Complex Dynamics
in Semilinear Parabolic PDEs

CANDIDATE
Martino Prizzi

SUPERVISOR
Prof. Krzysztof P. Rybakowski

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CONTENTS

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INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and L be a second order elliptic differential operator on Ω. Consider the semilinear parabolic equation

$$u_t = Lu + f(x, u, \nabla u), \quad t > 0,\ x \in \Omega \tag{0.0.4}$$

with Dirichlet boundary condition

$$u = 0, \qquad t > 0,\ x \in \partial\Omega \tag{0.0.5}$$

or Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega. \tag{0.0.6}$$

Here $f\colon (x, s, w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x, s, w) \in \mathbb{R}$ is some nonlinearity.

For $p \geq 1$, the operator $-L$ with Dirichlet or Neumann boundary condition on $\partial\Omega$ defines a sectorial operator on $X = L^p(\Omega)$ with the corresponding family X^α of fractional power spaces. If $p > N$, then α can be chosen such that $X^\alpha \subset C^1(\overline{\Omega})$ and then the solutions of (0.0.4)–(0.0.5) and (0.0.4)–(0.0.6) define a local semiflow on X^α.

It is well known that for $N = 1$ the dynamics of the semiflow generated by (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) is fairly simple, as all bounded solutions are convergent [14].

If the nonlinearity f in (0.0.4) is independent of gradient terms (i.e. $f = f(x, s)$), then the semiflow generated by (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) admits a global Ljapunov functional, namely

$$V(u) := (1/2)\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega F(x, u(x))dx, \quad u \in X^\alpha,$$

where F is a primitive of f with respect to s, i.e. $\partial_s F = f$. In particular, the ω-limit set $\omega(u)$ of an arbitrary bounded solution u of (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) consists only of stationary solutions. Moreover, neither periodic solutions nor homoclinic solutions can occurr in (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6).

If $N > 1$ and the nonlinearity f does depend in a nontrivial way on gradient terms, the situation changes drastically: it has recently been proved that the dynamics of (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) can be very complicated, in fact even 'arbitrary'. A first result of this kind was given by Poláčik in [18]. More specifically, he proved that every finite jet of a vector field on $\mathbb{R}^n$ can be imbedded (realized) on the center manifold of (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) with appropriate nonlinearity f provided the kernel of the operator L (with Dirichlet boundary conditions on $\partial\Omega$) has dimension n and the corresponding eigenfunctions satisfy a certain nondegeneracy condition (called Poláčik condition). In this case $n = N$ or $n = N + 1$ and Poláčik also gave examples of operators satisfying this condition, both with $n = N$ (and Ω being the unit ball) and $n = N + 1$ (with Ω being smooth and smoothly diffeomorphic to the unit ball), and with L of the form $L = \Delta + a(x)$. In [33] Rybakowski showed that under the Poláčik condition actually all sufficiently smooth and sufficiently small vector fields v on $\mathbb{R}^n$ can be realized on the center manifold of (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) with appropriate nonlinearity f. The method of proof used in [33] (the Nash-Moser implicit mapping theorem) leads to a typical loss of derivatives: g is less smooth that v. In [25] Poláčik and Rybakowski proved that if L has analytic coefficients and Poláčik condition holds then a vector field realization result holds without loss of derivatives. They also showed that there are real analytic functions a on $\mathbb{R}^N$ such that the operator $Lu = \Delta u + a(x)u$ satisfies the Poláčik condition on a ball of $\mathbb{R}^N$ with $n = N + 1$. These results lead to a restriction in the space dimension of (0.0.4): to get realizability of any vector field of $\mathbb{R}^n$ we have to choose $n = N$ or $n = N+1$. Therefore the question arises what is the least possible space dimension that allows arbitrary dynamics in (0.0.4)–(0.0.5) and (0.0.4)–(0.0.6). In [23] it was shown by P. Poláčik that every finite jet of a vector field on $\mathbb{R}^n$ can be realized on the center manifold of (0.0.4)–(0.0.5) or (0.0.4)–(0.0.6) with an appropriate polynomial nonlinearity f and an appropriate two-dimensional domain (close to a square). In [23] the form of the nonlinearity f involves high powers of the gradient of the solution u. By the way, when modelling scientific phenomena by equations (0.0.4)–(0.0.5) and (0.0.4)–(0.0.6), one usually tries to make the convection terms (i.e. the terms depending on ∇u) as simple as possible. Therefore the question arises if such systems can also exhibit complex dynamics. An affirmative answer was given by K. Rybakowski and the author of this thesis in [29], where it is shown that the polynomial f can have prescribed (e.g. linear) dependence on ∇u.

All of the above realization results have been proved only on very particular domains, diffeomophic to a ball or close to a square, and for operators of the form $L = \Delta + a(x)$. The natural question therefore arises if one can obtain such realization results on arbitrary (sufficiently regular) domains, and with general second order elliptic operators which can be written in divergence form.

A first affirmative answer to this question was given by K. Rybakowski and the author of this thesis in [30]. More specifically, in this paper it is shown that the vector field realization result from [25] is valid on arbitrary bounded domains Ω of class $C^{2,\gamma}$, $0 < \gamma < 1$. This is proved by showing that for every such Ω there is a polynomial function a on $\mathbb{R}^N$ such that the operator $Lu = \Delta u + a(x)u$ satisfies the Poláčik condition on Ω with $n = N + 1$. In [31] the author of this thesis shows that all the above realization results can be generalized to the case of a general second order elliptic operator in divergence form on an arbitrary (smooth) spatial domain.

In order to achieve these results, four general results are established, which are of independent interest:

(1) a general version of the surjective mapping theorem with all constant made explicit;
(2) the solvability of an abstract inverse eigenvalue problem;
(3) an eigenvalue convergence result;
(4) a "localization" result.

The thesis is organized in the following way:

In Chapter 1 we give a brief survey of the basic facts about semilinear parabolic PDEs, essentially following [12, Ch. 1–3]. We introduce sectorial operators and their semigroups in an abstract setting and we give an existence theorem for local solutions of the abstract semilinear parabolic equation

$$\dot{u} = Au + f(u);$$

then we come back to concrete equations of the form (0.0.4).

In Chapter 2 we recall some results about global and local invariant (center) manifolds of semilinear parabolic equations, following the approach of [37], [4] and [32].

In Chapters 3 and 4 we describe the main results contained in [18], [33], [25], [23] and [29]; we present these result in a slightly more general form and with some necessary modifications in order to apply the theorems given in the next chapters. We prove that all the results contained in these papers are true for a general second order elliptic operator on an arbitrary spatial domain, provided certain conditions are satisfied.

In Chapter 5 we deal with an abstract inverse eigenvalue problem: given a self-adjoint operator A on a Hilbert space H and a finite number $\lambda_{l+1} \leq \lambda_{l+2} \leq \cdots \leq \lambda_{l+p}$ of eigenvalues of A whose eigenvectors satisfy a certain hypothesis there is a constant $\alpha_0 > 0$, which does not depend on the operator A but only on the geometry of the spectrum of A, such that arbitrary tuples $\mu_1, \ldots, \mu_p$ lying in the α_0-neighborhood of $\lambda_{l+1}, \ldots, \lambda_{l+p}$ can be realized as eigenvalues of a suitable perturbed operator $A + B$.

In Chapter 6 we give an eigenvalue convergence result: if D is a subdomain of Ω and the functions $c_k(x)$ satisfy certain assumptions then the eigenvalues of the operators $Lu+c_k(x)u$ on Ω converge to the corresponding eigenvalues of L on the smaller domain D. We also obtain H^1-convergence of the corresponding eigenfunctions.

In Chapters 7 and 8, following [18], [25], [23] and [29], we show that the conditions given in Chapters 3 and 4 are actually satisfied on certain particular domains, with $L = \Delta + a(x)$ and with Dirichlet boundary condition.

In Chapter 9 we show that, thanks to the general results obtained in Chapters 5 and 6, the results in Chapters 7 and 8 can be extended to any smooth bounded domain and to any secon order elliptic operator which can be written in divergence form, both with Dirichlet and Neumann boundary condition.

In Chapter 10 we make a brief 'excursus' through other situations in which the technique of jet and vector field realization can be exploited to give examples of complex dynamics in various classes of dynamical systems and we address some open problems.

CHAPTER I

PRELIMINARIES ON SEMILINEAR PARABOLIC PDES

Let $\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary $\partial\Omega$. Consider the general second order differential operator

$$L := \sum_{|\alpha| \le 2} a_\alpha(x)\partial^\alpha$$

where the coefficients $a_\alpha : \overline{\Omega} \to \mathbb{R}$ are smooth functions. The principal part L' of L is the operator

$$L' := \sum_{|\alpha|=2} a_\alpha(x)\partial^\alpha$$

Definition 1.0.1. *We say that the operator L is strongly elliptic if there exists a constant $c > 0$ such that*

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge c|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \overline{\Omega}$. The formal adjoint of L is the operator

$$L^* := \sum_{|\alpha| \le 2} (-1)^{|\alpha|}\partial^\alpha \left(\overline{a_\alpha(x)}\ \cdot\right).$$

We say that L is formally selfadjoint if $L = L^*$. We say that L can be written in divergence form if there are coefficients $a_{ij}(\cdot)$, $i, j = 1,...,N$ such that

$$L = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j).$$

Remark. If an operator L has real coefficients and it can be written in divergence form with $a_{ij}(\cdot) = a_{ji}(\cdot)$, then L is formally selfadjoint.

Consider the following general semilinear parabolic equations with Dirichlet and Neumann boudary condition respectively:

$$\begin{aligned} u_t &= Lu + g(x, u, \nabla u), && t > 0,\ x \in \Omega \\ u(x,t) &= 0, && t > 0,\ x \in \partial\Omega, \end{aligned} \tag{1.0.1}$$

$$\begin{aligned} u_t &= Lu + g(x, u, \nabla u), && t > 0,\ x \in \Omega \\ \frac{\partial}{\partial \nu}u(x,t) &= 0, && t > 0,\ x \in \partial\Omega, \end{aligned} \tag{1.0.2}$$

with an appropriate nonlinearity $g\colon (x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto g(x,s,w) \in \mathbb{R}$, and with initial datum

$$u(0,x) = u_0(x),$$

where $u_0\colon \overline{\Omega} \to \mathbb{R}$ is a continuous (resp continuously differentiable up to the boundary) function satisfying the Dirichlet (resp. Neumann) condition on $\partial\Omega$.

Definition 1.0.2. *We say that a function $u\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ is a classical solution to (1.0.1) (respectively to (1.0.2)) with initial value u_0 if*

(1) *u is twice continuously differentiable with respect to $x \in \Omega$, for all $t \in]0,T[$;*
(2) *u is continuously differentiable with respect to $t \in]0,T[$, for all $x \in \overline{\Omega}$;*
(3) *u is continuous with respect to $(t,x) \in [0,T[\times\overline{\Omega}$ (resp. continuous with respect to $(t,x) \in [0,T[\times\overline{\Omega}$ and continuously differentiable up to the boundary with respect to $x \in \overline{\Omega}$ for all $t \in [0,T[$);*
(4) *$u(0,x) \equiv u_0(x)$ on $\overline{\Omega}$;*
(5) *$u(t,x)$ satisfies the Dirichlet (resp. Neumann) boundary condition on $\partial\Omega$ for all $t \in [0,T[$;*
(6) *(1.0.1) (resp. (1.0.2)) holds on $\Omega\times]0,T[$.*

In order to study equations (1.0.1) and (1.0.2), we need to introduce an abstract functional setting. This will be done in the following sections, where we collect some basic facts about parabolic PDEs. We refer the reader to [12, Ch. 1–3] for a detailed treatment of the subject.

1. Sectorial Operators and Analytic Semigroups

In what follows we indicate by $\mathbb{K}$ both $\mathbb{R}$ and $\mathbb{C}$.

Definition 1.1.1. *Let X be a Banach space over $\mathbb{K}$; let $A\colon D_A \subset X \to X$ be a $\mathbb{K}$-linear, closed, densely defined operator. If $\mathbb{K} = \mathbb{R}$, let $X_{\mathbb{C}}$ and $A_{\mathbb{C}}$ be respectively the complexification of X and A. We call A a sectorial operator if there exist $a \in \mathbb{R}$, $\phi \in]0,\pi/2[$ and $M \geq 1$ such that the sector*

$$S_{a,\phi} := \{\lambda \in \mathbb{C} \mid \phi \leq |\arg(\lambda - a)| \leq \pi,\ \lambda \neq a\}$$

is in the resolvent set $R(A)$ of A if $\mathbb{K} = \mathbb{C}$, in the resolvent set $R(A)$ of $A_{\mathbb{C}}$ if $\mathbb{K} = \mathbb{R}$, and, for all $\lambda \in S_{a,\phi}$,

$$\left\|(\lambda - A)^{-1}\right\| \leq \frac{M}{|\lambda - a|}$$

if $\mathbb{K} = \mathbb{C}$,

$$\left\|(\lambda - A_{\mathbb{C}})^{-1}\right\| \leq \frac{M}{|\lambda - a|}$$

if $\mathbb{K} = \mathbb{R}$.

Remark. If A is sectorial and B is bounded, then $A + B$ is sectorial.

Definition 1.1.2. *Let X be a Banach space over $\mathbb{K}$. An analytic semigroup on X is a family $T(t)$, $t \geq 0$ of continuous linear operators on X satisfying*

(1) $T(0) = I$, $T(t)T(s) = T(t+s)$ *for* $t \geq 0$, $s \geq 0$;
(2) $T(t)x \to x$ *as* $t \to 0^+$ *for each* $x \in X$;
(3) $t \mapsto T(t)x$ *is a real analytic function on* $]0, +\infty[$ *for each* $x \in X$.

Definition 1.1.3. *Let $T(t)$ be an analytic semigroup of operators on a Banach space X. The infinitesimal generator of $T(t)$ is the linear operator $A: D_A \subset X \to X$ defined by*

$$Ax := \lim_{t \to 0^+} \frac{1}{t}\left(T(t)x - x\right),$$

whose domain D_A consists of all $x \in X$ for which this limit exists. We write $T(t) = e^{At}$

Theorem 1.1.4. *If A is a sectorial operator over $\mathbb{K}$, then $-A$ is the infinitesimal generator of an analytic semigroup e^{-At}, $t \geq 0$, where e^{-At} is given by*

$$e^{-At} = \frac{1}{2\pi i}\int_{\Gamma} (\lambda + A)^{-1} e^{\lambda t} d\lambda$$

if $\mathbb{K} = \mathbb{C}$ and by the real part of

$$e^{-A_{\mathbb{C}}t} = \frac{1}{2\pi i}\int_{\Gamma} (\lambda + A_{\mathbb{C}})^{-1} e^{\lambda t} d\lambda$$

if $\mathbb{K} = \mathbb{R}$, and where Γ is a contour in $R(-A)$ with $\arg\lambda \to \pm\theta$ for some $\theta \in]\pi/2, \pi[$.

If $\mathrm{Re} S(A) > a$, i.e. if $\mathrm{Re}\lambda > a$ whenever λ is in the spectrum $S(A)$ of A, then for $t > 0$,

$$\left\|e^{-At}\right\| \leq Ce^{-at}$$

and

$$\left\|Ae^{-At}\right\| \leq \frac{C}{t}e^{-at}$$

for some constant C.

Finally, for $t > 0$,

$$\frac{d}{dt}e^{-At} = -Ae^{-At}.$$

Remark. Conversely, it can be proved that the infinitesimal generator of a semigroup is a sectorial operator.

2. Fractional Powers of Sectorial Operators

Definition 1.2.1. *Let A be a sectorial operator in a Banach space on $\mathbb{K}$, and suppose* $\mathrm{Re}S(A) > 0$*; then for every $\alpha > 0$ define*

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-At} dt.$$

It is easy to check that actually A^{-1} is the inverse of A in the usual sense.

Theorem 1.2.2. *Let A a sectorial operator in a Banach space over $\mathbb{K}$, with* $\mathrm{Re}S(A) > 0$*; then, for every $\alpha > 0$, $A^{-\alpha}$ is a one to one bounded linear operator on X; if $\alpha, \beta > 0$, then*

$$A^{-\alpha} A^{-\beta} = A^{-(\alpha+\beta)};$$

moreover, for $0 < \alpha < 1$,

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} \lambda^{-\alpha} (\lambda + A)^{-1} d\lambda.$$

Definition 1.2.3. *Let A be as above; define $A^0 := I_X$ and, for $\alpha > 0$,*

$$D_{A^\alpha} := R(A^{-\alpha})$$
$$A^\alpha := \text{inverse of } A^{-\alpha}.$$

Proposition 1.2.4.

(1) *If $\alpha > 0$, A^α is closed and densely defined;*
(2) *if $\alpha \geq \beta$ then $D_{A^\alpha} \subset D_{A^\beta}$;*
(3) *$A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ on D_{A^γ} where $\gamma = \max(\alpha, \beta, \alpha+\beta)$;*
(4) *$A^\alpha e^{-At} = e^{-At} A^\alpha$ on D_{A^α}.*

Theorem 1.2.5. *Let A be a sectorial operator and assume* $\mathrm{Re}S(A) > \delta > 0$*. Then for $\alpha \geq 0$ there exist constants $C_\alpha < +\infty$ such that*

$$\left\| A^\alpha e^{-At} \right\| \leq C_\alpha t^{-\alpha} e^{-\delta t} \text{ fot } t > 0,$$

and, if $0 < \alpha \leq 1$, $x \in D_{A^\alpha}$,

$$\left\| (e^{-At} - I)x \right\| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha \left\| A^\alpha x \right\|.$$

Theorem 1.2.6. *If* $0 \le \alpha \le 1$, $x \in D_A$, *then*

$$\|A^\alpha x\| \le C \|Ax\|^\alpha \|x\|^{1-\alpha}$$

or equivalently

$$\|A^\alpha x\| \le \epsilon \|Ax\| + C'\epsilon^{-\alpha/(1-\alpha)} \|x\|$$

for all $\epsilon > 0$. *The constants* C, C' *are independent of* α.

Proposition 1.2.7. *Let* $\alpha = \theta\beta + (1-\theta)\gamma$, *where* $0 \le \theta \le 1$, $\beta \ge 0$ $\gamma \ge 0$; *then there exists a constant* C *such that*

$$\|A^\alpha x\| \le C \left\|A^\beta x\right\|^\theta \|A^\gamma x\|^{1-\theta}$$

for all $x \in D_{A^\delta}$, *where* $\delta := \max(\beta, \gamma)$.

Corollary 1.2.8. *If* A *is a sectorial operator with* $\mathrm{Re}S(A) > 0$ *and if* B *is a linear operator such that* $BA^{-\alpha}$ *is bounded on* X *for some* α, $0 \le \alpha < 1$, *then* $A + B$ *is sectorial.*

3. Interpolation Spaces

Theorem 1.3.1. *Let* A, B *be sectorial operators in a Banach space* X, *with* $D_A = D_B$, *with* $\mathrm{Re}S(A) > 0$, $\mathrm{Re}S(B) > 0$, *and assume, for some* $\alpha \in [0, 1[$, $(A - B)A^{-\alpha}$ *is bounded on* X. *Then, for any* $\beta \in [0, 1]$, $A^\beta B^{-\beta}$ *and* $B^\beta A^{-\beta}$ *are bounded.*

Definition 1.3.2. *Let* A *be a sectorial operator in a Banach space X; choose* $a \in \mathbb{R}$ *so that* $\mathrm{Re}S(A_1) > 0$, *where* $A_1 := A + aI$. *Define, for each* $\alpha \ge 0$,

$$X^\alpha := D_{A^\alpha};$$

X^α, *endowed with the graph norm*

$$\|x\|_\alpha := \|A_1^\alpha x\|$$

is a Banach space. Different choices of a *give equivalent norms on* X^α *by Theorem 1.3.1, so we can supress the dependence on the choice of* a.

Proposition 1.3.3. *For* $\alpha \ge \beta \ge 0$, X^α *is a dense subspace of* X^β *with continuous inclusion. If* A *has compact resolvent, the inclusion* $X^\alpha \subset X^\beta$ *is compact when* $\alpha > \beta \ge 0$. *If* A_1, A_2 *are sectorial operators in* X *with the same domain and* $\mathrm{Re}S(A_j) > 0$ *for* $j = 1, 2$, *and if* $(A_1 - A_2)A_1^{-\alpha}$ *is a bounded operator for some* $0 \le \alpha < 1$, *then with* $X_j^\beta := D_{A_j^\beta}$ $(j = 1, 2)$, $X_1^\beta = X_2^\beta$ *with equivalent norms for* $0 \le \beta \le 1$.

Proposition 1.3.4. *Let A be a sectorial operator in a Banach space X, with $\mathrm{Re}S(A) > 0$; let Y be another Banach space and let*

$$B: D_B \subset X \to Y$$

be a linear map. Assume $D_B \supset D_A$ and, for some $\alpha \in [0,1[$ and some constants C, K, for all $x \in D_A$,

$$\|Bx\|_Y \leq C \|Ax\|^\alpha \|x\|^{1-\alpha}$$

or equivalently

$$\|Bx\|_Y \leq \epsilon \|Ax\| + K\epsilon^{-\alpha/(1-\alpha)} \|x\|$$

for all $\epsilon > 0$. Then, for any $\beta \in]\alpha, 1]$, B has a unique extension to a continuous linear operator from X^β to Y, i.e. $BA^{-\beta}$ is continuous.

4. Invariant Subspaces and Exponential Estimates

Definition 1.4.1. *Let A be a linear operator with domain and range in a Banach space X over $\mathbb{K}$; let $S(A)$ be the spectrum of A (of $A_\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$); a set $S \subset S(A) \cup \{\infty\} =: \hat{S}(A)$ is a spectral set if both S and $\hat{S}(A) \backslash S$ are closed in the extended plane $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.*

Theorem 1.4.2. *Let A be a linear closed operator in X and let S_1 be a bounded spectral set; let $S_2 := S(A) \backslash S_1$, so that $S_2 \cup \infty$ is another spectral set. Then there are projections $P_j: X \to X$, $j = 1, 2$, such that, if we set $X_j := P_j(X)$, $j = 1, 2$, then*

1. *$X = X_1 \oplus X_2$;*
2. *X_j is A-invariant for $j = 1, 2$;*
3. *let $A_j: D_A \cap X_j \to X_j$, $j = 1, 2$; then $A_1: X_1 \to X_1$ is bounded, $S(A_1) = S_1$ and $S(A_2) = S_2$;*

the projection P_1 is given by

$$P_1 = \frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} d\lambda$$

if $\mathbb{K} = \mathbb{C}$ and by the real part of

$$\frac{1}{2\pi i} \int_\Gamma (\lambda - A_\mathbb{C})^{-1} d\lambda$$

if $\mathbb{K} = \mathbb{R}$, and $P_2 = I - P_1$. If in addition A is sectorial, then A_2 is sectorial on X_2, $X_2^\alpha = X^\alpha \cap X_2$ for any $\alpha \geq 0$ and $\|\cdot\|_{X_2^\alpha} = \|\cdot\|_{X^\alpha}$. Finally, if $(\lambda - A)^{-1}$ is compact, X_1 is finite dimensional.

Theorem 1.4.3. *Let A be a sectorial operator, let S_1 be a bounded spectral set and let A_1, A_2 be as above. Then*

(1) *if* $\mathrm{Re}S_1 < \mu$, *then* $\left\|e^{-A_1 t}\right\| \le Ce^{-\mu t}$ *for* $t \le 0$;
(2) *if* $\mathrm{Re}S_2 > \nu$, *then, for* $t > 0$, $\left\|e^{-A_2 t}\right\| \le Ce^{-\nu t}$ *and* $\left\|A_2 e^{-A_2 t}\right\| \le Ct^{-1}e^{-\nu t}$.

Theorem 1.4.4. *Let A be a sectorial operator, let S_1 be a bounded spectral set for A, $S_2 = S(A)\backslash S_1$; assume $\mathrm{Re}S_2 > \mu$; let A_j, $j = 1, 2$, be as above. Assume also B is a sectorial operator with $D_B = D_A$, $\mathrm{Re}S(B) > 0$, $(B - A)B^{-\alpha}$ is bounded for some $\alpha < 1$. Then, using the norm $\|x\|_\beta := \left\|B^\beta x\right\|$, $0 \le \beta \le 1$, there exist a constant C_1 such that, for $x \in X_2 \cap D_{B^\beta}$ and $t > 0$,*

(1) $\left\|e^{-A_2 t}x\right\|_\beta \le C_1 \|x\| t^{-\beta} e^{-\mu t}$;
(2) $\left\|e^{-A_2 t}x\right\|_\beta \le C_1 \|x\|_\beta e^{-\mu t}$.

5. The Abstract Cauchy Problem

Let A be a sectorial operator in a Banach space X over $\mathbb{K}$; consider the abstract linear homogeneus equation

$$\frac{dx}{dt} + Ax = 0, \qquad 0 < t < T, \qquad x(0) = x_0, \tag{1.5.1}$$

where $x_0 \in X$ is given.

Definition 1.5.1. *A solution of (1.5.1) on $]0, T[$ is a continuous function $x\colon [0, T[\to x$ such that*

(1) *x is continuously differentiable on the open interval $]0, T[$;*
(2) *$x(t) \in D_A$ for $0 < t < T$;*
(3) *$x(t) \to x_0$ in X as $t \to 0^+$;*
(4) *$x(t)$ satisfies (1.5.1) on $]0, T[$.*

By Theorem 1.1.4 it is clear that $x(t) = e^{-At}x_0$ is a solution of (1.5.1) on $[0, +\infty[$. It is easy to prove that this is actually the only soution.

Next we consider the nonhomogeneus equation

$$\frac{dx}{dt} + Ax = f(t), \quad 0 < t < T, \qquad x(0) = x_0. \tag{1.5.2}$$

Definition 1.5.2. *A solution of (1.5.2) on* $]0, T[$ *is a continuous function* $x: [0, T[\to X$ *such that*

(1) x *is continuously differentiable on the open interval* $]0, T[$*;*
(2) $x(t) \in D_A$ *for* $0 < t < T$*;*
(3) $x(t) \to x_0$ *in* X *as* $t \to 0^+$*;*
(4) $x(t)$ *satisfies (1.5.2) on* $]0, T[$*.*

Lemma 1.5.3. *Let* $f:]0, T[\to X$ *be locally Hölder continuos, with*

$$\int_0^\rho \|f(s)\| \, ds < +\infty$$

for some $\rho > 0$*. For* $0 \le t < T$*, define*

$$F(t) := \int_0^t e^{-A(t-s)} f(s) ds.$$

Then $F(\cdot)$ *is continuous on* $[0, T[$*, continuos differentiable on* $]0, T[$*, with* $F(t) \in D_A$ *for* $0 < t < T$*,* $F(t) \to 0$ *in* X *as* $t \to 0^+$ *and*

$$\frac{dF}{dt}(t) + AF(t) = f(t), \quad 0 < t < T.$$

Theorem 1.5.4. *Let* A *be a sectorial operator in* X*, let* $x_0 \in X$*,* $f:]0, T[\to X$ *be locally Hölder continuous, with*

$$\int_0^\rho \|f(s)\| \, ds < +\infty$$

for some $\rho > 0$*; then there exists a unique solution* $x(\cdot)$ *of (1.5.2), namely*

$$x(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} f(s) ds.$$

Let us consider the nonlinear equation

$$\begin{aligned} \frac{dx}{dt} + Ax &= f(t, x), \quad t > t_0 \\ x(t_0) &= x_0, \end{aligned} \tag{1.5.3}$$

where A is a sectorial operator so that the fractional powers of $A_1 := A + aI$ are well defined, and the spaces X^α with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ are defined for $\alpha \ge 0$. We assume f maps some open set $U \subset \mathbb{R} \times X^\alpha$ into X, for some $0 \le \alpha < 1$, and f is locally Hölder continuous in t and locally Lypschitz in x on U; more precisely, if $(t_1, x_1) \in U$, there exists a neighborhood $V \subset U$ of (t_1, x_1) such that for $(t, x) \in V$, $(s, y) \in V$

$$\|f(t, x) - f(s, y)\| \le L \left(|t - s|^\theta + \|x - y\|_\alpha \right),$$

for some constants $L > 0$, $0 < \theta < 1$.

Definition 1.5.5. *A solution of the initial value problem (1.5.3) on* $]t_0, t_1[$ *is a continuos function* $x\colon [t_0, t_1[\to X$ *such that*

(1) $x(t) \in D_A$ *for* $t_0 < t < t_1$;
(2) $(t, x(t)) \in U$ *for* $t_0 < t < t_1$;
(3) $\frac{dx}{dt}(t)$ *exists on* $]t_0, t_1[$;
(4) *the map* $t \mapsto f(t, x(t))$ *is locally Hölder continuous on* $]t_0, t_1[$;
(5) $\int_{t_0}^{t_0+\rho} \|f(t, x(t))\|\, dt < +\infty$ *for some* $\rho > 0$;
(6) $x(t_0) = x_0$;
(7) $x(t)$ *satisfies (1.5.3) on* $]t_0, t_1[$.

Lemma 1.5.6. *If* x *is a solution of (1.5.3) on* $]t_0, t_1[$, *then* x *is a continuous function from* $]t_0, t_1[$ *into* X^α *and*

$$x(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-s)} f(s, x(s))ds. \tag{1.5.4}$$

Conversely, if x *is a continuous function from* $]t_0, t_1[$ *into* X^α *and from* $[t_0, t_1[$ *into* X, $\int_{t_0}^{t_1+\rho} \|f(s, x(s))\|\, ds < +\infty$ *for some* $\rho > 0$ *and the integral equation (1.5.4) holds for* $t_0 < t < t_1$, *then* $x(\cdot)$ *is a solution of the differential equation (1.5.3) on* $]t_0, t_1[$.

Theorem 1.5.7. *Assume* A *is a sectorial operator,* $0 \le \alpha < 1$, *and* $f\colon U \to X$, *where* $U \subset \mathbb{R} \times X^\alpha$ *is an open set, and* f *is locally Hölder continuous in* t *and locally Lypschitz in* x *on* U; *then for every* $(t_0, x_0) \in U$ *there exists* $T = T(t_0, x_0)$ *such that (1.5.3) has a unique solution on* $]t_0, t_0 + T[$ *with initial value* $x(t_0) = x_0$.

Theorem 1.5.8. *Assume* A *and* f *are as in Theorem 1.5.6, and assume that for every closed bounded set* $B \subset U$ *the image* $f(B)$ *is bounded in* X. *If* x *is a solution of (1.5.3) on* $]t_0, t_1[$ *and* t_1 *is maximal, so that there is no solution of (1.5.3) on* $]t_0, t_2[$ *if* $t_2 > t_1$, *then either* $t_1 = +\infty$ *or else there exists a sequence* $t_n \to t_1^-$ *as* $n \to \infty$ *such that* $(t_n, x(t_n)) \to \partial U$ *(if* U *is unbounded, the point at infinity is included in* ∂U*).*

Corollary 1.5.9. *Suppose* A *is sectorial,* $U =]\tau, +\infty[\times X^\alpha$, f *is locally Hölder continuous in* t, *locally Lipschitz continuous in* x *for* $(t, x) \in U$, *and also*

$$\|f(t, x)\| \le K(t)\,(1 + \|x\|_\alpha)$$

for all $(t, x) \in U$, *where* $K(\cdot)$ *is continuous on* $]\tau, +\infty[$. *If* $t_0 > \tau$, $x_0 \in X^\alpha$, *the unique solution of (1.5.3) through* (t_0, x_0) *exists for all* $t \ge t_0$.

Theorem 1.5.10. *Suppose A is a sectorial operator, $U \subset \mathbb{R} \times X^\alpha$ is an open set for some $0 \leq \alpha < 1$, $f: U \to X$ is locally Hölder continuous in t, locally Lipschitz continuous in x; let (t_n, x_n) be a sequence of points in U, $(t_n, x_n) \to (t_0, x_0)$ in $\mathbb{R} \times X^\alpha$ as $n \to \infty$, $(t_0, x_0) \in U$; let $\phi_n: [t_n, t_n + T_n[\to X^\alpha$ be the maximally defined solution of (1.5.3) with initial value $\phi_n(t_n) = x_n$, and let $\phi_0: [t_0, t_0 + T_0[\to X^\alpha$ be the maximally defined solution of (1.5.3) with initial value $\phi_0(t_0) = x_0$; then $T_0 \geq \limsup_{n\to\infty} T_n$, and $\phi_n(t) \to \phi_0(t)$ on compact subintervals of $]t_0, t_0 + T_0[$ as $n \to \infty$; if $t_n = t_0$ for all n, then $\phi_n(t) \to \phi_0(t)$ uniformly on compact subintervals of $[t_0, t_0 + T_0[$ as $n \to \infty$.*

More precise results on continuous and differentiable dependence of solutions can be found in [12, Sect. 3.4].

Definition 1.5.11. *Let Y be a topological space, D an open set in $\mathbb{R}^+ \times Y$ and let $\pi: D \to Y$ be a mapping. We write $y\pi t := \pi(y, t)$. We call π a local semiflow on Y if the following properties are satisfied:*

(1) *π is continuous;*
(2) *for every $y \in Y$ there is an ω_y, $0 < \omega_y \leq +\infty$, such that $(t, y) \in D$ if and only if $0 \leq t < \omega_y$;*
(3) *$y\pi 0 = y$ for all $y \in Y$;*
(4) *If $(t, y) \in D$ and $(s, y\pi t) \in D$, then $(t + s, y) \in D$ and $y\pi(t + s) = (y\pi t)\pi s$.*

If $\omega_y = +\infty$ for every $y \in Y$, π is called a global semiflow.

Let us consider the autonomous equation

$$\frac{dx}{dt} + Ax = f(x), \quad t > 0, \tag{1.5.5}$$

where A is a sectorial operator, $U \subset X^\alpha$ is an open set and $f: U \to X$ is a locally Lipschitz continuous function; as a consequence of Theorems 1.5.7 and 1.5.10 we have that the map $\pi: \mathbb{R}^+ \times X^\alpha \to X^\alpha$, which associates to the couple $(t, y) \in \mathbb{R}^+ \times U$ the solution of the initial value problem

$$\frac{dx}{dt} + Ax = f(x), \quad t > 0$$
$$x(0) = y,$$

evaluated at the time t, is a local semiflow on X^α. We refere to this semiflow as the semiflow defined by the differential equation (1.5.5) .

6. Smoothing

Lemma 1.6.1. *Suppose A is sectorial, $g:]0, T[\to X$ has $\|g(t) - g(s)\| \leq K(s)(t - s)^\gamma$ for $0 < s < t < T < +\infty$, where $K(\cdot)$ is continuous on $]0, T[$ and $\int_0^T K(s)ds < +\infty$. Then* *the function*

$$G(t) := \int_0^t e^{-A(t-s)} g(s)ds, \quad 0 < t < T,$$

is continuously differentiable on the open interval $]0, T[$ *into* X^β, *provided* $0 < \beta < \gamma$, *and*

$$\left\| \frac{dG}{dt}(t) \right\|_\beta \leq Mt^{-\beta} \|g(t)\| + M \int_0^t (t-s)^{\gamma-\beta-1} K(s)ds$$

for $0 < t < T$, *where* M *is a constant independent of* γ, β *and* $g(\cdot)$. *Further* $t \mapsto dG(t)/dt$ *is locally Hölder continuous from* $]0, T[$ *into* X^β, *if* $\int_0^h K(s)ds = O(h^\delta)$ *as* $h \to 0^+$, *for some* $\delta > 0$.

Theorem 1.6.2. *Assume* A *is sectorial,* $f: U \to X$ *is locally Lipschitzian on an open set* $U \subset \mathbb{R} \times X^\alpha$, *for some* $0 \leq \alpha < 1$. *Suppose* $x(\cdot)$ *is a solution on* $]t_0, t_1[$ *of*

$$\frac{dx}{dt} + Ax = f(t, x)$$
$$x(t_0) = x_0$$

and $(t_0, x_0) \in U$. *Then if* $\gamma < 1$, $t \to dx(t)/dt$ *is locally Hölder continuous from* $]t_0, t_1[$ *to* X^γ *and*

$$\left\| \frac{dx}{dt} \right\|_\gamma \leq C(t-t_0)^{\alpha-\gamma-1}$$

for some constant C.

7. Back to Semilinear Parabolic PDEs

Let us come back to equations (1.0.1) and (1.0.2) . We start by considering the second order strongly elliptic differential operator

$$L = -\sum_{|\alpha| \leq 2} a_\alpha(x) \partial^\alpha \tag{1.7.1}$$

with Dirichlet (resp. Neumann) boundary condition on $\partial\Omega$. Assume the coefficients $a_\alpha(\cdot)$ are of class $C^{0,\gamma}(\overline{\Omega})$ and the boundary $\partial\Omega$ of Ω is of class $C^{2,\gamma}$. For every $p > 1$, let D_p be the closure in $W^{2,p}(\Omega)$ of the set of all functions $u \in C^2(\overline{\Omega})$ that satisfy the Dirichlet (resp. Neumann) boundary condition on $\partial\Omega$. In the case of Dirichlet condition, $D_p = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, in the case of Neumann condition $D_p = W_N^{2,p}(\Omega)$, i.e. the space of all functions in $W^{2,p}(\Omega)$ that satisfy the Neumann condition in the sense of traces. In both cases, define, for $u \in D_p$,

$$(A_p u)(x) := Lu(x) = -\sum_{|\alpha| \leq 2} a_\alpha(x) \partial^\alpha u(x). \tag{1.7.2}$$

Then $A_p: D_p \subset L^p(\Omega) \to L^p(\Omega)$ is a closed operator, and the following theorem holds:

Theorem 1.7.1. *The operator A_p is sectorial on $X_p := L^p(\Omega)$. Moreover, A_p has compact resolvent and the spectrum $S(A_p)$ of A_p is independent of p and consists of a sequence $\{\lambda_k\}$, $k = 0, 1, \ldots$, with $\mathrm{Re}\lambda_0 \le \mathrm{Re}\lambda_1 \le \ldots$, of distinct eigenvalues such that $|\lambda_k| \to \infty$ as $k \to \infty$. If A_2 is selfadjoint, then each eigenvalue is real and has the same geometric and algebraic multiplicity.*

By Proposition 1.3.4, by the Sobolev imbedding theorems and by Gagliardo-Nirenberg inequalities the following result obtains:

Theorem 1.7.2. *Suppose $\Omega \subset \mathbb{R}^N$ is an open set with smooth boundary, $1 \le p < \infty$, and A is a sectorial operator in $X := L^p(\Omega)$ with $D_A = X^1 \subset W^{2,p}(\Omega)$. Then, for $0 \le \alpha \le 1$,*

$$X^\alpha \subset W^{k,q}(\Omega) \quad \text{when} \quad k - n/q < 2\alpha - n/p, \quad q \ge p,$$
$$X^\alpha \subset C^{[\nu],\nu-[\nu]}(\overline{\Omega}) \quad \text{when} \quad 0 \le \nu < 2\alpha - n/p.$$

Let $\Omega \subset \mathbb{R}^N$ be an open set with smooth boundary, and let

$$f\colon (t, x, s, w) \in \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to f(t, x, s, w) \in \mathbb{R}$$

be a map. Let

$$(u, v)\colon x \in \overline{\Omega} \mapsto (u(x), v(x)) \in \mathbb{R} \times \mathbb{R}^N$$

be another map. We can define the Nemitski operator

$$\check{f}(t, (u, v))(x) := f(t, x, u(x), v(x)).$$

It is easy to prove that if f is locally Hölder continuous in t, continuous in x and locally Lipschitz continuous in (s, w), then

$$\check{f}\colon \mathbb{R} \times C^0(\overline{\Omega}, \mathbb{R} \times \mathbb{R}^N) \to C^0(\overline{\Omega})$$

is locally Hölder continuous in t and locally Lipschitz continuous in (u, v). This implies that the operator

$$\hat{f}\colon (t, u) \in \mathbb{R} \times C^1(\overline{\Omega}) \mapsto \check{f}(t, (u, \nabla u)) \in C^0(\overline{\Omega}),$$

that is $\hat{f}(t, u)(x) := f(t, x, u(x), \nabla u(x))$, is locally Hölder continuous in t and locally Lipschitz in u. Moreover it can be proved that if f has derivatives $D^j_{(s,w)} f(t, x, s, w)$, $j = 1, \ldots, k$, which are continuous on $\mathbb{R} \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, then $D^j_u \hat{f}$, $j = 1, \ldots, k$, exists and is continuous on $\mathbb{R} \times C^1(\Omega)$. By Theorem 1.7.2, it follows that, if $p > N$ and $(N + p)/Np < \alpha < 1$, then, with $X = L^p(\Omega)$, $X^\alpha \subset C^1(\overline{\Omega})$ and consequently all the above conclusions hold for the map $\hat{f}: \mathbb{R} \times X^\alpha \to X$.

Finally, we come back to the general semilinear parabolic equation

$$u_t = Lu + g(x, u, \nabla u), \quad t > 0,\ x \in \Omega, \tag{1.7.3}$$

with Dirichlet or Neumann boundary condition on $\partial\Omega$. We assume L is a second order strongly elliptic differential operator with smooth coeficients, $\partial\Omega$ is smooth, f is locally Hölder continuous in t, continuous in x, locally Lipschitz continuous in (s, w). We choose $p > N$ and we set $X := L^p(\Omega)$. Then the above equation can be rewritten abstractly as

$$\frac{du}{dt} + A_p u = \hat{f}(t, u). \tag{1.7.4}$$

It is easy to check that all the hypotheses of Theorem 1.5.7 are satisfied, so that the Cauchy problem for (1.7.4) with initial datum in X^α is well posed. Thus we have a solution of the abstract form of the original equation (1.7.3). Finally, combining Theorem 1.6.2 with classical results in regularity theory for elliptic equations (see e.g. [10]) and with Sobolev imbedding theorems, we conclude that the solution of the abstract equation is a solution of the original equation in the sense of Definition 1.0.1 .

CHAPTER II

GLOBAL AND LOCAL CENTER MANIFOLDS

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and let L be a second order differential operator of the form

$$Lu := \sum_{i,j=1}^{N} \partial_i (a_{ij}(x)\partial_j u) + a(x)u, \tag{2.0.1}$$

where $a_{ij}, a: \overline{\Omega} \to \mathbb{R}$, $i, j = 1,\ldots,N$ are smooth and the matrix $(a_{ij})_{i,j}$ is symmetric and positive definite. As we have seen in the previous chapter, such an L is strongly elliptic and formally selfadjoint.

Consider the following general semilinear parabolic equations with Dirichlet and Neumann boundary condition respectively:

$$\begin{aligned} u_t &= Lu + f(x,u,\nabla u), && t>0,\ x\in\Omega \\ u(x,t) &= 0, && t>0,\ x\in\partial\Omega. \end{aligned} \tag{2.0.2}$$

$$\begin{aligned} u_t &= Lu + f(x,u,\nabla u), && t>0,\ x\in\Omega \\ \frac{\partial}{\partial\nu}u(x,t) &= 0, && t>0,\ x\in\partial\Omega. \end{aligned} \tag{2.0.3}$$

with a nonlinearity $f: (x,s,w) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \mapsto f(x,s,w) \in \mathbb{R}$, which we assume to be differentiable with respect to (s,w), with $(x,s,w) \mapsto f(x,s,w)$ and $(x,s,w) \mapsto D_{(s,w)}f(x,s,w)$ continuous from $\Omega\times\mathbb{R}\times\mathbb{R}^N$ to $\mathbb{R}$ and $\mathcal{L}(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ respectively. We choose $p > N$, we set $X := L^p(\Omega)$ and we consider the sectorial operators A_p defined by $-L$ with Dirichlet and Neumann condition, whose domains are $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and $W_N^{2,p}(\Omega)$ respectively. We choose α, $(N+p)/2p < \alpha < 1$, so that $X^\alpha \subset C^1(\overline{\Omega})$. We consider the abstract equation

$$\dot{u}(t) + A_p u(t) = \hat{f}(u(t)) \tag{2.0.4}$$

defined by (2.0.2) or (2.0.3) . As we have seen in the previous chapter, (2.0.4) defines a local semiflow π_f in the phase space X^α. In the sequel, if it does not generate any ambiguity, we supress the dependence of A_p on the Lebesgue exponent p and we just write A.

Definition 2.0.1. *Let Y and $\tilde{Y}$ be Banach spaces and π (resp. $\tilde{\pi}$) be a local semiflow on Y (resp. $\tilde{Y}$). We say that $\tilde{\pi}$ imbeds in π if there is an imbedding $\Lambda: \tilde{Y} \to Y$ such that whenever I is an interval in $\mathbb{R}$ and $z: I \to \tilde{Y}$ is an orbit of $\tilde{\pi}$, then $\lambda \circ z: I \to Y$ is an orbit of π. Here by imbedding we mean that Λ is injective, of class C^1, $\Lambda^{-1}: \Lambda(\tilde{Y}) \to \tilde{Y}$ is continuous, and for every $\tilde{y} \in \tilde{Y}$, $D\Lambda(\tilde{y})$ is injective and its image splits, i.e. admits a topological complement.*

In this case $M := \Lambda(\tilde{Y})$ is a C^1 submanifold of Y which is invariant for the local semiflow π and π restricted to M is C^1-conjugated to $\tilde{\pi}$.

Now suppose $Y = X^\alpha$, $\pi = \pi_f$ for some nonlinearity f, $\tilde{Y} = \mathbb{R}^n$ and $\tilde{\pi} = \pi_v$ is generated by an ordinary differential equation

$$\dot{\xi} = v(\xi), \quad \xi \in \mathbb{R}^n, \tag{2.0.5}$$

where $v: \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1.

Definition 2.0.2. *If π_v imbeds in π_f we say that the ODE (2.1.1) imbeds in the PDE (2.0.2) (resp. (2.0.3)), or that the PDE (2.0.2) (resp. (2.0.3)) realizes the vector field v (on the invariant manifold M).*

In this case the qualitative behaviour of the ODE is completely simulated by the abstract equation (2.0.4) (restricted to the invariant manifold M). An important candidate for the manifold M is the (global) center manifold.

1. Center Manifolds

Definition 2.1.1. *A set $S \subset X^\alpha$ is a local invariant manifold for (2.0.4) provided for any $u_0 \in S$ there exists a solution $u(\cdot)$ of the differential equation on an open interval $]t_0, t_1[$ containing 0 with $u(0) = u_0$ and $u(t) \in S$ for $t_1 < t < t_2$. S is a global invariant manifold if we can always choose $]t_1, t_2[=]-\infty, +\infty[$*

Remark. Note that S being an invariant manifold implies for all $u \in S$ there is at least one solution of the backward Cauchy problem.

Now define

$$X_0 := \ker A_p$$

and suppose $n := \dim X_0 \geq 1$. Since A_2 is selfadjoint, actually, for all p, X_0 is the invariant subspace corresponding to the spectral set $\{0\}$; hence, if $\phi_1,\ldots,\phi_n$ is a $L^2(\Omega)$-orthonormal basis of $\ker A_2$, then, for all p, the spectral projection P_0 on X_0 is given by the formula

$$P_0 u := \sum_{j=1}^{n} \phi_j \int_\Omega u(x)\phi_j(x)dx.$$

Write

$$\phi(x) := (\phi_1(x), ..., \phi_n(x)).$$

Note the assignement

$$Q: \mathbb{R}^n \to X_0, \quad Q\xi := \xi \cdot \phi = \sum_{i=1}^{n} \xi_i \phi_i$$

is a linear isomorphism.

Now, let X_+ (resp. X_-) be the eigenspace of all positive (resp. negative) eigenvalues of A. As we have seen in the previous chapter, X_+ and X_- are A invariant, and $X = X_- \oplus X_0 \oplus X_+$. Let P_+ (resp. X_-) be the spectral projection onto X_+ (resp. X_-), and let $P_h := P_+ + P_-$, $X_h := P_h(X)$. Set

$$A_+ := A|X_+ \quad A_- := A|X_-.$$

Note X_- is finite dimensional, so $-A_-$ is bounded on X_- and hence it generates a C^0-group $e^{-A_- t}$, $t \in \mathbb{R}$, of linear operators. Moreover $-A_+$ is sectorial on X_+ and so it generates an analytic semigroup $e^{-A_+ t}$, $t \geq 0$, of linear operators. Let c be such that

$$0 < c < \min \{ |\mathrm{Re} S(A_-)|,\ \mathrm{Re} S(A_+) \} ;$$

Then the following estimates hold:

$$\begin{aligned}
\|e^{-A_- t}\|_{\mathcal{L}(X_-, X_-)} &\leq M e^{ct}, & t &\leq 0, \\
\|e^{-A_+ t}\|_{\mathcal{L}(X_+, X_+)} &\leq M e^{-ct}, & t &\geq 0, \\
\|e^{-A_+ t}\|_{\mathcal{L}(X_+, X_+^\alpha)} &\leq M t^{-\alpha} e^{-ct}, & t &> 0.
\end{aligned}$$

Definition 2.1.2. *The global center manifold $\mathcal{M}_f$ of equation (2.0.4) is the set of all $u_0 \in X^\alpha$ for which there exists a solution $u:]-\infty, +\infty[\to X^\alpha$ of equation (2.0.4) satisfying $u(0) = u_0$ and such that*

$$\sup_{t \in \mathbb{R}} \|P_h u(t)\|_\alpha < \infty.$$

Remark. Obviously, the global center manifold of equation (2.0.4) is a global invariant set.

In this section we state the well known center manifold theorem and give some comments about the main ideas of its proof. First, we introduce some notations and terminology.

For $m \in \mathbb{N}_0$ let $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all maps

$$v: \mathbb{R}^n \to \mathbb{R}^n$$

such that for all $0 \leq k \leq m$ the Frechet derivative $D^k v$ exists and is continuous and bounded on $\mathbb{R}^n$. $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ is a linear space which becomes a Banach space when endowed with the norm

$$|v|_m := \sup_{y \in \mathbb{R}^n} \sup_{0 \leq k \leq m} \left| D^k v(y) \right|_{\mathcal{L}((\mathbb{R}^n)^k, \mathbb{R}^n)}.$$

Furthermore, let Y_m be the set of all functions

$$f\colon (x, s, w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x, s, w) \in \mathbb{R}$$

such that for all $0 \leq k \leq m$ the Frechet derivative $D^k_{(s,w)} f$ exists and is continuous and bounded on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Y_m is a linear space which becomes a Banach space when endowed with the norm

$$|f|_m := \sup_{(x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N} \sup_{0 \leq k \leq m} \left| D^k_{(s,w)} f(x, s, w) \right|_{\mathcal{L}^k((\mathbb{R} \times \mathbb{R}^N)^k, \mathbb{R})}.$$

As we have seen in the previous chapter, for $f \in Y^m$, the formula

$$\hat{f}(u)(x) := f(x, u(x), \nabla u(x)), \quad u \in X^\alpha, \ x \in \overline{\Omega},$$

defines the Nemitski operator

$$\hat{f}\colon X^\alpha \to X$$

of class C_b^m.

For $\delta > 0$ and $m \geq 1$ define

$$\mathcal{V}(\delta) := \left\{ f \in Y_1 \mid \sup_{(x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N} \left| D_{(s,w)} f(x, s, w) \right|_{\mathcal{L}((\mathbb{R} \times \mathbb{R}^N), \mathbb{R})} < \delta \right\}$$

and

$$\mathcal{V}_m(\delta) := \mathcal{V}(\delta) \cap Y_m.$$

Now the following result obtains:

Theorem 2.1.3. *For every $m \in \mathbb{N}$ there is a positive constant δ_m and a map*

$$\Lambda\colon (f, \xi) \in \mathcal{V}_m(\delta_m) \times \mathbb{R}^n \mapsto \Lambda(f, \xi) \in X^\alpha$$

satisfying the following properties:

(1) $P_0 \Lambda(f, \xi) \equiv Q\xi$ *and* $\Lambda(0, \xi) \equiv Q\xi$;

(2) Λ *is of class* C^m;

(3) *For every* $f \in \mathcal{V}_m(\delta_m)$ *the partial map* $\Lambda_f = \Lambda(f, \cdot)\colon \mathbb{R}^n \to X^\alpha$ *is an imbedding and the set*

$$\mathcal{M}_f := \{\Lambda(f,\xi) \mid \xi \in \mathbb{R}^n\}$$

is the global center manifold of (2.0.4) . Moreover, if $v_f\colon \mathbb{R}^n \to \mathbb{R}^n$ *is defined by*

$$v_f(\xi) := Q^{-1}P_0\hat{f}(\Lambda_f(\xi)), \quad \xi \in \mathbb{R}^n,$$

then the ODE defined by v_f *imbeds, via* Λ_f*, in (2.0.4) .*

We say that an imbedding satisfying property (1) is canonical.

We shall briefly recall the main steps of the proof and we refere the reader to the papers [37], [4] and [32] for details.

For Y a Banach space, $y\colon \mathbb{R} \to Y$ and $\zeta > 0$, define

$$\|y\|_\zeta := \sup_{t\in\mathbb{R}} e^{-\zeta|t|}\,\|y(t)\|\,.$$

Let $BC^\zeta(Y)$ be the set of all continuous $y\colon \mathbb{R} \to Y$ such that $\|y\|_\zeta < \infty$; $BC^\zeta(Y)$ is a linear space which becomes a Banach space if endowed with the norm $\|\cdot\|_\zeta$.

Choose $b > 1$ and $\eta > 0$ such that $mb\eta < c$. For $y\colon \mathbb{R} \to X$ set

$$(2.1.1) \qquad (Ky)(t) := \int_0^t P_0 y(s)ds + \int_{-\infty}^t e^{-A_+(t-s)}P_+y(s)ds + \int_\infty^t e^{-A_-(t-s)}P_-y(s)ds$$

whenever the right hand side of (2.1.1) is defined. It is not difficult to show that for $\eta \le \zeta \le mb\eta$ the map K restricts to a bounded linear operator

$$K\colon BC^\zeta(X) \to BC^\zeta(X^\alpha)$$

with

$$\sup_{\eta\le\zeta\le mb\eta} \|K\|_\zeta < \infty.$$

The following lemma is cited from [37]:

Lemma 2.1.4. *If* $f \in Y_1$*, the following properties are equivalent:*

(1) u *is a solution of (2.0.4) on* $\mathbb{R}$ *such that*

$$\sup_{t\in\mathbb{R}} \|P_h u(t)\|_{X^\alpha} < \infty;$$

(2) u *is a solution of (2.0.4) on* $\mathbb{R}$ *and* $u \in BC^\eta(X^\alpha)$*;*

(3) *$u \in BC^\eta(X^\alpha)$ and for every $t \in \mathbb{R}$*

$$u(t) = P_0 u(0) + Ky(t),$$

where

$$y(s) := \hat{f}(u(s)), \quad s \in \mathbb{R}.$$

Consequently, $\mathcal{M}_f$ is the set of all points $w \in X^\alpha$ such that $\phi(f,\xi)(0) = w$, where $\phi(f,\xi) \in BC^\eta(X^\alpha)$ satisfies the equation

$$\phi(f,\xi)(t) = Q\xi + K(s \mapsto \hat{f}(\phi(f,\xi)(s)))(t), \quad t \in \mathbb{R}. \tag{2.1.2}$$

We can write (2.1.2) as a fixed point equation in the space $BC^\eta(X^\alpha)$, namely

$$\phi(f,\xi) = F(f,\xi,\phi(f,\xi)), \tag{2.1.3}$$

where

$$F(f,\xi,\cdot)\colon BC^\eta(X^\alpha) \to BC^\eta(X^\alpha)$$
$$F(f,\xi,y)(t) := \xi + K(s \mapsto \hat{f}(y(s)))(t), \quad t \in \mathbb{R}.$$

If $\delta_m > 0$ is small enough, for every $f \in \mathcal{V}_m(\delta_m)$, the Lipshitz constant

$$\left\|\hat{f}\right\|_{\mathrm{Lip}} := \sup\left\{ \frac{\left\|\hat{f}(u) - \hat{f}(\tilde{u})\right\|_X}{\|u-\tilde{u}\|_{X^\alpha}} \mid u, u \in X^\alpha,\ u \neq \tilde{u} \right\}$$

satisfies the estimate

$$\kappa := \sup\left\{ \|K\|_{s\eta} \mid s \in S \right\} \left\|\hat{f}\right\|_{\mathrm{Lip}} < 1,$$

where

$$S := \{1, 2, ..., m\} \cup \{b, 2b, ..., mb\}.$$

This implies that $F(f,\xi,\cdot)$ is a contraction uniformly in $(f,\xi) \in \mathcal{V}_m(\delta_m)$ so by the contraction mapping principle for every $(f,\xi) \in \mathcal{V}_m(\delta_m) \times \mathbb{R}^n$ there is a unique solution $\phi(f,\xi)$ of equation (2.1.3) in $BC^\eta(X^\alpha)$. This defines a map

$$\phi\colon \mathcal{V}_m(\delta_m) \times \mathbb{R}^n \to BC^\eta(X^\alpha).$$

Define the map

$$\Lambda\colon (f,\xi) \in \mathcal{V}_m(\delta_m) \times \mathbb{R}^n \mapsto \Lambda(f,\xi) \in X^\alpha$$
$$\Lambda(f,\xi) := \phi(f,\xi)(0)$$

It follows that Λ satisfies property (1) of Theorem 2.1.3.

The proof of differentiability of Λ would follow from differentiability of ϕ. However, in general, F is not differentiable as a map into $BC^{\eta}(X^{\alpha})$ so the conclusion cannot be achieved via the usual implicit function theorem. In fact F is of class C^1 only as a map into $BC^{b\eta}(X^{\alpha})$, and of class C^2 as a map into $BC^{2b\eta}(X^{\alpha})$ etc. This observation suggest that ϕ is of class C^m only as a map into $BC^{mb\eta}(X^{\alpha})$. This is actually the case, but the proof is far from straightforward; it is based on the tecnique of scales of Banach spaces and it requires a lot of careful estimates and precise formulas for higher order derivatives of composite maps. The reader is referred to [37], [4] and [32] for details. Part (3) of Theorem 2.1.3 follows from Lemma 2.1.4.

Remark. If the function f depends smoothly on some parameter σ in a normed space Σ, it is very easy to prove, exploiting the abstract arguments contained in [32], that the map Λ in Theorem 2.1.3 depends on σ with the same degree of smoothness.

2. Local Center Manifolds

In this section we show how Theorem 2.1.3 can be used to prove existence of a local invariant manifold at an equilibrium of (2.0.4) when the map f is smooth but not globally bounded.

In the proof of the global center manifold theorem we used the boundedness of f and of its derivatives to deduce that the Nemitski operator $\hat{f}$ is in $C_b^m(X^{\alpha}, X)$. In the abstract contest the center manifold theorem holds with an arbitrary nonlinear map $g\colon X^{\alpha} \to X$ of class C_b^m, provided the Lipschitz constant of g is sufficiently small. If we are given a nonlinearity $f = f(x, s, w)$ in equation (2.0.2) and we only assume that, for any k, $0 \le k \le m$, the Frechet derivative $D^k_{(s,w)}f$ exists and is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, without any boundedness assumption, the Nemitski operator $\hat{f}\colon X^{\alpha} \to X$ is defined as well, and is of class C^m. If $f(x, 0, 0) \equiv 0$, $D_{(s,w)}f(x, 0, 0) \equiv 0$, then $\hat{f}(0) = 0$ and $D\hat{f}(0) = 0$; it is a standard argument to modify the abstract equation (2.0.4) by mean of some cut-off function defined on the finite dimensional space X_0 in such a way that the hypothesis of the abstract center manifold theorem are satisfied by the abstract modified equation (see e.g. [32]). For our pourposes, however, we are lead to allow $D_{(s,w)}f(x, 0, 0) \not\equiv 0$; what we need is a slightly different version of the local center manifold theorem in the case when $f(x, 0, 0) \equiv 0$ and $D_{(s,w)}f(x, s, w)$ is small for $(x, s, w) \in \overline{\Omega} \times K$, where K is a compact neighborhood of 0 in $\mathbb{R}^{N+1}$.

First, we observe that, since

$$X^{\alpha} = X_h^{\alpha} \oplus X_0^{\alpha} \subset C^1(\overline{\Omega}),$$

we can find some $R > 0$ such that, whenever $u \in X^\alpha$ and

$$\sup \left\{ \|P_h u\|_\alpha, \left\|Q^{-1} P_0 u\right\|_{\mathbb{R}^n} \right\} < 1,$$

then

$$\|(u(x), \nabla u(x))\|_{\mathbb{R}^{N+1}} < R \quad \forall x \in \overline{\Omega}.$$

Now the following result obtains:

Theorem 2.2.1. *For all $m \in \mathbb{N}$ there is a posittive constant $\tilde{\delta}_m$ with the following property: if $f \colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is such that*

(1) *for all k, $0 \leq k \leq m$, the Frechet derivative $D^k_{(s,w)} f(x, s, w)$ exists and is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$;*
(2) *$f(x, 0, 0) \equiv 0$ on $\overline{\Omega}$;*
(3) *$\sup_{(x,s,w) \in \overline{\Omega} \times B^{N+1}_{2R}(0)} \left\|D_{(s,w)} f(x, s, w)\right\| < \tilde{\delta}_m$, where $B^{N+1}_{2R}(0)$ is the ball of radius $2R$ centered at 0 in $\mathbb{R}^{N+1}$;*

then there is a map

$$\Lambda_f \colon \xi \in B^n_1(0) \subset \mathbb{R}^n \mapsto \Lambda(\xi) \in X^\alpha$$

with the following properties:

(1) *$P_0 \Lambda_f(\xi) \equiv Q\xi$ and $\Lambda_0(\xi) \equiv Q\xi$;*
(2) *Λ_f is of class C^m;*
(3) *the map Λ_f is an imbedding and the set*

$$\mathcal{M}^{\text{loc}}_f := \{\Lambda_f(\xi) \mid \xi \in B^n_1(0)\}$$

is a local invariant manifold of (2.0.4). Moreover, if $v_f \colon B^n_1(0) \to \mathbb{R}^n$ is defined by

$$v_f(\xi) := Q^{-1} P_0 \hat{f}(\Lambda_f(\xi)), \quad \xi \in B^n_1(0),$$

then the ODE defined by v_f imbeds, via Λ_f, in (2.0.4).

Proof. Take a C^∞ cut-off function $\chi \colon \mathbb{R} \times \mathbb{R}^N$ such that

$$\chi(s, w) \equiv 1, \quad (s, w) \in B^{N+1}_R(0)$$
$$\chi(s, w) \equiv 0, \quad (s, w) \in \mathbb{R}^{N+1} \setminus B^{N+1}_R(0);$$

define

$$g(x, s, w) := \chi(s, w) f(x, s, w);$$

then we have that

(1) $g \in Y_m$;

(2) if $\tilde{\delta}_m$ is sufficiently small and

$$\sup_{(x,s,w)\in\overline{\Omega}\times B^{N+1}_{2R}(0)} \left\|D_{(s,w)}f(x,s,w)\right\| < \tilde{\delta}_m,$$

then $g \in \mathcal{V}_m(\delta_m)$;

(3) $\hat{g}(u) = \hat{f}(u)$ for all $u \in X^\alpha$ such that $\sup\left\{\|P_h u\|_\alpha, \left\|Q^{-1}P_0 u\right\|_{\mathbb{R}^n}\right\} < 1$.

We can apply Theorem 2.1.3 to obtain a global center manifold

$$\mathcal{M}_g := \{\Lambda_g(\xi) \mid \xi \in \mathbb{R}^n\}$$

for equation

$$\dot{u}(t) + Au(t) = \hat{g}(u(t));$$

moreover, by formula (2.1.2) (with f replaced by g) and by the mean value theorem, we obtain that, if $\tilde{\delta}_m$ is sufficiently small and

$$\sup_{(x,s,w)\in\overline{\Omega}\times B^{N+1}_{2R}(0)} \left\|D_{(s,w)}f(x,s,w)\right\| < \tilde{\delta}_m,$$

then the following esimate holds:

$$\|P_h\Lambda_g(\xi)\|_\alpha < 1, \quad \xi \in \mathbb{R}^n.$$

We set $\Lambda_f := \Lambda_g | B^n_1(0)$ and we have concluded. $\square$

Remark. Assume Z is a Banach space of functions $f\colon \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $f(x,0,0) \equiv 0$, such that for all k, $0 \le k \le m$, the Frechet derivative $D^k_{(s,w)}f$ exists and is continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$; suppose that the topology of Z is at least as strong as the topology of locally uniform convergence of all derivatives $D^k_{(s,w)}f$, $k = 0, \ldots, m$, on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$; then the map $f \in Z \mapsto g = \chi f \in Y_m$ defined above is linear bounded; let $\mathcal{U}(\tilde{\delta}_m)$ be the set of all functions $f \in Z$ such that

$$\sup_{(x,s,w)\in\overline{\Omega}\times B^{N+1}_{2R}(0)} \left\|D_{(s,w)}f(x,s,w)\right\| < \tilde{\delta}_m;$$

then, again by Theorem 2.1.3, the assignement $(f,\xi) \in \mathcal{U}(\tilde{\delta}_m) \times B^n_1(0) \mapsto \Lambda_f(\xi) \in X^\alpha$ is of class C^m. An example of a Banach space Z with the above properties is given by the space of functions $f(x,s,w)$ which are polynomials in (s,w) of degree r for some $r \in \mathbb{N}$, with x-dependent coefficients in $C^0(\overline{\Omega})$ and which satisfy $f(x,0,0) \equiv 0$.

CHAPTER III

VECTOR FIELD REALIZATIONS

We have seen in the previous chapter that if $f \in \mathcal{V}_m(\delta_m)$ then the ODE

$$\dot{\xi} = v_f(\xi), \quad \xi \in \mathbb{R}^n,$$

where

$$v_f(\xi) = Q^{-1} \circ P_0 \circ \hat{f} \circ \Lambda_f(\xi),$$

imbeds in

$$\dot{u} = Au + \hat{f}(u),$$

via the canonical imbedding Λ_f. We shall now study the following inverse problem: given a vector field v, find a nonlinearity $f \in \mathcal{V}_m(\delta_m)$ such that $v = v_f$. Thus we want to "realize" the vector field v on the center manifold of some parabolic equation with Dirichlet or Neumann Boundary condition. Defining the nonlinear operator

$$\Psi(f) := Q^{-1} \circ P_0 \circ \hat{f} \circ \Lambda_f \tag{3.0.1}$$

we are therefore looking for solutions of the equation

$$\Psi(f) = v. \tag{3.0.2}$$

Since v_f is globally small, the problem makes sense only for small vector fields v. Thus we restrict our problem by looking for small solutions f of (3.0.2). Therefore the first idea that comes to one's mind is to apply the surjective mapping theorem:

Theorem 3.0.1. *Let X and Y be Banach spaces, U be open in X, $0 \in U$ and $\Psi: U \to Y$ be a C^1 map such that $\Psi(0) = 0$ and $D\Psi(0): X \to Y$ is surjective. Then $\Psi(U)$ is a neighborhood of 0 in Y.*

1. Vector Field Realizations via the Surjective Mapping Theorem

Let us try to apply Theorem 3.0.1 to equation (3.0.2). Without specifying the spaces X and Y let us compute the formal derivative $D\Psi(0)$. By calculating directional derivatives pointwise, we thus obtain:

$$\begin{aligned} D\Psi(0)f(\xi) &= \lim_{s\to 0} \frac{\Psi(sf)(\xi) - \Psi(0)(\xi)}{s} \\ &= \lim_{s\to 0}(Q^{-1} \circ P_0 \circ \hat{f})(\Lambda(sf,\xi)) = (Q^{-1} \circ P_0 \circ \hat{f})(\Lambda(0,\xi)) \end{aligned} \tag{3.1.1}$$

We know that $\Lambda_0(\xi) \equiv Q\xi = \xi \cdot \phi$. Consequently, given a vector field v, the function f satisfies the equation

$$D\Psi(0)f = v$$

if and only if the components v_i, $i = 1, \dots, n$, satisfy the equation

$$v_i(\xi) = \int_\Omega \phi_i(x) f(x, \xi \cdot \phi(x), \xi \cdot \nabla\phi(x)) dx. \tag{3.1.2}$$

Now if for every $x \in \overline{\Omega}$ the linear map

$$M(x): \mathbb{R}^n \to \mathbb{R}^{N+1}$$
$$M(x)\xi := (s, w) = (\xi \cdot \phi(x), \xi \cdot \nabla\phi(x))$$

were invertible, then we could define, for a given vector field v,

$$f(x, s, w) := \sum_{k=1}^{n} \phi_k(x) v_k(M(x)^{-1}(s, w)).$$

Then f would satisfy equation (3.1.2).

Now, since $\phi(x) = 0$ (or $\frac{\partial\phi}{\partial\nu} = 0$) on $\partial\Omega$, $M(x)$ cannot be invertible at all points of $\overline{\Omega}$. However, to solve equation (3.1.2), one actually needs only the invertibility of $M(x)$ at some point $x = x_0$. Thus we are led to introduce the following concept: let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with smooth boundary, and let

$$L := \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

be a second order strongly elliptic symmetric differential operator with smooth coeficients; let A_p be the sectorial operator in $X = L^p(\Omega)$, defined by $-L$ with Dirichlet or Neumann boundary condition on $\partial\Omega$. The following definition is independent of p.

Definition 3.1.1. *We say that the operator L satisfies the Poláčik condition on Ω if $\dim\ker A = N+1$ and for some (hence every) basis $\phi_1, \dots, \phi_{N+1}$ of $\ker A$, $R(x) \neq 0$ for some $x \in \Omega$, where*

$$R(\phi_1, \dots, \phi_{N+1})(x) := \det \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega.$$

Remark. We have $n = N+1$ in case the Poláčik condition holds. One can also define a (weaker and less interesting) version of the Poláčik condition with $n = N$.

Remark. Since $R(x)$ is a continuous function on $\overline{\Omega}$, when the Poláčik condition is satisfied, there exists an open set $G \subset \Omega$ such that $R(x) \neq 0$ for all $x \in G$.

Now we show how Poláčik condition implies solvability of equation (3.1.2). First we need a technical result:

Proposition 3.1.2. *Let Ω_0 be an arbitrary open subset of Ω. Then there are functions $b_i \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} b_i \subset \Omega_0$, $i = 1, \ldots, n$, such that*

$$\int_\Omega b_i(x)\phi_j(x)dx = \delta_{ij}, \quad i,j = 1,\ldots,n.$$

Proof. Let us first notice that $\phi_1|\Omega_0, \ldots, \phi_n|\Omega_0$ are linearly independent, by the unique continuation theorem (see e.g. [15]). Consequently, the Green matrix

$$G := \left(\int_{\Omega_0} \phi_i(x)\phi_j(x)dx\right)_{ij}$$

is invertible. It follows that the linear operator

$$T\colon L^2(\Omega_0) \to \mathbb{R}^n$$
$$Tb := \left(\int_{\Omega_0} b(x)\phi_1(x)dx, \ldots, \int_{\Omega_0} b(x)\phi_n(x)dx\right)$$

is surjective. Infact, for an arbitrary $c \in \mathbb{R}^n$, let $a := cG^{-1}$; then $b := a \cdot \phi$ satisfies $Tb = c$. Now, since $C_0^\infty(\Omega_0)$ is dense in $L^2(\Omega_0)$ and T is bounded, $T(C_0^\infty(\Omega_0))$ is dense in $\mathbb{R}^n$. But $T(C_0^\infty(\Omega_0))$ is a linear subspace of $\mathbb{R}^n$, hence it is closed. It follows that $T(C_0^\infty(\Omega_0)) = \mathbb{R}^n$ and the proposition is proved. □

Now assume L satisfies the Poláčik condition; let G be an open subset of Ω such that $M(x)$ is invertible on G. Let b_i, $i = 1, \ldots, n$ satisfy the assertions of proposition 3.1.2 with Ω_0 replaced by G. With $n = N+1$, define

$$f(x,s,w) := \begin{cases} \sum_{k=1}^n b_k(x)v_k(M(x)^{-1}(s,w)) & \text{if } x \in G; \\ 0 & \text{otherwise;} \end{cases} \tag{3.1.3}$$

then simple computations show f satisfies equation (3.1.2). Note that, if $v \in C_b^m(\mathbb{R}^n, \mathbb{R}^n)$, then $f \in Y_m$. Hence, for every $m \in \mathbb{N}$, the formal derivative $D\Psi(0)$ is surjective as a map from Y_m to $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$. However, to solve our realization problem we cannot use the surjective mapping theorem since, unfortunately, Ψ is not a C^1 map from $\mathcal{V}_m(\delta_m)$ to $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$. Infact, if we calculate the formal derivative at an arbitrary point f_0, then we obtain

$$D\Psi(f_0)f(\xi) = \lim_{s\to 0} \frac{\Psi(f_0 + sf(\xi)) - \Psi(f_0)(\xi)}{s}$$
$$= (Q^{-1} \circ P_0 \circ \hat{f})(\Lambda(f_0, \xi)) + (Q^{-1} \circ P_0) \cdot [Df_0(\Lambda(f_0, \xi)) \cdot (D_f\Lambda(f_0, \xi)f)].$$

Since this formula contains the derivative of f_0 it will, in general, no longer define a linear map from Y_m to $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ but only from Y_m to the larger space $C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n)$. Actually, it is not difficult to prove that Ψ is a C^1 map from $\mathcal{V}_m(\delta_m)$ to $C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n)$ with derivative given by the above expression. However, it is obvious that $D\Psi(0)$ cannot be surjective as a map from Y_m to $C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n)$.

The surjective mapping theorem can still be used to solve a weaker realization result (see [18]). Let $k \in \mathbb{N}_0$ and define $J_0^k = J_0^k(\mathbb{R}^n, \mathbb{R}^n)$ to be the (finite dimensional) Banach space of all jets in $\mathbb{R}^n$ with 0 as target and source, i.e. the linear space of all polynomial functions $h: \mathbb{R}^n \to \mathbb{R}^n$ of degree $\leq k$, with $v(0) = 0$, endowed with an arbitrary norm $|h|$.

Theorem 3.1.3. *Assume the Poláčik condition for L on Ω. Then for every $k \in \mathbb{N}_0$ and every $m \in \mathbb{N}$ with $m \geq k+1$ there is an $\epsilon > 0$ such that for every k-jet $h: \mathbb{R}^n \to \mathbb{R}^n$ of degree k with $|h| < \epsilon$ there is a map $f \in \mathcal{V}_m(\delta_m)$ such that h is the k-th order Taylor polynomial of the vector field v_f at 0*

Proof. The map

$$T^k: C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n) \to J_0^k$$
$$(T^k v)(\xi) = \sum_{i=0}^{k} \frac{1}{i!} D^i v(0)\xi^i, \quad v \in C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n),\ \xi \in \mathbb{R}^n$$

is linear and bounded. Since

$$\Psi: \mathcal{V}_m(\delta_m) \subset Y_m \to C_b^{m-1}(\mathbb{R}^n, \mathbb{R}^n)$$

is of class C^1, it follows that

$$T^k \circ \Psi: \mathcal{V}_m(\delta_m) \to J_0^k$$

is of class C^1. We only need to show that $D(T^k \circ \Psi)(0): Y_m \to J_0^k$ is surjective, because then an application of the surjective mapping theorem will conclude the proof. Take an arbitrary $h \in J_0^k$. Then choose $v \in C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ with $T^k v = h$. Define f as in (3.1.3). Then $f \in Y_m$ and $D\Psi(0)f = v$. Consequently

$$D(T^k \circ \Psi)(0)f = T^k(D\Psi(0)f) = T^k v = h.$$

This concludes the proof. □

A result of this type is called a jet realization result, since it shows that, for every $k \in \mathbb{N}_0$, every sufficiently small k-jet of a vector field on $\mathbb{R}^n$ can be realized on the center manifold of an appropriate semilinear parabolic equation

$$u_t - Lu = f(x, u, \nabla u)$$

with Dirichlet (or Neumann) boundary condition on a bounded domain $\Omega \subset \mathbb{R}^N$, where $N = n - 1$, provided the operator L satisfies the Poláčik condition on $\partial\Omega$. Later on we will come back to jet realizations to give more precise and satisfactory results.

For the vector field realization problem, a very strong theorem was proved in [33]:

Theorem 3.1.4. *Assume the Poláčik condition for L on Ω. Then for every $m \geq 13$ there exists an $\epsilon_m > 0$ such that for every vector field $v_0 \in C^{m+11}(\mathbb{R}^n, \mathbb{R}^n)$ with $|v_0|_{m+11} < \epsilon_m$ there is a nonlinearity $f_0 \in \mathcal{V}_m(\delta_m)$ such that*

$$Q^{-1} P_0 \hat{f}_0(\Lambda_{f_0}(\xi)) = v_0(\xi), \quad \xi \in \mathbb{R}^n.$$

The proof of Theorem 3.1.4 is very complicated: it is based on Nash-Moser iteration scheme and involves a loss of derivatives. If we do not require the imbedding to be canonical, life is considerably simpler. This will be discussed in the next section.

2. Vector Field Realizations via Noncanonical Imbedding

In this section we will prove that, essentially under the same hypotheses of Theorem 3.1.4, it is possible to realize all sufficiently small vector fields on $\mathbb{R}^n$ on the center manifold of an appropriate semilinear parabolic equation, without any loss of derivatives; the price to pay is that the imbedding is not canonical. The main result is the following theorem, which is a generalization of Theorem 2 in [25].

Theorem 3.2.1. *Let L be as above and let $\kappa > 1$; assume:*

1. *L (with Dirichlet or Neumann boundary condition on $\partial\Omega$) satisfies the Poláčik codition on Ω;*
2. *$G \subset \Omega$ is an open set;*

(3) $R(x) \neq 0$ *for all* $x \in G$;

(4) *there is a function* $b \in C^\infty(\overline{\Omega})$ *with* $\operatorname{supp} b \subset G$ *such that*

$$\lambda < -\kappa$$

for every eigenvalue λ *of the operator* $L + b$ *on* Ω *with Dirichlet (or Neumann) boundary condition on* $\partial\Omega$.

Then there is a $\delta_1 > 0$ *such that for every* $v \in C_b^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ *with* $|v|_{C_b^1} < \delta_1$ *there is a nonlinearity* $f = f_v \in Y_1$ *with the property that equation (2.0.4) realizes the vector field* v *on an invariant manifold* $M = M_v$ *via an imbedding* $\Lambda = \Lambda_v \colon \mathbb{R}^{N+1} \to X^\alpha$ *of class* C^1. *Moreover, for each* $m \geq 1$ *there exists a* $\delta_m > 0$ *such that if* $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ *and* $|v|_{C_b^m} < \delta_m$ *then* f_v *can be chosen such that* $f_v \in Y_m$ *and the imbedding* $\Lambda_v \colon \mathbb{R}^{N+1} \to X^\alpha$ *is of class* C^m.

In Ch. 9 we will prove that, given an open set Ω and a principal part

$$\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j),$$

both for Dirichlet and Neumann boundary condition on $\partial\Omega$, it is possible to construct a potential $a(x)$ in such a way that the operator

$$L = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

satisfies the Poláčik condition and assumption (4) of Theorem 3.2.1.

Before proving Theorem 3.2.1 we state two lemmas. The proof is very simple and is left to the reader. First, we introduce some notation and terminology. For every globally Lipschitzian map $v \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ let $\pi_v \colon \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be the global flow generated by v. By differentiating the equation

$$\dot{\xi} = v(\xi)$$

and using Gronwall's inequality we derive:

Lemma 3.2.2. *For every* $m \in \mathbb{N}$ *there is a constant* $\tilde{c}_m$ *such that for every vector field* $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ *the flow* π_v *is of class* C^m *and for every* $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$

$$\left| D_\xi^m \pi_v(t, \xi_0) \right|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \leq \tilde{c}_m \exp(mL|t|)$$

where $L := |v|_{C_b^m}$.

Applying the higher-order chain rule to the composite map $v \circ \pi_v$ and using Lemma 3.2.2 we obtain

Lemma 3.2.3. *For every $m \in \mathbb{N}$ there is a constant c_m such that for every vector field $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and for every $(t, \xi_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$*

$$\left|D_\xi^m (v \circ \pi_v)(t, \xi_0)\right|_{\mathcal{L}^m((\mathbb{R}^{N+1})^m, \mathbb{R}^{N+1})} \leq c_m L \exp(mL|t|)$$

where $L := |v|_{C_b^m}$.

Proof of Theorem 3.2.1. Given $v: \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, with $|v|_{C_b^1}$ sufficiently small, we have to find a nonlinearity f and an imbedding $\Lambda: \mathbb{R}^{N+1} \to X^\alpha$ such that for each solution $\xi(t)$ of the ode

$$\dot{\xi}(t) = v(\xi(t)) \tag{3.2.1}$$

the function $u(t, x) = \Lambda(\xi(t))(x)$ is a solution to equation (2.0.2) (or (2.0.3)). The latter means (dropping the argument t)

$$D_\xi \Lambda(\xi)(x) v(\xi) - L\Lambda(\xi)(x) = f(x, \Lambda(\xi)(x), \nabla\Lambda(\xi)(x)). \tag{3.2.2}$$

We look for Λ in the form

$$\Lambda(\xi)(x) = \Phi(x) \cdot \xi + \Gamma(\xi)(x), \tag{3.2.3}$$

where $\Phi(x) = (\phi_1(x), \ldots, \phi_{N+1}(x))$ with $\phi_1(x)$, $\ldots$, $\phi_{N+1}(x)$ as in Definition 3.1.1, and $\Gamma: \mathbb{R}^{N+1} \to X^\alpha$. The construction of f and Γ is based on the following idea. If $\Gamma(\xi)$ is "sufficiently small" then for each $x \in G$ the mapping $\xi \mapsto (\Lambda(\xi)(x), \nabla\Lambda(\xi)(x))$ is a diffeomorphism of $\mathbb{R}^{N+1}$. Thus for $x \in G$ we can choose f such that $f(x, \Lambda(\xi)(x), \nabla\Lambda(\xi)(x))$ equals any given function of ξ; we shall require this function to equal to $b(x)\Gamma(\xi)(x)$, where Γ is still to be found. For $x \notin G$ we set $f(x, s, w) = 0$. Substituting this expression for f and (3.2.3) into (3.2.2), we obtain that $\Gamma(\xi)(x)$ must satisfy

$$D_\xi \Gamma(\xi)(x) v(\xi) - L\Gamma(\xi)(x) - b(x)\Gamma(\xi)(x) = \Phi(x) \cdot v(\xi)$$

(we have used the fact that the ϕ_i are in the kernel of L). Equivalently, we have to find $\Gamma(\xi)$ such that for each solution $\xi(t)$ of (3.2.1) the function $\tilde{u}(t, x) = \Gamma(\xi(t))(x)$ satisfies

$$\tilde{u}_t - (L + b)\tilde{u} = \Phi(x) \cdot \tilde{u}(\xi(t)).$$

As we also require that $\tilde{u}$ be defined for each t and bounded, the variation of constant easily leads to a formula for $\tilde{u}$, hence for Γ. This formula is a starting point in the detailed construction that follows next. We verify that it yelds Γ and f of class C^m if $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$.

Let $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ be such that $|v|_{C_b^1} < \delta_1$, where δ_1 is a constant to be specified below. Denote $L_j = |v|_{C_b^j}$, $j = 0, \ldots, m$. Assume $L_1 + 1 < \kappa$ and $mL_m < \kappa$. Let $T(t)$, $t \geq 0$, be the analytic semigroup on X generated by the operator $-(L + b)$ on Ω with Dirichlet (or Neumann) boundary condition on $\partial\Omega$. We have the estimates

$$|T(t)u|_\alpha \leq C_\alpha t^{-\alpha} e^{-\kappa t}|u|, \quad t > 0, \ u \in X^\alpha. \tag{3.2.4}$$

Let $|\Phi| = |(\phi_1, \ldots, \phi_{N+1})| := \sum_{l=1}^{N+1} |\phi_l|_\alpha$. For $\xi \in \mathbb{R}^{N+1}$ define

$$\Gamma(\xi) = \Gamma_v(\xi) := -\int_0^\infty T(s)\Phi \cdot v(\pi(-s, \xi))ds \tag{3.2.5}$$

where $\pi = \pi_v$ is the flow generated by (3.2.1). The integrand in (3.2.4) is continuous into X^α and its X^α-norm is bounded by the function

$$g(s) := C_\alpha |\Phi| L_0 s^{-\alpha} e^{-\kappa s}.$$

This latter function is integrable, so the integral in (3.2.5) converges in X^α. Thus

$$\Gamma \colon \mathbb{R}^{N+1} \to X^\alpha$$

is defined and bounded globally. Moreover, in view of Lemma 3.2.3, for every j with $1 \leq j \leq m$ and every $\xi \in \mathbb{R}^{N+1}$ the j-th order Frechet derivative at ξ of the integrand in (3.2.5) is bounded in the $\mathcal{L}^j((\mathbb{R}^{N+1})^j, X^\alpha)$-norm by the function

$$g_j(s) := C_\alpha c_j |\Phi| L_j s^{-\alpha} e^{-(\kappa - jL_j)s}.$$

Since this function is integrable, it follows that $\Gamma \in C_b^m(\mathbb{R}^{N+1}, X^\alpha)$. Moreover,

$$\int_0^\infty g_j(s)ds \leq C_\alpha c_j |\Phi| L_j (1/(1-\alpha) + (1/(\kappa - jL_j))). \tag{3.2.6}$$

Define the map

$$\Lambda = \Lambda_v \colon \mathbb{R}^{N+1} \to X^\alpha$$

by

$$\xi \mapsto \Phi \cdot \xi + \Gamma_v(\xi).$$

Now let U be an open set with $\operatorname{supp} b \subset U \subset \overline{U} \subset G$. For every $x \in G$ the map

$$M(x) \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$$

$$\xi \mapsto (\Phi(x) \cdot \xi, \nabla\Phi(x) \cdot \xi)$$

is a linear isomorphism. Since $\overline{U}$ is compact

$$\bar{M} := \sup_{x \in \overline{U}} \left|(M(x))^{-1}\right| < \infty.$$

Using (3.2.6) with $j = 1$ and the relation $\kappa > L_1 + 1$, we see that there is a constant δ_1, $0 < \delta_1 \leq L_1$ (independent of v) such that whenever $|v|_{C_b^1} < \delta_1$

$$\sup_{x \in \overline{U}, \xi \in \mathbb{R}^{N+1}} (|D_\xi \Gamma(\xi)(x)| + |\nabla D_\xi \Gamma(\xi)(x)|) < 1/\bar{M}.$$

For such a v, the contraction mapping principle and the implicit function theorem imply that, for $x \in \overline{U}$ the map

$$\psi_x : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1},$$
$$\xi \mapsto (\Lambda(\xi)(x), \nabla\Lambda(\xi)(x))$$

is a diffeomorphism of class C^m and, for all j with $0 \leq j \leq m$, the map

$$(x, z) \in \overline{U} \times \mathbb{R}^{N+1} \mapsto D^j(\psi_x)^{-1}(z) \in \mathcal{L}^j((\mathbb{R}^{N+1})^j, \mathbb{R}^{N+1})$$

is continuous and bounded. In particular we obtain that $\xi \mapsto \Lambda(\xi)$ is an imbedding of $\mathbb{R}^{N+1}$ into X^α. Define for $z = (s, w) \in \mathbb{R}^{N+1}$ and $x \in \overline{\Omega}$

$$f(x, z) := \begin{cases} 0, & \text{if } x \notin \operatorname{supp} b; \\ b(x)\Gamma((\psi_x)^{-1}(z))(x), & \text{if } x \in U; \end{cases}$$

since $\operatorname{supp} b \subset U$, the definition of f is unambiguous and the smoothness properties proved so far imply that $f \in Y_m$. We shall show that f satisfies the assertions of Theorem 3.2.1. To this end, first note that

$$\Gamma(\xi) = \Gamma_v(\xi) = -\int_{-\infty}^0 T(-s)\Phi \cdot v(\pi(s, \xi))ds$$

for all ξ. Hence for all $t_0, t \in \mathbb{R}$ with $t_0 < t$

$$\begin{aligned} \Gamma(\pi(t, \xi)) &= -\int_{-\infty}^0 T(-s)\Phi \cdot v(\pi(t + s, \xi))ds \\ &= -\int_{-\infty}^t T(t - s)\Phi \cdot v(\pi(s, \xi))ds \\ &= -T(t - t_0)\int_{-\infty}^{t_0} T(t_0 - s)\Phi \cdot v(\pi(s, \xi))ds + \\ &\quad -\int_{t_0}^t T(t - s)\Phi \cdot v(\pi(s, \xi))ds \\ &= T(t - t_0)\Gamma(\pi(t_0, \xi)) - \int_{t_0}^t T(t - s)\Phi \cdot v(\pi(s, \xi))ds. \end{aligned}$$

Since the function $s \mapsto \Phi \cdot v(\pi(s,\xi))$ is locally Hölder continuous into X it follows by Theorem 1.5.4 and Lemma 1.6.1 that the function

$$\tilde{u}: t \in \mathbb{R} \mapsto \Gamma(\pi(t,\xi)) \in X^\alpha$$

is differentiable and

$$\dot{\tilde{u}}(t) = (L+b)\tilde{u}(t) - \Phi \cdot v(\pi(t,\xi)), \quad t \in \mathbb{R}.$$

Therefore the definition of f implies that the function

$$u: t \in \mathbb{R} \mapsto \Lambda(\pi(t,\xi)) \in X^\alpha$$

solves the equation

$$\dot{u}(t) = Lu(t) + f(\cdot, u(t)(\cdot), \nabla u(t)(\cdot)), \quad t \in \mathbb{R}.$$

The theorem is proved. □

It was not realized in [25] that, for δ_1 small enough, the manifold M in the above theorem is actually the global center manifold of (2.0.4), although the imbedding Λ, in general, is not the canonical imbedding; this was proved in [29]:

Theorem 3.2.4. *Assume that L is such that all hypotheses of Theorem 3.2.1 are satisfied. Then there is a $\delta_1 > 0$ such that for every $v \in C_b^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ with $|v|_{C_b^1} < \delta_1$ there is a nonlinearity $f = f_v \in Y_1$ with the property that equation (2.0.4) realizes the vector field v on its global center manifold $\mathcal{M}_f$ via a (not necessarily canonical) imbedding $\Lambda = \Lambda_v: \mathbb{R}^{N+1} \to X^\alpha$ of class C^1. If in addition $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ then f_v can be chosen such that $f_v \in Y_m$ and the imbedding $\Lambda_v: \mathbb{R}^{N+1} \to X^\alpha$ is of class C^m.*

Proof. Theorem 3.2.1 implies the present theorem provided that we prove the following

Claim: For $\delta_1 > 0$ sufficiently small the imbedding Λ constructed in the proof of Theorem 3.2.1 is an imbedding onto the global center manifold of (2.0.4).

The imbedding Λ is constructed in the form

$$\Lambda(\xi)(x) = \Phi(x) \cdot \xi + \Gamma(\xi)(x), \quad \xi \in \mathbb{R}^{N+1}, \, x \in \Omega$$

with $\Gamma = \Gamma_v: \mathbb{R}^{N+1} \to X^\alpha$ of class C^1 (and of class C^m if $v \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$) and satisfies the estimate

$$\text{(3.2.7)} \qquad \sup_{\xi \in \mathbb{R}^{N+1}} (|\Gamma_v(\xi)|_{X^\alpha} + |D\Gamma_v(\xi)|_{\mathcal{L}(\mathbb{R}^{N+1}, X^\alpha)}) \le C|v|_{C_b^1}, \quad v \in C_b^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1}),$$

with some constant C independent of $v \in C_b^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$. Now (3.2.7) and the definition of the function f given in the proof of Theorem 3.2.1 imply that

$$f_v \to 0 \quad \text{in } Y_1 \text{ as } |v|_{C_b^1} \to 0.$$

Thus the center manifold theorem implies that for δ_1 small enough and $|v|_{C_b^1} < \delta_1$ the global center manifold $\mathcal{M}_{f_v}$ of (2.0.4) is given by

$$\mathcal{M}_{f_v} = \Lambda_{f_v}(\mathbb{R}^{N+1})$$

where

$$\Lambda_{f_v} \colon \mathbb{R}^{N+1} \to X^\alpha$$

is the canonical center manifold imbedding. Fix such a v and let $f := f_v$. For $\xi \in \mathbb{R}^{N+1}$ let $t \in \mathbb{R} \mapsto \underline{\xi}(t)$ be the solution of (2.0.4) through ξ. Then

$$P_h \Lambda_v(\underline{\xi}(t)) = P_h \Gamma_v(\underline{\xi}(t))$$

so by (3.2.7) $t \mapsto P_h \Lambda_v(\underline{\xi}(t))$ is bounded in X^α. This implies that $\Lambda(\xi) \in \mathcal{M}_{f_v}$ so

$$M_v \subset \mathcal{M}_{f_v}.$$

Since Λ_{f_v} is the canonical imbedding,

$$\Lambda_{f_v}^{-1} \circ \Lambda_v(\xi) = \xi + Q^{-1} P_0 \Gamma_v(\xi), \quad \xi \in \mathbb{R}^{N+1}.$$

If $|v|_{C_b^1}$ is small enough then (3.2.7) implies that the map $\xi \in \mathbb{R}^{N+1} \mapsto Q^{-1} P_0 \Gamma_v(\xi) \in \mathbb{R}^{N+1}$ is a contraction so $\Lambda_{f_v}^{-1} \circ \Lambda_v \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is surjective. This immediately implies that

$$M_v \supset \mathcal{M}_{f_v}.$$

The proof is complete. □

CHAPTER IV

JET REALIZATIONS AND DENSITY RESULTS

In the previous chapter we have proved that all sufficiently small vector fields in $\mathbb{R}^n$ can be realized on the center manifold of an appropriate semilinear parabolic equation on a N-dimensional open set Ω, in case $n = N + 1$. We have also given a jet realization result under the same restriction on the space dimension of the PDE. In this chapter we deal with the following problem: what can we say if we do not impose any restriction on the space dimension? It is well known that, if $N = 1$, equation

$$\begin{aligned} &u_t = a(x)u_{xx} + f(x, u, u_x), && t > 0,\ x \in]0, 1[; \\ &u = 0, && t > 0,\ x = 0, 1, \end{aligned} \tag{4.0.1}$$

or

$$\begin{aligned} &u_t = a(x)u_{xx} + f(x, u, u_x), && t > 0,\ x \in]0, 1[; \\ &u_x = 0, && t > 0,\ x = 0, 1, \end{aligned} \tag{4.0.2}$$

admits a Ljapunov functional, so that the dynamics is far from complicated; in terms of realizations, the kernel of a strongly elliptic differential operator with Dirichlet or Neumann boundary condition in one space dimension is necessarily one dimensional, and hence the same is true for center manifolds of equations like (4.0.1) or (4.0.2). The situation is completely different as soon as $N \geq 2$. In this case it is possible to find operators with kernels of arbitrarily high dimensions. We will see that, if the eigenfuctions of such an operator satisfy certain algebraic independence conditions, it is possible to prove realizability of all sufficiently small jets even in two space dimensions; moreover, again in two space dimensions, it is possible to prove realizability, up to flow equivalence, of a dense (in the C^1 topology) subset of vector fields. We can also impose restrictions on the form of the nonlinearity and make it as simple as possible (e.g. linear in the gradient), whereas it is known that, if the nonlinearity does not depend on the gradient at all, again the parabolic equation admits a Ljapunov functional.

1. Jet Realizations

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let

$$L := \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

be a strongly elliptic second order differential operator with symmetric smooth coefficients. Now fix $k \in \mathbb{N}$ and arbitrary integers $q_1, \dots, q_k$ such that $1 \le q_l \le l$ for $l = 1, \dots, k$. Let $\mathcal{E} = \mathcal{E}(q_1, \dots, q_k)$ be the set of all functions $f: \mathbb{R}^4 \to \mathbb{R}$ of the form

$$f(x,y,s,w) = \sum_{l=1}^{k} a_l(x,y) s^{l-q_l} w^{q_l}, \quad (x,y,s,w) \in \mathbb{R}^4,$$

where $a_l \in H^2(\Omega)$ for $l = 1, \dots, k$. For $f \in \mathcal{E}$ and $\varpi \in \mathbb{R}^2$, consider the equations

$$\begin{aligned} u_t &= Lu + f(x,y,u,u_\varpi), && t > 0,\ (x,y) \in \Omega \\ u &= 0, && t > 0,\ (x,y) \in \partial\Omega \end{aligned} \tag{4.1.1}$$

and

$$\begin{aligned} u_t &= Lu + f(x,y,u,u_\varpi), && t > 0,\ (x,y) \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0, && t > 0,\ (x,y) \in \partial\Omega, \end{aligned} \tag{4.1.2}$$

where $u_\varpi := \varpi \cdot \nabla u$.

Set $X = L^p(\Omega)$, $p > 2$, and let $A: D_A \subset X \to X$ be the sectorial operator induced by L with Dirichlet or Neumann boundary condtion on $\partial\Omega$, where $D_A = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ in the first case, and $D_A = W_N^{2,p}(\Omega)$ in the second case. As usual we rewrite (4.1.1) and (4.1.2) as abstract equations in X:

$$\dot{u} + Au = \hat{f}^\varpi(u), \tag{4.1.3}$$

where $\hat{f}^\varpi(u)(x) := f(x, u(x), u_\varpi(x))$. Note $\hat{f}^\varpi \in C^\infty(X^\alpha, X)$. Assume

$$\dim \ker A = n.$$

We can identify $\mathcal{E}$ with $\left(H^2(\Omega)\right)^k$; with the norm induced by this identification, $\mathcal{E}$ becomes a Banach space whose topology is stronger than the topology of locally uniform convergence of all derivatives $D^h_{(s,w)} f(x,y,s,w)$, $h = 0, \dots, k+1$ on $\overline{\Omega} \times \mathbb{R}^2$. We are exactly in the situation described in the remark following Theorem 2.2.1, with Z replaced by $\mathcal{E}$ and m replaced by $k+1$. Thus we can find an open neighborhood $\mathcal{U}$ in $\mathcal{E}$, $0 \in \mathcal{U}$, and a map

$$\Lambda: \mathcal{U} \times B_1^n(0) \subset \mathcal{E} \times \mathbb{R}^n \to X^\alpha$$

with the following properties:

(1) $P_0 \Lambda(f, \xi) \equiv Q\xi$ and $\Lambda(0, \xi) \equiv Q\xi$;

(2) Λ is of class C^{k+1};

(3) the map $\Lambda_f(\cdot) := \Lambda(f,\cdot)$ is an imbedding and the set

$$\mathcal{M}_f^{\text{loc}} := \{\Lambda_f(\xi) \mid \xi \in B_1^n(0)\}$$

is a local invariant manifold of (4.1.3). Moreover, if $v_f\colon B_1^n(0) \to \mathbb{R}^n$ is defined by

$$v_f(\xi) := Q^{-1} P_0 \hat{f}^{\varpi}(\Lambda_f(\xi)), \quad \xi \in B_1^n(0),$$

then the ODE defined by v_f imbeds, via Λ_f, in (4.1.3).

As in Section 3.1, we define a map

$$\begin{gathered}\Psi\colon \mathcal{U} \subset \mathcal{E} \to C_b^k(B_1^n(0), \mathbb{R}^n)\\ \Psi(f)(\xi) := Q^{-1} \circ P_0 \circ \hat{f}^{\varpi} \circ \Lambda_f(\xi), \quad \xi \in B_1^n(0).\end{gathered}$$

Simple computation shows that Ψ is of class C^1 and

$$D\Psi(0)f(\xi) = (Q^{-1} \circ P_0 \circ \hat{f}^{\varpi})(Q\xi).$$

We are interested in jet realizations. Thus we introduce the linear bounded operator

$$\begin{gathered}T^k\colon C_b^k(B_1^n(0), \mathbb{R}^n) \to J_0^k\\ (T^k v)(\xi) = \sum_{i=0}^{k} \frac{1}{i!} D^i v(0), \quad v \in C_b^k(B_1^n(0), \mathbb{R}^n),\ \xi \in B_1^n(0).\end{gathered}$$

Our aim is to find a condition which guarantees that $D(T^k \circ \Psi)(0)$ is surjective onto J_0^k. Our starting point is the abstract surjectivity condition

(SC) For every polynomial function $h\colon \mathbb{R}^n \to \mathbb{R}^n$ of degree $\leq k$, $h(0) = 0$, there is an $f \in \mathcal{E}(q_1, \ldots, q_k)$ such that

$$T^k(D\Psi(0)f) = h. \tag{4.1.4}$$

To give a more explicit expression of this condition, we take an $L^2(\Omega)$-orthonormal basis of $\ker A$, namely $\phi_1, \ldots, \phi_n$. Then (4.1.4) reads

$$T^k \int_\Omega \phi_j(x,y) f\left(x, y, \sum_{i=1}^n \xi_i \phi_i(x,y), \sum_{i=1}^n \xi_i \phi_{i\varpi}(x,y)\right) dxdy = h_j(\xi), \quad j = 1, \ldots, n. \tag{4.1.5}$$

The nonlinearity f has the form

$$f(x, y, s, w) = \sum_{l=1}^{k} a_l(x, y) s^{l-q_l} w^{q_l}$$

and the polynomial vector field h has the form

$$h_j(\xi) = \sum_{l=1}^{k} \sum_{|\beta|=l} \rho_{j\beta} \xi^{\beta}, \quad j = 1, \ldots, n.$$

By substituting these expressions into (4.1.5), we finally get

$$\text{(4.1.6)} \quad \int_{\Omega} \phi_j \sum_{l=1}^{k} a_l \left(\sum_{j=1}^{n} \xi_j \phi_j \right)^{l-q_l} \left(\sum_{i=1}^{n} \xi_i \phi_{i\varpi} \right)^{q_l} dxdy = \sum_{l=1}^{k} \sum_{|\beta|=l} \rho_{j\beta} \xi^{\beta}, \qquad j = 1, ..., n.$$

Consequently (SC) is satisfied, provided we can find functions $a_1, \ldots, a_k \in H^2(\Omega)$ such that (4.1.6) holds for all $(\xi_1, ..., \xi_n) \in \mathbb{R}^n$. The left hand side of (4.1.6) can be manipulated in the following way:

$$\begin{aligned}
&\int_{\Omega} \phi_j \sum_{l=1}^{k} a_l \left(\sum_{j=1}^{n} \xi_j \phi_j \right)^{l-q_l} \left(\sum_{i=1}^{n} \xi_i \phi_{i\varpi} \right)^{q_l} dxdy = \\
&= \int_{\Omega} \phi_j \sum_{l=1}^{k} a_l \left(\sum_{|\alpha|=l-q_l} \frac{(l-q_l)!}{\alpha!} \xi^{\alpha} \phi^{\alpha} \right) \left(\sum_{|\gamma|=q_l} \frac{q_l!}{\gamma!} \xi^{\gamma} \phi_{\varpi}^{\gamma} \right) dxdy = \\
&= \sum_{l=1}^{k} \sum_{|\alpha|=l-q_l} \sum_{|\gamma|=q_l} \frac{(l-q_l)!}{\alpha!} \frac{q_l!}{\gamma!} \xi^{\alpha+\gamma} \int_{\Omega} a_l \phi_j \phi^{\alpha} \phi_{\varpi}^{\gamma} \, dxdy = \\
&= \sum_{l=1}^{k} \sum_{|\beta|=l} \left(\sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)! q_l!}{\alpha! \gamma!} \int_{\Omega} a_l \phi_j \phi^{\alpha} \phi_{\varpi}^{\gamma} \, dxdy \right) \xi^{\beta}
\end{aligned}$$

Equating coefficients we therefore see that (SC) is satisfied if and only if for every $l = 1, \ldots, k$, for all $\rho_{j\beta} \in \mathbb{R}$, $j = 1, \ldots, n$, $\beta \in \mathbb{N}_0^n$, $|\beta| = l$, there is $a_l(x, y) \in H^2(\Omega)$ such that, for all $j = 1, \ldots, n$, and for all $\beta \in \mathbb{N}_0^n$ with $|\beta| = l$,

$$\text{(4.1.7)} \qquad \int_{\Omega} a_l \left(\sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)! q_l!}{\alpha! \gamma!} \phi_j \phi^{\alpha} \phi_{\varpi}^{\gamma} \right) dxdy = \rho_{j\beta}.$$

Using the density of $H^2(\Omega)$ in $L^2(\Omega)$ it is easy to see that this latter condition is satisfied if and only if for for every $l = 1, \ldots, k$ the functions

$$\left\{ \sum_{\substack{\alpha+\gamma=\beta \\ |\alpha|=l-q_l, |\gamma|=q_l}} \frac{(l-q_l)!q_l!}{\alpha!\gamma!} \phi_j \phi^\alpha \phi_\varpi^\gamma \right\}_{\substack{j=1,\ldots,n \\ |\beta|=l}}$$

are linearly independent.

Now we introduce the following notations: given $\gamma, \beta \in \mathbb{N}_0^n$, we say that $\gamma \leq \beta$ iff $\gamma_i \leq \beta_i$, $i = 1, \ldots, n$. Moreover, set

$$\epsilon_j := (\underbrace{0, ..., 0, 1}_{j}, 0, ..., 0) \in \mathbb{N}_0^n.$$

With these notations, the independence condition reads (cf. [29]):

(IC) For every $l = 1, \ldots, k$ and for every q, $1 \leq q \leq l$, the functions

$$\left\{ \sum_{\substack{\gamma \leq \beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j} \phi_\varpi^\gamma \right\}_{\substack{j=1,\ldots,n \\ |\beta|=l}}$$

are linearly independent.

Remark. If (IC) is satisfied then the functions a_l in (4.1.7) can actually be chosen to belong to any dense subspace of $L^2(\Omega)$, e.g. they can be chosen as smooth as Ω is. On the other hand, these functions cannot be chosen to be constant, in general. The reason for this is that the subspace of functions in $\mathcal{E}(q_1, \ldots, q_k)$ with spatially constant coefficients has dimension k, while, for $\dim X_1 > 1$, the space of jets $J_0^k(X_1)$ has dimension $> k$ so the surjectivity condition (SC) is not satisfied in this case.

The above considerations together with the classical surjective mapping theorem yeald to the following

Theorem 4.1.1. *Let n and $k \in \mathbb{N}$. Assume $\dim\ker A = n$ and assume there is an $L^2(\Omega)$-orthonormal basis $\phi_1, \ldots, \phi_n$ of $\ker A$ and a vector $\varpi \in \mathbb{R}^2$ such that (IC) is satisfied up to the order k. Then there is an open neighborhood $\mathcal{B}$ of 0 in $J_0^k(\mathbb{R}^n)$ such that every jet $h \in \mathcal{B}$ can be realized in (4.1.4) by a a nonlinearity $f \in \mathcal{E}$.*

Remark. Choosing $q_l = l$ for all $l = 1, \ldots, k$, we obtain as a particular case Poláčik result ([23, Th. 2.2]. On the other hand, choosing $q_l = 1$ for all $l = 1, \ldots, k$, we obtain a jet realization result for nonlinearities which are polynomials in u and which are linear functions of ∇u: in other words, we obtain a jet realization result in the class of equations of the form

$$u_t = Lu + f(x,u)\varpi \cdot \nabla u$$

where $f(x,u)$ is a polynomial in u with x-dependent coefficients and $\varpi \in \mathbb{R}^2$.

In the next section we will show that condition (IC) can also be used to obtain realizability of a dense subset of vector fields.

2. Density Results

Let n and k be fixed. Assume $\ker A$ is spanned by $L^2(\Omega)$-orthonormal functions ϕ_1, $\ldots$,ϕ_n satisfyng (IC) up to the ordr k for some $\varpi \in \mathbb{R}^2$. Let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial function of degree k, with $h(0)=0$. Then we have seen that there exists a function $f \in \mathcal{E}$ such that

$$(Q^{-1}P_0\hat{f}^{\varpi})(Q\xi) = h(\xi), \quad \xi \in \mathbb{R}^n.$$

Keep this f fixed and, for every $\epsilon > 0$, consider equations

$$\text{(4.2.1}_\epsilon) \qquad \begin{aligned} u_t &= Lu + \epsilon f(x,y,u,u_{\varpi}), && t>0,\ x\in\Omega \\ u &= 0, && t>0,\ x \in \partial\Omega \end{aligned}$$

and

$$\text{(4.2.2}_\epsilon) \qquad \begin{aligned} u_t &= Lu + \epsilon f(x,y,u,u_{\varpi}), && t>0,\ x\in\Omega \\ \frac{\partial u}{\partial \nu} &= 0, && t>0,\ x \in \partial\Omega \end{aligned}$$

and the corresponding abstract equation

$$\text{(4.2.3}_\epsilon) \qquad \dot{u} + Au = \epsilon \hat{f}^{\varpi}(u).$$

Following the terminology of the previous section, we have that, for all sufficiently small ϵ, $\epsilon f \in \mathcal{U}$; then the map $\Lambda_{\epsilon f}\colon B_1^n(0) \subset \mathbb{R}^n \to X^\alpha$ is an imbedding and the set

$$\mathcal{M}_{\epsilon f}^{\text{loc}} := \{\Lambda_{\epsilon f}(\xi) \mid \xi \in B_1^n(0)\}$$

is a local invariant manifold of $(4.2.3)_\epsilon$. Morover, if $v_{\epsilon f}\colon B_1^n(0) \to \mathbb{R}^n$ is defined by

$$v_{\epsilon f}(\xi) := \epsilon Q^{-1}P_0\hat{f}^{\varpi}(\Lambda_{\epsilon f}(\xi)), \quad \xi \in B_1^n(0),$$

then the ODE defined by $v_{\epsilon f}$ imbeds, via $\Lambda_{\epsilon f}$, in $(4.2.3)_\epsilon$. Observe that the ODE

$$\dot{\xi} = v_{\epsilon f}(\xi)$$

is equivalent, up to a time rescaling, to the ODE

$$\dot{\xi} = h_\epsilon(\xi),$$

where

$$h_\epsilon(\xi) := Q^{-1} P_0 \hat{f}^{\varpi}(\Lambda_{\epsilon f}(Q\xi)), \quad \xi \in B_1^n(0).$$

Now, since the map

$$\Lambda : \mathcal{U} \times B_1^n(0) \subset \mathcal{E} \times \mathbb{R}^n \to X^\alpha$$

is of class C^{k+1} and $\Lambda_0(\xi) \equiv Q\xi$, we obtain that

$$h_\epsilon = Q^{-1} P_0 \hat{f}^{\varpi}(\Lambda_{\epsilon f}(Q\cdot)) \to Q^{-1} P_0 \hat{f}^{\varpi}(Q\cdot) = h(\cdot) \quad \text{as } \epsilon \to 0$$

in $C_b^k(B_1^n(0), \mathbb{R}^n)$. Thus we have reached the following result: given a polynomial function $h: \mathbb{R}^n \to \mathbb{R}^n$ of degree k with $h(0) = 0$ and given an $\eta > 0$, we can find a nonlinearity $g \in \mathcal{E}$ such that the vector field $v_g: B_1^n(0) \to \mathbb{R}^n$ is equivalent, up to a time rescaling, to a vector field $h_g: B_1^n(0) \to \mathbb{R}^n$ such that

$$\sup_{0 \le l \le k} \sup_{\xi \in B_1^n(0)} |D^l h_g(\xi) - D^l h(\xi)| < \eta.$$

In Chapter 9 we will prove that, given an open set $\Omega \subset \mathbb{R}^2$, a principal part

$$\sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j),$$

and numbers $n, k \in \mathbb{N}$, both for Dirichlet and Neumann boundary condition on $\partial\Omega$ it is possible to construct a potential $a(x)$ in such a way that the kernel of the operator

$$\sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

is spanned by $L^2(\Omega)$-orthonormal functions $\phi_1, \ldots, \phi_n$ satisfying (IC) up to the order k, for some $\varpi \in \mathbb{R}^2$.

Assume for the moment this is true. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, let $a_{ij}: \overline{\Omega} \to \mathbb{R}$ be smooth functions, $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, 2$, and

$$\sum_{i,j=1}^{2} a_{ij}(x)\zeta_i\zeta_j \ge c|\zeta|^2, \quad x \in \overline{\Omega},\ \zeta \in \mathbb{R}^2$$

for some $c > 0$. Then the following result obtains:

Theorem 4.2.1. *Let $n \in \mathbb{N}$ be arbitrary, let B be the unit ball centered at 0 in $\mathbb{R}^n$ and let ${}_0C^1(\overline{B})$ be the Banach space of all C^1 maps $h\colon \overline{B} \to \mathbb{R}^n$ with $h(0) = 0$, endowed with the C^1 norm. There is a dense set $\mathcal{D}$ in ${}_0C^1(\overline{B})$ with the property that for every $h \in \mathcal{D}$ there is a potential $a\colon \overline{\Omega} \to \mathbb{R}$, a nonlinearity g of the form*

$$g(x, y, s, w) = \sum_{l=1}^{k} a_l(x, y) s^{l-1} w, \quad (x, y, s, w) \in \overline{\Omega} \times \mathbb{R}^2,$$

for some $k \in \mathbb{N}$, and a vector $\varpi \in \mathbb{R}^2$ such that the flow of the equation

$$\dot{\xi} = h(\xi), \quad \xi \in B$$

is C^1 equivalent to the flow of equation

$$\dot{u} + A = \hat{g}^{\varpi}(u)$$

restricted to some (n-dimensional) local center manifold, where A is the sectorial operator generated by

$$L = \sum_{ij=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

with Dirichlet (or Neumann) condition on $\partial\Omega$.

Proof. Let $h\colon \overline{B} \to \mathbb{R}^n$ be a C^1 vector field. Since polynomial functions are dense in $C^1(\overline{B})$, it is not a restriction to assume h is a polynomial, say of degree k. We take a potential a such that the kernel of the operator L, with Dirichlet (or Neumann) boundary condition on $\partial\Omega$, is spanned by $L^2(\Omega)$-orthonormal functions $\phi_1, \dots, \phi_n$ satisfying (IC) up to the order k for some $\varpi \in \mathbb{R}^2$. Choose $(q_1, \dots, q_k) = (1, \dots, 1)$. Take $\eta > 0$; as we have seen above, we can find a nonlinearity $g \in \mathcal{E}(1, \dots, 1)$ such that the vector field $v_g\colon \overline{B} \to \mathbb{R}^n$ is equivalent, up to a time rescaling, to a vector field $h_g\colon \overline{B} \to \mathbb{R}^n$ such that

$$\max \left\{ \sup_{\xi \in \overline{B}} |h_g(\xi) - h(\xi)|, \sup_{\xi \in \overline{B}} |Dh_g(\xi) - Dh(\xi)| \right\} < \eta.$$

This concludes the proof. □

Remark. The results described in this section are essentially a restatement of results contained in [23] and [29]; however here we have given a much simpler proof, based on a kind of "singular perturbation" approach instead of the one contained in [23], based on the method of realization of C^m-families of jets. The present approch was suggested by Prof. P. Poláčik to the author of this thesis while he was visiting the institute of Mathematics at Comenius University in Bratislava.

Remark. All results in this chapter remain valid if the space dimension is any $N \geq 2$.

CHAPTER V

PERTURBATION OF EIGENVALUES OF SELFADJOINT OPERATORS

In this chapter we will consider perturbations of a finite number $\lambda_{l+1} \leq \lambda_{l+2} \leq \cdots \leq \lambda_{l+p}$ of eigenvalues of a self-adjoint operator A on a Hilbert space. We shall prove in Theorem 5.2.3 below that if the eigenvectors corresponding to these eigenvalues satisfy a certain nondegeneracy assumption (part (4) of Definition 5.2.2 below) then for some $\alpha_0 > 0$ all tuples $\mu_1, \ldots, \mu_p$ lying in the α_0-neighborhood of $\lambda_{l+1}, \ldots, \lambda_{l+p}$ can be realized as eigenvalues of a suitable perturbed operator $A + B$, B symmetric and bounded. It is an essential part of this result that the constant α_0 does not depend on the operator A but only on a bound for $\lambda_{l+p} - \lambda_{l+1}$, on the gap between $\lambda_{l+1}, \ldots, \lambda_{l+p}$ and the rest of the spectrum of A and on the constant contained in the above mentioned nondegeneracy assumption.

Theorem 5.2.3 solves a general, abstract inverse problem. Together with a convergence result established in the next chapter it will enable us to constuct concrete differential operators satisfying the assumptions of Theorem 3.2.1 and 4.1.1 above.

1. The Surjective Mapping Theorem

We begin by stating the following general result, which is essentially a version of the surjective mapping theorem with all constants made explicit.

Theorem 5.1.1. *Suppose X is a Banach space, Y is a normed space and α, θ, ρ are numbers satisfying α, $\theta > 0$ and $0 < \rho < 1$. Let $c \in X$ and $f\colon \mathrm{B}_\alpha(c) \subset X \to Y$ and $S \in \mathcal{L}(X, Y)$ be such that and*

(5.1.1) $$S(\mathrm{B}_1) \supset \mathrm{B}_\theta$$

and

(5.1.2) $$|f(b) - f(a) - S(b-a)| \leq \rho\theta|b-a|, \qquad \text{for all } a,\, b \in \mathrm{B}_\alpha(c).$$

Under these assumptions,

$$f(\mathrm{B}_\alpha(c)) \supset \mathrm{B}_{\theta(1-\rho)\alpha}(f(c)).$$

Proof. Replacing f by the map $f(\cdot + c) - f(c)$ we may assume that $c = 0$ and $f(c) = 0$. Property (5.1.1) and the homogeneity of S imply that for every $\beta > 0$,

(5.1.3) $$S(\mathrm{B}_{\theta^{-1}\beta}) \supset \mathrm{B}_\beta.$$

Let $y \in \mathrm{B}_{\theta(1-\rho)\alpha}$ be arbitrary. Choose δ with

$$|y| < \delta < \theta(1-\rho)\alpha.$$

We claim that there is a sequence $(x_k)_{k\in\mathbb{N}_0}$ such that $x_0 = 0$ and

$$(5.1.4) \qquad \begin{aligned} &|x_k| < \theta^{-1}\rho^{k-1}\delta, \\ &Sx_k = y - f(x_1 + \cdots + x_{k-1}), \quad \text{for } x \in \mathbb{N} \end{aligned}$$

(here, $x_1+\cdots+x_0 := 0$.) In fact, set $x_0 = 0$. By (5.1.3) there is an $x_1 \in X$ with $|x_1| < \theta^{-1}\delta$ and $Sx_1 = y$. Thus x_1 satisfies (5.1.4) with $k = 1$. Suppose that $n \geq 1$ and x_k, $1 \leq k \leq n$ are chosen so that (5.1.4) is satisfied for $1 \leq k \leq n$. Then

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n| \leq \theta^{-1}(1-\rho)^{-1}\delta < \alpha$$

so $f(x_1 + \cdots + x_n)$ is defined. Moreover, using (5.1.2) we have

$$\begin{aligned} |y - f(x_1 + \cdots + x_n)| = |y - f(x_1 + \cdots + x_{n-1}) \\ +f(x_1 + \cdots + x_{n-1}) - f(x_1 + \cdots + x_n)| \\ \leq |f(x_1 + \cdots + x_n) - f(x_1 + \cdots + x_{n-1}) - Sx_n| \leq \rho\theta|x_n| < \rho^n\delta. \end{aligned}$$

Hence, by (5.1.3), there is an $x_{n+1} \in X$ with $|x_{n+1}| < \theta^{-1}\rho^n\delta$ with $Sx_{n+1} = y - f(x_1 + \cdots + x_n)$. Thus (5.1.4) is satisfied with $k = n+1$. Now obvious recursion proves existence of a sequence $(x_k)_{k\in\mathbb{N}_0}$ with the desired properties. These properties and the completeness of the norm on X imply, on the one hand that the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent and so it has a limit $x \in X$, and, on the other hand, that $|x| < \alpha$ and $f(x) = y$. This completes the proof. □

Corollary 5.1.2. *Suppose X is a Banach space, Y is a normed space and θ, ρ are numbers satisfying $\theta > 0$ and $0 < \rho < 1$. Let S and $T \in \mathcal{L}(X, Y)$ be such that*

$$S(\mathrm{B}_1) \supset \mathrm{B}_\theta$$

and

$$|T - S| \leq \rho\theta.$$

Then

$$T(\mathrm{B}_1) \supset \mathrm{B}_{\theta(1-\rho)}$$

Proof. Just apply the preceding theorem with $\alpha = 1$ and $f = T|\mathrm{B}_1$. □

2. The Main Result

In the sequel, following the terminology of [30], we use the following notation: $\mathbb{R}$ are the real numbers, $\mathbb{C}$ the complex numbers and $\mathbb{K}$ denotes both $\mathbb{R}$ and $\mathbb{C}$. Most norms will be denoted by single bars: $|\cdot|$, with indices or without. This will not lead to confusion. If X is a normed space and $r > 0$, then $\mathrm{B}_r(c)$ denotes the open ball in X of radius r centered at c. Moreover, $\mathrm{B}_r := \mathrm{B}_r(0)$. Given normed spaces X and Y over the same field $\mathbb{K}$, we denote by $\mathcal{L}(X,Y)$ (resp. by $\mathcal{L}^p(X^p,Y)$) the space of all bounded $\mathbb{K}$-linear (resp. $\mathbb{K}$-p-linear) operators from X (resp. from X^p) to Y, endowed with the operator norm. Given a Hilbert space H over $\mathbb{K}$, $\mathcal{L}_{\mathrm{sym}}(H,H)$ is the (closed) $\mathbb{R}$-linear subspace of $\mathcal{L}(H,H)$ consisting of all symmetric operators.

By $\mathcal{S}_p$ we denote the (finite dimensional) space of all real symmetric $p \times p$-matrices, endowed with an arbitrary norm. The spectrum of A is denoted by $\operatorname{spec} A$.

Definition 5.2.1. We say that the pair (H, A) is $\mathbb{K}$-*admissible* if and only if the following properties hold:

(1) H is a infinite dimensional Hilbert space over $\mathbb{K}$.
(2) $A\colon \operatorname{dom} A \to H$ is $\mathbb{K}$-linear, symmetric, bounded below and such that for some $\mu \in \mathbb{R}$ the operator $(\mu - A)^{-1} \in \mathcal{L}(H,H)$ exists and is compact.

Note that (H, A) is $\mathbb{K}$-admissible if and only if H is a infinite dimensional Hilbert space over $\mathbb{K}$ and $A\colon \operatorname{dom} A \to H$ is $\mathbb{K}$-linear, self-adjoint, bounded below and with compact resolvent.

If (H, A) is $\mathbb{K}$-admissible then it follows that the spectrum of A is a countable set of real eigenvalues of finite multiplicity. This set is bounded below. We can therefore uniquely define a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of A, each one repeated according to its multiplicity. We call $(\lambda_n)_{n\in\mathbb{N}}$ *the repeated sequence of eigenvalues of* A. A pair (H, A) is $\mathbb{R}$-admissible if and only if $(H_{\mathbb{C}}, A_{\mathbb{C}})$ is $\mathbb{C}$-admissible where $H_{\mathbb{C}}$ is the complexification of H and $A_{\mathbb{C}}$ is the complexification of A. Moreover, in this case the spectra of A and $A_{\mathbb{C}}$ coincide and the real multiplicity of an eigenvalue λ of A is equal to the complex multiplicity of λ as an eigenvalue of $A_{\mathbb{C}}$.

Definition 5.2.2. We say that the triple $(H, \mathcal{G}, A)$ *is of type* $[p, M, \eta, \theta]$ if and only if the following properties hold:

(1) the pair (H, A) is $\mathbb{R}$-admissible and $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(H,H)$.
(2) p is a positive integer, M, η and θ are positive reals.
(3) Let $(\lambda_n)_{n\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of A. There exist real

numbers γ_1 and γ_2 and $l \in \mathbb{N}_0$ such that, setting $\lambda_0 = -\infty$,

$$0 < \gamma_2 - \gamma_1 < M,$$

$$\lambda_l < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1} \leq \lambda_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}.$$

(4) There exists an H-orthonormal set of vectors ϕ_j, $j = 1, \ldots, p$, in $\operatorname{dom} A$ such that $A\phi_j = \lambda_{l+j}\phi_j$, $j = 1, \ldots, p$, and such that the operator $T: \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B\phi_i, \phi_j \rangle)_{ij}$$

is such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta,$$

i.e. the image of the unit ball (at zero) in $\mathcal{G}$ contains the θ-ball (at zero) in $\mathcal{S}_p$.

We can now state the main result of this section:

Theorem 5.2.3. *For every $(p, M, \eta, \theta) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ there exists a positive number $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ with the following property:*

whenever the triple $(H, \mathcal{G}, A)$ is of type $[p, M, \eta, \theta]$, l, γ_1 and γ_2 are as in Definition 5.2.2 (with respect to the triple $(H, \mathcal{G}, A)$), $0 < \alpha \leq \alpha_0$ and $(\mu_1, \ldots, \mu_p) \in \mathbb{R}^p$ is nondecreasing with $|\mu_j - \lambda_{l+j}| < \alpha$ for $j = 1, \ldots, p$, and if $\mathcal{D}$ is an arbitrary linear dense subspace of $\mathcal{G}$, then there exists a $B \in \mathcal{D}$ with $|B| < (1/2)\theta\alpha$, such the pair $(H, A + B)$ is admissible and if $(\lambda_n(B))_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $A + B$ and $\lambda_0(B) := -\infty$, then

$$\lambda_l(B) < \gamma_1 - 3\eta < \gamma_1 - \eta < \lambda_{l+1}(B) \leq \lambda_{l+p}(B) < \gamma_2 + \eta < \gamma_2 + 3\eta < \lambda_{l+p+1}(B) \tag{5.2.1}$$

and

$$\lambda_{l+j}(B) = \mu_j, \quad j = 1, \ldots, p.$$

We shall obtain the proof of Theorem 5.2.3 as a consequence of a series of lemmas.

Lemma 5.2.4.

Let X be Banach space over $\mathbb{K}$ and $A : \operatorname{dom} A \subset X \to X$ be $\mathbb{K}$-linear and closed. Suppose $\zeta \in \mathbb{K}$ is in the resolvent set of A and K is a real number with $|(\zeta - A)^{-1}| \leq K$.

Then for every $B \in \mathcal{L}(X, X)$ the following statements hold:

(1) *The map $A + B: \operatorname{dom} A \to X$ is closed;*

(2) *If* $|B| = |B|_{\mathcal{L}(X,X)} < 1/K$ *then the number* ζ *is in the resolvent set of* $A+B$ *and*

$$(\zeta - A - B)^{-1} = \sum_{n=0}^{\infty} ((\zeta - A)^{-1}B))^n (\zeta - A)^{-1} \tag{5.2.2}$$

with the above series converging in $\mathcal{L}(X,X)$ *and*

$$|(\zeta - A - B)^{-1}| \leq K(B) := K/(1 - K|B|). \tag{5.2.3}$$

(3) *Finally,*

$$(A+B)(\zeta - A - B)^{-1} = \zeta(\zeta - A - B)^{-1} - \mathrm{id}_X. \tag{5.2.4}$$

Proof. All statements are well-known and very easy to prove. □

Lemma 5.2.5. *Let* X *be complex Banach space and* $A : \operatorname{dom} A \subset X \to X$ *be* $\mathbb{C}$*-linear and closed. Let* Γ *be simple counterclockwise oriented closed curve in* $\mathbb{C}$*, parametrized by some piecewise* C^1*-function* $\gamma\colon [0,1] \to \mathbb{C}$*. Suppose* Γ *is disjoint from the spectrum of* A *and* K *is a real number with* $|(\zeta - A)^{-1}| \leq K$ *for all* ζ *lying on* Γ*.*

Then:

(1) *for every* $B \in \mathcal{L}(X,X)$ *with norm* $|B| < 1/K$ *the function* $\zeta \mapsto (\zeta - A - B)^{-1}$ *from* Γ *to* $\mathcal{L}(X,X)$ *is continuous and the map* $P(B) \in \mathcal{L}(X,X)$ *defined by*

$$P(B) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - A - B)^{-1} \, \mathrm{d}\zeta$$

is a well-defined projection operator. Morever, $P(B)$ *maps* X *into* $\operatorname{dom} A$ *and*

$$(A+B)P(B) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \zeta(\zeta - A - B)^{-1} \, \mathrm{d}\zeta. \tag{5.2.5}$$

(2) *For every* $B_1 \in \mathcal{L}(X,X)$ *with operator norm* $|B| < 1/K(B)$ *the following power series representations hold (with* $\mathcal{L}(X,X)$*-convergent series):*

$$P(B+B_1) = \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_\Gamma ((\zeta - A - B)^{-1}B_1))^n (\zeta - A - B)^{-1} \, \mathrm{d}\zeta \tag{5.2.6}$$

$$(A+B+B_1)P(B+B_1) = \sum_{n=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \zeta((\zeta - A - B)^{-1}B_1))^n (\zeta - A - B)^{-1} \, \mathrm{d}\zeta. \tag{5.2.7}$$

(3) *The maps* $F_k\colon \mathrm{B}_{1/K} \subset \mathcal{L}(X,X) \to \mathcal{L}(X,X)$, $k = 1, 2$, *given by*

$$F_1(B) = P(B)$$

$$F_2(B) = (A+B)P(B)$$

are (complex) analytic. The (complex) dimension of the range of $P(B)$ *equals the (complex) dimension of the range of* $P(0)$. *For every number* M_1 *with*

$$(1/2\pi)\max\{ \sup_{t\in[0,1]} |\gamma(t)|, \sup_{t\in[0,1]} |\gamma(t)||\gamma'(t)| \} \le M_1,$$

for every $B \in \mathrm{B}_{1/K}$ *and every nonnegative integer* n:

$$|\mathrm{D}^n F_k(B)| \le M_1 n! (K/(1 - K|B|))^{n+1}, \quad k = 1, 2. \tag{5.2.8}$$

Finally, for every $B \in \mathrm{B}_{1/K}$ *and* $B_1 \in \mathcal{L}(X,X)$

$$\mathrm{D}F_1(B)B_1 = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - A - B)^{-1} B_1 (\zeta - A - B)^{-1}\,\mathrm{d}\zeta \tag{5.2.9}$$

and

$$\mathrm{D}F_2(B)B_1 = (A+B)(\mathrm{D}F_1(B)B_1) + B_1 F_1(B). \tag{5.2.10}$$

Proof. Part (1) follows from Lemma 5.2.4. In particular, formula (5.2.5) follows from (5.2.4) and the closedness of $A + B$. Formulas (5.2.6) and (5.2.7) follow from part (1), formula (5.2.2) (with A replaced by $A + B$ and B replaced by B_1) and the dominated convergence theorem. This proves part (2). The convergent power series expansions (5.2.6) and (5.2.7) together with the estimate (5.2.3) (with A replaced by $A + B$ and B replaced by B_1) show that F_1 and F_2 are complex analytic and that (5.2.8) holds. The constancy of the dimension of $P(B)$ follows from the continuity of F_1. Taking in (5.2.6) and (5.2.7) the summands corresponding to $n = 1$ implies formula (5.2.9) and shows that

$$\mathrm{D}F_2(B)B_1 = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \zeta(\zeta - A - B)^{-1} B_1 (\zeta - A - B)^{-1}\,\mathrm{d}\zeta.$$

Thus, by (5.2.4)

$$\begin{aligned}
&\mathrm{D}F_2(B)B_1 \\
&\quad = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (A+B)(\zeta - A - B)^{-1} B_1 (\zeta - A - B)^{-1}\,\mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_\Gamma B_1 (\zeta - A - B)^{-1}\,\mathrm{d}\zeta \\
&\quad = \frac{1}{2\pi \mathrm{i}} (A+B) \int_\Gamma (\zeta - A - B)^{-1} B_1 (\zeta - A - B)^{-1}\,\mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} B_1 \int_\Gamma (\zeta - A - B)^{-1}\,\mathrm{d}\zeta \\
&\qquad = (A+B)(\mathrm{D}F_1(B)B_1) + B_1 F_1(B).
\end{aligned}$$

This proves (5.2.10). □

Lemma 5.2.6. *Assume the hypotheses of Lemma 5.2.5. In addition, suppose that X is a (complex) Hilbert space ($\langle\cdot,\cdot\rangle$ denoting the corresponding inner product), A is symmetric and ϕ (resp. ψ) is an eigenvector corresponding to an eigenvalue λ (resp. μ) of A lying inside of Γ.*

Then for every $B_1 \in \mathcal{L}(X, X)$

$$\langle(\mathrm{D}F_1(0)B_1)\phi, \psi\rangle = 0.$$

Proof. From (5.2.9) and the symmetry of A we obtain

$$\langle(\mathrm{D}F_1(0)B_1)\phi, \psi\rangle = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \langle B_1(\zeta - A)^{-1}\phi, (\overline{\zeta} - A)^{-1}\psi\rangle\,\mathrm{d}\zeta.$$

Since

$$(\zeta - A)^{-1}\phi = (\zeta - \lambda)^{-1}\phi$$

and

$$(\overline{\zeta} - A)^{-1}\psi = (\overline{\zeta} - \mu)^{-1}\psi$$

it follows (assuming, w.l.o.g. that the inner product in H is linear in the first argument) that

$$\langle(\mathrm{D}F_1(0)B_1)\phi, \psi\rangle = \frac{1}{2\pi\mathrm{i}}\int_\Gamma (\zeta - \lambda)^{-1}(\zeta - \mu)^{-1}\,\mathrm{d}\zeta \cdot \langle B_1\phi, \psi\rangle.$$

Using our hypothesis and the Residue theorem we easily obtain that

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma (\zeta - \lambda)^{-1}(\zeta - \mu)^{-1}\,\mathrm{d}\zeta = 0.$$

This completes the proof. $\square$

Given a $\mathbb{K}$-admissible pair (H, A) and $S \subset \mathbb{R}$ we denote by $\mathrm{m}_{\mathbb{K}}(A, S)$ the total multiplicity of all the eigenvalues of A lying in S.

Lemma 5.2.7. *Let (H, A) be $\mathbb{K}$-admissible and $B \in \mathcal{L}(H, H)$ be symmetric. Then:*

(1) *If $\zeta \in \mathbb{K}$ and $\zeta \notin \operatorname{spec} A$ then*

$$|(\zeta - A)^{-1}| = 1/\operatorname{dist}(\zeta, \operatorname{spec} A).$$

(2) *The pair $(H, A + B)$ is $\mathbb{K}$-admissible.*

(3) *If a, $b \in \mathbb{R}$ and $\delta > 0$ are such that $a < b$, $\operatorname{dist}(a, \operatorname{spec} A) \geq \delta$, $\operatorname{dist}(b, \operatorname{spec} A) \geq \delta$ and $|B| < \delta$ then a, $b \notin \operatorname{spec} A$ and*

$$\mathrm{m}_{\mathbb{K}}(A + B, [a, b]) = \mathrm{m}_{\mathbb{K}}(A, [a, b]).$$

(4) *If* $b \in \mathbb{R}$ *and* $\delta > 0$ *are such that* $\operatorname{dist}(b, \operatorname{spec} A) \geq \delta$ *and* $|B| < \delta$ *then* $b \notin \operatorname{spec} A$ *and*

$$\mathrm{m}_{\mathbb{K}}(A + B,]{-\infty}, b]) = \mathrm{m}_{\mathbb{K}}(A,]{-\infty}, b]).$$

Proof. All the statements are well-known and easy to prove. For instance, in order to prove part (3), we first assume that $\mathbb{K} = \mathbb{C}$. Let Γ be the rectangle with vertices $a \pm \mathrm{i}\delta$ and $b \pm \mathrm{i}\delta$, parametrized in the obvious way so that it is oriented counterclockwise. Then by part (1), $|(\zeta - A)|^{-1} \leq \delta$ for all ζ lying on Γ and so an application of part (3) of Lemma 5.2.5 proves part (3) of the present lemma in the complex case. The real case is dealt with by first complexifying and then using the remarks following Definition 5.2.1. □

Let us note the following obvious consequence of the higher order chain rule for composite mappings:

Lemma 5.2.8. *Assume that* X, Y *and* Z *are normed spaces,* $n \in \mathbb{N}$, $f \colon \operatorname{dom} f \subset X \to Y$ *is* n*-times differentiable at* a, $g \colon \operatorname{dom} g \subset Y \to Z$ *is* n*-times differentiable at* $b = f(a)$, $K \in \mathbb{R}_+$,

$$\sup_{1 \leq k \leq n} |\mathrm{D}^k f(a)| \leq K \text{ and } \sup_{1 \leq k \leq n} |\mathrm{D}^k g(b)| \leq K$$

then

$$\sup_{1 \leq k \leq n} |\mathrm{D}^k (g \circ f)(a)| \leq C(n, K)$$

for some constant $C(n, K)$ *depending only on* n *and* K.

Lemma 5.2.9. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be a Hilbert space over* $\mathbb{K}$. *For every positive integer* p *let*

$$U_p = \{\, (x_1, \ldots, x_p) \in H^p \mid x_1, \ldots, x_p \text{ are } \mathbb{K}\text{-linearly independent} \,\}.$$

Define the Gram-Schmidt operators

$$Q_p \colon U_p \to H^p$$

as follows:

$$Q_1(x_1) = x_1 / |x_1|, \quad x_1 \in U_1 = H \setminus \{0\}$$

$$Q_{p+1}(x_1, \ldots, x_{p+1}) = (y_1, \ldots, y_p, y_{p+1}), \quad (x_1, \ldots, x_{p+1}) \in U_{p+1}$$

where

$$(y_1, \ldots, y_p) = Q_p(x_1, \ldots, x_p)$$

$$y_{p+1} = (x_{p+1} - \sum_{k=1}^{p} \langle x_{p+1}, y_k \rangle y_k) / |x_{p+1} - \sum_{k=1}^{p} \langle x_{p+1}, y_k \rangle y_k|.$$

Then for every p the following statements hold

(1) *The set U_p is open in H^p. (U_p is empty for p larger than the dimension of H.)*

(2) *The map $Q_p: U_p \to H^p$ is well-defined. Let $Q_{p,r}: U_p \to H$, $r = 1, \dots, p$, be the r-th component of Q_p. The vectors $Q_{p,r}(x_1, \dots, x_p)$, $r = 1, \dots, p$, are orthonormal. There are uniquely determined functions $a_{p,rm}: U_p \to \mathbb{K}$, $r = 1, \dots, p$, $m = 1, \dots, r$, such that*

$$Q_{p,r}(x_1, \dots, x_p) = \sum_{m=1}^{r} a_{p,rm}(x_1, \dots, x_p) x_m.$$

The functions $a_{p,rm}$ (and so $Q_{p,r}$) depend only on the first r vectors $(x_1, \dots, x_r)$. Both Q_p and all the functions $a_{p,rm}$ are real analytic on U_p. Furthermore, there are constants $\epsilon(p)$ and $C(p,n)$ such that for every $n \in \mathbb{N}_0$, every orthonormal p-tuple $(e_1, \dots, e_p) \in U_p$ and every $(x_1, \dots, x_p) \in H^p$ such that $|x_k - e_k| < \epsilon(p)$ for $k = 1, \dots, p$ it follows that $(x_1, \dots, x_p) \in U_p$ and

$$|\mathrm{D}^n Q_p(x_1, \dots, x_p)| \leq C(p,n),$$

$$|\mathrm{D}^n a_{p,rm}(x_1, \dots, x_p)| \leq C(p,n), \qquad r = 1, \dots, p, \ m = 1, \dots, r.$$

Proof. A simple argument proves that U_p is open in H^p.

The map

$$\beta: x \in H \setminus \{0\} \mapsto 1/|x| = (\langle x, x \rangle)^{-1/2}$$

is the composite of the map $\alpha_1: x \mapsto \langle x, x \rangle$ and the real function $\alpha_2: s \in \mathbb{R}_+ \mapsto s^{-1/2}$. The function α_2 is real analytic and since the inner product of H is real bilinear and bounded it follows that α_1 is real analytic as well. Thus β is real analytic. Moreover, calculating the derivatives of α_1 and α_2 and using Lemma 5.2.8 we obtain that for every $n \in \mathbb{N}$ and a, $b \in \mathbb{R}_+$

$$|\mathrm{D}^n \beta(x)| \leq C_1(n,a,b) \quad \text{whenever } a < |x| < b \tag{5.2.11}$$

for some constant $C_1(n,a,b)$ depending only on n, a and b. These remarks imply all assertions of part (2) for $p = 1$. (Take $\epsilon(1) = 1/2$.)

For induction, suppose part (2) of the lemma holds for some p. If $(x_1, \dots, x_{p+1}) \in U_{p+1}$ then $(x_1, \dots, x_p) \in U_p$ so the map

$$\alpha: (x_1, \dots, x_{p+1}) \in U_{p+1} \mapsto x_{p+1} - \sum_{k=1}^{p} \langle x_{p+1}, Q_{p,k}(x_1, \dots, x_p) \rangle Q_{p,k}(x_1, \dots, x_p) \in H$$

is well-defined and real analytic. Since the vectors $y_k = Q_{p,k}(x_1, \dots, x_p)$, $k = 1, \dots, p$ are linear combinations of the vectors $x_1, \dots, x_p$ with real analytic coefficients it follows that $\alpha(x_1, \dots, x_{p+1}) \neq 0$ so $Q_{p+1,p+1}(x_1, \dots, x_{p+1})$ is well-defined and

$$Q_{p+1,p+1} = \alpha \cdot (\beta \circ \alpha). \tag{5.2.12}$$

This shows that Q_{p+1} is well-defined, real analytic and the vectors

$$y_r = Q_{p+1,r}(x_1, \dots, x_{p+1}) \quad r = 1, \dots, p+1,$$

are orthonormal and that they are linear combinations of the vectors $x_1, \dots, x_{p+1}$ with coefficients $a_{p+1,rm}(x_1, \dots, x_{p+1})$ given by:

$$a_{p+1,rm}(x_1, \dots, x_{p+1}) = a_{p,rm}(x_1, \dots, x_p) \quad r \leq p,\ m \leq r,$$

$$\begin{aligned} &a_{p+1,p+1,m}(x_1, \dots, x_{p+1}) \\ &\quad = -(\beta \circ \alpha)(x_1, \dots, x_{p+1}) \sum_{k=m}^{p} \langle x_{p+1}, Q_{p,k}(x_1, \dots, x_p) \rangle a_{p,km}(x_1, \dots, x_p) \quad m \leq p, \end{aligned}$$

and

$$a_{p+1,p+1,p+1}(x_1, \dots, x_{p+1}) = (\beta \circ \alpha)(x_1, \dots, x_{p+1}).$$

It follows that these coefficients are real analytic functions on U_{p+1}. Now let e_k, $k = 1, \dots, p+1$, be orthonormal. Let $(x_1, \dots, x_{p+1}) \in H^{p+1}$ be such that $|x_k - e_k| < \epsilon(p+1)$ for $k = 1, \dots, p+1$, with $\epsilon(p+1) < \epsilon(p)$ to be determined later. Since $Q_{p,k}(e_1, \dots, e_p) = e_k$ it follows from the mean-value theorem and the induction hypothesis that

$$|Q_{p,k}(x_1, \dots, x_p) - e_k| \leq C(p,1) \left(\sum_{j=1}^{p} |x_j - e_j|^2 \right)^{1/2} < \sqrt{p} C(p,1) \epsilon(p+1),\ \ k = 1, \dots, p.$$

Using this estimate and setting $z_k = Q_{p,k}(x_1, \dots, x_p) - e_k$, $k = 1, \dots, p$, and $z_{p+1} = x_{p+1} - e_{p+1}$ we obtain

$$\alpha(x_1, \dots, x_{p+1}) = e_{p+1} + z_{p+1} - \sum_{k=1}^{p} t_k Q_{p,k}(x_1, \dots, x_p)$$

where

$$t_k = \langle z_{p+1}, e_k \rangle + \langle e_{p+1}, z_k \rangle + \langle z_{p+1}, z_k \rangle.$$

Thus

$$\alpha(x_1, \dots, x_{p+1}) = e_{p+1} + w$$

where

$$|w| \le \epsilon(p+1) + p(1 + \sqrt{p}C(p,1)\epsilon(p+1))(2\epsilon(p+1) + (\epsilon(p+1))^2)$$

Now choose $\epsilon(p+1) < \epsilon(p)$ such that

$$\epsilon(p+1) + p(1 + \sqrt{p}C(p,1)\epsilon(p+1))(2\epsilon(p+1) + (\epsilon(p+1))^2) < 1/2.$$

Then for all $(x_1, \dots, x_{p+1}) \in H^{p+1}$ such that $|x_k - e_k| < \epsilon(p+1)$ for $k = 1, \dots, p+1$, it follows that

$$1/2 < |\alpha(x_1, \dots, x_{p+1})| < 1 + 1/2.$$

Thus (5.2.12), (5.2.11) and Lemma 5.2.8 together with the induction assumption imply that

$$|\mathrm{D}^n Q_{p+1}(x_1, \dots, x_{p+1})| \le C(p+1, n)$$

for some constants $C(p+1,n)$. By the above formulas for the coefficients $a_{p+1,rm}$ we can choose the constants $C(p+1,n)$ such that

$$|\mathrm{D}^n a_{p+1,rm}(x_1, \dots, x_{p+1})| \le C(p+1,n), \quad r = 1, \dots, p+1,\ m = 1, \dots, r.$$

The lemma is proved. $\square$

Lemma 5.2.10. *Assume the hypotheses of Lemma 5.2.5. Besides assume that X is a (complex) Hilbert space. Set*

$$c(p,n,M_1,K) = \min\{1/(2K), \epsilon(p)/(4M_1K^2)^{-1}\}.$$

Let $(\phi_1, \dots, \phi_p)$ be an orthonormal p-tuple in X such that

$$P(0)\phi_j = \phi_j, \quad j = 1, \dots, p.$$

Define the maps $\alpha_j\colon \mathrm{B}_{c(p,n,M_1,K)} \subset \mathcal{L}(X,X) \to X$, $\beta_j\colon \mathrm{B}_{c(p,n,M_1,K)} \subset \mathcal{L}(X,X) \to X$, $j = 1, \dots, p$, as

$$\alpha_j(B) = Q_{p,j}(P(B)\phi_1, \dots, P(B)\phi_p)$$

and

$$\beta_j(B) = (A+B)Q_{p,j}(P(B)\phi_1, \dots, P(B)\phi_p) = (A+B)\alpha_j(B).$$

Moreover, define the map $g\colon \mathrm{B}_{c(p,n,M_1,K)} \subset \mathcal{L}(X,X) \to \mathbb{C}(p)$ *as*

$$g_{jk}(B) = \langle \beta_j(B), \alpha_k(B) \rangle, \quad j,\, k = 1,\, \dots\, ,\, p.$$

Then:

(1) α_j, β_j *and* g *are well-defined real analytic maps.*

$$|\mathrm{D}^n \alpha_j(B)| \le C(p,n,M_1,K), \quad |\mathrm{D}^n \beta_j(B)| \le C(p,n,M_1,K) \text{ and}$$
$$|\mathrm{D}^n g(B)| \le C(p,n,M_1,K), \quad B \in \mathrm{B}_{c(p,n,M_1,K)},\, n \in \mathbb{N} \tag{5.2.13}$$

for some constants $C = C(p,n,M_1,K)$ *depending only on* (p,n,M_1,K)*. Furthermore,*

$$|g(B_1) - g(B_2) - \mathrm{D}g(0)(B_1 - B_2)| \le C(p,2,M_1,K)\max\{|B_1|,|B_2|\}|B_1 - B_2|,$$
$$B_1,\, B_2 \in \mathrm{B}_{c(p,n,M_1,K)} \tag{5.2.14}$$

and

$$\mathrm{D}\beta_j(B)B_1 = B_1\alpha_j(B) + (A+B)(\mathrm{D}\alpha_j(B)B_1), \quad B \in \mathrm{B}_{c(p,n,M_1,K)},\, B_1 \in \mathcal{L}(X,X). \tag{5.2.15}$$

(2) *If, in addition,* A *is symmetric and there are numbers* λ_k, $k = 1,\, \dots\, ,\, p$ *inside of* Γ *such that*

$$A\phi_k = \lambda_k \phi_k, \quad k = 1,\, \dots\, ,\, p$$

then

$$\langle \mathrm{D}\alpha_j(0)B_1, \phi_k \rangle = 0, \quad B_1 \in \mathcal{L}(X,X),\, k = 1,\, \dots\, ,\, p \tag{5.2.16}$$

and

$$\mathrm{D}g_{jk}(0)B_1 = \langle B_1\phi_j, \phi_k \rangle, \quad B_1 \in \mathcal{L}(X,X),\, j,\, k = 1,\, \dots\, ,\, p \tag{5.2.17}$$

Proof. If $|B| < c(p,n,M_1,K)$ then by Lemma 5.2.5, and the mean-value theorem

$$\begin{aligned} |P(B)\phi_j - \phi_j| = |(F_1(B) - F_1(0))\phi_j| &\le \sup_{\tau \in [0,1]} |\mathrm{D}F_1(\tau B)||B| \\ &\le M_1(K/(1 - K|B|))^2|B| < 4M_1K^2|B| \le \epsilon(p). \end{aligned}$$

Therefore, by Lemma 5.2.9, $\alpha_j(B)$ and $\beta_j(B)$ are defined and

$$\alpha_j(B) = \sum_{m=1}^{j} a_{p,jm}(P(B)\phi_1, \dots, P(B)\phi_p)F_1(B)\phi_m$$

and

$$\beta_j(B) = \sum_{m=1}^{j} a_{p,jm}(P(B)\phi_1, \dots, P(B)\phi_p)F_2(B)\phi_m$$

These formulas together with Lemmas 5.2.5 and 5.2.9 prove part (1) of the present lemma. In particular, (5.2.14) follows from the third estimate in (5.2.13) by twice applying the mean-value theorem to the map g. Moreover, (5.2.15) follows from (5.2.10).

Let us now prove (5.2.16) by induction on j, $j = 1, \dots, p$. Suppose that $1 \le j \le p$ and (5.2.16) (with j replaced by m) holds for all $m \le j-1$. Let β be as in the proof of Lemma 5.2.9. Then

$$\alpha_j(B) = (\beta \circ \xi)(B) \cdot \xi(B),$$

where

$$\xi(B) := F_1(B)\phi_j - \sum_{m=1}^{j-1} \langle F_1(B)\phi_j, \alpha_m(B)\rangle \alpha_m(B).$$

(Here, as usual $\sum_{k=1}^{0} a_k = 0$.) It is clear that $\xi(0) = \phi_j$ so, by a straightforward calculation,

$$\begin{aligned} \langle \mathrm{D}\alpha_j(0)B_1, \phi_k\rangle &= \beta(\xi(0))\langle \mathrm{D}\xi(0)B_1, \phi_k\rangle + \mathrm{D}(\beta\circ\xi)(0)B_1\langle \xi(0), \phi_k\rangle \\ &= \langle \mathrm{D}\xi(0)B_1, \phi_k\rangle + \delta_{jk}\mathrm{D}(\beta\circ\xi)(0)B_1 \\ &= \langle \mathrm{D}\xi(0)B_1, \phi_k\rangle - \delta_{jk}\,\mathrm{Re}\langle \mathrm{D}\xi(0)B_1, \phi_j\rangle. \end{aligned} \tag{5.2.18}$$

The definition of ξ implies that

$$\begin{aligned} \mathrm{D}\xi(0)B_1 = \mathrm{D}F_1(0)B_1\phi_j - \sum_{m=1}^{j-1} \big((\langle \mathrm{D}F_1(0)B_1\phi_j, \alpha_m(0)\rangle + \langle F_1(0)\phi_j, \mathrm{D}\alpha_m(0)B_1\rangle)\alpha_m(0) \\ + \langle F_1(0)\phi_j, \alpha_m(0)\rangle \mathrm{D}\alpha_m(0)B_1\big). \end{aligned}$$

Now

$$\langle \mathrm{D}F_1(0)B_1\phi_j, \alpha_m(0)\rangle = \langle \mathrm{D}F_1(0)B_1\phi_j, \phi_m\rangle = 0$$

by Lemma 5.2.6,

$$\langle F_1(0)\phi_j, \mathrm{D}\alpha_m(0)B_1\rangle = \langle \phi_j, \mathrm{D}\alpha_m(0)B_1\rangle = 0$$

by the induction hypothesis, and

$$\langle F_1(0)\phi_j, \alpha_m(0)\rangle = \langle \phi_j, \phi_m\rangle = 0$$

by orthonormality.

Consequently,

$$\mathrm{D}\xi(0)B_1 = \mathrm{D}F_1(0)B_1\phi_j$$

so

$$\langle \mathrm{D}\xi(0)B_1, \phi_k\rangle = \langle \mathrm{D}F_1(0)B_1\phi_j, \phi_k\rangle = 0$$

by Lemma 5.2.6. Therefore, by (5.2.18)

$$\langle \mathrm{D}\alpha_j(0)B_1, \phi_k\rangle = 0.$$

This proves (5.2.16). Using our hypotheses on A, (5.2.15) and (5.2.16) we now easily obtain (5.2.17). The proof is complete. □

Lemma 5.2.11. *Let H be a Hilbert space, Y a normed space of finite dimension q, $T \in \mathcal{L}(H, Y)$ and θ, δ be arbitrary positive numbers with $\delta < \theta$. Suppose that*

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

Then for every linear dense subspace $\mathcal{D}$ of H there is q-dimensional linear subspace $\mathcal{D}_q$ of $\mathcal{D}$ such that

$$T(\mathrm{B}_1 \cap \mathcal{D}_q) \supset \mathrm{B}_{\theta-\delta}.$$

Proof. Let X be the orthogonal complement of the kernel of T. Since orthogonal projections have norm 1, it follows that

$$T(\mathrm{B}_1 \cap X) \supset \mathrm{B}_\theta.$$

Letting L denote the inverse of $T|X$ we see that $L \in \mathcal{L}(Y, H)$ and

$$|L|_{\mathcal{L}(Y,H)} \leq 1/\theta < 1/(\theta - \delta). \tag{5.2.19}$$

Let x_k, $k = 1, \dots, q$, be a basis of H_1 and set $y_k = Tx_k$, $k = 1, \dots, q$. Then y_k, $k = 1, \dots, q$, is a basis of Y. There are sequences $(x_{kn})_{n\in N} \in \mathcal{D}$, $k = 1, \dots, q$, such that

$$x_{kn} \to x_k, \quad k = 1, \dots, q.$$

Setting $y_{kn} = Tx_{kn}$, $n \in \mathbb{N}$, $k = 1, \dots, q$, we see that

$$y_{kn} \to y_k, \quad k = 1, \dots, q.$$

Hence for all n large enough, the vectors x_{kn}, $k = 1, \ldots, q$, are linearly independent in H and the vectors y_{kn}, $k = 1, \ldots, q$, are a basis of Y. Thus there are uniquely determined scalars b_{lkn}, $n \in \mathbb{N}$, k, $l = 1, \ldots, q$, such that

$$y_k = \sum_{l=1}^{q} b_{lkn} y_{ln}, \quad k = 1, \ldots, q.$$

The sequence of matrices $(b_{lkn})_{lk}$ converges to the identity matrix since its inverse obviously does. Define X_n to be the subspace of H spanned by x_{kn}, $k = 1, \ldots, q$. Then $T|X_n$ has an inverse L_n given by

$$L_n(y_{kn}) = x_{kn}, \quad k = 1, \ldots, q.$$

Then $L_n \in \mathcal{L}(X, Y)$ and

$$L_n y_k = \sum_{l=1}^{q} b_{lkn} L y_{ln} = \sum_{l=1}^{q} b_{nlk} x_{ln} \to x_k = L y_k, \text{ as } n \to \infty.$$

for $k = 1, \ldots, q$. Thus $|L_n - L| \to 0$ so by (5.2.19)

$$|L_n| < 1/(\theta - \delta) \quad \text{for } n \text{ large enough.}$$

Choosing such a n and defining $\mathcal{D}_q = X_n$ we conclude the proof. $\square$

We can finally give a

Proof of Theorem 5.2.3. Assume that the triple $(H, \mathcal{G}, A)$ is of type $[p, M, \eta, \theta]$. Let $(H_{\mathbb{C}}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$ be the complexification of $(H, \langle\cdot,\cdot\rangle)$. We use the symbol $\langle\cdot,\cdot\rangle$ to denote both the inner product in H and its extension to $H_{\mathbb{C}}$. Let $l \in \mathbb{N}_0$, γ_1 and γ_2 be as in the Definition 5.2.2 with respect to the triple $(H, \mathcal{G}, A)$. Suppose $|B| < \eta$. Applying Lemma 5.2.7, part (4) with $b = \gamma_1 - 3\eta$ and $b = \gamma_1 + 3\eta$ respectively, we obtain that

$$\lambda_l(B) < \gamma_1 - 3\eta \quad \text{ and } \gamma_1 + 3\eta < \lambda_{l+p+1}(B).$$

Applying part (3) of this lemma with $a = \gamma_1 - \eta$ and $b = \gamma_1 + \eta$ we see that

$$\mathrm{m}_{\mathbb{R}}(A + B,]\gamma_1 - \eta, \gamma_1 + \eta[) = \mathrm{m}_{\mathbb{R}}(A + B, [\gamma_1 - \eta, \gamma_1 + \eta]) = p.$$

This immediately implies that (5.2.1) holds for all B with $|B| < \eta$. Assume first that

$$\gamma_1 - 2\eta = 0. \tag{5.2.20}$$

Let Γ be the rectangle with the vertices $\pm \mathrm{i}2\eta$ and $\gamma_2 + 2\eta \pm \mathrm{i}2\eta$. Noting that

$$\gamma_2 + 2\eta = \gamma_2 - \gamma_1 + 4\eta$$

we easily see that there is a constant $M_1(M, \eta)$ depending only on (M, η) and a parametrization γ of Γ such that

$$(1/2\pi) \max\{ \sup_{t\in[0,1]} |\gamma(t)|, \sup_{t\in[0,1]} |\gamma(t)||\gamma'(t)| \} \leq M_1(M, \eta).$$

Using the notation of Lemma 5.2.10 define

$$\alpha_0(p, M, \eta, \theta) = \min\{ c(p, 2, M_1, K), \theta/(4C(p, 2, M_1, K) \}$$

with $M_1 = M_1(M, \eta)$, $K = 2\eta$. Then using Lemma 5.2.7 part (1) we see that all hypotheses of Lemma 5.2.10 hold. For $0 < \alpha \leq \alpha_0(p, M, \eta, \theta)$ define the map $f\colon \mathrm{B}_\alpha \subset \mathcal{G} \to \mathcal{S}_p$ by

$$f(B) = g(B_{\mathbb{C}})$$

where g is as in Lemma 5.2.10 and $B_{\mathbb{C}}$ is the complexification of B. To show that f is well-defined we must prove that $g_{jk} \in \mathbb{R}$ for all j, $k = 1, \ldots, p$. Now the curve Γ is symmetric with respect to the real axis, so for every $\phi \in H$ it follows that $P(B_{\mathbb{C}})\phi \in H$. Therefore by Lemma 5.2.9

$$\alpha_j(B_{\mathbb{C}}) = Q_{p,j}(P(B_{\mathbb{C}})\phi_1, \ldots, P(B_{\mathbb{C}})\phi_p) \in H$$

and so

$$\begin{aligned}\beta_j(B_{\mathbb{C}}) = (A_{\mathbb{C}} + B_{\mathbb{C}})&Q_{p,j}(P(B_{\mathbb{C}})\phi_1, \ldots, P(B_{\mathbb{C}})\phi_p) \\ &= (A + B)Q_{p,j}(P(B_{\mathbb{C}})\phi_1, \ldots, P(B_{\mathbb{C}})\phi_p) \in H.\end{aligned}$$

Since H is a real Hilbert space it therefore follows that

$$g_{jk}(B) = \langle \beta_j(B), \alpha_k(B) \rangle \in \mathbb{R}, \quad j,\, k = 1, \ldots, p.$$

Therefore, indeed, $f(B) \in \mathcal{S}_k$ and so f is well-defined as a map into $\mathcal{S}_k$. We also have, in view of (5.2.17), that

$$T(B) = \mathrm{D}g(0)B_{\mathbb{C}}, \quad B \in \mathcal{G}$$

where T is as in Definition 5.2.2 with respect to the triple $(H, \mathcal{G}, A)$. Moreover, given any linear dense subspace $\mathcal{D}$ of $\mathcal{G}$ we obtain, in view of Lemma 5.2.11, a finite dimensional linear subspace $\tilde{\mathcal{D}}$ of $\mathcal{D}$ such that

$$T(\mathrm{B}_1 \cap \tilde{\mathcal{D}}) \supset \mathrm{B}_{\theta/2}.$$

Finally, we have for $B_1,\ B_2 \in \mathrm{B}_\alpha \subset \mathcal{G}$,

$$|f(B_1) - f(B_2) - T(B_1 - B_2)| = |g(B_{1\mathbb{C}}) - g(B_{2\mathbb{C}}) - \mathrm{D}g(0)(B_{1\mathbb{C}} - B_{2\mathbb{C}})|$$
$$\leq C(p, 2, M_1, K) \max\{|B_{1\mathbb{C}}|, |B_{2\mathbb{C}}|\}|B_{1\mathbb{C}} - B_{2\mathbb{C}}| = C(p, 2, M_1, K) \max\{|B_1|, |B_2|\}|B_1 - B_2|$$
$$\leq (1/2)(\theta/2)|B_1 - B_2|.$$

Now an application of Theorem 5.1.1 implies that whenever $(\mu_1, \dots, \mu_p) \in \mathbb{R}^p$ is non-decreasing and such that $|\mu_j - \lambda_{l+j}| < \alpha$ for $j = 1, \dots, p$, then there exists a $B \in \tilde{\mathcal{D}} \subset \mathcal{D}$ with $|B| < (1/2)\theta\alpha$ such that

$$f(B) = \mathrm{diag}(\mu_1, \dots, \mu_p).$$

Since

$$\langle \alpha_j(B_\mathbb{C}), \alpha_k(B_\mathbb{C})\rangle = \delta_{jk}, \quad j,\ k = 1, \dots, p,$$

it follows that

$$f_{jk}(B) = \mu_j \langle \alpha_j(B_\mathbb{C}), \alpha_k(B_\mathbb{C})\rangle, \quad j,\ k = 1, \dots, p,$$

or

$$\langle (A + B)\alpha_j(B_\mathbb{C}) - \mu_j \alpha_j(B_\mathbb{C}), \alpha_k(B_\mathbb{C})\rangle = 0, \quad j,\ k = 1, \dots, p.$$

Since the the vectors $\alpha_j(B_\mathbb{C})$, $j = 1, \dots, p$, are a basis of the range of $P(B_\mathbb{C})$ and this range is mapped by $A_\mathbb{C} + B_\mathbb{C}$ into itself it follows that

$$(A + B)\alpha_j(B_\mathbb{C}) = \mu_j \alpha_j(B_\mathbb{C}), \quad j = 1, \dots, p.$$

Since $\alpha \leq \eta$, the inequalities (5.2.1) (proved above) imply that

$$\lambda_{l+j}(B) = \mu_j, \quad j = 1, \dots, p.$$

Therefore the theorem is proved if $\gamma_1 - 2\eta = 0$. In the general case replace the operator A by the operator $A - (\gamma_1 - 2\eta)$ and apply the special case of the theorem just proved. The proof is complete. $\square$

AN EIGENVALUE CONVERGENCE RESULT

Let Ω and D be bounded domains in $\mathbb{R}^N$ with $\overline{D} \subset \Omega$. Consider a strongly elliptic second order differential operator L on Ω. Define the following sequence of differential operators on Ω:

$$\begin{aligned} L_k u &= Lu + \beta_k b_k(x), \quad && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned}$$

or

$$\begin{aligned} L_k u &= Lu + \beta_k b_k(x), \quad && x \in \Omega \\ \frac{\partial u}{\partial \nu}(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Here L is a second order symmetric strongly elliptic differential operator, β_k, $k \in \mathbb{N}$, are positive real numbers and b_k, $k \in \mathbb{N}$, are (coefficient) functions. We shall prove in Theorem 6.2.2 that under appropriate hypotheses on β_k and b_k in both cases the eigenvalues of L_k converge, as $k \to \infty$, to the eigenvalues of the following 'limit' differential operator L_∞ on D:

$$\begin{aligned} L_\infty u &= Lu, \quad && x \in D \\ u(x) &= 0, && x \in \partial D. \end{aligned}$$

We also obtain H^1-convergence of the corresponding eigenfunctions. Our hypotheses are, essentially, that $\beta_k b_k(x)$ is very small on D but very large outside of D. Since Theorem 6.2.2 holds for both Dirichlet and Neumann boudary condition, and even for general mixed boundary conditions on $\partial\Omega$, provided the corresponding operators are self-adjoint, it is more convenient to work not with differential operators but rather with the corresponding bilinear forms or even with certain abstract bilinear forms as we shall now explain.

1. Symmetric Bilinear Forms and Their Variational Properties

In this chapter, all vector spaces are over the reals.

Definition 6.1.1. Let V be a vector space and $a : V \times V \to \mathbb{R}$ be symmetric bilinear form on V. If $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$ satisfy

$$a(u, v) = \lambda \langle u, v \rangle \quad \text{for all } v \in V$$

then we say that λ is *a proper value of* a and u is *a proper vector of* a, *corresponding to* λ. The dimension of the span of all proper vectors of a corresponding to λ is called *the multiplicity of* λ. If the set of proper values of a is countably infinite and if each proper value has finite multiplicity then *the repeated sequence of the proper values of* a is the uniquely determined nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the proper values of a and the number of occurrences of each proper value in this sequence is equal to its multiplicity.

The following result is well-known and is stated here for easy reference.

Proposition 6.1.2. *Let V, H be two infinite dimensional Hilbert spaces. Suppose $V \subset H$ with compact inclusion, and V is dense in H. Let $\|\cdot\|$ and $|\cdot|$ denote the norms of V and H respectively, and $\langle\cdot,\cdot\rangle$ denote the inner product of H. Let $a : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V. Assume that there are constants d, C, $\alpha \in \mathbb{R}$, $\alpha > 0$, such that, for all u, $v \in V$,*

$$|a(u,v)| \leq C\|u\|\|v\|$$
$$a(u,u) \geq \alpha\|u\|^2 - d|u|^2.$$

Then the set of proper values of a is countably infinite and each proper value has finite multiplicity. Let $(\lambda_n)_{n\in\mathbb{N}}$ be the repeated sequence of the proper values of a.

(1) *There exists an H-orthogonal sequence $(u_n)_{n\in\mathbb{N}}$ of corresponding proper vectors.*
(2) *Whenever $(u_n)_{n\in\mathbb{N}}$ is an H-orthogonal sequence such that for every $n \in \mathbb{N}$, u_n is a proper vector of a corresponding to λ_n, then $(u_n)_{n\in\mathbb{N}}$ is H-complete and for every $k \in \mathbb{N}$:*

$$\lambda_k = a(u_k,u_k)/|u_k|^2 = \min\{\, a(u,u)/|u|^2 \mid u \in V \setminus \{0\},\ \langle u, u_j\rangle = 0,\ j = 1, \ldots, k-1\}.$$

(3) *Whenever $(\mu_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of real numbers and $(v_n)_{n\in\mathbb{N}}$ is an H-complete and H-orthogonal sequence such that for every $n \in \mathbb{N}$, μ_n is a proper value of a and u_n is a proper vector of a corresponding to μ_n, then $\mu_n = \lambda_n$ for every $n \in \mathbb{N}$.*
(4) *Let $k \in \mathbb{N}$ and for $j \in \mathbb{N}$ with $j < k$ let μ_j be a proper value of a and v_j be a corresponding proper vector. Then there exists $v_k \in V \setminus \{0\}$ such that*

$$\langle v_k, v_j\rangle = 0, \qquad \text{for } j = 1, \ldots, k-1 \tag{6.1.1}$$

and

$$a(v_k,v_k)/|v_k|^2 = \min\{\, a(u,u)/|u|^2 \mid u \in V \setminus \{0\},\ \langle u, v_j\rangle = 0,\ j = 1, \ldots, k-1\}. \tag{6.1.2}$$

Whenever $v_k \in V \setminus \{0\}$ satisfies (6.1.1) and (6.1.2) *then $\mu_k := a(v_k,v_k)/|v_k|^2$ is a proper value of a and v_k is a corresponding proper vector.*

(5) *Whenever* $(\mu_n)_{n\in\mathbb{N}}$ *is a sequence of real numbers and* $(v_n)_{n\in\mathbb{N}}$ *is an* H*-orthogonal sequence in* $V\setminus\{0\}$ *such that for every* $k\in\mathbb{N}$:

$$\mu_k = a(v_k, v_k)/|v_k| = \min\{\, a(u,u)/|u|^2 \mid u\in V\setminus\{0\},\ \langle u, v_j\rangle = 0,\ j=1,\ \dots,\ k-1\},$$

then for every $n\in\mathbb{N}$, $\mu_n=\lambda_n$ *and* v_n *is a proper vector of* a *corresponding to* μ_n.

□

2. The Main Result

We begin by recalling the the following well known fact:

Lemma 6.2.1. *Let* D *be a bounded Lipschitz domain in* $\mathbb{R}^N$ *and* $u\in H^1(\mathbb{R}^N)$ *such that* $u=0$ *a.e. in* $\mathbb{R}^N\setminus D$. *Then* $u|D\in H_0^1(D)$.

Proof. Although this result is known and follows from trace theory, it is not easy to come by a proof in the literature. The quickest way is to follow the proof of the existence of traces of functions in $H^1(D)$ (see e.g. the proof of [1, A 5.7, pp. 190-192]) to show that under the present assumptions $u|\partial D=0$. This implies the lemma since elements of $H_0^1(D)$ are exactly those elements of $H^1(D)$ whose trace is zero (see e.g. [1, A 5.11, p. 196]).

We can state the main result of this chapter:

Theorem 6.2.2. *Assume the following hypotheses:*

(1) $\Omega\subset\mathbb{R}^N$ *is a bounded domain and* $D\subset\mathbb{R}^N$ *is a Lipschitz domain with* $\overline{D}\subset\Omega$. *Given a function* u *defined on* D, $u^\sim$ *denotes the trivial extension of* u *to* Ω.

(2) b, $b_k\colon\overline{\Omega}\to\mathbb{R}$, $k\in\mathbb{N}$, *are continuous functions and* β_k, $k\in\mathbb{N}$ *are positive real numbers. Moreover,* $b(x)>0$ *for* $x\in\Omega\setminus D$, $b_k\to b$ *uniformly on* $\overline{\Omega}$, $\beta_k\to\infty$, $\inf_{\substack{x\in\Omega\\ k\in\mathbb{N}}}\{\beta_k b_k(x)\}>-\infty$ *and* $\sup_{x\in D}\{\beta_k|b_k(x)|\}\to 0$.

(3) V *is a closed linear subspace of* $H^1(\Omega)$ *such that whenever* $u\in H_0^1(D)$ *then* $u^\sim\in V$. V *is endowed with the scalar product of* $H^1(\Omega)$.

(4) $\|\cdot\|_D$ *(resp.* $\|\cdot\|$*) denotes the* $H^1(D)$*- (resp. the* $H^1(\Omega)$*-) norm,* $|\cdot|_D$ *(resp.* $|\cdot|$*) denotes the* $L^2(D)$*- (resp. the* $L^2(\Omega)$*-) norm and* $\langle\cdot,\cdot\rangle_D$ *(resp.* $\langle\cdot,\cdot\rangle$*) denotes the* $L^2(D)$*- (resp. the* $L^2(\Omega)$*-) scalar product.*

(5) $a\colon V\times V\to\mathbb{R}$ *is a symmetric bilinear form and there are constants* d, C, $\alpha\in\mathbb{R}$, $\alpha>0$, *such that, for all* u, $v\in V$,

$$|a(u,v)|\le C\|u\|\|v\|$$
$$a(u,u)\ge\alpha\|u\|^2-d|u|^2.$$

Let $a_\infty: H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ be the restriction of a to $H_0^1(D)$. For $k \in \mathbb{N}$ let $(\lambda_n^k)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of the symmetric bilinear form $a_k: V \times V \to \mathbb{R}$ defined by

$$a_k(u,v) = a(u,v) + \beta_k \int_\Omega b_k(x)u(x)v(x)\,\mathrm{d}x$$

and $(u_n^k)_{n\in\mathbb{N}}$ be an $L^2(\Omega)$-orthonormal sequence of corresponding proper vectors of a_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of a_∞.

Then there is an increasing function $\phi: \mathbb{N} \to \mathbb{N}$ and a sequence $(v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, v_n is a proper vector of a_∞ corresponding to μ_n, the subsequence $(\lambda_n^{\phi(k)})_{k\in\mathbb{N}}$ of $(\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and the subsequence $(u_n^{\phi(k)})_{k\in\mathbb{N}}$ of $(u_n^k)_{k\in\mathbb{N}}$ converges to $v_n{}^\sim$ in V, as $k \to \infty$.

Remark. The forms a_k, $k \in \mathbb{N}$ and a_∞ obviously satisfy the hypotheses of Proposition 6.1.2 so the sequences $(\lambda_n^k)_{n\in\mathbb{N}}$, $k \in \mathbb{N}$ and $(\mu_n)_{n\in\mathbb{N}}$ are well-defined. Moreover, note that whenever the sequences $(b_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ satisfy the hypotheses of Theorem 6.2.2 then so do the subsequences $(b_{\phi(k)})_{k\in\mathbb{N}}$ and $(\beta_{\phi(k)})_{k\in\mathbb{N}}$, where $\phi: \mathbb{N} \to \mathbb{N}$ is an arbitrary increasing function.

Corollary 6.2.3. *Under the assumptions of Theorem 6.2.2 for every $n \in \mathbb{N}$, $\lambda_n^k \to \mu_n$ as $k \to \infty$.*

Proof of the corollary. Suppose the corollary does not hold. Then, using the remark following Theorem 6.2.2 and passing to a subsequence if necessary, we may assume that there is an $m \in \mathbb{N}$ and an $\epsilon > 0$ such that

$$|\lambda_m^k - \mu_m| \geq \epsilon, \quad \text{for all } k \in \mathbb{N}.$$

But then no subsequence of $(\lambda_m^k)_{k\in\mathbb{N}}$ can converge to μ_m, a contradiction to the statement of Theorem 6.2.2. □

Proof of Theorem 6.2.2. Using the remark following Theorem 6.2.2 together with induction and Cantor diagonal procedure, we easily see that Theorem 6.2.2 follows from Lemma 6.2.4 below. □

Lemma 6.2.4. *Assume the hypotheses of Theorem 6.2.2. In addition, suppose that $m \in \mathbb{N}$ and for every $l \in \mathbb{N}$ with $l < m$ there is a $v_l \in H_0^1(D)$ such that v_l is a proper vector of a_∞ corresponding to μ_l, $\lambda_l^k \to \mu_l$ and $u_l^k \to v_l{}^\sim$ in V, as $k \to \infty$.*

Then there is an increasing function $\phi: \mathbb{N} \to \mathbb{N}$ and a $v_m \in H_0^1(D)$ such that v_m is a proper vector of a_∞ corresponding to μ_m, the subsequence $(\lambda_m^{\phi(k)})_{k\in\mathbb{N}}$ of $(\lambda_m^k)_{k\in\mathbb{N}}$ converges to μ_m and the subsequence $(u_m^{\phi(k)})_{k\in\mathbb{N}}$ of $(u_m^k)_{k\in\mathbb{N}}$ converges to $v_m{}^\sim$ in V, as $k \to \infty$.

Proof. Define $\gamma \in \mathbb{R}$ by

$$-\gamma = \inf_{\substack{x\in\Omega \\ k\in\mathbb{N}}} \{\beta_k b_k(x)\}.$$

It follows that

$$\lambda_m^k = a_k(u_m^k, u_m^k) \geq a(u_m^k, u_m^k) - \gamma \geq \alpha \|u_m^k\|^2 - d - \gamma > -d - \gamma. \tag{6.2.1}$$

By Proposition 6.1.2 there exists $w \in H_0^1(D)$ with

$$|w|_D = 1, \quad \langle w, v_l\rangle_D = 0 \quad \text{for } l = 1, \ldots, m-1$$

and such that

$$\begin{aligned}\mu_m &= a_\infty(w,w)/|w|_D^2 \\ &= \min\{\, a_\infty(u,u)/|u|_D^2 \mid u \in H_0^1(D)\setminus\{0\},\ \langle u, v_l\rangle_D = 0,\ l = 1, \ldots, m-1\}.\end{aligned}$$

Let

$$\xi^k := w^\sim - \sum_{l=1}^{m-1} \langle w^\sim, u_l^k\rangle u_l^k, \quad k \in \mathbb{N}.$$

It follows that $\xi^k \in V$, $k \in \mathbb{N}$. By our assumptions, $\xi^k \to w^\sim$ in V as $k \to \infty$. In particular, $|\xi^k| \to 1$. Thus for all k large enough

$$\begin{aligned}\lambda_m^k &\leq a_k(\xi^k,\xi^k)/|\xi^k|^2 = a(\xi^k,\xi^k)/|\xi^k|^2 + (\beta_k \int_\Omega b_k \xi^k \xi^k \,\mathrm{d}x)/|\xi^k|^2 \\ &= a(\xi^k,\xi^k)/|\xi^k|^2 + (\beta_k \int_D b_k \xi^k\xi^k \,\mathrm{d}x)/|\xi^k|^2 + (\beta_k \int_{\Omega\setminus D} b_k\xi^k\xi^k\,\mathrm{d}x)/|\xi^k|^2.\end{aligned} \tag{6.2.2}$$

Now

$$a(\xi^k,\xi^k)/|\xi^k|^2 \to a(w^\sim, w^\sim)/|w^\sim|^2 = a_\infty(w,w)/|w|_D^2 = \mu_m,$$

$$|(\beta_k \int_D b_k\xi^k\xi^k\,\mathrm{d}x)/|\xi^k|^2| \leq \sup_{x\in D}\{\beta_k |b_k(x)|\} \to 0, \quad \text{for } k \to \infty,$$

and, since $w^\sim \equiv 0$ on $\Omega \setminus D$, we obtain from the definition of ξ^k

$$\beta_k \int_{\Omega\setminus D} b_k \xi^k\xi^k\,\mathrm{d}x = \sum_{j,l=1}^{m-1} \langle w^\sim, u_j^k\rangle\langle w^\sim, u_l^k\rangle \beta_k \int_{\Omega\setminus D} b_k u_j^k u_l^k\,\mathrm{d}x.$$

Now, by our assumptions,

$$\beta_k \int_\Omega b_k u_j^k u_l^k \,\mathrm{d}x = \lambda_j^k \langle u_j^k, u_l^k\rangle - a(u_j^k, u_l^k) \to \mu_j \langle v_j, v_l\rangle_D - a_\infty(v_j, v_l) = 0.$$

Since

$$\beta_k \int_D b_k \xi^k \xi^k \, \mathrm{d}x \to 0$$

it follows that

$$\beta_k \int_{\Omega \setminus D} b_k \xi^k \xi^k \, \mathrm{d}x \to 0.$$

From (6.2.1) and (6.2.2) we now conclude that

$$-d - \gamma \leq \lambda_m^k \leq \mu_m + \epsilon_k, \quad k \in \mathbb{N}, \quad \text{with } \epsilon_k \to 0. \tag{6.2.3}$$

Thus, passing to a subsequence if necessary we may assume that

$$\lambda_m^k \to \zeta \quad \text{as } k \to \infty, \text{ where } \zeta \in \mathbb{R}. \tag{6.2.4}$$

Now

$$\langle\!\langle\!\langle u, v \rangle\!\rangle\!\rangle := a(u, v) + d\langle u, v \rangle, \quad u, v \in V,$$

is a scalar product on V whose corresponding norm $|\!|\!| \cdot |\!|\!|$ is equivalent to $|| \cdot ||$. From (6.2.1) and (6.2.3) we obtain

$$\langle\!\langle\!\langle u_m^k, u_m^k \rangle\!\rangle\!\rangle \leq \lambda_m^k + d + \gamma \leq \mu_m + \epsilon_k + d + \gamma$$

i.e. the sequence $(\langle\!\langle\!\langle u_m^k, u_m^k \rangle\!\rangle\!\rangle)_{k \in \mathbb{N}}$ is bounded. Since $(V, \langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle)$ is a Hilbert space which is compactly included in H we may assume, again passing to a subsequence if necessary, that there is a vector $\omega \in V$ such that

$$u_m^k \rightharpoonup \omega \quad \text{in } (V, \langle\!\langle\!\langle \cdot, \cdot \rangle\!\rangle\!\rangle), \text{ as } k \to \infty \tag{6.2.5}$$

and

$$u_m^k \to \omega \quad \text{in } H, \text{ as } k \to \infty. \tag{6.2.6}$$

We have

$$-\gamma \leq \beta_k \int_\Omega b_k u_m^k u_m^k \, \mathrm{d}x = \lambda_m^k - \langle\!\langle\!\langle u_m^k, u_m^k \rangle\!\rangle\!\rangle + d.$$

Thus

$$\sup_{k \in \mathbb{N}} \beta_k \left| \int_\Omega b_k u_m^k u_m^k \, \mathrm{d}x \right| < \infty$$

so

$$\int_\Omega b_k u_m^k u_m^k \, \mathrm{d}x \to 0 \quad \text{as } k \to \infty.$$

However, by (6.2.6) and our hypotheses,

$$\int_\Omega b_k u_m^k u_m^k \,\mathrm{d}x \to \int_\Omega b\omega\omega \,\mathrm{d}x \qquad \text{as } k \to \infty.$$

Since $b(x) > 0$ for $x \in \Omega \setminus D$ it follows that $\omega(x) = 0$ a.e. in $\Omega \setminus D$. Let $v_m = \omega|D$ so $\omega = v_m{}^\sim$. Since the trivial extension of ω to $\mathbb{R}^N$ lies in $H^1(\mathbb{R}^N)$, Lemma 6.2.1 implies that $v_m \in H_0^1(D)$. From (6.2.6) we obtain

$$|v_m|_D = |\omega| = 1 \quad \text{and } \langle v_m, v_l\rangle_D = 0 \text{ for } l = 1, \dots, m-1. \tag{6.2.7}$$

Now by (6.2.5),

$$\|\omega\| \le \liminf_{k\to\infty} \|u_m^k\|.$$

Let $\epsilon > 0$ be arbitrary. It follows that there is k_0 such that

$$\|\omega\| < \|u_m^k\| + \epsilon \quad \text{for } k \ge k_0.$$

Therefore we obtain, for $k \ge k_0$,

$$\begin{aligned}
\mu_m &= \min\{\, a_\infty(v,v)/|v|_D^2 \mid v \in H_0^1(D) \setminus \{0\},\ \langle v, v_l\rangle_D = 0,\ l = 1, \dots, m-1\} \\
&\le a_\infty(v_m, v_m) = a(\omega,\omega) = \|\omega\| - d < \|u_m^k\| + \epsilon - d = a(u_m^k, u_m^k) + \epsilon \\
&= a_k(u_m^k, u_m^k) - \beta_k \int_\Omega b_k u_m^k u_m^k \,\mathrm{d}x + \epsilon = \lambda_m^k - \beta_k \int_\Omega b_k u_m^k u_m^k \,\mathrm{d}x + \epsilon \\
&\le \lambda_m^k + \sup_{\substack{x\in D\\ k\in\mathbb{N}}} \{\beta_k |b_k(x)|\} + \gamma \int_{\Omega\setminus D} u_m^k u_m^k \,\mathrm{d}x + \epsilon \\
&\le \mu_m + \epsilon_k + \sup_{\substack{x\in D\\ k\in\mathbb{N}}} \{\beta_k |b_k(x)|\} + \gamma \int_{\Omega\setminus D} u_m^k u_m^k \,\mathrm{d}x + \epsilon.
\end{aligned} \tag{6.2.8}$$

Now

$$\int_{\Omega\setminus D} u_m^k u_m^k \,\mathrm{d}x \to 0 \quad \text{as } k \to \infty$$

so using (6.2.4), letting $k \to \infty$ in (6.2.8) and then letting $\epsilon \to 0$ we see that

$$\begin{aligned}
\zeta = \mu_m &= \min\{\, a_\infty(v,v)/|v|_D^2 \mid v \in H_0^1(D) \setminus \{0\},\ \langle v, v_l\rangle_D = 0,\ l = 1, \dots, m-1\} \\
&= a_\infty(v_m, v_m).
\end{aligned} \tag{6.2.9}$$

and

$$a(u_m^k, u_m^k) \to a(\omega,\omega) \qquad \text{as } k \to \infty.$$

Thus

$$\langle\!\langle\!\langle u_m^k, u_m^k \rangle\!\rangle\!\rangle \to \langle\!\langle\!\langle \omega, \omega \rangle\!\rangle\!\rangle \quad \text{as } k \to \infty. \tag{6.2.10}$$

(6.2.5) and (6.2.10) imply that

$$u_m^k \to \omega \quad \text{in } V, \text{ as } k \to \infty \tag{6.2.11}$$

Now (6.2.4), (6.2.7), (6.2.9) and (6.2.11) together with part (4) of Proposition 6.1.2 imply the assertions of the lemma. □

3. C^1-convergence of Eigenfunctions

In this section we prove that, if the bilinear form in Theorem 6.2.2 arises from the variational formulation of a linear elliptic equation, then, for all n, $u_n^{\phi(k)}|D \to v_n$ in $C^1_{\text{loc}}(D)$ as $k \to \infty$:

Theorem 6.3.1. *Assume the same hypotheses of Theorem 6.2.2. Moreover, assume*

$$a(u,v) = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega a(x)u(x)v(x)dx,$$

where $A(x) := (a_{ij}(x))_{i,j}$ is a symmetric $N \times N$-matrix, $A(x)\xi \cdot \xi \geq c|\xi|^2$ for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$ for some $c > 0$, $a_{ij}: \overline{\Omega} \to \mathbb{R}$ are of class $C^{1,\gamma}(\overline{\Omega})$ and $a, b, b_k: \overline{\Omega} \to \mathbb{R}$ are of class $C^{0,\gamma}(\overline{\Omega})$. Then, for all n,

$$u_n^{\phi(k)} \to v_n \quad \text{as } k \to \infty$$

in $C^1_{\text{loc}}(D)$.

Before proving Theorem 6.3.1 we need to introduce some notation and to prove a technical lemma. For $N \in \mathbb{N}$, $p \in \mathbb{R}$, $1 < p < +\infty$, $p < N$, the Sobolev exponent p^* is defined by

$$p^* := \frac{pN}{N-p}.$$

For $p < N$, we define inductively

$$\begin{aligned} p^{0*} &:= p \\ p^{l*} &:= (p^{(l-1)*})^*, \quad l > 0. \end{aligned}$$

for all l such that $p^{(l-1)*} < N$.

Lemma 6.3.2. *There are $l \in \mathbb{N}$ and $p \in \mathbb{R}$, $1 < p \leq 2$ such that*

$$p^{l*} < N < p^{(l+1)*}.$$

Proof. First we prove that there exists an l such that $2^{l*} \geq N$. Assume by contradiction that $2^{l*} < N$ for all l; since

$$\frac{1}{2^{l*}} - \frac{1}{2^{(l+1)*}} = \frac{1}{N},$$

then for all l

$$\frac{1}{N} \leq \frac{1}{2^{l*}} = \frac{1}{2^{(l-1)*}} - \frac{1}{N} = \cdots = \frac{1}{2^{*}} - \frac{l}{N},$$

a contradiction. So we have proved that there is some l such that $2^{l*} \geq N$. If $2^{l*} > N$, we have concluded with $p = 2$. Otherwise we define, for all $\epsilon > 0$, $p_\epsilon := 2 - \epsilon$. It is clear that, for all $\epsilon > 0$,

$$(2-\epsilon)^{l*} < 2^{l*} = N$$

and that

$$(2-\epsilon)^{l*} \nearrow N \quad \text{as } \epsilon \to 0;$$

This implies that $(2-\epsilon)^{(l+1)*}$ is defined for all $\epsilon > 0$ and

$$(2-\epsilon)^{(l+1)*} = \frac{(2-\epsilon)^{l*} N}{N - (2-\epsilon)^{l*}} \to +\infty$$

as $\epsilon \to 0^+$; we take $p := p_\epsilon$ with $\epsilon > 0$ sufficiently small and we have concluded. □

Proof of Theorem 6.3.1. Fix n and $D' \subset\subset D$; it is not a restriction to assume that D' has smooth boundary. Let M be a positive constant such that

$$\sup_{i,j=1,\dots,N} \sup_{x \in \overline{\Omega}} |a_{ij}(x)| < M,$$
$$\sup_{i,j=1,\dots,N} \sup_{x,y \in \overline{\Omega}} |a_{ij}(x) - a_{ij}(y)|/|x-y|^\gamma < M,$$
$$\sup_{x \in \overline{\Omega}} |a(x)| < M,$$
$$\sup_{k \in \mathbb{N}} \sup_{x \in D} \beta_k |b_k(x)| < M,$$
$$\sup_{k \in \mathbb{N}} |\lambda_n^k| < M.$$

For all k, $u_n^{\phi(k)}$ satisfies:

$$\int_\Omega A(x) \nabla u_n^{\phi(k)}(x) \cdot \nabla v(x) dx + \int_\Omega \left(a(x) + \beta_{\phi(k)} b_{\phi(k)}(x) - \lambda_n^{\phi(k)} \right) u_n^{\phi(k)}(x) v(x) dx = 0$$
$$\text{for all } v \in V.$$

In particular,

$$\int_D A(x)\nabla u_n^{\phi(k)}(x)\cdot\nabla v(x)dx+\int_D\left(a(x)+\beta_{\phi(k)}b_{\phi(k)}(x)-\lambda_n^{\phi(k)}\right)u_n^{\phi(k)}(x)v(x)dx=0$$
$$\text{for all } v\in H_0^1(D).$$

By classical regularity results for elliptic equations (see e.g. [10, Thms. 8.8, 9.16]), it follows that, for all k,

$$u_n^{\phi(k)}(x)\in W^{2,p}_{\text{loc}}(D)\quad\text{for all } p$$

and

$$-\operatorname{div}(A(x)\nabla u_n^{\phi(k)}(x))+\left(a(x)+\beta_{\phi(k)}b_{\phi(k)}(x)-\lambda_n^{\phi(k)}\right)u_n^{\phi(k)}(x)=0\quad\text{a.e. in } D.$$

Now take $p\in\mathbb{R}$, $1<p\leq 2$, and $l\in\mathbb{N}$ such that

$$p^{l*}<N<p^{(l+1)*}$$

(this is possible thanks to Lemma 6.3.2). Fix open sets with smooth boundary D_j, $j=0$, $\dots$, l, such that

$$D':=D_{l+1}\subset\subset D_l\subset\subset\cdots\subset\subset D_1\subset\subset D_0\subset\subset D=:D_{-1}.$$

By [10, Th. 9.11], there are constants $C_j=C(N,M,D,D_j,p^{j*})$, $j=0,\dots,l+1$, such that, for all $k\in\mathbb{N}$ and all $j=0,\dots,l+1$,

$$\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{j*}}(D_j)}\leq C_j\left\|u_n^{\phi(k)}\right\|_{L^{p^{j*}}(D_{j-1})};$$

moreover, by the Sobolev imbedding theorems, there exist constants $K_j=K(N,D_j,p^{j*})$, $j=0,\dots,l+1$, such that, for all $k\in\mathbb{N}$ and all $j=0,\dots,l+1$,

$$\left\|u_n^{\phi(k)}\right\|_{L^{p^{j*}}(D_{j-1})}\leq K_j\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{(j-1)*}}(D_{j-1})}.$$

These inequalities together imply that there exists a constant C such that, for all k,

$$\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{(l+1)*}}(D')}\leq C\left\|u_n^{\phi(k)}\right\|_{L^p(D)}. \tag{6.3.1}$$

Now, since $u_n^{\phi(k)}\to v_n{}^{\sim}$ in $H^1(\Omega)$, we have that the sequence $u_n^{\phi(k)}$ is bounded in $L^p(D)$; then, by (6.3.1), we deduce that the sequence $u_n^{\phi(k)}|D'$ is bounded also in $W^{2,p^{l+1}*}(D')$. Since $p^{(l+1)*}>N$, the Sobolev imbedding theorem implies that

$$W^{2,p^{(l+1)*}}(D')\hookrightarrow C^1(\overline{D}')$$

with compact inclusion. Then we conclude that

$$u_n^{\phi(k)}\to v_n\quad\text{as } k\to\infty$$

in $C^1(\overline{D}')$. Since D' was arbitrary, we finally conclude that

$$u_n^{\phi(k)}\to v_n\quad\text{as } k\to\infty$$

in $C^1_{\text{loc}}(D)$. $\square$

CHAPTER VII

THE POLÁČIK CONDITION FOR THE LAPLACIAN

Let Ω be the open unit ball centered at 0 in $\mathbb{R}^N$. In this chapter, following [18] and [25], we will construct a potential $a\colon\overline{\Omega}\to\mathbb{R}$ in such a way that the operator $\Delta + a(x)$ on Ω with Dirichlet boundary condition on $\partial\Omega$ satisfies all the assumptions in Theorem 3.2.1, that is:

(1) $\Delta + a(x)$ satisfies the Poláčik condition on Ω;
(2) $G\subset\Omega$ is an open set and $\kappa > 1$;
(3) $R(x)\neq 0$ for all $x\in G$;
(4) there is a function $b\in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b\subset G$ such that

$$\lambda < -\kappa$$

for every eigenvalue λ of the operator $\Delta + a(x) + b(x)$ on Ω with Dirichlet boundary condition on $\partial\Omega$.

1. Eigenvalues for Radially Symmetric Potentials on the Ball

In this section we will prove that there is an analitic potential $a(x)$ such that the operator $\Delta + a(x)$ satisfies the Poláčik condition on Ω.

We take a radially symmetric potential $a(x) = a(|x|)$, where $r\mapsto a(r)$ is a a real analytic function on $\mathbb{R}$. Let us consider the eigenvalue problem

$$\begin{aligned} \Delta u(x) + a(|x|)u(x) - \mu u(x) &= 0, \quad x\in\Omega,\\ u(x) &= 0, \quad x\in\partial\Omega. \end{aligned} \tag{7.1.1}$$

Let $\mathbb{S}^{N-1} := \{x\in\mathbb{R}^N \mid |x| = 1\}$. We introduce the spherical cohordinates in $\mathbb{R}^N$ by $P\colon [0,\infty[\times\mathbb{S}^{N-1}\to\mathbb{R}^N$, where

$$x = P(r,\sigma) = r\sigma.$$

On $\mathbb{R}^N\setminus\{0\}$ we have the inverse map

$$Q(x) = (|x|, x/|x|),$$

and if $\tau\colon U\to\mathbb{R}^{N-1}$ is any chart on $\mathbb{S}^{N-1}$ then a chart θ is determined on the open cone $P(]0,\infty[\times U)$ in $\mathbb{R}^N$ by the formula

$$\theta(x) = (|x|, \tau(x/|x|)) = (r, \tau(\sigma)).$$

Then one has (see e.g. [3, p. 34]) that in spherical cohordinates problem (7.1.1) takes the form

$$u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u + a(r)u - \mu u = 0, \qquad u|_{r=1} = 0, \tag{7.1.2}$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplacian on $\mathbb{S}^{N-1}$ (with respect to the Riemanniann metric induced by the metric of $\mathbb{R}^N$). A standard separation of variables argument (see e.g. [36, p. 257–]) shows that the eigenvalues of (7.1.2) form a sequence

$$\mu_{ml}(a), \quad m = 0, 1, \ldots, \quad l = 1, 2, \ldots$$

and for $\mu = \mu_{ml}(a)$ the corresponding eigenspace is spanned by the functions

$$w_{ml}(r)v(\sigma), \quad v(\sigma) \in Y_m,$$

where $w_{ml}(r)$ is a nontrivial solution of

$$w_{rr} + \frac{N-1}{r} w_r + \left(a(r) - \frac{m(m+N-2)}{r^2} - \mu_{ml} \right) w = 0 \tag{7.1.3}$$

$$w(1) = 0 \qquad w \text{ regular at } r = 0 \tag{7.1.4}$$

and Y_m is the space of the spherical harmonics of order m in N variables. By definition, Y_m consists of the restrictions to $\mathbb{S}^{N-1}$ of all harmonic polynomials on $\mathbb{R}^N$ of degree m. Any element of $Y_m \backslash \{0\}$ is an eigenfunction of $\Delta_{\mathbb{S}^{N-1}}$ with the eigenvalue $-m(m+N-2)$. Thus, for a fixed m, the sequence

$$\mu_{m1}(a) < \mu_{m2}(a) < \ldots$$

is the sequence of the eigenvalues of the one dimensional problem (7.1.3). A Sturm comparison argument shows that

$$\mu_{01}(a) < \mu_{11}(a) < \mu_{21}(a) < \ldots$$

Note that

$$\dim Y_0 = 1 \quad \text{and} \quad Y_0 = \operatorname{span} \langle 1 \rangle$$

$$\dim Y_1 = m \quad \text{and} \quad Y_m = \operatorname{span} \left\langle x_1 | \mathbb{S}^{N-1}, \ldots, x_N | \mathbb{S}^{N-1} \right\rangle$$

So, if we can find a potential a in such a way that

$$\mu_{02}(a) = \mu_{11}(a),$$

then we have that the eigenvalue $\bar{\mu} = \mu_{02}(a) = \mu_{11}(a)$ has multiplicity $N+1$ and a basis of the corresponding eigenspace is given (in Cartesian cohordinates) by

$$w_{02}(|x|), \quad w_{11}(|x|)\frac{x_i}{|x|}, \quad i = 1, \dots, N.$$

We claim that

(1) there exists an analytic potential $a(r)$ such that $\mu_{02}(a) = \mu_{11}(a)$;
(2) $R(w_{02}(|x|), w_{11}(|x|)x_1/|x|, \dots, w_{11}(|x|)x_N/|x|)|_{x=0} \neq 0$.

In order to find an analytic function $a(r)$ such that $\mu_{11}(a) = \mu_{02}(a)$, we argue as in [22]. If $a_1(r)$ and $a_2(r)$ are analytic functions such that $\mu_{11}(a_1) < \mu_{02}(a_1)$ and $\mu_{11}(a_2) > \mu_{02}(a_2)$, then, by a standard continuity argument, $\mu_{11}(a) = \mu_{02}(a)$ for some a of the form $a = sa_1 + (1-s)a_2$. Smooth functions a_1, a_2 that satisfy the above relations and in addition are constant near $r = 0$ were found in [22] (see the proof of proposition 3.2 and Remark A.2 in [22]). We can approximate a_1, a_2 by real analytic radially symmetric functions such that the inequalities remain unchanged. The resulting function a is then real analytic as desired.

Finally we have to check that the Poláčik condition is satisfied. Analyticity of $a(r)$ and (7.1.4) imply that $w_{ml}(r)$ are analytic up to 0. We claim that the following relations are satisfied:

$$w_{02}(0) \neq 0 \quad w_{11}(0) = 0 \quad w'_{11}(0) \neq 0. \tag{7.1.5}$$

In fact, if we take $m = 0$ in (7.1.3)–(7.1.4), multiplying (7.1.3) by r and letting $r \to 0$, we obtain $w'_{02}(0) = 0$; if $w_{02}(0) = 0$, these two equalities together with (7.1.3) imply that all derivatives of w_{02} at 0 vanish, hence $w_{02} \equiv 0$. But this is impossible for an eigenfunction, thus $w_{02}(0) \neq 0$. Next, if we take $m = 1$ in (7.1.3)–(7.1.4), multiplying (7.1.3) by r^2 and letting $r \to 0$, we obtain $w_{11} = 0$; again, $w'_{11}(0) = 0$ leads to the contradiction $w_{11} \equiv 0$, and we have proved the claim. Now we can show that

$$R(w_{02}(|x|), w_{11}(|x|)x_1/|x|, \dots, w_{11}(|x|)x_N/|x|)|_{x=0} \neq 0.$$

Let

$$\phi_0(x) := w_{02}(|x|), \quad \phi_i(x) := \frac{w_{11}(|x|)}{|x|}x_i, \quad i = 1, \dots, N.$$

Then, for $i, j = 1, \dots, N$,

$$\begin{aligned}\partial_i \phi_j(x) &= \frac{w_{11}(|x|)}{|x|}\delta_{ij} + \frac{w'_{11}(|x|)|x| - w_{11}(|x|)}{|x|^3}x_i x_j \\ &= \frac{w_{11}(|x|)}{|x|}\delta_{ij} + \frac{(w'_{11}(|x|) - w'_{11}(0))|x| + O(|x|^2)}{|x|}\frac{x_i x_j}{|x|^2};\end{aligned}$$

this equality, together with (7.1.5), implies that, for $i, j = 1, \dots, N$,

$$\partial_i \phi_j(0) = w'_{11}(0)\delta_{ij}.$$

So we obtain

$$\begin{pmatrix} \phi_0(0) & \nabla\phi_0(0) \\ \phi_1(0) & \nabla\phi_1(0) \\ \vdots & \vdots \\ \phi_N(0) & \nabla\phi_N(0) \end{pmatrix} = \begin{pmatrix} w_{02}(0) & \nabla\phi_0(0) \\ & \\ 0 & w'_{11}(0)\mathrm{I}_N \\ & \end{pmatrix},$$

whence $R(\phi_0, \phi_1, \dots, \phi_N)(0) = w'_{11}(0)^N w_{02}(0) \neq 0$, by (7.1.5).

2. Moving Eigenvalues by Compact Support Perturbations of the Potential

In this section Ω is the open unit ball in $\mathbb{R}^N$, $a(r)$ is the potential constructed in the previous section, $\phi_1, \dots, \phi_{N+1}$ is an $L^2(\Omega)$-orthonormal basis of the kernel of the operator $\Delta + a(|x|)$ on Ω with Dirichlet boundary condition on $\partial\Omega$ and $R(x)$ is the *Poláčik determinant* of $\phi_1, \dots, \phi_{N+1}$. As in [25], we begin with the following:

Lemma 7.2.1. *Let G be the set of all $x \in \Omega$ such that $R(x) \neq 0$. Then G is open and $\overline{\Omega}\backslash G$ has N-dimensional measure zero.*

Proof. Since the eigenfunctions ϕ_i, $i = 1, \dots, N+1$, and hence also R, are real analytic on Ω (e.g. by pp.207-210 in [2]), the result follows from the fact that $R(x) \not\equiv 0$ and from the well known general result that the zero set of a nontrivial real analytic function defined on an open connected subset of $\mathbb{R}^N$ has measure zero. □

Finally, we can construct a potential b in such a way that property (4) in Theorem 3.2.1 is satisfied:

Lemma 7.2.2. *Let G be as in Lemma 7.2.1. For every $k \in \mathbb{N}$ there is a function $b \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b \subset G$ such that*

$$\lambda < k$$

for every eigenvalue λ of the operatoor $\Delta + a + b$ on Ω with Dirichlet boundary condition on $\partial\Omega$.

Proof. Let

$$c := \max_{x \in \overline{\Omega}} |a(x)|.$$

For $\epsilon > 0$ let G_ϵ be the set of all $x \in \Omega$ with $\operatorname{dist}(x, \overline{\Omega} \backslash G) \geq \epsilon$. Choose a function $b_\epsilon \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b_\epsilon \subset G$ and such that

$$b_\epsilon(x) \equiv -c-k-1, \quad x \in G_\epsilon$$
$$-c-k-1 \leq b_\epsilon(x) \leq 0, \quad x \in \overline{\Omega}.$$

We shall show that the lemma holds with b replaced by b_ϵ for $\epsilon > 0$ sufficiently small. Suppose the claim is not true. Then there are sequences $(\lambda_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, $(\epsilon_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0$ as $n \to \infty$ and

$$\Delta u_n = -(a+b_n)u_n + \lambda_n u_n \quad \text{on } \Omega \tag{7.2.1}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$
$$\lambda_n \geq -k \tag{7.2.2}$$

for all $n \in \mathbb{N}$, where $b_n := b_{\epsilon_n}$. We may assume that

$$(u_n \mid u_n) \equiv 1$$

where $(\cdot \mid \cdot)$ denotes the scalar products on both $L^2(\Omega, \mathbb{R})$ and $L^2(\Omega, \mathbb{R}^N)$. It follows that

$$-(\nabla u_n \mid \nabla u_n) + ((a+b_n)u_n \mid u_n) = \lambda_n \tag{7.2.3}$$

for all $n \in \mathbb{N}$. Since $a + b_n \leq c$ on $\overline{\Omega}$, this implies that

$$\lambda_n \leq c$$

for all $n \in \mathbb{N}$; so by (7.2.2)

$$-k \leq \lambda_n \leq c$$

for all $n \in \mathbb{N}$. Thus the right hand side of (7.2.1) is bounded in $L^2((\Omega)$, so $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega)$. Passing to a subsequence if necessary, we may therefore assume that there is a $\bar{u} \in H^1(\Omega)$ such that

$$u_n \to \bar{u} \quad \text{in } H^1(\Omega)$$

as $n \to \infty$. In particular,

$$(\bar{u} \mid \bar{u}) = 1. \tag{7.2.4}$$

Moreover, by Sobolev imbedding theorems, there is a $q > 2$ such that

$$u_n \to \bar{u} \quad \text{in } L^q(\Omega).$$

Now set

$$\bar{b}(x) \equiv -c - k - 1, \quad x \in \overline{\Omega}.$$

For every $x \in G$, $b_n(x) \to \bar{b}(x)$. Thus, by Lemma 7.2.1, $b_n \to \bar{b}$ a.e. on Ω. Since $(b_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, it follows from the dominated convergence theorem that

$$a + b_n \to a + \bar{b} \quad \text{in } L^r(\Omega)$$

for every r with $1 \le r < \infty$. Define r such that

$$(2/q) + (1/r) = 1.$$

It follows from Hölder's inequality that

$$((a + b_n)u_n \mid u_n) \to ((a + \bar{b})\bar{u} \mid \bar{u})$$
$$(\nabla u_n \mid \nabla u_n) \to (\nabla \bar{u} \mid \nabla \bar{u})$$

as $n \to \infty$. Thus, from (7.2.3) and (7.2.4),

$$\lambda_n \to -(\nabla \bar{u} \mid \nabla \bar{u}) + ((a + \bar{b})\bar{u} \mid \bar{u}) \le -k - 1,$$

contradicting (7.2.2). The lemma is proved. $\square$

CHAPTER VIII

THE ALGEBRAIC INDEPENDENCE CONDITION FOR THE LAPLACIAN

In this chapter we show that, if we are given any smooth domain Ω in $\mathbb{R}^2$ and numbers $n, k \in \mathbb{N}$, then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, it is possible to construct a potential $a\colon \overline{\Omega} \to \mathbb{R}$ such that the operator $\Delta + a(x)$ has an n-dimensional kernel spanned by $L^2(\Omega)$-orthonormal functions satisfying (IC) in Chapter 4 up to the order k and with $\varpi = (0, 1)$.

1. A Construction on the Square

As in [23], we begin by considering the eigenvalue problem

$$\text{(8.1.1)} \qquad \begin{aligned} u_{xx} + u_{yy} + a(x)u + b(y)u + \lambda u = 0 \quad &\text{in } \Omega \\ u = 0 \quad &\text{in } \partial\Omega \end{aligned}$$

on the square $\Omega =]0, \pi[\times]0, \pi[$. Let $\mu_i(a)$, $i \in \mathbb{N}$, be the increasing sequence of the eigenvalues of the problem

$$\text{(8.1.2)} \qquad \begin{aligned} u_{xx} + a(x)u + \mu u = 0 \quad &\text{in }]0, \pi[\\ u = 0 \quad &\text{in } \{0, \pi\} \end{aligned}$$

and $\psi_i(a)$, $i \in \mathbb{N}$, be the corresponding $L^2(0, \pi)$-normalized eigenfunctions. Analogously, let $\nu_j(b)$, $j \in \mathbb{N}$, be the increasing sequence of the eigenvalues of the problem

$$\text{(8.1.3)} \qquad \begin{aligned} u_{yy} + b(y)u + \nu u = 0 \quad &\text{in }]0, \pi[\\ u = 0 \quad &\text{in } \{0, \pi\} \end{aligned}$$

and $\chi_j(b)$, $j \in \mathbb{N}$, be the corresponding $L^2(0, \pi)$-normalized eigenfunctions. Then the eigenvalues of (8.1.1) are exactly all the sums $\lambda_{i,j} = \mu_i(a) + \nu_j(b)$, $i, j \in \mathbb{N}$, and the corresponding eigenfunctions are $\phi_{i,j}(x, y) = \psi_i(a)(x)\chi_j(b)(x)$. We want to adjust a and b in such a way that $\mu_i(a) + \nu_{n+1-i}(b) = 0$, $i = 1, \ldots, n$, so that problem (8.1.1) has $\lambda = 0$ as an eigenvalue of multiplicity n; moreover, we want that the corresponding eigenfunctions $\phi_{i,n+1-i}(x, y) = \psi_i(a)(x)\chi_{n+1-i}(b)(y)$, $i = 1, \ldots, n$, satisfy (IC) in Chapter 4. We procede as in [23]. First, we recall the following result on inverse eigenvalue problems in one dimension. The reader is referred to [28] for a detailed treatment of the subject.

Theorem 8.1.1. *The sequence $(\sigma_1, \sigma_2, \dots)$ is the spectrum of (8.1.2) for some function a in $L^2(0, \pi)$ if and only if it is real, strictly increasing, and of the form*

$$\sigma_i = \sigma_0 + i^2\pi^2 + \tau_i,$$

where $\sigma_0 \in \mathbb{R}$ and $(\tau_1, \tau_2, \dots)$ is a square summable sequence of real numbers. For $a \in L^2(0, \pi)$, and $i \in \mathbb{N}$, let

$$M_i(a) := \left\{a' \in L^2(0, \pi) \mid \mu_i(a') = \mu_i(a)\right\};$$

then every finite intersection

$$M_{i_1}(a) \cap M_{i_2}(a) \cap \dots \cap M_{i_l}(a),$$

$i_1 < i_2 < \dots < i_l$, is a real analytic submanifold of $L^2(0, \pi)$ of codimension l.

Now we take positive rationally independent real numbers $\mu_1, \dots, \mu_n$ and we denote

$$M' := \left\{a \in L^2(0, \pi) \mid \mu_i(a) = \mu_i^2,\ i = 1, \dots, n\right\}$$

and

$$M'' := \left\{b \in L^2(0, \pi) \mid \nu_i(b) = -\mu_{n+1-i}^2,\ i = 1, \dots, n\right\}.$$

By Theorem 8.1.1, both M' and M'' are real analytic submanifolds of $L^2(0, \pi)$ of codimension n. Let

$$H_0 := \{c \in C^\infty([0, \pi]) \mid c(x) = 0 \text{ on some interval } [0, \delta],\ \delta > 0\};$$

H_0 is dense in $L^2(0, \pi)$ and then, since M' and M'' have finite codimension, both $M' \cap H_0$ and $M'' \cap H_0$ are nonempty (see [23], Lemma 4.2). We take $a \in M' \cap H_0$ and $b \in M'' \cap H_0$; with this choice, we obviously get that problem (8.1.1) has 0 as an eigenvalue of multiplicity n. Now we show that the corresponding eigenfunctions satisfy (IC) in Chapter 4 up to any order k and with $\varpi = (0, 1)$. First of all, we observe that, since $a(x) = 0$ on an interval $[0, \delta]$, $\delta > 0$, then $\psi_i(a)$ satisfies the equation

$$w_{xx} + \mu_i^2 w = 0, \quad x \in [0, \delta]$$
$$w(0) = 0.$$

This implies that, for $i = 1, \dots, n$,

$$\psi_i(a)(x) = d_i \sinh(\mu_i x), \quad x \in [0, \delta],$$

for some $d_i \neq 0$ (otherwise $\psi_i \equiv 0$ on $[0, \pi]$ by the unique continuation theorem). Similarly, since $b(y) = 0$ on an interval $[0, \delta]$, $\delta > 0$, then $\chi_{n+1-i}(b)$ satisfies the equation

$$w_{yy} - \mu_i^2 w = 0, \quad y \in [0, \delta]$$
$$w(0) = 0.$$

This implies that, for $i = 1, \ldots, n$,

$$\chi_{n+1-i}(b)(y) = e_{n+1-i} \sin(\mu_i y), \quad y \in [0, \delta],$$

for some $e_{n+1-i} \neq 0$. So we have that, for $i = 1, \ldots, n$,

$$\phi_i(x, y) = \eta_i(x)\xi_i(y), \quad (x, y) \in [0, \delta] \times [0, \delta],$$

where

$$\phi_i(x, y) = \phi_{i,n+1-i}(x, y)$$

and

$$\eta_i(x) = d_i \sinh(\mu_i x)$$
$$\xi_i(y) = e_i \sin(\mu_i y).$$

Set

$$\phi(x, y) = (\phi_1(x, y), \ldots, \phi_n(x, y))$$

and

$$\eta(x) = (\eta_1(x), \ldots, \eta_n(x))$$
$$\xi(y) = (\xi_1(y), \ldots, \xi_n(y)).$$

Lemma 8.1.2. *Fix $q \in \mathbb{N}$; the functions*

$$\{\eta^c := \eta_1^{c_1} \cdots \eta_n^{c_n} \mid c \in \mathbb{N}_0^n, \ |c| = q\}$$

are linearly independent on $\mathbb{R}$.

Proof. We have that

$$\eta_i = \frac{d_i}{2} e^{\mu_i x}(1 - e^{-2\mu_i x}).$$

Therefore, up to a nonzero constant,

$$\eta^c = e^{\alpha_c x} Q_c(x),$$

where

$$\alpha_c = c_1\mu_1 + \cdots c_n\mu_n$$

and

$$Q_c(x) = (1 - e^{-2\mu_1 x})^{c_1} \cdots (1 - e^{-2\mu_n x})^{c_n}.$$

Since $\mu_1, \ldots, \mu_n$ are rationally independent, we have that, if $c \neq c'$, then $\alpha_c \neq \alpha_{c'}$ and hence the functions

$$e^{\alpha_c x}, \quad |c| = q$$

are linearly independent. Since $Q_c(x) \to 1$ as $x \to \infty$ for all c, we finally get that the functions η_c, $|c| = n$, are linearly independent. □

Lemma 8.1.3. *Fix $p \in \mathbb{N}$; the functions*

$$\left\{ \frac{\xi_y^\gamma}{\xi^\gamma} := \frac{\xi_{1y}^{\gamma_1}}{\xi_1^{\gamma_1}} \cdots \frac{\xi_{ny}^{\gamma_n}}{\xi_n^{\gamma_n}} \mid \gamma \in \mathbb{N}_0^n, \ |\gamma| = p \right\}$$

are linearly independent on every open interval $I \subset \mathbb{R}$ on which these functions are defined.

Proof. Suppose that

$$\sum_{|\gamma|=p} \alpha_\gamma \frac{\xi_y^\gamma}{\xi^\gamma} \equiv 0$$

on some open interval I; then, on I, we have

$$\sum_{|\gamma|=p} \alpha_\gamma \prod_{i=1}^{n} (\operatorname{cotan}(\mu_i y))^{\gamma_i} \equiv 0. \tag{8.1.4}$$

By analyticity of cotan on $\mathbb{C}\backslash\{k\pi, \ k \in \mathbb{Z}\}$, we have that (8.1.4) actually holds on $\mathbb{R}\backslash\mathcal{C}$, where $\mathcal{C} := \{k\pi/\mu_i, \ k \in \mathbb{Z}, \ i = 1, ..., n\}$. We consider the polynomial $z(y_1, ..., y_m) : \mathbb{R}^n \to \mathbb{R}$,

$$z(y_1, ..., y_n) := \sum_{|\gamma|=p} \alpha_\gamma y^\gamma;$$

we want to prove that $z = 0$. It is sufficient to show that z vanishes on a dense subset of $\mathbb{R}^n$. We know by hypothesis that z vanishes on the set

$$\mathcal{G} := \{(\operatorname{cotan}(\mu_1 y), \ldots, \operatorname{cotan}(\mu_n y)), \ y \in \mathbb{R}\backslash\mathcal{C}\} \subseteq \mathbb{R}^n.$$

The set $\mathcal{G}$ is the set $\mathcal{G} = \mathcal{T}(\mathcal{D})$, where

$$\mathcal{D} := \{([\mu_1 y] \operatorname{mod}\pi, \ldots, [\mu_n y] \operatorname{mod}\pi), \ y \in \mathbb{R}\} \cap]0,\pi[^n \subseteq]0,\pi[^n$$

and

$$\mathcal{T} \colon]0,\pi[^n \to \mathbb{R}^n$$

$$(w_1, ..., w_n) \mapsto (\operatorname{cotan}(w_1), ..., \operatorname{cotan}(w_n)).$$

Since $\mu_1, \ldots, \mu_n$ are rationally independent, $\mathcal{D}$ is dense in $]0,\pi[^n$, and since $\mathcal{T}(]0,\pi[^n) = \mathbb{R}^n$, we conclude that $\mathcal{G} = \mathcal{T}(\mathcal{D})$ is dense in $\mathbb{R}^n$. □

We need also the following result of convex analysis:

Lemma 8.1.4. *If $s \in \mathbb{N}$, $p \in \mathbb{N}$ and $c \in \mathbb{N}^s$ are such that $p < |c|$ then there exist vectors $x^1, \ldots, x^s \in \mathbb{N}_0^s$ such that:*

(1) *$x^1, \ldots, x^s$ are linearly independent;*
(2) *$x^i \leq c$, $i = 1, \ldots, s$;*
(3) *$|x^i| = p$, $i = 1, \ldots, s$.*

Proof. Define

$$C := \{ x \in \mathbb{R}^s \mid |x| := \sum_{j=1}^{s} x_j = p, \text{ and } 0 \leq x \leq c \}.$$

C is convex, closed and bounded, so C is the convex hull of C^*, where C^* is the set of the extremal points of C.

We claim that for every $x \in C^*$ there is an index $i = i(x)$ such that $x_j \in \{0, c_j\}$ for all j with $j \neq i$. In fact suppose there is an $x \in C^*$ and indices i and j with $i \neq j$, $0 < x_i < c_i$ and $0 < x_j < c_j$. Define y, $z \in \mathbb{R}^s$ in the following way:

$$y_k = z_k = x_k, \quad k \neq i, j$$
$$y_i = x_i + \epsilon, \quad y_j = x_j - \epsilon$$
$$z_i = x_i - \epsilon, \quad z_j = x_j + \epsilon.$$

We have $|y| = |z| = p = |x|$, and, if ϵ is sufficiently small, y, $z \in C$; but $x = (1/2)(y + z)$, contradicting the fact that $x \in C^*$. This proves the claim. Since $c \in \mathbb{N}_0^s$, the claim implies for every $x \in C^*$ that $x_j \in \mathbb{N}_0$ for all $j \neq i$, where $i = i(x)$. Since $|x| = p \in \mathbb{N}$ it also follows that $x_i \in \mathbb{N}_0$. So we conclude that, if $x \in C^*$, then $x \in \mathbb{N}_0^s$. If we show that $\dim \operatorname{span} C^* = s$, then the lemma will be proved. Suppose that $\dim \operatorname{span} C^* = s$. Consider the hyperplane

$$H := \{ x \in \mathbb{R}^s \mid |x| = \sum_{j=1}^{s} x_i = 0 \}.$$

clearly $\dim H = s - 1$. Let $\lambda := p/|c|$ (notice that $0 < \lambda < 1$), and set $a := \lambda c$; then $|a| = p$, and, since $0 < a_i < c_i$ for all i it follows that $a \leq c$, so that $a \in C$. Now write C^* as $C^* = \{x_\alpha\}_{\alpha \in A}$. Then there are $\{\lambda_\alpha\}_{\alpha \in A}$, $\lambda_\alpha \geq 0$ for all α, $\lambda_\alpha = 0$ for almost all α, $\sum_\alpha \lambda_\alpha = 1$, such that $a = \sum_\alpha \lambda_\alpha x_\alpha$. Let $y^1, ..., y^{s-1} \in H$ be linearly independent. Choose a sufficiently small $\delta > 0$, so that $0 < \delta y_i^j + a_i < c_i$, for all $j = 1, \ldots, s-1$, $i = 1, \ldots, s$. Then $\delta y^j + a \in C$ for $j = 1, \ldots, s-1$. For every $j = 1, \ldots, s-1$ there are $\{\lambda_\alpha^j\}_{\alpha \in A}$ such that $\lambda_\alpha^j \geq 0$ for all α, $\lambda_\alpha^j = 0$ for almost all α, $\sum_\alpha \lambda_\alpha^j = 1$ and $\delta y^j + a = \sum_\alpha \lambda_\alpha^j x_\alpha$. It follows that

$$y^j = \sum_\alpha \frac{(\lambda_\alpha^j - \lambda_\alpha)}{\delta} x_\alpha,$$

so that $y^j \in \operatorname{span} C^*$ for $j = 1, \ldots, s-1$. If $a \in \operatorname{span}\{ y_1, ..., y_{s-1} \}$, then $a \in H$, and then $|a| = 0$, but this is impossible since $|a| = p > 0$. Thus we obtain

$$\operatorname{span} C^* = \operatorname{span}\{ a, y_1, ..., y_{s-1} \} = \mathbb{R}^s.$$

This concludes the proof. $\square$

Now we can finally prove:

Theorem 8.1.5. *Let a and b as above. Then the eigenfunctions*

$$\phi_i(x,y) = \psi_i(a)(x)\chi_{n+1-i}(b)(y), \quad i = 1, \ldots, n, \ (x,y) \in \Omega$$

satisfy the independence condition (IC) up to any order k. Moreover, the functions $\phi_i\phi_j$, $1 \le i \le j \le n$ are linearly independent.

Proof. Fix $l \in \mathbb{N}$ and suppose

$$\sum_{\substack{j=1,\ldots,n \\ |\beta|=l}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!}\phi^{\beta-\gamma+\epsilon_j}\phi_y^{\gamma} = 0.$$

Thus

$$\sum_{\substack{c\in\mathbb{N}^n \\ |c|=l+1}} \sum_{\substack{j=1,\ldots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!}\phi^{\beta-\gamma+\epsilon_j}(x,y)\phi_y^{\gamma}(x,y)$$
$$= \sum_{\substack{c\in\mathbb{N}^n \\ |c|=l+1}} \eta^c(x) \sum_{\substack{j=1,\ldots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!}\xi^{c-\gamma}(y)\xi_y^{\gamma}(y) = 0$$
$$\text{for } (x,y) \in [0,\delta]\times[0,\delta].$$

By Lemma 8.1.2 the functions $(\eta^c)_{|c|=l+1}$ are analytic and linearly independent, so

$$\sum_{\substack{j=1,\ldots,n \\ |\beta|=l \\ \beta+\epsilon_j=c}} C_{j\beta} \sum_{\substack{\gamma\le\beta \\ |\gamma|=q}} \frac{1}{\gamma!(\beta-\gamma)!}\xi^{c-\gamma}(y)\xi_y^{\gamma}(y)$$
$$= \sum_{|\gamma|=q} \left(\sum_{\substack{j=1,\ldots,n \\ |\beta|=l,\beta\ge\gamma \\ \beta+\epsilon_j=c}} C_{j\beta}\frac{1}{\gamma!(\beta-\gamma)!} \right) \xi^{c-\gamma}(y)\xi_y^{\gamma}(y) = 0$$
$$\text{for } |c| = l+1 \text{ and } y \in \mathbb{R}.$$

By Lemma 8.1.3 the functions $(\xi^{c-\gamma}\xi_y^\gamma)_{|\gamma|=q}$ are linearly independent, so

$$\sum_{\substack{j=1,\dots,n\\|\beta|=l,\beta\geq\gamma\\\beta+\epsilon_j=c}} C_{j\beta}\frac{1}{(\beta-\gamma)!}=0 \quad \text{for } |\gamma|=q \text{ and } |c|=l+1.$$

Now it is easy to see that

$$\begin{aligned}\sum_{\substack{j=1,\dots,n\\|\beta|=l,\beta\geq\gamma\\\beta+\epsilon_j=c}} C_{j\beta}\frac{1}{(\beta-\gamma)!} &= \sum_{\substack{j=1,\dots,n\\\gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}\frac{1}{(c-\epsilon_j-\gamma)!}\\ &= \sum_{\substack{j=1,\dots,n\\\gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}\frac{(c_j-\gamma_j)}{(c-\gamma)!}\end{aligned}$$

so

$$\text{(8.1.5)} \qquad \sum_{\substack{j=1,\dots,n\\\gamma+\epsilon_j\leq c}} C_{j,c-\epsilon_j}(c_j-\gamma_j)=0 \quad \text{for } |\gamma|=q \text{ and } |c|=l+1.$$

Now fix $c\in\mathbb{N}_0^n$, $|c|=l+1$. If $\gamma\not\leq c$, then the sum in (8.1.5) is over an empty set of elements; if $\gamma\leq c$, but $\gamma+\epsilon_j\not\leq c$, then $\gamma_j+1\not\leq c_j$ and so $\gamma_j=c_j$. Thus (8.1.5) is equivalent to the statement that for every $c\in\mathbb{N}_0^n$ with $|c|=\sum_{i=1}^n c_i=l+1$

$$\text{(8.1.6)} \qquad \sum_{\substack{j=1,\dots,n\\\epsilon_j\leq c}} C_{j,c-\epsilon_j}(c_j-\gamma_j)=0 \quad \text{whenever } \gamma\leq c \text{ and } |\gamma|=q.$$

Let $c\in\mathbb{N}_0^n$ with $|c|=l+1$ be arbitrary. We will show that (8.1.6) implies that $C_{j,c-\epsilon_j}=0$ for all j such that $c\geq\epsilon_j$. This will conclude the proof of the proposition. Permuting components, we may assume that, for some s, $1\leq s\leq n$, $c_j\geq 1$ if $1\leq j\leq s$ and $c_j=0$ if $s+1\leq j\leq n$. Then whenever $\gamma\leq c$, we also have $\gamma_j=0$ for $s+1\leq j\leq n$. Therefore we only have to prove the following assertion:

(A). *If $s\in\mathbb{N}$, $q\in\mathbb{N}$, $c\in\mathbb{N}^s$ and $a\in\mathbb{R}^s$ are such that $q<|c|$ and*

$$\sum_{j=1}^s a_j(c_j-\gamma_j)=0 \qquad \textit{whenever } \gamma\in\mathbb{N}_0^s,\ \gamma\leq c \textit{ and } |\gamma|=q,$$

then $a_j=0$ for all $j=1,\dots,s$.

In order to prove (A) we apply Lemma 8.1.4 with $p=|c|-q$. Let x^i, $i=1,\dots,s$, be as in that lemma, and set $\gamma^i:=c-x^i$, $i=1,\dots,s$. It follows that $|\gamma^i|=p$ and $\gamma^i\leq c$ for all i and that the matrix $\left(c_j-\gamma_j^i\right)_{1\leq i,j\leq s}=\left(x_j^i\right)_{1\leq i,j\leq s}$ is regular. Assertion (A) follows immediately.

Finally, the linear independence of the functions $\phi_i\phi_j$, $1\leq i\leq j\leq n$ is an immediate consequence of Lemma 8.1.2. The theorem is proved. □

2. Arbitrary Smooth Domains

The construction on the square, combined with Theorem 6.2.2 and Theorem 5.2.3 yelds to the following:

Theorem 8.2.1. *Let $\Omega \subset \mathbb{R}^2$ be an arbitrary bounded domain with smooth ($C^{2,\gamma}$) boundary, and let $n, h \in \mathbb{N}$. Then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, there exists a potential $c: \overline{\Omega} \to \mathbb{R}$ of class $C^\infty(\overline{\Omega})$ with the following properties:*

(1) *the operator $\Delta + c(x)$ has a n-dimensional kernel;*

(2) *there exists an $L^2(\Omega)$-orthonormal basis $u_1, \ldots, u_n$ of the kernel of $\Delta + c(x)$ such that the algebraic independence condition (IC) in section 4.1 is satisfied up to the order h with $\varpi = (0, 1)$; moreover, the functions $u_i u_j$, $1 \le i \le j \le n$ are linearly independent.*

Proof. Let $D \subset\subset \Omega$ be a square; by the construction in the previous section, it is possible to find a potential $c: \overline{D} \to \mathbb{R}$ such that the assertions of the present theorem are true for the operator $\Delta + c(x)$ on D with Dirichlet boundary condition on ∂D. In paricular,

(8.2.1) for every basis $\phi_1, \ldots, \phi_n$ of the kernel of the operator $\Delta + c(x)$ on D the functions $\phi_i \phi_j$, $1 \le i \le j \le n$, are linearly independent.

Extend c to a continuous function on $\overline{\Omega}$. Let $H := L^2(\Omega)$, $V := H^1_0(\Omega)$ if we are working with Dirichlet boundary condition on $\partial\Omega$, $V := H^1(\Omega)$ if we are working with Neumann boundary condition on $\partial\Omega$; if $b: \mathbb{R}^N \to \mathbb{R}$ with $b|\overline{\Omega} \in C^0(\overline{\Omega})$ define $L_b := \Delta + b$ and $a_b: V \times V \to \mathbb{R}$ by

$$a_b(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega buv \, \mathrm{d}x.$$

If $b \in C^{0,\gamma}(\overline{\Omega})$, regularity theory of PDEs implies that λ is an eigenvalue of L_b on Ω and u is a corresponding eigenvector if and only if λ is a proper value of a_b and u is a corresponding proper vector. (In fact, every proper vector of a_b lies in $C^{2,\gamma}(\Omega)$.)

Let $b: \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $b(x) = 0$ for $x \in D$ and $b(x) > 0$ for $x \notin D$. Furthermore, let $(\beta_k)_{k \in \mathbb{N}}$ be an arbitrary sequence of positive numbers tending to ∞. Finally, for $k \in \mathbb{N}$ let $b_k: \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $c + \beta_k b_k$ is a polynomial function and $\sup_{x \in \Omega} |b_k(x) - b(x)| < 1/k\beta_k$. The existence of such a function b_k is obvious.

Let $a_k := a_{c+\beta_k b_k}$ and let a_∞ be the restriction of a_c to $H^1_0(D)$. We are now in a position to apply Theorem 6.2.2: for $k \in \mathbb{N}$ let $(\lambda^k_n)_{n \in \mathbb{N}}$ be the repeated sequence of proper values a_k and $(u^k_n)_{n \in \mathbb{N}}$ be an H-orthonormal sequence of corresponding proper vectors of

a_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of a_∞. Even if the boundary of ∂D is not of class $C^{2,\gamma}$, separation of variables shows that μ is an eigenvalue of $\Delta + c(x)$ on D with Dirichlet boundary condition on ∂D and v is a corresponding eigenvector if and only if μ is a proper value of a_∞ and v is a corresponding proper vector.

Then, using Theorem 6.2.2 and passing to a subsequence if necessary we may assume that there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, v_n is a proper vector of a_∞ corresponding to μ_n, $(\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(u_n^k)_{k\in\mathbb{N}}$ converges to $v_n{}^\sim$ in V, as $k \to \infty$. Set $p = n$. There are numbers γ_1 $\gamma_2 \in \mathbb{R}$, M, $\eta \in \mathbb{R}_+$ and $l \in \mathbb{N}_0$, such that, setting $\mu_0 = -\infty$, we have

$$0 < \gamma_2 - \gamma_1 < M,$$

$$\mu_l < \gamma_1 - 4\eta < \gamma_1 < 0 = \mu_{l+1} = \mu_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \mu_{l+p+1}.$$

For $h \in C^0(\overline{\Omega})$ let $B_h \in \mathcal{L}_{\text{sym}}(H, H)$ be the map

$$(Bu)(x) = h(x)u(x), \qquad u \in H,\, x \in \Omega.$$

Note that

$$|B_h|_{\mathcal{L}(H,H)} = |h|_{C^0(\overline{\Omega})}. \tag{8.2.2}$$

Let $\mathcal{G}$ be the set of all B_h with $h \in C^0(\overline{\Omega})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\text{sym}}(H, H)$.

Now (8.2.1) easily implies that the operator $T\colon \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B(v_i{}^\sim), v_j{}^\sim\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k\colon \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle Bu_i^k, u_j^k\rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$ so by Corollary 5.1.2

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \qquad \text{for } k \text{ large enough.}$$

Moreover, setting $\lambda_0^k = -\infty$, we have

$$\lambda_l^k < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^k \le \lambda_{l+p}^k < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}^k, \ k \text{ large enough.} \tag{8.2.3}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in Theorem 5.2.3. For all large k, there is an $\alpha_k > 0$ such that $|\lambda^k_{l+j}| < \alpha_k < \alpha_0$ for $j = 1, \ldots, p$ and $\alpha_k \to 0$ as $k \to 0$. Thus by Theorem 5.2.3 (with $A := \Delta + c(x) = \beta_k b_k(x)$, $\mu_j := 0$, $\lambda_{l+j} := \lambda^k_{l+j}$ for $j = 1, \ldots, p$ and $\mathcal{D}$ equal to the set of all B_h where h is the restriction of a polynomial function to $C^0(\overline{\Omega})$) there exists, for each large k, a polynomial function $h_k : \mathbb{R}^N \to \mathbb{R}$ such that $|h_k|_{C^0(\overline{\Omega})} < (1/2)\theta\alpha_k$ and such that if $(\hat{\lambda}^k_n)_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $\Delta + c + h_k + \beta_k b_k$, $(\hat{u}^k_n)_{n\in\mathbb{N}}$ is an H-orthogonal sequence of the corresponding eigenfunctions and $\hat{\lambda}^k_0 := -\infty$, then

$$(8.2.4)\qquad \hat{\lambda}^k_l < \gamma_1 - 3\eta < \gamma_1 - \eta < \hat{\lambda}^k_{l+1} \le \hat{\lambda}^k_{l+p} < \gamma_2 + \eta < \gamma_2 + 3\eta < \hat{\lambda}^k_{l+p+1}$$

and

$$(8.2.5)\qquad \hat{\lambda}^k_{l+j} = 0, \quad j = 1, \ldots, p.$$

Now the assumptions of Theorem 6.2.2 are satisfied with b_k replaced by $(1/\beta_k)h_k + b_k$. Therefore using Theorem 6.2.2 again and passing to a subsequence if necessary we may assume that there is a sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ in $H^1_0(D)$ such that for every $n \in \mathbb{N}$, $\hat{v}_n$ is a proper vector of a_∞ corresponding to μ_n, $(\hat{\lambda}^k_n)_{k\in\mathbb{N}}$ converges to μ_n and $(\hat{u}^k_n)_{k\in\mathbb{N}}$ converges to $\hat{v}_n{}^\sim$ in V, as $k \to \infty$. Let $\phi_1, \ldots, \phi_n$ be the basis of the kernel of $\Delta + c(x)$ constructed in the previous section. Then there exists an orthogonal $n \times n$-matrix $(r_{ij})_{i,j}$ such that

$$\phi_i = \sum_{j=1}^{n} r_{ij}\hat{v}_{l+j}, \quad i = 1, \ldots, n.$$

Then, for all k,

$$\phi^k_i := \sum_{j=1}^{n} r_{ij}\hat{u}^k_{l+j}, \quad i = 1, \ldots, n$$

is an $L^2(\Omega)$-orthonormal basis of the kernel of the operator $L_{c+h_k+\beta_k b_k}$, and $(\phi^k_i)_{k\in\mathbb{N}}$ converges to $\phi_i{}^\sim$ in V. Since V is a closed subspace of $H^1(\Omega)$, then, up to a subsequence, we have convergence almost everywhere of ϕ^k_i to $\phi_i{}^\sim$ and of $\nabla\phi^k_i$ to $\nabla\phi_i{}^\sim$ as k tends to infinity. This easily implies that, for sufficiently large k, the eigenfunctions ϕ^k_i, $i = 1, \ldots, n$, satisfy (IC) up to the order h with $\varpi = (0, 1)$. Moreover, the functions $\phi^k_i\phi^k_j$, $1 \le i \le j \le n$, are linearly independent. For every such k, $c + h_k + \beta_k b_k$ is a polynomial function and the conclusion follows with c replaced by $c + h_k + \beta_k b_k$. This proves the theorem. □

3. Generalization to Higher Space Dimension

The results contained in this chapter can be generalized to any space dimension $N \ge 2$; infact, if $N \ge 3$, we use the space variable $z = (x, y, x_3, \ldots, x_N)$; with $a(x)$ and $b(y)$ as in

Section 4.1, we consider the operator

$$\Delta + a(x) + b(y) + \frac{N-2}{\delta^2}$$

on the open set

$$\Omega =]0,\pi[\times]0,\pi[\times]0,\delta[^{N-2};$$

separation of variables shows that, for sufficiently small δ, this operator with Dirichlet boundary condition on $\partial\Omega$ has an n-dimensional kernel spanned by the functions

$$\phi_i(x,y)\sin(x_3/\delta)\cdots\sin(x_N/\delta), \quad i = 1,\dots,n.$$

It is very easy to see that these functions satisfy (IC) up to any order, with $\varpi = (0,1,0)$. Finally, the arguments in Section 4.2 extend directly to the present case and we conclude that, if $N \geq 3$ and $\Omega \subset \mathbb{R}^3$ is any smooth bounded domain, then, fixed $n, h \in \mathbb{N}$, both for Dirichlet and Neumann boundary condition on $\partial\Omega$ we can construct a potential $c: \overline{\Omega} \to \mathbb{R}$ such that the operator $\Delta + c(x)$ has an n-dimensional kernel spanned by $L^2(\Omega)$-orthonormal functions satisfying (IC) up to the order h with $\varpi = (0,1,0)$.

CHAPTER IX

GENERAL PRINCIPAL PARTS AND ARBITRARY DOMAINS

In chapters 6 and 7 we have shown that on suitable domains it is possible to find a potential a in such a way that the operator $\Delta + a$ has a kernel of some prescribed dimension, spanned by eigenfunctions satisfying the Poláčik condition or the algebraic independence condition. In this chapter we will show that this can be extended to the case of any smooth bounded domain and any *principal part* $L' = \sum_{ij} \partial_i(a_{ij}(x)\partial_j)$.

1. Localization

We begin with the following "localization" result:

Lemma 9.1.1. *Let $\Omega, S \subset \mathbb{R}^N$ be open bounded domains; assume S has $C^{2,\gamma}$ boundary. Let $a_{ij}: \Omega \to \mathbb{R}$ be of class $C^{1,\gamma}$, $i, j = 1, \ldots, N$, $a_{ij} \equiv a_{ji}$, $i, j = 1, \ldots, N$, and*

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c|\xi|^2, \quad x \in \Omega, \ \xi \in \mathbb{R}^N$$

for some $c > 0$. Consider the differential operator

$$L' = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j).$$

Let us suppose there exists a $C^\gamma(\overline{S})$ potential $a_0: \overline{S} \to \mathbb{R}$ such that the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel, spanned by $L^2(S)$-orthonormal eigenfunctions $\phi_1, \ldots, \phi_n$, and that the set of functions

$$\{\phi_i\phi_j, \ 1 \leq i \leq j \leq n\}$$

is linearly independent. Then for every $\epsilon > 0$ there exist an invertible affine transformation $W: \mathbb{R}^N \to \mathbb{R}^N$, an open bounded domain $D \subset\subset \Omega$ and a potential $a: \overline{\Omega} \to \mathbb{R}$, $a \in C^\gamma(\overline{\Omega})$, with the following properties:

1. *$D = W(S)$;*
2. *$L' + a(x)$ on D with Dirichlet boundary condition on ∂D has an n-dimensional kernel spanned by $L^2(D)$-orthonormal functions $\psi_1, \ldots, \psi_n$;*
3. *$\left\|(\det DW)^{1/2}\psi_i(W(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{D})} < \epsilon$, $i = 1, \ldots, n$.*

Moreover, if there exists a $C^\gamma(\overline{S})$ function $b_0\colon \overline{S} \to \mathbb{R}$ and a positive constant κ such that the operator $\Delta + a_0(x) + b_0(x)$ on S with Dirichlet boundary condition on ∂S has all eigenvalues $< -\kappa$, then W, D and $a(x)$ above can be chosen in such a way that, setting $b(x) := \rho^{-2} b_0(W^{-1}(x))$ for an appropriate $\rho > 0$, the operator $L' + a(x) + b(x)$ on D with Dirichlet boundary condition on ∂D has all eigenvalues $< -\kappa$.

Proof. First we introduce some notation; we indicate by λ_i, $i \in \mathbb{N}$, the repeated sequence of the eigenvalues of the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S; in the hypothesis, we have assumed that this operator has an n-dimensional kernel, so there is an $l > 1$ such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_{l+n} = 0 < \lambda_{l+n+1}$.

We procede in several steps:

1^{st} *step*: Take $\bar{x} \in S$ and $x_0 \in \Omega$; let $G_0 := G(x_0)$, where $G(x) := (a_{ij}(x))_{ij}$; G_0 is a symmetric positive definite $N \times N$-matrix, so we can take an invertible $N \times N$-matrix Q such that $G_0 = QQ^T$. We define the affine transformation

$$\begin{aligned} Z\colon \mathbb{R}^N &\to \mathbb{R}^N \\ x &\mapsto x_0 + Q(x - \bar{x}) \end{aligned}$$

and we set $D_1 := Z(S)$; finally, we define

$$\begin{aligned} \tilde{a}\colon D_1 &\to \mathbb{R} \\ \tilde{a}(x) &:= a_0(Z^{-1}(x)). \end{aligned}$$

The operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has the same repeated sequence of eigenvalues of the operator $\operatorname{div}(G_0 \nabla) + \tilde{a}(x)$ on D_1 with Dirichlet boundary condition on ∂D_1. In particular, this last operator has an n-dimensional kernel spanned by the $L^2(D_1)$-orthonormal functions

$$\tilde{\phi}_i(x) := (\det Q)^{-1/2} \phi_i(Z^{-1}(x)), \quad i = 1, \ldots, n.$$

Obviously, the set of functions

$$\left\{ \tilde{\phi}_i \tilde{\phi}_j,\ 1 \le i \le j \le n \right\}$$

is linearly independent.

2^{nd} *step*: For $\rho \ge 0$ sufficiently small, we consider the differential operators

$$L_\rho := \operatorname{div}(G(x_0 + \rho(x - x_0))\nabla) + \tilde{a}(x)$$

on D_1 with Dirichlet boundary condition on ∂D_1; note $L_0 = \mathrm{div}(G_0\nabla) + \tilde{a}(x)$. We indicate by λ_i^ρ, $i \in \mathbb{N}$, the repeated sequence of eigenvalues of L_ρ.

Let A_ρ be the sectorial operator in $L^2(D_1)$ corresponding to L_ρ; since the boundary of D_1 is of class $C^{2,\gamma}$ and the coefficients a_{ij} are in $C^{1,\gamma}$, it follows that, for all ρ, the domain of A_ρ is $H^2(D_1) \cap H_0^1(D_1)$. Morever, the map

$$\rho \mapsto A_\rho$$
$$[0, \rho_0[\to \mathcal{L}(H^2(D_1) \cap H_0^1(D_1), L^2(D_1))$$

is continuous. This implies that $\lambda_i^\rho \to \lambda_i$ as $\rho \to 0$ for all i; then we can find some $\eta > 0$ such that, for all sufficiently small ρ,

$$\lambda_l^\rho < -4\eta < -\eta < \lambda_{l+1}^\rho \leq \cdots \leq \lambda_{l+n}^\rho < \eta < 4\eta < \lambda_{l+n+1}^\rho;$$

in particular, the set

$$\{\lambda_{l+1}^\rho, \ldots, \lambda_{l+n}^\rho\}$$

is a spectral set of A_ρ and we can consider the corresponding spectral projection P_ρ and the corresponding spectral invariant subspace X_ρ. By the general formula

$$P_\rho = \frac{1}{2\pi i} \int_\Gamma (\zeta - A_\rho)^{-1} d\zeta,$$

it follows that the map

$$\rho \mapsto P_\rho$$
$$[0, \rho_0[\to \mathcal{L}(L^2(D_1), H^2(D_1) \cap H_0^1(D_1))$$

is continuous. By using the spectral projection P_ρ together with the Grahm-Schmidt orthonormalization algorithm, we can find, for all ρ, an $L^2(D_1)$-orthonormal basis $\tau_1^\rho, \ldots,$ τ_n^ρ of X_ρ, with

$$\tau_i^\rho \to \tilde{\phi}_i \quad \text{as } \rho \to 0$$

in $H^2(D_1) \cap H_0^1(D_1)$ for all $i = 1, \ldots, n$. In order to apply Theorem 5.2.3, we need a *basis of eigenfunctions*; to overcome this difficulty, we procede in the following way: for all $\rho > 0$ we can find an orthogonal $n \times n$-matrix $R_\rho = (r_{ij}^\rho)_{ij}$ such that the functions

$$\chi_i^\rho := \sum_{j=1}^n r_{ij}^\rho \tau_j^\rho, \quad i = 1, \ldots n,$$

are an $L^2(D_1)$-orthonormal basis of eigenfunctions of X_ρ, with

$$A_\rho \chi_i^\rho = \lambda_i^\rho \chi_i^\rho, \quad i = 1, \ldots, n.$$

By compactness, we can find a sequence $(\rho_k)_{k\in\mathbb{N}}$, with $\rho_k \to 0$ as $k \to \infty$, and an orthogonal matrix $R = (r_{ij})_{ij}$, such that

$$R^{\rho_k} \to R \quad \text{as } k \to \infty.$$

It follows that, for all $i = 1, \ldots, n$,

$$\chi_i^{\rho_k} \to \sum_{j=1}^{n} r_{ij}\tilde{\phi}_j =: \chi_i \quad \text{as } k \to \infty$$

in $H^2(D_1) \cap H_0^1(D_1)$. Of course $\chi_1, \ldots, \chi_n$ are an orthonormal basis of the n-dimensional kernel of $L_0 = \operatorname{div}(G_0\nabla) + \tilde{a}(x)$. Moreover the set of functions

$$\{\chi_i\chi_j,\ 1 \le i \le j \le n\}$$

is still linearly independent.

For $c \in C^0(\overline{D_1})$ let $B_c \in \mathcal{L}_{\rm sym}(L^2(D_1), L^2(D_1))$ be the map

$$(Bu)(x) = c(x)u(x), \qquad u \in L^2(D_1),\ x \in \mathrm{D}_1.$$

Note that

$$|B_c|_{\mathcal{L}(L^2(D_1),L^2(D_1))} = |c|_{C^0(\overline{\mathrm{D}_1})}. \tag{9.1.1}$$

Let $\mathcal{G}$ be the set of all B_c with $c \in C^0(\overline{D_1})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\rm sym}(L^2(D_1), L^2(D_1))$. Now, since the functions $\{\chi_i\chi_j,\ 1 \le i \le j \le n\}$ are linearly independent, it is easy to see that the operator $T\colon \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B\chi_i, \chi_j\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k\colon \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle B\chi_i^{\rho_k}, \chi_j^{\rho_k}\rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$ so it is easy to see that

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \quad \text{for } k \text{ large enough.}$$

Moreover we have

$$\lambda_l^{\rho_k} < -4\eta < -\eta < \lambda_{l+1}^{\rho_k} \le \lambda_{l+n}^{\rho_k} < \eta < 4\eta < \lambda_{l+n+1}^{\rho_k}, \ k \text{ large enough.} \tag{9.1.2}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in Theorem 5.2.3. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^{\rho_k}| < \alpha_k < \alpha_0$ for $j = 1, \ldots, n$ and $\alpha_k \to 0$ as $k \to 0$. Thus by Theorem 5.2.3 (with $A := A_{\rho_k}$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^{\rho_k}$ for $j = 1, \ldots, n$ and $\mathcal{D}$ equal to the set of all B_c where c is a $C^\gamma(\mathbb{R}^N)$ function) there exists, for each large k, a $C^\gamma(\mathbb{R}^N)$ function $c_k\colon \mathbb{R}^N \to \mathbb{R}$ such that $|c_k|_{C^0(\overline{D}_1)} < (1/2)\theta\alpha_k$ and such that if $(\hat\lambda_n^{\rho_k})_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $L_{\rho_k} + c_k$, then

$$\hat\lambda_l^{\rho_k} < -3\eta < -\eta < \hat\lambda_{l+1}^{\rho_k} \le \hat\lambda_{l+n}^{\rho_k} < \eta < 3\eta < \hat\lambda_{l+n}^{\rho_k} \tag{9.1.3}$$

and

$$\hat\lambda_{l+j}^{\rho_k} = 0, \quad j = 1, \ldots, n. \tag{9.1.4}$$

So we have found a sequence of potentials $c_k\colon \mathbb{R}^N \to \mathbb{R}$, $c_k \in C^\gamma(\mathbb{R}^N)$, $c_k \to 0$ in $C^0(\overline{D_1})$ as $k \to \infty$, such that, for all (sufficiently large) k, the operator $L_{\rho_k} + c_k(x)$ on D_1 with Dirichlet boundary condition on ∂D_1 has an n-dimensional kernel.

3^{rd} *step*: For $c \in C^0(\overline{D_1})$ let $B_c \in \mathcal{L}(L^p(D_1), L^p(D_1))$ be the map

$$(Bu)(x) = c(x)u(x), \qquad u \in L^p(D_1),\ x \in \mathrm{D}_1.$$

Note that

$$|B_c|_{\mathcal{L}(L^p(D_1), L^p(D_1))} = |c|_{C^0(\overline{\mathrm{D}_1})}. \tag{9.1.5}$$

Let A_ρ be the sectorial operator in $L^p(D_1)$ corresponding to L_ρ; since the boundary of D_1 is of class $C^{2,\gamma}$ and the coefficients a_{ij} are in $C^{1,\gamma}$, it follows that, for all ρ, for all $c \in C^0(\overline{D}_1)$ and for all $p > 1$, the domain of $A_\rho + B_c$ is $W^{2,p}(D_1) \cap W_0^{1,p}(D_1)$. Moreover

$$A_{\rho_k} + B_{c_k} \to A_0 \quad \text{as } k \to \infty$$

in $\mathcal{L}(W^{2,p}(D_1) \cap W_0^{1,p}(D_1), L^p(D_1))$. We choose $p > N$, so that $W^{2,p}(D_1) \subset C^1(\overline{D_1})$. Again by using the spectral projection P_{ρ_k} on the kernel of $A_{\rho_k} + B_{c_k}$ in $L^p(D_1)$ together with the Grahm-Schmidt $L^2(D_1)$-orthonormalization algorithm, we can find an $L^2(D_1)$-orthonormal basis $\tilde{\phi}_1^{\rho_k}, \ldots, \tilde{\phi}_n^{\rho_k}$ of $\ker(A_{\rho_k} + B_{c_k})$ with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $C^1(\overline{D_1})$ for all $i = 1, \ldots, n$.

Summarising, we have found a sequence of positive numbers ρ_k, $\rho_k \to 0$ as $k \to \infty$, and a sequence of $C^\gamma(\mathbb{R}^N)$ functions $c_k: \mathbb{R}^N \to \mathbb{R}$, $c_k \to 0$ in $C^0(\overline{D_1})$ as $k \to \infty$, such that, for all (sufficiently large) k, the operator

$$\operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + \tilde{a}(x) + c_k(x)$$

on D_1 with Dirichlet boundary condition on ∂D_1 has an n-dimensional kernel spanned by $L^2(D_1)$-orthonormal functions $\tilde{\phi}_1^{\rho_k}, \ldots, \tilde{\phi}_n^{\rho_k}$, with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $C^1(\overline{D_1})$ for $i = 1, \ldots, n$.

4^{th} *step*: For all $\rho > 0$ we define the homothety

$$\begin{aligned} O_\rho: \mathbb{R}^N &\to \mathbb{R}^N \\ x &\mapsto x_0 + \rho(x - x_0) \end{aligned}$$

and we define

$$D_\rho := O_\rho(D_1) = \left\{ y \in \mathbb{R}^N \mid y = x_0 + \rho(x - x_0),\ x \in D_1 \right\}.$$

If ρ is sufficiently small, then $\overline{D_\rho} \subset \Omega$. So, for sufficiently large k, we can consider the operator

$$\begin{aligned} &\operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}(x_0 + (\rho_k)^{-1}(x - x_0)) + (\rho_k)^{-2}c_k(x_0 + (\rho_k)^{-1}(x - x_0)) = \\ &= \operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho_k)^{-2}c_k((O_{\rho_k})^{-1}(x)) \end{aligned} \tag{9.1.6}$$

on D_{ρ_k} with Dirichlet boundary condition on ∂D_{ρ_k}. This operator has the same repeated sequence of eigenvalues of the operator

$$(\rho_k)^{-2} \operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + (\rho_k)^{-2}\tilde{a}(x) + (\rho_k)^{-2}c_k(x) \tag{9.1.7}$$

on D_1 with Dirichlet boundary condition on ∂D_1. In particular, the operator (9.1.6) has an n-dimensional kernel spanned by the $L^2(D_{\rho_k})$-orthonormal functions

$$\begin{aligned}\psi_i^{\rho_k}(x) :&= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}(x_0 + (\rho_k)^{-1}(x - x_0)) \\ &= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k})^{-1}(x)),\end{aligned}$$

$i = 1, \ldots, n$. Now we define $W_k := O_{\rho_k} \circ Z$, $D_k := W_k(S) = D_{\rho_k}$ and

$$\begin{aligned}a_k(x) :=&(\rho^k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)) \\ =&(\rho^k)^{-2}a_0((W_k)^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)).\end{aligned}$$

We finally estimate, for $i = 1, \ldots, n$,

$$\begin{aligned}&\left\|(\det DW_k)^{1/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{S})} \\ =&\left\|(\det Q)^{1/2}(\rho_k)^{N/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{S})} \\ =&\left\|(\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k}^{-1} \circ W_k(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{S})} \\ =&\left\|(\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{S})} \\ =&(\det Q)^{1/2}\left\|\tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \tilde{\phi}_i(Z(\cdot))\right\|_{C^1(\overline{S})} \to 0\end{aligned}$$

as $k \to \infty$.

Now, fixed $\epsilon > 0$, we choose a sufficiently large k and we set $W := W_k$, $D := D_k$ and $a := a_k$ and we have concluded the proof of the first part of the theorem.

5^{th} *step*: In order to conclude the proof of the theorem, we observe that, for all k, the operator

$$\begin{aligned}(9.1.8)\quad &\mathrm{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + \\ &\qquad + (\rho_k)^{-2}c_k((O_{\rho_k})^{-1}(x)) + (\rho_k)^{-2}b_0(Z^{-1} \circ (O_{\rho_k})^{-1}(x))\end{aligned}$$

on D_k with Dirichlet boundary condition on ∂D_k has the same repeated sequence of eigenvalues of the operator

$$(9.1.9)\quad (\rho_k)^{-2}\,\mathrm{div}(G(x_0 + \rho_k(x - x_0))\nabla) + (\rho_k)^{-2}\tilde{a}(x) + (\rho_k)^{-2}c_k(x) + (\rho_k)^{-2}b_0(Z^{-1}(x))$$

on D_1 with Dirichlet boundary condition on ∂D_1, which is obtained multiplying by $(\rho_k)^{-2}$ the eigenvalues of the operator

$$(9.1.10)\qquad \mathrm{div}(G(x_0 + \rho_k(x - x_0))\nabla) + \tilde{a}(x) + c_k(x) + b_0(Z^{-1}(x))$$

on D_1 with Dirichlet boundary condition on ∂D_1. As $k \to \infty$, the first eigenvalue of (9.1.10) tends to the first eigenvalue of

$$\operatorname{div}(G_0\nabla) + \tilde{a}(x) + b_0(Z^{-1}(x))$$

on D_1 with Dirichlet boundary condition on ∂D_1, that is the same as the first eigenvalue of the operator

$$\Delta + a(x) + b_0(x)$$

on S with Dirichlet boundary condition on ∂S. So, if k is sufficiently large, the first eigenvalue of (9.1.10) is $< -\kappa$, and since $(\rho_k)^{-2} \to \infty$ as $k \to \infty$, the first eigenvalue of (9.1.8) is $< -\kappa$ and we have concluded. $\square$

2. The Poláčik Condition

Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with $C^{2,\gamma}$ boundary. Let $a_{ij}\colon \overline{\Omega} \to \mathbb{R}$, $i, j = 1, \ldots, N$, be of class $C^{1,\gamma}$, $a_{ij} \equiv a_{ji}$, $i, j = 1, \ldots, N$, and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^N$$

for some $c > 0$. Let us consider the differential operator

$$L' = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j). \tag{9.2.1}$$

In this section we want to prove that, for both Dirichlet and Neumann boundary condition on $\partial\Omega$, we can construct a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class C^∞ such that all assumptions in Theorem 3.2.1 are satisfied with

$$L = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j) + a(x). \tag{9.2.2}$$

We will prove the following:

Theorem 9.2.1. *Let L' as above and let $\kappa > 1$; then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, there exists a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class $C^\infty(\overline{\Omega})$ with the following properties:*

(1) *the operator L in (9.2.2) satisfies the Poláčik condition on Ω;*
(2) *$U \subset \Omega$ is an open set;*

(3) *$R(x) \neq 0$ for all $x \in U$;*
(4) *there is a function $b \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b \subset U$ such that*

$$\lambda < -\kappa$$

for every eigenvalue λ of the operator $L + b$ on Ω with Dirichlet (or Neumann) boundary condition on $\partial\Omega$.

Proof. Our starting point is the existence (extablished in Chapter 7) of such a potential when $\Omega = B$ is the unit ball in $\mathbb{R}^N$, $a_{ij}(x) \equiv \delta_{ij}$, i.e. $L' = \Delta$, and we take the Dirichlet condition on ∂B. In this case there is a basis of $\ker L$ given by functions

$$\phi_i(x) = \frac{w(|x|)}{|x|} x_i, \quad x \in B,\ i = 1, \dots, N$$

and

$$\phi_{N+1}(x) = v(|x|), \quad x \in B$$

where $w, v\colon \mathbb{R} \to \mathbb{R}$ are analytic functions such that

$$w(0) = 0,\ w'(0) \neq 0,\ v(0) \neq 0,\ v'(0) = 0. \tag{9.2.3}$$

We claim that

the functions $\phi_i\phi_j$, $1 \leq i \leq j \leq N+1$, are linearly independent.

In fact, let ρ_{ij}, $1 \leq i \leq j \leq N+1$, be real numbers with

$$\sum_{1 \leq i \leq j \leq N+1} \rho_{ij}\phi_i\phi_j \equiv 0.$$

Evaluating this expression at $x = 0$ and using (9.2.3) we obtain $\rho_{N+1,N+1} = 0$. Thus

$$\frac{w(|x|)^2}{|x|^2} \sum_{1 \leq i \leq j \leq N} \rho_{ij} x_i x_j \equiv -\frac{w(|x|)v(|x|)}{|x|} \sum_{1 \leq i \leq N} \rho_{i,N+1} x_i \quad \text{for } x \neq 0.$$

Since

$$\frac{w(|x|)^2}{|x|^2} \neq 0 \quad \text{and} \quad \frac{w(|x|)v(|x|)}{|x|} \neq 0 \quad \text{for } |x| \text{ small},$$

it follows that

$$\Big| \sum_{1 \leq i \leq N} \rho_{i,N+1} x_i \Big| = o(|x|) \quad \text{for } x \to 0.$$

However, this implies that $\rho_{i,N+1} = 0$ for $i = 1, \ldots, N$. Hence

$$\sum_{1\le i\le j\le N} \rho_{ij} x_i x_j \equiv 0$$

which immediately implies that $\rho_{ij} = 0$ for $1 \le i \le j \le N$. The claim is proved.

Now we can apply Lemma 9.1.1 with $S = B$, $n = N+1$ and a_0, b_0 given by the construction in Chapter 7. We claim that, if we choose a sufficiently small ϵ, than the corresponding operators $L_a = L' + a$ and $L_{a+b} = L' + a + b$ on $D = W(S)$ with Dirichlet boundary condition on ∂D satisfy properties 1)–4) of the present theorem. First, we observe that, for a fixed invertible affine transformation W, on S we have

$$\begin{pmatrix} \psi_1(W(x)) & \nabla_x \psi_1(W(x)) \\ \vdots & \vdots \\ \psi_{N+1}(W(x)) & \nabla_x \psi_{N+1}(W(x)) \end{pmatrix} = \\ = \begin{pmatrix} \psi_1(W(x)) & (\nabla\psi_1)(W(x)) \\ \vdots & \vdots \\ \psi_{N+1}(W(x)) & (\nabla\psi_{N+1})(W(x)) \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & DW(x) \end{pmatrix}.$$

Since $DW(x)$ is constant and invertible, we have that, if $x \in D = W(S)$,

$$R(\psi_1, \ldots, \psi_{N+1})(x) \neq 0 \\ \text{if and only if} \\ R(\psi_1(W(\cdot)), \ldots, \psi_{N+1}(W(\cdot)))(W^{-1}x) \neq 0. \tag{9.2.4}$$

Let

$$U_0 := \{x \in B \mid R(\phi_1, \ldots, \phi_{N+1})(x) \neq 0\};$$

U_0 is open and by construction $\operatorname{supp} b_0 \subset U_0$; take an open set U_0', such that $\operatorname{supp} b_0 \subset U_0' \subset\subset U_0$ and set $U' := W(U_0')$; then $U' \subset D$ is open and, since by definition $b(x) = \rho^{-2} b_0(W^{-1}(x))$, it follows that $\operatorname{supp} b \subset U'$. But now property 3) in Lemma 9.1.1 implies that, if ϵ is sufficiently small, then

$$R(\psi_1(W(\cdot)), \ldots, \psi_{N+1}(W(\cdot)))(x) \neq 0$$

for all $x \in U_0'$, and hence, by (9.2.4),

$$R(\psi_1, \ldots, \psi_{N+1}) \neq 0$$

for all $x \in U'$. This proves the claim. The same argument shows that, if ϵ is sufficiently small, then the functions $\psi_i\psi_j$, $1 \le i \le j \le N+1$ are linearly independent.

Summarising, we have obtained the following intermediate result: we have found open sets $U' \subset\subset D \subset \Omega$ and two $C^{0,\gamma}(\overline{\Omega})$ functions $a, b: \overline{\Omega} \to \mathbb{R}$, $\operatorname{supp} b \subset U'$, such that:

(1) the operator $L_a = L' + a$ satisfies the Poláčik condition on D, with Dirichlet boundary condition on ∂D;

(2) $R(\psi_1, \ldots, \psi_{N+1}) \neq 0$ for all $x \in U'$, where $\psi_1, \ldots, \psi_{N+1}$ is any $L^2(D)$-orthonormal basis of the kernel of L_a on D with Dirichlet boundary condition on ∂D;

(3) $\lambda < -\kappa$ for every eigenvalue λ of the operator $L_{a+b} = L' + a + b$ on D with Dirichlet boundary condition on ∂D.

Moreover,

(9.2.5) for every basis $\psi_1, \ldots, \psi_{N+1}$ of the kernel of L_a on D with Dirichlet boundary condition on ∂D the functions $\psi_i\psi_j$, $1 \le i \le j \le N+1$ are linearly independent.

Now let $H := L^2(\Omega)$, $V := H^1_0(\Omega)$ if we are working with Dirichlet boundary condition on $\partial\Omega$, $V := H^1(\Omega)$ if we are working with Neumann boundary condition on $\partial\Omega$; if $d: \mathbb{R}^N \to \mathbb{R}$ with $d|\overline{\Omega} \in C^0(\Omega)$, define $g_d: V \times V \to \mathbb{R}$ by

$$g_d(u, v) = \int_\Omega G(x)\nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega duv \, \mathrm{d}x,$$

where $G(x) := (a_{ij}(x))_{i,j}$. If $d \in C^{0,\gamma}(\overline{\Omega})$, regularity theory of PDEs implies that, both for Dirichlet and Neumann boundary condition, λ is an eigenvalue of $L_d = L' + d$ and u is a corresponding eigenvector if and only if λ is a proper value of g_d and u is a corresponding proper vector. (In fact, every proper vector of g_d lies in $C^{2,\gamma}(\Omega)$.) Let $c: \mathbb{R}^N \to \mathbb{R}$ be of class $C^{0,\gamma}$ and such that $c(x) = 0$ for $x \in D$ and $c(x) > 0$ for $x \notin D$. Furthermore, let $(\beta_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of positive numbers tending to ∞. Finally, for $k \in \mathbb{N}$ let $c_k: \mathbb{R}^N \to \mathbb{R}$ be a $C^{0,\gamma}(\overline{\Omega})$ function such that $a + \beta_k c_k$ is the restriction to $\overline{\Omega}$ of a polynomial function and $\sup_{x\in\Omega} |c_k(x) - c(x)| < 1/k\beta_k$. The existence of such a function c_k is obvious.

Let $L_k := L_{a+\beta_k c_k}$, $g_k := g_{a+\beta_k c_k}$ and let g_∞ be the restriction of g_a to $H^1_0(D)$. We are now in a position to apply Theorem 6.2.2: For $k \in \mathbb{N}$ let $(\lambda^k_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_k and $(u^k_n)_{n\in\mathbb{N}}$ be an H-orthonormal sequence of corresponding proper vectors of g_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_∞.

Then, using Theorem 6.2.2 and passing to a subsequence if necessary we may assume that there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $H^1_0(D)$ such that for every $n \in \mathbb{N}$, v_n is a proper vector of g_∞ corresponding to μ_n, $(\lambda^k_n)_{k\in\mathbb{N}}$ converges to μ_n and $(u^k_n)_{k\in\mathbb{N}}$ converges to $v_n{}^\sim$ in V, as $k \to \infty$. Set $p = N + 1$. There are numbers γ_1 $\gamma_2 \in \mathbb{R}$, M, $\eta \in \mathbb{R}_+$ and $l \in \mathbb{N}_0$, such that, setting $\mu_0 = -\infty$, we have

$$0 < \gamma_2 - \gamma_1 < M,$$

$$\mu_l < \gamma_1 - 4\eta < \gamma_1 < 0 = \mu_{l+1} = \mu_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \mu_{l+p+1}.$$

For $h \in C^0(\overline{\Omega})$ let $B_h \in \mathcal{L}_{\rm sym}(H,H)$ be the map

$$(Bu)(x) = h(x)u(x), \qquad u \in H,\ x \in \Omega.$$

Note that

$$|B_h|_{\mathcal{L}(H,H)} = |h|_{C^0(\overline{\Omega})}. \tag{9.2.6}$$

Let $\mathcal{G}$ be the set of all B_h with $h \in C^0(\overline{\Omega})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\rm sym}(H,H)$. Now (9.2.5) easily implies that the operator $T\colon \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B(v_i{}^\sim), v_j{}^\sim\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k\colon \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle Bu_i^k, u_j^k\rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$ so by Corollary 5.1.2

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \qquad \text{for } k \text{ large enough.}$$

Moreover, setting $\lambda_0^k = -\infty$, we have

$$\lambda_l^k < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^k \le \lambda_{l+p}^k < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}^k,\ k \text{ large enough.} \tag{9.2.7}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in Theorem 5.2.3. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^k| < \alpha_k < \alpha_0$ for $j = 1, \ldots, p$ and $\alpha_k \to 0$ as $k \to 0$. Thus by Theorem 5.2.3 (with $A := L_k$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^k$ for $j = 1, \ldots, p$ and $\mathcal{D}$ equal to the set of all B_h where h is the restriction of a polynomial function to $C^0(\overline{\Omega})$) there exists, for each large k, a polynomial function $h_k\colon \mathbb{R}^N \to \mathbb{R}$ such that $|h_k|_{C^0(\overline{\Omega})} < (1/2)\theta\alpha_k$ and such that if $(\hat{\lambda}_n^k)_{n\in\mathbb{N}}$

denotes the repeated sequence of eigenvalues of $L_{a+h_k+\beta_k c_k}$, $(\hat{u}_n^k)_{n\in\mathbb{N}}$ is an H-orthogonal sequence of the corresponding eigenfunctions and $\hat{\lambda}_0^k := -\infty$, then

$$\hat{\lambda}_l^k < \gamma_1 - 3\eta < \gamma_1 - \eta < \hat{\lambda}_{l+1}^k \le \hat{\lambda}_{l+p}^k < \gamma_2 + \eta < \gamma_2 + 3\eta < \hat{\lambda}_{l+p+1}^k \tag{9.2.8}$$

and

$$\hat{\lambda}_{l+j}^k = 0, \quad j = 1, \ldots, p. \tag{9.2.9}$$

Now the assumptions of Theorem 6.2.2 are satisfied with c_k replaced by $(1/\beta_k)h_k + c_k$. Therefore using Theorem 6.2.2 again and passing to a subsequence if necessary we may assume that there is a sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, $\hat{v}_n$ is a proper vector of g_∞ corresponding to μ_n, $(\hat{\lambda}_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(\hat{u}_n^k)_{k\in\mathbb{N}}$ converges to $\hat{v}_n{}^\sim$ in V, as $k \to \infty$. Moreover, by Theorem 6.3.1, $(\hat{u}_n^k|D)_{k\in\mathbb{N}}$ converges to $\hat{v}_n$ in $C^1_{\mathrm{loc}}(D)$ as $k \to \infty$. It follows that, if $U \subset D$ is an open set, $\operatorname{supp} b \subset U \subset\subset U'$, then, for all k large enough, $R(\hat{u}_{l+1}^k, \ldots, \hat{u}_{l+p}^k)(x) \neq 0$ for all $x \in U$. Finally, again by Theorem 6.2.2, if k is sufficiently large, all the eigenvalues of $L' + a + h_k + \beta_k c_k + b$ are $< -\kappa$. For every such k, $a + h_k + \beta_k c_k$ is a polynomial function and the conclusion follows with a replaced by $a + h_k + \beta_k c_k$. This proves the theorem. □

3. The Algebraic Independence Condition

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{2,\gamma}$ boundary. Let $a_{ij}: \overline{\Omega} \to \mathbb{R}$, $i, j = 1, 2$, be of class $C^{1,\gamma}$, $a_{ij} \equiv a_{ji}$, $i, j = 1, 2$, and

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i i\xi_j \ge |\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^2$$

for some $c > 0$. Let us consider the differential operator

$$L' = \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j). \tag{9.3.1}$$

In this section we want to prove that, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, we can construct a potential $a: \overline{\Omega} \to \mathbb{R}$ of class C^∞ such that the operator

$$L = L' + a(x) = \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x) \tag{9.3.2}$$

has a kernel of a prescribed dimension, spanned by eigenfunctions satisfying the algebraic independence condition (IC) in Section 4.1 up to a prescribed order k with an appropriate $\varpi \in \mathbb{R}^2$. We will prove the following:

Theorem 9.3.1. *Let L' as above and let $n, k \in \mathbb{N}$. then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, there exists a potential $a: \overline{\Omega} \to \mathbb{R}$ of class $C^\infty(\overline{\Omega})$ with the following properties:*

(1) *the operator L in (9.3.2) has an n-dimensional kernel;*

(2) *there exists a vector $\varpi \in \mathbb{R}^2$ and an $L^2(\Omega)$-orthonormal basis $u_1, \ldots, u_n$ of the kernel of L such that the algebraic independence condition (IC) in Section 4.1 is satisfied up to the order k.*

Proof. As in the proof of Theorem 9.2.1, our starting point is the existence (extablished in Chapter 8) of such a potential for any smooth bounded domain when $a_{ij}(x) \equiv \delta_{ij}$, i.e. $L' = \Delta$, and with $\varpi = (0, 1)$. So we can always take a bounded smooth domain S and a smooth potential $a_0: \overline{S} \to \mathbb{R}$ such that:

(1) the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel;

(2) there is an $L^2(S)$-orthonormal basis $\phi_1, \ldots, \phi_n$ of the kernel of $\Delta + a_0(x)$ such that (IC) is satisfied up to the order k with $\varpi = (0, 1)$, i.e. for every $l = 1, \ldots, k$ and every q, $1 \le q \le l$, the functions

$$\left\{ \sum_{\substack{\gamma \le \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta - \gamma)!} \phi^{\beta - \gamma + \epsilon_j} \phi_y^\gamma \right\}_{\substack{j=1,\ldots,n \\ |\beta| = l}}$$

are linearly independent.

Moreover, the functions $\phi_i \phi_j$, $1 \le i \le j \le n$ are linearly independent. Now, as in the proof of Theorem 9.2.1, we apply Lemma 9.1.1; we obtain that, if we choose a sufficiently small ϵ, then, for some potential a, the kernel of $L + a$ on D with Dirichlet condition on ∂D is spanned by $L^2(D)$-orthonormal functions $\psi_1, \ldots, \psi_n$ such that for every $l = 1, \ldots, k$, and for every q, $1 \le q \le l$, the functions

$$\left\{ \sum_{\substack{\gamma \le \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta - \gamma)!} \psi(W(\cdot))^{\beta - \gamma + \epsilon_j} \psi(W(\cdot))_y^\gamma \right\}_{\substack{j=1,\ldots,n \\ |\beta| = l}}$$

are linearly independent on S. Since, for $i = 1, \ldots n$,

$$\psi_i(W(\cdot))_y = (\nabla \psi_i)(W(\cdot)) \cdot \varpi,$$

where ϖ is the second column of the (constant) matrix $DW(\cdot)$, we reach that, for every $l = 1, \ldots, k$ and for every q, $1 \leq q \leq l$, the functions

$$\left\{ \sum_{\substack{\gamma \leq \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta - \gamma)!} \psi^{\beta - \gamma + \epsilon_j} \psi_{\varpi}^{\gamma} \right\}_{\substack{j=1,\ldots,n \\ |\beta| = l}}$$

are linearly independent. Moreover, the functions $\psi_i \psi_j$, $1 \leq i \leq j \leq n$ are linearly independent on D. Finally, we conclude arguing exactly as in the proof of Theorem 8.2.1, applying Theorem 6.2.2, Theorem 6.3.1 and Theorem 5.2.3. $\square$

Remark. The present result generalizes naturally to any space dimension $N \geq 2$ (see Section 8.3).

CHAPTER X

REMARKS AND PROBLEMS

Vector field and jet realizations are a useful tool for giving examples of complex dynamics in different classes of dynamical systems. All of them are generated by equations of a very particular form and presence of complicated dynamics is not obvious.

First, consider the equation

$$u_t = u_{xx} + c(x)\alpha(u) + f(x,u), \quad x \in]0,1[, \tag{10.0.1}$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $c\colon [0,1] \to \mathbb{R}$ are sufficiently regular functions and $u \mapsto \alpha(u)$ is a nonlocal linear functional on $L^2(0,1)$ of the form

$$\alpha(\theta) = \int_0^1 \nu(x)\theta(x)dx. \tag{10.0.2}$$

Equation (10.0.1) under the boundary condition

$$u(0,t) = u(1,t) = 0 \tag{10.0.3}$$

has been studied in [8], where a jet realization result has been proved. Specifically, any jet whose linear part has simple eigenvalues can be realized in (10.0.1)–(10.0.3). However, the restriction on the linear part does not allow to apply the scaling arguments of [23, Sect. 2] in order to obtain realizability of a dense subset of vector fields. Adding another integral term $d(x)\beta(u)$ to (10.0.1), then any family of small jets in $\mathbb{R}^3$ can be realized, and this implies presence of chaos in some equation of this form.

In [27] Poláčik and Šošovička consider a nonlocal equation of the form

$$u_t = u_{xx} + F(u, \alpha(u)), \quad -1 < x < 1, \; t > 0, \tag{10.0.4}$$

where $F\colon \mathbb{R}^2 \to \mathbb{R}$ is smooth and

$$\alpha(u) := \int_{-1}^1 u(x)dx, \quad u \in L^2(-1,1);$$

they have proved that, both with Dirichlet and Neumann boundary condition, for any given set of complex numbers, one can arrange, choosing the equation properly, that this set is contained in the spectrum of the linearisation. Moreover, they have proved that equations of the above form can undergo a supercritical Hopf bifurcation to an asymptotically stable periodic solution.

Another special class of equations where vector field and jet realizations problems have been studied is the class of scalar delay equations. In [11] Hale investigated the reduced equation representing the flow on a p-dimensional center manifold of the equation with $p-1$ delays

$$\dot{x} = f(x(t), x(t-r_1), \ldots, x(t-r_{p-1})). \tag{10.0.5}$$

It is interesting to find what kind of reduced equations can be obtained. As shown by Hale, this cannot be an arbitrary ODE; it is always a scalar ODE of p-th order. However, within this limit, the reduced equation can be arbitrary, as shown by Rybakowski in [34]. Several result discussing the numbers of delays needed for realization of certain bifurcation problems have been obtained by Faria and Magalhaes in [7].

Returning to scalar parabolic PDEs, we briefly discuss some results which are in some sense related to the subject of this thesis. In the introduction we have pointed out that a semilinear parabolic equation with a nonlinearity which is independent of gradient terms admits a global Ljapunov functional. This excludes chaotic behaviour of solutions. However one can ask whether, as in the one dimensional case, all bounded trajectories converge. The answer is negative: in [26], Poláčik and Rybakowski have shown that a nonlinearity $f = f(x, u)$ can be found in such a way that the equation

$$u_t = \Delta u + f(x, u) \tag{10.0.6}$$

on the ball of $\mathbb{R}^2$ with Dirichlet boundary condition admits a solution whose ω-limit is diffeomorphic to $\mathbb{S}^1$.

Another interesting problem deals with equations without explicit x-dependence, that is

$$u_t = \Delta u + f(u, \nabla u). \tag{10.0.7}$$

In [6], Dancer and Poláčik give a jet realization result for this class of equations.

We obtain another important class of equations if we introduce periodic time dependence in (10.0.6), that is we consider

$$u_t = \Delta u + f(t, x, u), \tag{10.0.8}$$

where $f(t + \tau, \cdot, \cdot) = f(t, \cdot, \cdot)$ for some $\tau > 0$. In [5] Dancer proved that, if the space dimension is larger or equal to 2 or if the space dimension is 1 but we take periodic boundary conditions, then a nonlinearity f can be found such that (10.0.8) has a two-dimensional invariant torus which contains no periodic solutions. In [1], Poláčik announce that arbitrary (chaotic) dynamics can be found in the period map of (10.0.8) by adjusting the nonlinearity f. For the more general equation

$$u_t = u_{xx} + f(t, x, u, u_x), \tag{10.0.9}$$

Fiedler and Sandstede have proved in [9], using a vector field realization result, that shift dynamics can be found in an appropriate problem (10.0.9) with periodic boundary conditions.

We conclude by discussing some open problems. First, we remark that, since parabolic equations belong to the class of strongly monotone dynamical systems (see e.g. [17] and [14]), any interesting invariant set that we find in

$$u_t = \Delta u + f(x, u, \nabla u) \tag{10.0.10}$$

is unstable. It would be nice if one could find such invariant sets with the least possible instability (that is, with unstable manifold of dimension 1). This leads to the problem of finding an elliptic operator L on some open set Ω whose second eigenvalue has high multiplicity. None of the methods exploited up to now seems to give any result in this direction.

A second question is the following: as we have explained in Ch. 4, two space dimensions are enough to obtain realizability of a dense subset of vector fields in $\mathbb{R}^n$ for any n. It seems rather reasonable to conjecture that actually any vector field in $\mathbb{R}^n$ can be realized in an equation of type (10.0.10) on a two-dimensional domain, but up to now a result of this kind seems to be hardly reachable.

References

1. H. W. Alt, *Lineare Funktionalanalysis*, 2. Auflage, Springer-Verlag, Berlin Heidelberg New York, 1992.
2. L. Bers, F. John, M. Schechter, *Partial Differential Equations*, Interscience Publishers, NY, 1964.
3. I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, inc., Orlando, Florida, 1984.
4. S. N. Chow, K. Lu, *Invariant Manifolds for Flows in Banach Spaces*, J. Differential Equations **81** (1988), 285–317.
5. E. N. Dancer, *On the Existence of Two-dimensional Invariant Tori for Scalar Parabolic Equations with Time Periodic Coefficients*, Annali Scuola Norm. Sup. Pisa **43** (1991), 455 – 471.
6. E. N. Dancer and P. Poláčik, *Realization of Vector Fields and Dynamics of Spatially Homogeneous Parabolic Equations*, preprint.
7. T. Faria and L. Magalhães, *Realization of Ordinary Differential Equations by Retarded Functional Differential Equations in Neighborhoods of Equilibrium Points*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 759–770.
8. B. Fiedler and P. Poláčik, *Complicated Dynamics of Scalar Reaction-Diffusion Equations with a Nonlocal Term*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), 167–192.
9. B. Fiedler and B. Sandstede, *Dynamics of Periodically Forced Parabolic Equations on the Circle*, Ergodic Theory Dynamical Systems **12** (1992), 559–571.
10. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin Heidelberg, 1983.
11. J. K. Hale, *Flows on Centre Manifolds for Scalar Functional Differential Equations*, Proc. Roy. Soc. Edinburgh Sect. A **101** (1985), 193–201.
12. D. Henri, *Geometric Theory of Parabolic Equations*, Lecture notes in mathematics, Vol 840, Springer-Verlag, NY, 1981.
13. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin Heidelberg, 1976.
14. H. Matano, *Convergence of Solutions of One-dimensional Semilinear Parabolic Equations*, J. Math. Kyoto Univ. **18** (1978), 221–227.
15. C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer-Verlag, Berlin Heidelberg.
16. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, NY, 1983.
17. P. Poláčik, *Convergence in Strongly Monotone Flows Defined by Semilinear Parabolic Equations*, J. Differential Equations **79** (1989), 89–110.
18. P. Poláčik, *Complicated Dynamics in Scalar Semilinear Parabolic Equations in Higher Space Dimension*, J. Differential Equations **89** (1991), 244–271.
19. P. Poláčik, *Imbedding of Any Vector Field in a Scalar Semilinear Parabolic Equation*, Proc. Amer. Math. Soc. **115** (1992), 1001–1008.
20. P. Poláčik, *Realization of any Finite Jet in a Scalar Semilinear Equation on the Ball in* $\mathbb{R}^3$, Ann. Scuola Norm. Sup. Pisa **XVII** (1991), 83–102.
21. P. Poláčik, *Realization of the Dynamics of ODEs in Scalar Parabolic PDEs*, Tatra Mountains Math. Publ. **4** (1994), 179–185.
22. P. Poláčik, *Transversal and Non Transversal Intersections of Stable and Unstable Manifolds in Reaction Diffusion Equations on Symmetric Domains*, Diff. Int. Equations **7** (1994), 1527–1545.
23. P. Poláčik, *High-dimensional* ω*-limit Sets and Chaos in Scalar Parabolic Equations*, J. Differential Equations **119** (1995), 24–53.
24. P. Poláčik, *Reaction-diffusion Equations and Realization of Gradient Vector Fields*, Proc. Equadiff. (1995) (to appear).
25. P. Poláčik and K. P. Rybakowski, *Imbedding Vector Fields in Scalar Parabolic Dirichlet BVPs*, Ann. Scuola Norm. Sup. Pisa **XXI** (1995), 737–749.
26. P. Poláčik and K. P. Rybakowski, *Nonconvergent Bounded Trajectories in Semilinear Heat Equations*, J. Differential Equations **124** (1995), 472–494.

27. P. Poláčik and V. Šošovička, *Stable Periodic Solutions of a Spatially Homogeneous Nonlocal Reaction-diffusion Equation*, Proc. Roy. Soc. Edinburgh **126A** (1996), 867–884.
28. J. Pöschel, E. Trubowitz, *Inverse Spectral Theory*, Academic Press, inc., Orlando, 1987.
29. M. Prizzi and K. P. Rybakowski, *Complicated Dynamics of Parabolic Equations with Simple Gradient Dependence*, to appear, Trans. Am. Math. Soc..
30. M. Prizzi and K. P. Rybakowski, *Inverse Problems and Chaotic Dynamics of Parabolic Equations on Arbitrary Spatial Domains*, to appear, J. Differential Equations.
31. M. Prizzi, *Realization of Jets and Vector Fields in Scalar Parabolic Equations*, preprint SISSA, submitted.
32. K. P. Rybakowski, *An Abstract Approach to Smoothness of Invariant Manifolds*, Applicable Analysis **49** (1993), 119-150.
33. K. P. Rybakowski, *Realization of Arbitrary Vector Fields on Center Manifolds of Parabolic Dirichlet BVPs*, J. Differential Equations **114** (1994), 199-221.
34. K. P. Rybakowski, *Realization of Arbitrary Vector Fields on Invariant Manifolds of Delay Equations*, J. Differential Equations **114** (1994), 222–231.
35. K. P. Rybakowski, *The Center Manifold Technique and Complex Dynamics of Parabolic Equations*, Topological Methods in Differential Equations and Inclusions (A. Granas M. Frigon, eds.), NATO ASI Series, vol. 472, Kluwer Academic Publishers, Dordrecht/Boston/London, 1995, pp. 411–446.
36. W. A. Strauss, *Partial Differential Equations*, John Wiley and Sons, inc., NY, 1992.
37. A. Vanderbauwhede, S. A. Van Gils, *Center Manifolds and Contractions on a Scale of Banach Spaces*, J. Func. Analysis **72** (1987), 209–224.