make strong assumptions on the regularity of the external data, i.e. the volume force f and the boundary condition w. The second existence result is instead simpler, since we make the stronger assumption that A_1 is positive definite. With this hypothesis we are allowed to weaken the assumptions on the external data. The proof of this second result is actually very similar to the first one and is not discussed in detail. Indeed all the preliminary results to the proof of Theorem 1.2.1 can be adapted and obtained with some stronger regularity on the solutions.

Before proving Theorem 1.2.1, in analogy with the energy method for rate independent processes developed by Mielke (see [47] and the references therein), we prove that system (1.0.3) has a weak formulation expressed in terms of an energy balance together with a stability condition (Theorem 1.2.4). Then the proof of the existence of a solution to this weak formulation is obtained by time discretization. This standard procedure consists in dividing the time interval $[0, T]$ into N equal subintervals of length $\tau := T/N$, and then to solve a suitable incremental minimum problem at every discrete times. A piecewise affine interpolation of these minima will give rise to approximating functions that will converge to a solution when we let the time step τ tend to 0.

In Section 1.3.4 we analyze the behavior of the solution to system (1.0.3) as the data of the problem become slower and slower. Before performing such analysis we introduce the concept of quasistatic evolution in perfect plasticity and prove some preliminary results (Section 1.3). As usual in the theory of rate independent systems, an energetic formulation is given to a quasistatic evolution. This formulation consists of two conditions: an equilibrium condition, that says that at every time of the evolution the solution is a minimum of the energy functional, and an energy balance, which expresses the fact that during the evolution there are no dissipations due to non-conservative phenomena, and at every time the energy equals the initial energy plus the work done by the external forces on the system. For the development of quasistatic evolution in perfect plasticity we refer to [73] and [20]. The energetic formulation is given in Definition 1.3.4. Instead the strong formulation of the quasistatic evolution is expressed by the system

$$Eu = e + p, \tag{1.0.4a}$$
$$\sigma = A_0 e, \tag{1.0.4b}$$
$$-\operatorname{div}\sigma = f, \tag{1.0.4c}$$
$$\sigma_D \in K \ \text{ and } \ \dot{p} \in N_K \sigma_D, \tag{1.0.4d}$$

where $N_K\sigma_D$ denotes the normal cone to K at σ_D.

Dynamic approximation of the quasistatic evolution. Section 1.3.4 is devoted to show the already mentioned convergence result. The quasistatic evolution is obtained as the limit of approximate dynamic evolutions that are given by a suitable rescaling of time. The rescaling leads us to a suitable change of variables. More precisely, we start from an external load $f(t)$, a boundary datum $w(t)$ defined on the interval $[0, T]$, and initial conditions u_0, e_0, p_0, and v_0. We then consider the rescaled problem with external load $f_\epsilon(t) := f(\epsilon t)$, boundary condition $w_\epsilon(t) = w(\epsilon t)$ on the interval $[0, T/\epsilon]$, and initial conditions $u_\epsilon(0) = u_0$, $e_\epsilon(0) = e_0$, $p_\epsilon(0) = p_0$, and $\dot{u}_\epsilon(0) = \epsilon v_0$. The dynamic solutions of the corresponding systems (1.0.3) are denoted by $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$.

To study the limit behavior of $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$ on the whole interval $[0, T/\epsilon]$ it is convenient to consider the rescaled functions $(u^\epsilon(t), e^\epsilon(t), p^\epsilon(t), \sigma^\epsilon(t))$ $:= (u_\epsilon(t/\epsilon), e_\epsilon(t/\epsilon), p_\epsilon(t/\epsilon), \sigma_\epsilon(t/\epsilon))$, defined on $[0, T]$, and to study their limit as $\epsilon \downarrow 0$. A straightforward change of variables shows that $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ will satisfy the following system of equations on $[0, T]$

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{1.0.5a}$$

$$\sigma^\epsilon = A_0 e^\epsilon + \epsilon\mu A_1 \dot{e}^\epsilon_{A_1}, \tag{1.0.5b}$$

$$\epsilon^2 \rho \ddot{u}^\epsilon - \text{div}\sigma^\epsilon = f, \tag{1.0.5c}$$

$$\epsilon \dot{p}^\epsilon = \sigma^\epsilon - \pi_K \sigma^\epsilon, \tag{1.0.5d}$$

with the same boundary and initial conditions. As it can be seen, this system is equal to (1.0.3) with the only difference that the new mass density is $\epsilon^2\rho$ and the viscosity is $\epsilon\mu$. The rescaling of time can be interpreted as follows: we slow down the speed of the process and we see what happen to the solutions. What we show is that the dissipative and inertial effects disappear as ϵ tends to 0. This analysis is equivalent to compute a vanishing inertia and viscosity at the same time. More precisely, as observed, we analyze the behavior of the solutions when the viscosity tends to 0 as ϵ and the mass density tends to 0 as ϵ^2.

The main result of this section is stated by Theorem 1.3.10 where, under suitable assumptions, we show that the solutions $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (1.0.5) tend to a solution of the quasistatic evolution problem in perfect plasticity, according to Definition 1.3.4. The proof of this convergence result is obtained using the energetic formulation of (1.0.3) expressed by energy balance and stability condition (see Theorem 1.2.4). We show that we can pass to the limit obtaining the energy formulation of (1.0.4) developed in [20]. A remarkable difficulty in this proof is due to the fact that problems (1.0.3) and (1.0.4) are formulated in completely different function spaces (see Theorem 1.2.1 and Definition 1.3.4). Theorem 1.3.10 can be applied also to solutions of Theorem 1.2.9 with slightly weaker assumptions. This is not discussed in detail since it is a straightforward analysis that can be obtained following the lines of the proof of Theorem 1.3.10.

The delamination model. The model discussed in Section 1.4.1 consists of two elastic bodies Ω_1 and Ω_2 glued by an adhesive on an interface Γ. External forces and high stresses due to elastic deformations of the bodies may break the macromolecules of the adhesive, weakening its effect. Such process is irreversible, in the sense that the deteriorated adhesive cannot be restored. The state of the adhesive is described by the delamination coefficient z, that is a function defined on the interface which takes values in $[0, 1]$ (see Section 1.4.1). Until the glue is effective the movements of the bodies at the interface are constrained. Moreover some constrains at the interface are always considered due to the non-interpenetrability of the two bodies or to the pressure of the system (see Section 1.4.1). In our model we consider both inertia and viscosity in the bulk, and also the evolution of the internal variable z is not rate-independent since we consider the viscous effects related to the deterioration of the adhesive. In the bulk we neglect the thermal effects. In [63] and [62] it is considered a system where also thermal effects are analyzed, while no viscosity of the delamination coefficient is considered. Terms related to friction of the adhesive have been studied in different settings where inertia is neglected (see, e.g., [15], [10]).

As in the plasticity model, the constitutive law for the stress is

$$\sigma = A_0 Eu + \mu A_1 E\dot{u}. \tag{1.0.6a}$$

In contrast to (1.0.1) the elastic part of the strain e is here replaced by the whole symmetric gradient Eu. This is due to the absence of plasticity. As usual

A_0 is symmetric and positive definite, and in this case we also assume that A_1 is symmetric and positive definite. The balance of momentum is still expressed by the equation

$$\rho\ddot{u} - \mathrm{div}\sigma = f. \tag{1.0.6b}$$

The equations above are supplemented by boundary conditions, by initial conditions, and by a condition of interaction of the two bodies

$$\sigma\nu = -\nabla V([u])z \quad \text{on } \Gamma, \tag{1.0.6c}$$

where ν is the unit normal to Γ, V is the energy introduced in Section 1.4.1, and $[u]$ is the jump of the displacement at the interface. As it can be seen, the interaction between Ω_1 and Ω_2 depends on $[u]$, that is the difference between the two traces of u from Ω_1 and Ω_2, and it is null when $z = 0$, i.e., when the glue is no more effective. As for the evolution of the delamination coefficient z, the flow rule is expressed by the following system of equations

$$\dot{z} \leq 0, \tag{1.0.6d}$$
$$d \leq -\mu\dot{z}, \tag{1.0.6e}$$
$$\dot{z}(d + \mu\dot{z}) = 0, \tag{1.0.6f}$$
$$d \in \partial I_{[0,1]} + V([u]) - \alpha, \tag{1.0.6g}$$

valid on Γ. Here α is a bounded positive function on Γ connected with the dissipative effects of the delamination, $\partial I_{[0,1]}$ is the subdifferential of the characteristic function of $[0, 1]$, and $\mu > 0$ is the viscosity of the adhesive.

As usual in delamination problems, it is natural to consider constrains on the sign of $[u] \cdot \nu$. To avoid interpenetration of the two bodies one is led to require that $[u] \cdot \nu \leq 0$. In some models a bilateral constrain is required, arising in the condition $[u] \cdot \nu = 0$. These are models under high pressure, where no cavitation phenomena are allowed, and then the two bodies cannot separate.

In Section 1.4 we prove a result of existence for solutions to (1.0.6) (Theorem 1.4.9), without constrains on the jump $[u]$ at the interface. As in the case of plasticity, we formulate the problem in a weak form, consisting of three weak equations, and prove the existence of a solution by time discretization. In this case, at each time step, we solve a minimum problem for the displacement, and then solve a minimum problem expressed in terms of this solution for the delamination parameter. Again the piecewise affine interpolations of these minima provide the approximate solutions. Also in this case the solutions turn out to satisfy an energy balance that is proved in a second step. We finally prove the existence of an evolution with a bilateral constraint at the interface in Theorem 1.4.11.

The dynamic approximation. In Section 1.5 we want to study the asymptotic behavior of the dynamic solutions obtained by rescaling the time as in the case of plasticity. The rescaling leads us to a dynamic solution (u^ϵ, z^ϵ) satisfying the equations

$$\sigma^\epsilon = A_0 E u^\epsilon + \epsilon\mu A_1 E\dot{u}^\epsilon \tag{1.0.7a}$$
$$\epsilon^2\rho\ddot{u}^\epsilon - \mathrm{div}\sigma^\epsilon = f. \tag{1.0.7b}$$

with

$$\sigma^\epsilon \nu = -\nabla V([u^\epsilon]) z^\epsilon \quad \text{on } \Gamma, \tag{1.0.7c}$$

and the flow rules

$$\dot{z}^\epsilon \leq 0, \tag{1.0.7d}$$
$$d \leq -\epsilon\mu\dot{z}^\epsilon, \tag{1.0.7e}$$
$$\dot{z}^\epsilon(d + \epsilon\mu\dot{z}^\epsilon) = 0, \tag{1.0.7f}$$
$$d \in \partial I_{[0,1]} + V([u^\epsilon]) - \alpha. \tag{1.0.7g}$$

The convergence result is state in Theorem 1.5.1 whose proof is only sketched, since it is very similar to the proof of the main result of [64], where the same asymptotic limit is analyzed for solutions of a dynamic process where the viscosity on the adhesive is neglected. A different argument and proof are needed to prove the following lemmas, which describe the limit flow rule. In contrast with the case of plasticity, the limit of the rescaled dynamic evolutions is not a quasistatic evolution in delamination. Indeed we prove that the limit satisfies an equilibrium condition, while in general an energy balance does not hold, but only an energy inequality, showing a lack of energy that takes the form of residual dissipation. This residual dissipation is the limit of the viscous dissipations of the dynamic evolutions, and it is expressed by two nonnegative Borel measures μ_b and μ_z which concentrate in the product spaces $[0,T] \times (\Omega_1 \cup \Omega_2)$ and $[0,1] \times \Gamma$, respectively. As a consequence, the quasistatic limit shows discontinuities in time, where it jumps from a minimum of the energy of the system to another.

In Subsection 1.5.1 we focus our analysis on the one-dimensional case, where we give a finer description of the behavior of the solutions at the limit. We prove that, in most the cases, the evolution shows a jump where the delamination coefficient switches instantaneously from 1 to 0. In some sense, Theorem 1.5.14 shows that the dynamic solutions cannot approximate a quasistatic evolution in delamination (in the sense of [39]), with the only exception of very particular (and unrealistic) cases.

1.1 Notation

Vectors and Matrices. If $a, b \in \mathbb{R}^n$, their scalar product is defined by $a \cdot b := \sum_i a_i b_i$, and $|a| := (a \cdot a)^{1/2}$ is the norm of a. If $\eta = (\eta_{ij})$ and $\xi = (\xi_{ij})$ belong to the space $\mathbb{M}^{n\times n}$ of $n \times n$ matrices with real entries, their scalar product is defined by $\eta \cdot \xi := \sum_{ij} \eta_{ij}\xi_{ij}$. Similary $|\eta| := (\eta \cdot \eta)^{1/2}$ is the norm of η. $\mathbb{M}^{n\times n}_{\text{sym}}$ is the subspace of $\mathbb{M}^{n\times n}$ composed of symmetric matrices. Moreover $\mathbb{M}^{n\times n}_D$ denotes the subspace of symmetric matrices with null trace, i.e., $\eta \in \mathbb{M}^{n\times n}_D$ if η is symmetric and $\text{tr}\eta = \sum_i \eta_{ii} = 0$. The space $\mathbb{M}^{n\times n}_{\text{sym}}$ can be split as

$$\mathbb{M}^{n\times n}_{\text{sym}} = \mathbb{M}^{n\times n}_D \ \oplus \ \mathbb{R}I,$$

where I is the identity matrix, so that every $\eta \in \mathbb{M}^{n\times n}_{\text{sym}}$ can be written as $\eta = \eta_D + \frac{\text{tr}\eta}{n} I$, where η_D, called the deviatoric part of η, is the projection of η into $\mathbb{M}^{n\times n}_D$.

Duality and Norms. If X is a Banach space and $u \in X$, we usually denote the norm of u by $\|u\|_X$. If X is $L^p(\Omega)$, $L^p(\Omega;\mathbb{R}^n)$, $L^p(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, or $L^p(\Omega;\mathbb{M}^{n\times n}_D)$ the norm is denoted by $\|u\|_{L^p}$. In general, if X is a Banach space, X' is its dual space and $\langle u, v\rangle_X$ denotes the duality product between $u \in X'$ and $v \in X$. The subscript X is sometimes omitted, if it is clear from the context.

If V represents the space $\mathbb{R}$, $\mathbb{R}^k$, $\mathbb{M}^{n\times n}_{\rm sym}$, $\mathbb{M}^{n\times n}_D$, then the symbol $\mathcal{M}_b(A, V)$ denotes the space of Radon measures on the open set A with values in V.

If $\Omega \subset \mathbb{R}^n$ is an open set and u is a function in $W^{1,1}(\Omega, \mathbb{R}^n)$, then the its symmetrized gradient Eu is defined as

$$Eu := \frac{1}{2}(\nabla u + (\nabla u)^T).$$

If u represents the displacement of a body, Eu is its linearized strain. Sometimes we will deal with displacements u whose derivatives are not in $L^1(\Omega)$. In these cases the linearized strain is also denoted by Eu and is defined as the $\mathbb{M}^{n\times n}_{\rm sym}$-valued distribution with components $E_{ij}u = \frac{1}{2}(D_i u_j + D_j u_i)$.

1.2 Visco-elasto-plastic evolution

1.2.1 Kinematical setting

The Reference Configuration. The reference configuration is a bounded connected open set Ω in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We suppose that $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \partial\Gamma$, where Γ_0, Γ_1, and $\partial\Gamma$ are pairwise disjoint, Γ_0 and Γ_1 are relatively open in $\partial\Omega$, and $\partial\Gamma$ is the relative boundary in $\partial\Omega$ both of Γ_0 and Γ_1. We assume that $\Gamma_0 \neq \emptyset$ and that $\mathcal{H}^{n-1}(\partial\Gamma) = 0$, where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. On Γ_0 we will prescribe a Dirichlet condition on the displacement u, while on Γ_1 we will impose a Neumann condition on the stress σ.

Elastic and Plastic Strain. If u is the displacement, the linearized strain Eu is decomposed as the sum of the elastic strain e and the plastic strain p. Given $w \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple (u, e, p) is kinematically admissible for the visco-elasto-plastic problem with boundary datum w if $u \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, $p \in L^2(\Omega;\mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \qquad \text{on } \Omega, \tag{1.2.1a}$$
$$u|_{\Gamma_0} = w \qquad \text{on } \Gamma_0. \tag{1.2.1b}$$

We denote the set of these triples by $A(w)$. It is convenient to introduce the subspace of $H^1(\Omega;\mathbb{R}^n)$ defined by

$$H^1_{\Gamma_0}(\Omega;\mathbb{R}^n) := \{u \in H^1(\Omega;\mathbb{R}^n) : u|_{\Gamma_0} = 0\}$$

and its dual space, denoted by $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$. It is clear that $(u, e, p) \in A(w)$ if and only if $u - w \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $Eu = e + p$, with $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$ and $p \in L^2(\Omega;\mathbb{M}^{n\times n}_D)$.

Stress and External Forces. In the visco-elasto-plastic model the stress σ depends linearly on the elastic part e of the strain Eu and on its time derivative

$\dot{e}$. To express this dependence we introduce the elastic tensor A_0 and the visco-elastic tensor A_1, which are symmetric linear operators of $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ into itself. We assume that there exist positive constants α_0, β_0, and β_1 such that

$$|A^i\xi| \le \beta_i|\xi|, \quad \text{for } i = 1, 2, \tag{1.2.2a}$$

$$A_0\xi\cdot\xi \ge \alpha_0|\xi|^2 \quad \text{and } A_1\xi\cdot\xi \ge 0, \tag{1.2.2b}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$. Note that $A_1 = 0$ is allowed. Inequalities (1.2.2) imply

$$|A^i\xi|^2 \le \beta_i A^i\xi\cdot\xi, \tag{1.2.2c}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and for $i = 1, 2$.

For every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ let ξ_{A_1} be the orthogonal projection of ξ onto the image of A_1. Then there exists a constant $\alpha_1 > 0$ such that

$$A_1\xi\cdot\xi \ge \alpha_1|\xi_{A_1}|^2 \tag{1.2.3}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$.

The stress satisfies the constitutive relation

$$\sigma = A_0 e + A_1\dot{e}. \tag{1.2.4}$$

The term $A_1\dot{e}$ in the equation above is the component of the stress due to internal frictions. To express the energy balance it is useful to introduce the quadratic forms

$$Q_0(\xi) = \frac{1}{2}A_0\xi\cdot\xi \quad \text{and} \quad Q_1(\xi) = A_1\xi\cdot\xi.$$

For every $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ we define

$$\mathcal{Q}_0(e) = \int_\Omega Q_0(e)dx \quad \text{and} \quad \mathcal{Q}_1(e) = \int_\Omega Q_1(e)dx.$$

These function turn out to be lower semicontinuous with respect to the weak topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. $\mathcal{Q}_0(e)$ represents the *stored elastic energy* associated to $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ while $\mathcal{Q}_1(\dot{e})$ represents the rate of visco-elastic dissipation.

We assume that the time dependent body force $f(t)$ belongs to $L^2(\Omega;\mathbb{R}^n)$ and that the time dependent surface force $g(t)$ belongs to $L^2(\Gamma_1,\mathcal{H}^{n-1};\mathbb{R}^n)$. It is convenient to introduce the total load $\mathcal{L}(t) \in H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ of external forces acting on the body, defined by

$$\langle\mathcal{L}(t), u\rangle := \langle f(t), u\rangle_\Omega + \langle g(t), u\rangle_{\Gamma_1}, \tag{1.2.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$, $\langle\cdot,\cdot\rangle_\Omega$ denotes the scalar product in $L^2(\Omega;\mathbb{R}^n)$, while $\langle\cdot,\cdot\rangle_{\Gamma_1}$ denotes the scalar product in $L^2(\Gamma_1,\mathcal{H}^{n-1};\mathbb{R}^n)$.

When dealing with the visco-elasto-plastic problem, we will only suppose that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$, without assuming the particular form (1.2.5). The hypotheses on the functions $t \mapsto \mathcal{L}(t)$ and $t \mapsto w(t)$ and the regularity of $t \mapsto (u(t), e(t), p(t))$ will be made precise in the statement of Theorems 1.2.1 and 1.2.4 below.

The law which expresses the second law of dynamic is

$$\ddot{u}(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{1.2.6}$$

where we assume that the mass density of the elasto-plastic body is 1. Equation (1.2.6) is supplemented with the boundary conditions

$$u(t) = w(t) \quad \text{on } \Gamma_0, \tag{1.2.7a}$$
$$\sigma(t)\nu = g(t) \quad \text{on } \Gamma_1. \tag{1.2.7b}$$

To deal with (1.2.6) and (1.2.7), it is convenient to introduce the continuous linear operator $\mathrm{div}_{\Gamma_0} : L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \to H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ defined by

$$\langle \mathrm{div}_{\Gamma_0}\sigma, \varphi\rangle := -\langle \sigma, E\varphi\rangle \tag{1.2.8}$$

for every $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$.

If $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, Γ_0, and Γ_1 are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (1.2.5), then we can prove, using integration by parts, that (1.2.6) and (1.2.7b) are equivalent to

$$\ddot{u}(t) - \mathrm{div}_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{1.2.9}$$

interpreted as equality between elements of $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In other words (1.2.9) is satisfied if and only if

$$\langle \ddot{u}(t), \varphi\rangle + \langle \sigma(t), E\varphi\rangle = \langle \mathcal{L}(t), \varphi\rangle \tag{1.2.10}$$

for every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In the irregular case, equation (1.2.10) represents the weak formulation of problem (1.2.6) with boundary condition (1.2.7b).

Plastic Dissipation. The elastic domain K is a convex and compact set in $\mathbb{M}^{n\times n}_D$. We will suppose that there exist two positive real numbers $r_1 < R_1$ such that

$$B(0, r_1) \subseteq K \subseteq B(0, R_1). \tag{1.2.11}$$

It is convenient to introduce the set

$$\mathcal{K}(\Omega) := \{\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D) : \xi(x) \in K \text{ for a.e. } x \in \Omega\}. \tag{1.2.12}$$

If π_K denotes the minimal distance projection of $\mathbb{M}^{n\times n}_D$ into K, and $\pi_{\mathcal{K}(\Omega)}$ denotes the projection of $L^2(\Omega; \mathbb{M}^{n\times n}_D)$ into $\mathcal{K}(\Omega)$, then it is easy to check that

$$(\pi_{\mathcal{K}(\Omega)}\xi)(x) = \pi_K \xi(x) \quad \text{for a.e. } x \in \Omega, \tag{1.2.13}$$

for every $\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$.

The evolution of the plastic strain $p(t, x)$ will be expressed by the Maximum Dissipation Principle (Hill's Principle of Maximum Work, see, e.g., [34], [45], [73]): if σ is the stress, then p will satisfy the following

$$(\sigma_D(t, x) - \xi) \cdot \dot{p}(t, x) \geq 0 \quad \text{for every } \xi \in K \text{ and a.e. } x \text{ in } \Omega$$
$$\sigma_D(t, x) - \dot{p}(t, x) \in K, \quad \text{for a.e. } x \text{ in } \Omega,$$

where we assume for simplicity that the viscosity coefficient is 1. Thanks to the characterization of the projection onto convex sets (see, e.g., [36]), this condition is satisfied if and only if $\sigma_D(t,x) - \dot{p}(t,x)$ coincides with $\pi_K \sigma_D(t,x)$, for a.e. $x \in \Omega$. By (1.2.13), this can be written as

$$\dot{p}(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)} \sigma_D(t). \tag{1.2.14}$$

We define the support function $H : \mathbb{M}^{n\times n}_D \to [0, +\infty[$ of K by

$$H(\xi) = \sup_{\zeta \in K} \zeta \cdot \xi. \tag{1.2.15}$$

It turns out that H is convex and positively homogeneous of degree one. In particular it satisfies the triangle inequality

$$H(\xi + \zeta) \leq H(\xi) + H(\zeta)$$

and the following inequality, due to (1.2.11):

$$r_1|\xi| \leq H(\xi) \leq R_1|\xi|. \tag{1.2.16}$$

We define $\mathcal{H} : L^2(\Omega; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ by

$$\mathcal{H}(p) = \int_\Omega H(p(x))dx. \tag{1.2.17}$$

If $p \in H^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D))$ and $\dot{p}(t)$ is its time derivative, then $\mathcal{H}(\dot{p})$ represents the rate of plastic dissipation, so that,

$$\int_0^T \mathcal{H}(\dot{p})dt \tag{1.2.18}$$

is the total plastic dissipation in the time interval $[0,T]$.

We notice that, by the definition of H, the subdifferential of H satisfies (see e.g. [61, Theorem 13.1])

$$\partial H(0) = K. \tag{1.2.19}$$

From (1.2.19), it easily follows

$$\partial \mathcal{H}(0) = \mathcal{K}(\Omega), \tag{1.2.20}$$

where $\partial \mathcal{H}(\xi)$ denotes the subdifferential of $\mathcal{H}$ at ξ.

1.2.2 Existence results for elasto-visco-plastic evolutions

Given an elasto-visco-plastic body satisfying all the properties described in the previous section, we fix an external load $\mathcal{L}$ and a Dirichlet boundary datum w, and look for a solution of the dynamic equation (1.2.9) and of the flow rule (1.2.14), with stress σ defined by (1.2.4) and strain satisfying equation (1.2.1). Our existence result for an elasto-visco-plastic evolution is given by the following theorem.

Theorem 1.2.1. *Let $T > 0$, let $\mathcal{L} \in AC([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n))$, and let w be a function such that*

$$w \in L^\infty([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{1.2.21a}$$

$$\dot{w} \in C^0([0,T]; L^2(\Omega; \mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{1.2.21b}$$

$$\ddot{w} \in L^2([0,T]; L^2(\Omega; \mathbb{R}^n)). \tag{1.2.21c}$$

Then for every $(u_0, e_0, p_0) \in A(w(0))$ *and* $v_0 \in L^2(\Omega; \mathbb{R}^n)$ *there exists a unique quadruple* (u, e, p, σ) *of functions, with*

$$u \in L^\infty([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{1.2.22a}$$
$$\dot{u} \in L^\infty([0,T]; L^2(\Omega; \mathbb{R}^n)), \tag{1.2.22b}$$
$$\ddot{u} \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)), \tag{1.2.22c}$$
$$e \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \tag{1.2.22d}$$
$$p \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{D})), \tag{1.2.22e}$$
$$\dot{e}_{A_1} \in L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \tag{1.2.22f}$$
$$\dot{p} \in L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{D})), \tag{1.2.22g}$$
$$\sigma \in L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \tag{1.2.22h}$$

such that for a.e. $t \in [0,T]$ *we have*

$$Eu(t) = e(t) + p(t), \tag{1.2.23a}$$
$$\sigma(t) = A_0 e(t) + A_1 \dot{e}_{A_1}(t), \tag{1.2.23b}$$
$$\ddot{u}(t) - \mathrm{div}_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{1.2.23c}$$
$$\dot{p}(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)}\sigma_D(t), \tag{1.2.23d}$$

and

$$u(t) = w(t) \quad \text{on } \Gamma_0, \tag{1.2.24}$$

$$u(0) = u_0, \quad p(0) = p_0, \tag{1.2.25a}$$
$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|e(t) - e_0\|^2_{L^2} dt = 0, \qquad \lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot{u}(t) - v_0\|^2_{L^2} dt = 0. \tag{1.2.25b}$$

In (1.2.22f) *and in the rest of the paper the symbol* $\dot{e}_{A_1}$ *denotes the time derivative (in the sense of distributions) of the function* e_{A_1} *defined before* (1.2.3).

Moreover (u, e, p, σ) *satisfies the equilibrium condition*

$$-\mathcal{H}(q) \le \langle \sigma(t), \eta\rangle + \langle \dot{p}(t), q\rangle + \langle \ddot{u}(t), \varphi\rangle - \langle \mathcal{L}(t), \varphi\rangle \le \mathcal{H}(-q), \tag{1.2.26}$$

for a.e. $t \in [0,T]$ *and for every* $(\varphi, \eta, q) \in A(0)$*, where* $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ *and* $H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ *in the terms containing* $\ddot{u}$ *and* $\mathcal{L}$*, while it denotes the scalar product in* L^2 *in all other terms.*

Remark 1.2.2. In view of (1.2.21) and (1.2.22) we see that u, w, $\dot{u}$, $\dot{w}$, e_{A_1}, and p are absolutely continuous in time, more precisely,

$$w \in AC([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{1.2.27a}$$
$$u, \dot{w} \in AC([0,T]; L^2(\Omega; \mathbb{R}^n)), \tag{1.2.27b}$$
$$\dot{u} \in AC([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)), \tag{1.2.27c}$$
$$e_{A_1} \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{1.2.27d}$$
$$p \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{D})) \tag{1.2.27e}$$

(see, e.g., [12], Proposition A.3 and following Corollary). Properties (1.2.27b) and (1.2.27e) give a precise meaning to the initial conditions (1.2.25a).

Moreover since u is bounded in $H^1(\Omega;\mathbb{R}^n)$ by (1.2.22a), we deduce from (1.2.27b) that $t\mapsto u(t)$ is weakly continuous into $H^1(\Omega;\mathbb{R}^n)$. Similarly, thanks to (1.2.27c) and since $\dot u\in L^\infty([0,T];L^2(\Omega;\mathbb{R}^n))$ by (1.2.22b), it follows that $t\mapsto \dot u(t)$ is weakly continuous into $L^2(\Omega;\mathbb{R}^n)$. Moreover, $e=Eu-p\in H^1([0,T];H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ by (1.2.22a), (1.2.22b), (1.2.22e), and (1.2.22g), thus $e\in AC([0,T];H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$. Since we have also $e\in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ by (1.2.22d), we conclude that $t\mapsto e(t)$ is weakly continuous into $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$. In particular for every $t\in[0,T]$ the functions $u(t)$, $e(t)$, $p(t)$, $\dot u(t)$ are univocally defined as elements of $H^1(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, $L^2(\Omega;\mathbb{M}^{n\times n}_D)$, and $L^2(\Omega;\mathbb{R}^n)$, respectively.

Remark 1.2.3. From (1.2.22), (1.2.23a), and (1.2.25) it follows that

$$\lim_{h\to0^+}\frac{1}{h}\int_0^h \|u(t)-u_0\|^2_{H^1}dt=0. \tag{1.2.28}$$

Indeed, by (1.2.27b) we have $\frac{1}{h}\int_0^h\|u(t)-u_0\|^2_{L^2}dt\to0$, and (1.2.22g), (1.2.23a), while (1.2.25b) give $\frac{1}{h}\int_0^h\|Eu(t)-Eu_0\|^2_{L^2}dt\to0$.

Before proving Theorem 1.2.1 we will first state the following result, which characterizes the solutions of equations (1.2.23c) and (1.2.23d).

Theorem 1.2.4. *Under the hypotheses of Theorem 1.2.1, let us assume that* (u,e,p,σ) *satisfies* (1.2.22), (1.2.23a), (1.2.23b), (1.2.24), *and* (1.2.25). *Then* (u,e,p,σ) *satisfies* (1.2.23c) *and* (1.2.23d) *for a.e.* $t\in[0,T]$ *if and only if both the following conditions hold:*

(a) Energy balance: for a.e $t\in[0,T]$ *we have*

$$\begin{aligned}&\mathcal{Q}_0(e(t))+\frac12\|\dot u(t)-\dot w(t)\|^2_{L^2}+\int_0^t\mathcal{Q}_1(\dot e_{A_1})ds+\int_0^t\|\dot p\|^2_{L^2}ds+\int_0^t\mathcal{H}(\dot p)ds=\\&=\mathcal{Q}_0(e_0)+\frac12\|v_0-\dot w(0)\|^2_{L^2}+\int_0^t\langle\sigma,E\dot w\rangle ds-\int_0^t\langle\ddot w,\dot u-\dot w\rangle ds\\&+\langle\mathcal{L}(t),u(t)-w(t)\rangle-\langle\mathcal{L}(0),u_0-w(0)\rangle-\int_0^t\langle\dot{\mathcal{L}},u-w\rangle ds,\end{aligned} \tag{1.2.29}$$

(b) For a.e. $t\in[0,T]$ *the equilibrium condition* (1.2.26) *holds for every* $(\varphi,\eta,q)\in A(0)$.

Moreover, if the two previous conditions are satisfied, then

$$\langle\sigma_D(t)-\dot p(t),\dot p(t)\rangle=\mathcal{H}(\dot p(t))\quad \textit{for a.e. } t\in[0,T]. \tag{1.2.30}$$

Remark 1.2.5. If A_1 is positive definite, then (1.2.27d), (1.2.27e), and the Korn inequality, imply that $u\in AC([0,T];H^1(\Omega;\mathbb{R}^n))$. If moreover the data w and $\mathcal{L}$ are sufficiently regular, $\mathcal{L}$ has the form (1.2.5), then we can integrate by parts the terms $\int_0^t\langle\ddot w,\dot u\rangle ds$ and $\int_0^t\langle\ddot w,\dot w\rangle ds$ obtaining that we can rewrite the energy balance as follows:

$$\begin{aligned}&\mathcal{Q}_0(e(t))+\frac{1}{2}\|\dot u(t)\|^2_{L^2}+\int_0^t \mathcal{Q}_1(\dot e)ds+\int_0^t\|\dot p\|^2_{L^2}ds+\int_0^t\mathcal{H}(\dot p)ds=\\&=\int_0^t\langle\sigma,E\dot w\rangle ds+\int_0^t\langle f,\dot u-\dot w\rangle ds+\int_0^t\langle g,\dot u-\dot w\rangle_{\Gamma_1}ds\\&+\int_0^t\langle\ddot u,\dot w\rangle ds+\mathcal{Q}_0(e_0)+\frac{1}{2}\|v_0\|^2_{L^2},\end{aligned}$$

which becomes, using $\ddot u=\mathrm{div}_{\Gamma_0}\sigma+\mathcal{L}$:

$$\begin{aligned}&\mathcal{Q}_0(e(t))+\frac{1}{2}\|\dot u(t)\|^2_{L^2}+\int_0^t \mathcal{Q}_1(\dot e)ds+\int_0^t\|\dot p\|^2_{L^2}ds+\int_0^t\mathcal{H}(\dot p)ds=\\&=\int_0^t\langle\sigma\nu,\dot u\rangle_{\Gamma_0}ds+\int_0^t\langle f,\dot u\rangle ds+\int_0^t\langle g,\dot u\rangle_{\Gamma_1}ds+\mathcal{Q}_0(e_0)+\frac{1}{2}\|v_0\|^2_{L^2},\end{aligned}$$

where we have used $\dot u=\dot w$ on Γ_0. This is the usual formulation of the energy balance. Indeed $\mathcal{Q}_0(e(t))$ is the stored elastic energy, $\frac{1}{2}\|\dot u(t)\|^2_{L^2}$ is the kinetic energy, $\int_0^t\mathcal{Q}_1(\dot e(t))ds$ is the visco-elastic dissipation, $\int_0^t\|\dot p\|^2_{L^2}ds$ is the visco-plastic dissipation, and $\int_0^t\mathcal{H}(\dot p)ds$ is the plastic dissipation. On the right-hand side the terms $\int_0^t\langle\sigma\nu,\dot u\rangle_{\Gamma_0}ds$, $\int_0^t\langle g,\dot u\rangle_{\Gamma_1}ds$, and $\int_0^t\langle f,\dot u\rangle ds$ represent the work done by the external forces on the Dirichlet boundary, on the Neumann boundary, and on the body itself, while the two terms $\mathcal{Q}_0(e_0)$ and $\frac{1}{2}\|v_0\|^2_{L^2}$ are the stored elastic energy and the kinetic energy at the initial time.

Lemma 1.2.6. *Let $T>0$, let $\mathcal{L}\in AC([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, let w satisfy* (1.2.21)*, and let (u,e,p,σ) be a quadruple satisfying* (1.2.22)*,* (1.2.23a)*,* (1.2.23b)*,* (1.2.23c)*,* (1.2.24)*, and* (1.2.25)*. Then*

$$\begin{aligned}&\mathcal{Q}_0(e(t))-\mathcal{Q}_0(e_0)+\int_0^t\mathcal{Q}_1(\dot e_{A_1})ds-\int_0^t\langle\sigma,E\dot w\rangle ds+\int_0^t\langle\sigma_D,\dot p\rangle ds\\&+\frac{1}{2}\|\dot u(t)-\dot w(t)\|^2_{L^2}-\frac{1}{2}\|v_0-\dot w(0)\|^2_{L^2}=-\int_0^t\langle\ddot w,\dot u-\dot w\rangle ds\\&+\langle\mathcal{L}(t),u(t)-w(t)\rangle-\langle\mathcal{L}(0),u_0-w(0)\rangle-\int_0^t\langle\dot{\mathcal{L}},u-w\rangle ds,\end{aligned}\tag{1.2.31}$$

for a.e. $t\in[0,T]$.

Proof. Given a function ϑ from $[0,T]$ into a Banach space X, for all $h>0$ we define the difference quotient $s^h\vartheta:[0,T-h]\to X$ as $s^h\vartheta(t):=\frac{1}{h}(\vartheta(t+h)-\vartheta(t))$. By (1.2.21), (1.2.22), and (1.2.24) for a.e. $t\in[0,T]$ the function $\varphi:=s^hu(t)-s^hw(t)$ belongs to $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. We use this function in (1.2.10) first at time t and then at time $t+h$. Summing the two expressions we get

$$\begin{aligned}&\langle\ddot u(t+h)-\ddot w(t+h)+\ddot u(t)-\ddot w(t),s^hu(t)-s^hw(t)\rangle+\langle hs^h\sigma(t),s^hp(t)-s^hEw(t)\rangle\\&+\langle A_0e(t+h)+A_1\dot e_{A_1}(t+h)+A_0e(t)+A_1\dot e_{A_1}(t),s^he(t)\rangle=\\&=-\langle\ddot w(t+h)+\ddot w(t),s^hu(t)-s^hw(t)\rangle+\langle\mathcal{L}(t+h)+\mathcal{L}(t),s^hu(t)-s^hw(t)\rangle.\end{aligned}\tag{1.2.32}$$

We now integrate in time on the interval $[0,t]$. An integration by parts in time gives that the first term is equal to

$$\begin{aligned}&\langle \dot u(t+h)-\dot w(t+h), s^h u(t)-s^h w(t)\rangle + \langle \dot u(t)-\dot w(t), s^h u(t)-s^h w(t)\rangle\\&-\langle \dot u(h)-\dot w(h), s^h u(0)-s^h w(0)\rangle - \langle \dot u(0)-\dot w(0), s^h u(0)-s^h w(0)\rangle\\&-\frac{1}{h}\int_t^{t+h}\|\dot u(r)-\dot w(r)\|^2_{L^2}dr+\frac{1}{h}\int_0^h\|\dot u(r)-\dot w(r)\|^2_{L^2}dr.\end{aligned}\tag{1.2.33}$$

As for the third term we find that it is equal to

$$\frac{2}{h}\int_t^{t+h}\mathcal{Q}_0(e(r))dr-\frac{2}{h}\int_0^h\mathcal{Q}_0(e(r))dr+\int_0^t\langle A_1(\dot e_{A_1}(r+h)+\dot e_{A_1}(r)), s^h e_{A_1}(r)\rangle dr,\tag{1.2.34}$$

while the last one is equal to

$$\begin{aligned}&\frac{2}{h}\int_t^{t+h}\langle \mathcal{L}(r), u(r)-w(r)\rangle dr-\frac{2}{h}\int_0^h\langle \mathcal{L}(r), u(r)-w(r)\rangle dr\\&-\int_0^t\langle s^h\mathcal{L}(r), u(r+h)-w(r+h)+u(r)-w(r)\rangle dr.\end{aligned}\tag{1.2.35}$$

Now (1.2.21), (1.2.25b), (1.2.27b), the weak continuity of $\dot u$ on $[0,T]$ into $L^2(\Omega;\mathbb{R}^n)$ (see Remark 1.2.2), and the Lebesgue mean value Theorem, allow us to pass to the limit as $h\to 0$ in (1.2.33) for a.e. $t\in[0,T]$. By similar arguments, using (1.2.21), (1.2.22), (1.2.25b), (1.2.28), and the weak continuity of u on $[0,T]$ into $H^1(\Omega;\mathbb{R}^n)$ (see Remark 1.2.2), we pass to the limit in (1.2.34), (1.2.35), and in the other terms of (1.2.32), so that we obtain (1.2.31) for a.e. $t\in[0,T]$. □

Proof of Theorem 1.2.4. Let us suppose that the quadruple (u,e,p,σ) satisfies (1.2.26) and (1.2.29); let us prove (1.2.23c). Let $\varphi\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$; since $(\varphi, E\varphi, 0)\in A(0)$, we can choose $\eta=E\varphi$ and $q=0$ in (1.2.26) and for a.e. $t\in[0,T]$ we get

$$\langle A_0 e(t)+A_1\dot e_{A_1}(t), E\varphi\rangle+\langle \ddot u(t),\varphi\rangle-\langle\mathcal{L}(t),\varphi\rangle=0,\tag{1.2.36}$$

which is equivalent to (1.2.23c), thanks to (1.2.10) and (1.2.23b).

It remains to prove (1.2.23d). Choosing $(0,q,-q)\in A(0)$ in (1.2.26) for some $q\in L^2(\Omega,\mathbb{M}^{n\times n}_D)$, for a.e. $t\in[0,T]$ we get

$$-\mathcal{H}(-q)\le\langle A_0 e(t)+A_1\dot e_{A_1}(t), q\rangle-\langle\dot p(t),q\rangle\le\mathcal{H}(q),\tag{1.2.37}$$

which, by (1.2.23b), says that

$$\sigma_D(t)-\dot p(t)\in\partial\mathcal{H}(0)=\mathcal{K}(\Omega)\tag{1.2.38}$$

thanks to the arbitrariness of q.

Now we observe that (u,e,p,σ) satisfies the hypotheses of Lemma 1.2.6, so (1.2.31) holds for a.e. $t\in[0,T]$. This, together with the energy balance (1.2.29), implies that (1.2.30) holds for a.e. $t\in[0,T]$. As a consequence, by the definition of $\mathcal{H}$, we deduce that for a.e. $t\in[0,T]$ and for every $\xi\in\mathcal{K}(\Omega)$ we have

$$\langle\sigma_D(t)-\dot p(t),\dot p(t)\rangle\ge\langle\xi,\dot p(t)\rangle,$$

which is equivalent to

$$\langle \sigma_D(t) - (\sigma_D(t) - \dot{p}(t)), \xi - (\sigma_D(t) - \dot{p}(t)) \rangle \leq 0.$$

Thanks to (1.2.38), $\sigma_D(t) - \dot{p}(t)$ belongs to $\mathcal{K}(\Omega)$; therefore the arbitrariness of ξ and the well-known characterization of the projection onto convex sets (see, e.g., [36], Chapter 1.2) give that $\sigma_D(t) - \dot{p}(t) = \pi_{\mathcal{K}(\Omega)}\sigma_D(t)$ for a.e. $t \in [0, T]$.

Conversely, suppose (u, e, p, σ) to be a solution of the system of equations (1.2.23). Then (1.2.23d) implies (1.2.38), which in turn gives (1.2.37). On the other hand (1.2.23b) and (1.2.23c) give (1.2.36). Subtracting (1.2.37) from (1.2.36) term by term and taking into account (1.2.1a), we get (1.2.26).

In order to obtain the energy balance we first prove that, if a function (u, e, p, σ) satisfies (1.2.23), then (1.2.30) holds. Indeed, if $\xi \in \mathcal{K}(\Omega)$, then from the properties of convex sets it follows that for a.e. $t \in [0, T]$

$$\begin{aligned}&(\sigma_D - \dot{p}) \cdot \dot{p} = \pi_K \sigma_D \cdot (\sigma_D - \pi_K \sigma_D) \geq \\ &\geq \pi_K \sigma_D \cdot (\sigma_D - \pi_K \sigma_D) + (\xi - \pi_K \sigma_D) \cdot (\sigma_D - \pi_K \sigma_D) = \xi \cdot (\sigma_D - \pi_K \sigma_D)\end{aligned}$$

almost everywhere in Ω, that is $(\sigma_D - \dot{p}) \cdot \dot{p} \geq H(\sigma_D - \pi_K \sigma_D) = H(\dot{p})$ thanks to the definition of H. Since $\sigma_D - \dot{p} \in K$ a.e. in Ω and for a.e. $t \in [0, T]$ by (1.2.23d), the definition of H gives also the opposite inequality. So integrating on Ω we get (1.2.30).

Now since (u, e, p, σ) satisfies the hypotheses of Lemma 1.2.6, we obtain (1.2.31), which together with (1.2.30) gives the energy balance (1.2.29) for a.e. $t \in [0, T]$. □

Proof of Theorem 1.2.1. The proof is reminiscent of that of [7, Theorem 3.1], with some important differences. In [7, Theorem 3.1] only Dirichlet conditions are considered and the data of the problem are more regular than ours: the external load f belongs to $AC([0, T]; L^2(\Omega; \mathbb{R}^n))$ and the boundary datum w belongs to $H^2([0, T]; H^1(\Omega; \mathbb{R}^n)) \cap H^3([0, T]; L^2(\Omega; \mathbb{R}^n))$. Moreover, the model discussed in [7] is slightly different from ours: in [7] the plastic component of the strain plays a role in the viscous part of the stress, while we assume that the component $\dot{p}$ of the strain rate does not affect the viscous stress, which only depends on $\dot{e}$. This leads to a different flow rule, whose strong form cannot be proved directly from the approximate flow rules as in [7]; for this reason we prefer to prove first the energy balance and then to derive the flow rule from it.

As in [7] we will obtain the solution by time discretization, considering the limit of approximate solutions constructed by solving incremental minimum problems. Given an integer $N > 0$ we define $\tau = T/N$ and subdivide the interval $[0, T)$ into N subintervals $[t_i, t_{i+1})$, $i = 0, \ldots, N-1$ of length τ, with $t_i = i\tau$. Let us set

$$\begin{aligned}&u_{-1} = u_0 - \tau v_0, \quad w_{-1} = w_0 - \tau \dot{w}(0), \\ &w_i = w(t_i), \qquad \mathcal{L}_i = \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \mathcal{L}(s) ds.\end{aligned}$$

We construct a sequence (u_i, e_i, p_i) with $i = 0, 1, \ldots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data in (1.2.25). Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \ldots, i$. Then

$(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the unique minimizer on $A(w_{i+1})$ of the functional

$$V_i(u,e,p) = \frac{1}{2}\langle A_0 e, e\rangle + \frac{1}{2\tau}\langle A_1(e-e_i), e-e_i\rangle + \frac{1}{2\tau}\|p-p_i\|^2_{L^2} + \mathcal{H}(p-p_i) + \frac{1}{2}\Big\|\frac{u-u_i}{\tau} - \frac{u_i-u_{i-1}}{\tau}\Big\|^2_{L^2} - \langle \mathcal{L}_i, u\rangle, \qquad (1.2.39)$$

which turns out to be coercive and strictly convex on $A(w_{i+1})$.

To obtain the Euler conditions we observe that $(u_{i+1}, e_{i+1}, p_{i+1}) + \lambda(\varphi, \eta, q) \in A(w_{i+1})$ for every $(\varphi, \eta, q) \in A(0)$, and for every $\lambda \in \mathbb{R}$. Evaluating V_i in this point and differentiating with respect to λ at $0^\pm$ we get

$$-\mathcal{H}(q) \le \langle A_0 e_{i+1}, \eta\rangle + \frac{1}{\tau}\langle A_1(e_{i+1}-e_i), \eta\rangle + \frac{1}{\tau}\langle p_{i+1}-p_i, q\rangle + \frac{1}{\tau}\langle v_{i+1}-v_i, \varphi\rangle - \langle \mathcal{L}_i, \varphi\rangle \le \mathcal{H}(-q), \qquad (1.2.40)$$

where we have set

$$v_j = \frac{1}{\tau}(u_j - u_{j-1}). \qquad (1.2.41)$$

We now define the piecewise affine interpolation $u_\tau, e_\tau, p_\tau, w_\tau$ on $[0,T]$ by

$$u_\tau(t) = u_i + \frac{u_{i+1}-u_i}{\tau}(t-t_i) \quad \text{if } t \in [t_i, t_{i+1}) \qquad (1.2.42a)$$

$$e_\tau(t) = e_i + \frac{e_{i+1}-e_i}{\tau}(t-t_i) \quad \text{if } t \in [t_i, t_{i+1}) \qquad (1.2.42b)$$

$$p_\tau(t) = p_i + \frac{p_{i+1}-p_i}{\tau}(t-t_i) \quad \text{if } t \in [t_i, t_{i+1}) \qquad (1.2.42c)$$

$$w_\tau(t) = w_i + \frac{w_{i+1}-w_i}{\tau}(t-t_i) \quad \text{if } t \in [t_i, t_{i+1}) \qquad (1.2.42d)$$

To simplify the notation we also set $\omega_i = \frac{1}{\tau}(w_i - w_{i-1}) = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} \dot{w}(s)ds$ and define, for $t \in [0,T]$,

$$\omega_\tau(t) = \omega_i + (\omega_{i+1} - \omega_i)\frac{t-t_i}{\tau} \quad \text{if } t \in [t_i, t_{i+1}), \qquad (1.2.43a)$$

$$v_\tau(t) = v_i + (v_{i+1} - v_i)\frac{t-t_i}{\tau} \quad \text{if } t \in [t_i, t_{i+1}). \qquad (1.2.43b)$$

The proof now is divided into five steps: in the first one we prove that a subsequence of (u_τ, e_τ, p_τ) has a limit (u, e, p) as $\tau \to 0$, and we show that this limit satisfies the regularity conditions (1.2.22). In the second step we pass to the limit in (1.2.40), obtaining the equilibrium condition (1.2.26). In the third step we obtain the energy balance (1.2.29) for (u, e, p). In the fourth step we prove that (u, e, p) satisfies the initial conditions (1.2.25). From this and Theorem 1.2.4 it will follow that (u, e, p) satisfies the required equations (1.2.23). In the last step we prove the uniqueness.

Step 1. Since $\ddot{w} \in L^2([0,T]; L^2(\Omega;\mathbb{R}^n))$ and $\dot{w} \in L^2([0,T]; H^1(\Omega;\mathbb{R}^n))$, we see that

$$w_\tau \to w \ \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \qquad (1.2.44a)$$

$$\dot{w}_\tau \to \dot{w} \ \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \qquad (1.2.44b)$$

$$\omega_\tau \to \dot{w} \ \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \qquad (1.2.44c)$$

$$\dot{\omega}_\tau \to \ddot{w} \ \text{ strongly in } L^2([0,T]; L^2(\Omega;\mathbb{R}^n)). \qquad (1.2.44d)$$

The proof of the first three properties is straightforward. To prove (1.2.44d) we first put $\tilde{w}_\tau(t) := \frac{1}{\tau}\int_{t_i}^{t_{i+1}}\ddot{w}(s)ds \in L^2(\Omega;\mathbb{R}^n)$ for $t \in [t_i, t_{i+1})$. Since $\tilde{w}_\tau$ tends to $\ddot{w}$, it suffices to show that $\tilde{w}_\tau - \dot{\omega}_\tau$ tends to 0 strongly in $L^2([0,T];L^2(\Omega;\mathbb{R}^n))$. So we write

$$\begin{aligned}\|\dot{\omega}_\tau - \tilde{w}_\tau\|^2_{L^2(L^2)} &= \sum_{i=0}^{N-1}\tau\Big\|\frac{1}{\tau}\int_{t_i}^{t_{i+1}}\Big(\frac{1}{\tau}\int_{s-\tau}^{s}\ddot{w}(r)dr - \ddot{w}(s)\Big)ds\Big\|^2_{L^2} \le \\ &\le \frac{1}{\tau}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\int_{s-\tau}^{s}\|\ddot{w}(r)-\ddot{w}(s)\|^2_{L^2}drds \le \\ &\le \frac{1}{\tau}\sum_{i=0}^{N-1}\int_{t_{i-1}}^{t_{i+1}}\int_{t_{i-1}}^{t_{i+1}}\|\ddot{w}(r)-\ddot{w}(s)\|^2_{L^2}drds,\end{aligned}$$

where we set $\ddot{w}(s) = 0$ for $s < 0$. Defining $W(r,s) = \|\ddot{w}(r) - \ddot{w}(s)\|^2_{L^2}$, we see that the integral in the last line is bounded by

$$\frac{2}{\tau}\int_{-2\tau}^{2\tau}dh\int_0^T W(r, r+h)dr,$$

that turns out to go to 0 as $\tau \to 0$, because $h \mapsto \int_0^T W(r, r+h)dr$ is continuous and vanishes at $h = 0$.

We shall use the three following identities:

$$\langle A^0 e_{i+1}, e_{i+1} - e_i\rangle = \int_{t_i}^{t_{i+1}}\langle A^0 e_\tau, \dot{e}_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}}\langle A^0\dot{e}_\tau, \dot{e}_\tau\rangle ds, \quad (1.2.45)$$

$$\begin{aligned}&\langle A^0 e_{i+1}, Ew_{i+1} - Ew_i\rangle = \\ &\quad= \int_{t_i}^{t_{i+1}}\langle A^0 e_\tau, E\dot{w}_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}}\langle A^0\dot{e}_\tau, E\dot{w}_\tau\rangle ds, \quad (1.2.46)\\ &\langle (v_{i+1} - v_i) - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle = \\ &\quad= \frac{1}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} - \frac{1}{2}\|v_i - \omega_i\|^2_{L^2} + \frac{\tau}{2}\int_{t_i}^{t_{i+1}}\|\dot{v}_\tau - \dot{\omega}_\tau\|^2_{L^2}ds. \quad (1.2.47)\end{aligned}$$

We put

$$\begin{aligned}&\varphi = u_{i+1} - u_i - (w_{i+1} - w_i),\\ &\eta = e_{i+1} - e_i - (Ew_{i+1} - Ew_i),\\ &q = p_{i+1} - p_i, \qquad (1.2.48)\end{aligned}$$

into (1.2.40) and take the sum over $i = 0, \ldots, j-1$. Using (1.2.45)-(1.2.47) we get

$$
\begin{aligned}
&\int_0^{t_j} \langle A^0 e_\tau, \dot e_\tau\rangle ds + \frac{\tau}{2}\int_0^{t_j} \langle A^0 \dot e_\tau, \dot e_\tau\rangle ds + \int_0^{t_j} \langle A^1 \dot e_\tau, \dot e_\tau\rangle ds \\
&+ \int_0^{t_j} \|\dot p_\tau\|^2_{L^2} ds + \frac{\tau}{2}\int_0^{t_j} \|\dot v_\tau - \dot\omega_\tau\|^2 ds + \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1}))\|^2 \le \\
&\le \int_0^{t_j} \mathcal{H}(-\dot p_\tau) ds + \langle \mathcal{L}(t_j), u_\tau(t_j) - w_\tau(t_j)\rangle - \langle \mathcal{L}(0), u_\tau(0) - w_\tau(0)\rangle \\
&- \int_0^{t_j} \langle \dot{\mathcal{L}}, u_\tau - w_\tau\rangle ds - \int_0^{t_j} \langle \dot\omega_\tau, \dot u_\tau - \dot w_\tau\rangle ds \\
&+ \frac{1}{2}\|v_0 - \omega_0\|^2 + \int_0^{t} \langle A^0 e_\tau + A^1 \dot e_\tau + \frac{\tau}{2} A^0 \dot e_\tau, E\dot w_\tau\rangle ds, \qquad (1.2.49)
\end{aligned}
$$

By (1.2.16) there exists a constant C such that $\mathcal{H}(q) \le C\|q\|_{L^2}$ for every $q \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$. Therefore we obtain

$$
\begin{aligned}
&\frac{1}{2}\langle A^0 e_\tau(t_j), e_\tau(t_j)\rangle + \frac{\tau}{2}\int_0^{t_j} \langle A^0 \dot e_\tau, \dot e_\tau\rangle ds + \int_0^{t_j} \langle A^1 \dot e_\tau, \dot e_\tau\rangle ds \\
&+ \int_0^{t_j} \|\dot p_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1})\|^2_{L^2} \le \\
&\le C\int_0^{t_j} \|\dot p_\tau\|_{L^2} dt + \frac{1}{2\lambda}\int_0^{t_j} \|\dot\omega_\tau\|^2_{L^2} ds + \frac{\lambda}{2}\int_0^{t_j} \|\dot u_\tau - \dot w_\tau\|^2_{L^2} ds + \frac{1}{2\lambda}\|\mathcal{L}(t_j)\|^2_{H^{-1}_{\Gamma_0}} \\
&+ \frac{\lambda}{2}\|u_\tau - w_\tau\|^2_{H^1_{\Gamma_0}} + \frac{1}{2\lambda}\int_0^{t_j} \|\dot{\mathcal{L}}_\tau\|^2_{H^{-1}_{\Gamma_0}} ds + \frac{\lambda}{2}\int_0^{t_j} \|u_\tau - w_\tau\|^2_{H^1_{\Gamma_0}} ds \\
&+ \frac{3}{2\lambda}\int_0^{t_j} \|E\dot w_\tau\|^2_{L^2} ds + \frac{\lambda}{2}\int_0^{t_j} \|A^0 e_\tau\|^2_{L^2} ds \\
&+ \frac{\lambda}{2}\int_0^{t_j} \|A^1 \dot e_\tau\|^2_{L^2} ds + \frac{\tau\lambda}{4}\int_0^{t_j} \|A^0 \dot e_\tau\|^2_{L^2} ds + D, \qquad (1.2.50)
\end{aligned}
$$

where λ is an arbitrary positive number, that we will choose later, and C and D are positive constants independent of λ.

From (1.2.44) and the hypothesis on $\mathcal{L}$ we see that the term

$$
\int_0^{t_j} \|\dot\omega_\tau\|^2_{L^2} ds + + \frac{1}{2\lambda}\|\mathcal{L}(t_j)\|^2_{H^{-1}_{\Gamma_0}} + \int_0^{t_j} \|\dot{\mathcal{L}}_\tau\|^2_{H^{-1}_{\Gamma_0}} ds + \int_0^{t_j} \|E\dot w_\tau\|^2_{L^2} ds
$$

is bounded from above. By Poincaré and Korn inequalities there exists a constant γ such that

$$
\|u_\tau - w_\tau\|^2_{H^1_{\Gamma_0}} \le \gamma(\|e_\tau\|^2_{L^2} + \|p_\tau\|^2_{L^2} + \|Ew_\tau\|^2_{L^2}).
$$

Since for some constant $C_1 > 0$

$$
C\int_0^{t_j} \|\dot p_\tau\|_{L^2} ds \le C_1 + \frac{1}{2}\int_0^{t_j} \|\dot p_\tau\|^2_{L^2} ds,
$$

writing $p_\tau(t_j) = \int_0^{t_j} \dot p_\tau ds$ and then using the Cauchy inequality and formula

(1.2.2), we get from (1.2.50)

$$\frac{\alpha_0}{2}\|e_\tau(t_j)\|^2_{L^2} + \frac{\alpha_0\tau}{2}\int_0^{t_j}\|\dot e_\tau\|^2_{L^2}ds + \alpha_1\int_0^{t_j}\|(\dot e_\tau)_{A_1}\|^2_{L^2}ds + \frac{1}{2}\int_0^{t_j}\|\dot p_\tau\|^2_{L^2}ds$$
$$+\frac{1}{2}\|\dot u_\tau(t_j^-) - \dot w_\tau(t_j^-)\|^2_{L^2} \leq$$
$$\leq \frac{\lambda\gamma}{2}\|e_\tau(t_j)\|^2_{L^2} + \frac{\lambda\gamma T}{2}\int_0^{t_j}\|\dot p_\tau\|^2_{L^2}ds + \frac{\lambda\beta_0^2}{2}\int_0^{t_j}\|e_\tau\|^2_{L^2}ds$$
$$+\frac{\lambda}{2}\int_0^{t_j}\|\dot u_\tau - \dot w_\tau\|^2_{L^2}ds + \frac{\lambda\beta_1^2}{2}\int_0^{t_j}\|(\dot e_\tau)_{A_1}\|^2_{L^2}ds + \tau\frac{\lambda\beta_0^2}{4}\int_0^{t_j}\|\dot e_\tau\|^2_{L^2}ds + M_\lambda,$$

where M_λ is a constant depending on λ. Choosing now λ in such a way that $2\lambda\gamma < \alpha_0$, $2\lambda\gamma T < 1$, $\lambda\beta_1^2 < \alpha_1$, $2\lambda\beta_0^2 < \alpha_0$, and $\lambda < 1$ we obtain

$$\frac{\alpha_0}{4}\|e_\tau(t_j)\|^2_{L^2} + \frac{\alpha_0\tau}{4}\int_0^{t_j}\|\dot e_\tau\|^2_{L^2}ds + \frac{\alpha_1}{2}\int_0^{t_j}\|(\dot e_\tau)_{A_1}\|^2_{L^2}ds + \frac{1}{4}\int_0^{t_j}\|\dot p_\tau\|^2_{L^2}ds$$
$$+\frac{1}{2}\|\dot u_\tau(t_j^-) - \dot w_\tau(t_j^-)\|^2_{L^2} \leq \frac{\alpha_0}{4}\int_0^{t_j}\|e_\tau\|^2_{L^2}ds + \frac{1}{2}\int_0^{t_j}\|\dot u_\tau - \dot w_\tau\|^2_{L^2}ds + M_\lambda. \tag{1.2.51}$$

Now neglecting some non-negative terms in the left-hand side we get

$$\frac{\alpha_0}{2}\|e_\tau(t)\|^2_{L^2} + \|\dot u_\tau^-(t) - \dot w_\tau^-(t)\|^2_{L^2} \leq K + \int_0^t \frac{\alpha_0}{2}\|e_\tau\|^2_{L^2} + \|\dot u_\tau^- - \dot w_\tau^-\|^2_{L^2}ds \tag{1.2.52}$$

for all $t \in [0,T]$, where K is a positive constant independent of τ and $\dot u_\tau^- - \dot w_\tau^-$ denotes the left-continuous representative of the piecewise constant function $\dot u_\tau - \dot w_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n))$. So we can use Gronwall lemma to obtain that e_τ and $\dot u_\tau$ are bounded in $L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ and $L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n))$ respectively, uniformly with respect to τ. Going back to (1.2.51) we also obtain:

$$\dot u_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{1.2.53a}$$
$$e_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.53b}$$
$$(\dot e_\tau)_{A_1} \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.53c}$$
$$\dot p_\tau \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{1.2.53d}$$

and $\dot u_\tau, e_\tau, (\dot e_\tau)_{A_1}, \dot p_\tau$ are bounded in these spaces uniformly with respect to τ. For the first condition above we have used that ω_τ is uniformly bounded in $L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n))$, as a consequence of (1.2.21) and (1.2.43). Moreover from the same estimate we find that

$$\tau^{\frac{1}{2}}\dot e_\tau \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.54}$$

uniformly with respect to τ. We can then pass to the limit as τ tends to 0 in a subsequence, and find functions v, e, h and q such that

$$\dot u_\tau \rightharpoonup v \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{1.2.55a}$$
$$e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.55b}$$
$$(\dot e_\tau)_{A_1} \rightharpoonup h \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.55c}$$
$$\dot p_\tau \rightharpoonup q \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)). \tag{1.2.55d}$$

From (1.2.55b) we see that $(e_\tau)_{A_1} \rightharpoonup e_{A_1}$ weakly* in $L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}))$, and writing $(e_\tau)_{A_1}(t) = \int_0^t (\dot e_\tau)A_1 ds + (e_0)_{A_1}$ we see that

$$(e_\tau)_{A_1} \text{ is bounded in } L^\infty([0,T]; L^2(\Omega; \mathbb{R}^n)), \tag{1.2.56a}$$
$$(e_\tau)_{A_1}(t) \rightharpoonup e_{A_1}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.56b}$$
$$(\dot e_\tau)_{A_1} \rightharpoonup \dot e_{A_1} \quad \text{weakly in } L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})), \tag{1.2.56c}$$

From the estimates (1.2.55) and from the equalities $u_\tau(t) = \int_0^t \dot u_\tau ds + u_0$ and $p_\tau(t) = \int_0^t \dot p_\tau ds + p_0$ it follows that

$$u_\tau \text{ is bounded in } L^\infty([0,T]; L^2(\Omega; \mathbb{R}^n)), \tag{1.2.56d}$$
$$p_\tau \text{ is bounded in } L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D)), \tag{1.2.56e}$$
$$u_\tau(t) \rightharpoonup u(t) := \int_0^t v(s)ds + u_0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n), \tag{1.2.56f}$$
$$p_\tau(t) \rightharpoonup p(t) := \int_0^t q(s)ds + p_0 \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_D) \tag{1.2.56g}$$

for every $t \in [0,T]$. Note that, by (1.2.55a) and (1.2.55d) we deduce that

$$u \in L^\infty([0,T]; L^2(\Omega; \mathbb{R}^n)), \tag{1.2.57a}$$
$$p \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D)). \tag{1.2.57b}$$

and in particular

$$p_\tau \rightharpoonup p \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D)),$$

In view of (1.2.56) we see that u and p are absolutely continuous and that their derivatives with respect to t coincide with v and q almost everywhere in $[0,T]$, in other words

$$\dot u_\tau \rightharpoonup \dot u \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega; \mathbb{R}^n)),$$
$$\dot p_\tau \rightharpoonup \dot p \quad \text{weakly in } L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D)).$$

Now, the identity

$$Eu_\tau(t) = e_\tau(t) + p_\tau(t), \tag{1.2.58}$$

together with conditions (1.2.53b) and (1.2.56e), implies that $Eu_\tau(t)$ is bounded in $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ uniformly with respect to τ and t. Then the Korn inequality implies that $Du_\tau(t)$ is actually uniformly bounded in $L^2(\Omega; \mathbb{M}^{n\times n})$, so since $u_\tau(t) \rightharpoonup u(t)$ weakly in $L^2(\Omega; \mathbb{R}^n))$, we get $u(t) \in H^1(\Omega; \mathbb{R}^n)$ and

$$u_\tau(t) \rightharpoonup u(t) \text{ weakly in } H^1(\Omega; \mathbb{R}^n) \text{ and strongly in } L^2(\Omega; \mathbb{R}^n) \tag{1.2.59}$$

for all $t \in [0,T]$. Hence (1.2.58) passes to the limit giving

$$Eu(t) = e(t) + p(t) \tag{1.2.60}$$

for all $t \in [0,T]$. To summarize the previous discussion, we have obtained the following convergences

$$u_\tau \rightharpoonup u \quad \text{weakly* in } \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.61a}$$
$$\dot u_\tau \rightharpoonup \dot u \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{1.2.61b}$$
$$e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{1.2.61c}$$
$$(\dot e_\tau)_{A_1} \rightharpoonup \dot e_{A_1} \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{1.2.61d}$$
$$p_\tau \rightharpoonup p \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{1.2.61e}$$
$$\dot p_\tau \rightharpoonup \dot p \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)). \tag{1.2.61f}$$

Let $\varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Putting $\eta = E\varphi$ and $q = 0$ in (1.2.40) we get

$$-\text{div}_{\Gamma_0}(A_0 e_{i+1}) - \text{div}_{\Gamma_0}(A_1 \frac{e_{i+1}-e_i}{\tau}) + \frac{v_{i+1}-v_i}{\tau} = \mathcal{L}_i,$$

which allows us to deduce from (1.2.53b) and (1.2.53c) that $\dot v_\tau = \frac{v_{i+1}-v_i}{\tau}$ is bounded in $L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ uniformly with respect to τ, thanks to the continuity of the operator div_{Γ_0}.

So, using the Hölder inequality, we estimate

$$\|v_\tau(t) - v_\tau(t_{i+1})\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2} M \quad \text{for } t \in [t_i, t_{i+1}),$$

for some positive constant M independent of τ, t, and i. Since $\dot u_\tau(t) = v_\tau(t_{i+1})$ for $t \in [t_i, t_{i+1})$ we have

$$\|v_\tau(t) - \dot u_\tau(t)\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2} M,$$

so that $v_\tau - \dot u_\tau$ tends to 0 strongly in $L^\infty([0,T], H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$. From this it easily follows that the two sequences v_τ and $\dot u_\tau$ must have the same weak* limit in $L^\infty([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, so

$$v_\tau \rightharpoonup \dot u \quad \text{weakly* in } L^\infty([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{1.2.62}$$

The boundness condition proved above implies that $\dot v_\tau$ tends, up to a subsequence, to a function ζ weakly in $L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and it easily follows that $\zeta = \ddot u$. Therefore

$$\dot v_\tau \rightharpoonup \ddot u \quad \text{weakly in } L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{1.2.63}$$

We now define $\sigma(t) := A_0 e(t) + A_1 \dot e_{A_1}(t)$. The results proved so far imply that (u, e, p, σ) satisfies (1.2.22).

Step 2. In order to show that the functions above satisfy (1.2.23) we need to pass to the limit in (1.2.40). We consider the piecewise constant interpolation $\tilde e_\tau$ defined by

$$\tilde e_\tau(t) = e_{i+1} \quad \text{if } t \in [t_i, t_{i+1}).$$

We want to prove that

$$\tilde e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})). \tag{1.2.64}$$

Since $\tilde{e}_\tau$ is bounded in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ it is not restrictive to assume that $\tilde{e}_\tau \rightharpoonup \tilde{e}$ weakly* in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$. Since $e_\tau = Eu_\tau - p_\tau$, by (1.2.61b) and (1.2.61c) we have that

$$(e_\tau)_{\tau>0} \text{ is bounded } H^1([0,T];H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\rm sym})). \tag{1.2.65}$$

Therefore, using the Hölder inequality, we obtain

$$\|e_\tau(t) - e_\tau(t_{i+1})\|_{H^{-1}} \leq \tau^{1/2} M \quad \text{for } t \in [t_i, t_{i+1}),$$

for some constant $M > 0$ independent of τ, t, and i. Since $\tilde{e}_\tau(t) = e_\tau(t_{i+1})$ for $t \in [t_i, t_{i+1})$, we have

$$\|e_\tau(t) - \tilde{e}_\tau(t)\|_{H^{-1}} \leq \tau^{1/2} M \quad \text{for all } t \in [0,T].$$

This implies $e = \tilde{e}$ and concludes the proof of (1.2.64).

We also define the piecewise affine interpolation $\mathcal{L}_\tau$ by

$$\mathcal{L}_\tau(t) = \mathcal{L}_i + (\mathcal{L}_{i+1} - \mathcal{L}_i)\frac{t-t_i}{\tau} \quad \text{if } \ t \in [t_i, t_{i+1}),$$

where $\mathcal{L}_i := \mathcal{L}(t_i)$. By standard properties of L^2 functions and of their approximation by averaging on subintervals, we have that

$$\mathcal{L}_\tau \to \mathcal{L} \quad \text{strongly in } L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{1.2.66a}$$
$$\dot{\mathcal{L}}_\tau \to \dot{\mathcal{L}} \quad \text{strongly in } L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{1.2.66b}$$

For fixed τ (1.2.40) says that for a.e. $t \in [0,T]$ we have

$$-\mathcal{H}(q) \leq \langle A_0\tilde{e}_\tau, \eta\rangle + \langle A_1(\dot{e}_\tau)_{A_1}, \eta\rangle + \langle \dot{p}_\tau, q\rangle + \langle \dot{v}_\tau, \varphi\rangle - \langle \mathcal{L}_\tau, \varphi\rangle \leq \mathcal{H}(-q)$$

for every $(\varphi,\eta,q) \in A(0)$. All terms in the formula above converge weakly in $L^1([0,T])$ as $\tau \to 0$, thanks to (1.2.61d), (1.2.61e), (1.2.63), (1.2.64), and (1.2.66). So for every $(\varphi,\eta,q) \in A(0)$ we can pass to the limit obtaining

$$-\mathcal{H}(q) \leq \langle A_0 e, \eta\rangle + \langle A_1\dot{e}_{A_1}, \eta\rangle + \langle \dot{p}, q\rangle + \langle \ddot{u}, \varphi\rangle - \langle \mathcal{L}, \varphi\rangle \leq \mathcal{H}(-q) \tag{1.2.67}$$

for a.e. $t \in [0,T]$. Since the space $A(0)$ is separable, we can construct a set of full measure in $[0,T]$ such that (1.2.67) holds in this set for every $(\varphi,\eta,q) \in A(0)$, which gives (1.2.26).

Step 3. We will now prove the energy balance (1.2.29). Let $\lambda \in (0,1)$ and put $\varphi = u_{i+1} - \lambda(u_{i+1} - u_i) + \lambda(w_{i+1} - w_i)$, $\eta = e_{i+1} - \lambda(e_{i+1} - e_i) + \lambda(Ew_{i+1} - Ew_i)$, and $q = p_{i+1} - \lambda(p_{i+1} - p_i)$, so by the minimality of $(u_{i+1}, e_{i+1}, p_{i+1})$ for the functional V_i defined by (1.2.39) we have $V_i(u_{i+1}, e_{i+1}, p_{i+1}) \leq V_i(\varphi,\eta,q)$.

This implies

$$
\begin{aligned}
&\frac{1}{2}\langle A_0 e_{i+1}, e_{i+1}\rangle + \frac{1}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle + \frac{1}{2\tau}\|p_{i+1}-p_i\|^2_{L^2}\\
&+\mathcal{H}(p_{i+1}-p_i) + \frac{1}{2}\|v_{i+1}-v_i\|^2_{L^2} - \langle \mathcal{L}_i, u_{i+1}\rangle \leq\\
&\leq \frac{(1-\lambda)^2}{2}\langle A_0 e_{i+1}, e_{i+1}\rangle + \lambda(1-\lambda)\langle A_0 e_{i+1}, e_i\rangle + \frac{\lambda^2}{2}\langle A_0 e_i, e_i\rangle\\
&+\frac{\lambda^2}{2}\langle A_0(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle + \lambda\langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle\\
&-\lambda^2\langle A_0(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle + \frac{(1-\lambda)^2}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle\\
&+\frac{\lambda^2}{2\tau}\langle A_1(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle\\
&+\frac{\lambda(1-\lambda)}{\tau}\langle A_1(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle\\
&+\frac{(1-\lambda)^2}{2\tau}\|p_{i+1}-p_i\|^2_{L^2} + (1-\lambda)\mathcal{H}(p_{i+1}-p_i) + \frac{1}{2}\|v_{i+1}-v_i\|^2_{L^2}\\
&+\frac{\lambda^2}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2} - \lambda\langle v_{i+1}-v_i-(\omega_{i+1}-\omega_i), v_{i+1}-\omega_{i+1}\rangle\\
&-\langle \mathcal{L}_i, u_{i+1}\rangle + \lambda\tau\langle \mathcal{L}_i - \frac{\omega_{i+1}-\omega_i}{\tau}, v_{i+1}-\omega_{i+1}\rangle.
\end{aligned}
$$

Dividing by λ we get

$$
\begin{aligned}
&\frac{2-\lambda}{2}(A_0 e_{i+1}, e_{i+1}) - (1-\lambda)\langle A_0 e_{i+1}, e_i\rangle\\
&-\langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle + \lambda\langle A_0(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle\\
&-\frac{\lambda}{2}\langle A_0(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle + \frac{2-\lambda}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle\\
&+\frac{2-\lambda}{2\tau}\|p_{i+1}-p_i\|^2_{L^2} + \mathcal{H}(p_{i+1}-p_i) - \frac{\lambda}{2\tau}\langle A_1(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle\\
&-\frac{1-\lambda}{\tau}\langle A_1(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle + \langle v_{i+1}-v_i-(\omega_{i+1}-\omega_i), v_{i+1}-\omega_{i+1}\rangle\\
&-\tau\langle \mathcal{L}_i - \frac{\omega_{i+1}-\omega_i}{\tau}, v_{i+1}-\omega_{i+1}\rangle \leq \frac{\lambda}{2}\langle A_0 e_i, e_i\rangle + \frac{\lambda}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2}.
\end{aligned}
$$

Since $\langle A_0 e_{i+1}, e_{i+1}\rangle \geq 0$ and $\lambda \in (0,1)$ it follows that

$$
\begin{aligned}
&(1-\lambda)\langle A_0 e_{i+1}, e_{i+1}-e_i\rangle + \frac{2-\lambda}{2}\tau\langle A_1\frac{e_{i+1}-e_i}{\tau}, \frac{e_{i+1}-e_i}{\tau}\rangle\\
&-\langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle + \lambda\tau^2\langle A_0\frac{e_{i+1}-e_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle\\
&-\tau^2\frac{\lambda}{2}\langle A_0\frac{Ew_{i+1}-Ew_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle\\
&-(1-\lambda)\tau\langle A_1\frac{e_{i+1}-e_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle + \frac{2-\lambda}{2}\tau\|\frac{p_{i+1}-p_i}{\tau}\|^2_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\tau\mathcal{H}(\frac{p_{i+1}-p_i}{\tau})+\langle(v_{i+1}-v_i)-(\omega_{i+1}-\omega_i),v_{i+1}-\omega_{i+1}\rangle\leq\\
&\leq\tau\langle\mathcal{L}_i-\frac{\omega_{i+1}-\omega_i}{\tau},v_{i+1}-\omega_{i+1}\rangle\\
&+\frac{\lambda}{2}\langle A_0e_i,e_i\rangle+\frac{\lambda}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2}+\frac{\lambda\tau}{2}\langle A_1\frac{Ew_{i+1}-Ew_i}{\tau},\frac{Ew_{i+1}-Ew_i}{\tau}\rangle.
\end{aligned}
$$

Now, thanks to (1.2.45)-(1.2.47), from the last inequality we get

$$
\begin{aligned}
&(1-\lambda)\int_{t_i}^{t_{i+1}}\langle A_0e_\tau,\dot e_\tau\rangle ds+\frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}}\langle A_1\dot e_\tau,\dot e_\tau\rangle ds\\
&+\frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}}\|\dot p_\tau\|^2_{L^2}ds+\int_{t_i}^{t_{i+1}}\mathcal{H}(\dot p_\tau)ds\\
&+\frac{\tau}{2}\int_{t_i}^{t_{i+1}}\|\dot v_\tau-\dot\omega_\tau\|^2_{L^2}ds+\frac{1}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2}-\frac{1}{2}\|v_i-\omega_i\|^2_{L^2}\leq\\
&\leq-\int_{t_i}^{t_{i+1}}\langle\dot\omega_\tau,\dot u_\tau-\dot w_\tau\rangle ds-\int_{t_i}^{t_{i+1}}\langle\dot{\mathcal{L}},u_\tau-w_\tau\rangle ds\\
&+\langle\mathcal{L}(t_{i+1}),u_{i+1}-w_{i+1}\rangle-\langle\mathcal{L}(t_i),u_i-w_i\rangle-\frac{6-7\lambda}{12}\tau\int_{t_i}^{t_{i+1}}\langle A_0\dot e_\tau,\dot e_\tau\rangle ds\\
&+\frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}}\|\dot u_\tau-\dot w_\tau\|^2_{L^2}ds+\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\langle\tau A_0E\dot w_\tau+A_1E\dot w_\tau,E\dot w_\tau\rangle ds\\
&+\int_{t_i}^{t_{i+1}}\langle A_0e_\tau+A_1\dot e_\tau,E\dot w_\tau\rangle ds+\int_{t_i}^{t_{i+1}}\langle(\frac{\tau}{2}-\lambda\tau)A_0\dot e_\tau-\lambda A_1\dot e_\tau,E\dot w_\tau\rangle ds\\
&+\frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}}\langle A_0e_\tau,e_\tau\rangle ds-\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\langle A_0e_\tau,\dot e_\tau\rangle ds,
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
\frac{\lambda}{2}\langle A_0e_i,e_i\rangle=&\frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}}\langle A_0e_\tau,e_\tau\rangle ds\\
&-\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\langle A_0e_\tau,\dot e_\tau\rangle ds+\frac{\lambda\tau}{12}\int_{t_i}^{t_{i+1}}\langle A_0\dot e_\tau,\dot e_\tau\rangle ds.
\end{aligned}
$$

We now sum over $i=0,\dots,j$ and we obtain

$$
\begin{aligned}
&\frac{1-\lambda}{2}\langle A_0e_\tau(t_{j+1}),e_\tau(t_{j+1})\rangle-\frac{1-\lambda}{2}\langle A_0e_0,e_0\rangle\\
&+\frac{2-\lambda}{2}\int_0^{t_{j+1}}\langle A_1(\dot e_\tau)_{A_1},(\dot e_\tau)_{A_1}\rangle ds+\frac{2-\lambda}{2}\int_0^{t_{j+1}}\|\dot p_\tau\|^2_{L^2}ds+\int_0^{t_{j+1}}\mathcal{H}(\dot p_\tau)ds\\
&+\frac{\tau}{2}\int_0^{t_{j+1}}\|\dot v_\tau-\dot\omega_\tau\|^2_{L^2}ds+\frac{1}{2}\|v_{j+1}-\omega_{j+1}\|^2_{L^2}-\frac{1}{2}\|v_0-\omega_0\|^2_{L^2}\leq
\end{aligned}
$$

$$
\begin{aligned}
&\le -\int_0^{t_{j+1}} \langle \dot\omega_\tau, \dot u_\tau - \dot w_\tau\rangle ds - \int_0^{t_{j+1}} \langle \dot{\mathcal{L}}_\tau, u_\tau - w_\tau\rangle ds + \langle \mathcal{L}_\tau(t_{j+1}), u_{j+1} - w_{j+1}\rangle \\
&- \langle \mathcal{L}(0), u_0 - w(0)\rangle + \int_0^{t_{j+1}} \langle A_0 e_\tau + A_1(\dot e_\tau)_{A_1}, E\dot w_\tau\rangle ds \\
&+ \frac{\lambda}{2\tau}\int_0^{t_{j+1}} \langle A_0 e_\tau, e_\tau\rangle ds + \frac{\lambda}{2\tau}\int_0^{t_{j+1}} \|\dot u_\tau - \dot w_\tau\|^2_{L^2} ds \\
&- \frac{6-7\lambda}{12}\tau \int_0^{t_{j+1}} \langle A_0 \dot e_\tau, \dot e_\tau\rangle ds + \frac{\lambda}{2}\int_0^{t_{j+1}} \langle \tau A_0 E\dot w_\tau + A_1 E\dot w_\tau, E\dot w_\tau\rangle ds \\
&+ \int_0^{t_{j+1}} \langle (\frac{\tau}{2} - \lambda\tau) A_0 \dot e_\tau - \lambda A_1(\dot e_\tau)_{A_1}, E\dot w_\tau\rangle ds - \frac{\lambda}{2}\int_0^{t_{j+1}} \langle A_0 e_\tau, \dot e_\tau\rangle ds.
\end{aligned}
$$

We now take $\lambda = o(\tau)$ and then pass to the limit as $\tau \to 0$. To this aim we fix $t \in [0,T]$ and, for every $\tau > 0$, we define $\hat t_\tau = t_{j+1}$, where j is the unique index such that $t_j \le t < t_{j+1}$. For the third, fourth, and fifth term in the left-hand side of the previous inequality we just use the lower semicontinuity with respect to the convergences in (1.2.61); the sixth term is nonnegative; to deal with the first and the seventh term we apply Lemma 1.2.7 below taking into account (1.2.44c), (1.2.44d), (1.2.61c), (1.2.62), (1.2.63), and (1.2.65), obtaining

$$
\begin{aligned}
&e_\tau(t_{j+1}) = e_\tau(\hat t_\tau) \rightharpoonup e(t) \text{ weakly in } H^{-1}(\Omega; \mathbb{M}^{n\times n}_{\rm sym}), \\
&v_{j+1} - \omega_{j+1} = v_\tau(\hat t_\tau) - \omega_\tau(\hat t_\tau) \rightharpoonup \dot u(t) - \dot w(t) \text{ weakly in } H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n).
\end{aligned}
$$

Since the L^2 norm is lower semicontinuous with respect to weak convergence in H^{-1} and $H^{-1}_{\Gamma_0}$ (this can be proved by a duality argument), we obtain a lower semicontinuity inequality also for these terms.

As for the right-hand side of the previous inequality, we can pass to the limit in the first two terms thanks to (1.2.44), (1.2.61a), (1.2.61b), and (1.2.66), which implies also that $u_\tau \rightharpoonup u$ weakly in $H^1([0,T]; L^2(\Omega;\mathbb{R}^n))$. This implies by Lemma 1.2.7 that $u_{j+1} = u_\tau(\hat t_j) \rightharpoonup u(t)$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Since u_{j+1} is bounded in $H^1(\Omega;\mathbb{R}^n)$ by (1.2.61a) we deduce that $u_{j+1} \rightharpoonup u(t)$ weakly in $H^1(\Omega;\mathbb{R}^n)$. We can now pass to the limit in the third term of the right-hand side thanks to (1.2.44) and (1.2.65), and in the fifth term thanks to (1.2.44b), (1.2.61c), and (1.2.61d). The eighth has a negative coefficient, while all other terms tend to 0 by (1.2.44), (1.2.54), and (1.2.61). Thus we obtain

$$
\begin{aligned}
&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e(0)) + \int_0^t \mathcal{Q}_1(\dot e_{A_1}) ds + \int_0^t \|\dot p\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p) ds + \frac{1}{2}\|\dot u(t) - \dot w(t)\|^2_{L^2} \\
&- \int_0^t \langle A_0 e + A_1 \dot e_{A_1}, E\dot w\rangle ds - \frac{1}{2}\|v_0 - \dot w(0)\|^2_{L^2} + \int_0^t \langle \ddot w, \dot u - \dot w\rangle ds \\
&+ \int_0^t \langle \dot{\mathcal{L}}, u - w\rangle ds - \langle \mathcal{L}(t), u(t) - w(t)\rangle + \langle \mathcal{L}(0), u_0 - w(0)\rangle \le 0. \qquad (1.2.68)
\end{aligned}
$$

To prove the energy balance (1.2.29) we need to show that also the opposite inequality holds. To this aim, for a.e. $t \in [0,T]$, we use the first inequality of (1.2.26) with $\varphi = s^h u(t) - s^h w(t)$, $\eta = s^h e(t) - s^h E w(t)$, and $q = s^h p(t)$, we sum this expression to the one obtained from (1.2.26) at time $t+h$, using the same test functions. Then, using an argument similar to the one employed in (1.2.31), we get the opposite inequality in (1.2.68) for a.e. $t \in [0,T]$.

Step 4. Equalities (1.2.25a) follow easily from (1.2.61) and from the initial conditions satisfied by the approximate solutions (u_τ, e_τ, p_τ). Moreover by (1.2.65) the functions e_τ converge to e weakly in $H^1([0,T]; H^{-1}(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}))$ as $\tau \to 0$. Since $e_\tau(0) = e_0$ for all τ, we conclude that $e(0) = e_0$. Since $t \to e(t)$ is weakly continuous into $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ by Remark 1.2.2, we deduce that

$$e(t) \rightharpoonup e_0 \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}) \quad \text{as } t \to 0. \tag{1.2.69}$$

Similarly, using (1.2.63), we find that $v_\tau \to \dot{u}$ weakly in $H^1([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n))$. Since $v_\tau(0) = v_0$ for every τ, we conclude that $\dot{u}(0) = v_0$. Since $t \to \dot{u}(t)$ is weakly continuous into $L^2(\Omega; \mathbb{R}^n)$ by Remark 1.2.2, we deduce that

$$\dot{u}(t) \rightharpoonup v_0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n) \quad \text{as } t \to 0. \tag{1.2.70}$$

In order to deduce from (1.2.69) and (1.2.70) the stronger conditions (1.2.25b) we use the energy equality (1.2.29). Let t_k be a sequence in $[0,T]$ converging to 0 such that (1.2.29) holds for $t = t_k$. Then

$$\frac{1}{2}\|\dot{u}(t_k) - \dot{w}(t_k)\|^2_{L^2} + \mathcal{Q}_0(e(t_k)) \to \frac{1}{2}\|v_0 - \dot{w}(0)\|^2_{L^2} + \mathcal{Q}_0(e_0). \tag{1.2.71}$$

Since $\dot{w} \in C^0([0,T]; L^2(\Omega; \mathbb{R}^n))$, the weak convergence (1.2.69) and (1.2.70) together with (1.2.71) imply that $e(t_k) \to e_0$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ and $\dot{u}(t_k) \to v_0$ strongly in $L^2(\Omega; \mathbb{R}^n)$. Equalities (1.2.25b) follow now from the arbitrariness of the sequence t_k.

We are now in a position to apply Theorem 1.2.4: since the quadruple (u, e, p, σ) satisfies (1.2.26) and (1.2.29), it satisfies also equations (1.2.23c) and (1.2.23d) .

Step 5. It only remains to prove that the solution is unique. Let us suppose that $(u_1, e_1, p_1, \sigma_1)$ and $(u_2, e_2, p_2, \sigma_2)$ are solutions. We set $u := u_2 - u_1$, $e := e_2 - e_1$, $p := p_2 - p_1$, $\sigma := \sigma_2 - \sigma_1$, and observe that the quadruple (u, e, p, σ) satisfies the hypotheses of Lemma 1.2.6, implying that (1.2.31) holds for a.e. $t \in [0,T]$. Since the map $\xi \to \xi - \pi_K \xi$ is a monotone operator from $\mathbb{M}^{n\times n}_D$ into itself (see e.g. [12, Chapter 2]), it follows from (1.2.23d) that

$$\langle \sigma_D(t), \dot{p}(t)\rangle ds \geq 0$$

for a.e. $t \in [0,T]$. Using this inequality in (1.2.31) we obtain that

$$\mathcal{Q}_0(e(t)) + \int_0^t \mathcal{Q}_1(\dot{e}_{A_1}) ds + \frac{1}{2}\|\dot{u}(t)\|^2_{L^2} = 0$$

for a.e. $t \in [0,T]$, taking into account the initial and boundary conditions satisfied by u. This implies by standard arguments that $u(t) = 0$ for all $t \in [0,T]$, concluding the proof. □

Here we prove the lemma we have used in the previous proof.

Lemma 1.2.7. *Let X be a Banach space. Assume that q_τ tends to q_0 weakly in $H^1([0,T]; X)$ as τ tends to zero. Then*

$$q_\tau(t_\tau) \rightharpoonup q_0(t_0) \quad \textit{weakly in } X \tag{1.2.72}$$

for every $t_\tau, t_0 \in [0,T]$ with $t_\tau \to t_0$ as $\tau \to 0$.

Proof. Since $H^1([0,T];X)$ is continuously embedded in $C^{0,1/2}([0,T];X)$, we have $q_\tau \rightharpoonup q_0$ weakly in $C^{0,1/2}([0,T];X)$. This implies in particular that

$$q_\tau(t) \rightharpoonup q_0(t) \text{ weakly in } X \tag{1.2.73}$$

for all $t \in [0,T]$. If $t_\tau \to t_0$ we have

$$\|q_\tau(t_\tau) - q_\tau(t_0)\| \leq \int_{t_0}^{t_\tau} \|\dot{q}_\tau\| dt \leq M(t_\tau - t_0)^{1/2},$$

where $\|\cdot\|$ is the norm in X and M is an upper bound for the norm of q_τ in $H^1([0,T];X)$. Now (1.2.72) follows from the previous inequality and (1.2.73). □

Theorem 1.2.8. *Let (u,e,p,σ) be the solution of the problem considered in Theorem 1.2.1. Then $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^n))$, $e \in C^0([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, $\dot{u} \in C^0([0,T];L^2(\Omega;\mathbb{R}^n))$, and the energy balance (1.2.29) holds for all $t \in [0,T]$.*

Proof. We may assume that w and $\mathcal{L}$ are defined on $[0,T+1]$ and satisfy the hypotheses of Theorem 1.2.1 with T replaced by $T+1$. As for w, it is enough to set $w(t) := w(T) + (t-T)\dot{w}(T)$ for $t \in (T,T+1]$, noticing that $\dot{w}(T)$ can be univocally defined as an element of $H^1(\Omega;\mathbb{R}^n)$ arguing as in Remark 1.2.2. By Theorem 1.2.1 the solution on $[0,T]$ can be extended to a solution on $[0,T+1]$ still denoted by (u,e,p,σ).

Let us fix $t^* \in [0,T]$. Thanks to Remark 1.2.2, the functions $u(t^*)$, $e(t^*)$, $p(t^*)$, $\dot{u}(t^*)$ are univocally defined as elements of $H^1(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $L^2(\Omega;\mathbb{M}^{n\times n}_D)$, and $L^2(\Omega;\mathbb{R}^n)$, respectively. Therefore we can consider the solution (u^*,e^*,p^*,σ^*) of the problem of Theorem 1.2.1, with $[0,T]$ replaced by $[t^*,T+1]$ and initial data $u(t^*)$, $e(t^*)$, $p(t^*)$, and $\dot{u}(t^*)$ in the sense of (1.2.25), with 0 replaced by t^*. It is easy to see that the function defined by (u,e,p,σ) on $[0,t^*)$ and by (u^*,e^*,p^*,σ^*) on $[t^*,T+1]$ is a solution of the problem considered in Theorem 1.2.1 on $[0,T+1]$, with initial data u_0, e_0, p_0, and v_0. By uniqueness $(u^*,e^*,p^*,\sigma^*) = (u,e,p,\sigma)$ on $[t^*,T+1]$.

In view of Theorem 1.2.4, we can fix $\hat{t} \in (t^*,T+1]$ such that the energy balance (1.2.29) between 0 and $\hat{t}$ holds for (u,e,p,σ) and the energy balance between t^* and $\hat{t}$ holds for (u^*,e^*,p^*,σ^*). Since $(u^*,e^*,p^*,\sigma^*) = (u,e,p,\sigma)$ on $[t^*,\hat{t}]$, by difference we obtain the energy balance for (u,e,p,σ) between $[0,t^*]$. Since t^* is arbitrary, this implies that the energy balance holds for all $t \in [0,T]$.

Now the energy balance, together with the continuity of $\mathcal{L}$ and the weak continuity of $u-w$, implies that the term $\mathcal{Q}_0(e) + \|\dot{u}-\dot{w}\|^2_{L^2}$ is a continuous function on $[0,T]$. Then for all $t \in [0,T]$ and any sequence $t_k \to t \in [0,T]$ we have

$$\mathcal{Q}_0(e(t)) + \|\dot{u}(t) - \dot{w}(t)\|^2_{L^2} = \lim_{k\to\infty} \mathcal{Q}_0(e(t_k)) + \|\dot{u}(t_k) - \dot{w}(t_k)\|^2_{L^2}.$$

This and the weak continuity of e and $\dot{u}-\dot{w}$, thanks to the fact that $\mathcal{Q}_0$ is equivalent to the norm on $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, imply that $e(t_k) \to e(t)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, and $\dot{u}(t_k) - \dot{w}(t_k) \to \dot{u}(t) - \dot{w}(t)$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Thanks to (1.2.21b), this implies that $e \in C^0([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ and also that $\dot{u} \in C^0([0,T];L^2(\Omega;\mathbb{R}^n))$. Instead, using (1.2.21b) and (1.2.27e) we conclude that $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^n))$. □

1.2.3 Existence result with A_1 positive definite

When A_1 is a positive definite tensor it is possible to prove the following existence result, where the hypotheses on the data $\mathcal{L}$ and w are weakened.

Theorem 1.2.9. *Let $T > 0$, let $\mathcal{L} \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and let w be a function such that*

$$w \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.74a}$$

$$\dot{w} \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.74b}$$

$$\ddot{w} \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{1.2.74c}$$

Then for every $(u_0, e_0, p_0) \in A(w(0))$ and $v_0 \in L^2(\Omega;\mathbb{R}^n)$ there exists a unique quadruple (u, e, p, σ) of functions, with

$$u \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.75a}$$

$$\dot{u} \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.75b}$$

$$\ddot{u} \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{1.2.75c}$$

$$e \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{1.2.75d}$$

$$p \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{D})), \tag{1.2.75e}$$

$$\dot{e} \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{1.2.75f}$$

$$\dot{p} \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{D})), \tag{1.2.75g}$$

$$\sigma \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{1.2.75h}$$

such that for a.e. $t \in [0,T]$ we have

$$Eu(t) = e(t) + p(t), \tag{1.2.76a}$$

$$\sigma(t) = A^0 e(t) + A^1 \dot{e}(t), \tag{1.2.76b}$$

$$\ddot{u}(t) - \operatorname{div}_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{1.2.76c}$$

$$\dot{p}(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)}\sigma_D(t), \tag{1.2.76d}$$

and

$$u(t) = w(t) \quad \text{on } \Gamma_0, \tag{1.2.77}$$

$$u(0) = u_0, \quad e(0) = e_0, \quad p(0) = p_0, \quad \dot{u}(0) = v_0. \tag{1.2.78}$$

Moreover (u, e, p, σ) satisfies the equilibrium condition

$$\begin{aligned} -\mathcal{H}(q) \le \langle A^0 e(t), \eta\rangle + \langle A^1 \dot{e}(t), \eta\rangle + \langle \dot{p}(t), q\rangle \\ + \langle \ddot{u}(t), \varphi\rangle - \langle \mathcal{L}(t), \varphi\rangle \le \mathcal{H}(-q), \end{aligned} \tag{1.2.79}$$

for a.e. $t \in [0,T]$ and for every $(\varphi, \eta, q) \in A(0)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ in the terms containing $\ddot{u}$ and $\mathcal{L}$, while it denotes the scalar product in L^2 in all other terms.

Remark 1.2.10. In view of (1.2.74) and (1.2.75) we see that u, w, $\dot{u}$, $\dot{w}$, e and p are absolutely continuous, i.e.,

$$u, w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.2.80a}$$

$$\dot{u}, \dot{w} \in AC([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{1.2.80b}$$

$$e \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{1.2.80c}$$

$$p \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{D})) \tag{1.2.80d}$$

(see, e.g., [12], Proposition A.3 and following Corollary). Moreover [81, Proposition 23.23] implies that

$$\dot u - \dot w \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{1.2.81a}$$

$$\|\dot u - \dot w\|^2_{L^2} \in AC([0,T]), \tag{1.2.81b}$$

$$\tfrac{d}{dt}\|\dot u - \dot w\|^2_{L^2} = 2\langle \ddot u - \ddot w, \dot u - \dot w\rangle \text{ a.e. } t \in [0,T], \tag{1.2.81c}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Since $\dot w \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n))$, from (1.2.81a) we obtain

$$\dot u \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)). \tag{1.2.82}$$

This property gives a precise meaning to the initial conditions (1.2.78).

1.3 Perfect plasticity

In this and in the next sections we study the behavior of the solutions of (1.2.23) when the data of the problem, i.e., the external load and the boundary conditions, vary very slowly. We are going to prove that the inertial and viscosity terms become negligible in the limit, and that the solutions of the dynamic problems actually approach the quasistatic evolution for perfect plasticity. To this aim we provide in this section the mathematical setting and tools to formulate and solve the perfect plasticity problem. In Section 1.3.4 we will then rescale the time as described in the preamble and study the system of equations

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{1.3.1a}$$

$$\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot e^\epsilon_{A_1}, \tag{1.3.1b}$$

$$\epsilon^2 \ddot u^\epsilon - \operatorname{div}_{\Gamma_0}(\sigma^\epsilon) = \mathcal{L}, \tag{1.3.1c}$$

$$\epsilon \dot p^\epsilon = \sigma^\epsilon - \pi_K \sigma^\epsilon, \tag{1.3.1d}$$

with boundary and initial conditions

$$u^\epsilon(t) = w(t) \quad \text{on } \Gamma_0 \quad \text{for every } t \in [0,T], \tag{1.3.2}$$

$$u^\epsilon(0) = u_0, \quad e^\epsilon(0) = e_0, \quad p^\epsilon(0) = p_0, \quad \dot u^\epsilon(0) = v_0. \tag{1.3.3}$$

We shall prove (Theorem 1.3.10) that, under suitable assumptions, the solutions $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (1.3.1) tend to a solution of the quasistatic evolution problem in perfect plasticity, according to Definition 1.3.4.

1.3.1 Preliminary tools

Space BD. In perfect plasticity the displacement u belongs to the space of functions with bounded deformation on Ω, defined as

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : Eu \in \mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})\}.$$

Here and henceforth, if V is a finite dimensional vector space and A is a locally compact subset of $\mathbb{R}^n$, the symbol $\mathcal{M}_b(A;V)$ denotes the space of V-valued bounded Radon measures on A, endowed with the norm $\|\lambda\|_{\mathcal{M}_b} := |\lambda|(A)$, where $|\lambda|$ is the variation of λ.

The space $BD(\Omega)$ is endowed with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_{\mathcal{M}_b}.$$

Besides the strong convergence, we shall also consider a notion of weak* convergence in $BD(\Omega)$. We say that a sequence u_k converges to u weakly* in $BD(\Omega)$ if and only if u_k converges to u weakly in $L^1(\Omega;\mathbb{R}^n)$ and Eu_k converges to Eu weakly* in $\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. Every function u in $BD(\Omega)$ has a trace in $L^1(\partial\Omega;\mathbb{R}^n)$, that we will still denote by u, or sometimes by $u|_{\partial\Omega}$. By [75, Proposition 2.4 and Remark 2.5] there exists a constant C depending only on Ω such that

$$\|u\|_{L^1(\Omega)} \le C(\|u\|_{L^1(\Gamma_0)} + \|Eu\|_{\mathcal{M}_b(\Omega)}). \tag{1.3.4}$$

For technical reasons related to the stress-strain duality, in addition to the assumption already introduced in Section 2.1, we now suppose that

$$\partial\Omega \text{ and } \partial\Gamma \text{ are of class } C^2. \tag{1.3.5}$$

Elastic and Plastic Strain. In perfect plasticity the plastic strain p belongs to $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. The singular part of this measure describes plastic slips. Given $w \in H^1(\Omega;\mathbb{R}^n)$, we say that a triple (u,e,p) is kinematically admissible for the perfectly plastic problem with boundary datum w if $u \in BD(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \quad \text{ on } \Omega, \tag{1.3.6a}$$

$$p = (w-u)\odot\nu\mathcal{H}^{n-1} \quad \text{ on } \Gamma_0, \tag{1.3.6b}$$

where ν denotes the outer unit normal to $\partial\Omega$ and $\odot$ denotes the symmetrized tensor product.

The set of these triples will be denoted by $A_{BD}(w)$. Note that in this definition of kinematical admissibility, the Dirichlet boundary condition (1.2.1b) is replaced by the relaxed condition (1.3.6b), which represents a plastic slip occurring at Γ_0. It is also easily seen that the inclusion $A(w) \subset A_{BD}(w)$ holds, so that every admissible triple for the visco-elasto-plastic problem is also admissible for the perfectly plastic problem.

The following closure property is proved in [20, Lemma 2.1].

Lemma 1.3.1. *Let w_k be a sequence in $H^1(\Omega;\mathbb{R}^n)$ and $(u_k,e_k,p_k) \in A_{BD}(w_k)$. Let us suppose that $w_k \rightharpoonup w_\infty$ weakly in $H^1(\Omega;\mathbb{R}^n)$, $u_k \rightharpoonup u_\infty$ weakly* in $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p_k \rightharpoonup p_\infty$ weakly* in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. Then $(u_\infty,e_\infty,p_\infty) \in A_{BD}(w_\infty)$.*

Stress. In addition to the assumptions of Section 2.1, we now suppose that the elastic tensor A_0 maps the orthogonal spaces $\mathbb{M}^{n\times n}_D$ and $\mathbb{R}I$ into themselves. This is equivalent to require that there exist a positive definite symmetric operator $A_{0D} : \mathbb{M}^{n\times n}_D \to \mathbb{M}^{n\times n}_D$ and a positive constant κ^0 such that

$$A_0\xi = A_{0D}\xi_D + \kappa^0(\mathrm{tr}\xi)I. \tag{1.3.7}$$

In the perfectly plastic model the stress σ is related to the strain by the equation

$$\sigma = A_0 e \tag{1.3.8}$$

where e is the elastic component of the strain Eu. Therefore if (u, e, p) is kinematically admissible, then σ belongs to $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$.

In perfect plasticity the stress satisfies the constraint

$$\sigma_D \in \mathcal{K}(\Omega), \tag{1.3.9}$$

where $\mathcal{K}(\Omega)$ is defined in (1.2.12). In particular

$$\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D). \tag{1.3.10}$$

Convex Functions of Measures. In perfect plasticity we need to define the functional (1.2.17) for $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. This is done by using the theory of convex functions of measures (see [32] and [75]): for every $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we consider the nonnegative Radon measure $H(p)$ on $\Omega \cup \Gamma_0$ defined by

$$H(p)(B) := \int_B H(p/|p|)d|p| \tag{1.3.11}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$, where $p/|p|$ is the Radon-Nikodym derivative of p with respect to its variation $|p|$. We also define

$$\mathcal{H}(p) := H(p)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H(p/|p|)d|p|.$$

The function $p \mapsto \mathcal{H}(p)$ turns out to be lower semicontinuous with respect to the weak* topology of $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and satisfies the triangle inequality. Moreover if $p_k \rightharpoonup p$ weakly* and $|p_k|(\Omega\cup\Gamma_0) \to |p|(\Omega\cup\Gamma_0)$, then $\mathcal{H}(p_k) \to \mathcal{H}(p)$.

Stress-Strain Duality. If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$, with $\mathrm{div}\sigma \in L^2(\Omega; \mathbb{R}^n)$, we define the distribution $[\sigma\nu]$ on $\partial\Omega$ by setting

$$\langle [\sigma\nu], \varphi\rangle_{\partial\Omega} := \langle \mathrm{div}\sigma, \varphi\rangle + \langle \sigma, E\varphi\rangle, \tag{1.3.12}$$

for each $\varphi \in H^1(\Omega; \mathbb{R}^n)$. It turns out that $[\sigma\nu] \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ (see e.g. [75, Theorem 1.2, Chapter I]). We define the normal and tangential part of $[\sigma\nu]$ by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu, \qquad [\sigma\nu]^\perp_\nu := [\sigma\nu] - [\sigma\nu]_\nu, \tag{1.3.13}$$

and we have that $[\sigma\nu]_\nu$ and $[\sigma\nu]^\perp_\nu$ belong to $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ thanks to the regularity assumption (1.3.5) on $\partial\Omega$. If $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, by [40, Lemma 2.4] we also have that $[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega; \mathbb{R}^n)$ and

$$\|[\sigma\nu]^\perp_\nu\|_{\infty,\partial\Omega} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty}. \tag{1.3.14}$$

The set of admissible stresses for the perfectly plastic problem is defined by

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}) : \mathrm{div}\sigma \in L^n(\Omega; \mathbb{R}^n) \text{ and } \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}.$$

The set of admissible plastic strains $\Pi_{\Gamma_0}(\Omega)$ is the set of all $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ such that there exist $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfying $(u, e, p) \in A_{BD}(w)$.

If $\sigma \in \Sigma(\Omega)$ it turns out that $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ for all $r < +\infty$ (see [76, Proposition 2.5]). For every $u \in BD(\Omega)$ with $\mathrm{div} u \in L^2(\Omega)$ we define the distribution $[\sigma_D \cdot E_D u]$ by

$$\langle [\sigma_D \cdot E_D u], \varphi \rangle = -\langle \mathrm{div}\sigma, \varphi u \rangle - \frac{1}{n} \langle \mathrm{tr}\sigma, \varphi \mathrm{div} u \rangle - \langle \sigma, u \odot \nabla\varphi \rangle \tag{1.3.15}$$

for every $\varphi \in C^\infty_c(\Omega)$. As proved in [76, Theorem 3.2] the distribution $[\sigma_D \cdot E_D u]$ is a bounded Radon measure in Ω.

As in [20], if $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define the bounded Radon measure $[\sigma_D \cdot p]$ on $\Omega \cup \Gamma_0$ by setting

$$\begin{aligned} [\sigma_D \cdot p] &:= [\sigma_D \cdot E_D u] - \sigma_D \cdot e_D \qquad &&\text{on } \Omega, \\ [\sigma_D \cdot p] &:= [\sigma\nu]^\perp_\nu \cdot (w - u)\mathcal{H}^{n-1} \qquad &&\text{on } \Gamma_0, \end{aligned}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfy $(u, e, p) \in A_{BD}(w)$, and we notice that this definition does not depend on the particular choice of u, e, w (see [20, page 250]). We also define the duality pairing between $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$ by

$$\langle \sigma_D, p \rangle := [\sigma_D \cdot p](\Omega \cup \Gamma_0). \tag{1.3.16}$$

The following inequalities between measures hold (see [20, (2.33) and Proposition 2.4]):

$$|[\sigma_D \cdot p]| \le \|\sigma_D\|_{L^\infty} |p| \quad \text{on } \Omega \cup \Gamma_0, \tag{1.3.17}$$

$$[\sigma_D \cdot p] \le H(p) \quad \text{on } \Omega \cup \Gamma_0, \tag{1.3.18}$$

where $H(p)$ is the measure introduced in (1.3.11). The following integration by parts formula is proved in [20, Proposition 2.2] when $\varphi \in C^1(\bar\Omega)$. The extension to Lipschitz functions is straightforward.

Proposition 1.3.2. *Let $\sigma \in \Sigma(\Omega)$, $f \in L^n(\Omega; \mathbb{R}^n)$, $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and suppose $(u, e, p) \in A_{BD}(w)$ with $w \in H^1(\Omega; \mathbb{R}^n)$. If $-\mathrm{div}\sigma = f$ on Ω and $[\sigma\nu] = g$ on Γ_1, then it holds*

$$\langle \sigma_D, p \rangle + \langle \sigma, e - Ew \rangle = \langle f, u - w \rangle + \langle g, u - w \rangle_{\Gamma_1}. \tag{1.3.19}$$

Moreover

$$\begin{aligned} &\langle [\sigma_D \cdot p], \varphi \rangle + \langle \sigma \cdot (e - Ew), \varphi \rangle + \langle \sigma, \nabla\varphi \odot (u - w) \rangle = \\ &= \langle f, \varphi(u - w) \rangle + \langle g, \varphi(u - w) \rangle_{\Gamma_1}, \end{aligned} \tag{1.3.20}$$

for every $\varphi \in C^{0,1}(\bar\Omega)$.

As a consequence of the formula above we obtain the following lemma.

Lemma 1.3.3. *Let $\sigma_k, \sigma \in \Sigma(\Omega)$, $w_k, w \in H^1(\Omega; \mathbb{R}^n)$, $(u_k, e_k, p_k) \in A_{BD}(w_k)$, and $(u, e, p) \in A_{BD}(w)$ be such that*

$$\begin{aligned} &\sigma_k \to \sigma \textit{ strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \\ &\mathrm{div}\sigma_k \to \mathrm{div}\sigma \textit{ strongly in } L^n(\Omega; \mathbb{R}^n), \\ &(\sigma_k)_D \textit{ are uniformly bounded in } L^\infty(\Omega; \mathbb{M}^{n\times n}_D), \\ &u_k \rightharpoonup u \textit{ weakly in } L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n), \\ &w_k \rightharpoonup w \textit{ weakly in } H^1(\Omega; \mathbb{R}^n), \\ &e_k \rightharpoonup e \textit{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \end{aligned}$$

then $\langle [(\sigma_k)_D \cdot p_k], \varphi \rangle \to \langle [\sigma \cdot p], \varphi \rangle$ *for every* $\varphi \in C_c^{0,1}(\Omega \cup \Gamma_0)$.

Proof. Our hypotheses imply that $\sigma_k \to \sigma$ strongly in $L^n(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ by [76, Proposition 2.5]. The conclusion follows now from (1.3.20). □

1.3.2 Hypotheses on the data

We discuss here the hypotheses on the data for the quasistatic evolution problem in perfect plasticity.

External Load. In contrast to the dynamic case, in perfect plasticity it is not enough to assume that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$. Instead, we assume that $\mathcal{L}(t)$ takes the form (1.2.5), with $f(t) \in L^n(\Omega; \mathbb{R}^n)$ and $g(t) \in L^\infty(\Gamma_1; \mathbb{R}^n)$, so that now the duality $\langle \mathcal{L}(t), u\rangle$ is well defined by (1.2.5) for every $u \in BD(\Omega)$.

The balance equations for the forces are

$$-\operatorname{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{1.3.22}$$

$$[\sigma(t)\nu] = g(t) \quad \text{on } \Gamma_1, \tag{1.3.23}$$

where $[\sigma(t)\nu]$ denotes the normal component of $\sigma(t)$, which can be defined as a distribution according to (1.3.12), since $\operatorname{div}\sigma(t) \in L^2(\Omega; \mathbb{R}^n)$ by (1.3.22). As for the time dependence, we assume that

$$f \in AC([0,T]; L^n(\Omega; \mathbb{R}^n)), \tag{1.3.24a}$$

$$g \in AC([0,T]; L^\infty(\Gamma_1; \mathbb{R}^n)). \tag{1.3.24b}$$

This implies that for a.e. $t \in [0,T]$ there exists an element of the dual of $BD(\Omega)$, denoted by $\dot{\mathcal{L}}(t)$, such that

$$\langle \dot{\mathcal{L}}(t), u\rangle = \lim_{s\to t} \langle \frac{\mathcal{L}(s) - \mathcal{L}(t)}{s-t}, u\rangle \tag{1.3.25}$$

for every $u \in BD(\Omega)$ (see [20, Remark 4.1]).

As usual in perfect plasticity problems, we assume a uniform safe-load condition: there exist a function $\varrho : [0,T] \to L^2(\Omega, \mathbb{M}^{n\times n}_{\rm sym})$ and a positive constant δ such that for every $t \in [0,T]$ we have

$$-\operatorname{div}\varrho(t) = f(t) \ \text{ on } \Omega, \tag{1.3.26a}$$

$$[\varrho(t)\nu] = g(t) \ \text{ on } \Gamma_1, \tag{1.3.26b}$$

and

$$\varrho_D(t) + \xi \in \mathcal{K}(\Omega) \ \text{ for every } \xi \in \mathbb{M}^{n\times n}_D \text{ with } |\xi| \le \delta. \tag{1.3.27}$$

Moreover we require that

$$t \mapsto \varrho(t) \ \text{ and } \ t \mapsto \varrho_D(t) \quad \text{are absolutely continuous} \tag{1.3.28}$$

from $[0,T]$ to $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and $L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ respectively, so that the function $t \mapsto \dot\varrho(t)$ belongs to $L^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}))$ and

$$\frac{\varrho_D(t) - \varrho_D(s)}{t-s} \to \dot\varrho_D(s) \ \text{ weakly* in } L^\infty(\Omega; \mathbb{M}^{n\times n}_D) \quad \text{as } t \to s, \tag{1.3.29}$$

for a.e. $s \in [0, T]$, and

$$t \mapsto \|\dot{\varrho}(t)\|_{L^\infty} \text{ belongs to } L^1([0, T]) \tag{1.3.30}$$

(see [20, Theorem 7.1]).

Using (1.3.17) and (1.3.28) we see that for every $p \in \Pi_{\Gamma_0}(\Omega)$ the function

$$t \mapsto \langle \varrho_D(t), p \rangle \quad \text{belongs to} \quad AC([0, T]). \tag{1.3.31}$$

Moreover, by (1.3.24a), (1.3.26a), (1.3.27), and (1.3.28), we obtain

$$\tfrac{d}{dt} \langle \varrho_D(t), p \rangle = \langle \dot{\varrho}_D(t), p \rangle \quad \text{for a.e. } t \in [0, T], \tag{1.3.32}$$

thanks to [20, formula (2.38)].

Boundary Conditions. The boundary condition on Γ_0 is given in the relaxed form considered in (1.3.6b) with a time dependent function $t \to w(t)$. We assume that

$$w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n)). \tag{1.3.33}$$

Plastic Dissipation. In the energy formulation for the quasistatic evolution problem for perfect plasticity, it is not convenient to use formulas like (1.2.18), because they require the existence of the time derivative of $p(t)$. Instead, for an arbitrary function $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we define the plastic dissipation in $[a, b] \subset [0, T]$ as

$$\mathcal{D}_H(a, b; p) := \sup \sum_{i=0}^{N-1} \mathcal{H}(p(t_{i+1}) - p(t_i)), \tag{1.3.34}$$

where the supremum is taken over all the possible choices of the integer $N > 0$ and of the real numbers $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$. One can prove (see [20, Chapter 7]) that, if $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ is absolutely continuous, then

$$\mathcal{D}_H(a, b; p) = \int_a^b \mathcal{H}(\dot{p}(t))dt, \tag{1.3.35}$$

where $\dot{p}$ is the derivative of p defined by

$$\dot{p}(t) := w^*\text{-}\lim_{s\to t} \frac{p(s) - p(t)}{s - t}. \tag{1.3.36}$$

As a consequence of the safe-load condition (1.3.27) we can easily prove that for every $t \in [0, T]$

$$\mathcal{H}(q) - \langle \varrho(t), q \rangle \geq \gamma \|q\|_{\mathcal{M}_b}, \tag{1.3.37}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$, where the positive constant γ is independent of q and t (see [20, Lemma 3.2]). Moreover we have that

$$H(q) - \varrho(t) \cdot q \geq 0 \text{ a.e. in } \Omega, \tag{1.3.38}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$.

1.3.3 Quasistatic evolution in perfect plasticity

We recall here the energy formulation of a perfectly plastic quasistatic evolution.

Definition 1.3.4. Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, and $p_0 \in M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_D)$. Suppose that f, g, $\mathcal{L}$, ϱ, and w satisfy (1.2.5), (1.3.24), (1.3.26), (1.3.27), (1.3.28), and (1.3.33). A *quasistatic evolution in perfect plasticity* with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 is a function (u,e,p,σ) from $[0,T]$ into $BD(\Omega,\mathbb{R}^n) \times L^2(\Omega,\mathbb{M}^{n\times n}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_D) \times L^2(\Omega,\mathbb{M}^{n\times n}_{\rm sym})$, with

$$u(0) = u_0, \quad e(0) = e_0, \quad p(0) = p_0, \tag{1.3.39}$$

$$\sigma(t) = A_0 e(t) \quad \text{for every } t \in [0,T], \tag{1.3.40}$$

such that $t \mapsto p(t)$ has bounded variation and the following two conditions are satisfied for every $t \in [0,T]$:

(a) $(u(t),e(t),p(t)) \in A_{BD}(w(t))$ and

$$\mathcal{Q}_0(e(t)) - \langle \mathcal{L}(t), u(t)\rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(t), \varphi\rangle + \mathcal{H}(q - p(t)) \tag{1.3.41}$$

for every $(\varphi,\eta,q) \in A_{BD}(w(t))$;

(b) $$\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \mathcal{D}_H(p;0,t) = \int_0^t \langle \sigma, E\dot{w}\rangle ds - \int_0^t \langle \mathcal{L}, \dot{w}\rangle ds$$
$$+\langle \mathcal{L}(t), u(t)\rangle - \langle \mathcal{L}(0), u_0\rangle - \int_0^t \langle \dot{\mathcal{L}}, u\rangle ds, \tag{1.3.42}$$

where $\mathcal{D}_H(p;0,t)$ is defined by (1.3.34).

The integrals in the right-hand side of (1.3.42) are well defined thanks to [20, Theorem 3.8 and Remark 4.3].

If $(u_0,e_0,p_0) \in A_{BD}(w(0))$ satisfies the following stability condition

$$\mathcal{Q}_0(e_0) - \langle \mathcal{L}(0), u_0\rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(0), \varphi\rangle + \mathcal{H}(q - p_0) \tag{1.3.43}$$

for every $(\varphi,\eta,q) \in A_{BD}(w(0))$, then there exists a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 (see [20, Theorem 4.5]). Moreover the function $t \mapsto (u(t),e(t),p(t))$ is absolutely continuous from $[0,T]$ into $BD(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_D)$ ([20, Theorem 5.1]).

In our analysis of the behavior of the solutions $(u^\epsilon,e^\epsilon,p^\epsilon,\sigma^\epsilon)$ of (1.3.1) as $\epsilon \to 0$ we find that $(u^\epsilon,e^\epsilon,p^\epsilon,\sigma^\epsilon)$ converges to a function (u,e,p,σ) which satisfies conditions (1.3.41) and (1.3.42) only for a.e. $t \in [0,T]$. The following theorem shows that this is enough to guarantee that (u,e,p,σ) is a quasistatic evolution, according to Definition 1.3.4.

Theorem 1.3.5. *Let u_0, e_0, p_0, f, g, $\mathcal{L}$, w, and ϱ be as in Definition 1.3.4. Let S be a subset of $[0,T]$ of full $\mathcal{L}^1$ measure containing 0 and let $(u,e,\sigma) : S \to BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ be a bounded and measurable function satisfying (1.3.39) and (1.3.40) for all $t \in S$. Suppose that $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_D)$ has bounded variation and that conditions (a) and (b) of Definition 1.3.4 are satisfied for every $t \in S$. Then there exists an absolutely continuous* *function* $(u, e, \sigma) : [0, T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ *which extends* (u, e, σ). *Moreover* p *is absolutely continuous and* (u, e, p, σ) *is a quasistatic evolution in perfect plasticity with initial conditions* u_0, e_0, p_0, *and boundary condition* w *on* Γ_0.

Remark 1.3.6. Let $t \in S$, $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and $\sigma(t) := A_0 e(t)$. As shown in [20, Theorem 3.6] the following conditions are equivalent:

(a) Inequality (1.3.41) is satisfied for every $(\varphi, \eta, q) \in A_{BD}(w(t))$;

(b) $-\mathcal{H}(q) \le \langle A_0 e(t), \eta\rangle - \langle \mathcal{L}(t), v\rangle \le \mathcal{H}(-q)$ for every $(v, \eta, q) \in A_{BD}(0)$;

(c) $\sigma(t) \in \Sigma(\Omega)$, $\sigma_D(t) \in \mathcal{K}(\Omega)$, $-\mathrm{div}\sigma(t) = f(t)$ in Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1.

The following lemma gives an elementary but useful tool for the proof of Theorem 1.3.5.

Lemma 1.3.7. *Let* $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *be a function with bounded variation and let* $\psi(t) := \mathcal{D}_H(p; 0, t)$ *for* $t \in [0, T]$. *Assume that there exists a set* $S \subseteq [0, T]$ *of full* $\mathcal{L}^1$ *measure such that* $p|_S$ *and* $\psi|_S$ *are absolutely continuous on* S. *Then* p *is absolutely continuous on* $[0, T]$.

Proof. The absolute continuity on S implies that

$$\lim_{\substack{s\to t^-\\ s\in S}} \psi(s) = \lim_{\substack{s\to t^+\\ s\in S}} \psi(s)$$

for every $t \in [0, T]$. Since ψ is non-decreasing, we deduce that the common value of the limit coincides with $\psi(t)$. This shows that ψ is continuous on $[0, T]$. Since

$$\|p(t_1) - p(t_2)\|_{\mathcal{M}_b} \le \mathcal{D}_H(p; t_1, t_2) = \psi(t_2) - \psi(t_1)$$

for every $0 \le t_1 \le t_2 \le T$, we conclude that also p is continuous on $[0, T]$. Moreover the fact that the restriction of p to S is absolutely continuous implies that it is absolutely continuous on $[0, T]$ as well. ☐

Proof of Theorem 1.3.5. We first prove that the functions e, p and u are absolutely continuous on S. We argue as in the proof of [20, Theorem 5.2] using only times t_1, t_2 and s in the set S, and we obtain that for any t_1, $t_2 \in S$ with $t_1 < t_2$ we have that

$$\|e(t_2) - e(t_1))\|^2_{L^2} \le \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_{L^2}\phi(s)ds + (\int_{t_1}^{t_2} \phi(s)ds)^2,$$

where ϕ is a suitable nonnegative integrable function. As a consequence of [20, Lemma 5.3] we get that $\|e(t_2) - e(t_1))\|_{L^2} \le \frac{3}{2}\int_{t_1}^{t_2} \phi(s)ds$ so that $t \mapsto e(t)$ is absolutely continuous from S into $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$. Continuing as in the proof of [20, Theorem 5.2] we obtain also that p and u are absolutely continuous on S. From equation (1.3.42) it follows that $t \mapsto \mathcal{D}_H(p; 0, t)$ is absolutely continuous on S, so that, applying Lemma 1.3.7, we get that p is absolutely continuous on $[0, T]$. Now (u, e) admits an absolutely continuous extension to $[0, T]$ that we still denote by (u, e). By continuity this extension satisfies (1.3.41) and (1.3.42) for every $t \in [0, T]$. This completes the proof. ☐

Remark 1.3.8. Under the hypotheses of Definition 1.3.4, for every $t \in [0,T]$ condition (b) of Definition 1.3.4 is equivalent to the following condition:

(b') The function $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and

$$\begin{aligned}&\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho_D(t), p(t)\rangle = \\ &= \mathcal{Q}_0(e_0) - \langle \varrho(0), e(0) - Ew(0)\rangle - \langle \varrho_D(0), p(0)\rangle + \int_0^t \langle \sigma, E\dot{w}\rangle ds \\ &- \int_0^t \langle \dot{\varrho}, e - Ew\rangle ds - \int_0^t \langle \dot{\varrho}_D, p\rangle ds. \end{aligned} \tag{1.3.44}$$

This is proved in [20, Theorem 4.4] using the integration by parts formula (1.3.19). Note that the duality product $\langle \dot{\varrho}_D(t), p(t)\rangle$ is well defined for a.e. $t \in [0,T]$ by (1.3.24a), (1.3.26a), (1.3.28), and (1.3.29).

1.3.4 Limit of dynamic visco-elasto-plastic evolutions

Here we prove the main result of the Section, which state that solutions to (1.3.1) approach to a quasistatic evolution in perfect plasticity. We state this result starting from the solutions provided by Theorem 1.2.1. However, under suitable hypotheses, the same proof works if we start from solutions given by Theorem 1.2.9. It is easy to see that such hypotheses are actually the ones ensuring existence of solution in Theorem 1.2.9, and then corresponding hypotheses on the data as $\epsilon \to 0$.

Hypotheses on the Data. The regularity assumptions on the data considered in the dynamical problem are not sufficient to study the limit of the solutions of (1.3.1). Therefore we introduce a new set of hypotheses, which includes also the case of data depending on ϵ and converging in a suitable way as ϵ tends to 0.

Let $M > 0$ be a constant. For $\epsilon \in (0,1)$ we consider the following assumptions.

(i) Hypotheses on w^ϵ and w:

$$w^\epsilon \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.3.45a}$$
$$\dot{w}^\epsilon \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.3.45b}$$
$$\ddot{w}^\epsilon \in L^2([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{1.3.45c}$$
$$w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.3.45d}$$
$$w^\epsilon \to w \text{ strongly in } W^{1,1}([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{1.3.45e}$$
$$\epsilon \|\dot{w}^\epsilon(0)\|_{L^2} \to 0, \tag{1.3.45f}$$
$$\epsilon \|\dot{w}^\epsilon(t)\|_{L^2} \le M \text{ for all } t \in [0,T], \tag{1.3.45g}$$
$$\epsilon \int_0^T \|\dot{w}^\epsilon\|^2_{H^1} dt \to 0, \tag{1.3.45h}$$
$$\epsilon^2 \int_0^T \|\ddot{w}^\epsilon\|^2_{L^2} dt \to 0. \tag{1.3.45i}$$

(ii) Hypotheses on f^ϵ, g^ϵ, f, and g: we assume that there exist ϱ^ϵ and ϱ satisfying (1.3.26) and (1.3.27) with f^ϵ, g^ϵ and f, g respectively, and with

δ independent of ϵ. We also suppose that

$$f^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{1.3.46a}$$
$$g^\epsilon \in AC([0,T]; H^{-\frac{1}{2}}(\Gamma_1;\mathbb{R}^n)), \tag{1.3.46b}$$
$$\varrho^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.3.46c}$$
$$f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{1.3.46d}$$
$$g \in AC([0,T]; L^\infty(\Gamma_1;\mathbb{R}^n)), \tag{1.3.46e}$$
$$\varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.3.46f}$$
$$\varrho_D \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_{D})), \tag{1.3.46g}$$
$$f^\epsilon \to f \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{1.3.46h}$$
$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})). \tag{1.3.46i}$$

The functionals $\mathcal{L}^\epsilon(t)$ and $\mathcal{L}(t)$ are defined by (1.2.5) with $f^\epsilon(t)$, $g^\epsilon(t)$ and $f(t)$, $g(t)$ respectively.

(iii) Hypotheses on the initial data $(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon)$, (u_0, e_0, p_0), and v_0^ϵ.

$$(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon) \in A(w^\epsilon(0)), \tag{1.3.47a}$$
$$(u_0, e_0, p_0) \in A_{BD}(w(0)), \tag{1.3.47b}$$
$$(u_0, e_0, p_0) \text{ satisfies the stability condition (1.3.43)}, \tag{1.3.47c}$$
$$u_0^\epsilon \to u_0 \text{ strongly in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n), \tag{1.3.47d}$$
$$e_0^\epsilon \to e_0 \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}), \tag{1.3.47e}$$
$$p_0^\epsilon \rightharpoonup p_0 \text{ weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D), \tag{1.3.47f}$$
$$v_0^\epsilon \in L^2(\Omega;\mathbb{R}^n) \text{ and } \epsilon\|v_0^\epsilon\|_{L^2} \to 0. \tag{1.3.47g}$$

Remark 1.3.9. If we assume that

$$\varrho_D^\epsilon \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)), \tag{1.3.48a}$$
$$\int_0^T \|\dot\varrho_D^\epsilon - \dot\varrho_D\|_{L^\infty}\, dt \to 0, \tag{1.3.48b}$$

then we can replace (1.3.46c), (1.3.46f), and (1.3.46i) by the weaker conditions

$$\varrho_\epsilon,\ \varrho \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{1.3.48c}$$
$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})). \tag{1.3.48d}$$

Indeed using [76, Proposition 2.5] (see also [75, Chapter 2, Proposition 7.1]) from (1.3.30), (1.3.46h), and (1.3.48) we deduce that ϱ^ϵ, $\varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ and that (1.3.46i) holds.

We now state the main result.

Theorem 1.3.10. *Assume hypotheses (i)-(iii) above. Let $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ be the solution of* (1.3.1)*, with $\mathcal{L}$ replaced by $\mathcal{L}^\epsilon$, satisfying the boundary condition w^ϵ on Γ_0 for every $t\in[0,T]$, and the initial data*

$$u^\epsilon(0) = u_0^\epsilon,\ e^\epsilon(0) = e_0^\epsilon,\ p^\epsilon(0) = p_0^\epsilon \quad \dot u^\epsilon(0) = v_0^\epsilon.$$

Then there exist a quasistatic evolution in perfect plasticity (u, e, p, σ), with initial conditions (u_0, e_0, p_0) and boundary condition w on Γ_0, and a subsequence of $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$, not relabeled, such that

$$u^\epsilon(t) \rightharpoonup u(t) \quad \textit{weakly* in } BD(\Omega), \tag{1.3.49}$$

$$e^\epsilon(t) \to e(t) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \tag{1.3.50}$$

for a.e. $t \in [0, T]$, and

$$p^\epsilon(t) \rightharpoonup p(t) \ \textit{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D), \tag{1.3.51}$$

for all $t \in [0, T]$. Moreover there exists $M > 0$ such that

$$\|u^\epsilon(t)\|_{L^1} + \|e^\epsilon(t)\|_{L^2} + \|p^\epsilon(t)\|_{\mathcal{M}_b} \le M \tag{1.3.52}$$

for every $\epsilon \in (0, 1)$ and every $t \in [0, T]$.

Proof. From Theorem 1.2.4 we get the energy balance formula

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds = \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \langle f^\epsilon(t), u^\epsilon(t) - w^\epsilon(t)\rangle - \langle f^\epsilon(0), u^\epsilon(0) - w^\epsilon(0)\rangle \\
&- \int_0^t \langle \dot f^\epsilon, u^\epsilon - w^\epsilon\rangle ds + \langle g^\epsilon(t), u^\epsilon(t) - w^\epsilon(t)\rangle_{\Gamma_1} - \langle g^\epsilon(0), u^\epsilon(0) - w^\epsilon(0)\rangle_{\Gamma_1} \\
&- \int_0^t \langle \dot g^\epsilon, u^\epsilon - w^\epsilon\rangle_{\Gamma_1} ds - \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e^\epsilon_0) + \frac{\epsilon^2}{2}\|v^\epsilon_0 - \dot w^\epsilon(0)\|^2_{L^2},
\end{aligned} \tag{1.3.53}$$

where $\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot e^\epsilon_{A_1}$. Using the safe-load condition (1.3.26) and (1.3.27) and integrating by parts in space, we get

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds = \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \langle \varrho^\epsilon(t), Eu^\epsilon(t) - Ew^\epsilon(t)\rangle - \langle \varrho^\epsilon(0), Eu^\epsilon(0) - Ew^\epsilon(0)\rangle \\
&- \int_0^t \langle \dot\varrho^\epsilon, Eu^\epsilon - Ew^\epsilon\rangle ds - \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e^\epsilon_0) + \frac{\epsilon^2}{2}\|v^\epsilon_0 - \dot w^\epsilon(0)\|^2_{L^2}.
\end{aligned} \tag{1.3.54}$$

By (1.2.2), (1.3.45e), (1.3.45g),(1.3.45i), (1.3.46i), (1.3.47e), and (1.3.47g), using the Cauchy inequality, we get a positive constant D_0 such that

$$\begin{aligned}
&\frac{\alpha_0}{2}\|e^\epsilon(t)\|^2_{L^2} + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2} ds \\
&+ \int_0^t \mathcal{H}(\dot p^\epsilon)ds \le \beta_0\int_0^t \|e^\epsilon\|_{L^2}\|E\dot w^\epsilon\|_{L^2} ds + \epsilon\int_0^t \|A_1\dot e^\epsilon_{A_1}\|_{L^2}\|E\dot w^\epsilon\|_{L^2} ds \\
&+ \langle \varrho^\epsilon(t), e^\epsilon(t)\rangle - \langle \varrho^\epsilon(0), e^\epsilon(0)\rangle - \int_0^t \langle \dot\varrho^\epsilon, e^\epsilon\rangle ds + \int_0^t \langle \varrho^\epsilon_D, \dot p^\epsilon\rangle ds \\
&+ \frac{\epsilon^2}{2}\int_0^t \|\dot u^\epsilon - \dot w^\epsilon\|^2_{L^2} ds + D_0,
\end{aligned} \tag{1.3.55}$$

for every $\epsilon \in (0,1)$, where we have integrated by parts in time the term $\int_0^t \langle \dot{\varrho}^\epsilon, p^\epsilon \rangle$. Using again the Cauchy inequality and the inequality $\|e^\epsilon\|_{L^2} \leq 1 + \|e^\epsilon\|^2_{L^2}$, we obtain that for every $\lambda > 0$ the right-hand side of (1.3.55) can be estimated from above by

$$\beta_0 \int_0^t \|e^\epsilon\|^2_{L^2} \|E\dot{w}^\epsilon\|_{L^2} ds + \epsilon\lambda \int_0^t \|A_1 \dot{e}^\epsilon_{A_1}\|^2_{L^2} ds + \lambda \|e^\epsilon(t)\|^2_{L^2} + \int_0^t \|\dot{\varrho}^\epsilon\|_{L^2} \|e^\epsilon\|^2_{L^2} ds$$
$$+ \int_0^t \langle \varrho^\epsilon_D, \dot{p}^\epsilon \rangle ds + \frac{\epsilon^2}{2} \int_0^t \|\dot{u}^\epsilon - \dot{w}^\epsilon\|^2_{L^2} ds + D_\lambda, \tag{1.3.56}$$

for a suitable constant D_λ independent of ϵ that can be obtained using (1.3.45e), (1.3.45h), (1.3.46i), and (1.3.47e). Taking $\lambda = \min\{\frac{\alpha_0}{4}, \frac{1}{2\beta_1}\}$ and recalling that $\|A_1 \dot{e}^\epsilon_{A_1}\|^2_{L^2} \leq \beta_1 \mathcal{Q}_1(\dot{e}^\epsilon_{A_1})$ by (1.2.2c), from (1.3.37), (1.3.55), and (1.3.56) we get

$$\frac{\alpha_0}{4} \|e^\epsilon(t)\|^2_{L^2} + \frac{\epsilon^2}{2} \|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|^2_{L^2} + \frac{\epsilon}{2} \int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1}) ds + \epsilon \int_0^t \|\dot{p}^\epsilon\|^2_{L^2} dt$$
$$+ \gamma \int_0^t \|\dot{p}^\epsilon\|_{L^1} ds \leq \int_0^t \psi^\epsilon \|e^\epsilon\|^2_{L^2} ds + \frac{\epsilon^2}{2} \int_0^t \|\dot{u}^\epsilon - \dot{w}^\epsilon\|^2_{L^2} ds + D_\lambda, \tag{1.3.57}$$

where $\psi^\epsilon = \beta_0 \|E\dot{w}^\epsilon\|_{L^2} + \|\dot{\varrho}^\epsilon\|_{L^2}$. Since ψ^ϵ is bounded in $L^1([0,T])$ by (1.3.45e) and (1.3.46i), using the Gronwall Lemma we obtain that $\|e^\epsilon(t)\|_{L^2}$ and $\frac{\epsilon^2}{2}\|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|^2_{L^2}$ are bounded by some constant independent of t and ϵ. Together with (1.3.45g) and (1.3.57), this gives

$$\|e^\epsilon(t)\|_{L^2} \leq M \quad \text{for all } t \in [0,T], \tag{1.3.58a}$$

$$\epsilon \|\dot{u}^\epsilon(t)\|_{L^2} \leq M \quad \text{for all } t \in [0,T], \tag{1.3.58b}$$

$$\epsilon \int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1}) ds \leq M, \tag{1.3.58c}$$

$$\epsilon \int_0^T \|\dot{p}^\epsilon\|^2_{L^2} ds \leq M, \tag{1.3.58d}$$

$$\int_0^T \|\dot{p}^\epsilon\|_{L^1} ds \leq M, \tag{1.3.58e}$$

for all $\epsilon \in (0,1)$ and some constant $M > 0$ independent of t and ϵ.

Since $L^1(\Omega; \mathbb{M}^{n\times n}_D)$ is naturally embedded into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, the functions p^ϵ are actually continuous functions from $[0,T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and inequality (1.3.58e) says that the total variation of p^ϵ is bounded uniformly with respect to ϵ. Taking into account (1.3.47f), we can employ a generalization of Helly Theorem (see [20, Lemma 7.2] and [8, Theorem 3.5, Chapter 1]), which implies that there exist a subsequence, still denoted by p^ϵ, and a function $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, with bounded variation, such that, as $\epsilon \to 0$,

$$p^\epsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \quad \text{for every } t \in [0,T]. \tag{1.3.59}$$

It then follows that $p(t)$ is bounded in $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ uniformly with respect to t.

From (1.3.58a) we also get, possibly passing to another subsequence, that there exists $e \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}))$ such that

$$e^\epsilon \rightharpoonup e \text{ weakly* in } L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})), \tag{1.3.60}$$

as $\epsilon \to 0$.

Writing $E(u^\epsilon - w^\epsilon) = e^\epsilon + p^\epsilon - Ew^\epsilon$, by (1.3.45e), (1.3.47f), (1.3.58a), and (1.3.58e), we see that $E(u^\epsilon - w^\epsilon)$ is bounded in $L^\infty([0,T]; L^1(\Omega; \mathbb{M}^{n\times n}_{\rm sym}))$ uniformly with respect to ϵ, so that, thanks to (1.3.4), $u^\epsilon - w^\epsilon$ is bounded in $L^\infty([0,T]; BD(\Omega, \mathbb{R}^n))$ uniformly with respect to ϵ. Then, as a consequence of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$, there exists $u \in L^\infty([0,T]; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n))$ such that

$$u^\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T]; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)), \tag{1.3.61}$$

again for a suitable subsequence, as $\epsilon \to 0$. Using the equality $Eu^\epsilon = e^\epsilon + p^\epsilon$, from (1.3.59) and (1.3.60) we obtain that $u \in L^\infty([0,T]; BD(\Omega))$ and $Eu = e+p$.

By (1.2.26) we see that the function $(u^\epsilon, e^\epsilon, p^\epsilon)$ satisfies the equilibrium condition

$$\begin{aligned} -\mathcal{H}(q) \le \langle A_0 e^\epsilon(t), \eta\rangle + \langle \epsilon A_1 \dot e^\epsilon_{A_1}(t), \eta\rangle + \langle \epsilon \dot p^\epsilon(t), q\rangle \\ + \langle \epsilon^2 \ddot u^\epsilon(t), \varphi\rangle - \langle f^\epsilon(t), \varphi\rangle - \langle g^\epsilon(t), \varphi\rangle_{\Gamma_1} \le \mathcal{H}(-q), \end{aligned} \tag{1.3.62}$$

for every $(\varphi, \eta, q) \in A(0)$ and a.e. $t \in [0,T]$.

Let us fix a smooth and nonnegative real function ψ on $[0,T]$. Multiplying the previous formula by ψ and integrating on $[0,T]$ we get

$$\begin{aligned} -\int_0^T \mathcal{H}(q)\psi(s)ds \le \int_0^T \langle A_0 e^\epsilon(s), \eta\rangle\psi(s)ds + \int_0^T \langle \epsilon A_1 \dot e^\epsilon_{A_1}(s), \eta\rangle \psi(s) ds \\ + \int_0^T \langle \epsilon \dot p^\epsilon(s), q\rangle\psi(s)ds + \int_0^T \langle \epsilon^2 \ddot u^\epsilon(s), \varphi\rangle\psi(s)ds - \int_0^T \langle f^\epsilon(s), \varphi\rangle\psi(s)ds \\ - \int_0^T \langle g^\epsilon(s), \varphi\rangle_{\Gamma_1}\psi(s)ds \le \int_0^T \mathcal{H}(-q)\psi(s)ds, \end{aligned} \tag{1.3.63}$$

for every $(\varphi, \eta, q) \in A(0)$. It is easily seen that, if ψ has compact support, thanks to (1.3.58b) the term

$$\int_0^T \langle \epsilon^2 \ddot u^\epsilon(s), \varphi\rangle\psi(s)ds = -\epsilon^2 \int_0^T \langle \dot u^\epsilon(s), \varphi\rangle \dot\psi(s)ds$$

vanishes as $\epsilon \to 0$, and the same is true for the term

$$\int_0^T \langle \epsilon \dot p^\epsilon(s), q\rangle \psi(s)ds$$

thanks to (1.3.58d).

By (1.2.2c) we have

$$\epsilon \int_0^T \|A_1 \dot e^\epsilon_{A_1}\|_{L^2} ds \le \epsilon \beta_1^{\frac12} \int_0^T \mathcal{Q}_1(\dot e^\epsilon_{A_1})^{\frac12} ds \le \Big(\epsilon^2 \beta_1 \int_0^T \mathcal{Q}_1(\dot e^\epsilon_{A_1}(s))ds\Big)^{\frac12}.$$

By (1.3.58c) this shows that

$$\epsilon A_1 \dot e^\epsilon_{A_1} \to 0 \text{ strongly in } L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})), \tag{1.3.64}$$

as $\epsilon \to 0$. This implies that the term

$$\int_0^T \langle \epsilon A_1 \dot{e}^\epsilon_{A_1}(s), \eta\rangle \psi(s) ds$$

vanishes as $\epsilon \to 0$.

Since $(\varphi, \eta, q) \in A(0)$, by (1.3.26) we can write

$$\int_0^T (\langle f^\epsilon(s), \varphi\rangle + \langle g^\epsilon(s), \varphi\rangle_{\Gamma_1})\psi(s) ds = \int_0^T \langle \varrho^\epsilon(s), \eta + q\rangle \psi(s) ds,$$

and, thanks to (1.3.46i), we obtain that the last expression tends to

$$\int_0^T \langle \varrho(s), \eta + q\rangle \psi(s) ds = \int_0^T (\langle f(s), \varphi\rangle + \langle g(s), \varphi\rangle_{\Gamma_1})\psi(s) ds.$$

So from (1.3.60) and (1.3.63) we get

$$\begin{aligned} -\int_0^T \mathcal{H}(q)\psi(s) ds \leq \int_0^T \langle A_0 e(s), \eta\rangle \psi(s) ds - \int_0^T \langle f(s), \varphi\rangle \psi(s) ds \\ -\int_0^T \langle g(s), \varphi\rangle_{\Gamma_1} \psi(s) ds \leq \int_0^T \mathcal{H}(-q)\psi(s) ds, \end{aligned}$$

and thanks to the arbitrariness of ψ we conclude that

$$-\mathcal{H}(q) \leq \langle A_0 e(t), \eta\rangle - \langle f(t), \varphi\rangle - \langle g(t), \varphi\rangle_{\Gamma_1} \leq \mathcal{H}(-q), \tag{1.3.65}$$

for a fixed $(\varphi, \eta, q) \in A(0)$ and for a.e. $t \in [0, T]$. The fact that $A(0)$ is separable allows us to prove that for a.e. $t \in [0, T]$ inequalities (1.3.65) hold for every $(\varphi, \eta, q) \in A(0)$.

Let us define $\sigma(t) := A_0 e(t)$. For each $q \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$, since $(0, q, -q) \in A(0)$, we see that

$$-\mathcal{H}(-q) \leq \langle \sigma(t), q\rangle \leq \mathcal{H}(q), \tag{1.3.66}$$

which says that $\sigma_D(t) \in \partial\mathcal{H}(0) = \mathcal{K}(\Omega)$ (see (1.2.20)). Moreover, since for each $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ we have $(\varphi, E\varphi, 0) \in A(0)$, from (1.3.65) we obtain

$$\langle \sigma(t), E\varphi\rangle - \langle f(t), \varphi\rangle = \langle g(t), \varphi\rangle_{\Gamma_1} \quad \text{for all } \varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n). \tag{1.3.67}$$

From this we get $\text{div}\sigma(t) = f(t)$ a.e. in Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1. Therefore, $(u(t), e(t), p(t))$ satisfies condition (c) of Remark 1.3.6. This implies that for a.e. $t \in [0, T]$, $(u(t), e(t), p(t))$ satisfies the minimality condition (1.3.41) for all $(\varphi, \eta, q) \in A_{BD}(w(t))$. We now set $S := \{0\} \cup \{t \in (0, T] : \text{ (1.3.41) is satisfied}\}$ and we define $u(0) := u_0$ and $e(0) := e_0$. Since $p(0) = p_0$ by (1.3.47f) and (1.3.59), we deduce from (1.3.47c) that condition (1.3.41) is also satisfied for $t = 0$.

Since $t \mapsto p(t)$ has bounded variation from $[0, T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, it is globally bounded and there exists a countable set $N \subset [0, T]$ such that for every $t \in [0, T] \setminus N$

$$p(s) \to p(t) \quad \text{strongly in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \text{ as } s \to t. \tag{1.3.68a}$$

By the minimality property of $(u(s), e(s), p(s))$ for $s \in S$ we can apply [20, Theorem 3.8] and for every $t \in S \setminus N$ we obtain

$$e(s) \to e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}) \text{ as } s \to t, \tag{1.3.68b}$$

$$u(s) \to u(t) \quad \text{strongly in } BD(\Omega) \text{ as } s \to t. \tag{1.3.68c}$$

By the continuity of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ we also get

$$u(s) \to u(t) \quad \text{strongly in } L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \text{ as } s \to t. \tag{1.3.68d}$$

In order to prove the energy balance (1.3.42) we fix $t \in S \setminus (N \cup \{0\})$. For every k let $0 = t^k_0 < t^k_1 < ... < t^k_k = t$ be elements of $(S \setminus N) \cup \{0\}$ such that $\max_i(t^k_i - t^k_{i-1}) \to 0$ as $k \to \infty$. Then, since $(u(t^k_i) - (w(t^k_i) - w(t^k_{i-1})), e(t^k_i) - (Ew(t^k_i) - Ew(t^k_{i-1})), p(t^k_i)) \in A_{BD}(w(t^k_{i-1}))$ by (1.3.41), we have

$$\begin{aligned}
&\mathcal{Q}_0(e(t^k_{i-1})) - \langle f(t^k_{i-1}), u(t^k_{i-1})\rangle - \langle g(t^k_{i-1}), u(t^k_{i-1})\rangle_{\Gamma_1} \le \mathcal{Q}_0(e(t^k_i)) \\
&- \langle A_0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \mathcal{Q}_0(Ew(t^k_i)) - Ew(t^k_{i-1})) \\
&- \langle f(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle \\
&- \langle g(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle_{\Gamma_1} + \mathcal{H}(p(t^k_i) - p(t^k_{i-1})).
\end{aligned}$$

Employing the integration by parts formula (1.3.19) and then summing up over $i = 1, \dots, k$, we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \sum_{i=1}^k \mathcal{H}(p(t^k_i) - p(t^k_{i-1})) + \sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1})) \ge \\
&\ge \sum_{i=1}^k \langle A_0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho(0), e(0) - Ew(0)\rangle \\
&+ \langle \varrho_D(t), p(t)\rangle - \langle \varrho_D(0), p(0)\rangle - \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), e(t^k_i)\rangle \\
&+ \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), Ew(t^k_i)\rangle - \sum_{i=1}^k \langle \varrho_D(t^k_i) - \varrho_D(t^k_{i-1}), p(t^k_i)\rangle.
\end{aligned} \tag{1.3.69}$$

By (1.3.31), (1.3.32), (1.3.45d), (1.3.46f), (1.3.46g), and (1.3.68) we can apply Lemmas 1.3.11 and 1.3.12, with S replaced by $S \setminus (N \cup \{0\})$, and we obtain that the four Riemann sums in the right-hand side of (1.3.69) converge to

$$\int_0^t \langle \sigma, Ew\rangle ds, \quad \int_0^t \langle \dot{\varrho}, e\rangle ds, \quad \int_0^t \langle \dot{\varrho}, Ew\rangle ds, \quad \int_0^t \langle \dot{\varrho}_D, p\rangle ds.$$

Moreover we see that $\sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1}))$ tends to 0 as $k \to \infty$, thanks to the absolute continuity of $t \mapsto Ew(t)$. Therefore, passing to the limit in (1.3.69) we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p; 0, t) - \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho_D(t), p(t)\rangle \ge \\
&\ge \mathcal{Q}_0(e_0) - \langle \varrho(0), e(0) - Ew(0)\rangle - \langle \varrho_D(0), p(0)\rangle + \int_0^t \langle \sigma, E\dot{w}\rangle ds \\
&- \int_0^t \langle \dot{\varrho}, e - Ew\rangle ds - \int_0^t \langle \dot{\varrho}_D, p\rangle ds,
\end{aligned} \tag{1.3.70}$$

for a.e. $t \in [0,T]$, where $\sigma = A_0 e$.

We want to show that actually equality holds. In order to prove the opposite inequality we consider equation (1.3.54).

Thanks to the semicontinuity of $\mathcal{Q}_0(\cdot)$, by (1.3.60) we have

$$\int_a^b \mathcal{Q}_0(e(t))dt \leq \liminf_{\epsilon\to 0} \int_a^b \mathcal{Q}_0(e^\epsilon(t))dt \tag{1.3.71}$$

for all $0 < a < b < T$. We claim that

$$\begin{aligned}
&\int_a^b \Big(\mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot{\varrho}_D, p\rangle ds\Big) dt \leq \\
&\leq \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon\rangle ds\Big)dt,
\end{aligned} \tag{1.3.72}$$

for all $0 < a < b < T$. This, together with (1.3.71), implies

$$\begin{aligned}
&\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot{\varrho}_D, p\rangle ds\Big) dt \leq \\
&\leq \liminf_{\epsilon\to 0} \int_a^b \Big(\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|_{L^2}^2 + \epsilon \int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1})ds \\
&+ \epsilon \int_0^t \|\dot{p}^\epsilon\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon\rangle ds\Big)dt = \\
&= \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot{w}^\epsilon\rangle ds + \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle \\
&- \langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle - \int_0^t \langle \dot{\varrho}^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds \\
&- \epsilon^2 \int_0^t \langle \ddot{w}^\epsilon, \dot{u}^\epsilon - \dot{w}^\epsilon\rangle ds + \mathcal{Q}_0(e_0^\epsilon) + \frac{\epsilon^2}{2}\|v_0^\epsilon - \dot{w}^\epsilon(0)\|_{L^2}^2\Big)dt,
\end{aligned} \tag{1.3.73}$$

where the equality follows from (1.3.54) after an integration by parts in time.

Using (1.3.45f), (1.3.45g), (1.3.45i), (1.3.47g), and (1.3.58b) it is easily seen that

$$\epsilon^2 \int_a^b \Big(\int_0^t \langle \ddot{w}^\epsilon, \dot{u}^\epsilon - \dot{w}^\epsilon\rangle ds\Big) dt \to 0, \tag{1.3.74a}$$

$$\epsilon^2 \|v_0^\epsilon - \dot{w}^\epsilon(0)\|_{L^2}^2 \to 0, \tag{1.3.74b}$$

while

$$\int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot{w}^\epsilon\rangle ds\Big)dt \to \int_a^b \Big(\int_0^t \langle \sigma, E\dot{w}\rangle ds\Big)dt, \tag{1.3.74c}$$

$$\mathcal{Q}_0(e_0^\epsilon) \to \mathcal{Q}_0(e_0), \tag{1.3.74d}$$

$$\int_a^b \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle dt \to \int_a^b \langle \varrho(t), e(t) - Ew(t)\rangle dt, \tag{1.3.74e}$$

$$\langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle \to \langle \varrho(0), e(0) - Ew(0)\rangle, \tag{1.3.74f}$$

$$\int_a^b \Big(\int_0^t \langle \dot{\varrho}^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds\Big) dt \to \int_a^b \Big(\int_0^t \langle \dot{\varrho}, e - Ew\rangle ds\Big)dt, \tag{1.3.74g}$$

thanks to (1.3.45e), (1.3.45h), (1.3.46i), (1.3.47e), (1.3.60), and (1.3.64). This implies that

$$\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot\varrho_D, p\rangle ds\Big) dt \leq$$
$$\leq \int_a^b \Big(\int_0^t \langle \sigma, E\dot w\rangle ds + \mathcal{Q}_0(e_0) + \langle \varrho(t), e(t) - Ew(t)\rangle$$
$$- \langle \varrho(0), e(0) - Ew(0)\rangle - \int_0^t \langle \dot\varrho, e - Ew\rangle ds\Big) dt. \qquad (1.3.75)$$

From the arbitrariness of a and b and from (1.3.70) for a.e. $t \in [0,T]$ we obtain (1.3.44), which is equivalent to (1.3.42).

It remains to prove claim (1.3.72). This will be done by adapting the proof of [20, Theorem 4.5]. Let $\varphi : [0,+\infty) \to \mathbb{R}$ be a nonnegative C^∞ function such that $\phi(s) = 0$ for $s \leq 1$ and $\phi(s) = 1$ for $s \geq 2$. For $\delta > 0$ we define $\psi_\delta(x) := \phi(\frac{1}{\delta}\mathrm{dist}(x,\Gamma_1))$ for $x \in \bar\Omega$.

Since H is positively 1-homogeneous and satisfies (1.3.38) we have that

$$\int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon \psi_\delta\rangle ds \leq \int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds. \qquad (1.3.76)$$

Integrating by parts with respect to time and using then (1.3.20), this is equivalent to

$$\int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon) ds - \int_0^t \langle \dot\varrho^\epsilon, (e^\epsilon - Ew^\epsilon)\psi_\delta\rangle ds + \int_0^t \langle \dot f^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds$$
$$- \int_0^t \langle \dot\varrho^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta\rangle ds - \langle [\varrho_D^\epsilon(t) \cdot p^\epsilon(t)], \psi_\delta\rangle + \langle [\varrho_D^\epsilon(0) \cdot p^\epsilon(0)], \psi_\delta\rangle \leq$$
$$\leq \int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds. \qquad (1.3.77)$$

The lower semicontinuity of the variation, together with (1.3.35) and (1.3.59), implies

$$\mathcal{D}_H(\psi_\delta p; 0, t) \leq \liminf_{\epsilon\to 0} \int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon(s)) ds. \qquad (1.3.78)$$

By (1.3.27), (1.3.45e), (1.3.46h), (1.3.46i),(1.3.47d), and (1.3.47e), using Lemma 1.3.3 we obtain

$$\langle [\varrho_D^\epsilon(0) \cdot p^\epsilon(0)], \psi_\delta\rangle \to \langle [\varrho_D(0) \cdot p(0)], \psi_\delta\rangle. \qquad (1.3.79)$$

For what concerns the term $\langle [\varrho_D^\epsilon(t) \cdot p^\epsilon(t)], \psi_\delta\rangle$, we fix $0 \leq a < b \leq T$ and integrate on $[a,b]$ with respect to time. Using (1.3.20) we write

$$\int_a^b \langle [\varrho_D^\epsilon \cdot p^\epsilon], \psi_\delta\rangle ds = -\int_a^b \langle \varrho^\epsilon \cdot (e^\epsilon - Ew^\epsilon), \psi_\delta\rangle ds$$
$$+ \int_a^b \langle f^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds - \int_a^b \langle \varrho^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta\rangle ds,$$

where we have used the fact that ψ_δ is zero in a neighborhood of Γ_1. The last three terms pass to the limit thanks to (1.3.45e), (1.3.46h), (1.3.46i), (1.3.60), and (1.3.61). Therefore, using again (1.3.20) we obtain

$$\int_a^b \langle[\varrho_D^\epsilon \cdot p^\epsilon], \psi_\delta\rangle ds \to \int_a^b \langle[\varrho_D \cdot p], \psi_\delta\rangle ds. \tag{1.3.80}$$

We now integrate in (1.3.77) with respect to time. By (1.3.45e), (1.3.46h), (1.3.46i), (1.3.60), (1.3.61), and (1.3.78)-(1.3.80) we get

$$\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0, t) - \int_0^t \langle \dot\varrho \cdot (e - Ew), \psi_\delta\rangle ds + \int_0^t \langle \dot f, \psi_\delta(u - w)\rangle ds$$
$$- \int_0^t \langle \dot\varrho, (u - w) \odot \nabla\psi_\delta\rangle ds - \langle[\varrho_D(t) \cdot p(t)], \psi_\delta\rangle + \langle[\varrho_D(0) \cdot p(0)], \psi_\delta\rangle\Big) dt \le$$
$$\le \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big) dt. \tag{1.3.81}$$

Using (1.3.20) we get

$$\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0, t) - \langle[\varrho_D(t)\cdot p(t)], \psi_\delta\rangle + \langle[\varrho_D(0)\cdot p(0)], \psi_\delta\rangle + \int_0^t \langle[\dot\varrho_D \cdot p], \psi_\delta\rangle ds\Big) dt \le$$
$$\le \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big) dt.$$

Letting $\delta \to 0$ and using the semicontinuity of $\mathcal{D}_H$ we then obtain (1.3.72). This concludes the proof of (1.3.42) for a.e. $t \in [0, T]$.

Since (1.3.41) and (1.3.42) are satisfied for a.e. $t \in [0, T]$, and in particular for $t = 0$, we can apply Theorem 1.3.5. We obtain that $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ is absolutely continuous and we can redefine $u(t)$ and $e(t)$ on a set of times with measure zero so that $u : [0, T] \to BD(\Omega)$ and $e : [0, T] \to L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$ are absolutely continuous and the function (u, e, p, σ), with $\sigma(t) = A_0 e(t)$, is a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0.

From (1.3.74) and from the energy balance (1.3.42) it follows that the inequality in (1.3.73) is actually an equality and that the liminf is a limit. So, since

$$\int_a^b \Big(\frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|_{L^2}^2 + \epsilon \int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1}) ds + \epsilon \int_0^t \|\dot p^\epsilon\|^2_{L^2} ds\Big) dt \ge 0,$$

it follows that equality holds also in (1.3.71) and (1.3.72), and that the liminf is a limit also in this formulae. In particular

$$\int_0^T \mathcal{Q}_0(e^\epsilon(t)) dt \to \int_0^T \mathcal{Q}_0(e(t)) dt, \tag{1.3.82}$$

Since $e^\epsilon \rightharpoonup e$ weakly by (1.3.60), from (1.3.82) it follows that

$$e^\epsilon \to e \quad \text{strongly in } L^2([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{1.3.83}$$

which gives (1.3.50) for a suitable subsequence. From this and (1.3.59) we conclude that

$$Eu^\epsilon(t) \rightharpoonup Eu(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{\mathrm{sym}}), \tag{1.3.84}$$

for a.e. $t \in [0,T]$.

Let us fix t for which (1.3.50) and (1.3.84) hold. Since $u^\epsilon(t) \in A(w^\epsilon(t))$, it follows from (1.3.4) that $u^\epsilon(t)$ is bounded in $BD(\Omega)$ uniformly with respect to ϵ. Up to a subsequence we may assume that $u^\epsilon(t)$ converges weakly* in $BD(\Omega)$ to a function v. By Lemma 1.3.1 it follows that $(v, e(t), p(t)) \in A_{BD}(w(t))$. Since we have also $(u(t), e(t), p(t)) \in A_{BD}(w(t))$, we deduce that $Ev = Eu(t)$ in Ω and $(w(t) - v) \odot \nu = (w(t) - u(t)) \odot \nu$ $\mathcal{H}^{n-1}$-almost everywhere on Γ_0. This implies that $v = u(t)$ $\mathcal{H}^{n-1}$ almost everywhere on Γ_0, and applying inequality (1.3.4) to $v - u(t)$ we obtain that $v = u(t)$ almost everywhere in Ω. This concludes the proof of (1.3.49).

□

1.3.5 Appendix

This section contains the proof of two technical results concerning the convergence of suitable Riemann sums for functions with values in Banach spaces.

Lemma 1.3.11. *Let X be a Banach space, let $\phi \in W^{1,1}([0,T];X)$, let $S \subset (0,T]$ be a set of full measure containing T and let $\psi : S \to X'$ be a bounded weakly* continuous function. For every $k > 0$ let $\{t_i^k\}_{0 \le i \le k}$ be a subset of $S \cup \{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{k} \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k) \rangle = \int_0^T \langle \psi(t), \dot\phi(t) \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between X' and X.

Proof. Let $\psi_k : [0,T] \to X'$ be the piecewise constant function defined by $\psi_k(t) = \psi(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Then

$$\sum_{i=1}^{k} \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k) \rangle = \int_0^T \langle \psi_k(t), \dot\phi(t) \rangle dt.$$

Since $\psi_k(t) \rightharpoonup \psi(t)$ weakly* for every $t \in S$ we have $\langle \psi_k(t), \dot\phi(t) \rangle \to \langle \psi(t), \dot\phi(t) \rangle$ for a.e. $t \in [0,T]$. The conclusion follows from the Dominated Convergence Theorem. □

The next lemma extends the previous result to the case of the duality product introduced in (1.3.16).

Lemma 1.3.12. *Let ϱ be the function introduced in the safe-load condition* (1.3.26)-(1.3.28) *and let $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ be a bounded function. Assume that there exists a set $S \subset (0,T]$ of full measure containing T such that for every $t \in S$ the function p is continuous at t with respect to the strong topology of $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ and $p(t) \in \Pi_{\Gamma_0}(\Omega)$. For every $k > 0$ let $\{t_i^k\}_{0\le i\le k}$ be a subset of $S \cup \{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{k} \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k) \rangle = \int_0^T \langle \dot\varrho_D(t), p(t) \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product introduced in (1.3.16).

Proof. Let $p_k : [0,T] \to \Pi_{\Gamma_0}(\Omega)$ be the piecewise constant function defined by $p_k(t) = p(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Using (1.3.31) and (1.3.32) we obtain that

$$\sum_{i=1}^{k} \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k)\rangle = \int_0^T \langle \dot{\varrho}_D(t), p_k(t)\rangle dt =$$
$$= \int_0^T \langle \dot{\varrho}_D(t), p_k(t) - p(t)\rangle dt + \int_0^T \langle \dot{\varrho}_D(t), p(t)\rangle dt. \tag{1.3.85}$$

By (1.3.17) we have

$$\int_0^T |\langle \dot{\varrho}_D(t), p_k(t) - p(t)\rangle| dt \le \int_0^T \|\dot{\varrho}_D(t)\|_{L^\infty} \|p_k(t) - p(t)\|_{\mathcal{M}_b} dt$$

Since $\|p_k(t) - p(t)\|_{\mathcal{M}_b} \to 0$ for a.e. $t \in S$ by our continuity assumption and $t \mapsto \|\dot{\varrho}(t)\|_{L^\infty}$ belongs to $L^1([0,T])$ (see [20, Theorem 7.1]), we obtain

$$\lim_{k\to\infty} \int_0^T |\langle \dot{\varrho}_D(t), p_k(t) - p(t)\rangle| dt = 0 \tag{1.3.86}$$

by the Dominated Convergence Theorem. The conclusion follows from (1.3.85) and (1.3.86). $\square$

1.4 The delamination problem

1.4.1 Preliminaries

Reference configuration and notation. We consider an hyperelastic body that occupies a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with Lipschitz boundary. We suppose that

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2,$$

where Γ is a Lipschitz surface which is the common boundary of the two disjoint connected and open sets Ω_1 and Ω_2. The body is perfectly elastic on $\Omega_1 \cup \Omega_2$ while the surface Γ represents the interface where Ω_1 and Ω_2 are glued and where delamination may occur. We denote by ν the normal to Γ that points from Ω_1 into Ω_2. We also suppose that the boundary $\partial\Omega$ writes as the union

$$\partial\Omega := \partial_D\Omega \cup \partial_N\Omega,$$

where $\partial_D\Omega$ and $\partial_N\Omega$ are the closure in $\partial\Omega$ of two disjoint open sets with common boundary. We assume that $\partial_D\Omega$ has positive $(n-1)$-Hausdorff measure and that it has nonnegligible intersection with both $\partial\Omega_1$ and $\partial\Omega_2$.

Stress and Strain. The class of admissible displacements of the delamination problem is the space $H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$. It is convenient to define

$$H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^n)) := \{u \in H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)) : u = 0 \text{ on } \partial_D\Omega\}. \tag{1.4.1}$$

The jump on Γ of a displacement u is denoted by $[u] = u_2 - u_1$ where u_1 and u_2 are, respectively, the trace on Γ of $u \in H^1(\Omega_1, \mathbb{R}^n)$ and $u \in H^1(\Omega_2, \mathbb{R}^n)$. The continuity of the trace operator from $H^1(\Omega_i, \mathbb{R}^n)$ into $H^{\frac{1}{2}}(\Gamma, \mathbb{R}^n)$ reads

$$\|u\|_{H^{1/2}(\Gamma)} \leq \frac{\gamma}{2}\|u\|_{H^1(\Omega_i)}, \tag{1.4.2}$$

for a positive constant γ, and then we have

$$\|[u]\|_{H^{1/2}} \leq \gamma\|u\|_{H^1_D}. \tag{1.4.3}$$

In $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^n))$ the following Korn inequality holds

$$\|u\|_{H^1} \leq \beta\|Eu\|_{L^2} \qquad \text{for every } u \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.4}$$

for a positive constant β.

The two elasticity tensors A_0 and A_1 are symmetric and positive definite. As in the case of plasticity we assume that there exist positive constants α_i and β_i such that

$$\alpha_0|\eta|^2 \leq \langle A_0\eta, \eta\rangle \leq \beta_0|\eta|^2, \tag{1.4.5a}$$
$$\alpha_1|\eta|^2 \leq \langle A_1\eta, \eta\rangle \leq \beta_1|\eta|^2, \tag{1.4.5b}$$

for all $\eta \in \mathbb{M}^{n\times n}$. Also in this case it is convenient to introduce the following notations

$$\mathcal{Q}_0(e) = \frac{1}{2}\langle A_0 e, e\rangle, \tag{1.4.6}$$
$$\mathcal{Q}_1(e) = \langle A_1 e, e\rangle, \tag{1.4.7}$$

for all $e \in L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\rm sym})$.

The stress σ satisfies the constitutive relation

$$\sigma = A_0 Eu + \mu A_1 E\dot{u}, \tag{1.4.8}$$

where $\mu > 0$ is the viscosity parameter in the bulk. Then the second principle of dynamics reads

$$\rho\ddot{u}(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{ in } \Omega, \tag{1.4.9}$$

where we assume that the mass density of the elastic body is the constant $\rho > 0$. Together with (1.4.9) we require that the following boundary conditions are satisfied

$$u(t) = w(t) \quad \text{on } \partial_D\Omega, \tag{1.4.10a}$$

$$\sigma(t)\nu = g(t) \quad \text{on } \partial_N\Omega, \tag{1.4.10b}$$

$$\sigma(t)\nu = -\nabla V([u(t)])z(t) \quad \text{on } \Gamma, \tag{1.4.10c}$$

where V and z are the potential and the delamination coefficient introduced in the next Section. We can define the total external load of the system $\mathcal{L}(t) \in H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ by

$$\langle \mathcal{L}(t), \varphi\rangle := \langle f(t), \varphi\rangle + \langle g(t), \varphi\rangle_{\partial_N\Omega}, \tag{1.4.11}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$. To deal with (1.4.9) and (1.4.10), we define the continuous linear operator $\mathrm{div}_D : L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}) \to H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ by

$$\langle \mathrm{div}_D\sigma, \varphi\rangle := \langle \sigma, E\varphi\rangle, \tag{1.4.12}$$

for every $\sigma \in L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\rm sym})$ and every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$. Hence, if $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, $\partial_D\Omega$, and $\partial_N\Omega$ are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (1.4.11), then (1.4.9), (1.4.10b), and (1.4.10c) are equivalent to

$$\rho\ddot{u}(t) - \mathrm{div}_D\sigma(t) = \mathcal{L}(t) + T(u, z), \tag{1.4.13}$$

where equality holds in $H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$, and where $T(u, z)$ is the linear operator on $H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ defined by (1.4.17) below. In weak form (1.4.13) reads as

$$\langle \rho\ddot{u}(t), \varphi\rangle + \langle \sigma(t), E\varphi\rangle = \langle \mathcal{L}(t), \varphi\rangle - \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{1.4.14}$$

for every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$.

Delamination parameter and energy stored by the adhesive. As in the modelling approach by M. Frémond (see [27], [28]), at a fixed time the state of the glue on the interface Γ is described by the variable $z : \Gamma \to [0, 1]$. The state $z(x) = 1$ means that the adhesive is completely undestroyed, while $z(x) = 0$ means that the molecular links are all broken and the interface is totally debonded at $x \in \Gamma$. The deterioration of the glue is considered irreversible, that is the variable z is a nonincreasing function of the time. This turns into the condition

$$\dot{z} \le 0.$$

The class of admissible delamination parameters is denoted by

$$\mathcal{Z} := \{z \in L^2(\Gamma) : 0 \le z \le 1\}.$$

During the evolution of the system the energy needed to delaminate is denoted by $\alpha \in L^\infty(\Gamma)$, and such energy is dissipated in two ways, by heat production, whose cost we denote by $a_1 = a_1(x) > 0$, $x \in \Gamma$, and by creation of new delaminated surfaces, whose cost we denote by $a - a_1 := a_0 = a_0(x) > 0$, $x \in \Gamma$. Hence the dissipation due to these effects in the time interval $[t_1, t_2]$ reads

$$\mathcal{D}_a(t_1, t_2) := -\int_{t_1}^{t_2} \langle a_0 + a_1, \dot{z}(s)\rangle_\Gamma ds. \tag{1.4.15}$$

When evolution is quite fast we also consider the dissipation due to the viscosity of the glue. We consider a parameter $\mu = \mu(x) > 0$, $x \in \Gamma$, for which the energy dissipated by viscosity effects during the delamination process in the interval $[t_1, t_2]$ reads

$$\mathcal{D}_\tau(t_1, t_2) := \int_{t_1}^{t_2} \langle \mu \dot{z}(s), \dot{z}(s)\rangle_\Gamma ds. \tag{1.4.16}$$

In the sequel we will adopt the simpler (but not restrictive) hypothesis that μ is constant on Γ and coincides with the friction μ introduced in (1.4.8).

The energy stored in Γ by the adhesive is modeled as follows: let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonnegative and convex map such that

(i) $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. In particular $x = 0$ is the only local minimum of V.

(ii) $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant $L > 0$.

(iii) There exists $1 \le \delta \le \delta^*$ and $C > 0$ such that $|V(x)| \le C(|x| + 1)^\delta$ for all $x \in \mathbb{R}^n$.

Here $\delta^* = +\infty$ for $d \le 2$ and $\delta^* = \frac{d-1}{d-2}$ for $d > 2$. Since from (i) ∇V must vanish at the origin, property (ii) has the following consequence

(iv) For all $x \in \mathbb{R}^n$ it holds $|\nabla V(x)| \le L|x|$.

The energy stored on Γ at a fixed time then reads:

$$\mathcal{E}_\Gamma(u, z) := \langle V([u]), z\rangle_\Gamma.$$

We remark that in dimension $d \le 3$ we can take $V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u]$ where $\mathbb{K}$ is called elastic coefficient of the adhesive. Such matrix is supposed positive definite and symmetric. With this choice we see that the growth of V in (iii) above is $\delta = 2$. In higher dimension such a choice cannot be done for compactness reasons that will be clear in the proof of Theorem 1.4.1.

For all $u \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ and $z \in L^\infty(\Gamma)$ we define $T(u, z) \in H^{-1}_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ as

$$\langle T(u, z), \varphi\rangle := \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{1.4.17}$$

for every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$, so that, from (1.4.3), one has

$$|\langle T(u, z), \varphi\rangle| \le \|\nabla V([u])\|_{L^2} \|[\varphi]\|_{L^2} \|z\|_{L^\infty} \le 2L\gamma \|u\|_{H^1_D} \|\varphi\|_{H^1_D} \|z\|_{L^\infty},$$

which implies that there exists a positive constant C such that

$$\|T(u,z)\|_{H_D^{-1}} \le C\|u\|_{H_D^1}\|z\|_{L^\infty}. \tag{1.4.18}$$

Mechanical constraints and delamination process. When delamination occurs on the interface Γ it may happen that the two parts Ω_1 and Ω_2 of the body separate. In particular cavitation phenomena or shear movements may occur. Such phenomenon arises by the appearance of a non-zero jump of the displacement on Γ. Since interpenetration of Ω_1 and Ω_2 must be avoided, classically such jump is constrained to have a nonnegative normal component. Such condition is known in literature as Signorini contact condition. A generalization of the Signorini condition is usually considered, in the following way. Let $D(x) \subset \mathbb{R}^n$ be a convex and closed cone, possibly depending on $x \in \Gamma$. This induces an ordering relation on the set of functions $v : \Gamma \to \mathbb{R}^n$, as follows,

$$v_1 \preceq v_2 \text{ if and only if } v_2(x) - v_1(x) \in D(x) \text{ for a.e. } x \in \Gamma.$$

The dual ordering $\preceq^*$ induced by the negative polar cone to D is given by

$$\zeta \preceq^* 0 \text{ if and only if } \zeta(x) \le 0 \text{ for all } w \in D(x), \text{ for a.e. } x \in \Gamma.$$

Possible choices for the cone $D(x)$ are the following,

$$D(x) = \{v \in \mathbb{R}^n : v \cdot \nu(x) \ge 0\}, \tag{1.4.19a}$$
$$D(x) = \{v \in \mathbb{R}^n : v \cdot \nu(x) = 0\}, \tag{1.4.19b}$$

the first case being the classical unilateral Signorini contact condition, the latter being considered when cavitation cannot occur, for instance in systems under high pressure. The delamination mode (1.4.19a) and (1.4.19b) are usually referred to as *Mode I* and *Mode II* respectively. The constraint on the jump $[u]$ and the normal stress $t(\sigma) := \sigma\nu$ on Γ than reads

$$[u] \succeq 0, \tag{1.4.20a}$$
$$t(\sigma) + T(u,z) \succeq^* 0, \tag{1.4.20b}$$
$$(t(\sigma) + T(u,z)) \cdot [u] = 0. \tag{1.4.20c}$$

The behavior of the variable z is strictly connected to the evolution of $[u]$. Whenever $[u]$ varies this has the effect of destroying molecular links on Γ, that turns into a decrease of the corresponding glue state z. When the glue is completely erased, that is $z = 0$, any change of $[u]$ will not require energetic cost due to delamination. This is expressed by the constitutive equations

$$\dot z \le 0, \tag{1.4.21a}$$
$$d \le -\mu \dot z, \tag{1.4.21b}$$
$$\dot z(d + \mu \dot z) = 0, \tag{1.4.21c}$$
$$d \in \partial I_{[0,1]}(z) + V([u]) - \alpha, \tag{1.4.21d}$$

where $\partial I_{[0,1]}$ is the subdifferential of the function $I_{[0,1]}$, that is the function with equals 0 on $[0,1]$ and $+\infty$ on $\mathbb{R} \setminus [0,1]$. The parameter $\mu > 0$ is the viscosity of the adhesive. Let us remark that as soon as $z = 0$ equations (1.4.21b)-(1.4.21d) lose their significance and system (1.4.21) reduces to $z \equiv 0$, and no restriction to the evolution of $[u]$ is prescribed. At the same time, when $z > 0$ system (1.4.21) reads

$$\dot{z} \le 0, \tag{1.4.22a}$$
$$\dot{z}(V([u]) + \mu\dot{z} - \alpha) = 0. \tag{1.4.22b}$$

Since z is a function defined on the interface Γ, equations (1.4.21) and (1.4.22) must be intended to hold everywhere on Γ.

1.4.2 Existence of unconstrained dynamic solutions

Theorem 1.4.1. *Let $\mathcal{L} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^n))$, $u_0, v_0 \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$, and $z_0 \in \mathcal{Z}$. Then there exists a triple (u, σ, z) with*

$$u \in L^\infty([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.23a}$$
$$\dot{u} \in L^2([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)) \cap L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.23b}$$
$$\ddot{u} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.23c}$$
$$\sigma \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{sym}), \tag{1.4.23d}$$
$$z \in H^1([0,T], L^2(\Gamma)) \cap L^\infty([0,T], L^\infty(\Gamma)), \tag{1.4.23e}$$

satisfying, for a.e. $t \in [0,T]$,

$$\rho\ddot{u}(t) - \mathrm{div}_D\sigma(t) = \mathcal{L}(t) + T(u,z), \tag{1.4.24a}$$
$$\sigma(t) = A_0 E u(t) + \mu A_1 E\dot{u}(t), \tag{1.4.24b}$$

the flow rule

$$\dot{z}(t) \le 0, \tag{1.4.25a}$$
$$\dot{z}(t)(kV([u(t)]) + \mu\dot{z}(t) - \alpha) = 0, \tag{1.4.25b}$$

on Γ,

$$V([u(t)]) + \mu\dot{z}(t) - \alpha \le 0, \tag{1.4.26}$$

on $\Gamma \cap \{z(t) > 0\}$, and the initial data

$$u(0) = u_0, \qquad \dot{u}(0) = v_0, \qquad z(0) = z_0. \tag{1.4.27}$$

Remark 1.4.2. Let us remark that, when $\mathcal{L}$ takes the form (1.4.11), in the regular case, (1.4.24a) means that

$$\sigma(t)\nu = -\nabla V([u(t)])z(t), \tag{1.4.28}$$

on Γ, and

$$\rho\ddot{u}(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{1.4.29a}$$
$$\sigma(t)\nu = g(t) \quad \text{on } \partial_N\Omega. \tag{1.4.29b}$$

This is proved as follows. Integrating by parts (1.4.14) we get

$$\begin{aligned} &\langle \rho\ddot{u}, \varphi\rangle - \langle \mathrm{div}\sigma, \varphi\rangle - \langle \mathcal{L}, \varphi\rangle = \\ &- \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma - \langle \sigma\nu, [\varphi]\rangle_\Gamma - \langle \sigma\nu, \varphi\rangle_{\partial_N\Omega}, \end{aligned} \tag{1.4.30}$$

where ν represents both the normal versor to Γ pointing from Ω_1 into Ω_2 and the outer normal to $\partial_N\Omega$. If we set $[\varphi]=0$ we obtain the strong form (1.4.29), which together with (1.4.30) implies

$$\langle \nabla V([u])\cdot[\varphi], z\rangle_\Gamma = -\langle \sigma\nu, [\varphi]\rangle_\Gamma,$$

that is (1.4.28).

To prove Theorem 1.4.1 we proceed by time discretization, and solve a minimum problem at every discrete time. For all integer $n>0$ we divide the interval $[0,T]$ in n equal subintervals of length $\tau := T/n$. We set $t_i^n := i\tau$,

$$u_0^n = u_0, \qquad u_{-1}^n := u_0 - \tau v_0, \qquad z_0^n := z_0,$$

and define $\mathcal{L}_i^n := \frac{1}{\tau}\int_{t_i^n}^{t_{i+1}^n} \mathcal{L}(s)ds$ for all $n>0$. Then for $1\le i\le n$ we recursively define $u_i^n \in H_D^1(\Omega_1\cup\Omega_2;\mathbb{R}^n))$ as a minimizer of

$$\begin{aligned} U_i^n(u) :=& \frac{\rho}{2}\|\frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2 + \mathcal{Q}_0(Eu) + \langle V([u]), z_{i-1}^n\rangle_\Gamma \\ &+ \frac{\mu}{2}\langle A_1(Eu - Eu_{i-1}^n), Eu - Eu_{i-1}^n\rangle - \langle \mathcal{L}_i^n, u\rangle, \end{aligned} \tag{1.4.31}$$

and $z_i^n \in \mathcal{Z}$ as the minimizer of

$$W_i^n(z) := \frac{\mu}{2\tau}\|z - z_{i-1}^n\|_{L^2}^2 + \langle V([u_i^n]), z\rangle_\Gamma - \langle \alpha, z\rangle_\Gamma. \tag{1.4.32}$$

Computing variations in the variable u at the minimum u_i^n of (1.4.31) we get

$$\begin{aligned} &\frac{\rho}{\tau}\langle \frac{u-u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}, \varphi\rangle + \langle A_0 Eu_i^n, E\varphi\rangle \\ &+ \frac{\mu}{\tau}\langle A_1(Eu_i^n - Eu_{i-1}^n), E\varphi\rangle = \langle \mathcal{L}_i^n, \varphi\rangle - \langle \nabla V([u_i^n])\cdot[\varphi], z_{i-1}^n\rangle_\Gamma, \end{aligned} \tag{1.4.33}$$

for every $\varphi \in H^1(\Omega_1\cup\Omega_2;\mathbb{R}^n)$. Instead taking variations η of the minimum z_i^n of (1.4.32), and taking into account that z_i must be non-negative, we get

$$\langle V([u_i^n]), \eta\rangle_{\Gamma\cap\{z_i>0\}} + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \ge 0, \tag{1.4.34}$$

for every $\eta \le 0$.

Moreover, if the variation $\eta\le 0$ is such that $z_i \pm \epsilon\eta \in [0, z_{i-1}]$ for some $\epsilon>0$, then we will have equality. Denoting by $\mathcal{V}(z_i)$ the set of such variations, we have

$$\langle V([u_i^n]), \eta)\rangle_\Gamma + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma = 0, \tag{1.4.35}$$

for all $\eta \in \mathcal{V}(z_i)$. Now we set $v_i^n := \frac{u_i^n - u_{i-1}^n}{\tau}$ and define the following piecewise linear (or constant) functions

$$\begin{aligned} u_\tau(t) &:= u_i^n + (t-t_i^n)\frac{u_{i+1}^n - u_i^n}{\tau} && \text{for } t\in[t_i^n, t_{i+1}^n), \\ z_\tau(t) &:= z_i^n + (t-t_i^n)\frac{z_{i+1}^n - z_i^n}{\tau} && \text{for } t\in[t_i^n, t_{i+1}^n), \\ v_\tau(t) &:= v_i^n + (t-t_i^n)\frac{v_{i+1}^n - v_i^n}{\tau} && \text{for } t\in[t_i^n, t_{i+1}^n), \\ \mathcal{L}_\tau(t) &:= \mathcal{L}_i^n && \text{for } t\in[t_i^n, t_{i+1}^n), \end{aligned} \tag{1.4.36}$$

for $i = 0, \ldots, n-1$. The fact that

$$\mathcal{L}_\tau \to \mathcal{L} \quad \text{strongly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.37}$$

is standard and will often be tacitly used in the sequel. The following statement holds.

Proposition 1.4.3. *There are a function $u \in H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ and a function $z \in L^\infty([0,T], L^2(\Gamma)) \cap \mathcal{Z}$ such that*

$$\begin{aligned}
&u_\tau \rightharpoonup u \quad \textit{weakly in } H^1([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), && \text{(1.4.38a)}\\
&u_\tau(t) \rightharpoonup u(t) \quad \textit{weakly in } H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n),\ \textit{for every } t \in [0,T], && \text{(1.4.38b)}\\
&z_\tau \rightharpoonup z \quad \textit{weakly* in } L^\infty([0,T], L^2(\Gamma)), && \text{(1.4.38c)}\\
&z_\tau(t) \rightharpoonup z(t) \quad \textit{weakly* in } L^\infty(\Gamma) \textit{ for every } t \in [0,T],, && \text{(1.4.38d)}
\end{aligned}$$

as $\tau \to 0$. Moreover $\dot{u} \in H^1([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$, $z \in H^1([0,T], L^2(\Gamma))$, and

$$\begin{aligned}
&v_\tau \rightharpoonup \dot{u} \quad \textit{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), && \text{(1.4.38e)}\\
&\dot{v}_\tau \rightharpoonup \ddot{u} \quad \textit{weakly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), && \text{(1.4.38f)}\\
&\dot{z}_\tau \rightharpoonup \dot{z} \quad \textit{weakly in } L^2([0,T], L^2(\Gamma)). && \text{(1.4.38g)}
\end{aligned}$$

Proof. We choose $\varphi = u_i^n - u_{i-1}^n$ and $\eta = z_i^n - z_{i-1}^n$ in (1.4.33) and sum it with (1.4.35), we get

$$\begin{aligned}
&\frac{\rho}{2}\|\frac{u_i^n - u_{i-1}^n}{\tau}\|_{L^2}^2 - \frac{\rho}{2}\|\frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2 + \frac{\rho}{2}\|\frac{u_i^n - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\|_{L^2}^2\\
&+ \mathcal{Q}_0(Eu_i^n) - \mathcal{Q}_0(Eu_{i-1}^n) + \frac{1}{2}\langle A_0(Eu_i^n - Eu_{i-1}^n), Eu_i^n - Eu_{i-1}^n\rangle\\
&+ \frac{\mu}{\tau}\langle A_1(Eu_i^n - Eu_{i-1}^n), Eu_i^n - Eu_{i-1}^n\rangle - \langle \mathcal{L}_i^n, u_i^n - u_{i-1}^n\rangle\\
&- \langle \alpha, (z_i^n - z_{i-1}^n)\rangle_\Gamma + \langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma\\
&+ \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma + \frac{\mu}{\tau}\|z_i^n - z_{i-1}^n\|_{L^2}^2 \le 0.
\end{aligned} \tag{1.4.39}$$

Using the notations introduced in (1.4.36) and keeping into account that

$$\begin{aligned}
&\langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma + \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma =\\
&= \langle V([u_i^n]), z_i^n\rangle_\Gamma - \langle V([u_{i-1}^n]), z_{i-1}^n\rangle_\Gamma\\
&- \langle \int_{t_{i-1}}^{t_i} \nabla V([u_\tau]) \cdot [\dot{u}_\tau] - \nabla V([u_i^n]) \cdot [\dot{u}_\tau] dt, z_{i-1}^n\rangle_\Gamma,
\end{aligned} \tag{1.4.40}$$

we can rewrite (1.4.39) as follows

$$\frac{\rho}{2}\|v_\tau(t_i^n)\|_{L^2}^2 - \frac{\rho}{2}\|v_\tau(t_{i-1}^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_{t_{i-1}^n}^{t_i^n}\|\dot v_\tau\|_{L^2}^2 dt + \mathcal{Q}_0(Eu_\tau(t_i^n))$$
$$- \mathcal{Q}_0(Eu_\tau(t_{i-1}^n)) + \tau\int_{t_{i-1}^n}^{t_i^n}\mathcal{Q}_0(E\dot u_\tau)dt + \mu\int_{t_{i-1}^n}^{t_i^n}\mathcal{Q}_1(E\dot u_\tau)dt + \mu\int_{t_{i-1}^n}^{t_i^n}\|\dot z_\tau\|_{L^2}^2 dt$$
$$- \int_{t_{i-1}^n}^{t_i^n}\langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_i^n)]), z_\tau(t_i^n)\rangle - \langle V([u_\tau(t_{i-1}^n)]), z_\tau(t_{i-1}^n)\rangle$$
$$\le \int_{t_{i-1}^n}^{t_i^n}\langle \mathcal{L}_\tau, \dot u_\tau\rangle dt + \int_{t_{i-1}^n}^{t_i^n}\langle \nabla V([u_\tau])\cdot[\dot u_\tau] - \nabla V([u_i^n])\cdot[\dot u_\tau], z_{i-1}^n\rangle_\Gamma dt. \quad (1.4.41)$$

Using the fact that ∇V is Lipschitz, the continuity of the trace operator (1.4.3), and the Korn inequality (1.4.4) we write

$$|\int_{t_{i-1}^n}^{t_i^n}\langle \nabla V([u_\tau])\cdot[\dot u_\tau] - \nabla V([u_i^n])\cdot[\dot u_\tau], z_{i-1}^n\rangle_\Gamma dt| \le \tau k L\int_{t_{i-1}^n}^{t_i^n}\|[\dot u_\tau]\|_2^2 dt$$
$$\le \tau L\gamma^2\int_{t_{i-1}^n}^{t_i^n}\|\dot u_\tau\|_{H^1}^2 dt \le \tau L\gamma^2\beta^2\int_{t_{i-1}^n}^{t_i^n}\|E\dot u_\tau\|_{L^2}^2 dt. \quad (1.4.42)$$

Summing over $i = 1, \ldots, j$ expression (1.4.41) and then using (1.4.5), one gets

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_0^{t_j^n}\|\dot v_\tau\|_{L^2}^2 dt + \frac{\alpha_0}{2}\|Eu_\tau(t_j^n)\|_{L^2}^2$$
$$+ \frac{\alpha_0\tau}{2}\int_0^{t_j^n}\|E\dot u_\tau\|_{L^2}^2 dt + \alpha_1\mu\int_0^{t_j^n}\|E\dot u_\tau\|_{L^2}^2 dt + \mu\int_0^{t_j^n}\|\dot z_\tau\|_{L^2}^2 dt$$
$$- \int_0^{t_j^n}\langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle$$
$$\le \int_0^{t_j^n}\langle \mathcal{L}_\tau, \dot u_\tau\rangle dt + \tau L\gamma^2\beta^2\int_0^{t_j^n}\|E\dot u_\tau\|_{L^2}^2 dt + C, \quad (1.4.43)$$

for a constant $C > 0$ depending on u_0, v_0, z_0, μ, ρ, but independent of τ. Now we write

$$\int_0^{t_j^n}\langle \mathcal{L}_\tau, \dot u_\tau\rangle dt \le \frac{\lambda^{-1}}{2}\int_0^{t_j^n}\|\mathcal{L}_\tau\|_{H_D^{-1}}^2 dt + \frac{\lambda}{2}\int_0^{t_j^n}\|\dot u_\tau\|_{H^1}^2 dt$$
$$\le \frac{\lambda\beta^2}{2}\int_0^{t_j^n}\|E\dot u_\tau\|_{L^2}^2 dt + C, \quad (1.4.44)$$

where we have used the Korn inequality (1.4.4), $C > 0$ is a constant depending on the squared norm of $\mathcal{L} \in L^2([0,T], H_D^{-1}(\Omega_1\cup\Omega_2, \mathbb{R}^n))$ and on a fixed arbitrary positive number λ, but independent of τ. Then (1.4.43) implies

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_0^{t_j^n}\|\dot v_\tau\|_{L^2}^2 dt + \frac{\alpha_0}{2}\|Eu_\tau(t_j^n)\|_{L^2}^2 + \mu\int_0^{t_j^n}\|\dot z_\tau\|_{L^2}^2 dt$$
$$+ (\alpha_1\mu - \delta)\int_0^{t_j^n}\|E\dot u_\tau\|_{L^2}^2 dt - \int_0^{t_j^n}\langle \alpha, \dot z_\tau\rangle dt + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle \le C, \quad (1.4.45)$$

where $\delta := \frac{\lambda\beta^2}{4} + \tau L\gamma^2\beta^2$ and C is a positive. Since for λ sufficiently small and τ small enough all the terms in the left hand side are positive, we entail that all such terms are bounded. In particular there is a constant $L > 0$ such that

$$\|e_\tau(t)\|_{L^2}^2 \leq L, \tag{1.4.46}$$

for all $t \in [0,T]$, n and $\tau = \tau(n)$. So that there are an increasing sequence n_k and a function $e \in L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\mathrm{sym}}))$ such that

$$Eu_{\tau(n_k)} \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{1.4.47a}$$

as $k \to \infty$. We will write $\tau \to 0$ for $k \to \infty$. Using the Korn inequality, from (1.4.46) we get for all $t \in [0,T]$

$$\|u_\tau(t)\|_{H^1} \leq C, \tag{1.4.47b}$$

for some constant $C > 0$. This implies that, up to a subsequence, there is a function $u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ such that

$$u_\tau \rightharpoonup u \quad \text{weakly* in } L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.47c}$$

as $\tau \to 0$. (1.4.47c) also implies that $Eu(t) = e(t)$ for a.e. $t \in [0,T]$. Moreover (1.4.45) gives, up to passing to another subsequence,

$$E\dot{u}_\tau \rightharpoonup l \quad \text{weakly in } L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{1.4.47d}$$

$$v_\tau \rightharpoonup v \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.47e}$$

$$z_\tau \rightharpoonup \hat{z} \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{1.4.47f}$$

$$\dot{z}_\tau \rightharpoonup h \quad \text{weakly in } L^2([0,T], L^2(\Gamma)), \tag{1.4.47g}$$

as $\tau \to 0$, for functions $l \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{M}^{n\times n}_{\mathrm{sym}}))$, $v \in L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$, $\hat{z} \in L^\infty([0,T], \mathcal{Z})$, and $h \in L^2([0,T], L^2(\Gamma))$. Moreover z_τ are all functions with bounded variation on $[0,T]$, and their variations are bounded by the same constant. A generalization of Helly Theorem (see Lemma 7.2 of [20]) then implies that

$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma), \tag{1.4.47h}$$

for all $t \in [0,T]$ as $\tau \to 0$, for a function $z \in L^2([0,T], \mathcal{Z})$.

Writing $z_\tau(t) = z_0 + \int_0^t \dot{z}_\tau(s)ds$ and multiplying by a test function in $L^2(\Gamma)$ we see that $h(t) = \dot{z}(t)$. Multiplying z_τ by a test function in $L^1([0,T], L^2(\Gamma))$ it is easily seen that it must be $\hat{z} = z$. A similar argument shows that $l(t) = E\dot{u}(t)$ for a.e. $t \in [0,T]$. The Korn inequality and (1.4.47d) implies that there is a function $\hat{u} \in L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ such that

$$\dot{u}_\tau \rightharpoonup \hat{u} \quad \text{weakly in } L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.47i}$$

and writing $u_\tau(t) = u_0 + \int_0^t \dot{u}_\tau(s)ds$, arguing as before, we entail that u in (1.4.47c) belongs to $L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$, that $\hat{u} = \dot{u}$, and also that

$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n), \tag{1.4.48}$$

for all $t \in [0,T]$.

From (1.4.33) it follows

$$\rho\dot{v}_\tau(t) = -\mathrm{div}_D(A_0 e_\tau(t_i^n) + \mu A_1 \dot{e}_\tau(t)) + \mathcal{L}_i^n - T(u_\tau(t_i^n), z_\tau(t_i^n)), \quad (1.4.49)$$

for all $t \in [t_i^n, t_{i+1}^n]$ and all i. From the continuity of the operators div_D and T, and from the convergences (1.4.47) we see that the right-hand side of the last expression is uniformly bounded in $L^2([0,T], H_D^{-1}(\Omega;\mathbb{R}^n))$, so that the same is true for $\dot{v}_\tau$ and, up to subsequences, there exists $\hat{v} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ such that

$$\dot{v}_\tau \rightharpoonup \hat{v} \quad \text{weakly in } L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n)). \quad (1.4.50)$$

Now, $v_\tau(t) - \dot{u}_\tau(t) = (\tau - (t - t_i^n))\dot{v}_\tau(t)$ when $t \in [t_i^n, t_{i+1}^n]$, for all i, so that $\int_0^T \|v_\tau - \dot{u}_\tau\|^2_{H_D^{-1}} ds = \frac{\tau^2}{3}\int_0^T \|\dot{v}_\tau\|^2_{H_D^{-1}} ds$, which, for the boundedness of $\dot{v}_\tau$, tends to zero. In particular, by (1.4.47i), since $\hat{u} = \dot{u}$, we find out that $\dot{u}(t) = v(t)$ for a.e. $t \in [0,T]$ and

$$v_\tau, \dot{u}_\tau \rightharpoonup \dot{u} \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2, \mathbb{R}^n)). \quad (1.4.51)$$

Finally we write $v_\tau(t) = v_0 + \int_0^t \dot{v}_\tau(s)ds$ and multiplying by a test function in $L^2([0,T], H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ we get $\dot{u}(t) = v_0 + \int_0^t \hat{v}(s)ds$, and then we get that $\dot{u}$ is differentiable in time and $\ddot{u} = \hat{v} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$. This concludes the proof. □

Corollary 1.4.4. *For the same subsequence of Theorem 1.4.1, it holds*

$$[u_\tau] \rightharpoonup [u] \quad \text{weakly* in } L^\infty([0,T], H^{\frac{1}{2}}(\Gamma)), \quad (1.4.52a)$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{weakly in } H^{\frac{1}{2}}(\Gamma), \text{ for every } t \in [0,T], \quad (1.4.52b)$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{strongly in } L^q(\Gamma), \text{ for every } t \in [0,T], \quad (1.4.52c)$$

for every $1 \le q < q^*$ with $\frac{1}{q^*} = \frac{d-2}{2(d-1)}$ if $d > 2$, $q^* = +\infty$ otherwise.

Proof. (1.4.52a) and (1.4.52b) are straightforward consequence of (1.4.38a), (1.4.38b), and continuity of the trace operator. (1.4.52c) follows instead from (1.4.38b) and the fact that the embedding $H^{\frac{1}{2}} \hookrightarrow L^q$ is compact for all $q < q^*$. □

Let us introduce the piecewise constant functions

$$\begin{aligned} \tilde{u}_\tau &= u_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \\ \tilde{z}_\tau &= z_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \end{aligned} \quad (1.4.53)$$

for all $i \le (n-1)$. It is easy to show that convergences (1.4.38a), (1.4.38b), and (1.4.38d) holds true also for $\tilde{u}_\tau$ and $\tilde{z}_\tau$ in place of u_τ and z_τ. Now we are ready to prove the first Euler condition.

Proposition 1.4.5. *Let u and z be the functions obtained in Proposition 1.4.3. Then it holds*

$$\langle \rho\ddot{u}, \varphi\rangle + \langle A_0 Eu + \mu A_1 E\dot{u}, E\varphi\rangle - \langle \mathcal{L}, \varphi\rangle + \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma = 0, \quad (1.4.54)$$

for all $\varphi \in H_D^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ and for a.e. $t \in [0,T]$.

Proof. We start from (1.4.33), that with the notation introduced above reads

$$\rho\langle \dot{v}_\tau, \varphi\rangle + \langle A_0 E\tilde{u}_\tau + \mu A_1 E\dot{u}_\tau, E\varphi\rangle - \langle \mathcal{L}_\tau, \varphi\rangle + \langle \nabla V([\tilde{u}_\tau])\cdot[\varphi], \tilde{z}_\tau\rangle_\Gamma = 0. \tag{1.4.55}$$

For $\psi \in C_c^\infty((0,T))$ we write

$$\begin{aligned}&\int_0^T \big(\langle A_0 E\tilde{u}_\tau + \mu A_1 E\dot{u}_\tau, E\varphi\rangle - \langle \mathcal{L}_\tau, \varphi\rangle + \langle \nabla V([\tilde{u}_\tau])\cdot[\varphi], \tilde{z}_\tau\rangle_\Gamma\big)\psi dt\\ &= -\int_0^T \rho\langle v_\tau, \varphi\rangle\dot{\psi} dt,\end{aligned} \tag{1.4.56}$$

and letting $\tau \to 0$, thanks to (1.4.47) we get

$$\begin{aligned}&\int_0^T \big(\langle A_0 Eu + \mu A_1 E\dot{u}, E\varphi\rangle - \langle \mathcal{L}, \varphi\rangle + \langle \nabla V([u])\cdot[\varphi], z\rangle_\Gamma\big)\psi dt\\ &= -\int_0^T \rho\langle \dot{u}, \varphi\rangle\dot{\psi} dt.\end{aligned} \tag{1.4.57}$$

Arbitraryness of ψ then implies (1.4.54). □

In order to prove the next Lemma we need to recall the Fréchet-Kolmogorov Theorem. For all $h \in \mathbb{R}^d$ we introduce the h-translation in $\mathbb{R}^n$, that is the function $s_h : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ defined by $s_h(f)(x) := f(x+h)$ for all $x \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$. Then the following Theorem holds true.

Theorem 1.4.6 (Fréchet-Kolmogorov)**.** *Let B be a subset of $L^1(\mathbb{R}^n)$ such that for all $f \in B$ it holds $f = 0$ out of a bounded set $U \subset \mathbb{R}^n$. Then B is a relatively compact set in $L^1(\mathbb{R}^n)$ if and only if there exists a continuous non-negative function $\omega : \mathbb{R}^n \to \mathbb{R}$ such that $\omega(0) = 0$ and $\|f - s_h(f)\|_1 \le \omega(h)$, for all $f \in B$ and for all $h \in \mathbb{R}^n$.*

See, e.g., [13] for a proof.

Lemma 1.4.7. *For all $q \ge 1$ and $t \in [0,T]$ we have*

$$z_\tau(t) \to z(t) \quad \textit{strongly in } L^q(\Gamma). \tag{1.4.58}$$

Proof. Since

$$z_i = \operatorname{argmin}_{0\le z\le z_{i-1}} \langle V([u_i]) - \alpha, z\rangle_\Gamma + \frac{\mu}{2\tau}\|z - z_{i-1}\|^2_{L^2(\Gamma)},$$

we see that the value of $z_i(x)$ in $x \in \Gamma$ is exactly the minimizer in $[0, z_{i-1}(x)]$ of

$$\langle V([u_i(x)]) - \alpha(x), z\rangle_\Gamma + \frac{\mu}{2\tau}|z - z_{i-1}(x)|^2, \tag{1.4.59}$$

so that, denoting $a(x) := V([u_i(x)]) - \alpha(x)$, we can explicitly compute the value of $z_i(x)$. If $\hat{z}(x) := -\frac{\tau}{\mu}a(x) + z_{i-1}(x)$ is the minimizer of (1.4.59) on $\mathbb{R}$, then we have, omitting the symbol x,

$$\begin{cases} \hat{z} > z_{i-1} & \Leftrightarrow & a < 0 & \Rightarrow & z_i = z_{i-1}, \\ 0 \le \hat{z} \le z_{i-1} & \Leftrightarrow & 0 \le a < \frac{\mu}{\tau}z_{i-1} & \Rightarrow & z_i = -\frac{\tau}{\mu}a + z_{i-1}, \\ \hat{z} < 0 & \Leftrightarrow & a > \frac{\mu}{\tau}z_{i-1} & \Rightarrow & z_i = 0, \end{cases} \tag{1.4.60}$$

from which it follows

$$\mu\dot{z}_\tau = -(a \wedge \frac{\mu}{\tau} z_{i-1})^+, \quad \text{and } z_i = z_{i-1} - (\frac{\tau}{\mu} a \wedge z_{i-1})^+. \tag{1.4.61}$$

From (1.4.52c) and the definition of V we see that $V([u_\tau])(t)$ is a converging sequence in $L^1(\Gamma, \mathbb{R}^n)$. So that from Theorem 1.4.6 we get a function $\omega : \Gamma \cong \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\omega(0) = 0$ and

$$\|[u_\tau]^2(t) - s_h([u_\tau]^2(t))\|_1 \leq \omega(h), \tag{1.4.62}$$

for all $h \in \mathbb{R}^{d-1}$ and for all τ and $t \in [0, T]$. Without loss of generality we can also suppose that $\|a - s_h(a)\|_1 \leq \omega(h)$, since $\alpha \in L^\infty(\Gamma)$.

For fixed τ, let us prove by induction on i that $\|z_i - s_h(z_i)\|_1 \leq \frac{i\tau}{\mu}\omega(h)$. Indeed, using the expression of z_i obtained above, we have

$$\begin{aligned}
\|z_i - s_h(z_i)\|_{L^1} &= \|z_{i-1} - (\frac{\tau}{\mu} a \wedge z_{i-1})^+ - \big(s_h(z_{i-1}) - (\frac{\tau}{\mu} s_h(a) \wedge s_h(z_{i-1}))^+\big)\|_{L^1} \\
&\leq \|z_{i-1} - s_h(z_{i-1})\|_{L^1} + \|\frac{\tau}{\mu} a - \frac{\tau}{\mu} s_h(a)\|_{L^1} \\
&\leq \frac{(i-1)\tau}{\mu}\omega(h) + \frac{\tau}{\mu}\omega(h) = \frac{i\tau}{\mu}\omega(h),
\end{aligned} \tag{1.4.63}$$

where the first inequality follows by the fact that the function $(x, y) \mapsto x - (x \wedge y)^+$ is 1-Lipschitz in both the two real variables, and the second inequality follows by the inductive hypothesis. Now, recalling that $\tau = \frac{T}{N}$, (1.4.63) implies that for all τ and $t \in [0, T]$ it holds $\|z_\tau(t) - s_h(z_\tau(t))\|_1 \leq \frac{T}{\mu}\omega(h)$. Since $z_\tau(t) \in [1, 0]$, we have $|z_\tau(t) - s_h(z_\tau(t))| \leq 1$, and then also

$$\|z_\tau(t) - s_h(z_\tau(t))\|_q^q \leq \frac{T}{\mu}\omega(h). \tag{1.4.64}$$

Using (1.4.47h), the last formula implies (1.4.58). □

We are now ready to prove the conditions governing the flow rule.

Proposition 1.4.8. *Let $u \in L^\infty([0, T], H^1(\Omega))$ and $z \in L^2([0, T], L^\infty(\Gamma))$ be the functions defined in* (1.4.47c) *and* (1.4.47h). *Then for a.e. $t \in [0, T]$ it holds*

$$\langle V([u(t)]), \dot{z}(t)\rangle_\Gamma + \mu\|\dot{z}(t)\|_{L^2}^2 - \langle \alpha, \dot{z}(t)\rangle_\Gamma = 0, \tag{1.4.65}$$

and

$$\langle V([u(t)]), \eta\rangle_{\{z(t)>0\}} + \mu\langle \dot{z}(t), \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \geq 0, \tag{1.4.66}$$

for all $\eta \in L^\infty(\Gamma)$, $\eta \leq 0$.

Proof. Let us fix $t \in [0, T]$, and for all τ we decompose the interface Γ as the union of the three sets $\Gamma = A_\tau^t \cup B_\tau^t \cup C_\tau^t$ where, if $t \in [t_i - 1, t_i)$, then $A_\tau^t := \{z_i = 0 < z_{i-1}\}$, $B_\tau^t := \{z_i = z_{i-1}\}$, $C_\tau^t := \{0 < z_i < z_{i-1}\}$. We recognize these three cases as the three options of (1.4.60), so that it is readily seen that

$$V([u_\tau])\dot{z}_\tau + \mu|\dot{z}_\tau|^2 - \alpha\dot{z}_\tau = 0, \tag{1.4.67}$$

on B_τ^t and C_τ^t, while on A_τ^t

$$V([u_\tau]) + \mu\dot{z}_\tau - \alpha \geq 0. \tag{1.4.68}$$

The latter being true for all $t \in [0,T]$. In particular, for every positive smooth function φ on $[0,T]$, recalling that $\dot z_\tau \le 0$, we have

$$\int_0^T \big(\langle V([u_\tau]), \dot z_\tau\rangle_\Gamma + \mu\|\dot z_\tau\|_2^2 - \langle \alpha, \dot z_\tau\rangle_\Gamma\big)\varphi dt \le 0. \tag{1.4.69}$$

We would like to pass to the limit in (1.4.69). To this aim, we first observe that from (1.4.52c) and the definition of V we see that actually $V([u_\tau])(t)$ is converging in $L^2(\Gamma,\mathbb{R})$. Thus we have $V([u_\tau]) \to V([u])$ strongly in $L^2([0,T],L^2(\Gamma))$. This together with (1.4.38g) and the Fatou lemma implies

$$\int_0^T \big(\langle V([u]), \dot z\rangle_\Gamma + \mu\|\dot z\|_2^2 - \langle \alpha, \dot z\rangle_\Gamma\big)\varphi dt \le 0. \tag{1.4.70}$$

Now formula (1.4.34) provides

$$\int_0^T \big(\langle V([\tilde u_\tau]), \eta\rangle_{\Gamma\cap\{\tilde z_\tau>0\}} + \mu\langle \dot z_\tau, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma\big)\varphi dt \ge 0. \tag{1.4.71}$$

for all $\eta \le 0$. We note that, by definitions of z_τ and $\tilde z_\tau$ it holds $\chi_{\{\tilde z_\tau>0\}} = \chi_{\{z_\tau>0\}}$. From Lemma 1.4.7 we know that $z_\tau \to z$ strongly in $L^1(\Gamma\times[0,T])$, so that we can suppose it converges almost everywhere in $\Gamma\times[0,T]$. As a consequence we entail

$$\limsup \chi_{\{z_\tau>0\}} \ge \chi_{\{z>0\}}.$$

Then, from (1.4.71), taking into account that $\eta \le 0$ and that $V([u_\tau]) \to V([u])$ strongly in $L^1(\Gamma\times[0,T])$, we obtain

$$\int_0^T \big(\langle V([u]), \eta\rangle_{\{z>0\}} + \mu\langle \dot z, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma\big)\varphi dt \ge 0, \tag{1.4.72}$$

for every smooth nonnegative function φ on $[0,T]$, and for all $\eta \le 0$. From arbitrariness of φ we get (1.4.66). Now, plugging $\eta = \dot z$ we recover the opposite inequality of (1.4.70) provided $\dot z = 0$ almost everywhere on the set $\{z = 0\}$. But this is a straightforward consequence of the fact that z is non-negative, then (1.4.65) follows and the Proposition is proved.

□

Proof of Theorem 1.4.1. Let us prove that conditions (1.4.54), (1.4.65), and (1.4.66) implies equations (1.4.24) and (1.4.25). Equation (1.4.24b) holds by definition, while (1.4.24a) is expressed by (1.4.14), that is exactly (1.4.54). From arbitrariness of η equation (1.4.66) readily implies

$$V([u(t)]) + \mu\dot z(t) - \alpha \le 0 \qquad \text{a.e. on } \Gamma\cap\{z(t)>0\},$$

that is (1.4.26), while (1.4.65) implies (1.4.25a) and (1.4.25b), keeping into account that z is nonnegative and nonincreasing. To prove (1.4.27), we use (1.4.38b), (1.4.38d), and the fact that $u_\tau(0) = u_0$ and $z_\tau(0) = z_0$ for all τ. It remains to show that $\dot u(0) = v_0$. We first note that (1.4.38e) and (1.4.38f) imply that

$$v_\tau \rightharpoonup \dot u \quad \text{weakly in } H^1([0,T], H_D^{-1}(\Omega_1\cup\Omega_2,\mathbb{R}^n)),$$

so that we entail $v_\tau(t) \rightharpoonup \dot u(t)$ weakly in $H_D^{-1}(\Omega_1\cup\Omega_2,\mathbb{R}^n)$ for all $t\in[0,T]$. Thesis follows since by definition $v_\tau(0) = v_0$ for all τ. □

When we deal with nonhomogeneous boundary datum the existence theorem is stated as follows:

Theorem 1.4.9. *Let* $\mathcal{L} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^n))$*,* $u_0, v_0 \in H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$*,* $z_0 \in \mathcal{Z}$*, and let* $w \in H^1([0,T], H_D^1(\Omega, \mathbb{R}^n))$ *with* $\dot{w} \in H^1([0,T], H_D^{-1}(\Omega, \mathbb{R}^n))$ *be such that* $w(0) = u_0$ *and* $\dot{w}(0) = v_0$ *on* $\partial_D \Omega$*. Then there exists a triple* (u, σ, z) *with*

$$u \in L^\infty([0,T], H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^n)), \tag{1.4.73a}$$
$$\dot{u} \in L^2([0,T], H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^n)) \cap L^\infty([0,T], L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^n)), \tag{1.4.73b}$$
$$\ddot{u} \in L^2([0,T], H_D^{-1}(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.73c}$$
$$\sigma \in L^2([0,T], L^2(\Omega_1 \cup \Omega_2; \mathbb{M}^{n\times n}_{sym}), \tag{1.4.73d}$$
$$z \in H^1([0,T], L^2(\Gamma)) \cap L^\infty([0,T], \mathcal{Z}), \tag{1.4.73e}$$

satisfying

$$\rho\ddot{u}(t) - \mathrm{div}_D \sigma(t) = \mathcal{L}(t) + T(u,z), \tag{1.4.74a}$$
$$\sigma(t) = A_0 E u(t) + \mu A_1 E \dot{u}(t), \tag{1.4.74b}$$

for a.e. $t \in [0,T]$*, the Dirichlet condition*

$$u(t) = w(t) \ \text{ on } \partial_D \Omega, \tag{1.4.74c}$$

for a.e. $t \in [0,T]$*, the relations*

$$\dot{z}(t) \leq 0, \tag{1.4.75a}$$
$$\dot{z}(t)(V([u(t)]) + \mu \dot{z}(t) - \alpha) = 0, \tag{1.4.75b}$$

on Γ*,*

$$V([u(t)]) + \mu \dot{z}(t) - \alpha \leq 0, \tag{1.4.76}$$

on $\Gamma \cap \{z(t) > 0\}$*, for a.e.* $t \in [0,T]$*, and the initial data*

$$u(0) = u_0, \qquad \dot{u}(0) = v_0, \qquad z(0) = z_0. \tag{1.4.77}$$

The proof is essentially the same of Theorem 1.4.1, that can be easily arranged.

Proof. We set $w_{-1}^n := w(0) - \tau \dot{w}(0)$, $w_i^n = w(t_i^n)$, $\omega_i^n := \frac{w_i^n - w_{i-1}^n}{\tau}$ for $i = 0, \ldots, n$, then we define the piecewise affine functions

$$w_\tau = w_i^n + (t - t_i^n)\frac{w_{i+1}^n - w_i^n}{\tau} \qquad \text{for } t \in [t_i^n, t_{i+1}^n), \tag{1.4.78a}$$
$$\omega_\tau = v_i^n + (t - t_i^n)\frac{\omega_{i+1}^n - \omega_i^n}{\tau} \qquad \text{for } t \in [t_i^n, t_{i+1}^n), \tag{1.4.78b}$$

for $i = 0, \ldots, n-1$. The fact that

$$w_\tau \to w \quad \text{strongly in } H^1([0,T], H^1(\Omega, \mathbb{R}^n)), \tag{1.4.79a}$$
$$\omega_\tau \to \dot{w} \quad \text{strongly in } H^1([0,T], H_D^{-1}(\Omega, \mathbb{R}^n)), \tag{1.4.79b}$$

is standard and easily checked. We also define the piecewise affine function $l_\tau : [0,T] \to H^{-1}_D(\Omega,\mathbb{R}^n))$ by setting

$$l_\tau := \rho\dot{\omega}_\tau - \mathrm{div}_D(A_0 E w_\tau + \mu A_1 E\dot{w}_\tau), \tag{1.4.80}$$

so that property (1.4.5), the continuity of div_D, and (1.4.79) imply that

$$l_\tau \to l \quad \text{strongly in } L^2([0,T], H^{-1}_D(\Omega,\mathbb{R}^n)), \tag{1.4.81}$$

where $l := \rho\ddot{w} - \mathrm{div}_D(A_0 E w + \mu A_1 E\dot{w})$. Arguing as in the proof of Theorem 1.4.1 we solve the minimum problems (1.4.31) and (1.4.32) with $\mathcal{L}^n_i - l(t^n_i)$ in place of $\mathcal{L}^n_i$ and denote by u^n_i and z^n_i their minimizers. Standard arguments taking into account relation (1.4.81) ensure one that the same estimates (1.4.45) hold for the functions u'_τ, z_τ, v'_τ defined as in (1.4.36). So that we found functions $u' \in H^1([0,T], H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^n))$ with $\dot{u}' \in H^1([0,T], H^{-1}_D(\Omega_1\cup\Omega_2,\mathbb{R}^n))$ and $z \in L^\infty([0,T], L^2(\Gamma)) \cap H^1([0,T],\mathcal{Z})$ such that

$$u'_\tau \rightharpoonup u' \quad \text{weakly in } H^1([0,T], H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^n)), \tag{1.4.82a}$$

$$u'_\tau(t) \rightharpoonup u'(t) \quad \text{weakly in } H^1_D(\Omega_1\cup\Omega_2,\mathbb{R}^n),\ \text{for every } t\in[0,T], \tag{1.4.82b}$$

$$z_\tau \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{1.4.82c}$$

$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t\in[0,T],, \tag{1.4.82d}$$

$$v'_\tau \rightharpoonup \dot{u}' \quad \text{weakly* in } L^\infty([0,T], L^2(\Omega_1\cup\Omega_2,\mathbb{R}^n)), \tag{1.4.82e}$$

$$\dot{v}'_\tau \rightharpoonup \ddot{u}' \quad \text{weakly in } L^2([0,T], H^{-1}_D(\Omega_1\cup\Omega_2,\mathbb{R}^n)), \tag{1.4.82f}$$

$$\dot{z}_\tau \rightharpoonup \dot{z} \quad \text{weakly in } L^2([0,T], L^2(\Gamma)). \tag{1.4.82g}$$

Moreover we also get (1.4.34), (1.4.35), while (1.4.33) is replaced by the following

$$\begin{aligned}&\rho\langle \dot{v}'_\tau,\varphi\rangle + \langle A_0 E\tilde{u}'_\tau + \mu A_1 E\dot{u}'_\tau, E\varphi\rangle + \langle \nabla V([\tilde{u}'_\tau])\cdot[\varphi], \tilde{z}_\tau\rangle_\Gamma \\ &= \langle \tilde{\mathcal{L}}_\tau - \tilde{l}_\tau, \varphi\rangle,\end{aligned} \tag{1.4.83}$$

for all $\varphi \in H^1_D$ and for a.e. $t\in[0,T]$. Arguing as in Proposition 1.4.5 we see that (1.4.83) passes to the limit as $\tau \to 0$ and leads one to

$$\begin{aligned}&\rho\langle \ddot{u}',\varphi\rangle + \langle A_0 E\tilde{u}'_\tau + \mu A_1 E\dot{u}'_\tau, E\varphi\rangle + \langle \nabla V([u'])\cdot[\varphi], z\rangle_\Gamma \\ &= \langle \mathcal{L} - l, \varphi\rangle,\end{aligned} \tag{1.4.84}$$

for all $\varphi \in H^1_D$ and for a.e. $t\in[0,T]$. If we define $u := u' + w$, observing that, since $w \in H^1(\Omega,\mathbb{R}^n)$, $[w] = 0$ on Γ, then (1.4.84) reads

$$\rho\langle \ddot{u},\varphi\rangle + \langle A_0 Eu + \mu A_1 E\dot{u}, E\varphi\rangle + \langle \nabla V([u])\cdot[\varphi], z\rangle_\Gamma = \langle \mathcal{L},\varphi\rangle. \tag{1.4.85}$$

At the same time (1.4.34) and (1.4.35) pass to the limit like in the case of homogeneous boundary datum, and give rise to the same equations (1.4.65) and (1.4.66). The conclusion easily follows. □

The following Proposition provides the energy balance of the system.

Proposition 1.4.10. *Let u be the solution of Theorem 1.4.9. Then for all $0 \le t_1 < t_2 \le T$, the following energy balance holds*

$$\frac{\rho}{2}\|\dot u(t_2) - \dot w(t_2)\|^2_{L^2} + \mathcal{Q}_0(Eu(t_2)) + \langle V([u(t_2)]), z(t_2)\rangle_\Gamma + \mu\int_{t_1}^{t_2}\mathcal{Q}_1(E\dot u)ds$$
$$+\mu\int_{t_1}^{t_2}\|\dot z\|^2_{L^2}ds - \langle \alpha, z(t_2)\rangle_\Gamma = \frac{\rho}{2}\|\dot u(t_1) - \dot w(t_1)\|^2_{L^2} + \mathcal{Q}_0(Eu(t_1)) - \langle \alpha, z(t_1)\rangle_\Gamma$$
$$+ \langle V([u(t_1)]), z(t_1)\rangle_\Gamma + \int_{t_1}^{t_2}\langle \sigma, E\dot w\rangle ds + \int_{t_1}^{t_2}\langle \mathcal{L} - \rho\ddot w, \dot u - \dot w\rangle ds, \qquad (1.4.86)$$

where $\sigma = A_0 Eu + \mu A_1 E\dot u$.

Proof. We put $\varphi = \dot u - \dot w$ in (1.4.84) and sum this expression with (1.4.65). Integrating in time on $[t_1, t_2]$ we get (1.4.86). □

1.4.3 Processes in Mode II

In order to prove the existence of solution of the problem in Theorem 1.4.9 which also satisfy constrains as in (1.4.20), we use a standard argument dealing with a penalization term.

Let $D \subset \mathbb{R}^n$ be the convex and closed cone defined in (1.4.19b). Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative and convex map such that

(i) $\Phi(0) = 0$ and $\Phi(x) > 0$ if $x \neq 0$.

(ii) The derivative Φ' of Φ is Lipschitz with constant $L > 0$.

(iii) There exists $1 \le \delta < q^*$ and $C > 0$ such that $|\Phi(x)| \le C(|x| + 1)^\delta$ for all $x \in \mathbb{R}$.

Here $q^* = +\infty$ for $d \le 2$ and $q^* = \frac{2(d-1)}{d-2}$ for $d > 2$. As for V, property (ii) has the following consequence

(iv) For all $x \in \mathbb{R}$ it holds $|\Phi'(x)| \le L|x|$.

Now we define $\bar V : \mathbb{R}^n \times \Gamma \to \mathbb{R}$ the function $\bar V(y, x) := \Phi(\mathrm{dist}(y, D(x))$. We then define $\tilde V : L^1(\Gamma) \to L^1(\Gamma)$ as $\tilde V([u(x)]) := \bar V([u(x)], x)$ when $[u] \in L^1(\Gamma)$. Finally, for all positive integers $h > 0$, we set $\tilde V_h := h\tilde V$.

Let us remind the constraint conditions on the jump of $[u]$ that we want to satisfy. They read

$$[u(t)] \succeq 0, \qquad (1.4.87a)$$
$$\sigma(t)\nu + T(u(t), z(t)) \succeq^* 0, \qquad (1.4.87b)$$
$$(\sigma(t)\nu + T(u(t), z(t))) \cdot [u(t)] = 0. \qquad (1.4.87c)$$

for a.e. $t \in [0, T]$. Since $\sigma(t)$ is not in general an element of $L^1(\Gamma, \mathbb{R}^n)$, we prove a theorem where the solutions satisfy (1.4.87) in a weak form.

Theorem 1.4.11. *Let D be the cone in* (1.4.19b) *and let $\mathcal{L}$, u_0, v_0, z_0, and w be as in Theorem 1.4.9. Then there exists a couple* (u, z) *satisfying* (1.4.73), (1.4.74c), (1.4.77), *and such that, for a.e.* $t \in [0,T]$, *it satisfies conditions* (1.4.65), (1.4.66), *and*

$$u(t) \in D, \tag{1.4.88a}$$

$$\langle \rho \ddot{u}, \varphi\rangle + \langle \mu A_1 E\dot{u} + A_0 Eu, E\varphi\rangle + \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma = \langle \mathcal{L}, \varphi\rangle, \tag{1.4.88b}$$

for all $\varphi \in H^1_D$ *with* $[\varphi]\cdot\nu = 0$.

We will give a sketch of the proof, being it very similar to the one of Theorem 1.4.1. Moreover, for simplicity, we will only treat the case with homogeneous boundary datum.

Proof. Let u_i^n be the minimum of the potential

$$\begin{aligned} U_i^n(u) :=& \frac{\rho}{2}\|\frac{u-u^n_{i-1}}{\tau} - \frac{u^n_{i-1}-u^n_{i-2}}{\tau}\|^2_{L^2} + \mathcal{Q}(Eu) + \langle V([u]), z^n_{i-1}\rangle_\Gamma \\ &+ \frac{\mu}{2}\langle A_1(Eu - Eu^n_{i-1}), Eu - Eu^n_{i-1}\rangle - \langle \mathcal{L}^n_i, u\rangle + \|\tilde{V}_h([u]\cdot\nu)\|_{L^1(\Gamma)}, \end{aligned} \tag{1.4.89}$$

and z_i^n the minimum of (1.4.32). The discrete Euler condition then is

$$\begin{aligned} &\frac{\rho}{\tau}\langle\frac{u-u^n_{i-1}}{\tau} - \frac{u^n_{i-1}-u^n_{i-2}}{\tau}, \varphi\rangle + \langle A_0 Eu^n_i, E\varphi\rangle + \langle \tilde{V}_h'([u^n_i]\cdot\nu), [\varphi]\cdot\nu\rangle_\Gamma \\ &+ \frac{\mu}{\tau}\langle A_1(Eu^n_i - Eu^n_{i-1}), E\varphi\rangle - \langle \mathcal{L}^n_i, \varphi\rangle + \langle \nabla V([u^n_i])\cdot[\varphi], z^n_i\rangle_\Gamma = 0, \end{aligned} \tag{1.4.90}$$

for all $\varphi \in H^1_D(\Omega_1\cup\Omega_2, \mathbb{R}^n)$. Arguing in the same way as in proof of Proposition 1.4.3 we obtain the same bounds and convergences (1.4.38) and the further information

$$\|\tilde{V}([u_\tau(t^n_j)]\cdot\nu)\|_{L^1(\Gamma)} \le C. \tag{1.4.91}$$

Passing to the limit as $\tau \to 0$ we find that the functions $u_h \in L^\infty([0,T], H^1(\Omega_1\cup \Omega_2;\mathbb{R}^n))$ and $z_h \in H^1([0,T], L^2(\Gamma))$ satisfies (1.4.65), (1.4.66), and, in place of (1.4.54),

$$\begin{aligned} &\langle \rho\ddot{u}_h, \varphi\rangle + \langle A_0 Eu_h + \mu A_1 E\dot{u}_h, E\varphi\rangle + \langle \nabla V([u_h])\cdot[\varphi], z_h\rangle_\Gamma \\ &= \langle \mathcal{L},\varphi\rangle - \langle \tilde{V}_h'([u_h]\cdot\nu), [\varphi]\cdot\nu\rangle_\Gamma, \end{aligned} \tag{1.4.92}$$

for all $\varphi \in H^1_D(\Omega_1\cup\Omega_2, \mathbb{R}^n)$ and for a.e. $t \in [0,T]$.

The same argument for Proposition 1.4.10 gives the following energy balance

$$\begin{aligned} &\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L^2} + \mathcal{Q}_0(Eu_h(t)) + \langle V([u_h(t)]), z_h(t)\rangle_\Gamma + \mu\int_0^t \mathcal{Q}_1(E\dot{u}_h)ds \\ &+ \mu\int_0^t \|\dot{z}_h\|^2_{L^2}ds - \int_0^t \langle \alpha, \dot{z}_h\rangle_\Gamma + \|\tilde{V}_h([u_h(t)]\cdot\nu\|_{L^1(\Gamma)} \\ &= \frac{\rho}{2}\|v_0\|^2_{L^2} + \mathcal{Q}_0(Eu_0) + \langle V([u_0]), z_0\rangle_\Gamma - \int_0^t \langle \mathcal{L}, \dot{u}_h\rangle ds, \end{aligned} \tag{1.4.93}$$

for all $t \in [0,T]$ We write

$$\int_0^t \langle \mathcal{L}, \dot{u}_h\rangle ds \le \frac{1}{2\lambda}\int_0^t \|\mathcal{L}\|^2_{H^{-1}_D}ds + \frac{\beta\lambda}{2}\int_0^t \|E\dot{u}_h\|^2_2 ds,$$

where $\lambda = \frac{\mu\alpha_1}{2\beta}$, so that, plugging this into the energy balance (1.4.93) and using (1.4.5) we obtain that there is a positive constant C independent of h such that

$$\frac{\rho}{2}\|\dot{u}_h(t)\|_{L^2}^2 + \frac{\alpha_0}{2}\|Eu_h(t)\|_2^2 + \langle V([u_h(t)]), z(t)\rangle_\Gamma + \frac{\mu\alpha_1}{4}\int_0^t \|E\dot{u}_h\|_2^2 ds$$
$$+ \mu\int_0^t \|\dot{z}_h\|_2^2 ds - \int_0^t \langle \alpha, \dot{z}_h\rangle_\Gamma ds + \|\tilde{V}_h([u_h(t)]\cdot\nu)\|_{L^1(\Gamma)} \le C. \tag{1.4.94}$$

Thanks to this a-priori estimate we have that there exists $u \in H^1([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n))$ and $z \in H^1([0,T], \mathcal{Z})$ such that, up to a subsequence,

$$u_h \rightharpoonup u \quad \text{weakly in } H^1([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.4.95a}$$
$$u_h(t) \rightharpoonup u(t) \quad \text{weakly in } H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n), \text{ for every } t \in [0,T], \tag{1.4.95b}$$
$$z_h \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], L^2(\Gamma)), \tag{1.4.95c}$$
$$z_h(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t \in [0,T], \tag{1.4.95d}$$
$$\dot{z}_\tau \rightharpoonup \dot{z} \quad \text{weakly in } L^2([0,T], L^2(\Gamma)). \tag{1.4.95e}$$

as $h \to +\infty$. The proof of this fact is identical to the proof of Proposition 1.4.3. Moreover, since the Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^q(\Gamma)$ is compact for all $1 \le q < q^*$, (1.4.95b) implies

$$[u_h(t)] \to [u(t)] \quad \text{strongly in } L^q(\Gamma), \text{ for every } t \in [0,T], \tag{1.4.95f}$$

for all $1 \le q < q^*$ as $h \to +\infty$. By definition of $\tilde{V}_h$, one has $\|\tilde{V}_h([u_h(t)] \cdot \nu)\|_{L^1(\Gamma)} = h\|\tilde{V}([u_h(t)]\cdot\nu)\|_{L^1(\Gamma)}$, so that (1.4.94) implies

$$\tilde{V}([u_h(t)]\cdot\nu) \to 0 \quad \text{strongly in } L^1(\Gamma), \text{ for every } t \in [0,T], \tag{1.4.95g}$$

as $h \to +\infty$, and in particular we get that $\chi_{D^c}([u_h(t)]\cdot\nu)[u_h(t)]\cdot\nu \to 0$ almost everywhere on Γ for all $t \in [0,T]$. This implies the important condition

$$[u(t)] \in D. \tag{1.4.96}$$

Thanks to convergences (1.4.95) it is now easy to pass to the limit as $h \to +\infty$ in (1.4.65) and (1.4.66). Indeed, passing to the limit in the first one, we get the inequality (1.4.70), thanks to (1.4.95d) and (1.4.95f). To get (1.4.66) we argue as in the proof of Proposition 1.4.8, getting also equality in (1.4.70), and then (1.4.65). Instead (1.4.92) passes to the limit in the case that $[\varphi]\cdot\nu = 0$ providing condition (1.4.88b). This concludes the proof. □

Corollary 1.4.12. *Let* (u, z) *be a solution of* (1.4.23)*,* (1.4.65)*, and* (1.4.88)*. Then the energy balance* (1.4.86) *holds.*

Proof. The proof is the same as Proposition 1.4.10, since $\dot{u}$ satisfies the constraint $[\dot{u}]\cdot\nu = 0$ and we can employ (1.4.88b) with $\varphi = \dot{u} - \dot{w}$. □

1.5 Limit of solutions in rescaled time

In this section we study the asymptotic behavior of dynamic evolutions when the rate of the external loads and the boundary conditions become slower and slower. This can be done through a suitable rescaling of the data as described in the introduction. This procedure shows that the new rescaled solution (u_ϵ, z_ϵ) solves the same equations of (u, z), with a scalar ϵ appearing besides all the terms with one time derivative, and ϵ^2 appearing beside the second derivative. In other words, this rescaling provides that (u_ϵ, z_ϵ) are the solutions of the beginning delamination problem with a density mass equal to $\rho\epsilon^2$ and a viscosity parameter equal to $\mu\epsilon$. For simplicity in what follows we simply replace ρ by ϵ^2 and μ by ϵ.

Now we are ready to compute the limit of (u_ϵ, z_ϵ) as the parameter ϵ vanishes. We will restrict to the dimension $n \leq 3$, and we will assume that the potential $V([u])$ has the form

$$V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u],$$

where $\mathbb{K}$ is called the elastic coefficient of the adhesive, and is constant on Γ. We assume also that $\mathbb{K}$ is positive definite, that is $\langle \mathbb{K}[u] \cdot [u] \rangle_\Gamma$ is a an equivalent norm on $L^2(\Gamma, \mathbb{R}^n)$. Such hypothesis are classical in literature. Moreover we will need to assume more regularity on the data. In particular we suppose that $w \in H^2([0,T], H^1_D(\Omega, \mathbb{R}^n))$ and $\mathcal{L} \in H^1([0,T], H^{-1}_D(\Omega, \mathbb{R}^n))$.

We first state the Theorem in the case of homogeneous boundary datum.

Theorem 1.5.1. *Let $\mathcal{L} \in H^1([0,T], H^{-1}_D(\Omega, \mathbb{R}^n))$ and u_0, v_0, z_0 as in Theorem 1.4.1. Let (u_ϵ, z_ϵ) be a solution of the problem in Theorem 1.4.1, then there exist $u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2))$ and $z \in L^2([0,T], \mathcal{Z})$ such that, up to a subsequence,*

$$u_\epsilon \to u \text{ strongly in } L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.5.1a}$$

$$z_\epsilon \rightharpoonup z \quad \text{weakly* in } L^\infty([0,T], \mathcal{Z}), \tag{1.5.1b}$$

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{for all} t \in [0,T], \tag{1.5.1c}$$

as $\epsilon \to 0$. There also exist two nonnegative Borel measures $\mu_z \in \mathcal{M}([0,T] \times \Gamma)$ and $\mu_b \in \mathcal{M}([0,T] \times \Omega)$ such that, for the same subsequence

$$\epsilon \dot{z}_\epsilon^2 \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}([0,T] \times \Gamma), \tag{1.5.1d}$$

$$\epsilon A_1 E\dot{u}_\epsilon \cdot E\dot{u}_\epsilon \rightharpoonup \mu_b \text{ weakly* in } \mathcal{M}([0,T] \times \Omega). \tag{1.5.1e}$$

as $\epsilon \to 0$. Moreover (u, z) satisfies for a.e. $t \in [0,T]$ the semistability condition

$$\langle A_0 Eu(t), E\varphi \rangle + \langle \mathbb{K}[u(t)] \cdot [\varphi], z(t) \rangle_\Gamma = \langle \mathcal{L}(t), \varphi \rangle, \tag{1.5.2}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$, and the energy equality

$$\begin{aligned} &\mathcal{Q}_0(Eu(t_2)) + \langle \frac{1}{2}\mathbb{K}[u(t_2)] \cdot [u(t_2)], z(t_2) \rangle_\Gamma - \langle \alpha, z(t_2) \rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) \rangle \\ &= \mathcal{Q}_0(Eu(t_1)) + \langle \frac{1}{2}\mathbb{K}[u(t_1)] \cdot [u(t_1)], z(t_1) \rangle_\Gamma - \langle \alpha, z(t_1) \rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) \rangle \\ &\quad + \mu_z(]t_1, t_2] \times \Gamma) + \mu_b(]t_1, t_2] \times \Omega) + \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}, u \rangle ds, \end{aligned} \tag{1.5.3}$$

for a.e. $0 \leq t_1 < t_2 \leq T$.

The proof of the theorem is essentially the same of [64, Proposition 3.2], with the only difference that we have the addition of viscosity in the adhesive. We summarize some important steps and emphasize some differences, and then refer to [64] for a complete discussion.

Proof. **Step 1: apriori bounds**. We recall the energy balance for the solution (u_ϵ, z_ϵ), that is

$$\frac{\epsilon^2}{2}\|\dot u_\epsilon(t)\|^2_{L^2} + \mathcal{Q}_0(Eu_\epsilon(t)) + \langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t)\rangle_\Gamma + \epsilon\int_0^t \mathcal{Q}_1(E\dot u_\epsilon)ds$$
$$+\epsilon\int_0^t \|\dot z_\epsilon\|^2_{L^2}ds - \int_0^t\langle\alpha, \dot z_\epsilon\rangle_\Gamma$$
$$= \epsilon^2\|u_0\|^2_{L^2} + \mathcal{Q}_0(Eu_0) + \langle\frac{1}{2}\mathbb{K}[u_0]\cdot[u_0], z_0\rangle_\Gamma + \int_0^t\langle\mathcal{L}, \dot u_\epsilon\rangle ds. \tag{1.5.4}$$

Integrating by parts in time and then using the Cauchy and the Korn inequalities, we see that the right-hand side of (1.5.4) is bounded by the quantity

$$\frac{C_0}{\lambda} + \frac{\beta\lambda}{2}\|Eu_\epsilon(t)\|^2_2 + C_1\int_0^t \|Eu_\epsilon\|^2_2 ds,$$

for some constants $C_0,\ C_1 > 0$ depending on the data of the problem but independent of ϵ, and for an arbitrary constant $\lambda > 0$. Setting $\lambda = \frac{\alpha_0}{2\beta}$, from (1.5.4) we obtain

$$\frac{\epsilon^2}{2}\|\dot u_\epsilon(t)\|^2_{L^2} + \frac{\alpha_0}{4}\|Eu_\epsilon(t)\|^2_2 + \langle\frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t)\rangle_\Gamma + \epsilon\alpha_1\int_0^t\|E\dot u_\epsilon\|^2_2 ds$$
$$+\epsilon\int_0^t\|\dot z_\epsilon\|^2_{L^2}ds - \int_0^t\langle\alpha, \dot z_\epsilon\rangle_\Gamma \le \frac{2\beta C_0}{\alpha_0} + C_1\int_0^t\|Eu_\epsilon\|^2_2 ds, \tag{1.5.5}$$

and in particular, since all the term in the left-hand side are non-negative, we entail

$$\|Eu_\epsilon(t)\|^2_2 \le C + C\int_0^t \|Eu_\epsilon\|^2_2 ds, \tag{1.5.6}$$

for some constant $C > 0$ independent of ϵ. The Gronwall Lemma then implies that the right-hand side of (1.5.5) is bounded by a constant. This provides the following estimates: there exists a constant $C > 0$ such that

$$\|Eu_\epsilon(t)\|^2_2 \le C \quad \text{for all } t\in[0,T], \tag{1.5.7a}$$
$$\langle\frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t))\rangle_\Gamma \le C \quad \text{for all } t\in[0,T], \tag{1.5.7b}$$
$$\epsilon\|\dot u_\epsilon(t)\|_2 \le C \quad \text{for all } t\in[0,T], \tag{1.5.7c}$$
$$\int_0^T \epsilon\|E\dot u_\epsilon\|^2_2 ds \le C, \tag{1.5.7d}$$
$$\int_0^T \epsilon\|\dot z_\epsilon\|^2_2 ds \le C. \tag{1.5.7e}$$

and arguing as in [64, Proposition 3.2] we find $z \in L^\infty([0,T], \mathcal{Z})$ such that

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma), \tag{1.5.8}$$

for all $t\in[0,T]$. The boundness

$$\|u_\epsilon(t)\|_{H^1} \le C \quad \text{for all } t\in[0,T], \tag{1.5.9}$$

implies that there exists $u \in L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ such that, up to a subsequence,

$$u_\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)), \tag{1.5.10a}$$

$$[u]_\epsilon \rightharpoonup [u] \text{ weakly* in } L^\infty([0,T], H^{\frac{1}{2}}(\Gamma, \mathbb{R}^n)), \tag{1.5.10b}$$

as $\epsilon \to 0$. Finally, the bounds (1.5.7d) and (1.5.7e) show that the functions $\epsilon \dot{z}_\epsilon^2$ and $\epsilon A_1 E\dot{u}_\epsilon \cdot E\dot{u}_\epsilon$ are uniformly bounded in $L^1([0,T] \times \Gamma)$ and $L^1([0,T] \times \Omega)$ respectively, so that there exist two nonnegative Borel measures μ_z and μ_b such that, up to a subsequence,

$$\epsilon \dot{z}_\epsilon^2 \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}([0,T] \times \Gamma), \tag{1.5.10c}$$

$$\epsilon A_1 E\dot{u}_\epsilon \cdot E\dot{u}_\epsilon \rightharpoonup \mu_b \text{ weakly* in } \mathcal{M}([0,T] \times \Omega). \tag{1.5.10d}$$

Step 2. The two following key lemma is proved in [64, Proposition 3.2].

Lemma 1.5.2. *For all $\varphi \in H^1(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ and all ψ compactly supported real smooth function on $[0,T]$, it holds*

$$\lim_{\epsilon \to 0} \int_0^T \langle \mathbb{K}[u_\epsilon(s)]\psi(s) \cdot [\varphi], z_\epsilon(s)\rangle_\Gamma = \int_0^T \langle \mathbb{K}[u(s)]\psi(s) \cdot [\varphi], z(s)\rangle_\Gamma. \tag{1.5.11}$$

Lemma 1.5.3. *It holds*

$$\int_0^t \langle \mathbb{K}[u(s)] \cdot [u(s)], z(s)\rangle_\Gamma ds \le \liminf_{\epsilon \to 0} \int_0^t \langle \mathbb{K}[u_\epsilon(s)] \cdot [u_\epsilon(s)], z_\epsilon(s)\rangle_\Gamma ds. \tag{1.5.12}$$

Step 3. Let ψ be a smooth and compactly supported positive function on $[0,T]$. Multiplying equation (1.4.85) by ψ and integrating in time on $[0,T]$ we obtain

$$\begin{aligned} &\int_0^T \big(\langle C^0 E u_\epsilon + \epsilon A_1 E\dot{u}_\epsilon, E\varphi\rangle + \langle \mathbb{K}[u_\epsilon] \cdot [\varphi], z_\epsilon\rangle_\Gamma\big)\psi ds \\ &= \int_0^T \langle \epsilon^2 \dot{u}_\epsilon, \varphi\rangle \dot{\psi} + \langle \mathcal{L}, \varphi\rangle \psi ds. \end{aligned} \tag{1.5.13}$$

Lemma 1.5.2 allows us to pass to the limit obtaining, thanks to (1.5.7c), (1.5.7d), (1.5.7e), (1.5.8), (1.5.10), and the arbitrariness of ψ,

$$\langle A_0 E u(t), E\varphi\rangle + \langle \mathbb{K}[u(t)] \cdot [\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{1.5.14}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ and a.e. $t \in [0,T]$. The

Taking $\varphi = u_\epsilon$ in (1.4.84) and then integrating in time on $[0,t]$ we obtain

$$\begin{aligned} &\epsilon^2 \langle \dot{u}_\epsilon(t), u_\epsilon(t)\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(E u_\epsilon(t)) + \int_0^t \epsilon^2 \|\dot{u}\|^2 + \mathcal{Q}_0(E u_\epsilon) ds \\ &= \epsilon^2 \langle v_0, u_0\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(E u_0)) - \int_0^t \langle \mathbb{K}[u_\epsilon] \cdot [u_\epsilon], z_\epsilon\rangle_\Gamma + \langle \mathcal{L}, u_\epsilon\rangle ds, \end{aligned} \tag{1.5.15}$$

and taking into account the bounds (1.5.7c), (1.5.7d), and (1.5.7e), letting $\epsilon \to 0$, we entail

$$\lim_{\epsilon \to 0} \int_0^t \mathcal{Q}_0(E u_\epsilon) + \langle \mathbb{K}[u_\epsilon] \cdot [u_\epsilon], z_\epsilon\rangle_\Gamma ds = \int_0^t \langle \mathcal{L}, u\rangle ds. \tag{1.5.16}$$

From (1.5.14) with $\varphi = u$, the right-hand side equals $\int_0^t \mathcal{Q}_0(Eu) + \langle \mathbb{K}[u] \cdot [u], z\rangle_\Gamma ds$. Now,

$$\int_0^t \mathcal{Q}_0(Eu) \leq \liminf_{\epsilon\to 0} \int_0^t \mathcal{Q}_0(Eu_\epsilon)ds,$$

and, from Lemma 1.5.3,

$$\int_0^t \langle \mathbb{K}[u] \cdot [u], z\rangle_\Gamma ds \leq \liminf_{\epsilon\to 0} \int_0^t \langle \mathbb{K}[u_\epsilon] \cdot [u_\epsilon], z_\epsilon\rangle_\Gamma ds,$$

so that by (1.5.16) we entail that equalities hold, and hence

$$u_\epsilon \to u \text{ strongly in } L^2([0,T], H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)). \tag{1.5.17}$$

In particular this gives that for a.e. $t \in [0,T]$ one has

$$u_\epsilon(t) \to u(t) \text{ strongly in } H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n), \tag{1.5.18}$$

$$[u_\epsilon](t) \to [u](t) \text{ strongly in } H^{\frac{1}{2}}(\Gamma, \mathbb{R}^n), \tag{1.5.19}$$

so that, thanks to (1.5.8), we also have

$$\langle \mathbb{K}[u_\epsilon] \cdot [u_\epsilon], z_\epsilon(t)\rangle_\Gamma \to \langle \mathbb{K}[u] \cdot [u], z(t)\rangle_\Gamma, \tag{1.5.20}$$

for a.e. $t \in [0,T]$. This allows us to pass to the limit as $\epsilon \to 0$ in (1.4.86), getting (1.5.25).

Step 4. The same argument of [64, Proposition 3.2] applies to prove (1.5.3). □

Theorem 1.5.1 easily generalizes to the case of nonhomogeneous boundary datum. Let us remark that in this case $u \in H^1$ and no longer in H^1_D, so convergences (1.5.1) hold with this difference.

Theorem 1.5.4. *Let $\mathcal{L} \in H^1([0,T], H^{-1}_D(\Omega, \mathbb{R}^n))$, $w \in H^2([0,T], H^1_D(\Omega, \mathbb{R}^n))$, and u_0, v_0, z_0 as in Theorem 1.4.1. Let (u_ϵ, z_ϵ) be the solution given by Theorem 1.4.1, then there exist $u \in L^\infty([0,T], H^1(\Omega_1 \cup \Omega_2))$ with $u(t) = w(t)$ on $\partial_D\Omega$, and $z \in L^2([0,T], \mathcal{Z})$ such that for a subsequence (1.5.1) hold as $\epsilon \to 0$ and for a.e. $t \in [0,T]$ the semistability condition holds*

$$\langle A_0 Eu(t), E\varphi\rangle + \langle \mathbb{K}[u(t)] \cdot [\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{1.5.21}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$. Moreover the energy equality

$$\begin{aligned}
&\mathcal{Q}_0(Eu(t_2)) + \langle \frac{1}{2}\mathbb{K}[u(t_2)] \cdot [u(t_2)], z(t_2)\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) - w(t_2)\rangle \\
&= \mathcal{Q}_0(Eu(t_1)) + \langle \frac{1}{2}\mathbb{K}[u(t_1)] \cdot [u(t_1)], z(t_1)\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) - w(t_1)\rangle - \langle \alpha, z(t_1)\rangle_\Gamma \\
&\quad + \mu_z(]t_1, t_2] \times \Gamma) + \mu_b(]t_1, t_2] \times \Omega) - \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}, u - w\rangle ds + \int_{t_1}^{t_2} \langle \sigma, E\dot{w}\rangle ds,
\end{aligned} \tag{1.5.22}$$

is true for a.e. $0 \leq t_1 < t_2 \leq T$, where $\sigma := A_0 Eu$.

Proof. The following energy balance holds

$$\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)-\dot{w}(t)\|^2_{L^2}+\mathcal{Q}_0(Eu_\epsilon(t))+\langle\frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)],z_\epsilon(t)\rangle_\Gamma+\epsilon\int_0^t\mathcal{Q}_1(E\dot{u}_\epsilon)ds$$
$$+\epsilon\int_0^t\|\dot{z}_\epsilon\|^2_{L^2}ds-\langle\alpha,z_\epsilon(t)\rangle_\Gamma=\epsilon^2\|v_0-\dot{w}_0\|^2_{L^2}+\mathcal{Q}_0(Eu_0)-\langle\alpha,z(0)\rangle_\Gamma$$
$$+\langle\frac{1}{2}\mathbb{K}[u_0]\cdot[u_0],z_0\rangle_\Gamma+\int_0^t\langle\sigma_\epsilon,E\dot{w}\rangle ds+\int_0^t\langle\mathcal{L}-\epsilon^2\ddot{w},\dot{u}_\epsilon-\dot{w}\rangle ds, \tag{1.5.23}$$

where $\sigma_\epsilon=A_0Eu_\epsilon+\epsilon A_1E\dot{u}_\epsilon$. We then write

$$\int_0^t\langle\mathcal{L},\dot{u}_\epsilon-\dot{w}\rangle ds\le|\langle\mathcal{L}(t),u_\epsilon(t)-w(t)\rangle|+\int_0^t\|\dot{\mathcal{L}}\|_{H^{-1}}\|u_\epsilon-w\|_{H^1}ds+C$$
$$\le C\|\mathcal{L}(t)\|_{H^{-1}}\|Eu_\epsilon(t)-Ew(t)\|_2+C\int_0^t\|\dot{\mathcal{L}}\|_{H^{-1}}\|Eu_\epsilon-Ew(t)\|_2ds+C$$
$$\le C+C\|Eu_\epsilon(t)\|_2+C\int_0^t\|Eu_\epsilon\|_2ds$$
$$\le C+\frac{\alpha_0}{4}\|Eu_\epsilon(t)\|_2^2+C\int_0^t\|Eu_\epsilon\|_2^2ds,$$

for some constant $C>0$ possibly different from line to line. Moreover

$$\int_0^T\langle\sigma_\epsilon,E\dot{w}\rangle ds\le C+C\int_0^T\|Eu_\epsilon\|_2^2ds+\frac{\epsilon\alpha_1}{2}\int_0^T\|E\dot{u}_\epsilon\|_2^2ds,$$

and

$$|\int_0^t\langle\epsilon^2\ddot{w},\dot{u}_\epsilon-\dot{w}\rangle ds|\le C+\epsilon^2\int_0^T\|\ddot{w}\|_{H^{-1}}\|E\dot{u}\|_2ds\le C+\epsilon^2\int_0^T\|E\dot{u}\|_2^2ds.$$

Hence the right-hand side of (1.5.23) is bounded by

$$C+\frac{\alpha_0}{4}\|Eu_\epsilon(t)\|_2^2+C\int_0^t\|Eu_\epsilon\|_2^2ds+\frac{\epsilon\alpha_1}{2}\int_0^T\|E\dot{u}_\epsilon\|_2^2ds+\epsilon^2\int_0^T\|E\dot{u}\|_2^2ds,$$

and we are lead to

$$\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|^2_{L^2}+\frac{\alpha_0}{4}\|Eu_\epsilon(t)\|_2^2+\langle\frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)],z_\epsilon(t)\rangle_\Gamma+\epsilon\int_0^t\|\dot{z}_\epsilon\|^2_{L^2}ds$$
$$+\frac{\epsilon\alpha_1-2\epsilon^2}{2}\int_0^t\|E\dot{u}_\epsilon\|_2^2ds-\int_0^t\langle\alpha,\dot{z}_\epsilon\rangle_\Gamma\le C+C\int_0^t\|Eu_\epsilon\|_2^2ds, \tag{1.5.24}$$

for some $C>0$. This again implies (1.5.6) and the a-priori bounds (1.5.7). The proof now is very similar to the previous and can be arranged straightforwardly. □

An immediate consequence of (1.5.3) is the following:

Corollary 1.5.5. *Let* (u, z) *be the evolution obtained in the previous theorem. Then*

$$\begin{aligned}
&\mathcal{Q}_0(Eu(t_2)) + \langle \frac{1}{2}\mathbb{K}[u(t_2)] \cdot [u(t_2)], z(t_2)\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) - w(t_2)\rangle \\
&\le \mathcal{Q}_0(Eu(t_1)) + \langle \frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) - w(t_1)\rangle \\
&- \langle \alpha, z(t_1)\rangle_\Gamma - \int_{t_1}^{t_2} \langle \dot{\mathcal{L}}, u - w\rangle ds + \int_{t_1}^{t_2} \langle \sigma, E\dot{w}\rangle ds, \qquad (1.5.25)
\end{aligned}$$

for a.e. $0 \le t_1 < t_2 \le T$.

Remark 1.5.6 (Limit of processes in mode II). The limit of evolution with constrains as provided by Theorem 1.4.11 is straightforwardly arranged. The limit (u, z) will satisfy for a.e. $t \in [0, T]$ the property

$$u(t) \in D, \qquad (1.5.26)$$

while the semistability condition (1.5.21) is replaced by

$$\langle A_0 Eu(t), E\varphi\rangle + \langle \mathbb{K}[u(t)] \cdot [\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}, \varphi\rangle, \qquad (1.5.27)$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ with $[\varphi] \cdot \nu = 0$.

We are now in position to discuss the flow rule of the limit evolution (u, z). The presence of the viscosity term $\dot{z}$ in the flow rules (1.4.65) and (1.4.66), in contrast to [64] where the flow rule is rate-independent, makes the following analysis necessary.

Lemma 1.5.7. *For a.e.* $(x, t) \in \Gamma \times [0, T]$ *it holds*

$$\frac{1}{2}\mathbb{K}[u(x, t)] \cdot [u(x, t)] - \alpha(x) \le 0 \text{ or } z(x, t) = 0. \qquad (1.5.28)$$

Proof. By (1.4.26), for all $\epsilon > 0$ it holds

$$(\frac{1}{2}\mathbb{K}[u_\epsilon] \cdot [u_\epsilon] - \epsilon\dot{z}_\epsilon - \alpha)\chi_{\{z_\epsilon > 0\}} \le 0.$$

Up to a subsequence we have that $\chi_{\{z_\epsilon > 0\}} \rightharpoonup \zeta$ weakly* in $L^\infty([0, T] \times \Gamma)$ for some $\zeta \in L^\infty([0, T] \times \Gamma)$. Thanks to (1.5.7e) we know that $\epsilon\dot{z}_\epsilon \to 0$ strongly in $L^2([0, T], L^2(\Gamma))$, while thanks to (1.5.9) and (1.5.19) we know that $\frac{1}{2}\mathbb{K}[u_\epsilon] \cdot [u_\epsilon] \to \frac{1}{2}\mathbb{K}[u] \cdot [u]$ strongly in $L^1([0, T], L^1(\Gamma))$, so that at the limit as $\epsilon \to 0$ the previous relation gives rise to

$$(\frac{1}{2}\mathbb{K}[u] \cdot [u] - \alpha)\zeta \le 0, \qquad (1.5.29)$$

almost everywhere on $[0, T] \times \Gamma$. Now the thesis follows if we prove that $\zeta > 0$ on the set $\{z > 0\}$. Let $A := \{(t, x) \in [0, T] \times \Gamma : 0 = \zeta(t, x) < z(x, t)\}$, and let us prove that $|A| = 0$. Then suppose $|A| > 0$. From the fact that $z_\epsilon(t) \rightharpoonup z(t)$ weakly* in $L^\infty(\Gamma)$ for all $t \in [0, T]$, the Fubini Theorem and the Dominated Convergence Theorem implies that

$$0 < \int_A z = \lim_{\epsilon \to 0} \int_A z_\epsilon,$$

but, on the other side we see that the right-hand side must be zero. Indeed we claim that $z_\epsilon \to 0$ strongly in $L^1(A)$. Since $z_\epsilon \leq 1$, the claim follows if we prove that $|\{z_\epsilon > 0\} \cap A| \to 0$. But this is true since $|\{z_\epsilon > 0\} \cap A| = \int_A \chi_{\{z_\epsilon>0\}} \to \int_A \zeta = 0$ by hypothesis, and the lemma is proved.

□

Now we prove that there is a representative $\bar{z} : [0,T] \times \Gamma \to [0,1]$ in the class of $z \in L^1([0,T] \times \Gamma)$ such that for a.e. $(t,x) \in [0,T] \times \Gamma$ there exists the time derivative $\frac{d}{dt}\bar{z}(t,x) \in \mathbb{R}$. Let us define

$$\bar{z}(t,x) := \liminf_{\delta\to 0} \int_{B_{x,\delta}} z(t,y)dy, \tag{1.5.30}$$

where $B_{x,\delta}$ is the ball in Γ centered at x and with radius $\delta > 0$. It turns out that such limit exists and coincides with $z(t,x)$ for a.e. $(t,x) \in [0,T] \times \Gamma$. Moreover for all x and all $0 \leq t_1 < t_2 \leq T$ it holds $\bar{z}(t_1,x) \leq \bar{z}(t_2,x)$, since this inequality holds for z_ϵ and we have $\int_{B_{x,\delta}} z(t,y)dy = \lim_{\epsilon\to 0} \int_{B_{x,\delta}} z_\epsilon(t,y)dy$ for all $\delta > 0$ by (1.5.8). In particular for all fixed $x \in \Gamma$ the function $t \to \bar{z}(t,x)$ is nonincreasing so that it is differentiable almost everywhere on $[0,T]$. Note also that with such definition for all $t \in [0,T]$ the function $\bar{z}(t,\cdot)$ coincides with $z(t,\cdot)$ almost everywhere on Γ, that is $\bar{z}(t)$ is a particular representative of $z(t)$ in $L^\infty(\Gamma)$.

For $\bar{z}$ the following is true.

Lemma 1.5.8. *For a.e.* $(t,x) \in [0,T] \times \Gamma$ *it holds*

$$(\frac{1}{2}\mathbb{K}[u(t,x)] \cdot [u(t,x)] - \alpha(x))\dot{\bar{z}}(t,x) = 0. \tag{1.5.31}$$

Proof. For all real numbers $0 \leq a < b \leq T$ and all open set $A \subset \Gamma$ we can define the total variation of z_ϵ on $[a,b] \times A$ as

$$\mathrm{Var}(z_\epsilon, [a,b] \times A) := \langle \chi_A, z_\epsilon(a) - z_\epsilon(b)\rangle_\Gamma, \tag{1.5.32}$$

that defines a nonnegative measure on the Borel subsets of $[0,T] \times \Gamma$. Defining similarly the total variation of z we see that $\mathrm{Var}(z_\epsilon,\cdot) \rightharpoonup \mathrm{Var}(z,\cdot)$ weakly* in the space of nonnegative Radon measures $\mathcal{M}_b([0,T] \times \Gamma)$. Writing $z_\epsilon(a) - z_\epsilon(b) = -\int_a^b \dot{z}_\epsilon(s)ds$ and similarly $z(a) - z(b) = -\int_a^b D_t\bar{z}(s)ds$ where D_t is the distributional derivative in time, we also obtain that for all Borel set $B \subset [0,T] \times \Gamma$,

$$-\int_B \dot{\bar{z}} \leq \mathrm{Var}(\bar{z}, B) \leq \mathrm{Var}(z_\epsilon, B) = -\int_B \dot{z}_\epsilon, \tag{1.5.33}$$

where the first inequality is due to the fact that $-\dot{\bar{z}}$ is only the part of $-D_t\bar{z}$ that is absolutely continuous with respect to the Lebesgue measure, while the second one follows by the lower semicontinuity of the mass.

Now from the fact that $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] \to \frac{1}{2}\mathbb{K}[u]\cdot[u]$ strongly in $L^1([0,T], L^1(\Gamma))$ we have that $\frac{1}{2}\mathbb{K}[u_\epsilon(t,x)] \cdot [u_\epsilon(t,x)] \to \frac{1}{2}\mathbb{K}[u(t,x)] \cdot [u(t,x)]$ for a.e. $(t,x) \in [0,T] \times \Gamma$. Let us define $C := \{(t,x) \in [0,T] \times \Gamma : \dot{\bar{z}}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)] \cdot [u(t,x)] - \alpha(x) \neq 0\}$. From the fact that $\bar{z}$ is nonnegative and nonincreasing it is straightforward that $\dot{\bar{z}} = 0$ on the set $\bar{z} = 0$, so that condition (1.5.28) tells us that $|C\Delta C'| = 0$, with $C' := \{(t,x) \in [0,T] \times \Gamma : \dot{\bar{z}}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)] \cdot$

$[u(t,x)] - \alpha(x) < 0\}$. Let us then prove that $|C'| = 0$. Suppose it is not the case, so that for some $n > 0$ it holds that $|C_n| > 0$, with $C_n := \{(t,x) \in [0,T] \times \Gamma : \dot{z}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)] \cdot [u(t,x)] - \alpha(x) < -\frac{1}{n}\}$. Thanks to the pointwise convergence of $\frac{1}{2}\mathbb{K}[u_\epsilon] \cdot [u_\epsilon]$ to $\frac{1}{2}\mathbb{K}[u] \cdot [u]$ we can find a subset $B \subset C_n$ with positive measure and a number ϵ_0 such that for all $\epsilon < \epsilon_0$ and all $(t,x) \in B$ it holds $\mathbb{K}[u_\epsilon(t,x)] \cdot [u_\epsilon(t,x)] - \alpha(x) < 0$. This means that, thanks to (1.4.25b), $\dot{z}_\epsilon(t,x) = 0$ for all $\epsilon < \epsilon_0$ and all $(t,x) \in B$. So that

$$0 = -\lim_{\epsilon \to 0} \int_B \dot{z}_\epsilon \geq -\int_B \dot{\bar{z}},$$

where we have used (1.5.33). But since $-\dot{\bar{z}}$ is nonnegative we find $\dot{\bar{z}} = 0$ almost everywhere on B, contradicting the hypothesis. □

Let us define $\mathcal{E} : [0,T] \to \mathbb{R}$ the energy of the limit evolution (u,z) obtained in Theorem 1.5.1 as

$$\begin{aligned}\mathcal{E}(t) :=& \mathcal{Q}_0(Eu(t)) + \langle \frac{1}{2}\mathbb{K}[u(t)] \cdot [u(t)], z(t)\rangle_\Gamma \\ &- \langle \alpha, z(t)\rangle_\Gamma - \langle \mathcal{L}(t), u(t)\rangle + \int_0^t \langle \dot{\mathcal{L}}, u\rangle ds, \end{aligned} \tag{1.5.34}$$

for all $t \in [0,T]$. Inequality (1.5.25) says exactly that $\mathcal{E}$ is an essentially nonincreasing function. Essentially means that there exists a negligible set $N \subset [0,T]$ such that $\mathcal{E}$ is nonincreasing on $[0,T] \setminus N$. We can then always extend it to a (unique) left-continuous nonincreasing function on the whole $[0,T]$. As a consequence the new $\mathcal{E}$ is discontinuous on an at most countable set $J_E \subset [0,T]$, and this set does not depend on the value of $\mathcal{E}$ on N. We will also denote by J_z the subset of $[0,T]$ where the function z is discontinuous with respect to the strong topology of $L^1(\Gamma)$. Since z is a nonincreasing function with values in $[0,1]$, we see that J_z is at most countable as well.

Theorem (1.5.1) shows that the evolution (u,z) limit of (u_ϵ, z_ϵ) satisfies the stability condition almost everywhere on $[0,T]$. The next Lemma gives a more precise description of the set of times where stability holds, and at the same time tells us that we can change the map $u \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^n))$ on the negligible set N in such a way that the energy $\mathcal{E}$ is globally nonincreasing.

Lemma 1.5.9. *Suppose $\bar{t} \in [0,T] \setminus (J_E \cup N)$ is such that z is continuous at $\bar{t}$ with respect to the strong topology of $L^1(\Gamma)$, i.e. $\bar{t} \notin J_z$. Then the stability condition* (1.5.2) *holds at such $\bar{t}$.*

Moreover there exists a representative of $u \in L^\infty([0,T], L^2(\Omega,\mathbb{R}^n))$, still denoted by u, such that the stability condition (1.5.2) *holds at all $t \in [0,T] \setminus J_z$ and the corresponding energy* (1.5.34) *is nonincreasing and continuous at all $t \in [0,T] \setminus J_z$.*

Proof. Condition (1.5.2) tells us that $u(t)$ is the (unique) minimizer in $H^1_\Gamma(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$ of the potential

$$W_t(u) := \mathcal{Q}_0(Eu) + \langle \frac{1}{2}\mathbb{K}[u] \cdot [u], z(t)\rangle_\Gamma - \langle \mathcal{L}(t), u\rangle. \tag{1.5.35}$$

Let us denote by $M(t) := \min W_t$. The fact that z is continuous at $\bar{t}$ entails that also M is continuous at $\bar{t}$. Let us choose a sequence t_n such that $t_n \notin N$

and $u(t_n)$ satisfies the stability condition (1.5.2) for all $n > 0$, then we have

$$\lim_{n\to\infty} \mathcal{E}(t_n) = \lim_{n\to\infty} \big(M(t_n) + \langle \alpha, z(t_n)\rangle - \int_0^{t_n} \langle \dot{\mathcal{L}}, u\rangle ds\big)$$
$$= M(\bar{t}) + \langle \alpha, z(\bar{t})\rangle - \int_0^{\bar{t}} \langle \dot{\mathcal{L}}, u\rangle ds = \mathcal{E}(\bar{t}), \tag{1.5.36}$$

where the last equality follows from the continuity of $\mathcal{E}$. This says that $W_{\bar{t}}(u(\bar{t})) = M(\bar{t})$, which, thanks to the uniqueness of the minimizer of $W_{\bar{t}}$, entails that $u(\bar{t})$ is such minimizer, so that it also satisfies (1.5.2), and the first part of the statement is proved.

Let us now fix $t \in [0,T] \setminus J_z$, if we choose t_n such that $t_n \to t$ and $u(t_n)$ satisfies the stability condition (1.5.2), formula (1.5.36) still holds with $\bar{t}$ replaced by t thanks to the continuity of z and proves that we can redefine u at all points $t \in N \setminus J_z$ as the minimizer of W_t. We see that the new u coincides with the old one almost everywhere and satisfies (1.5.2) at all $t \in N \setminus J_z$ by definition. This concludes the proof, noting that the new $\mathcal{E}$ corresponding to the new u is continuous on $[0,T] \setminus J_z$. □

Remark 1.5.10. A consequence of Lemma 1.5.9 is that the set of times $t \in [0,T]$ such that the new $u(t)$ does not satisfy the stability condition (1.5.2) is an at most countable set. Let us denote it by S_u. Lemma 1.5.9 then reads

$$(S_u \cup J_E) \subset J_z.$$

Another consequence of this fact is that at any time where z is continuous, also u is continuous with respect to the strong topology of $H^1_\Gamma(\Omega_1 \cup \Omega_2, \mathbb{R}^n)$. If we denote by J_u the set of times where u is discontinuous, then J_u is at most countable and $J_u \subset J_z$.

Another consequence of Lemma (1.5.9) is that the definition of the new u implies that for all $t \in [0,T] \setminus J_z$ relation (1.5.28) holds true for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma_1$.

Let us finally remark that, with the new definition of $\mathcal{E}$, the energy inequality (1.5.25) holds for all $t_1, t_2 \in [0,T] \setminus J_z$.

Theorem 1.5.11. *Suppose that there exists $0 < s \le T$ such that $z(t,x) > 0$ at a.e. $x \in \Gamma$ for all $0 \le t \le s$. Then the energy $\mathcal{E}$ is constant on $[0,s] \setminus J_z$, i.e. $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \in [0,s] \setminus J_z$. In particular $\mu_z = 0$ on $[0,s] \times \Gamma$ and $\mu_b = 0$ on $[0,s] \times \Omega$.*

Proof. Taking into account (1.5.25), it suffices to show that $\mathcal{E}(0) \le \mathcal{E}(s)$. To prove this, for all integers $n > 0$ let us choose a sequence of times $0 = t_0 < t_1 < \cdots < t_n = s$ such that $t_i \in [0,T] \setminus S_u$ for all $i \le n$ and such that $\max_{i<n} |t_{i+1} - t_i| \to 0$ as $n \to \infty$. The minimality of W_{t_i} at $u(t_i)$ implies $W_{t_i}(u(t_i)) \le W_{t_i}(u(t_{i+1}))$ for all $0 \le i < n$. This is equivalent to

$$\mathcal{Q}_0(Eu(t_i)) - \mathcal{Q}_0(Eu(t_{i+1})) - \langle \mathcal{L}(t_i), u(t_i)\rangle + \langle \mathcal{L}(t_{i+1}), u(t_{i+1})\rangle$$
$$+ \langle \frac{1}{2}\mathbb{K}[u(t_i)] \cdot [u(t_i)], z(t_i)\rangle_\Gamma - \langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_{i+1})\rangle_\Gamma$$
$$\le \langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_i) - z(t_{i+1})\rangle_\Gamma + \langle \mathcal{L}(t_{i+1}), u(t_{i+1}) - u(t_i)\rangle$$
$$\le \langle \alpha, z(t_i) - z(t_{i+1})\rangle_\Gamma + \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1})\rangle, \tag{1.5.37}$$

where in the last inequality we have used (1.5.28) with Remark 1.5.10. Summing this expression for $i = 0, \dots, n-1$ we obtain

$$\begin{aligned} &\mathcal{Q}_0(Eu(0)) - \mathcal{Q}_0(Eu(s)) - \langle \mathcal{L}(0), u(0)\rangle + \langle \mathcal{L}(s), u(s)\rangle \\ &+ \langle \frac{1}{2}\mathbb{K}[u(0)] \cdot [u(0)], z(0)\rangle_\Gamma - \langle \frac{1}{2}\mathbb{K}[u(s)] \cdot [u(s)], z(s)\rangle_\Gamma \\ &\leq \langle \alpha, z(0)\rangle_\Gamma - \langle \alpha, z(s)\rangle_\Gamma + \sum_{i=0}^{n-1} \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1})\rangle, \end{aligned} \tag{1.5.38}$$

but the last term tends to $\int_0^s \langle \dot{\mathcal{L}}, u\rangle ds$ as $n \to \infty$ thanks to the regularity of $\mathcal{L}$ and the fact that J_u is at most countable. So that the inequality above implies exactly $\mathcal{E}(0) \leq \mathcal{E}(s)$, and the thesis follows. □

Remark 1.5.12. If we do not redefine the functions $\mathcal{E}$ and u as in Lemma 1.5.9, Theorem 1.5.11 still holds, with the only difference that the equality $\mathcal{E}(t) = \mathcal{E}(0)$ holds only for a.e. $t \in [0, s] \setminus (N \cup J_z)$. To see this it suffices to apply the same proof with the only difference that we have to choose the times t_i in the set where (1.5.21) holds for the original u.

1.5.1 The one-dimensional case

In this section we consider the case $d = 1$. Without loss of generality we set $\Omega_1 :=]0, 1[$, $\Omega_2 :=]-1, 0[$, $\Gamma := \{0\}$ and $\partial_D\Omega := \{-1, 1\}$ and assume that $A_0 = 1$ and $\mathbb{K} = 1$. We denote by u the displacement, and we want to study an evolution with Dirichlet conditions $u(t, 1) = a_1(t)$ and $u(t, -1) = a_{-1}(t)$ for all $t \in [0,]$, and external forces $\mathcal{L}(t, x)$. This arises imposing $w(t, x) := a_{-1}(t) + \frac{x+1}{2}(a_1(t) - a_{-1}(t))$. We assume that at the initial time we have $z_0 = 1$.

Let us first state the following preliminary fact:

Lemma 1.5.13. *$\mathcal{L} \in H_D^{-1}(]-1, 0[\cup]0, 1[, \mathbb{R})$ if and only if there exists $F \in L^2(]-1, 0[\cup]0, 1[)$ such that $\langle \mathcal{L}, \varphi\rangle = -\langle F, \varphi_x\rangle$, for all $\varphi \in H_D^1(]-1, 0[\cup]0, 1[, \mathbb{R})$.*

Proof. We can write

$$\langle \mathcal{L}, \varphi\rangle \leq C_1 \|\varphi\|_{H^1} \leq C_2 \|\varphi_x\|_2,$$

thanks to the Poincaré inequality. In particular, since the linear map $A : H_D^{-1} \to L^2(]-1, 0[\cup]0, 1[)$ given by $A(\varphi) = \varphi_x$ is bijective, we see that the map $\mathcal{L} \circ A^{-1}$ belongs to the dual of $L^2(]-1, 0[\cup]0, 1[)$, and then there exists $F \in L^2(]-1, 0[\cup]0, 1[)$ such that $\mathcal{L} \circ A^{-1}(\psi) = -\langle F, \psi\rangle$ for all $\psi \in L^2(]-1, 0[\cup]0, 1[)$. The claim follows by writing $\psi = \varphi_x$. □

Lemma 1.5.9 guarantees that (u, z) satisfies (1.5.21) and (1.5.25) everywhere on $[0, T] \setminus J_z$. Now we prove that, up to suitably change the function $t \to (u(t), z(t))$ on a negligible set, we can assume that such conditions are satisfied for all $t \in [0, T[$. In the one-dimensional case $z(t)$ is just a real number, and convergence (1.5.1c) ensures that z is nonincreasing, and then coincides with $\bar{z}$ defined in Lemma 1.5.8. We define

$$\tilde{z}(t) := \lim_{s \to t^-} z(s).$$

In particular $\tilde{z}$ is left-continuous. Let $S_u \subset [0,T]$ be the set of all t at which (1.5.21) does not hold. Then for all $t \in S_u$ we define $u'(t)$ as the (unique) solution of problem (1.5.21) with $z(t)$ replaced by $\tilde{z}(t)$ and boundary datum $w(t)$. Then we set

$$\tilde{u}(t) := \begin{cases} u'(t) & \text{if } t \in L \\ u(t) & \text{otherwise.} \end{cases}$$

Not to weight up the notation since now on we will still denote $(\tilde{u}, \tilde{z})$ by (u, z). Let us remark that, thanks to Lemma 1.5.9, the fact that z is left-continuous at all $t \in [0, t_1]$, it is easily seen that the energy (1.5.34) turns out to be globally nonincreasing, i.e. it is a nonincreasing function on the whole interval $[0, t_1]$.

In other words we have first redefined z in order that it is left-continuous, and then we have redefine u as in Lemma 1.5.9. Thanks to the left-continuity of z we see that the proof of Lemma 1.5.9 provides that the new u satisfies (1.5.21) on the whole $[0, t_1]$.

When (t, z) are fixed, (1.5.21) is equivalent to the fact that u is the minimizer of the functional

$$u \to \frac{1}{2}\langle u_x, u_x\rangle + \frac{1}{2}[u]^2 z - \langle \mathcal{L}, u\rangle,$$

among all the functions $u \in H^1(]-1,0[\cup]0,1[)$ with $u(1) = w(t,-1)$ and $u(-1) = w(t,1)$. Equivalently, this is expressed by the following system of equations

$$\begin{cases} -u_{xx}(t,x) = \mathcal{L}(t,x) & \text{on} \quad]-1,0[\ \cup\ ]0,1[, \\ u_x(t,0) = [u(t,0)]z(t) \\ u(1) = w(t,-1) \\ u(-1) = w(t,1). \end{cases} \tag{1.5.39}$$

It is not difficult to compute explicitly the solutions of such system. Let $F \in H^1([0,T], L^2(]-1,0[\cup]0,1[))$ be the function, provided by Lemma 1.5.13, such that $\langle \mathcal{L}(t), \varphi\rangle = -\langle F(t), \varphi_x\rangle$ and set $G(t,x) := \int_0^x F(t)(y)dy$ for all $x \in]-1,0[\cup]0,1[$, the solution $u = u(t,x)$ of (1.5.39) takes the form

$$u(t,x) = \begin{cases} G(t,1) - G(t,x) + g(t)\frac{z(t)}{1+2z(t)}(x-1) + w(t,1) & \text{if } x > 0 \\ G(t,-1) - G(t,x) + g(t)\frac{z(t)}{1+2z(t)}(x+1) + w(t,-1) & \text{if } x < 0, \end{cases} \tag{1.5.40}$$

where $g(t) := G(t,1) - G(t,-1) + w(t,1) - w(t,-1)$. We can compute

$$[u(t)] := \frac{g(t)}{1+2z(t)}. \tag{1.5.41}$$

Let us define

$$\begin{aligned} t_0 &:= \inf_t\{\frac{1}{2}[u(t)]^2 - \alpha \geq 0\}, \\ t_1 &:= \inf_t\{\frac{1}{2}[u(t)]^2 - \alpha > 0\}, \end{aligned} \tag{1.5.42}$$

and let these values be T if the corresponding infima are computed on empty sets. Obviously we have $t_0 \leq t_1$. We see that the times t_0 and t_1 depend only on g and the value of z, in particular

$$t_0 := \inf_t \{z(t) \leq \frac{g(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}}\},$$
$$t_1 := \inf_t \{z(t) < \frac{g(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}}\}. \tag{1.5.43}$$

The energy (1.5.34) reads

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\langle u_x(t,x), u_x(t,x)\rangle + \frac{1}{2}[u(t)]^2 z(t) - \alpha z(t)\\ &+ \langle F(t), u_x(t) - w_x(t)\rangle - \int_0^t \langle \dot{F}(s), u_x(s) - w_x(s)\rangle ds\\ &- \int_0^t \langle u_x(s), \dot{w}_x(s)\rangle ds,\end{aligned}$$

and plugging the formulae found above in this expression we obtain

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\frac{g(t)^2 z(t)}{1+2z(t)} - \alpha z(t) - \frac{(G(0,1) - G(0,-1))(w(0,1) - w(0,-1))}{2}\\ &- \int_0^t \frac{g(s)\dot{g}(s)}{1+2z(s)} z(s).\end{aligned}$$

We will now employ a standard formula providing the expression of the distributional derivative of the composition of a smooth function with a function with bounded variation (see, e.g., [78], or [4]). If $z : [0,T] \to \mathbb{R}$ is a BV function and $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth, such formula applied to the function $t \to f(t, z(t))$ reads

$$\begin{aligned}&D_t f(\cdot, z(\cdot)) =\\ &\quad f_1(\cdot, z(\cdot))\mathcal{L}^1 + f_2(\cdot, \bar{z}(\cdot)) D_t z \llcorner C_z + \sum_{s\in\mathbb{R}^+} [f(s, z(s^+)) - f(s, z(s^-))]\delta_s,\end{aligned} \tag{1.5.44}$$

where f_i is the derivative of f with respect to the i-th variable, $\mathcal{L}^1$ is the Lebesgue measure on $\mathbb{R}$, $\bar{z}$ is the continuous representative of z on the set C_z, the set where z is continuous, $z(s^+)$ (resp. $z(s^-)$) is the limit from the right (resp. left) of z at $s \in \mathbb{R}$, and δ_s is the Dirac delta at $s \in \mathbb{R}$. We use this formula to compute the distributional derivative of $\mathcal{E}$. Let us recall that the function z itself is continuous at every t except at the jump times. Therefore we find

$$\begin{aligned}D_t\mathcal{E}(t) =& (\frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha)(\dot{z} + \dot{z}^c)\\ &+ \sum_{s\in[0,T]} \Big(\frac{1}{2}\frac{g(s^+)z(s^+)}{(1+2z(s^+))} - \frac{1}{2}\frac{g(s^-)z(s^-)}{(1+2z(s^-))} - \alpha z(s^+) + \alpha z(s^-)\Big)\delta_s,\end{aligned} \tag{1.5.45}$$

where $\dot{z}$ and $\dot{z}^c$ are the absolutely continuous part of $D_t z \llcorner C_z$ with respect to $\mathcal{L}^1$ and the Cantor part respectively. We can write the jumps of (1.5.45) in the following equivalent way

$$-\sum_{s\in[0,T]} \Big(\int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha dr\Big)\delta_s. \tag{1.5.46}$$

From the energy inequality we know that the energy is a nonincreasing function, so that its total derivative (1.5.45) must be a nonpositive measure on $[0, T]$. Since the absolutely continuous, the Cantor and the jump part of this measure are mutually singular, they must all be nonpositive. This applied to the jumps implies that the integrals appearing in the sum (1.5.46) are all nonnegative. On the other hand we have

$$\int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha dr \leq \int_{z(t^+)}^{z(t^-)} \frac{1}{2}\frac{g(s)^2}{(1+2z(t^+))^2} - \alpha ds \leq 0,$$

where the first inequality follows from the fact that $r \to \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha$ is nonincreasing, and the second inequality follows until $t \in [0, t_1[$. Moreover, the first inequality is strict if $g(s) \neq 0$, since $r \to \frac{1}{2}\frac{g(s)^2}{(1+2r)^2} - \alpha$ is strictly decreasing in this case, while if $g(s) = 0$ the second inequality is strict since $\alpha > 0$. In particular we find out that no jump can occur in the interval $[0, t_1[$.

We claim that, if there is a jump of z, than such jump is unique and takes place at $t = t_1$. Moreover $z(t) = 0$ for $t > t_1$. Without loss of generality suppose $t_1 < T$. Since z is left-continuous, the function $\frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha$ is left-continuous, so that by definition of t_1 it holds $\frac{1}{2}\frac{g(t_1)^2}{(1+2z(t_1))^2} - \alpha \leq 0$, and there is a sequence $t_k \searrow t_1$ such that $f(t_k) > 0$ for all k. Again, since f is left-continuous we obtain that for all $\delta > 0$ the set of all t such that $f(t) > 0$ has positive Lebesgue measure on $[t_1, t_1 + \delta]$. This, thanks to (1.5.28), implies that $z(t) = 0$ for $t > t_1$, getting the claim.

Let us now consider the Cantor and absolutely continuous part of (1.5.45). We see that $\dot{z}$ and $\dot{z}^c$ might concentrate only on the set $A := \{t \in [0, t_1] : \frac{1}{2}\frac{g(t)^2}{(1+2z(t))^2} - \alpha = 0\} = \{t \in [0, t_1] : z(t) = \frac{g(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}\}$. This is the set where the continuous function $g(t)$ coincides with $f(t) := \sqrt{2\alpha}(1 + 2z(t))$. We claim that the distributional derivatives of the BV functions g and f coincide on A. It is a particular case of a more general fact provided by [18, Theorem A.1]. As a consequence we get

$$\dot{g} = 2\sqrt{2\alpha}(\dot{z} + \dot{z}^c),$$

which implies that $\dot{z}^c = 0$ since the right-hand side is absolutely continuous with respect to the Lebesgue measure. Moreover we find out that $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{g}$.

We can summarize our discussion with the following results, which holds in the 1-dimensional case:

Theorem 1.5.14 (1-dimensional case)**.** *Let (u, z) be the limit of dynamic processes given by Theorem 1.5.4. Then there is a representative of z that is left-continuous. Let t_0, t_1 be as in* (1.5.43)*. Then there is a representative of u such that $u(t)$ is the solution of* (1.5.39) *for all $t \in [0, t_1]$. For these representatives, still denoted by (u, z), it holds that z is constant on the interval $[0, t_0]$ and it is such that $z(t) \equiv 0$ for $t > t_1$. Moreover z can jump only at $t = t_1$, $\dot{z}^c \equiv 0$ on $[0, T]$, and $\dot{z}$ is concentrated on the set*

$$A := \{t \in [t_0, t_1] : z(t) = \frac{g(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}}\}, \tag{1.5.47}$$

where it also holds $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{g}$, *with* $g(t) := G(t,1) - G(t,-1) + w(t,1) - w(t,-1)$. *In formula*

$$\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{g}\chi_A.$$

In terms of the data of the problem we can state the following:

Theorem 1.5.15. *Let* (u,z) *be the limit of dynamic processes given by Theorem 1.5.4 with initial condition* $z(0) = z_0 > 0$ *and suppose* z *is left-continuous. Let*

$$\tilde{t}_0 := \inf_{t\in[0,T]}\{g(t) \geq (1+2z_0)\sqrt{2\alpha}\}, \qquad \tilde{t}_1 := \inf_{t\in[0,T]}\{g(t) > (1+2z_0)\sqrt{2\alpha}\},$$

then it holds $z(t) = z_0$ *if* $t \leq \tilde{t}_0$, $z(t) = 0$ *if* $t > \tilde{t}_1$, $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{g}\chi_A$, *and* z *can jump only at* $t = t_1$.

Corollary 1.5.16. *If* $g(t)$ *is strictly increasing and is such that* $g(0) < (1 + 2z_0)\sqrt{2\alpha}$, *then there is only one solution* $\bar{t} > 0$ *of* (1.5.47) *and* $z(t) = z_0$ *for* $t \leq \bar{t}$, *while* $z(t) = 0$ *for* $t > \bar{t}$.

Proof. In such a case $t_0 = t_1 = \bar{t}$. Note that hypothesis $g(0) < (1+2z_0)\sqrt{2\alpha}$ prevents that $\bar{t} = 0$. □

The last statement proves that the function (u,z) given by an external load and boundary condition as in the example of [64, Section 4] coincides with the couple of such example. We emphasize that Theorem 1.5.14 refers to a couple (u,z) which evolves without constrains on the jump. However, if the jump remains positive, as in the example of [64, Section 4], the evolution itself satisfies the constraint of mode I.

We conclude the section with the following remark, that show that the conditions we have obtained by the analysis of the limit (u,z) is not sufficient to conclude whether jumps occur or not.

Remark 1.5.17. Suppose that the function $g \in \mathcal{C}^\infty(\mathbb{R})$ be such that $g(0) = 0$, $g(1) = 3\sqrt{2\alpha}$, $g(2) = \sqrt{2\alpha}$, and g is strictly monotone in the intervals $[0,1]$ and $[1,2]$. Let then $z = 1$ on $[0,1]$, $z(t) = \frac{g(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}$ for $t \in [1,2]$, and $z(t) = 0$ for $t > 2$. Then let $u(t)$ be the solution of (1.5.39), i.e. the function in (1.5.40). For such (u,z) we see that (1.5.21) holds by definition while (1.5.45) shows that (1.5.22) holds true with $\mu_b = \mu_z = 0$. This is an example of an evolution satisfying the conditions of the limit of dynamic processes with initial condition $z_0 = 1$, and which does not show any jump, actually being smooth in time. However it is still not clear if there exists some dynamic process whose limit is such function. In particular it is not clear if the measures μ_b and μ_z must be strictly positive, as in the case of Corollary 1.5.16, or may vanish.

Chapter 2

A geometric approach to dislocations in single crystals

Preamble

Dislocations are material defects that arise as small closed curves, called loops, or also long path going through the body and connecting two points of its boundary. Their presence is responsible of many nonconservative and dissipative effects, first of all the plastic behavior of the material. Such phenomena are concentrated on the dislocation lines, whereas, at the mesoscopic scale, the body is perfectly elastic outside them. Let us consider a single dislocation loop L in a continuum medium Ω. The set L is a one-dimensional singularity set for the extensive fields such as stress and strain. If Ω is assumed as a single crystal (as opposed to a polycrystal with internal boundaries) then the family of dislocations are free to move in the bulk and through part of the boundary, and hence are likely to form geometrically complex structures, called clusters. This phenomenon is enhanced if the crystal is considered at high temperature or subjected to high temperature gradients, since the constrained motion of dislocations on predefined glide planes only holds for moderate temperature ranges. In Section 2.2 we model dislocations by mean of integer multiplicity 1-currents. In particular, the assumption that all the dislocation curves are closed or begin and terminate at $\partial\Omega$ implies that these currents are closed. With any point of the dislocation line it is associated a vector, called *Burgers vector*, that is linked to the type of discontinuity of the strain in a neighborhood of the dislocation. For this reason it is convenient to introduce the concept of integral 1-currents with coefficients in $\mathbb{R}^3$. The Burgers vector associated with a dislocation loop being constant along the dislocation, the integral currents are assumed closed in the set occupied by the crystal.

As already introduced, an intrisic difficulty of mesoscopic dislocations is that there is no unambiguous definiton of the displacement field (whatever the reference configuration) in the whole body. In the linear elastic model this amounts to observe that the displacement field as defined by line integration of appropriate combinations of the strain and strain curl is path-dependent, rendering the displacement field multiple valued and hence uneasy to properly handle in a mathematical model. The Burgers vector measures the mismatch of the displacement when we compute an integration along a closed path around a dislocation, so this turns out to be a multiple of the width of the atom layers (see Section 2.1.1). Hence the Burgers vectors must belong to a discrete lattice, that is assumed to be $2\pi\mathbb{Z}^3$. This has the important consequence that every closed path integration of the displacement gradient F gives rise to a mismatch that belongs to the lattice $2\pi\mathbb{Z}^3$ (specifically, the result of the path integration is the sum of the Burgers vectors of all the dislocation the path winds around). This allows us to univocally define a deformation u which takes values in the three dimensional torus $\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$. The rigorous theory of torus-valued Sobolev maps which describe the deformations in the presence of dislocations is developed in Section 2.1.3 and Section 2.3.

The minimum problem. After the dislocation lines have been properly defined, we face the main subject of the chapter, which is the following minimum problem. We consider an energy $\mathcal{W}$ which depends on the strain and on the density of the dislocations, and we want to minimize it among a suitable class of strains and dislocations $\mathcal{F} \times \mathcal{D}$. The minimum problem reads

$$\min_{(F,\mathcal{L})\in\mathcal{F}\times\mathcal{D}} \mathcal{W}(F,\mathcal{L}), \tag{2.0.1}$$

where the strain F and the dislocation $\mathcal{L}$ are constrained by the relation

$$-\operatorname{Curl} F = \Lambda_{\mathcal{L}}, \tag{2.0.2}$$

with $\Lambda_{\mathcal{L}}$ the density induced by the dislocation $\mathcal{L}$. As usual in variational problems where elastic bodies are considered, we assume that the energy can be written as $\mathcal{W} = \mathcal{W}_e + \mathcal{W}_{defect}$, where $\mathcal{W}_e$ is the bulk energy and depends only on the strain $\mathcal{W}_e(F) = \int_\Omega W_e(F)dx$. The defect part of the energy $\mathcal{W}_{defect}$, called *core energy*, is a function of the density of the dislocation.

We assume that this energy depends only on the Curl of the strain, and then is concentrated on the dislocation line L. Indeed continuum models with linear constitutive elastic laws cannot be used since the energy turns out to be unbounded at the line. Some continuum approaches therefore consider that a tube around the dislocation is removed (the core) in order for linear elasticity to hold. We disagree with this approach from a mathematical standpoint, although it might be justified from a physical point of view. In fact it seems to unnatural to allow classical treatment and modeling. Moreover, the physics of the core is atomistic, whereas in the bulk, continuum models are perfectly suited. The interrelation between these models at various scales of matter description is very delicate, since it requires careful matched asymptotic analysis at the core interface. Let us mention the results in terms of core modeling by quasicontinnum models (let us here quote the pioneer work by Tadmor et al [74]). We then prefer to consider that distributional quantities are concentrated in the line and develop appropriate tools and functional spaces for their study. This is the core of the present research.

If a linear elastic constitutive law is chosen, classical examples show that the stress and the strain are not square integrable (see [35] and [82]), and hence that the strain energy is unbounded near L. This shows that, although linear elasticity with a quadratic energy is perfectly valid away from the line (from the core), it is not valid on the line. This suggests to consider finite elasticity near the line with a less-than-quadratic strain energy, possibly matched with a linear law at some distance from the singularities, since it is also known that linear elasticity and the small strain assumption are perfectly valid to describe the single crystal away from the dislocations (see [52]). We then consider a bulk energy which has a p-growth with $1 \leq p < 2$, so we have also to consider deformations $F \in L^p$, $1 \leq p < 2$. Moreover, with a view to a global model, cavitation solutions cannot be ruled out, since they are at the origin of the nucleation of dislocations from the growth of micro-voids in the bulk (see [60]). Here, classical examples show that deformations allowing for radial cavitation are such that $\mathrm{cof}F \in L^q$ with $1 \leq q < 3/2$ (see [33]). Thus, one cannot restrict to the interval $3/2 \leq p < 2$ where some existence results in finite elasticity exists (see [57]), and must allow F, $\mathrm{cof}F \in L^p$ in the whole range $1 \leq p < 2$. Moreover, nucleation resulting from the collapse of a void will provoke locally high pressure gradient and hence the behavior of the Jacobian $J = \det F$ must be controlled. Therefore, classical pointwise conditions on J will be considered: these are the nonnegativeness (to ensure orientation preserving deformation and non-interpenetration of matter) or the fact that $J \to 0^+$ is precluded by finite energy states. Finally, to avoid any spurious, i.e., concentrated and dissipative effects away from the dislocation set, we will assume not only that $\det F$, $\mathrm{cof}F \in L^p$ but also that their distributional counterpart have no s-dimensional $(0 \leq s < 3)$ singular parts in $\Omega \backslash L$, that is, $\mathrm{Det}F$, $\mathrm{Cof}F \in L^p$ locally away from L (see [53]). As a consequence, the strain energy $\mathcal{W}_e$ will depend on F, $\mathrm{cof}F$ and $\det F$, and be assumed polyconvex, with a growth bounded from below, writing for instance as

$$W_e(F) \geq C(|F|^p + |\,\mathrm{cof}F|^p + |\det F|^p) - \beta$$

for some $C, \beta > 0$. The defect part of the energy controls pathological behaviors of dislocation clusters, so it satisfies a growth condition of the type

$$W_{\mathrm{defect}}(\Lambda) \geq C\|\Lambda\|_{\mathcal{M}(\Omega)}.$$

This variational framework was inspired by the pioneering paper by S. Muller and M. Palombaro [57], where a single and fixed dislocation loop was considered, and hence minimization was achieved only with respect to the deformation tensor F. We propose three existence results for the solutions of problem (2.0.1) where minimization is made also with respect to the dislocation lines. Moreover we will impose boundary conditions in terms of dislocation density and of the strain too.

To achieve the proof of existence, we will apply the direct method of calculus of variations. This needs suitable properties of closure of the class of admissible strains and dislocations. As for the latter, we define and carefully analyze two classes of dislocations, at the *mesoscopic* and at the *continuum* scales. The desired closure property is then provided by the compactness of integral 1-currents and of compact connected 1-sets. As for the strains, a more delicate analysis must be done. We define a suitable class of admissible displacements whose gradients will coincide with the strains F away from the dislocation lines, and then we introduce the concept of graphs of these maps, seen as rectifiable currents. The assumption that these displacements are locally Cartesian maps allows us to employ the well-known closure and compactness theorems for them (see [31]).

Then, the two existence results are given in Theorems 2.6.6 and Theorem 2.6.8, respectively for the class of mesoscopic and continuum dislocations.

In order to obtain an existence result for a more general class of dislocations (i.e., not only mesoscopic neither continuum), much more work is needed. In particular, we apply a stronger closure theorem for maps whose graphs is a rectifiable current. This result (Theorem 2.1.12) states that the class of torus-valued Sobolev maps whose graphs are uniformly bounded integral currents is actually a closed class. To apply it, in Section 2.7.1 we explicitly compute the boundaries of displacements generating dislocations. We emphasize that many preliminary results are needed, in particular we have to solve some systems involving elliptic PDEs and to study the properties of particular harmonic maps which shows a jump on a prescribed surface; these results are achieved in Section 2.4.1 and Section 2.4.2. Hence we provide the third existence result for solutions to (2.0.1) (Theorem 2.7.10). Here we must restrict to dislocations whose Burgers vectors are all multiple of a fixed generating $b \in 2\pi\mathbb{Z}^3$, i.e., to dislocation densities of the form $\Lambda = b \otimes \mathcal{L}$, with $\mathcal{L}$ an integral 1-current. Moreover we assume that $\mathcal{W}$ is a second order energy, that is, it depends also on the gradient of F. In particular we assume that it has a growth bounded from below by the norm of its divergence. At the same time, this result is more general since it also consider competitors whose dislocations are locally dense or completely disconnected in the bulk.

Some hypotheses on the energy considered in the first two existence results are very important from a mathematical standpoint, although can be questioned from a physical point of view. A discussion of the model is carried out in Section 2.7.3 and throughout the chapter. We get rid of these hypotheses in the third existence result, where, however, we need to consider a second grade energy, that is, the energy depends on the strain and on its gradient too. We emphasize that even if some hypotheses remain questionable from a physical standpoint, existence results for minimizers in a general setting where dislocations are free in the bulk and edge and screw dislocations are considered at the same time are quite new (to the knowledge of the authors), and many improvements can be still be done in the future developments.

Variations of the energy at minima and the Peach-Köhler force. In Section 2.8 we consider variations of the energy at the minima obtained by the existence results mentioned above. The variation, by a formal chain rule, can be written as

$$\delta_L \mathcal{W}(F, \Lambda_{\mathcal{L}}) = \delta_F \mathcal{W}(F, \Lambda_{\mathcal{L}}) \delta_L F + \delta_\Lambda \mathcal{W}(F, \Lambda_{\mathcal{L}}) \delta_L \Lambda_{\mathcal{L}}.$$

As we have seen, $\mathcal{W}$ is the sum of a deformation and a defect part, the first one depending on F, the second on $\Lambda_{\mathcal{L}} = -(\operatorname{Curl} F)^{\mathrm{T}}$. However both variables are related to L in a specific manner. Moreover the constraint (2.0.2) must be expressed in some simpler form, for instance providing a one-to-one correspondence between strains and dislocations densities. This requires to invert the curl operator, as obtained in Section 2.4.1. Moreover, we need a Helmholtz-type decomposition of the strain since in general the maps associating to each density the corresponding strain is not onto.

Finally, computing $\delta_{\mathcal{L}}\mathcal{W}$ amounts to consider that a certain (configurational) force exerted on the dislocations is vanishing. Therefore, a moving dislocation will evolve with a velocity proportional to this force (this is a well known fact in dislocation theories [1,35]), force that originates from the variation of the deformation part of the energy. In the final Theorem 2.8.5 we show that at optimality, there is a balance of forces, one of which being the Peach-Köhler force, while the other is a line-tension term provided by variation of the defect part of the energy.

Future developments. The aforementioned analysis is done with view to a future study of the evolution of dislocations. In particular the existence results for minima of the energy is the start point for the analysis of the quasistatic evolution. We emphasize that there are still open questions even about the obtained existence results. For instance, the third existence result (Theorem 2.7.10) holds true only for deformations whose dislocation densities are associated to only one generating Burgers vector. The main challenging task is to generalize this result to a general dislocation density. Moreover there are still open questions about the nature of the distributional determinant and adjunct of F, that, for physical reasons, it would be good to be Radon measures with a singular part concentrated on the dislocation lines. A more detailed description of these open problems is done in Section 2.7.3 and 2.8.6.

2.1 Notation and preliminaries

2.1.1 The displacement in the presence of dislocations

As explained in the introduction, in crystals with dislocations a displacement cannot be defined by path integration of the strain F. This is due to the fact that the integration depends on the chosen path. The path dependence is a consequence of the nonvanishing of the elastic strain incompatibility $\operatorname{inc}\mathcal{E} := \operatorname{Curl}\,(\operatorname{Curl}\mathcal{E})^T$ with $\mathcal{E} = \mathbb{S}\sigma$, σ the stress tensor, and $\mathbb{S}$ the compliance tensor. Let us assume for a while that there are no dislocations and that the current configuration Ω is simply connected. In finite elasticity, frame-indifference implies that the strain energy will depend on $\mathbb{C}$, the metric tensor in Ω. Then it is known that $\mathbb{C}$ can be written as $\mathbb{C} = \nabla\phi^T\nabla\phi$ for some reference configuration Ω' and some smooth immersion $\phi : \Omega' \to \mathbb{R}^3$ such that $\phi(\Omega') = \Omega$ if and only if the Riemannian curvature tensor associated to $\mathbb{C}$ vanishes identically in Ω (see [16]). Let us emphasize that the Riemannian curvature is the finite-elasticity counterpart of the aforementioned incompatibility tensor. By eigendecomposition one has $\mathbb{C} = F^TF$ for some F and hence $\mathbb{C} = \nabla\phi^T\nabla\phi$ for some ϕ as soon as $\operatorname{Curl} F = 0$ in Ω. In this case the displacement field is defined as $u := \Phi - \mathrm{Id}$ and $F = \nabla\Phi = I + \nabla u$ is called the deformation gradient associated to Ω. Otherwise, $\operatorname{Curl} F$ and the Riemann curvature are nonvanishing, which is a specific geometrical constraint for the deformation in the presence of dislocations. The dislocations which generate curvature are called geometrically necessary (see [54,56]) and will be given a precise mathematical meaning in Section 2.2, together with their companion geometrically unnecessary (called "statistically stored" in the engineering literature) which solely contribute to plastic strain in the absence of strain gradients.

The precise expression of $\operatorname{Curl} F$ in the presence of dislocations will now be described with some detail, since the concepts of displacement, deformation and reference configuration become uncomfortable in the presence of dislocations. First, we emphasize that no perfect, that is, dislocation-free reference configuration can be considered. Second, the fundamental issue is that even if the reference configuration is needed to consider finite elasticity, the dislocation line is better defined in the current configuration. It is worth describing what happens in the presence of a dislocation loop in finite elasticity (the following discussion is illustrated in Fig. 2.1). Consider the current configuration $\Omega(t)$ at time t (a bounded simply connected open set) with a single dislocation loop L and any dividing smooth surface S_L containing L. The set $\Omega(t)\setminus L$ is not simply connected, but the upper and lower partition of $\Omega(t)$, $\Omega^+(t)$ and $\Omega^-(t)$ divided by S_L, are simply connected and in each it holds $\operatorname{inc}\mathcal{E} = 0$. Thus there exists a linear-elasticity displacement field $u_{S_L} = u^{\pm}_{S_L}$ such that $\mathcal{E} = \nabla u_{S_L}$ in $\Omega^{\pm}(t)$. For any smooth one-to-one φ, the map $\phi := \varphi \circ (\,\mathrm{Id}\ - u_{S_L})$ defines a reference configuration. It turns out that in the presence of a dislocation the map ϕ is multivalued, i.e., there is a mismatch in the reference configuration due to presence of the dislocation. Indeed, let $\Omega^{\pm} := \phi(\Omega^{\pm}(t))$ define the lower and upper parts of the reference configuration while $F = F^{\pm} = \nabla\phi$ are the associated deformation gradients. Now take two points P and Q in S_L, respectively outside and inside L, and consider a curve α^+ in $\Omega^+(t)$ and α^- in $\Omega^-(t)$ both with start point P and endpoint Q. We assume that the value $\phi(P)$ is prescribed. We can compute the value of ϕ by path integration of F starting from P. However integrating F along $\alpha^{\pm}$ we get two different values $\phi(Q)^{\pm}$, whose difference defines the nonzero Burgers vector b attached to L, $b := \int_{\alpha^+} F^+dx - \int_{\alpha^-} F^-dx$. Thus S_L is mapped into two surfaces which match outside L (i.e., at P), but do not coincide inside (i.e., at Q). The region of S_L inside L is denoted by S_L°, and it is observed by Stokes Theorem that b is independent of $Q \in S_L^{\circ}$.

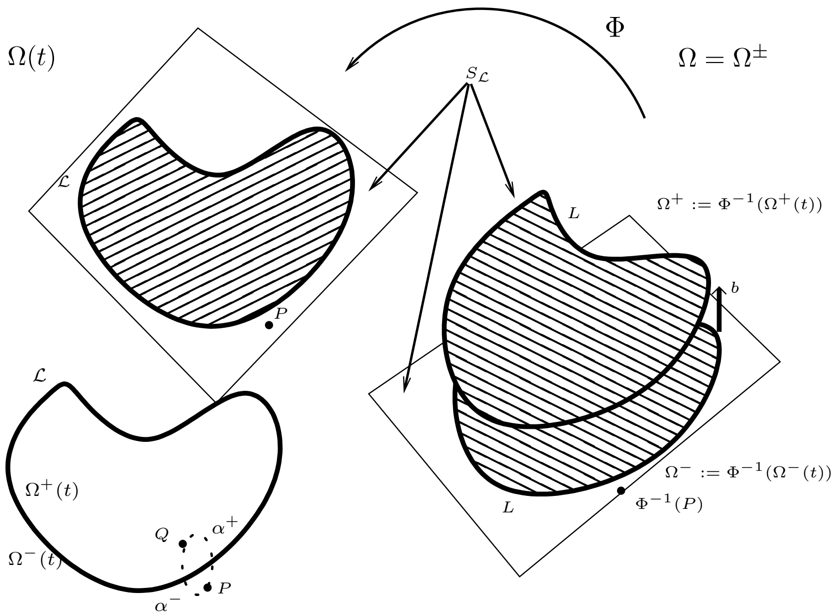


Figure 2.1: Current and reference configurations show a jump of the displacement due to the presence of dislocations.

Summarizing, this procedure ”à la Volterra” yields

$$b = \int_{\alpha^\pm} \nabla\phi dl(x) = \int_{\alpha^\pm} F dl, \tag{2.1.1}$$

otherwise said, ϕ shows a jump of amplitude b in S_L°, while $F = \nabla\phi$ in $S_L \setminus S_L^\circ$. Hence, if ν represents the unit normal to S_L, its distributional derivative can be written as $D\phi = F + b \otimes \nu\mathcal{H}^2_{\llcorner S_L^\circ}$ and it holds $-$ Curl $F =$ Curl $(b \otimes \nu\mathcal{H}^2_{\llcorner S_L^\circ})$. Thus by Stokes theorem and written in terms of the dislocation density

$$\Lambda := \tau \otimes b\mathcal{H}^1_{\llcorner L},$$

we find

$$-\operatorname{Curl} F = \Lambda^T, \tag{2.1.2}$$

whereby (2.1.1) is equivalent to (2.1.2). The fact that Curl F is a measure concentrated on L can therefore be understood as L preventing F to be globally the gradient of a deformation and hence preventing the right Cauchy-Green tensor $\mathbb{C}$ to write as $\mathbb{C} = \nabla^T\phi\nabla\phi$ for some immersion ϕ. This is the main difficulty when dealing with minimization problems involving deformations in the presence of dislocations. In the present chapter we propose an original approach consisting of defining the deformation in a suitable space in such a way that we can deal with it as it was the standard gradient of a Sobolev map.

2.1.2 General tools

Distributions. In the following of the chapter for $x \in \mathbb{R}^n$ we will denote it in euclidean coordinates by $x = (x_1, x_2, \ldots, x_n)$. Let $U \subset \mathbb{R}^n$ be an open set. The space of real distributions on U is denoted by $\mathcal{D}(U)$. Let the symbol $\mathcal{R}$ stand for either $\mathcal{R} = \mathbb{R}^{n\times n}$ or $\mathcal{R} = \mathbb{R}^n$. Often we will consider $\mathcal{R}$-valued distributions whose space is denoted by $\mathcal{D}(U, \mathcal{R})$. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality product between distributions and C_c^∞-functions. If X is a Banach space with dual X', we denote by $\langle\cdot,\cdot\rangle_X$ the duality product between X' and X. Often we will omit the symbol X when it is clear from the context. In general, if U is a n-dimensional manifold and f, g are square integrable functions on U, the symbol $\langle\cdot,\cdot\rangle$ is used to note the classical inner product in $L^2(U)$, i.e. the integration with respect to the n-dimensional Hausdorff measure,

$$\langle f, g\rangle := \int_U fg d\mathcal{H}^n.$$

Accordingly, if it is not clear from the context, we will write $\langle\cdot,\cdot\rangle_U$ to stress that the integration takes place on U. We will sometimes use the same notation $\langle\cdot,\cdot\rangle_U$ to mean the duality pairing in the sense of distributions on U, or in a Banach space of functions or measures defined on U.

Solenoidal measures. Let $U \subset \mathbb{R}^3$ be an open set. In the following the symbol $\mathcal{R}$ stands for either $\mathcal{R} = \mathbb{R}^{3\times 3}$ or $\mathcal{R} = \mathbb{R}^3$, while $\mathcal{R}'$ stands for $\mathbb{R}^3$ or $\mathbb{R}$, respectively. The space of solenoidal Radon measures in U is defined as

$$\mathcal{M}_{\mathrm{div}}(U, \mathcal{R}) := \{\mu \in \mathcal{M}_b(U, \mathcal{R}) \text{ s.t. } \langle \mu, D\varphi\rangle = 0 \quad \forall\varphi \text{ in } \quad C_0^1(U, \mathcal{R}')\}, \tag{2.1.3}$$

with D denoting the distributional derivative, and where the duality product yields a real-valued tensor whose components read $(\langle\mu_{ij}, D_j\varphi_k\rangle)_{ik}$. Recall that

$\varphi \in C^1_0(U,\mathcal{R}')$ if φ and $D\varphi$ are continuous and if for every $\epsilon > 0$ there exists a compact $K \subset U$ such that $\|\varphi(x)\|_\infty$ and $\|D\varphi(x)\|_\infty$ are smaller than ϵ for any $x \in U \setminus K$. Observe that $\mathcal{M}_{\rm div}(U,\mathcal{R})$ is a closed subset of $\mathcal{M}_b(U,\mathcal{R})$ an hence is a Banach space, endowed with the norm of total variation $|\mu|(U) = \sup\{\langle \mu,\varphi\rangle : \varphi \in C^0(U,\mathcal{R}), \|\varphi\|_\infty \le 1\}$ (see [4] for details on vector- and tensor-valued Radon measures).

Functions with bounded variation. Let $\Omega \subset \mathbb{R}^n$ be an open set. The symbol $BV(\Omega)$ denotes the space of real functions on Ω with bounded variations, i.e., the space of summable functions whose distributional gradient is a Radon measure in $\mathcal{M}_b(\Omega,\mathbb{R}^n)$. If $u \in BV(\Omega)$, then we write $Du = D^a u + D^s u$, with $D^a u$ denoting the part of the gradient of u that is absolutely continuous with respect to the Lebesgue measure, and with $D^s u$ being the singular part. It is well know that the singular part can be decomposed as $D^s u = D^c u + D^j u$, the Cantor and the jump part, respectively. The space $SBV(\Omega)$, called space of special functions with bounded variation, is the subspace of $BV(\Omega)$ consisting of those functions whose Cantor part of the gradient is null. If $u \in BV(\Omega)$ then $D^j u$ can be written as $D^j u = [u]\nu \cdot \mathcal{H}^2 \llcorner_{J_u}$, where J_u is a $(n-1)$-rectifiable set with unit normal ν pointing from the side $-$ to the side $+$, and $[u] := u^+ - u^-$ is the jump of u on J_u, i.e., the difference between the traces $u^\pm$ of u.

Curl and matrices. Let A be a $\mathbb{R}^{3\times 3}$-valued field. The curl of the tensor A is defined componentwise as $(\,\mathrm{Curl}\ A)_{ij} = \epsilon_{jkl} D_k A_{il}$ where D is the symbol for the distributional derivative. In particular one has

$$\langle\,\mathrm{Curl}\ A,\psi\rangle = -\langle A_{il},\epsilon_{jkl}D_k\psi_{ij}\rangle = \langle A_{il},\epsilon_{lkj}D_k\psi_{ij}\rangle = \langle A,\ \mathrm{Curl}\ \psi\rangle, \qquad (2.1.4)$$

for every $\psi \in \mathcal{D}(\Omega,\mathbb{R}^{3\times 3})$. Note that with this convention one has Div Curl $A = 0$ in the sense of distributions, since componentwise the divergence is classically defined as $(\,\mathrm{Div}\ A)_i = D_j A_{ij}$. If N is a vector, we use the convention that $(N \times A)_{ij} = -(A \times N)_{ij} = -\epsilon_{jkl} A_{ik} N_l$. In general, if ψ has not compact support and Ω has smooth boundary with outer normal N, formula (2.1.4) becomes

$$\langle\,\mathrm{Curl}\ A,\psi\rangle = \langle A,\ \mathrm{Curl}\ \psi\rangle + \int_{\partial\Omega} (N \times A)\cdot \psi d\mathcal{H}^2. \qquad (2.1.5)$$

Let $u \in W^{1,p}(\Omega;\mathbb{R}^3)$, and suppose $u_i D u_j \in L^1(\Omega,\mathbb{R}^3)$ for all $i \neq j$, we define the *distributional cofactor* of Du, the distribution $\mathrm{Cof}Du$ writing componentwise

$$(\,\mathrm{Cof}Du)_{ij} := D_{j+1}(u_{i+1}Du_{(i+2)(j+2)}) - D_{j+2}(u_{i+1}Du_{(i+2)(j+1)})$$

with indices $i,j \in \{1,2,3\}$ (taken mod 3 when summed and with the derivatives intended in the sense of distributions). Moreover, $\mathrm{Adj}Du$ is the *distributional adjunct* of Du, that is the transpose matrix of the distributional cofactor $\mathrm{Cof}Du$. In general it is not true that the pointwise and distributional adjuncts coincide. Suppose $u_1(\mathrm{adj}Du)^1 \in L^1(\Omega,\mathbb{R}^3)$, with $(\mathrm{adj}Du)^1 := (\mathrm{adj}(Du)_{11}, \mathrm{adj}(Du)_{21}, \mathrm{adj}(Du)_{31})$ being the first column of $\mathrm{adj}Du$. The *distributional determinant* of Du is the distribution $\mathrm{Det}Du$ given taking the distributional divergence of $u_1(\mathrm{adj}Du)^1$, i.e.,

$$\langle \mathrm{Det}Du,\varphi\rangle := \int_\Omega u_1(\mathrm{adj}Du)^1 D\varphi dx, \qquad \forall \varphi \in C^\infty_c(\Omega,\mathbb{R}^3).$$

As for the adjunct, in general $\mathrm{Det}Du$ and $\det Du$ differ.

An ordered (increasing) subset α of $\{1, 2, \dots, n\}$ is said to be a multi-index. We denote by $|\alpha|$ the cardinality of α, and we denote by $\bar{\alpha}$ the complementary set of α, i.e., the multi-index given by the ordered set $\{1, 2, \dots, n\} \setminus \alpha$.

If A is a $n \times n$ matrix with real entries and α and β are multi-indices such that $|\alpha| + |\beta| = n$, $M_{\bar{\alpha}}^{\beta}(A)$ denotes the determinant of the submatrix of A obtained by erasing the i-th columns and the j-th rows, for all $i \in \alpha$ and $j \in \bar{\beta}$. Moreover, the symbol $M(A)$ denotes the n-vector in $\Lambda_n \mathbb{R}^{2n}$ given by

$$M(A) := \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar{\alpha}) M_{\bar{\alpha}}^{\beta}(A) e_{\alpha} \wedge \varepsilon_{\beta},$$

where $\{e_i, \varepsilon_i\}_{i \leq n}$ is the Euclidean basis of $\mathbb{R}^{2n}$ and $\sigma(\alpha, \bar{\alpha})$ denotes the sign of the ordered set $\{\alpha, \bar{\alpha}\}$ seen as a permutation of the set $\{1, 2, \dots, n\}$. Accordingly, it holds

$$|M(A)| := (1 + \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|>0}} |M_{\bar{\alpha}}^{\beta}(A)|^2)^{1/2}.$$

For a matrix $A \in \mathbb{R}^{3\times 3}$, the symbols adj A and $\det A$ denote the adjunct, i.e., the transpose of the matrix of the cofactors of A, and the determinant of A, respectively. Explicitly,

$$M_j^i(A) = A_{ij}, \quad M_{\bar{j}}^{\bar{i}}(A) = (\operatorname{cof} A)_{ij} \quad M_{\{1,2,3\}}^{\{1,2,3\}}(A) = \det A. \tag{2.1.6}$$

Moreover,

$$|M(A)| = \big(1 + \sum_{i,j} A_{ij}^2 + \sum_{i,j} \operatorname{cof}(A)_{ij}^2 + \det(A)^2\big)^{1/2}. \tag{2.1.7}$$

It is convenient also to introduce the following notation

$$\mathcal{M}(A) := (A, \operatorname{adj} A, \det A). \tag{2.1.8}$$

Compact sets. Let C be a compact set in $\mathbb{R}^n$. We define $\mathcal{K}(C)$ as the family of compact and nonempty subsets of C. We define the Gromov-Hausdorff distance $d_H(\cdot, \cdot)$ in $\mathcal{K}(C)$ by

$$d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

for all $A, B \in \mathcal{K}(C)$. If A is a set in $\mathbb{R}^n$, we denote by A_ϵ the set of points at distance less than ϵ from A, i.e.,

$$A_\epsilon := \{x \in \mathbb{R}^n : d(x, A) < \epsilon\}.$$

It is known that the Gromov-Hausdorff distance satisfies

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset B_\epsilon \text{ and } B \subset A_\epsilon\},$$

for all $A, B \in \mathcal{K}(C)$, and hence the latter can be taken as an equivalent definition. The following theorem is a standard result, whose proof can be found, for instance, in [5, 14].

Theorem 2.1.1. *(Blaschke) Let $C \subset \mathbb{R}^n$ be a compact set. Then the space $\mathcal{K}(C)$ endowed with the Gromov-Hausdorff distance d_H is sequentially compact.*

In particular, if K_n is a sequence in $\mathcal{K}(C)$ converging to K, then K is a compact set. Moreover, it holds (for the proof see, e.g., [5, 14]):

Theorem 2.1.2. (Golab) *Let $\{K_n\}$ be a sequence of connected sets in $\mathcal{K}(C)$ converging to K, such that $\mathcal{H}^1(K_n) < \lambda < \infty$. Then K is connected, has finite 1-Hausdorff measure, and*

$$\mathcal{H}^1(K) \le \liminf_{n\to\infty} \mathcal{H}^1(K_n). \tag{2.1.9}$$

2.1.3 Currents

Some definitions are given here. For more detail our main references are [26, 31, 42].

Let k, n be integers with $0 \le k \le n$. We denote by $\Lambda^k\mathbb{R}^n$ and $\Lambda_k\mathbb{R}^n$ the vector spaces of k-covectors and k-vectors respectively. The duality product between covectors and vectors is denoted by $\langle\cdot,\cdot\rangle$. A k-vector ξ is said to be simple if it can be written as a single wedge product of k vectors in $\mathbb{R}^n$, i.e., $\xi = v_1 \wedge v_2 \wedge \cdots \wedge v_k$.

Let $\Omega \subset \mathbb{R}^n$ be an open set, then $\mathcal{D}^k(\Omega)$ denotes the topological vector space of compactly supported smooth k-forms on Ω, that is the topological vector space of compactly supported and smooth maps on Ω with values in $\Lambda^k\mathbb{R}^n$. The dual space of $\mathcal{D}^k(\Omega)$, denoted by $\mathcal{D}_k(\Omega)$, is called the space of *k-currents* on Ω. Since $\mathcal{D}_k(\Omega)$ is defined as a dual space, it is endowed with a natural weak topology. More precisely, the currents $\mathcal{T}_h \in \mathcal{D}_k(\Omega)$ are said to weakly converge to $\mathcal{T} \in \mathcal{D}_k(\Omega)$ if and only if

$$\mathcal{T}_k(\omega) \to \mathcal{T}(\omega)$$

for every $\omega \in \mathcal{D}^k(\Omega)$. For all $\mathcal{T} \in \mathcal{D}_k(\Omega)$ the *mass* of $\mathcal{T}$ is the number $M(\mathcal{T}) \in [0, +\infty]$ defined as

$$M(\mathcal{T}) := \sup_{\omega\in\mathcal{D}^k(\Omega),\ |\omega|\le 1} \mathcal{T}(\omega).$$

If $V \subset \Omega$ is an open set, we can consider the mass of $\mathcal{T}$ in V, i.e.,

$$|\mathcal{T}|_V := \sup_{\substack{\omega\in\mathcal{D}^M(\Omega), |\omega|\le 1,\\ \mathrm{spt}\omega\subset V}} \mathcal{T}(\omega). \tag{2.1.10}$$

If $M(\mathcal{T}) < +\infty$ then $\mathcal{T}$ turns out to be a Borel measure in $\mathcal{M}_b(\Omega, \Lambda_k\mathbb{R}^n)$, and its mass coincides with $M(\mathcal{T})$. Moreover the mass is lower semicontinuous with respect to the weak topology in $\mathcal{D}_k(\Omega)$. Indeed if $\limsup_{h\to\infty} M(\mathcal{T}_h) < +\infty$ and $\mathcal{T}_h \rightharpoonup \mathcal{T}$ then we also find that $\mathcal{T}$ is a Borel measure and $\mathcal{T}_h \rightharpoonup \mathcal{T}$ weakly* in $\mathcal{M}_b(\Omega, \Lambda^k\mathbb{R}^n)$, so that the lower-semicontinuity of the mass follows from the lower-semicontinuity of the mass in $\mathcal{M}_b(\Omega, \Lambda^k\mathbb{R}^n)$.

The *boundary* of a current $\mathcal{T} \in \mathcal{D}_k(\Omega)$ is the current $\partial\mathcal{T} \in \mathcal{D}_{k-1}(\Omega)$ defined by

$$\partial\mathcal{T}(\omega) := \mathcal{T}(d\omega) \quad \text{for } \omega \in \mathcal{D}^{M-1}(\Omega),$$

where $d\omega$ is the external derivative of ω.

We also define the quantity

$$N(\mathcal{T}) := M(\mathcal{T}) + M(\partial\mathcal{T}),$$

for every $\mathcal{T} \in \mathcal{D}_k(\Omega)$. We remark that this number, which measures both the mass of a current and of its boundary, is not a norm.

If S is a k-dimensional oriented submanifold in $\mathbb{R}^n$ and $\vec{S} : S \to \Lambda_k(\mathbb{R}^n)$ is a k-vector giving the orientation, the symbol $[\![S]\!] \in \mathcal{D}_k(\mathbb{R}^n)$ denotes the current obtained by integration on S, i.e.,

$$[\![S]\!](\omega) = \int_S \langle \omega, \vec{S} \rangle d\mathcal{H}^k \qquad \text{for } \omega \in \mathcal{D}^k(\Omega), \tag{2.1.11}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $F : U \to V$ be a smooth map. Then the *push-forward* of a current $\mathcal{T} \in \mathcal{D}_k(U)$ through F is defined as

$$F_\sharp \mathcal{T}(\omega) := \mathcal{T}(\zeta F^\sharp \omega) \qquad \text{for } \omega \in \mathcal{D}^k(V),$$

where $F^\sharp\omega$ is the standard pull-back of ω and ζ is any C^∞ function that is equal to 1 on $\mathrm{spt}\mathcal{T} \cap \mathrm{spt}F^\sharp\omega$. It turns out that $F_\sharp\mathcal{T} \in \mathcal{D}_k(V)$ does not depend on ζ and satisfies

$$\partial F_\sharp \mathcal{T} = F_\sharp \partial \mathcal{T}. \tag{2.1.12}$$

Rectifiable and integral currents. Let $0 \le k \le n$. A set $S \subset \mathbb{R}^n$ is said to be $\mathcal{H}^k$-rectifiable if it is contained in the union of a negligible set and a countable family of C^1-submanifolds of dimension k. We also say that a $\mathcal{H}^k$-rectifiable set is a k-set if it has finite $\mathcal{H}^k$-measure. It is well known that at $\mathcal{H}^k$-a.e. point x of a $\mathcal{H}^k$-rectifiable set S, there exists an approximate tangent space defined as the k-dimensional plane T_xS in $\mathbb{R}^n$ such that

$$\lim_{\lambda\to 0} \int_{\eta_{x,\lambda}(S)} \varphi(y) d\mathcal{H}^k(y) = \int_{T_xS} \varphi(y) d\mathcal{H}^k(y),$$

for all $\varphi \in C^0_c(\mathbb{R}^n)$, where $\eta_{x,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ is the map defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$ with $x, y \in \mathbb{R}^n$ and $\lambda > 0$. Moreover, if $\tau : S \to \Lambda_k(\mathbb{R}^n)$ and $\theta : S \to \mathbb{R}$ are $\mathcal{H}^k$-integrable functions with $\tau(x) \in T_xS$ a simple unit k-vector for $\mathcal{H}^k$-a.e. $x \in S$, then we can define the current $\mathcal{T}$ as

$$\mathcal{T}(\omega) = \int_S \langle \omega(x), \tau(x) \rangle \theta(x) d\mathcal{H}^k(x) \qquad \text{for } \omega \in \mathcal{D}^k(\Omega). \tag{2.1.13}$$

Every current for which there exists S, τ, and θ as before is said to be *rectifiable* current. If also its boundary $\partial\mathcal{T}$ is rectifiable, then we adopt the following notation

$$\mathcal{T} \equiv \{S, \tau, \theta\}. \tag{2.1.14}$$

The current $\mathcal{T} \in \mathcal{D}_k(\Omega)$ is *rectifiable with integer multiplicity* if it is rectifiable, it has rectifiable boundary, and the function θ in (2.1.13) is integer-valued. A integer multiplicity current $\mathcal{T}$ such that $N(\mathcal{T}) < \infty$ is said to be *integral current.*

The following compactness theorem for integral currents holds:

Theorem 2.1.3 (Compactness for i.m. currents)**.** *Let $\{\mathcal{T}_i\} \subset \mathcal{D}_k(\Omega)$ be a sequence of integer multiplicity currents such that*

$$N_U(\mathcal{T}) < C \qquad \textit{for all } i \textit{ and } U \subset\subset \Omega,$$

with $C > 0$. Then there exist an integer multiplicity current $\mathcal{T} \in \mathcal{D}_k(\Omega)$ and a subsequence, still denoted by $\{\mathcal{T}_i\}_i$, such that $\mathcal{T}_i \rightharpoonup \mathcal{T}$ weakly in the sense of currents on Ω as $i \to \infty$.

An integer multiplicity current $\mathcal{T} \in \mathcal{D}_k(\mathbb{R}^n)$ is said to be indecomposable if there exists no integral current $\mathcal{R}$ such that $\mathcal{R} \neq 0 \neq \mathcal{T} - \mathcal{R}$ and

$$N(\mathcal{T}) = N(\mathcal{R}) + N(\mathcal{T} - \mathcal{R}).$$

The following theorem provides the decomposition of every integral current and the structure of integer-multiplicity indecomposable 1-current (see [26, Section 4.2.25]).

Theorem 2.1.4. *For every integer-multiplicity current $\mathcal{T}$ there exists a sequence of indecomposable integral currents $\mathcal{T}_i$ such that*

$$\mathcal{T} = \sum_i \mathcal{T}_i \qquad \textit{and} \qquad N(\mathcal{T}) = \sum_i N(\mathcal{T}_i).$$

Suppose $\mathcal{T}$ is an indecomposable integer multiplicity 1*-current on $\mathbb{R}^n$. Then there exists a Lipschitz function $f : [0, M(\mathcal{T})] \to \mathbb{R}^n$ with* $\mathrm{Lip}(f) = 1$ *such that*

$$f \llcorner [0, M(\mathcal{T})) \textit{ is injective and } \mathcal{T} = f_\sharp [\![0, M(\mathcal{T})]\!].$$

Moreover $\partial\mathcal{T} = 0$ if and only if $f(0) = f(M(\mathcal{T}))$.

Graphs and Cartesian maps. For a measurable set $A \subset \mathbb{R}^n$ its upper density $\theta^*(A, x)$ at $x \in \mathbb{R}^n$ is defined by

$$\theta^*(A, x) := \limsup_{r \to 0} \frac{|B(x, r) \cap A|}{|B(x, r)|}.$$

Similarly one defines its lower density $\theta_*(A, x)$ at $x \in \mathbb{R}^n$ as

$$\theta_*(A, x) := \liminf_{r \to 0} \frac{|B(x, r) \cap A|}{|B(x, r)|}.$$

Whenever the upper and lower densities of a set A at x coincides we define the density of A at x by their common value $\theta(A, x) := \theta^*(A, x) = \theta_*(A, x)$.

Let $A \subset \mathbb{R}^n$ be a measurable set and suppose $u : A \to \mathbb{R}^m$ is a measurable function. Let $x \in A$ be a point of positive density in A. Then we say that $l \in \mathbb{R}^m$ is the *approximate limit* of u as y tends to x in A if for all $\epsilon > 0$ the set $A_\epsilon := \{y \in A : |u(y) - l| > \epsilon\}$ has density 0 at x. In such a case we write

$$l := \operatorname*{aplim}_{\substack{y \to x \\ y \in A}} u(y).$$

We say that the function $u : A \to \mathbb{R}^m$ is approximatively continuous at $x \in A$ if $u(x) = \operatorname{aplim}_{\substack{y \to x \\ y \in A}} u(y)$. Among the properties of the approximate limit, we point out that, whenever it exists, it is unique (for more details see [31]). It turns out that if $u : A \to \mathbb{R}^m$ is measurable, then it is approximatively continuous almost everywhere in A with respect to the Lebesgue measure. Moreover, we say that the function u is *approximatively differentiable* at x if there exists a matrix $L \in \mathbb{R}^{m \times n}$ such that for all $\epsilon > 0$ the set $A_\epsilon := \{y \in A : |\frac{u(y) - u(x) - L(y - x)}{|y - x|}| > \epsilon\}$ has density 0 at x. If u is approximatively differentiable at x we call L the approximate differential of u at x. It turns out that if $u \in W^{1,1}(A, \mathbb{R}^m)$, then it is approximatively differentiable almost everywhere in A and its approximate differential coincides almost everywhere with Du, its distributional gradient.

Let $\Omega \subset \mathbb{R}^n$ be an open set and consider the space $\Omega \times \mathbb{R}^n$. Denoting $x = (x_1, x_2, \dots, x_n)$ for $x \in \Omega$ and $y = (y_1, y_2, \dots, y_n)$ for $y \in \mathbb{R}^n$, every n-form $\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^n)$ can be decomposed as

$$\omega(x, y) = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}(x, y))dx^\alpha \wedge dy^\beta. \tag{2.1.15}$$

with $\omega_{\alpha\beta} \in C_c^\infty(\Omega \times \mathbb{R}^n)$, and where the sum is computed over all multi-indices α and β such that $|\alpha| + |\beta| = n$.

For $1 \le p < +\infty$ we define

$$\begin{aligned} \mathcal{A}_p(\Omega) := \{u \in L^p(\Omega, \mathbb{R}^n) : u \text{ is approx. diff. a.e. on } \Omega, \text{ and} \\ M_{\bar\alpha}^\beta(Du) \in L^p(\Omega) \text{ for all } |\alpha| + |\beta| = n\}, \end{aligned}$$

where Du is the approximate differential of u. We set

$$\|u\|_{\mathcal{A}^p} := \|u\|_p + \|M(Du)\|_p,$$

which is not a norm on $\mathcal{A}^p(\Omega, \mathbb{R}^n)$. In other words, a function $u \in \mathcal{A}^p(\Omega, \mathbb{R}^n)$ if and only if $u \in L^p(\Omega, \mathbb{R}^n)$, $Du \in L^p(\Omega, \mathbb{R}^{n\times n})$, adj $Du \in L^p(\Omega, \mathbb{R}^{n\times n})$, and $\det Du \in L^p(\Omega)$.

A weak convergence is defined on $\mathcal{A}_p(\Omega)$ when $p > 1$. We say that the sequence $u_h \in \mathcal{A}_p(\Omega)$ converges to $u \in \mathcal{A}_p(\Omega)$ weakly in $\mathcal{A}_p(\Omega)$ if $u_h \rightharpoonup u$ weakly in $L^p(\Omega)$ and $M_{\bar\alpha}^\beta(Du_h) \rightharpoonup M_{\bar\alpha}^\beta(Du)$ weakly in $L^p(\Omega)$ for all multi-indices α and β with $|\alpha| + |\beta| = n$.

Given $u \in \mathcal{A}_1(\Omega, \mathbb{R}^n)$, we define its graph $G_u \subset \Omega \times \mathbb{R}^n$ as

$$G_u := \{(x, u(x)) : x \in \Omega\}.$$

Note that this set is defined up to a $\mathcal{L}^n$-negligible set. Let us consider the map $(\mathrm{Id} \times u) : \Omega \to \Omega \times \mathbb{R}^n$ defined by $(\mathrm{Id} \times u)(x) := (x, u(x))$. If $u \in \mathcal{A}_1(\Omega; \mathbb{R}^n)$ and $\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^n)$, we can extend the definition of pull-back also to the map $\mathrm{Id} \times u$, i.e.,

$$(\mathrm{Id} \times u)^\sharp \omega = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar\alpha)\omega_{\alpha\beta}(u, u(x)) M_{\bar\alpha}^\beta(Du(x)) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

This allows us to extend the definition of push-forward of a current $\mathcal{T}$ also throughout the map $\mathrm{Id} \times u$, provided $u \in \mathcal{A}_1(\Omega; \mathbb{R}^n)$. Let us consider the current $[\![\Omega]\!]$, the canonical current given by integration on Ω. We define the *current carried by the graph* of u as follows

$$\mathcal{G}_u := (Id \times u)_\sharp [\![\Omega]\!]. \tag{2.1.16}$$

Explicitly we have

$$\mathcal{G}_u(\omega) = \int_\Omega \sigma(\alpha, \bar\alpha)\omega_{\alpha\beta}(x, u(x)) M_{\bar\alpha}^\beta(Du(x)) dx, \tag{2.1.17}$$

for all $\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^n)$.

The following Theorem (proven in [31]) says that $\mathcal{G}_u$ is an integer multiplicity current whenever $u \in \mathcal{A}_1(\Omega; \mathbb{R}^n)$.

Theorem 2.1.5. *If $u \in \mathcal{A}^1(\Omega, \mathbb{R}^n)$ then $\mathcal{G}_u$ is an integer multiplicity current with multiplicity* 1 *and support given by the rectifiable set G_u whose orientation is given by the n-form*

$$\vec{G}_u(x, u(x)) := \frac{M(Du(x))}{|M(Du(x))|},$$

which turns out to be almost everywhere orthogonal to the approximate tangent plane to G_u.

In symbols,

$$\mathcal{G}_u(\omega) = \int_\Omega \langle \omega, \frac{M(Du(x))}{|M(Du(x))|} \rangle d\mathcal{H}^n \llcorner_{G_u}. \tag{2.1.18}$$

Lemma 2.1.6. *Let $p > 1$. Let u_ϵ, $u \in \mathcal{A}_p(\Omega)$ be such that $u_\epsilon \rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega)$, then $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as currents.*

Proof. This is a straightforward consequence of formula (2.1.17). □

For $1 \leq p < +\infty$ the class of p-Cartesian maps is the subset of $\mathcal{A}_p(\Omega, \mathbb{R}^3)$ defined as follows

$$\mathrm{Cart}^p(\Omega, \mathbb{R}^3) := \{u \in \mathcal{A}_p(\Omega, \mathbb{R}^3) : \partial \mathcal{G}_u = 0\}. \tag{2.1.19}$$

If $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $p \geq 3$, then it is easy to see that $u \in \mathrm{Cart}^1(\Omega, \mathbb{R}^3)$. See [31, Section 3.2.2] for details.

The following closure theorem for Cartesian maps holds (see [31, Section 3.3.3]):

Theorem 2.1.7. *Let $p > 1$. Let $u_k \in \mathrm{Cart}^p(\Omega, \mathbb{R}^n)$ a sequence such that*

$$\begin{aligned} &u_k \rightharpoonup u \quad \text{weakly in } L^p(\Omega, \mathbb{R}^n), \\ &M^\beta_{\bar{\alpha}}(Du_k) \rightharpoonup v^\beta_{\bar{\alpha}} \quad \text{weakly in } L^p(\Omega), \end{aligned}$$

for all α, β with $|\alpha| + |\beta| = n$, then $u \in \mathrm{Cart}^p(\Omega, \mathbb{R}^n)$ and $v^\beta_{\bar{\alpha}} = M^\beta_{\bar{\alpha}}(Du)$.

The crucial point for our purposes is that for Cartesian maps in dimension 3 it is always true that $\mathrm{Det}Du = \det Du$ and $\mathrm{Adj}Du = \mathrm{adj}Du$. In particular $\mathrm{Det}Du \in L^p(\Omega)$ and $\mathrm{Adj}Du \in L^p(\Omega, \mathbb{R}^{n\times n})$.

For maps that are not Cartesian the following closure Theorem holds true:

Theorem 2.1.8. *Let $p > 1$. Let u_k be a sequence in $\mathcal{A}_p(\Omega, \mathbb{R}^n)$ such that $u_k \to u$ strongly in $L^p(\Omega, \mathbb{R}^3)$ and suppose that there exist functions $v^\alpha_\beta \in L^p(\Omega)$ such that $M^\beta_{\bar{\alpha}}(Du_k) \rightharpoonup v^\beta_\alpha$ for all multi-indices α and β with $|\alpha| + |\beta| = n$. If*

$$M(\partial \mathcal{G}_{u_k}) < C < +\infty \tag{2.1.20}$$

for all $k > 0$, then $u \in \mathcal{A}_p(\Omega, \mathbb{R}^3)$ and $v^\beta_\alpha = M^\beta_{\bar{\alpha}}(Du)$.

This is proved in Theorem 2 of [31, Section 3.3.2].

Graphs of maps with values in the torus. We introduce the torus $\mathbb{T} \cong \mathbb{R}/\sim$, where $\sim$ denotes the equivalent relation given by $a \sim b$ iff $a - b \in 2\pi\mathbb{Z}$.

Now we will consider graphs of maps $u : \Omega \to \mathbb{T}^n$. These turn out to be n-rectifiable currents in $\Omega \times \mathbb{T}^n$. Note that the space of n-forms in $\Omega \times \mathbb{T}^n$, i.e. $\mathcal{D}^n(\Omega \times \mathbb{T}^3)$, are exactly the space of n-forms in $\Omega \times \mathbb{R}^n$ that have coefficients which are smooth and 2π-periodic (with all their derivatives) in the last three variables (actually, they do not have compact support). As a consequence, if $\mathcal{T}$ is a n-current in $\mathcal{D}_n(\Omega \times \mathbb{R}^n)$ that has compact support in $\bar{\Omega} \times \mathbb{R}^n$, then it is well-defined the current $T(\mathcal{T}) \in \mathcal{D}_n(\Omega \times \mathbb{T}^n)$ defined as

$$T(\mathcal{T}) := \mathcal{T}_{\llcorner \mathcal{D}^n(\Omega \times \mathbb{T}^3)}. \tag{2.1.21}$$

Moreover, since in general smooth functions in $\Omega \times \mathbb{R}^n$ are not periodic in the last three variables, it turns out that $M(T(\mathcal{T})) \leq M(\mathcal{T})$.

Let $u \in \mathcal{A}_p(\Omega, \mathbb{R}^n)$, then we define $T(u) : \Omega \to \mathbb{R}^n$ by mean of the standard projection $\pi_T : \mathbb{R} \to \mathbb{T}$, i.e. $T(u) := \pi_T(u)$. It is easily seen that, $\mathbb{T}$ being locally isomorphic to $\mathbb{R}$, $T(u)$ is almost everywhere approximatively differentiable with the same approximate derivatives of u. As a consequence $\mathcal{G}_{T(u)}$ is a n-rectifiable current in $\Omega \times \mathbb{T}^n$. It is also easy to see that in such a case $\mathcal{G}_{T(u)} = T(\mathcal{G}_u)$. This fundamental identity follows from the fact that the approximate differential of u and $T(u)$ coincides almost everywhere and from (2.1.17), where we use that if ω is 2π-periodic in the second variable, then $\omega(x, u(x)) = \omega(x, T(u(x)))$.

We introduce the space $\mathcal{A}_p(\Omega, \mathbb{T}^n)$ as follows:

Definition 2.1.9.

$$\begin{aligned} \mathcal{A}_p(\Omega, \mathbb{T}^n) := \{u \in L^p(\Omega, \mathbb{T}^n) : u \text{ is approx. diff. a.e. on } \Omega, \text{ and} \\ M_{\bar{\alpha}}^{\beta}(Du) \in L^p(\Omega) \text{ for all } |\alpha| + |\beta| = n\} \end{aligned} \tag{2.1.22}$$

In such a way we see that for all $u \in \mathcal{A}_p(\Omega, \mathbb{T}^n)$ the graph $\mathcal{G}_u$ is well-defined as n-rectifiable current. A consequence of the fact that the mass of a current does not increase when we compose with T is that, if there exists $\bar{u} \in \mathcal{A}_p(\Omega, \mathbb{R}^n)$ such that $T(\bar{u}) = u$ and $\mathcal{G}_{\bar{u}}$ is an integral current, then $\mathcal{G}_u$ is an integral current. Note that it might happen that such $\bar{u}$ does exist with $\partial \mathcal{G}_{\bar{u}}$ unbounded, while $M(\partial \mathcal{G}_u) < \infty$.

Remark 2.1.10. The fact that $u \in \mathcal{A}_p(\Omega, \mathbb{T}^n)$ does not imply, in general, that there exists $\bar{u} \in \mathcal{A}_p(\Omega, \mathbb{R}^n)$ such that $T(\bar{u}) = u$. However, in Section 2.3, we provide a condition under which such correspondence holds true. In particular Theorem 2.3.9, together with Remark 2.3.10, have the consequence that if $u \in \mathcal{W}^{1,p}(\Omega, \mathbb{T}^3)$ satisfies Curl $Du \in \mathcal{M}_b(\Omega, \mathbb{R}^3)$, then there exists $\bar{u} \in SBV^p(\Omega, \mathbb{R}^3)$ with $T(\bar{u}) = u$.

Lemma (2.1.6) readily applies to the case of maps with value in $\mathbb{T}^n$.

Lemma 2.1.11. *Let $p > 1$. Let u_ϵ, $u \in \mathcal{A}_p(\Omega, \mathbb{T}^n)$ be such that $u_\epsilon \rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega, \mathbb{T}^n)$, then $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as currents.*

Proof. This is again a consequence of formula (2.1.17) and the fact that currents in $\mathcal{D}_n(\Omega, \mathbb{R}^n)$ belong also to $\mathcal{D}_n(\Omega, \mathbb{T}^n)$. □

Theorem 2.1.8, being a consequence of the compactness theorem for integral currents, straightforwardly applies also to the case of maps with values in $\mathbb{T}^n$.

Theorem 2.1.12. *Let $p > 1$. Let u_k be a sequence in $\mathcal{A}_p(\Omega, \mathbb{T}^n)$ such that $u_k \to u$ strongly in $L^p(\Omega, \mathbb{T}^n)$ and suppose that there exist functions $v_{\bar{\alpha}}^{\beta} \in L^p(\Omega)$ such that $M_{\bar{\alpha}}^{\beta}(Du_k) \rightharpoonup v_{\bar{\alpha}}^{\beta}$ for all multi-indices α and β with $|\alpha| + |\beta| = n$. If*

$$M(\partial \mathcal{G}_{u_k}) < C < +\infty \tag{2.1.23}$$

for all $k > 0$, then $u \in \mathcal{A}_p(\Omega, \mathbb{T}^n)$ and $v_{\bar{\alpha}}^{\beta} = M_{\bar{\alpha}}^{\beta}(Du)$.

2.2 Dislocations as currents

A dislocation in an elasto-plastic body arises as a closed arc, or a path connecting two points of the boundary, to which a Burgers vector $b \in \mathbb{R}^3$ and a measure concentrated on the dislocation line (the dislocation density) are associated. Since dislocation densities fulfill linear additivity when dislocation lines overlap, and since to each dislocation 2 preferential directions are associated, which is also linked to its density, we will describe dislocations by the tool of integer-multiplicity 1-currents with coefficients in a group, that in the crystallographic case is assumed isomorphic to $\mathbb{Z}^3$. The multiplicity θ represents the Burgers vector with its multiplicity, and the fact that it is constant on any dislocation and that the dislocations are closed correspond to the requirement that such currents are boundaryless (i.e., that the density is divergence free). Moreover, integer-multiplicity 1-currents, thanks to Theorem 2.2.22, are essentially Lipschitz curves, and hence a description of dislocations without using the notion of currents is also possible. However the notion of currents, as we will see, simplifies some descriptions and provides more direct proofs of some of the following statements.

We point out that a similar description of dislocations by use of currents has been given in [17], where no variational problems are considered, but a relaxation result on the *core energy*, that we will treat in the next paragraphs (see Section 2.6).

Let Ω be a bounded and connected open set in $\mathbb{R}^3$, with smooth boundary. Let $\mathcal{I} \subset \mathbb{N}$ be a family of indices.

Definition 2.2.1. A *dislocation* is a couple $\mathcal{L}_{\mathcal{I}} := (\mathcal{L}_i, b_i)_{i \in \mathcal{I}}$, where $\mathcal{L}_i$ are closed integer-multiplicity 1-currents in Ω, and b_i are vectors of $\mathbb{R}^3$. We define $\mathcal{B}_{\mathcal{I}} = \{b_i\}_{i \in \mathcal{I}}$ the set of *Burgers vectors* of $\mathcal{L}_{\mathcal{I}}$. Each dislocation $\mathcal{L}_{\mathcal{I}}$ can be represented by means of the quadruple $\{L_i, \tau_i, \theta_i, b_i\}_{i \in \mathcal{I}}$.

In many applications, the Burgers vector is constrained by crystallographic properties to belong to a lattice. For simplicity this lattice will be assumed isomorphic to $\mathbb{Z}^3$. Let the lattice basis $\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ be fixed, and define the set of *admissible Burgers vectors* as

$$\mathcal{B} := \{b \in \mathbb{R}^3 : \exists \beta \in \mathbb{Z}^3 \text{ such that } b = \sum_{k=1}^{3} \beta_i \bar{b}_i\}. \tag{2.2.1}$$

Accordingly, if $\mathcal{B}_{\mathcal{I}} \subset \mathcal{B}$, then $\mathcal{L}_{\mathcal{I}}$ is called *crystallographic dislocation.* Without loss of generality we will assume that $\bar{b}_i = e_i$, that is $\mathcal{B} := \mathbb{Z}^3$. With this definition we can identify each dislocation with a current with coefficients in the group $\mathbb{Z}^3$. Specifically, given a dislocation $\mathcal{L}_\mathcal{I}$, for all $i \in \mathcal{I}$ we define the current

$$\hat{\mathcal{L}}_i := \{L_i, \tau_i, \theta_i b_i\}, \tag{2.2.2}$$

which has multiplicity in $\mathbb{Z}^3$. In other words if ω is a 1-form with vector-valued coefficients, i.e. $\omega_j = \omega_{kj} dx_k$, $j = 1, 2, 3$ (with Einstein summation convention on repeated indices), then, for every fixed i,

$$\hat{\mathcal{L}}_i(\omega) := \mathcal{L}_i(\omega b_i),$$

where $\omega b_i = \omega_{kj}(b_i)_j dx_k$. Accordingly, the current associated to the dislocation is defined by

$$\hat{\mathcal{L}}_\mathcal{I} := \sum_{i \in \mathcal{I}} \hat{\mathcal{L}}_i. \tag{2.2.3}$$

In the sequel the space of 1-forms with vector-valued smooth and compactly supported coefficients will be denoted by $\mathcal{D}^1(\Omega, \mathbb{R}^3)$.

The *density of a dislocation* is a key measure associated to the dislocation current.

Definition 2.2.2. The *density* associated to $\mathcal{L}_\mathcal{I}$ is the linear functional $\Lambda_\mathcal{L}$ defined by

$$\langle \Lambda_\mathcal{L}, w \rangle := \sum_{i \in \mathcal{I} \subset \mathbb{N}} \mathcal{L}_i((wb^i)^*), \tag{2.2.4}$$

for every $w \in C^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3})$, where in the right-hand side $\omega := (wb)^*$ is the covector writing componentwise $(wb)^* := w_{kj} b_j dx_k$.

If $\sum_{i \in \mathcal{I}} M(\mathcal{L}_i)|b_i| < \infty$ then $\Lambda_\mathcal{L}$ is well defined as a Radon measure, and we write $\Lambda_\mathcal{L} \in \mathcal{M}_b(\bar{\Omega}, \mathbb{R}^{3\times 3})$.

Definition 2.2.3 (Equivalence between dislocations)**.** Two dislocations $\mathcal{L}_\mathcal{I}$ and $\mathcal{L}'_\mathcal{I}$ are said *geometrically equivalent* if

$$\Lambda_\mathcal{L} = \Lambda_{\mathcal{L}'}. \tag{2.2.5}$$

Definition 2.2.4 (Geometrically necessary dislocation set)**.** The geometric necessary dislocation set $L^\star$ is the support of $\Lambda_\mathcal{L}$. In particular there are $\tau^\star$ and $\mathcal{I}^\star$, such that $\{L^\star, \tau^\star, 1, \mathcal{B}_{\mathcal{I}^\star}\}$ is said to be the *minimal dislocation* equivalent to $\mathcal{L}_\mathcal{I}$.

Under suitable assumptions $L^\star$ turns out to be a $\mathcal{H}^1$-rectifiable compact set. In the sequel we discuss some sufficient assumptions in order that $L^\star$ has this regularity.

We will need some specific characterization of dislocations which are physically admissible. This is why we need to introduce finer classes of dislocations in the sequel.

Regular dislocations. We introduce the following definition.

Definition 2.2.5 (*b*-dislocation)**.** Let $b \in \mathcal{B}$. A *b*-dislocation $\mathcal{L}^b$ is a dislocation $\mathcal{L}_\mathcal{I}$ such that (i) $b_i = b$ for all $i \in \mathcal{B}_\mathcal{I}$, (ii) $\mathcal{I}$ is finite with cardinality k_b, (iii) there exist k_b Lipschitz functions $\varphi_i^b : [0, T_i] \to \bar{\Omega}$ with $\mathrm{Lip}(\varphi_i^b) \le 1$ such that

$$\mathcal{L}_i = \varphi^b_{i\sharp}[\![[0, T_i]]\!]. \tag{2.2.6}$$

Moreover, for all $i \le k_b$ we have either $\varphi_i^b(0) = \varphi_i^b(T_i)$ or $\varphi_i^b(0), \varphi_i^b(T_i) \in \partial\Omega$. We set

$$\mathcal{L}^b = \sum_{i \in \mathcal{I}} \mathcal{L}_i. \tag{2.2.7}$$

The current $\hat{\mathcal{L}}^b$ defined by

$$\hat{\mathcal{L}}^b(\omega) := \mathcal{L}^b(\omega b), \tag{2.2.8}$$

for all 1-form with vector-valued coefficients $\omega \in \mathcal{D}^1(\Omega, \mathbb{R}^3)$, is called *b-dislocation current* associated to $\mathcal{L}^b$.

In particular, with this definition, we require that a *b*-dislocation is always closed in Ω.

From Theorem 2.1.4, one can always decompose $\mathcal{L}^b$ as follows

$$\mathcal{L}^b = \sum_{i \in \mathcal{I}^b} \mathcal{L}_i^b, \tag{2.2.9}$$

with $\mathcal{L}_i^b$ indecomposable 1-current such that $\sum_{i \in \mathcal{I}^b} N(\mathcal{L}_i^b) = N(\mathcal{L}^b)$. The components $\mathcal{L}_i^b$ are called *current loops*. Thanks to the Lipschitzianity of the functions φ_j^b one has $\sum_{j=1}^{k_b} l_j^b := \sum_{j=1}^{k_b} \int_0^{T_j} \|\dot{\varphi}_j^b\| dt < \infty$, meaning that the total length of the supporting set of the current $\mathcal{L}^b$ (counted with overlapping) is finite (here l_j^b is the length of the current given by φ_j^b).

We remark that even if the word loop usually refers to a closed path, we use the same word when we refer to a no-closed path. However the closeness property of the current implies that in such a case one has that the boundary of the path belongs to $\partial\Omega$.

By definition of rectifiable current, if $\mathcal{L}^b$ is a *b*-dislocation then there is a 1-set called *dislocation set* that we denote by L^b, such that

$$\mathcal{L}^b(\omega) = \int_{L^b} \langle \omega(x), \tau^b(x) \rangle \theta^b(x) d\mathcal{H}^1(x) \qquad \text{for } \omega \in \mathcal{D}^1(\Omega). \tag{2.2.10}$$

We can choose

$$L^b := \bigcup_{j=1}^{k_b} \varphi_j^b([0, T_j]), \tag{2.2.11}$$

for the rectifiable set supporting the current $\mathcal{L}^b$, and we will also write $\mathcal{L}^b = \{L^b, \tau^b, \theta^b\}$. With such a choice L^b is a compact set. Note that with this choice for the dislocation set, in general L^b does not coincide with the geometrically necessary dislocation set $L^\star$, since somewhere on L^b it may happen that $\theta^b = 0$. Indeed, with this notation, θ^b may also take the value 0 in a set of $\mathcal{H}^1$ positive measure. If $\mathcal{L}_i^b$ are the indecomposable components of $\mathcal{L}^b$ in (2.2.9), we write $\mathcal{L}_i^b = \{L_i^b, \tau^b, \theta^b\}$, in such a way that it holds $L^b = (\cup_{i \in \mathcal{I}^b} L_i^b) \cup \Xi^b$, where Ξ^b is defined as the set $\{x \in L^b : \theta^b(x) = 0\}$.

As for general dislocations, to any *b*-dislocation we associate a *density*.

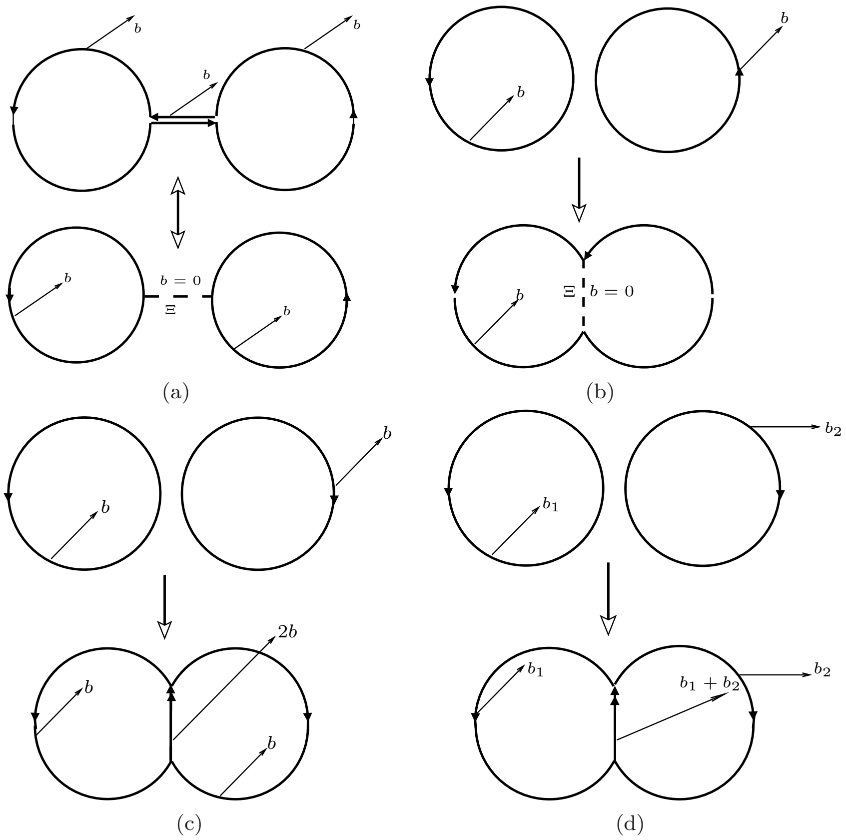
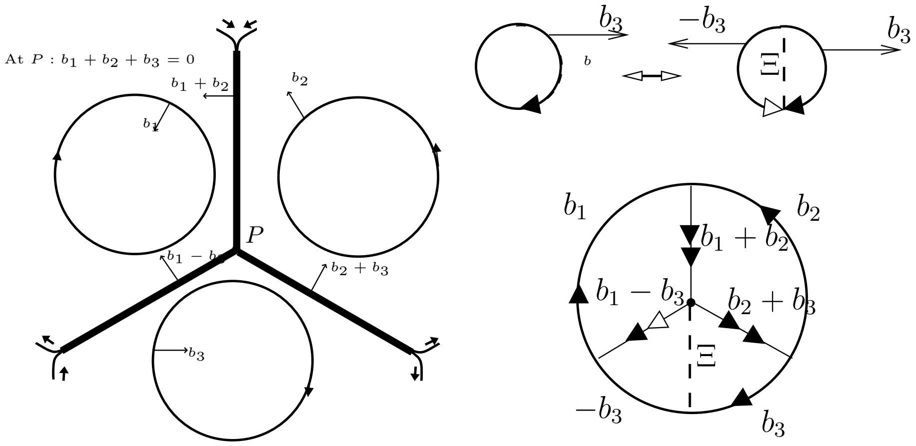
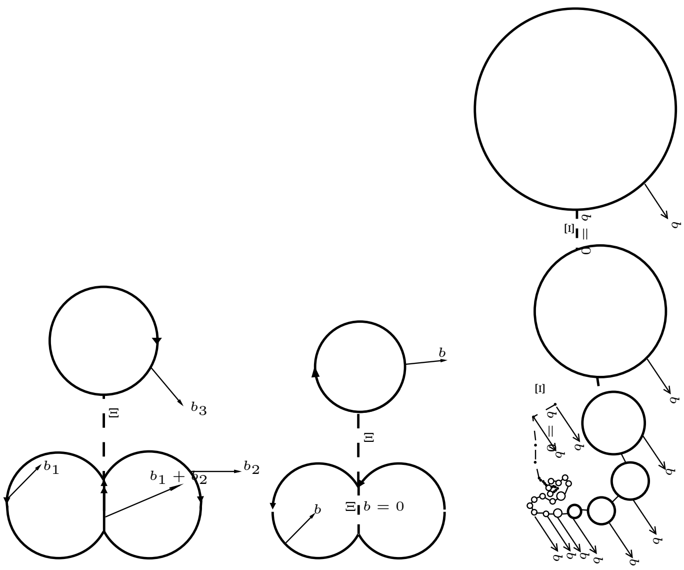


Figure 2.2: Typical indecomposable dislocation loops and the resulting dislocation currents: in (a), a single b-dislocation loops is equivalently viewed as two indecomposable b-loops with opposite directions and connected by a geometrically unnecessary arc Ξ; the inverse property is observed in (b) where two identical b-loops give rise to a single connected b-dislocation loops and a geometrically unnecessary arc Ξ where $\Lambda = 0$; in contrast, (c) describes two b-loops with opposite direction which provide a simple cluster showing subarcs with Burgers vectors b and $2b$; the general case is shown in (d) where the cluster is due to the union of two loops with distinct Burgers vectors obeying to Frank rule.



(a) (b)

Figure 2.3: For certain combinations of Burgers vectors, the three separated loops of (a) might intersect and form the cluster element of (b) where the Frank law at the intersection points is satisfied.



(a) (b) (c)

Figure 2.4: Different kinds of cluster components: in (a) the sum of b-current dislocations $\mathcal{L}^{b_1} + \mathcal{L}^{b_2} + \mathcal{L}^{b_3}$ is depicted, whereas (b) shows a single b-current constituted of three elementary b-loops. In (c) a b-dislocation cluster writing as $\mathcal{L}^b = \varphi^b_\sharp[\![0, T]\!]$ is shown: it can be viewed as a countable chain of indecomposable b-loops interconnected with geometrically unnecessary arcs.

Definition 2.2.6. The *density* of a b-dislocation $\mathcal{L}^b$ is the measure $\Lambda_{\mathcal{L}^b} \in \mathcal{M}_b(\bar{\Omega}, \mathbb{R}^{3\times 3})$ defined by

$$\langle \Lambda_{\mathcal{L}^b}, w \rangle := \mathcal{L}^b((wb)^*), \tag{2.2.12}$$

for every $w \in C_c^\infty(\Omega, \mathbb{R}^{3\times 3})$, where in the right-hand side $\omega := (wb)^*$ is the covector writing componentwise $(wb)^* := w_{kj} b_j dx_k$.

Note that, by (2.2.8), if we identify smooth compactly supported tensor-valued fields with smooth 1-forms with vector-valued coefficients, the density and the current associated to a dislocation becomes the same object.

Since k_b is finite $\Lambda_{\mathcal{L}^b}$ is always a Radon measure. In the sequel we will use the following shortcut notation from (2.2.10) and (2.2.12):

$$\Lambda_{\mathcal{L}^b} = \mathcal{L}^b \otimes b = \tau^b \otimes b\theta^b \, \mathcal{H}^1 \llcorner L^b. \tag{2.2.13}$$

Definition 2.2.7 (Regular dislocation)**.** A *regular dislocation* is a sequence of b-dislocation $\mathcal{L}_{\mathcal{B}} := \{\mathcal{L}^b\}_{b\in\mathcal{B}}$ whose total density (or associated current) has finite mass. According to the previous definitions, the *dislocation current*, still denoted by $\hat{\mathcal{L}}$, and the *dislocation density* $\Lambda_{\mathcal{L}}$, are given by

$$\hat{\mathcal{L}} := \sum_{b\in\mathcal{B}} \hat{\mathcal{L}}^b, \qquad \Lambda_{\mathcal{L}} := \sum_{b\in\mathcal{B}} \Lambda_{\mathcal{L}^b}. \tag{2.2.14}$$

The *dislocation set* L is defined as

$$L := \bigcup_{b\in\mathcal{B}} L^b, \tag{2.2.15}$$

so that we can write $\hat{\mathcal{L}} = \{L, \tau, \theta\}$ with

$$\tau \in \mathrm{Tan}L, \qquad \theta = \sum_{b\in\mathcal{B}} \mathrm{sg}(\tau^b)\theta^b b, \tag{2.2.16}$$

where $\mathrm{sg}(\tau^b)$ being 1 or -1, chosen in such the way that $\tau = \mathrm{sg}(\tau^b)\tau^b$ (note that $\theta \in \mathbb{Z}^3$, while $\theta^b \in \mathbb{Z}$).

Note that, in general, the multiplicity θ of the dislocation current $\mathcal{L}$ may be also zero in some non-negligible set. Moreover, the dislocation current $\mathcal{L} = \{L, \tau, \theta\}$ is said to be connected if L is a connected set. By (2.2.7), every dislocation current can also be written as

$$\hat{\mathcal{L}}(\omega) = \sum_{b\in\mathcal{B}} \hat{\mathcal{L}}^b(\omega) = \sum_{b\in\mathcal{B}} \sum_{1\le j\le k_b} \varphi^b_{j\sharp}[\![0, T_j]\!](\omega b), \tag{2.2.17}$$

for all $\omega \in \mathcal{D}^1(\Omega, \mathbb{R}^3)$, and, enumerating the family of generating functions $\{\varphi^b_j\}$, we construct a set of indices $\mathcal{J} = \mathcal{J}(\mathcal{L})$ such that

$$\sum_{b\in\mathcal{B}} \sum_{1\le j\le k_b} \varphi^b_{j\sharp}[\![0, T_j]\!] = \sum_{j\in\mathcal{J}} \varphi_{j\sharp}[\![0, T_j]\!]. \tag{2.2.18}$$

Moreover, setting $S_i := \varphi_i([0, T_i])$, from (2.2.11) and (2.2.15) we also have

$$L = \bigcup_{j\in\mathcal{J}} S_j. \tag{2.2.19}$$

Canonical regular dislocations. Among all geometrically equivalent dislocations there exists one representation which is sharp in the sense that it is expressed in terms of the independent elementary Burgers vectors. Let $\mathcal{L}_\mathcal{B}$ be a regular dislocation. Since a b-dislocation $\mathcal{L}^b$ with $b = (\beta_1, \beta_2, \beta_3)$ has integer multiplicity, it can be written by means of projections. Recalling definition (2.2.1) and notation (2.1.14), we introduce

$$\mathcal{L}^{b,i} := \{L^b, \tau^b, \beta_i \theta^b\}, \tag{2.2.20}$$

with the corresponding density $\Lambda_{\mathcal{L}^{b,i}} := \mathcal{L}^{b,i} \otimes e_i = \mathcal{L}^b \otimes \beta_i e_i$. Hence to any regular dislocation $\mathcal{L}_\mathcal{B}$ we associate univocally three currents $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, with

$$\mathcal{L}_i := \sum_{b \in \mathcal{B}} \mathcal{L}^{b,i}, \tag{2.2.21}$$

so that $\mathcal{L}_i = \{L, \tau, \theta_i\}$, θ_i defined by

$$\theta_i := \sum_{b \in \mathcal{B}} \mathrm{sg}(\tau^b) \beta_i \theta^b, \qquad \text{with } b = (\beta_1, \beta_2, \beta_3),$$

and $\mathrm{sg}(\tau^b)$ being such that $\tau = \mathrm{sg}(\tau^b)\tau^b$. We then define the *canonical dislocation current associated to* $\mathcal{L}_\mathcal{B}$:

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_2 + \hat{\mathcal{L}}_3, \tag{2.2.22}$$

where $\hat{\mathcal{L}}_i$ is the i-th component of $\hat{\mathcal{L}}$ defined as

$$\hat{\mathcal{L}}_i(\omega) := \mathcal{L}_i(\omega e_i) = \mathcal{L}_i(\omega_i), \tag{2.2.23}$$

for all $\omega \in \mathcal{D}^1(\Omega, \mathbb{R}^3)$, and fixed $i = 1, 2, 3$. In other words $\hat{\mathcal{L}}_i = \{L, \tau, \theta_i e_i\}$.

A useful property of the decomposition (2.2.22) is that the three measures $\{\Lambda_{\mathcal{L}_i}\}_{i=1}^3$ operate on different (pointwise) orthogonal subspaces of $C_c^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})$.

Lemma 2.2.8. *The following assertions hold true:*

(a) The currents $\mathcal{L}_i$ ($i = 1, 2, 3$) are integer-multiplicity currents in Ω. As a consequence $\hat{\mathcal{L}}_i$ are integral currents with coefficients in $\mathbb{Z}^3$.

(b) The mass of the current and the total variation of the associated measure are related by

$$|\mathcal{L}_i|_\Omega = |\hat{\mathcal{L}}_i|_\Omega = \|\Lambda_{\mathcal{L}_i}\|_{\mathcal{M}(\Omega)} \le \|\Lambda_\mathcal{L}\|_{\mathcal{M}(\Omega)}, \tag{2.2.24}$$

for $i = 1, 2, 3$.

(c) The geometrically necessary dislocation set reads $L^\star := \bigcup_{i=1}^3 \mathrm{spt}(\mathcal{L}_i) \subset \bar{L}$ and coincides with the support of the density $\Lambda_\mathcal{L}$.

Proof. Assertion (a) follows by Theorem 2.1.3 since $\sum_{b \in \mathcal{B}} N(\mathcal{L}^{b,i}) < \infty$ by definition of regular dislocation.

To prove (b), observe first that for fixed b it holds

$$\sum_{i=1}^{3} \Lambda_{\mathcal{L}^{b,i}} = \sum_{i=1}^{3} \mathcal{L}^{b,i} \otimes e_i = \sum_{i=1}^{3} \tau^b \otimes \beta_i e_i \, \theta^b \mathcal{H}^1 \llcorner L^b = \Lambda_{\mathcal{L}^b}.$$

Thus it also holds

$$\Lambda_{\mathcal{L}} = \sum_{b \in \mathcal{B}} \Lambda_{\mathcal{L}^b} = \sum_{i=1}^{3} \Lambda_{\mathcal{L}_i} = \Lambda_{\hat{\mathcal{L}}}, \tag{2.2.25}$$

and explicitly,

$$\Lambda_{\hat{\mathcal{L}}} = \sum_{i=1}^{3} \tau \otimes e_i \theta_i \, \mathcal{H}^1 \llcorner L = \sum_{i=1}^{3} \mathcal{L}_i \otimes e_i, \tag{2.2.26}$$

(recall that τ and θ_i are functions of $x \in L$). Note that

$$\|\Lambda_{\mathcal{L}}\|_{\mathcal{M}} = \|\Lambda_{\hat{\mathcal{L}}}\|_{\mathcal{M}} \geq \|\Lambda_{\mathcal{L}_i}\|_{\mathcal{M}} \qquad \text{for } i = 1, 2, 3, \tag{2.2.27}$$

and since $\Lambda_{\mathcal{L}_i} = \mathcal{L}_i \otimes e_i$, it holds $\|\Lambda_{\mathcal{L}_i}\|_{\mathcal{M}(\Omega)} = |\hat{\mathcal{L}}_i|_\Omega = |\mathcal{L}_i|_\Omega$ so that yields (2.2.24).

To prove (c), observe first that $\mathcal{L}_i = \{L, \tau, \theta_i\}$ and by definition of $\mathcal{L}_i$ and $\Lambda_{\mathcal{L}_i}$ it easily follows that $\mathrm{spt}\mathcal{L}_i = \mathrm{spt}\Lambda_{\mathcal{L}_i}$. So we only need to prove that $\mathrm{spt}\Lambda_{\mathcal{L}} = \cup_{i=1}^{3} \mathrm{spt}\Lambda_{\mathcal{L}_i}$. But this is a direct consequence of the fact that $\Lambda_{\mathcal{L}_i}$ acts on orthogonal subspaces of $C_c^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3})$. □

Definition 2.2.9 (Unnecessary dislocations)**.** The *set of unnecessary dislocations* Ξ is defined as $\bar{L} \setminus L^\star$.

Let us remark that L defined in (2.2.19) depends on the generating loops of Definition 2.2.5.

2.2.1 Classes of admissible dislocations

Two classes of dislocations will now be introduced, the first being useful if one wishes to follow (for instance, with time) each line as it deforms, intersect with others etc., whereas the second will be more appropriate if the model relevant quantity is the dislocation density, and not the single lines. In the latter case dislocations are determined up to the equivalence relation (2.2.5) and the clusters might exhibit locally dense subsets of unnecessary dislocations.

The class of dislocations at the mesoscopic scale. At the mesoscopic scale, it is considered that every dislocation $\mathcal{L}_{\mathcal{B}}$ has been generated by a finite number of b-dislocation currents $\mathcal{L}^b$.

Assumption 2.2.10 (Finite generation)**.**

$$k_{\mathcal{L}} := \sum_{b \in \mathcal{B}} k_b < \infty, \tag{2.2.28}$$

with k_b defined in Definition 2.2.5.

Let us recall that a finite number of generating b-dislocation currents does not imply that the dislocation density $\Lambda_{\mathcal{L}}$ is associated to a finite number of distinct Burgers vectors, since the multiplicity on each arc of L is not limited and since countably intersections of arcs may take place (in other words, the resulting Burgers vector might be very large, provided it is attached to an arc which is small enough). Moreover, the cluster of Fig. 2.4(c) made of countably many loops whose lengths are summable and interconnected by unnecessary segments, is a mesoscopic dislocation since it can be generated by a single b-loop.

From the definitions above and Assumption 2.2.10 the following lemma is readily proved.

Lemma 2.2.11. *The following properties hold for dislocations at the mesoscopic scale:*

(a) The density of a dislocation $\Lambda_{\mathcal{L}}$ is a bounded Radon measure since

$$\|\Lambda_{\mathcal{L}}\|_{\mathcal{M}(\bar{\Omega})} \leq \sum_{\substack{b\in\mathcal{B}^{\mathcal{L}} \\ i=1,\dots,k_b}} |b| l_i^b < \infty. \tag{2.2.29}$$

with $\mathcal{B}^{\mathcal{L}} := \{b \in \mathbb{Z}^3 : k_b \neq 0\}$ (Recall l_i^b is the length of the dislocation loop φ_i^b).

(b) The dislocation current $\hat{\mathcal{L}}$ is an integral current with coefficients in $\mathbb{Z}^3$ satisfying

$$\|\Lambda_{\mathcal{L}}\| = M(\hat{\mathcal{L}}) \leq \sum_{\substack{b\in\mathcal{B}^{\mathcal{L}} \\ i=1,\dots,k_b}} |b| l_i^b < \infty, \tag{2.2.30}$$

with $\mathcal{B}^{\mathcal{L}} := \{b \in \mathbb{Z}^3 : k_b \neq 0\}$. In particular θ and θ_i, for $i = 1, 2, 3$ are all summable functions with respect to $\mathcal{H}^1 \llcorner L$.

(c) The dislocation set L of the current $\mathcal{L}$ (defined in (2.2.15)) is a closed set with finite $\mathcal{H}^1$-measure. In particular $L^\star \subseteq L$ and $L = L^\star \cup \Xi$.

Proof. To prove (a), observe that $\mathcal{L} = \{\mathcal{L}^b\}_{b\in\mathcal{B}^{\mathcal{L}}}$ and hence $\|\Lambda_{\mathcal{L}}\| \leq \sum_{b\in\mathcal{B}^{\mathcal{L}}} \|\mathcal{L}^b \otimes b\| \leq \sum_{\substack{b\in\mathcal{B}^{\mathcal{L}} \\ i=1,\dots,k_b}} \|\mathcal{L}_i^b \otimes b\| = \sum_{\substack{b\in\mathcal{B}^{\mathcal{L}} \\ i=1,\dots,k_b}} |b| l_i^b$, which is finite since the sum is finite by the mesoscopicity Assumption 2.2.10. Statement (b) follows as a direct consequence of the definition of b-dislocation current and from (a) and property (b) of Lemma 2.2.8. Property (c) is a straightforward consequences of the fact that $\mathcal{H}^1(L) \leq \sum_{\substack{b\in\mathcal{B} \\ i=1,\dots,k_b}} l_i^b = \sum_{\substack{b\in\mathcal{B} \\ i=1,\dots,k_b}} \int_0^{T_i} \|\dot{\varphi}_i^b\| dt < \infty$ by the mesoscopicity Assumption 2.2.10.

□

From the preceding results, we are ready to define the class of *admissible dislocations at the mesoscale*.

Definition 2.2.12. [Admissible mesoscopic dislocation]

$$\mathcal{MD} := \{\mathcal{L} = \{\mathcal{L}^b\}_{b\in\mathcal{B}} : \mathcal{L}^b \text{ takes the form (2.2.7)} \text{ and satisfies Assumption 2.2.10.}\}. \tag{2.2.31}$$

Dislocations at the continuum scale. A set in $\mathbb{R}^n$ is said to be a continuum if it is the finite union of connected and compact 1-sets with finite $\mathcal{H}^1$ measure. Let us recall that the geometric necessary dislocation set $L^\star$ is the support of $\Lambda_{\mathcal{L}}$. The space of *admissible dislocations at the continuum scale* is introduced as follows:

Definition 2.2.13. [Admissible continuum dislocation]

$$\mathcal{CD} := \{\mathcal{L}_{\mathcal{I}}, \mathcal{I} \subset \mathbb{N} : \text{there exists a continuum } \mathcal{K} \text{ such that } L^\star \subset \mathcal{K}\}. \quad (2.2.32)$$

When the context is clear, we will write $\mathcal{L} = \mathcal{L}_{\mathcal{I}}$ and the set of continua $\mathcal{K}$ for which $L^\star \subset \mathcal{K}$ will be denoted by $\mathcal{C}_{\mathcal{L}} = \mathcal{C}_{\mathcal{L}_{\mathcal{I}}}$.

In particular every $\mathcal{L}$ such that the support $L^\star$ of $\Lambda_{\mathcal{L}}$ consists of finitely many connected 1-sets is an admissible dislocation at the continuum scale. Remark that contrarily to mesoscopic dislocations (cf. Lemma 2.2.11 (b)), the density of a continuum dislocation must not be finite (this might for an unconstrained family of Burgers vectors).

In the applications, the notion of continuum dislocations is usefull to study the cases in which Assumption 2.2.10 is not satisfied. However, if one is not interested in the particular dislocation current associated to a given dislocation density, mesoscopic dislocations become a superfluous notion. In fact, crystallographic mesoscopic dislocations turn out to be equivalent to continuum dislocations, in the sense that, for any continuum dislocation $\mathcal{L}$, there is a mesoscopic dislocation $\mathcal{L}'$ such that $\mathcal{L} \equiv \mathcal{L}'$. The proof of this fact is based on the following theorem

Theorem 2.2.14. *Let $\mathcal{L}$ be a closed integer-multiplicity current with finite mass and whose support $L^\star$ is contained in a connected and compact set $\mathcal{K}$ with finite $\mathcal{H}^1$-measure. Then there exists a Lipschitz function $\alpha : S^1 \to \mathcal{K}$ such that $\mathcal{L} = \alpha_\sharp [\![S^1]\!]$.*

To prove Theorem 2.2.14 we need some preliminary Lemmas:

Lemma 2.2.15. *Let K be a compact connected set in $\mathbb{R}^n$ such that $\mathcal{H}^1(K) < \infty$. Then there exists a Lipschitz map $\psi : S^1 \to K$ that is onto and is homotopic to the constant map.*

Proof. In the following we consider S^1 as a subset of the complex plane $\mathbb{C}$. Let $P \in K$ and let us consider the set

$$\mathcal{S} := \{\phi : S^1 \to K \text{ satisfying the following three properties}\} \quad (2.2.33)$$

(i) $\phi(1) = P$.

(ii) ϕ is homotopic to the constant map $\phi \equiv P$.

(iii) Letting $C = \phi(S^1)$ and $L_C = \mathcal{H}^1(C)$, the curve ϕ is Lipschitz with constant $\frac{L_C}{\pi}$.

It is easily seen that, since K is a rectifiable set, $\mathcal{S}$ is non-empty. Given a $\phi \in \mathcal{S}$ we want to enlarge its range in order to get an onto map. To this aim we define the following order relation in $\mathcal{S}$: we say that $\phi < \phi'$ if and only if

$\phi(S^1) = C \subseteq C' = \phi'(S^1)$. Let $\{\phi_j\}_{j\in J\subset\mathbb{R}}$ be a chain in $\mathcal{S}$ (assumed ordered by the corresponding ordering of the indices in $\mathbb{R}$), and set $L_j := \mathcal{H}^1(\phi_j(S^1))$. Then the sequence $\{L_j\}_{j\in J}$ is non-decreasing and bounded by $\mathcal{H}^1(K)$, so that, since the maps $\{\phi_j\}$ are uniformly continuous in j, there is an increasing sequence $j_k \to \sup J$ and a map ϕ such that $\phi_{j_k} \to \phi$ uniformly on S^1. We claim that ϕ is an upper bound for $\{\phi_j\}_{j\in J}$. Indeed, denoting $C_j = \phi_j(S^1)$, the increasing sequence $\{C_j\}$ converges to a compact set $C \subseteq K$ with respect to the Gromov-Hausdorff distance. Since $j_k \to \sup J$ we see that for each $k \in J$ we have $C_k \subseteq C$, so that we only have to prove that ϕ belongs to the family $\mathcal{S}$. Setting $L := \mathcal{H}^1(C)$, we have $L \le \mathcal{H}^1(K)$, and since $L_j \le L$ the uniform convergence and the uniform bound $\mathrm{Lip}(\phi_j) \le \frac{L}{\pi}$ implies that $\mathrm{Lip}(\phi) \le \frac{L}{\pi}$. So (i) and (iii) are readily fulfilled. Also (ii) is easy to see: let Φ_j be the homotopy map between $\Phi_j(\cdot,1) = \phi_j$ and the constant $\Phi_j(\cdot,0) \equiv P$, and up to a rescaling, we suppose that for all $x \in S^1$ the map $\Phi_j(x,\cdot)$ is Lipschitz with $\mathrm{Lip}(\Phi_j(x,\cdot)) \le L$, so that it readily turns out that Φ_j are uniformly continuous in j, and uniformly converge to a map Φ; now it is straightforward that Φ is a homotopy between ϕ and P, and the claim is proved.

We now are in the hypotheses of the Zorn's Lemma, so that we get a maximal element ψ for the class $\mathcal{S}$. It remains to show that ψ is onto. Suppose it is not the case. We set $C_\psi := \psi(S^1)$ and suppose $X \in K \setminus C_\psi$. Since C_ψ is closed and K is connected, there is a Lipschitz continuous arc $\alpha : [0,1] \to K$ such that $\alpha(0) \in C_\psi$, $\alpha(1) = X$, and $\alpha(y) \in K \setminus C_\psi$ for $y > 0$. Let $x \in \psi^{-1}(\alpha(0))$, and split $S^1 = [1,x] \cup [x,1]$. Consider the restriction of ψ to this two intervals, ψ_1 and ψ_2. Then it is readily seen that the arc $\psi_1 \star \alpha \star \alpha_{-1} \star \psi_2$, if suitably rescaled as a function on S^1, is a map in $\mathcal{S}$ that is strictly greater than ψ, contradicting the maximality of ψ. Hence the thesis follows. □

Lemma 2.2.16. *Let K be a compact 1-set and $\psi : S^1 \to K$ be a Lipschitz continuous map homotopic to a constant map. Then $\psi_\sharp[\![S^1]\!] = 0$.*

Proof. Suppose for simplicity $K \subset \mathbb{R}^2$. Since K is compact, K^c is an open set, with only one unbounded connected component A. If $X \in B := K^c \setminus A$, there exists an open ball centered in X that does not intersect K, so that it follows that any connected component of B has positive Lebesgue measure. As a consequence there are at most countably many connected components in B. Let X_i be a point in the i-th connected component of B. The homotopic group of Lipschitz closed arcs in K coincides with the free group on the generators $\{X_i\}_{i\in\mathbb{N}}$.

Now, if the current carried by ψ is nonzero, the decomposition theorem implies that there exists $T = \alpha_\sharp[\![S^1]\!]$ an undecomposable component of the 1-current $\psi_\sharp[\![S^1]\!]$. If $X = \psi(a) = \psi(b)$, then, since ψ is homotopic to the constant, we can replace ψ with $\hat\psi$, setting $\hat\psi\llcorner[a,b] \equiv X$ and $\hat\psi\llcorner[a,b]^c = \psi$, getting a map that is still homotopic to the constant. Moreover the homotopy class of a loop in K does not change under homotopy in the space K, so that the operation above does not change the homotopy class of the current. In this way we find out that α must belong to the same homotopy class of ψ. On the other hand, since α is an injective loop, its homotopy class is $\prod_{X_j\in\Delta} X_j$, with Δ being the bounded connected set with boundary α. Thus the homotopy class of ψ is nonzero, contradicting the hypothesis that ψ was homotopic to a constant map. □

Now we can prove Theorem (2.2.14).

Proof of Theorem 2.2.14. By the decomposition Theorem there are loops β_j such that $\mathcal{L} = \sum_j \beta_{j\sharp}[\![S^1]\!]$. Consider a function ψ like in Lemma 2.2.15, so that there are points $x_j \in S^1$ such that $\psi(x_j) = \beta_j(1)$. Suppose for simplicity $x_1 = 1$ and x_j are clockwise ordered on S^1. Setting $\psi_j := \psi \llcorner [x_j, x_{j+1}]$, then the chain

$$\alpha := \beta_1 \star \psi_1 \star \beta_2 \star \psi_2 \star \ldots \beta_j \star \psi_j \ldots ,$$

suitably rescaled, will match the required conditions, since ψ, being homotopic to the constant, is such that $\psi_\sharp[\![S^1]\!] = 0$ from Lemma 2.2.16. □

The precise equivalence theorem is stated as follows.

Theorem 2.2.17. *Let $\mathcal{L}_\mathcal{I}$ be a continuum dislocation such that $\mathcal{B}_\mathcal{I} \subset \mathbb{Z}^3$ and $\Lambda_{\mathcal{L}_\mathcal{I}}$ is finite. Then $\mathcal{L}_\mathcal{I}$ is a mesoscopic dislocation.*

Proof. Considering the canonical dislocation current $\hat{\mathcal{L}}$ equivalent to $\mathcal{L}_\mathcal{I}$ (cf. Eq. (2.2.22)), the thesis follows from Eq. (2.2.24) and Theorem 2.2.14. Indeed the latter provides three Lipschitz functions α_i $(i = 1, 2, 3)$ such that $\alpha_{i\sharp}[\![S^1]\!] = \mathcal{L}_i$ so it follows $\Lambda_\mathcal{L} = \sum_i \alpha_{i\sharp}[\![S^1]\!] \otimes e_i$. □

In particular Theorem 2.2.17 tells us that continuum and mesoscopic dislocation are equivalent if the energy $\mathcal{W}$ of the system does not depend on the particular dislocation current, but only on its dislocation density. We remark that the thesis does not hold true if we do not make the assumption that the set of Burgers vectors $\mathcal{B}$ is crystallographic (i.e., isomorphic to $\mathbb{Z}^3$).

Boundary conditions for dislocations. Let U be a bounded open set such that $U \cap \partial\Omega = \partial_D\Omega$.

Definition 2.2.18 (Boundary conditions)**.** A *boundary condition* is a triple $(N, \mathcal{P}, \alpha_D)$ satisfying:

(i) $N \geq 0$ is a natural number.

(ii) $\mathcal{P}$ is a triple $(P_i, Q_i, \mathcal{B}_P)_{0\leq i\leq N}$ with $\{P_i\}$ and $\{Q_i\}$ sequences of points in $\partial_D\Omega$, and $\mathcal{B}_P = \{b_{Pi}\}_{0<i\leq N}$ a sequence of vectors belonging to $\mathcal{B}$. We associate to $\mathcal{P}$ the 0-current with coefficients in $\mathbb{Z}^3$ as $\hat{T}_P := \sum_{0<i\leq N} \delta_{P_i} b_{Pi} - \delta_{Q_i} b_{Pi}$, with δ_P the Dirac mass at P.

(iii) $\alpha_D := \alpha + \alpha'$ is the sum of two mesoscopic dislocations in U. We suppose α is a closed current with support in $\partial_D\Omega$ consisting of $M < \infty$ loops α_i and Burgers vector b^i_α, while α' consists of the union of N dislocation loops α_i with support in $\bar{U} \setminus \Omega$, such that for all i, α_i has boundary $\partial\alpha_i = \delta_{Q_i} - \delta_{P_i}$ and associated Burgers vector $b_{Pi} \in \mathcal{B}_D$.

From (iii) we can define $\Lambda_{\alpha_D} = \sum_{0\leq i\leq M} \alpha_{b^i_\alpha} \otimes b^i_\alpha + \sum_{0\leq i\leq N} \alpha_{b_{\hat{\alpha}i}} \otimes b_{Pi}$ to be the density of the dislocation current α. According to the definitions of dislocation currents given above we denote by $\hat{\alpha}_D$, $\hat{\alpha}$, and $\hat{\alpha}'$ the corresponding currents with coefficient in $\mathbb{Z}^3$.

Definition 2.2.19. We say that the boundary condition $(N_P, \mathcal{P}, \alpha_D)$ is *admissible* if the following condition is satisfied: there exists a regular dislocation $\mathcal{L}$ such that $\partial\hat{\mathcal{L}} = \hat{T}_P$. We say that a dislocation $\mathcal{L}$ satisfies the admissible boundary condition $(N, \mathcal{P}, \alpha_D)$ if it satisfies the previous property.

As a consequence of the previous definition, it turns out that $\hat{\alpha}_D + \hat{\mathcal{L}}$ is closed in $\bar{U} \cup \bar{\Omega}$.

Some remarks. So far, dislocations are mathematically represented by currents but it is crucial to keep in mind their physical origin and formation. A dislocation loop in the bulk results from nucleation, that is, the collapse of a void (i.e., a cavitation formed by aggregation of vacancies) which has become unstable. Another source of dislocations is the flux of vacancies or interstitials at the crystal boundary. In each case, the basic dislocation is a loop which is associated to a single Burgers vector that depends on the crystal structure. Submitted to thermal and mechanical forces, to diffusion, anihilation, recombination and any kind of mutual interactions, these loops might in turn deform and move inside the crystal and through its boundary, but also form clusters which themselves will either evolve or behave as fixed obstacle to the motion of other loops, provoking material hardening.

These considerations are at the basis of the notion of *regular dislocation* introduced above. According to the dislocation physics, the basic object will be the loops associated to a given Burgers vector b, i.e., the functions φ_j^b introduced in Definition 2.2.5. These simple *generator* loops will then be smoothly deformed and summed (in the sense of currents) in order to form dislocation clusters. Moreover, it should be emphasized that the limited number of Burgers vectors of the generating loops might increase significantly as clusters are considered since Frank law applies at dislocation junctions [35]. For this reason, our restriction to finite families of regular loops associated to a finite number of distinct Burgers vectors (Assumption 2.2.10) does not preclude the formation of complex structures. As a consequence, a dislocation of this kind might be formed by countably regular loops connected by arcs which are effectless in terms of the intrinsic geometry of the crystal, and therefore referred to as *geometrically unnecessary* Ξ (Definition 2.2.9). Moreover, though being 1-sets, the clusters might exhibit complex geometries at the countable intersections or at the sets of accumulation points of their generating loops. It should nevertheless be clarified that since overlapping of dislocations is not acceptable from a physical viewpoint, it should be equivalently understood as a non-overlapping curve associated to a scalar multiple of the Burgers vector.

Let us now describe a dislocation cluster which is not a mesoscopic dislocation. Consider the cluster of Fig. 2.4(c) but instead of assuming that each loop possesses the same Burgers vector b, suppose that the family $\mathcal{B}_{\mathcal{I}} \nsubseteq \mathbb{Z}^3$ of Burgers vectors is non-crystallographic, that means that if $\mathcal{B}_{\mathcal{I}} = \{b_i\}_{i\in\mathbb{N}}$ then the ratios b_i/b_j is never rational for every $i \neq j$. Thus, it clearly appears that this cluster cannot be made of regular dislocations without violating Assumption 2.2.10. Instead, it turns out that the broader notion of continuum dislocation holds for this kind of pathological cluster, as long as the sum of the length of the loops is finite. We emphasize that from a strictly mesoscopic standpoint allowing the Burgers vectors to take countably many values ($\mathcal{B}_{\mathcal{I}} \nsubseteq \mathbb{Z}^3$ non-crystallographic) is not physical, all the more for bounded crystals. However it can become important to permit this limit case, for instance if one considers homogenization, or from a statistical viewpoint, ensemble averaging of dislocations.

If $\mathcal{L}$ is a regular mesoscopic dislocation, the fact that $\mathcal{L} \in \mathcal{CD}$ does not imply that $\mathcal{H}^1(L) < \infty$, even if $\Lambda_{\mathcal{L}}$ is finite. Indeed *continuum dislocations* in $\mathcal{CD}$ might be quite wild, since they can consist of countable fully disconnected loops and may admit geometrically unnecessary arcs which are locally dense, i.e., $\mathcal{H}^1(\bar{\Xi}) = \infty$. Moreover, since disconnected pieces of a dislocation can be connected by adding geometrically unnecessary arcs Ξ (cf. Fig 2.4), it might also happen that $\mathcal{H}^1(\Xi) = \infty$.

The introduction of continuum dislocations might be convenient for some other reasons. First, considering time-evolution of dislocations, this latter class, as opposed to the former, allows us to consider an evolution of the unnecessary part $\Xi(t)$ such that $\mathcal{H}^1(\Xi(t)) \to \infty$ (or $\mathcal{H}^1\left(\bar{\Xi}(t)\right) \to \infty$) as t converges to some limit time. Time-evolution of some subset of $\mathcal{K}$ to a pathological Ξ is also possible within this setting, and it might be taken into account since unnecessary dislocations play an effective role in dynamics (as obstacle to motion, i.e. *hardening*), whereas they do not contribute to the dislocation density. Second, continuum dislocations conceptually suits better engineer models of dislocations in which necessary and unnecessary dislocations are treated by distinct, though coupled, equations.

2.3 Maps in $\mathbb{T}^3$ with bounded curl

In this section we prove a characterization of the maps in $L^p(\Omega, \mathbb{R}^{3\times 3})$ whose Curl is the density of a regular dislocation (This is the space $\mathcal{BC}^{p,\Lambda}(\Omega, \mathbb{R}^{3\times 3})$ introduced in (2.5.15) below). We prove that to each such map F corresponds a function u in the Sobolev space $W^{1,p}(\Omega, \mathbb{T}^3)$ (or, equivalently, in $W^{1,p}(\Omega, (S^1)^3)$, where S^1 is the set of points in $\mathbb{R}^2$ with norm equal to 1) such that $\nabla u = F$. Moreover, if α represents an inverse of the projection $\pi : \mathbb{R} \to S^1$, then we prove that to each map u in $W^{1,p}(\Omega, \mathbb{T}^3)$ such that the gradient of $\alpha \circ u$ is a map in $L^p(\Omega, \mathbb{R}^{3\times 3})$ with curl a Radon measure, then such Radon measure must be the density of some dislocation. We will state our result for vector-valued fields ($\mathcal{BC}^p(\Omega, \mathbb{R}^3)$) instead of $\mathbb{R}^{3\times 3}$-valued tensors, which will correspond to maps in $W^{1,p}(\Omega, \mathbb{T})$. In order to achieve the proof of this correspondence we need some preliminary tools of measure theory.

Preliminary facts. Let $u \in W^{1,1}(\Omega, \mathbb{R}^m)$, then we define the set of regular points of u as the set $\mathcal{R}_u$ of all $x \in \Omega$ such that x is a Lebesgue point for both u and Du, and the value $u(x)$ concides with its Lebesgue value. Of course, the complementary set of $\mathcal{R}_u$ is negligible in Ω.

Theorem 2.3.1. *Let $u \in W^{1,1}(\Omega, \mathbb{R}^m)$. Then there exist a countable family of closed sets $F_k \subset \Omega$ and Lipschitz functions $u_k : \Omega \to \mathbb{R}^m$ such that $u \equiv u_k$ on F_k, $\mathcal{R}_u \subset \cup_k F_k$, and for all k we have $Du(x) = Du_k(x)$ for a.e. $x \in F_k$. Moreover $u_k \to u$ strongly in $W^{1,1}(\Omega, \mathbb{R}^m)$.*

This statement is proved in [31, Theorem 4 of Section 3.1.3] with the help of [31, Proposition 1 of Section 3.1.1].

Also the following Theorem is needed. We refer to [31, Theorem 2 of Section 2.2.7].

Theorem 2.3.2 (Boundary Rectifiability Theorem)**.** *Let $\mathcal{S}$ be a m-integer multiplicity current such that $\partial\mathcal{S}$ has finite mass. Then $\partial\mathcal{S}$ is a $(m-1)$-integer multiplicity current.*

Definition 2.3.3. For every 1-form $\omega \in \mathcal{D}^1(\mathbb{R}^3)$ we identify $\omega = \omega_i dx_i$ with the vector field $w = (w_1, w_2, w_3) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by setting $w_i := \omega_i$. Moreover we identify every 2-form $\omega = \omega_i dx_{\bar{i}} \in \mathcal{D}^2(\mathbb{R}^3)$ with another vector field $w_i = (-1)^{i+1}\omega_i$. With this convention we can see the external derivative $d\omega$ of a 1-form ω as the curl of the corresponding vector w, i.e., $w_i dx_i = \omega_i$ and $d\omega = (\operatorname{Curl} w)_i dx_{\bar{i}}$.

As a consequence of this identification, if $\mathcal{L}$ is a 1-current with finite mass, then it is a measure in $\mathcal{M}_b(\Omega, \mathbb{R}^3)$. The same holds true for 2-dimensional currents $\mathcal{S}$. In particular the boundary of a current corresponds to the curl of the correspondent measure since

$$\partial\mathcal{S}(\omega) = \mathcal{S}(d\omega) = \langle \mathcal{S}, \operatorname{Curl} w \rangle = \langle \operatorname{Curl} \mathcal{S}, w \rangle. \tag{2.3.1}$$

The following Theorem is a classical result, whose proof is given in [51].

Theorem 2.3.4. *Let Ω be a bounded and simply connected open set. Let $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^3)$ be a Radon measure such that* $\operatorname{Curl} \mu = 0$ *as a distribution. Then there exists a function with bounded variation $u \in BV(\Omega)$ such that $Du = \mu$.*

Moreover we need the following Theorem which provides a chain rule to compute the derivative of the composition of a smooth function with a function with bounded variation (see [4] or [78]).

Theorem 2.3.5. *Let $u \in BV(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded open set, and let $f \in C^1(\Omega)$. Then the distributional derivative of $f \circ u$ is given by*

$$D(f \circ u) = Df(u) D^a u \mathcal{L}^n + Df(\tilde{u}) D^c u + (f(u^+) - f(u^-)) \nu_{J_u} \mathcal{H}^2 \llcorner_{J_u}, \tag{2.3.2}$$

where $\tilde{u}$ is the Lebesgue representant of u, i.e., $\tilde{u}(x)$ is the Lebesgue value of u at x.

Let Ψ be a Lipschitz map between two Riemannian manifolds M^m and N^n of dimension m and n respectively. The differential $D\Psi(x)$ of Ψ at a point $x \in M^m$ is a linear map between the tangent space $T_x M$ and $T_{\Psi(x)} N$. We can then define the Jacobian $J\Psi(x)$ of Ψ at x as

$$J\Psi(x) := \sqrt{D\Psi(x) D\Psi(x)^T}.$$

The classical Coarea Formula for Lipschitz maps is given by the following.

Theorem 2.3.6. *Let M^m and N^n two Riemannian manifolds of dimension m and n respectively, with $m > n$. Let $\Psi : M^m \to N^n$ be a Lipschitz map. Then for all functions $g \in L^1(M^m, \mathbb{R})$ it holds*

$$\int_{M^m} g(x) |J\Psi(x)| d\mathcal{H}^m(x) = \int_{N^n} \Big(\int_{\Psi^{-1}(y)} g(x) d\mathcal{H}^{m-n}(x) \Big) d\mathcal{H}^n(y). \tag{2.3.3}$$

The space of Sobolev functions in the torus. We denote by $W^{1,1}(\Omega, S^1)$ the space of all $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ such that $|u(x)| = 1$ for a.e. $x \in \Omega$. Obviously $W^{1,1}(\Omega, S^1)$ is a closed subspace of $W^{1,1}(\Omega, \mathbb{R}^2)$.

For all $\beta \in [0, 2\pi)$ we denote by $r^\beta : \mathbb{R}^2 \to \mathbb{R}^2$ the anticlockwise rotation of an angle β, i.e.,

$$r^\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

We define the function $\alpha : \mathbb{R}^2 \to [0, 2\pi)$ as

$$\alpha(x) := \begin{cases} \arctan(-\frac{x_1}{x_2}) + \frac{\pi}{2} & \text{if } x_2 > 0, \\ 0 & \text{if } x_1 \geq 0,\ x_2 = 0, \\ \pi & \text{if } x_1 < 0,\ x_2 = 0, \\ \arctan(-\frac{x_1}{x_2}) + \frac{3\pi}{2} & \text{if } x_2 < 0. \end{cases} \tag{2.3.4}$$

We see that α is smooth outside the set $\{x_1 \geq 0, x_2 = 0\}$, and it has a jump of width 2π on this set. For convenience we also introduce, for all real number $\beta \in [0, 2\pi)$, the function $\alpha^\beta : \mathbb{R}^2 \to [0, 2\pi)$ defined by $\alpha^\beta := r^{-\beta} \circ \alpha$. In particular, composing with $r^{-\beta}$, we have moved the jump of α to the set $\{\alpha(x) = \beta\}$.

Let $\varphi \in C^\infty(\Omega, \mathbb{R}^2)$. For all real numbers $\beta \in [0, 2\pi)$ we define the α^β-lifting of φ as $\alpha^\beta \circ \varphi : \Omega \to [0, 2\pi)$. Computing the differential of $\alpha^\beta \circ \varphi$ at a point x such that $\varphi(x)$ is not on the jump set of α^β we get

$$D(\alpha^\beta \circ \varphi)(x) = r^\beta \Big(\frac{-u_2(x) D_1 \varphi(x)}{|u(x)|^2}, \frac{u_1(x) D_2 \varphi(x)}{|u(x)|^2} \Big)^T. \tag{2.3.5}$$

The aforementioned assumption on $u \in W^{1,1}(\Omega, S^1)$ that the gradient of $\alpha \circ u$ is a map in $L^p(\Omega, \mathbb{R}^3)$ with Curl a Radon measure is rigorously expressed in the following form (see also Remark 2.3.10 below):

Assumption 2.3.7. *There exists a Radon measure $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^3)$ such that*

$$\langle -u_2 Du_1 + u_1 Du_2, \operatorname{Curl} \varphi \rangle = \int_\Omega \varphi d\mu, \tag{2.3.6}$$

for every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$.

Main result. The following Theorem states that each strain F in the presence of dislocations can be written as the gradient of a Sobolev torus-valued map.

Theorem 2.3.8. *Let Ω be a bounded and simply connected open set. Let $\mathcal{L} \in \mathcal{D}_1(\Omega)$ be a closed 1-integer multiplicity current and suppose $F \in L^1(\Omega, \mathbb{R}^3)$ is such that* Curl $F = \mathcal{L}$ *(with the identification* (2.3.1)*). Then there exists $u \in W^{1,1}(\Omega, S^1)$ such that $-u_2 Du_1 + u_1 Du_2 = F$ on Ω.*

Proof. Since $\mathcal{L}$ is a closed 1-integer multiplicity current, there exists a 2-integer multiplicity current $\mathcal{S}$ with finite mass and such that $-\partial\mathcal{S} = \mathcal{L}$. Let us now define the distribution $\mu \in \mathcal{D}'(\Omega, \mathbb{R}^3)$ as follows

$$\mu(\varphi) := \mathcal{S}(\varphi) + \langle F, \varphi \rangle,$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$, where we have identified the map φ with the 2-form $\sum_{i=1}^3 (-1)^i \varphi_i dx_{\bar{i}}$ as in Definition 2.3.3. The distribution μ is easily seen to be a Radon measure with finite mass. We compute the rotation of μ, that is

$$\langle \text{Curl } \mu, \varphi\rangle = \mathcal{S}(\text{ Curl } \varphi) + \langle F, \text{ Curl } \varphi\rangle = \partial\mathcal{S}(\varphi) + \mathcal{L}(\varphi) = 0,$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$, by definition of $\mathcal{S}$. Then Theorem 2.3.4 implies that then exists $v \in SBV(\Omega)$ such that $Dv = \mu = \mathcal{S} + F$. Since $\mathcal{S}$ is an integer multiplicity current, there exist a 2-rectifiable set S with unit normal ν and an integer-valued function $\theta \in L^1(S, \mathcal{H}^2)$ such that $\mathcal{S} = (S, \nu, \theta)$. In particular we see that the jump of v is given by the measure $\theta\nu \cdot \mathcal{H}^2 \llcorner_S$, while the absolutely continuous part of the gradient Dv is F. We then set

$$u(x) = (u_1(x), u_2(x)) := (\cos(2\pi v(x)), \sin(2\pi v(x))).$$

The map $t \to 2\pi t$ is of class C^1 on $\mathbb{R}$, so formula (2.3.2) applies and we obtain $D^j u_1 = (\cos(2\pi v^+(x)) - \cos(2\pi v^-(x)))\nu\mathcal{H}^2 \llcorner_S = 0$, since $v^+ - v^- = \theta \in \mathbb{Z}$, and we conclude that u_1 has not jump part, and then it belongs to $W^{1,1}(\Omega)$. The same being true for u_2, we get $u \in W^{1,1}(\Omega, S^1)$. Moreover $Du_1 = -\sin(2\pi v)F$ and $Du_2 = \cos(2\pi v)F$ so that $-u_2 Du_1 + u_1 Du_2 = F$ and we have concluded. □

The main result of this section states that also the opposite of Theorem 2.3.8 holds true.

Theorem 2.3.9. *Let $u \in W^{1,1}(\Omega, S^1)$ and assume that u satisfies hypothesis 2.3.7. Then there exists a closed integral 1-current $\mathcal{L}$ such that* Curl $(-u_2 Du_1 + u_1 Du_2) = 2\pi\mathcal{L}$.

Remark 2.3.10. Note that S^1 is obviously isometric to $\mathbb{T}$. The isometry is the function $\alpha : S^1 \to \mathbb{T}$ given by (2.3.4). This isometry is smooth and has differential $(-x_2, x_1)^T$. So that, if $u \in W^{1,1}(\Omega, S^1)$ and $\hat{u} := \alpha(u) \in W^{1,1}(\Omega, \mathbb{T})$ is the corresponding map with values in the torus, then the quantity $-u_2 Du_1 + u_1 Du_2$ is exactly the gradient of $\hat{u}$. In particular, in terms of maps with values in the torus, the previous Theorem states that if $u \in W^{1,1}(\Omega, \mathbb{T})$ is not constant and Curl $\nabla\hat{u}$ is a Radon measure, then Curl $\nabla\hat{u}$ is represented by a closed integral 1-current.

Without lose of generality we suppose the set $\{x \in \Omega : u_2(x) = 0\}$ is negligible (otherwise it suffices to choose another basis of $\mathbb{R}^2$). Let $\{F_k\}_{k>0}$ be the closed sets provided by Theorem 2.3.1. By this Theorem we know that u is Lipschitz on F_k and we can extend $u \llcorner_{F_k}$ to a Lipschitz map $u^k : \Omega \to \mathbb{R}^2$ (with the same Lipschitz constant). Moreover $u^k \to u$ strongly in $W^{1,1}(\Omega, \mathbb{R}^2)$.

We prefer to divide the proof of Theorem 2.3.9 in several steps. We provide some lemmas and propositions, the first one being the following.

Lemma 2.3.11. *There is a positive real number $r < 1$ such that the level set $\{|u^k| = r\}$ has finite 2-dimensional Hausdorff measure for all $k > 0$, and $\mathcal{H}^2(\{|u^k| = r\}) \to 0$ as $k \to \infty$.*

Proof. Since the maps $|u^k|$ are Lipschitz, the coarea formula applies and gives

$$\int_{\Omega\cap\{u^k\neq 0\}} |\frac{u^k(x)}{|u^k(x)|} Du^k(x)|dx = \int_0^{+\infty} \mathcal{H}^2(\{|u^k| = t\})\Big)dt. \tag{2.3.7}$$

The Dominated Convergence Theorem implies that $\frac{u^k}{|u^k|}Du^k \to \frac{u}{|u|}Du = D|u| = 0$ as $k \to \infty$, so that we obtain that the functions $t \to \mathcal{H}^2(\{|u^k| = t\})$ are converging to zero strongly in $L^1(\mathbb{R})$. In particular they converge to zero almost everywhere, and then there exists $0 < r < 1$ such that $\mathcal{H}^2(\{|u^k| = r\}) \to 0$. □

It is convenient to introduce the following notation. We set

$$N_k := \{|u^k| < r\}$$

and, up to choosing another r in the previous lemma, we can assume that $\{|u^k| = r\} = \partial N_k \cup M_k$ with M_k a $\mathcal{H}^2$-negligible set for all $k > 0$. We also set

$$\Omega_k := \Omega \setminus \{|u^k| \leq r\}, \tag{2.3.8}$$

so that $\Omega_k \to \Omega$. Consider the closed sets $S_0^k := \{u_2^k = 0\}$. We now prove the following Lemma where we deal with the function α introduced in (2.3.4).

Lemma 2.3.12. *There is a real number $t \in (0, 2\pi)$ such that the level set $\{\alpha \circ u^k = t\} \cap \Omega_k$ has finite 2-dimensional Hausdorff measure for all $k > 0$ and*

$$\mathcal{H}^2(\{\alpha \circ u^k = t\} \cap \Omega_k) \to \mathcal{H}^2(\{\alpha \circ u = t\} \cap (\cup_k F_k)) < +\infty. \tag{2.3.9}$$

Proof. The maps $\alpha \circ u^k$ are locally Lipschitz on $U_k := \Omega_k \setminus S_0^k$, therefore the Coarea formula applies and gives

$$\int_{U_k} |D(\alpha \circ u^k)(x)| g(x) dx = \int_0^{2\pi} \Big(\int_{\{\alpha \circ u^k = t\}} g(y) d\mathcal{H}^2(y) \Big) dt, \tag{2.3.10}$$

for all functions $g \in L^1(\Omega)$. Setting first $g = \chi_{F_k}$ and then $g = \chi_{F_k^c}$ we find

$$\begin{aligned} \int_{\Omega_k \cap F_k} |D(\alpha \circ u^k)(x)| dx &= \int_0^{2\pi} \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap \Omega_k \cap F_k) dt \\ &= \int_0^{2\pi} \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap F_k) dt, \end{aligned} \tag{2.3.11}$$

since $F_k \subset \Omega_k$, and

$$\int_{\Omega_k \setminus F_k} |D(\alpha \circ u^k)(x)| dx = \int_0^{2\pi} \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap \Omega_k \setminus F_k) dt. \tag{2.3.12}$$

Now, since $|u^k| \geq r > 0$ in Ω_k by (2.3.8), it holds $D(\alpha \circ u^k) = \frac{-u_2^k Du_1^k + u_1^k Du_2^k}{|u^k|^2}$, so we see that $|D(\alpha \circ u^k)|\chi_{U_k} \leq | - u_2^k Du_1^k + u_1^k Du_2^k| r^{-2} \chi_{\Omega_k} \to | - u_2 Du_1 + u_1 Du_2| r^{-2}$ strongly in $L^1(\Omega)$, so that the left-hand side of (2.3.12) is converging to 0 since $\Omega_k \setminus F_k \subset \Omega \setminus F_k \downarrow \Omega \setminus \cup_k F_k$, that is negligible. This proves that the function $t \mapsto \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap \Omega_k \setminus F_k)$ is converging to 0 strongly in $L^1((0, 2\pi))$, and in particular tends to 0 for a.e. t. Let us denote by Σ_1 the subset of $(0, 2\pi)$ where pointwise convergence holds.

The left-hand side of (2.3.11) is uniformly bounded by a constant. Moreover the functions $t \mapsto \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap F_k)$ are increasing and pointwise converging to $t \mapsto \mathcal{H}^2(\{\alpha \circ u = t\} \cap \cup_k F_k)$ thanks to the fact that F_k are increasing and that $\{\alpha \circ u^k = t\} \cap F_k = \{\alpha \circ u = t\} \cap F_k$ by definition of u^k. So the Monotone

Convergence Theorem implies that the maps $t \mapsto \mathcal{H}^2(\{\alpha\circ u^k = t\}\cap\Omega_k\cap F_k)$ are converging to $t \mapsto \mathcal{H}^2(\{\alpha\circ u^k = t\}\cap(\cup_k\Omega_k)\cap(\cup_k F_k)) = \mathcal{H}^2(\{\alpha\circ u^k = t\}\cap(\cup_k F_k))$ strongly in $L^1((0,2\pi))$. Again there is a subset Σ_2 of $(0,2\pi)$ of full measure where pointwise convergence holds.

Writing

$$\{\alpha\circ u^k = t\}\cap\Omega_k = (\{\alpha\circ u^k = t\}\cap F_k)\bigcup(\{\alpha\circ u^k = t\}\cap\Omega_k\cap F_k^c), \quad (2.3.13)$$

we deduce from the observations made so far that if $t\in\Sigma_1\cap\Sigma_2$ is such that $\mathcal{H}^2(\{\alpha\circ u = t\}\cap(\cup_k F_k)) < +\infty$, then

$$\mathcal{H}^2(\{\alpha\circ u^k = t\}\cap F_k)\uparrow\mathcal{H}^2(\{\alpha\circ u = t\}\cap(\cup_k F_k)), \quad (2.3.14)$$

and

$$\mathcal{H}^2(\{\alpha\circ u^k = t\}\cap\Omega_k\cap F_k^c)\to 0. \quad (2.3.15)$$

In particular (2.3.13) implies that (2.3.9) holds and the proof is complete. □

Let us define $S^k := \Omega_k\cap\{\alpha\circ u^k = t\}$. Let $\gamma := \pi - t\in(0,2\pi)$, and consider the maps $\alpha^\gamma\circ u^k$, so that by the definition of α^γ the maps $\alpha^\gamma\circ u^k$ have a jump of high 2π on the set S^k. Hence the maps $(\alpha^\gamma\circ u^k)\chi_{\Omega_k}$ belong to $SBV(\Omega)$, being Sobolev on the set $\Omega_k\setminus S_k$.

Proposition 2.3.13. *The map $\alpha^\gamma\circ u$ belongs to $SBV(\Omega)$ and $(\alpha^\gamma\circ u^k)\chi_{\Omega_k}$ converges to $\alpha^\gamma\circ u$ strongly in $SBV(\Omega)$. Moreover it holds*

$$\mathcal{S}(\varphi) := \int_S \nu\cdot\varphi d\mathcal{H}^2 = \frac{1}{2\pi}\langle -u_2 Du_1 + u_1 Du_2, \varphi\rangle_\Omega + \frac{1}{2\pi}\langle\alpha^\gamma\circ u, \text{Div } \varphi\rangle_\Omega, \quad (2.3.16)$$

for all $\varphi\in C_c^\infty(\Omega,\mathbb{R}^3)$, where $S := \{\alpha\circ u = t\}\cap(\cup_k F_k)$.

Proof. The Divergence Theorem on the open set $\Omega_k\setminus S^k$ provides

$$\int_{S^k}[\alpha^\gamma\circ u^k]\varphi\cdot\nu d\mathcal{H}^2 + \int_{\partial N_k}\alpha^\gamma\circ u^k\varphi\cdot\nu d\mathcal{H}^2 = \langle\frac{-u_2^k Du_1^k + u_1^k Du_2^k}{|u^k|^2}, \varphi\rangle_{\Omega_k} + \langle\alpha^\gamma\circ u^k, \text{Div } \varphi\rangle_{\Omega_k}, \quad (2.3.17)$$

for all $\varphi\in C_c^\infty(\Omega,\mathbb{R}^3)$, where $[\alpha^\gamma\circ u^k] = (\alpha^\gamma\circ u^k)^+ - (\alpha^\gamma\circ u^k)^-$ is the jump of $\alpha^\gamma\circ u^k$ on the two faces of S^k, and ν is the normal to S^k pointing from the face with trace $(\alpha^\gamma\circ u^k)^+$ to the face with trace $(\alpha^\gamma\circ u^k)^-$, and let ν denote also the normal to ∂N_k pointing inside N_k. It is seen that the two traces on S^k are the two constants 0 and 2π, so assume $(\alpha^\gamma\circ u^k)^\pm = \pi\pm\pi$ and hence $[\alpha^\gamma\circ u^k] = 2\pi$. In particular

$$\int_{S^k}[\alpha^\gamma\circ u^k]\varphi\cdot\nu d\mathcal{H}^2 = 2\pi\int_{S^k}\varphi\cdot\nu d\mathcal{H}^2.$$

Now we define the distributions $\mathcal{S}^k$ and $\mathcal{N}^k$ by

$$\mathcal{S}^k(\varphi) := \int_{S^k}\varphi\cdot\nu d\mathcal{H}^2 \quad\text{and}\quad \mathcal{N}^k(\varphi) := \int_{\partial N_k}\alpha^\gamma\circ u^k\varphi\cdot\nu d\mathcal{H}^2,$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$. Let us first note that $\mathcal{S}^k$ and $\mathcal{N}^k$ are Radon measures, more precisely $\mathcal{S}^k = \nu \cdot \mathcal{H}^2 \llcorner_{S_k}$ and $\mathcal{N}^k = (\alpha \circ u^k)\nu \cdot \mathcal{H}^2 \llcorner_{\partial N_k}$. Thanks to the fact that $\alpha^\gamma \circ u^k$ is bounded, Lemma 2.3.11 shows that

$$\mathcal{N}^k \rightharpoonup 0 \quad \text{weakly* in } \mathcal{M}_b(\Omega, \mathbb{R}^3), \tag{2.3.18}$$

while Lemma 2.3.12 implies that there exists $\mathcal{S} \in \mathcal{M}_b(\Omega, \mathbb{R}^3)$ such that

$$\mathcal{S}^k \rightharpoonup \mathcal{S} \quad \text{weakly* in } \mathcal{M}_b(\Omega, \mathbb{R}^3), \tag{2.3.19}$$

as $k \to \infty$. Moreover we claim that $\mathcal{S} = \nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha\circ u = t\} \cap (\cup_k F_k)}$ and that the limit in the previous formula holds with respect to the strong topology of $\mathcal{M}_b(\Omega, \mathbb{R}^3)$. To see this we set $\mathcal{S}' = \nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha\circ u = t\} \cap (\cup_k F_k)}$ and using (2.3.9) we write

$$|\mathcal{S}' - \mathcal{S}^k| \leq |\nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha\circ u = t\} \cap (F \setminus F_k)}| + |\nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha \circ u^k = t\} \cap \Omega_k \cap F_k^c)}|,$$

where we have set $F := \cup_h F_h$, and both the terms in the right-hand side vanishes as $k \to \infty$. The first one vanishes since we can write $\mathcal{H}^2(\{\alpha \circ u = t\} \cap (F \setminus F_k)) = \mathcal{H}^2(\{\alpha \circ u = t\} \cap F - \mathcal{H}^2(\{\alpha \circ u^k = t\} \cap F_k)$ and use (2.3.14), the latter term vanishes thanks to (2.3.15). This proves that $\mathcal{S} = \mathcal{S}' = \nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha \circ u = t\} \cap F}$.

By definition of Ω_k we know that $|u^k| > r$ on Ω_k so that $\frac{-u_2^k Du_1^k + u_1^k Du_2^k}{|u^k|^2}\chi_{\Omega_k}$ converges to $\frac{-u_2 Du_1 + u_1 Du_2}{|u|^2} = -u_2 Du_1 + u_1 Du_2$ while $\alpha^\gamma \circ u^k$ converges to $\alpha^\gamma \circ u$ strongly in $L^1(\Omega)$. We can then pass to the limit as $k \to \infty$ in (2.3.17) and using (2.3.18) and (2.3.19) we obtain

$$2\pi \int_\Omega \varphi d\mathcal{S} = \langle -u_2 Du_1 + u_1 Du_2, \varphi \rangle_\Omega + \langle \alpha^\gamma \circ u, \text{Div } \varphi \rangle_\Omega, \tag{2.3.20}$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$. Thanks to the fact that $\mathcal{S} = \nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha\circ u = t\} \cap (\cup_k F_k)}$ the previous formula is equivalent to (2.3.16).

Formula (2.3.17) also shows that the absolutely continuous part of the gradient of $(\alpha^\gamma \circ u^k)\chi_{\Omega_k}$ is $\frac{-u_2^k Du_1^k + u_1^k Du_2^k}{|u^k|^2}\chi_{\Omega_k}$, and that the jump part is the measure $\pi \mathcal{S}_k + \mathcal{N}_k$. Now formula (2.3.16) shows that the distributional derivative of $\alpha^\gamma \circ u$ is given by the measure $\mathcal{S} - u_2 Du_1 + u_1 Du_2$. But we have proved so far that $D(\alpha^\gamma \circ u^k)\chi_{\Omega_k}$ converges to $\mathcal{S} - u_2 Du_1 + u_1 Du_2$ strongly as measure, so $(\alpha^\gamma \circ u^k)\chi_{\Omega_k}$ converges to $\alpha^\gamma \circ u$ strongly in $SBV(\Omega)$. □

Lemma 2.3.14. *The measure $\mathcal{S} = \nu \cdot \mathcal{H}^2 \llcorner_{\{\alpha\circ u = t\} \cap (\cup_k F_k)}$ is a 2-dimensional integral current, i.e., $\partial \mathcal{S}$ is a 1-integer multiplicity current.*

Proof. The fact that $\mathcal{S}$ is a rectifiable current is straightforward and it has finite mass (formula (2.3.9)). Moreover, from (2.3.16), we can write

$$\partial \mathcal{S}(\varphi) = \int_S \nu \cdot \text{ Curl } \varphi d\mathcal{H}^2 = \frac{1}{2\pi} \langle -u_2 Du_1 + u_1 Du_2, \text{ Curl } \varphi \rangle_\Omega, \tag{2.3.21}$$

where we have identify the 1-form $\varphi_i dx_i$ with the vector field φ. So from (2.3.6) it follows that $\partial \mathcal{S} \in \mathcal{M}_b(\Omega, \mathbb{R}^3)$, and in particular it has finite mass. Now Theorem 2.3.2 implies that $\partial \mathcal{S}$ is an integer multiplicity current, and the thesis follows. □

The proof of Theorem 2.3.9 is now complete thanks to formula (2.3.21) that shows that Curl $(D(\alpha \circ u))$ = Curl $(-u_2 Du_1 + u_1 Du_2) = 2\pi \partial \mathcal{S}$, with $\partial \mathcal{S}$ an integral 1-current.

2.4 Functional properties of the strain in the presence of dislocations

2.4.1 L^p-fields with bounded measure curl

Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. We introduce the vector space of tensor-valued fields

$$\mathcal{BC}^p(\Omega, \mathbb{R}^{3\times 3}) := \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) \text{ s.t. Curl } F \in \mathcal{M}_{\text{div}}(\Omega, \mathbb{R}^{3\times 3})\}, \quad (2.4.1)$$

which, as endowed with norm

$$\|F\|_{\mathcal{BC}^p} := \|F\|_p + |\operatorname{Curl} F|(\Omega), \quad (2.4.2)$$

turns out to be a Banach space. Here the curl of $F \in \mathcal{BC}^p(\Omega, \mathbb{R}^{3\times 3})$ is intended in the sense of distributions.

Remark 2.4.1. The "antinormal" tensor $F \times N = (F\tau_A) \otimes \tau^B - (F\tau_B) \otimes \tau^A$ is distinct from the tangent projection $F - FN \otimes N = (F\tau_A) \otimes \tau^A + (F\tau_B) \otimes \tau^B$ with (τ^A, τ^B) the 2 tangent vectors of $\partial\Omega$.

Helmholtz decomposition for tensor fields. The following Lemma is a direct tensor extension of the theorems of existence and uniqueness of Neumann problem as shown in [72] (see also [30, Lemma III.1.2 and Theorem III.1.2]).

Lemma 2.4.2. *Let $G \in L^p(\Omega, \mathbb{R}^{3\times 3})$ with $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected open set with boundary of class C^1. Then there exists a unique solution (up to a constant) $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ of*

$$\begin{cases} -\Delta u &= \operatorname{Div} G \quad \text{in } \Omega \\ \partial_N u &= -GN \quad \text{on } \partial\Omega. \end{cases} \quad (2.4.3)$$

Moreover the solution satisfies $\|Du\|_p \leq C\|G\|_p$.

Let us remark that equation (2.4.3) is a formal strong form meaning that the following weak form is solved (see [80]):

$$-\langle \nabla u, \nabla \varphi \rangle = \langle G, \nabla \varphi \rangle \quad \forall \varphi \in W^{1,p'}(\Omega, \mathbb{R}^{3\times 3}). \quad (2.4.4)$$

Indeed GN is not well defined on $\partial\Omega$. This issue will be addressed by Lemma 2.4.3.

Let us define

$$\begin{aligned} L^p_{\text{div}}(\Omega, \mathbb{R}^{3\times 3}) &:= \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) \quad \text{s.t.} \quad \operatorname{Div} F = 0\} \\ &= \operatorname{adh}_{L^p}\{F \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3}) \quad \text{s.t.} \quad \operatorname{div} F = 0\}, \quad (2.4.5) \\ L^p_{\text{curl}}(\Omega, \mathbb{R}^{3\times 3}) &:= \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) \quad \text{s.t.} \quad \operatorname{Curl} F = 0\} \\ &= \operatorname{adh}_{L^p}\{F \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3}) \quad \text{s.t.} \quad \operatorname{curl} F = 0\}. \quad (2.4.6) \end{aligned}$$

Let $1 < p < \infty$. Let $V \in L^p(\Omega, \mathbb{R}^3)$ with $\operatorname{Div} V \in L^p(\Omega, \mathbb{R})$, then there exists $VN \in W^{-1/p,p}(\partial\Omega) := \left(W^{1/p,p'}(\partial\Omega)\right)'$. Moreover, if $V \in L^p(\Omega, \mathbb{R}^3)$ with $\operatorname{Curl} V \in L^p(\Omega, \mathbb{R}^3)$, there exists $V \times N \in W^{-1/p,p}(\partial\Omega, \mathbb{R}^3)$. These properties straightforwardly apply to tensor-valued maps, where VN (componentwise,

$V_{ij}N_j$) and $V \times N$ (componentwise, $\epsilon_{jlp}V_{il}N_p$) mean, with an abuse of notation, the bounded normal and antinormal traces of V in $W^{-1/p,p}(\partial\Omega,\mathbb{R}^3)$ and $W^{-1/p,p}(\partial\Omega,\mathbb{R}^{3\times 3})$, respectively. In particular, these traces are well defined for tensors belonging to the spaces $L^p_{\rm div}(\Omega,\mathbb{R}^{3\times 3})$ and $L^p_{\rm curl}(\Omega,\mathbb{R}^{3\times 3})$ (see [41] and references therein). Specifically, the following Lemma holds.

Lemma 2.4.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded open set with boundary of class C^1 and let $F\in L^p(\Omega,\mathbb{R}^{3\times 3})$ be such that* Div $F\in L^p(\Omega,\mathbb{R}^3)$. *Then there exists $FN\in W^{-1/p,p}(\partial\Omega,\mathbb{R}^3):=\left(W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)\right)'$ such that*

$$\langle FN,\gamma(\varphi)\rangle := \langle \,\mathrm{Div}\ F,\varphi\rangle + \langle F, D\varphi\rangle \tag{2.4.7}$$

for all $\varphi\in W^{1,p'}(\Omega,\mathbb{R}^3)$, with $\gamma(\varphi)\in W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)$ the boundary trace of φ.

Proof. Let us define the linear functional on $W^{1/p,p'}(\partial\Omega)$ by

$$\langle FN,\gamma\rangle := \langle \,\mathrm{Div}\ F,\varphi\rangle + \langle F, D\varphi\rangle,$$

where $\varphi\in W^{1,p'}(\Omega,\mathbb{R}^3)$ has trace γ on $\partial\Omega$. First observe that FN does not depend on the particular φ chosen as extension of $\gamma(\varphi)$. If φ_1,φ_2 are two such extensions, then their difference has zero trace and

$$0 = \langle \,\mathrm{Div}\ F,\varphi_1-\varphi_2\rangle + \langle F, D(\varphi_1-\varphi_2)\rangle,$$

by definition of the distributional divergence. Thus we chose a lifting operator $\mathcal{L}_{\partial\Omega}: W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)\to W^{1,p'}(\Omega,\mathbb{R}^3)$ so that by its linearity and continuity (cf. [23]), it holds

$$\begin{aligned} |\langle FN,\gamma\rangle| &\le C\left(\|F\|_p + \|\,\mathrm{Div}\ F\|_{L^p}\right)\|\mathcal{L}_{\partial\Omega}(\gamma)\|_{W^{1,p'}(\Omega)} \\ &\le C\left(\|F\|_p + \|\,\mathrm{Div}\ F\|_{L^p}\right)\|\gamma\|_{W^{1/p,p'}(\partial\Omega)}, \end{aligned}$$

achieving the proof. □

A similar proof provides the following:

Lemma 2.4.4. *Let $\Omega\subset\mathbb{R}^3$ be a bounded open set with boundary of class C^1 and let $F\in L^p(\Omega,\mathbb{R}^{3\times 3})$ be such that* Curl $F\in L^p(\Omega,\mathbb{R}^{3\times 3})$. *Then there exists $F\times N\in W^{-1/p,p}(\partial\Omega,\mathbb{R}^3):=\left(W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)\right)'$ such that*

$$\langle F\times N,\gamma(\varphi)\rangle := \langle \,\mathrm{Curl}\ F,\varphi\rangle - \langle F,\ \mathrm{Curl}\ \varphi\rangle \tag{2.4.8}$$

for all $\varphi\in W^{1,p'}(\Omega,\mathbb{R}^3)$, with $\gamma(\varphi)\in W^{1/p,p'}(\partial\Omega,\mathbb{R}^3)$ the boundary trace of φ.

Let us introduce the spaces

$$\mathcal{V}^p(\Omega) := \left\{V\in L^p_{\rm div}(\Omega,\mathbb{R}^{3\times 3}):\ \mathrm{Curl}\ V\in L^p(\Omega,\mathbb{R}^{3\times 3}), V\times N=0 \text{ on } \partial\Omega\right\}, \tag{2.4.9}$$

$$\tilde{\mathcal{V}}^p(\Omega) := \left\{V\in L^p_{\rm div}(\Omega,\mathbb{R}^{3\times 3}):\ \mathrm{Curl}\ V\in L^p(\Omega,\mathbb{R}^{3\times 3}), VN=0 \text{ on } \partial\Omega\right\}. \tag{2.4.10}$$

Lemma 2.4.5. *Let $\Omega\subset\mathbb{R}^3$ be a bounded open set with boundary of class C^1 and $V\in\mathcal{V}^p(\Omega)$. Then* (Curl $V)N=0$ *in the sense of Lemma 2.4.3.*

Proof. Take any $\varphi \in W^{1,p'}(\Omega, \mathbb{R}^3)$. By part integration (equations (2.4.7) and (2.4.8)), it holds

$$\langle (\operatorname{Curl} V)N, \varphi\rangle_{\partial\Omega} = \langle \operatorname{Curl} V, D\varphi\rangle = \langle V \times N, D\varphi\rangle_{\partial\Omega} = 0.$$

Since φ is arbitrary, the proof is achieved. □

The following estimate can be found in [41].

Lemma 2.4.6 (Kozono-Yanagisawa)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with boundary of class C^1 and assume $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then $F \in W^{1,p}(\Omega, \mathbb{R}^{3\times 3})$ and it holds*

$$\|\nabla F\|_p \le C\left(\|\operatorname{Curl} F\|_p + \|F\|_p\right). \tag{2.4.11}$$

This shows that $\mathcal{V}^p(\Omega)$ and $\tilde{\mathcal{V}}^p(\Omega)$ are closed subspaces in $W^{1,p}(\Omega, \mathbb{R}^{3\times 3})$. In simply connected and bounded domains the following better estimate can be obtained (see [79]). Note that this is a classical result for smooth functions with compact support.

Lemma 2.4.7 (von Wahl)**.** *Let Ω be a simply connected and bounded domain and let $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then it holds*

$$\|\nabla F\|_p \le C\|\operatorname{Curl} F\|_p. \tag{2.4.12}$$

As a direct consequence the following result holds.

Lemma 2.4.8. *Let $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then* $\operatorname{Curl} F = 0 \iff F = 0$.

We remark that, when $F \in \tilde{\mathcal{V}}^p(\Omega)$, Lemma 2.4.8 amounts to proving the uniqueness property of Lemma 2.4.2. Moreover, in [41], a more general statement is established without the simply connectedness assumption. In general, for Ω a smooth and bounded subset of $\mathbb{R}^3$, $\operatorname{Curl} F = \operatorname{Div} F = 0$, coupled with the boundary condition $V \times N = 0$ or $VN = 0$, has a non-trivial solution. In particular Kozono and Yanagisawa [80] show that the solutions belong to a subspace of $C^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3})$ with positive finite dimension, depending on the Betti number of Ω.

The following result is well known in the Hilbertian case L^2 but is not classical for the general Banach space L^p. It is basically proven with help of Lemma 2.4.2 (for a complete proof see [41, 80], cf. also [30, 55]).

Theorem 2.4.9 (Helmholtz-Weyl-Hodge-Yanagisawa)**.** *Let $1 < p < \infty$ and let Ω be a bounded, simply connected and smooth open set in $\mathbb{R}^3$. For every $F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, then the two following statements hold true:*

(i) There exist $u_0 \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ and $V \in \tilde{\mathcal{V}}^p(\Omega)$, such that

$$F = Du_0 + \operatorname{Curl} V. \quad \Big(L^p(\Omega, \mathbb{R}^{3\times 3}) = \nabla W_0^{1,p}(\Omega, \mathbb{R}^3) \oplus \operatorname{Curl} \tilde{\mathcal{V}}^p(\Omega)\Big) \tag{2.4.13}$$

(ii) There exist $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $\partial_N u = FN$ on $\partial\Omega$, and $V_0 \in \mathcal{V}^p(\Omega)$, such that

$$F = Du + \operatorname{Curl} V_0. \quad \big(L^p(\Omega, \mathbb{R}^{3\times 3}) = \nabla W^{1,p}(\Omega, \mathbb{R}^3) \oplus \operatorname{Curl} \mathcal{V}^p(\Omega)\big) \tag{2.4.14}$$

Moreover the decompositions are unique, in the sense that u_0*,* V*,* V_0 *are uniquely determined, while* u *is unique up to a constant, and it holds* $\|Du_0\|_p, \|Du\|_p \leq C\|F\|_p$*, respectively.*

By Lemma 2.4.5, the potential u of (2.4.14) is found by solving (2.4.3) with $G = -F$, this also gives a meaning to the condition $\partial_N u = FN$.

Remark 2.4.10. When F is smooth with compact support, decompositions such as (2.4.13) and (2.4.14) are classically given by Stokes theorem and explicit formulae involving the divergence and the curl of F (see [79], [9]). Notice that no boundary data for F is here given.

Remark 2.4.11. Let $F \in C^1(\Omega, \mathbb{R}^{3\times3})$. In the particular case $\mathrm{Curl}\ F = 0$ the Helmholtz decomposition is trivial if Ω is a simply connected domain. Indeed it is well-known that in such a case there exists $u \in C^2(\Omega, \mathbb{R}^3)$ satisfying $F = Du$. This result extends for $F \in L^p$ with $1 < p < +\infty$ as shown in [30]. See [41] for a complete treatment of Helmholtz decomposition in L^p, relying on the pioneer paper [29]. Moreover, if $\mathrm{Div}\ F = 0$ then, by Theorem 2.4.9, $F = \mathrm{Curl}\ V$ with $V \in \widetilde{\mathcal{V}}^p(\Omega)$. We remark that for smooth functions F, this result holds for any simply connected domain with Lipschitz boundary.

Remark 2.4.12. Smoothness of the boundary is a strong requirement which is needed for the following reason: (2.4.13) and (2.4.14) require in principle to solve a Poisson equation $\Delta u = \mathrm{Div}\ F$ with the right-hand side in some distributional (viz., Sobolev-Besov) space for which smoothness of the boundary is needed. It is known [25] that for a Lipschitz boundary the solution holds for restricted p (namely $3/2 - \epsilon \leq p \leq 3 + \epsilon$) for some $\epsilon = \epsilon(\Omega) > 0$. Note that for $p = 2$ a Lipschitz boundary would be sufficient.

Invertibility of the curl. A key equation behind the results of this work is the following system:

$$\left\{ \begin{array}{rcll} -\mathrm{Curl}\ F & = & \mu^{\mathrm{T}} & \text{in } \Omega \\ \mathrm{Div}\ F & = & 0 & \text{in } \Omega \\ FN & = & 0 & \text{on } \partial\Omega, \end{array} \right. , \tag{2.4.15}$$

with μ^{T} a Radon measure in $\mathcal{M}_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$. Existence and uniqueness of a solution is given by the following Theorem 2.4.13, for which Lemma 2.4.2 (or Lemma 2.4.8) will be required. Let us introduce the following linear subspace of $\mathcal{BC}^p(\Omega, \mathbb{R}^{3\times3})$:

$$\mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3}) := \{F \in \mathcal{BC}^p(\Omega, \mathbb{R}^{3\times3}) : \mathrm{Div}\ F = 0 \text{ and } FN = 0 \text{ on } \partial\Omega\}. \tag{2.4.16}$$

In the last definition the divergence is intended in the sense of distributions. The following results generalize to the case of measures the result in [11].

Theorem 2.4.13 (Biot-Savart)**.** *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded and simply connected open set with boundary of class* C^1*. Let* μ *be a tensor-valued Radon measure such that* $\mu^{\mathrm{T}} \in \mathcal{M}_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$*, and let it be extended by zero outside* Ω*. Then there exists a unique* F *in* $\mathcal{BC}^1_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$ *solution to* (2.4.15)*. Moreover* F *belongs to* $\mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$ *for all* p *with* $1 \leq p < 3/2$ *and for all such* p *there exists a constant* $C > 0$ *satisfying*

$$\|F\|_p \leq C|\mu|(\Omega). \tag{2.4.17}$$

Proof. **Step 1.** Let Φ be the fundamental solution of the Laplacian in $\mathbb{R}^3$ (i.e., $\Delta\Phi = \delta_0$) and let $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. It holds

$$\begin{aligned}\int_{\mathbb{R}^3} \varphi(x) d\mu_{ij}(x) &= \langle \mu_{ij}, \varphi \rangle = \langle \mu_{ij}^x, \langle \Delta\Phi(x-\cdot), \varphi(\cdot)\rangle\rangle \\ &= \langle \mu_{ij}^x, \langle D\Phi(x-\cdot), D\varphi(\cdot)\rangle\rangle, \end{aligned} \tag{2.4.18}$$

Here the subscript x means that the field on which it is appended is a function of x.

Let $\bar\varphi(x) := D\Phi \star \varphi(x) = \int_{\mathbb{R}^3} D\Phi(x-\xi)\varphi(\xi)d\xi = -\int_{\mathbb{R}^3} D\Phi(\xi - x)\varphi(\xi)d\xi = -\int_{\mathbb{R}^3} D\Phi(y)\varphi(x+y)dy \in \mathcal{C}_0^1(\mathbb{R}^3, \mathbb{R}^3)$, where we have used the odd and asymptotically decreasing properties of $D\Phi$. By solenoidal property of μ^{T}, one has $\langle \mu_{ij}, D_i\bar\varphi_k\rangle = 0$, and recalling that $\epsilon_{iqm}\epsilon_{mkl} = \delta_{ik}\delta_{ql} - \delta_{il}\delta_{qk}$ while $D\Phi(\xi - x) = -D\Phi(x-\xi)$, we rewrite (2.4.18) as

$$\begin{aligned}\langle \mu_{ij}, \varphi\rangle &= -\langle \mu_{kj}, \epsilon_{iqm}\epsilon_{mkl} D_q\bar\varphi_l\rangle = \langle \mu_{kj}^x, \epsilon_{iqm}\epsilon_{mkl}\langle D_l\Phi(\cdot), D_q\varphi(\cdot + x)\rangle\rangle \\ &= \epsilon_{iqm}\epsilon_{mkl}\int_{\mathbb{R}^3} d\mu_{kj}(x)\int_{\mathbb{R}^3} D_l\Phi(y) D_q\varphi(x+y)dy \\ &= \epsilon_{iqm}\epsilon_{mkl}\int_{\mathbb{R}^3} D_l\Phi(\xi - x)d\mu_{kj}(x)\int_{\mathbb{R}^3} D_q\varphi(\xi)d\xi,\end{aligned}$$

which by definition of the convolution between distributions (cf. [71, Théorème 1,VI,2;5]) reads $\langle \epsilon_{mkl}D_l\Phi \star \mu_{kj}, \epsilon_{iqm}D_q\varphi\rangle = \langle \epsilon_{iqm}D_q\left(\epsilon_{mlk}D_l\Phi \star \mu_{kj}\right), \varphi\rangle$, implying that

$$\mu_{ij} = \epsilon_{iqm}D_q\left(\epsilon_{mlk}D_l\Phi \star \mu_{kj}\right), \tag{2.4.19}$$

as a distribution.

Therefore the solution G writes componentwise as

$$G_{jm}(x) := -\epsilon_{mlk}\left(D_l\Phi \star \mu_{kj}\right)(x) = \int_{\mathbb{R}^3} \epsilon_{mlk} D_l\Phi(\xi - x)d\mu_{kj}(\xi), \tag{2.4.20}$$

and satisfies, by (2.4.19),

$$-\operatorname{Curl} G = \mu^{\mathrm{T}} \quad \text{in} \quad \Omega. \tag{2.4.21}$$

Step 2. First observe that $D\Phi \in L^p(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ with $1 \le p < 3/2$, since $D\Phi(x) = O(|x|^{-2})$, and hence, posing $R := |x-y|$, while $\bar R$ is the radius of a ball centered in 0 and containing $\bar\Omega$,

$$\|D\Phi\|_p^p \le \int_0^{\bar R} R^{-2p} R^2 dR, \tag{2.4.22}$$

where the last factor in the right-hand side is bounded as long as $1 \le p < 3/2$. The boundedness in L^p (hence the continuity) now follows from Minkowski's inequality since for some $C > 0$, it holds (see also [71, VI.I;4]),

$$\|G\|_p \le C|\mu|(\Omega).$$

Now, taking $\psi \in W^{1,p'}(\mathbb{R}^3, \mathbb{R}^3))$, we have

$$\begin{aligned}\langle G_{jm}, D_m\psi_j\rangle &= \langle \epsilon_{mlk}\langle D_l^y\Phi(\xi - x), \mu_{kj}(\xi)\rangle, D_m\psi_j(x)\rangle \\ &= -\int_{\mathbb{R}^3} \epsilon_{mlk}\langle D_l\Phi(\xi - x), D_m\psi_j(x)\rangle d\mu_{kj}(\xi) \\ &= \int_{\mathbb{R}^3} \epsilon_{mlk}\langle D_m D_l\Phi(\xi - x), \psi_j(x)\rangle d\mu_{kj}(\xi) = 0,\end{aligned}$$

where the last equality follows from the smoothness of Φ. In other words $\mathrm{Div}\ G = 0$.

Step 3. Let us now prove that $FN = 0$. By Lemma 2.4.3 (with Ω in place of Ω), since $\mathrm{Div}\ G = 0$ the normal trace of G, denoted as GN, exists as an element of $W^{-1/p,p}(\partial\Omega, \mathbb{R}^3)$ and satisfies

$$\langle GN, \varphi\rangle_{\partial\Omega} := \langle G, \nabla\varphi\rangle + \langle \mathrm{Div}\ G, \varphi\rangle = \langle G, \nabla\varphi\rangle \quad \forall\varphi \in W^{1,p'}(\mathbb{R}^3, \mathbb{R}^3) \tag{2.4.23}$$

If $\phi \in W^{1,p}(\Omega)$ we have that $F := G + D\phi$ also satisfies $-\mathrm{Curl}\ F = \mu^{\mathrm{T}}$. Thus, Lemma 2.4.2 (with Ω in place of Ω) provides a solution ϕ such that $\mathrm{Div}\ F = 0$ in Ω, $FN = 0$ on $\partial\Omega$, and such that (2.4.17) holds.

Step 4. We now prove the uniqueness of solution. Assume that there exist two solutions and denote by $H \in L^p$ their difference, one has $\mathrm{Curl}\ H = \mathrm{Div}\ H = 0$ in Ω while $HN = 0$ on $\partial\Omega$. From Remark 2.4.11 there exists $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $H = \nabla u$. Taking the divergence one gets $\Delta u = 0$ in Ω. Moreover from $HN = 0$ we also have $\partial_N u = 0$ on $\partial\Omega$. By Lemma 2.4.2 this implies that there is a constant c with $u \equiv c$ in Ω, whereby $H = 0$, achieving the proof of uniqueness. The proof is complete. □

By uniqueness, there exists a linear one-to-one and onto correspondence between $\nu \in \mathcal{M}_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$ and $F \in \mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$. Thus the map

$$\mathrm{Curl}^{-1} : \mathcal{M}_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3}) \to \mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3}), \quad \nu \mapsto F = -\mathrm{Curl}^{-1}(\nu) \tag{2.4.24}$$

is well defined and linear. Therefore, we may write

$$\mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3}) := \mathrm{Curl}^{-1}\left(\mathcal{M}_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})\right). \tag{2.4.25}$$

Moreover, for any $F \in \mathcal{BC}^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times3})$ we recover by Eq. (2.4.17) the L^p-counterpart of Maxwell relation in L^2 [55], that is,

$$\|F\|_p \le C|\,\mathrm{Curl}\ F|(\Omega). \tag{2.4.26}$$

Remark 2.4.14. In case Ω is not simply connected the uniqueness of solution of problem (2.4.15) does not hold. In such a case, Lemma 2.4.8 would also not hold, since the problem might exhibit non-trivial solutions, as shown in [80].

2.4.2 Harmonic maps with prescribed jump on a surface

Lemma 2.4.15. *Let C be a smooth closed curve in $\mathbb{R}^3$ and let S be a smooth bounded surface with boundary C and unit normal N. Let $b \in \mathbb{R}$. The solution of*

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \setminus S \\ [u] := u^+ - u^- = b & \text{on} \quad S \\ [\partial_N u] := \partial_N u^+ - \partial_N u^- = 0 & \text{on} \quad S \end{cases} \tag{2.4.27}$$

is given by (up to a harmonic map on $\mathbb{R}^3$)

$$u(x) = -b\int_S \partial_N \Gamma(x' - x)\,dS(x'), \tag{2.4.28}$$

for $x \in \mathbb{R}^3 \setminus S$, where Γ is the solution in $\mathbb{R}^3$ of $\Delta\Gamma = \delta_0$.

Proof. Let S_V be a thin ellipsoid enclosing S and V the set containing S whose boundary is S_V with outer unit normal N. Let u be an arbitrary smooth real function. We have the identities in V

$$\int_V \partial_k(\partial_l u(x')\Gamma(x'-x))dx' = \int_{S_V} \partial_l u(x')\Gamma(x'-x)N_k(x')dS(x')$$

and

$$\int_V \partial_l(u(x')\partial_k\Gamma(x'-x))dx' = \int_{S_V} u(x')\partial_k\Gamma(x'-x)N_l(x')dS(x').$$

Thus by subtraction it holds

$$\int_V \partial_k\partial_l u(x')\Gamma(x'-x))dx' - \int_V u(x')\partial_k\partial_l\Gamma(x'-x))dx' =$$
$$\int_{S_V} (\partial_l u(x'))^- \Gamma(x'-x)N_k(x')dS(x') - \int_{S_V} u_i^-(x')\partial_k\Gamma(x'-x)N_l(x')dS(x').$$

Moreover, the same identities in $\mathbb{R}^3 \setminus \bar{V}$ yield

$$\int_{\mathbb{R}^3\setminus\bar{V}} \partial_k\partial_l u(x')\Gamma(x'-x))dx' - \int_{\mathbb{R}^3\setminus\bar{V}} u(x')\partial_k\partial_l\Gamma(x'-x))dx' =$$
$$-\int_{S_V} (\partial_l u(x'))^+ \Gamma(x'-x)N_k(x')dS(x') + \int_{S_V} u_i^+(x')\partial_k\Gamma(x'-x)N_l(x')dS(x').$$

and hence, by summing,

$$\int_{\mathbb{R}^3\setminus S_V} \partial_k\partial_l u(x')\Gamma(x'-x))dx' - \int_{\mathbb{R}^3\setminus S_V} u(x')\partial_k\partial_l\Gamma(x'-x))dx' =$$
$$-\int_{S_V} [\partial_l u(x')]\Gamma(x'-x)N_k(x')dS(x') + \int_{S_V} [u(x')]\partial_k\Gamma(x'-x)N_l(x')dS(x').$$

Contracting with δ_{kl} yields

$$\int_{\mathbb{R}^3\setminus S_V} \Delta u(x')\Gamma(x'-x))dx' - \int_{\mathbb{R}^3\setminus S_V} u(x')\Delta\Gamma(x'-x))dx' =$$
$$-\int_{S_V} [\partial_N u(x')]\Gamma(x'-x)dS(x') + \int_{S_V} [u(x')]\partial_N\Gamma(x'-x)dS(x'), \tag{2.4.29}$$

that is, for $x \in \mathbb{R}^3 \setminus S_V$,

$$\int_{\mathbb{R}^3\setminus S_V} \Delta u(x')\Gamma(x'-x))dx' - u(x) =$$
$$-\int_{S_V} [\partial_N u(x')]\Gamma(x'-x)dS(x') + \int_{S_V} [u(x')]\partial_N\Gamma(x'-x)dS(x'). \tag{2.4.30}$$

Since u is arbitrary, let us set

$$u = -b\int_S \partial_N\Gamma(y-\cdot)dS(y),$$

so that u is seen to be harmonic in $\mathbb{R}^3 \setminus S$, $\Delta u(x) = 0$ for $x \in \mathbb{R}^3 \setminus S$, and hence, by (2.4.30)

$$u(x) = \int_{S_V} [\partial_N u(x')]\Gamma(x'-x)dS(x') - \int_{S_V} [u(x')]\partial_N \Gamma(x'-x)dS(x'),$$

for all $x \in \mathbb{R}^3 \setminus S_V$. Consider now any smooth tensor function with compact support φ in place of the tensor Γ. By (2.4.29), it holds

$$\begin{aligned} \int_{\mathbb{R}^3\setminus S_V} u(x')\Delta\varphi(x')dx' &= \int_{\mathbb{R}^3} u(x')\Delta\varphi(x')dx' \\ = \int_{S_V} [\partial_N u(x')]\varphi(x')dS(x') &- \int_{S_V} [u(x')]\partial_N \varphi(x')dS(x'). \quad (2.4.31) \end{aligned}$$

Let S_V^+ and S_V^- be the upper and lower faces of S_V with respect to S. Define the distribution $\gamma[b]$ concentrated on S_V^+ as

$$\langle \gamma[b], \varphi \rangle := -b \int_{S_V^+} \partial_N \varphi(y) d\mathcal{H}^2(y).$$

By definition, $u(x) = -b \int_{S_V^+} \partial_N \Gamma(x-y)d\mathcal{H}^2(y) = -\langle \gamma[b], \Gamma(x-\cdot)\rangle$. Observe that

$$\Delta u = -\gamma[b]$$

holds in the distribution sense, since for any smooth test function with compact support φ, by definition of the convolution between distributions [71], it holds

$$\begin{aligned} \langle u, \Delta\varphi \rangle = \langle \Delta u, \varphi \rangle &= -\langle \langle \gamma[b], \Gamma(x-\cdot)\rangle, \Delta\varphi(x)\rangle = -\langle \gamma[b], \langle \Delta\Gamma(x-\cdot), \varphi(x)\rangle\rangle \\ &= -\langle \gamma[b], \varphi \rangle. \quad (2.4.32) \end{aligned}$$

Subtracting (2.4.32) from (2.4.31) yields

$$0 = \int_{S_V} [\partial_N u(x')]\varphi(x')dS(x') - \int_{S_V^+} \left([u(x')] - b\right) \partial_N \varphi(x')dS(x'),$$

thereby proving that

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \setminus S_V^+ \\ u^+ - u^- = b & \text{on} \quad S_V^+ \\ \partial_N u^+ - \partial_N u^- = 0 & \text{on} \quad S_V^+ \end{cases}.$$

Now, letting $S_V^+ = S$ (while the lower face S_V^- can be arbitrarily located below S) entails that

$$u(x) = -b \int_S \partial_N \Gamma(x'-x)dS(x')$$

satisfies (2.4.27), achieving the proof. □

By (2.4.28), it holds

$$\partial_i u(x) \quad = \quad -b \int_S \left(\frac{N_i}{|x-x'|^3} - 3\frac{N \cdot (x-x')(x_i - x_i')}{|x-x'|^5} \right) dS'. \quad (2.4.33)$$

Lemma 2.4.16. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Let C be a closed Lipschitz curve in Ω and let $b \in 2\pi\mathbb{Z}^3$. Then for any Lipschitz surface S with boundary C, if $u \in BV^p(\Omega, \mathbb{R}^3)$ has components u_i satisfying* (2.4.27) *with $b = b_i$, then* $\text{Div } \nabla u = 0$ *and* $-\text{ Curl } \nabla u = b \otimes \mathcal{L}$ *as distributions, with ∇u the part of the gradient of u that is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let u be a solution to (2.4.27). Then $\nabla u \in L^p(\Omega, \mathbb{R}^{3\times 3})$. It has been shown that u is smooth outside S where it has a jump of amplitude b. In particular u belongs to $SBV(\Omega, \mathbb{R}^3)$ and its distributional derivative is given by

$$\langle Du, \varphi \rangle := -\langle u, \text{ Div } \varphi \rangle = S(\varphi) + \langle \nabla u, \varphi \rangle, \tag{2.4.34}$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^{3\times 3})$, where S is the distribution $S(\varphi) = -\int_S N_j b_i \varphi_{ij} d\mathcal{H}^2$.

Let us prove that $-\text{ Curl } \nabla u = \mathcal{L} \otimes b$. To this aim let us take $\psi \in \mathcal{D}(\Omega, \mathbb{R}^{3\times 3})$ and write

$$\begin{aligned} -\langle \text{ Curl } \nabla u, \psi \rangle &:= -\langle \nabla u, \text{ Curl } \psi \rangle = -\langle Du, \text{ Curl } \psi \rangle + S(\text{ Curl } \psi) \\ &= \int_C \tau_j b_i \psi_{ij} d\mathcal{H}^1 = b \otimes \mathcal{L}(\psi), \end{aligned}$$

where the second equality follows from (2.4.34) with $\varphi = \text{ Curl } \psi$, and the third one by Stokes theorem.

We now prove that $\text{Div } \nabla u = 0$. Again, we take $\psi \in \mathcal{D}(\Omega, \mathbb{R}^3)$ and write

$$-\langle \text{ Div } \nabla u, \psi \rangle := \langle \nabla u, \nabla \psi \rangle = \langle Du, \nabla \psi \rangle - S(\nabla \psi), \tag{2.4.35}$$

and using the explicit formula (2.4.28) for u we obtain

$$\begin{aligned} &\langle Du, \nabla \psi \rangle = b_k \langle D_i \int_S \partial'_N \Gamma'_k (x' - \cdot) dS(x'), D_i \psi \rangle = \\ &- \int_S b_k \langle \Delta \Gamma'_k (x' - \cdot), D_j \psi N_j \rangle dS(x') = -b_k \int_S \partial'_N \psi_k (x') dS(x') = S(\nabla \psi), \end{aligned}$$

where $\Gamma'(x')(x) := \Gamma(x - x')$ for $x \in \mathbb{R}^3$. So that plugging the last identity in (2.4.35) we obtain $\text{Div } \nabla u = 0$. ☐

In order to prove that $\text{Div } \nabla u = 0$, we might also argue as follows. Let $\hat{S} \supset S$ such that $\hat{S}$ separates Ω in two parts Ω^- and Ω^+. Then for every test function $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$ it holds

$$\begin{aligned} &\int_\Omega \nabla u \nabla \varphi dx = \int_{\Omega^+} \nabla u \nabla \varphi dx + \int_{\Omega^-} \nabla u \nabla \varphi dx = \\ &- \int_{\Omega^+} \text{ Div } \nabla u \varphi dx - \int_{\Omega^-} \text{ Div } \nabla u \varphi dx + \int_{\hat{S}^+} \partial_N u^+ \nabla \varphi dx - \int_{\hat{S}^-} \partial_N u^- \varphi dx = 0. \end{aligned}$$

Remark 2.4.17. The statement of Lemma 2.4.16 readily applies to the case of C being a finite union of Lipschitz curves.

Lemma 2.4.18. *Let C and S be as in Lemma 2.4.15 and u be the explicit solution of* (2.4.27) *given by* (2.4.28)*. Then it holds*

$$|\partial_i u(x)| \leq 8\pi \frac{|b|}{d(x, C)}. \tag{2.4.36}$$

Proof. Let us first prove that the value of the derivative $\partial_i u(x)$ does not depend on the surface S appearing in (2.4.27). Let indeed S' be another smooth surface that does not contain the point x and has C as boundary. For simplicity let us suppose it is disjoint from S. Let u' be the solution of (2.4.27) with S' replacing S and let A be the open set enclosed by S and S'. By formula (2.4.28), $(u-u')(x) = c + b\int_{\partial A}\partial'_N\Gamma(x'-x)dS(x') = c + b\chi_A(x)$, the first equality being a consequence of the Divergence theorem. In particular we see that $u-u'$ is constant in a neighborhood of x, so that $\partial_i u(x) = \partial_i u'(x)$. By approximation, we can also extends this to the case of Lipschitz surface S', and then to every rectifiable current S' with $\partial S' = C$ and whose support is outside a neighborhood of x. Let $B_d(x)$ be a ball with radius $d = d(x,C)$ and center x. Let $\pi_d : \mathbb{R}^3 \to \partial B_d(x)$ be the orthogonal projection onto the sphere $\partial B_d(x)$ and let C_d be the image of C throughout π_d. Let us consider the Lipschitz homotopy $\Phi : [0,1]\times[0,l] \to \Omega$ such that $\Phi(0,[0,l]) = C$, $\Phi(1,[0,l]) = C_d$, and $\Phi(\cdot,t)$ is affine for all $t\in[0,l]$. Then $E := \Phi_\sharp[\![0,1]\times[0,l]]\!]$ is a rectifiable current with boundary $C\cup C_d$. Let D be the spherical cap on $\partial B_d(x)$ bounded by C_d with minimal area[1]. The rectifiable current $S' := E + D$ has boundary C, so we can consider the map u' solution of (2.4.27) with S replaced by S'. Now $\int_E \partial'_N\Gamma(x'-x)dS(x') = 0$ since Γ is radial with respect to x and N is always orthogonal to $x'-x$ by construction of E, so that by (2.4.28),

$$u(x) = -b\int_D \partial_N\Gamma(x'-x)dS(x'),$$

that explicitly gives

$$\begin{aligned}|\partial_i u(x)| &= |b\int_D\left(\frac{N_i}{|x-x'|^3} - 3\frac{N\cdot(x-x')(x_i-x'_i)}{|x-x'|^5}\right)d\mathcal{H}^2(x')| \\ &\le \frac{4|b|}{d(x,C)^3}\int_D d\mathcal{H}^2(x') \le \frac{8\pi|b|}{d(x,C)},\end{aligned}$$

since $\mathcal{H}^2(D)\le 2\pi d(x,C)^2$, and the thesis follows. □

Remark 2.4.19. Actually, if C_d is not simple, D is not well-defined. In general it is a rectifiable current that can be constructed as follows. Let $P\notin C_d$ be a point on $\partial B_d(x)$. We construct an homotopy $\Psi^P : [0,1]\times[0,2\pi]\to\partial B_d(x)$ that satisfies $\Psi^P(0,\cdot)\equiv P$ and $\Psi^P(1,[0,2\pi]) = C_d$, and we can consider the current $\Psi^P_\sharp[\![0,1]\times[0,2\pi]]\!]$. Then we can set $D := \Psi^P_\sharp[\![0,1]\times[0,2\pi]]\!]$, where P is chosen in such a way that $\Psi^P_\sharp[\![0,1]\times[0,2\pi]]\!]$ has minimal mass.

Remark 2.4.20. In Lemma 2.4.18 we also proved that the integral in (2.4.28) does not depend on the particular surface S, but only on its boundary C.

Corollary 2.4.21. *Let C be the union of $N>0$ Lipschitz closed curves C_k, let S be the union of the corresponding surfaces S_k with boundary C_k respectively, and let u be the solution to* (2.4.27) *in* (2.4.28). *Then* (2.4.36) *holds true.*

Proof. Actually the same proof of Lemma 2.4.18 applies, since we can always consider $N-1$ smooth curves connecting all the C_k, so that it is not restrictive to assume that $N=1$. □

[1] See following Remark.

Lemma 2.4.22. *Let $b \in 2\pi\mathbb{Z}$. Then the solution u of* (2.4.27) *belongs to $C^\infty(\Omega \setminus C, \mathbb{T})$ and it is harmonic in $\Omega \setminus C$.*

Proof. As we have proved in Corollary 2.4.18 if we choose a surface S' with boundary C disjoint from S, and denote by u' the corresponding solution of (2.4.27), then $u - u' = b\chi_A$, with A the open set with boundary $S \cup S'$. Since $b \in \mathbb{Z}$ we see that $u = u'$ as maps into $\mathbb{T}$. Moreover if $x \notin S$ then u is smooth at x, so in particular, up to change the surface S, we obtain that it belongs to $C^\infty(\Omega \setminus C, \mathbb{T})$ and u is harmonic at x for all $x \notin C$. □

Lemma 2.4.23. *Let S as above and let u be the solution of the elliptic problem*

$$\begin{cases} \Delta u = 0 & on \quad \mathbb{R}^3 \setminus S \\ u^+ - u^- = 1 & on \quad S \\ \partial_N^+ u - \partial_N^- u = 0 & on \quad S. \end{cases} \qquad (2.4.37)$$

Then, if U is a tubular neighborhood of C, for all $(\rho, \theta, z) \in U$ with $\theta \neq 0$ there exists the limit $\lim_{\epsilon \to 0^+} u(\epsilon\rho, \theta, z) = \frac{\theta}{2\pi} + c$, where c is a fixed arbitrary constant. Moreover $|\lim_{\epsilon \to 0^+} \partial_z u(\epsilon\rho, \theta, z)| < c < +\infty$ for some constant $c > 0$ that depends only on the curve C.

Proof. With no loss of generality we can suppose that the curve C that represents the boundary of S passes through the origin of an euclidean coordinate system and that it is tangent to the z-axis in such a point. Moreover we choose the coordinates x_1 and x_2 in such a way that $x_1 = \rho\cos\theta$ and $x_2 = \rho\sin\theta$, so that it follows that the point $(\epsilon\rho, \theta, z)$ coincides with $(\epsilon x_1, \epsilon x_2, z)$. For simplicity we take $z = 0$ and denote $x = (x_1, x_2, 0)$, while S is orthogonal to the x_2-axis in 0. From Lemma 2.4.15 we have the explicit formula

$$u(\epsilon\rho, \theta, 0) = u(\epsilon x_1, \epsilon x_2, 0) = -\int_S \partial_N \Gamma(x' - \epsilon x, y' - \epsilon y, z') dS(x', y', z'),$$

with the change of variables $(\epsilon x_1'', \epsilon x_2'', \epsilon z'') = (x_1', x_2', z')$ we obtain

$$u(\epsilon x_1, \epsilon x_2, 0) = -\int_{\frac{1}{\epsilon}S} \partial_N \Gamma(x_1'' - x_1, x_2'' - x_2, z'') dS(x_1'', x_2'', z''),$$

where we have used the explicit expression of Γ, with ∂_N being the partial derivative in the new variable. Letting ϵ go to zero we obtain

$$\lim_{\epsilon \to 0^+} u(\epsilon\rho, \theta, z) = -\int_{\Pi_0} \partial_N \Gamma(x'' - x) dS(x''),$$

where Π_0 is the half-plane $\{z = x_2 = 0, x_1 > 0\}$ and we have used the shorter notation $x'' = (x_1'', x_2'', z'')$. Thanks to Lemma 2.4.15, we see that the right-hand side coincides with $u(x_1, x_2, 0)$, where u is the solution of (2.4.37) with $S = \Pi_0$. But it is well known that such solutions are given by, in cylindrical coordinates, $u(\rho, \theta, z) = \frac{\theta}{2\pi} + c$ for an arbitrary constant c. In particular we have $\lim_{\epsilon \to 0^+} u(\epsilon\rho, \theta, z) = \frac{\theta}{2\pi} + c$.

To prove the last statement we use the explicit expression (2.4.33), which reads, after the change of variables $x' = \epsilon x''$,

$$\partial_z u(\epsilon x_1, \epsilon x_2, 0) = -\frac{1}{\epsilon} \int_{\frac{1}{\epsilon}S} \left(\frac{N_z}{|x - x''|^3} - 3\frac{N \cdot (x - x'')(z - z'')}{|x - x''|^5} \right) dS''. \quad (2.4.38)$$

We fix $R > 0$ and consider the ball B_ϵ with center $(\epsilon x_1, \epsilon x_2, 0)$ and radius R. We then write the last integral as

$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon}\left(\frac{N_z}{|x-x''|^3} - 3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)dS''$$
$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon^c}\left(\frac{N_z}{|x-x''|^3} - 3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)dS'',$$

and thanks to Remark 2.4.20, up to choose R small enough, we can assume that the surface S is everywhere orthogonal to the vector $(\epsilon x - x')$ in B_ϵ, that is, to $(x - x'')$ in $\frac{1}{\epsilon}B_\epsilon$, so that the integral above becomes

$$-\frac{1}{\epsilon}\int_{\frac{1}{\epsilon}S\cap B_\epsilon}\frac{N_z}{|x-x''|^3} - \left(\frac{N_z}{|x-x''|^3} - 3\frac{N\cdot(x-x'')(z-z'')}{|x-x''|^5}\right)dS''. \quad (2.4.39)$$

Let us now estimate the second term in (2.4.39). In B_ϵ^c it holds $|\epsilon x - x'| > R$, that is, $|x - x''| > \epsilon^{-1}R$, so it is easy to see that this term can be estimate by

$$\frac{\mathcal{H}^2(S)}{R^3} \le \gamma\frac{|C|^2}{R^3}, \quad (2.4.40)$$

where $|C|$ is the length of C and $\gamma > 0$ is the constant of the isoperimetric inequality.

It remains to estimate the first term. Let us consider the plane Π passing through 0 and tangent to the versor $\bar{z}$ and $\overline{x-0}$. Let Π^+ be the half-plane in Π bounded by the axis $\hat{z}$ and not containing the point x. Thanks to the smoothness of C, we can assume that there exists a smooth one-to-one map $\Phi : \Pi^+ \cap B_\epsilon \to S \cap B_\epsilon$, so that also $N \circ \Phi : \Pi^+ \ni \hat{x}' \mapsto N(x')$ is smooth, and then in $B_\epsilon \cap \Pi^+$ we can use the Taylor expansion of $N \circ \Phi$ at 0. Going back to the variable $x' = \epsilon x''$ (and $\hat{x}' := \epsilon\hat{x}''$), we find that the first term in (2.4.39) reads

$$-\int_{S\cap B_\epsilon}\frac{N_z(x')}{|\epsilon x - x'|^3}dS'' = -\int_{S\cap B_\epsilon}\frac{\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'}{|\epsilon x - x'|^3} + \frac{r_N(|\hat{x}'|^2)}{|\epsilon x - x'|^3}dS'.$$

The Taylor expansion of Φ at 0 provides $x' = \hat{x}' + \nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')$ and if R is small enough we can assume that $|\nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')| < \frac{1}{2}|\hat{x}'|$. Note that, since C is smooth, we can find such a $R > 0$ satisfying the last inequality globally, i.e., R is independent of the point x. In particular we find $|\epsilon x - x'| > |\epsilon x - \hat{x}'| - |\nabla^2\Phi(0)\hat{x}'\cdot\hat{x}' + r_\Phi(\hat{x}')| > |\hat{x}'| - \frac{1}{2}|\hat{x}'| = \frac{1}{2}|\hat{x}'|$ for all $\epsilon > 0$, so that the integral is bounded by

$$\int_{\Pi^+\cap B_\epsilon}\frac{|\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'|}{|\hat{x}'|^3} + \frac{r_N(|\hat{x}'|^2)}{|\hat{x}'|^3}dS',$$

and taking into account that $R > 0$ can be small as we want, we assume that $|r_N(\hat{x}')| < |\nabla^2 N_z(0)\hat{x}'\cdot\hat{x}'|$, whereby the last integral can be estimated by

$$C_0\int_{\Pi^+\cap B_\epsilon}\frac{1}{|\hat{x}'|}dS(\hat{x}'),$$

where the constant C_0 is independent of R and x, and whose limit as $\epsilon \to 0$ reads by the monotone convergence theorem

$$C_0 \int_{\Pi^+ \cap B(0,R)} \frac{1}{|\hat{x}'|} dS(\hat{x}'),$$

which is uniformly bounded. Now, since the value of R is independent of the point x but only depends on the geometry of the curve C, we achieved the proof. □

Remark 2.4.24. Let us point out that Lemma 2.4.23 still holds true if we do not assume that C is connected. Indeed if C is the union of a finite family of smooth closed curves, the surface S will be the union of a finite family of smooth surfaces and the arguments used in the proof of Lemma 2.4.23 still work.

2.5 Minimizers of a continuum dislocation energy

In this Section we focus on the main object of discussion of the Chapter, that is the minimum problem

$$\min_{(F,\mathcal{L})\in\mathcal{A}} \mathcal{W}(F,\Lambda_{\mathcal{L}}), \tag{2.5.1}$$

where the energy $\mathcal{W}$ satisfies some appropriate convexity and coerciveness conditions, while $\mathcal{A}$ is the space of admissible couples of deformations and dislocation currents.

We will provide three existence results with some important differences on the hypotheses on the energy and on the class of admissible dislocations. In particular, we first solve problem (2.5.1) in a narrow class of couples $(F,\Lambda_{\mathcal{L}})$, where $\mathcal{L}$ is a dislocation generated by a finite number of Lipschitz curves (mesoscopic dislocation). As a second step we solve the problem among a class of deformations whose curl is the density of a continuum dislocation. In contrast with the first existence result, here the energy depends on the total length of the dislocation set L which is assumed connected.

Let us describe a pathological case which we do not treat in this first analysis. Consider a countable family of loops $L_{i\in\mathcal{I}}$ of lengths $l_{i\in\mathcal{I}}$, with $\sum_{i\in\mathcal{I}}\mathcal{H}^1(l_i)$ is finite. If the set $L := \cup L_{i\in\mathcal{I}}$ turns out to be somewhere dense in Ω, then mesoscopicity assumption will be violated since for some points outside L there is no ball centered at them with empty intersection with L. Moreover every connected set C that contains the dislocation set L turns out to have unbounded $\mathcal{H}^1$ measure. In order to solve problem (2.5.1) in a class of competitors where also this case is taken into account, a more detailed analysis of the deformations graphs is needed. This is accomplished in Section 2.7.1. This leads to the third, more general, existence result provided in the following Section.

2.5.1 Preliminaries

It is convenient, with no loss of generality, to assume that Ω is bounded and simply connected, with smooth boundary. Recall that U be a bounded open set such that $U\cap\partial\Omega = \partial_D\Omega$. Moreover we assume:

Assumption 2.5.1. *Let $\hat{\Omega}$ be a bounded and simply connected open set with smooth boundary such that $(\bar{U}\cup\bar{\Omega})\subset\hat{\Omega}$.*

In order to simplify the notation, let us also assume that the atomic spacing of the material is a multiple of 2π, that is, we are supposing that the crystallographic assumption holds with the lattice $2\pi\mathbb{Z}$.

We will restrict our attention to the class of *continuum dislocations* (c.d.), defined by Definition 2.2.13. We have seen, thanks to Theorem 2.2.17, that $\mathcal{L}$ is a continuum dislocation if, for $i = 1,2,3$, there exists a 1-Lipschitz map $\lambda^i : [0,M^i]\to\hat{\Omega}$ such that $\mathcal{L}_i = \lambda^i_\sharp[\![0,M^i]\!]$. Moreover, since all such currents are boundaryless by definition, we can rescale the functions λ^i and suppose they are defined on S^1. In such a case, the density of a continuum dislocation in $\bar{\Omega}$ can be written as the sum of the three measures

$$\Lambda_{\mathcal{L}} = \sum_{i=1}^{3}\Lambda_i = \sum_{i=1}^{3}\lambda^i_\sharp[\![S^1]\!]_{\llcorner\bar{\Omega}} \otimes e_i, \tag{2.5.2}$$

that we can equivalently write as $\Lambda_i = (\dot\lambda^i \otimes e_i)\lambda^i_\sharp \mathcal{H}^1$, where $\lambda^i_\sharp\mathcal{H}^1$ is the push-forward of the 1-dimensional Hausdorff measure on S^1 through λ^i.

We then introduce the class of *dislocation density measures* in $\hat\Omega$ as

$$\mathcal{M}_\Lambda(\hat\Omega,\mathbb{R}^{3\times3}) := \{ \quad \nu \in \mathcal{M}(\hat\Omega,\mathbb{R}^{3\times3}) : \exists\, \mathcal{L} \text{ c.d. with density } -(\Lambda_{\mathcal{L}})^{\mathrm{T}} = \nu\}. \tag{2.5.3}$$

Setting $\varphi^\lambda_{ij}(s) := \varphi_{ij}(\lambda^j(s)) = \delta_{jk}\varphi_{ij}(\lambda^k(s))$ for every $\varphi \in \mathcal{C}_c(\hat\Omega,\mathbb{R}^{3\times3})$ (where no sum is meant in the second term), the density $\mu_\lambda := -(\Lambda_{\hat{\mathcal{L}}})^{\mathrm{T}}$ which is associated to λ reads

$$\begin{aligned}-\langle \mu_\lambda,\varphi\rangle &= \sum_{k=1}^3 \int_{S^1} \varphi(\lambda^k(s))\cdot(e_k\otimes\dot\lambda^k(s))d\mathcal{H}^1(s)\\ &= \int_{S^1} \varphi^\lambda_{ij}(s)\left(\dot\lambda^i\right)_j(s)ds = \int_L \varphi_{ij}\tau_j\theta_i d\mathcal{H}^1,\end{aligned} \tag{2.5.4}$$

with $L = \cup_{i=1}^3 \lambda^i(S^1)$ be the dislocation set. In (2.5.4), we have introduced

$$\begin{aligned}\theta_i(P) :=& \sharp\{s\in(\lambda^i)^{-1}(P) : \frac{\dot\lambda^i}{|\lambda^i|}(s) = \tau(P)\}\\ &- \sharp\{s\in(\lambda^i)^{-1}(P) : \frac{\dot\lambda^i}{|\lambda^i|}(s) = -\tau(P)\},\end{aligned}$$

for every $P\in L$, which stands for the multiplicity of the dislocation with Burgers vector e_i. Here $\tau_j\theta_i d\mathcal{H}^1 = (\dot\lambda^i)_j ds$. The correspondence between the arcs λ and the Burgers vectors of the dislocation will appear clearer in Remark 2.5.2 below.

If $\mathcal{L}$ is a continuum dislocation, then there exists a set $\mathcal{C}_{\mathcal{L}}\subset\hat\Omega$ containing the support of the density $\Lambda_{\mathcal{L}}$ which is a continuum, i.e., a finite union of connected compact sets with finite 1-dimensional Hausdorff measure. Note that such a set is not unique, and that we can always take, for example, $\mathcal{C}_{\mathcal{L}} = \cup_{i=1}^3\lambda^i(S^1) = L$.

Remark 2.5.2. When we deal with a dislocation $\mathcal{L}$ generated by only one loop with Burgers vector $b = (\beta_1,\beta_2,\beta_3) = \beta_i e_i, \beta_i \in 2\pi\mathbb{Z}$ ($b\neq 0$) (i.e., e_i is the lattice spacing), then we have a Lipschitz function $\gamma^b\in W^{1,1}(S^1,\mathbb{R}^3)$ such that $\mathcal{L} = \gamma^b_\sharp[\![S^1]\!]_{\llcorner\bar\Omega}$ and $-\mu^{\mathrm{T}}_{\gamma^b} = \Lambda_{\mathcal{L}} = \mathcal{L}\otimes b$, that is the measure such that

$$\begin{aligned}-\langle\mu_{\gamma^b},\varphi\rangle &= \int_{S^1}\varphi(\gamma^b(s))\cdot(b\otimes\dot\gamma^b(s))ds = \int_{S^1}\varphi_{ij}(\gamma^b(s))b_i\dot\gamma^b_j(s)ds\\ &= \int_L \varphi_{ij}\tau_j b_i\theta d\mathcal{H}^1,\end{aligned} \tag{2.5.5}$$

where $\theta(P)$ represents the multiplicity of the dislocation and is defined for every $P\in L$ as

$$\begin{aligned}\theta(P) :=& \sharp\{s\in(\gamma^b)^{-1}(P) : \frac{\dot\gamma^b}{|\gamma^b|}(s) = \tau(P)\}\\ &- \sharp\{s\in(\gamma^b)^{-1}(P) : \frac{\dot\gamma^b}{|\gamma^b|}(s) = -\tau(P)\}.\end{aligned} \tag{2.5.6}$$

For every $\mu \in \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})$ it is easy to check that $\operatorname{Div} \mu = 0$ in $\hat{\Omega}$, since $\mathcal{L}_i$ are closed integral currents. In fact for all $\psi \in C_c^\infty(\hat{\Omega}, \mathbb{R}^3)$ one has $-\langle D\psi, \mu\rangle = \langle D\psi, \sum_{k=1}^3 e_k \otimes \dot{\lambda}^k(\lambda^k_\sharp \mathcal{H}^1)\rangle = \sum_{i=1}^3 \int_{S^1} D_j\psi_i(\lambda^i)\dot{\lambda}^i_j ds = \int_{S^1} D_t\psi_k(\lambda^k)dt = 0$. We then get $\mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3}) \subset \mathcal{M}_{\text{div}}(\hat{\Omega}, \mathbb{R}^{3\times 3})$.

We can now identify the space $\mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})$ with $W^{1,1}(S^1, \hat{\Omega}^3)$, through the map $T : W^{1,1}(S^1, \hat{\Omega}^3) \to \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})$ given by

$$T(\lambda) = \mu_\lambda \text{ defined in (2.5.4).} \tag{2.5.7}$$

The map T is by definition onto, while for every $\lambda \in W^{1,1}(S^1, \hat{\Omega}^3)$ it holds

$$\|T(\lambda)\|_{\mathcal{M}} \le \|\dot{\lambda}\|_{L^1}, \tag{2.5.8}$$

implying the continuity of T. However T is not an injective map. We now define an equivalence relation $\sim$ in $W^{1,1}(S^1, \hat{\Omega}^3)$ by writing $\lambda \sim \lambda'$ if and only if $T(\lambda) = T(\lambda')$. Then we set $\dot{W}^{1,1}(S^1, \hat{\Omega}^3) := W^{1,1}(S^1, \hat{\Omega}^3)/\!\sim\, = W^{1,1}(S^1, \hat{\Omega}^3)/\ker(T)$, and so we may define the inverse of T as $T^{-1} : \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3}) \to \dot{W}^{1,1}(S^1, \hat{\Omega}^3))$. If we define a new norm $\|\cdot\|_\sim$ on $\dot{W}^{1,1}(S^1, \hat{\Omega}^3)$, given by $\|\lambda\|_\sim = \inf_{\lambda'\sim\lambda} \|\dot{\lambda}'\|_{L^1}$ then by virtue of the open mapping theorem, T^{-1} is also linear and bounded, whereas with the norm of $W^{1,1}(S^1, \hat{\Omega}^3)$ an inverse of T is in general not continuous. For every $\mu \in \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})$ we set

$$m(\mu) := \inf_{\lambda'\sim\lambda} \|\dot{\lambda}'\|_{L^1} = \|\lambda\|_\sim, \tag{2.5.9}$$

As a consequence,

$$T\left(\dot{W}^{1,1}(S^1, \hat{\Omega}^3)\right) = \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3}), \tag{2.5.10}$$

$$T^{-1}\left(\mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})\right) = \dot{W}^{1,1}(S^1, \hat{\Omega}^3). \tag{2.5.11}$$

where the infimum is taken over all $\lambda \in T^{-1}(\mu)$. Introduce also

$$\mathcal{BC}^{p,\Lambda}(\hat{\Omega}, \mathbb{R}^{3\times 3}) := \{F \in \mathcal{BC}^p(\hat{\Omega}, \mathbb{R}^{3\times 3}) : \operatorname{Curl} F \in \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})\}, \tag{2.5.12}$$

and its proper subspace

$$\mathcal{BC}^{p,\Lambda}_{\text{div}}(\hat{\Omega}, \mathbb{R}^{3\times 3}) := \{F \in \mathcal{BC}^p_{\text{div}}(\hat{\Omega}, \mathbb{R}^{3\times 3}) : \operatorname{Curl} F \in \mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})\} \tag{2.5.13}$$

in such a way that by Theorem 2.4.13 and (2.5.11), it holds

$$\mathcal{BC}^{p,\Lambda}_{\text{div}}(\hat{\Omega}, \mathbb{R}^{3\times 3}) := \operatorname{Curl}^{-1}\Big(\mathcal{M}_\Lambda(\hat{\Omega}, \mathbb{R}^{3\times 3})\Big) = \operatorname{Curl}^{-1}\Big(T\Big(\dot{W}^{1,1}(S^1, \hat{\Omega}^3)\Big)\Big). \tag{2.5.14}$$

Moreover we introduce

$$\mathcal{BC}^{p,\Lambda}(\Omega, \mathbb{R}^{3\times 3}) := \{F \in \mathcal{BC}^p(\Omega, \mathbb{R}^{3\times 3}) : \exists \bar{F} \in \mathcal{BC}^{p,\Lambda}(\hat{\Omega}, \mathbb{R}^{3\times 3}) : F = \bar{F}_{\llcorner\Omega}\}. \tag{2.5.15}$$

2.5.2 The class of admissible deformations

Let us fix an admissible boundary condition $(N, \mathcal{P}, \alpha_D)$. In the sequel, whenever we consider an admissible dislocation $\mathcal{L}$, it is always supposed that such $\mathcal{L}$ satisfies the boundary condition $(N, \mathcal{P}, \alpha_D)$, and hence it will be convenient to still denote the dislocation $\mathcal{L}' := \mathcal{L} + \alpha$ by $\mathcal{L}$. In other words, when referring to an admissible dislocation current, it is intended that it has been already summed with $\hat{\alpha}$. We also fix a map $\bar{F} \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ such that $-\operatorname{Curl} \bar{F} = (\Lambda_\alpha)^{\mathrm{T}}$ on U.

Definition 2.5.3.

$$\mathcal{F} := \{(F,\mathcal{L}) \in L^p(\Omega,\mathbb{R}^{3\times 3}) \times \mathcal{MD} : F \text{ satisfies (i)-(iii) below}\} \qquad (2.5.16)$$

(i) The dislocation current $\hat{\mathcal{L}} = \{L,\tau,\theta\}$ satisfies the boundary condition and the function $\hat{F} := \chi_{\hat{\Omega}\setminus\Omega}\bar{F} + \chi_\Omega F \in L^p(\hat{\Omega},\mathbb{R}^{3\times 3})$ is such that $-\operatorname{Curl}\hat{F} = (\Lambda_{\mathcal{L}})^{\mathrm{T}}$ in $\hat{\Omega}$.

(ii) We require that for every point $x \in \Omega\setminus L$ there is a ball $B \subset \Omega\setminus L$ centered at x such that there exists a function $\phi \in \mathrm{Cart}^p(B;\mathbb{R}^3)$ with $F = D\phi$ in B.

(iii) $\det F > 0$ almost everywhere in Ω.

Let us recall that if $F = Du$ is the gradient of a Cartesian map, then it is readily satisfied that the distributional determinant $\mathrm{Det}(F)$ and adjoint $\mathrm{Adj}(F)$ of F are elements of $L^1(U,\mathbb{R}^{3\times 3})$ and coincide with $\det(Du)$ and $\mathrm{adj}(Du)$ respectively. It is also straightforward that smooth functions $u \in C^1(U,\mathbb{R}^3)$ are Cartesian.

We denote by

$$\mathcal{AD}^p(\hat{\Omega}) := \{F \in \mathcal{BC}^{p,\Lambda}(\hat{\Omega},\mathbb{R}^{3\times 3}) : F \text{ satisfies } (ii) \text{ above}\}, \qquad (2.5.17)$$

$$\mathcal{AD}^{p,\Lambda}(\Omega) := \{F \in \mathcal{BC}^p(\Omega,\mathbb{R}^{3\times 3}) : \exists \hat{F} \in \mathcal{AD}^p(\hat{\Omega}) : \ F = \hat{F}_{\llcorner\Omega}\}. \qquad (2.5.18)$$

Thanks to the equivalence between mesoscopic dislocation and continuum dislocation we see that the class $\mathcal{F}$ concides with the set of functions in $\mathcal{AD}^{p,\Lambda}(\Omega)$ satisfying also (iii).

Remark 2.5.4. As a consequence of the crystallographic assumption, that is, the hypothesis that the Burgers vectors belong to the lattice $2\pi\mathbb{Z}^3$, it turns out that $\mathcal{AD}^p(\Omega)$ is not a linear subspace of $\mathcal{BC}^p(\Omega,\mathbb{R}^{3\times 3})$. Indeed it is easy to see that if $F \in \mathcal{AD}^p(\Omega)$ has density $-(\operatorname{Curl} F)^{\mathrm{T}}$, then ηF, with η an irrational real number, has density $-(\eta \operatorname{Curl} F)^{\mathrm{T}}$ which has not Burgers vectors in $2\pi\mathbb{Z}^3$.

We also introduce the proper subset of $\mathcal{AD}^p(\hat{\Omega})$

$$\mathcal{AD}^p_{\mathrm{div}}(\hat{\Omega}) := \{F \in \mathcal{BC}^{p,\Lambda}_{\mathrm{div}}(\hat{\Omega},\mathbb{R}^{3\times 3}) : F \text{ satisfies } (ii) \text{ above}\}. \qquad (2.5.19)$$

The following regularity result holds:

Theorem 2.5.5. *Let $\mu \in \mathcal{M}_\Lambda(\hat{\Omega},\mathbb{R}^{3\times 3})$, then the solution $F := \operatorname{Curl}^{-1}(\mu)$ of (2.7.19) in $\hat{\Omega}$ satisfies (ii) above. In other words*

$$\mathcal{AD}^p_{\mathrm{div}}(\hat{\Omega}) \equiv \mathcal{BC}^{p,\Lambda}_{\mathrm{div}}(\hat{\Omega},\mathbb{R}^{3\times 3}).$$

Proof. By hypothesis there is a $\lambda \in W^{1,1}(S^1,\hat{\Omega}^3)$ such that $\mu^{\mathrm{T}} = -\sum_{i=1}^3 \dot{\lambda}^i \otimes e^i \lambda^i_\sharp \mathcal{H}^1$. Let $C_\mu = \cup_{i=1}^3 \lambda^i(S^1)$, that is a closed set of finite length. Let us fix a ball B with $\bar{B} \subset (\hat{\Omega} \setminus C_\mu)$. We first see that the function $x \mapsto G(x)$ defined in (2.4.20) turns out to be $C^\infty(B)$, thanks to the fact that for fixed x, the map $y \mapsto \nabla\phi(x-y)$ and all its derivatives are uniformly continuous on C_μ. Now $F = G + D\psi$ where ψ is the solution of (2.4.3) with $\operatorname{Div} G = 0$. Since C_μ does not intersect $\partial\hat{\Omega}$ we see that GN is smooth on $\partial\hat{\Omega}$, so that ψ is smooth in $\hat{\Omega}$ and we find out that F is smooth on any ball $B \subset \hat{\Omega} \setminus C_\mu$. In particular, in any such ball, since F is curl-free, it is the gradient of a smooth map, and thus the gradient of a Cartesian map. □

We will now show that there exists at least one element in $\mathcal{F}$ with an admissible $\mathcal{L}$ coinciding with α in $\partial\Omega_D$. In the following theorem, we will use the general fact:

$$-\operatorname{Curl} F = b \otimes \tau\, \mathcal{H}^1 \llcorner L \quad \text{if and only if} \quad \int_{C_L} F\, \underline{e}_\theta d\mathcal{H}^1 = b. \tag{2.5.20}$$

for all Lipschitz-continuous closed path C_L in Ω enclosing once L and with unit tangent vector $\underline{e}_\theta$. To check identity (2.5.20), simply observe that, if S_L is a Lipschitz and closed surface in Ω with boundary L and normal ν, $\Omega \setminus S_L$ is simply connected and hence there exists a function $\phi \in W^{1,p}(\Omega \setminus S_L)$ such that $F = \nabla\phi$ in $\Omega \setminus S_L$. By (2.5.20), ϕ has a constant jump on $\S_L$ (i.e., $[\![\phi]\!]_{S_L} = b$). Thus the distributional derivative of ϕ writes as $D\phi = \nabla\phi + b \otimes \nu\mathcal{H}^2_{\llcorner S_L}$. Multiplying by a test function ψ one has by (2.1.4) that $\langle \operatorname{Curl}(b \otimes \nu\mathcal{H}^2_{\llcorner S_L}), \psi\rangle = \langle b \otimes \nu\mathcal{H}^2_{\llcorner S_L}, \operatorname{Curl}\psi\rangle$. Componentwise, by Stokes theorem, it reads as

$$\int_{S_L} n_i b_j \epsilon_{ikl} \partial_k \psi_{jl} d\mathcal{H}^2 = b_j \int_L \tau_p \psi_{jp} d\mathcal{H}^1,$$

and hence $\langle \operatorname{Curl}(b \otimes \nu\mathcal{H}^2_{\llcorner S_L}), \psi\rangle = \langle (b \otimes \tau\mathcal{H}^1_{\llcorner L}), \psi\rangle$.

Theorem 2.5.6. *The set $\mathcal{F}$ is non-empty for $1 \le p < 2$.*

Proof. We first construct an admissible function for a simple geometry. Consider the circle $L := \{(x, y, z) \in \mathbb{R}^3 : |x|^2 + |y|^2 = R^2,\ z = 0\}$ as a dislocation loop with Burgers vector $b = \beta_1\underline{e}_1 + \beta_2\underline{e}_2 + \beta_3\underline{e}_3 = \beta_R\underline{h}_R + \beta_l\underline{h}_l + \beta_z\underline{h}_z$, with the local basis on L, $\{\underline{h}_R, \underline{h}_l, \underline{h}_z\} = Q(l)\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ where $Q(l)$ is the matrix of rotation around $\underline{e}_3 = \underline{h}_z$ and with angle l (see Fig. 2.5(a)). Let V_δ be a tubular neighborhood of L with radius $\delta > 0$, and let $(r, \theta, l) \in [0, 2\delta] \times [0, 2\pi] \times [0, 2\pi R]$ be a system of cylindrical coordinates in V_δ chosen in the following way: the origin of θ is chosen in such a way that all points $(x, y, z) \in V_\delta$ with $z = 0$ and $|x|^2 + |y|^2 < R^2$ satisfy $\theta = a + \pi/4$ for some constant $a > 0$ which fix the orientation of the solid angle of amplitude $\pi/2$ constructed on L (cf. the black triangle on the box below right of Fig. 2.5(a) denoted as S or V in the sequel), while the coordinate r is the distance from the set L, and l, as before, R times the angle around z axis. In V_δ we denote by $\underline{g} := (\underline{g}_r, \underline{g}_\theta, \underline{g}_l)$, with $\underline{g}_l = \underline{h}_l$, the local cylindrical basis defined on the normal sections ∂V_δ, corresponding to such coordinates. We then consider the function F inside V_δ whose components in the basis $\{\underline{h}_R, \underline{h}_l, \underline{h}_z\}$ read

$$F(r, \theta, l) = \zeta(\theta) \begin{pmatrix} -\frac{\sin\theta}{r}\beta_R & +\frac{\cos\theta}{r}\beta_R & 0 \\ -\frac{\sin\theta}{r}\beta_l & +\frac{\cos\theta}{r}\beta_l & 0 \\ -\frac{\sin\theta}{r}\beta_z & +\frac{\cos\theta}{r}\beta_z & 0 \end{pmatrix}, \tag{2.5.21}$$

where (r, θ, l) are the coordinates associated to the basis system $\underline{g}$, and ζ is a smooth function on $[0, 2\pi)$ which is non-negative in $(a, a+\pi/2)$, zero outside, and has integral equal to 1. It is readily checked that curl F $= 0$ in $V_\delta \setminus \gamma$. It is known that there exists a solution to equation $F = \nabla\phi_\delta$ in the simply connected domain $S := \{(r, \theta, l) : a < \theta < a + \pi/2, 0 < r < \delta\}$ with $0 \le l \le 2\pi$, and in order to fix the arbitrary constant, set $\phi_\delta = 0$ on $S \cap \{\theta = a\}$ and $\phi_\delta = b$ on $\bar{S} \cap \{\theta = a+\pi/2\}$. Let V be the solid of revolution around the z-axis generated by S. Considering the axis-symmetry we then extend ϕ_δ over the whole V and note that U is constant on the sets $C_{\bar\theta} := \{(\delta, \bar\theta, l) : 0 \le l \le 2\pi R\}$ for every $a < \bar\theta < a + \pi/2$. Let $D_{\bar\theta}$ be the disk with boundary $C_{\bar\theta}$ where for every $x \in D_{\bar\theta}$, $\phi_\delta(x)$ is defined as $\phi_\delta(x) = \phi_\delta(y)$ with $y \in C_{\bar\theta}$; define also $D := \bigcup_{\theta\in(a,a+\pi/2)} D_\theta$. We set $\phi_\delta = 0$ in $\Omega \setminus V \setminus D$ and observe that it is smooth everywhere except at the interface I between V and D and on $J := \bar D_{a+\pi/2} \cup (V \cap \{\theta = a + \pi/2\})$ where it has a constant jump of magnitude b (cf. Fig. 2.5(b) above). Therefore we introduce $\tilde\phi_\delta$, a C^∞-regularization of ϕ_δ in a set $D \cap \mathcal{V}$, with $\mathcal{V}$ a neighborhood of I, in such a way that $\|\nabla\tilde\phi_\delta\|_{L^\infty(D\cap\mathcal{V})} \le 2\|\nabla\phi_\delta\|_{L^\infty(D\cap\mathcal{V})}$ and define $F := \nabla\tilde\phi_\delta$, the absolutely continuous part of the distributional gradient $D\tilde\phi_\delta$ (i.e., the pointwise gradient of $\tilde\phi_\delta$), while in the jump set J, the jump part of $D\tilde\phi_\delta$ reads $b\otimes\nu\ \mathcal{H}^2 \llcorner J$. Moreover, (2.5.20) and (2.5.21) together entail that $-\operatorname{Curl} F = b \otimes \tau\, \mathcal{H}^1$ on L. As a consequence, we have constructed a function F which is smooth outside L and vanishes outside $T := V \cup D$, while from expression (2.5.21), $F \in L^p(\Omega)$ for $p \in [1, 2)$, since

$$\|F\|^p_{L^p(\Omega)} \le C|b|(R\delta^{2-p} + \delta^{1-p}R^2), \tag{2.5.22}$$

for some positive constant C independent of R and δ. Moreover, by adding to F an appropriate multiple of the identity it is readily seen that $\det(F + cI) > 0$ for some $c > 0$, while $\det(F + cI)$, $\operatorname{adj}(F + cI)$ also belong to $L^p(\Omega)$ for $p \in [1, 2)$.

Finally, fix a ball $B \subseteq \Omega \setminus L$: in such a ball the function F is smooth and has null rotation and hence there exists a $\phi \in C^\infty(B)$ such that $D\phi = F$. In particular we can take $\phi = \tilde\phi_\delta$ when the ball does not intersect the jump set J, otherwise, if it does, we sum to $\tilde\phi_\delta$ the constant b at all points of B which are below J, thereby nullifying the discontinuity due to the jump. Thus ϕ is smooth, and hence, is a cartesian map.

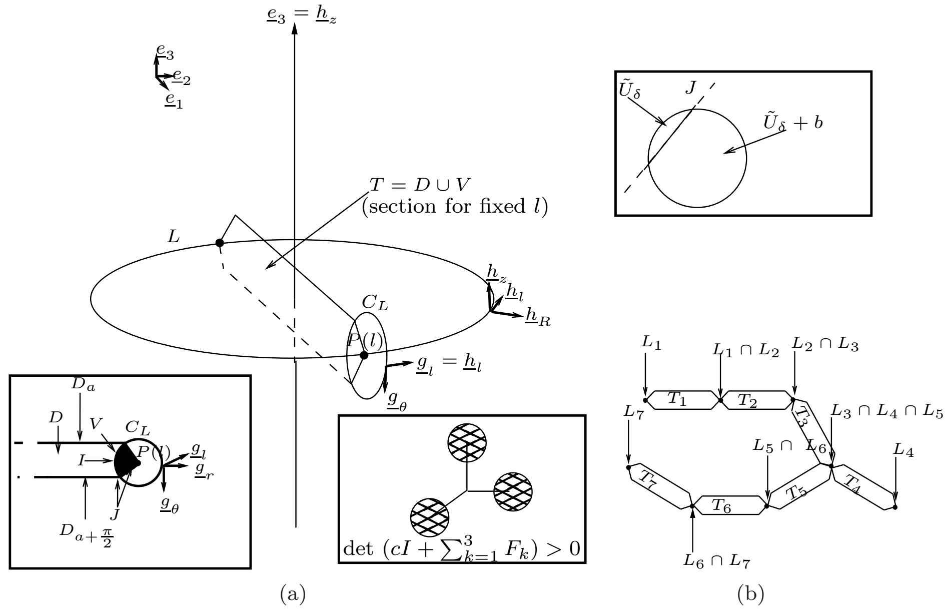


Figure 2.5: Picture of the tube construction for the proof (a); the case of finitely many boundary dislocation segments (b)

Let us now reproduce this argument for a finite number of circles with possible mutual intersection in $\partial\Omega$, and show that the constant $c > 0$ can be chosen in such a way that the determinant of the resulting deformation still remains non-negative. Let us consider a finite number of loops L_k with $1 \leq k \leq K$ with the associated $T_k := V_k \cup D_k$ constructed as described above, and observe that (by possibly adapting the amplitude of the solid angle S_k, i.e., replacing $\pi/2$ by π/N) the T_k's only intersect at points in L_k for some k's, while keeping the V_k's with empty mutual intersection (cf. Fig 2.5(b) below left). Let F_k be defined as (2.5.21) with β_k in place of β and $a_k = \hat{a}_k(l)$ in place of a such that $f_k(\theta, l) := \beta_l^k(l)\cos\theta - \beta_R^k(l)\sin\theta = \beta_2^k \cos(\theta + \frac{l}{R}) - \beta_1^k \sin(\theta + \frac{l}{R}) \geq 0$ (for instance, if $\beta_1, \beta_2 > 0$ then $a_k := \frac{3\pi}{2} - \frac{l}{R}$). Defining $F := \sum_{k=1}^{K} F_k + cI$, (2.5.22) entails that F, $\det F$, $\operatorname{adj} F$ belong to L^p and also that

$$\det F = \frac{c^2}{r} f_k(\theta, l)\zeta(\theta) + c^3 \geq 0 \quad \text{in } V_k, \tag{2.5.23}$$

while in D_k, one has $\det F > 0$ provided $c > 3\max_k\{\|F_k\|_{L^\infty(D_k)}\}$ (cf. box below right in Fig. 2.5a).

Since the arguments presented above for a finite family of circular loops remain valid for a finite family of Lipschitz deformation of such loops, with appropriate Lipschitz deformations of the T_ks. In particular, it holds for the boundary current α and for any finite family of curves joining P_i's to the Q_i's without self-intersections and prolonged by a geometrically unnecessary arc in $\partial\Omega$ (an admissible F can be constructed as above in $\hat{\Omega} \supset \Omega$ and then restricted to Ω with its curl restricted to $\bar{\Omega}$). Thus the proof is achieved.

□

2.6 Existence of minimizers

We propose two models in which the energy does not depend on the particular currents generating the dislocations but only on the density. However, we remark that in general, energies depending on the loops per se may also be considered (this was considered beyond the scope of this paper). In the first existence result the model variables are the deformation and the family of mesoscopic dislocations. In the second existence result, the model variable is the sole deformation, while the dislocations are sought at the continuum scale and hence are only found in an equivalence class.

2.6.1 Existence result in $\mathcal{F} \times \mathcal{MD}$

Here we study the existence of solutions to the problem

$$\inf_{\substack{(F,\mathcal{L}) \in \mathcal{F} \times \mathcal{MD} \\ -\operatorname{Curl} F = \Lambda_{\mathcal{L}}^{\mathrm{T}}}} \mathcal{W}(F, \Lambda_{\mathcal{L}}), \tag{2.6.1}$$

where the energy $\mathcal{W} : \mathcal{F} \times \mathcal{MD} \to \bar{\mathbb{R}}$ is such that there are positive constants C and β for which

$$\mathcal{W}(F, \mathcal{L}) := \int_\Omega \mathrm{W_e}(F)dx + \mathcal{W}_{\mathrm{defect}}(\Lambda_{\mathcal{L}}) \geq \\ C\big(\|\mathcal{M}(F)\|_{L^p} + \sum_{j \leq k_{\mathcal{L}}} b^j \|\dot{\varphi}_j\|_{L^1} + k_{\mathcal{L}}\big) - \beta. \tag{2.6.2}$$

Let us recall that $k_{\mathcal{L}}$ is defined in (2.2.28), $\{\varphi_j\}_{j \leq k_{\mathcal{L}}}$ are the generating loops defined in 2.2.7, and $\mathcal{M}(F)$ is the vector defined in (2.1.8). Here, W_{e} is an integrable function and $\mathcal{W}_{\mathrm{defect}}$ a functional defined on Radon measures. For the elastic part of the energy we introduce also the following notation

$$\mathcal{W}_e(F) := \int_\Omega \mathrm{W_e}(F)dx.$$

It is also assumed that

(W1) $W_e(F) \geq h(\det F)$, for a continuous real function h such that $h(t) \to \infty$ as $t \to 0$,

(W2) W_e is polyconvex, i.e., there exists a convex function $g : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}^+ \to \bar{\mathbb{R}}$ s.t. $W_e(F) = g(\mathcal{M}(F)),\ \forall F \in \mathcal{F}$,

(W3) $\mathcal{W}_{\mathrm{defect}} := \mathcal{W}^1_{\mathrm{defect}} + \mathcal{W}^2_{\mathrm{defect}}$, with $\mathcal{W}^1_{\mathrm{defect}}(\Lambda_{\mathcal{L}}) \geq \kappa_1|\Lambda_{\mathcal{L}}|$ and $W^2_{\mathrm{defect}}(\Lambda_{\mathcal{L}}) = \kappa_2(\sum_{1\leq j\leq k_{\mathcal{L}}} b^j\|\dot{\varphi}_j\|_{L^1} + k_{\mathcal{L}})$, for some constitutive material parameters κ_1 and κ_2.

(W4) $\mathcal{W}^1_{\mathrm{defect}}$ is weakly* lower semicontinuous, that is $\liminf_{k\rightharpoonup\infty} \mathcal{W}^1_{\mathrm{defect}}(\Lambda^k) \geq \mathcal{W}^1_{\mathrm{defect}}(\Lambda)$ as $\Lambda^k \rightharpoonup \Lambda$ weakly* in $\mathcal{M}_b(\bar{\Omega}, \mathbb{R}^{3\times3})$.

Note that assumption $(W2)$ implies that also $\mathcal{W}_e$ is weakly lower semicontinuous, i.e., $\liminf_{k\to\infty} \mathcal{W}_e(F^k) \geq \mathcal{W}_e(F)$ as $\mathcal{M}(F^k) \to \mathcal{M}(F)$ weakly in $L^p(\Omega, \mathbb{R}^{3\times3}) \times L^p(\Omega, \mathbb{R}^{3\times3}) \times L^p(\Omega)$.

Remark 2.6.1. A simple example for the functional $\mathcal{W}^1_{\mathrm{defect}}$ to satisfy condition $(W4)$ is just

$$\mathcal{W}^1_{\mathrm{defect}}(\Lambda) := c|\Lambda|(\Omega).$$

In [17] it is consider the following form for the functional $\mathcal{W}^1_{\mathrm{defect}}$ satisfying $(W4)$:

$$\mathcal{W}^1_{\mathrm{defect}}(\Lambda) = \int_L \psi(\theta b, \tau) d\mathcal{H}^1, \tag{2.6.3}$$

with the function ψ satisfying some properties of smoothness and convexity, and where b, θ, and τ represent the Burgers vector, its multiplicity, and the tangent vector to the dislocation loop L, respectively. We will deal with this functional in Section 2.8 below (see formula (2.8.4)).

Remark 2.6.2. The term involving $\|\dot{\varphi}_j\|_{L^1}$ in the energy bound is mandatory for mesoscopic dislocations, since it controls the length of the lines. In fact, minimizing sequences of Lipschitz maps (describing minimizing sequences of lines) might become locally dense, a phenomenon which should be prohibited to get existence. Moreover, recalling (2.2.29), this term implies a bound on the densities. From a physical viewpoint this term is questionable since dense arcs of the dislocation cluster might be nonnecessary, and hence admissible from an energetical standpoint. This drawback is addressed in the second existence result for continuum dislocations in Section 2.6.2. See also the discussion about the model in Section 2.7.3.

Before stating the existence of minimizers of the problem (2.6.1) some technical results should be stated and proven.

Lemma 2.6.3. *Let* $(F_k, \mathcal{L}_k)$ *be a minimizing sequence for the problem* (2.6.1)*, and suppose* $\det F_k \rightharpoonup D$ *weakly in* $L^p(\Omega)$*. Then* $D > 0$ *a.e. in* Ω*.*

Proof. Let $A := \{D = 0\}$ and suppose A has positive Lebesgue measure. We have $\det F_k \rightharpoonup 0$ weakly in $L^p(A)$, which since $\det F_k \geq 0$ on A implies that $\liminf \det F_k = 0$ almost everywhere in A. Indeed, if $B := \{x \in A : \liminf \det F_k(x) > 0\}$ has positive measure, then $\liminf \int_A \det F_k > 0$ since $\chi_A \in L^q(A)$, a contradiction.

Hence from condition $(W1)$ we must have $\mathcal{W}(F_k, \Lambda_{\mathcal{L}_k}) \geq \int_A W_e(F_k, \Lambda_{\mathcal{L}_k})dx \geq \int_A h(\det F_k)dx$. By Fatou's Lemma and the fact that $(F_k, \mathcal{L}_k)$ is a minimizing sequence, the contradiction follows, so A must be negligible, achieving the proof. □

Lemma 2.6.4. *Let* γ_n *be a sequence of* 1*-currents inside* $\bar{\Omega}$ *such that* $\gamma_n = \varphi_{n\sharp}[\![0, M]\!]$ *for Lipschitz functions* φ_n *with* $\mathrm{Lip}(\varphi_n) \leq 1$ *for all* n*. Then, there is* 1*-current* γ *such that, up to subsequence,* $\gamma_n \rightharpoonup \gamma$*, and* $\gamma = \varphi_\sharp[\![0, M]\!]$ *for a Lipschitz function* φ *with* $\mathrm{Lip}(\varphi) \leq 1$*.*

Proof. The functions φ_n are equibounded and equicontinuous on $[0, M]$, and by the Ascoli-Arzelà Theorem there is a map $\varphi : [0, M] \to \mathbb{R}^3$ with $\mathrm{Lip}(\varphi) \leq 1$ such that, up to subsequence, $\varphi_n \to \varphi$ uniformly. So it easily follows that $\gamma_n \rightharpoonup \gamma := \varphi_\sharp[\![0, M]\!]$. □

Lemma 2.6.5. *Let* $\hat{\mathcal{L}}_n = \{S_n, \tau_n, \theta_n\}$ *be a sequence of equibounded dislocation currents of the form* (2.2.22) *all satisfying the same boundary condition. Then there is a dislocation current* $\hat{\mathcal{L}}$ *such that* $\hat{\mathcal{L}}_n$ *weakly converges to* $\hat{\mathcal{L}}$ *in the sense of currents and that* $\Lambda_n := \Lambda_{\mathcal{L}_n}$*, the sequence of densities of* $\mathcal{L}_n$*, weakly* converges to* $\Lambda \in \mathcal{M}(\bar{\Omega}, \mathbb{R}^{3\times 3})$*. Moreover* $\hat{\mathcal{L}}$ *satisfies the boundary condition, it has density equal to* $\Lambda = \Lambda_{\mathcal{L}}$*, and for all* $i = 1, 2, 3$*,* $\mathcal{L}_i^n \rightharpoonup \mathcal{L}_i$*,* $\Lambda_i^n \rightharpoonup \Lambda_i$*, and* $\Lambda_i = \mathcal{L}_i \otimes e_i$ *(with the notation* (2.2.13)*).*

Proof. As in (2.2.22) we write $\hat{\mathcal{L}}_n = \hat{\mathcal{L}}_n^1 + \hat{\mathcal{L}}_n^2 + \hat{\mathcal{L}}_n^3$, and $\Lambda_n = \Lambda_n^1 + \Lambda_n^2 + \Lambda_n^3$, with $\Lambda_n^i = \mathcal{L}_n \otimes e_i$. By the assumption we have that also $\mathcal{L}_n^i$ are boundaryless in Ω and, thanks to (2.2.24), we have that $N(\mathcal{L}_n^i)$ are uniformly bounded, so that, by Theorem 2.1.3, we deduce the existence of three closed integer multiplicity currents $\{\mathcal{L}^i\}_{i=1}^3$ such that $\mathcal{L}_n^i \rightharpoonup \mathcal{L}^i$. Since

$$\hat{\mathcal{L}}_n(\omega) = \sum_{i=1}^{3} \mathcal{L}_n^i(\omega_i) \to \sum_{i=1}^{3} \mathcal{L}^i(\omega_i), \tag{2.6.4}$$

for all $\omega \in \mathcal{D}^1(\Omega, \mathbb{R}^3)$, we get $\hat{\mathcal{L}}_n \rightharpoonup \hat{\mathcal{L}} := \sum_{i=1}^3 \hat{\mathcal{L}}^i$. The fact that $\hat{\mathcal{L}}$ satisfies the boundary condition follows from the fact that $\partial\hat{\mathcal{L}}_n \rightharpoonup \partial\hat{\mathcal{L}}$. Identifying $\mathcal{D}^1(\Omega, \mathbb{R}^3)$

with $C_c^\infty(\Omega, \mathbb{R}^{3\times3})$ it is straightforward that $\Lambda_n \rightharpoonup \Lambda = \Lambda^1 + \Lambda^2 + \Lambda^3$ weakly* in $\mathcal{M}(\bar{\Omega}, \mathbb{R}^{3\times3})$, with $\Lambda_n^i \rightharpoonup \Lambda^i$ weakly* in $\mathcal{M}(\bar{\Omega}, \mathbb{R}^{3\times3})$, and that $\Lambda^i = \mathcal{L}^i \otimes e_i$ for all $i = 1, 2, 3$, achieving the proof. $\square$

Now we are ready to solve Problem (2.6.1).

Theorem 2.6.6 (Existence in $\mathcal{F}\times\mathcal{MD}$)**.** *Under assumptions* $(W1)-(W4)$ *and assuming that there exists an admissible* $(F, \mathcal{L}) \in \mathcal{F}\times\mathcal{MD}$ *such that* $\mathcal{W}(F, \Lambda_\mathcal{L}) < \infty$*, there is at least a* $(F, \mathcal{L})$ *solution of the minimum problem* (2.6.1).

Proof. Let $(F_n, \mathcal{L}_n)$ be a minimizing sequence in $\mathcal{F}$. Then $\|F_n\|_{L^p}$, $\|\mathrm{adj}F_n\|_{L^p}$, $\|\det F_n\|_{L^p}$ are uniformly bounded, so that there exist $F,\ A \in L^p(\Omega, \mathbb{R}^{3\times3})$, $D \in L^p(\Omega)$ such that

$$F_n \rightharpoonup F \quad \text{weakly in } L^p(\Omega, \mathbb{R}^{3\times3}), \tag{2.6.5a}$$

$$\mathrm{adj}\ F_n \rightharpoonup A \quad \text{weakly in } L^p(\Omega, \mathbb{R}^{3\times3}), \tag{2.6.5b}$$

$$\det F_n \rightharpoonup D \quad \text{weakly in } L^p(\Omega). \tag{2.6.5c}$$

Since we consider extensions $\hat{F}_n$ of F on $\hat{\Omega}$, it is straightforward that we can suppose the same boundedness for $\hat{F}_n$ on $\hat{\Omega}$ as for F_n on Ω, so that $\hat{F}$, $\hat{A}$, and $\hat{D}$ are such that (2.6.5a)-(2.6.5c) hold for $\hat{F}_n$, $\hat{F}$, $\hat{A}$, and $\hat{D}$. Moreover, since F_n satisfy the same boundary condition, it is obvious that $\hat{F}_n = \hat{F} = \bar{F}$ on $\hat{\Omega}\setminus\Omega$, so $\hat{F}$ satisfies the boundary condition.

By the uniform bound on $\sum_{j\le k_\mathcal{L}} b^j \|\dot{\varphi}_j\|_{L^1}$ in (2.6.2) and by (2.2.29), it holds a uniform bound on $\Lambda_n^T := -\,\mathrm{Curl}\ \hat{F}_n$, and there is a measure $\Lambda \in \mathcal{M}(\bar{\Omega}, \mathbb{R}^{3\times3})$ such that

$$\Lambda_n \rightharpoonup \Lambda \quad \text{weakly* in } \mathcal{M}(\bar{\Omega}, \mathbb{R}^{3\times3}). \tag{2.6.5d}$$

The result will follow by the direct method of the calculus of variations and classical semicontinuity results for convex functionals, since conditions $(W1)-(W4)$ hold, provided the found minimizer is admissible.

Since the energies at $(F_n, \mathcal{L}_n)$ are uniformly bounded by $k_\mathcal{L}$ in (2.6.2), we can suppose that the dislocation currents $\hat{\mathcal{L}}_n$ are generated by the same number k of 1-Lipschitz functions $\{\varphi_n^j\}_{j=1}^k$, i.e.,

$$\hat{\mathcal{L}}_n(\omega) = \sum_{j=1}^k \varphi^j_{n\sharp}[\![0, M]\!](\omega b^j) \qquad \text{and} \qquad \Lambda_n = \sum_{j=1}^k \varphi^j_{n\sharp}[\![0, M]\!] \otimes e_i. \tag{2.6.6}$$

for all $\omega \in \mathcal{D}^1(\hat{\Omega}, \mathbb{R}^3)$. So by Lemma 2.6.4 we can suppose that for every j we have

$$\varphi^j_{n\sharp}[\![0, M]\!] \rightharpoonup \varphi^j_\sharp[\![0, M]\!],$$

for some 1-Lipschitz functions $\{\varphi^j\}_{j=1}^k$. If we set $\hat{\mathcal{L}}(\omega) := \sum_j \varphi^j_\sharp[\![0, M]\!](\omega b^j)$ for all $\omega \in \mathcal{D}^1(\hat{\Omega}, \mathbb{R}^3)$, by Lemma 2.6.5 we have $\hat{\mathcal{L}}_n \rightharpoonup \hat{\mathcal{L}}$, $\Lambda_n \rightharpoonup \sum_j \varphi^j_\sharp[\![0, M]\!] \otimes b^j$ weakly* in $\mathcal{M}(\hat{\Omega}, \mathbb{R}^{3\times3})$, so from (2.6.5d) we get

$$\Lambda = \sum_j \varphi^j_\sharp[\![0, M]\!] \otimes b^j. \tag{2.6.7}$$

Now, for a test function $w \in C_c^\infty(\hat{\Omega}, \mathbb{R}^{3\times3})$, it holds

$$\langle\,\mathrm{Curl}\ \hat{F}_n, w\rangle = \langle \hat{F}_n, \mathrm{Curl}\ w\rangle \to \langle \hat{F}, \mathrm{Curl}\ w\rangle = \langle\,\mathrm{Curl}\ \hat{F}, w\rangle. \tag{2.6.8}$$

Since the first term in the left-hand side of (2.6.8) also tends to $\langle -\Lambda^T, w\rangle$, we finally get

$$- \operatorname{Curl} \hat{F} = \sum_j b^j \otimes \varphi^j_\sharp [\![0, M]\!]. \tag{2.6.9}$$

Let us set $L_n := \cup_{j=1}^k \varphi_n^j([0,M])$ and $L := \cup_{j=1}^k \varphi^j([0,M])$. We now want to show that for every point $x \in \Omega \setminus L$ there is a ball $B \subset \Omega \setminus L$ centered at x and a map $u \in \mathrm{Cart}^p(B, \mathbb{R}^n)$ such that $Du = F$ in B. Let x be such a point, since $\varphi_n^j \to \varphi^j$ uniformly, it follows that L_n tends to L in the Gromov-Hausdorff topology, so that we have $B \cap L_n = \varnothing$ for n sufficiently large. In such a ball, by hypotheses, there are maps $u_n \in \mathrm{Cart}^p(B, \mathbb{R}^n)$ satisfying $Du_n = F_n$, and, up to summing suitable constants to u_n, we can also suppose u_n have all zero average in B. So that the Poincaré's inequality provides u such that $u_n \rightharpoonup u$ weakly in $W^{1,p}$. Now Theorem 2.1.7 implies that $A = \mathrm{adj}F$ and $D = \det F$, so the thesis follows from (2.6.5a)-(2.6.5c) and Lemma 2.6.3. ☐

We remark that with the formulation (2.6.2) the potential $W(F, \Lambda_{\mathcal{L}})$ depends explicitly on the dislocation current.

An example. Let $\Omega \subset \mathbb{R}^3$ be the open set defined, in cylindrical coordinates, by

$$\Omega := \{0 < \rho < R, z \in (-h, h)\}.$$

Let $\hat{\Omega}$ be a ϵ-neighborhood of Ω and set $U := \hat{\Omega} \setminus \Omega$.

With this example we would like to show that provided a boundary condition for the dislocation density, the dislocation of the minimizers will not be in U but will stay inside Ω.

Then we consider the map $\bar{F} : \hat{\Omega} \to \mathbb{R}^{3\times 3}$ defined as

$$\bar{F}(\rho, \theta, z) = \zeta(\theta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sin\theta}{\rho}\beta & \frac{\cos\theta}{\rho}\beta & 1 \end{pmatrix}, \tag{2.6.10}$$

for some suitable smooth functions ζ, so that it turns out that

$$- \operatorname{Curl} \bar{F} = b \otimes e_z \mathcal{H}^1 \llcorner_{\hat{z} \cap U},$$

that is $\bar{F}$ shows a screw dislocation on the z-axis $\hat{z}$ with Burgers vector $b = (0, 0, \beta)$. We want to minimize the energy (2.6.2) satisfying (W1)-(W4)

$$\mathcal{W}(F, \Lambda_{\mathcal{L}}) := \int_\Omega \mathrm{W_e}(F) dx + W_{\mathrm{defect}}(\Lambda_{\mathcal{L}}),$$

among all the deformations F belonging to the class (2.5.16) with $\bar{F}$ as boundary condition. Let us suppose that the defect part of the energy takes the form

$$W_{\mathrm{defect}}(\Lambda_{\mathcal{L}}) = \gamma \int_0^1 \|\dot{\varphi}(s)\| ds + \sum_{1 \le i < k_{\mathcal{L}}} \gamma \int_{S^1} \|\dot{\varphi}_i(s)\| ds + \mu |\Lambda_{\mathcal{L}}(\Omega)|, \tag{2.6.11}$$

where the mesoscopic dislocation $\mathcal{L}$ is the image of $k_{\mathcal{L}}$ closed loops φ_i with Burgers vector b_i and of φ which is a dislocation with endpoints $P := (0, 0, h)$

and $Q := (0,0,-h)$ and Burgers vector b. Then let us consider an admissible deformation which shows only one dislocation path φ^0 coinciding with the segment $\overline{PQ}$. In this case $k_{\mathcal{L}} = 1$ and the energy is

$$\mathcal{W}(F^0) = \int_\Omega \mathrm{W_e}(F^0)dx + \gamma \int_0^1 \|\dot{\varphi}^0(s)\|ds + \mu|\Lambda_{\mathcal{L}^0}(\Omega)| = \\ = \int_\Omega \mathrm{W_e}(F^0)dx + 2h\gamma + 2h\mu\beta. \quad (2.6.12)$$

Let us now take another admissible deformation F^1 which has the dislocation path φ^1 connecting P and Q which has an intermediate point at $\varphi(t) = (x_t, y_t, z_t) \in \Omega$ with $R_t := (x_t^2 + y_t^2)^{1/2} > 0$. In this case we have

$$W_{\mathrm{defect}}(\mathcal{L}^1) \geq \gamma \int_0^1 \|\dot{\varphi}^1(s)\|ds + \mu|\Lambda_{\mathcal{L}^1}(\Omega)| \\ \geq 2\gamma(R_t^2 + h^2)^{1/2} + 2h\mu\beta, \quad (2.6.13)$$

so that, if $2\gamma(R_t^2+h^2)^{1/2} > \int_\Omega \mathrm{W_e}(F^0)dx + 2h\gamma$ it turns out that $\mathcal{W}(F^0) < \mathcal{W}(F^1)$. This may happen if

$$R > R_t > \bar{R} := \frac{1}{2\gamma}\left((\int_\Omega \mathrm{W_e}(F^0)dx + 2h\gamma)^2 - h^2 \right)^{1/2}$$

so that in this case we see that the minimizer of the energy must have the dislocation path connecting P and Q inside the cylinder $\{x^2 + y^2 < \bar{R}, z \in (-h,h)\} \subsetneq \Omega$. In the contrary, if $R < \bar{R}$ then the dislocation of the minimizer could lie outside Ω. In particular we see that with our choice of boundary datum dislocations tends to remain inside the body Ω and not to escape from the boundary.

2.6.2 Second existence result

We now prove an existence result with $\mathcal{W}$ a function of F only, and where the dislocations associated to the optimal F are *geometrically equivalent* to a 1-set. This means that the dislocation itself can be locally dense and of infinite length. As for the first result, we fix a boundary condition α and a map $\bar{F} \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ such that $-\operatorname{Curl} \bar{F} = (\Lambda_\alpha)^{\mathrm{T}}$ on U. We redefine the set of admissible functions:

$$\mathcal{F}' := \{F \in L^p(\Omega, \mathbb{R}^{3\times 3}) : F \text{ satisfies (i)-(iii) below}\} \quad (2.6.14)$$

(i) There exists a continuum dislocation $\mathcal{L} := \mathcal{L}_{\mathcal{I}} \in \mathcal{CD}$ satisfying the boundary condition such that $\hat{F} := \bar{F}\chi_{\hat{\Omega}\setminus\Omega} + F\chi_\Omega \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ satisfies $-\operatorname{Curl} \hat{F} = (\Lambda_{\mathcal{L}})^{\mathrm{T}}$ in $\hat{\Omega}$.

(ii) There is a continuum $\mathcal{C}$ such that $L^\star \subset C$ and such that for every $x \in \Omega\setminus\mathcal{C}$ there is a ball $B \subset \Omega \setminus \mathcal{C}$ centered at x and a function $\phi \in \mathrm{Cart}^p(B; \mathbb{R}^3)$ satisfying $F = D\phi$ in B.

(iii) $\det F > 0$ almost everywhere in Ω.

We consider a slightly different set of assumptions on $\mathcal{W} : \mathcal{F}' \to \bar{\mathbb{R}}$:

(W5) there is a positive constant C such that

$$\mathcal{W}(F) \geq C\big(\||\mathcal{M}(F)|\|_{L^p} + \| \operatorname{Curl} \hat{F}\|_{\mathcal{M}(\bar{\Omega})} + G(\mathcal{L})\big) - \beta,$$

with

$$G(\mathcal{L}) := \inf_{\mathcal{K}\in\mathcal{C}_{\mathcal{L}}} \left(\mathcal{H}^1(\mathcal{K}) + \kappa\#\mathcal{K}\right), \tag{2.6.15}$$

where $\#\mathcal{K}$ represents the number of connected components of the embedding continuum $\mathcal{K}$. Note that by Golab theorem G is also lower semicontinuous.

(W6) there exists a convex and weakly lower semicontinuous function g and a weakly lower semicontinuous functional $\mathcal{W}_{\text{defect}}$ such that

$$\mathcal{W}(F) = \mathcal{W}_e(F) + \mathcal{W}_{\text{defect}}(-(\operatorname{Curl} \hat{F})^{\mathrm{T}}).$$

It is also assumed that $\mathcal{W}_e(F) = \int_\Omega g(\mathcal{M}(DF))dx$ with g as in (W2) above and $g(\mathcal{M}(DF)) \geq h(\det F)$, for some continuous real function h such that $h(t) \to \infty$ as $t \to 0$.

As mentioned for the first minimum problem, again we can assume $\mathcal{W}_{\text{defect}} = \mathcal{W}^1_{\text{defect}} + \mathcal{W}^2_{\text{defect}}$, with, for instance, $\mathcal{W}^2_{\text{defect}} = \kappa G$ for some $\kappa > 0$, whereas a typical example for $\mathcal{W}^1_{\text{defect}}$ is the form

$$\mathcal{W}^1_{\text{defect}}(\Lambda) = \int_L \psi(\theta b, \tau) d\mathcal{H}^1. \tag{2.6.16}$$

Under suitable hypotheses on the function ψ, this is proved to be lower semicontinuous in the sense of $(W6)$ (see [17] and formula (2.8.4) below). As for the function g, hypothesis $(W2)$ fulfills the requirements.

Remark 2.6.7. Again, the term $\mathcal{W}^1_{\text{defect}}$ involving the function G is slightly unnatural from a physical point of view. However it has a crucial role and is mathematically necessary. The physical interpretation of $G(\mathcal{L})$ is the following. To create a new loop at some finite distance d from the current dislocation $\mathcal{L}$, it is worth to nucleate (i.e., add a connected component) rather than deforming the existent dislocation, as soon as $d > \kappa$. However it should be recognized that (2.6.15) is at this stage a mathematical assumption whose physical meaning remains to be elucidated. It basically means that the continuum dislocation lies in a compact 1-set which keeps as minimal the balance between the number of its connected subsets (of the continuum, not of the dislocation cluster) and its length.

In the third existence result we get rid of this mathematical tool, but we need a more delicate analysis of the strain.

Since $\mathcal{F}'$ is not empty, we now solve the minimum problem with these new assumptions.

Theorem 2.6.8 (Existence in $\mathcal{F}'$)**.** *Under assumption $(W5)$ and $(W6)$ and assuming that there exists an admissible $F \in \mathcal{F}'$ such that $\mathcal{W} := \int_\Omega W(F) < \infty$, there exists a minimizer of problem $\inf_{\mathcal{F}'} \mathcal{W}$.*

Proof. Let F_n be a minimizing sequence in $\mathcal{F}'$. We denote the dislocation currents associated to F_n by $\hat{\mathcal{L}}_n$, and their densities by $\Lambda_n = \Lambda_{\mathcal{L}_n}$. Without loss of generality, if we deal as in the proof of Theorem 2.6.6, we can assume F_n and $\hat{\mathcal{L}}_n$ be defined on the whole $\hat{\Omega}$. By $(W5)$, F_n converges weakly to F in L^p and Λ_n converges weakly-* to a Radon measure Λ. Thanks to (2.2.24) $\{\hat{\mathcal{L}}_n\}$ is equibounded, so that one has by Theorem 2.1.3 the existence of an integer multiplicity current $\hat{\mathcal{L}}$ such that $\hat{\mathcal{L}}_n \to \hat{\mathcal{L}}$, while by Lemma 2.6.5, $\Lambda = \Lambda_{\hat{\mathcal{L}}} = -\operatorname{Curl} \hat{F}$ in the distribution sense. Moreover, by admissibility, one can associate to every $\hat{\mathcal{L}}_n$ a continuum $\mathcal{K}_n \subset \hat{\Omega}$ such that $G(\hat{\mathcal{L}}_n) = \big(\mathcal{H}^1(\mathcal{K}_n) + k(\mathcal{K}_n)\big)$. By $(W5)$, Blaschke and Golab theorems, there is convergence in the Gromov-Hausdorff sense to a continuum $\mathcal{K}$. Now we see that the support $L^\star$ of $\hat{\mathcal{L}}$ is a subset of $\mathcal{K}$. Indeed, for all forms $\omega \in \mathcal{D}^1(\hat{\Omega}, \mathbb{R}^3)$ whose support is contained in $\hat{\Omega} \setminus \mathcal{K}$, it holds $\lim_{n\to\infty} \hat{\mathcal{L}_n}(\omega) = 0$, thanks to the fact that $\hat{\mathcal{L}_n}$ has support in $\mathcal{K}_n$ which converges to $\mathcal{K}$ in the Gromov-Hausdorff topology. So we find out that $\hat{\mathcal{L}} = (\hat{L}, \tau, \theta)$ is admissible since $L^\star := \operatorname{supp} \Lambda \subset \mathcal{K}$. Taking now any ball in $\Omega \setminus \mathcal{K}$, we conclude as in the proof of Theorem 2.6.6. □

2.7 Third existence result

2.7.1 Boundary of graphs of harmonic maps

We introduce the following notation. For all $b \in \mathbb{R}^3$ we define the 1-current $\vec{b} \in \mathcal{D}_1(\mathbb{T}^3)$ as

$$\vec{b}(\omega) := \int_0^{2\pi} \langle \omega(\frac{b_1\theta}{2\pi}, \frac{b_2\theta}{2\pi}, \frac{b_3\theta}{2\pi}), b\rangle d\theta, \tag{2.7.1}$$

for any 1-form $\omega \in \mathcal{D}^1(\mathbb{T}^3)$. It is easy to see that $M(\vec{b}) = 2|b|$. The fact that we are on the torus, i.e., ω is 2π-periodic on $\mathbb{R}^3$, implies that $\vec{b}$ is a closed current whenever $b \in 2\pi\mathbb{Z}$.

Let C be a closed loop of class C^1. There is a cylindrical neighborhood U with cylindrical coordinates $(\rho, \theta, z) \in [0, R] \times [0, 2\pi] \times [0, h]/\sim$, where $\sim$ means that the coordinate $\theta = 0$ (and $z = 0$) is identified with $\theta = 2\pi$ (resp. $z = h$). The neighborhood U is also parametrized by the coordinates (x, y, z) setting $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Let S be a smooth surface with boundary C and such that $S \cap U$ coincides with the set $\{\theta = 0\}$.

In the sequel we will use the notation $\Phi := Id \times u : \Omega \to \Omega \times \mathbb{T}^3$.

Proposition 2.7.1. *Let S be a smooth surface in Ω whose boundary C is a smooth and closed curve in Ω. Let $b = (b_1, b_2, b_3) \in 2\pi\mathbb{Z}^3$ and let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be the map with u_i given by (2.4.28) with $b = b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and its boundary is given by*

$$\partial\mathcal{G}_u(\omega) = \mathcal{L} \wedge \vec{b}(\omega), \tag{2.7.2}$$

for all $\omega \in \mathcal{D}^2(\Omega \times \mathbb{T}^3)$.

Before proving Proposition 2.7.1 we state the following preliminary fact:

Lemma 2.7.2. *Let u be as in Theorem 2.7.1. Then $u \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$ for all $p < 2$.*

Proof. Lemma (2.4.22) shows that u is well-defined in $\mathbb{T}^3$. In order to prove that it belongs to $\mathcal{A}_p(\Omega, \mathbb{T}^3)$ we need to show that all its minors $M_{\bar{\alpha}}^{\beta}(Du)$ belong to $L^p(\Omega)$. Thanks to Lemma (2.4.18) it is easy to see that every 1×1-minor belongs to $L^p(\Omega)$. Moreover from Lemma 2.4.15 we have that u_1, u_2, and u_3 differ from a constant, so that the rows of the matrix Du are linearly dependent. In particular all the minors greater than 1×1 vanish, and the thesis follows. □

Proof of Proposition 2.7.1. Let u_ϵ be the restriction of the map u to $\Omega_\epsilon := \Omega \backslash \bar{D}_\epsilon$, $u_\epsilon := u \llcorner_{\Omega_\epsilon}$, where $D_\epsilon := \{(\rho, \theta, z) \in [0, R] \times [0, 2\pi] \times [0, h]/ \sim = V : \rho < \epsilon\}$. The graph $\mathcal{G}_{u_\epsilon}$ is the restriction of the graph $\mathcal{G}_u$ to the open set $\Omega_\epsilon \times \mathbb{T}^3$. Formula (2.1.17) and the Dominated Convergence Theorem readily implies that $\mathcal{G}_{u_\epsilon} \rightharpoonup \mathcal{G}_u$ as current. As a consequence we find

$$\partial \mathcal{G}_{u_\epsilon} \rightharpoonup \partial \mathcal{G}_u.$$

In order to compute explicitly the boundary of $\mathcal{G}_u$ we write $\partial \mathcal{G}_{u_\epsilon}(\omega) = \mathcal{G}_{u_\epsilon}(d\omega)$, for $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^3)$. Lemma 2.4.22 implies that u is smooth outside a neighborhood of C, so that we can apply Stokes theorem and find

$$\partial \mathcal{G}_{u_\epsilon}(\omega) = \int_{\partial D_\epsilon} \langle \omega \circ \Phi, \frac{\partial \Phi}{\partial \tau} \wedge \frac{\partial \Phi}{\partial x_3} \rangle dx,$$

where (τ, x_3) is an orthogonal coordinate system in the tangent space to ∂D_ϵ. The gradient of Φ reads

$$D(\Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_1}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \tag{2.7.3}$$

Let $\partial D_\epsilon \cong [0, 2\pi] \times [0, h]/ \sim$ for all $(\theta, z) \in \partial D_\epsilon$. In the coordinate system $(\rho, \tau, x_3, y_1, y_2, y_3)$ it holds

$$D(\Phi \llcorner_{\partial D_\epsilon}) = (\frac{\partial \Phi}{\partial \tau}, \frac{\partial \Phi}{\partial x_3}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \frac{\partial u_1}{\partial \tau} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial \tau} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial \tau} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \tag{2.7.4}$$

If $\omega = \omega_{ij} dz_i \wedge dz_j$, with $1 \leq i < j \leq 6$, where we have defined $z_1 = \rho$, $z_2 = \tau$, $z_3 = x_3$, and $z_{k+3} = y_k$ for $k = 1, 2, 3$, we can write

$$\int_{\partial D_\epsilon} \langle \omega \circ \Phi, \frac{\partial \Phi}{\partial \tau} \wedge \frac{\partial \Phi}{\partial x_3} \rangle dx = \int_{\partial D_\epsilon} \sigma(i, \bar{i}) \omega_{ij}(x, u(x)) \tilde{M}_i^j (D(\Phi \llcorner_{\partial D_\epsilon}(x))) dx, \tag{2.7.5}$$

with $\tilde{M}_i^j(D(\Phi \llcorner_{\partial D_\epsilon}(x)))$ being the minor of $D(\Phi \llcorner_{\partial D_\epsilon})$ given by the i-th and j-th rows. From (2.7.4) we see that the (2×2)-minors of $D(\Phi \llcorner_{\partial D_\epsilon})$ which are nonzero are the only ones involving either the second or third row. So (2.7.5) reads

$$\int_{\partial D_\epsilon} \omega_{23}(x,u(x)) - \sum_{k=4}^{6}(\omega_{2k}(x,u(x))\frac{\partial u_{k-3}}{\partial x_3}(x) + \omega_{3k}(x,u(x))\frac{\partial u_{k-3}}{\partial \tau}(x))dx =$$
$$\int_{\partial D_\epsilon} \omega_{23}(x,u(x))dx - \sum_{k=4}^{6}\int_0^{2\pi}\int_0^l \epsilon(\tilde\omega_{2k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))\frac{\partial u_{k-3}}{\partial x_3}(\epsilon,\theta,x_3)dx_3 d\theta$$
$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi} \epsilon\tilde\omega_{3k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))\frac{\partial u_{k-3}}{\partial \tau}(\epsilon,\theta,x_3))d\theta dx_3, \tag{2.7.6}$$

where $\tilde\omega := \omega \det\Psi$, with $\Psi : [0,\epsilon]\times[0,2\pi]\times[0,l] \to D_\epsilon$ is the map of change of variables. Note that by the assumption of smoothness of C, we have that Ψ is smooth and $\det\Psi = 1$ on C. Now the first term vanishes as $\epsilon \to 0$ since ω is bounded and $\mathcal{H}^2(\partial D_\epsilon) \to 0$. Integrating by parts the second term and using Lemma (2.4.23) we obtain

$$\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l \frac{\partial\tilde\omega_{2k}}{\partial x_3}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))u_{k-3}(\epsilon,\theta,x_3)dx_3 d\theta =$$
$$\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l \frac{\partial\tilde\omega_{2k}}{\partial x_3}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))(\frac{\theta b_{k-3}}{2\pi}+o(1))dx_3 d\theta \tag{2.7.7}$$
$$+\sum_{k=4}^{6}\int_0^{2\pi}\epsilon\int_0^l\sum_{h=1}^{3} \frac{\partial\tilde\omega_{2k}}{\partial x_{3+h}}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))\frac{\partial u_h}{\partial x_3}(\epsilon,\theta,x_3)(\frac{\theta b_{k-3}}{2\pi}+o(1))dx_3 d\theta,$$

where $o(1) \to 0$ as $\epsilon \to 0$, so that its absolute value can be estimated by $\|\frac{\partial\tilde\omega_{2k}}{\partial x_3}\|_\infty o(1)$, and thus also this term vanishes. As for the third term of (2.7.5), we first set

$$R(\epsilon) := -\sum_{k=4}^{6}\int_0^{2\pi}\int_0^l \epsilon\Delta\tilde\omega_{3k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))\frac{\partial u_{k-3}}{\partial\tau}(\epsilon,\theta,x_3))dx_3 d\theta,$$

with $\Delta\tilde\omega_{3k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3)) := \tilde\omega_{3k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3)) - \tilde\omega_{3k}(0,\theta,x_3,u(\epsilon,\theta,x_3))$. Since $\frac{\partial}{\partial\tau} = \frac{1}{\epsilon}\frac{\partial}{\partial\theta}$, we obtain

$$-\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\tilde\omega_{3k}(0,\theta,x_3,u^+(0,\theta,x_3))\frac{\partial u_{k-3}}{\partial\theta}(\epsilon,\theta,x_3))d\theta dx_3 + R(\epsilon) =$$
$$-\sum_{k=4}^{6}\int_0^l \tilde\omega_{3k}(0,\theta,x_3,\frac{b_1\theta}{2\pi},\frac{b_2\theta}{2\pi},\frac{b_3\theta}{2\pi})u_{k-3}(\epsilon,\theta,x_3)\Big|_0^{\theta=2\pi}dx_3$$
$$+\sum_{k=4}^{6}\int_0^l\int_0^{2\pi}\frac{d}{d\theta}\tilde\omega_{3k}(0,\theta,x_3,\frac{b_1\theta}{2\pi},\frac{b_2\theta}{2\pi},\frac{b_3\theta}{2\pi})u_{k-3}(\epsilon,\theta,x_3))d\theta dx_3 + R(\epsilon). \tag{2.7.8}$$

Using Lemma 2.4.18, we estimate

$$\begin{aligned}|R(\epsilon)| &\le \sum_{k=4}^{6}\int_0^{2\pi}\int_0^{l}\epsilon\|\Delta\tilde{\omega}_{3k}(x,u(x))\|_{L^\infty(\partial D_\epsilon\cap\{\theta=\tilde{\theta}\})}\left|\frac{\partial u_{k-3}}{\partial\tau}(\epsilon,\tilde{\theta},x_3)\right|dx_3d\tilde{\theta}\\ &\le 16\pi^2(b_1+b_2+b_3)l\int_0^{2\pi}\|\Delta\tilde{\omega}_{3k}(x,u(x))\|_{L^\infty(\partial D_\epsilon\cap\{\theta=\tilde{\theta}\})}d\tilde{\theta}\to 0,\end{aligned}$$

as $\epsilon\to 0$, being $\tilde{\omega}_{3k}(\epsilon,\theta,x_3,u(\epsilon,\theta,x_3))$ uniformly continuous at $\epsilon=0$, again thanks to Lemma 2.4.23 and the fact that C is compact. So that letting $\epsilon\to 0$ in (2.7.8), using Lemma 2.4.23 and integrating by parts again, (2.7.8) becomes

$$\begin{aligned}&\sum_{k=4}^{6}\int_0^{l}\int_0^{2\pi}\omega_{3k}(0,0,x_3,\frac{b_1\theta}{2\pi},\frac{b_2\theta}{2\pi},\frac{b_3\theta}{2\pi})b_{k-3}d\theta dx_3\\ &=\int_0^{l}\int_0^{2\pi}\langle\omega(0,0,x_3,\frac{b_1\theta}{2\pi},\frac{b_2\theta}{2\pi},\frac{b_3\theta}{2\pi}),\tau\wedge b\rangle d\theta dx_3\\ &=\mathcal{L}\wedge\vec{b}(\omega\circ\Phi),\end{aligned}\tag{2.7.9}$$

and the proof is completed. □

Corollary 2.7.3. *Let S be a Lipschitz surface in Ω whose boundary C is a Lipschitz and closed curve in Ω. Let $b=(b_1,b_2,b_3)\in 2\pi\mathbb{Z}^3$ and let $u=(u_1,u_2,u_3):\Omega\to\mathbb{R}^3$ be a map with u_i satisfying formula* (2.4.28) *with $b=b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega\times\mathbb{T}^3)$ and* (2.7.2) *holds.*

Proof. We proceed by approximation. Let $\{C_k\}_{k>0}$ be a sequence of smooth closed curves approximating C (uniformly and in the sense of 1-currents) and let $\{S_k\}_{k>0}$ be smooth surfaces with boundary $\{C_k\}_{k>0}$ and converging (uniformly and in the sense of currents) to S. Let u_k be maps as in theorem 2.7.1 with C replaced by C_k and S replaced by S_k. Thanks to the uniform convergence of S_k to S and using formula (2.4.28) we see that u_k converges pointwise to u, and then strongly in $L^p(\Omega,\mathbb{T}^3)$. Since C_k are converging uniformly to C whose length is finite, the lengths of C_k are uniformly bounded so (2.4.36) gives a uniform bound in $L^p(\Omega)$, for some $1<p<2$, for the 1×1 minors of Du_k, while the greater minors are all null. Therefore there are maps $v^{\beta}_{\bar{\alpha}}\in L^p(\Omega)$ such that, up to a subsequence, $M^{\beta}_{\bar{\alpha}}(Du_k)\rightharpoonup v^{\beta}_{\bar{\alpha}}$ weakly in $L^p(\Omega)$. Finally, the lengths of C_k being uniformly bounded, Theorem 2.7.1 provides an uniform bound on the masses of $\partial\mathcal{G}_{u_k}$. Now Theorem 2.1.12 applies and implies that $u\in\mathcal{A}_p(\Omega,\mathbb{T}^3)$, $p>1$. In particular we have that $u_k\rightharpoonup u$ weakly in $\mathcal{A}_p(\Omega,\mathbb{T}^3)$, thus Lemma 2.1.6 implies that $\partial\mathcal{G}_{u_k}\rightharpoonup\partial\mathcal{G}_u$ as currents, and the fact that for u_k the explicit form (2.7.2) holds true implies that it holds also at the limit, concluding the proof. □

Theorem 2.7.4. *Let S be the union of $N>0$ Lipschitz surfaces S_k in Ω whose boundary is C, the union of the corresponding boundaries C_k, that are closed curves in Ω. Let $b=(b_1,b_2,b_3)\in 2\pi\mathbb{Z}^3$ and let $u=(u_1,u_2,u_3):\Omega\to\mathbb{R}^3$ be a map with u_i satisfying* (2.4.28) *with $b=b_i$. Then $\mathcal{G}_u$ is an integral current in $\mathcal{D}_3(\Omega\times\mathbb{T}^3)$ and* (2.7.2) *holds.*

Proof. Let us first suppose that S_k and C_k are smooth and that the curves C_k are mutually disjoint. Then we will obtain the general result by approximation by mean of Theorem 2.1.12, arguing as in the proof of Corollary 2.7.3. Since N is finite, we see that C is compact and there is a tubular neighborhood around C. We can then argue as in the proof of Lemma 2.7.1, obtaining a formula similar to (2.7.8). Now the same estimates hold thanks to Lemma 2.4.23 and Remark 2.4.24. The thesis follows. □

Lemma 2.7.5. *Let S, C, b and u as in Theorem 2.7.4, and let $v \in C^1(\bar{\Omega}, \mathbb{R}^3)$. Then $\mathcal{G}_{u+v}$ is an integral current in $\mathcal{D}_3(\Omega \times \mathbb{T}^3)$ and it holds*

$$M(\partial \mathcal{G}_{u+v}) \leq (1 + 64\sqrt{3}\pi^2 \|Dv\|_{L^\infty(\Omega)})|\mathcal{L} \otimes b|(\Omega). \tag{2.7.10}$$

Proof. As in Proposition 2.7.1, we first prove the result for a smooth loop C and then we obtain the general case arguing as in Theorem 2.7.4. Let us check that $u+v \in \mathcal{A}_p(\Omega, \mathbb{T}^3)$. To this aim let us prove that $\operatorname{adj}(Du+Dv)$ and $\det(Du+Dv)$ are summable functions. Since the rows of Du are linearly dependent and recalling the identity $\det A = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$, it follows that

$$\det(Du + Dv) = \det\begin{pmatrix} Dv_1 \\ Dv_2 \\ Dv_3 \end{pmatrix} + \det\begin{pmatrix} Du_1 \\ Dv_2 \\ Dv_3 \end{pmatrix} + \det\begin{pmatrix} Dv_1 \\ Du_2 \\ Dv_3 \end{pmatrix} + \det\begin{pmatrix} Dv_1 \\ Dv_2 \\ Du_3 \end{pmatrix}.$$

Since $Dv_i \in C^0(\bar{\Omega}, \mathbb{R}^3)$, in particular it is bounded, so that all the determinants belong to $L^p(\Omega, \mathbb{R}^3)$ thanks to (2.4.36). A similar arguments applies for $\operatorname{adj}(Du + Dv)$.

To compute the boundary of $\mathcal{G}_{u+v}$ we proceed as in the proof of Proposition 2.7.1, obtaining (2.7.5). This formula, setting $w := u + v$, takes the form

$$\begin{aligned}
&\int_{\partial D_\epsilon} \omega_{23}(x, w(x)) - \sum_{k=1}^{3} (\omega_{2k}(x, w(x)) \frac{\partial u_k}{\partial x_3}(x) + \omega_{3k}(x, w(x)) \frac{\partial u_k}{\partial \tau}(x)) dx \\
&- \int_{\partial D_\epsilon} \sum_{k=1}^{3} (\omega_{2k}(x, w(x)) \frac{\partial v_k}{\partial x_3}(x) + \omega_{3k}(x, w(x)) \frac{\partial v_k}{\partial \tau}(x)) dx + \\
&+ \sum_{4 \leq i,j \leq 6} \int_{\partial D_\epsilon} \omega_{ij}(x, w(x)) \tilde{M}_i^j (D(\mathrm{Id} \times v)\llcorner_{\partial D_\epsilon}(x)) dx \\
&+ \sum_{4 \leq i,j \leq 6} \int_{\partial D_\epsilon} \omega_{ij}(x, w(x)) (\frac{\partial u_i}{\partial x_3} \frac{\partial v_j}{\partial \tau} - \frac{\partial u_i}{\partial \tau} \frac{\partial v_j}{\partial x_3} + \frac{\partial u_j}{\partial x_3} \frac{\partial v_i}{\partial \tau} - \frac{\partial u_j}{\partial \tau} \frac{\partial v_i}{\partial x_3}) dx.
\end{aligned} \tag{2.7.11}$$

The first row, as seen, tends to (2.7.9), the second and the third ones vanish as $\epsilon \to 0$ since they tend to $\partial G_v(\omega)$ and since v is smooth $\partial G_v = 0$. The last row can be estimated, by virtue of (2.4.36), by

$$\begin{aligned}
&(8\pi)^2 l(|b_1| + |b_2| + |b_3|) \|Dv\|_{L^\infty(\Omega)} \\
&\leq 64\sqrt{3}\pi^2 l |b| \|Dv\|_{L^\infty(\Omega)} = 64\sqrt{3}\pi^2 |\mathcal{L} \otimes b|(\Omega) \|Dv\|_{L^\infty(\Omega)}.
\end{aligned} \tag{2.7.12}$$

This together with (2.7.9) gives (2.7.10), taking into account that $|\mathcal{L} \wedge \vec{b}| = |\mathcal{L} \otimes b|$. The proof is complete.

□

2.7.2 The minimum problem

For the rest of this section we assume that U is a neighborhood of the whole $\partial\Omega$ (i.e., $\partial_D\Omega = \partial\Omega$), so that $\Omega \Subset \hat{\Omega} := U \cup \Omega$. Moreover $\hat{\Omega}$ is simply connected and has smooth boundary. As usual we fix a boundary condition in U, so that in particular, for all admissible dislocations $\mathcal{L}$, it is controlled the distance $d(L, \partial\hat{\Omega})$, with $L := \text{supp}\mathcal{L}$. We deal with an energy $\mathcal{W}$ with the form

$$\mathcal{W}(F) := \mathcal{W}_e(F, \operatorname{Div} F) + \mathcal{W}_{\text{defect}}(\operatorname{Curl} F), \tag{2.7.13}$$

where we assume the following properties on $\mathcal{W}_e$ and $\mathcal{W}_{\text{defect}}$:

(i) The following coerciveness condition holds: there exists positive constants α_0, α_1, β_0, and β_1 such that

$$\mathcal{W}_e(F, \operatorname{Div} F) \geq \beta_1(\|F\|^p_{L^p} + \|\text{adj}F\|^p_{L^p} + \|\det F\|^p_{L^p} + \|\operatorname{Div} F\|^q_{L^q}) - \beta_0,$$
$$\mathcal{W}_{\text{defect}}(\Lambda) \geq \alpha_1|\Lambda|(\Omega) - \alpha_0.$$

(ii) $\mathcal{W}_{\text{defect}}$ is a function on $\mathcal{M}_b(\Omega, \mathbb{R}^{3\times3})$ which is lower semicontinuous with respect to the weak* convergence.

(iii) $\mathcal{W}_e$ is a function of $M(F)$ (i.e., of F, adj F, and $\det F$) and $\operatorname{Div} F$, it is lower semicontinuous in $M(F)$ with respect to the weak convergence in L^p, and is lower semicontinuous in $\operatorname{Div} F$ with respect to the weak convergence in L^q.

For instance, in order that (iii) be satisfied, we can choose $\mathcal{W}_e(F, \operatorname{Div} F) = \int_\Omega g(\mathcal{M}(F), \operatorname{Div} F)dx$, with g a convex function that has p-growth in $\mathcal{M}(F)$ and q-growth in $\operatorname{Div} F$.

Remark 2.7.6. The form (2.7.13) of the energy shows that a particular form of gradient elasticity is chosen, namely with the rotational part of the strain derivatives incorporated in the defect contribution of the energy, whereas the divergence part is related its elastic part. In general we can attribute the present model to the general second grade models, where the energy is a function of the strain and of its gradient, i.e., $\mathcal{W} = \mathcal{W}(F, \nabla F)$.

Remark 2.7.7. Hypotheses (ii) is weaker than the corresponding hypotheses on $\mathcal{W}_{\text{defect}}$ given in the previous minimum problems. Indeed, here we can get rid of the term $\mathcal{W}^2_{\text{defect}}$, and we can consider functionals of the form

$$\mathcal{W}_{\text{defect}}(\mu) = \mathcal{W}^1_{\text{defect}}(\mu) = \int_L \psi(\theta b, \tau)d\mathcal{H}^1,$$

again with suitable assumptions on ψ.

As for the previous minimum problems, we fix a boundary condition α for the dislocation and a map $\hat{F} \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ with $-\operatorname{Curl} \hat{F} = \Lambda_\alpha$ on $\hat{\Omega}$, and the additional property that $\operatorname{Div} \hat{F} \in L^q(\hat{\Omega}, \mathbb{R}^3)$. Let $q > 1$ and let $b \in 2\pi\mathbb{Z}^3$ a fixed Burgers vector, then we define the class of admissible functions as

$$\mathcal{F}^{p,q}_b := \{F \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3}) : -\operatorname{Div} F \in L^q(\hat{\Omega}, \mathbb{R}^3),\ -\operatorname{Curl} F = b \otimes \mathcal{L}$$
$$\text{for some closed integral 1-current } \mathcal{L},\ \text{and } F = \hat{F} \text{ on } \hat{\Omega} \setminus \Omega\}. \tag{2.7.14}$$

Let $1 < p < 2$, let $F \in \mathcal{BC}^p(\hat\Omega, \mathbb{R}^{3\times3})$ be such that $-\operatorname{Curl} F = b \otimes \mathcal{L}$, with $b \in 2\pi\mathbb{Z}^3$ and $\mathcal{L}$ a 1-integer multiplicity current which is closed in $\hat\Omega$ and its support is compact in $\hat\Omega$. The Helmholtz decomposition in $L^p(\hat\Omega, \mathbb{R}^3)$ provides $v \in W^{1,p}(\hat\Omega, \mathbb{R}^3)$ with $\partial_N v = FN$ on $\partial\hat\Omega$ and $G \in \mathcal{V}^p(\hat\Omega)$ such that

$$F = Dv + \operatorname{Curl} G. \tag{2.7.15}$$

If we set $V := \operatorname{Curl} G$, then of course $\operatorname{Div} V = 0$, while since $-\operatorname{Curl} F = b \otimes \mathcal{L}$, we also have $-\operatorname{Curl} V = b \otimes \mathcal{L}$. Moreover $VN = 0$ on $\partial\hat\Omega$. Thanks to the decomposition theorem for 1-integer multiplicity currents (Theorem 2.1.4) we find a sequence of Lipschitz maps

$$f_k : S^1 \to \hat\Omega \quad \text{such that} \quad \mathcal{L} = \sum_{k>0} f_{k\sharp}[\![S^1]\!]. \tag{2.7.16}$$

Let us denote by C_k the closed Lipschitz curves $f_k(S^1)$.

Theorem 2.7.8. *Let $\mathcal{L}$ be a closed integral current with compact support in $\hat\Omega$, and let $V \in L^p(\hat\Omega, \mathbb{R}^{3\times3})$ be such that $-\operatorname{Curl} V = b \otimes \mathcal{L}$ in $\hat\Omega$. Then there exists a map $\tilde u \in \mathcal{A}_p(\hat\Omega, \mathbb{T}^3)$ such that $\nabla\tilde u = V$ almost everywhere in $\hat\Omega$, and*

$$M(\partial\mathcal{G}_{\tilde u}) \le C|\mathcal{L} \otimes b|(\hat\Omega)(1 + |\mathcal{L} \otimes b|(\hat\Omega)), \tag{2.7.17}$$

with $C > 0$ a constant depending only on $\hat\Omega$ and on $d(L, \partial\hat\Omega) > 0$, $L := supp\mathcal{L}$. Moreover $\tilde u = u - v$ with $v \in C^1(\bar{\hat\Omega}, \mathbb{T}^3)$, $u \in \mathcal{A}_p(\hat\Omega, \mathbb{T}^3)$, and

$$\partial\mathcal{G}_u(\omega) = \mathcal{L} \wedge \vec b(\omega), \tag{2.7.18}$$

for all $\omega \in \mathcal{D}^3(\hat\Omega \times \mathbb{T}^3)$.

Proof. To prove the Theorem we will use the fact that the following system

$$\begin{cases} -\operatorname{Curl} U &= \mu & \text{in } \hat\Omega \\ \operatorname{Div} U &= 0 & \text{in } \hat\Omega \\ UN &= 0 & \text{on } \partial\hat\Omega, \end{cases} \tag{2.7.19}$$

has a unique solution that also satisfies $\|U\|_{L^p} \le C|\mu|(\hat\Omega)$, with $C = C(\hat\Omega)$. This is proved in [58].

Another key fact is the following: if $\hat\Omega$ is a bounded open set with smooth boundary, $g \in C^0(\partial\hat\Omega, \mathbb{R}^3)$ with $\int_{\partial\hat\Omega} gNdS = 0$, and $v \in C^1(\hat\Omega, \mathbb{R}^3)$ is the zero-mean value solution to

$$\begin{cases} \Delta v &= 0 & \text{in } \hat\Omega \\ \partial_N v &= g & \text{on } \partial\hat\Omega, \end{cases} \tag{2.7.20}$$

then it holds $\|v\|_{C^1} \le C\|g\|_{C^0}$, with $C = C(\hat\Omega)$.

We use the decomposition (2.7.16) for $\mathcal{L}$ and we first suppose that the maps f_k are smooth. The general case will follow from this by using an approximation argument and proceeding as in the proof of Theorem 2.7.4. If C_k is a smooth closed curve, we can choose a smooth surface S_k with boundary C_k. Then we set $S := \cup_k S_k$ and $C := \cup_k C_k$, we want to find u as a solution of (2.4.27) with these S and C. Let us also set $\hat S_n := \cup_{k=0}^n S_k$ and $\hat C_n := \cup_{k=0}^n C_k$. For

$i = 1, 2, 3$, let u_i^n be the solution of (2.4.27) with $\hat{S}_n$, $\hat{C}_n$, and b_i. Lemma 2.4.16 and Remark 2.4.17 show that the distributional divergence of ∇u^n is zero, while the curl is given by $-b \otimes \sum_{k=0}^{n} f_{k\sharp}[\![S^1]\!]$. Up to subtracting a constant to u^n, we also suppose it has zero mean value.

By hypotheses it holds $\inf_k d(C_k, \partial\hat{\Omega}) > 0$, and then u^n are of class C^∞ on $\partial\hat{\Omega}$, and their C^h norms are uniformly bounded with respect to n for all $h > 0$ (taking into account that the set $C = \cup_k C_k$ has finite length, and then $S = \cup_k S_k$ has finite $\mathcal{H}^2$ measure). Let v^n be the solution to (2.7.20) with $g := \partial_N u^n$. From the estimates of this solution we find $\|v^n\|_{C^1} \le C_1 \|\partial_N u^n\|_{C^0} < C_2$, for some constant C_2 independent of n. Setting $\tilde{u}^n := u^n - v^n$, we see that $\nabla\tilde{u}^n$ solves system (2.7.19) with $\mu = \mu^n := b \otimes \sum_{k=0}^{n} f_{k\sharp}[\![S^1]\!]$, so that we also have $\|\nabla\tilde{u}^n\|_p \le |\mu^n|(\hat{\Omega}) < C_3$, with C_3 independent of n. In particular we get $\|u^n\|_{W^{1,p}} \le \|v^n\|_{W^{1,p}} + \|\tilde{u}^n\|_{W^{1,p}} \le C$, for a constant $C > 0$ independent of n. Therefore $u^n \rightharpoonup u$ weakly in $W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$, for some $u \in W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$. Similarly $\tilde{u}^n \rightharpoonup \tilde{u}$ and $v^n \rightharpoonup v$ weakly in $W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$, with $u = \tilde{u} + v$. Since the rows of ∇u^n are equal up to a multiplicative factor, we also get that all the minors of u^n are uniformly bounded in L^p. Then, by Theorem 2.1.12, u^n weakly converge in $\mathcal{A}_p(\hat{\Omega}, \mathbb{T}^3)$ to u. Moreover Theorem 2.7.4 implies that for all $n > 0$ equation (2.7.2) holds for u^n, with $\mathcal{L}$ replaced by $\sum_{k=0}^{n} f_{k\sharp}[\![S^1]\!]$. Lemma 2.1.6 implies that $\mathcal{G}_u$ is an integral current and its boundary satisfies

$$\partial\mathcal{G}_u(\omega) = \sum_{k=1}^{\infty} f_{k\sharp}[\![S^1]\!] \wedge \vec{b}(\omega), \tag{2.7.21}$$

for all $\omega \in \mathcal{D}^3(\hat{\Omega} \times \mathbb{T}^3)$. To conclude the proof it suffices to observe that the maps $\tilde{u}^n$ are smooth in a neighborhood of $\partial\hat{\Omega}$ with $\partial_N\tilde{u}^n$ vanishing, and hence $\partial_N\tilde{u}$ also vanishes, in such a way that $\nabla\tilde{u}$ satisfies (2.7.19) with $\mu := b \otimes \mathcal{L}$. By the smoothness properties of v^n, it is also true that v satisfies (2.7.20) with a bounded and smooth $g = \partial_N u$, so it is smooth in $\hat{\Omega}$ and Lemma 2.7.5 applies (2.7.10). Since we can compute g by using formula (2.4.33) and $d(L, \partial\hat{\Omega}) > 0$, we find a constant $C_4 > 0$ such that $\|g\|_{C^1} \le C_4 |b \otimes \mathcal{L}|(\hat{\Omega})$, so that the inequality $\|v\|_{C^1} \le C\|g\|_{C^0}$ together with (2.7.10) gives (2.7.17). □

Remark 2.7.9. By definition of the u^k, we have observed that for all k the three components u_i^k, $i = 1, 2, 3$, differ in a multiplication factor. In particular we have seen that their gradients ∇u_i^k (i.e., the rows of the matrix ∇u) are linearly dependent differing in a multiplication factor. As a consequence the same is true for the gradients $\nabla\tilde{u}_i$. Thus, the three components of the harmonic function v have as boundary data $\partial_N\tilde{u}_i$ three linearly dependent vector fields. This implies, by the uniqueness of solution of elliptic equations, that also ∇v_i are linearly dependent. So that we find that the final matrix $V = \nabla u = \nabla\tilde{u} - \nabla v$ has linearly dependent rows and its pointwise adjunct and determinant are constantly zero.

The existence of a minimizer of $\mathcal{W}$ in $\mathcal{F}_b^{p,q}$ is provided by the following:

Theorem 2.7.10. *Let $p > 1$ and $q > 3$. If $\mathcal{W}$ satisfies (i), (ii), and (iii), then there exists a minimizer $F \in \mathcal{F}_b^{p,q}$ of $\mathcal{W}$.*

Proof. We will apply the direct method. Let $\{F_k\}_{k>0}$ be a minimizing sequence. From the coerciveness (i) we see that there exists $F \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$,

$A \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$, and $D \in L^p(\hat{\Omega})$ such that

$$F_k \rightharpoonup F \quad \text{weakly in } L^p(\hat{\Omega}, \mathbb{R}^{3\times 3}), \tag{2.7.22a}$$

$$\text{adj} F_k \rightharpoonup A \quad \text{weakly in } L^p(\hat{\Omega}, \mathbb{R}^{3\times 3}), \tag{2.7.22b}$$

$$\det F_k \rightharpoonup D \quad \text{weakly in } L^p(\hat{\Omega}), \tag{2.7.22c}$$

$$\text{Div } F_k \rightharpoonup R \quad \text{weakly in } L^q(\hat{\Omega}, \mathbb{R}^3). \tag{2.7.22d}$$

Moreover we find a measure $\Lambda \in \mathcal{M}_b(\hat{\Omega}, \mathbb{R}^{3\times 3})$ with

$$\Lambda_k \rightharpoonup \Lambda \quad \text{weakly* in } \mathcal{M}_b(\hat{\Omega}, \mathbb{R}^{3\times 3}), \tag{2.7.23}$$

where we have set $\Lambda_k = \Lambda_{\mathcal{L}_k} = -\text{Curl } F_k$. As [59, Lemma 7.5] shows, there exists a regular dislocation current $\mathcal{L}$ such that $(\mathcal{L}_k)_i \rightharpoonup \mathcal{L}_i$ in $\mathcal{D}_1(\hat{\Omega})$ and $\Lambda = \Lambda_{\mathcal{L}} = \sum_{i=1}^3 \mathcal{L}_i \otimes e_i$. In order to prove the Theorem we have to show that $\text{Div } F = R$, $-\text{Curl } F = \Lambda_{\mathcal{L}}$, $A = \text{adj} F$, and $D = \det F$.

The Helmholtz decomposition gives

$$F_k = Dw_k + \text{Curl } G_k, \tag{2.7.24}$$

with $w_k \in W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$ which satisfies $-\Delta w_k = -\text{Div } F_k$ with $\partial_N w_k = F_k N = \tilde{F} N$ on $\partial\hat{\Omega}$, and $G_k \in \tilde{\mathcal{V}}^p(\hat{\Omega})$. Since $\text{Div } F_k \in L^q(\hat{\Omega}, \mathbb{R}^3)$, with $q > 3$, by the regularity theory of elliptic problems and the Sobolev embedding Theorem, we find that $w_k \in C^1(\hat{\Omega}, \mathbb{R}^{3\times 3})$ and that the L^∞ norm of their gradients are bounded by a constant,

$$\|Dw_k\|_\infty < C. \tag{2.7.25}$$

Moreover we have, up to a subsequence, that

$$w_k \rightharpoonup w \quad \text{weakly in } W^{1,q}(\hat{\Omega}, \mathbb{R}^3), \tag{2.7.26}$$

for some $w \in W^{1,q}(\hat{\Omega}, \mathbb{R}^3)$.

Let us set $V_k := \text{Curl } G_k$. Now $-\text{Curl } V_k = \Lambda_{\mathcal{L}_k}$, and Theorem 2.7.8 provides functions $u_k \in \mathcal{A}_p(\hat{\Omega}, \mathbb{T}^3)$ and $v_k \in C^1(\hat{\Omega}, \mathbb{T}^3)$ such that $\nabla u_k - \nabla v_k = V_k$ satisfying

$$\partial \mathcal{G}_{u_k}(\omega) = \mathcal{L}_k \wedge \vec{b}(\omega), \tag{2.7.27}$$

for all $\omega \in \mathcal{D}^3(\hat{\Omega} \times \mathbb{T}^3)$, and

$$\|Dv_k\|_\infty \le C|b \otimes \mathcal{L}_k|(\hat{\Omega}). \tag{2.7.28}$$

Thanks to (2.7.22a), (2.7.26), and (2.7.28), we can assume that there exist $u \in W^{1,p}(\hat{\Omega}, \mathbb{T}^3)$ and $v \in W^{1,p}(\hat{\Omega}, \mathbb{R}^{3\times 3})$ such that $u_k \to u$ and $v_k \to v$ strongly in $L^p(\hat{\Omega}, \mathbb{R}^3)$,

$$\nabla u_k \rightharpoonup \nabla u \quad \text{weakly in } L^p(\hat{\Omega}, \mathbb{R}^{3\times 3}). \tag{2.7.29}$$

and

$$\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in } L^p(\hat{\Omega}, \mathbb{R}^{3\times 3}). \tag{2.7.30}$$

Thanks to estimates (2.7.25) and (2.7.28), Lemma 2.7.5 applies providing

$$M(\partial \mathcal{G}_{w+u_k-v_k}) \le C(1 + |\Lambda_{\mathcal{L}_k}|(\hat{\Omega}))|\Lambda_{\mathcal{L}_k}|(\hat{\Omega}) < C. \tag{2.7.31}$$

This allows us to apply Theorem 2.1.12, obtaining

$$\operatorname{adj}(D\psi_k) \rightharpoonup \operatorname{adj}(D\psi) \quad \text{weakly in } L^p(\hat{\Omega}, \mathbb{R}^{3\times 3}), \tag{2.7.32}$$

$$\det(D\psi_k) \rightharpoonup \det(D\psi) \quad \text{weakly in } L^p(\hat{\Omega}), \tag{2.7.33}$$

with $\psi_k := w_k + u_k - v_k$ and $\psi := w + u - v$. As a consequence of (2.7.22a), convergences (2.7.32) and (2.7.33) read $\operatorname{adj}F_k \rightharpoonup \operatorname{adj}F$ weakly in $L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ and $\det F_k \rightharpoonup \det F$ weakly in $L^p(\hat{\Omega})$. Therefore $A = \operatorname{adj}F$ by (2.7.22b), and $\det F = D$ from (2.7.22c). Moreover, for every test function $\varphi \in C_c^\infty(\hat{\Omega}, \mathbb{R}^3)$ we have

$$\langle F, \nabla\varphi\rangle = \langle \nabla w, \nabla\varphi\rangle = \lim_{k\to\infty} \langle \nabla w_k, \nabla\varphi\rangle = \lim_{k\to\infty} \langle \operatorname{Div} F_k, \nabla\varphi\rangle,$$

and from (2.7.22d) it follows $R = \operatorname{Div} F$. Finally we write

$$\langle \operatorname{Curl} F_k, \varphi\rangle = \langle F_k, \operatorname{Curl} \varphi\rangle \to \langle F, \operatorname{Curl} \varphi\rangle = \langle \operatorname{Curl} F, \varphi\rangle,$$

for all $\varphi \in \mathcal{D}(\hat{\Omega}, \mathbb{R}^{3\times 3})$, and by (2.7.23) we conclude $-\operatorname{Curl} F = \Lambda_{\mathcal{L}}^T$ achieving the proof. □

2.7.3 Some remarks on the variational approach and open questions

In the last section we proposed three existence results for minima of the energy of a single crystal containing dislocations. In the first and second theorems we consider an energy depending on the strain F and its curl. Actually, the defect part of the energy, which coincides with the high order part of the energy, do not depend only on the total density of the dislocations, but also on their representation by mean of currents. Even if the current used to parametrize the dislocation has not a direct effect on the physics of the body, it can be related on the history of nucleation and annihilation of dislocations. To be precise, let us consider two single, simple, and disjoint loops in a body. In this case, the defect part of energy taken into account in Theorems 2.6.6 and 2.6.8 might be exactly a multiple of the total mass of the density. At the same time, if we consider a slow evolution of this system with the two dislocation loops (having equal Burgers vectors with opposite sign) overlapping until they coincide, then the density becomes null, and so the energy of the system, while, representing the dislocations still by two Lipschitz curves, the energy still counts 2 times the length of the single loop. This is an advantage if we are looking for an energy conservation law during the (quasistatic) evolution. In other words, the unnecessary dislocation obtained by the overlapping of two identical loops takes its part in the energy balance of the evolution, since otherwise, for instance, after their overlapping we might see a mysterious lost of energy.

As for the term considered in (2.6.15), it reflects the fundamental issue that nucleation of dislocations has a discrete and strictly positive cost. Indeed, the configuration of two single loops with identical Burgers vectors can be obtained either by two nucleations and dilations of dislocations, or by a single nucleation followed by the division of the single loop. It is quite natural to think that, if the given loops are far enough, it is energetically cheaper to proceed with the former evolution, while, if they are very close, it is too expensive to have a new nucleation.

On the contrary, we get rid of the representation of the dislocations in the third existence result. This fits quite better in order to study the absolute minima of the energy. As we already focused out, this result is stated for dislocations whose Burgers vectors belong to a 1-dimensional lattice, whereas its generalization is still an open problem. Moreover, in Theorem 2.7.10, we make the assumption that the energy is of the second order type, that is, it depends on F and on its first derivative. Since the norm of the gradient can be controlled by the norms of the divergence and the curl, it is natural to assume that this energy depends directly on these objects.

Moreover we emphasize that in the first two results we assume also a strong requirement on the regularity of the admissible deformations outside the dislocation lines. However we recover this regularity in the third result by the assumption on the integrability of the divergence, which, being quite natural, justifies the previous assumption. We remind that Cartesian maps are considered to represent the deformation F, so that its adjunct and determinant are only locally defined away from a continuum, that is $\mathrm{Cof}F = \mathrm{cof}F \in L^p_{loc}(\Omega \setminus \mathcal{K})$ and $\mathrm{Det}F = \det F \in L^p_{loc}(\Omega \setminus \mathcal{K})$. The fact that the adjunct and the determinant might be concentrated distributions on $\mathcal{K}$ means that the continuum represents a singular set where spurious effects might take place, such as cavitation, and hence nucleation of elementary dislocation loops. This makes sense from a physical standpoint, since dislocations at the mesoscale are by essence the location of field singularities.

It is yet an open question to elucidate the structure of the distributional determinant, which one would like for physical reasons to be a Radon measure (i.e., an extensive field) on $\mathcal{K}$. To our knowledge few results exist about this issue, without the too restrictive assumptions of field boundedness, high space dimension, and with the current range of p between 1 and 2.

The described mathematical framework will be considered for future work in order to describe evolution problems involving the dissipation due to dislocation motion. Here a preliminary step before the complete dynamics will be the quasistatic problem. The role of higher-order strains acting as constrain reactions to the geometrical condition $-\,\mathrm{Curl}\ F = \Lambda^T$ is studied in the next section.

Two other extensions of this work are the analysis of the distributional determinant at the continuum $\mathcal{K}$, in particular to address the open question whether it is a measure, and homogenization of a countable family to the continuum to the macroscale where Γ-convergence tools may be considered (see, eg., [22]). About the latter problem let us mention that our setting at the continuum scale, allowing for countable many dislocations, was thought with a view to homogenization, since limit passage from finite to countable families must unavoidably be faced.

2.8 Configurational forces at minimizing dislocation clusters

Certain forces apply on the dislocation clusters, solutions to the minimization problems considered in the previous Section. They are due to the combined effect of the deformation and defect part of the energy. The line having no mass, these forces must be understood as being of configurational nature. All results of previous sections will allow us to prove Theorem 2.8.5, which consists of a balance of forces at minimality. The keypoint to obtain this balance law is to perform variations around the minima of Problem (2.6.1) in the largest possible functional spaces. As far as the deformation part of the energy is concerned, this amounts to proving the existence of an appropriate Lagrange multiplier to account for the constraint

$$- \operatorname{Curl} F = \Lambda_{\mathcal{L}}.$$

This will be achieved thanks to Theorem 2.8.4. In principle, variations can be made with respect to (i) F, (ii) the dislocation density Λ, and (iii) the dislocation set L. In the first case one recovers the equilibrium equations, where the Piola-Kirchhoff stress is written as the curl of the constraint reaction. The second case is more delicate since the space of variations is not a linear space (due to the so-called crystallographic assumption), thus creating a series of difficulties which we do not address further. Most interesting is the variation with respect to the line, that is, with respect to infinitesimal Lipschitz variations at the optimal dislocation cluster $L^\star$. The difficulty here is that both F and Λ depend on L. In the case of Λ, the dependence is explicit since L is in some sense the support of $\Lambda = \Lambda_{\mathcal{L}}$ (see (2.5.2)). In the case of F, the dependence is implicit since it holds

$$F = \nabla u + F^\circ, \tag{2.8.1}$$

where F implicitly depends on $\mathcal{L}$ through F° solution of $\operatorname{Curl} F^\circ = -(\Lambda_{\mathcal{L}})^{\mathrm{T}}$. Therefore, since the energy consists of one term in F and another in Λ, variation of the energy w.r.t. to $\mathcal{L}$ will require an appropriate version of the chain rule. This computation is the main objective of this Section, which to be carried out carefully requires a series of preliminary steps.

2.8.1 Review of the existence of minimizers

We will compute the variation at minima of problem (2.5.1). We write in a slightly different setting the already stated existence results, and then we will employ some variational tecnique to the solutions.

Let us define

$$\mathcal{F}'' := \{F \in \mathcal{AD}^p(\Omega) : \tilde{F} := F\chi_\Omega + \hat{F}\chi_V \in \mathcal{AD}^p(\hat{\Omega}), - \operatorname{Curl} \tilde{F} = \left(\Lambda_{\hat{\mathcal{L}}}\right)^{\mathrm{T}} \text{ on } \hat{\Omega}$$
$$\text{for some closed and connected dislocation current } \hat{\mathcal{L}} \text{ in } \hat{\Omega}\}. \tag{2.8.2}$$

We remark that, asking for the connection of the dislocation current $\hat{\mathcal{L}}$, amounts to requiring that the dislocation set L is connected, that physically corresponds to the presence of a unique dislocation cluster. Connection is equivalent to the fact that we always suppose that, for an admissible F, there exists

$\lambda \in W^{1,1}(S^1, \hat{\Omega}^3)$ such that $-\operatorname{Curl} F = (\Lambda)^{\mathrm{T}} = T(\lambda)$. In general, this does not imply that a unique Burgers vector b is assigned to $\mathcal{L}$.

The coerciveness conditions on the energy $\mathcal{W}$ are summarized as follows. We assume that there are positive constants C and β for which

$$\hat{\mathcal{W}}(F) = \mathcal{W}(F, \Lambda_{\mathcal{L}}) := \mathcal{W}_e(F) + \mathcal{W}_{\text{defect}}(\Lambda_{\mathcal{L}}) \geq C\big(\|M(F)\|_{L^p} + m(\Lambda_{\mathcal{L}})\big) - \beta, \tag{2.8.3}$$

with $F \in \mathcal{F}''$ and $M(F) = (F, \operatorname{adj} F, \det F)$. Moreover we consider the following assumption

(H) We assume that

$$\mathcal{W}_{\text{defect}}(\Lambda_{\mathcal{L}}) := \mathcal{W}^1_{\text{defect}}(\Lambda_{\mathcal{L}}) + \mathcal{W}^2_{\text{defect}}(\Lambda_{\mathcal{L}}),$$

the second term being bounded from below by $m(\Lambda_{\mathcal{L}})$, defined in (2.5.9). Moreover $\mathcal{W}_{\text{defect}}$ is lower semicontinuous with respect to the weak* topology in $\mathcal{M}_b(\hat{\Omega}, \mathbb{R}^{3\times 3})$.

In [17] (where no variational problem is solved) an expression for $\mathcal{W}_{\text{defect}}$ is suggested as

$$\mathcal{W}^1_{\text{defect}}(\mu) = \int_L \psi(\theta b, \tau)\, d\mathcal{H}^1, \tag{2.8.4}$$

whenever $\mu = b \otimes \gamma_\sharp[\![S^1]\!] = b \otimes \theta\tau\mathcal{H}^1 \llcorner L$ is the dislocation density of a cluster generated by the loop $\gamma \in W^{1,1}(S^1, \mathbb{R}^3)$ and Burgers vector $b = \beta_i e_i, \beta_i \in 2\pi\mathbb{Z}$ $(b \neq 0)$, while it takes the value $+\infty$ if μ is not of this type. Here $\psi : 2\pi\mathbb{Z}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a nonnegative function satisfying $\psi(0, \cdot) = 0$ and $\psi(b, t) \geq c|b|$ for a constant $c > 0$. Note that the first entry of ψ is given by the Burgers vector of the dislocation, that in general does not coincide with b everywhere, but is a multiple of it since also the multiplicity of the current must be taken into account. We remark that within our formalism the multiplicity, defined in (2.5.6), depends on the image of the curve γ.

Remark 2.8.1. In general such a $\mathcal{W}_{\text{defect}}$ is not lower semicontinuous. However, the main result of [17] stated that its relaxation also writes in integral form

$$\overline{\mathcal{W}}_{\text{defect}}(\mu) = \int_L \bar{\psi}(\theta b, \tau)\, d\mathcal{H}^1, \tag{2.8.5}$$

for a function $\bar{\psi}$ satisfying some properties (see for details [17]).

According to Remark 2.8.1, we introduce the following alternative assumptions to (H):

(W4') $\mathcal{W}_{\text{defect}}(\Lambda_{\mathcal{L}}) := \mathcal{W}^1_{\text{defect}}(\Lambda_{\mathcal{L}}) + \mathcal{W}^2_{\text{defect}}(\Lambda_{\mathcal{L}})$, the second term being bounded from below by $m(\Lambda_{\mathcal{L}})$, defined in (2.5.9), and the first being of the form (2.8.4) (if it is already semicontinuous) or (2.8.5) (else).

(W5') g and $\mathcal{W}^2_{\text{defect}}$ are weakly lower semicontinuous.

The proofs of Theorems 2.6.6 and 2.6.8 actually provide the following:

Theorem 2.8.2. *Let $\hat{\mathcal{W}}$ be a potential satisfying assumptions* (2.8.3) *and either (W1), (W2), (W4), and (W5), or (W1), (W2), (W4'), and (W5'). Then there exists a minimizer of $\hat{\mathcal{W}}$ in $\mathcal{F}''$.*

Remark 2.8.3. As opposed to Theorem 2.6.8, in Hypothesis (2.8.3) we do not consider the term of the energy which accounts for the number of connected components of the dislocation set. This hypothesis is replaced by the restriction on the set of admissible deformations $\mathcal{F}''$ to show only one dislocation cluster as singularity set. Moreover since projections have been considered in (2.5.2) and (2.5.4), a bound on $m(\Lambda_{\mathcal{L}})$ in (2.8.3) is, contrarily to Theorem 2.6.8 and thanks to (2.5.8), sufficient to prove existence.

2.8.2 Existence of a constraint reaction

In the next sections we will deal with a linear and continuous map,

$$\Phi : \mathcal{BC}^p(\hat{\Omega}, \mathbb{R}^{3\times3}) \to \mathbb{R}, \tag{2.8.6}$$

such that $|\Phi(F)| \le C\|F\|_p$ for some $C > 0$, and satisfying

$$L^p_{\mathrm{curl}}(\hat{\Omega}, \mathbb{R}^{3\times3}) \subset \ker \Phi. \tag{2.8.7}$$

An important result for maps of this kind is the following.

Theorem 2.8.4. *Let $1 < p < 3/2$ and let Φ be a linear and continuous map on $L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$ satisfying $\Phi(Du) = 0$ for every $u \in W^{1,p}(\hat{\Omega}, \mathbb{R}^3)$. Then there exist two linear and continuous maps $\mathbb{L}$, $\tilde{\mathbb{L}} : \mathcal{M}_{\mathrm{div}}(\hat{\Omega}, \mathbb{R}^{3\times3}) \to \mathbb{R}$ belonging to $C(\overline{\hat{\Omega}}, \mathbb{R}^{3\times3}) \cap W^{1,p'}(\hat{\Omega}, \mathbb{R}^{3\times3})$, with $3 < p' < \infty$, $1/p + 1/p' = 1$, such that, for every $F \in L^p(\hat{\Omega}, \mathbb{R}^{3\times3})$,*

$$\Phi(F) = \langle \mathrm{Curl}\ \mathbb{L}, F\rangle = \langle \mathrm{Curl}\ \tilde{\mathbb{L}}, F\rangle, \tag{2.8.8}$$

and satisfying $\mathrm{Div}\ \mathbb{L} = \mathrm{Div}\ \tilde{\mathbb{L}} = 0$ *in* $\hat{\Omega}$, $N \times \mathbb{L} = 0$ *and* $\tilde{\mathbb{L}}N = 0$ *on* $\partial\hat{\Omega}$. *Moreover it holds*

$$\Phi(F) = \langle \mathbb{L}, \mathrm{Curl}\ F\rangle = \langle \tilde{\mathbb{L}}, \mathrm{Curl}\ F\rangle + \langle N \times \tilde{\mathbb{L}}, F\rangle_{\partial\hat{\Omega}}. \tag{2.8.9}$$

Proof. Since Φ is linear and continuous it holds

$$\Phi(F) = \langle \mathbb{T}, F\rangle, \tag{2.8.10}$$

for some $\mathbb{T} \in L^{p'}(\hat{\Omega}, \mathbb{R}^{3\times3})$. Now, for every $\varphi \in C^\infty(\bar{\hat{\Omega}}, \mathbb{R}^3)$ we have $\langle \mathbb{T}, D\varphi\rangle = \Phi(D\varphi) = 0$, proving that (i) $\mathrm{Div}\ \mathbb{T} = 0$ in $\hat{\Omega}$ and, by integration by parts, that (ii) $\mathbb{T}N = 0$ on $\partial\hat{\Omega}$. By Theorem 2.4.9 (Eq. (2.4.13) or (2.4.14)), there exist a unique $\mathbb{L} \in L^{p'}_{\mathrm{div}}(\hat{\Omega}, \mathbb{R}^{3\times3})$ satisfying $N \times \mathbb{L} = 0$ on $\partial\hat{\Omega}$ and a unique $\tilde{\mathbb{L}} \in L^{p'}_{\mathrm{div}}(\hat{\Omega}, \mathbb{R}^{3\times3})$, with $\tilde{\mathbb{L}}N = 0$ on $\partial\hat{\Omega}$, such that

$$\mathrm{Curl}\ \mathbb{L} + Du = \mathrm{Curl}\ \tilde{\mathbb{L}} + Du_0 = \mathbb{T}. \tag{2.8.11}$$

Since $\mathrm{Div}\ \mathbb{T} = 0$ in $\hat{\Omega}$, one has $u_0 \equiv 0$, and from $\mathrm{Curl}\ \mathbb{L}N = \mathbb{T}N = 0$ on $\partial\hat{\Omega}$, we get $Du \equiv 0$. By Maxwell-Friedrich-type inequality (i.e., the generalization of (2.4.11), see [80]), i.e.,

$$\|\nabla\mathbb{L}\|_{p'} \le C\left(\|\mathrm{Curl}\ \mathbb{L}\|_{p'} + \|\mathrm{Div}\ \mathbb{L}\|_{p'} + \|\mathbb{L}\|_{p'}\right), \tag{2.8.12}$$

the fact that $\mathbb{L} \in L^{p'}(\hat{\Omega}, \mathbb{R}^{3\times 3})$ with $\operatorname{Curl} \mathbb{L} \in L^{p'}(\hat{\Omega}, \mathbb{R}^{3\times 3})$ and $\operatorname{Div} \mathbb{L} = 0$, implies that $\mathbb{L} \in W^{1,p'}(\hat{\Omega}, \mathbb{R}^{3\times 3})$, which, since $3 < p' \leq \infty$, entails by Sobolev embedding that

$$\mathbb{L} \in C(\overline{\hat{\Omega}}, \mathbb{R}^{3\times 3}). \tag{2.8.13}$$

The same is true for $\tilde{\mathbb{L}}$. Integrating by parts the identities (2.8.8) we get, since $N \times \mathbb{L} = 0$ on $\partial\hat{\Omega}$,

$$\Phi(F) = \langle \operatorname{Curl} \mathbb{L}, F\rangle = \langle \mathbb{L}, \operatorname{Curl} F\rangle,$$

and similarly

$$\Phi(F) = \langle \operatorname{Curl} \tilde{\mathbb{L}}, F\rangle = \langle \tilde{\mathbb{L}}, \operatorname{Curl} F\rangle + \langle N \times \tilde{\mathbb{L}}, F\rangle_{\partial\hat{\Omega}},$$

achieving the proof by (2.8.11). □

In the applications, Φ will be the defect part of the energy. In the sequel we will restrict to those variations whose deformation curl is concentrated in a closed curve, and, specifically, is associated to some dislocation density measure.

2.8.3 Internal variations at minimality

The energy functional $\mathcal{W}_e$ introduced in (2.8.3) can be extended to $L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$ by setting

$$\mathcal{W}_e(F) := \int_{\hat{\Omega}} W_e(F)\chi_\Omega dx,$$

for all $F \in L^p(\hat{\Omega}, \mathbb{R}^{3\times 3})$. Let us fix $F^\star \in \mathcal{AD}^p(\hat{\Omega})$ and $(\Lambda^\star)^{\mathrm{T}} = -\operatorname{Curl} F^\star$ on $\hat{\Omega}$. Variations F of the deformation $F^\star$ still satisfying the constraint $-\operatorname{Curl}(F^\star + F) = (\Lambda^\star)^{\mathrm{T}}$ must belong to $\mathcal{AD}^p_{\mathrm{curl}}(\hat{\Omega}) := \{F \in \mathcal{AD}^p(\hat{\Omega}) \text{ s.t. } \operatorname{Curl} F = 0\}$. Moreover, such variations at the minimum points of the energy $\mathcal{W}$ must provide a vanishing variation of $\mathcal{W}$. Within our formalism, it is assumed that W depends on the density Λ via the $W^{1,1}(S^1, \hat{\Omega}^3)$-field $\lambda := (\lambda^1, \lambda^2, \lambda^3)$ as defined in (2.5.7), viz.,

$$\mathcal{W}(F, \Lambda) = \mathcal{W}^\circ(F, \lambda), \tag{2.8.14}$$

with $\lambda \in T^{-1}(-\Lambda^{\mathrm{T}})$.

Specifically, by Theorem 2.4.13, Eq. (2.5.11) and Theorem 2.5.5, it holds

$$\mathcal{AD}^p_{\mathrm{div}}(\hat{\Omega}) = \operatorname{Curl}^{-1}\Big(T\Big(\dot{W}^{1,1}(S^1, \hat{\Omega}^3)\Big)\Big),$$

while for any admissible deformation $F \in \mathcal{AD}^p(\hat{\Omega})$ it holds by Helmholtz decomposition (Theorem 2.4.9) that $F = Du + F^\circ$ with $F^\circ \in \mathcal{AD}^p_{\mathrm{div}}(\hat{\Omega})$, so that

$$\hat{\mathcal{W}}(F) = \mathcal{W}(F, \Lambda) = \mathcal{W}_{\mathrm{e}}(Du + F^\circ) + \mathcal{W}^\circ_{\mathrm{defect}}(\operatorname{Curl} F^\circ) := W^{\circ\circ}(u, \lambda), \tag{2.8.15}$$

the latter being well defined, since F° is associated to a unique curve λ by (2.8.15). Here $\mathcal{W}^\circ_{\mathrm{defect}}$ is defined by $\mathcal{W}^\circ_{\mathrm{defect}}(A) := \mathcal{W}_{\mathrm{defect}}\left(-A^{\mathrm{T}}\right)$.

We make the assumption that the energy density $\mathcal{W}_{\mathrm{e}} : L^p(\Omega) \to \mathbb{R}$ of (2.6.2) is Fréchet differentiable in $L^p(\Omega)$ with the Fréchet derivative of $F \mapsto W(F, \Lambda^\star)$ denoted by $W_F \in L^{p'}(\Omega)$. As a consequence, for every $F \in L^p(\Omega)$, it holds

(A1) $\delta\mathcal{W}^\star(F) := \frac{d}{d\epsilon}\mathcal{W}(F^\star + \epsilon F, \Lambda^\star)|_{\epsilon=0} = \int_\Omega W_F^\star \cdot F dx = \delta\mathcal{W}_{\mathrm{e}}(F^\star)[F]$,

where $W_F^\star := W_F(F^\star, \Lambda^\star)$. Since $F^\star$ is a minimum point, for every $F \in L^p_{\mathrm{curl}}(\hat{\Omega})$, it holds

$$\delta\mathcal{W}^\star(F) = 0. \tag{2.8.16}$$

From (A1), (2.8.16) and Theorem 2.8.4 it results that there exists $\mathbb{L}^\star$ such that

$$\mathbb{P} := W_F^\star = \text{ Curl } \mathbb{L}^\star, \tag{2.8.17}$$

and hence

$$\begin{cases} -\operatorname{Div} \mathbb{P} &= 0 \quad \text{in} \quad \hat{\Omega} \\ \quad \mathbb{P}N &= 0 \quad \text{on} \quad \partial\hat{\Omega} \end{cases}, \tag{2.8.18}$$

in such a way that $W_F^\star$ is identified with the first Piola-Kirchhoff stress.

Recalling (2.4.9) and observing that (A1) means that $\delta\mathcal{W}_{\mathrm{e}}(F^\star) = W_F^\star \in L^{p'}(\Omega, \mathbb{R}^{3\times 3})$, we make the additional assumption that

(A2) $W_F^\star \in \tilde{\mathcal{V}}^{p'}(\Omega)$,

and in particular that Curl $W_F^\star$ (extended by zero in $\hat{\Omega} \setminus \Omega$) belongs to $L^{p'}(\hat{\Omega})$.

2.8.4 Shape variation at minimizers

Let L be a single smooth enough dislocation loop with tangent vector τ, normal vector ν, curvature κ, and total Burgers vector B. We introduce

$$\mathcal{F}^\star := \left(W_F^\star \times \tau\right)^{\mathrm{T}} B\delta_L,$$
$$\mathcal{G}^\star := \kappa\left(\psi(b,\tau) - \nabla\psi(b,\tau)\cdot\tau + \nabla\nabla\psi(b,\tau)\cdot\nu\otimes\nu\right)\nu\delta_L,$$

the so-called deformation-induced Peach-Köhler force and line tension, respectively, where ψ is the energy density as introduced in (2.8.4).

At minumum points, the following theorem holds. Note that restricting to a single generating loop with Burgers vector b is chosen for the simplicity of the exposition, and the reader interested by generality, can find the complete force formulae in the proof developments.

Theorem 2.8.5. *Assume that $\mathcal{W}_{defect}$ satisfies (W4') and (W5') with $\psi, \bar{\psi} : 2\pi\mathbb{Z}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$, of class C^2. Under Assumptions (A1),(A2), let $F^\star$ be a minimum point of $\hat{\mathcal{W}}(F)$, and assume that the optimal cluster $\mathcal{L}^\star$ associated to $F^\star$ was generated by a single loop with Burgers vector b, associated to a density $\Lambda_{\mathcal{L}^\star} = b\otimes\gamma^\star_\sharp[\![S^1]\!]$ with $\gamma^\star \in W^{2,\infty}(S_1)$. Then, the associated Peach-Köhler force $\mathcal{F}^\star$ is balanced by a defect-induced force $\mathcal{G}^\star$ in $L^\star$, i.e.,*

$$\mathcal{F}^\star + \mathcal{G}^\star = 0.$$

If the line is straight then $\mathcal{F}^\star = 0$. Moreover at $F^\star$ it holds Div $W_F^\star = 0$ *in Ω and $W_F^\star N = 0$ on $\partial\Omega$.*

Note: We recall that $L^\star$ is the dislocation set of the current $\mathcal{L}^\star$ with Curl $F^\star = -\left(\Lambda_{\mathcal{L}^\star}\right)^{\mathrm{T}}$.

Remark 2.8.6. In the proof we will consider the core energy $\mathcal{W}_{defect}$ as given only by $\mathcal{W}^1_{defect}$, in contrast with the hypotheses (W3) and (W6) in Section 2.6. However, since we are computing variations which are continuous with respect to the line, the presence of such terms do not change the computation. In particular it is not restrictive to set $\mathcal{W}^2_{defect} = 0$ and the statement of Theorem 2.8.5 still holds true for energies as in Theorems 2.6.6 and 2.6.8. See also Remark 2.8.8.

Proof. We define the linear map

$$S : W^{1,1}(S^1, \hat{\Omega}^3) \to \mathcal{BC}^{p,\Lambda}_{\text{div}}(\hat{\Omega}, \mathbb{R}^{3\times 3}) : \quad S := \text{Curl}^{-1} \circ T,$$

where Curl^{-1} is the solution of (2.4.15). By Theorem 2.8.2 there exists $F^\star \in \mathcal{BC}^{p,\Lambda}(\hat{\Omega}, \mathbb{R}^{3\times 3})$ a minimizer of the energy (2.8.15) with $\text{Curl}\, F^\star = T(\lambda^\star)$, where $F^\star = Du^\star + S(\lambda^\star)$ by (2.4.14). We then define, for all $\lambda \in W^{1,\infty}(S^1, \hat{\Omega}^3)$,

$$\mathcal{W}^\circ(\lambda) := \mathcal{W}^\circ_{\text{e}}(\lambda) + \mathcal{W}^{\circ\circ}_{\text{defect}}(\lambda), \tag{2.8.19}$$

where $\mathcal{W}^\circ_{\text{e}}(\lambda) := \mathcal{W}_{\text{e}}(Du^\star + S(\lambda))$ and $\mathcal{W}^{\circ\circ}_{\text{defect}}(\lambda) := \mathcal{W}^\circ_{\text{defect}}(T(\lambda))$. In particular, $\mathcal{W}^\circ(\lambda^\star) = \mathcal{W}(F^\star)$. We now want to perform variations in $W^{1,\infty}(S^1, \hat{\Omega}^3)$ of $\mathcal{W}^\circ(\lambda^\star)$. For $\lambda \in W^{1,\infty}(S^1, \hat{\Omega}^3)$ we have

$$\delta\mathcal{W}^\circ(\lambda^\star)[\lambda] = \delta\mathcal{W}^\circ_{\text{e}}(\lambda^\star)[\lambda] + \delta\mathcal{W}^{\circ\circ}_{\text{defect}}(\lambda^\star)[\lambda]. \tag{2.8.20}$$

Here $\delta\mathcal{W}^\circ_{\text{e}}(\lambda^\star)[\lambda] = \langle W^\star_F, DS(\lambda^\star)[\lambda]\rangle$ and $DS(\lambda^\star)$ is the Fréchet derivative of S in $\lambda^\star$. Let us recall the notation $\varphi^\lambda_{ij}(s) := \varphi_{ij}(\lambda^j(s)) = \delta_{jk}\varphi_{ij}(\lambda^k(s))$ (with no sum in the second term), valid for every $\varphi \in C^1(\Omega, \mathbb{R}^{3\times 3})$. Then,

$$\begin{aligned} DS(\lambda^\star)[\lambda] &= \lim_{\epsilon\to 0} \big(\text{Curl}^{-1}(T(\lambda^\star + \epsilon\lambda)) - \text{Curl}^{-1}(T(\lambda^\star))\big)/\epsilon = \\ &\text{Curl}^{-1}\left(DT(\lambda^\star)[\lambda]\right), \end{aligned}$$

while, from (2.5.4) and (2.5.7), we entail by a Taylor expansion of φ that

$$\langle DT(\lambda^\star)[\lambda], \varphi\rangle = \int_{S^1} \varphi^{\lambda^\star}_{ij}(s)\dot{\lambda}^i_j(s) + D_k\varphi^{\lambda^\star}_{ij}(s)(\dot{\lambda}^\star)^i_j(s)\lambda^i_k(s)ds, \tag{2.8.21}$$

for all $\varphi \in C^1(\Omega, \mathbb{R}^{3\times 3})$.

Recall that $\text{Curl}\ \text{Curl}\, A = D\,\text{Div}\, A - \Delta A$, and hence

$$\left\{\begin{array}{ccccc} \text{Curl}\ \text{Curl}\, \mathbb{L}^\star = -\Delta\mathbb{L}^\star & = & \text{Curl}\, W^\star_F & \text{in} & \hat{\Omega} \\ \text{Div}\, \mathbb{L}^\star & = & 0 & \text{on} & \partial\hat{\Omega} \\ \mathbb{L}^\star \times N & = & 0 & \text{on} & \partial\hat{\Omega} \end{array}\right., \tag{2.8.22}$$

which is an elliptic system in the sense of Agmon-Douglis and Nirenberg (see [2]), and has a unique solution (see [41, Theorem 10.5]). Remark that the right-hand side of (2.8.22) being in $L^{p'}(\hat{\Omega}, \mathbb{R}^{3\times 3})$, the solution $\mathbb{L}^\star$ is by elliptic regularity results in $W^{2,p'}(\hat{\Omega}, \mathbb{R}^{3\times 3})$, and by Sobolev's embedding Theorem, since $p' > 3$, in $C^1(\hat{\Omega}, \mathbb{R}^{3\times 3})$. Therefore, it holds

$$\delta\mathcal{W}^\circ_{\text{e}}(\lambda^\star)[\lambda] = \langle W^\star_F, \text{Curl}^{-1}\left(DT(\lambda^\star)[\lambda]\right)\rangle, \tag{2.8.23}$$

which, since $W_F^\star = \operatorname{Curl} \mathbb{L}^\star$ by (2.8.17), is equal to

$$\begin{aligned}\delta\mathcal{W}_{\mathrm{e}}^{\circ}(\lambda^\star)[\lambda] &= \langle DT(\lambda^\star)[\lambda], \mathbb{L}^\star\rangle \\ &= \int_{S^1} (\mathbb{L}^\star)_{ij}^{\lambda^\star}(s)\dot{\lambda}_j^i(s) + D_k(\mathbb{L}^\star)_{ij}^{\lambda^\star}(s)(\dot{\lambda}^\star)_j^i(s)\lambda_k^i(s)ds,\end{aligned}$$

where we have used (2.8.21). Integrating by parts the last expression we get

$$\begin{aligned}&\delta\mathcal{W}_{\mathrm{e}}^{\circ}(\lambda^\star)[\lambda] = \\ &\int_{S^1} D_k(\mathbb{L}_{ij}^\star)^{\lambda^\star}(s)(\dot{\lambda}^\star)_k^i(s)\lambda_j^i(s) - D_k(\mathbb{L}_{ij}^\star)^{\lambda^\star}(s)(\dot{\lambda}^\star)_j^i(s)\lambda_k^i(s)ds = \\ &\int_{S^1} \Big(D_k(\mathbb{L}_{ij}^\star)^{\lambda^\star}(s) - D_j(\mathbb{L}_{ik}^\star)^{\lambda^\star}(s)\Big)(\dot{\lambda}^\star)_k^i(s)\lambda_j^i(s)ds. \qquad (2.8.24)\end{aligned}$$

Let us now particularize (2.8.24) to the case where the density $\Lambda^\star$ is generated by one single loop $\gamma^\star \in W^{1,\infty}(S^1, \hat{\Omega})$ with Burgers vector $b = \beta_i e_i, \beta_i \in 2\pi\mathbb{Z}$ $(b \neq 0)$ (cf. Remark 2.5.2). For variations of the form $\gamma^\star + \epsilon\gamma$ with $\gamma \in W^{1,\infty}(S^1, \hat{\Omega})$, (2.8.24) becomes

$$\delta\mathcal{W}_{\mathrm{e}}^{\circ}(\lambda^\star)[\lambda] = \int_{S^1} \Big(D_k(\mathbb{L}_{ij}^\star)^{\gamma^\star}(s) - D_j(\mathbb{L}_{ik}^\star)^{\gamma^\star}(s)\Big)(\dot{\gamma}^\star)_k(s)b_i\gamma_j(s)ds. \quad (2.8.25)$$

For a dislocation density of the form $\mu = b \otimes \gamma_\sharp[\![S^1]\!]$, (2.8.4) writes as

$$\mathcal{W}_{\mathrm{defect}}^{\circ}(\mu) = \int_{S^1} \psi(b, \frac{\dot{\gamma}}{|\dot{\gamma}|}(s))|\dot{\gamma}(s)|ds. \qquad (2.8.26)$$

We can now compute the first variation of the energy (2.8.26) at the point $\gamma^\star \in W^{1,1}(S^1, \hat{\Omega})$. Setting $\hat{\mathcal{W}}_{\mathrm{defect}} := \mathcal{W}_{\mathrm{defect}}^{\circ} \circ T$, it holds

$$\begin{aligned}&\delta\hat{\mathcal{W}}_{\mathrm{defect}}(\gamma^\star)[\gamma] = \\ &= \int_{S^1} D_k\psi(b, \frac{\dot{\gamma}^\star}{|\dot{\gamma}^\star|}(s))\big(\frac{\dot{\gamma}_k|\dot{\gamma}^\star|^2 - \dot{\gamma}_k^\star\dot{\gamma}_j^\star\dot{\gamma}_j}{|\dot{\gamma}^\star|^2}(s)\big) + \psi(b, \frac{\dot{\gamma}^\star}{|\dot{\gamma}^\star|}(s))\big(\frac{\dot{\gamma}_j^\star\dot{\gamma}_j}{|\dot{\gamma}^\star|}(s)\big)ds,\end{aligned}$$

where $D_k\psi$ is the derivative of ψ with respect to the k-th component of its second variable. Denoting $\tau = \frac{\dot{\gamma}^\star}{|\dot{\gamma}^\star|}$ and $\dot{\tau} = \frac{\ddot{\gamma}^\star}{|\dot{\gamma}^\star|} - \tau\tau_j\frac{\ddot{\gamma}_j^\star}{|\dot{\gamma}^\star|}$, we integrate by parts to obtain

$$\begin{aligned}&\delta\hat{\mathcal{W}}_{\mathrm{defect}}(\gamma^\star)[\gamma] = \\ &\int_{S^1} -D_jD_k\psi(b,\tau)(\dot{\tau}_k\gamma_j - \dot{\tau}_k\tau_j\tau_p\gamma_p) - \big(\psi(b,\tau) - \tau_kD_k\psi(b,\tau)\big)\dot{\tau}_j\gamma_jds = \\ &-\int_{S^1} \Big(\psi(b,\tau)\dot{\tau}_j - D_k\psi(b,\tau)\tau_k\dot{\tau}_j + D_jD_k\psi(b,\tau)\dot{\tau}_k - D_pD_k\psi(b,\tau)\dot{\tau}_k\tau_p\tau_j\Big)\gamma_jds,\end{aligned}$$

where we dropped the variable s. Plugging the last expression in (2.8.20) and using (2.8.25), we obtain

$$\begin{aligned}\delta\mathcal{W}(\gamma^\star)[\gamma] = \int_{S^1} \Big(&D_k(\mathbb{L}_{ij}^\star)^{\gamma^\star}\dot{\gamma}_k^\star b_i - D_j(\mathbb{L}_{ik}^\star)^{\gamma^\star}\dot{\gamma}_k^\star b_i - \psi(b,\tau)\dot{\tau}_j + D_k\psi(b,\tau)\tau_k\dot{\tau}_j \\ &- D_jD_k\psi(b,\tau)\dot{\tau}_k + D_pD_k\psi(b,\tau)\dot{\tau}_k\tau_p\tau_j\Big)\gamma_j(s)ds. \qquad (2.8.27)\end{aligned}$$

From the condition

$$\delta\mathcal{W}^\circ(\gamma^\star)[\gamma] = 0 \qquad \text{for all } \gamma \in W^{1,\infty}(S^1,\mathbb{R}^3), \tag{2.8.28}$$

due to the minimality of $\gamma^\star$, we then get from (2.8.27),

$$\int_{S^1} \Big(D_k(\mathbb{L}^\star_{ij})^{\gamma^\star}\dot\gamma^\star_k b_i - D_j(\mathbb{L}^\star_{ik})^{\gamma^\star}\dot\gamma^\star_k b_i - \psi(b,\tau)\dot\tau_j + D_k\psi(b,\tau)\tau_k\dot\tau_j \\ - D_jD_k\psi(b,\tau)\dot\tau_k + D_pD_k\psi(b,\tau)\dot\tau_k\tau_p\tau_j\Big)\gamma_j(s)ds = 0, \tag{2.8.29}$$

for all $\gamma \in W^{1,\infty}(S^1,\mathbb{R}^3)$, which implies, by arbitraryness of $\gamma \in W^{1,\infty}(S^1,\hat\Omega)$, that

$$D_k(\mathbb{L}^\star_{ij})^{\gamma^\star}(s)\dot\gamma^\star_k(s) b_i - D_j(\mathbb{L}^\star_{ik})^{\gamma^\star}(s)\dot\gamma^\star_k(s) b_i - \psi(b,\tau)\dot\tau_j + D_k\psi(b,\tau)\tau_k\dot\tau_j \\ - D_jD_k\psi(b,\tau)\dot\tau_k + D_pD_k\psi(b,\tau)\dot\tau_k\tau_p\tau_j = 0 \quad \text{ for all } s \in S^1. \tag{2.8.30}$$

Equivalently, recalling that $\dot\tau_i = \kappa\nu_i$ and $D_jD_k\psi(b,\tau)\dot\tau_k = \tau_j\tau_pD_pD_k\psi(b,\tau)\dot\tau_k + \nu_j\nu_pD_pD_k\psi(b,\tau)\dot\tau_k$, it holds for every $s \in S^1$ that

$$\begin{aligned}
\mathcal{G}^\star_j(s) &:= \psi(b,\tau)\dot\tau_j - D_k\psi(b,\tau)\tau_k\dot\tau_j + D_jD_k\psi(b,\tau)\dot\tau_k - D_pD_k\psi(b,\tau)\dot\tau_k\tau_p\tau_j \\
&= \psi(b,\tau)\dot\tau_j - D_k\psi(b,\tau)\tau_k\dot\tau_j + D_pD_k\psi(b,\tau)\dot\tau_k\nu_p\nu_j \\
&= \kappa\left(\psi(b,\tau) - D_k\psi(b,\tau)\tau_k + D_pD_k\psi(b,\tau)\nu_p\nu_k\right)\nu_j \\
&= \epsilon_{pjk}(\,\mathrm{Curl}\,\mathbb{L})^{\gamma^\star}_{ip}(s) b_i\dot\gamma^\star_k(s).
\end{aligned}$$

Recalling (2.5.5), the last formula becomes

$$\mathcal{F}^\star_j(P) + \mathcal{G}^\star_j(P) = 0, \tag{2.8.31}$$

at every point $P \in L$, with

$$\begin{aligned}
\mathcal{F}^\star_j(P) &:= \epsilon_{jpk}(\,\mathrm{Curl}\,\mathbb{L}^\star)_{ip}(P)\theta_P b_i\tau_k(P), \\
\mathcal{G}^\star_j(P) &:= \kappa\left(\psi(\theta_Pb,\tau) - D_k\psi(\theta_Pb,\tau)\tau_k + D_pD_k\psi(\theta_Pb,\tau)\nu_p\nu_k\right)\nu_j,
\end{aligned}$$

where $\theta_P = \theta(P)$ is defined by (2.5.6) and stands for the multiplicity of the dislocation (accounting for the loops of the cluster whose Burgers vector is a multiple of b). The result follows by writing

$$\mathcal{F}^\star := (W^\star_F \times \tau)^{\mathrm{T}} B = (\,\mathrm{Curl}\,\mathbb{L}^\star \times \tau)^{\mathrm{T}} B, \tag{2.8.32}$$

where $B := \theta_P b$ is the total Burgers vector (i.e., associated to a nonnecessary unit multiplicity). □

Remark 2.8.7. Actually, (2.8.31) holds at $\mathcal{H}^1$-a.e. $P \in L$, and not at all P. This is due to the fact that it might happen that a point $P \in L$ is the overlapping of parts of γ which, although having the same tangent vector τ, do not have the same curvature κ nor the same ortogonal vector ν.

2.8.5 A modeling example

Let us analyze equation (2.8.31) in an explicit situation. In [17] it is considered a potential $\mathcal{W}_{\text{defect}}$ of the form (2.8.4) with

$$\psi(b,\tau) := |b|^2 + \eta\langle b,\tau\rangle^2, \tag{2.8.33}$$

where $\eta > 0$ is a constant.

In the particular case where $b = \beta e_1, \beta \geq 1$, it is shown that ψ and $\bar{\psi}$ share the same expression up to the multiplicative factor β. In particular, they have the same regularity, i.e., are both smooth. In such a case, the above computations entail that

$$\mathcal{G}_j^\star(P) = \Big(|b|^2 - \eta\langle b,\tau\rangle^2 + 2\eta\langle b,\nu\rangle^2\Big)\kappa\nu_j,$$

so that at minimum of the energy, it holds

$$\theta_P^2\Big((1-\eta)\langle b,\tau\rangle^2 + (1+2\eta)\langle b,\nu\rangle^2\Big)\kappa\nu_j = \epsilon_{jpk}(\,\text{Curl}\;\mathbb{L}^\star)_{ip}(P)\theta_P b_i\tau_k(P).$$

Note that the line curvature at equilibrium is given by

$$\begin{aligned}\kappa &= \Big((1-\eta)\langle b,\tau\rangle^2 + (1+2\eta)\langle b,\nu\rangle^2\Big)^{-1}\epsilon_{jpk}\nu_j(\,\text{Curl}\;\mathbb{L}^\star)_{ip}(P)\theta_P^{-1}b_i\tau_k(P)\\ &= \Big((1-\eta)\langle b,\tau\rangle^2 + (1+2\eta)\langle b,\nu\rangle^2\Big)^{-1}(\,\text{Curl}\;\mathbb{L}^\star)_{iz}(P)b_i\theta_P^{-1},\end{aligned}$$

the latter equation holding for a plane loop.

Remark 2.8.8. Let us note that energy (2.8.33) alone does not satisfy the hypothesis (W4') necessary to have existence of minimizers among the class $\mathcal{AD}^p(\Omega)$. In particular in such a case $\mathcal{W}^2_{\text{defect}} = 0$, that is, such energy does not take account of the number of connected components of the dislocation. In order to recover an existence theorem in the class $\mathcal{AD}^p(\Omega)$ we can still add a term $\mathcal{W}^2_{\text{defect}}$ as in (W4'), justifying its presence by the fact that such term takes account of the number of nucleations needed to form the clusters, which are known to be energetically expensive. Note however that the term with $|b|^2$ in (2.8.33) permits a control of the length of the curve γ; without such control, it might happen that the dislocation would form countable many small disconnected branches which turn out to be dense in some part of the crystal, even if their total mass remains small [59]. Let us emphasize that such term does not change the expression of the Peach-Köhler force, since it is not dislocation depending, rather it depends only on the history of the crystal evolution.

2.8.6 Concluding remarks

On the way to mathematically understand time evolution of dislocations, Theorem 2.8.5 introduces two forces balancing each other at optimality, the first deriving from the elastic part of the energy and named after Peach and Köhler (and well-known in dislocation models [35]), and the second deriving by shape variation of the defect part of the energy. Here crucial use has been made of the decomposition $F = \nabla u + F^\circ$ where F° and Curl F° depend of the line.

It turns out that the sum of these two forces naturally provides an expression of the velocity of the dislocation (for instance, a linear law is acceptable under certain working assumptions, see [1]). Of course, a nonvanishing velocity, i.e., a nonzero force, means that the solution does not coincide with energy minimization, as well-known for evolution problems. A proper task in the future is to determine the dissipative effects, the balance equations, and analyze in detail the evolutionary scheme.

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