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**GRAVITY BEYOND GENERAL RELATIVITY:
NEW PROPOSALS AND THEIR PHENOMENOLOGY**

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*“The most beautiful thing we can experience is the mysterious.
It is the source of all true art and science.”*

Albert Einstein

Abstract

GRAVITY BEYOND GENERAL RELATIVITY: NEW PROPOSALS AND THEIR PHENOMENOLOGY

by **Daniele Vernieri**

This Thesis is devoted to the study of phenomenologically viable gravitational theories, in order to address the most pressing open issues both at very small and very large energy scales. Lovelock's theorem singles out General Relativity as the only theory with second-order field equations for the metric tensor. So, two possible ways to circumvent it and modify the gravitational sector are taken into account. The first route consists in giving up diffeomorphism invariance, which generically leads to extra propagating degrees of freedom. In this framework Hořava gravity is discussed, presenting two restrictions, called respectively “projectability” and “detailed balance”, which are imposed in order to reduce the number of terms in the full theory. We introduce a new version of the theory assuming detailed balance but not projectability, and we show that such theory is dynamically consistent as both the spin-0 and spin-2 gravitons have a well behaved dynamics at low-energy. Moreover three-dimensional rotating black hole solutions are found and fully studied in the context of Hořava gravity, shedding light on its causal structure. A new concept of black hole horizon, dubbed “universal horizon”, arises besides the usual event horizon one, since in Lorentz-violating gravity theories there can be modes propagating even at infinite speed. The second route which is considered, consists in adding extra fields to the gravitational action while diffeomorphism invariance is preserved. In this respect we consider the less explored option that such fields are auxiliary fields, so they do not satisfy dynamical equations but can be instead algebraically eliminated. A very general parametrization for these theories is constructed, rendering also possible to put on them very tight, theory-independent constraints. Some insight about the cosmological implications of such theories is also given. Finally in the conclusions we discuss about the future challenges that the aforementioned gravity theories have to face.

Preface

The main goal of this Thesis is to present the research work which has been conducted mainly in SISSA - International School for Advanced Studies, during the period November 2010 - October 2014. The Thesis is organized as follows.

In the Introduction the foundations of General Relativity are given, and its limitations and problems are discussed. A first way to modify General Relativity is by considering theories characterized by the presence of higher-order derivatives. Nevertheless in doing so, one has to take into account the theorem of Ostrogradski, which highlights from a very general theoretical perspective why the laws of Nature are expected to contain not more than second time derivatives of the fundamental fields. In fact, when higher-order time derivatives are taken into account, the Hamiltonians of the related physical systems are shown to be unbounded from below, leading to pathological instabilities. If one is willing to modify the gravitational sector avoiding such instabilities, two choices are left: breaking diffeomorphism invariance or adding extra fields.

In Chapter 2 the first option is considered. In this context Hořava gravity is discussed and its connection with Einstein-Æther theory is also shown in the infrared limit, once the æther vector field is restricted to be hypersurface orthogonal at the level of the action.

In Chapter 3, the various restricted versions of Hořava gravity are presented, focusing on two simplifications, called respectively “projectability” and “detailed balance”, which have been imposed in order to limit the proliferation of terms which are present in the full theory. A new version of the theory which assumes detailed balance but not projectability is introduced, and it is shown to be dynamically consistent.

In Chapter 4, rotating black hole solutions with anti-de Sitter asymptotics are found and then fully studied in the context of three-dimensional Hořava gravity, shedding light on the causal structure of the theory. In particular the existence of a new kind of horizon, named “universal horizon”, is highlighted for Lorentz-violating gravity theories, where the intrinsic notion of a black hole is modified due to the presence of degrees of freedom propagating at any speed, even infinite. Then the notion of event horizon relinquishes its role as an absolute causal boundary.

Chapter 5 is devoted to the exploration of a further possibility to modify General Relativity, which is to consider extra non-dynamical fields in the gravitational action. A general parametrization for this class of theories is built through a gradient expansion up to fourth-order derivatives, and it is shown that only two free parameters control all the theories within this class. This approach can be used to put very tight, theory-independent constraints on such theories, as it is demonstrated by using the Newtonian limit as an example. Particular attention is also paid to the implications they can have at very large (cosmological) length scales, where these theories generically lead to a modified phenomenology with respect to General Relativity.

Finally, in Chapter 6 the conclusions of this work are given, with remarks and comments about the future challenges that the aforementioned gravity theories have to face.

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Collaborations

This Thesis contains the results obtained through the scientific collaborations stated below, except where explicit reference is made to the results of others.

The content of this Thesis is based on the following research papers published in refereed Journals or refereed Conference Proceedings, and on a paper in preparation:

1) **“Hořava-Lifshitz Gravity: Detailed Balance Revisited”**

D. Vernieri and T. P. Sotiriou

Phys. Rev. D **85**, 064003 (2012) [arXiv:1112.3385 [hep-th]]

2) **“Hořava-Lifshitz Gravity with Detailed Balance”**

D. Vernieri and T. P. Sotiriou

J. Phys. Conf. Ser. **453**, 012022 (2013) [arXiv:1212.4402 [hep-th]]

3) **“Gravity with Auxiliary Fields”**

P. Pani, T. P. Sotiriou and D. Vernieri

Phys. Rev. D **88**, 121502 (2013) [arXiv:1306.1835 [gr-qc]]

4) **“Rotating Black Holes in Three-Dimensional Hořava Gravity”**

T. P. Sotiriou, I. Vega and D. Vernieri

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5) **“Phenomenology of Gravity with Auxiliary Fields”**

P. Pani, T. P. Sotiriou and D. Vernieri

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Dedicated to Noemi ...

Chapter 1

Introduction

1.1 Foundations of General Relativity

1.1.1 Lovelock's Theorem

General Relativity (GR) is the cornerstone of our theoretical knowledge about the gravitational interaction, and its predictions are in excellent agreement with all weak-field experiments.

In 1915 – 1916 Einstein completed the formulation of GR, whose field equations have the following form

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad (1.1)$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (1.2)$$

is the so called “Einstein tensor”, $R_{\mu\nu}$ and R are respectively the Ricci tensor and Ricci scalar of the metric $g_{\mu\nu}$, G_N is the Newton's constant and $T_{\mu\nu}$ is the matter stress-energy tensor. Moreover the speed of light c has been set to 1.

In 1922, a theorem developed by Cartan [1], Weyl [2] and Vermeil [3] provided further motivation for using $G_{\mu\nu}$ as the left-hand side of the gravitational field equations (1.1). They found that the Einstein tensor of a metric $g_{\mu\nu}$, is the only 2-covariant tensor, up to the addition of a cosmological constant term $\Lambda g_{\mu\nu}$, which satisfies the following properties:

1. The components $G^{\mu\nu}$ are functions of the coefficients of the metric tensor $g_{\mu\nu}$, its first and second derivatives, *i.e.*,

$$G^{\mu\nu} = G^{\mu\nu}(g_{\alpha\beta}, \partial_\sigma g_{\alpha\beta}, \partial_\gamma \partial_\sigma g_{\alpha\beta}); \quad (1.3)$$

2. It is divergence-free, *i.e.*,

$$\nabla_\mu G^{\mu\nu} = 0, \quad (1.4)$$

where ∇_μ is the covariant derivative defined as the Levi-Civita connection of the metric $g_{\alpha\beta}$;

3. It is symmetric, *i.e.*,

$$G^{\mu\nu} = G^{\nu\mu}; \quad (1.5)$$

4. The components $G^{\mu\nu}$ are linear functions in the second derivatives of $g_{\alpha\beta}$.

It is interesting to notice that none of these remarks depend on the dimension n of the underlying pseudo-Riemannian manifold.

Later on, in 1971, the theorem was revisited and greatly improved by Lovelock [4, 5] who demonstrated that once the assumption 4 is relaxed, additional terms have to be considered in higher dimensions [4]. Moreover he also showed that in four dimensions the assumptions 3 and 4 are unnecessary conditions to prove the theorem [5].

Since the very first days of the theory, this theorem has represented one of the milestones for the construction and the theoretical justification of the field equations of GR.

1.1.2 Action of General Relativity and Field Equations

With the mild additional requirement that the field equations for the gravitational field and the matter fields be derived by a diffeomorphism-invariant action, where no fields other than the metric and the matter are present, the arguments above single out in four dimensions the action of GR with a cosmological constant term:

$$S_{GR} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_M[g_{\mu\nu}, \psi_M], \quad (1.6)$$

where G_N is the Newton's constant, g the determinant of the spacetime metric $g_{\mu\nu}$, R is the Ricci scalar of the metric, Λ is the bare cosmological constant and S_M is the matter action. Moreover ψ_M collectively denotes the matter fields which couple to $g_{\mu\nu}$, so that S_M is understood to reduce to the Standard Model (SM) action in the local frame.

The variation of the action (1.6) with respect to the metric gives rise to the field equations of GR in presence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad (1.7)$$

where

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \quad (1.8)$$

Using the contracted Bianchi identity one gets from Eq. (1.7) that $T_{\mu\nu}$ has to be divergence-free, that is

$$\nabla^\mu T_{\mu\nu} = 0. \quad (1.9)$$

1.1.3 Stress-Energy Tensor Conservation and Geodesic Motion

We want to demonstrate now that the conservation of the stress-energy tensor implies geodesic motion of test particles. For this purpose let us consider for example non-interacting particles moving in spacetime, whose idealized description is given in terms of a pressure-less perfect fluid (dust). The stress-energy tensor of such fluid can then be written as

$$T^{\mu\nu} = \rho u^\mu u^\nu, \quad (1.10)$$

where ρ is the energy density and u^μ the four-velocity of an infinitesimal volume element of the fluid. Asking that the stress-energy tensor is conserved, *i.e.*, that Eq. (1.9) holds, one gets

$$u_\nu \nabla^\mu (\rho u_\mu) + \rho u_\mu \nabla^\mu u_\nu = 0. \quad (1.11)$$

Since the two terms in Eq. (1.11) are 4-orthogonal one can deduce the continuity equation in a general curved background (in analogy with the one for a Newtonian fluid in a flat spacetime), *i.e.*,

$$\nabla^\mu (\rho u_\mu) = 0. \quad (1.12)$$

We are then left only with the second term in Eq. (1.11). As $u^\mu = dx^\mu/d\tau$, where x^μ denote the coordinates and τ is the proper time along the path of the particles, we immediately get

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma_{\sigma\mu}^\nu \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} = 0, \quad (1.13)$$

where $\Gamma_{\sigma\mu}^\nu$ are the Christoffel symbols associated to the Levi-Civita connection ∇^ν . We have thus found the geodesic equation describing the free-fall motion of the test particles constituting the fluid.

This means that the conservation of the stress-energy tensor, $\nabla^\mu T_{\mu\nu} = 0$, implies that test particles follow geodesics. It therefore guarantees that the weak equivalence principle, *i.e.*, the universality of free-fall, holds.

1.1.4 Diffeomorphism Invariance of the Matter Action

The action of GR (1.6) is diffeomorphism invariant when considered in vacuum, where no matter fields are present. Instead, if we want the whole action to be diffeomorphism invariant we have to ask the matter action S_M to be as well [6]. This will reveal another

interesting aspect concerning the conservation of the stress-energy tensor, as we shall see below.

We can write the variation of S_M under a diffeomorphism as

$$\delta S_M = \frac{\delta S_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\delta S_M}{\delta \psi_M} \delta \psi_M. \quad (1.14)$$

Assuming the matter fields are on shell, $\delta S_M / \delta \psi_M$ will vanish for any variation since the gravitational part of the action doesn't involve the matter fields. So, in order to have a diffeomorphism invariant action we must ask that

$$\frac{\delta S_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = 0. \quad (1.15)$$

If the vector field V^μ is the diffeomorphism generator, then the infinitesimal change of the metric is given by the Lie derivative \mathcal{L}_V acting along V^μ . After some manipulations one gets

$$\delta g^{\mu\nu} = \mathcal{L}_V g^{\mu\nu} = 2\nabla^{(\mu} V^{\nu)}, \quad (1.16)$$

where the round brackets denote symmetrization.

Then, from Eqs. (1.15)-(1.16), and using the definition (1.8) for the stress-energy tensor one gets

$$\int d^4x \frac{\delta S_M}{\delta g^{\mu\nu}} \nabla^\mu V^\nu = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \nabla^\mu V^\nu = 0. \quad (1.17)$$

Finally integrating by parts and asking that V^μ vanishes on the boundary yields

$$\int d^4x \sqrt{-g} (\nabla^\mu T_{\mu\nu}) V^\nu = 0. \quad (1.18)$$

Demanding that Eq. (1.18) holds for any diffeomorphism generated by an arbitrary vector field V^μ that vanishes on the boundary, one gets exactly the stress-energy tensor conservation equation $\nabla^\mu T_{\mu\nu} = 0$.

This calculation shows that asking for a diffeomorphism invariant matter action allows one to link the requirement that the matter fields are on shell with the universality of free-fall. Moreover the results obtained above rely only on the matter action, so they are not specific to GR but apply to any theory.

1.2 General Relativity and Quantum Field Theory

The gravitational interaction is so weak compared to the other interactions that the characteristic energy scale at which one would expect to experience relevant modifications to the classical gravity picture is the Planck energy scale of 10^{19} GeV. What happens to the gravitational interaction at these extremely high energies where we expect that

quantum effects should become relevant?

One needs a theory which is able to explain these effects and at the same time to recover the classical results at sufficiently small energies.

However, at the moment, there are many conceptual reasons for which GR and Quantum Field Theory (QFT) have not been found to merge consistently in an unified picture.

QFT is naturally built on a fixed rigid (flat or curved) spacetime inhabited by quantum fields. On the other hand GR considers the spacetime as being dynamical, without taking into account the quantum nature of matter fields.

Another conceptual difference is represented by the Heisenberg uncertainty principle which is at the basis of any quantum theory while being completely absent in GR which is instead a classical theory.

It is interesting to note that GR considers the background to be dynamical, but at the same time, when such a configuration is being determined by the matter content (for simplicity we can think about a stationary spacetime), then the knowledge of the spacetime can furnish a detailed record of all the information about past, present and future.

On the contrary, in QFT, although one works within a fixed background scenario, there is, for reasons intrinsic to the theory, a degree of uncertainty for the position of any event in the spacetime.

1.2.1 The Problem of Renormalizability in Gravity

GR, by its own nature, is a classical geometric description of spacetime, and by straightforward inspection it reveals all its limits at quantum scales. At such scales it is not adequate to describe the gravitational interaction or the spacetime itself (depending on the perspective), and one cannot consistently construct its quantum counterpart by means of conventional quantization techniques.

In fact, in 1962 Utiyama and De Witt [7] showed that renormalization at one-loop demands that the gravitational action of GR should be supplemented by higher-order curvature terms, such as R^2 and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, so that these new actions are indeed renormalizable.

Later on, in 1977, Stelle [8] showed that higher-order theories are in effect renormalizable but a very high price must be paid for that: they contain ghost degrees of freedom and are, therefore, not unitary. This happens because the addition of higher-order curvature invariants in the action leads to higher-order time derivatives. If one allows the equations of motion to be higher than second-order partial differential equations, more degrees of freedom would be generically introduced, and these can be pathological.

The problem of renormalizability represents a strong motivation to consider modifications of GR, and the corrections (counter-terms) one adds to let GR be renormalizable must be such as not to call into play unphysical degrees of freedom.

1.2.2 The Cosmological Constant

It was Einstein who, in 1917, introduced for the first time a cosmological constant term in his equations. In fact, he was attempting to apply his new theory to the whole Universe, and his guiding principle was that the universe is static: “The most important fact that we draw from experience is that the relative velocities of the stars are very small as compared with the velocity of light” [9].

Soon after Einstein developed his static theory (which is unstable under small perturbations), observations by Edwin Hubble suggested that the Universe was expanding. These observations were also found to be consistent with a cosmological solution to the original GR equations that had been discovered by the mathematician Friedmann. Einstein then retracted his proposal of a cosmological constant, referring to it as “biggest blunder”. However, the cosmological constant remained a subject of theoretical and empirical interest, and it is even more interesting today that we know with certainty that the Universe is expanding.

If we move the bare cosmological constant term (that we call here Λ_b) in Eq. (1.7) to the right-hand side, we can consider it as a kind of energy-momentum tensor given by

$$T_{\mu\nu}^b = -\rho_{\Lambda_b} g_{\mu\nu}, \quad (1.19)$$

where

$$\rho_{\Lambda_b} = \frac{\Lambda_b}{8\pi G_N}. \quad (1.20)$$

Apart from this bare contribution to the cosmological constant, we also expect another very large one coming from particles that we know to exist in the SM. In order to understand this, we can take the simple example of a scalar field ϕ with potential energy $V(\phi)$. The matter action for such a scalar field can be written as

$$S_M^\phi = - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (1.21)$$

and the corresponding stress-energy tensor, using the definition in Eq. (1.8), is found to be

$$T_{\mu\nu}^\phi = \partial_\mu \phi \partial_\nu \phi - \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right] g_{\mu\nu}. \quad (1.22)$$

The configuration of minimal energy is the one with no contribution from the kinetic term, that is when $\partial_\mu\phi = 0$. In such a configuration the stress-energy tensor becomes

$$T_{\mu\nu}^\phi = V(\phi_0)g_{\mu\nu}, \quad (1.23)$$

where ϕ_0 is the value of the scalar field which minimizes $V(\phi)$.

Since there is no reason in principle for which $V(\phi_0)$ should vanish, one should expect that matter fields give a non-zero vacuum energy contribution $\langle\rho\rangle$ to the effective cosmological constant. As we know, vacuum carries energy and momentum even in the ground state, then one can estimate the size of $\langle\rho\rangle$ by considering matter fields of the SM as a collection of harmonic oscillators sitting at their zero-point energy. So, considering the vacuum energy as the sum of the energies E_k of the ground states oscillations of all these fields, one gets

$$\langle\rho\rangle = \int_0^{\Lambda_{UV}} \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar E_k \sim \int_0^{\Lambda_{UV}} dk k^2 \sqrt{k^2 + m^2} \sim \Lambda_{UV}^4, \quad (1.24)$$

where Λ_{UV} is the cut-off of the theory. Since the SM is extremely well tested up to the weak energy scale $\Lambda_{UV} \sim 1\text{TeV}$, one finds a theoretical expectation value for the vacuum energy to be

$$\Lambda_{\text{th}} \sim (\text{TeV})^4 \sim 10^{-60} M_{\text{pl}}^4. \quad (1.25)$$

On the other side, if we consider the Universe as described by an effective local quantum field theory down to the Planck energy scale, we would expect the cosmological constant to have a natural scale of the order of M_{pl}^4 .

The observed value for the cosmological constant in terms of Planck units is of the order of $10^{-120} M_{\text{pl}}^4$. It differs from both the above estimates and from any estimate provided with a realistic cut-off, at best for 60 orders of magnitude. This falls off under the name of “the old cosmological constant problem”: a Universe with a large cosmological constant would expand too fast, so preventing galaxy formation [10]. However, there is no known natural way to derive from particle physics the tiny observed value for the cosmological constant. It could be that the bare cosmological constant Λ_b appearing in the Einstein’s field equations gives a contribution that exactly cancels Λ_{th} , leaving behind the tiny residual we currently get from observations. Nevertheless a no-go theorem by Weinberg proves, under certain assumptions, that it cannot really happen dynamically [9].

In Refs. [11, 12] Weinberg’s no-go theorem is evaded by relaxing the condition of Poincaré invariance in the scalar sector, and it is shown that the cosmological constant problem can be solved classically in the context of scalar-tensor theories.

A new promising mechanism which removes the vacuum energy contributions from the field equations leaving behind a naturally small effective cosmological constant in our Universe, has been recently provided in Refs. [13, 14]. Moreover, since no new propagating mode appears in such a theory, the latter looks just like standard GR, but without

a large cosmological constant.

1.3 General Relativity, Astrophysics and Cosmology

1.3.1 Dark Matter

The theory of Big Bang Nucleosynthesis (BBN) [15–18] and the Cosmic Microwave Background (CMB) radiation data [19] imply that no more than about 5% of the energy density of the universe can consist of any material with which we are presently familiar, and only a fraction of that is observed. What remains is almost the whole part, and represents the mirror of our ignorance about what’s going on at large length scales.

In order to make GR in agreement with the observed dynamics of galaxies and galactic clusters we must postulate that about five times the mass of ordinary matter comes in the form of non-baryonic Cold Dark Matter (CDM).

The first evident proof for the existence of DM dates back to 1933, when Zwicky [20] observed that the visible mass was not sufficient by its own to explain the individual dispersion velocity of galaxies in the Coma Cluster. Later on, starting from 1970, the observations of the rotation curves of the galaxies provided further confirmation for the existence of this dark component [21, 22]. In particular, according to Newtonian gravity, it is found that for a galaxy the potential generated by the matter distribution should have the profile

$$V(r) = -G_N \frac{M(r)}{r}, \quad (1.26)$$

where $M(r)$ is the total mass at a fixed radius r . So we would expect a fall off of the potential at sufficiently large radii. Nevertheless observations show that well beyond the galaxy core the potential follows an almost constant profile [23], and such a behaviour can be explained by taking into account an extra mass contribution

$$M(r) \propto r. \quad (1.27)$$

From this, one obtains the picture of a galaxy as an astrophysical object entirely surrounded by a DM halo and residing in its core, providing a motivation for the existence of this exotic non-baryonic matter component. Moreover further investigations of the features of spiral galaxies have also led to a very challenging discovery. In fact the rotation velocity profiles of stars in such galaxies were found to share a universal profile, leading to the so-called Universal Rotation Curves (URC) paradigm [24, 25].

The candidate particles [26–28] which have been sought for a long time, can be divided in two main groups: hot DM, made by non-baryonic particles which move (ultra-)relativistically, and CDM, made by non-baryonic particles which move non-relativistically.

The most favourite particle belonging to the first class is the neutrino. However, because strong motivations coming from astrophysical observations suggest that DM has clumped to form structures on rather small length scales, it cannot consist of particles moving at very high velocities as they would indeed suppress the clustering [26, 29, 30]. Hence, CDM is currently the most favoured nominee. Its non-baryonic nature seems to find the perfect candidate in the Super Symmetric (SUSY) extension of the SM. In fact, the lightest SUSY particle, the neutralino, is stable, weakly interacting with the particles of the SM, and is able to provide the correct structure formation process and the right observed DM relic abundance. Although there are several possible candidates for what DM might be, until now no Earth-bound laboratory has yet succeeded in detecting it, not even the Large Hadron Collider (LHC), which is at the moment the preferred way to directly detect DM. In fact, as DM is very weakly interacting, it is very unlikely to detect it via interaction with SM particles, while one could try to create it directly by means of high energy collisions between SM particles. Another way to detect DM particles may be through the detection of the products of their annihilation in high density regions like the core of the Sun or DM dominated galaxies. But, as already stated, nothing has been detected until now.

1.3.2 Dark Energy

The mystery of DM is not the only open issue GR has to face. In fact the most challenging problem for which a satisfactory explanation has not yet been found comes at very large length scales.

In order for GR to be in agreement with the observed ongoing Universe accelerated expansion, that is the Hubble plots of distant Type Ia supernovae [31–43], with the power spectrum of anisotropies in the CMB [19, 44–52] and with Large Scale Structure (LSS) surveys [53–60], one must accept the existence of an additional exotic component, usually referred to as “Dark Energy” (DE), that must be about fourteen times larger than that of ordinary matter.

If this picture is accepted to be true, then it would mean that about the 95% of the current Universe’s energy exists in forms which have so far only been detected gravitationally, but that results to be completely obscure to all the other forms of detection we have experimented until now.

The standard cosmological model, referred to as Λ CDM, is able to reproduce the ongoing accelerated expansion phase of the Universe in the framework of GR, through the presence of an effective cosmological constant Λ . The main assumption used to build up this model, is that at length scales larger than about 100 Mpc, corresponding to the largest ever observed structures, the Universe is nearly homogeneous and isotropic. This

assumption at very large scales is confirmed by the high level of isotropy of the CMB radiation and by the distribution of LSS and it is often referred to as the Copernican or Cosmological Principle. Moreover these symmetries are only seen by observers who are at rest with respect to the Universe expansion, otherwise observers who are moving with respect to this frame would experience a dipolar anisotropy.

The isotropy and homogeneity of the Universe as seen by such a comoving observer, can be translated into the following ansatz for the metric written in spherical coordinates:

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (1.28)$$

This line-element is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, where $a(t)$ is the scale factor, which is the only degree of freedom describing the geometry of the Universe, and k is the spatial curvature, normalized in such a way that it can take only three values: -1 for an hyperspherical Universe, 0 for a spatially flat one, and 1 for an hyperbolic one.

The stress-energy tensor for a perfect fluid is written as

$$T^{\mu\nu} = (P + \rho) u^\mu u^\nu - P g^{\mu\nu}, \quad (1.29)$$

where u^μ is the four-velocity of an observer comoving with the fluid, while P and ρ are respectively the pressure and density of the fluid. Inserting the metric (1.28) and the stress-energy tensor (1.29) into Eq. (1.7), one gets the so-called Friedmann equations

$$H^2 = \frac{8\pi G_N}{3} \rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \quad (1.30a)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G_N}{3} (\rho + 3P), \quad (1.30b)$$

where $H \equiv \dot{a}/a$ is the Hubble function, and the overdot indicates differentiation with respect to the coordinate time t .

Considering a model for the Universe where the spatial curvature k is zero, and the matter content is dominated by the cosmological constant, Friedmann equations admit as solution

$$a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t}, \quad (1.31)$$

where a_0 is the value of the scale factor at the present time.

This solution describes a Universe undertaking a phase of accelerated expansion since

$$\ddot{a} > 0, \quad (1.32)$$

and it is usually referred to as de Sitter (dS) Universe. So in the framework of GR it is possible to get an accelerated expansion of the Universe, as provided by current

cosmological observations. Nevertheless, as already mentioned in Sec. (1.2.2), there is a huge discrepancy between the value observed for Λ and the one predicted by particle physics. So, despite of the success of Λ CDM model, one is brought to explore further scenarios.

In the absence of a cosmological constant term, one can see from Eq. (1.30b) that the only way to get a positive acceleration, is to have a perfect fluid for which

$$(\rho + 3P) < 0 \quad \implies \quad P < -\frac{1}{3}\rho. \quad (1.33)$$

Assuming that the density is positive definite, this implies that the overall pressure of the fluid must be negative, acting as a repulsive force. Since it is not possible for any kind of baryonic matter to satisfy Eq. (1.33), the only way to have a period of accelerated expansion of the Universe in the framework of GR, where the effective value of the cosmological constant is zero, is to consider some new form of matter fields with very special characteristics.

Let us consider for example an homogeneous scalar field $\phi = \phi(t)$, whose dynamics is described by the action given in Eq. (1.21). By means of the stress-energy tensor (1.22), one finds that the scalar acts as a perfect fluid with an equation of state $P_\phi = w_\phi \rho_\phi$, where the energy density ρ_ϕ and pressure P_ϕ are respectively given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (1.34a)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (1.34b)$$

and the barotropic index w_ϕ is found to be

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}, \quad (1.35)$$

which is generically time varying.

In order to obtain a negative pressure and also to source the observed expansion, we must have $w_\phi \simeq -1$, which requires a very slowly-rolling field such that $\dot{\phi}^2 \ll V(\phi)$. Notice that this condition is very similar to the one required for inflation, even if the latter also places a constraint for $\ddot{\phi}$ which is asked to be negligible.

These dynamical models, dubbed as DE or quintessence models, represent the simplest extension of Λ CDM and have been extensively studied [61–68]. Their most interesting feature is that they have a very rich phenomenology. It is indeed possible, by suitably choosing the potential, that the energy density of the cosmological scalar field is such that the Universe history during radiation/matter eras is reproduced at early times, then growing to dominate the energy budget at very late times [69–75].

So, for the DE problem as well some extra mysterious exotic field must be taken into

account in order to explain the aforementioned accelerated expansion of the Universe. Anyway this is not the unique possibility one can think about. Indeed one can also argue that these observed cosmic phenomena constitute further arguments in favour of a modification of GR, and it is then possible to predict them without the need for DE, and perhaps even without the need for DM.

1.4 The No-Go Theorem of Ostrogradski

On the way of modifying GR, one has to take into account a very powerful theorem which constrains fundamental theory: the theorem of Ostrogradski [76].

It essentially shows that Newton was right to expect that the laws of physics involve no more than two time derivatives of the fundamental dynamical variables. In fact it is demonstrated that the Hamiltonians associated with Lagrangians depending upon more than one time derivative have a linear dependence on the momenta. So, one gets physical systems whose energy states are not bounded from below, and they are therefore unstable.

The theorem is very general and we can briefly present it in the case of a single, one dimensional point particle whose position as a function of time is given by $q(t)$, and the Lagrangian of the system involves N time derivatives [77].

Let us consider a Lagrangian $L(q, \dot{q}, \dots, q^{(N)})$, which depends up to the N -th derivative of $q(t)$. The Euler-Lagrange equations can be written as

$$\sum_{i=0}^N \left(-\frac{d}{dt}\right)^i \frac{\partial L}{\partial q^{(i)}} = 0, \quad (1.36)$$

which in general contains $q^{(2N)}$. The assumption that $\partial L/\partial q^{(N)}$ depends upon $q^{(N)}$ is known as *non-degeneracy*. So, we need $2N$ coordinates to build up the canonical phase space, and we use the ones Ostrogradski singled out:

$$Q_i \equiv q^{(i-1)}, \quad P_i \equiv \sum_{j=i}^N \left(-\frac{d}{dt}\right)^{j-i} \frac{\partial L}{\partial q^{(j)}}. \quad (1.37)$$

If the Lagrangian is not degenerate, then we can solve for $q^{(N)}$ in terms of P_N and the Q_i 's. This means that there exists a function $\mathcal{A}(Q_i, P_N)$ such that

$$\left. \frac{\partial L}{\partial q^{(N)}} \right|_{q^{(i-1)}=Q_i; q^{(N)}=\mathcal{A}} = P_N(Q_i, \mathcal{A}), \quad (1.38)$$

where the existence (locally) of $\mathcal{A}(Q_i, P_N)$ is justified by the implicit function theorem. It is now possible to write the Ostrogradski Hamiltonian as obtained by a Legendre

transformation:

$$\begin{aligned} H &\equiv \sum_{i=1}^N P_i q^{(i)} - L \\ &= P_1 Q_2 + P_2 Q_3 + \dots + P_{N-1} Q_N + P_N \mathcal{A} - L(Q_1, \dots, Q_N, \mathcal{A}). \end{aligned} \quad (1.39)$$

We can immediately observe that the Hamiltonian above is linear in P_1, P_2, \dots, P_{N-1} and hence, being not bounded from below, it is obviously unstable.

Notice that GR is not plagued by such instability because its Lagrangian is linear in the second-order derivatives of the metric, then it violates the non-degeneracy assumption.

This is one way to avoid the Ostrogradski instability, *i.e.* by considering a degenerate Lagrangian. However there is also another way to escape such instability, which is *e.g.* the case of theories whose action is a general function of the Ricci scalar (*i.e.* $f(R)$ theories). In fact these theories, which belong to the most general class of higher-order theories, can be re-written as second-order derivative theories with an extra scalar field, as we shall see in the next Section. Theories for which this is possible obviously evade the Ostrogradski instability as well.

1.4.1 The Exception: $f(R)$ Gravity Theories

Let us consider the theories coming from a straightforward generalization of the GR action, that is by replacing the Ricci scalar R with a generic function $f(R)$:

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} f(R) + S_M[g_{\mu\nu}, \psi_M]. \quad (1.40)$$

These theories, referred to as $f(R)$ gravity theories (see Ref. [78] and references therein), generically lead to higher-order field equations that can be written as follows:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] f'(R) = 8\pi G_N T_{\mu\nu}, \quad (1.41)$$

where the prime indicates differentiation with respect to the argument. It is straightforward to see from the equation above that these theories involve up to fourth-order derivatives of the metric, then it seems that they naturally fall in the class of theories plagued by Ostrogradski instability. Nevertheless, as already anticipated in the previous Section, this is not the case.

If we introduce an extra field χ , it is possible to write the following dynamically equivalent action

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [f(\chi) + f'(\chi)(R - \chi)] + S_M[g_{\mu\nu}, \psi_M]. \quad (1.42)$$

Taking its variation with respect to χ leads to

$$f''(\chi)(R - \chi) = 0. \quad (1.43)$$

If $f''(\chi) \neq 0$, then $\chi = R$, and the original action (1.40) is indeed recovered.

Through the redefinition of the field χ by $\phi = f'(\chi)$, and by defining

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \quad (1.44)$$

one can rewrite the action (1.42) as

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [\phi R - V(\phi)] + S_M[g_{\mu\nu}, \psi_M]. \quad (1.45)$$

The action above is known as the Jordan frame representation of the action of a Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$.

The field equations coming from the variation of this action with respect to the metric and the field ϕ are respectively

$$G_{\mu\nu} = \frac{8\pi G_N}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \phi, \quad (1.46a)$$

$$R = V'(\phi). \quad (1.46b)$$

We can already note that the equations above are second-order field equations both for the metric and the scalar field.

By taking the trace of Eq. (1.46a), and using it to eliminate R from Eq. (1.46b), one gets

$$3\square\phi + 2V(\phi) - \phi V'(\phi) = 8\pi G_N T, \quad (1.47)$$

which is the dynamical equation for ϕ .

Thus, having started with the action (1.40) leading to fourth-order field equations, we have ended up with the second-order theory (1.45), containing an extra propagating scalar degree of freedom. These theories then obviously evade the Ostrogradski instability.

1.5 Beyond General Relativity: Breaking Lorentz Invariance or Adding Extra Fields?

1.5.1 Higher-Dimensional Spacetimes

How can we modify GR still preserving second-order field equations? Since Lovelock's theorem singles out GR as the only theory with second-order field equations for the metric tensor, in order to circumvent it we necessarily need to violate one of its implicit assumptions. The first possibility consists in considering more than four dimensions, though until now we have experimentally detected only four. However, one can expect that for any higher-dimensional theory, a four-dimensional effective field theory can be derived at low-energy. Going beyond the four-dimensional effective description will be a necessary step to fully explain all the characteristics of the theory, and, first of all, to understand its theoretical motivations. Nevertheless, the four-dimensional effective description should anyway be adequate to study the low-energy phenomenology and viability. Since we are only interested in the phenomenological description at low-energy as given by viable theories in four-dimensions, we shall not consider explicitly higher-dimensional theories.

1.5.2 Giving Up Diffeomorphism Invariance

Another route we can undertake consists in giving up diffeomorphism invariance, which generically leads to extra propagating degrees of freedom because of less symmetry. In fact it is well known that symmetries can be restored by introducing extra fields, and this procedure is known as the Stueckelberg mechanism (see Ref. [79] for a review). Hence, one can think of theories that give up diffeomorphism invariance as diffeomorphism invariant theories with extra Stueckelberg fields. This scenario obviously implies the necessity to build up specific mechanisms able to hide these extra fields in regimes where GR is well tested and no extra degrees of freedom have been detected so far. At the same time they should allow them to exist and lead to different phenomenology in other regimes.

In the next Chapters we will extensively discuss about this possibility. We will consider theories built within a preferred spacetime foliation, where Lorentz symmetry is broken at very high energies and the Lorentz violations are instead taken under the experimental bounds at enough small energies. Moreover it will be shown how these theories are interestingly able to bridge classical gravity towards a possible quantum renormalizable theory at Planckian energy scales.

1.5.3 Adding Extra Fields

The last possibility one can take into account in order to circumvent Lovelock's theorem, is to add extra dynamical fields in the gravitational action while preserving diffeomorphism invariance. Notice that, instead of adding explicitly extra fields, one may also consider theories whose field equations are higher than second-order partial differential equations. This would generically correspond to introducing more degrees of freedom, as it has been shown for example in the case of $f(R)$ gravity.

In the last decades a plethora of such theories has been proposed in the attempt to give answers to the most pressing unsolved problems, both at very small and very large energy scales. In this respect Horndeski, in 1974, found the most general class of theories with second-order field equations for the metric tensor and an extra scalar field [80]. This work has been resurrected only some years ago in Ref. [11], and has attracted a lot of attention, leading to a very quick proliferation of publications about the phenomenology of this most general class of second-order theories. However, the necessity to tame the behaviour of the extra fields at low-energies still persists in these theories. This is the reason why we will finally take into account the possibility to modify the gravitational action by adding extra fields that are non-dynamical, then called auxiliary fields. After having implemented such hypothesis, we will study in a very generic fashion the modifications to the field equations of GR arising once the extra fields are eliminated in favour of the metric, the matter fields and their derivatives.

In the next Chapter we will start discussing about the former route. We will first show how Lorentz symmetry breaking can lead to a modification of the graviton propagator, so rendering the theory power-counting renormalizable. Then we will consider specific theories which have been proposed so far, Hořava gravity and Einstein-Æther theory, studying their features and the existing relation between them.

Chapter 2

Hořava Gravity: A Bridge between Gravity and Quantum Field Theory

2.1 Lorentz Invariance Breaking as Field Theory Regulator

We have already mentioned that theories including invariants quadratic in the curvature are renormalizable but nevertheless contain ghost degrees of freedom, and are therefore not unitary. And we have also stated that this is due to the presence of higher-order time derivatives.

So, one can think to modify the graviton propagator adding only higher-order spatial derivatives without adding higher-order time derivatives. It is conceivable that this could lead to a theory with improved ultraviolet (UV) behaviour without the problems related to the presence of higher-order time derivatives. This procedure intrinsically requires to treat space and time on different footing and to inevitably break Lorentz invariance. Nevertheless, because the modified behaviour of the propagator is strictly needed in the UV where Lorentz symmetry is explicitly broken, one has to ask that Lorentz invariance is recovered in the infrared (IR), or at least that Lorentz violations in the IR stay below current experimental bounds.

Let us now consider the simple example of a scalar field theory that explicitly violates Lorentz symmetry [81, 82]. The theory is said to be “power-counting renormalizable” if all of its interaction terms scale like momentum to some non-positive power, as in this case Feynman diagrams are expected to be convergent or have at most a logarithmic divergence.

We take the scalar field action to be of the following form

$$S_\phi = \int dt dx^d \left[\dot{\phi}^2 - \sum_{m=1}^z a_m \phi (-\Delta)^m \phi + \sum_{n=1}^N b_n \phi^n \right], \quad (2.1)$$

where $\cdot \equiv \partial/\partial t$, $\Delta \equiv \vec{\nabla}^2$ is the Laplacian, z and N are positive integers, a_m and b_n are coupling constants of suitable dimensions.

Moreover the theory is constructed such that space and time scale anisotropically, that is, measuring canonical dimensions of all objects in the units of spatial momentum k , space and time coordinates have the following dimensions:

$$[dt] = [k]^{-z}, \quad [dx] = [k]^{-1}. \quad (2.2)$$

In accord with the requirement that the action is dimensionless, we are looking for the conditions to impose on the constants appearing in the action, so that the theory results to be power-counting renormalizable. The dimensions for the scalar field are then immediately derived to be

$$[\phi] = [k]^{(d-z)/2}. \quad (2.3)$$

Since the action has to be dimensionless, requiring that the interaction terms scale like momentum to some non-positive power is equivalent to the requirement that the couplings of these interaction terms scale like momentum to some non-negative power.

It can be easily verified that

$$[a_m] = [k]^{2(z-m)}, \quad [b_n] = [k]^{d+z-n(d-z)/2}. \quad (2.4)$$

It follows that a_m has non-negative momentum dimension for all values of m , while b_n for $z \geq d$ has non-negative momentum dimensions for all values of n . Moreover, if $z < d$, b_n has non-negative momentum dimension only when $n \leq [2(d+z)]/(d-z)$.

In general the dispersion relation one gets for such a Lorentz violating field theory is of the following form

$$\omega^2 = m^2 + k^2 + \sum_{n=2}^z a_n \frac{k^{2n}}{K^{2n-2}}, \quad (2.5)$$

where K is the momentum-scale suppressing the higher-order operators. The resulting Quantum Field Theory (QFT) propagator is then

$$G(\omega, k) = \frac{1}{\omega^2 - [m^2 + k^2 + \sum_{n=2}^z a_n k^{2n}/K^{2n-2}]}. \quad (2.6)$$

The very rapid fall-off as $k \rightarrow \infty$ improves the behaviour of the integrals one encounters in the QFT Feynman diagram calculations.

In any $(d+1)$ -dimensional scalar QFT with $z = d$, with arbitrary polynomial self-interactions of the scalar field, it is found that this is indeed enough to keep all the Feynman diagrams finite [81, 83]. We have considered the simple example of a scalar field. But what happens if instead of a scalar field we wanted to consider a graviton?

In the case of the scalar field, momenta did not enter self-interaction vertices, while

the graviton self-interaction vertices contain up to $2z$ factors of momentum (spatial derivatives) [81]. This introduces some further complication, but it is not enough to spoil the power-counting renormalizability of the theory as long as $z \geq d$, *i.e.* as long as the action contains operators with at least $2d$ spatial derivatives [84].

Although renormalizability in the gravity sector is still an open issue, power-counting arguments are a strong indication that a gravity theory is indeed renormalizable.

2.2 A Quantum Field Theory of Gravity: Hořava Gravity

2.2.1 Hořava Gravity

In 2009 Hořava put all these arguments in a rigorous framework making it possible to test whether they are indeed valid for gravity theories. The resulting theory has been referred to as Hořava gravity [85, 86].

The theory is naturally built within the framework of an Arnowitt–Deser–Misner (ADM) decomposition,

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (2.7)$$

where N is the lapse, N_i the shift and h_{ij} the induced metric on the spacelike hypersurfaces. Using the ADM decomposition is indeed very natural because of the fundamentally non-relativistic nature of the theory we are constructing.

The theory is built such as to be compatible with the anisotropic scaling of the coordinates (see Eq. (2.2)). In the case of general z , one can postulate the scaling dimensions of the fundamental fields to be as follows:

$$[h_{ij}] = [1], \quad [N^i] = [k]^{z-1}, \quad [N] = [1]. \quad (2.8)$$

Since the time dimension plays a privileged role in the anisotropically scaling structure of the theory, the latter encodes the special role of time by assuming that spacetime is foliated by a set of constant time hypersurfaces. So, the action of Hořava gravity is not invariant under diffeomorphisms, but is instead invariant under the subclass of diffeomorphisms that leave the foliation intact, that is

$$x^i \rightarrow \tilde{x}^i(t, x^j), \quad t \rightarrow \tilde{t}(t). \quad (2.9)$$

These coordinate transformations are referred to as foliation-preserving diffeomorphisms. The most general action of the theory can be written as

$$S_H = S_K - S_V. \quad (2.10)$$

S_K is the kinetic term, which is given by the most general expression quadratic in the first-order time derivatives \dot{h}_{ij} of the spatial metric, and invariant under foliation-preserving diffeomorphisms. It is found to be

$$S_K = \frac{2}{k^2} \int dt d^3x \sqrt{h} N (K_{ij} K^{ij} - \lambda K^2) = \frac{2}{k^2} \int dt d^3x \sqrt{h} N K_{ij} G^{ijkl} K_{kl}, \quad (2.11)$$

where k is a coupling of suitable dimensions, λ is a dimensionless coupling, K_{ij} is the extrinsic curvature of the spacelike hypersurfaces given by

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (2.12)$$

$K \equiv h^{ij} K_{ij}$ is the trace of the extrinsic curvature, ∇_i is the covariant derivative associated with h_{ij} , and

$$G^{ijkl} = \frac{1}{2} (h^{ik} h^{jl} + h^{il} h^{jk}) - \lambda h^{ij} h^{kl}, \quad (2.13)$$

is the generalized DeWitt “metric on the space of metrics”.

Notice that in GR the invariance under all spacetime diffeomorphisms forces $\lambda = 1$, while here λ is a running coupling constant.

S_V is the potential term, which contains all the terms in the action which do not contain time derivatives but depend on spatial derivatives, and are compatible with the subclass of diffeomorphisms that leave the foliation intact. One can write it as

$$S_V = \frac{k^2}{8} \int dt d^3x \sqrt{h} N V[h_{ij}, N]. \quad (2.14)$$

In order to pin down the theory fully, one needs to specify $V[h_{ij}, N]$. In principle all the terms compatible with the symmetry of the theory and built with h_{ij} and N would have to be taken into account in V . Moreover the requirement of power-counting renormalizability leads to $z \geq d$, implying, for $d = 3$, the presence of operators with at least six spatial derivatives in the potential term.

This leads to a very large number of terms and an equally large number of independent couplings (order of magnitude $\sim 10^2$) which are compatible with the symmetry of the theory and can be therefore included in the potential term. Let us list for example the sort of terms one can include in the potential [87]:

$$R, a_i a^i, R_{ij} R^{ij}, R^2, R \nabla_i a^i, a_i \Delta a^i, (\nabla_i R_{jk})^2, (\nabla_i R)^2, \Delta R \nabla_i a^i, a_i \Delta^2 a^i, \dots, \quad (2.15)$$

where $a_i \equiv \partial_i \ln N$, and all the curvature invariants are understood to be the ones induced on the three-dimensional spacelike hypersurfaces.

2.2.2 Einstein-Æther Theory

There is also another well studied example of Lorentz-violating gravitational theory. This theory, referred to as Einstein-Æther theory (Æ-theory) [88, 89], is the most general theory one can construct by coupling a dynamical unit timelike vector field to the metric, with the requirement that the field equations are second-order in derivatives. The resulting theory is manifestly diffeomorphism invariant and the unit timelike vector field u^α , referred to as the *æther*, breaks local boost but not local rotation symmetries. Then Lorentz invariance is explicitly broken and the æther defines a preferred foliation in every solution.

The most general action for Æ-theory in vacuum, discarding boundary terms, is

$$S_{\text{æ}} = \frac{1}{16\pi G_{\text{æ}}} \int d^4x \sqrt{-g} (-R + L_{\text{æ}}), \quad (2.16)$$

where $G_{\text{æ}}$ is a coupling constant with dimensions of a length squared, R is the four-dimensional Ricci scalar of the spacetime metric $g_{\mu\nu}$, g is the determinant of the metric and

$$L_{\text{æ}} = -M^{\alpha\beta\mu\nu} \nabla_\alpha u_\mu \nabla_\beta u_\nu, \quad (2.17)$$

with $M^{\alpha\beta\mu\nu}$ defined as

$$M^{\alpha\beta\mu\nu} = c_1 g^{\alpha\beta} g^{\mu\nu} + c_2 g^{\alpha\mu} g^{\beta\nu} + c_3 g^{\alpha\nu} g^{\beta\mu} + c_4 u^\alpha u^\beta g^{\mu\nu}. \quad (2.18)$$

The c_i are dimensionless coupling constants, and u^ν is constrained to be a unit timelike vector, that is

$$g_{\mu\nu} u^\mu u^\nu = 1. \quad (2.19)$$

The constraint on the æther can also be explicitly imposed introducing a Lagrange multiplier term $\alpha (g_{\mu\nu} u^\mu u^\nu - 1)$ in the action (2.16). Moreover, in the Newtonian limit it is found [90] that the constant $G_{\text{æ}}$ is related to Newton's constant G_N by

$$G_N = \frac{G_{\text{æ}}}{1 - (c_1 + c_4)/2}. \quad (2.20)$$

Notice that, since the covariant derivative operator involves derivatives of the metric through the connection components, and since the unit vector cannot vanish anywhere, the terms quadratic in ∇u also introduce modifications to the kinetic term for the metric. Varying the action (2.16) with respect to the metric yields

$$G_{\alpha\beta} = T_{\alpha\beta}^{\text{æ}}, \quad (2.21)$$

where $T_{\alpha\beta}^{\text{æ}}$ denotes the æther stress-energy tensor

$$\begin{aligned} T_{\alpha\beta}^{\text{æ}} &= \nabla_{\mu} \left[J_{(\alpha}^{\mu} u_{\beta)} - J^{\mu}_{(\alpha} u_{\beta)} - J_{(\alpha\beta)} u^{\mu} \right] + c_1 \left[(\nabla_{\mu} u_{\alpha})(\nabla^{\mu} u_{\beta}) - (\nabla_{\alpha} u_{\mu})(\nabla^{\beta} u^{\mu}) \right] \\ &\quad + \left[u_{\nu}(\nabla_{\mu} J^{\mu\nu}) - c_4 \dot{u}^2 \right] u_{\alpha} u_{\beta} + c_4 \dot{u}_{\alpha} \dot{u}_{\beta} - \frac{1}{2} L_{\text{æ}} g_{\alpha\beta}, \end{aligned} \quad (2.22)$$

where

$$J^{\alpha}_{\mu} = M^{\alpha\beta}_{\mu\nu} \nabla_{\beta} u^{\mu}, \quad (2.23)$$

and $\dot{u}_{\nu} = u^{\mu} \nabla_{\mu} u_{\nu}$. Finally, variation with respect to u^{μ} yields

$$(\nabla_{\alpha} J^{\alpha\nu} - c_4 \dot{u}_{\alpha} \nabla^{\nu} u^{\alpha})(g_{\mu\nu} - u_{\mu} u_{\nu}) = 0. \quad (2.24)$$

An interesting property of \mathcal{A} -theory is that the action is formally invariant under the following redefinitions of the metric and the æther:

$$\begin{aligned} g'_{\alpha\beta} &= g_{\alpha\beta} + (\sigma - 1) u_{\alpha} u_{\beta}, \\ u'^{\alpha} &= \frac{1}{\sqrt{\sigma}} u^{\alpha}. \end{aligned} \quad (2.25)$$

This redefinition was first considered in Ref. [91], and under such transformation the action in terms of the primed fields retains its form taking on the coefficients c'_i that are related to the initial c_i through the following relations:

$$c'_{14} = c_{14}, \quad (2.26a)$$

$$c'_{123} = \sigma c_{123}, \quad (2.26b)$$

$$c'_{13} - 1 = \sigma(c_{13} - 1), \quad (2.26c)$$

$$c'_1 - c'_3 - 1 = \sigma^{-1}(c_1 - c_3 - 1), \quad (2.26d)$$

where $c_{ij} = c_i + c_j$, $c_{ijk} = c_i + c_j + c_k$, and the analogous holds for the primed coefficients. In the next Section we will see that Hořava gravity and \mathcal{A} -theory are equivalent in the IR limit if the æther is restricted to be hypersurface-orthogonal.

2.2.3 Relation Between Hořava Gravity and \mathcal{A} -Theory

The action of Hořava gravity in four dimensions,¹ can be generically written as follows:

$$S_H = \frac{1}{16\pi G_H} \int dT d^3x \sqrt{h} N \left[L_2 + \frac{1}{M_4^2} L_4 + \frac{1}{M_6^4} L_6 \right], \quad (2.27)$$

¹To write the action we assume there are no parity-violating terms.

where G_H is a coupling constant with dimensions of a length squared, T is the preferred time,

$$L_2 = K_{ij}K^{ij} - \lambda K^2 + \xi R + \eta a_i a^i, \quad (2.28)$$

L_4 and L_6 collectively denote all possible terms invariant under diffeomorphisms that leave the foliation untouched with respectively 4th- and 6th-order spatial derivatives, and M_4 and M_6 are the scales suppressing the corresponding operators, which do not coincide a priori with M_{pl} .

Let us now take into account the IR limit of the theory, that is only the L_2 part (2.28) of the action (2.27), which contains all the operators up to second-order in derivatives. It has been shown in Ref. [92] that this action is equivalent to the \mathcal{A} -theory one, once the æther is restricted to be hypersurface-orthogonal to the constant- T hypersurfaces, *i.e.* it can be written as

$$u_\alpha = \frac{\partial_\alpha T}{\sqrt{g^{\mu\nu} \partial_\mu T \partial_\nu T}}, \quad (2.29)$$

where the unit constraint on the æther has been taken into account.

To show this, let us start with the action (2.27) of \mathcal{A} -theory. By giving up part of the gauge freedom and choosing T as the time coordinate, the æther takes the form $u_\mu = N \delta_\mu^t$. Decomposing as usual the volume element it follows

$$\sqrt{-g} = N \sqrt{h}, \quad (2.30)$$

while decomposing the covariant derivative of u_μ yields

$$\nabla_\mu u_\nu = K_{\mu\nu} + u_\mu a_\nu, \quad (2.31)$$

where $K_{\mu\nu}$ is the extrinsic curvature of the surfaces orthogonal to u_ν , and $a_\nu \equiv u^\mu \nabla_\mu u_\nu$ is the acceleration of the orthogonal curves. Moreover, the extrinsic curvature and the acceleration satisfy the following conditions:

$$K_{\mu\nu} u^\nu = 0, \quad a_\nu u^\nu = 0, \quad (2.32)$$

and they do not have time-components in the preferred foliation.

Using the spatial coordinates x^i , the acceleration can be written as $a_i = \partial_i \ln N$. Then the lagrangian $L_{\mathcal{A}}$ takes the form

$$L_{\mathcal{A}} = -c_{13} K_{ij} K^{ij} - c_2 K^2 + c_{14} a_i a^i, \quad (2.33)$$

and, furthermore, the term $-R$ in the action (2.16) can be decomposed as usual

$$-R = K_{ij} K^{ij} - K^2 + {}^{(3)}R. \quad (2.34)$$

Finally, using Eqs. (2.30)-(2.33)-(2.34), the action (2.16) reduces to that of Hořava gravity (2.27) in the IR limit, with the following correspondence of parameters:

$$\frac{G_H}{G_{\mathfrak{a}}} = \xi = \frac{1}{1 - c_{13}}, \quad \frac{\lambda}{\xi} = 1 + c_2, \quad \frac{\eta}{\xi} = c_{14}. \quad (2.35)$$

\mathbb{A} -theory is well understood and perturbations have been fully studied [89, 93]. Besides the spin-2 graviton, \mathbb{A} -theory also propagates spin-1 and spin-0 degrees of freedom. If hypersurface-orthogonality is imposed at the level of the variation, one gets the IR limit of Hořava gravity written in a covariant fashion. The spin-1 degree of freedom cannot still be there because the æther is now written as the gradient of the scalar field T . The latter can be thought of as the Stueckelberg field one needs to introduce in order to restore full diffeomorphism invariance in Hořava gravity. In the covariant picture, T becomes an explicit extra degree of freedom – the scalar mode. It is forced to always have a non-trivial configuration, as is obvious from Eq. (2.29), and it defines the preferred foliation in every solution. If one considers the theory written in the preferred foliation, as in Eq. (2.27), then the scalar degree of freedom is no longer explicit but its existence can be expected because the theory has now less symmetry, as it is invariant under foliation-preserving diffeomorphisms only.

It is also interesting to notice that we started with the action (2.16), where the 4 coefficients $c_{1,2,3,4}$ were present, and we ended up with the action (2.33) with only three combinations of these coefficients. This is due to the fact that when the æther is restricted to be hypersurface-orthogonal, there is a relation between three of the terms in the Lagrangian (2.17). In fact, by defining the dual vector

$$\omega^\alpha = \varepsilon^{\alpha\beta\mu\nu} u_\beta \nabla_\mu u_\nu, \quad (2.36)$$

one finds the identity

$$\omega_\alpha \omega^\alpha = -2(\nabla_\beta u_\mu)(\nabla^{[\beta} u^{\mu]}) + (u^\mu \nabla_\mu u_\beta)(u^\nu \nabla_\nu u^\beta). \quad (2.37)$$

Since for an hypersurface-orthogonal æther ω^α identically vanishes, then the c_1 , the c_3 or the c_4 term in the Lagrangian (2.16) can be written in terms of the other two, leading to three independent combinations of parameters only, as in Eq. (2.33).

2.2.4 Perturbative Dynamics: Degrees of Freedom and Low-Energy Behaviour

Hořava gravity propagates a spin-2 mode (the usual graviton), and as mentioned earlier, since less symmetry generically means more degrees of freedom, also an extra scalar

degree of freedom.

Here we briefly discuss the IR behaviour of the spin-2 and spin-0 gravitons in the most general version of Hořava gravity, while in the next Chapter we will present extensively the low-energy dynamics of both the degrees of freedom in some restricted version of the theory.

One can study the linearized dynamics of the theory at low-energies, by perturbing the IR part of the action (2.27), *i.e.* the Lagrangian (2.28), at quadratic order around Minkowski space-time. It is found that the theory describes healthy excitations provided that two conditions are satisfied.

First, both the spin-2 and spin-0 gravitons must avoid exponential instabilities so as to be classically stable. Furthermore, to avoid the presence of ghosts one must require that their kinetic terms have the same sign. Both requests are satisfied for [87]

$$\lambda < \frac{1}{3} \quad \text{or} \quad \lambda > 1, \quad \text{and} \quad 0 < \eta < 2\xi. \quad (2.38)$$

Since there is a non-empty region of the parameter space where it is possible to satisfy the conditions above, it follows that the theory describes healthy excitations at low-energies.

2.2.4.1 Strong Coupling

The results discussed in the previous Section are obtained considering the linearized dynamics, as described by the part of the IR action quadratic in perturbations. Going to the next order in perturbation analysis, as given by the cubic interactions, the IR part of the action (2.27) exhibits strong-coupling for the scalar mode [94–96].

The strong-coupling scale M_{sc} is parametrically smaller than M_{pl} , and its size is controlled by the parameters λ , ξ and η . Since power-counting renormalizability arguments are based on the assumption that a perturbative treatment can be used to arbitrarily high-energies, strong coupling must be avoided. As proposed in Ref. [97], this requirement can be satisfied if

$$M_{sc} > M_{\star}, \quad (2.39)$$

where $M_{\star} \sim M_4 \sim M_6$ is the mass scale suppressing the higher-order derivative operators. One may be tempted to call M_{\star} the Lorentz-breaking scale, but the theory exhibits Lorentz-violations at all scales, as L_2 already contains Lorentz-violating operators.

Considering Cherenkov radiation constraints, one gets $\eta \sim |\lambda - 1|$, and the strong coupling scale is found to be $M_{sc} \sim \sqrt{\eta} M_{pl} \sim \sqrt{|\lambda - 1|} M_{pl}$ [95, 96]. Since absence of preferred frame effects in the Solar system observations requires $\eta, |\lambda - 1| \lesssim 10^{-7}$ [98], the strong coupling scale will be much lower than the Planck scale, *i.e.* $M_{sc} < 10^{16}$ GeV.

An important consequence of the effective constraint on M_{\star} coming from the requirement

to avoid strong coupling (first pointed out in Ref. [95]), and of the upper bound on M_{sc} , is that now M_\star becomes bound from both above and below.

The lower-bound on M_\star strongly depends on what observations one intends to use. Considering the mildest constraints coming from purely gravity-related observations, one would need $M_\star \gtrsim$ few meV for Lorentz violations to have remained undetected in sub-mm precision tests.

Much more stringent constraints can be obtained if one considers that Lorentz violations in gravity will percolate the matter sector. However, a strong constraint coming from synchrotron radiation in the Crab nebula has been recently derived in Ref. [99], which suffices to exclude the possibility that the scale suppressing the higher-order operators is of the same order of magnitude as the Lorentz breaking scale in the matter sector. This highlights the need for a mechanism that suppresses the percolation of Lorentz violations in the matter sector and is effective for higher-order operators as well (see Ref. [100] and references therein).

Moreover, if strong coupling is not to be a problem, now one has also an upper bound for M_\star . The combined set of constraints would leave a very large window open for M_\star within which Hořava gravity avoids strong coupling without exhibiting detectable Lorentz violations, at least with current experimental accuracy.

Chapter 3

Restricted Versions of Hořava Gravity

In the previous Chapter we have discussed how to construct the most general action of Hořava gravity. We have pointed out that it contains a very large number of terms compatible with the symmetry of the theory, then it seems to be not tractable.

In this Chapter we will discuss two great simplifications, called respectively “projectability” and “detailed balance”, which have been proposed in order to limit the proliferation of terms present in the full theory.

3.1 Projectable Version

Since the most general action of Hořava gravity contains a very large number of terms, in proposing the theory [85, 86], Hořava imposed a restriction called *projectability*, which sums up to the requirement that the lapse is just a function of time, *i.e.* $N = N(t)$, and then it is constant throughout each leaf of the foliation.

There is no fundamental principle behind such an assumption. The main motivation for considering it was that under this assumption one has enough gauge freedom to set $N = 1$, as in GR. The same cannot be done without projectability, as the symmetry of the action allows only space-independent time reparametrizations.

Once projectability is imposed the number of invariants one can include in the potential is drastically reduced, because now it is no longer possible to use $a_i = \partial_i \ln N$ in order to construct invariants under foliation-preserving diffeomorphisms. Then the potential will only depend on the metric and its spatial derivatives and this means that the action should include all of the curvature invariants constructed with h_{ij} and its spatial derivatives (up to six), so as to guarantee power-counting renormalizability. Moreover when

$N = N(t)$ there is another simplification. In fact N can now be pulled out of the integral over space, thus various invariants constructed with just the metric become related by total divergences.

The most general action one can write is the following [101, 102]:

$$\begin{aligned}
S_p = & \frac{M_{\text{pl}}^2}{2} \int d^3x dt \sqrt{h} N \left[K^{ij} K_{ij} - \lambda K^2 - d_0 M_{\text{pl}}^2 - d_1 R - d_2 M_{\text{pl}}^{-2} R^2 - d_3 M_{\text{pl}}^{-2} R_{ij} R^{ij} \right. \\
& - d_4 M_{\text{pl}}^{-4} R^3 - d_5 M_{\text{pl}}^{-4} R (R_{ij} R^{ij}) - d_6 M_{\text{pl}}^{-4} R^i{}_j R^j{}_k R^k{}_i - d_7 M_{\text{pl}}^{-4} R \nabla^2 R \\
& \left. - d_8 M_{\text{pl}}^{-4} \nabla_i R_{jk} \nabla^i R^{jk} \right], \tag{3.1}
\end{aligned}$$

where d_i are dimensionless couplings, and d_0 controls the value of the (bare) cosmological constant term.

In absence of matter it is possible to rescale the coordinates in such a way to set $d_1 = -1$, which is the value it has in GR. Also note that in the action above parity violating terms have been omitted. We are finally left with 9 free couplings which can be used to tune accordingly the scales which suppress the higher-order operators.

As we have already discussed, the theory is not fully diffeomorphism invariant, so we expect that breaking the full diffeomorphism group gives rise to extra propagating degrees of freedom other than the usual spin-2 graviton. This is indeed the case, and one finds that the theory propagates an extra scalar degree of freedom.

3.1.1 Dynamics and Low-Energy Behaviour

Let us now discuss the perturbations in order to study the low-energy dynamics of the theory. Discarding for simplicity the cosmological constant term, one can use flat space as a suitable background. Then, considering linearized perturbations around flat space one has

$$h_{ij} = \delta_{ij} + \epsilon p_{ij}, \quad N_i = 0 + \epsilon n_i, \quad N = 1 + \epsilon N(t). \tag{3.2}$$

It is useful to define

$$P_{ij} \equiv p_{ij} - \lambda \delta_{ij} p, \tag{3.3}$$

where $p = \delta^{ij} p_{ij}$. Then, by adopting the gauge-fixing

$$\partial^i p_{ij} - \lambda \partial_j p = 0, \tag{3.4}$$

P_{ij} becomes transverse,

$$\partial^i P_{ij} = 0. \tag{3.5}$$

In order to separate the individual physical modes, it is possible to decompose P_{ij} into the transverse traceless part \tilde{P}_{ij} and the trace part P :

$$P_{ij} = \tilde{P}_{ij} + \frac{1}{2} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) P, \quad (3.6)$$

where

$$\tilde{P}^i_i = 0, \quad \partial^i \tilde{P}_{ij} = 0, \quad P = P^i_i. \quad (3.7)$$

Then the spin-2 graviton is found to satisfy the following modified dispersion relation

$$\ddot{\tilde{P}}_{ij} = - \left[d_1 \partial^2 + d_3 M_{\text{pl}}^{-2} \partial^4 + d_8 M_{\text{pl}}^{-4} \partial^6 \right] \tilde{P}_{ij}. \quad (3.8)$$

In order to guarantee the stability of the spin-2 graviton at low-energy we have to require that d_1 is negative, as $\sqrt{-d_1}$ determines the speed $s_2^{(p)}$ of low-energy spin-2 gravitons [86, 101, 102]. We have already mentioned that, as long as we have not coupled matter to the theory, we are free to rescale the coordinates in order to set $d_1 = -1$. In this way we are making the speed $s_2^{(p)} = 1$.

Moreover, notice that the couplings d_3 and d_8 control the scale at which the higher-order Lorentz-violating terms become important.

As already mentioned there is also a spin-0 graviton which propagates besides the usual spin-2 graviton. It is found [102] that the linearized dynamics, described by the quadratic perturbations for the extra scalar degree of freedom, is governed by the action

$$S_p^{(2)} = -M_{\text{pl}}^2 \int d^3x dt \left[\frac{1}{c_p^2} \dot{p}^2 + p \partial^2 p + \frac{8d_2 + 3d_3}{M_{\text{pl}}^2} p \partial^4 p - \frac{8d_7 - 3d_8}{M_{\text{pl}}^4} p \partial^6 p \right], \quad (3.9)$$

where

$$c_p^2 = \frac{1 - \lambda}{3\lambda - 1}. \quad (3.10)$$

Notice that above we have used p instead of P , which by means of Eq. (3.3) are related as

$$P = (1 - 3\lambda) p. \quad (3.11)$$

Given the overall minus sign in Eq. (3.9), as the sign of the kinetic term for the spin-2 graviton is positive, the scalar mode is a ghost whenever $1/3 < \lambda < 1$, while it is classically unstable when $\lambda < 1/3$ and $\lambda > 1$ [102]. It follows that these two conditions cannot be evaded concurrently, so the theory is pathological at low-energies unless λ is sufficiently close to 1.

3.1.2 Strong Coupling

Furthermore, when the linearized action at third-order in the perturbations is taken into account, such a perturbative treatment breaks down when $\lambda \rightarrow 1$ and the scalar mode gets strongly coupled [94, 103, 104].

To see how this happens we have to consider the action for the cubic interactions of h , which is found to be

$$S_p^{(3)} = M_{\text{pl}}^2 \int dt d^3x \left\{ p(\partial p)^2 - \frac{2}{c_p^4} \dot{p} \partial_i p \frac{\partial^i}{\partial^2} \dot{p} + \frac{3}{2} \left[\frac{1}{c_p^4} p \left(\frac{\partial_i \partial_j}{\partial^2} \dot{p} \right)^2 - \frac{(2c_p^2 + 1)}{c_p^4} p \dot{p}^2 \right] \right\}. \quad (3.12)$$

Notice that in writing this action we have considered only the cubic interactions coming from the lower-order operators. By performing the redefinitions $\bar{t} = |c_h|t$ and $\bar{h} = c_h^{-1/2} M_{\text{pl}} h$, which canonically renormalize the lower-order part of the action (3.9), the cubic action reads

$$S_p^{(3)} = \frac{1}{|c_p|^{3/2} M_{\text{pl}}} \int d\bar{t} d^3x \left\{ c_p^2 \bar{p}(\partial \bar{p})^2 - 2\bar{p}' \partial_i \bar{p} \frac{\partial^i}{\partial^2} \bar{p}' + \frac{3}{2} \left[\bar{p} \left(\frac{\partial_i \partial_j}{\partial^2} \bar{p}' \right)^2 - (2c_p^2 + 1) \bar{p} (\bar{p}')^2 \right] \right\}. \quad (3.13)$$

where $' = \partial/\partial \bar{t}$. As we can see from the equation above, the cubic interactions of the spin-0 graviton are suppressed with respect to the quadratic ones by the scale $|c_p|^{3/2} M_{\text{pl}}$. Therefore, the theory becomes strongly coupled at the scale

$$M_{\text{sc}} = |c_p|^{3/2} M_{\text{pl}}. \quad (3.14)$$

Given that λ has to approach 1 in order to evade the stability constraints discussed above, such mass scale results to be phenomenologically unacceptably low (see Refs. [94, 103, 104]), as we know that we can treat gravity perturbatively at enough low-energies. Furthermore, there is also the issue of renormalizability: the arguments for power-counting renormalizability are based on the validity of the perturbative treatment at all energies. If there is strong coupling such arguments simply fail. In Refs. [105–107] a different position has been advocated regarding whether the strong coupling of the scalar graviton in the projectable case is truly a problem or an opportunity: it has been claimed that non-perturbative effects, exactly because of the strong coupling, lead to phenomenology very close to that of GR via the Vainshtein effect [108].

At the same time, strong coupling might lead to rapid running of the coupling constant λ and act as a remedy for the instability of the scalar mode.

A possible shortcoming in this way of arguing might be that the arguments used for power-counting renormalizability are essentially based upon the assumption that perturbative treatment does not break down. In Ref. [107], it has been claimed that it is enough to solve the momentum constraint non-perturbatively, in which case the power-counting

renormalizability arguments are still applicable.

3.2 Detailed Balance with Projectability

3.2.1 Superpotential and Action

Working in the projectable version of the theory where $N = N(t)$, Hořava also imposed an additional symmetry to the theory called *detailed balance* [85, 86], inspired by condensed matter systems, which sums up to the requirement that V should be derivable from a superpotential W as follows:

$$V = E^{ij} \mathcal{G}_{ijkl} E^{kl}. \quad (3.15)$$

E^{ij} is given in term of a superpotential W as

$$E^{ij} = \frac{1}{\sqrt{h}} \frac{\delta W}{\delta h_{ij}}, \quad (3.16)$$

\mathcal{G}_{ijkl} is the inverse of the DeWitt metric such that

$$\mathcal{G}_{ijmn} \mathcal{G}^{mnkl} = \frac{1}{2} \left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k \right), \quad (3.17)$$

and it explicitly reads

$$\mathcal{G}_{ijkl} = \frac{1}{2} (h_{ik} h_{jl} + h_{il} h_{jk}) + \frac{\lambda}{1 - 3\lambda} h_{ij} h_{kl}. \quad (3.18)$$

The superpotential is supposed to contain all of the possible terms up to a given order in derivatives which are invariant under the symmetry of the theory, *i.e.* invariant under foliation-preserving diffeomorphisms. The order in derivatives is dictated by the requirement that the theory be power-counting renormalizable. Minimally this requires sixth-order spatial derivatives in the action, which translates to third-order spatial derivatives in the superpotential.

The most general superpotential which satisfies these requirements is

$$W = \mu \int d^3x \sqrt{h} (R - 2\Lambda_W) + \frac{1}{w^2} \int \omega_3(\Gamma), \quad (3.19)$$

where μ , Λ_W and w are couplings of suitable dimensions, and

$$\omega_3(\Gamma) = \text{Tr} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) \equiv \varepsilon^{ijk} \left(\Gamma_{il}^m \partial_j \Gamma_{km}^\ell + \frac{2}{3} \Gamma_{il}^n \Gamma_{jm}^\ell \Gamma_{kn}^m \right) d^3x \quad (3.20)$$

is the gravitational Chern-Simons term. The variation with respect to h_{ij} of the last term in Eq. (3.19) yields a contribution to the field equations proportional to the Cotton tensor C^{ij} which is defined as

$$C^{ij} \equiv \varepsilon^{ikl} \nabla_k \left(R_\ell^j - \frac{1}{4} R \delta_\ell^j \right). \quad (3.21)$$

The Cotton tensor enjoys several symmetry properties: it is symmetric and traceless, transverse (*i.e.* covariantly conserved), and vanishes for conformally flat three-dimensional spaces.

Using Eqs. (3.15)-(3.16) and the superpotential W in Eq. (3.19), the corresponding potential V is found to be

$$V = -\frac{\mu^2}{(1-3\lambda)} \left(\frac{1-4\lambda}{4} R^2 + \Lambda_W R - 3\Lambda_W^2 \right) + \mu^2 R_{ij} R^{ij} - \frac{4\mu}{w^2} \varepsilon^{ijk} R_{il} \nabla_j R_k^l + \frac{4}{w^4} C_{ij} C^{ij}. \quad (3.22)$$

This potential gives rise to the following full action for Hořava gravity, where both projectability and detailed balance have been implemented:

$$S_H = \frac{2}{k^2} \int dt d^3x \sqrt{\hbar} N \left[(K_{ij} K^{ij} - \lambda K^2) + \frac{k^4 \mu^2}{16(1-3\lambda)} \left(\frac{1-4\lambda}{4} R^2 + \Lambda_W R - 3\Lambda_W^2 \right) - \frac{k^4 \mu^2}{16} R_{ij} R^{ij} + \frac{k^4 \mu}{4w^2} \varepsilon^{ijk} R_{il} \nabla_j R_k^l - \frac{k^4}{4w^4} C_{ij} C^{ij} \right]. \quad (3.23)$$

3.2.2 Known Problems and Potential Solutions

The version of the theory with detailed balance and projectability is known to be plagued by the following shortcomings:

1. There is a parity violating term, namely, the term which is fifth-order in derivatives. The presence of this term in the action (3.23) is inevitable if the latter has to contain sixth-order derivatives and come from a superpotential as defined above [101, 102].
2. The only sixth-order term is the square of the Cotton tensor, which vanishes for conformally flat three-dimensional spaces and, as such, it does not contribute to the propagator of the scalar graviton that the theory has. Hence, the scalar mode does not satisfy a sixth-order dispersion relation and is not power-counting renormalizable, unlike the spin-2 mode. This spoils the overall UV properties of the theory [86].
3. The IR behaviour of the scalar mode is plagued by instabilities and strong coupling at unacceptably low energies [94, 103].

4. The (bare) cosmological constant has the opposite sign from the observed value [101, 102].
5. The (bare) cosmological constant has to be large, much larger than the observed value [109].

The second problem had been already noticed by Hořava in Ref. [86] and a resolution has been proposed there, that is to add to the superpotential fourth-order terms. These would lead to eighth-order (super-renormalizable) terms in the action, as well as new sixth and lower-order operators. These terms would contribute to the propagator of the scalar mode and power-counting renormalizability would be restored.

In fact there are only two fourth-order terms one could add to the superpotential, $R^{ij}R_{ij}$ and R^2 . That is, the improvement in the UV behaviour of the scalar graviton comes at a relatively low cost in terms of proliferating the couplings.¹

Actually, once one has added these terms to cure the behaviour of the scalar graviton, a natural resolution to problem 1 emerges: imposing parity invariance explicitly does away with parity violating terms.

This would not allow for the presence of C^{ij} in E^{ij} , but the renormalizability of the spin-2 graviton is not compromised since there would be both sixth and eighth-order terms in V . Of course, one might be content with parity violations provided that they come at high enough energies to have remained undetectable so far. If this is the case, provided that the scale of parity violation can be tuned accordingly, problem 1 was not a problem in the first place.

Problem 3 is not one with an easy remedy. On the other hand, this is not actually a problem specific to detailed balance. In fact, as already discussed in Sec. (3.1), the most general action in Hořava gravity with projectability and without detailed balance exhibits similar problematic behaviour when it comes to the IR dynamics of the scalar graviton.

Therefore, we consider this problem to be related to projectability and not detailed balance, and we will argue below, in Sec. (3.3.2), that it can be addressed successfully in the same manner it has been addressed in the version without detailed balance, *i.e.* by doing away with projectability. In other words, we will argue that one needs not abandon detailed balance in order to resolve this issue, but projectability.

Lastly, there remain problems 4 and 5 regarding the sign and magnitude of the cosmological constant. We devote the next Section to a discussion of these problems.

¹Note that terms with two time and two spatial derivatives of h_{ij} , such as $(\nabla_\mu K)(\nabla^\mu K)$ appear to be eighth-order operators with $z = 3$ scaling but are actually tenth-order with $z = 4$ scaling.

3.2.3 The Size of the Cosmological Constant

To get a clearer picture on the various scales involved in the action (3.23), we perform the following redefinitions of the couplings (see Refs. [110, 111]):

$$\begin{aligned} M_{\text{pl}}^2 &= \frac{4}{k^2}, & M_6^2 &= \frac{w^2}{2} M_{\text{pl}}^2, \\ M_4^2 &= \frac{M_{\text{pl}}^4}{\mu^2}, & \xi &= \frac{\Lambda_W}{(1-3\lambda)M_4^2}, \end{aligned} \quad (3.24)$$

where M_{pl} , M_6 and M_4 have dimensions of a mass, whereas ξ is dimensionless. After these redefinitions the action (3.23) then takes the form

$$\begin{aligned} S_H &= \frac{M_{\text{pl}}^2}{2} \int dt d^3x \sqrt{h} N \left[K_{ij} K^{ij} - \lambda K^2 + \xi R - 2\Lambda - \frac{1}{M_4^2} R_{ij} R^{ij} + \frac{1-4\lambda}{4(1-3\lambda)} \frac{1}{M_4^2} R^2 \right. \\ &\quad \left. + \frac{2}{M_6^2 M_4} \epsilon^{ijk} R_{il} \nabla_j R_k^l - \frac{1}{M_6^4} C_{ij} C^{ij} \right], \end{aligned} \quad (3.25)$$

where the cosmological constant is

$$\Lambda = \frac{3}{2} \xi^2 (1-3\lambda) M_4^2. \quad (3.26)$$

Clearly GR corresponds to $\xi = \lambda = 1$ with the higher-order derivative terms being absent. Instead of being parametrized by k , w , μ and Λ_W the theory is now parametrized by M_{pl} , M_6 , M_4 , and ξ (and of course λ which is the parameter in the kinetic term in both cases).

On the other hand, the cosmological constant Λ is *not* a free parameter, but instead it is fully determined by the dimensionless parameters ξ and λ and the energy scale M_4 , which is the scale that suppresses both of the fourth-order operators in the action when $\lambda \sim 1$.

If we want the theory to be close to GR in the IR, then $\lambda, \xi \sim 1$ to high accuracy. It is already obvious that Λ has to be negative in this case, as has been pointed out in the literature (*e.g.* [101, 102]).

Less attention has been paid to the fact that, what seems to be determining the size of Λ is really M_4 . The latter will be the energy M_\star at which Lorentz-violating effects will become manifest as higher-order terms in the dispersion relations.

As discussed in Sec. (2.2.4.1) there are two classes of observational constraints on Lorentz violations that restrict the size of $M_\star \sim M_4$. Using only the mildest constraints coming from purely gravitational experiments, the value of the cosmological constant (taking into account the $M_{\text{pl}}^2/2$ overall factor in the action) would be (roughly) of the order of $10^{-60} M_{\text{pl}}^4$. If more stringent constraints coming from matter are to be imposed this value

will get even higher and perhaps larger than M_{pl}^4 .

Consequently, the value of the cosmological constant has to be so large that its negative sign seems to be a secondary problem only: there is at best a 60 orders of magnitude discrepancy between the value required by detailed balance and the observed value. Given that this is a bare cosmological constant, were it allowed to have an arbitrarily small yet negative value, one could just hope for it to be an irrelevant contribution to the total cosmological constant.

Note that the vacuum energy problem, or the “old cosmological constant problem”, is anyway still an open problem in Hořava gravity, and in a theory which proposes itself as a UV completion of GR finding a resolution is pertinent. However, and simply for comparison, the value of the bare cosmological constant when detailed balance is imposed, turns out to be at best comparable to the naive estimate of the vacuum energy obtained with the weak energy scale $\Lambda_{\text{UV}} \sim 1\text{TeV}$ as a cut-off, as given in Eq. (1.25).

The fact that the size of the cosmological constant will be related to the size of the energy scale suppressing the fourth-order operators has been previously pointed out in Ref. [109] and it was used there to argue that this leads to a bare cosmological constant that could potentially cancel out the contribution of an equally large vacuum energy. The approach followed there was to provide a heuristic estimate for the vacuum energy and then identify the value that M_4 would have to have in order for the aforementioned cancellation to work. This led to near-Planckian values for M_4 .

Here we took an orthogonal approach. We discussed the possible constraints on M_4 and derived corresponding constraints on the value of the cosmological constant. Our primary goal was to derive a rough but robust lower limit for the magnitude of the bare cosmological constant. The reason for this is twofold. First of all there is currently no precise and convincing argument that such a cancellation can indeed be achieved without fine tuning. Second, the projectable theory with detailed balance is anyway plagued by problem 3. Therefore, our main concern here was to argue beyond any doubt that the size of the cosmological constant is indeed unacceptably large (the lower bounds we have derived will persist in the non-projectable theory, as we will discuss shortly).

To conclude, the most important problem with the bare cosmological constant in Hořava gravity with detailed balance is not its sign but its magnitude: it has such a large value that, unless one is willing to allow a violation of detailed balance, some sort of self-tuning mechanism along the lines of Ref. [109] would be the only way to achieve sensible phenomenology.

3.3 Detailed Balance without Projectability

We have argued above that problems 1 and 2 in the list of Sec. (3.2.2), *i.e.* parity violations and the UV behaviour of the scalar mode, are not real problems, in the sense that they have a straightforward resolution within the framework of projectable Hořava gravity with detailed balance.

We have also shown that the main problem with the (bare) cosmological constant is not its sign but its size. Finally, we mentioned already in Sec. (3.2.2) that we consider problem 3, the IR behaviour of the scalar mode, to not be a problem stemming from detailed balance but from projectability and we claimed that it can find a resolution once the latter is abandoned, without having to abandon also the former. We provide support for this claim below.

3.3.1 Superpotential and Action

To the best of our knowledge there does not exist in the literature a consistent implementation and consideration of detailed balance without projectability. As has been pointed out in Ref. [87], having in mind the version of the theory without detailed balance, once projectability is abandoned one can use not only the Riemann tensor of h_{ij} and its derivatives, but also the vector

$$a_i = \partial_i \ln N, \quad (3.27)$$

in order to construct invariants under foliation-preserving diffeomorphisms. In the version of the theory without detailed balance this leads to a proliferation of terms $\sim \mathcal{O}(10^2)$. In fact all of these terms have to be taken into account as they would anyway be generated by radiative corrections.

On the other hand, there is, remarkably, only one term one can add to the superpotential W in the version with detailed balance: $a_i a^i$. One then has [110, 111]

$$W = \int d^3x \sqrt{h} [\mu (R - 2\Lambda_W) + \beta a_i a^i] + \frac{1}{w^2} \int \omega_3(\Gamma), \quad (3.28)$$

where β is the new coupling.

This new term has been repeatedly neglected in the literature and abandoning projectability within the framework of detailed balance had been restricted to simply allowing N to have a space dependence without modifying the action.

We will show that the presence of this term is crucial when it comes to the low-energy dynamics of the scalar mode (similarly to the version without detailed balance [87]).

The variation of the superpotential with respect to the metric leads to the following

additional contribution to E^{ij} ,

$$E_{extra}^{ij} = \frac{1}{\sqrt{h}} \frac{\delta W_{extra}}{\delta h_{ij}} = \beta \left(\frac{1}{2} h^{ij} a_k a^k - a^i a^j \right). \quad (3.29)$$

Defining the potential in the same way as before (we will return to this subtle issue shortly), according to Eq. (3.15) one gets the following additional contributions to the action

$$S_{extra} = \int dt d^3x \sqrt{h} N \left\{ \frac{k^2 \mu \beta}{4} \left[-R_{kl} a^k a^l + \frac{1-4\lambda}{4(1-3\lambda)} R a_k a^k + \frac{\Lambda_W}{2(1-3\lambda)} a_k a^k \right] + \frac{k^2 \beta}{2w^2} C_{kl} a^k a^l - \frac{k^2 \beta^2}{32} \frac{3-8\lambda}{1-3\lambda} (a_k a^k)^2 \right\}. \quad (3.30)$$

Using the coupling redefinitions of Eq. (3.24) and introducing the extra redefinition

$$\eta = \frac{\beta \xi M_4}{M_{\text{pl}}^2}, \quad (3.31)$$

leads to the total action

$$S_{DB} = \frac{M_{\text{pl}}^2}{2} \int dt d^3x \sqrt{h} N \left\{ K_{ij} K^{ij} - \lambda K^2 + \xi R - 2\Lambda + \eta a^i a_i - \frac{1}{M_4^2} R_{ij} R^{ij} + \frac{1-4\lambda}{4(1-3\lambda)} \frac{1}{M_4^2} R^2 + \frac{2\eta}{\xi M_4^2} \left[\frac{1-4\lambda}{4(1-3\lambda)} R a^i a_i - R_{ij} a^i a^j \right] - \frac{\eta^2}{4\xi^2 M_4^2} \frac{3-8\lambda}{1-3\lambda} (a^i a_i)^2 + \frac{2}{M_6^2 M_4} \epsilon^{ijk} R_{il} \nabla_j R_k^l + \frac{2\eta}{\xi M_6^2 M_4} C^{ij} a_i a_j - \frac{1}{M_6^4} C_{ij} C^{ij} \right\}. \quad (3.32)$$

Recovery of Lorentz symmetry would require $\eta \rightarrow 0$, as well as $\xi, \lambda \rightarrow 1$. The last term in the first line contributes to the low-energy limit of the theory.

Several comments are in order. First of all, the inclusion of the $a^i a_i$ term in the superpotential has no effect in the magnitude of the cosmological constant, so this problem will persist in the theory described by the action in Eq. (3.32).

Secondly, our implementation of detailed balance in the non-projectable version of the theory might seem too naive or simplistic. Why not generalize the DeWitt metric further? And why should one stick with Eq. (3.15)? Actually, there do not seem to be any terms one can create which are quadratic in time derivatives of N and are invariant under foliation-preserving diffeomorphism. This seems to exclude straightforward generalizations of the DeWitt metric. Note also that, in principle one could have new contributions that include one time derivative, such as $K_{ij} a^i a^j$. However, they can be avoided by imposing symmetry under time reversal.

Regarding the generalization of the definition of the potential, indeed a first thing that comes in mind is that $N(\sqrt{-g})^{-1}(\delta W/\delta N) \propto \nabla^2 \ln N$ is an invariant, which could potentially be used to create (higher-order) contributions to the action. In absence of a

generalized DeWitt metric an unambiguous generalization is, however, not obvious to us.

In Ref. [112] a possible generalization of detailed balance is proposed, in a version of Hořava gravity with an extra symmetry. Besides the usual superpotential W , the authors use an additional superpotential W_a built with a_i and its derivatives. But it is not clear to us what motivates the use of $\delta W_a/\delta a_i$ in the construction of the potential, given that N is actually the fundamental field in the action. On the other hand, in Ref. [113] generalizations of detailed balance that include the matter fields were considered. We will not follow this approach here.

Being left without a guiding principle, and in view of the fact that whether or not (any or which) version of detailed balance is robust against radiative corrections, we will take the most conservative and simple approach. Note that a generalization of detailed balance would lead to additional terms (and new couplings), but it is unlikely to exclude any of the terms already present in Eq. (3.32). So, we will proceed with the action at hand, considering it to be some sort of minimal consistent implementation of detailed balance in non-projectable Hořava gravity.

3.3.2 Linearization at Quadratic Order in Perturbations

The question that will be addressed next is whether the theory described by the action in Eq. (3.32) has improved behaviour when it comes to the dynamics of the scalar mode. However, the presence of a large cosmological constant continues to be both a practical complication in this discussion and a phenomenologically undesirable characteristic of the theory under scrutiny.

So, perhaps a much better motivated question is the following: if some resolution to the cosmological constant problem were to be found, which would allow one to tune down its magnitude to an acceptable level, would the theory then be free of pathologies when it comes to the dynamics of the scalar? If the answer to this question is positive, then the value of the cosmological constant becomes the only real shortcoming of detailed balance, and indeed this is what we show next.

Therefore, let us assume a *deus ex machina* resolution, simply setting $\Lambda = 0$ in Eq. (3.32), and ask again if the theory has improved behaviour when it comes to the dynamics of the scalar mode. At the low-energy limit, *i.e.* considering only the second-order operators, the answer is obviously yes. This is because, up to this order, the action (3.32) actually fully coincides with the most general action (2.27) considered in Ref. [87], which is known to have sensible scalar dynamics at low energies as we have briefly discussed in Sec. (2.2.4). Here we will demonstrate how the results presented back there were derived.

On the other hand, the theory with detailed balance that we are considering here has

significantly less couplings than the most general theory when it comes to higher-order terms, and in fact ξ and η do enter the coefficients of these terms as well.

So, in order to disperse all doubt and show that the theory has sensible dynamics at low-energies, we will linearize it around flat space. We start with the total action given in Eq. (3.32) and we perturb to quadratic order considering only scalar perturbations, since the theory does not have vector excitations and, moreover, the analysis of the IR limit for the spin-2 graviton in our case is already captured in the (well behaved) analysis of the projectable version as presented in Sec. (3.1.1), since the low-energy dynamics of the spin-2 mode is not affected by projectability or detailed balance. We, then, have

$$N = 1 + \alpha, \quad N_i = \partial_i y, \quad h_{ij} = e^{2\zeta} \delta_{ij}. \quad (3.33)$$

Our ansatz for the perturbation for h_{ij} differs from the most general scalar perturbation by the term $\partial_i \partial_j E$, but one can use part of the available gauge freedom to set $E = 0$. One obtains for the Ricci tensor and the Ricci scalar of h_{ij}

$$R_{ij} = -\partial_i \partial_j \zeta - \delta_{ij} \partial^2 \zeta + \partial_i \zeta \partial_j \zeta - \delta_{ij} \partial_k \zeta \partial^k \zeta, \quad (3.34)$$

$$R = -e^{-2\zeta} (4\partial^2 \zeta + 2(\partial\zeta)^2), \quad (3.35)$$

where $\partial^2 = \delta_{ij} \partial^i \partial^j$. The quantity K_{ij} appears only quadratically in the action, so we only need to compute it to first-order:

$$K_{ij}^{(1)} = \dot{\zeta} \delta_{ij} - \partial_i \partial_j y, \quad (3.36)$$

$$K^{(1)} = 3\dot{\zeta} - \partial^2 y. \quad (3.37)$$

The quadratic action then takes the form

$$S_{DB}^{(2)} = \frac{M_{\text{pl}}^2}{2} \int dt d^3x \left\{ 3(1-3\lambda)\dot{\zeta}^2 - 2(1-3\lambda)\dot{\zeta}(\partial^2 y) + (1-\lambda)(\partial^2 y)^2 + 2\xi(\partial\zeta)^2 - 4\xi\alpha\partial^2\zeta + \eta(\partial_i\alpha)(\partial^i\alpha) - \frac{2(1-\lambda)}{1-3\lambda} \frac{1}{M_4^2} (\partial^2\zeta)^2 \right\}. \quad (3.38)$$

The $C_{ij}C^{ij}$ term does not contribute because of the conformal properties of the Cotton tensor. The two fifth-order operators and the fourth-order operators that contain a_i do not contribute as well because they are zero to quadratic order.

Variation with respect to y yields

$$(1-\lambda)\partial^4 y - (1-3\lambda)\partial^2 \dot{\zeta} = 0, \quad (3.39)$$

which, assuming regularity, leads to

$$\partial^2 y = \frac{1-3\lambda}{1-\lambda} \dot{\zeta}. \quad (3.40)$$

Variation with respect to α yields

$$\eta \partial^2 \alpha + 2\xi \partial^2 \zeta = 0, \quad (3.41)$$

which, again imposing regularity, can be solved to give

$$\alpha = -\frac{2\xi}{\eta} \zeta. \quad (3.42)$$

We can now use Eqs. (3.40) and (3.42) to integrate out the non-dynamical fields y and α in favour of the dynamical field ζ . The quadratic action then reads

$$S_{DB}^{(2)} = \frac{M_{\text{pl}}^2}{2} \int dt d^3x \left\{ \frac{2(1-3\lambda)}{1-\lambda} \dot{\zeta}^2 + 2\xi \left(\frac{2\xi}{\eta} - 1 \right) \zeta \partial^2 \zeta - \frac{2(1-\lambda)}{1-3\lambda} \frac{1}{M_4^2} (\partial^2 \zeta)^2 \right\}. \quad (3.43)$$

The dispersion relation for the scalar is then given by

$$\omega^2 = \xi \left(\frac{2\xi}{\eta} - 1 \right) \frac{1-\lambda}{1-3\lambda} k^2 + \frac{1}{M_4^2} \left(\frac{1-\lambda}{1-3\lambda} \right)^2 k^4. \quad (3.44)$$

As expected, the low-energy dynamics of the scalar is satisfactory for a significant part of the parameter space, which is the same part as in the most general non-projectable theory [87]. In particular, for the scalar to have positive energy (given the positive sign of the kinetic term for the spin-2 graviton [86, 102, 114]) one needs

$$\lambda < \frac{1}{3} \quad \text{or} \quad \lambda > 1, \quad (3.45)$$

whereas classical stability requires that

$$\xi \left(\frac{2\xi}{\eta} - 1 \right) \frac{1-\lambda}{1-3\lambda} > 0. \quad (3.46)$$

As the low-energy dynamics of the spin-2 graviton is not affected by projectability or detailed balance, we know from the analysis of the projectable version presented in Sec. (3.1.1) that the spin-2 graviton is stable if $\xi > 0$ (notice that the overall coupling of the Ricci scalar was there called $-d_1$, and the condition to impose was $d_1 < 0$).

Given the constraints in Eqs. (3.45)-(3.46) and the fact that $\xi > 0$, one has

$$0 < \eta < 2\xi. \quad (3.47)$$

From the coefficient of the k^4 -term in the dispersion relation one sees directly that any choice for λ cannot lead to an instability at higher energies. However, it is also obvious that the scalar satisfies a fourth, and not a sixth, order dispersion relation. This was expected given our discussion about which terms have zero contribution to quadratic order.

So, same as in the projectable case, the arguments on which the discussion about the renormalizability properties of the theory is based are compromised, unless we actually go one order higher in the superpotential W . Adding fourth-order terms in W would lead to both sixth and eight-order terms for the scalar, rendering the theory power-counting renormalizable.

The fourth-order terms one could add in the superpotential W are

$$\begin{aligned} R^2, \quad R^{ij}R_{ij}, \quad R\nabla^i a_i, \quad R^{ij}a_i a_j, \\ R a_i a^i, \quad (a_i a^i)^2, \quad (\nabla^i a_i)^2, \quad a_i a_j \nabla^i a^j. \end{aligned} \quad (3.48)$$

These would add 8 new couplings to the theory.

Strictly speaking, given that how each new coupling will contribute to the coefficients in the dispersion relation is not obvious once detailed balance has been imposed, one still needs to calculate the full dispersion relations for both the spin-2 and the spin-0 gravitons in order to show without doubt that there is no issue with instabilities at high energies. However, the fairly large number of independent couplings is more than encouraging. After adding these terms one could also impose parity invariance without compromising the renormalizability properties of the spin-2 graviton.

In this case one would end up with 7 more couplings than the theory in Eq. (3.32). In total there would be 12 couplings (not including the coupling to matter). This is roughly an order of magnitude less than the number of couplings in the theory without detailed balance (and up to sixth-order operators) [86].

So, even after the addition of the fourth-order operators in W , detailed balance still provides a significant reduction in the number of couplings.

Let us finally mention that the discussion about strong coupling in the most general version of the theory, given in Sec. (2.2.4.1), will be qualitatively similar here, since the version of the theory with detailed balance that we have introduced does not restrict the low-energy action but only the higher-order operators.

3.4 Summary and Open Problems

We have revisited the idea of detailed balance in Hořava gravity, as a way to restrict the proliferation of independent couplings. We first considered the projectable version of

the theory, in which this principle had been initially implemented. We listed the various shortcomings usually associated with detailed balance and discussed some potential resolutions that have been proposed for some of them. The problems that cannot find a resolution within the framework of projectability and detailed balance were the sign and magnitude of the bare cosmological constant and the dynamical inconsistencies associated with the scalar mode.

We have shown that the latter of the two problems is actually related to projectability and not detailed balance. That is, we have shown that a non-projectable formulation of the theory with detailed balance would lead to sensible dynamics for the scalar mode and the same low-energy phenomenology as the version without detailed balance, were the magnitude problem of the cosmological constant to find a resolution. The theory would be required to have fourth-order derivative terms in the superpotential, but it would still have a number of independent couplings which would be roughly an order of magnitude lower than the version without detailed balance.

Could the magnitude and the sign of the bare cosmological constant, which are the only shortcomings that persist once projectability is abandoned, be blessings in disguise? That is, could the bare cosmological constant end up cancelling out the contribution of the vacuum energy, leaving behind a tiny residual that would account for the observed value? Certainly at this stage, and with the current poor level of understanding of the vacuum energy problem in Hořava gravity, such a statement is at the level of wishful thinking. Nevertheless, one cannot exclude the possibility. This provides some extra motivation to consider the vacuum energy problem, which is anyway a pressing problem for a theory that aspires to be a UV completion of GR.

In absence of a miraculous cancellation between bare cosmological constant and vacuum energy, one could consider the idea that the bare cosmological constant could be tuned down by a soft breaking of detailed balance, which raises the question whether such a soft breaking would not affect higher-order terms as well. In fact, this brings one back to the key issue of whether detailed balance can anyway be robust against radiative corrections. This question becomes much more interesting in the light of the fact that a dynamically consistent theory that satisfies detailed balance does indeed exist.

Chapter 4

Rotating Black Holes in Three-Dimensional Hořava Gravity

4.1 Black Holes in Hořava Gravity

One of the most impressive predictions of GR is the existence of black holes, as its field equations naturally allow for such solutions. It is also more extraordinary since the existence of black holes in Nature is provided by strong indications coming from current astrophysical observations [115–120].

The defining property of a black hole is the presence of an event horizon, that is a causal boundary separating the interior from the exterior of the black hole, so that any signal coming from the interior cannot reach the exterior.

The reason why spacetimes that possess an event horizon can exist in GR finds its roots in the causal structure of the theory. The latter is inherited from Special Relativity, and confines any signal, irrespective of their nature, to propagate within future-directed light cones with a velocity that cannot exceed the speed of light. This is a general feature of any theory for which Lorentz symmetry is respected.

But what happens to horizons and black holes if Lorentz invariance is not an exact symmetry of Nature?

Let us consider for example a theory with a preferred frame, but where all excitations have linear dispersion relations. So, Lorentz symmetry is broken by the existence of a preferred frame, and some modes may perhaps propagate faster than light there, but anyhow with a finite speed. In such a Lorentz-violating theory the causal structure remains qualitatively similar to GR and the modification that one may expect at the level of black holes is the presence of multiple horizons, *i.e.* modes which propagate at different

speeds can have in general different causal boundaries. However, once Lorentz symmetry has been given up, there is no particular reason why dispersion relations should be linear. This is indeed the case of Hořava gravity, as we have seen in the previous Chapters. So, excitations propagating with sufficiently low-momenta would experience the existence of an event horizon, but more energetic ones would be able to escape it. In a Lorentz-violating gravitational theory with higher-order dispersion relations, the event horizon relinquishes then its role as an absolute causal boundary, since modes propagating infinitely fast will be able to penetrate it. However, this role will be taken over by the *universal horizon* [121, 122]. Since in Hořava gravity the preferred foliation takes a special physical content, the leaves of such a foliation can be chosen as constant-time hypersurfaces for any solution, and any sort of physical process is presumed to proceed normally to them. In spherically-symmetric spacetimes, when a constant-time hypersurface coincides with a constant-radius hypersurface a universal horizon is indeed present. Since it can only be crossed in one direction then no signal can escape the interior of the universal horizon propagating to the exterior, no matter how fast it moves. So, the existence of a universal horizon is a strong indication that the notion of a black hole still makes sense in Lorentz-violating gravity theories.

People have tried to find spherically symmetric black hole solutions in four-dimensional Hořava gravity. The ones found in Ref. [121] are numerical and those of Ref. [122] are numerical and valid in the small-coupling limit. Explicit solutions in four dimensions are also known for specific, tuned choices of the parameters of the theory, but they are all static [123]. The only rotating solutions currently known in four dimensions are not entirely explicit, and moreover, rely on the assumption of slow rotation [124, 125].

The lack of general exact solutions in four dimensions is certainly due to the complexity of the field equations. This is the main reason why in what follows we will work in three dimensions, as working in a lower-dimensional spacetime will allow us to avoid approximations or numerics. In fact we will be able to find solutions which are explicit and exact, unlike their four-dimensional counterparts.

Our main goal will be to seek a Lorentz-violating version of the celebrated Banados-Teitelboim-Zanelli (BTZ) solution [126, 127], which is the unique black hole of GR in three dimensions, referred to as a solution with a Killing horizon hiding a “causal” singularity at $r = 0$. Before doing that, in the next Section we will briefly review this famous result.

4.2 BTZ Black Hole in Three-Dimensional General Relativity

The action of GR in a three-dimensional spacetime is

$$S_{GR} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g} [-R - 2\Lambda], \quad (4.1)$$

where R is the three-dimensional Ricci scalar of the spacetime metric $g_{\mu\nu}$ and Λ is a negative cosmological constant term, related to the characteristic curvature radius l by $-\Lambda = l^{-2}$. Variation of the action (4.1) with respect to the metric tensor $g_{\mu\nu}$ yields the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0. \quad (4.2)$$

In three dimensions, by using the symmetry and contraction properties of the Riemann tensor $R_{\lambda\mu\nu\kappa}$, one finds that

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu} - \frac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R. \quad (4.3)$$

So, by using the field equations (4.2), the Riemann tensor can be written as

$$R_{\lambda\mu\nu\kappa} = \Lambda(g_{\mu\nu}g_{\lambda\kappa} - g_{\mu\kappa}g_{\lambda\nu}), \quad (4.4)$$

describing a symmetric spacetime of constant positive curvature

$$R = -6\Lambda. \quad (4.5)$$

With the assumptions of stationarity and circular symmetry, that is by restricting the variational principle to a class of fields which possess a rotational Killing vector $\partial/\partial\phi$ and a timelike Killing vector $\partial/\partial t$, the line element can be written as

$$ds^2 = N^2(r)dt^2 - \frac{1}{f^2(r)}dr^2 - r^2(d\phi + N^\phi(r)dt)^2. \quad (4.6)$$

It is found [126, 127] that under the assumptions of time independence and circular symmetry, the field equations (4.2) are solved by

$$N^2(r) = f^2(r) = -M + \frac{J^2}{4r^2} - \Lambda r^2, \quad (4.7a)$$

$$N^\phi(r) = -\frac{J}{2r^2}, \quad (4.7b)$$

where M and J are the conserved charges associated respectively with asymptotic invariance under time displacements (mass) and rotational invariance (angular momentum).

Since Λ has been assumed to be negative, the BTZ spacetime is then found to be locally anti-de Sitter (AdS). Note that because of our convention, AdS spacetime gives $R > 0$. In stationary spacetimes, horizons are null, stationary surfaces and the normal to any stationary surface is proportional to $\partial_\alpha r$ which is null when

$$g^{\alpha\beta} (\partial_\alpha r) (\partial_\beta r) = g^{rr} = -f^2 = 0. \quad (4.8)$$

The function $f(r)$ vanishes for two values of the radius r given by

$$r_\pm = \sqrt{-\frac{M}{2\Lambda} \left(1 \pm \sqrt{1 + \frac{J^2 \Lambda}{M^2}} \right)}, \quad (4.9)$$

while g_{00} vanishes at

$$r_{\text{erg}} = \sqrt{-\frac{M}{\Lambda}}. \quad (4.10)$$

Of these three special values of the radius r , r_+ is the black-hole horizon, r_{erg} demarcates the ergosurface, and the region between r_+ and r_{erg} is the ergosphere. They also obey the relation

$$r_- \leq r_+ \leq r_{\text{erg}}. \quad (4.11)$$

Notice that, in order for the solution to describe a black hole, from Eq. (4.9) one obtains the following bounds on the mass and the angular momentum

$$M > 0, \quad J^2 \leq \frac{M^2}{|\Lambda|}, \quad (4.12)$$

and in the extreme case $J^2 |\Lambda| = M^2$, the two roots r_+ and r_- coincide.

The characterizing feature of such black hole spacetime is that it has a singularity at $r = 0$, which is neither a curvature nor a conical singularity, but is instead a ‘‘causal’’ singularity where the curvature is everywhere finite (and constant). In fact it is shown [126, 127] that continuing past $r = 0$ would bring in closed timelike lines.

Moreover, there is also an additional argument for considering the spacetime as ending at $r = 0$. In fact, it has been shown in some simple examples that the introduction of matter leads to a curvature singularity in $r = 0$. The first case concerns the collapse of a cloud of dust with $J = 0$, for which the matter is shown to generically reach infinite density at $r = 0$ [128]. In such a scenario, the dust ‘‘probes’’ only part of the spacetime, because only the part of the surface $r = 0$ intersecting the history of the dust becomes singular. However, in the case of a field, such as the electromagnetic field, all the spacetime is probed. As it was indicated in Ref. [126], restricting the electromagnetic field to depend only on the radial coordinate, the gauge invariant scalar $F_{\mu\nu} F^{\mu\nu}$ (where $F_{\mu\nu}$ is the electromagnetic tensor) is found to be proportional to r^{-2} . Then it is singular at all

points on the surface $r = 0$. Therefore, considering that these singularities are produced once matter couplings are turned on, one can generically exclude the region $r = 0$ from the spacetime.

For a number of years, such a black hole solution was deemed impossible, and for good reason. In three-dimensional GR, there are no local gravitational degrees of freedom: curvature is algebraically fixed by the matter content, which implies that in a true vacuum the spacetime can only be flat. With a non-vanishing cosmological constant, the field equations admit locally dS and AdS solutions, allowing Λ respectively to be positive or negative, but still preclude solutions with non-trivial curvature, which is indeed everywhere finite and constant. Hence, one was led to believe that black hole solutions in three-dimensional Einstein gravity are impossible.

This argument is evaded by noting that it relies solely on local considerations. Taking account of its global structure, a spacetime can contain a black hole in spite of being locally maximally symmetric. The BTZ solution is an example; it is locally AdS, but it is turned into a black hole spacetime by certain identifications of space-like related events. Ever since its discovery, the BTZ black hole has generated a considerable amount of attention, in large part due to its foreseen applications, particularly in addressing conceptual issues of quantum gravity that become more tractable in three dimensions.

4.3 Three-Dimensional Hořava Gravity

We hope that the three-dimensional black hole solutions in the framework of Hořava gravity presented in the next Sections, can be used as a playground for studying quantum field theory and quantum gravity effects in black hole spacetimes, as has been the case for the BTZ black hole. We also hope that we will gain some insight into the causal structure of black holes in the presence of Lorentz violations – at least the aspects that do not depend on the dimensionality.

The action of Hořava gravity in three dimensions takes the form:

$$S_H = \frac{1}{16\pi G_H} \int dT d^2x \sqrt{h} N \left[L_2 + \frac{1}{M_4^2} L_4 \right], \quad (4.13)$$

where

$$L_2 = K_{ij} K^{ij} - \lambda K^2 + \xi \left({}^{(2)}R - 2\Lambda \right) + \eta a_i a^i, \quad (4.14)$$

h is the determinant of the induced metric h_{ij} on the constant- T hypersurfaces, and $(K_{ij}, K, {}^{(2)}R)$ are its extrinsic, mean and scalar curvatures, respectively. L_4 collectively denotes a set of all terms with four spatial derivatives that are invariant under diffeomorphisms that leave the foliation untouched. Note that, because power-counting renormalizability

requires terms of order $2d$ in derivatives to be present, when $d = 2$ we only need terms with up to four derivatives (the terms collectively denoted as L_6 in Eq. (2.27) are not needed). The full list of such terms for $d = 2$ is given in Ref. [129] and we write them explicitly below:

$$L_4 = g_1 R^2 + g_2 \nabla^2 R + g_3 a_i a^i + g_4 R a_i a^i + g_5 a_i a^i (\nabla \cdot a) + g_6 (\nabla \cdot a)^2 + g_7 (\nabla_i a_j \nabla^i a^j), \quad (4.15)$$

where g_i are couplings with dimensions of an inverse mass squared.

The theory only propagates a scalar degree of freedom because in three dimensions there is no spin-2 graviton [129]. In fact applying the uniformization theorem [130], there is enough gauge freedom to turn the metric tensor h_{ij} in the one of a constant curvature spherical, Euclidean or hyperbolic 2-dimensional space, through a conformal transformation.

Linearizing the theory around flat space and setting $\Lambda = 0$, the dispersion relation for the scalar is found to be [129]

$$\omega^2 = \frac{2(1-\lambda)}{1-2\lambda} \left\{ \frac{\xi^2 k^2 - 2[2\eta g_1 + \xi g_2] k^4 + [g_2^2 - 4g_1(g_6 + g_7)] k^6}{2[\eta + (g_6 + g_7) k^2]} \right\}. \quad (4.16)$$

In the IR limit the latter becomes a standard linear dispersion relation

$$\omega^2 = \frac{\xi^2}{\eta} \frac{1-\lambda}{1-2\lambda} k^2, \quad (4.17)$$

and the low-energy phase velocity of the spin-0 graviton is

$$c_0 = \frac{\xi}{\sqrt{\eta}} \sqrt{\frac{1-\lambda}{1-2\lambda}}. \quad (4.18)$$

For what follows we will focus on the IR limit of the theory by neglecting the L_4 terms. This is expected to be a good approximation so long as the curvature remains small enough and the foliation is sufficiently smooth.

It is important to stress that when the L_4 terms are ignored, the scalar mode has a linear dispersion relation in flat space, whereas, in the full theory the dispersion relation is rational and well approximated by $\omega^2 \sim k^4$ for large momenta. So, excitations with sufficiently high momenta can reach arbitrarily high speeds. Moreover, the theory has an instantaneous mode even in the low-energy limit (see Ref. [122] for a discussion in four dimensions). Both of these facts are particularly relevant for black holes spacetimes. High-energy modes will be able to penetrate surfaces that appear as usual horizons in the low-energy limit of the theory. More importantly, even within the framework of the low-energy approximation, the presence of instantaneous, infinite speed, modes means that information can be transmitted through these horizons.

In what follows, we will still refer to any solution for which the metric that couples minimally to matter fields has a Killing horizon, as a black hole. We choose to do so because, for the matter fields, which we assume to be relativistic, the Killing horizon will be an event horizon. Hence, the spacetime will be a black hole in the conventional (GR) sense. It should be clear, however, that this is actually an abuse of terminology in the context of Hořava gravity, as perturbations that reside in the gravity sector can propagate infinitely fast. So one could have chosen to reserve the term black hole for solutions that have a universal horizon.

4.3.1 Reduced Action

We have already seen in Sec. (2.2.3) that the low-energy part of Hořava gravity can be formulated in a covariant fashion, *i.e.* \mathcal{A} -theory, with the \mathfrak{a} ether assumed to be hypersurface-orthogonal before the variation. We find it convenient to work with such covariantized version of the theory.

In (2 + 1) dimensions, \mathcal{A} -theory with a cosmological constant Λ is defined by the action

$$S_{\mathfrak{a}} = \frac{1}{16\pi G_{\mathfrak{a}}} \int d^3x \sqrt{-g} (-R - 2\Lambda + L_{\mathfrak{a}}), \quad (4.19)$$

where R is the three-dimensional Ricci scalar and $L_{\mathfrak{a}}$ is the one in Eq. (2.17).

Assuming stationarity and circular symmetry, let's write again the most general metric in (2+1) dimensions as

$$ds^2 = Z^2(r)dt^2 - \frac{1}{F^2(r)}dr^2 - r^2(d\phi + \Omega(r)dt)^2. \quad (4.20)$$

The \mathfrak{a} ether field is also just a function of r : $u_{\alpha}(x^{\beta}) = u_{\alpha}(r)$. We shall refer to these as BTZ coordinates.

In three dimensions, u_{α} is hypersurface-orthogonal if and only if $u_{[\alpha} \nabla_{\beta} u_{\gamma]} = 0$ (the square brackets denote anti-symmetrization), which in BTZ coordinates is explicitly $u_t \partial_r u_{\phi} = u_{\phi} \partial_r u_t$. A trivial solution to this is $u_{\phi} = 0$.

More generally, the hypersurface-orthogonality condition can be integrated to give $u_{\phi} = C u_t$, for some constant C . This must hold throughout the spacetime. Since from Eq. (2.29) we have that

$$u_{\alpha} = \frac{\partial_{\alpha} T}{\sqrt{g^{\mu\nu} \partial_{\mu} T \partial_{\nu} T}}, \quad (4.21)$$

if $C \neq 0$, then T (which is the scalar field defining the preferred foliation in every solution) will satisfy $\partial_{\phi} T = C \partial_t T$. This means that the dependence of T on t and ϕ can be only through the combination $\zeta = t + C\phi$.

In other words, we have $T(t, r, \phi) = f(r, \zeta) = f(r, t + C\phi)$, for some arbitrary function

$f(r, \zeta)$. But the coordinate ϕ runs along orbits of the spacelike axial Killing vector of the spacetime. Keeping all other coordinates fixed, there must then exist a constant p so that ϕ and $\phi + p$ refer to the same spacetime event. This means that $f(r, \zeta)$ will either be multivalued on each spacetime event, or it will have to be periodic in both ϕ and t . None of these options seem to be acceptable for a coordinate that is supposed to act as the preferred time of a global foliation. Hence, we shall only focus on æther configurations for which $C = 0$ or $u_\phi = 0$.

With $u_\phi = 0$, the unit norm constraint allows us to parametrize the æther as

$$u_t = \pm \sqrt{Z^2(r)(1 + F^2(r)U^2(r))}, \quad u_r = U(r), \quad (4.22)$$

where we denote u_r by the function $U(r)$ from now on.

With no loss of generality we shall choose the positive (+) branch for u_t , as choosing the alternative (−) branch yields the same reduced action.

Inserting Eqs. (4.20) and (4.22) into Eq. (4.19), discarding boundary terms, and using the Hořava parameters $\{\lambda, \xi, \eta\}$, we arrive at the reduced action [131]

$$S_r = \frac{1}{8G_H} \int dt dr L_r, \quad (4.23)$$

where

$$\begin{aligned} L_r = & \frac{r^3 F}{2Z} (\Omega')^2 - 2\xi Z \left(\Lambda \frac{r}{F} + F' \right) + \frac{r\eta F Z'^2}{Z} + \frac{(1-\lambda)F^3 Z U^2}{r} \\ & + \frac{rFZ \left[1 - \lambda + (1 + \eta - \lambda)F^2 U^2 \right]}{1 + F^2 U^2} (UF' + FU')^2 + r(1 + \eta - \lambda)F^2 U Z' \\ & \cdot \left[U \left(2F' + \frac{FZ'}{Z} \right) + 2FU' \right] + 2(\xi - \lambda)F^2 U \left[FUZ' + Z(UF' + FU') \right]. \end{aligned} \quad (4.24)$$

Requiring stationarity of the reduced action, $\delta S_r = 0$, then supplies the equations of motion. These are the Euler-Lagrange (EL) equations with respect to the functions Z, F, Ω and U .

Results obtained with the reduced action approach should always be treated and interpreted with some caution. Critical points with respect to symmetric variations of the action need not be stationary points with respect to general variations. Therefore, solutions to equations-of-motion that arise from symmetry-reduced actions need not satisfy the full field equations [132–136]. However, any symmetric solution to the full field equation ought to be a critical point with respect to symmetric variations. The equations of motion from symmetric variations then constitute necessary conditions for any solution

to the full field equations.

If one succeeds in integrating them (or a subset of them), one can simply check if the solutions indeed satisfy the full field equations [134]. This is the strategy we adopt here.

4.3.2 Ω Equation

From Eq. (4.24), the EL equation with respect to Ω is

$$\Omega'' + \left(\frac{3}{r} + \frac{F'}{F} - rFZ' \right) \Omega' = 0. \quad (4.25)$$

This can be integrated to give

$$\Omega(r) = c + \mathcal{J} \int \frac{Z(r)}{r^3 F(r)} dr, \quad (4.26)$$

for integration constants \mathcal{J} and c . With the coordinate transformation $\{t \rightarrow t', \phi \rightarrow \phi' - ct'\}$, we can set $c = 0$ without loss of generality. Substituting Ω into each of three remaining EL equations, we are left with a coupled non-linear system in the remaining unknowns $\{Z, F, U\}$, which are too lengthy to be usefully displayed here. In the remainder, we refer to the EL equation corresponding to Z as the Z -equation, and likewise for the others.

4.4 Anti-de Sitter and Asymptotically Anti-de Sitter Solutions

A natural starting point is to look for maximally symmetric solutions in three-dimensional Hořava gravity. After all, the (BTZ) black hole of three-dimensional GR belongs to this class of spacetimes (*i.e.* AdS), and we shall search for solutions that approach BTZ in the appropriate limit.

We shall discover in this Section that any asymptotically-AdS analogue in Hořava gravity can only exist in the $\eta = 0$ sector of the theory (see also Ref. [137]).

In three dimensions, a spacetime is (locally) maximally symmetric if

$$M_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{3} g_{\mu\nu} = 0. \quad (4.27)$$

When F does not vanish identically, one finds that $M_{rr} = 0$ and $M_{\phi\phi} = 0$ can be combined to give

$$Z (FZ' - ZF') = 0, \quad (4.28)$$

from which we conclude that $Z = \kappa F$ is necessary for maximal symmetry. κ is some constant, which we can always set to 1 without loss of generality by a time rescaling. Now, inserting $Z = F$ into Eq. (4.26), which is one of the EL equations, we get

$$\Omega(r) = -\frac{\mathcal{J}}{2r^2}. \quad (4.29)$$

This in turn reduces all of $M_{\alpha\beta} = 0$ into a single differential equation, which can be integrated to give

$$F^2(r) = Z^2(r) = \frac{\mathcal{J}^2}{4r^2} + A + Br^2. \quad (4.30)$$

For such metrics, the scalar curvature is $-6B$, and the geometry is either dS, AdS or flat when $B < 0$, $B > 0$, or $B = 0$, respectively. Note, however, that when $A \neq 1$, these spacetimes have a deficit angle. The literature has referred to these as “quasi asymptotically flat”, but for convenience, we shall call them simply “flat”.

With Eq. (4.26) being a necessary condition, Eq. (4.30) can be taken to be the most general form of a maximally symmetric spacetime in three-dimensional Hořava gravity. In what follows, we shall discover black hole solutions very similar in form.

To check whether metrics of this form indeed exist in three-dimensional Hořava gravity and, if so, to specify their corresponding æther configurations, we return to the EL equations. Since $Z = F$, these now form a coupled system of three non-linear differential equations for $\{F(r), U(r)\}$. For a solution to exist, these equations clearly must not all be independent of each other.

By systematically eliminating terms proportional to ξ and λ in the EL equations, they can be combined to give the equation

$$\eta u_t^3 (u_t' + r u_t'') = 0. \quad (4.31)$$

If $\eta \neq 0$, then we have

$$u_t(r) = c + d \ln r, \quad (4.32)$$

where c and d are arbitrary integration constants. Using Eq. (4.22), one gets

$$U(r) = \pm \sqrt{\frac{(c + d \ln r)^2 - F(r)^2}{F(r)^4}}. \quad (4.33)$$

For AdS space, which is our primary interest here, we have $F^2 \sim \alpha r^2$ with $\alpha > 0$, which clearly leads to an ill-defined æther because $r^2 \gg \ln r$ as $r \rightarrow \infty$.

We conclude from all this that AdS is not a solution in three-dimensional Hořava gravity when $\eta \neq 0$.

The restriction $Z = F$ might seem overly restrictive if we only want to require that the spacetime be AdS only asymptotically. In BTZ coordinates, boundary conditions for

asymptotically AdS spacetimes in three dimensions were previously identified in [138]. These read

$$\begin{aligned}
g_{tt} &= \frac{r^2}{\mathcal{L}^2} + O(1), \\
g_{tr} &= O(r^{-3}), \\
g_{t\phi} &= O(1), \\
g_{rr} &= -\frac{\mathcal{L}^2}{r^2} + O(r^{-4}), \\
g_{r\phi} &= O(r^{-3}), \\
g_{\phi\phi} &= r^2 + O(1),
\end{aligned} \tag{4.34}$$

where \mathcal{L} is the length scale associated with the asymptotic curvature, which is specified by an effective cosmological constant, $\bar{\Lambda} = -1/\mathcal{L}^2$.

These require that our metric functions behave asymptotically as

$$\begin{aligned}
\Omega &= O(r^{-2}), \\
Z &= \frac{r}{\mathcal{L}} + O(r^{-1}), \\
F &= \frac{r}{\mathcal{L}} + O(r^{-1}).
\end{aligned} \tag{4.35}$$

The solution for Ω in Eq. (4.26) satisfies this. Now if $U \sim U_0 r^m$ as $r \rightarrow \infty$, for some unspecified m , then the leading-order terms in the EL equations cannot simultaneously vanish unless $m = -1$ or $m = -3$. More importantly, for either choice of fall-off, it can be shown that η has to be zero.

A lengthy but straightforward demonstration can be found in Appendix (A.1). Furthermore we show in Appendix (A.2) that when $m = -1$, the æther is not orthogonal to constant- t surfaces (*i.e.* it does not align with the timelike Killing vector) asymptotically, but this does happen when $m = -3$.

4.5 Black Hole Solution for $\eta = 0$

4.5.1 The Solution

The considerations of the previous Section suggest that, in looking for a BTZ analogue, we ought to focus on the $\eta = 0$ sector of the theory. In this sector, the EL equations take the generic form

$$(\lambda - 1)(FUZ'' + FZU'' + ZUF'') + \mathcal{H}(Z, Z', F, F', U, U') = 0, \tag{4.36}$$

with \mathcal{H} being a non-linear algebraic function of the unknowns and their derivatives. One way to simplify the problem would be to choose $U = 0$, which would mean choosing a

configuration in which æther is globally aligned with the timelike Killing vector.

This approach was followed in Ref. [139] and parts of Ref. [140],¹ by working directly in the preferred foliation. (We will discuss the correspondence of the two approaches in Appendix (A.3)). Imposing global alignment trivializes the U -equation and kills all second-order derivatives in the remaining EL equations, paving an easier route to exact solutions. However, it is easy to argue that these solutions cannot represent black holes in Hořava gravity. The Killing vector (∂_t) is null at the Killing horizon (or the ergosurface) and spacelike inside it, but the æther has to be timelike everywhere if it is to define a foliation by spacelike hypersurfaces of constant preferred time. Global alignment is thus kinematically impossible in black hole spacetimes. The details about this are found in Appendix (A.4).

Without any a priori assumptions about U , the EL equations can nevertheless be combined to give

$$\frac{4\xi}{\Lambda} r^3 ZF (ZF' - FZ') = 0. \quad (4.37)$$

Since we wish to keep other coupling constants generic, and since neither Z nor F vanish identically, we can conclude that $\eta = 0$ necessitates $Z = \kappa F$, where again we shall set $\kappa = 1$ with no loss of generality.

Using this, the U -equation turns into

$$(\lambda - 1) \left\{ (FU'' + 2UF'') + \frac{1}{r^2 F} \left[2r^2 U F'^2 + 2r F F' (2rU' + U) + F^2 (rU' - U) \right] \right\} = 0, \quad (4.38)$$

and the Z - and F -equations collapse into a single equation (which we shall not display here due to its length). The special case $\lambda = 1$ is discussed in Appendix (A.5).

If $\lambda \neq 1$, with the change of variables

$$y = UF^2, \quad (4.39)$$

Eq. (4.38) turns into the simple differential equation

$$r^2 \frac{d^2 y}{dr^2} + r \frac{dy}{dr} - y = 0. \quad (4.40)$$

This turns out to be equivalent to the condition that constant preferred time surfaces have constant mean curvature.² (See Appendix (A.6)). The general solution to Eq.

¹Ref. [140] contains a collection of static (non-rotating) solutions for special values of the parameters ξ , λ , and η and/or restrictions in the metric ansatz. These special choices seem to be motivated by the fact that they simplify the calculations and make it easier to obtain analytic solutions. The diagonal solutions in the preferred foliation are not black holes for the reasons discussed above while the causal structure of the non-diagonal solutions and the behaviour of the corresponding foliation is left unexplored.

²The fact that constant preferred time surfaces have constant mean curvature is also a property of Cuscuton theory, which has been argued to be related to Hořava gravity [141, 142].

(4.40) is

$$y = UF^2 = \frac{a}{r} + br, \quad (4.41)$$

where a and b are integration constants.

Therefore, U and F have to be related in the following way:

$$U = \frac{1}{F^2} \left(\frac{a}{r} + br \right). \quad (4.42)$$

Inserting this into either the Z - or F -equation, we get

$$\frac{1}{2} \frac{d}{dr} (F^2) + \left[\frac{\mathcal{J}^2 + 4a^2(1-\xi)}{4\xi} \right] \frac{1}{r^3} - \left[b^2 \left(\frac{2\lambda - \xi - 1}{\xi} \right) - \Lambda \right] r = 0. \quad (4.43)$$

This leads to the metric functions

$$F^2(r) = Z^2(r) = -\mathcal{M} + \frac{\bar{\mathcal{J}}^2}{4r^2} - \bar{\Lambda}r^2, \quad (4.44a)$$

$$\Omega(r) = -\frac{\mathcal{J}}{2r^2}, \quad (4.44b)$$

where \mathcal{M} is an integration constant and

$$\bar{\mathcal{J}}^2 = \frac{\mathcal{J}^2 + 4a^2(1-\xi)}{\xi}, \quad (4.45)$$

$$\bar{\Lambda} = \Lambda - \frac{b^2(2\lambda - \xi - 1)}{\xi}. \quad (4.46)$$

In the limit to GR ($\lambda \rightarrow 1$, $\xi \rightarrow 1$), Eq. (4.44) gives the familiar BTZ metric. When $\xi = 1$, and thus $\bar{\mathcal{J}} = \mathcal{J}$, the solution becomes the BTZ metric with a shifted cosmological constant, $\bar{\Lambda} = \Lambda - 2b^2(\lambda - 1)$. Note that $\bar{\mathcal{J}}^2$ can be negative; this happens when either $\xi < 0$ or $\xi > 1$, $a^2 > \mathcal{J}^2/(4(\xi - 1))$.

The æther configuration for this metric is

$$u_r = \frac{1}{F^2} \left(\frac{a}{r} + br \right), \quad (4.47a)$$

$$u_t = \sqrt{F^2 + \left(\frac{a}{r} + br \right)^2}. \quad (4.47b)$$

Since a vanishing u_r signifies alignment of the æther with the timelike Killing vector, the constants a and b can be regarded as measures of æther misalignment. Of these two æther parameters, b is what dominates asymptotically and is what affects the asymptotic behaviour of the metric.

As shown in Appendix (A.2), if $b \neq 0$, then the æther does not align with the timelike Killing vector asymptotically. Thus, the parameter b can be understood to be a measure of asymptotic misalignment.

Taken together, Eqs. (4.44) and (4.47) give the most general metric and æther configuration in the $\eta = 0$ sector. It is a four-parameter family of solutions, specified by $\{\mathcal{M}, \mathcal{J}, a, b\}$.

Unless one imposes restrictions on the parameters, u_t can become imaginary in parts of the spacetime. That would signal a breakdown of the foliation. It is reasonable to restrict one's attention to solutions for which a foliation exists all the way to the singularity, as in Hořava gravity the existence of a spacelike foliation is a fundamental feature. This can be achieved by imposing the condition $F^2 + (a/r + br)^2 > 0$ or

$$\frac{1}{r^2} \left((b^2 - \bar{\Lambda}) r^4 + (2ab - \mathcal{M}) r^2 + \left(\frac{\bar{\mathcal{J}}^2}{4} + a^2 \right) \right) > 0. \quad (4.48)$$

As $r \rightarrow 0$, the combination $a^2 + \bar{\mathcal{J}}^2/4$ or $(\mathcal{J}^2/4 + a^2)/\xi$ dominates u_t^2 , and so it must be positive. Thus, in order to ensure the existence of a foliation close to the singularity, we are restricted to working in the domain $\xi > 0$.

At large r , the term whose coefficient is $(b^2 - \bar{\Lambda})$ dominates instead. This coefficient is always positive for AdS asymptotics, as $\bar{\Lambda} < 0$. For dS asymptotics one would have to impose that $b^2 > \bar{\Lambda}$ in order for the foliation to not end at some finite r .

4.5.2 Curvature Scalars and Asymptotics

A quick calculation of the scalar curvature leads to

$$R = -6\bar{\Lambda} + \frac{1}{2r^4} (\bar{\mathcal{J}}^2 - \mathcal{J}^2), \quad (4.49)$$

which is not constant and generically diverges at $r = 0$. When $\xi = 1$, we have $\bar{\mathcal{J}} = \mathcal{J}$, so the Ricci scalar is constant, but it can be of either sign depending on λ , Λ , and b . The Kretschmann scalar also diverges at $r = 0$:

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12\bar{\Lambda}^2 - \frac{2\bar{\Lambda}}{r^4} (\bar{\mathcal{J}}^2 - \mathcal{J}^2) + \frac{11}{4r^8} (\bar{\mathcal{J}}^2 - \mathcal{J}^2)^2. \quad (4.50)$$

These imply that $r = 0$ is a curvature singularity, unless $\bar{\mathcal{J}}^2 = \mathcal{J}^2$. This is in contrast to the BTZ black hole for which $r = 0$ is neither a curvature nor a conical singularity, but is instead a ‘‘causal’’ singularity where both the Ricci and Kretschmann scalars are finite and perfectly smooth.

The metric can be (quasi) asymptotically flat, dS, or AdS, irrespective of the sign of the (bare) cosmological constant, Λ (which will be negative, $\Lambda = -1/l^2$, for BTZ). The sign of the effective cosmological constant,

$$\bar{\Lambda} = \Lambda - \frac{b^2(2\lambda - \xi - 1)}{\xi}, \quad (4.51)$$

determines the asymptotic behaviour of the metric.

4.5.3 Setting $\xi = 1$ by Redefinitions

It is clear that $\xi = 1$ is a special value for the solution we have found. The metric reduces to the BTZ solution with an effective cosmological constant given by Eq. (4.51), and the curvature singularity disappears.

However, one can actually set $\xi = 1$ by means of field redefinitions. In the preferred frame picture, one can perform a constant rescaling of the lapse function N . If one sets the new lapse $N' = \sigma N$, action (2.27) (always restricting attention on the L_2 part only) remains invariant apart from an overall factor and after the following parameter rescaling

$$\xi' = \frac{\xi}{\sigma}, \quad (4.52a)$$

$$\eta' = \frac{\eta}{\sigma}, \quad (4.52b)$$

$$\lambda' = \lambda, \quad (4.52c)$$

$$\Lambda' = \frac{\Lambda}{\sigma}. \quad (4.52d)$$

This implies that, with the choice $\sigma = \xi$, any theory in the sector $\{\eta = 0, \xi > 0\}$ can be mapped onto $\{\eta = 0, \xi = 1\}$.

In the covariant picture, the corresponding redefinition is the one already presented in Eq. (2.25) with the same rescaling for Λ (where σ is restricted to be positive so that the new metric is Lorentzian).

In order to get an equivalent primed action, we only need to consider the first three relations between the coefficients c'_i and c_i in Eq. (2.26), because the æther is now restricted to be hypersurface orthogonal. Using the correspondence in Eq. (2.35) one can verify that choosing $\sigma = \xi$, one can set ξ to 1.

Clearly, using these redefinitions allows one to work with a more familiar spacetime, since the metric now reduces to the BTZ solution with an effective cosmological constant, which is free of curvature singularities, even if it does not actually simplify the derivation of the solution significantly.

However, we will choose not to follow this route. Such a redefinition is only allowed in vacuum. If other fields couple to the lapse, the shift and the induced metric (or the spacetime metric and the æther), then such a redefinition no longer leaves the action invariant. Additionally, one might be interested specifically in the spacetime structure of $g_{\mu\nu}$.

For instance, in four dimensions one can require that $g_{\mu\nu}$ couples minimally to the matter in order for the equivalence principle to be satisfied. This would make this metric

Case	$\bar{\mathcal{J}}^2 \bar{\Lambda} \geq 0$	$\bar{\mathcal{J}}^2 = 0$	$-\mathcal{M}^2 < \bar{\mathcal{J}}^2 \bar{\Lambda} < 0$	$\bar{\mathcal{J}}^2 \bar{\Lambda} = -\mathcal{M}^2$
$\mathcal{M} > 0, \bar{\Lambda} < 0$	horizons \tilde{r}_+ (b) singularity spacelike	$\sqrt{2}r_{(1/2)}$ (b) spacelike	r_{\pm} (b) timelike	$r_{(1/2)}$ (b) timelike
$\mathcal{M} > 0, \bar{\Lambda} > 0$	horizons \tilde{r}_- (c) singularity timelike	— spacelike	— spacelike	— spacelike
$\mathcal{M} < 0, \bar{\Lambda} < 0$	horizons \tilde{r}_- (b) singularity spacelike	— timelike	— timelike	— timelike
$\mathcal{M} < 0, \bar{\Lambda} > 0$	horizons \tilde{r}_+ (c) singularity timelike	$\sqrt{2}r_{(1/2)}$ (c) timelike	r_+ (c), r_- (b) spacelike	$r_{(1/2)}$ (c) spacelike

TABLE 4.1 Killing horizons and the nature of the curvature singularity for various cases. Each of the Killing horizons is denoted either by a **(c)** for de Sitter (cosmological) horizon, or **(b)** for black hole (event) horizon. Their locations are specified by: $r_{\pm}^2 := |\mathcal{M}/(2\bar{\Lambda})| [1 \pm (1 - |\bar{\mathcal{J}}^2 \bar{\Lambda}|/\mathcal{M}^2)^{1/2}]$, $\tilde{r}_{\pm}^2 := |\mathcal{M}/(2\bar{\Lambda})| [(1 + |\bar{\mathcal{J}}^2 \bar{\Lambda}|/\mathcal{M}^2)^{1/2} \pm 1]$, $r_{(1/2)}^2 := |\mathcal{M}/(2\bar{\Lambda})|$. $\bar{\Lambda} = 0$ is excluded from this table, simply because $\bar{\mathcal{J}}^2 \bar{\Lambda}$ vanishes and the sign of $\bar{\mathcal{J}}^2$ cannot be immediately inferred. In this case, an asymptotically flat black hole exists for $\{\mathcal{M} = -1, \bar{\mathcal{J}}^2 < 0\}$ and the horizon radius is $r_o := \sqrt{-\bar{\mathcal{J}}^2}/2$.

distinct.

Here we are considering three dimensions, but if we want to use our solutions to understand something about four-dimensional black holes it seems prudent to understand the structure of $g_{\mu\nu}$ itself. As we will see later on, the causal structure of the two metrics can also be different.

4.5.4 Metric Horizons and Causal Structure

In stationary spacetimes, horizons are null, stationary surfaces. The normal to any stationary surface must be proportional to $\partial_{\alpha} r$, and this is null when

$$g^{\alpha\beta} (\partial_{\alpha} r) (\partial_{\beta} r) = g^{rr} = -F^2 = 0, \quad (4.53)$$

or

$$g^{rr} = \frac{\bar{\Lambda}}{r^2} (r^2 - r_+^2) (r^2 - r_-^2) = 0, \quad (4.54)$$

where

$$r_{\pm} = \sqrt{-\frac{\mathcal{M}}{2\bar{\Lambda}} \left(1 \pm \sqrt{1 + \frac{\bar{\mathcal{J}}^2 \bar{\Lambda}}{\mathcal{M}^2}} \right)} \quad (4.55)$$

are the locations of the horizons.

For there to be two horizons (*i.e.* for both values in Eq. (4.55) to be real), both $\mathcal{M}/\bar{\Lambda}$

and $\bar{\mathcal{J}}^2 \bar{\Lambda}$ must at least be negative. In which case, we can write

$$\mathcal{M} = -\bar{\Lambda} (r_+^2 + r_-^2), \quad \bar{\mathcal{J}}^2 = -4\bar{\Lambda}(r_+ r_-)^2. \quad (4.56)$$

The case $\{\bar{\Lambda} < 0, \mathcal{M} > 0, \bar{\mathcal{J}}^2 > 0\}$ corresponds closely to the BTZ solution of GR. (The reader is referred to Fig. (4.1) for its causal structure).

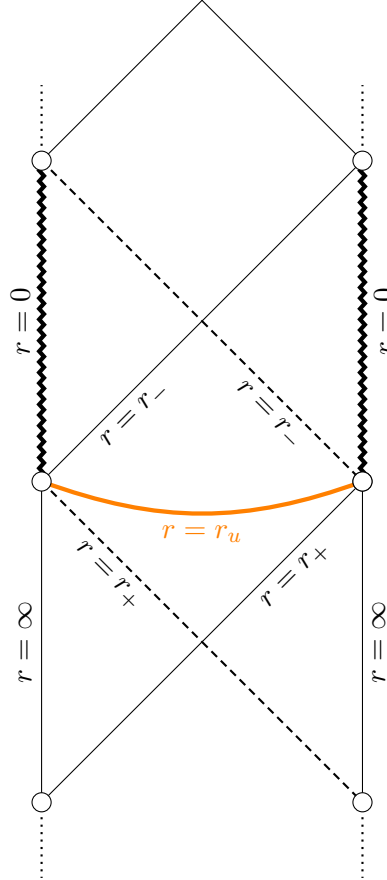


FIGURE 4.1 Penrose diagram for $\mathcal{M} > 0, \bar{\Lambda} < 0, \bar{\mathcal{J}}^2 > 0$. This is equivalent to the rotating BTZ case, except for the curvature singularity at $r = 0$. The dashed lines represent null leaves of the foliation. Along these surfaces, the æther diverges because it becomes lightlike. The orange solid curve represents the universal horizon (for choices of the parameters for which it is present).

For this BTZ-like branch of our solutions, there exists an analogous “angular momentum” bound

$$\bar{\mathcal{J}}^2 \leq \frac{\mathcal{M}^2}{|\bar{\Lambda}|}, \quad (4.57)$$

which guarantees that r_{\pm} are both real. These are the locations of the inner and outer horizons of the black hole. When the bound is saturated, the horizons coincide at $r = r_{(1/2)} := \sqrt{|\mathcal{M}/(2\bar{\Lambda})|}$. The inner horizon approaches $r = 0$ when $\bar{\mathcal{J}}^2 \rightarrow 0^+$, while keeping a fixed $\bar{\Lambda} < 0$.

As $\bar{\Lambda} \rightarrow 0^-$, while keeping $\bar{\mathcal{J}}^2 > 0$, r_+ gets pushed to infinity so that only the interior

of the black hole remains. This is similar to the situation in three-dimensional GR, where the black hole can only be asymptotically AdS, because the relevant parameter is a strictly non-negative \mathcal{J}^2 , rather than $\bar{\mathcal{J}}^2$.

Remarkably, there exist solutions with black hole horizons and dS (Fig. (4.2)) or flat asymptotics (Fig. (4.3)).

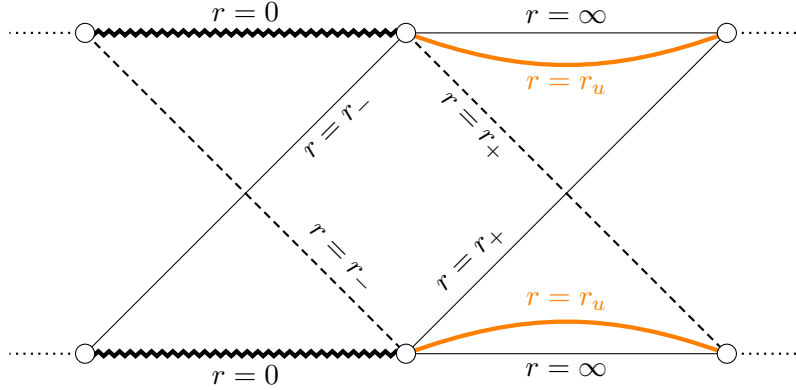


FIGURE 4.2 Penrose diagram for $\mathcal{M} < 0, \bar{\Lambda} > 0, \bar{\mathcal{J}}^2 < 0$. This is equivalent to the Penrose diagram for Schwarzschild-de Sitter spacetime. The dashed lines represent null leaves of the foliation. The orange solid curves represent universal horizons (for choices of the parameters for which they are present).

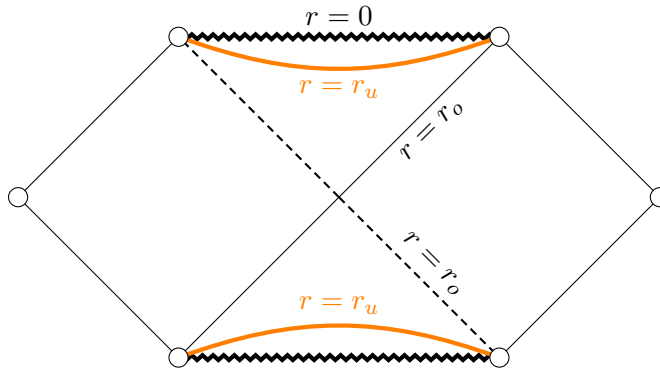


FIGURE 4.3 Penrose diagram for an asymptotically flat black hole, whose causal structure is essentially that of the Schwarzschild spacetime. The dashed line represents a null leaf of the foliation. The orange solid curves represent universal horizons (for choices of the parameters for which they are present).

In particular, when $\{\bar{\Lambda} > 0, \mathcal{M} < 0, \bar{\mathcal{J}}^2 < 0\}$, r_{\pm} are both still real and their associated hypersurfaces are both Killing horizons. But since $\bar{\Lambda} > 0$, r_+ corresponds to the dS horizon, and r_- takes the role of the black hole event horizon. For $\{\bar{\Lambda} = 0, \mathcal{M} = -1, \bar{\mathcal{J}}^2 < 0\}$, one obtains an asymptotically flat black hole with a horizon at $r = r_o := \sqrt{-\bar{\mathcal{J}}^2}/2$ (for $\mathcal{M} \neq -1$ the asymptotics would be “quasi asymptotically flat”). In GR such solutions do not exist because \mathcal{J}^2 plays the role of $\bar{\mathcal{J}}^2$, and \mathcal{J}^2 is strictly non-negative.

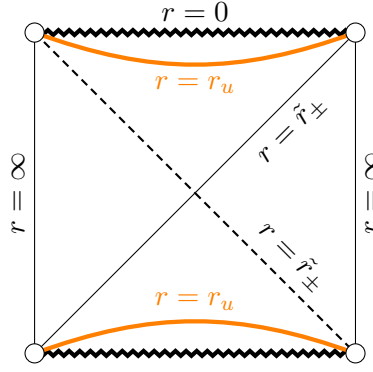


FIGURE 4.4 Penrose diagram for $\bar{\Lambda} < 0$, $\bar{\mathcal{J}}^2 < 0$, and for either sign of \mathcal{M} ($r = \tilde{r}_+$ for $\mathcal{M} > 0$ and $r = \tilde{r}_-$ for $\mathcal{M} < 0$). The dashed line represents a null leaf of the foliation. The orange solid curves represent universal horizons (for choices of the parameters for which they are present).

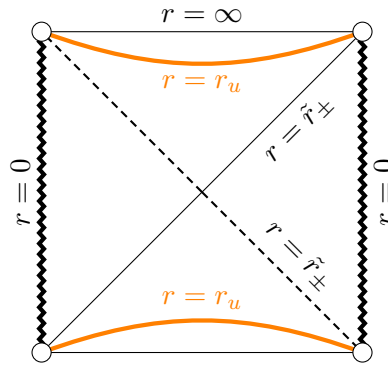


FIGURE 4.5 Penrose diagram for $\bar{\Lambda} > 0$, $\bar{\mathcal{J}}^2 > 0$ and for either sign of \mathcal{M} ($r = \tilde{r}_+$ for $\mathcal{M} < 0$ and $r = \tilde{r}_-$ for $\mathcal{M} > 0$). The dashed line represents a null leaf of the foliation. The orange solid curves represent universal horizons (for choices of the parameters for which they are present).

Other possibilities exist for which there is only one Killing horizon, which can be either an event horizon or a dS horizon, depending on the sign of the cosmological constant. Figs. (4.4) and (4.5) are examples of spacetimes with only one Killing horizon, the former being a black hole horizon and the latter a cosmological horizon. Many of these cases are summarized in Table (4.1), which also provides the respective positions of the Killing horizons for convenience.

We note as well that these spacetimes can have ergoregions. These are demarcated by $r = r_{\text{erg}}$, such that $g_{tt}(r_{\text{erg}}) = Z^2 - r_{\text{erg}}^2(\Omega)^2 = 0$. The ergosurfaces are thus located at

$$r_{\pm}^{\text{erg}} = \sqrt{-\frac{\mathcal{M}}{2\bar{\Lambda}} \left(1 \pm \sqrt{1 + \frac{\bar{\Lambda}}{\mathcal{M}^2} \Delta} \right)}. \quad (4.58)$$

This is essentially Eq. (4.55), with the replacement $\bar{\mathcal{J}}^2 \rightarrow \Delta := \bar{\mathcal{J}}^2 - \mathcal{J}^2$. The key parameter is then

$$\Delta := \bar{\mathcal{J}}^2 - \mathcal{J}^2 = \left(\frac{1-\xi}{\xi} \right) (\mathcal{J}^2 + 4a^2). \quad (4.59)$$

When $\xi = 1$, the ergosurface is uniquely at $r_{\text{erg}} = \sqrt{\mathcal{M}/|\bar{\Lambda}|}$. We thus recover the BTZ case in GR for which $r_- \leq r_+ \leq r_{\text{erg}}$. In the parameter region $0 < \xi \leq 1$ and for the BTZ-like case $\{\bar{\Lambda} < 0, \mathcal{M} > 0, \bar{\mathcal{J}}^2 > 0\}$, we have $\bar{\mathcal{J}}^2 > \Delta \geq 0$, and

$$r_-^{\text{erg}} \leq r_- \leq r_+ \leq r_+^{\text{erg}}. \quad (4.60)$$

Outside the parameter region $0 < \xi \leq 1$, Δ is negative, and r_-^{erg} becomes imaginary and so there is no ‘‘inner’’ ergosurface. Various other cases can be easily worked out, but they shall not be our concern for the rest of the solutions analysis.

Our next goal shall be to get a better sense of the spacetime’s causal structure, for which we shall also need to know the character of its singularity, in addition to identifying its horizons and the nature of its asymptotic infinities. This is generally controlled by $\bar{\mathcal{J}}^2$, whose sign dictates the behaviour of F^2 as $r \rightarrow 0$.

Consider first the case $\bar{\mathcal{J}}^2 \neq 0$. Then as $r \rightarrow 0$, $g^{rr} = g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = -F^2 \sim -\bar{\mathcal{J}}^2/(4r^2)$. The normal to constant- r surfaces is then spacelike when $\bar{\mathcal{J}}^2 > 0$ (like the rotating BTZ black hole) or timelike when $\bar{\mathcal{J}}^2 < 0$. When $\bar{\mathcal{J}}^2 = 0$ and $\mathcal{J}^2 \neq 0$, there will still exist a curvature singularity, but whether it is timelike or spacelike now depends on the sign of \mathcal{M} , since $g^{rr} \rightarrow \mathcal{M}$ as $r \rightarrow 0$. When $\mathcal{M} > 0$ ($\mathcal{M} < 0$), the singularity is spacelike (timelike). The spacelike nature of $r = 0$ in the positive- \mathcal{M} case corresponds to the non-rotating BTZ black hole.

We have already mentioned in the previous Section that the causal structure of the redefined metrics that lead to $\xi = 1$ is different from that of $g_{\mu\nu}$. This should be clear now, as, in a suitable coordinate system, the redefined metric takes the same form as $g_{\mu\nu}$ but with $\xi = 1$, so it is always a BTZ spacetime (potentially with different asymptotics than those of $g_{\mu\nu}$). Consider, for example, the asymptotically flat black holes that were discussed above and assume $J = 0$ and $b = 0$ (to avoid divergence of the æther asymptotically). The redefinition will lead to flat spacetime with a non-trivial æther.

4.5.5 Energy Conditions

It is now convenient to check energy conditions using the preferred rest frame of the æther, in order to prove the consistency of our solutions. The weak energy condition

(WEC) states that the energy density measured by an arbitrary observer must be positive. Taking this observer to be at rest with respect to the æther, we have

$$T_{\alpha\beta}^{\text{æ}} u^\alpha u^\beta = \frac{\bar{\mathcal{J}}^2 - \mathcal{J}^2 + 4\bar{\Lambda}r^4}{4r^4} \geq 0, \quad (4.61)$$

where $T_{\alpha\beta}^{\text{æ}} := R_{\alpha\beta} - (1/2)g_{\alpha\beta}R$, and u^α is the æther vector field.

Insisting that this condition is satisfied for all r requires $\bar{\Lambda} > 0$ and $\bar{\mathcal{J}}^2 \geq \mathcal{J}^2$. AdS asymptotics thus violates the WEC. On the other hand, in order to have black hole horizons in solutions with dS or flat asymptotics, $\bar{\mathcal{J}}^2$ has to be negative.

This means that the WEC is violated in these solutions as well. Hence, all our black hole solutions violate the WEC. Since the WEC is a necessary condition for the dominant energy condition (DEC), all our black holes violate the DEC as well.

Violating the DEC is to be expected from the work of Ida [143], which states that if the DEC is satisfied then the spacetime cannot have apparent horizons. Since apparent horizons are also event horizons in stationary spacetimes, this result precludes the existence of black holes when the DEC holds.

4.5.6 Foliation and Universal Horizons

4.5.6.1 Regularity of the Æther

In the previous Section, we focused mainly on the geometry of our solution, that is, on the metric and its properties. The causal structure of this metric is what is relevant to matter degrees of freedom minimally coupled to it. The second half of the solution is the æther field, or more precisely, the foliation it specifies.

We shall first look at how the æther behaves along the horizons in the maximal extension. For this it is sufficient to follow the Kruskal construction that brings the line-element to the form

$$ds^2 = \bar{\Omega}(r)^2 dUdV - r^2(d\bar{\phi}^2 + N^\phi(r)dt^2), \quad (4.62)$$

in terms of null coordinates U and V , where $t = t(U, V)$, $r = r(U, V)$, $\bar{\phi} = \bar{\phi}(\phi, t(U, V))$. Several charts are generally needed to cover the full manifold, depending on how many Killing horizons the spacetime has. Only one chart is needed for the asymptotically flat case ($\mathcal{M} = -1$, $\bar{\Lambda} = 0$, $\bar{\mathcal{J}}^2 < 0$), which in BTZ coordinates has

$$F^2(r) = \frac{1}{r^2} (r^2 - R_0^2), \quad (4.63)$$

where $R_0 = \sqrt{|\mathcal{J}^2|}/2$. The standard Kruskal coordinates are then

$$U = \mp e^{-\kappa u}, \quad (4.64a)$$

$$V = e^{\kappa v}, \quad (4.64b)$$

where $u = t - r^*$, $v = t + r^*$, $r^* := \int F^{-2} dr$, and $\kappa = 1/R_0 = 2/\sqrt{|\mathcal{J}^2|}$ is the surface gravity of the horizon. The upper sign (-) is for the region $r > R_0$ and the lower sign (+) is for $r < R_0$. In this case, r depends on U and V through

$$e^{2\kappa r} \left(\frac{\kappa r - 1}{\kappa r + 1} \right) = -UV, \quad (4.65)$$

and $\bar{\Omega}(r) = (1/\kappa) (1 + 1/(\kappa r)) \exp(-\kappa r)$. These coordinates are clearly regular through the Killing horizon.

The æther has components

$$u_U = \frac{1}{2\kappa U} \left[\left(\frac{a}{r} + br \right) \mp \sqrt{\left(\frac{a}{r} + br \right)^2 + F^2} \right], \quad (4.66a)$$

$$u_V = \frac{1}{2\kappa V} \left[\left(\frac{a}{r} + br \right) \pm \sqrt{\left(\frac{a}{r} + br \right)^2 + F^2} \right], \quad (4.66b)$$

where the upper signs hold for the future-pointing solution, $u_t > 0$, which we have chosen to work with in the text, while the lower signs hold for the past-pointing solution, $u_t < 0$, which we have hitherto disregarded.

Close to R_0 , one can verify that $F^2 \simeq 2\kappa(r - R_0)$ and $r^* \simeq (2\kappa)^{-1} \ln |\kappa(r - R_0)|$, which imply $F^2 \simeq -2UV$. Therefore, as $U \rightarrow 0, V \rightarrow 0$ we have

$$u_U \simeq \frac{1}{2\kappa U} \left[h_0 \mp |h_0| \left(1 - \frac{UV}{h_0^2} \right) \right], \quad (4.67a)$$

$$u_V \simeq \frac{1}{2\kappa V} \left[h_0 \pm |h_0| \left(1 - \frac{UV}{h_0^2} \right) \right], \quad (4.67b)$$

where $h_0 := a/R_0 + bR_0$, which we assume not to vanish. Moreover, we shall assume for now that $h_0 > 0$.

For the future-pointing solution, we therefore have

$$u_U \simeq \frac{V}{2\kappa h_0}, \quad (4.68a)$$

$$u_V \simeq \frac{h_0}{\kappa V}, \quad (4.68b)$$

as $r \rightarrow R_0$. The future-pointing solution is thus regular at the future event horizon ($U = 0$), but is divergent at the past event horizon ($V = 0$). This divergence arises because the foliation turns null. In the various Penrose diagrams, we mark the singularity

of the æther with dashed lines.

On the other hand, the past-pointing solution behaves like

$$u_U \simeq \frac{h_0}{\kappa U}, \quad (4.69a)$$

$$u_V \simeq \frac{U}{2\kappa h_0}, \quad (4.69b)$$

and is thus regular at the past event horizon ($V = 0$) but divergent at the future event horizon ($U = 0$).

This analysis also applies to the AdS case. For this, at least two charts are needed, each respectively in the neighborhoods of the two Killing horizons. The Kruskal coordinates for the flat space case carry over exactly to the region containing the outer horizon, except that the surface gravity is now $\kappa_+ := -\bar{\Lambda}(r_+^2 - r_-^2)/r_+$. Clearly then, the future-pointing æther field is again regular at the future event horizon and singular at the past event horizon. Around the inner horizon, one installs the usual coordinates, $U_- = \mp \exp(\kappa_- u)$, $V_- = -\exp(-\kappa_- v)$, where $\kappa_- := -\bar{\Lambda}(r_+^2 - r_-^2)/r_-$.

In these coordinates, the future-pointing æther can be seen to diverge at $V_- = 0$ and to remain regular at $U_- = 0$. A pattern thus emerges where the future-pointing æther diverges along past event horizons ($V^* = 0$) and is regular along future event horizons ($U^* = 0$), where $\{U^*, V^*\}$ are the outgoing/ingoing Kruskal coordinates adapted to an arbitrary Killing horizon. This holds for the dS spacetimes as well.

4.5.6.2 Universal Horizons

As already discussed, a universal horizon exists when a constant preferred time (constant- T) hypersurface coincides with a constant- r hypersurface. This constant- r hypersurface will then act as an absolute causal boundary for all the modes, irrespective of their speed, because any sort of physical process is presumed to proceed in the direction of increasing T . Therefore, any constant- r hypersurface that happens to coincide with a constant- T hypersurface (*i.e.*, a leaf of the foliation) can only be crossed in one direction.

Since $u_\phi = 0$, there will be a universal horizon when

$$\partial_\alpha r \propto u_\alpha, \quad (4.70)$$

or equivalently, when $u_\alpha t^\alpha = 0$, where t^α is the timelike Killing vector. For the class of solutions given by Eq. (4.44), the universal horizon is given by the surface $r(x^\alpha) = r_u$, where r_u satisfies

$$u_t^2 = F(r_u)^2 + \left(\frac{a}{r_u} + br_u \right)^2 = 0, \quad (4.71)$$

or

$$(b^2 - \bar{\Lambda})r^4 + (2ab - \mathcal{M})r^2 + \left(a^2 + \frac{\bar{\mathcal{J}}^2}{4}\right) = 0. \quad (4.72)$$

The roots are

$$(r_u^\pm)^2 = \frac{\mathcal{M} - 2ab}{2(b^2 - \bar{\Lambda})} \pm \frac{1}{2(b^2 - \bar{\Lambda})} \left[(\mathcal{M} - 2ab)^2 - (4a^2 + \bar{\mathcal{J}}^2)(b^2 - \bar{\Lambda}) \right]^{1/2}. \quad (4.73)$$

If the discriminant is negative then the roots will be imaginary and there will not be any universal horizon. If the discriminant is positive both roots in Eq. (4.73) will be real and distinct. But then there will exist a region, $r_u^- < r < r_u^+$, where the æther turns imaginary and the foliation will have to end on that largest of the two roots. So, for the foliation to extend all the way to the singularity and still have a universal horizon one needs to require that

$$\frac{(4a^2 + \mathcal{J}^2)(b^2 - \bar{\Lambda}(b))}{\xi(\mathcal{M} - 2ab)^2} = 1. \quad (4.74)$$

We can use this constraint to express a in terms of the other parameters $\{\mathcal{M}, \mathcal{J}, b\}$, thus reducing the dimension of the parameter space to three.

Assuming that the resulting r_u^2 is real (which imposes a further constraint on the parameters), the universal horizon is uniquely located at

$$r_u^2 = \frac{\mathcal{M} - 2a_\pm(\mathcal{M}, \mathcal{J}, b)b}{2(b^2 - \bar{\Lambda}(b))}, \quad (4.75)$$

where a is now understood to depend on the other parameters through Eq. (4.74). Because Eq. (4.74) is quadratic in a , there will generally be two values of a (which we denote by a_\pm) for every choice of $\{\mathcal{M}, \mathcal{J}, b\}$. Each particular triple $\{\mathcal{M}, \mathcal{J}, b\}$ can represent two distinct solutions, each possibly harboring a universal horizon.

4.5.6.3 Black Holes with Universal Horizons

For a BTZ-like solution with AdS asymptotics, the universal horizon is located between the outer and inner event horizons. This is illustrated in Fig. (4.6). Note that b and \mathcal{J} are dimensionful quantities ($[b] = 1/L$, $[\mathcal{J}] = L$); for the plots we use their dimensionless versions $\bar{r} := r/l$ and $\bar{b} := bl$, where l is the “bare” cosmological length scale, $l := 1/\sqrt{|\Lambda|}$. Fig. (4.6) shows the positions of the horizons as a function of \bar{b} , keeping other parameters fixed at $\{\mathcal{M} = 10, \mathcal{J}/l = 0.1\}$ and with the coupling constants set to be $\{\xi = 1/2, \lambda = 1\}$.

We have also chosen the sign of the bare cosmological constant to be negative, so that $\bar{\Lambda}(b=0) < 0$. To ensure that the æther represents a well-defined foliation at large r for any value of b , we need to work within the parameter region $\{\xi > 0, \lambda > 1/2\}$. With

$\Lambda < 0$, any choice from this region guarantees that $(b^2 - \bar{\Lambda}(b)) = -\Lambda + b^2(2\lambda - 1)/\xi > 0$ is non-negative for any value of b .

Moreover, if one chooses them such that $\lambda \geq (1 + \xi)/2$, then $\bar{\Lambda}$ is always negative for any b . Fig. (4.6) is such a case, where all values of b give regular AdS black holes. Fig. (4.1) shows the locations of the universal horizon in the Penrose diagram of an AdS black hole spacetime.

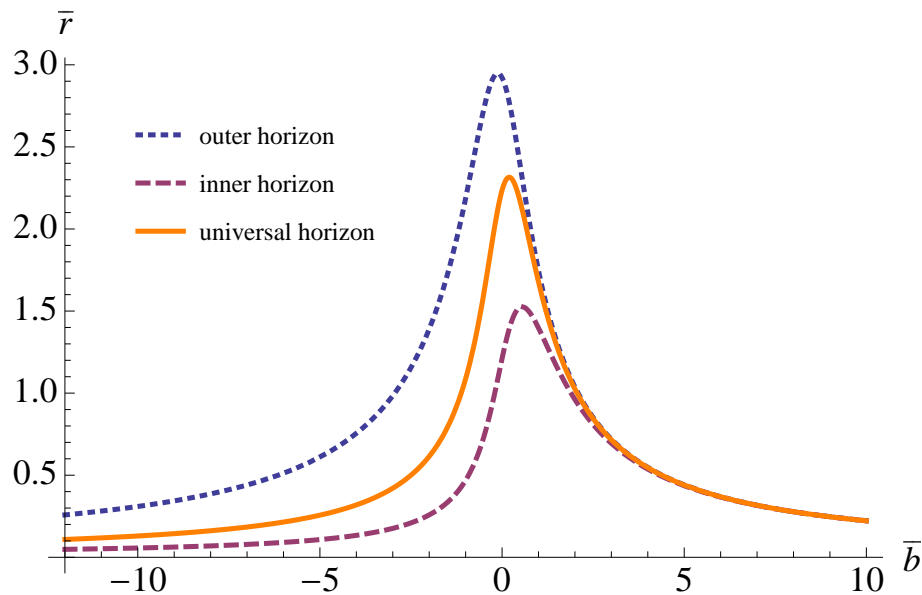


FIGURE 4.6 Radial positions of various horizons in a BTZ-like anti-de Sitter black hole.

Now if the coupling constants are such that $\{\xi > 0, \lambda > 1/2\}$ and $\lambda < (1 + \xi)/2$, then $\bar{\Lambda}$ will switch sign at some value of b . When this happens, the æther charge b radically changes the causal structure of the spacetime. In Fig. (4.7), we have an example of a spacetime starting with AdS asymptotics at $b = 0$ and turning asymptotically dS as b is increased. This plot is made with the parameters $\{\mathcal{M} = 10, \mathcal{J}/l = 0.1\}$, but with $\{\xi = 3/4, \lambda = 3/4\}$.

One can verify that the spacetime turns dS at $\bar{b} = \pm\sqrt{3}$. The shaded regions denote solutions that are asymptotically dS, but these solutions are not black holes since $\mathcal{M} > 0$ and $\bar{\mathcal{J}}^2 > 0$. (For $0 < \xi < 1$, $\bar{\mathcal{J}}^2$ is always positive). Only the unshaded regions – those with AdS asymptotics – are black holes.

Interestingly, within the AdS region, there is a kink in the curves, $\bar{r}_{\pm}(\bar{b})$. For this case, it occurs around $\bar{b} = -1.2247$, which is where $1 + \bar{\mathcal{J}}^2 \bar{\Lambda}/\mathcal{M}^2$ vanishes. We note that while both curves touch, they do not cross over. At this point, which is also where all horizons meet, $\bar{r}_{\pm}(\bar{b})$ are continuous but not differentiable with respect to the parameter \bar{b} .

As the transition from AdS to dS asymptotics is made, the outer horizon is pushed to $r = \infty$, leaving as the “outer” region of the asymptotically dS spacetime what was formerly the interior of the AdS black hole. At the same time, the inner horizon of

the AdS black hole turns into the dS event horizon. The universal horizon remains in between the inner and outer horizons of the AdS black hole, and can be found in the “outer” region of the dS spacetime.

That the universal horizon tends to be located beyond the dS horizon (*i.e.* at a larger value of r) appears to be a generic property of these solutions. Such a horizon can be thought of as a *cosmological universal horizon*.

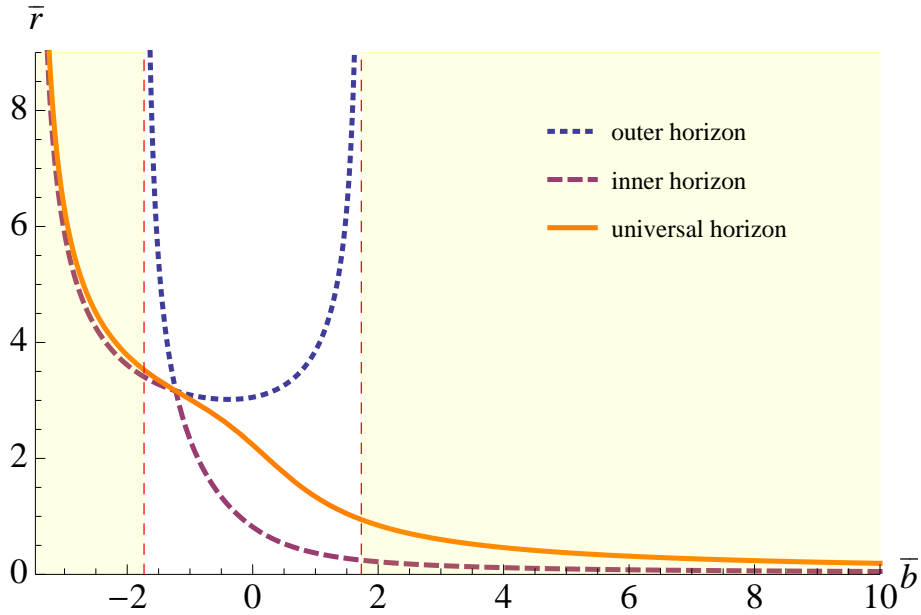


FIGURE 4.7 Transitioning from AdS to dS asymptotics. The yellow shaded regions are asymptotically de Sitter spacetimes, while the unshaded region represents an AdS black hole.

It is also of interest to look at the case of the dS black hole. Choosing the sign of the bare cosmological constant to be positive this time ($\Lambda > 0$), we now choose the other parameters to be $\{\mathcal{M} = -10, \mathcal{J} = 0.1\}$ and the coupling constants $\{\xi = 2, \lambda = 1\}$.

The coupling constants are chosen so that all values of b lead to dS asymptotics, which is $\lambda < (1 + \xi)/2$ for $\Lambda > 0$. However, to guarantee that the æther is real at large r (re: $b^2 > \bar{\Lambda}(b)$), we are limited to the region $\bar{b} \geq \sqrt{2}$. For all values of \bar{b} shown in Fig. (4.8), the spacetime is a dS black hole with an event horizon and a dS horizon.

However, for sufficiently large \bar{b} (not shown in the plot), $\bar{\mathcal{J}}^2$ becomes positive, and the event horizon ceases to exist. Again, we see here that the universal horizon is located beyond the dS horizon. In Fig. (4.2), the universal horizon is displayed in the Penrose diagram of a dS black hole spacetime. In the asymptotically flat case $\bar{\Lambda} = 0$, the æther charge b is fixed at a particular value:

$$b_{\text{flat}}^2 = \Lambda \left(\frac{\xi}{2\lambda - \xi - 1} \right). \quad (4.76)$$

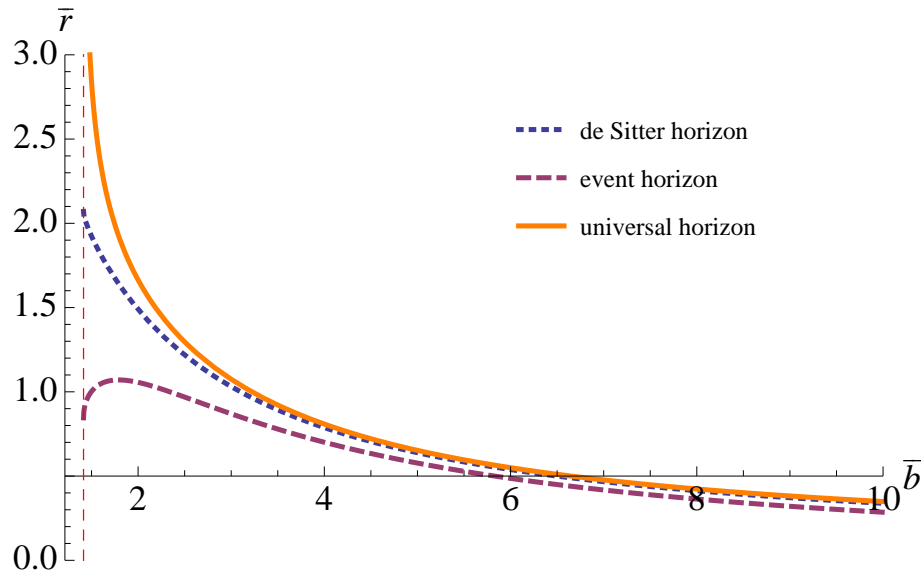


FIGURE 4.8 Radial positions of various horizons in a de Sitter black hole.

It is quite straightforward to choose parameters for which the universal horizon exists. Asymptotically flat solutions with universal horizons have no extra hair (*i.e.* independent æther charge) apart from \mathcal{M} and \mathcal{J} .

In Fig. (4.3), the universal horizon is displayed in the Penrose diagram of an asymptotically flat, black hole spacetime.

4.5.6.4 Black Holes without Universal Horizons

We have implicitly already stated two conditions for universal horizons to not exist at all: firstly, the discriminant in Eq. (4.73) can be negative, and secondly, r_u^2 can be negative. It is worth pointing out that these condition can be satisfied even in black hole solutions if the parameters are chosen appropriately.

Consider, as an example, the black hole with flat asymptotics, $\{\bar{\Lambda} = 0, \mathcal{M} = -1, \bar{\mathcal{J}}^2 < 0\}$, and assume, additionally, that $b = 0$ so that the æther asymptotically aligns with the timelike Killing vector. Eq. (4.76) requires that Λ has to vanish as well. One can then straightforwardly calculate the root of Eq. (4.72). This is

$$r_u^2 = - \left(\frac{J^2 + 4a^2}{\xi} \right), \quad (4.77)$$

and it is negative-definite ($\bar{\mathcal{J}}^2 < 0$ requires that $\xi > 1$). So, no universal horizon exists for black holes with flat asymptotics and an æther that asymptotically aligns with the timelike Killing vector.

As another example, let us consider black holes with AdS asymptotics. The negative

discriminant condition reads

$$\frac{(4a^2 + \mathcal{J}^2)(b^2 - \bar{\Lambda}(b))}{\xi(\mathcal{M} - 2ab)^2} > 1, \quad (4.78)$$

while the black hole bound given in Eq. (4.57) for $\bar{\Lambda}(b) < 0$ is

$$0 \leq \mathcal{M}^2 + \bar{\Lambda}(b)\bar{\mathcal{J}}(a)^2. \quad (4.79)$$

Finally, we also need to require that the æther is real at large r ($b^2 \geq \bar{\Lambda}(b)$) and small r ($\xi > 0$). All need to be satisfied for the parameters to represent regular black hole solutions without universal horizons.

We graphically demonstrate that a fairly large region of parameter space satisfies all these requirements. For the values $\{\mathcal{M} = 1, \Lambda l^2 = -1, \mathcal{J}/l = 1\}$, $\lambda = 2$ and $\xi = 0.9$, we display in Fig. (4.9) the values of $\{a, b\}$ satisfying (a) $\bar{\Lambda} < 0$, (b) the black hole bound in Eq. (4.57), (c) the negative discriminant condition in Eq. (4.78), and (d) the æther regularity constraint at large r . These all correspond to asymptotically AdS black holes with no universal horizons.

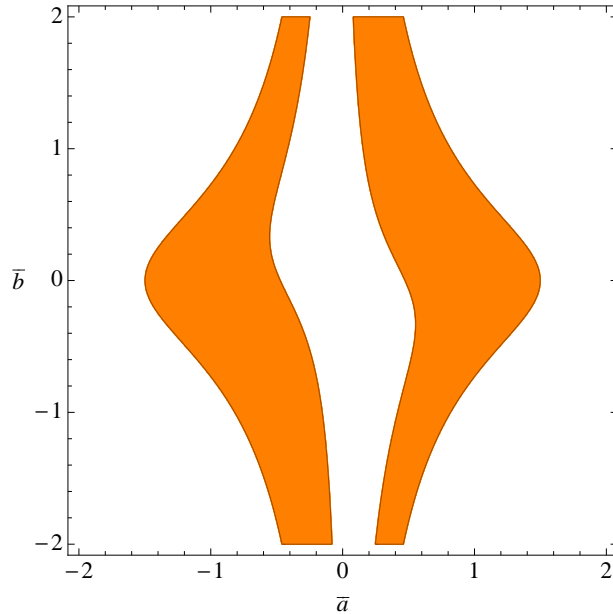


FIGURE 4.9 Region of parameter space corresponding to asymptotically anti-de Sitter black holes without universal horizons. These plots use $\{\mathcal{M} = 1, \Lambda l^2 = -1, \mathcal{J}/l = 1\}$, $\lambda = 2$, and $\xi = 0.9$. Here, $\bar{a} := a/l$ and $\bar{b} := bl$, with $l := 1/\sqrt{-\Lambda}$.

4.5.7 Non-Rotating Black Holes

One can choose to focus in the $\mathcal{J} = 0$ case which corresponds to a non-rotating black hole. In general the spacetime retains most of the features it had when $\mathcal{J} \neq 0$ provided that $a \neq 0$. Curvature invariants still diverge at $r = 0$ and the causal structure remains largely unaffected. Ergosurfaces now coincide with the metric horizons, as expected.

Nevertheless, it is worth pointing out that one can still have two black hole horizons in black hole solutions with AdS asymptotics.

As far as universal horizons are concerned, they can be present or absent, depending on the solutions.

When $\mathcal{J} = 0$, the constraint given by Eq. (4.74) reduces to

$$\frac{4a^2(b^2 - \bar{\Lambda}(b))}{\xi(\mathcal{M} - 2ab)^2} = 1. \quad (4.80)$$

One can readily identify two characteristic examples of non-rotating black holes that cannot satisfy this constraint and cannot have a universal horizon. The first is the asymptotically flat black hole with $b = 0$ (discussed above) and $J = 0$. The second is a black hole with AdS asymptotics and $a = 0$. This is actually a non-rotating BTZ black hole with a non-trivial æther configuration.

4.6 Summary of the Results and Some Remarks

Our intention was to find an analogue of the BTZ black hole in three-dimensional Hořava gravity. To this end we first considered whether AdS space or AdS asymptotics are admissible in this theory. Using the reduced action approach we have shown that this is only true if $\eta = 0$. We subsequently focused on the $\eta = 0$ sector of the theory. We have found the most general class of solutions in this sector, without imposing specific asymptotics. Remarkably, the black hole solutions in this class do not have exclusively AdS asymptotics, but there exist instead also black holes with dS and flat asymptotics, unlike GR.

The black hole solutions we found have very interesting properties. They harbour a curvature singularity, unlike their GR counterparts. They can have an inner and an outer metric (Killing) horizon and one or two ergosurfaces. But perhaps what is their most interesting feature within the context of Lorentz-violating gravity theories is that they can have universal horizons. Rotation does not seem to play a key role in the existence of these horizons. Depending on the configuration of the preferred foliation, there can be non-rotating black holes without universal horizons or rapidly rotating black holes with universal horizons. Some of our solutions also feature the existence of cosmological

universal horizons. These results demonstrate that the existence of universal horizons does not seem to depend on spherical symmetry or the number of spacetime dimensions and it is not specific to black hole spacetimes. At the same time, they also highlight the importance of the asymptotic behaviour of the foliation for the existence of universal horizons.

The $\eta = 0$ sector of three-dimensional Hořava gravity, to which the requirement of AdS asymptotics has restricted us, is likely to be a special theory. At the perturbative level the scalar mode that the theory propagates appears to travel at infinite speed as it is evident by looking at Eq. (4.18). Furthermore, in four dimensions choosing $\eta = 0$ leads to a physically (but not mathematically) inconsistent theory [144].

Nevertheless, we expect the black hole solutions presented here to be useful tools for applications such as quantum field theory near horizons in the presence of Lorentz violations and black hole thermodynamics, so long as one remains cautious about the interpretation of the results.

Moreover, the existence of black hole solutions with flat or dS asymptotics in the $\eta = 0$ sector of the theory suggests that it is likely for black hole solutions with these asymptotics to exist also when $\eta \neq 0$.

Chapter 5

Gravity with Auxiliary Fields

In the previous Chapters we have considered the option to modify GR by relaxing one of the implicit assumptions of Lovelock's theorem, *i.e.* diffeomorphism invariance. As we have seen, this procedure inevitably leads to the propagation of extra modes, as expected due to less symmetry. This is one way to circumvent Lovelock's theorem. The most natural and extensively explored one, is to add extra dynamical fields in the gravitational action while preserving diffeomorphism invariance. This is indeed what happens in the covariantized version of Hořava gravity in the IR limit, which does not violate the diffeomorphism invariance assumption any more, but includes explicitly an extra field (Stueckelberg field).

By doing so, however, one is inevitably introducing again extra propagating degrees of freedom. Such degrees of freedom remain undetected to date. Hence, a major problem for alternative theories of gravity has been to tame the behaviour of extra degrees of freedom, so as to evade current experimental constraints related to their existence [98]. This comes in addition to the fact that it is not at all straightforward to construct theories with extra fields non-minimally coupled to gravity that avoid instabilities associated to the new degrees of freedom [77].

What we wish to consider here is the much less explored option of adding *non-dynamical* extra fields. This is enough to circumvent Lovelock's theorem and, at the same time, it does not add extra propagating degrees of freedom. The extra fields will then have to be auxiliary, *i.e.* the field equations should allow them to be determined algebraically.

One might be tempted to consider a modification of GR which is not restricted by Lovelock's theorem, as it refers to the right-hand side of the equations. More specifically, one may add any rank-2 tensor that is solely constructed by the metric and the matter fields and is identically divergence-free, so as to not compromise the weak equivalence principle. However, it is unclear if such a tensor actually exists. Additionally it is reasonable to think that, if this theory is to come from an action, the corresponding

modification would amount to an addition of extra terms including the matter fields, hence introducing unacceptable modification to the equations of motion of the matter sector.

In what follows we will start with three specific and very well known examples of theories where auxiliary fields are present in the action. After that we will develop a very general framework for theories with auxiliary fields where also the aforementioned special cases will be captured.

5.1 Three Known Examples

5.1.1 Eddington-Inspired Born-Infeld Gravity

Eddington-inspired Born-Infeld (EiBI) gravity is described by the following action [145]

$$S_{EiBI} = \frac{2}{\kappa} \int d^4x \left(\sqrt{|\det(g_{\mu\nu} + \kappa \mathcal{R}_{(\mu\nu)})|} - (1 + \kappa\Lambda)\sqrt{g} \right) + S_M[g_{\mu\nu}, \Psi_M], \quad (5.1)$$

where κ is the extra EiBI parameter which has dimensions of length squared and $\mathcal{R}_{\mu\nu}$ is the Ricci tensor built with the connection $C_{\mu\nu}^\sigma$. In this Chapter we will work in Planck units $8\pi G_N = 1$.

EiBI gravity is naturally based on the Palatini formulation, so the connection $C_{\mu\nu}^\sigma$ is considered as an independent field and it does not enter the matter action.

Variation of the action (5.1) with respect to the metric and the independent connection respectively yields

$$\sqrt{q}q^{\mu\nu} = \sqrt{g}((1 + \kappa\Lambda)g^{\mu\nu} - \kappa T^{\mu\nu}), \quad (5.2a)$$

$$\tilde{\nabla}_\sigma[\sqrt{q}q^{(\mu\nu)}] - \tilde{\nabla}_\gamma[\sqrt{q}q^{(\mu\gamma)}\delta_\sigma^\nu] = 0, \quad (5.2b)$$

where $q_{\mu\nu} \equiv g_{\mu\nu} + \kappa \mathcal{R}_{(\mu\nu)}$, $\tilde{\nabla}_\mu$ is the covariant derivative defined with the connection $C_{\mu\nu}^\sigma$, and $T^{\mu\nu}$ is the stress-energy tensor defined as usual

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \quad (5.3)$$

After having manipulated Eq. (5.2b) it takes the form

$$C_{\mu\nu}^\sigma = \frac{1}{2} q^{\sigma\gamma} (\partial_\mu q_{\nu\gamma} + \partial_\nu q_{\mu\gamma} - \partial_\gamma q_{\mu\nu}). \quad (5.4)$$

Moreover, using Eq. (5.2a) one gets

$$q^{\mu\nu} = \frac{(1 + \kappa\Lambda)g^{\mu\nu} - \kappa T^{\mu\nu}}{\sqrt{g}\sqrt{|\det[(1 + \kappa\Lambda)g^{\alpha\beta} - \kappa T^{\alpha\beta}]|}}, \quad (5.5)$$

which can be rewritten as

$$\kappa \mathcal{R}_{(\mu\nu)} = \sqrt{g} \sqrt{|\det [(1 + \kappa\Lambda)g^{\alpha\beta} - \kappa T^{\alpha\beta}]|} [(1 + \kappa\Lambda)g^{\mu\nu} - \kappa T^{\mu\nu}]^{-1} - g_{\mu\nu}. \quad (5.6)$$

It follows that Eq. (5.5) determines $q_{\mu\nu}$ algebraically in terms of $g_{\mu\nu}$ and $T_{\mu\nu}$, whereas Eq. (5.4) determines $C_{\mu\nu}^\sigma$ as the Levi-Civita connection of $q_{\mu\nu}$. So, using Eqs. (5.4)-(5.5), it follows that one can write $C_{\mu\nu}^\sigma$ in terms of the metric, the stress-energy tensor and their derivatives. This means that the C 's are not dynamical fields but rather auxiliary fields.

Hence, one can use these equations to eliminate $C_{\mu\nu}^\sigma$ from Eq. (5.6), which then becomes a second-order partial differential equation in $g_{\mu\nu}$, and it also contains second-order derivatives of $T_{\mu\nu}$.

For what follows, it is also convenient to write below the field equations in the small-coupling limit, that is by expanding the action (5.1) at first-order in κ .

Using the fact that

$$q^{\mu\nu} = g^{\mu\nu} - \kappa \tau^{\mu\nu} + \mathcal{O}(\kappa^2), \quad (5.7)$$

where $\tau_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T + \Lambda g_{\mu\nu}$, the equations for the metric $g_{\mu\nu}$ are found to be [146]

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} + \kappa \left[S_{\mu\nu} - \frac{1}{4}S g_{\mu\nu} \right] + \frac{\kappa}{2} [\nabla_\mu \nabla_\nu \tau - 2\nabla^\sigma \nabla_{(\mu} \tau_{\sigma\nu)} + \square \tau_{\mu\nu}] + \mathcal{O}(\kappa^2), \quad (5.8)$$

where $S_{\mu\nu} = T^\sigma{}_\mu T_{\sigma\nu} - \frac{1}{2}T T_{\mu\nu}$.

Note that in the equation above $R_{\mu\nu}$ is now constructed with the Levi-Civita connection of $g_{\mu\nu}$ and ∇_μ is the covariant derivative associated to it.

GR is automatically recovered for $\kappa = 0$, while it is now evident that when $\kappa \neq 0$ Eq. (5.8) contains second derivatives of $T_{\mu\nu}$, *i.e.* at least third derivatives of matter fields, unless we consider a perfect fluid approximation of matter. This is in contrast with GR where usually only first derivatives appear on the right-hand side of the field equations.

5.1.2 Palatini $f(R)$ Gravity

Let us now consider the case of Palatini $f(R)$ gravity [78, 147], which is perhaps the most well-known example of a theory with auxiliary fields. Its action is written as

$$S_{Pal} = \frac{1}{2} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi_M). \quad (5.9)$$

Varying it independently with respect to the metric and the independent connection $C_{\mu\nu}^\sigma$, yields respectively

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = T_{\mu\nu}, \quad (5.10a)$$

$$-\bar{\nabla}_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) + \bar{\nabla}_\sigma(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu})\delta_\lambda^\nu = 0, \quad (5.10b)$$

where $\bar{\nabla}_\mu$ denotes the covariant derivative defined with the independent connection $C_{\mu\nu}^\sigma$. Taking the trace of Eq. (5.10b), it can be easily shown that

$$\bar{\nabla}_\sigma(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}) = 0, \quad (5.11)$$

from which the field equations can be recast in the form

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = T_{\mu\nu}, \quad (5.12a)$$

$$\bar{\nabla}_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0. \quad (5.12b)$$

Notice that if one takes $f(\mathcal{R}) = \mathcal{R}$, then trivially $f'(\mathcal{R}) = 1$, Eq. (5.12b) becomes the definition of the Levi-Civita connection of $g_{\mu\nu}$ and Eq. (5.12a) immediately leads to GR. This is consistent with the well known result that the Palatini approach for GR yields Einstein's equations [148, 149], with the only difference that the connection turns out to be the Levi-Civita one only dynamically instead of being assumed a priori.

Defining a metric which is conformal to $g_{\mu\nu}$ through

$$h_{\mu\nu} \equiv f'(\mathcal{R})g_{\mu\nu}, \quad (5.13)$$

one finds

$$\sqrt{-h}h^{\mu\nu} = \sqrt{-g}f'(\mathcal{R})g^{\mu\nu}, \quad (5.14)$$

and Eq. (5.12b) becomes

$$\bar{\nabla}_\lambda(\sqrt{-h}h^{\mu\nu}) = 0, \quad (5.15)$$

that is the definition of the Levi-Civita connection of $h_{\mu\nu}$. Thus solving it algebraically one gets

$$C_{\mu\nu}^\sigma = \frac{1}{2}h^{\sigma\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}), \quad (5.16)$$

which can be written in terms of $g_{\mu\nu}$ as

$$C_{\mu\nu}^\sigma = \frac{1}{2} \frac{1}{f'(\mathcal{R})} g^{\sigma\lambda} \left[\partial_\mu (f'(\mathcal{R})g_{\nu\lambda}) + \partial_\nu (f'(\mathcal{R})g_{\mu\lambda}) - \partial_\lambda (f'(\mathcal{R})g_{\mu\nu}) \right]. \quad (5.17)$$

Since taking the trace of Eq. (5.12a) yields

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = T, \quad (5.18)$$

which implies that for a given $f(\mathcal{R})$ one gets an algebraic equation relating \mathcal{R} to T , it follows that the independent connection (5.17) can be written in terms of the metric and the matter fields. So again, since the connection can be algebraically expressed only in terms of the latter two, it follows that it acts as an auxiliary field.

5.1.3 Brans–Dicke Theory with $\omega_0 = -3/2$

In order to see what this implies at the level of the field equations, we find it convenient and also instructive writing Palatini $f(R)$ gravity as Brans–Dicke theory with $\omega_0 = -3/2$, which is actually dynamically equivalent to Palatini $f(R)$ gravity [78, 147]. To this end, let us start from Eq. (5.9), and follow exactly the same steps of Sec. (1.4.1) that we re-propose here for clarity.

Introducing an extra field χ , one gets the following dynamically equivalent action

$$S_{Pal} = \frac{1}{2} \int d^4x \sqrt{-g} [f(\chi) + f'(\chi)(\mathcal{R} - \chi)] + S_M[g_{\mu\nu}, \psi_M]. \quad (5.19)$$

Taking its variation with respect to χ leads to

$$f''(\chi)(\mathcal{R} - \chi) = 0, \quad (5.20)$$

and, if $f''(\chi) \neq 0$ then $\chi = \mathcal{R}$, and the original action (5.19) is recovered.

Through the redefinition of the field χ by $\phi = f'(\chi)$, and by defining

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \quad (5.21)$$

one can rewrite the action (5.19) as

$$S_{Pal} = \frac{1}{2} \int d^4x \sqrt{-g} [\phi\mathcal{R} - V(\phi)] + S_M[g_{\mu\nu}, \psi_M]. \quad (5.22)$$

Let us now use Eq. (5.17) in order to relate the Ricci tensor $\mathcal{R}_{\mu\nu}$ associated to the connection $C_{\mu\nu}^\sigma$, to the Ricci tensor $R_{\mu\nu}$ associated to the Levi-Civita connection of $g_{\mu\nu}$. It follows that

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2} \frac{1}{(f'(\mathcal{R}))^2} (\nabla_\mu f'(\mathcal{R})) (\nabla_\nu f'(\mathcal{R})) - \frac{1}{f'(\mathcal{R})} \left(\nabla_\mu \nabla_\nu + \frac{1}{2} g_{\mu\nu} \square \right) f'(\mathcal{R}), \quad (5.23)$$

whose contraction with $g^{\mu\nu}$ yields

$$\mathcal{R} = R + \frac{3}{2(f'(\mathcal{R}))^2} (\nabla_\mu f'(\mathcal{R})) (\nabla^\mu f'(\mathcal{R})) - \frac{3}{f'(\mathcal{R})} \square f'(\mathcal{R}), \quad (5.24)$$

where $\mathcal{R} \equiv g^{\mu\nu} \mathcal{R}_{\mu\nu}$ and $R \equiv g^{\mu\nu} R_{\mu\nu}$.

Using now Eq. (5.24) in Eq. (5.22) with $\phi = f'(\mathcal{R})$, one finally gets, modulo surface terms, the action

$$S_{BD} = \frac{1}{2} \int d^4x \sqrt{-g} \left(\phi R + \frac{3}{2\phi} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right) + S_M(g_{\mu\nu}, \psi_M), \quad (5.25)$$

which is just the action of Brans–Dicke theory with $\omega_0 = -3/2$.

The corresponding field equations coming from the variation of the action (5.25) with respect to the metric and the scalar field ϕ , are respectively

$$G_{\mu\nu} = \frac{1}{\phi} T_{\mu\nu} - \frac{3}{2\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \phi \nabla^\sigma \phi \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \phi - \frac{V}{2\phi} g_{\mu\nu}, \quad (5.26a)$$

$$\square \phi = \frac{\phi}{3} (R - V') + \frac{1}{2\phi} \nabla_\sigma \phi \nabla^\sigma \phi. \quad (5.26b)$$

By taking the trace of Eq. (5.26a), and using it to eliminate R from Eq. (5.26b), one gets

$$2V(\phi) - \phi V'(\phi) = T. \quad (5.27)$$

So, it is possible to relate ϕ directly to the matter sources since, once the potential $V(\phi)$ is assigned, the scalar field can be algebraically solved as $\phi = \phi(T)$. It directly follows that it is a non-dynamical field.

Finally substituting $\phi = \phi(T)$ back into the field equations (5.26a), one gets higher-order derivatives of matter fields, as it was also the case for EiBI gravity.

5.2 Higher-Order Derivatives of Matter in the Field Equations

It has been shown in several works that major problems arise when perfect-fluid equilibrium structures are considered in the theories discussed above. In Ref. [146], static and spherically symmetric perfect-fluid stars with polytropic equations of state – where $P = K\rho_0^\Gamma$, with P being the pressure, ρ_0 the rest-mass density and K and Γ constants – are considered in the context of EiBI gravity. It is found that for any $\Gamma > 3/2$ the theory does not admit any regular solution as the scalar curvature diverges at the surface of these polytropic matter configurations. At least two physical matter configurations

are exactly described by a polytropic equation of state with $\Gamma > 3/2$: a degenerate gas of nonrelativistic electrons and a monoatomic isentropic gas, both having $\Gamma = 5/3$. Moreover there are also other examples of stars whose matter configuration resembles a polytrope, *e.g.* the atmosphere of white dwarfs is well approximated by a polytrope with $\Gamma = 10/3$ (see Refs. [150, 151]). These perfectly reasonable systems, which can also be described within Newtonian theory, have no description in EiBI gravity, so the theory is at best incomplete.

Similar results have also been obtained in the case of Palatini $f(R)$ gravity, in which case the divergence occurs only for $3/2 < \Gamma < 2$ [152–154], and in theories where matter is coupled to the Ricci scalar [155].

The key issue is that higher-order derivatives of matter fields, which appear both in the EiBI and Palatini $f(R)$ Gravity (or Brans–Dicke Theory with $\omega_0 = -3/2$) field equations, as a result of integrating out non-dynamical degrees of freedom, make the geometry unacceptably sensitive to sharp variations in the matter configuration.

In fact having a higher differential order in the metric than in the matter fields is what guarantees that the metric depends in a cumulative way on the matter. On the contrary, if this is not the case, then the metric loses its immunity to rapid changes in matter gradients since it is directly related to them instead of being an integral over them. Any matter configuration which is discontinuous or just not smooth enough will produce discontinuities in the metric and singularities in the curvature invariants, leading to unacceptable phenomenology.

The same problem should be generically expected for any theory which includes fields other than the metric for describing the gravitational interaction which are algebraically related to matter rather than dynamically coupled [156]. In this case one can always solve the field equations for the extra field and then insert the solution into the field equations for the metric, inducing a dependence of the metric on higher derivatives of matter fields, as we have seen in the examples discussed above.

In what follows we will develop a general framework which will indeed show that this problem appears to be a generic prerogative of gravitational theories which do not propagate any degree of freedom other than the massless spin-2 field, but instead contain auxiliary fields. Furthermore, it will also act as a nice parametrization for all such theories, rendering possible to study their features in full generality without referring to specific examples.

5.3 Constructing the Gravitational Field Equations

5.3.1 Hypothesis of the Argument

Keeping in mind what we have learned from the previous examples of theories with auxiliary fields, we want now to construct a very general parametrization able to include the entire class of such theories. In particular, our argument will not rely on the specific form of the auxiliary field appearing in the field equations, because, as seen before, the auxiliary field can be represented not only by a scalar field, but also by a more general geometrical object, *e.g.* the connection. The power of this parametrization lies precisely on the fact that it remains oblivious to the nature of the auxiliary fields and the way they enter in the action.

Our general argument [157] is based on the following hypotheses:

- i) The theory admits a covariant Lagrangian;
- ii) In this formulation any extra field is auxiliary;
- iii) The matter fields couple only to the metric in the usual way, so that the matter Lagrangian L_M reduces to that of the Standard Model in the local frame.

For concreteness we will focus on the simplest case of just one auxiliary field. However, this assumption is not crucial and the generalization to N fields is straightforward. The theory is described by the Lagrangian

$$L = L_g[\mathbf{g}, \phi] + L_M[\mathbf{g}, \psi]; \quad (5.28)$$

L_g is the gravitational part where the metric \mathbf{g} is possibly coupled to the extra field ϕ , for which the tensorial rank or other characteristics are left unspecified.

5.3.2 Field Equations

Variation with respect to \mathbf{g} and ϕ yields respectively to

$$E_{\mu\nu}[\mathbf{g}, \phi] = T_{\mu\nu}, \quad (5.29)$$

$$\Phi[\mathbf{g}, \phi] = 0, \quad (5.30)$$

where $E_{\mu\nu}$ is a generic rank-2 tensor. Variation with respect to ψ will yield the field equations for the matter fields. Our requirement that ϕ be an auxiliary field implies that, by using Eqs. (5.29) and (5.30) (and possibly their derivatives) in some particular combination, it is possible to obtain an algebraic equation for ϕ , which can be *schematically*

written as

$$\mathcal{F}[\phi, \mathbf{g}, \mathbf{T}] = 0, \quad (5.31)$$

where ϕ only appears at zeroth differential order. Note that we do not necessarily require to solve Eq. (5.31) in *closed* form. Indeed, it is sufficient to solve for ϕ *implicitly*, provided the implicit relation can be used to obtain a closed set of field equations for \mathbf{g} , where any dependence on ϕ has been eliminated. It is clear that the matter fields will appear in Eq. (5.31) only in the specific combination that forms the stress-energy tensor.

Let us assume that \mathcal{F} does not depend on the matter fields at all. Then ϕ can be algebraically determined in terms of the metric only through $\mathcal{F}[\phi, \mathbf{g}] = 0$.

Consistency requires that Eq. (5.30) be trivially satisfied, and Eq. (5.29) reduces to

$$E_{\mu\nu}[\mathbf{g}] = T_{\mu\nu}. \quad (5.32)$$

But then, if $E_{\mu\nu}$ contains up to second derivatives of the metric, Lovelock's theorem requires $E_{\mu\nu}[\mathbf{g}] \equiv G_{\mu\nu} + \Lambda g_{\mu\nu}$ and the theory has to be GR.

The case where $E_{\mu\nu}$ contains more than second derivatives of the metric does not concern us here, as the scope of adding auxiliary fields instead of dynamical ones was to avoid extra degrees of freedom.

On the other hand, if \mathcal{F} depends on the matter fields, eliminating ϕ from Eq. (5.29) will yield

$$E_{\mu\nu}[\mathbf{g}, \mathbf{T}] = T_{\mu\nu}, \quad (5.33)$$

which can be written without loss of generality as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} + S_{\mu\nu}[\mathbf{g}, \mathbf{T}]. \quad (5.34)$$

The precise form of $S_{\mu\nu}$ will obviously depend on the specific form of the auxiliary field and how it enters the Lagrangian. However, $S_{\mu\nu}$ has to have the following properties:

- i) It vanishes when $T_{\mu\nu} = 0$, as it was previously shown that when \mathcal{F} is independent of the matter fields $E_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$;
- ii) It is divergence-free, as a consequence of the contracted Bianchi identity and the fact that $T_{\mu\nu}$ is divergence-free when the matter fields satisfy their field equations.

Note that this latter property should hold identically, modulo the fact that $\nabla_\mu T^{\mu\nu} = 0$, as it should not impose any further restriction to the dynamics. This is consistent with the fact that $S_{\mu\nu}$ came after eliminating an auxiliary field and that matter is minimally coupled to \mathbf{g} in the Lagrangian (5.28).

Lastly, it is worth mentioning that Λ need not be identified with the cosmological constant

that was initially present in L_g , as the whole procedure of eliminating the auxiliary field might possibly affect its value.

5.3.3 Constructing $S_{\mu\nu}$

We have shown that, in theories with an auxiliary field, eliminating the latter generically corresponds to modifying Einstein's equations by adding a divergence-free tensor that vanishes in vacuum. This tensor depends on the metric, the stress-energy tensor, and their derivatives. One could now set its origin aside and just attempt to construct it from its constituents. We proceed by doing so, order by order in the derivatives of the fields.

The stress-energy tensor is generically second-order in derivative of the matter fields (fermions being an exception) and this defines the lowest-order.

The only term one could add at the lowest-order is $g_{\mu\nu}T$, where $T \equiv T^\mu{}_\mu$, so the equations can take the form

$$R_{\mu\nu} = T_{\mu\nu} - \alpha g_{\mu\nu}T + g_{\mu\nu}\Lambda + \dots, \quad (5.35)$$

α being an arbitrary coefficient. There are no terms with three derivatives one can construct. The terms with four derivatives are of three types: \mathbf{T}^2 , $\nabla^2\mathbf{T}$, and contractions between \mathbf{T} and the Riemann tensor. The only term that actually involves the Riemann tensor itself is $R_{(\mu|\nu\sigma|\gamma)}T^{\nu\sigma}$, which can be eliminated without loss of generality in favour of other terms since $\nabla_\sigma\nabla_\gamma T^{\mu\sigma} = R^\mu{}_{\nu\sigma\gamma}T^{\nu\sigma} + R_{\nu\gamma}T^{\mu\nu}$. Assuming that the perturbative expansion does not break down (which should be true at least in regimes where one expects to recover GR), one could use the lowest-order equation (5.35) in order to express $R_{\mu\nu}$ in terms of $T_{\mu\nu}$. Hence, up to fourth-order in derivatives, we obtain

$$\begin{aligned} S_{\mu\nu} = & \alpha_1 g_{\mu\nu}T + \alpha_2 g_{\mu\nu}T^2 + \alpha_3 T T_{\mu\nu} + \alpha_4 g_{\mu\nu}T_{\sigma\gamma}T^{\sigma\gamma} + \alpha_5 T^\sigma{}_\mu T_{\sigma\nu} \\ & + \beta_1 \nabla_\mu\nabla_\nu T + \beta_2 g_{\mu\nu}\square T + \beta_3 \square T_{\mu\nu} + 2\beta_4 \nabla^\sigma\nabla_{(\mu}T_{\nu)\sigma} + \dots, \end{aligned} \quad (5.36)$$

where α_i and β_j are coefficients with appropriate dimensions. In the expression above we are not considering possible parity violating terms which would involve the Levi-Civita tensor.

We still need to impose that $S_{\mu\nu}$ be divergence free, at least to the required order, and this condition will impose some bond between the various coefficients. At first it might seem that the only solution is the trivial one, $\alpha_i = \beta_j = 0$. However, this is not the case,

as using the relations

$$(\square \nabla_\nu - \nabla_\nu \square) T = R_{\mu\nu} \nabla^\mu T, \quad (5.37)$$

$$(\nabla^\mu \nabla^\sigma \nabla_\mu - \nabla^\sigma \square) T_{\sigma\nu} = R_{\mu\nu\sigma\gamma} \nabla^\gamma T^{\sigma\mu}, \quad (5.38)$$

$$\nabla^\mu R_{\mu\nu\sigma\gamma} = 2\nabla_{[\sigma} R_{\gamma]\nu}, \quad (5.39)$$

and the lowest-order Eq. (5.35) leads to cancellations between terms. Indeed, imposing $\nabla_\mu S^{\mu\nu} = 0$ we obtain:

$$\begin{aligned} \alpha_1 &= -\beta_1 \Lambda, & 4\alpha_2 &= (1 + 2\alpha_1)(\beta_1 - \beta_4), \\ \alpha_3 &= \beta_4(1 + 2\alpha_1) - \beta_1, & 2\alpha_4 &= \beta_4, \\ \alpha_5 &= -2\beta_4, & \beta_2 &= -\beta_1, & \beta_3 &= -\beta_4. \end{aligned} \quad (5.40)$$

The field equations finally read

$$\begin{aligned} G_{\mu\nu} &= T_{\mu\nu} - \Lambda g_{\mu\nu} - \beta_1 \Lambda g_{\mu\nu} T + \frac{1}{4} (1 - 2\beta_1 \Lambda) (\beta_1 - \beta_4) g_{\mu\nu} T^2 \\ &+ [\beta_4 (1 - 2\beta_1 \Lambda) - \beta_1] T T_{\mu\nu} + \frac{1}{2} \beta_4 g_{\mu\nu} T_{\sigma\gamma} T^{\sigma\gamma} - 2\beta_4 T^\sigma{}_\mu T_{\sigma\nu} + \beta_1 \nabla_\mu \nabla_\nu T \\ &- \beta_1 g_{\mu\nu} \square T - \beta_4 \square T_{\mu\nu} + 2\beta_4 \nabla^\sigma \nabla_{(\mu} T_{\nu)\sigma} + \dots, \end{aligned} \quad (5.41)$$

where all coefficients are expressed in terms of β_1 and β_4 .

Notice that similar quadratic-in- $T_{\mu\nu}$ terms also arise in so-called brane-world scenarios (see Ref. [158] for a review).

It is worth pointing out that, although the equations above have been obtained as a derivative expansion, they could be equivalently obtained as a double expansion in the small- \mathbf{T} and small- $\nabla\mathbf{T}$ limits. More precisely, introducing a further book-keeping parameter λ associated to each derivative of the stress-energy tensor, it can be easily verified that Eq. (5.41) is the most generic field equation which satisfies the aforementioned hypotheses to $\mathcal{O}(\mathbf{T}^2)$ and $\mathcal{O}(\lambda^2\mathbf{T})$. This equivalence hinges on the symmetries and on the tensorial rank of \mathbf{T} . Assigning a derivative order to \mathbf{T} itself simply allows one to have a single book-keeping parameter and simplifies the discussion.

Note also that, when one is working with an effective description of matter, such as fluids, quantities such as the energy density and pressure will not be of zeroth-order even though they do not explicitly appear to contain derivatives. That can be understood intuitively by the fact that a scalar field admits an effective description as a perfect fluid. Known theories with auxiliary fields do indeed fall within the parametrization developed above. EiBI gravity in the small coupling limit (see Eq. (5.8)) corresponds to

$\beta_1 = 0$ and $\beta_4 = -\kappa/2$. Generic Palatini $f(R)$ theories (and also Brans–Dicke theory with $\omega_0 = -3/2$) correspond to $\beta_4 = 0$, with Λ and β_1 being dependent on the model parameters. Interestingly, these two particular cases are in fact representative of two “orthogonal” classes of corrections.

Our analysis demonstrates that theories with auxiliary fields, as well as any modification that does not allow for extra degrees of freedom, inevitably lead to equations with more than second derivatives of the matter fields. In the absence of extra dynamical fields, such terms have already been shown to be a major shortcoming in the specific theories we have presented in Sec. (5.1)–(5.2), leading to curvature singularities when there are sharp changes in the energy density of matter [146, 152–155]. This problem will generically persist in the class of corrections we are discussing.

5.4 Newtonian Limit

An analysis of the Newtonian limit is quite illuminating. For simplicity we set $\Lambda = 0$. If the cosmological constant is to have the observed value then it can safely be considered as a higher post-Newtonian order contribution. In the limit of small velocities and small matter fields, one has

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \quad T_{\mu\nu} = \epsilon \rho \delta_{\mu}^0 \delta_{\nu}^0, \quad (5.42)$$

where $\epsilon \ll 1$ is a book-keeping parameter.

We define $\Psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, and indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$. By performing the infinitesimal transformation $x^\mu \rightarrow x^\mu + \epsilon \xi^\mu$ where ξ^μ satisfies

$$\epsilon \square \xi_\nu = \epsilon \partial_\mu \Psi_\nu^\mu - \zeta \partial_\nu T, \quad (5.43)$$

we can impose a gauge such that $\epsilon \partial_\mu \Psi_\nu^\mu = \zeta \partial_\nu T$.

Here ζ is a numerical coefficient that we shall fix later on. To first-order in ϵ , and after some manipulations, the field equations read

$$\nabla^2 h_{0i} = 0, \quad (5.44a)$$

$$-\frac{\nabla^2 h_{00}}{2} = \frac{\rho}{2} + \frac{1}{2}\beta_- \nabla^2 \rho, \quad (5.44b)$$

$$-\frac{\nabla^2 h_{ij}}{2} = \frac{\delta_{ij}}{2} [\rho - \beta_+ \nabla^2 \rho] - [\beta_1 - \zeta] \partial_i \partial_j \rho, \quad (5.44c)$$

where $\beta_\pm = \beta_1 \pm \beta_4$ and $i, j = 1, 2, 3$. It is now evident that setting $\zeta = \beta_1$ is the gauge choice that makes the spatial part of the metric diagonal [98].

The solutions of the equations above then are

$$h_{0i} = 0, \quad (5.45a)$$

$$h_{00} = \int d^3x' \frac{\rho}{4\pi|\vec{x} - \vec{x}'|} - \beta_- \rho, \quad (5.45b)$$

$$h_{ij} = \delta_{ij} \int d^3x' \frac{\rho}{4\pi|\vec{x} - \vec{x}'|} + \beta_+ \rho \delta_{ij}. \quad (5.45c)$$

The last terms in Eqs. (5.45b) and (5.45c) depend on the local value of the density. Thus, already at first-order, it is evident that such an expansion does not fit into the standard parametrized post-Newtonian framework [98]. The latter has to be extended to accommodate such corrections. It is also important to stress that the standard post-Newtonian expansion does not assume derivatives of the matter fields to be small. Therefore, a post-Newtonian expansion of Eq. (5.41) would be valid at most to order $\mathcal{O}(\mathbf{T}^2)$ and $\mathcal{O}(\lambda^2 \mathbf{T})$.

If one considers the metric outside a spherical source, such as the Sun, these local terms vanish and the metric would be identical to the Newtonian metric in GR. This reflects the fact that *in vacuo* Eq. (5.41) reduces to Einstein's equation. However, when one considers what happens inside matter, and more specifically near the surface of an object, then the deviation from GR is drastic.

In particular, consider a situation where the density has a discontinuity, as can happen on the surface of a solid (or even for fluids that are described adequately by polytropic equations of state near the surface [153]). Then the metric would be discontinuous there and the corresponding spacetime singular. In fact, the gauge transformation (5.43) would not even be admissible, as the right-hand side would diverge and it would be impossible to eliminate the off-diagonal term of the metric. It is worth noting that this is not a coordinate problem, neither a problem associated with the Newtonian approximation. One can use Eq. (5.41) to straightforwardly calculate invariants such as R or $R_{ab}R^{ab}$ and check that they diverge when \mathbf{T} is discontinuous, unlike GR.

One can argue that discontinuities in the density are not really physical and that the perfect-fluid description of matter would break down at very small densities, rendering our treatment inadequate. This is in principle true, but in practice it does not alleviate the problem. Indeed after introducing some microphysical description, there are no guaranties that the solution would be regular. Moreover, abandoning the fluid approximation would just increase the differential order of the field equations in the matter sector, rendering the curvature even more sensitive to abrupt variations of the matter fields.

5.5 Deriving Constraints

We know from everyday experience that very sharp transitions in density do exist in nature and one needs to go to very small scales to resolve them. This is enough to impose very tight constraints on β_1 and β_4 . Let us demonstrate this with a simple example, a calculation of the acceleration $\vec{a} = \vec{\nabla} h_{00}$ experienced within a thin layer in the interior and close to the surface of an object with Newtonian mass M and radius R_s . The total acceleration reads

$$\vec{a} = \vec{a}_N - \beta_- \vec{\nabla} \rho, \quad (5.46)$$

where \vec{a}_N is the standard Newtonian acceleration. We assume spherical symmetry and for simplicity we take the density of the object to be nearly constant, $\rho(r) \sim \rho_0$, everywhere apart from a thin layer of width $L \ll R_s$ near the surface and the object is otherwise in vacuum.

If the thin layer were absent and the density had a jump, *e.g.* $\rho(r) = \rho_0 \Theta(R_s - r)$ where Θ is the Heaviside function, then the correction $\vec{\nabla} \rho$ would introduce a Dirac delta contribution to the acceleration.

This is already indicative of the pathology associated with having higher-order derivatives of matter in the gravitational field equations. Suppose now that microphysics in the transition region would allow for a smoother transition that fails to be captured in the description above. Then one could consider the aforementioned layer to have the following density profile:

$$\rho(r) = \rho_0 [(R_s - r)/L]^n, \quad R_s - L < r < R_s. \quad (5.47)$$

This can be thought of as an effective description for the smoother transition, where L is the characteristic length scale at which microphysics would become important and n parametrizes the slope of the profile.

Using this profile, in the region $R_s - L < r < R_s$ we find:

$$\frac{a}{a_N} = 1 + \frac{3n}{4\pi R_s L} \beta_- [(R_s - r)/L]^{n-1}, \quad (5.48)$$

where we have used $a_N = M/R_s^2$, which is valid in the $L \ll R_s$ limit.

To not affect the standard Newtonian force to measurable levels in tabletop experiments, the last term on the right-hand side of the equation above must be much smaller than unity. Evaluating the acceleration at $r \sim R_s - L$ we obtain the constraint

$$(\beta_1 - \beta_4) \ll 4\pi R_s L / (3n). \quad (5.49)$$

Note that, once G_N is appropriately reinstated, each copy of T_{ab} carries with it a G_N , as they only appear in this combination before eliminating the auxiliary field. Modulo fine-tuning, one could think of β_1 and β_4 as numerical coefficients of order unity times a characteristic length scale λ_β squared. Then, choosing appropriate values for R_s and L one can turn the constraint above on a constraint on λ_β . For instance, if we choose – quite conservatively – R_s to be of the order of meters and L to be of the order of microns, then

$$\lambda_\beta \ll n^{-1/2} \text{ mm}. \quad (5.50)$$

Compared to typical astrophysical and cosmological length scales, this is an extremely tight constraint.

For comparison, the Hubble radius squared is roughly $\Lambda^{-1} \sim 10^{52} \text{m}^2$. One could hope to evade this constraint by fine-tuning the parameters. However, similar arguments can be made for the stresses $\sim \vec{\nabla} h_{ij}$, which would provide a constraint on $\beta_+ = \beta_1 + \beta_4$ in Eq. (5.45c). Fine-tuning would not suffice to evade both constraints.

It goes beyond the scope of our analysis to provide precise and exhaustive constraints on β_1 and β_4 . Our goal is to demonstrate that the theories we are discussing are unlikely to have any effect at large scales, if they are to be compatible with local experiments. Our analysis does not actually rule out the possibility that eliminating an auxiliary field can affect the value of the (effective) cosmological constant, which could perhaps be of some value in addressing the cosmological constant problem. However, the corresponding theory would have to accommodate at least two length scales apart from the Planck scale: λ_β and the effective cosmological constant scale. Keeping these scales separated without fine-tuning would not be an easy task.

5.6 Some Remarks

It is worth to notice that our approach has certain limitations. S_{ab} was constructed under the assumption that an expansion in derivatives (or, equivalently, a small- \mathbf{T} and a small- $\nabla\mathbf{T}$ expansions) is applicable, which does not have to hold in all regimes.

Our take on this is that such an expansion is expected to be valid in regimes where experiments verify the predictions of GR already. One could also wonder if going to the next order in derivatives could help relax the constraints. This is not the case, as adding more derivatives of \mathbf{T} would simply make the metric even more sensitive to abrupt changes in the energy density.

A subtle point is that coefficients of higher-order terms could “contaminate” the relations of Eq. (5.40). This is due to the use of the lowest-order Eq. (5.35) in order to express $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ – and specifically because of the presence of a cosmological constant.

This is already seen in Eq. (5.40), where higher-order coefficients multiplied by Λ are added to lower-order coefficients. The presence of Λ in such terms is required for reasons of dimensionality, and if G_N were to be reinstated, all such terms would appear to be suppressed by ΛG_N . This guarantees that their contribution will be negligible.

In summary, we have shown that gravity theories with auxiliary fields effectively lead to a modification of Einstein's equations by an addition of a divergence-free second rank tensor which is constructed solely with the usual stress-energy tensor of matter, the metric, and their derivatives. It would be interesting to interpret these corrections as an effective stress-energy tensor [159].

In these theories, the presence of higher-order derivatives of the matter fields is inevitable. We have developed a very general parametrization of auxiliary field theories and showed that, to next-to-leading order in derivatives of the matter fields, all auxiliary field theories can be described with only two parameters (apart from the cosmological constant). Finally, we have shown that these parameters can be severely constrained, as the presence of higher-order derivatives of the matter fields in the field equations renders the metric overly sensitive to abrupt changes of the matter energy density. The very tight constraint obtained on the length scale characterizing these theories, makes it particularly challenging to construct theories with auxiliary fields that could have any effect at very large length scales.

5.7 Cosmological Phenomenology of Gravity with Auxiliary Fields

We want here to get a deeper insight about the cosmological implications of gravity theories with auxiliary fields [160]. Let us then consider a FLRW background written in spherical coordinates in the comoving frame,

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (5.51)$$

Matter is described by a perfect fluid whose stress-energy tensor is of the form

$$T^{\mu\nu} = [\rho(t) + P(t)] u^\mu u^\nu - P(t) g^{\mu\nu}, \quad (5.52)$$

where $u^\mu = (1, 0, 0, 0)$ in the comoving frame. The stress-energy tensor conservation equation, $\nabla_\mu T^{\mu\nu} = 0$, reduces to the standard conservation law

$$\dot{\rho}(t) + 3H(t) [\rho(t) + P(t)] = 0. \quad (5.53)$$

For simplicity, we consider a single fluid, with a barotropic equation of state $P(t) = w\rho(t)$. Then, Eq. (5.53) can be immediately integrated to give

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)} \right)^{3w+3}, \quad (5.54)$$

where ρ_0 and a_0 are respectively the values of the density and of the scale factor at the present time.

Inserting the ansatz (5.51) into the field equations (5.41), and expanding the latter in terms of $\epsilon \equiv \beta_i \rho \ll 1$, this reduces to the following modified Friedmann equation:

$$\begin{aligned} 3H^2(t) = & \Lambda - \frac{3k}{a(t)^2} + \rho(t) \left\{ 1 + [9w(1+w) - 4] \beta_1 \Lambda + 3w(1+w) \beta_4 \Lambda \right. \\ & \left. + \frac{3k(1+w)(1-3w)(3\beta_1 + \beta_4)}{a(t)^2} \right\} - \frac{3}{4} (1+w)(1-3w)(3\beta_1 + \beta_4) \rho(t)^2 \\ & + \mathcal{O}(\rho^3), \end{aligned} \quad (5.55)$$

where we have included terms up to $\mathcal{O}(\beta_i)$. It is easy to show that Eqs. (5.53)-(5.55) completely solve the modified Einstein equations to the same order. This is a check of our derivation, since in this theory the conservation of the stress-energy tensor is a consequence of the contracted Bianchi identity, as in GR.

Interestingly, the corrections proportional to ρ^2 vanishes when $w = 1/3$ (radiation-dominated era), $w = -1$ (vacuum-dominated era) and for the special combination $\beta_4 = -3\beta_1$. Furthermore, when $\Lambda = 0$, the coefficients β_i appear only in the combination $\beta \equiv 3\beta_1 + \beta_4$. In particular, when $\Lambda = 0$ and for a spatially flat Universe ($k = 0$), Eq. (5.55) reduces to

$$3H^2(t) = \rho(t) - \frac{\rho(t)^2}{\rho_c} + \mathcal{O}(\rho^3), \quad (5.56)$$

where

$$\rho_c = \frac{4}{3(1+w)(1-3w)(3\beta_1 + \beta_4)}. \quad (5.57)$$

Therefore, following the evolution of the Universe backward in time, one finds that it is similar to the one of GR when $\rho \ll |\rho_c|$. However, if $\rho_c > 0$, as $\rho \rightarrow \rho_c$ the scale factor freezes and the evolution stops at finite time. Clearly, a bounce occurs when $\rho = \rho_c$.

Furthermore, in principle the existence of a minimal length scale is guaranteed whenever w is not exactly $1/3$ or -1 .

Although in the radiation-dominated era there is no correction to the order we are considering, it has been discussed in Ref. [145], which is a non-linear completion of our representation in the case of EiBI gravity, that a correction will be indeed present at higher-orders.

Moreover, if $\Lambda \neq 0$ (as the observations indicate), there are corrections to the coefficient of ρ , which is essentially the value of G_N once the latter is reinstated. For example, in

the case of a spatially flat Universe ($k = 0$), filled with dust ($w = 0$), we get from Eq. (5.55) a correction to G_N given by $(1 - 4\beta_1\Lambda)$.

So, we have obtained an effective modified Friedmann equation without taking into account extra degrees of freedom, that generically leads to a largest phenomenology with respect to GR. Interestingly, Eq. (5.56) looks like the modified Friedmann equation which is obtained in the very different context of Loop Quantum Cosmology [161, 162]. In that scenario, the critical density for the bounce is given by $\rho_c = \sqrt{3}/(16\pi^2 G_N \gamma^3 l_p^2)$, where γ is the Barbero-Immirzi parameter and l_p is the Planck length.

Moreover, it has been found in Ref. [163] that it is possible to reproduce the same effective Friedmann equation also in the context of higher-order gravity theories such as $f(R)$ gravity, using the order reduction technique developed in Refs. [164–166].

Chapter 6

Conclusions and Future Perspectives

6.1 Summary

In this Thesis we have started from the fact that GR plus a cosmological constant term is elegantly singled out from Lovelock's theorem. In fact it is the only theory which is built up solely with the metric tensor and has second order field equations. Nevertheless it is plagued by some major theoretical and phenomenological problems. On one side, we have mentioned the problems concerning its (non-)renormalizability and the size of the cosmological constant predicted by the SM of particle physics, much larger of the observed value. On the other side, we have discussed the necessity to take into account DM and DE components in order to match the astrophysical and cosmological observations at very large scales. The most common road taken in order to go beyond GR and find a solution to these shortcomings, has been to add extra dynamical fields in the gravitational action or to consider theories with higher-order derivatives of the metric tensor which generically corresponds to introduce more degrees of freedom as well. In fact, considering more propagating modes is the first possibility one can take into account, hoping that the desired phenomenology can be obtained by adding extra features to the theory.

However this procedure can lead to shortcomings and instabilities. First one has to be very careful in order to get a theory which is free of ghosts as dictated by the theorem of Ostrogradski. The latter demonstrates that theories with higher-order derivatives are generically plagued by instabilities. However, higher-order theories which can be explicitly re-written as second-order theories with more fields evade such instabilities.

Moreover, once stability issues have been addressed and the behaviour of the new degrees of freedom has been tamed, the next step is to find a mechanism that hides them in regimes where GR is well tested and no extra dynamical fields have been detected. Such

mechanisms should still allow them to be present and lead to different phenomenology in other regimes. In absence of a strong evidence for the existence of these hypothetical extra fields, it is due to explore further alternatives to GR.

In this respect, we have explored two possible ways to circumvent Lovelock's theorem in order to get extra phenomenological features into the game: breaking diffeomorphism invariance or adding non-dynamical extra fields.

In Chapter 2 we have discussed Hořava gravity, which is a theory built within a preferred spacetime foliation and so it does not enjoy invariance under full diffeomorphisms. It is a theory which breaks Lorentz invariance and in the last years has gained a lot of attention, as Lorentz symmetry breaking has been proven to act as a field theory regulator leading to improved UV behaviour. The theory also propagates an extra scalar degree of freedom besides the usual graviton because of less symmetry. Moreover we have also discussed \mathcal{A} -theory, and we have shown how the latter is related to Hořava gravity in the IR limit once the æther vector field is restricted to be hypersurface orthogonal.

In Chapter 3 we have studied the various restricted versions of Hořava gravity, which are compatible with the symmetry of the theory. In particular we have paid special attention to the dynamics of the extra propagating scalar degree of freedom, showing that most of the problems in this respect are caused by the assumption of projectability, *i.e.* the requirement that the lapse is just a function of time. We have introduced a new version of the theory which assumes detailed balance but not projectability, and we have shown that it enjoys well-behaved dynamics for both the spin-0 and spin-2 gravitons. Moreover we have shown that in such a version of the theory the bare cosmological constant is related to the mass scale suppressing the higher-order operators, and we have obtained some bounds for it. Interestingly such bounds are found to tune with the value of the vacuum energy contribution gained at the weak energy scale. With this in mind it is even more pressing to look for a resolution to the vacuum energy problem in the context of Hořava gravity, where a conclusive answer is still to be found.

We have then moved in Chapter 4 to study the causal properties of Hořava gravity by looking for three-dimensional rotating black hole solutions. Since in three-dimensional GR black hole solutions are only found with the presence of a negative cosmological constant, thus having AdS asymptotics, we have looked for its analogue in the IR limit of Hořava gravity. We have shown that solutions with AdS asymptotics can only exist in the $\eta = 0$ sector of the theory. So, within this sector we have found the most general AdS rotating black hole solutions and we have shown that black holes with flat or dS asymptotics also exist. Moreover, we have studied the properties of these black hole solutions, such as the singularities and the causal horizons.

In this respect, we have highlighted how in the context of Lorentz-violating gravity theories the concept of universal horizon arises as an absolute causal boundary for all signals, no matter how fast they propagate [121, 122].

Nevertheless we have also found that in three-dimensional Hořava gravity black holes do not always possess such a universal horizon, as we have pointed out by focusing on some specific examples. By virtue of this, the problem of defining black holes in Lorentz-violating gravity theories arises. In fact, black holes which do not possess a universal horizon are still black holes in the conventional (GR) sense, as the Killing horizon acts as an absolute causal boundary for all modes which propagate at finite speed. However, since perturbations that reside in the gravity sector can propagate infinitely fast in such a theory, the concept of black hole itself must be questioned.

It must be mentioned that, at the moment, the solutions we have found are the most general and exact rotating black hole solutions with AdS asymptotics existing in the context of Hořava gravity in a lower-dimensional spacetime.

The last part of this Thesis has been devoted to study another alternative way to modify the gravitational action of GR. Instead of adding extra dynamical fields, in Chapter 5 we have considered the less explored possibility of adding fields which are non-dynamical. We have assumed that matter fields couple only to the metric in the usual way (*i.e.* minimally), so the weak equivalence principle is automatically satisfied and the matter Lagrangian reduces to that of the SM in the local frame.

If the extra fields are not dynamical, they can be algebraically related to the metric, the matter fields and their derivatives, and then they can be eliminated from the field equations. Such fields are then called auxiliary fields, and the entire procedure through which they are eliminated leads to modifications of the field equations of GR.

We have developed in a very generic fashion a parametrization of these theories, irrespectively of the tensorial rank or other characteristics of the auxiliary field, by constructing the most general field equations up to second-order derivatives of the stress-energy tensor. The resulting outcome is that these theories generically lead to field equations containing higher-order derivatives of the matter fields, while the derivatives which act on the metric tensor are still second-order. Moreover we have shown that only two free parameters distinguish all the possible theories within this class, aside from a cosmological constant whose final value can be affected by the procedure through which the auxiliary field is removed from the field equations.

Our approach can be used to put stringent, theory-independent constraints on such theories, as we demonstrate using the Newtonian limit as an example. In fact we are able to estimate in a quite conservative way the characteristic length scale at which these theories are expected to have relevant effects. Such a length scale is found to be 30 orders of magnitude smaller than the Hubble radius, which is the typical cosmological length scale. This makes it particularly challenging to construct theories with auxiliary fields that could have any effect at very large length scales.

6.2 Remarks and Future Perspectives

As we have discussed, Lorentz invariance breaking can act as a field theory regulator rendering a theory power-counting renormalizable. This is indeed sufficient to keep all the Feynman diagrams of the graviton finite, even though momenta enter self-interaction vertices. Nevertheless, the issue of renormalizability beyond the power-counting arguments is still open for a theory like Hořava gravity which is candidate to be a UV completion of GR. Recently, a work about renormalization in Hořava gravity has appeared [167], where a one-loop calculation is presented in a toy model of the theory. In fact due to the complexity of the computation in the full theory, the authors consider the projectable version of Hořava gravity in $2 + 1$ dimensions. It is then claimed that in such a model the UV limit is singular at one-loop order. However, since in this lower-dimensional toy model the conformal mode is the only physical degree of freedom, one cannot make any conclusion about the full theory in higher-dimensional spacetimes and only an explicit calculation can tell if this picture might change.

Furthermore, in another very recent work [168], the renormalization group flow of projectable Hořava gravity coupled to n Lifshitz scalars is studied in four dimensions. It is shown that the theory is asymptotically free in the large- n expansion, providing a strong indication that it is perturbatively renormalizable.

We have already stated that a shortcoming of the full theory is the very large number of operators that are allowed by the symmetry. This makes the theory intractable and compromises predictability in the UV. In order to reduce the number of terms in the action one is in need of a principle or symmetry and, as discussed previously, the current suggestions of projectability and detailed balance do not seem optimal, so further proposals in this direction are needed.

Moreover, also the issue related to vacuum energy in Hořava gravity deserves further investigation. Surprisingly, if detailed balance is implemented, the size of the bare cosmological constant is comparable to the estimate for the vacuum energy obtained by using the weak energy scale as a cut-off. This may be interesting in the perspective in which the two contributions can cancel each other. An answer to this issue should be definitely found as well.

A further open problem in the scenario of Lorentz-violating gravity theories concerns black holes. Since their thermodynamics is presumed to represent the coarse-grained behaviour of the (as yet unknown) microstates of quantum gravity, they have a central role in the context of quantum gravity phenomenology. Of course, before one can investigate black hole thermodynamics, one first has to have a black hole solution and then ask if thermodynamical laws can be ascribed to it. This has been the case for the BTZ solution in GR. The BTZ solution has been so important for the field because quantizing GR in

three dimensions is more tractable than in four dimensions. As such, the hope was that the thermodynamics of the BTZ solution could be directly linked to the quantized version of three-dimensional GR, establishing the connection between BH thermodynamics and quantum gravity on a much more solid foundation. This reasoning carries over to any alternative theory – and in particular those that possibly possess a UV completion, such as Hořava gravity.

However, Lorentz-violating gravity presents some challenges to this overarching program. In GR, thermodynamics is built on the event horizon, but in Lorentz-violating gravity, it isn't even clear what the relevant horizon might be. In fact the universal horizon, as a structure that blocks even arbitrarily fast moving signals, emerges there as a replacement for the usual event horizon. So, the possibility of constructing a similar thermodynamical theory on whatever the relevant horizon may be, remains to be seen.

By presenting an exact black hole solution, we pave the way to all these investigations.

Finally, let us just give some remarks about gravity theories with auxiliary fields. Some shortcomings have been revealed in the context of Palatini $f(R)$ gravity theories, which are specific theories within the aforementioned class. For example, in [169] potential conflicts with the SM of particle physics have been pointed out. It is indeed claimed that integrating out the non-dynamical degree of freedom gives rise to additional interactions among the matter fields of the SM at unacceptably small energy scales, thus creating potential conflicts with particle physics experiments. Furthermore, other issues with averaging matters have also been suggested in [170], where Palatini gravity theories are indeed proved to affect the propagation of photons, thus changing the cosmic expansion of the Universe during radiation domination. Observational data on BBN could then place severe constraints on such models.

All these arguments provide further reasons to carry on a very detailed analysis of the phenomenology of gravity theories with auxiliary fields, in particular now that a general parametrization for such theories does indeed exist.

In addition to the issues raised above, an extension of the parametrized post-Newtonian framework [98], the cosmological applications [171, 172], and the analysis of the generalized Tolman-Oppenheimer-Volkoff equations for this class of theories deserve some special investigation in the near future.

Appendix A

Rotating Black Holes in Three-Dimensional Hořava gravity

A.1 Brown-Henneaux Asymptotic Conditions for Anti-de Sitter Spacetime

Inserting Brown-Henneaux boundary conditions into the EL equations results in rather complicated expressions, but our interest here is to investigate only the leading-order terms. For this it will suffice to consider just the numerators of the expressions. For example, consider the expression

$$f := \frac{ar^k + br^{(k-1)} + \dots}{cr^j + dr^{(j-1)} + \dots}, \quad (\text{A.1})$$

with $k > 0$ and $j > 0$ (for the sake of argument). Then as $r \rightarrow \infty$, the leading-order term of f is

$$\frac{ar^k + br^{(k-1)} + \dots}{cr^j + dr^{(j-1)} + \dots} \sim \frac{a}{c} r^{(k-j)}. \quad (\text{A.2})$$

Enforcing that f vanishes asymptotically to leading order requires only that $a = 0$, so it is sufficient to focus mainly on the numerator of f . We shall call this the *leading-order coefficient* (LOC). The denominator merely rescales the LOC by a constant and so it shall not play an important role in the leading-order asymptotic analysis.

The LOCs of the asymptotic EL equations will depend on m . For the Z -equation, the dominant term in the numerator is either $\sim r^{12}$ or $\sim r^{(4m+16)}$. When $m < -1$, r^{12} dominates. When $m > -1$, $r^{(4m+16)}$ dominates. And when $m = -1$, both terms (along with several others) scale with r in the same way (i.e., $\sim r^{12}$).

For the F -equation, you get something similar. The dominant term in the numerator is either $\sim r^{14}$ or $\sim r^{(6m+20)}$. When $m < -1$, r^{14} dominates. When $m > -1$, $r^{(6m+20)}$

dominates. And when $m = -1$, both terms (along with several others) scale with r in the same way (i.e., $\sim r^{14}$).

Finally, for the U -equation, the dominant term in the numerator is either $\sim r^{(m+14)}$ or $\sim r^{(5m+18)}$. When $m < -1$, $r^{(m+14)}$ dominates. When $m > -1$, $r^{(5m+18)}$ dominates. And when $m = -1$, both terms (along with several others) scale with r in the same way (i.e., $\sim r^{13}$).

Clearly, $m = -1$ is the critical value for the analysis. We shall investigate each of the cases in turn: $\{m > -1, m = -1, m < -1\}$.

A.1.1 Case I: $m > -1$

In this case, the LOCs of the U -, F -, and Z -equations (modulo harmless factors) are, respectively,

$$(3 + 4\eta - 3\lambda) + 4(1 + \eta - \lambda)m + (1 + \eta - \lambda)m^2, \quad (\text{A.3a})$$

$$(1 + 4\eta + 3\lambda) + [4\eta - 2(-2 + \lambda + \xi)]m + (1 + \eta - \lambda)m^2, \quad (\text{A.3b})$$

$$\left(\frac{11}{3} + 4\eta - 5\lambda + \frac{4}{3}\xi\right) + \frac{2}{3}(6 + 6\eta - 7\lambda + \xi)m + (1 + \eta - \lambda)m^2. \quad (\text{A.3c})$$

These clearly do not vanish simultaneously for generic coupling constants. For them to vanish simultaneously, the coupling constants will have to be especially chosen. This can only happen if the expressions are identical. The coefficients of the terms linear in m have to match. So,

$$4(1 + \eta - \lambda) = [4\eta - 2(-2 + \lambda + \xi)] = \frac{2}{3}(6 + 6\eta - 7\lambda + \xi). \quad (\text{A.4})$$

This is a system of three equations in three unknowns, for which the solution is simply

$$\xi = \lambda. \quad (\text{A.5})$$

The constant terms (i.e. $\mathcal{O}(m^0)$) also have to match:

$$(3 + 4\eta - 3\lambda) = (1 + 4\eta + 3\lambda) = \left(\frac{11}{3} + 4\eta - 5\lambda + \frac{4}{3}\xi\right), \quad (\text{A.6})$$

which gives

$$\xi = 0, \quad \lambda = \frac{1}{3}. \quad (\text{A.7})$$

Because these are incompatible with Eq. (A.5), we conclude that the leading-order terms of the EL equations cannot simultaneously vanish. This means that for $m > -1$, in $U \sim U_0 r^m$, AdS asymptotics for the metric are inadmissible.

A.1.2 Case II: $m = -1$

In this case, enforcing that the LOCs of the U -, F -, and Z -equations vanish (respectively), we have

$$\eta U_0 (\mathcal{L}^2 + U_0^2)^2 = 0, \quad (\text{A.8a})$$

$$2\mathcal{L}^4 \xi + (1/\Lambda) [\mathcal{L}^2 (\eta + 2\xi) - (-2 + \eta + 4\lambda - 2\xi) U_0^2] = 0, \quad (\text{A.8b})$$

$$2\mathcal{L}^4 \xi + (1/\Lambda) [\mathcal{L}^2 (3\eta + 2\xi) + (2 + 3\eta - 4\lambda - 2\xi) U_0^2] = 0, \quad (\text{A.8c})$$

where $\bar{\Lambda} = -1/\mathcal{L}^2$. The first of these equations demands that $\eta = 0$. When this is the case, the other two equations lead to the same solution

$$U_0^2 = \mathcal{L}^2 (1 + \Lambda \mathcal{L}^2) \frac{\xi}{2\lambda - \xi - 1}. \quad (\text{A.9})$$

This we can verify to be the first æther charge of our solution. As $r \rightarrow \infty$, our exact solution behaves like

$$U \sim \frac{br}{F^2} \sim \frac{br}{r^2/\mathcal{L}^2} = \frac{b\mathcal{L}^2}{r}. \quad (\text{A.10})$$

Therefore, since in our asymptotic analysis, $U \sim U_0/r$ (for $m = -1$), we must have $U_0 = b\mathcal{L}^2$.

On the other hand, from Eq. (4.46) we have

$$\bar{\Lambda} = \Lambda - \frac{b^2 (2\lambda - \xi - 1)}{\xi}, \quad (\text{A.11a})$$

$$-\frac{1}{\mathcal{L}^2} = \Lambda - \frac{U_0^2 (2\lambda - \xi - 1)}{\mathcal{L}^4 \xi}, \quad (\text{A.11b})$$

which is identical to Eq. (A.9). This demonstrates that the asymptotic analysis recovers one of the æther charges (i.e. b) for the case $m = -1$. What is most essential, however, is that $m = -1$ forces us to set $\eta = 0$.

A.1.3 Case III: $m < -1$

For this final case, the LOCs of the U -, F - and Z - equations give

$$(3 + \eta - 3\lambda) + 4(1 - \lambda)m + (1 - \lambda)m^2 = 0, \quad (\text{A.12a})$$

$$2\mathcal{L}^2 \xi + (1/\Lambda) (\eta + 2\xi) = 0, \quad (\text{A.12b})$$

$$2\mathcal{L}^2 \xi + (1/\Lambda) (3\eta + 2\xi) = 0. \quad (\text{A.12c})$$

The second and third of these equations imply again that η has to be zero. Putting this into the first equation gives

$$(m+1)(m+3)(-1+\lambda) = 0. \quad (\text{A.13})$$

Since $m < -1$ and since we wish to keep the coupling constants as generic as possible, we must choose $m = -3$. Moreover, the second and third equations give

$$2\xi(\mathcal{L}^2 + 1/\Lambda) = 0 \implies \mathcal{L}^2 = -1/\Lambda. \quad (\text{A.14})$$

In other words, the effective cosmological constant must be the bare one: $\bar{\Lambda} = \Lambda$. Again, however, this case shows that $\eta = 0$ is required.

To summarize, we have demonstrated in this appendix that Brown-Henneaux AdS boundary conditions forces us into the $\eta = 0$ sector. As an added bonus, we see that for AdS asymptotics, U can only scale as r^{-1} or r^{-3} at large values of r , indicating the existence of two asymptotic æther charges, which is precisely what we find in our exact solution.

A.2 Æther Alignment

Because the timelike Killing vector, $t^\alpha := \left(\frac{\partial}{\partial t}\right)^\alpha$, turns null in black hole spacetimes, the æther cannot be aligned with it everywhere. In this appendix, we work out how alignment is realized in terms of our unknown functions, $\{Z, F, U, \Omega\}$. If u^α is aligned with t^α , then $t_\alpha a^\alpha = 0$, where $a^\alpha := u^\beta \nabla_\beta u^\alpha$. Normalizing t^α to get $\hat{t}^\alpha := t^\alpha / \sqrt{g_{\mu\nu} t^\mu t^\nu}$, alignment is then equivalent to

$$\hat{t}^\alpha a_\alpha = -\frac{UF^3 [(U^2 F^2 + 1) Z^2]'}{Z \sqrt{F^2 (1 + U^2 F^2)} \sqrt{Z^2 - r^2 \Omega^2}} = 0. \quad (\text{A.15})$$

This is satisfied either when $F = 0$, $U = 0$, or

$$(U^2 F^2 + 1) Z^2 = C^2 \implies u_t = C, \quad (\text{A.16})$$

for some constant C .

This latter case just corresponds to a zero-acceleration æther, where the foliation is provided by the Painlevé-Gullstrand time. It does not represent alignment between the æther and the timelike Killing vector. Therefore, since F cannot vanish everywhere, the timelike Killing vector and the æther are aligned everywhere if and only if $U = 0$.

Now specializing to our solution, where $Z = F$, we can use Eq. (A.15) to check for asymptotic alignment. For AdS asymptotics, $F^2 \sim r^2$. As we have shown in Sec. (4.4) and Appendix (A.1), the æther component u_r (or U) can then only fall-off as $U \sim r^{-1}$

or $U \sim r^{-3}$.

It is easy to check that $\hat{t}^\alpha a_\alpha \sim r^0$ or $\hat{t}^\alpha a_\alpha \sim r^{-2}$, respectively. It aligns asymptotically only for the latter case. Thus, when $b \neq 0$ in Eq. (4.42), and hence $U \sim r^{-1}$, the æther does not become orthogonal to constant- t surfaces as $r \rightarrow \infty$. The parameter b is then a measure of the asymptotic misalignment of the æther.

A.3 Metric Ansatz in the Preferred Time

The approach in [139] amounts to setting $\eta = 0$ and $U = 0$ in Eq. (4.24). There, also an R^2 -term was included in the action but we do not consider this term and work fully within the IR sector. Doing so gives the much simplified Lagrangian

$$L_{\text{align}} = \frac{r^3 F}{2Z} (\Omega')^2 - 2\xi Z \left(\Lambda \frac{r}{F} + F' \right). \quad (\text{A.17})$$

However, the metric ansatz given in (4.20) is not the most general stationary metric when one works within the preferred foliation. In this case the æther is normal to constant- T hypersurfaces and thus u_T will be the only non-vanishing component in the preferred foliation. To bring the metric ansatz of Eq. (4.20) into the preferred frame, one needs to perform the coordinate transformation that puts the æther into this form, resulting in $u_{r_T} = 0$.

If the coordinate transformation is given by $T = h(t, r)$, $r_T = r$, $\phi_T = j(r, \phi)$, then $u_{r_T} = 0$ or $(\partial t / \partial r_T) u_t(r) + (\partial r / \partial r_T) u_r(r) = 0$ can be integrated to give

$$t = G(T) + \int K(r) dr, \quad (\text{A.18})$$

where $K(r) = -u_r(r)/u_t(r)$, and $G(T)$ is some general function of T .

Thus, $T = G^{-1}(t - \int K(r) dr)$, and using the fact that $T \rightarrow F(T)$ is a symmetry of the theory, we conclude that T has the general form $T = t + f(r)$. Explicitly, this is

$$T = t + \int \frac{u_r(r)}{u_t(r)} dr, \quad r_T = r. \quad (\text{A.19})$$

In terms of the unknown functions Z, F, Ω and U we use, the full transformation is

$$dt = dT - \frac{U}{N\sqrt{1 + F^2 U^2}} dr_T, \quad (\text{A.20a})$$

$$d\phi = d\phi_T, \quad (\text{A.20b})$$

$$dr = dr_T, \quad (\text{A.20c})$$

and the metric ansatz in the preferred frame is

$$ds^2 = Z^2 dT^2 - \frac{2ZU}{\sqrt{1+F^2U^2}} dT dr - \frac{1}{F^2(1+F^2U^2)} dr^2 - r^2 \left(d\phi + \Omega dT - \frac{\Omega U}{Z\sqrt{1+F^2U^2}} dr \right)^2. \quad (\text{A.21})$$

Inserting this metric ansatz directly into the preferred frame action (2.27), provides an equivalent strategy to the one we have adopted.

The metric in the preferred foliation will generally have a g_{Tr} and a $g_{r\phi}$ component because the æther will not be orthogonal to constant- t hypersurfaces, or equivalently, T and t do not generally coincide. Evidently, an aligned æther configuration is just a special case, which in our parametrization is $U = 0$.

A.4 Globally Aligned Æther

As previously mentioned, the case of a globally aligned æther is a significant simplification in the search for exact solutions. Assuming $U = 0$, the remaining N - and F -equations then respectively simplify to

$$\eta \left[N'' - \frac{N'^2}{2N} + \left(\frac{F'}{F} + \frac{1}{r} \right) N' \right] - \frac{(4r^3 \xi F F' + \mathcal{J}^2 - 4r^4 \xi \Lambda)}{4r^4 F^2} N = 0, \quad (\text{A.22a})$$

$$\eta N'^2 + \frac{2\xi F F'}{r} + \left(\frac{\mathcal{J}^2}{2r^4} + 2\xi \Lambda \right) \left(\frac{N}{F} \right)^2 = 0. \quad (\text{A.22b})$$

Assuming $N = F$ Eqs. (A.22a) and (A.22b) turn into

$$\frac{1}{F} \left(\frac{\mathcal{J}^2}{2r^3} + 2r\xi\Lambda \right) + 2\xi F' = -\eta \frac{rF'^2}{F} - 2\eta (F' + rF''), \quad (\text{A.23a})$$

$$\frac{1}{F} \left(\frac{\mathcal{J}^2}{2r^3} + 2r\xi\Lambda \right) + 2\xi F' = -\eta \frac{rF'^2}{F}. \quad (\text{A.23b})$$

From these, one can simply pick off the necessary condition

$$\eta (rF'' + F') = 0. \quad (\text{A.24})$$

This is satisfied if either $\eta = 0$ or $F = A + B \ln r$. Supposing $\eta \neq 0$, so that $F = A + B \ln r$, we will only have a solution if

$$-\mathcal{J}^2 - 2B(B\eta + 2A\xi)r^2 - 4\xi\Lambda r^4 - 4\xi B^2 r^2 \ln r = 0. \quad (\text{A.25})$$

This vanishes identically only if $B, \bar{\mathcal{J}}$, and Λ are all zero. In other words, the spacetime has to be static and have zero cosmological constant. In this case, $\{N = F = A, U = 0\}$ is a solution, but this is just flat spacetime with an aligned æther.

To find non-trivial solutions satisfying $N = F$, we have to require $\eta = 0$. In this case, the F - and N -equations both become

$$\frac{1}{2} \frac{d}{dr} [F(r)^2] + \frac{\mathcal{J}^2}{4r^3\xi} + \Lambda r = 0, \quad (\text{A.26})$$

which, by simple inspection, is readily integrated to give

$$F^2 = N^2 = -\mathcal{M} + \frac{\mathcal{J}^2}{4r^2\xi} - \Lambda r^2. \quad (\text{A.27})$$

This, together with $N^\phi = -\mathcal{J}/(2r^2)$, is the pure IR limit of the solution found in [139], which reduces to the BTZ solution of GR when $\xi = 1$. It is the unique solution to the EL equations, assuming $U = 0$ and $N = F$.

As noted in Sec. (4.5.1), this solution is problematic from the viewpoint of Hořava gravity. An æther configuration that becomes null at the event horizon is inadmissible because then it cannot represent a spacelike foliation, which in Hořava gravity is not merely a gauge artifact. Because the foliation carries physical content, it is essential to require its regularity throughout the spacetime manifold, and in particular, at the event horizon. To enforce this, one first needs to work in coordinates that are regular across the event horizon. We can easily transform to ingoing Einstein-Finkelstein coordinates, $\{v, \tilde{r}, \tilde{\phi}\}$, with the transformation

$$dt = dv - \frac{1}{NF} d\tilde{r}, \quad (\text{A.28a})$$

$$dr = d\tilde{r}, \quad (\text{A.28b})$$

$$d\phi = d\tilde{\phi} + \frac{N^\phi}{NF} d\tilde{r}. \quad (\text{A.28c})$$

In these coordinates, the metric appears as

$$ds^2 = N^2 dv^2 - \frac{2N}{F} dv dr - r^2 \left(d\tilde{\phi} + N^\phi dv \right)^2, \quad (\text{A.29})$$

and the æther components are

$$u_v = \sqrt{N^2 (1 + F^2 U^2)}, \quad (\text{A.30a})$$

$$u_{\tilde{r}} = U - \frac{1}{NF} \sqrt{N^2 (1 + F^2 U^2)}, \quad (\text{A.30b})$$

$$u_{\tilde{\phi}} = 0. \quad (\text{A.30c})$$

From this we see that, when $N = F$, a globally aligned æther would have Eddington-Finkelstein components, $u_v = \sqrt{F^2}$ and $u_{\tilde{r}} = -1/\sqrt{F^2}$. The latter component diverges at the event horizon (where $F=0$). The requirement of regularity invalidates any such solution, and therefore, any aligned æther solution (i.e. $U = 0$) cannot have a black hole.

In GR, we shrug off the irregularity of the metric at the event horizon as an artifact of the coordinate system, since there presumably exists a transformation $t \rightarrow \tilde{t} = \tilde{t}(t, r)$, $r \rightarrow \tilde{r} = \tilde{r}(t, r)$, $\phi \rightarrow \tilde{\phi} = \tilde{\phi}(t, r)$, such that the metric is regular at the zero of F . But such a transformation is not part of the symmetry group of Hořava theory. The transformation $t \rightarrow \tilde{t} = \tilde{t}(t, r)$ is *not* foliation-preserving. Hence, the irregularity of the metric therefore cannot be cured. At best, one can only claim that the solution is valid in the region that excludes the metric horizon. Either way, it cannot be a black hole spacetime.

We emphasize that $N = F$ is not a crucial assumption for this no-go statement. For solutions with global alignment, $U = 0$, the foliation is irregular at the black hole horizon. Therefore, one must look to misaligned æther configurations in order to find black hole solutions. As already mentioned in the main text, all of this should be obvious. Since the æther needs to be timelike everywhere, it cannot always be aligned with the timelike Killing vector, which turns null at the event horizon.

A.5 Special Choices of Hořava Parameters

Within the $\eta = 0$ sector, $\lambda = 1$ is special because we lose the constraint provided by Eq. (4.38) in Sec. (4.5.1). The U -equation is identically satisfied and one is left with an underdetermined system for the functions U and F .

The Z - and F -equations provide the sole constraint:

$$(\xi - 1) \left[\frac{d}{dr} (U^2) + 4 \left(\frac{F'}{F} \right) U^2 \right] + 2\xi \frac{F'}{F^3} + \left(\frac{\mathcal{J}^2}{2r^3} + 2r\xi\Lambda \right) \frac{1}{F^4} = 0, \quad (\text{A.31})$$

which can be integrated to give

$$(\xi - 1) U^2 = \frac{1}{F^2} \left[\mathcal{C} + \frac{\mathcal{J}^2}{4r^2} - \xi (\Lambda r^2 + F^2) \right], \quad (\text{A.32})$$

for some integration constant \mathcal{C} . When $\xi = 1$, Eq. (A.32) does not depend on U and becomes purely a condition on F . In this case, it returns for F the BTZ solution of GR, while U can be any function. This result is not surprising.

For $\eta = 0$, $\lambda = \xi = 1$ Hořava gravity in its covariant version is equivalent to GR with a hypersurface-orthogonal æther that only needs to satisfy the unit constraint without further dynamical restrictions. With our definitions the æther is indeed unit for an arbitrary U .

When $\xi \neq 1$, and since there are no more equations to satisfy, the functions F and U can be chosen so long as they are related according to Eq. (A.32). One can verify that no extra conditions arise when working with the full set of field equations instead of the reduced action equations of motion.

Note that the condition between U and F is different from Eq. (4.42). This result is consistent with the discussion in Sec. (4.5.3) about metric and æther redefinitions that set $\xi = 1$. One could think of generating the solution for an arbitrary ξ from a solution of the $\xi = 1$ theory by an inverse redefinition. Then, a suitable choice of U could lead to the desired F .

A.6 Constant- T Hypersurfaces Have Constant Mean Curvature when $\eta = 0$

In this appendix we derive the mean curvature, K , of a constant- T hypersurface. The extrinsic curvature is defined as

$$K_{\alpha\beta} := h_{\alpha}{}^{\gamma} h_{\beta}{}^{\delta} \nabla_{\gamma} u_{\delta}, \quad (\text{A.33})$$

where

$$h_{\alpha}{}^{\beta} := g^{\beta\delta} h_{\alpha\delta} = g^{\beta\delta} (g_{\alpha\delta} - u_{\alpha} u_{\delta}) \quad (\text{A.34})$$

are spatial projectors. Then the extrinsic curvature reads

$$K_{\alpha\beta} = \nabla_{\alpha} u_{\beta} - u_{\alpha} u^{\gamma} \nabla_{\gamma} u_{\beta}, \quad (\text{A.35})$$

and taking into account that u_{α} is constrained to be a unit vector, the mean curvature, defined as $K := g^{\alpha\beta} K_{\alpha\beta}$, reads

$$K = \nabla^{\beta} u_{\beta}. \quad (\text{A.36})$$

Finally, in terms of the functions $\{Z, F, U\}$, the mean curvature can be written as

$$K = -UF^2 \left[\frac{d}{dr} (\ln UZF) + \frac{1}{r} \right]. \quad (\text{A.37})$$

A straightforward calculation then reveals that the æther field in Eq. (4.42) defines a surface with constant mean curvature $K = -2b$. This turns out to be a necessary condition for any $\eta = 0$ solution. In fact, because $\eta = 0$ implies $F = Z$, in our parametrization the mean curvature according to Eq. (A.37) is just

$$K = -UF^2 \left[\frac{d}{dr} (\ln UF^2) + \frac{1}{r} \right] = -y \left(\frac{d}{dr} \ln y + \frac{1}{r} \right) = - \left(y' + \frac{y}{r} \right), \quad (\text{A.38})$$

where again we have used the substitution in Eq. (4.39), $y = UF^2$. Therefore,

$$r^2 K' = - (r^2 y'' + r y' - y), \quad (\text{A.39})$$

so Eq. (4.40) is equivalent to $K' = 0$ or $K = \text{constant} = -2b$. In other words, when $\eta = 0$, the æther defines hypersurfaces of constant mean curvature.

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