Curvature-type invariants of Geometrical Control Theory in problems of Hamiltonian Dynamics

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Contents

Introduction

The main subject of this thesis is the application of the geometric methods, originally introduced in the framework of the Optimal Control Theory, to the analysis of Hamiltonian Systems.

After the publication of the celebrated book by L.S.Pontryagin and his colleagues [23], the analysis of the solutions of certain Hamiltonian equations plays a key role in finding the extremals of an optimal control problem. From the geometrical view point the solutions of Hamiltonian equations are the integral curves of Hamiltonian vector fields in the cotangent bundle over the configuration space which is the state-space of the problem. As the cotangent bundle carries a natural symplectic structure, the language of Symplectic Geometry turns to be a natural language for the study of the extremals of an optimal control problem. Let us recall briefly the main idea of the Pontryagin Maximum Principle and its relation with the problems of classical Calculus of Variations.

Example 0.0.1. On a smooth manifold M consider a control problem

$$\dot{q}(t) = f(q(t), u(t))\,, \qquad q \in M, \tag{I.1}$$

where for the fixed parameter $u \in U$ vector fields $f(\cdot, u)$ are smooth, $f(q,u)$ is continuous w.r.t. the both variables and $t \to u(t) \in U$ are measurable locally bounded functions. The Optimal Control Problem with fixed end-points and fixed terminal time consists in finding the admissible trajectories of (I.1) satisfying the boundary conditions

$$q(0) = q_0, \qquad q(T) = q_1 \tag{I.2}$$

for some $q_0, q_1 \in M$ and $T \in \mathbb{R}$ fixed, and minimizing the *cost functional*

$$A^T[q(\cdot)] = \int_0^T L\big(q(t), u(t)\big)\, dt \mapsto \min \tag{I.3}$$

for some function $L:\ M \times U \mapsto \mathbb{R}$, usually satisfying the same regularity assumptions as vector fields f.

To the problem (I.1)-(I.3) one can associate the following function on T^*M:

$$h_u(p,q) = p(f(q,u)) - \nu L(q,u), \quad q \in M,\ \ p \in T_q^*M,\ \ \nu = \{0,1\} \tag{I.4}$$

called the *Hamiltonian function of the optimal control problem (I.1)-(I.3)*. Let us denote by σ the canonical symplectic form on T^*M, and by $\vec{h}_u$ the Hamiltonian vector field corresponding to the Hamiltonian h_u: $d_z h_u(\cdot) = \sigma_z(\cdot, \vec{h}_u)$ with $z \in T^*M$.

According to the Pontryagin Maximum principle (see [1], [23]), if the curve $\hat{q} : [0,T] \mapsto M$, satisfying (I.2), is an extremal trajectory of the problem (I.1)-(I.3) corresponding to the control function $\tilde{u}(t)$, then there exists a non-trivial pair $(\gamma(t), \nu) \in T^*M \times \mathbb{R}$, where the curve $\gamma : [0,T] \mapsto T^*M$, called *the extremal of the problem (I.1)-(I.3)*, is the integral curve of the Hamiltonian vector field $\vec{h}_{\tilde{u}(t)}$, such that $\hat{q}(t) = \pi\big(\gamma(t)\big)$ with $\pi : T^*M \mapsto M$ being the canonical projection of the cotangent bundle to the base manifold, and

$$h_{\tilde{u}(t)}(\gamma(t)) = \max_{u \in U} h_u(\gamma(t)), \qquad \text{for a.e. } t \in [0,T].$$

The problem of classical Calculus of Variations can be seen as a particular case of the Optimal Control problem for $f(q,u) = u$ and $U = T_qM$. In this case $L : TM \mapsto \mathbb{R}$, and throughout the thesis we assume that the function L is convex on each fiber. The Pontryagin Maximum Principle implies that the extremal trajectory $\hat{q} : [0,T] \mapsto M$, satisfying (I.2), is the extremal trajectory of the functional (I.3) if and only if $\hat{q}(t) = \pi\big(\gamma(t)\big)$ with $\gamma : [0,T] \mapsto T^*M$ being the integral curve of the Hamiltonian vector field $\vec{h}$ corresponding to the Hamiltonian function h, defined as the Legendre transform of L:

$$h(p,q) = (p\dot{q} - L(q,\dot{q}))\Big|_{\dot{q} \to p = \frac{\partial L}{\partial \dot{q}}}. \tag{I.5}$$

The function $L(q,\dot{q})$ is called *the Lagrangian of the problem (I.1)-(I.3)*, and the cost functional (I.3) is called *the action functional*. For the sake of completeness let us remark, that in the framework of Classical Mechanics this result is also known as the Least Action Principle.

In this thesis we consider the problems of Calculus of Variations and Classical Mechanics from the point of view of Optimal Control Theory which we have just described.

The curvature-type invariants related to the extremals of an optimal control problem were first introduced in the paper of A.Agrachev and R.Gamkrelidze [2]. The theory was developed then by the same authors [3]-[5] and in the papers by A.Agrachev and I.Zelenko [8], [9], [27]. These invariants: *the generalized curvature operator*, *generalized Ricci curvature* and the *curvature form* describe the invariant properties of one-parametric families of Lagrangian subspaces in a symplectic space with respect to the natural action of the symplectic group. They provide a kind of generalization of the notion of classical curvature tensor of Riemannian Geometry (see Example 1.4.8 in Chapter 1). As in Riemannian Geometry, the generalized curvature operator and the generalized Ricci curvature contain the intrinsic information about the extremals of an optimal control problem and describe the global behavior of such extremals. For example, these invariants can be used to localize the conjugate points along extremals of the corresponding variational problems (see Theorem1.3.5 in Chapter 1).

One of the most important examples of one-parametric families of Lagrangian subspaces is provided by a pair (Hamiltonian vector field, Lagrangian distribution), called *the dynamical Lagrangian distributions*. It can be described as follows.

Let W be a symplectic manifold with symplectic form σ. The Lagrangian distribution $\mathcal{D}$ on W is a smooth vector sub-bundle of the tangent bundle TW such that each fiber $\mathcal{D}_z$ is a Lagrangian subspace of the linear symplectic space T_zW, i.e., $\dim \mathcal{D}_z = \frac{1}{2}\dim W$ and $\sigma_z(v_1, v_2) = 0$ for all $v_1, v_2 \in \mathcal{D}_z$. If h is a smooth function on W and $\vec{h}$ is a corresponding Hamiltonian vector field, then the pair $(\vec{h}, \mathcal{D})$ defines the one-parametric family of Lagrangian distributions $\mathcal{D}(t) = e_*^{t\vec{h}}\mathcal{D}$ in TW. The pair $(\vec{h}, \mathcal{D})$ is called *the dynamical Lagrangian distribution*.

Dynamical Lagrangian distributions appear in a natural way in the framework of Calculus of Variations and Optimal Control Theory. As one can see from Example 0.0.1, a particular role is played by the so-called *vertical dynamical Lagrangian distribution* $(\vec{h}, \Pi)$ generated by a Hamiltonian vector field on the cotangent bundle T^*M of a smooth manifold M and the Lagrangian distribution Π such that each leaf of this distribution is a tangent space to the fiber, i.e $\Pi_z = T_z(T^*_{\pi(z)}M)$ for all $z \in T^*M$. Coming back to the situation described in Example 0.0.1, we say that the vertical dynamical Lagrangian distributions $(\vec{h}, \Pi)$ is *generated by the problem* (I.1)-(I.4)(or (I.5)). The point $\gamma(T)$ is conjugate to $\gamma(0)$ w.r.t. the pair $(\vec{h}, \Pi)$[1] if and only if q_1 is conjugate to q_0 along the extremal $\hat{q}(\cdot)$ in the classical variational sense for the problem (I.2)-(I.3), i.e. the second variation of the functional $A^T[\hat{q}]$ is degenerate ([1]).

In this thesis we develop an approach which is intended to be used in the framework of regular problems of Calculus of Variations and Classical Mechanics. Namely, we restrict our attention to the various aspects of reduction by first integrals in involution in Hamiltonian systems and analyze a special class of Hamiltonian systems, the systems with negative generalized curvature form. The presented material is organized in the following order.

Chapter 1 contains the basic definitions and facts concerning the symplectic invariants of Dynamical systems that will be used in the sequent chapters. We introduce the notion of the generalized curvature operator, the generalized Ricci curvature and the curvature form of a curve in the Lagrange Grassmannian over a linear even-dimensional space and consider a special class of such curves, the so-called *Jacobi curves* generated by the vertical dynamical Lagrangian distribution in the cotangent bundle of a smooth manifold along an orbit of the corresponding Hamiltonian vector field. We conclude this chapter by some basic examples of Classical Mechanics and Riemannian Geometry which illustrate the meaning of the generalized curvatures in classical cases.

In Chapter 2 we study the behavior of the curvature operator, the curvature form and the conjugate points of a dynamical Lagrangian distribution after its reduction by arbitrary first integrals in involution.

As it is well known in the theory of Dynamical systems, the existence of first integrals reduces the analysis of a system of ordinary differential equations to the analysis of a system of lower order. This is particularly important for the Hamiltonian systems: in this case

[1] The conjugate points for the dynamical Lagrangian distribution $(\vec{h}, \mathcal{D})$ are also called *focal points*.

the existence of one first integral can decrease the order of the system by two. In Section 2.2 we derive the explicit formulas which express the generalized curvature operator and curvature form of the reduced Hamiltonian system in terms of the Hamiltonian vector field and the first integrals of the original system (Theorems 2.2.3 and 2.2.6). It turns out that in the case of systems with Hamiltonians, convex on fibers, the curvature form of the generated Jacobi curve does not decrease after reduction. It worth to mention also that our technique avoids the use of any special canonical variables for the reduced system.

In Section 2.3 we study the relation between the conjugate points of some dynamical Lagrangian distribution and of its reduction by a group of first integrals in involution. Using the Maslov Index arguments, we prove (Theorem 2.3.1) that in the case of systems with Hamiltonians, convex on fibers, on every interval of time the number of conjugate points to the given point for the original problem is non greater than the number of conjugate points (to the same point) with respect to its reduction. In the case of one first integral the sets of conjugate points to a given point are alternating. Moreover, the first conjugate point corresponding to the reduced system comes before any conjugate point related to the original system. The presented results are contained in the original paper [7].

In Chapter 3 we study the vertical dynamical Lagrangian distribution $(\vec{h}, \Pi)$ in the cotangent bundle over some smooth manifold along integral curves of the generating vector field $\vec{h}$. We consider the case of strictly convex (concave) on fibers Hamiltonians. We show that, using the notion of the generalized curvature form, the classical results about the geodesic flows on compact Riemannian manifolds of negative sectional curvature can be generalized to a much larger class of problems of Hamiltonian Dynamics. Namely, we prove (Theorem 3.3.1) that any invariant compact subset of the level set of the Hamiltonian function h with negative-defined reduced curvature is hyperbolic and the corresponding Hamiltonian flow is an Anosov flow. Moreover, in Theorem 3.4.1 we show that if the curvature form of the distribution $(\vec{h}, \Pi)$ is strictly negative, then the only possible connected compact invariant subset of such a system is a hyperbolic equilibrium point. This result to our notion has no analogous in the classical Riemannian case. Both Theorems 3.3.1 and 3.4.1 open a wide possibility for searching of new examples of the hyperbolic dynamical systems having explicit physical interpretation. These results are published in [6].

In Chapter 4 we consider a survey of examples of the application of the developed theory to the classical N-body problem. We believe that some of our results can be of interest non only within the framework of the Geometrical Control Theory. Partly the results we present in this chapter are contained in [13].

Our interest to the N-body problem was caused by the recent success of variational methods in finding new periodic solutions of this problem. In the original work of A.Chenciner and R.Montgomery ([14]) the authors succeed in finding a periodic orbit of the 3-body problem with Newton's potential as a solution of a variational problem for the action functional, subject to some carefully chosen symmetry conditions. This paper was the first one in a series of papers containing further developments of this technique and plenty of new periodic solution of the N-body problem for different N. The nature of these new orbits, which are the minimizers of the action functional only on a certain part of their period, leaves without answer the question about their minimality on a bigger interval. In this thesis we present a technique based on the results of Chapters 1 and 2, which can be used for testing the minimality property of such orbits by localizing their conjugate points.

In this thesis we study the plane N-body problem with equal masses. First of all, in Section 4.1 we analyze the structure of the known first integrals of the problem and calculate the Ricci curvature of its reduction. In Section 4.2 we study in greater detail the minimality properties of the so-called 8-*shaped orbit* of the classical 3-body problem with equal masses. This orbit is the simplest example within the class of orbits mentioned above. Combining the methods presented in Chapters 1 and 2 with numerical computation we find conjugate points along this orbit for the original and reduced systems: it turns out that the first conjugate point (to the starting point) in the original space appears at ≈ 0.76 of the period T of the orbit, and at $\approx 0.52T$ in the reduced space. We also find numerically a new solution for the fixed-end problem in the reduced space having the action smaller than the 8-shaped orbit.

Chapter 1

Symplectic invariants of Dynamical systems

In this chapter we describe the basic objects we will deal with and make a short introduction to the theory of the curves in the Lagrange Grassmannian that will be our main tool in the next chapters.

In Section 1.1 we recall some basic facts from Symplectic Geometry and discuss the properties of curves in the Lagrange Grassmannian following mainly the ideas presented in [2] and [3].

Section 1.2 contains a short introduction to the Maslov Index Theory for the curves in the Lagrange Grassmannian in the form which will be used in Chapter 2. We define the notion of conjugate points for a curve in the Lagrange Grassmannian and discuss the relations between the Maslov Index of the curve and its conjugate points. For the detailed exposition on this subject one can consult [4] or [20].

In Section 1.3 we introduce the basic symplectic invariants of regular curves in the Lagrange Grassmannian: the generalized curvature operator and the generalized Ricci curvature. We illustrate the intrinsic meaning of these objects using the notion of the so-called *canonical moving frame* associated to a curve in the Lagrange Grassmannian.

In Section 1.4 we study the dynamical Lagrangian distributions generated by a Hamiltonian vector filed $\vec{h}$ in the tangent space to the cotangent bundle of a smooth manifold M. As we saw already (Example 0.0.1), these distributions naturally appear in the study of the dynamics in the cotangent bundle. As the model example we consider the vertical dynamical Lagrangian distribution $(\vec{h}, \Pi)$ and the *Jacobi curve* in the Lagrange Grassmannian generated by such distribution in the tangent spaces attached along an integral curve of the vector field $\vec{h}$ in the cotangent bundle. We give the Hamiltonian formulation of the theory which will be used in the sequent chapters.

1.1 Curves in the Lagrange Grassmannian: basic definitions and notations

In this thesis all smooth objects are supposed to be C^∞. The results remain valid for the class C^k with a finite and not large k but we prefer not to specify the minimal possible k.

Let Σ be a $2n$-dimensional linear space endowed with a symplectic form σ. The *Lagrange Grassmannian* $\mathcal{L}_n(\Sigma)$ of the symplectic space Σ is a set of all Lagrangian subspaces in Σ

$$\mathcal{L}_n(\Sigma) = \left\{\Lambda \in \Sigma \, : \, \Lambda^\angle = \Lambda\right\} ,$$

where

$$\Lambda^\angle = \{v \in \Sigma \, : \, \sigma(v, w) = 0 \;\; \forall w \in \Lambda\}$$

is a symplectic complement of Λ in Σ w.r.t. the form σ.

The symplectic form σ is non-degenerate and it vanishes on any Lagrangian subspace Λ of Σ, hence it induces the canonical isomorphism $\Lambda^* \cong (\Sigma/\Lambda)$ via the linear operator $\lambda \to \sigma(\cdot, \lambda)$, $\lambda \in \Lambda$. In particular, it follows that $\Lambda \cong (\Sigma/\Lambda)^*$.

The tangent space $T_\Lambda(\mathcal{L}_n(\Sigma))$ of the Lagrange Grassmannian at every point Λ has a natural identification with the space of quadratic forms on Λ via the following construction. Let $t \mapsto \Lambda_t$ be a curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$ such that $\Lambda_0 = \Lambda$. Take some smooth curve $\lambda(t) \in \Lambda_t$ with $\lambda(0) \in \Lambda$. We associate with the tangent vector $\frac{d}{dt}\Lambda$ the quadratic form $\dot\Lambda : \lambda(0) \to \sigma(\dot\lambda(0), \lambda(0))$. This form depends only on $\frac{d}{dt}\Lambda_t\big|_{t=0}$ and $\lambda(0)$, so $\dot\Lambda(\lambda(0))$ is correctly defined. This construction leads to the following definition.

Definition 1.1.1. *We say that the curve $t \mapsto \Lambda_t$ is a regular curve in $\mathcal{L}_n(\Sigma)$, if for every t the corresponding quadratic form $\dot\Lambda_t$ is non-degenerate.*

The structure of the tangent space to the Lagrange Grassmannian allows to use the terms "positive" and "negative" with respect to the derivatives of the curves in the Lagrange Grassmannian. We will call a regular curve *increasing (decreasing)* if the corresponding quadratic form is positively (negatively) definite.

Actually there exists a canonical isomorphism between spaces $T_\Lambda(\mathcal{L}_n(\Sigma))$ and the space of linear self-adjoint operators $L(\Lambda, \Sigma/\Lambda)$. Indeed, let $\Lambda_t \in \mathcal{L}_n(\Sigma)$, $\Lambda_0 = \Lambda$ and $V = \dot\Lambda_0$. To any $V \in T_\Lambda(\mathcal{L}_n(\Sigma))$ one can associate a linear operator

$$\overline{V} : \Lambda \to \Sigma/\Lambda \cong \Lambda^* ,$$

such that

$$\overline{V}\,\lambda(0) = \dot\lambda(0) + \Lambda \;\in \Sigma/\Lambda , \tag{1.1}$$

for any smooth curve $\lambda(t) \in \Lambda_t$ with initial condition $\lambda(0) \in \Lambda$. Since the subspace Λ is Lagrangian, the operator $\overline{V}$ is self -adjoint and $V \to \overline{V}$ realizes an isomorphism of linear spaces $T_\Lambda(\mathcal{L}_n(\Sigma))$ and $L(\Lambda, \Sigma/\Lambda)$.

Let us now give a coordinate version of the introduced objects. One can choose a basis in Σ such that

$$\Sigma \cong \mathbb{R}^n \times \mathbb{R}^n = \{(x, y) : x, y \in \mathbb{R}^n\},$$

and

$$\sigma((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Such a basis is called *symplectic* or *Darboux* basis.

Let $t \to \Lambda_t$ be a regular curve in $\mathcal{L}_n(\Sigma)$. Assume that $\Lambda_0 \cap \{(0, y),\, y \in \mathbb{R}^n\} = 0$. Then for any t sufficiently close to 0 there exits an $n \times n$ matrix S_t such that

$$\Lambda_t = \begin{pmatrix} x \\ S_t x \end{pmatrix}, \qquad x \in \mathbb{R}^n$$

with $S_0 = 0$. The matrix curve $t \mapsto S_t$ is called *the coordinate representation of the curve* Λ_t *w.r.t. the chosen Darboux basis in* Σ. Since the subspaces Λ_t are Lagrangian, from (1.2) it follows that the matrices S_t are symmetric.

The regularity of the curve Λ_t is equivalent to the non-degeneracy of the matrix $\dot{S}_t$. Indeed, let λ_t be a smooth curve in Λ_t so that $\lambda_t = (x_\lambda, S_t x_\lambda)^T$. Then

$$\dot{\Lambda}_t(\lambda_t) = \sigma(\dot{\lambda}_t, \lambda_t) =$$

$$= \sigma\left(\begin{pmatrix} \dot{x}_\lambda \\ S_t \dot{x}_\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{S}_t x_\lambda \end{pmatrix}, \begin{pmatrix} x_\lambda \\ S_t x_\lambda \end{pmatrix} \right) = -\langle \dot{S}_t x_\lambda, x_\lambda \rangle . \tag{1.3}$$

Thus the quadratic form defined by the symmetric matrix $-\dot{S}_t$ corresponds to the derivative of the curve Λ_t.

It is easy to see that $\dot{S}_t$ also gives the coordinate representation of the linear operator $\underline{\dot{\Lambda}_0}$ defined by (1.1).

1.2 Maslov Index and Conjugate points

Let Λ_0, $\Delta \in \mathcal{L}_n(\Sigma)$ be a pair of transversal Lagrangian subspaces: $\Sigma = \Lambda_0 \oplus \Delta$. Suppose that in some local coordinates $\Lambda_0 = \{(p, 0)^T,\, p \in \mathbb{R}^n\}$ and $\Delta = \{(0, q)^T,\, q \in \mathbb{R}^n\}$. Then any $\Lambda \in \Delta^{\pitchfork}$ can be presented as $\Lambda = \{(p, Sp)^T,\, p \in \mathbb{R}^n\}$ with S being a symmetric $n \times n$ matrix, and one can assign to Λ a quadratic form

$$q_\Lambda(p) = \langle Sp, p \rangle .$$

To the subspace Λ_0 there corresponds a zero quadratic form.

Definition 1.2.1. *The Maslov Index of the triple of the Lagrangian subspaces Λ_0, Δ and Λ is the signature of the quadratic form q_Λ (i.e the difference between the number of positive and negative squares in the diagonal form of q_Λ):*

$$\mu(\Delta, \Lambda_0, \Lambda) = \operatorname{sgn} q_\Lambda .$$

Let

$$\Lambda^{\pitchfork} = \{\Delta \in \mathcal{L}_n(\Sigma) : \ \Lambda \cap \Delta = 0\}$$

be a space of all subspaces of $\mathcal{L}_n(\Sigma)$ transversal to Λ. Denote by $\mathcal{M}_{\Lambda_0}$ the following subset of $\mathcal{L}_n(\Sigma)$:

$$\mathcal{M}_{\Lambda_0} = \mathcal{L}_n(\Sigma) \setminus \Lambda_0^{\pitchfork} = \{\Lambda \in \mathcal{L}_n(\Sigma) : \ \Lambda \cap \Lambda_0 \neq 0\}\,.$$

Following [10], we will call $\mathcal{M}_{\Lambda_0}$ the *train* of the Lagrangian subspace Λ_0. Essentially $\mathcal{M}_{\Lambda_0}$ is a hyper-surface in $\mathcal{L}_n(\Sigma)$ consisting of degenerate quadratic forms: to a subspace Λ such that $\dim(\Lambda \cap \Lambda_0) = k$ there corresponds a form with a k-dimensional kernel. The singularities of $\mathcal{M}_{\Lambda_0}$ consist of the Lagrangian subspaces Λ such that $\dim(\Lambda \cap \Lambda_0) \geq 2$. The set of singular points has a co-dimension 3 in $\mathcal{L}_n(\Sigma)$.

As we saw already, the tangent space $T_\Lambda \mathcal{L}_n(\Sigma)$ has a natural identification with the space of quadratic forms on Λ. If Λ is a non-singular point of the train $\mathcal{M}_{\Lambda_0}$, then those vectors from $T_\Lambda \mathcal{L}_n(\Sigma)$ that correspond to positive or negative definite quadratic forms are not tangent to the train. Hence one can define a canonical co-orientation of the hyper-surface $\mathcal{M}_{\Lambda_0}$ at a non-singular point Λ by taking as a positive side the side of $\mathcal{M}_{\Lambda_0}$ containing positive definite quadratic forms.

This construction leads to the notion of the *intersection number* of an arbitrary continuous curve in the Lagrange Grassmannian, having endpoints outside of $\mathcal{M}_{\Lambda_0}$: if the smooth curve $\Lambda(\cdot)$ intersects $\mathcal{M}_{\Lambda_0}$ transversally at non-singular points, then, as usual, every intersection point $\Lambda(t)$ with $\mathcal{M}_{\Lambda_0}$ adds $+1$ or -1 into the value of the intersection number according to the direction of the vector $\dot{\Lambda}(t)$ w.r.t. to the positive and negative side of $\mathcal{M}_{\Lambda_0}$.

Moreover, the intersection number is invariant w.r.t. any homotopy which leaves the endpoints of the curve outside of $\mathcal{M}_{\Lambda_0}$ (see [4] for details). So an arbitrary continuous curve $\Lambda(\cdot)$ with endpoints outside $\mathcal{M}_{\Lambda_0}$ by a small perturbation can be (homotopically) transformed into a curve which is smooth and transversally intersects $\mathcal{M}_{\Lambda_0}$ in non-singular points. Since the set of singular points of $\mathcal{M}_{\Lambda_0}$ has a co-dimension 3 in $\mathcal{L}_n(\Sigma)$, any two curves, obtained by such transformation, can be deformed one to another by a homotopy, which avoids the singularities of $\mathcal{M}_{\Lambda_0}$. Hence the intersection number of the curve, obtained by the small perturbation from the original curve, does not depend on the perturbation and can be taken as the intersection number of the original curve.

Let $t \mapsto \Lambda_t$, $t \in [t_0, t_1]$ be a curve in $\mathcal{L}_n(\Sigma)$ such that

$$\Lambda_{t_0} \notin \mathcal{M}_{\Lambda_0}\,, \qquad \Lambda_{t_1} \notin \mathcal{M}_{\Lambda_0}\,. \tag{1.4}$$

Denote by $\mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_{t_0}^{t_1}$ the intersection number of a curve Λ_t with $\mathcal{M}_{\Lambda_0}$ on the interval $[t_0, t_1]$ such that (1.4) holds. If $\Lambda_t \in \Delta^{\pitchfork}$ for some $\Delta \in \mathcal{L}_n(\Sigma)$, then

$$\begin{aligned} \mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_{t_0}^{t_1} &= \frac{1}{2}(\operatorname{sgn} q_{\Lambda_{t_0}} - \operatorname{sgn} q_{\Lambda_{t_1}}) = \\ &= \frac{1}{2}(\mu(\Delta, \Lambda_0, \Lambda_{t_0}) - \mu(\Delta, \Lambda_0, \Lambda_{t_1}))\,. \end{aligned} \tag{1.5}$$

We will refer to $\mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_{t_0}^{t1}$ as to the *Maslov index related to* Λ_0 *of the curve* Λ_t *on* $[t_0, t_1]$. It turns out that that the right-hand side of (1.5) does not depend on the choice of Δ. Moreover, from the homotopy invariance of the intersection number it follows that if the curve $t \mapsto \Lambda_t$ is monotone increasing on $[t_0, t_1]$ and satisfy (1.4), then

$$\mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_{t_0}^{t_1} = \sum_{t_0 \le t \le t_1} \dim\big(\Lambda_t \cap \Lambda_0\big) . \tag{1.6}$$

The proof of these facts can be find in [4].

Definition 1.2.2. *The points* t_0 *and* t_1 *such that* $t_0 \neq t_1$ *are said to be conjugate for the curve* $\Lambda(\cdot) \in \mathcal{L}_n(\Sigma)$ *if* $\Lambda(t_0) \cap \Lambda(t_1) \neq 0$. *The dimension of the intersection* $\Lambda(t_0) \cap \Lambda(t_1)$ *is called the multiplicity of the conjugate point* t_1.

Let $t \to \Lambda_t$ be a curve in $\mathcal{L}_n(\Sigma)$ with the end-points satisfying (1.4). Then from Definition 1.2.2 it follows that the points conjugate to 0 are exactly the points where the Maslov index of Λ_t related to Λ_0 changes. If, in addition, Λ_t is a monotone curve, then from (1.6) it follows that the number of conjugate (to 0) points on the interval $[t_0, t_1]$, counted with their multiplicities, coincides with $\mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_{t_0}^{t_1}$.

1.3 Regular curves

Now we are going to introduce the basic symplectic invariants of the curves in the Lagrange Grassmannian. In this thesis we will deal only with regular curves. Both regular and non-regular situations were studied in papers [2], [4]-[5], and [8]-[9]. As we will see later, the problems of Classical Mechanics and Hamiltonian Dynamics produce plenty of examples of regular curves.

We begin by investigating the intrinsic structural properties of the Lagrange Grassmannian.

1.3.1 Affine structures

Definition 1.3.1. *The space* A *is called an affine space over a linear space* L *if for all* $x, y, z \in A$ *there defined the operation of subtraction such that* $x - y \in L$ *and the following axioms hold:*
1. $(x - y) + (y - z) = x - z \in L$,
2. for every $y \in A$ *and* $v \in L$ *there exists a unique element* $x \in A$ *such that* $x - y = v$.

We say that the function $x : \mathbb{R} \to A$ has a pole at $t = \tau$ if the function $x(t) - \bar{x}$ has a pole (as a vector function in the linear space L) at $t = \tau$ for some $\bar{x} \in A$. Actually this property does non depend on the choice of $\bar{x}$. Indeed, assume that the function $x(t)$ has a pole at $t = \tau$, and hence it admits a Lorain expansion $x(t) - \bar{x} = a_0(\tau) + \sum_{i\neq 0} a_i(\tau)(t-\tau)^i$. Take some other $\tilde{x} \in A$. Then using the axioms of the affine space we get

$$x - \tilde{x} = (x(t) - \bar{x}) + (\bar{x} - \tilde{x}) = a_0(\tau) + (\bar{x} - \tilde{x}) + \sum_{i\neq 0} a_i(\tau)(t-\tau)^i =$$

$$= \tilde{a}_0(\tau) + \sum_{i\neq 0} a_i(\tau)(t-\tau)^i \,.$$

By second axion of Definition 1.3.1 there exists a unique $\tilde{x}$ such that $\bar{x} - \tilde{x} = -a_0(\tau)$. We will call such $\tilde{x}$ the *derivative element* of the curve $x(t)$ at $t = \tau$. Directly from the definition it follows that the derivative element $\tilde{x}$ is the unique element of A such that the free term of the Laurent expansion of $x(t) - \tilde{x}$ at $t = \tau$ vanishes.

1.3.2 Derivative curve

Let Λ and Δ be a pair of transversal Lagrangian subspaces such that $\Sigma = \Lambda \oplus \Delta$. Let $\Pi_{\Delta\Lambda}$ be a projector of Σ onto Λ parallel to Δ such that

$$\pi_{\Delta\Lambda}\Big|_{\Delta} = 0\,, \qquad \pi_{\Delta\Lambda}\Big|_{\Lambda} = \mathrm{Id}\,. \tag{1.7}$$

The space $\{\pi_{\Delta\Lambda} : \Delta \in \Lambda^{\pitchfork}\}$ has a structure of an affine subspace of $gl(\Sigma)$: the first axiom of (1.3.1) is trivially satisfied and for any Δ_1, $\Delta_2 \in \Lambda^{\pitchfork}$ and $\alpha \in \mathbb{R}$

$$\alpha\pi_{\Delta_1\Lambda} + (1-\alpha)\pi_{\Delta_2\Lambda} = \pi_{\Delta\Lambda}$$

is again a projector of Σ onto Λ along the subspace $\Delta = \ker(\alpha\pi_{\Delta_1\Lambda} + (1-\alpha)\pi_{\Delta_2\Lambda}) \in \Lambda^{\pitchfork}$.

If $t \to \Lambda_t$ is a regular curve, then for any t sufficiently close to zero $\Lambda_t \in \Lambda_0^{\pitchfork}$, and so the pair Λ_0, Λ_t defines a splitting of Σ. As before, we choose some local coordinates such that $\Lambda_t = \{(x, S_t x)^T,\ x \in \mathbb{R}^n\}$, $S_t = S_t^T$ and $S_0 = 0$. Then

$$\pi_{\Lambda_t\Lambda_0} = \begin{pmatrix} \mathrm{Id} & -S_t^{-1} \\ 0 & 0 \end{pmatrix}.$$

By our choice of coordinates $S_0 = 0$. Let us suppose in addition that 0 is a root of minimal possible order n of the scalar function $t \mapsto \det S_t$. Then the matrix curve S_t^{-1} has a pole of the first order at zero and one gets the following Laurent expansion:

$$S_t^{-1} = (t\dot{S}_0 + \frac{t^2}{2}\ddot{S}_0 + \dots)^{-1} = \frac{1}{t}(\mathrm{Id} + \frac{t}{2}\dot{S}_0^{-1}\ddot{S}_0 + \dots)^{-1}\dot{S}_0^{-1} =$$

$$= \frac{1}{t}(\mathrm{Id} - \frac{t}{2}\dot{S}_0^{-1}\ddot{S}_0 + \dots)\dot{S}_0^{-1} = \frac{1}{t}\dot{S}_0^{-1} - \frac{1}{2}\dot{S}_0^{-1}\ddot{S}_0\dot{S}_0^{-1} + O(t)\,.$$

The Laurent expansion for the corresponding operator-valued function $t \to \pi_{\Lambda_t\Lambda_0}$ reads

$$\pi_{\Lambda_0\Lambda_t} = \sum_{i\neq 0} t^i\pi_i + \pi_0\,, \tag{1.8}$$

where

$$\pi_0 = \begin{pmatrix} \mathrm{Id} & \frac{1}{2}\dot{S}_0^{-1}\ddot{S}_0\dot{S}_0^{-1} \\ 0 & 0 \end{pmatrix}.$$

As we saw, the free term of a power expansion of any function in the affine space behaves as an element of the affine space while the other terms can be regarded as the elements of some linear space. Therefore the free term π_0 defines a projector onto Λ_t along some subspace $\Lambda_0^\circ \in \Lambda^\pitchfork$. In coordinates Λ_0° reads

$$\Lambda_0^\circ = \begin{pmatrix} -\frac{1}{2}\dot{S}_0^{-1}\ddot{S}_0\dot{S}_0^{-1}\, y \\ y \end{pmatrix}, \qquad y \in \mathbb{R}^n \tag{1.9}$$

It easy to check that $\pi_i|_{\Lambda_0} = 0$ for $i \neq 0$. The subspace Λ_0° defines the *derivative element* to Λ_0 at zero.

An analogous construction applied to Λ_τ with $\tau > 0$ gives a new curve $\tau \to \Lambda_\tau^\circ$ called *the derivative curve* of the curve Λ_t.

By construction, $\Lambda_0^\circ \in \Lambda_0^\pitchfork$ and we say that the pair Λ_0, Λ_0° *defines the canonical splitting of* Σ. We remark that w.r.t. local coordinates such that

$$\Lambda_0 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \Lambda_0^\circ = \begin{pmatrix} 0 \\ y \end{pmatrix},$$

the regular curve $\Lambda_t = \{(x, S_t x)^T,\ x \in \mathbb{R}^n\}$ is represented by the matrix S_t such that $S_0 = 0$, $\ddot{S}_0 = 0$ and $\det(\dot{S}_0) \neq 0$.

From the coordinate representation (1.9) it is easy to see that to a regular curve there corresponds a smooth derivative curve.

1.3.3 Generalized curvature operator

Definition 1.3.2. *The linear operator*

$$R_\Lambda(t) = -\overline{\dot{\Lambda}_t^\circ} \circ \overline{\dot{\Lambda}_t}\,. \tag{1.10}$$

is called the generalized curvature operator of the curve Λ_t at a point t.

By the isomorphism mentioned above, $\Lambda_t^\circ \cong \Sigma/\Lambda_t = \Lambda_t^*$, $\Lambda_t \cong \Sigma/\Lambda_t^\circ = \Lambda_t^{\circ *}$ and $\overline{\dot{\Lambda}_t} : \Lambda_t \mapsto \Lambda_t^*$ is a self-adjoint linear operator. Therefore R_Λ is a linear operator on Λ_t.

Definition 1.3.3. *The trace of the generalized curvature operator is called the generalized Ricci curvature:*

$$\rho_\Lambda(t) = \operatorname{tr} R_\Lambda(t)\,. \tag{1.11}$$

Definition 1.3.4. *The quadratic form*

$$r_\Lambda(t)(\xi) = \left(\overline{\dot{\Lambda}_t} \circ R_\Lambda(t)\xi\right)(\xi)\,, \qquad \xi \in \Lambda_t \tag{1.12}$$

is called the curvature form of the curve $t \mapsto \Lambda_t$ at a point t.

In what follows, depending on the context, in order to simplify notations we will use the same notation $\dot{\Lambda}_t$ for the quadratic form, bilinear form and the linear mapping $\Lambda_t \to \Lambda_t^*$, corresponding to a curve Λ_t.

The curvature $R_\Lambda(t)$ can be seen as a kind of generalization of the curvature operator $R : \xi \to \mathcal{R}(\dot{\gamma}(t), \xi)\dot{\gamma}(t)$ that appears in Riemannian geometry and describes the Jacobi vector fields via the classical Jacobi equation

$$\ddot{\xi} + R(\xi) = 0 \tag{1.13}$$

along any geodesic $\gamma(t)$, with $\mathcal{R}$ being the Riemann's tensor. We will discuss this point in details in Chapter 3.

As in Riemannian Geometry, the generalized curvature operator and the generalized Ricci curvature provide information about the location of the conjugate points of a curve in the Lagrange Grassmannian. There takes place the following analogous of the classical Rauch Theorem (see [2] for the proof):

Theorem 1.3.5. *Assume $t \mapsto \Lambda_t$ is a smooth curve in $\mathcal{L}_n(\Sigma)$ and $\dot{\Lambda}_t$ is a positive - definite quadratic form for all $t > 0$. If $R_\Lambda(t) \leq C\,Id$ for some constant $C > 0$, then $|t_1 - t_0| \geq \frac{\pi}{\sqrt{C}}$ for every pair of conjugate points t_0 and t_1. In particular, if $R_\Lambda(t) \leq 0$, then there are no conjugate points.*

If for all $t > 0$ $\rho_\Lambda(t) \geq nC$ for some $C > 0$, then for arbitrary $t_0 \leq t$ the interval $[t, t + \frac{\pi}{\sqrt{C}}]$ contains a point conjugate to t_0.

In local coordinates in $\mathcal{L}_n(\Sigma)$ such that $\Lambda_t = \{(x, S_t x)^T,\ x \in \mathbb{R}^n\}$ the generalized curvature operator has the following representation:

$$R_t = \frac{1}{2}\dot{S}_t^{-1}S_t^{(3)} - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2 \,. \tag{1.14}$$

The proof of this formula can be found, for example, in [3]. Here we would like to make the following remark.

Remark 1.3.6. In the case $n = 1$ the operator

$$\mathbb{S}(S_t) = \frac{1}{2}\dot{S}_\tau^{-1}S_\tau^{(3)} - \frac{3}{4}(\dot{S}_\tau^{-1}\ddot{S}_\tau)^2 \tag{1.15}$$

is just the classical *Schwarzian derivative* or *Schwarzian* of the scalar function S_t. It is well known that for scalar functions the Schwarzian satisfies the following remarkable identity:

$$\mathbb{S}\left(\frac{a\varphi(t) + b}{c\varphi(t) + d}\right) = \mathbb{S}\big(\varphi(t)\big) \tag{1.16}$$

for any constants a, b, c, and d, $ad - bc \neq 0$.

Similarly, if $n > 1$ by choosing another symplectic basis in Σ, we obtain a new coordinate representation $t \mapsto \widetilde{S}_t$ of the curve Λ_t which is a matrix Möbious transformation of S_t,

$$\widetilde{S}_t = (C + DS_t)(A + BS_t)^{-1} \tag{1.17}$$

for some $n \times n$ matrices A, B, C, and D. It turns out that the matrix Schwarzian (1.15) is invariant w.r.t. matrix Möbious transformations (1.17) by analogy with identity (1.16) (the only difference is that instead of identity we obtain similarity of corresponding matrices). Therefore formula (1.14) actually does not depend on the choice of local coordinates in $\mathcal{L}_n(\Sigma)$. This fact shows again the intrinsic meaning of curvature operator.

1.3.4 Canonical moving frame

Let $t \to \Lambda_t$ be a regular curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$. Consider a family of Lagrangian subspaces $\Delta_t \in \mathcal{L}_n(\Sigma)$ such that for any t the pair (Λ_t, Δ_t) form a splitting of Σ: $\Sigma = \Lambda_t \oplus \Delta_t$. Let $\{e_1^0, \dots, e_n^0\}$ be a basis of the space Λ_0.

Lemma 1.3.7. *There exists a unique way to choose the basis $\{e_1(t), \dots, e_n(t)\}$ of Λ_t such that $e_i(0) = e_i^0$, $\dot{e}_i(t) \in \Delta_t$ and $\{e_i(t), \dot{e}_i(t)\}_{i=1}^n$ form a Darboux basis of Σ for every $t \in \mathbb{R}$.*

Proof. Let $\{e_i(t)\}_{i=1}^n$ be a basis of Λ_t such that $e_i(0) = e_i^0$ and let $\{f_i(t)\}_{i=1}^n$ be the complementary basis in Δ_t. Denote by $E = (e_1(t), \dots, e_n(t))^T$ the matrix whose rows are the vectors $e_i(t)$ and similarly let $F = (f_1(t) \dots, f_n(t))^T$[1].

The derivative of E can be presented as $\dot{E} = M_1(t)E + M_2(t)F$, where M_1, M_2 are some $n \times n$ matrices. Let $\widetilde{E} = (\tilde{e}_1(t), \dots, \tilde{e}_n(t))^T$ be another basis of Λ_t such that $\widetilde{E} = A_t E$ where $A_t : \Lambda_t \to \Lambda_t$ is a linear non-degenerate operator depending on t. Then the inverse of its transform defines a new basis in Δ_t, complementary to $\widetilde{E}$: $\widetilde{F} = (A_t)^{-T} F$, $\widetilde{F} = (\tilde{f}_1(t), \dots, \tilde{f}_n(t))^T$, so that the $\{\tilde{e}_i(t), \tilde{f}_i(t)\}_{i=1}^n$ is a new Darboux basis of Σ. Further,

$$\dot{\widetilde{E}} = \dot{A}_t E + A_t \dot{E} = (\dot{A}_t + A_t M_1(t))E + A_t M_2(t) F\,.$$

Hence if the linear operator A_t satisfies the matrix ordinary differential equation $\dot{A}_t + A_t M_1(t) = 0$ with some initial condition A_0, then the new basis $\{\tilde{e}_i(t)\}_{i=1}^n$ is the desired one. For example, one can take $A_0 = \mathrm{Id}$. The uniqueness follows from the classical theorem of existence and uniqueness of the solutions of the ordinary differential equations. □

Any splitting of Σ generates a moving frame described in Lemma 1.3.7. In particular, as a complementary space to a regular curve Λ_t one can take its derivative curve Λ_t°. To

[1] We call the basis F complementary to the basis E, if $\{e_i(t), f_i(t)\}_{i=1}^n$ form a Darboux basis of the whole symplectic space Σ. If $(\widetilde{E}, \widetilde{F})$ is some other Darboux basis of Σ, then the symplectic map Φ : $(E, F) \to (\widetilde{E}, \widetilde{F})$ has the form

$$\begin{pmatrix} \widetilde{E} \\ \widetilde{F} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix}$$

with $A^T C = C^T A$, $B^T D = D^T B$ and $A^T D - C^T B = \mathrm{Id}$.

this special splitting there corresponds a moving frame with particular properties which reflect the intrinsic meaning of the curvature operator of the corresponding curve in the Lagrange Grassmannian. More precisely, there takes place the following statement.

Lemma 1.3.8. *Let $\{e_i(t), f_i(t)\}_{i=1}^n$ be the Darboux basis from Lemma 1.3.7: for every $t \in \mathbb{R}$ there is a splitting $\Sigma = \Lambda_t \oplus \Lambda_t^\circ$ and*

$$\Lambda_t = \operatorname{span}\{e_1(t), \dots, e_n(t)\}, \qquad \Lambda_t^\circ = \operatorname{span}\{f_1(t) \dots, f_n(t)\}.$$

Then there exist symmetric matrices $\varrho(t) = \{\varrho_{ij}(t)\}_{i,j=1}^n$ and $r(t) = \{r_{ij}(t)\}_{i,j=1}^n$ such that

$$\dot e_i(t) = \sum_{j=1}^n \varrho_{ij}(t) f_j(t), \qquad \dot f_i(t) = \sum_{j=1}^n r_{ij}(t) e_j(t), \qquad i = 1, \dots, n. \tag{1.18}$$

Moreover, the matrix $\varrho(t)$ is constant: $\varrho(t) = \varrho$.

Proof. The first statement of the lemma follows from the properties of a Darboux basis. Indeed, for all t and every $i, j = 1, \dots, n$ we have

$$\begin{aligned} \sigma(e_i(t), e_j(t)) &= 0, \\ \sigma(f_i(t), f_j(t)) &= 0, \\ \sigma(e_i(t), f_j(t)) &= \delta_{ij}, \end{aligned} \tag{1.19}$$

where δ_{ij} is a Kronecker symbol.

From the Lemma1.3.7 we know that the derivatives of $e_i(t)$, $i = 1, \dots, n$ belong to the subspace Δ_t, so there exist $\varrho_{ij}(t)$ such that

$$\dot e_i(t) = \sum_{j=1}^n \varrho_{ij}(t) f_j(t).$$

Now we show that the derivatives of $f_i(t)$ belong to Λ_t. Indeed, assume that $\dot f_i(t) = \sum_{j=1}^n \alpha_{ij}(t) f_j(t) + r_{ij}(t) e_j(t)$ with some $\alpha(t) \in \mathbb{R}^{n\times n}$. Then from the last formula of (1.19) we get

$$0 = \sigma(\dot e_i(t), f_j(t)) + \sigma(e_i(t), \dot f_j(t)) =$$

$$= \sum_{k=1}^n \left(\varrho_{ik}(t)\sigma(f_k(t), f_i(t)) + r_{jk}(t)\sigma(e_i(t), e_k(t)) + \alpha_{jk}(t)\sigma(e_i(t), f_k(t))\right) = \alpha_{ji}(t).$$

Next, differentiating the first formula of (1.19) we get

$$0 = \sigma(\dot e_i(t), e_j(t)) + \sigma(e_i(t), \dot e_j(t)) =$$

$$= \sum_{k=1}^n \left(\varrho_{ik}(t)\sigma(f_k(t), e_j(t)) + \varrho_{jk}(t)\sigma(e_i(t), f_k(t))\right) = -\varrho_{ij}(t) + \varrho_{ji}(t),$$

and so the matrix $\varrho(t)$ is symmetric. Analogously from the second formula of (1.19) it follows the symmetry of the matrix $r(t)$.

Let us show that the matrix $\varrho(t)$ is actually constant. First of all we observe that $\dot\varrho_{ij}(t) = \sigma(\ddot e_i(t), e_j(t))$.

For any t one can choose some local coordinates in $\mathcal{L}_n(\Sigma)$ so that $\Lambda_\tau = \{(x, S_\tau x)^T,\ x \in \mathbb{R}^n\}$ with $S_t = \ddot S_t = 0$ and $\det(\dot S_t) \neq 0$. In these coordinates every $e_i(\tau) \in \Lambda_\tau$, $i = 1, \dots, n$ admits the following representation:

$$e_i(\tau) = \begin{pmatrix} x_i(\tau) \\ S_\tau x_i(\tau) \end{pmatrix}, \qquad x_i(\tau) \in \mathbb{R}^n\,,$$

and by differentiation we get

$$\dot e_i(\tau) = \begin{pmatrix} \dot x_i(\tau) \\ \dot S_\tau x_i(\tau) + S_\tau \dot x_i(\tau) \end{pmatrix}.$$

Since $\dot e_i(\tau) \in \Delta_\tau$ and $\dot S_t$ is non-degenerate, $\dot x_i(t) = 0$. Differentiating again we have

$$\ddot e_i(\tau) = \begin{pmatrix} \ddot x_i(\tau) \\ S_\tau \ddot x_i(\tau) \end{pmatrix} + \begin{pmatrix} 0 \\ 2\dot S_\tau \dot x_i(\tau) + \ddot S_\tau x_i(\tau) \end{pmatrix},$$

So $\ddot e_i|_{\tau=t} = (\ddot x_i(t), 0)^T \in \Lambda_t$ and hence $\dot\varrho(t) = 0$. □

The basis $\{e_i(t), f_i(t)\}_{i=1}^n$ satisfying equations (1.18) is called *the canonical moving frame associated with the curve* Λ_t. Equations (1.18) are called *the structural equations.*

From Lemma 1.3.8 and (1.10) it follows immediately that w.r.t. the canonical moving frame the curvature operator has the form

$$R_\Lambda(t) = -\varrho\, r(t)\,. \tag{1.20}$$

In particular, for the regular curve one can choose the basis $\{e_i\}_{i=1}^n$ so that $\varrho = \mathrm{Id}$, then $R(t) = R_\Lambda(t) = -r(t)$. We will call such a basis *the special canonical moving frame.* With respect to this frame equations (1.18) take the form

$$\ddot e_i(t) + \sum_{j=1}^n R_{ij}(t) e_j(t) = 0\,, \quad i = 1, \dots, n\,, \tag{1.21}$$

with $R_{ij}(t)$ being the elements of the coordinate representation of the operator $R_\Lambda(t)$ w.r.t. the special canonical moving frame. We also remark that w.r.t. such basis the matrix representations of the curvature operator and the curvature form coincide.

1.4 Jacobi curves

Let $M \in \mathbb{R}^n$ be a smooth n - dimensional manifold and T^*M be the cotangent bundle over M. Denote by π the canonical projection $\pi : T^*M \to M$. Let σ be a non-degenerate canonical closed 2-form on T^*M. The cotangent bundle T^*M endowed with σ has a structure of a symplectic manifold, so that the space $\Sigma_z = (T_z(T^*M), \sigma)$ attached to any $z \in T^*M$ is a $2n$-dimensional symplectic space.

Let $h \in C^\infty(T^*M)$ and $\vec{h}$ be the Hamiltonian vector field associated to h:

$$\sigma_z(\cdot, \vec{h}) = d_z h, \qquad \forall z \in T^*M\,.$$

Hereafter we assume that $\vec{h}$ is a complete vector field, in other words we assume that the solution of a Cauchy problem $\dot{z} = \vec{h}(z)$, $z(0) = z_0$ is well defined for all $t \in \mathbb{R}$ and all $z_0 \in T^*M$.

If $q = (q_1, \ldots, q_n)$ are local coordinates in some open subset $\mathcal{N}$ of M and $p = (p_1, \ldots, p_n)$ are the induced coordinates in the fiber of $T^*\mathcal{N}$, then the canonical symplectic form is given by

$$\sigma = \sum_{i=1}^{n} dp_i \wedge dq_i\,.$$

The last expression allows to identify $T^*\mathcal{N}$ with $\mathbb{R}^n \times \mathbb{R}^n = \{(p,q),\, p,q \in \mathbb{R}^n\}$ so that $\mathcal{N} = 0 \times \mathbb{R}^n$. Then the tangent space $T_z(T^*\mathcal{N})$ to $T^*\mathcal{N}$ at any $z \in T^*\mathcal{N}$ is identified with $\mathbb{R}^n \times \mathbb{R}^n$. Under this identification

$$\vec{h} = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial p_i} \partial_{q_i} - \frac{\partial h}{\partial q_i} \partial_{p_i} \right)\,. \tag{1.22}$$

We denote by $e^{t\vec{h}} : M \to M$ the Hamiltonian flow induced by $\vec{h}$ on T^*M and by

$$e^{t\vec{h}}_* : T_z(T^*M) \to T_{e^{t\vec{h}}(z)}(T^*M)$$

its differential. Being a flow generated by a Hamiltonian vector field, $e^{t\vec{h}}$ preserves the symplectic structure on T^*M:

$$e^{t\vec{h}*}\sigma = \sigma\,,$$

and therefore it transforms Lagrangian subspaces of Σ_z into Lagrangian subspaces of $\Sigma_{e^{t\vec{h}}z}$.

A *Lagrange distribution* $\mathcal{D} \subset T(T^*M)$ is a smooth vector sub-bundle of $T(T^*M)$ such that each fiber $\mathcal{D}_z = \mathcal{D} \cap \Sigma_z$, $z \in T^*M$ is a Lagrange subspace of the symplectic space Σ_z. The pair $(\vec{h}, \mathcal{D})$ is called the *dynamical Lagrange distribution*

The basic example of a Lagrange distribution on $T(T^*M)$ is the so-called *vertical* distribution $\Pi = \bigcup_{z \in T^*M} \Pi_z$ with

$$\Pi_z = T_z(T^*_{\pi(z)}M)\,,\ z \in T^*M$$

being the tangent space to the fiber $T^*_{\pi(z)}M$ at the point $z \in T^*M$. As we saw in Example 0.0.1, this distribution appears naturally in the problems of classical Calculus of Variations. It will play a particular role in our further analysis. We will call the pair $(\vec{h}, \Pi)$ *the vertical dynamical Lagrangian distribution*.

For any two vector field ξ_1, ξ_2 from the distribution $\mathcal{D}$ the number $\sigma_z([\vec{h},\xi_1],\xi_2)$ depends only on the vectors $\xi_1(z)$ and $\xi_2(z)$. Moreover, it defines a symmetric bilinear form on $\mathcal{D}$. Indeed, since the Hamiltonian flow preserves the symplectic structure,

$$0=\sigma(\xi_1,\xi_2)=\left(e^{t\vec{h}*}\sigma\right)(\xi_1,\xi_2)=\sigma(e_*^{t\vec{h}}\xi_1,e_*^{t\vec{h}}\xi_2)\,.$$

Differentiating this equality w.r.t. time and setting $t=0$ gives $0=\sigma([\vec{h},\xi_1],\xi_2)+\sigma(\xi_1,[\vec{h},\xi_2])$, and since the form σ is anti-symmetric,

$$\sigma([\vec{h},\xi_1],\xi_2)=\sigma([\vec{h},\xi_2],\xi_1)\,.$$

Definition 1.4.1. *We say that $\vec{h}$ is regular at $z\in T^*M$ with respect to the Lagrange distribution $\mathcal{D}$ if the quadratic form*

$$g_z^h(\xi(z),\xi(z))=\sigma_z([\vec{h},\xi],\xi),\qquad \xi\in\mathcal{D}$$

is non-degenerate. In this case the dynamical Lagrange distribution $(\vec{h},\mathcal{D})$ is called regular.

Analogously we say that the vector field $\vec{h}$ and the dynamical distribution $(\vec{h},\mathcal{D})$ are *monotone increasing (decreasing)* at $z\in T^*M$ if the quadratic form g_z^h is positive (negative) definite.

Definition 1.4.2. *The curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma_z)$ defined by the following expression*

$$J_z(t)=e_*^{-t\vec{h}}\Pi_{e^{t\vec{h}}(z)} \tag{1.23}$$

is called the Jacobi curve of the curve $t\to e^{t\vec{h}}z$ attached at the point $z\in T^*M$.

Essentially the Jacobi curve is just a family of the vertical Lagrangian subspaces translated along an integral curve of the field $\vec{h}$ and collected at a point z. All the information about the Jacobi curve is encoded in the generating vector field $\vec{h}$. Due to the invariance of the symplectic structure with respect to the symplectic transformation, all points $z(t)=e^{t\vec{h}}z$, $t\in\mathbb{R}$ are equivalent. Therefore knowing the Jacobi curve and the associated derivative curve at some instant of time, for example, at $t=0$ at any point z of a given trajectory $t\mapsto e^{t\vec{h}}z_0$, one can reconstruct the whole curves just by translation along the Hamiltonian flow $e^{t\vec{h}}$:

$$J_{z_0}(t)=e_*^{-t\vec{h}}J_z(0)\,,\qquad J_{z_0}^\circ(t)=e_*^{-t\vec{h}}J_z^\circ(0)\,. \tag{1.24}$$

The pair consisting of a Jacobi curve $J_z(t)$ and the corresponding derivative curve $J_z^\circ(t)$ defines a canonical splitting of Σ_z at any time t. The distribution

$$\Xi=\bigcup_{z\in T^*M}J_z^\circ(0)$$

is a direct complement to the vertical distribution Π, it is called the *canonical connection* associated to the field $\vec{h}$. The corresponding splitting $T(T^*M)=\Pi\oplus\Xi$ is called the *canonical splitting* of $T(T^*M)$. We will call vector fields from Ξ *horizontal.*

Proposition 1.4.3. *The regular w.r.t. the vertical distribution vector field generates a regular Jacobi curve.*

Proof. Recall that for any vector field ζ in T^*M one has

$$\frac{d}{dt}\left(e_*^{-t\vec{h}}\zeta\right) = e_*^{-t\vec{h}}[\vec{h},\zeta]\,. \tag{1.25}$$

Without loss of generality we can choose local coordinates on the cotangent bundle

$$T^*M = \{z = (p,q),\ p,q \in \mathbb{R}^n\}$$

such that $\Pi_z = \{(p,0)^T,\ p \in \mathbb{R}^n\}$. Take some $\lambda(t) \in J_z(t)$ such that $\lambda(0) = \xi(z)$ where $\xi \in \Pi$ and $\xi = \sum\limits_{i=1}^{n} \xi_i \partial_{p_i}$. Then

$$\dot{\lambda}(0) = \left[\vec{h}, \sum_{i=1}^{n} \xi_i \partial_{p_i}\right](z) = -\sum_{i,j=1}^{n} h_{p_i p_j}(z)\xi_j(z)\partial_{q_i} \quad \mathrm{mod}\ \Pi_{\mathrm{z}}\,,$$

and recalling the definition of the form g_z^h we get

$$\sigma(\dot{\lambda}_0, \lambda_0) = g_z^h(\xi(z), \xi(z)) = \langle h_{pp}\xi(z), \xi(z)\rangle\,, \tag{1.26}$$

where $h_{pp} = \{h_{p_i p_j}(z)\}_{i=1}^{n}$. The proposition now follows from the regularity of the field $\vec{h}$. □

Remark 1.4.4. From the last proposition it follows that the Hamiltonial vector field $\vec{h}$ is regular w.r.t. vertical distribution Π if and only if the Hessian of the restriction of the corresponding Hamiltonian function h to the fibers is non-degenerate. In particular, $\vec{h}$ is monotone increasing (decreasing) if and only if h is strictly convex (concave) on fibers.

Definition 1.4.5. *The point $z_1 = e^{t_1\vec{h}}z_0$ is called conjugate to z_0 w.r.t. the dynamical Lagrangian distribution $(\vec{h},\mathcal{D})$ along the integral curve $t \mapsto e^{t_1\vec{h}}z_0$ of the field $\vec{h}$ if and only if*

$$e_*^{t_1\vec{h}}\mathcal{D}_{z_0} \cap \mathcal{D}_{z_1} \neq 0\,. \tag{1.27}$$

Remark 1.4.6. From the definitions of conjugate points in the Lagrange Grassmannian (Definition 1.2.2) and Jacobi curve (Definition 1.4.2) it is easy to see that the point z_1 is conjugate to z_0 in the sense of Definition 1.4.5 for the vertical dynamical Lagrangian distribution $(\vec{h},\Pi)$ if and only if the time t_1 is conjugate to 0 for the Jacobi curve $J_{z_0}(\cdot)$ in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma_z)$.

One can describe all symplectic invariants associated with a Jacobi curve in terms of the generating vector field $\vec{h}$. Such formulation is particularly useful in practical computations.

First let us find an explicit expression of the canonical connection Ξ associated to the field $\vec{h}$. Let ξ be a vertical vector field: $\xi(z) \in \Pi_z$ for all $z \in T^*M$. Choose some local coordinates in $\mathcal{L}_n(\Sigma_z)$ such that

$$J_z(0) = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad J_z^\circ(0) = \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad x, y \in \mathbb{R}^n ,$$

and

$$J_z(t) = \begin{pmatrix} x \\ S_t x \end{pmatrix}, \quad S_t^T = S_t , \quad S_0 = \ddot{S}_0 = 0 , \quad \det(\dot{S}_0) \neq 0 .$$

Then any vector $\zeta \in \Sigma_z$ can be presented as a sum

$$\zeta = \zeta^h + \zeta^v ,$$

where $\zeta^h \in J_z(0)$, $\zeta^v \in J_z^\circ(0)$ are the horizontal and vertical parts of ζ. Take now some $\lambda_t \in J_z(t)$ such that $\lambda_0 = (x_\lambda, 0)^T = \xi(z)$ where $\xi \in \Pi$. Then

$$\dot{\lambda}_t = \begin{pmatrix} \dot{x}_\lambda \\ S_t \dot{x}_\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{S}_t x_\lambda \end{pmatrix} .$$

Comparing with (1.25) we get

$$[\vec{h}, \xi]^v(z) = \dot{x}_\lambda(0) , \qquad [\vec{h}, \xi]^h(z) = \dot{S}_0 x_\lambda(0) . \tag{1.28}$$

Differentiating one more time gives

$$[\vec{h}, [\vec{h}, \xi]]^h(z) = 2[\vec{h}, [\vec{h}, \xi]^v]^h(z) ,$$

and finally

$$[\vec{h}, [\vec{h}, \xi]^v]^h(z) = [\vec{h}, [\vec{h}, \xi]^h]^h(z) . \tag{1.29}$$

If now we assume that the horizontal basis in Σ_z has the form

$$J_z^\circ(0) = \operatorname{span}\left\{ \partial_{q_i} + \sum_{j=1}^n a_{ij} \partial_{p_j} , \ i = 1, \dots, n \right\} ,$$

where (p, q) are local coordinates in T^*M, then substituting this expression into (1.29) we get the elements of $n \times n$ symmetric matrix $A(z) = \{a_{ij}(z)\}_{i,j=1}^n$:

$$2h_{pp} \, A(z) \, h_{pp} = \{h, h_{pp}\} - h_{pq} h_{pp} - h_{pp} h_{qp} , \tag{1.30}$$

where

$$h_{pp} = \{h_{p_i p_j}(z)\}_{i,j=1}^n , \quad h_{pq} = \{h_{p_i q_j}(z)\}_{i,j=1}^n , \quad h_{qq} = \{h_{q_i q_j}(z)\}_{i,j=1}^n ,$$

and

$$\{h, h_{pp}\}_{ij} = \{h, h_{p_i p_j}\}(z) .$$

Here the matrix $\{h, h_{pp}\}$ is the matrix of Poisson brackets of h with elements of h_{pp}.

The similar computation applied to (1.10) (see [3] for the details) gives the formula for the curvature operator

$$R_{J_z}(0)\xi(z) = -[\vec{h}, [\vec{h}, \xi]^h]^v(z)\,. \tag{1.31}$$

We will call the operator $R_z^{\vec{h}} = R_{J_z}(0)$ and the quadratic form $r_z^{\vec{h}}(\xi) = g_z^{\vec{h}}(R_z^{\vec{h}}\xi, \xi)$, $\xi \in \Pi$ the *curvature operator* and the *curvature form* of the Hamiltonian vector field $\vec{h}$ at a point $z \in T^*M$. We say that the field $\vec{h}$ has a negative (positive) curvature at $z \in T^*M$ if its curvature form is negative (positive) definite quadratic form on Π_z.

We conclude this chapter with few basic examples.

Example 1.4.7. *(Natural Mechanical system)* A mechanical system with the Hamiltonian of the form

$$h(p,q) = \frac{\langle p,p\rangle}{2} + U(q)\,, \qquad p, q \in \mathbb{R}^n\,, \tag{1.32}$$

$U(q)$ being some function on $M = \mathbb{R}^n$, is called *the natural mechanical system.*[2] It provides the simplest example of a regular Jacobi curve. In this case $h_{pp} = \mathrm{Id}$ and the corresponding Hamiltonian vector field is monotone increasing (w.r.t. Π) together with the generated Jacobi curve. The splitting defined by (1.30) is trivial: $A(z) \equiv 0$. Moreover, the curvature form $r_z^{\vec{h}}$ is just the Hessian of the potential energy U:

$$r_z^{\vec{h}}(\partial p_i, \partial p_j) = \frac{\partial^2 U}{\partial q_i \partial q_j}(q)\,, \qquad z = (p,q) \in T^*M\,. \tag{1.33}$$

Example 1.4.8. *(Riemannian manifold)* Let M be a Riemannian manifold endowed with a Riemannian metric G that defines an inner product $G_q(\cdot,\cdot)$ on each tangent space T_qM, $q \in M$ and depends smoothly on q.

There exists a canonical isomorphism between T_q^*M and T_qM via $G_q(\cdot,\cdot)$. For any $q \in M$ and $p \in T_q^*M$ we will denote by $p^\uparrow$ the image of p under this isomorphism, namely, the vector $p^\uparrow \in T_qM$, satisfying

$$p = G_q(p^\uparrow, \cdot)\,. \tag{1.34}$$

This operation corresponds to the raising of indexes in the coordinate representation of co-vectors and vectors.

The standard Legendre transformation (see (I.5)) leads to the following Hamiltonian function on T^*M: $h(p,q) = \frac{1}{2}G_q^{-1}(p,p)$. The corresponding vector field $\vec{h}$ is regular provided the form G is non-degenerate. Note that $\vec{h}$ generates a geodesic flow on TM due to the natural isomorphism defined by (1.34).

It turns out that the generalized curvature operator is exactly the Riemann's operator from the Jacobi equation (1.13):

$$(R_z^{\vec{h}}\xi(z))^\uparrow = \mathcal{R}(z^\uparrow, \xi^\uparrow)z^\uparrow\,, \qquad \xi \in \Pi\,,\ z \in T^*M\,.$$

[2]Usually the function U represents the potential energy of a mechanical system.

Here we identified the vertical space Π_z with $T^*_{\pi(z)}M$ using the fact that the fibers of T^*M are linear spaces.

Example 1.4.9. *(Mechanical system with holonomic constrains)* The motion of a mechanical system subject to some holonomic constrains can be modeled as a motion of a point on a Riemannian manifold M endowed with the metric G from the previous example in the potential filed described by some function U on M. In this case the Hamiltonian function takes the form

$$h(p,q) = \frac{1}{2}G_q^{-1}(p,p) + U(q)\,,\ (p,q) \in T^*M\,, \tag{1.35}$$

and it can be shown that

$$(R_z^{\vec{h}}\xi(z))^{\uparrow} = \mathcal{R}(z^{\uparrow},\xi^{\uparrow})z^{\uparrow} + \nabla_{\xi^{\uparrow}}(\nabla U)(\pi(z))\,, \qquad \xi \in \Pi\,,\ z \in T^*M\,, \tag{1.36}$$

where ∇_x denotes the covariant derivative along some vector field $x \in TM$, defined by the Levi-Civita connection on the Riemannian manifold M.

Chapter 2

Reduction by first integrals

In this chapter we study regular dynamical Lagrangian distributions generated by Hamiltonian systems with first integrals. Namely, we study the change of curvature operator, curvature form and the behavior of conjugate points related to the reduction of a dynamical system by an arbitrary number of first integrals in involution.

The existence of first integrals plays an important role in the analysis of the global behavior of the solutions of a Hamiltonian system. As it is well known from the course of Dynamical Systems, the presence of first integrals permits to reduce the number of degrees of freedom of the system by restriction the original phase space to the level set of the first integrals. The new *reduced* dynamical system has less unknown variables and often it is easier to deal with. The class of systems with first integrals is quite large, for example, it includes all autonomous Hamiltonian systems: the Hamiltonian function itself is a first integral. Another classical example are the mechanical systems with rotational symmetries (so-called *cyclic integrals*), like the Rigid Body or the system of N bodies interacting gravitationally.

We start by a short discussion about the geometrical and variational meaning of the reduction by first integrals. In Section 2.1 we give the definition of a reduced dynamical Lagrangian distribution and illustrate the variational meaning of conjugate points of a reduced distribution illustrating our exposition by the most common situations of Hamiltonian Dynamics.

In Section 2.2 we derive an explicit formula for the curvature operator and the curvature form of a reduced dynamical Lagrangian distribution. First we define the *reduction of a curve in the Lagrange Grassmannian* and make the proof for this case. Then we reformulate the results in terms of dynamical Lagrangian distributions. It turns out that reduction in a monotone increasing Lagrangian dynamical distributions does not decrease its curvature form. The method of calculation which we use allows to avoid the introduction of the new canonical variables for the reduced system.

In Section 2.3 we analyze the relative behavior of conjugate points of the original and reduced systems. An interesting phenomenon that we find is that the sets of conjugate points to a given point w.r.t. the monotone increasing dynamical Lagrangian distribution and w.r.t. its reduction by one integral are alternating. Moreover, the first conjugate point (to a given point) of the reduced Lagrangian distribution comes before any conjugate point related to the original dynamical distribution.

2.1 Variational and geometrical aspects of reduction

Consider a dynamical Lagrangian distribution $(\vec{h}, \mathcal{D})$ generated by a regular Hamiltonian vector field $\vec{h} \in T(T^*M)$. Let $g_1, \ldots, g_s$ be the first integrals in involution of the Hamiltonian h, i.e., s functions on T^*M such that

$$\{h, g_i\} = 0\,, \qquad \{g_i, g_j\} = 0\,, \qquad i, j = 1, \ldots, s\,. \tag{2.1}$$

Let $\mathcal{G} = (g_1, \ldots, g_s)$ and consider the set

$$\mathcal{D}_z^{\mathcal{G}} = \overline{\mathcal{D}_z^{\mathcal{G}}} + \operatorname{span}\left(\vec{g}_1(z), \ldots, \vec{g}_s(z)\right)\,, \qquad \overline{\mathcal{D}_z^{\mathcal{G}}} = \left(\bigcap_{i=1}^{s} \ker\, d_z g_i\right) \cap \mathcal{D}_z\,. \tag{2.2}$$

It is easy to see that the subspace $\mathcal{D}_z^{\mathcal{G}}$ is Lagrangian, $\dim \mathcal{D}_z^{\mathcal{G}} = n$ and $\mathcal{D}^{\mathcal{G}}$ is a Lagrangian distribution. The pair $(\vec{h}, \mathcal{D}^{\mathcal{G}})$ is called *the reduction by the s-tuple $\mathcal{G}$ of first integrals of in involution* or shortly *the $\mathcal{G}$-reduction* of the dynamical Lagrangian distribution $(\vec{h}, \mathcal{D})$.

Let us consider more in detail the geometrical meaning of the reduction in the case of one first integral. In this case $\mathcal{G} = g$ with $g \in C^\infty(T^*M)$ such that $\{h, g\} = 0$. The following example describes the standard reduction of the Hamiltonian systems on the level set of the first integral linear on fibers, commonly used in Mechanics.

Example 2.1.1. Assume that the first integral g is such that the corresponding Hamiltonian vector field $\vec{g}$ preserves the distribution $\mathcal{D}$, namely,

$$e_*^{t\vec{g}} \mathcal{D} = \mathcal{D}\,. \tag{2.3}$$

Fixing some value c of g, one can define (at least locally) the following quotient manifold:

$$W_{g,c} = g^{-1}(c)/\mathcal{C}\,,$$

where $\mathcal{C}$ is the line foliation of the integral curves of the vector field $\vec{g}$. It is easy to see that $\dim W_{g,c} = n - 1$ and it is a symplectic manifold with the symplectic form induced by the symplectic form σ on T^*M. Moreover, if we denote by

$$\Phi : g^{-1}(c) \mapsto W_{g,c}$$

the canonical projection on the quotient set, then the vector field $\Phi_* \vec{h}$ is a well defined Hamiltonian vector field on $W_{g,c}$, because by our assumptions the vector fields $\vec{h}$ and $\vec{g}$ commute. Thus by (2.3), $\Phi_*(\mathcal{D}^g)$ is well defined Lagrangian distribution on $W_{g,c}$. So, to any dynamical Lagrangian distribution $(\vec{h}, \mathcal{D})$ on T^*M one can associate the dynamical Lagrangian distribution $(\Phi_* \vec{h}, \Phi_* \mathcal{D}^g)$ on the symplectic manifold $W_{g,c}$ of smaller dimension.

The model example of a first integral satisfying (2.3) is a first integral linear on fibers, i.e a function g such that

$$g(p,q) = p\left(V(q)\right), \quad q \in M,\ p \in T_q^*M \tag{2.4}$$

with V being a vector field on M. The first integrals satisfying (2.4) are called *cyclic*. By the Theorem on the straightening of vector fields ([12]), one can choose the canonical variables in the phase space T^*M is such a way that $V = \partial q_i$ for some i, then q_i is a cyclic variable (i.e. the Lagrangian function and, consequently, the Hamiltonian do not depend on q_i explicitly) and the corresponding conjugate impulse $p_i = g(p,q)$.

If we denote by $\mathcal{V}$ the line foliation of integral curves of the vector field V, then $W_{g,c}$ can be identified with $T^*(M/\mathcal{V})$. So, after reduction we work with the dynamical Lagrangian distribution $\left(\Phi_*\vec{h}, \mathcal{P}\right)$ on the reduced phase space $T^*(M/\mathcal{V})$ with $\mathcal{P} = \overline{\Pi^g}$ being the vertical sub-bundle of $T^*(M/\mathcal{V})$.

In view of the previous example the following analogue of the notion of the conjugate points along the extremal of the $\mathcal{G}$-reduction of the pair $(\vec{h}, \mathcal{D})$ is natural: the point $z_1 = e^{t_1\vec{h}}z_0$ is called *conjugate to* z_0 for the $\mathcal{G}$-reduction of the pair $(\vec{h}, \mathcal{D})$ along the integral curve $t \to e^{t\vec{h}}z_0$ of the Hamiltonian field $\vec{h}$, if

$$\left((e^{t_1\vec{h}})_*\mathcal{D}^{\mathcal{G}}_{z_0} \cap \mathcal{D}^{\mathcal{G}}_{z_1}\right) / \mathrm{span}\{\vec{g}_1(z_1), \dots, \vec{g}_s(z_1)\} \neq 0\,. \tag{2.5}$$

In the situation, described in Example 2.1.1, the point $z_1 = e^{t_1\vec{h}}z_0$ is conjugate to z_0 for the g-reduction of the pair $(\vec{h}, \mathcal{D})$ along the curve $t \to e^{t\vec{h}}z_0$ if and only if the points $\Phi(z_1)$ and $\Phi(z_0)$ are conjugate w.r.t. the the pair $\left(\Phi_*\vec{h}, \Phi_*\mathcal{D}^g\right)$ along the curve $t \to \Phi(e^{t\vec{h}}z_0)$ in the reduced space $W_{g,c}$.

We illustrate the variational meaning of the conjugate points of the reduction on the following two important examples. In both examples $L \in \mathcal{C}^\infty(TM)$ is a given Lagrangian function convex on T_qM for any $q \in M$, and $h : T^*M \to \mathbb{R}$ is the corresponding Hamiltonian defined via the Legendre transformation (I.4).

Example 2.1.2. *(Reduction by a cyclic integral)* Assume that the Hamiltonian h admits a first integral g satisfying (2.4). The following optimal control problem with fixed terminal time and free terminal point illustrates the variational meaning of conjugate points of the reduced problem in this case.

Let $\mathcal{V}_1 : \mathbb{R} \to M$ be an integral curve of the vector field V and $a(\cdot)$ be a function on $\mathcal{V}_1$ such that

$$a(\mathcal{V}_1(s)) = s, \quad s \in \mathbb{R}.$$

Fix some constant $c \in \mathbb{R}$. Then for a given point q_0 and time T consider the minimizing problem

$$A_c^T[q(\cdot)] = \int_0^T L\left(q(t), u(t)\right)\, dt - c\, a\left(q(T)\right) \to \min, \tag{2.6}$$

$$q(0) = q_0, \quad q(T) \in \mathcal{V}_1, \quad u \in T_q M\,. \tag{2.7}$$

The curve $\hat{q} : [0,T] \mapsto M$, satisfying (2.7), is an extremal of the problem (2.6)-(2.7) if and only if there exists an integral curve $\gamma : [0,T] \mapsto g^{-1}(c)$ of $\vec{h}$, such that $\hat{q}(t) = \pi\left(\gamma(t)\right)$ for all $0 \le t \le T$. In this case the point $\gamma(0)$ is conjugate to $\gamma(T)$ for the g-reduction of the pair $(\vec{h}, \Pi)$ if and only if the point q_0 is conjugate to the point $q(T)$ along the extremal $\hat{q}(\cdot)$ in the classical variational sense for the problem (2.6)-(2.7), i.e. the second variation of the functional $A_c^T[\hat{q}]$ is degenerate.

The next example shows that actually condition (2.4) is not restrictive and the reduction makes sense also for the non-cyclic integrals, like the Hamiltonian function itself.

Example 2.1.3. *(Reduction by the energy integral)* Suppose now that $g = h$. This case can be modeled by the following optimal control problem with fixed end-points and free terminal time : for given real $c \in \mathbb{R}$ and points q_0, $q_1 \in M$

$$A_h^T[q(\cdot)] = \int_0^T L\left(q(t), u(t)\right)\, dt - cT \to \min, \quad T \text{ is free}, \tag{2.8}$$

$$q(0) = q_0, \quad q(T) = q_1, \quad u \in T_q M\,. \tag{2.9}$$

The curve $\hat{q} : [0,T] \mapsto M$, satisfying (2.9), is an extremal of the problem (2.8)-(2.9) if and only if there exists an integral curve $\gamma : [0,T] \mapsto h^{-1}(c)$ of the field $\vec{h}$, such that $\hat{q}(t) = \pi\left(\gamma(t)\right)$ for all $0 \le t \le T$. In this case the point $\gamma(0)$ is conjugate to $\gamma(T)$ for the h-reduction of the pair $(\vec{h}, \Pi)$ if and only if the point q_0 is conjugate to the point q_1 along the extremal $\hat{q}(\cdot)$ in the classical variational sense for the problem (2.8)-(2.9).

Note that the present case can be seen as a generalization of the situation considered in Example 2.1.2: one can pass to the extended configurational space $\widetilde{M} = M \times \mathbb{R}$ instead of M and take the following function $\widetilde{L} : T\widetilde{M} \mapsto \mathbb{R}$ as the new Lagrangian:

$$\widetilde{L}(\tilde{q}, \widetilde{u}) = L\left(q, \frac{u}{y}\right) y\,.$$

Here $\tilde{q} \in \widetilde{M}$ so that $\tilde{q} = (q,t)$, $q \in M$, $t \in \mathbb{R}$ and $\widetilde{u} \in T_{\tilde{q}}\widetilde{M}$ is such that $\widetilde{u} = (u, y)$, $u \in T_q M$, $y \in T_t\mathbb{R} \cong \mathbb{R}$. This construction reflects the well known fact ([17]) that (t, H) is the pair of conjugate variables for the new Lagrangian function $\widetilde{L}$, so that the vector filed ∂_t plays the role of the filed V of the previous example. [1]

[1] This is the Optimal Control version of the so-called Mopertui Least Action Principle well known is Classical Mechanics.

Remark 2.1.4. Observe that in both cases of Examples 2.1.2 and 2.1.3 the number of free parameters of the variational problem for the reduced system is not less than for the original system: making reduction together with decreasing the number of degrees of freedom of the system adds some new free parameters, like the terminal point in the Example 2.1.2 and the terminal time in Example 2.1.3.

2.2 Reduced curvature

2.2.1 Reduction and the curves in the Lagrange Grassmannian

Consider a s-tuple of vectors $(l_1, \dots l_s)$ in the $2n$-dimensional symplectic space Σ such that

$$\sigma(l_i, l_j) = 0\,, \qquad i, j = 1, \dots, s\,. \tag{2.10}$$

Let $\ell = \operatorname{span}\{l_1, \dots l_s\}$. For any $\Lambda \in \mathcal{L}_n(\Sigma)$ define

$$\Lambda^\ell = \Lambda \cap \ell^\angle + \ell, \qquad \overline{\Lambda^\ell} = \Lambda^\ell / \ell\,. \tag{2.11}$$

Actually, $\overline{\Lambda^\ell}$ is a Lagrangian subspace of $2n - 2s$ dimensional symplectic space $\widetilde{\Sigma} = (\ell^\angle/\ell)$ (with symplectic form induced by σ).

Let $\Lambda(\cdot)$ be a regular curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$. We will call the curve $\Lambda(\cdot)^\ell$ *the reduction by the s-tuple ℓ*, satisfying (2.10), or shortly *the ℓ-reduction* of the curve $\Lambda(\cdot)$. Note that by (2.11) $\ell \subset \Lambda(t)^\ell$ for any t. Therefore the curve $\Lambda(\cdot)^\ell$ is not regular and the constructions of Chapter 1 cannot be applied to it directly. Instead, suppose that the curve $\overline{\Lambda(\cdot)^\ell}$ is a regular curve in the Lagrange Grassmannian $\mathcal{L}_{n-s}(\widetilde{\Sigma})$. Then the curvature operator $R_{\overline{\Lambda^\ell}}(t)$ of this curve is well-defined linear operator on the space $\overline{\Lambda(t)^\ell}$.

Denote by $\phi : \Sigma \mapsto \Sigma/\ell$ the canonical projection on the factor-space.

Definition 2.2.1. *The curvature operator $R_{\Lambda^\ell}(t)$ of the ℓ-reduction $\Lambda(\cdot)^\ell$ at a point t is the linear operator on $\Lambda(t)^\ell$, satisfying*

$$R_{\Lambda^\ell}(t)(\xi) = \left(\phi\big|_{\Lambda(t)\cap\ell^\angle}\right)^{-1} \circ R_{\overline{\Lambda^\ell}}(t) \circ \phi(\xi), \qquad \xi \in \Lambda(t)^\ell. \tag{2.12}$$

The curvature form $r_{\Lambda^\ell}(t)$ of the ℓ-reduction $\Lambda(\cdot)^\ell$ at a point t is the quadratic form on $\Lambda(t)^\ell$, satisfying

$$r_{\Lambda^\ell}(t)(\xi) = \frac{d}{dt}\left(\Lambda(t)^\ell\right)\left(R_{\Lambda^\ell}(t)\xi, \xi\right)\,, \qquad \xi \in \Lambda(t)^\ell\,. \tag{2.13}$$

The reduction in the Lagrange Grassmannian is a model of the reduction by first integrals in the phase space of a dynamical system which we discussed in Section 2.1. Indeed, let $\mathcal{G} = (g_1, \dots, g_s)$ be a set of s involutive first integrals. Then at any $z \in T^*M$ the span of the corresponding to $\mathcal{G}$ Hamiltonian vector fields $\vec{g}_1, \dots, \vec{g}_s$ is an isotropic subspace of Σ_z, actually it is nothing but the subspace ℓ.

Summing up we see that to any regular Hamiltonian vector field $\vec{h}$ one can associate two Jacobi curves in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma_z)$ attached at a point $z \in T^*M$: the curve $J_z(\cdot)$ generated by the standard vertical dynamical distribution $(\vec{h}, \Pi)$ and the curve

$$J_z^{\mathcal{G}}(t) = e_*^{-t\vec{h}} \Pi^{\mathcal{G}}_{e^{t\vec{h}}z}, \qquad z \in T^*M$$

generated by the pair $(\vec{h}, \Pi^{\mathcal{G}})$ with $\Pi^{\mathcal{G}}$ defined by (2.2). We will denote by $R_z^{\vec{h},\mathcal{G}}$, $r_z^{\vec{h},\mathcal{G}}$ and $\rho_z^{\vec{h},\mathcal{G}}$ the *reduced curvature operator*, the *reduced curvature form* and the *reduced Ricci curvature* corresponding to the curve $J_z^{\mathcal{G}}(\cdot)$.

The meaning of the reduced curvature form is particularly clear when the fields $\vec{g}_i$, $i = 1, \ldots, s$ preserve fibers, i.e. in the situation described in Example 2.1.1. For simplicity let us consider the case of one first integral g, satisfying (2.3): $\mathcal{G} = g$. Let symplectic manifold $W_{g,c}$ and a mapping $\Phi : g^{-1}(c) \mapsto W_{g,c}$ be as in this example. Then

$$r_z^{\vec{h},g} = r_{\Phi(z)}^{\Phi_*\vec{h}}, \qquad \text{where} \quad r_{\Phi(z)}^{\Phi_*\vec{h}} : \Phi_*\Pi^g \mapsto \mathbb{R}, \tag{2.14}$$

i.e. the curvature form of $J_z^g(t)$ at $z \in g^{-1}(c)$ is equal to the pull-back by Φ of the curvature form of the curve generates by $\Phi_*\vec{h}$ in the reduced symplectic space $W_{g,c}$.

The natural question is what is the relation between the curvature forms and operators of a Jacobi curve $J_z(\cdot)$ and it reduction $J_z^{\mathcal{G}}(\cdot)$ on the common space of their definition $\overline{\Pi_z^{\mathcal{G}}}$. The following example shows that even in the simplest situation the answer is not obvious.

Example 2.2.2. *(Kepler's problem)* Consider a natural mechanical system on $M = \mathbb{R}^2$ with the potential energy $U = -r^{-1}$, where r is the distance between a moving point in a plane and some fixed point. This system describes the motion of the center of masses of two gravitationally interacting bodies in the plane of their motion ([11]). Let $q = (r, \varphi)$ be the polar coordinates in $\mathbb{R}^2$. Then the Hamiltonian function of the problem takes the form

$$h = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} - \frac{1}{r}, \tag{2.15}$$

where p_r and p_φ are the canonical impulses conjugated to r and φ. If $z = (p, q)$, where $q \in M$, $p \in T_q^*M$, then $p_r(z) = p\big(\partial_r(q)\big) = d_q r$, $p_\varphi(z) = r^2 p\big(\partial_\varphi(q)\big) = r^2 d_q\varphi$. Observe that $g = p_\varphi$ is nothing but the angular momentum of the moving point w.r.t. the origin, and from (2.15) we immediately see that it is a first integral of the system: φ is a cyclic variable. Let us compare the curvature forms $r_z^{\vec{h}}$ and $r_z^{\vec{h},g}$ on the common space of their definition

$$\overline{\Pi_z^g} = \Pi_z \cap \ker d_z g = \mathbb{R}\partial_{p_r}(z)\,.$$

First, according to formula (1.33) of Example 1.4.7, the curvature form of $J_z(\cdot)$ is equal to the Hessian of U at q. In particular, it implies that

$$r_z^{\vec{h}}(\partial_{p_r}) = \frac{\partial^2}{\partial r^2}U(q) = -\frac{2}{r^3}. \tag{2.16}$$

Now fix some $c \in \mathbb{R}$ and consider the level set $g(z) = c$. Note that g satisfies the condition (2.4) of Example 2.1.1 with $V = r^2\partial_\varphi$. Let $W_{g,c}$ and Φ be as in Example 2.1.1. In the present case $W_{g,c} \cong T^*\mathbb{R}^+$ and the dynamical Lagrangian distribution $(\Phi_*\vec{h}, \Phi_*\mathbb{R}\partial_{p_r})$ is equivalent (symplectomorphic) to the dynamical Lagrangian distribution associated to the natural mechanical system with the configurational space $\mathbb{R}^+$ and the potential energy

$$U_a = \frac{c^2}{2r^2} - \frac{1}{r}.$$

U_a is the so - called *amended* potential energy ([11]), it comes from the following identity: $h\big|_{g^{-1}(c)} = \frac{p_r^2}{2} + U_a(r)$. Hence by (2.14)

$$r_z^{\vec{h},g}(\partial_{p_r}) = \frac{d^2}{dr^2}U_a(r) = \frac{3c^2}{r^4} - \frac{2}{r^3} = r_z^{\vec{h}}(\partial_{p_r}) + \frac{3c^2}{r^4}. \tag{2.17}$$

Note that from (2.17) it follows that on the common space of the definition the reduced curvature form is grater than the curvature form itself. As we will show later (Corollary 2.2.7), it turns out that this is a general fact.

2.2.2 Variation of the curvature after reduction in Lagrange Grassmannian

In this subsection we derive the relation between the curvature forms of some regular curve $\Lambda(\cdot)$ and its ℓ-reduction $\Lambda(\cdot)^\ell$ in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$. As before, denote by $\ell = \mathrm{span}\{l_1, \dots, l_s\}$ the span of s vectors, satisfying (2.10).

First we introduce some more notations. Let $B_\Lambda : \Sigma \mapsto \Lambda^*$ be such that for a given $w \in \Sigma$

$$B_\Lambda(w)(v) = \sigma(w, v), \qquad v \in \Lambda. \tag{2.18}$$

Let $a_i(t) \in \Lambda(t)$, $i = 1, \dots, s$ be such that

$$\sigma(\dot{a}_i(t), v) = \sigma(l_i, v), \quad \forall v \in \Lambda(t), \tag{2.19}$$

or, equivalently,

$$\dot{a}_i(t) \equiv l_i \mod \Lambda(t). \tag{2.20}$$

Recalling the definition of the linear operator $\overline{\dot{\Lambda}}$ we notice that actually

$$a_i(t) = \left(\overline{\dot{\Lambda}(t)}\right)^{-1} \circ B_{\Lambda(t)}(l_i).$$

Denote by $A(t)$ the $s \times s$ matrix such that

$$A_{km}(t) = \sigma\left(l_k, a_m(t)\right), \qquad k, m = 1, \dots, s. \tag{2.21}$$

By (2.20)

$$A_{km}(t) = \sigma\left(\dot{a}_k(t), a_m(t)\right). \tag{2.22}$$

Notice that the matrix $A(t)$ is symmetric. Now we are ready to formulate the main result of the subsection.

Theorem 2.2.3. *Let $\Lambda(\cdot)$ be a regular curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$ and ℓ be a span of s vectors $l_i \in \Sigma$ such that (2.10) holds and*

$$\det A(\tau) \neq 0$$

at some point τ.

Then the curvature form $r_\Lambda(\tau)$ of the curve $\Lambda(\cdot)$ and the curvature form $r_{\Lambda^\ell}(\tau)$ of its ℓ-reduction $\Lambda(\cdot)^l$ at the point τ satisfy the following identity:

$$\left(r_{\Lambda^\ell}(\tau) - r_\Lambda(\tau)\right)(\xi) = \frac{3}{4}\sum_{k,m=1}^{s} \left(A(\tau)^{-1}\right)_{km} \sigma\left(\ddot{a}_k(\tau), \xi\right) \sigma\left(\ddot{a}_m(\tau), \xi\right), \qquad \forall \xi \in \Lambda(\tau) \cap \ell^\angle, \tag{2.23}$$

where $\left(A(\tau)^{-1}\right)_{km}$ is the km-element of the matrix $A(\tau)^{-1}$.

Moreover, for the curvature operator $R_\Lambda(\tau)$ of the curve $\Lambda(\cdot)$ and the curvature operator $R_{\Lambda^\ell}(\tau)$ of its ℓ-reduction $\Lambda(\cdot)^l$ at the point τ there holds the following identity[2]*:*

$$\left[R_{\Lambda^\ell}(\tau) - R_\Lambda(\tau)\right]\Big|_{\Lambda(\tau)\cap\ell^\angle} = \frac{3}{4}\sum_{k,m=1}^{s} (A(\tau)^{-1})_{km} B_{\Lambda(\tau)}\ddot{a}_m(\tau) \otimes \left(\left(\dot{\Lambda}(\tau)\right)^{-1} \circ B_{\Lambda(\tau)}\ddot{a}_k(\tau)\right). \tag{2.24}$$

Proof.

First let us prove identity (2.23). Observe that since $\det A(\tau) \neq 0$

$$\operatorname{span}\{a_1(\tau), \dots, a_s(\tau)\} \cap \ell^\angle = 0. \tag{2.25}$$

Denote be ϵ_i the i-th vector of a standard basis in $\mathbb{R}^n$. We can choose a Darboux basis in Σ in such a way that $\Sigma = \operatorname{span}\{e_1, \dots, e_n, f_1, \dots, f_n\}$ where

$$\Lambda(\tau) = \operatorname{span}\{e_1, \dots, e_n\},$$

$$e_i = \begin{pmatrix}\epsilon_i \\ 0\end{pmatrix}, \qquad f_i = \begin{pmatrix}0 \\ \epsilon_i\end{pmatrix}, \qquad i = 1, \dots, n,$$

and the following relations hold

$$\Lambda(\tau) \cap \ell^\angle = \operatorname{span}\{e_1, \dots, e_{n-s}\}, \tag{2.26}$$

$$a_i \in \operatorname{span}\{e_{n-s+1}, \dots, e_n\}, \tag{2.27}$$

$$l_i = f_{n-s+i}. \tag{2.28}$$

Note that by construction

$$l^\angle = \operatorname{span}\{e_1, \dots, e_{n-s}, f_1, , \dots, f_n\}.$$

[2]As usual, for a given linear functional ξ and a given vector v by $\xi \otimes v$ we denote the following rank 1 linear operator $\xi \otimes v(\cdot) = \xi(\cdot)v$.

Therefore

$$\widetilde{\Sigma} = \ell^{\angle}/\ell \cong \mathrm{span}\{e_1, \ldots, e_{n-s}, f_1, \ldots, f_{n-s}\}. \tag{2.29}$$

Since by definition $a_i(t) \in \Lambda(t)$, then there exists $p_i(t) \in \mathbb{R}^n$ such that

$$a_i(t) = \begin{pmatrix} p_i(t) \\ S_t p_i(t) \end{pmatrix}.$$

Differentiating we get

$$\dot{a}_i(t) = \begin{pmatrix} p_i \\ S_t \dot{p}_i(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{S}_t p_i(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{S}_t p_i(t) \end{pmatrix} \operatorname{mod} \Lambda(t). \tag{2.30}$$

This, together with (2.20) and (2.28), implies that

$$l_i = \begin{pmatrix} 0 \\ \dot{S}_t p_i(t) \end{pmatrix}$$

and hence

$$p_i(t) = \dot{S}_t^{-1} \epsilon_{n-s+i}\,, \tag{2.31}$$

provided the curve $\Lambda(\cdot)$ is regular.

On the other hand, from (2.27) and (2.26), one can obtain that

$$p_i(\tau) = \sum_{j=1}^{s} \sigma\left(a_i(\tau), l_j\right) \epsilon_{n-s+j} = -\sum_{j=1}^{s} A_{ij}(\tau) \epsilon_{n-s+j}\,. \tag{2.32}$$

For a given $n \times n$ matrix S let us denote by $\overline{S}$ the $(n-s) \times (n-s)$ matrix, obtained from S by erasing the last s columns and rows. Then from (2.31), (2.32) and symmetry of $\dot{S}_\tau$ it follows that $\dot{S}_\tau^{-1}$ has the following blocked form:

$$\dot{S}_\tau^{-1} = \left(\begin{array}{ccc|ccc} & & & 0 & \ldots & 0 \\ & \overline{\dot{S}_\tau^{-1}} & & \vdots & & \vdots \\ & & & 0 & \ldots & 0 \\ \hline 0 & \ldots & 0 & & & \\ \vdots & & \vdots & & -A(\tau) & \\ 0 & \ldots & 0 & & & \end{array}\right), \tag{2.33}$$

Consider now the curve $\overline{\Lambda(\cdot)^\ell}$ in the Lagrange Grassmannian $\mathcal{L}_{n-s}(\widetilde{\Sigma})$. By construction, if S_t is the coordinate representation of the curve $\Lambda(t)$ w.r.t. the chosen symplectic basis, then $\overline{S}_t$ is a coordinate representation of the curve $\overline{\Lambda(\cdot)^\ell}$ w.r.t. the basis of $\widetilde{\Sigma}$, indicated in (2.29). Since $\det A(\tau) \neq 0$, from (2.33) it follows that the germ at τ of the curve $\overline{\Lambda(\cdot)^\ell}$ is regular. In particular, the curvature form of the ℓ-reduction $\Lambda(\cdot)^\ell$ is well defined at τ. Using formula (1.15), we obtain that the quadratic form

$$\left(r_{\Lambda^\ell}(\tau) - r_\Lambda(\tau)\right)\Big|_{\Lambda(\tau)\cap\ell^{\angle}}$$

in the basis $\{e_i\}_{i=1}^{n-s}$ is represented by the matrix

$$-\dot{\overline{S}}_\tau \mathbb{S}\left(\overline{S}_\tau\right) + \overline{\dot{S}_\tau \mathbb{S}(S_\tau)}\,.$$

Observing that $\dot{S}_\tau$ has the same blocks as $\dot{S}_\tau^{-1}$, from (1.15) and (2.33) we easily get

$$-\dot{\overline{S}}_\tau \mathbb{S}(\overline{S}_\tau) + \overline{\dot{S}_\tau \mathbb{S}(S_\tau)} = -\frac{3}{4}\left\{\sum_{k,m=n-s+1}^{n} (\ddot{S}_\tau)_{ik}(\dot{S}_\tau^{-1})_{km}(\ddot{S}_\tau)_{mj}\right\}_{i,j=1}^{n-s}. \tag{2.34}$$

In order to prove (2.23), it is remains to prove the following technical Lemma.

Lemma 2.2.4. *The restriction on $\Lambda(\tau)\cap l^{\angle}$ of the quadratic form*

$$\xi \mapsto \sum_{k,m=1}^{s} (A(\tau)^{-1})_{km}\sigma\left(\ddot{a}_k(\tau),\xi\right)\sigma\left(\ddot{a}_m(\tau),\xi\right) \tag{2.35}$$

in the basis $\{e_i\}_{i=1}^{n-s}$ is represented by the matrix with ij-entry equal to

$$-\sum_{k,m=n-s+1}^{n} (\ddot{S}_\tau)_{ik}(\dot{S}_\tau^{-1})_{km}(\ddot{S}_\tau)_{mj}\,.$$

Proof. From the symmetry of S_t and (2.33) it follows that

$$\sum_{k,m=n-s+1}^{n} (\ddot{S}_\tau)_{ik}(\dot{S}_\tau^{-1})_{km}(\ddot{S}_\tau)_{mj} = \sum_{k,m=n-s+1}^{n} (\ddot{S}_\tau\dot{S}_\tau^{-1})_{ik}(\dot{S}_\tau)_{km}(\ddot{S}_\tau\dot{S}_\tau^{-1})_{jm} = \tag{2.36}$$

$$= -\sum_{k,m=1}^{s} (\ddot{S}_\tau\dot{S}_\tau^{-1})_{i,n-s+k}\left(A(\tau)^{-1}\right)_{km}(\ddot{S}_\tau\dot{S}_\tau^{-1})_{j,n-s+m}\,.$$

On the other hand, by differentiation of (2.30) we get

$$\ddot{a}_i(t) = \begin{pmatrix} 0 \\ \ddot{S}_t p_i(t)\end{pmatrix} + \begin{pmatrix} 0 \\ 2\dot{S}_t\dot{p}_i(t)\end{pmatrix} + \begin{pmatrix} \ddot{p}_i(t) \\ S_t\ddot{p}_i(t)\end{pmatrix}. \tag{2.37}$$

Since $\dot{S}_\tau$ has the same blocked structure as $\dot{S}_\tau^{-1}$, (2.31) implies that the second term of (2.37) belongs to ℓ. From here it follows that

$$\sigma(\ddot{a}_i(\tau),e_j) = -(\ddot{S}_\tau\dot{S}_\tau^{-1})_{j,n-s+i}, \qquad \forall 1\le j\le n-s,\ 1\le i\le s\,. \tag{2.38}$$

The last identity together with (2.36) implies the statement of the lemma and also formula (2.23). □

Finally, identity (2.24) follows directly from the definition of the curvature form. This completes the proof of the theorem. □

Note that if the curve $\Lambda(\cdot)$ is monotone increasing or decreasing, the definition of vectors $a_i(t)$ implies that condition $\det A(\tau) \neq 0$ is equivalent to the following condition

$$\Lambda(\tau) \cap \ell = 0 \,. \tag{2.39}$$

Taking into account this fact one has the following Corollary of formula (2.23).

Corollary 2.2.5. *Let the curve $\Lambda(\cdot)$ be monotone increasing and $\ell = \operatorname{span}\{l_1, \dots, l_s\}$ with $l_i \in \Sigma$ be such that condition (2.39) holds for some t. Then the quadratic form*

$$\xi \mapsto \left(r_{\Lambda^\ell}(t) - r_\Lambda(t)\right)(\xi)\,, \qquad \xi \in \Lambda(t) \cap \ell^\angle\,,$$

is a non-negative quadratic form of rank not greater than s.

2.2.3 Variation of the curvature: Hamiltonian setting

Let us now reformulate the results of Theorem 2.2.3 for the Jacobi curves generated by a regular Hamiltonian vector field $\vec{h}$ on T^*M. Let (p,q) be some local coordinates in T^*M such that $\sigma = \sum_{i=1}^{n} dp_i \wedge dq_i$ and $\vec{h}$ has the form (1.22).

Suppose that the Hamiltonian h admits a s-tuple $\mathcal{G} = (g_1, \dots, g_s)$ of involutive first integrals. Similarly to (2.18) denote by $\mathcal{B}_z : \ \Sigma_z \mapsto \Pi_z^*$ the linear mapping such that for given $Y \in \Sigma_z$

$$\mathcal{B}_z Y(Z) = \sigma(Y, Z), \qquad \forall Z \in \Pi_z\,. \tag{2.40}$$

Then using regularity of the filed $\vec{h}$ by analogy with (2.19) we construct s vector fields $\mathcal{X}_i$ on T^*M:

$$\mathcal{X}_i(z) = \left(\dot{J}_z(0)\right)^{-1} \circ \mathcal{B}_z\left(\vec{g}_i(z)\right) = \left(h_{pp}^{-1}\frac{\partial g_i}{\partial p}, 0\right)^T(z)\,. \tag{2.41}$$

Actually $\mathcal{X}_i$ is a unique vertical vector field such that

$$[\vec{h}, \mathcal{X}_i](z) \equiv \vec{g}_i(z) \mod \Pi_z \tag{2.42}$$

for all $z \in T^*M$. Finally, let $\Upsilon(z)$ be the $s \times s$ matrix with the entries

$$\Upsilon(z)_{km} = \sigma_z\left(\vec{g}_k, \mathcal{X}_m\right), \qquad k, m = 1, \dots s\,. \tag{2.43}$$

Now we can reformulate Theorem 2.2.3 in terms of regular Jacobi curves.

Theorem 2.2.6. *Let $\vec{h}$ be a regular Hamiltonian vector field on T^*M, and suppose that the Hamiltonian h admits a s-tuple $\mathcal{G} = (g_1, \dots, g_s)$ of involutive first integrals such that $\det \Upsilon(z) \neq 0$. Then the curvature form $r_z^{\vec{h}}$ of the curve $J_z(\cdot)$ and the curvature form $r_z^{\vec{h},\mathcal{G}}$ of its $\mathcal{G}$-reduction $J_z^{\mathcal{G}}(\cdot)$ for all $\xi \in \overline{\Pi_z^{\mathcal{G}}}$ satisfy the following identity*

$$\left(r_z^{\vec{h},\mathcal{G}} - r_z^{\vec{h}}\right)(\xi) = \frac{3}{4}\sum_{k,m=1}^{s}\left(\Upsilon(z)^{-1}\right)_{km}\sigma_z\left(\left[\vec{h}, [\vec{h}, \mathcal{X}_k]\right], \xi\right)\sigma_z\left(\left[\vec{h}, [\vec{h}, \mathcal{X}_m]\right], \xi\right)\,, \tag{2.44}$$

while the curvature operators $R_z^{\vec{h}} = R_{J_z}(0)$ *and the* $R_z^{\vec{h},\mathcal{G}} = R_{J_z^{\mathcal{G}}}(0)$ *satisfy the identity*

$$[R_z^{\vec{h},\mathcal{G}} - R_z^{\vec{h}}]\Big|_{\overline{\Pi_z^{\mathcal{G}}}} = \frac{3}{4}\sum_{k,m=1}^{s}\left(\Upsilon(\lambda)^{-1}\right)_{km}\mathcal{B}_z\left[\vec{h},[\vec{h},\mathcal{X}_m]\right](z)\otimes\left(h_{pp}^{-1}(z)\circ\mathcal{B}_z\left[\vec{h},[\vec{h},\mathcal{X}_k]\right](z)\right). \tag{2.45}$$

Also, by analogy with Corollary 2.2.5 we have

Corollary 2.2.7. *If Hamiltonian vector field* $\vec{h}$ *is monotone increasing and the Hamiltonian* h *admits a s-tuple* $\mathcal{G} = (g_1, \ldots, g_s)$ *of involutive first integrals such that*

$$\Pi_z \cap \operatorname{span}(\vec{g}_1(z), \ldots, \vec{g}_s(z)) = 0, \tag{2.46}$$

then the curvature form

$$\xi \mapsto (r_z^{\vec{h},\mathcal{G}} - r_z^{\vec{h}})(\xi), \qquad \xi \in \overline{\Pi_z^{\mathcal{G}}}$$

is a non-negative definite quadratic form of rank not greater than s.

Theorem 2.2.6 provides an effective method for the computation of the reduced curvature without introducing any new canonical coordinates in the reduced phase space. This method is particularly useful when the original system is a natural mechanical system. In this case, according to Example 1.4.7, $T^*M = \mathbb{R}^n \times \mathbb{R}^n$, the canonical splitting is trivial, the curvature is defined by the Hessian of the potential energy and the reduced curvature can be calculated via the formula (2.44) using the Cartesian coordinates on T^*M. We will illustrate the effectiveness of our method in Chapter 4, where we will discuss in details the reduction in the classical N-body problem.

2.3 Reduction by first integrals and conjugate points

In the present section we study the relation between the sets of conjugate points to the given point for a monotone increasing Jacobi curve and for its reduction by first integrals.

Let $t \mapsto \Lambda_t$ be a monotone increasing curve in $\mathcal{L}_n(\Sigma)$ defined on the interval $[0,T]$. Let $I \in [0,T]$. Denote by $\#\mathrm{conj}\,\Lambda_t\big|_I$ the number of conjugate to 0 times on the subset I, counted with their multiplicities:

$$\#\mathrm{conj}\,\Lambda_t\big|_I = \sum_{t\in I}\dim\Lambda_t \cap \Lambda_0\,. \tag{2.47}$$

Since Λ_t is monotone increasing, the conjugate points do not accumulate and this number is finite. As we saw in Chapter1, the right hand-side of (2.47) is nothing but the Maslov index of the curve Λ_t related to Λ_0 on the interval I.

Let $\ell = \operatorname{span}\{l_1\ldots,l_s\}$ be a span of s linearly independent vectors in Σ, satisfying (2.10). The time t_1 is called *conjugate to* 0 *for the* ℓ*-reduction* Λ_t^ℓ *of the curve* Λ_t, if $\Lambda_{t_1}^\ell \cap \Lambda_0^\ell \neq \ell$ or, equivalently, t_1 is the conjugate time to 0 for the curve $\overline{\Lambda_t^\ell}$ in the Lagrange

Grassmannian $\mathcal{L}_{n-s}(\widetilde{\Sigma})$ with $\widetilde{\Sigma} = \ell^{\angle}/\ell$. The multiplicity of the conjugate time t_1 to 0 for the ℓ-reduction Λ_t^ℓ is equal by definition to

$$\dim\left(\overline{\Lambda_{t_1}^\ell} \cap \overline{\Lambda_0^\ell}\right) = \dim\left(\Lambda_{t_1}^\ell \cap \Lambda_0^\ell\right) - s. \tag{2.48}$$

It follows from the coordinate presentation which we used in the previous section, that if Λ_t is monotone increasing, then $\overline{\Lambda_t^\ell}$ is monotone increasing as well. So the number of conjugate times to 0 for the ℓ-reduction is also finite. Therefore

$$\#\mathrm{conj}\, \Lambda_t^\ell\big|_I = \#\mathrm{conj}\, \overline{\Lambda_t^\ell}\big|_I . \tag{2.49}$$

Now we are ready to formulate the main result of this section.

Theorem 2.3.1. *Let $\Lambda_t : [0,T] \mapsto \mathcal{L}_n(\Sigma)$ be a monotone increasing curve and $\ell = \{l_1 \ldots, l_s\}$ be a span of s linearly independent vectors in Σ, satisfying (2.10) and (2.39) at $t = 0$. Then on the set $]0,T]$ the difference between the number of conjugate times to 0 for the ℓ-reduction Λ_t^ℓ and the number of conjugate times to 0 for the curve Λ_t itself, counted with their multiplicity, is non-negative and does not exceed s, namely*

$$0 \le \#\mathrm{conj}_0\, \overline{\Lambda_t^\ell}\big|_{]0,T]} - \#\mathrm{conj}_0\, \Lambda_t\big|_{]0,T]} \le s\,. \tag{2.50}$$

Proof. Since the curves Λ_t and $\overline{\Lambda_t^\ell}$ are monotone increasing, for sufficiently small $\varepsilon > 0$ the set $]0,\varepsilon]$ does not contain times conjugate to 0 for both of these curves. Also, without loss of generality, one can assume that the terminal time T is not conjugate to 0 for both of these curves. Otherwise one can extend Λ_t as a monotone increasing curve to a slightly bigger interval $[0, T+\tilde{\varepsilon}]$ such that $\Lambda_{T+\tilde{\varepsilon}} \cap \Lambda_0 = 0$ and $\overline{\Lambda_{T+\tilde{\varepsilon}}^\ell} \cap \overline{\Lambda_0^\ell} = 0$. Taking into account this observation, according to Section 1.2 we have

$$\#\mathrm{conj}_0\, \Lambda_t\big|_{]0,T]} = \mathrm{Ind}_{\Lambda_0} \Lambda_t\big|_\varepsilon^T . \tag{2.51}$$

Further, from (2.11) it follows that

$$\dim\left(\Lambda_t^\ell \cap \Lambda_0^\ell\right) - s = \dim\left(\Lambda_t^\ell \cap \Lambda_0\right). \tag{2.52}$$

Hence, combining (2.47), (2.49) and (1.6) with (2.48) and (2.52), we get

$$\#\mathrm{conj}_0\, \overline{\Lambda_t^\ell}\big|_{]0,T]} = \mathrm{Ind}_{\Lambda_0} \Lambda_t^\ell\big|_\varepsilon^T . \tag{2.53}$$

Let us prove the theorem for the case $s = 1$. In this case $\ell = l_1 \equiv l$. We will use the invariance of the intersection number under homotopy, which preserves the end-points.

Let $a(t)$ be as in (2.19):

$$\dot{a}(t) = l \mod \Lambda(t).$$

Denote

$$F(\tau, t) = \mathrm{span}\big(\Lambda_t \cap l^{\angle}, (1-\tau)a(t) + \tau l\big)\,. \tag{2.54}$$

Note that all subspaces $F(\tau,t)$ are Lagrangian. Let $\Phi_\tau : [0,T] \mapsto \mathcal{L}_n(\Sigma)$ and $\Gamma_t : [0,1] \mapsto \mathcal{L}_n(\Sigma)$ be the curves, satisfying

$$\begin{aligned} \Phi_\tau(\cdot) &= F(\tau,\cdot), \quad 0 \le \tau \le 1; \\ \Gamma_t(\cdot) &= F(\cdot,t), \quad 0 \le t \le T. \end{aligned} \tag{2.55}$$

Then $\Phi_0(t) = \Lambda_t$, $\Phi_1(t) = \Lambda_t^l$, and the curves

$$\Gamma_\varepsilon(\cdot)\big|_{[0,\tau]} \cup \Phi_\tau(\cdot)\big|_{[\varepsilon,T]} \cup \left(-\Gamma_T(\cdot)\big|_{[0,\tau]}\right)$$

define the homotopy between $\Lambda_t\big|_{[\varepsilon,T]}$ and $\Gamma_\varepsilon(\cdot) \cup \Lambda_t^l\big|_{[\varepsilon,T]} \cup \left(-\Gamma_T(\cdot)\right)$, preserving the end-points.[3] Therefore,

$$\mathrm{Ind}_{\Lambda_0}\Lambda_t\big|_\varepsilon^T = \mathrm{Ind}_{\Lambda_0}\Gamma_\varepsilon(\tau)\big|_0^1 + \mathrm{Ind}_{\Lambda_0}\Lambda_t^l\big|_\varepsilon^T - \mathrm{Ind}_{\Lambda_0}\Gamma_T(\tau)\big|_0^1 .$$

Using (2.51) and (2.53), the last relation can be presented in the form

$$\#\mathrm{conj}_0\, \overline{\Lambda_t^l}\big|_{]0,T]} - \#\mathrm{conj}_0\, \Lambda_t\big|_{]0,T]} = \mathrm{Ind}_{\Lambda_0}\Gamma_T(\tau)\big|_0^1 - \mathrm{Ind}_{\Lambda_0}\Gamma_\varepsilon(\tau)\big|_0^1 . \tag{2.56}$$

So, in order to prove the theorem for $s = 1$ it is sufficient to prove the following two relations:

$$0 \le \mathrm{Ind}_{\Lambda_0}\Gamma_T(\tau)\big|_{\tau=0}^1 \le 1 \tag{2.57}$$

$$\exists \varepsilon_0 > 0 : \qquad \mathrm{Ind}_{\Lambda_0}\Gamma_\varepsilon(\tau)\big|_{\tau=0}^1 = 0 \qquad \forall \varepsilon \in]0,\varepsilon_0] . \tag{2.58}$$

a) First let us prove (2.57). If $l \in \Lambda_t$, then by definition $\Gamma_T(\tau) \equiv \Lambda_T$. Since by our assumptions, $\Lambda_T \cap \Lambda_0 = 0$, we obviously have $\mathrm{Ind}_{\Lambda_0}\Gamma_T(\tau)\big|_0^1 = 0$.

If $l \notin \Lambda_t$, then $\dim(\Lambda_0 + \Lambda_t \cap l^\angle) = 2n-1$. In particular, it implies that for any $\tau \in [0,1]$

$$0 \le \dim\left(\Gamma_T(\tau) \cap \Lambda_0\right) \le 1. \tag{2.59}$$

Further, let

$$p : \Sigma \mapsto \Sigma/(\Lambda_0 + \Lambda_T \cap l^\angle)$$

be the canonical projector on the factor space. Then from (2.54) and (2.55), by standard arguments of Linear Algebra it follows that $\Gamma_T(\tau) \cap \Lambda_0 \neq 0$ if and only if

$$(1-\tau)p\big(a(T)\big) + \tau p(l) = 0 . \tag{2.60}$$

Since, by assumptions, $\Gamma_T(0) \cap \Lambda_0 = 0$ (recall that $\Gamma_T(0) = \Lambda_t$), equation (2.60) has at most one solution on the interval $[0,1]$. In other words, the curve $\Gamma_T(\cdot)$ intersects the train $\mathcal{M}_{\Lambda_0}$ at most ones and according to (2.59) the point of intersection is non-singular.

[3] Here by $-\Gamma(\cdot)$ we mean the curve, obtained from a curve $\Gamma(\cdot)$ by inverting orientation.

Moreover, the curve $\Gamma_T(\cdot)$ is monotone non-decreasing, i.e. for any $\tau \in [0,1]$ its velocities $\frac{d}{d\tau}\Gamma_T(\tau)$ are non-negative definite quadratic forms. Indeed, fix $t = T$. Since $\Lambda_T \cap l^\angle$ is the common space for all $\Gamma_T(\tau)$, we have

$$\frac{d}{d\tau}\Gamma_T(\tau)\big|_{\Lambda_T \cap l^\angle} \equiv 0 .$$

On the other hand, if we denote by $c(\tau) = (1-\tau)a(t) + \tau l$, then

$$\frac{d}{d\tau}\Gamma_T(\tau)\big(c(\tau)\big) = \sigma\big(\frac{d}{d\tau}c(\tau), c(\tau)\big) = \sigma\big(l - a(T), (1-\tau)a(T) + \tau l\big) =$$

$$= \sigma(l, a(T)) = \sigma(\dot a(T), a(T)) > 0 ,$$

provided Λ_t is monotone increasing.

So, $\frac{d}{d\tau}\Gamma_T(\tau)$ are non-negative definite quadratic forms. Hence in the only point of intersection of $\Gamma_T(\cdot)$ with the train $\mathcal{M}_{\Lambda_0}$ the intersection index becomes equal to $+1$. This proves (2.57).

b) Let us now prove (2.58). Take a Lagrangian subspace Δ such that $l \in \Delta$ and $\Delta \cap \Lambda_0 = 0$. Then there exists ε_0 such that

$$\Lambda_t\big|_{[0,\varepsilon_0]} \subset \Delta^{\pitchfork}. \tag{2.61}$$

By the same arguments as in **a)**, for any $0 < \varepsilon \leq \varepsilon_0$ the curve $\Gamma_\varepsilon(\cdot)$ intersects the train $\mathcal{M}_\Delta$ once. But by construction this unique intersection occurs at $\tau = 1$. Indeed, $\Gamma_\varepsilon(1) = \Lambda_\varepsilon^l$, hence $l \in \Gamma_\varepsilon(1) \cap \Delta$. In other words,

$$\Gamma_\varepsilon(\cdot)\big|_{[0,1[} \subset \Delta^{\pitchfork}. \tag{2.62}$$

On the other hand one can choose a Darboux basis in Σ such that $\Sigma = \mathbb{R}^n \times \mathbb{R}^n$, where $\Lambda_0 = 0 \times \mathbb{R}^n$ and and $\Delta = \mathbb{R}^n \times 0$. By (2.61) and (2.62), there exist two one parametric families of symmetric matrices S_t, $0 \leq t \leq \varepsilon_0$ and C_τ, $0 \leq \tau < 1$ such that $\Lambda_t = \{(S_t x, x) : x \in \mathbb{R}^n\}$ and $\Gamma_\varepsilon(\tau) = \{(C_\tau x, x) : x \in \mathbb{R}^n\}$. Since the curve Λ_t is monotone increasing and the curve $\Gamma_\varepsilon(\tau)$ is monotone non-decreasing, for any $0 \leq \tau < 1$ the quadratic forms $x \mapsto \langle C_\tau x, x\rangle$ are positive definite, while $S_0 = 0$. This implies that

$$\Gamma_\varepsilon(\tau) \cap \Lambda_0 = 0 , \qquad \tau \in [0,1[. \tag{2.63}$$

Note also that for sufficiently small $\varepsilon > 0$

$$\Gamma_\varepsilon(1) \cap \Lambda_0 = 0. \tag{2.64}$$

Indeed, $\Gamma_\varepsilon(1) = \Lambda_\varepsilon^l$ and sufficiently small $\varepsilon > 0$ is not a conjugate time for the l-reduction Λ_t^l, which according to (2.52) is equivalent to the fact that $\Lambda^l(\varepsilon) \cap \Lambda_0 = 0$ and hence to (2.64). From (2.63) and (2.64) it follows that for $\varepsilon > 0$ sufficiently small the curve $\Gamma_\varepsilon(\cdot)$

does not intersect the train $\mathcal{M}_{\Lambda_0}$. Hence (2.58) is proved, and this completes the proof of the theorem in the case $s = 1$.

If $s \neq 1$ then obviously

$$\Lambda_t^{(l_1,\dots,l_s)} = \left(\Lambda_t^{(l_1,\dots,l_{s-1})}\right)^{l_s}. \tag{2.65}$$

So, starting with $s = 1$ and proceeding by induction we get (2.50) for $s > 1$ as well. □

Remark 2.3.2. If $s = 1$ from the sets of conjugate (to 0) times for a monotone increasing curve in $\mathcal{L}_n(\Sigma)$ and its reduction are alternating.

Moreover, for any s the first conjugate (to 0) time for a reduction precedes the first conjugate time of the curve itself.

We conclude this section reformulating the statement of Theorem 2.3.1 for a monotone increasing Jacobi curve and its reduction by first integrals.

Let $\vec{h}$ be a monotone increasing Hamiltonian vector filed. As before, denote by $\mathcal{G} = (g_1, \dots, g_s)$ the s-tuple of involutive first integrals. Note that $z = e^{t_1\vec{h}}z_0$ is conjugate to z_0 for the $\mathcal{G}$-reduction of the pair $(\vec{h}, \Pi)$ along the integral curve $t \mapsto e^{t\vec{h}}z_0$ of $\vec{h}$ if and only if t_1 is conjugate to 0 for $\big(\vec{g}_1(z_0), \dots, \vec{g}_s(z_0)\big)$-reduction of the Jacobi curve $J_{z_0}^{\mathcal{G}}(\cdot)$ attached at z_0. Translating Theorem 2.3.1 into the terms of Jacobi curves, we have immediately the following

Corollary 2.3.3. *If the convex on fibers Hamiltonian h admits a tuple $\mathcal{G} = (g_1, \dots, g_s)$ of s involutive first integrals satisfying (2.46), then along any segment of the integral curve of $\vec{h}$ the difference between the number of conjugate points to the starting point of the segment w.r.t. the $\mathcal{G}$-reduction of the pair $(\vec{h}, \Pi)$ and the number of conjugate points to the starting point of the segment w.r.t. the pair $(\vec{h}, \Pi)$ itself, counted with their multiplicity, is non-negative and does not exceed s.*

Chapter 3

Hamiltonian Systems of Negative Curvature

In this chapter we generalize the classical result about the geodesic flows on compact Riemannian manifolds with negative sectional curvature for the case of Hamiltonian flows with negative-definite curvature form. In the Riemannian case the geodesic flow $\phi^t : M \to M$ on a compact manifold M with negative sectional curvature is an Anosov flow, i.e. the manifold M is a hyperbolic set for the flow ϕ^t. The detailed proof of this fundamental result can be found in [16]. The key role in the proof is played by the Jacobi equation (1.13) which describes the dynamics of the Jacobi field along the geodesic. The negativity of the curvature provides the existence of invariant expanding and contracting subsets in the tangent spaces along geodesics.

In the case of Hamiltonian flows the dynamics of the canonical moving frame (see (1.21)) suggests to expect a kind of similar behavior for the orbits of the generating Hamiltonian vector field. This means that the hyperbolic theory can be applied to a much larger class of problems of Hamiltonian Dynamics.

This chapter is organized as follows. In Section 3.1 we introduce the basic notions of Hyperbolic Theory for flows following [16]. We give the definitions of hyperbolic sets, Anosov flows and state the Cone Criterion which provides the necessary conditions for a set to be hyperbolic. This criterion is based on the existence of the so-called *invariant expanding and contracting cones*.

In Section 3.2 we analyze the structure of the canonical moving frame associated to a regular Jacobi curve. We show that the analysis of the dynamics in the tangent space to a manifold along an orbit of a regular Hamiltonian vector field reduces to the study of the dynamics of the associated special canonical moving frame at any point of the orbit. Moreover, we show that this dynamics is described by a set of second order equations of the same type as the Jacobi equations in the classical Riemannian case. This fact creates a far-going analogy between these two problems and allows to use the classical arguments of the Hyperbolic Theory for the study of the dynamics generated by a monotone Hamiltonian vector field in the cotangent bundle.

In Section 3.3 we consider the flow generated by a monotone Hamiltonian flow $\vec{h}$ on the level set of the corresponding Hamiltonian function. We prove that the negative definiteness of the h-reduced curvature implies the existence of the invariant hyperbolic splitting in the tangent space to any compact connected invariant subset of the level set of the Hamiltonian. This result generalizes the already mentioned classical theorem for the geodesic flows on Riemannian manifolds.

In Section 3.4 we show that the only possible compact invariant subset of the Hamiltonian flow generated by a monotone Hamiltonian vector field $\vec{h}$ with negative definite curvature form is a hyperbolic equilibrium point. To our notion, this fact has no analogous in the classical Riemannian case.

We give the detailed proofs of the theorems using the classical technique from the already cited book [16] by A.Katok and B.Hasselblatt.

3.1 Elements of Hyperbolic Theory for flows

Let M be a smooth manifold, V be a smooth vector field on M and $\phi^t : M \to M$ be a flow generated by V:

$$\frac{d}{dt}\phi^t(q) = V(\phi^t(q)), \qquad \phi^0(q) = q.$$

Definition 3.1.1. *We say that the point $q \in M$ is a hyperbolic equilibrium point for the vector field V iff $V(q) = 0$ and D_qV has no eigenvalues on the imaginary axis.*

Recall that the mapping D_qV is nothing but the right-hand side of the linearized (at q) system associated with the dynamical system $\dot{q}(t) = V(q(t))$ generated by the vector filed V on M:

$$\dot{v} = A(q)v, \qquad q \in M, \qquad v \in T_qM,$$

where $A(q) = D_qV$.

Let $\Omega \in M$ be a compact invariant subset for ϕ^t.

Definition 3.1.2. *The set Ω is called a hyperbolic set for the flow ϕ^t if there exist a Riemannian metric on an open neighborhood $U \in \Omega$ and $\lambda < 1 < \mu$ such that*

$$T_qM = E_q^+ \oplus E_q^- \oplus E_q^0, \qquad q \in \Omega,$$

$\frac{d}{dt}\big|_{t=0}\phi^t(q) \in E_q^0/\{0\}$, $\dim E_q^0 = 1$, *and the subspaces $E_q^\pm$ are invariant w.r.t. $D\phi^t$:*

$$D\phi^t(E_q^+) = E_{\phi^t(q)}^+, \qquad D\phi^{-t}(E_q^-) = E_{\phi^{-t}(q)}^-.$$

Moreover,

$$\|D\phi^t\big|_{E_q^-}\| \le \lambda^t, \qquad \|D\phi^{-t}\big|_{E_q^+}\| \le \mu^{-t}. \tag{3.1}$$

If the whole compact manifold M is a hyperbolic set for a $\mathcal{C}^1$ flow $\phi^t : M \to M$, then the flow ϕ^t is called an *Anosov flow* on M.

The following construction provides a useful instrument that helps to distinguish the hyperbolic sets.

Definition 3.1.3. *For every $q \in \mathbb{R}^n$ the following subsets*

$$H_q^\gamma = \{(x,y) \in T_q\mathbb{R}^n : \ \|y\| \le \gamma\|x\|\},$$

$$V_q^\gamma = \{(x,y) \in T_q\mathbb{R}^n : \ \|x\| \le \gamma\|y\|\}$$

are called the standard horizontal and standard vertical γ-cones.

Any closed convex cone contained in the standard horizontal (vertical) cone and containing the corresponding coordinate neighborhood inside is called *the horizontal (vertical) cone.*

There takes place the following criterion which gives a sufficient condition for the invariant set to be hyperbolic.

Proposition 3.1.4. *(The Cone Criterion) Let $\phi^t : M \to M$ be a smooth flow on M. A compact ϕ^t-invariant set $\Lambda \in M$ is hyperbolic, if there exist constants $\lambda < 1 < \mu$ such that for all $q \in \Lambda$ there exists a decomposition*

$$T_qM = E_q^0 \oplus S_q \oplus T_q,$$

a family of invariant horizontal cones $H_q \supset S_q$ associated with the decomposition $S_q \oplus (E_q^0 \oplus T_q)$ and a family of invariant vertical cones $V_q \supset T_q$ associated with the decomposition $(S_q \oplus E_q^0) \oplus T_q$ such that for $t > 0$

$$D\phi^t H_q \subset \mathrm{Int} H_{\phi^t(q)}, \qquad D\phi^{-t} V_q \subset \mathrm{Int} V_{\phi^{-t}(q)},$$

and

$$\frac{d}{dt}\|D\phi^t\xi\| \ge \|\xi\| \log\mu, \quad \text{for} \quad \xi \in H_q,$$

$$\frac{d}{dt}\|D\phi^{-t}\xi\| \ge \|\xi\| \log\lambda, \quad \text{for} \quad \xi \in V_q.$$

The proof of this proposition can be found in [16] (Proposition 17.4.4).

3.2 Hamiltonian flows of negative curvature

3.2.1 Canonical moving frame of a regular Jacobi curve

Let $\vec{h}$ be a monotone Hamiltonian vector field on T^*M, M, as usual, being a smooth n-dimensional manifold, and let $J_z(\cdot)$, $z \in T^*M$ be the corresponding monotone Jacobi curve in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma_z)$ over the symplectic space $\Sigma_z = T_z(T^*M)$.

According to Lemmas 1.3.7 and 1.3.8 of Chapter 1, at any point $z \in T^*M$ there exists a canonical moving frame $\{e_z^i(\cdot), f_z^i(\cdot)\}_{i=1}^n$ associated with the curve $J_z(\cdot)$ so that $J_z(t) = \mathrm{span}\{e_z^1(t), \ldots, e_z^n(t)\}$, $J_z^\circ(t) = \mathrm{span}\{f_z^1(t), \ldots, f_z^n(t)\}$ for any $t \in \mathbb{R}$ and the structural equations (1.18) hold. By Definition 1.4.2

$$e_{z_t}^i(0) = e_*^{t\vec{h}} e_{z_0}^i(t), \qquad f_{z_t}^i(0) = e_*^{t\vec{h}} f_{z_0}^i(t), \qquad i = 1, \ldots, n, \tag{3.2}$$

where $z_t = e^{t\vec{h}} z_0$, and (1.25) implies that

$$f_z^i(t) = [\vec{h}, e^i(t)](z) .$$

In particular from (1.18) and (3.2) it follows that the curvature operator satisfy the following identity along an orbit of the field $\vec{h}$:

$$R_{J_{z_0}}(t) = e_*^{-t\vec{h}} R_{J_{z_t}}(0) e_*^{t\vec{h}} . \tag{3.3}$$

By assumption, the generating Hamiltonian vector field $\vec{h}$ is monotone. Without loss of generality we can assume that h_{pp} is positive definite. Then it defines a Euclidean structure on $J_z(0)$ via the quadratic form g_z^h: it is enough to choose the basis $\{e_z^i(0)\}_{i=1}^n$ to be orthonormal w.r.t. the form $h_{pp}(z)$:

$$\|e_z^i(0)\|_{\vec{h}}^2 = \sigma(e_z^i(0), \dot{e}_z^i(0)) = \langle h_{pp}(z) p_i, p_i \rangle .$$

The adjoint basis $\{f_z^i(0)\}_{i=1}^n$ becomes normalized accordingly: $\|f_z^i(0)\|_{\vec{h}}^2 = \langle h_{pp}^{-1}(z) q_i, q_i \rangle$. Since the field $\vec{h}$ is regular, the new norm $\|\cdot\|_{\vec{h}}$, is smooth in M, while the orthonormal frame $\{e_z^i(0), f_z^i(0)\}_{i=1}^n$ is smooth along the integral trajectories of the field $\vec{h}$. By construction the basis $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$, is a special canonical moving frame. So, as we already mentioned at the end of Section 1.3.4, the matrix representation of the curvature operator w.r.t. this basis is a symmetric matrix which coincides with the matrix representation of the corresponding curvature form.

We have defined the basis $X_1, \dots, X_{2n}$ of vertical and horizontal vector fields along an orbit of the vector field $\vec{h}$. By definition

$$X_i(z) = e_z^i(0) , \qquad X_{i+n}(z) = f_z^i(0) , \qquad i = 1, \dots, n .$$

This basis is orthonormal along the orbit of $\vec{h}$ w.r.t. the norm $\|\cdot\|_{\vec{h}}$. In what follows we will consider a matrix $\Gamma(t) = \{\gamma_{ij}\}_{i,j=1}^n$ such that $e_*^{-t\vec{h}} X_i = \sum_{j=1}^{2n} \gamma_{ij}(t) X_j$.

Fix now some point $z \in \{e^{t\vec{h}} z_0 \, , z_0 \in T^*M, \, t \in \mathbb{R}\}$. Let $Y \in \Sigma_z$ be a constant vector attached at z so that $Y = \sum_{i=1}^{2n} y_i X_i(z)$. Let $\eta(t), \xi(t) \in \mathbb{R}^n$ be the coordinates of Y w.r.t. the special canonical moving frame $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$ attached at z, i.e. we assume that

$$Y = \sum_{i=1}^n \eta_i(t) e_z^i(t) + \xi_i(t) f_z^i(t)$$

with $y = (\eta(0), \xi(0))^T$. Using (3.2), we have

$$Y = \sum_{i=1}^n \eta_i(t) e_z^i(t) + \xi_i(t) f_z^i(t) = \sum_{i=1}^n \eta_i(t) e_*^{-t\vec{h}} \left(X_i(e^{t\vec{h}} z) \right) + \xi_i(t) e_*^{-t\vec{h}} \left(X_{i+n}(e^{t\vec{h}} z) \right) =$$

$$= \sum_{i,j=1}^{n} \left(\eta_i(t)\gamma_{ij}(t) + \xi_i(t)\gamma_{i+n\,j}(t)\right) X_j(z) = \sum_{i,j=1}^{2n} (\Gamma^T(t)J\alpha(t))_{ij} X_j(z) = \sum_{i=1}^{2n} y_i X_i(z)\,,$$

where

$$J = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

is the standard unit symplectic $2n \times 2n$ matrix and

$$\alpha(t) = -J\begin{pmatrix}\eta(t)\\ \xi(t)\end{pmatrix}, \qquad J\alpha(0) = y\,. \tag{3.4}$$

By definition, the matrix $\Gamma(t)$ is symplectic, i.e. $\Gamma^T(t)J\Gamma(t) = J$. Therefore

$$y = \Gamma^T(t)J\alpha(t) = J\Gamma^{-1}(t)\alpha(t)\,,$$

and hence

$$\alpha(t) = \Gamma(t)\alpha(0)\,. \tag{3.5}$$

Equations (3.5) provide the relation between the coordinates of vectors w.r.t. the fixed and moving frames attached at the same point of the orbit.

Taking into account the structural equations (1.21), one can derive the differential equations which describe the dynamics of the coordinates $\alpha(t) \in \mathbb{R}^{2n}$. Indeed,

$$\dot{Y} = 0 = \sum_{i=1}^{n} \dot{\eta}_i(t)e_z^i(t) + \dot{\xi}_i(t)f_z^i(t) + \eta_i(t)\dot{e}_z^i(t) + \xi_i\dot{f}_z^i(t) =$$

$$= \sum_{i=1}^{n}\left(\dot{\eta}_i(t) - \sum_{j=1}^{n}\xi_j(t)R_{jt}(t)\right)e_z^i(t) + \left(\dot{\xi}_i(t) + \eta_i(t)\right)f_z^i(t)\,. \tag{3.6}$$

Here we denote by $R(t) = \{R_{ij}(t)\}_{i,j=1}^n$ the matrix representation of the curvature operator $R_{J_z}(t)$ w.r.t. the canonical moving frame. Recalling that $R(t)$ is a symmetric matrix, from (3.6) we get the following second order differential equation

$$\ddot{\xi}(t) + R(t)\xi(t) = 0\,. \tag{3.7}$$

In particular it follows that $\alpha(t) = (\xi(t), \dot{\xi}(t))^T$. Letting Y to be one of the vectors of the fixed basis $\{X_i(z)\}_{i=1}^{2n}$ we get $2n$ copies of equations (3.7) in $\mathbb{R}^n$ which describe the dynamics of vectors $X_i(z)$ in Σ_z in w.r.t. the moving frame.

Proposition 3.2.1. *For any $t \in \mathbb{R}$ and any $z \in T^*M$ the coordinates of the vectors in Σ_z w.r.t. the special canonical moving frame $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$ are equal to the coordinates of their images in $\Sigma_{e^{t\vec{h}}z}$ w.r.t. the fixed basis $X_i(e^{t\vec{h}}z)$ under the action of the flow $e^{t\vec{h}}$.*

Proof. Fix some point $z_0 \in T^*M$ and let $\alpha_{z_0}(t)$ satisfying (3.4) be the $2n$-tuple of the coordinates of some vector $Y \in \Sigma_{z_0}$ w.r.t. the special canonical moving frame $\{e^i_{z_0}(t), f^i_{z_0}(t)\}_{i=1}^n$. Notice that $J\alpha_{z_0}(0) = y(z_0)$ are the coordinates of Y w.r.t. the fixed basis $\{X_i(z_0)\}_{i=1}^{2n}$. Then

$$e_*^{t\vec{h}} Y = \sum_{i=1}^{2n} y_i(z_0) e_*^{t\vec{h}} (X_i(z_0)) = \sum_{i=1}^{2n} y_i(z_0) \left(e_*^{t\vec{h}} X_i\right) (e^{t\vec{h}} z_0) =$$

$$= \sum_{i=1}^{2n} y_i(z_0) (\Gamma^{-1}(t))_{ij} X_j(z_t) = \sum_{i=1}^{2n} y_i(z_t) X_i(z_t) \,.$$

Therefore

$$y(z_t) = J\alpha_{z_t}(0) = \Gamma^{-T}(t) y(z_0) = \Gamma^{-T}(t) J\alpha_{z_0}(0) = J\Gamma(t)\alpha_{z_0}(0) \,,$$

and hence

$$\alpha_{z_t}(0) = \Gamma(t)\alpha_{z_0}(0) \,. \tag{3.8}$$

Now the standard theorem of existence and uniqueness of the solutions of ordinary differential equations implies

$$\alpha_{z_t}(0) = \alpha_{z_0}(t) = (\xi(t), \dot{\xi}(t))^T \tag{3.9}$$

where $\xi(t) \in \mathbb{R}^n$ satisfies the second order equation (3.7). □

So, it turns out that equations (3.7) describe the transformation of the vector fields X_i along an orbit of the field $\vec{h}$ under the action of the flow $e^{t\vec{h}}$. In what follows we will use the same notation $\alpha(z_t)$ for the coordinates of the image of some vector $Y \in \Sigma_z$ under the action of the Hamiltonian flow $e^{t\vec{h}}$ and for its coordinates at the initial point z w.r.t. the corresponding canonical moving frame:

$$\alpha(z_t) = \alpha_{e^{t\vec{h}} z_0}(0) \equiv \alpha_{z_0}(t) \,.$$

Summing up, we see that the structural equations (1.18), which describe the dynamics of the vectors of the canonical moving frame along any orbit of the generating field $\vec{h}$ can be reduced to the $2n$ copies of equations (3.7) in $\mathbb{R}^n$ for the coordinates of the fields X_i. Observe that the second order ODE (3.7) have the same form as the Jacobi equations (1.13) which define the structure of the geodesics on a Riemannian manifold. This observation creates a far-going analogy between the Riemannian geometry and the Hamiltonian Dynamics. In the next sections we will see that equations (3.7) play the crucial role in the analysis of the behavior of the orbits of the generating vector field $\vec{h}$.

3.2.2 Existence of invariant cones

Let us consider more in details the second order equations of the form (3.7) in $\mathbb{R}^{2n}$.

Assume $\Omega \in \mathbb{R}^n$ is a compact subset of $\mathbb{R}^n$ and R is a negative-definite quadratic form on Ω. Then there exist constants $k, \kappa > 0$ such that

$$\langle R\xi, \xi\rangle \le k\langle \xi, \xi\rangle\,, \qquad \|R\xi\|^2 \le \frac{1}{\kappa^2}\|\xi\|\,, \quad \text{for } \xi \in \mathbb{R}^n\,. \tag{3.10}$$

Define the following norm on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\|\xi, \dot{\xi}\|^2 = \|\xi\|^2 + \varepsilon\|\dot{\xi}\|^2\,,$$

with some $0 < \varepsilon < \min\{\frac{1}{\kappa}, \frac{1}{k}\}$, and $\| \;\; \|^2 = \langle \cdot, \cdot\rangle$ being a standard Euclidean norm in $\mathbb{R}^n$.

Consider the following subset:

$$C_\delta = \left\{ (\xi, \dot{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2} \ge \delta \right\}.$$

Proposition 3.2.2. *For any $\delta, \varepsilon > 0$ C_δ is a horizontal cone in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Since $\|\xi\|^2 + \varepsilon\|\dot{\xi}\|^2 \le \|\xi\|^2$ for any $\varepsilon > 0$, using Cauchy-Schwartz inequality we get

$$\delta \le \frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2} \le \frac{\sqrt{\langle \xi, \xi\rangle\langle \dot{\xi}, \dot{\xi}\rangle}}{\|\xi, \dot{\xi}\|^2} \le \frac{\|\dot{\xi}\|}{\|\xi\|}\,,$$

and hence $\|\dot{\xi}\| \ge \delta\|\xi\|$. Now from (3.4) and Definition 3.1.3 it follows that C_δ defines a horizontal cone. □

Actually one can choose parameters δ and ε in such a way that the family of cones C_δ is invariant and contains exponentially expanding vectors for $t > 0$.

Lemma 3.2.3. *For $\delta < \frac{k}{1+\kappa^{-3/2}}$ the family of horizontal cones C_δ is strictly invariant.*

Proof. Let $\delta < \frac{1}{2}$ and consider the boundary of the cone

$$B_\delta = \left\{ \xi \in \mathbb{R}^n : \frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|} = \delta \right\}.$$

We show that $\frac{d}{dt}\frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|}$ is positive whenever $\xi \in B_\delta$. Indeed,

$$\begin{aligned} \frac{d}{dt}\frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2} &= \frac{(\langle \dot{\xi}, \dot{\xi}\rangle + \langle \xi, \ddot{\xi}\rangle)\|\xi, \dot{\xi}\|^2 - 2\langle \xi, \dot{\xi}\rangle(\langle \xi, \dot{\xi}\rangle + \varepsilon\langle \ddot{\xi}, \dot{\xi}\rangle)}{\|\xi, \dot{\xi}\|^4} \\ &= \frac{\|\dot{\xi}\|^2 - \langle R(t)\xi, \xi\rangle}{\|\xi, \dot{\xi}\|^2} - 2\frac{\langle \xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2}\frac{\langle \xi - \varepsilon R(t)\xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2} \\ &\ge \frac{\|\dot{\xi}\|^2 + k\|\xi\|^2}{\|\xi, \dot{\xi}\|^2} - 2\delta\left(\delta - \varepsilon\frac{\langle R(t)\xi, \dot{\xi}\rangle}{\|\xi, \dot{\xi}\|^2}\right). \end{aligned}$$

Let us estimate the first term of the last expression. To simplify the notations we denote $a = \|\xi\|^2$, $b = \|\dot{\xi}\|^2$. Then

$$\frac{ka+b}{a+\varepsilon b} = k\left(1+\frac{\frac{1}{k}-\varepsilon}{\frac{a}{b}+\varepsilon}\right) \geq k$$

provided $\varepsilon \leq \frac{1}{k}$. So

$$\frac{\|\dot{\xi}\|^2 + k\|\xi\|^2}{\|\xi,\dot{\xi}\|^2} \geq k\,, \tag{3.11}$$

On the other hand, since

$$\frac{\sqrt{\langle\xi,\xi\rangle\langle\dot{\xi},\dot{\xi}\rangle}}{\|\xi,\dot{\xi}\|^2} = \frac{\sqrt{\langle\xi,\xi\rangle\langle\dot{\xi},\dot{\xi}\rangle}}{\langle\xi,\xi\rangle+\varepsilon\langle\dot{\xi},\dot{\xi}\rangle} \leq \frac{1}{2\sqrt{\varepsilon}}\,,$$

and applying again the Cauchy-Schwartz inequality we get

$$\delta+\varepsilon\frac{\langle -R(t)\xi,\dot{\xi}\rangle}{\|\xi,\dot{\xi}\|^2} \leq \delta+\varepsilon\frac{\sqrt{\langle R(t)\xi,R(t)\xi\rangle,\langle\dot{\xi},\dot{\xi}\rangle}}{\|\xi,\dot{\xi}\|^2} \leq \delta+\frac{\varepsilon}{\kappa}\frac{\sqrt{\langle\xi,\xi\rangle\langle\dot{\xi},\dot{\xi}\rangle}}{\|\xi,\dot{\xi}\|^2} \leq \delta+\frac{\sqrt{\varepsilon}}{2\kappa}\,. \tag{3.12}$$

Putting together (3.11) and (3.12) and recalling that $\delta < \frac{1}{2}$ we have

$$\frac{d}{dt}\frac{\langle\xi,\dot{\xi}\rangle}{\|\xi,\dot{\xi}\|^2} \geq k-2\delta\left(\delta+\frac{\sqrt{\varepsilon}}{2\kappa}\right) > k-\delta\left(1+\frac{1}{\kappa^{-3/2}}\right) > 0$$

provided $\delta < \frac{k}{1+\kappa^{-3/2}}$. □

Lemma 3.2.4. *For $\varepsilon < 4\kappa^2\delta^2$ the family of horizontal cones C_δ is exponentially expanding.*

Proof. Take some $(\xi,\dot{\xi}) \in C_\delta$. Then

$$\frac{d}{dt}\ln\|\xi,\dot{\xi}\| = \frac{\frac{d}{dt}\|\xi,\dot{\xi}\|}{\|\xi,\dot{\xi}\|} = \frac{\langle\xi,\dot{\xi}\rangle+\varepsilon\langle\ddot{\xi},\dot{\xi}\rangle}{\|\xi,\dot{\xi}\|^2} = \frac{\langle\xi,\dot{\xi}\rangle}{\|\xi,\dot{\xi}\|^2} - \varepsilon\frac{\langle R(t)\xi,\dot{\xi}\rangle}{\|\xi,\dot{\xi}\|^2} \geq \tag{3.13}$$

$$\geq \delta-\varepsilon\frac{\sqrt{\langle R(t)\xi,R(t)\xi\rangle\langle\dot{\xi},\dot{\xi}\rangle}}{\|\xi,\dot{\xi}\|^2} \geq \delta-\frac{\varepsilon}{\kappa}\frac{\sqrt{\langle\xi,\xi\rangle,\langle\dot{\xi},\dot{\xi}\rangle}}{\|\xi,\dot{\xi}\|^2} \geq \delta-\frac{\sqrt{\varepsilon}}{2\kappa} = \mu > 0\,,$$

provided $\varepsilon < 4\kappa^2\delta^2$. Hence $\|\xi(t),\dot{\xi}(t)\| \geq e^{\mu t}\|\xi(0),\dot{\xi}(0)\|$ and C_δ defines an exponentially expanding cone for $t > 0$. □

Remark 3.2.5. Let $\alpha(z_t) = (\xi(t),\dot{\xi}(t))^T$ with $\alpha(z_0) = (\xi(0),\dot{\xi}(0))^T$. Then the last inequality of Lemma 3.2.4 implies

$$\|\alpha(z_t)\| = \|\Gamma(t)\alpha(z_0)\| \geq e^{\mu t}\|\alpha(z_0)\|\,, \qquad \alpha(z_0) \in C_\delta\,, \tag{3.14}$$

while from (3.13) one can easily see that for any Y in Σ_{z_0} such that $Y = \sum_{i=1}^{2n}(J\alpha(z_0))_i X_i(z_0)$ with $\alpha(z_0) \in C_\delta$ there takes place the following inequality:

$$\frac{d}{dt}\|e_*^{t\vec{h}}Y\|_{\vec{h}} \geq \mu e^{\mu t}\|Y\|_{\vec{h}}\,. \tag{3.15}$$

The analogous construction shows the existence of the invariant cones expanding in negative time.

3.3 Anosov flows on the level sets of the Hamiltonian

In this section we consider the dynamics induced by a monotone increasing Hamiltonian vector field $\vec{h} \in T(T^*M)$ on the level set of the corresponding Hamiltonian function. Following the construction presented in Chapter 2, we will consider the reduced Jacobi curve $\overline{J_z^h(\cdot)}$ in the Lagrange Grassmannian $Gr_{n-1}(\widetilde{\Sigma}_z)$ over the factor-space $\widetilde{\Sigma}_z = \Sigma_z \cap \vec{h}^{\angle}/\mathbb{R}\vec{h}$. There takes place the following theorem.

Theorem 3.3.1. *Let $\vec{h}$ be a monotone field, S be a compact invariant subset of the flow $e^{t\vec{h}}$ contained in a fix level set of h, $S \subset h^{-1}(c)$, and $d_z h \neq 0$ $\forall z \in S$. If $\vec{h}$ has a negative reduced curvature at each point of S, then S is a hyperbolic set of the flow $e^{t\vec{h}}\Big|_{h^{-1}(c)}$.*

Proof. Without loss of generality one can make the proof for the case $S = h^{-1}(c)$.

Let $c \in \mathbb{R}$ and denote by $\mathcal{H}_c = h^{-1}(c)$ the level set of the Hamiltonian $h \in C^\infty(T^*M)$. Since $d_z h \neq 0$ for all $z \in T^*M$, $\mathcal{H}_c$ is a regular hyper-surface in T^*M of dimension $2n-1$ and $T\mathcal{H}_c = T(T^*M) \cap \vec{h}^{\angle}$.

The vector field $\vec{h}$ generates the h-reduced Jacobi curve $\overline{J_z^h(\cdot)}$ in $Gr_{n-1}(\widetilde{\Sigma}_z)$. This curve is monotone and the canonical moving frame associated to $\overline{J_z^h(\cdot)}$ is well defined on $\mathcal{H}_c$ at any $z \in \{e^{t\vec{h}} z_0 : \quad t \in \mathbb{R}, \quad z_0 \in \mathcal{H}_c\}$. The corresponding curvature operator $\overline{R(t)} = R_{\overline{J_z^h}}(t)$ is well defined and satisfy a priori bounds (3.10).

First of all we note that since the field $\vec{h} \in T\mathcal{H}_c$ is invariant w.r.t. the action of the corresponding Hamiltonian flow (i.e. $e_*^{t\vec{h}}\vec{h} = \vec{h}$), in order to show that $\mathcal{H}_c$ is hyperbolic, it is enough to show that the factor space $\widetilde{\Sigma}_z$ spanned by the vectors of the canonical moving frame, associated to $\overline{J_z^h(\cdot)}$, admits a splitting into exponentially contracting and expanding invariant subspaces.

According to Section 3.2, the special canonical moving frame $\{e_z^i(t), f_z^i(t)\}_{i=1}^{n-1}$ associated with the reduced Jacobi curve $\overline{J_z^h(\cdot)}$ satisfy equations (1.21) with $i = 1, \ldots, n-1$. Hence the coordinates of the vectors $e_z^i(0)$, $f_z^i(0)$ w.r.t. the basis $\{e_z^i(t), f_z^i(t)\}_{i=1}^{n-1}$ satisfy the second order ODE of the form (3.7). Now from Proposition 3.2.1 and Lemmas 3.2.3 and 3.2.4 it follows the existence of the invariant contracting and expanding invariant cones $C_\delta^\pm$ in the tangent space $T_{e^{t\vec{h}}z}(\mathcal{H}_c)/\mathbb{R}\vec{h}$ along an orbit of the vector field $\vec{h}$. Indeed, for any $Y \in T_z(\mathcal{H}_c)/\mathbb{R}\vec{h}$ such that $Y = \sum_{i=1}^{2n-2} (J\alpha(z))_i X_i(z)$ one has $\|Y\|_{\vec{h}} = \|\alpha(z)\|$. So, using (3.14), one can easily see that

$$H_z = \operatorname{span}\left\{ Y^+ \in (T_z\mathcal{H}_c)/\mathbb{R}\vec{h} : \, Y^+ = \sum_{i=1}^{2n} \alpha^+(z) X_{i_k}(z), \, \alpha^+(z) \in C_\delta^+ \right\},$$

$$V_z = \operatorname{span}\left\{ Y^- \in (T_z\mathcal{H}_c)/\mathbb{R}\vec{h} : Y^- = \sum_{i=1}^{2n} \alpha^-(z) X_{i_k}(z),\ \alpha^-(z) \in C_\delta^- \right\}$$

form a couple of invariant expanding and contracting cones in the tangent space at any point $z \in \mathcal{H}_c$.

Taking into account the $e^{t\vec{h}}$-invariance of the field $\vec{h}$, from Proposition 3.1.4 we deduce now that $\mathcal{H}_c$ is a hyperbolic set for the Hamiltonian flow $e^{t\vec{h}}$. □

Example 3.3.2. *(Mechanical system on a Riemannian manifold)* Let us come back to the situation described in Example 1.4.9. Consider a mechanical system with the potential energy U on a Riemannian manifold M, endowed with the positive definite quadratic form G, which defines an inner product $G_q(\cdot,\cdot)$ on T_qM, $q \in M$. We will denote by $\|\cdot\|$ the norm on TM defined by the inner product G_q. The Hamiltonian vector field $\vec{h}$ defined by (1.35) is monotone increasing together with the bilinear form $g_z^h = G^{-1}_{\pi(z)}$ which defines a Euclidean structure on Π_z.

Let us derive the formula of the curvature form of the field $\vec{h}$ restricted to the level set $\mathcal{H}_c = h^{-1}(c)$. First of all from (2.19) we find $\mathcal{X} = \left(h_{pp}^{-1}h_p, 0\right)^T$. Let $\kappa_q(\xi^\uparrow, z^\uparrow)$ be the sectional curvature of the section $\operatorname{span}\{\xi^\uparrow, z^\uparrow\} \subset T_qM$. Applying (1.36) and (2.44) to $\vec{h}$ and $\mathcal{X}$ we find that for all $\xi \in \overline{\Pi_z^h}$

$$r_z^{\vec{h},h}(\xi) = \kappa_q(\xi^\uparrow, z^\uparrow)(\|z^\uparrow\|^2\|\xi^\uparrow\|^2 - G_q(\xi^\uparrow, z^\uparrow)^2) + G_q(\nabla_{\xi^\uparrow}(\nabla U), \xi^\uparrow) + \frac{3G_q^{-1}(\nabla U, \xi)^2}{2(c - U(q))}, \quad (3.16)$$

where in local coordinates on T_qM with $q = \pi(z)$ one has $z^\uparrow = \sum_{i=1}^{n} p_i \partial_{q_i}$, and $G_q(\xi^\uparrow, z^\uparrow) = G_q(\nabla_q U, \xi^\uparrow)$ provided $\xi \in T_z\mathcal{H}_c$. It turns out that $\|z^\uparrow\|^2 = G_q^{-1}(p,p) = 2(c - U(q))$.

Let κ_q be the maximal sectional curvature of the Riemannian manifold M at $q \in M$. Then from (3.16) it is easy to see that any compact invariant set S of the flow $e^{t\vec{h}}\big|_{h^{-1}(c)}$ such that the projection of S to M is contained in the domain

$$\left\{ q \in M : \kappa_q < 0,\ \|\nabla_q^2 U\| + \left(|\kappa_q| + \frac{3}{2(c - U(q))} \right) \|\nabla_q U\|^2 \le 2|\kappa_q|(c - U(q)) \right\}$$

is hyperbolic. In particular, if M is a compact Riemannian manifold of a negative sectional curvature, then $e^{t\vec{h}}\big|_{h^{-1}(c)}$ is an Anosov flow for any big enough c. This fact generalizes a classical result on geodesic flows.

3.4 Hyperbolic equilibrium

Now let us consider a Hamiltonian flow on the whole T^*M generated by a monotone increasing vector field $\vec{h}$. Assume that the curvature operator $R_{\vec{h}}(\cdot)$ associated with the field $\vec{h}$ is negative. Then there takes place the following statement.

Theorem 3.4.1. *Let $\vec{h}$ be a monotone field and $z_0 \in T^*M$. Assume that the semi-trajectory $\{e^{t\vec{h}}(z_0) : t \geq 0\}$ has a compact closure and $\vec{h}$ has a negative curvature at each point of its closure. Then there exists $z_\infty = \lim_{t\to+\infty} e^{t\vec{h}}(z_0)$, where $\vec{h}(z_\infty) = 0$ and z_∞ is a hyperbolic equilibrium point for the field $\vec{h}$.*

Proof. The curvature operator of $\vec{h}$ satisfy the a priori bounds (3.10) at any point of the semi-orbit $\{e^{t\vec{h}}(z_0) : \ t \geq 0\}$ provided it has a compact closure. The same arguments as in Theorem 3.3.1 show that at any point $z \in \{e^{t\vec{h}}(z_0) : \ t \geq 0\}$ there exists a splitting of Σ_z into invariant exponentially expanding and contracting cones $C^{\pm}_{\delta}$ for some $\delta > 0$.

Now let us analyze more in details the dynamics of the vector field $\vec{h}$ under the action of the corresponding Hamiltonian flow $e^{t\vec{h}}$. According to (3.4)

$$\vec{h}(z_t) = \sum_{i_k=1}^{2n} h_i(z_t)X_i(z_t) = \sum_{i=1}^{2n}(J\alpha(z_t))_iX_i(z_t)$$

with $z_t = e^{t\vec{h}}z_0$. Then

$$\|\vec{h}(z_t)\|_{\vec{h}} = \|\alpha(z_t)\| \leq C \tag{3.17}$$

for some constant $C > 0$, provided the Hamiltonian h and the norm $\|\cdot\|_{\vec{h}}$ are smooth on the compact set $clos\{e^{t\vec{h}}(z_0) : t \geq 0\}$.

Without loss of generality we can assume that z_0 is not an equilibrium point for the field $\vec{h}$. Then $\alpha(z_0) \neq 0$ and $\alpha(z_t) = \alpha(z_0)^+ + \alpha(z_0)^-$ where $\alpha(z_0)^{\pm} \in C^{\pm}_{\delta}$. According to (3.14), $\alpha(z_t)^+$ grows exponentially as $t \to +\infty$, which contradicts (3.17). Hence $\alpha(z_0)^+ = 0$. Passing to the limit as $t \to +\infty$ we get that $\|\vec{h}(z_\infty)\|_{\vec{h}} = 0$ with $z_\infty = \lim_{t\to+\infty} e^{t\vec{h}}z_0$ and hence $\vec{h}(z_\infty) = 0$.

It remains to show that z_∞ is a hyperbolic equilibrium point. Observe that the point z_∞ is a compact $e^{t\vec{h}}$-invariant subset of T^*M. Since the coordinates of the vectors w.r.t. the special canonical moving frame attached at z_∞ satisfy equations (3.7), which are autonomous in this case, Lemmas 3.2.2-3.2.4 imply the existence of the invariant splitting of $\Sigma_{z_\infty} = H_{z_\infty} \oplus V_{z_\infty}$ into expanding and contracting subspaces. Therefore $D_{z_\infty}\vec{h}$ cannot have any eigenvalue on the imaginary axis, and hence z_∞ is a hyperbolic equilibrium point in the sense of Definition 3.1.1. □

Corollary 3.4.2. *Let $\vec{h}$ be a monotone field, S be a compact $e^{t\vec{h}}$-invariant subset contained in a fix level set of h, $S \subset h^{-1}(c)$, and $d_zh \neq 0$ for all $z \in S$. Assume that $\vec{h}$ admits a first integral $g \in C^\infty(T^*M)$, such that $\vec{g}(z) \neq 0$ and $\vec{h}(z) \wedge \vec{g}(z) \neq 0$ for all $z \in S$. Then the h-reduced curvature of $\vec{h}$ cannot be negative-definite everywhere on S.*

Proof. Assume that the h-reduced curvature of $\vec{h}$ is negative-definite on S. Then by Theorem 3.3.1, S is a hyperbolic set for the flow $e^{t\vec{h}}$, and in particular there exists a splitting $T_z(\mathcal{H}_c)/\mathbb{R}\vec{h} = E^+_z \oplus E^-_z$ into exponentially expanding and contracting subspaces. Since g is a first integral of $\vec{h}$, the corresponding Hamiltonian vector field $\vec{g}$ is invariant w.r.t. the action of the Hamiltonian flow $e^{t\vec{h}}$: $e_*^{t\vec{h}}\vec{g} = \vec{g}$. Now the same arguments as in the proof of Theorem 3.4.1 show the existence of a singular point for the field $\vec{g}$, which leads to a contradiction. □

Example 3.4.3. A natural mechanical system on $M = \mathbb{R}^n$ (see Example 1.4.7) with a concave potential energy provides an example of a Hamiltonian system with negative-definite curvature in $T^*\mathbb{R}^n$. In this case if the potential energy has a maximum at some point $q_0 \in M$, then the point $z_0 = (0, q_0) \in T^*\mathbb{R}^n$ is a hyperbolic equilibrium point, i.e. z_0 is a global saddle for the orbits of the field $\vec{h}$. Otherwise $e^{t\vec{h}}$ does not have any compact invariant subset.

Chapter 4

The N-body problem

In the present chapter we discuss the application of the theory, developed in Chapters 1 and 2, to the classical plane N-body problem. In Section 4.1 we recall briefly the main facts concerning the known first integrals of the dynamical system which describe the motion of N particles in $\mathbb{R}^3$ interacting gravitationally, and calculate the Ricci curvature of the reduced plane N-body problem with equal masses, applying the method developed in Chapter 2.

Section 4.2 contains a detailed analysis of the curvatures and the conjugate points of the so-called 8-*shaped solution of the* 3-*body problem with equal masses*, or just *the Eight*, discovered in 2000 by A.Chenciner and R.Montgomery (see [14]). The aim of this study was to test the minimality property of this orbit on the intervals of time bigger then the fundamental domain of its symmetry group. We begin with discussion about the geometrical meaning of the reduction by the angular momentum integral in the 3-body problem and recall briefly the characteristic properties of the 8-shaped orbit on the plane and in the so-called *shape space*, which is the configuration space of the 3-body problem reduced by the triple of first integrals in involution, consisting of the linear and angular momentum integrals. We present the result of the numerical calculation of the sets of conjugate points for the Jacobi curve along the Eight and for its reduction, and perform a detailed analysis of the structure of the projection of the Eight on the shape space. We find another solution of the reduced problem with fixed end-points which has the value of the action functional smaller than the Eight.

4.1 First integrals of the classical N-body problem

Let us consider a system of N bodies of unit mass in $\mathbb{R}^3$ endowed with the standard Cartesian coordinates, so that r_i represents the radius vector of the i-th body with respect to some inertial frame. We assume that the bodies interact gravitationally according to Newton's gravitational low. Then the motion of this system is described by a natural mechanical system on $M = \mathbb{R}^{3N}$ with potential energy

$$U(r_1, \ldots, r_N) = -\sum_{i<j}^{N} \frac{1}{r_{ij}}, \qquad r_{ij} = \|r_i - r_j\| . \tag{4.1}$$

The general N-body problem possesses three vector and one scalar first integrals ([15], [26]). These integrals consist of two vector integrals of center of mass c_1 and c_2, the angular momentum integral g and the energy integral h. In the case of unite masses these integrals take the form

$$\sum_{i=1}^{N} \dot{r}_i = c_1 , \qquad \sum_{i=1}^{N} r_i - c_1 t = c_2 , \tag{4.2}$$

$$\sum_{i=1}^{N} r_i \times \dot{r}_i = g , \tag{4.3}$$

$$\frac{1}{2} \sum_{i=1}^{N} \|\dot{r}_i\|^2 + U(r_1, \ldots, r_N) = h . \tag{4.4}$$

It turns out that not all of these integrals are in involution (see [15]). Actually it is possible to show that in some special canonical coordinates in the configuration space (the so-called *Jacobi coordinates*) the integrals of group (4.2) form a pair of conjugate variables consisting of cyclic integral $\sum\limits_{i=1}^{N} \dot{r}_i$ and the corresponding state variable $\sum\limits_{i=1}^{N} r_i$. All together first integrals (4.2) -(4.4) permit to reduce the number of degrees of freedom by 6 in the spatial case and by 4 in the planar case.

In this chapter we will analyze the plane N-body problem with unit masses. In this case $M = \mathbb{R}^{2N}$, $T^*M \cong \mathbb{R}^{2N} \times \mathbb{R}^{2N} = \{(p, q),\ p, q \in \mathbb{R}^{2N}\}$ and $p_1 \ldots, p_{2N}$ are the canonical impulses conjugated to $q_1, \ldots, q_{2N}$ ($p_i \sim \dot{q}_i$), such that $r_i = (q_{2i-1}, q_{2i})$.

The first vector integral of (4.2) reads

$$g_1 = \sum_{i=1}^{N} p_{2i-1} , \qquad g_2 = \sum_{i=1}^{N} p_{2i} , \tag{4.5}$$

We will refer to these first integrals as to the *linear momenta integrals*. The angular momentum integral for the planar problem is actually a scalar function. In Cartesian coordinates on T^*M it takes the form

$$g = \sum_{i=1}^{N} (p_{2i} q_{2i-1} - p_{2i-1} q_{2i}) .$$

The Hamiltonian vector field $\vec{h}$ in the chosen coordinates has the following expression

$$\vec{h} = \sum_{i-1}^{2N} p_i \partial_{q_i} - \frac{\partial U}{\partial q_i} \partial_{p_i} = (-U_q, p)^T , \tag{4.6}$$

and the dynamics of the system is defined by the following system of Hamiltonian equations:

$$\dot{p}_i = -U_{q_i}\,, \qquad \dot{q}_i = p_i\,, \qquad i = 1, \ldots, 2N\,. \tag{4.7}$$

An easy computation shows that the first integrals g_1, g_2 and g are in involution. By *the reduced N-body problem* we will mean the N-body problem reduced by these three scalar integrals.

From Example 1.4.7 we know that the generalized curvature form of the dynamical Lagrangian distribution $(\vec{h}, \Pi)$ is just the Hessian of the potential energy U, and consequently its generalized Ricci curvature is just the Laplacian of U, which can be calculated without difficulties:

$$\rho_z^{\vec{h}} = \mathrm{tr} R_z^{\vec{h}} = \Delta\, U(z) = -2 \sum_{i<j}^{N} \frac{1}{r_{ij}^3}, \qquad z = (p, q). \tag{4.8}$$

Now let us compute the Ricci curvature of the reduced N-body problem. First of all we note that the reduction by the linear momenta integrals (4.5) preserve the Ricci curvature. Indeed, using (2.41) we compute the vertical vector fields

$$\mathcal{X}^{g_1} = \sum_{i=1}^{N} \partial_{p_{2i-1}}\,, \qquad \mathcal{X}^{g_2} = \sum_{i=1}^{N} \partial_{p_{2i}}$$

corresponding to the integrals (4.5). These vector fields are constant, hence the additional term defined by (2.44), related to these integrals, is equal to zero. Moreover, the following simple computation shows that also the restriction of the curvature operator to the space $\Pi_z \cap \ker d_z g_1 \cap \ker_z g_2$ does not change the Ricci curvature of the problem:

$$\mathrm{tr}\left[R_z^{\vec{h}} \Big|_{\Pi_z \cap \ker d_z g_1 \cap \ker_z g_2} \right] = \Delta\, U - \frac{r_z^{\vec{h}}(\mathcal{X}^{g_1})}{g_z^{\vec{h}}(\mathcal{X}^{g_1}, \mathcal{X}^{g_1})} - \frac{r_z^{\vec{h}}(\mathcal{X}^{g_2})}{g_z^{\vec{h}}(\mathcal{X}^{g_2}, \mathcal{X}^{g_2})} =$$

$$= \Delta\, U - \frac{1}{N} \sum_{i,j=1}^{N} \left(U_{q_{2i-1} q_{2j-1}} + U_{q_{2i} q_{2j}} \right) =$$

$$= \Delta\, U - \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\partial}{\partial q_{2i-1}} \left(\sum_{j=1}^{N} U_{q_{2j-1}} \right) + \frac{\partial}{\partial q_{2i}} \left(\sum_{j=1}^{N} U_{q_{2j}} \right) \right) = \Delta\, U,$$

since by (4.5) and (4.7)

$$\sum_{j=1}^{N} U_{q_{2j-1}} = -\frac{d}{dt} \sum_{j=1}^{N} p_{2j-1} = 0\,, \qquad \sum_{j=1}^{N} U_{q_{2j}} = -\frac{d}{dt} \sum_{j=1}^{N} p_{2j} = 0\,.$$

Here we used the fact that by definition of the vectors $\mathcal{X}_i(z)$, they are orthogonal to the subspace $\overline{\Pi_z^{\mathcal{G}}}$ w.r.t. the inner product defined by the bilinear form $g_z^{\vec{h}}$.

So, it turn out that in order to obtain the Rici curvature of the reduced N-body problem it is sufficient to calculate the Ricci curvature of the g-reduction of the dynamical

Lagrangian distribution $(\vec{h}, \Pi)$ by the angular momentum integral g. We will use formula (2.44). In our case $s = 1$. Let $\mathcal{X}$ be as in (2.41) with g instead of g_i. Then

$$\rho_z^{\vec{h},g} = \Delta U(z) - \frac{r_z^{\vec{h}}(\mathcal{X})}{g_z^{\vec{h}}(\mathcal{X}, \mathcal{X})} + \operatorname{tr}\left[\left(R_z^{\vec{h},g} - R_z^{\vec{h}}\right)\Big|_{\Pi_z \cap \ker d_z g}\right]. \tag{4.9}$$

The Hamiltonian vector field corresponding to the function g is given by $\vec{g} = (Jp, -Jq)^T$ with J being the unit symplectic $2N \times 2N$ matrix:

$$J = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{pmatrix}.$$

Applying formula (2.41) we find that

$$\mathcal{X} = (Jq, 0)^T. \tag{4.10}$$

Denote $\overline{\mathcal{X}} = Jq$. Using again (1.33), from(4.1) one can obtain by direct computation that

$$r_z^{\vec{h}}(\mathcal{X}) = \langle U_{qq}\overline{\mathcal{X}}, \overline{\mathcal{X}}\rangle = -U\,. \tag{4.11}$$

Further, from (2.44) (or (2.45)) and (4.10), it follows that

$$\operatorname{tr}\left[\left(R_z^{\vec{h},g} - R_z^{\vec{h}}\right)\Big|_{\Pi_z \cap \ker d_z g}\right] = \tag{4.12}$$

$$= \frac{3}{4\sigma(\mathcal{X}, \vec{g})}\left(\sum_{i=1}^{2N} \sigma([\vec{h}, [\vec{h}, \mathcal{X}]], \partial_{p_i})^2 - \frac{\sigma([\vec{h}, [\vec{h}, \mathcal{X}]], \mathcal{X})^2}{g_z^{\vec{h}}(\mathcal{X}, \mathcal{X})}\right) = \frac{3}{4}\left(\frac{\|p\|^2}{\|q\|^2} - \frac{\langle p, q\rangle^2}{\|q\|^4}\right).$$

Finally, substituting (4.8), (4.11) and (4.12) into (4.9) we have

$$\rho_z^{\vec{h},g} = -2\sum_{i<j}^{N} \frac{1}{r_{ij}^3} - \frac{U}{I} + \frac{3}{I^2}(2KI - \frac{1}{4}\{h, I\}^2)\,, \tag{4.13}$$

where $I = \|q\|^2$, $K = \frac{1}{2}\|p\|^2$ are the central momentum of inertia and the kinetic energy of the system of N bodies. The last term in (4.13) contains the right-hand side of the famous Sundman's inequality $2KI - \frac{1}{4}\{h, I\}^2 \geq 0$. On the other hand, by Corollary 2.2.7 we know that it has to be non-negative. Actually this term is nothing but the generalized area of the parallelogram formed by two $2N$-dimensional vectors p and q.

4.2 The 8-shaped solution of the 3-body problem

Let us consider now the case $N = 3$. Already in this case the N-body problem is unsolved: the known first integrals (4.2) - (4.4) are not sufficient for the integrability. Nevertheless some particular solutions for the 3-body problem were discovered already by Euler(1765) and Lagrange(1772). Following the classical terminology of Celestial Mechanics we will refer to them as to *the Euler* and *Lagrange solutions*. Assume that the origin of the inertial frame is fixed in the common center of mass of the bodies. Then these known solutions can be briefly described as follows.

The Euler solutions are such that all the bodies are placed on the same straight line which rotates around the origin so that each of the bodies moves along its own ellipse with one of the foci in the origin. The ratios of the distances between the bodies and the origin are defined by the ratios of masses and so remain constant. In the case of equal masses one of the bodies "sits" in the origin, the other two being placed symmetrically. In particular, the bodies can collide in the origin.

Along the Lagrange solution the bodies form an equilateral triangle with variable size, which evolves around the origin so that the orbit of each of the bodies is again a Keplerian ellipse. The special case of Lagrange solutions is the homothetic *triple collision-ejection* when each body moves along the collision-ejection orbit with one of the ends in the origin.

4.2.1 The shape space of the 3-body problem

Hereafter we will suppose that all the bodies have the unite mass. Needless to say that in the case $N = 3$ the original configuration space is $M = \mathbb{R}^6$.

As it is well known in Celestial Mechanics, the reduction by first integrals (4.2), (4.3) can be performed in two steps ([11]): first by passing to the barycentric coordinates and getting rid of translations (reduction on the integrals of group (4.2)), and then by fixing the value of the angular momentum and making quotient by rotations around the constant angular momentum vector. In terms of configuration space one has

$$M \longrightarrow M_0 \longrightarrow \overline{M} \,,$$

with

$$M_0 = \left\{ q = (r_1, r_2, r_3) \in \mathbb{R}^6 \,,\ \sum_{i=1}^{3} r_i = 0 \right\} \,, \qquad \overline{M} = M_0 / SO(2) \,.$$

The reduced configuration space $\overline{M}$ is homeomorphic to $\mathbb{R}^3$ ([19]). In what follows we consider in detail the second step of the reduction: $M_0 \to \overline{M}$.

Essentially the space $\overline{M}$ is nothing but the space of congruence classes of triangles and it is called the *shape space*. Topologically it is a cone over a 2-sphere $I = 1$ whose points correspond to the similarity classes of triangles; the cone point is the point of the triple collision. This sphere is usually called the *shape sphere* (see Fig.4.1) [1]. The poles of the

[1] We took this beautiful picture from the article of A.Chenciner and R.Montgomery ([14]).

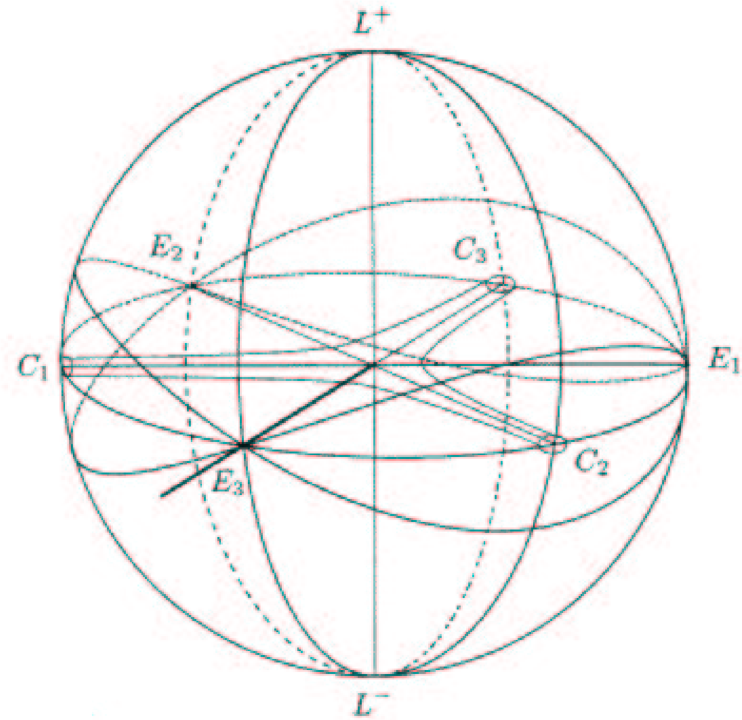

Figure 4.1: The shape space

shape sphere correspond to the Lagrange configurations L^+ and L^-, and there are three special meridians M_i, $i = 1, 2, 3$ which represent the three different (w.r.t. the order of the bodies) types of isosceles triangles. Each of these meridians intersects the equator in the Euler point E_i and in its antipodal double-collision point C_i. Here we have assumed that the bodies are numerated and we use E_i (M_i) for the Euler (isosceles) configuration with the i-th body in the middle (or, respectively, in the top).

4.2.2 Description of the Eight

In 2000, A. Chenciner and R.Montgomery proved the existence of a new periodic solution of the planar 3-body problem with equal masses - the 8-shaped orbit or just *the Eight* ([14]). Here we give a short description of this orbit.

Let $T > 0$ be a positive real number and $H^1([0, T], M_0)$ be the Sobolev space of functions which are square integrable in the sense of distributions together with their first derivatives. The orbit of Chenciner and Montgomery is a smooth collision - free curve which consists of 12 pieces such that each of them minimizes the Lagrangian action functional

$$A^T[\gamma] = \int_0^T K_0 - U dt\,. \tag{4.14}$$

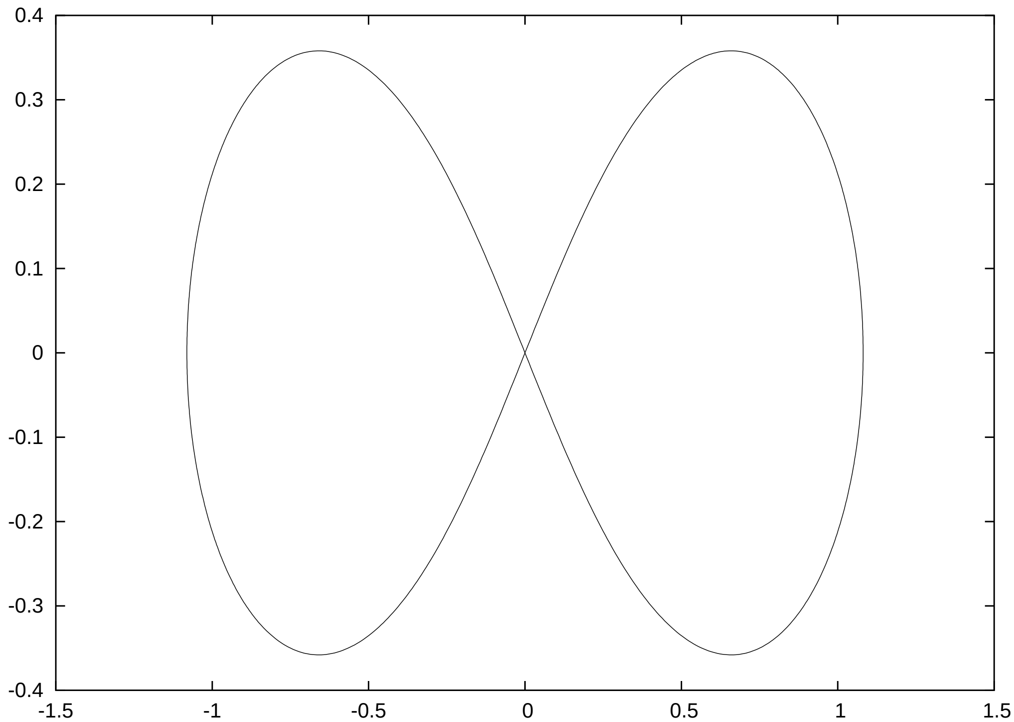

Figure 4.2: The Eight in the Cartesian coordinates in M_0

over subspace

$$\Upsilon = \left\{ \gamma \in H^1([0,T], M_0) : \ \gamma(0) = E_i\,,\ \gamma(T/12) \in M_k\,,\ \{i,j,k\} = \{1,2,3\} \right\}$$

on the interval $[0, T/12]$. Here K_0 is the kinetic energy of the system in the space M_0.

It turns out that the angular momentum $g \equiv 0$ along the Eight, hence actually it minimizes the reduced action functional

$$\int_0^{\frac{T}{12}} (K_{red} + U)dt$$

over the paths lying in the reduced configurational space $\overline{M}$. Here K_{red} is the deformation part of the kinetic energy $K_{red} = K_0 - g^2 I^{-1}$, it defines a metric structure on the shape space $\overline{M}$.

In the space M_0 the Eight is a symmetrical planar 8-shaped curve with the intersection point in the origin (see Fig.4.2). During the full period of motion the bodies six times form collinear Euler configurations and isosceles triangles so that each body passes through the origin and the top of each triangle twice.

The projection of the Eight on the shape sphere (see Fig.4.1) is a closed path consisting of 12 pieces such that each of them connects an Euler point with an Euler meridian so that the curve makes a double turn around the equator of the shape sphere. It intersects each of Euler meridians M_i transversally at points $M_i^{\pm}$, and all the pieces of this orbit are gluing together smoothly. Anyway it is not clear whether the longer then 1/12-th piece of the Eight do minimize the action functional.

In the next sections we analyze the minimizing properties of the Eight in the shape space in details, combining the theory developed in Chapters 1 and 2 with numerical computations.

4.2.3 Reduced curvatures of the Eight

There are several possibilities to choose coordinates in the reduced space $\overline{M}$. Following [14], we will use the standard spherical coordinates $r = \sqrt{I}$, θ and ϕ. The advantage of such a choice is that these coordinates have quite simple physical meaning: by definition, $r^2 = I$ is the central momentum of inertia of the bodies, the parallels $\phi = \text{const}$ of the shape sphere are essentially the similarity classes of triangles with the same ellipse of inertia up to rotation, they can be characterized by their common area; the meridians $\theta = \text{const}$ are defined by some linear relations between the squares of the mutual distances ([18]).

In the spherical coordinates (r, θ, ϕ) the potential energy and the reduced kinetic energy have the following expressions:

$$U = -\frac{U_0}{r}, \quad U_0 = \frac{1}{\sqrt{(1+\cos\theta\cos\phi)}} + \frac{1}{\sqrt{(1+\cos(\theta+\frac{2\pi}{3})\cos\phi)}} + \frac{1}{\sqrt{(1+\cos(\theta+\frac{4\pi}{3})\cos\phi)}}, \tag{4.15}$$

$$K_{red} = \frac{\dot{r}^2}{2} + \frac{r^2}{8}(\cos\phi^2\dot{\theta}^2 + \dot{\phi}^2)\,.$$

Here U_0 is a scaled potential defined on the shape sphere: $U_0 = -U|_{I=1}$. The singularities of U are the points of double collision $\theta = \pm\pi/3, \pi$ and the point of total triple collision $r = 0$.

The Hamiltonian function $H = h|_{T^*\overline{M}}$ of the reduced 3-body problem takes the form

$$H = \frac{p_1^2}{2} + \frac{2p_2^2}{r^2\cos^2\phi} + \frac{2p_3^2}{r^2} - \frac{U_0}{r}\,, \tag{4.16}$$

where

$$p_1 = \dot{r}\,, \quad p_2 = \frac{1}{4}r^2\cos^2\phi\dot{\theta}\,, \quad p_3 = \frac{1}{4}r^2\dot{\phi}$$

are the canonical impulses conjugated to r, θ and ϕ. So, the orbits of the reduced 3-body problem with equal masses satisfy the following Hamiltonian system

$$\dot{p}_1 = \frac{4p_2^2}{r^3\cos^2\phi} + \frac{4p_3}{r^3} - \frac{U_0}{r^2}\,,$$

$$\dot{p}_2 = \frac{1}{r}\frac{\partial U_0}{\partial \theta}\,, \qquad \dot{p}_3 = -\frac{4p_2 \sin\phi}{r^2\cos^3\phi} + \frac{1}{r}\frac{\partial U_0}{\partial \phi}\,, \tag{4.17}$$
$$\dot{r} = p_1\,, \qquad \dot{\theta} = \frac{4p_2}{r^2\cos^2\phi}\,, \qquad \dot{\phi} = \frac{4p_3}{r^2}\,,$$

with U_0 from (4.15).

Note that the reduced 3-body problem in the shape space is not a natural mechanical system, and in order to compute its curvature operator directly in the shape space one should first find the canonical splitting using (1.30), and then apply (1.31) to find the curvature operator related to the Hamiltonian vector field $\vec{H}$.

The easier way is to compute everything in the original space T^*M applying (2.44) and (2.45), as we did in section 4.1. Using (4.13) we rewrite the expression for the reduced Ricci curvature in the polar coordinates (r, θ, ϕ):

$$\rho_z^{\vec{H}} = \frac{12\,p_2^2}{r^4\cos\phi^2} + \frac{12\,p_3^2}{r^4} - \frac{1}{r^3}\left(U_0 + 2U_1\right)\,, \tag{4.18}$$

where

$$U_1 = \frac{1}{(1+\cos\theta\cos\phi)^{\frac{3}{2}}} + \frac{1}{(1+\cos(\theta+\frac{2\pi}{3})\cos\phi)^{\frac{3}{2}}}$$
$$+\frac{1}{(1+\cos(\theta+\frac{4\pi}{3})\cos\phi)^{\frac{3}{2}}}\,.$$

and $z = (p_1, p_1, p_3, r, \theta, \phi)$.

The function $\rho_z^{\vec{H}}$ inherits its singularities at the points of double and triple collisions from the potential energy U. The direct computation shows that the Euler points and isosceles configurations are the extrema of function $\rho_z^{\vec{H}}$. We omit this simple technical proof here.

In order to test the minimality of the Eight in the shape space we need to localize somehow the conjugate (to 0) times for the Jacobi curve $J_z^{\vec{H}}(\cdot)$, $z \in T^*\overline{M}$ along the projection of the Eight on $\overline{M}$. Note that the result of Chenciner and Montgomery ([14]) just guaranties the existence of the 8-shaped orbit. On the other hand, this orbit was already studied numerically (see, for example [24]). This is why the use of numerical methods looks quite natural.

4.2.4 Conjugate points: numerical analysis

First of all let us say a couple of words about the algorithms we were using. All the calculations were made in the double precision standard. Our numerical results are based on the numerical integration of equations (4.17) and their linearizion along the Eight, for this we used the standard Runge-Kutta method of 5-th order with accuracy 10^{-8}. For the calculation of integrals of action we were using the trapezoid method. The initial data for the Eight are taken from [14] for the period of the Eight $T = 6.32591398$.

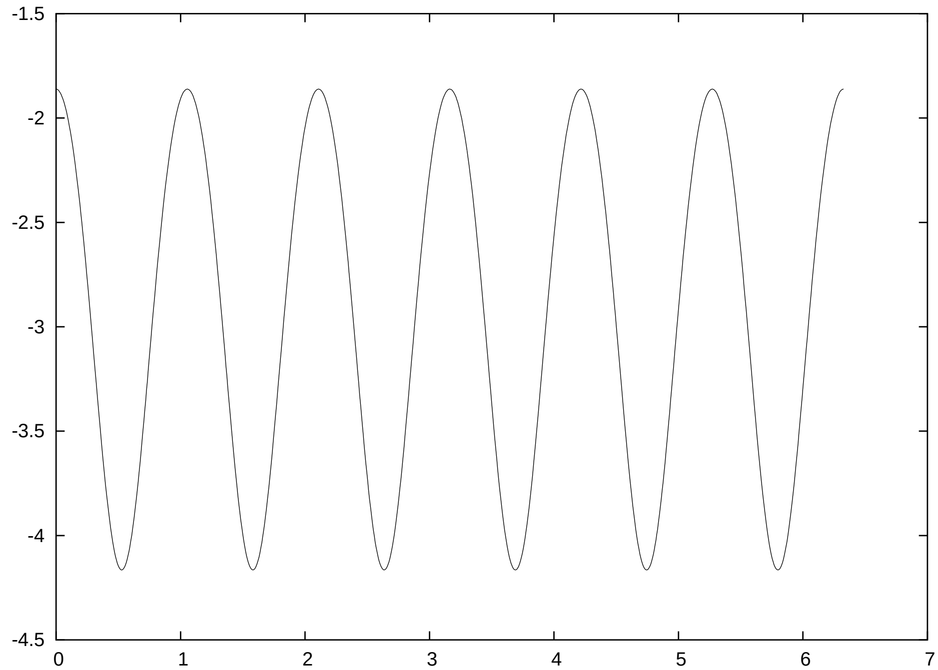

Figure 4.3: $\rho_{z_t}^{\vec{H}}$ along the Eight for $t \in [0, T]$.

In the Fig.4.3 it is shown the graph of the generalized Ricci curvature $\rho_z^{\vec{H}}$ along the Eight, the horizontal axis represents the time-line. All the symmetries of the Eight are easily visible: the graph of $\rho_z^{\vec{H}}$ consists of 12 equal pieces, with the extrema in the points of Euler and isosceles configurations. It turns out that $\rho_z^{\vec{H}} < -1.86$. On the other hand the numerical computation of the co-factors of the whole curvature operator $R_z^{\vec{H}}$ along the Eight shows that the corresponding quadratic form is sign - indefinite, so that in the present case the Theorem 1.3.5 gives no information about the location of conjugate points.

Observe that according to Definition 1.4.5 and Remark 1.4.6, the time t is conjugate to 0 for the Jacobi curve $J_{z_0}^{\vec{H}}(\cdot)$ attached at some point $z_0 \in T^*M$ if and only if

$$\psi^{\vec{h}}(t) = \det\left(\pi_*(e_*^{t\vec{h}}\partial_{p_1}), \ldots, \pi_*(e_*^{t\vec{h}}\partial_{p_n})\right) = 0\,. \tag{4.19}$$

Thus in order to find conjugate (to 0) points for the Jacobi curve attached at $z_0 \in T^*M$ and generated by a Hamiltonian vector filed $\vec{h}$ along some orbit of this field, it is enough to integrate the linearized system corresponding to the Hamiltonian system generated by $\vec{h}$ along the orbit of reference, taking the initial conditions for the linearized system from Π_{z_0}. Then the zeros of function $\psi^{\vec{h}}(t)$ give the conjugate times for the Jacobi curve generated by the field $\vec{h}$ along the orbit of reference.

In Fig.4.4 it is shown the graph of function $\psi^{\vec{H}}(t)$ along the projection of the Eight on the shape space. As usual, the horizontal axis corresponds to the time direction. We

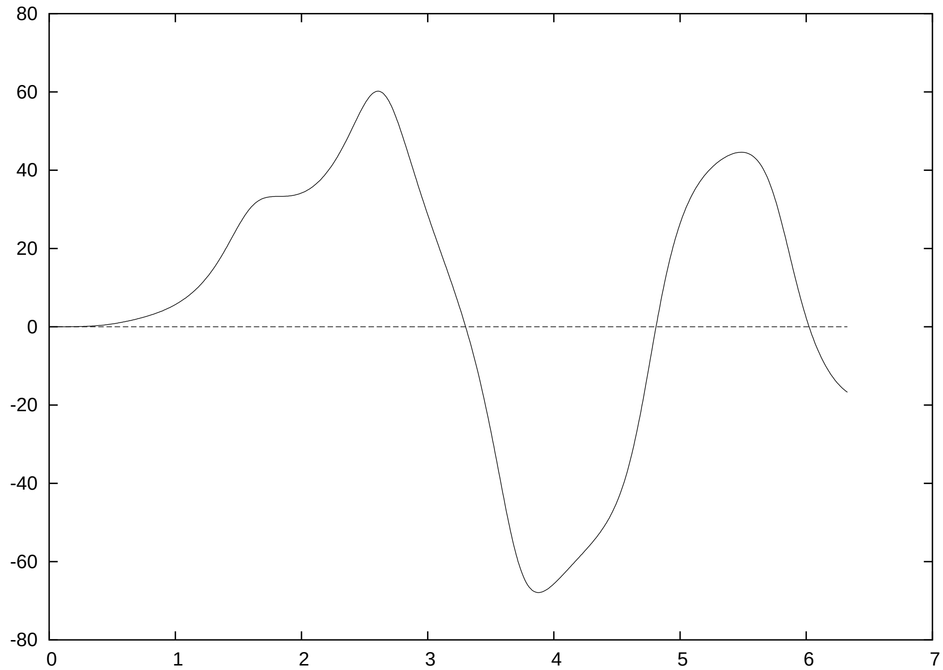

Figure 4.4: $\psi^{\vec{H}}(t)$ along the Eight on $[0,T]$

started integration from the point E_3 toward the point M_1^+. As it can be seen in Fig.4.4, the Jacobi curve of the reduced 3-body problem has 3 conjugate to 0 times along the Eight on $[0,T]$, the first one occurs after the first half of the period: $t_1 \approx 0.52T \in [T/2, 7T/12]$, it belongs to the piece of the Eight connecting the point E_3 with the point M_1^-

In view of Theorem 2.3.1 it is interesting to compare the conjugate to 0 times for the Jacobi curve along the Eight in the space M_0 generated by the field $\vec{h}$ defined by (4.6) and the conjugate to 0 times for the reduced Jacobi curve generated by the field $\vec{H}$ along the projection of the Eight on $\overline{M}$. It turns out that for the problem in M_0 the first conjugate to 0 time occurs at $\tau_1 \approx 0.76$ of the period. In Fig.4.5 we show the graphs of functions $\psi^{\vec{H}}(t)$ (dashed line) and $\psi^{\vec{h}}(\tau)$ (continuous line) for $t, \tau \in [0, 3T]$ found by numerical integration of the corresponding Hamiltonian systems.

The table below contains the approximate values of the conjugate (to 0) times. By τ_i we denote the zeros of conjugate (to 0) times along the Eight in M_0, and by t_i we denote the conjugate to 0 times of the reduced problem. It turns out that all this times have multiplicity 1.

i	1	2	3	4	5	6	7	8	9	10	11
$t_i/T \approx$	0.52	0.76	0.95	1.08	1.52	1.56	1.88	2.05	2.29	2.49	2.65
$\tau_i/T \approx$	0.76	0.95	1.08	1.42	1.54	1.88	2.05	2.28	2.45	2.65	

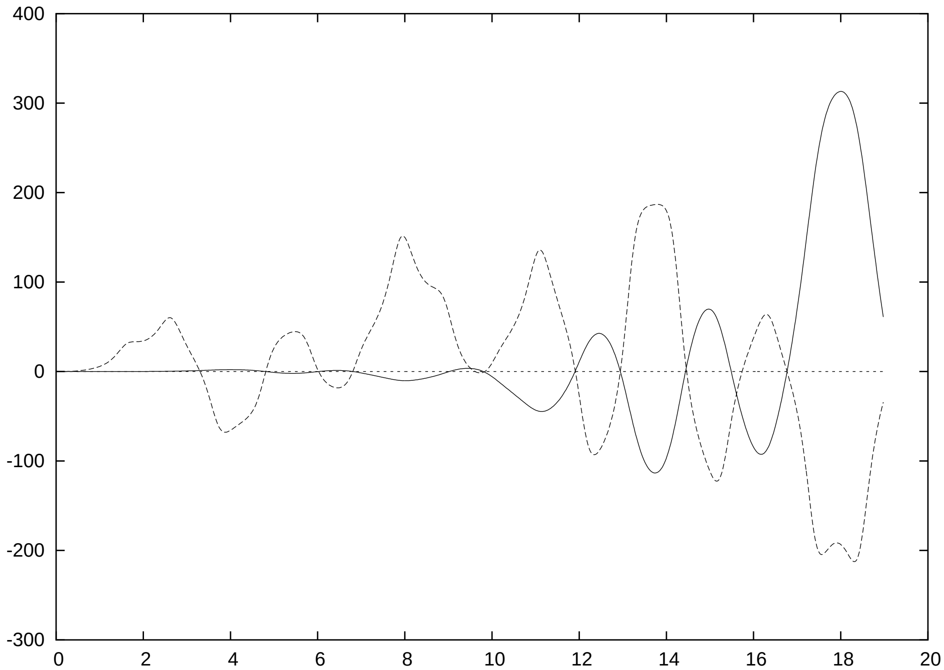

Figure 4.5: Conjugate to 0 times along the Eight on $[0, 3T]$

Observe that $t_i \leq \tau_i \leq t_{i+1}$, $1 \leq i \leq 10$, as it was expected by Theorem 2.3.1, and the set of conjugate times $\{t_i\}$ and $\{\tau_i\}$ are alternating.

4.2.5 New orbit in the shape space

Let us analyze more in detail the minimizing properties of the Eight in the shape space $\overline{M}$. The presence of conjugate points along the Eight in $\overline{M}$ means that after the half of the period it stops to be a minimizer of the action functional (4.14). The natural question is whether there exists another curve, which is a solution of equations (4.17), satisfies the same boundary conditions as the Eight at $t = 0$ and $t = 7T/12$, and has the action smaller than the Eight on the interval $[0, 7T/12]$, i.e. on the interval containing the first conjugate (to 0) time for the Jacobi curve along the Eight in the shape space.

In order to answer this question we consider the following boundary value problem. Denote by $\gamma_8(t) = (p_8(t), q_8(t)) \in T^*\overline{M}$ the lifting of the Eight in $T^*\overline{M}$. Let $\overline{T} = 7T/12$, where T is a period of the Eight. Our aim is to find a curve $\gamma(t) = (p(t), q(t)) \in T^*\overline{M}$ such that

$$\mathcal{A}^{\overline{T}}[\gamma] < \mathcal{A}^{\overline{T}}[\gamma_8]\,,$$

$$q(0) = q_8(0) = E_3\,, \qquad q(\overline{T}) = q_8(\overline{T}) = M_1^-\,. \tag{4.20}$$

.

The problem (4.20) is a standard two point boundary value problem with the restriction on the functional. The numerical algorithm we were using is based on the standard Newton-Raphson method (see [22]) with the following modification.

Recall that the standard m-dimensional problem for the Newton-Raphson algorithm consists of finding a vector $x \in \mathbb{R}^m$ that solves m equations of the type

$$F_i(x) = 0\,, \qquad i = 1, \ldots, m\,, \tag{4.21}$$

where the vector function $F = (F_1, \ldots, F_m)$ is supposed to be differentiable such that $\det\left(\frac{\partial F}{\partial x}\right) \neq 0$. Usually equations (4.21) represent a discrepancy vector for some boundary value problem. The Newton - Raphson method solves these equations by the shooting procedure. The new step is defined via the following linear equations:

$$x_{new} = x_{old} + \lambda \delta x_0\,,$$

$$\delta x_0 = -\left(\frac{\partial F}{\partial x}\right)^{-1} \cdot F\,,$$

so that the direction of the new step is the descent direction of the quadratic functional $f_0 = \frac{1}{2}\|F\|^2$. The parameter λ is to be chosen such that it brings f_0 sufficiently close to zero (we were using the standard back-tracking procedure to detect λ on each step of the algorithm).

In our case $m = 3$, the unknown variable is $x = p(0) \in \mathbb{R}^3$ and boundary conditions (4.20) define the discrepancy vector $F = q(\overline{T}) - q_8(\overline{T})$.

The main problem of the realization of such method in the present case is to find a "good" initial shooting direction, i.e. the initial direction such that the algorithm converges to some solution of (4.17) which gives a value of the action functional smaller than $\mathcal{A}^{\overline{T}}[\gamma_8]$.

We start the shooting procedure in the descent direction of $\mathcal{A}^{\overline{T}}$

$$x_1 = -\mu_0 \frac{\partial \mathcal{A}^{\overline{T}}}{\partial p_0}\,, \qquad p_0 = p(0)\,,$$

with constant $\mu_0 = 0.01$ being chosen experimentally to get the best convergence. The gradient of the action functional can be calculated via the following formula

$$\frac{\partial \mathcal{A}^{\overline{T}}}{\partial p_0} = p(\overline{T}) \cdot \left(\frac{\partial q}{\partial p_0}\right)(\overline{T})$$

where $\left(\frac{\partial p}{\partial p_0}, \frac{\partial q}{\partial p_0}\right)^T$ is the solution of the linearized (along the Eight) system corresponding to the Hamiltonian system (4.17).

The idea is to perform the Newton - Raphson algorithm in two steps. First we find some orbit which comes sufficiently close to the target point on the boundary and has the value of the action functional smaller than $\mathcal{A}^{\overline{T}}[\gamma_8]$. The problem here is to find a solution which is really different from the Eight: it was shown numerically that the Eight is stable (see [14] and references therein), so the problem of finding a good initial guess for the new solution is not that trivial. Using the initial data of such intermediate orbit as a first approximation for the standard Newton-Raphson procedure we hope to find the true solution of the boundary value problem (4.20).

In order to realize this idea on the first step of the algorithm we consider the following corrected functional

$$f = f_0 + \frac{\mu}{\mathcal{A}^{\overline{T}}[\gamma_8] - \mathcal{A}^{\overline{T}}[\gamma]} \,.$$

Here γ is the orbit corresponding the current initial data $\gamma(0) = (x, q_8(0))$, and μ is a correction weight to be chosen experimentally in order to get the best convergence, we were using $\mu = 0.01$. One can see that f is a positive function in the directions we are interested in, but the exact solutions of the boundary value problem (4.20) do not send it to zero, moreover, the second term makes it grow in the directions close to the Eight. The new shooting step corresponding to f is

$$\delta x = \delta x_0 - \frac{\mu}{(\mathcal{A}^{\overline{T}}[\gamma_8] - \mathcal{A}^{\overline{T}}[\gamma])^2} \left(\frac{\partial q}{\partial p_0}\right)^{-1} \cdot x(\overline{T}) \,.$$

We keep going with this step in the descent direction of the functional f as long as it brings us closer to the target point. In the realization of this algorithm we stopped when the difference between two consecutive values of f became 10^{-5}. Then we were using the Newton-Raphson procedure with the standard functional f_0 until $f < 10^{-13}$.

The presented method allowed us to find an orbit which is quite different from the Eight (see dashed curve in Fig.4.6 and Fig.4.7). The new action is $\mathcal{A}^{\overline{T}}[\gamma_{new}] \approx 14.2371416$, which is smaller than $\mathcal{A}^{\overline{T}}[\gamma_8] \approx 14.2491093$.

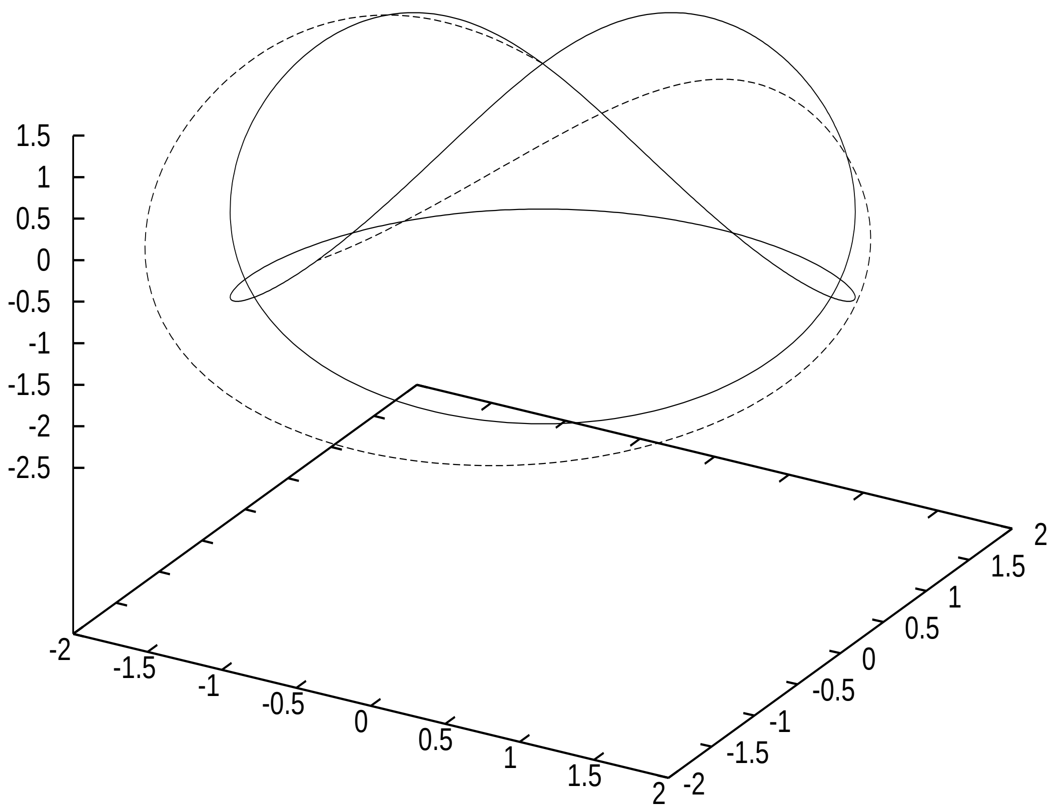

Figure 4.6: The Eight and the new curve in the shape space

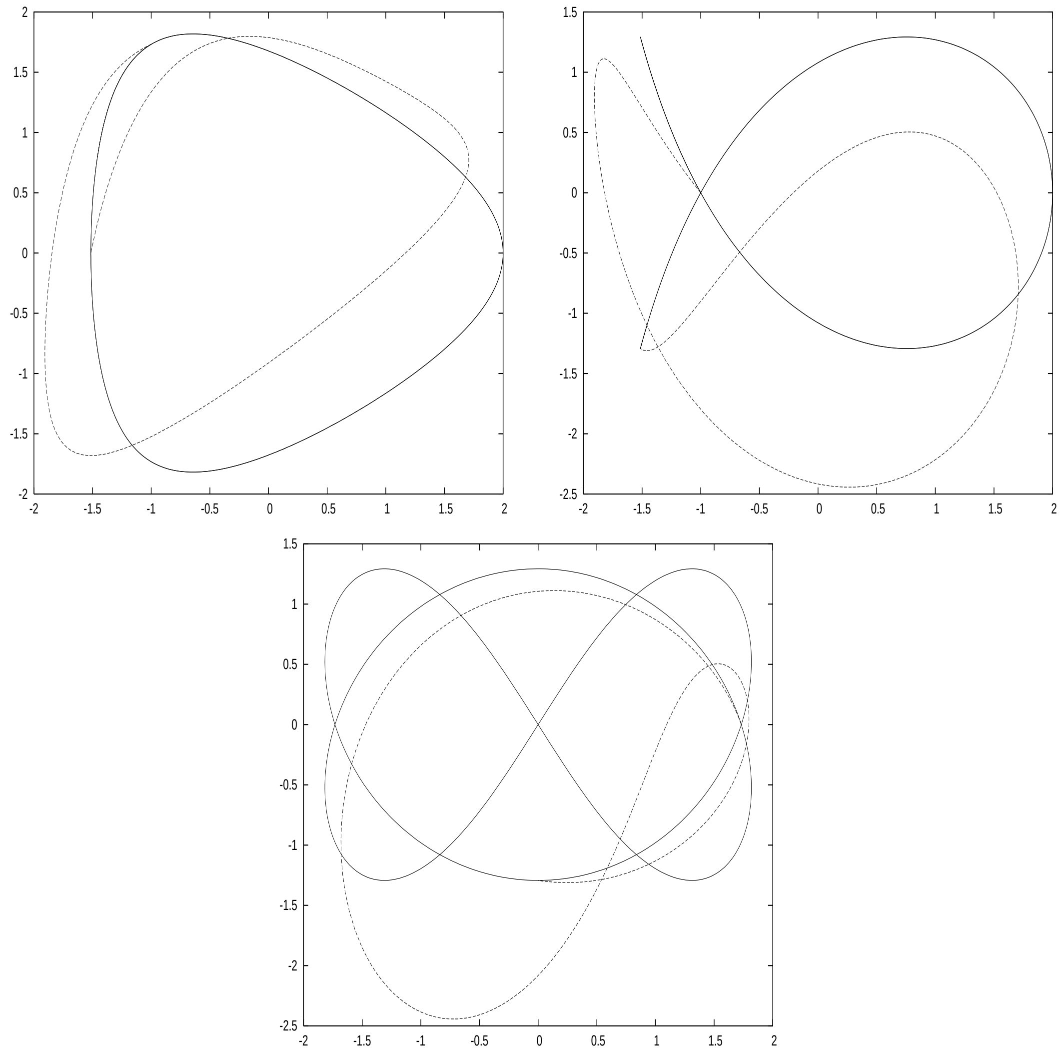

Figure 4.7: Projections of the Eight and the new curve on the coordinate planes in the shape space

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