Stability and Reconstruction for the Determination of Boundary Terms by a Single Measurement

CANDIDATE
Eva Sincich

SUPERVISOR
Prof. Giovanni Alessandrini

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Chapter 1

Introduction

Given a mathematical problem, the corresponding inverse problem is one where the roles of the data and the unknown are reversed. The study of inverse problems had a great development in the last twenty years, in connection with applied sciences and technology. Indeed the analysis of a phenomenon arising in mathematical physics requires the knowledge of parameters, which in the classical model, described by the direct problem, are usually assumed to be given, whereas in practice they are not available. Thus the models need a preliminary treatment in which parameters are recovered by the measurements on the fields, which in theory are considered as unknown. It often happens that the quantities of main interest are the parameters rather than the fields, indeed they are related to the internal properties of the material subject to those fields, or remote objects, that are out of reach. The direct problem consists in finding the fields when the parameters are known, whereas the corresponding inverse problem is to determine the parameters from the measurements on the fields.
On the other hand, a common difficulty which occurs in treating most of inverse problems and which characterizes them, is due to their ill-posedness, that means, in the classical Hadamard sense, that the solution either does not exists, is not unique or does not depend continuously on the data. In many inverse problems the availability of some additional informations on the solutions, as for instance bounds on the size, knowledge on the smoothness or on the shape may reduce the range of admissible parameters. One of the main topic addressed by inverse problems is to establish when such additional information enable to restore for instance the continuous dependence on the data and to quantify the rate of such a dependence. The main issues for an accurate discussion of inverse problems are uniqueness and stability, indeed uniqueness and stability results constitute a fundamental step in treating these problems, since they provide informations to establish whether or not a reconstruction procedure is applicable or a parameter can be recovered in a stable manner.
A fundamental example of inverse problem is given by the *inverse conductivity problem*, which is related with the Electrical Impedance Tomography, an imaging technique that has applications in medical imagining, underground prospec-

tion and non destructing testing. The aim is to detect the internal electrical conductivity of a conductor body by taking repeated electrical measurements from its surface. The corresponding direct problem consists in a well posed Dirichlet problem for a linear elliptic partial differential equation. Namely, if the conductivity is known, then for every voltage potential on the boundary one can determine the current density at the boundary. In other terms, one can recover the so-called Dirichlet to Neumann map which associates to every boundary voltage the corresponding current density. The inverse conductivity problem consists in determining the conductivity from the electrical measurements taken from the boundary, that is from the knowledge of the Dirichlet to Neumann map. Contrary to the direct problem this is an highly non linear problem. The mathematical model of such a problem has been introduced by Calderon [22] and developed with uniqueness results by Kohn and Vogelius [47], Sylvester and Uhlmann [67] and later by Nachman [62]. The stability issue was resolved by Alessandrini [7] and, more recently, Mandache in [59] has proved that the logarithmic type of stability obtained in [7] is optimal.
Among the variety of inverse problems present in the literature, let us examine in more detail two problems concerning the determination of inaccessible boundary terms. Such type of problem arise in non-destructive testing. Indeed they are related, for instance, to the phenomenon of corrosion in metals. In applied contexts the surface portion of the metal specimen where the corrosion takes place in not accessible. Thus to investigate whether the material is corroded or not one has to solve the inverse problem of recovering an unknown boundary term, which models the presence of corrosion, by the available measurements. The study of such a problem has been discussed by many authors, among them let us mention the following Alessandrini, Del Piero, Rondi [9], Chaabane, Fellah, Jaoua, Leblond [23, 24], Bryan, Kavian, Vogelius, Xu [18, 45, 70], Fasino, Inglese [35]. The same boundary value problem models also the phenomenon of the stationary heat conduction, as introduced by Chaabane and Jaoua in [25]. Moreover, the inverse problem of detecting unknown boundary terms arises also in the inverse scattering literature. Indeed, it often happens that hostile objects are coated by a material with unknown surface impedance. This phenomenon is modeled by a boundary condition where the boundary impedance plays the role of the unknown. The main contributions to this problem are due to Cakoni, Colton, Kress, Monk, Piana [6, 20, 21, 29, 30].
In this framework we shall treat two kinds of inverse problems, concerning the determination of *unknown boundary terms*.
We shall focus our attention on the *stability* issue, that is the continuous dependence of the unknown boundary term upon the measurements. Actually, we shall deal with the conditional stability, that means to study such a dependence under some additional assumptions on the data of the problems and especially under the a priori information on the boundary terms themselves. For a general theoretical setting on conditional stability, see for example, [55]. Let us also stress that we are interested not only in a qualitative stability analysis, but also in a quantitative one. In fact we shall exhibit an explicit evaluation of the modulus of continuity of such a dependence, which will turn out to be of logarithmic

type.
Furthermore, as a consequent step of the stability analysis, we shall discuss the *reconstruction* issue, that is the approximate identification of the boundary term by the approximate measurements.

An inverse corrosion problem

We shall discuss an inverse boundary value problem arising in *corrosion* detection. The aim of such a problem is to determine a *nonlinear* term in a boundary condition, which models the possible presence of corrosion damage, by performing a finite number of current and voltage measurements on the boundary. This means to apply a nontrivial current density on a suitable portion of the boundary of the conductor and to measure the corresponding voltage potential on the same portion.
In Chapter 3 and Chapter 4 we shall discuss respectively the stability and the reconstruction issues for this inverse problem, obtaining a stability result and proposing a reconstruction method under some suitable a priori assumptions on the data of the problem, which are the conductor and the prescribed current density and under a priori bounds on the nonlinear term itself.
Before discussing the details of this topic, let us overview the main contributions to this kind of problem given in recent years, pointing out their common formulation as well as the different choices of the boundary term which models the electrochemical phenomenon of surface corrosion in metals.
The physical problem is modeled as follows. A bounded Lipschitz domain Ω in $\mathbb{R}^n$ represents the region occupied by the electrostatic conductor which contains no sources and no sinks and this is modeled by the Laplace operator, so that the voltage potential u satisfies

$$\Delta u = 0 \quad \text{in } \Omega. \tag{1.1}$$

The simplified model of corrosion appearance reduces to the problem of recovering a coefficient $\varphi = \varphi(x)$ in a *linear* boundary condition of the type

$$\frac{\partial u}{\partial \nu} = -\varphi u, \tag{1.2}$$

where ν is the outward unit normal at the boundary and $\varphi \geq 0$ is the so-called *Robin coefficient*. The study of such a problem has been developed by many authors, among them, let us illustrate the following. Alessandrini, Del Piero, Rondi [9] and Chaabane, Fellah, Jaoua, Leblond [23] have established a stability result for the Robin problem in a two dimensional setting using tools of analytic function theory and quasiconformal mappings. Chaabane, Jaoua and Leblond [24] have provided a constructive procedure in order to solve the Robin problem by means of complex analysis. Chaabane and Jaoua [25] have obtained a Lipschitz stability estimate provided the Robin coefficient depends on a scalar parameter only. Let us also refer to Fasino and Inglese [35, 36, 37], who have introduced numerical methods relied on the thin-plate approximation

and the Galerkin method, beyond a logarithmic stability estimate and results concerning the relation between stability of the solution and thickness of the domain.
A more accurate model of corrosion requires a nonlinear relationship between voltage and current density on the corroded surface. A model of this kind, known as the Butler and Volmer model, postulates the boundary condition

$$\frac{\partial u}{\partial \nu} = \lambda(\exp(\alpha u) - \exp(-(1-\alpha)u)). \tag{1.3}$$

Such a *nonlinear* boundary value problem, has been recently discussed by Bryan, Kavian, Vogelius and Xu in [18, 45, 70]. The authors have examined the questions of the existence and the uniqueness of the solution of the problem with a given nonlinearity of the type (1.3). Namely, they have assumed to know the nonlinearity (1.3), by prescribing the coefficients λ and α in suitable ranges, and they have discussed the existence and the uniqueness issues for the direct problem.
In this thesis, motivated by these studies, we have considered a more general choice of the nonlinear profile, namely of the form

$$\frac{\partial u}{\partial \nu} = f(u) \ , \tag{1.4}$$

and we have dealt with the inverse problem. In other terms, we have considered the issue of the identification of the nonlinearity f, which is indeed unknown in practical applications.
Let us also observe that a further aspect arising in the study of the corrosion phenomenon consists in the recovery of the shape of the boundary where corrosion occurs. In this respect our results on the determination of the boundary coefficients involved in the corrosion model, can be read as one step in the process of the treatment of the full inverse problem. Indeed the main steps of such a treatment can be outlined as follows. The first one relies in the determination of the nonlinearity f when the geometry of the conductor is prescribed, which is indeed one of the topics discussed in this thesis. Once the nonlinearity f has been recovered, the second step consists in the determination of the shape and the location of the defect by the knowledge of the boundary condition satisfied by the potential on the unknown surface. For instance such type of problems have been discussed by Alessandrini and Rondi in [13, 14, 65, 64] for the identification of cracks, cavities and material losses at the boundary.
Let us now give the formulation of our problem. We assume that the boundary of the conductor, which is modeled by the domain Ω, is decomposed in three open, nonempty and disjoint portions $\Gamma_1, \Gamma_2, \Gamma_D$, one of which, say Γ_2, is accessible to the electrostatic measurements, whereas the portion Γ_1, where the corrosion takes place, is out of reach. The remaining portion Γ_D, which separates Γ_1 from Γ_2, is assumed to be grounded.
We prescribe a current density on the accessible part of the boundary Γ_2, given by an Hölder function $g \in C^{0,\alpha}(\Gamma_2)$ with Hölder exponent α, $0 < \alpha < 1$, satisfying furthermore a lower bound to be stated later on. Moreover, as already

remarked, the possible presence of corrosion damage is modeled by a nonlinear term in a boundary condition of the form (1.4), such that the profile f satisfies an a priori bound on its Lipschitz continuity as well as a compatibility condition to be specified in the sequel.
Then the *direct* problem amounts to find the harmonic potential u in the metal specimen Ω, given the current density g and the nonlinear profile f, from the following mixed boundary value problem

$$\begin{cases} \Delta u = 0 \ , & \text{in } \Omega \ , \\ \dfrac{\partial u}{\partial \nu} = g \ , & \text{on } \Gamma_2 \ , \\ \dfrac{\partial u}{\partial \nu} = f(u) \ , & \text{on } \Gamma_1 \ , \\ u = 0 \ , & \text{on } \Gamma_D \ . \end{cases} \tag{1.5}$$

Let us observe that, according with the result in [18, 45, 70], we have that also under the previous mild assumption on the nonlinearity f, the direct problem (1.5) might not be well-posed. This is, for instance, the case when $f(u) = pu$ where $p > 0$ is an eigenvalue for the Steklov type eigenvalue problem

$$\begin{cases} \Delta v = 0 \ , & \text{in } \Omega \ , \\ \dfrac{\partial v}{\partial \nu} = 0 \ , & \text{on } \Gamma_2 \ , \\ \dfrac{\partial v}{\partial \nu} = pv \ , & \text{on } \Gamma_1 \ , \\ v = 0 \ , & \text{on } \Gamma_D \ . \end{cases} \tag{1.6}$$

The *inverse* problem reads as follows. We assume that the conductor, modeled by the domain Ω, and the decomposition into the three portions $\Gamma_1, \Gamma_2, \Gamma_D$, are given. We impose a non trivial current density g on the accessible part of the conductor Γ_2 and we measure the corresponding electrostatic potential u, solution to the problem (1.5) upon the same portion of the boundary. By the pair of boundary measurements $\{u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}|_{\Gamma_2}\}$, we want to recover the unknown nonlinear profile f on the inaccessible portion Γ_1 of the conductor.

An inverse scattering problem

In Chapter 5 an inverse scattering problem arising in target identification is considered. Indeed, in order to avoid detection by radar, hostile targets are typically coated by some material on a portion of the boundary designed to reduced the radar cross section of the scattered wave. We want to recover the surface *impedance* of a partially coated obstacle by collecting a finite number of measurements of the far field pattern. In practice, this corresponds to prescribe an incoming plane wave which is scattered by an obstacle and to measure the amplitude of the corresponding scattered wave.
We are concerned with the stability issue for this problem, limiting our study to the three dimensional case. Indeed we shall prove a stability result up to assume some a priori hypothesis on the data of the problem, which are the obstacle, the

wave number and the incident direction of the incoming wave, beyond some a priori assumptions on the unknown surface impedance.
Such a problem has been already discussed by many authors, among them, let us cite Akduman, Cakoni, Colton, Kress, Piana [6, 20, 29, 30], who have extensively developed the reconstruction issue. Such a problem, in two dimensions, has been recently studied by Cakoni and Colton in [20]. The authors have provided a variational method for the determination of the essential supremum of the surface impedance when the far field data are available. Moreover, they have extended such a result to the vector case of the Maxwell's equation, beyond considering several numerical examples when the surface impedance is constant. Moreover, Colton, Kress and Piana [29, 30] have considered the problem of determining lower bounds for the surface impedance, while in [6] Akduman and Kress have introduced a potential theoretic method for determining the surface impedance when the obstacle is completely coated. On the contrary the stability issue under *mild* a priori assumptions, as far as we know, has not yet been studied.
Let us now illustrate the mathematical model which describes the phenomenon of the scattering of an incident wave by a partially coated obstacle. A bounded Lipschitz domain D in $\mathbb{R}^3$ represents the region occupied by the impenetrable object. We consider the scattering of a given acoustic incident time-harmonic plane wave, at a given wave number $k > 0$ and at a given incident direction $\omega \in \mathbb{S}^2$, by the obstacle D. The total field u, given as the sum of the scattered wave u^s and the incident plane wave $\exp(ikx \cdot \omega)$, satisfies the Helmholtz equation in the exterior of the domain D. Moreover, we assume that the boundary has a Lipschitz dissection in two open, connected and disjoint portions Γ_I and Γ_D, such that on Γ_I the total field satisfies an impedance boundary condition of the form

$$\frac{\partial u}{\partial \nu} + i\lambda(x)u = 0 \ , \tag{1.7}$$

which characterizes obstacle for which the normal velocity on the boundary is proportional to the excess of pressure on the boundary. The surface impedance λ satisfies an a priori bound on its Lipschitz continuity and a technical condition that will be specified in the course of the exposition. On the remaining part of the boundary the tangential component of the total field vanishes.
Then the *direct* problem is to find the total field $u = u^s + \exp(ikx \cdot \omega)$ from the following mixed boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} + i\lambda(x)u = 0, & \text{on } \Gamma_I. \end{cases} \tag{1.8}$$

Moreover, the scattered field u^s is required to satisfy the so-called *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \qquad r = \|x\|, \tag{1.9}$$

which guarantees that the scattered wave is outgoing. The well-posedness of the direct problem (1.8) has been proved, in two dimensions and for a constant λ, by Cakoni, Colton and Monk in [21]. However, we shall observe in the sequel that the arguments of potential theory developed in [21], can be adapted to our setting.
It is well-known that the radiation condition (1.9) yields the following asymptotic behavior

$$u^s(x) = \frac{\exp(ikr)}{r}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{r}\right)\right\} \ , \tag{1.10}$$

as r tends to ∞, uniformly with respect to $\hat{x} = \frac{x}{\|x\|}$ and where u_∞ is the so-called far field pattern of the scattered wave, (see for instance [32]).
The *inverse* problem is the following. We assume that the scatterer, modeled by the domain D with boundary decomposed in two portions Γ_I and Γ_D, is given. We prescribe an incident plane wave and we measure the corresponding scattering amplitude u_∞. Our aim is to recover the unknown surface impedance λ by using this additional measurement on u_∞.

Stability and reconstruction results for the inverse corrosion problem

In Chapter 3 and Chapter 4 we shall collect the stability and reconstruction results for the inverse corrosion problem obtained in [15] and in [16]. Let us start the analysis of such a problem by discussing the stability issue. As already pointed out, in the context of Inverse Problems stability means the continuous dependence of the unknown boundary terms upon the electrostatic measurements.
The main cause of ill-posedness of the present problem consists in the solution of a Cauchy problem, which, as it is well-known by the work of Hadamard (see for instance [40]), is severely ill-posed. Indeed, in the inverse corrosion problem, the knowledge of the solution is restricted to the Cauchy data $\{u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}|_{\Gamma_2}\}$ on the accessible portion Γ_2 of the conductor, thus, to recover the needed information, first in the interior of the domain and consequently on the inaccessible portion Γ_1, a Cauchy problem has to be solved.
Hence to restore stability we have to require some suitable additional assumptions on the data of the problems and particularly we have to assume some a priori information on the unknown boundary terms that we wish to recover.
Infact, since the direct problem (1.5) might not be well-posed, it seems natural to require an a priori energy bound on the electrostatic potential u within the conductor,

$$\int_\Omega |\nabla u(x)|^2 \leq E^2 \ . \tag{1.11}$$

Next, we require an a priori bound of the Lipschitz continuity of f, namely

$$|f(u) - f(v)| \leq L|u - v| \ , \text{ for every } u, v \ \in \mathbb{R} \ . \tag{1.12}$$

Moreover, in order to treat this inverse problem, we shall assume the knowledge of some additional information on the measured current density g on the accessible part of the boundary Γ_2. More precisely, we assume a bound on the Hölder continuity of g

$$\|g\|_{C^{0,\alpha}(\Gamma_2)} \leq G \ . \tag{1.13}$$

Also, we shall require a lower bound on the same current density g. Namely, we shall prescribe that, for a given inner portion $\Gamma_2^{2r_0}$ of Γ_2, and a given number $m > 0$, we have

$$\|g\|_{L^\infty(\Gamma_2^{2r_0})} \geq m > 0 \ . \tag{1.14}$$

Let us now overview the main features of the inverse corrosion problem , focusing our attention on the main difficulties that arise as well as the methods used to overcome them.
The reasons of the ill-posedness are essentially two and they can be summarized as follows.

i) The first one, that, as already observed, embodies one of the main causes of their ill-posedness, consists in the solution of a severely ill-posed Cauchy problem;

ii) the other cause of ill-posedness is due to the problem of determining f and the domain on which it is defined.

Our stability estimates will be mainly achieved by combining the results obtained treating step i) and step ii).
We shall approach the issue of step i) by considering a stability estimate near the boundary for a Cauchy problem. Infact, since we have access only to the Cauchy data on Γ_2 for a solution u to the problem (1.5), we shall need to evaluate how much the error on such data can affect the interior values of u near Γ_2. We obtain such an evaluation by handling an inequality due to Payne [63] and then developed by Trytten [69]. As a consequent step, we shall study the propagation of the error in the interior of the domain. Such a study leads to an Hölder type stability result, which will be obtained by means of quantitative estimates of unique continuation as the iterated use of the *three spheres inequality*. Such an Hölder stability estimate holds, as long as we consider interior values of the solution in the domain. Hence to obtain a stability result up to the boundary we shall deal with a minimization argument, that, of course, makes worse the estimate leading to a logarithmic type one. Let us also stress that the minimal Lipschitz assumption on the regularity of the domain Ω is actually needed since it ensures the uniform cone condition which will play a crucial role in the proof of the stability result up to the boundary. Let us also remark, that as a preliminary analysis on the direct problem, we shall prove by means of the Moser iteration techniques, that the solution is Hölder continuous with its first order derivatives in a suitable neighborhood of the inaccessible portion Γ_1 of the boundary.
For what concerns step ii), it has to be noticed that, since one can expect to identify the corrosion profile f only on the range of values taken by the voltage

potential u on the corroded part of the boundary and since it is not a priori given, it follows that the unknown of the problem are indeed the domain upon which f may be determined, beyond the profile f on such a domain. Thus as preliminary step of the treatment of this inverse problem, we shall prove a lower bound on the oscillation of u on Γ_1, namely

$$\operatorname*{osc}_{\Gamma_1} u \geq \mathit{const.} \exp\Big(-\big(\mathit{const.}\ m\big)^{-\gamma}\Big) , \tag{1.15}$$

where γ is a positive exponent such that $\gamma > 1$. The proof of such a result shares the same spirit of the one used in treating step i).
By the lower bound on the oscillation we obtain a quantitative control from below of the tangential gradient of the solution along its steepest descent direction. Such a control will allow us to state a local monotonicity property for the solution along a suitable curve on Γ_1, as well as an evaluation of its length. The set of the images of the solution on such a curve will constitute the range of values where the nonlinearity f will be identify. Infact we will show that if u_1 and u_2 are two potentials corresponding to nonlinearities f_1 and f_2 whose Cauchy data are close

$$\|u_1 - u_2\|_{L^2(\Gamma_2)} \leq \varepsilon \ ,$$
$$\left\|\frac{\partial u_1}{\partial \nu} - \frac{\partial u_1}{\partial \nu}\right\|_{L^2(\Gamma_2)} \leq \varepsilon \ ,$$

then the ranges of u_1 and u_2 on Γ_1 agree on an interval V, such that

$$\text{length of } V \ \sim \ \exp\left[-\left(\frac{m}{c}\right)^{-\gamma}\right]. \tag{1.16}$$

As a consequence of the above result and the local monotonicity property, we shall consider the inverse functions of u_1 and u_2 restricted to the interval V of the common values of u_1 and u_2. Hence, by inverting u_1 and u_2 respectively we can pass from a value u in V to a point x_1 and x_2 on Γ_1 and viceversa. This connection shall provide us a useful tool to express the difference between the scalar functions f_1 and f_2 defined on the real interval V in terms of the difference between the normal derivatives of u_1 and u_2 evaluated in the points x_1 and x_2 on Γ_1. By this relation, we will be able to prove that the nonlinearities f_1 and f_2 agree up to an error of the type

$$\left|\log\left(\frac{1}{\varepsilon}\right)\right|^{-\theta} \ , \tag{1.17}$$

where $0 < \theta < 1$.

For what concerns the reconstruction issue, let us recall that we shall term reconstruction the inverse problem of the approximate identification of the nonlinear term f by the approximate electrostatic measurements $\{u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}|_{\Gamma_2}\}$, u

being the solution to (1.5), under some suitable a priori assumptions on the data of the problem and some a priori bounds on the nonlinearity f. Indeed the Cauchy data will be affected by errors since they are given by finitely many samples. Thus, as a consequence, we can expect to recover the nonlinearity f only in an approximate manner.
In this setting the stability analysis just discussed, can be understood as a preliminary result for the reliability of the reconstruction procedure.
As already observed, the main cause of the ill-posedness of such an inverse problem relies on the solution of a the severely ill-posed Cauchy problem with Cauchy data $\{u|_{\Gamma_2}, \frac{\partial u}{\partial \nu}|_{\Gamma_2}\}$. Hence to ensure the feasibility of the reconstruction procedure we shall keep the same assumptions on the data and the same a priori bounds on the nonlinearity required for the stability analysis.
The aim of Chapter 4 is to suggest a method to reconstruct the nonlinear profile f in terms of the Cauchy data on the accessible portion Γ_2 of the domain Ω. This will be achieved in two steps that can be outlined as follows.

i) The first step is to solve the Cauchy problem for u with Cauchy data on Γ_2, determining the corresponding Cauchy data for u on the inaccessible portion of the boundary Γ_1;

ii) the second step consists in proposing a procedure for the identification of the nonlinear term f by the Cauchy data on Γ_1 provided by the step i).

Before discussing our approach in treating step i), let us mention the most recent contributions to the approximate solution of the Cauchy problem due to Berntsson, Cheng, Eldén, Elliott, Engl, Fomin, Hào, Heggs, Hon, Ingham, Kabanikhin, Karchevskiĭ, Kozlov, Marin, Maz'ya, Leit˜ao, Lesnic, Maz'ya, Wei, [17], [27], [33], [41], [44], [49], [50], [56], [60], [61]. The method that we shall propose is based on the reformulation of the Cauchy problem to a regularized inversion of a suitable compact operator, fitting our problem in the widely developed theory of regularization for equations of the first kind. Indeed, with appropriate reductions of the problem, we will prove that the operator that maps the unknown Cauchy data on Γ_1 into the Cauchy data on Γ_2, is compact. Such a compactness result is strongly based on well-known regularity property for solution of elliptic equations. This reformulation allows the method of singular value decomposition and the approximate inversion by the technique of Tikhonov regularization.
In step ii), we shall suggest an approximate expression of the nonlinearity f. Indeed by a formal computation we shall select a candidate minimizer of the so-called best-fit functional (4.72). Moreover, as a novelty with respect the results achieved in [16], we shall add the proof of the pointwise convergence of such candidate minimizers to the exact nonlinearity f. The proof shall need some further a priori assumptions on the solution u to (1.5), see Section (4.5).

The stability result for the inverse scattering problem

In Chapter 5 we shall discuss the result contained in [66] concerning the stability

issue for the inverse scattering problem.
The major cause of ill-posedness consists in estimating how the error on the interior values of the solution propagates up to the boundary. Such an evaluation can be read as a step of the solution of a Cauchy problem, which, as it has been already pointed out in the previous section, is severely ill-posed. It turns out then, that the inverse scattering problem and the inverse corrosion one share some common features that will be outlined in the course of exposition.
In order to recover stability, we shall make use of some a priori assumptions on the unknown surface impedance. The additional a priori information that we shall require on the unknown surface impedance λ, is an a priori bound on its Lipschitz continuity, that is we shall assume that for a given positive constant Λ, the following holds

$$\|\lambda\|_{C^{0,1}(\Gamma_I)} \leq \Lambda. \tag{1.18}$$

Moreover, we shall prescribe the following uniform lower bound

$$\lambda(x) \geq \lambda_0, \quad \text{for every } x \in \Gamma_I, \tag{1.19}$$

where λ_0 is a given positive constant.
The treatment of the inverse scattering problem shall need an accurate preliminary analysis of the direct one. Indeed, following the arguments of potential theory developed in [21], we shall observe that the direct scattering problem is well posed. The proof relies on the fact that the mixed boundary value problem (1.8) can be reformulated as a system of boundary integral equations. Moreover, in analogy with the inverse corrosion problem, also for the inverse scattering one we shall prove a regularity result showing that the solution and their first order derivatives are Hölder continuous in a neighborhood of the portion Γ_I, where the impedance takes place. As a final step of this preliminary analysis, we shall obtain a uniform lower bound for the total field u on sets away from the obstacle.
Let us now illustrate the underlying ideas and the main tools that shall lead to the stability result. The reasons why such a problem lacks of well-posedness can be outlined as follows.

i) The first one consists in evaluating how much the error on the far field can affect the values of the field near the scatterer;

ii) the second one concerns a stability estimate of the field at the boundary in terms of the near field;

iii) finally, the last one relies on the problem of determining the impedance λ by the values of the field at the boundary.

Let us start the analysis of the inverse problem illustrating the arguments introduced in the step iii) of the list above.
By the impedance condition in (1.8) we can formally compute λ as

$$\lambda(x) = \frac{i}{u(x)} \frac{\partial u(x)}{\partial \nu(x)} \ . \tag{1.20}$$

Since u may vanish in some points of Γ_I, it follows that the quotient in (1.20) may be undetermined. In this respect, we shall evaluate the local vanishing rate of the solution on the boundary. To establish such a control we shall make use of quantitative estimates of unique continuation, of the form of the doubling inequality, which have been first introduced by Garofalo and Lin [38] for the unique continuation in the interior. Here we need estimates of the same sort, but which allow to evaluate the unique continuation property at boundary points where some kind of homogeneous boundary condition holds. For Dirichlet and Neumann homogeneous boundary conditions, results of this kind are due to Adolfsson, Escauriaza, Kukavica, Kenig and Nyström [3, 51, 52]. Here, assuming the impedance boundary condition in (1.8), we first obtain a *volume doubling inequality* at the boundary, namely

$$\int_{\Gamma_{I,2\rho}(x_0)} |u|^2 \leq const. \int_{\Gamma_{I,\rho}(x_0)} |u|^2 \ , \tag{1.21}$$

where $\Gamma_{I,\rho}(x_0)$ and $\Gamma_{I,2\rho}(x_0)$ are the portions of the balls centered at the boundary point x_0 of radius ρ and 2ρ respectively, contained in $\mathbb{R}^3 \setminus \overline{D}$.
In order to obtain the formula in (1.21), we shall adapt the arguments developed by Adolfsson and Escauriaza in [2] for the more general setting of complex valued solutions which is required by the boundary value problem (1.8).
A further difficulty that will arise in dealing with such arguments is due to the fact that the techniques used in [2] apply to an homogeneous Neumann boundary condition. We shall overcome such a difficulty by performing a suitable change of the independent variable, that fits our problem under the assumptions required in [2]. Moreover, well-known stability estimates for the Cauchy problem [69], will allow us to reformulate the *volume* doubling inequality at the boundary deriving a new one on the boundary, that is a *surface doubling inequality*

$$\int_{\Delta_{I,2\rho}(x_0)} |u|^2 \leq const. \int_{\Delta_{I,\rho}(x_0)} |u|^2 \ , \tag{1.22}$$

where $\Delta_{I,\rho}(x_0)$ and $\Delta_{I,2\rho}(x_0)$ are the portions of the boundary of $\Gamma_{I,\rho}(x_0)$ and $\Gamma_{I,2\rho}(x_0)$ respectively, which have non empty intersection with ∂D.
The surface doubling inequality will allow us to apply the theory of *Muckenhoupt weights* [28] which, in particular, implies the existence of some exponent $p > 1$ such that $|u|^{-\frac{2}{p-1}}$ is integrable on an inner portion of Γ_I. This integrability property, as well as the Hölder continuity of the normal derivative, justifies the computation made in (1.20) in the $L^{\frac{2}{p-1}}$ sense.
Let us carry over our analysis by discussing the evaluation introduced in the step i). Such an evaluation, introduced by V. Isakov [42, 43], and then developed by I. Bushuyev [19], concerns a stability estimate for the *near field* in terms of the measurements of the *far field*.
It means that if u_1 and u_2 are two acoustic fields corresponding to impedances λ_1 and λ_2 such that their scattering amplitudes, $u_{1,\infty}$ and $u_{2,\infty}$ respectively, are close

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leq \varepsilon, \tag{1.23}$$

then u_1 and u_2 satisfy

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))} \leq const.\varepsilon^{\alpha(\varepsilon)}, \tag{1.24}$$

where $R_1 > 0$ is a suitable radius such that $B_{R_1}(0) \supset \overline{D}$ and $\alpha(\varepsilon)$ is the following function

$$\alpha(\varepsilon) = \frac{1}{1 + \log(\log(\varepsilon^{-1}) + e)} \ . \tag{1.25}$$

As last step of this treatment we provide the stability estimate introduced in ii). The proof is based on the same arguments of quantitative unique continuation, as the iterated use of the *three spheres inequality*, that we have yet outlined for the case of the inverse corrosion problem. This procedure shall lead to the following estimate

$$\|u_1 - u_2\|_{C^1(\Gamma_I^\rho)} \leq const.|\log(\|u_1 - u_2\|^{-1}_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))})|^{-2\theta}, \tag{1.26}$$

where $\theta > 0$ and where Γ_I^ρ is a given inner portion of Γ_I .
By combining the stability estimates listed in i) and ii), we shall obtain a stability result for the total field at the boundary in terms of the measurements of the far field.
Finally, as a consequence of the previous achievements, we shall formulate the main result of Chapter 5, that consists in a stability estimate of the surface impedance by the far field measurements. Assuming that (1.23) holds, we have shown that the impedances λ_1, λ_2 agree up to an error

$$\big|\log(\varepsilon)\big|^{-\theta}. \tag{1.27}$$

For a sake of completeness, let us point out that Labreuche [53] has proved a stability result for this inverse problem under the much stronger assumption of analyticity of the boundary, whereas in the present thesis we shall deal with the more concrete case of a priori bounds on finitely many derivatives, that is we shall assume that Γ_I is a $C^{1,1}$ portion of ∂D.

Chapter 2

Quantitative estimates of unique continuation

The aim of this chapter is to collect the main tools and the methods on which are based the proofs of the stability results contained in this thesis. Here and in the sequel we shall refer to those techniques as *quantitative estimates of unique continuation*.
In Section 2.1 we shall introduce the quantitative notions of smoothness of the geometry that we shall consider. Moreover, we shall fix some notations that will be used throughout the thesis.
In Section 2.2 we shall deal with the stability issue for the following Cauchy problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \Sigma, \\ \sigma\nabla u \cdot \nu = g & \text{on } \Sigma\ . \end{cases} \tag{2.1}$$

The proof of the stability Theorem 2.7 will be obtained by combining the results contained in each one of the three subsections.
We will start the analysis of the problem by formulating the main hypothesis. We shall assume that the domain Ω is of Lipschitz class and we will require that the Cauchy surface Σ is $C^{1,\alpha}$ smooth. Moreover, we will specify the space where the Cauchy data are taken and we will require that the background conductivity σ satisfies an ellipticity condition as well a Lipschitz continuity assumption. We prescribe also an a priori energy bound on the solution u itself.
In Subsection 2.2.1 we shall approach the treatment of the solution of the Cauchy problem by stating an inequality, see Lemma 2.3, first discussed by Payne [63] and then developed by Trytten [69]. This inequality consists in an upper bound for the solution to the Cauchy problem (2.1) near the boundary in terms of the L^2 norm of u and its gradient on the Cauchy surface Σ.
By handling the inequality provided by Lemma 2.3, we will derive in Theorem 2.4 a stability estimate for the solution u to (2.1) near the boundary in terms of the Cauchy data. Indeed, we shall make use of the regularity assumptions

on the Cauchy data, as well as those made on the portion Σ, to reformulate the stability estimate due to Trytten in a new version, where, roughly speaking, the L^2 norm of the gradient is replaced by the L^2 norm of the normal derivative.
In Subsection 2.2.2 we shall discuss estimates of unique continuation from the interior as the *three spheres inequality*, see Lemma 2.5. This is a classic tool arising in unique continuation, which generalizes the Hadamard's three circles theorem. We recall the proof by Landis [54] based on Carlemann estimates and Agmon [4] relying on arguments of logarithmic convexity. We shall refer also to Garofalo and Lin [38] and to Kukaviza [51]. In Theorem 2.6 we will exhibit a useful application of the three spheres inequality based on an iterative procedure. Such a theorem will allow us to evaluate how much the error on the solution propagates in the interior.
Finally, we will conclude the study of the Cauchy problem by providing in Subsection 2.2.3 a stability result of logarithmic type up to the portion of the boundary Γ, being $\Gamma = \partial\Omega \setminus \bar{\Sigma}$. In order to prove such a result, we shall need to require some further a priori assumptions on the solution u itself, as the Hölder regularity of the solution together with its first order derivatives in a neighborhood of Γ as well a further regularity assumption on the portion Γ. The proof of Theorem 2.7 mostly relies on the techniques introduced in the previous two subsections, beyond the use of the cone condition which is guaranteed by the Lipschitz regularity of the boundary. Such a condition will allow us to carry over the iterated techniques of the three spheres inequality within the cone. By this trick, we will prove that the rate of stability is of log type.
In Section 2.3 we shall treat a quite recent tool of unique continuation as the *doubling inequality*. In Proposition 2.8 we shall state a doubling inequality in the interior, that has been introduced by Garofalo and Lin [38], whereas in Proposition 2.9, we shall state a doubling inequality at the boundary when an homogeneous Neumann boundary condition applies. The study of this tool has been introduced by Adolfsson, Escauriaza and Kenig [3], developed by Kukavika and Nyström [52] and Adolfsson and Escauriaza [2], to whom we shall refer. Let us also stress that these kinds of inequalities shall provide a useful tool to evaluate the local vanishing rate of a solution, and as its consequence allows to apply the theory of the Muckenhoupt weights [28].

2.1 Definitions and notations

We shall make a repeated use throughout the thesis of quantitative notions of smoothness for the boundary of the domain Ω. Let us introduce the following notations and definitions.
In several places it will be useful to isolate one privilege coordinate direction, to this purpose, we shall use the following notions for points $x \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}, n \geq 2$, $x = (x', x_n)$, $x' = (x'', x_{n-1})$, with $x'' \in \mathbb{R}^{n-2}$ and $x_n, x_{n-1} \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we shall denote with $B_r(x)$, $B_r^{'}(x)$, $B_r^{''}(x)$ the ball in $\mathbb{R}^n$, $\mathbb{R}^{n-1}$, $\mathbb{R}^{n-2}$ respectively, centered in x with radius r.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. We shall say* *that the boundary* $\partial\Omega$ *of* Ω *is of* Lipschitz class with constants $r_0,\ M > 0$ *if, for every* $x_0 \in \partial\Omega$*, there exists a rigid transformation of coordinates under which,*

$$\Omega \cap B_{r_0}(x_0) = \{(x', x_n) : x_n > \gamma(x')\} \tag{2.2}$$

where

$$\gamma : B'_{r_0}(x_0) \subset \mathbb{R}^{n-1} \to \mathbb{R}\ ,$$

satisfying $\gamma(0) = 0$ *and*

$$\|\gamma\|_{C^{0,1}(B'_{r_0}(x_0))} \leq M r_0\ ,$$

where we denote by

$$\|\gamma\|_{C^{0,1}(B'_{r_0}(x_0))} = \|\gamma\|_{L^\infty(B'_{r_0}(x_0))} + r_0 \sup_{\substack{x,y \in B'_{r_0}(z_0) \\ x \neq y}} \frac{|\gamma(x) - \gamma(y)|}{|x - y|}\ .$$

Definition 2.2. *Given an integer* $k \geq 1$ *and* $\alpha,\ 0 < \alpha \leq 1$*, we shall say that a portion* S *of* $\partial\Omega$ *is of* class $C^{k,\alpha}$ with constants $r_0,\ M > 0$ *if for any* $z_0 \in S$*, there exists a rigid transformation of coordinates under which,*

$$\Omega \cap B_{r_0}(z_0) = \{(x', x_n) : x_n > \varphi(x')\} \tag{2.3}$$

where

$$\varphi : B'_{r_0}(z_0) \subset \mathbb{R}^{n-1} \to \mathbb{R} \tag{2.4}$$

is a $C^{k,\alpha}$ *function satisfying for every multi-index* $0 \leq |\beta| \leq k$

$$|D^\beta \varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^{k,\alpha}(B'_{r_0}(z_0))} \leq M r_0\ , \tag{2.5}$$

where we denote

$$\begin{aligned} \|\varphi\|_{C^{k,\alpha}(B'_{r_0}(z_0))} &= \sum_{j=0}^{k} {r_0}^j \sum_{|\beta|=j} \|D^\beta \varphi\|_{L^\infty(B'_{r_0}(z_0))} + \\ &+ {r_0}^{k+\alpha} \sum_{|\beta|=j} \sup_{\substack{x,y \in B'_{r_0}(z_0) \\ x \neq y}} \frac{|D^\beta \varphi(x) - D^\beta \varphi(y)|}{|x - y|^\alpha}\ . \end{aligned} \tag{2.6}$$

We introduce some notations that we shall use in the present chapter as well as in Chapter 3 and Chapter 4.
Let S be a portion of $\partial\Omega$, then for every $\rho > 0$, we set

$$U_\rho^S = \{x \in \bar{\Omega} : \mathrm{dist}(x, \partial\Omega \setminus S) > \rho\}\ , \tag{2.7}$$

$$S^\rho = U_\rho^S \cap S\ , \tag{2.8}$$

$$\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \rho\}\ , \tag{2.9}$$

$$H_0^1(\Omega, S) = \{\eta \in H^1(\Omega)\ :\ \eta|_S = 0\}. \tag{2.10}$$

2.2 Stability for the Cauchy problem

In this section we shall deal with the stability issue for the Cauchy problem (2.1). We shall prove that, under suitable a priori assumptions, the dependence of the solution to (2.1) upon the Cauchy data is of logarithmic type.
Let us formulate the main hypothesis.

Assumptions on the domain

Given $D > 0$, we assume that

$$\text{the diameter of } \Omega \text{ is bounded by } D. \tag{2.11}$$

Given $r_0,\ M > 0$ we assume that

$$\Omega \text{ is of Lipschitz class with constants } r_0, M. \tag{2.12}$$

Moreover, given $0 < \alpha \leq 1$, we assume that the portion of the boundary

$$\Sigma \text{ is of class } C^{1,\alpha} \text{ with constants } r_0, M. \tag{2.13}$$

Assumptions on the Cauchy data

We shall assume the following on the Dirichlet datum ψ

$$\psi \in H^{\frac{1}{2}}(\Sigma)\ , \tag{2.14}$$

where $H^{\frac{1}{2}}(\Sigma)$ is the interpolation space $[H^1(\Sigma), L^2(\Sigma)]_{\frac{1}{2}}$ see [58, Chap. 1] for details.
Concerning the Neumann datum g we shall assume

$$g \in L^2(\Sigma). \tag{2.15}$$

Assumptions on the conductivity

We shall assume that the conductivity σ is a function from $\mathbb{R}^n$ with values in an $n \times n$ symmetric matrix $\sigma(x) = (\sigma_{ij}(x))_{i,j=1}^n$ satisfying the ellipticity condition

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} \sigma_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2, \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad \xi \in \mathbb{R}^n, \tag{2.16}$$

and the Lipschitz condition

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq K|x-y|, \quad \text{for all } i,j = 1,\dots,n \text{ and } x,y \in \Omega, \tag{2.17}$$

where $K > 0, \mu \geq 1$ are prescribed constants.

A priori bound on the energy

Given $E > 0$, we assume that

$$\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leq E \; . \tag{2.18}$$

In the course of the chapter the constants $r_0, M, D, \alpha, \mu, K, E$ will be referred as the *a priori data*.
Let us recall that, given $\psi \in H^{\frac{1}{2}}(\Sigma)$ and $g \in L^2(\Sigma)$, a weak solution to the Cauchy problem (2.1) is a function $u \in H^1(\Omega)$ such that $u|_\Sigma = \psi$ in the trace sense and

$$\int_\Omega \sigma \nabla u \cdot \nabla \eta = \int_\Sigma g\eta \; , \tag{2.19}$$

for every $\eta \in H^1_0(\Omega, \Gamma)$, where

$$\Gamma = \partial\Omega \setminus \bar{\Sigma}. \tag{2.20}$$

2.2.1 Stability estimates of continuation from Cauchy data

We state below a classical estimate of continuation from the boundary due to Trytten [69]. We also outline a sketch of the proof.

Lemma 2.3 (Trytten). *Let Ω be a domain satisfying* (2.12). *Let $u \in H^1(\Omega)$ be a weak solution to* (2.1) *and let* (2.14)-(2.17) *be satisfied.*
Then, for every $P_1 \in \Sigma^{r_0}$

$$\begin{aligned} \|u\|_{L^2(B_\rho(P_0)\cap U^\Sigma_{r_0})} \;\; \leq \;\; & C_1 \left(\|\psi\|_{L^2(\Sigma^\rho)} + \; \|\nabla u\|_{L^2(\Sigma^\rho)} \; + \|u\|_{H^1(\Omega)}\right)^{1-\eta} \cdot \\ & \cdot \left(\|\psi\|_{L^2(\Sigma^\rho)} + \; \|\nabla u\|_{L^2(\Sigma^\rho)}\right)^{\eta} \end{aligned} \tag{2.21}$$

where $\rho \in \left(\frac{M}{4\sqrt{1+M^2}} r_0, \frac{3M}{4\sqrt{1+M^2}} r_0\right)$ and $P_0 = P_1 + \frac{M}{4\sqrt{1+M^2}} r_0 \cdot \nu$, where ν is the outer unit normal to Ω at P_1 and $C_1 > 0$, $0 < \eta < 1$ are constants depending on the a priori data *and on ρ only.*

Proof. We shall give a sketch of the proof using the stability estimate for the Cauchy problem for elliptic equations in divergence form with Lipschitz coefficients proved by Trytten [69], see also Payne [63].
Let us define

$$\rho_1 = \frac{M}{4\sqrt{1+M^2}} r_0, \tag{2.22}$$

$$\rho_2 = \frac{3M}{4\sqrt{1+M^2}} r_0. \tag{2.23}$$

We can deduce from [69] that, for every $\rho \in (\rho_1, \rho_2)$, there exists an exponent $p > 1$ and a constant $\tilde{K}$ depending on the *a priori data* and on ρ only, such that

$$F\left(\frac{\rho}{2}\right) \leq \frac{C}{{r_0}^p} \left(\int_{\Sigma^\rho} u^2 + r_0^2 \int_{\Sigma^\rho} |\nabla u|^2 \right)^{\eta} \cdot \left(\int_{\Sigma^\rho} u^2 + r_0^2 \int_{\Sigma^\rho} |\nabla u|^2 + r_0 \int_{U^\Sigma_{r_0}} \sigma \nabla u \cdot \nabla u \right)^{1-\eta} \tag{2.24}$$

where

$$F(\rho)=\int_{\rho_1}^{\rho} r^{-p}\int_{B_r(P_0)\cap U^{\Sigma}_{2r_0}}\sigma\nabla u\cdot\nabla u+\frac{\tilde{K}}{r_0^p}\left(\int_{\Sigma^{\rho_1}}u^2+r_0^2\int_{\Sigma^{\rho_1}}|\nabla u|^2\right)\ ,\quad (2.25)$$

with $\eta,\ 0<\eta<1$ and $C>0$ constants depending only on the *a priori data* and on ρ only.
On the other hand the arguments in [69, p. 226] ensures the existence of a constant $c_1>0$ only depending on on the *a priori data* and on ρ such that

$$F\left(\frac{\rho}{2}\right)\geq c_1\int_{B_\rho(P_0)\cap U^{\Sigma}_{r_0}}u^2. \qquad (2.26)$$

Thus, combining (2.24) and (2.26) the thesis follows. □

In the following theorem, we shall elaborate the inequality (2.21) obtaining a stability estimate of continuation from Cauchy data.

Theorem 2.4 (Stability near the boundary). *Let Ω and Σ be a domain and a portion of its boundary satisfying* (2.12) *and* (2.13) *respectively. Let $u\in H^1(\Omega)$ be a weak solution to* (2.1) *and let* (2.14)-(2.17) *be satisfied.*
Then, we have that for every $P_1\in\Sigma^{2r_0}$, u satisfies the following estimate

$$\begin{aligned}\|u\|_{L^2(B_\rho(P_0)\cap U^{\Sigma}_{2r_0})}\quad\leq\quad & C\left(\|\psi\|_{L^2(\Sigma^\rho)}+\ \|g\|_{L^2(\Sigma^\rho)}\ +\|u\|_{H^1(\Omega)}\right)^{1-\delta}\cdot\\ &\cdot\left(\|\psi\|_{L^2(\Sigma^\rho)}+\ \|g\|_{L^2(\Sigma^\rho)}\right)^{\delta}\end{aligned}$$

where $\rho\in\left(\frac{M}{4\sqrt{1+M^2}}r_0,\frac{3M}{4\sqrt{1+M^2}}r_0\right)$, $P_0=P_1+\frac{M}{4\sqrt{1+M^2}}r_0\cdot\nu$, ν is the outer unit normal to Ω at P_1 and $C>0$, $0<\delta<1$ are constants depending on the a priori data *and on ρ only.*

Proof. Let us define the function $\tilde{g}\in L^2(\partial\Omega)$ as follows

$$\tilde{g}(x)=\begin{cases} g(x), & \text{for a.e. } x\in\Sigma^{r_0}\ ,\\ -\dfrac{1}{|\partial\Omega\setminus\Sigma|}\displaystyle\int_{\Sigma^{r_0}}g\ , & \text{for a.e. } x\in\partial\Omega\setminus\Sigma\ ,\\ 0\ , & \text{otherwise}\ .\end{cases}$$

Let us consider the following Neumann problem

$$\begin{cases}\mathrm{div}(\sigma\nabla z)=0, & \text{in }\Omega\ ,\\ \sigma\nabla z\cdot\nu=\tilde{g}, & \text{on }\partial\Omega\ .\end{cases}\qquad (2.27)$$

Note that $\int_{\partial\Omega}\tilde{g}=0$, hence a weak solution $z\in H^1(\Omega)$ exists and it is unique up to an additive constant. We select the solution z of (2.27) with zero average, it is well-known that for such a z the following holds

$$\|z\|_{H^1(\Omega)}\leq C_2\|\tilde{g}\|_{L^2(\partial\Omega)}\leq C_3\|g\|_{L^2(\Sigma^{r_0})}$$

where C_2 and C_3 are positive constants depending on the *a priori data* only. Let us set $w = u - z$, thus w solves the following Cauchy problem

$$\begin{cases} \mathrm{div}(\sigma\nabla w) = 0, & \text{in } \Omega, \\ w = \psi - z, & \text{on } \Sigma^{r_0}, \\ \sigma\nabla w \cdot \nu = 0, & \text{on } \Sigma^{r_0}. \end{cases} \tag{2.28}$$

By a standard boundary regularity estimate (see for instance [5, p.667]), we have that $w \in C^{1,\beta}(U^{\Sigma}_{\frac{3}{2}r_0})$ and the following holds

$$\|w\|_{C^{1,\beta}(U^{\Sigma}_{\frac{3}{2}r_0})} \leq C_4 \|w\|_{H^1(\Omega)}, \tag{2.29}$$

where $0 < \beta < 1$ and $C_4 > 0$ are constants depending on the *a priori data* only. By an interpolation inequality, (see for instance [8, p.777]) we have that

$$\|\nabla w\|_{L^2(\Sigma^{2r_0})} \leq C_5 \|w\|^{1-\gamma}_{C^{1,\beta}(U^{\Sigma}_{\frac{3}{2}r_0})} \|w\|^{\gamma}_{L^2(\Sigma^{2r_0})} , \tag{2.30}$$

where $C_5 > 0$ and $0 < \gamma < 1$ are constants depending on the *a priori data* only. Moreover,

$$\|w\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} + \|z\|_{H^1(\Omega)} \leq C_6 \left(\|u\|_{H^1(\Omega)} + \|g\|_{L^2(\Sigma^{r_0})}\right) , \tag{2.31}$$

where $C_6 = \max\{1, C_3\}$. From (2.29),(2.30) and (2.31) it follows that

$$\begin{aligned} \|\nabla w\|_{L^2(\Sigma^{2r_0})} \leq C_7 \left(\|u\|_{H^1(\Omega)} + \|g\|_{L^2(\Sigma^{r_0})}\right)^{1-\gamma} \cdot \\ \left(\|\psi\|_{L^2(\Sigma^{2r_0})} + \|z\|_{L^2(\Sigma^{2r_0})}\right)^{\gamma} . \end{aligned} \tag{2.32}$$

Applying (2.21) to w and using (2.32) we obtain

$$\begin{aligned} \|u\|_{L^2(B_\rho(P_0)\cap U^{\Sigma}_{2r_0})} \quad \leq \quad & C \left(\|\psi\|_{L^2(\Sigma^{r_0})} + \ \|g\|_{L^2(\Sigma^{r_0})} \ + \|u\|_{H^1(\Omega)}\right)^{1-\gamma\eta} \cdot \\ & \cdot \left(\|\psi\|_{L^2(\Sigma^{r_0})} + \ \|g\|_{L^2(\Sigma^{r_0})}\right)^{\gamma\eta} . \end{aligned}$$

And the thesis follows with $\delta = \gamma\eta$. □

2.2.2 The three spheres inequality

In this subsection we shall consider a solution u to the elliptic equation

$$\mathrm{div}(\sigma\nabla u) = 0 \quad \text{in } \Omega. \tag{2.33}$$

We state the following classical inequality in connection with unique continuation.

Lemma 2.5 (Three spheres inequality). *Let Ω be a bounded domain satisfying* (2.11),(2.12) *and let the conductivity tensor σ satisfies the ellipticity condition* (2.16) *and the Lipschitz regularity assumption* (2.17).
Let u be a solution to (2.33). *Then for every $r_1, r_2, r_3, \bar{r},\ 0 < r_1 < r_2 < r_3 \leq \bar{r}$ and for every $x_0 \in \Omega_{\bar{r}}$, we have that*

$$\int_{B_{r_2}(x_0)} u^2 \leq C \left(\int_{B_{r_1}(x_0)} u^2 \right)^{\tau} \cdot \left(\int_{B_{r_3}(x_0)} u^2 \right)^{1-\tau} , \tag{2.34}$$

where $C > 0$ and $\tau,\ 0 < \tau < 1$ only depending on $\mu, K, \frac{r_1}{r_3}, \frac{r_2}{r_3}$.

Proof. For the proof we refer to Kukavika [51] and also to Korevaar and Meyers [48]. □

By the iterated use of the three spheres inequality we obtain a stability estimate of continuation from the interior, as follows.

Theorem 2.6. *Let the hypothesis of Lemma 2.5 be satisfied. Let $\rho_0 > 0$ and let $x_0, y_0 \in \Omega_{4\rho_0}$, then*

$$\int_{B_{\rho_0}(y_0)} u^2 \leq C \left(\int_{B_{3\rho_0}(x_0)} u^2 \right)^{\tau^s} \cdot E^{(1-\tau^s)} . \tag{2.35}$$

where $C > 0$ and $\tau,\ 0 < \tau < 1$ are constants only depending on μ, K, whereas s is a positive constant such that $s < \frac{|\Omega|}{\omega_n \rho_0^n}$.

Proof. Following Lieberman [57], we introduce a regularized distance $\tilde{d}$ from the boundary of Ω. We have that there exists $\tilde{d}$ such that $\tilde{d} \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$, satisfying the following properties

i) $\gamma_0 \leq \dfrac{\text{dist}(x, \partial\Omega)}{\tilde{d}(x)} \leq \gamma_1$,

ii) $|\nabla \tilde{d}(x)| \geq c_1$, for every x such that $\text{dist}(x, \partial\Omega) \leq b r_0$,

iii) $\|\tilde{d}\|_{C^{0,1}} \leq c_2 r_0$,

where $\gamma_0, \gamma_1, c_1, c_2, b$ are positive constants depending on M only, (see also [8, Lemma 5.2]).
Let us define for every $\rho > 0$

$$\tilde{\Omega}_\rho = \{x \ \in \Omega \ \ : \ \ \tilde{d}(x) > \rho\} \ .$$

It follows that, there exists a, $0 < a \leq 1$, only depending on M such that for every ρ, $0 < \rho \leq a r_0$, $\tilde{\Omega}_\rho$ is connected with boundary of class C^1 and

$$\tilde{c}_1 \rho \leq \text{dist}(x, \partial\Omega) \leq \tilde{c}_2 \rho \quad \text{for every} \ \ x \ \in \ \partial\tilde{\Omega}_\rho \cap \Omega, \tag{2.36}$$

where $\tilde{c}_1, \tilde{c}_2$ are positive constants depending on M only. By (2.36) it follows that

$$\Omega_{\tilde{c}_2 \rho} \subset \tilde{\Omega}_\rho \subset \Omega_{\tilde{c}_1 \rho} \ .$$

Let γ be a path in $\tilde{\Omega}_{\frac{4\rho_0}{\tilde{c}_1}}$ joining x_0 to y_0 and let us define $\{y_i\}$, $i = 0, \dots, s$ as follows, $y_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \ : |\gamma(t) - y_i| = 2\rho_0\}$ if $|x_0 - y_i| > 2\rho_0$ otherwise let $i = s$ and stop the process. Now by Lemma (2.5) we have that

$$\int_{B_{3\rho_0}(y_0)} u^2 \leq C \left(\int_{B_{\rho_0}(y_0)} u^2 \right)^{\tau} \cdot \left(\int_{B_{4\rho_0}(y_0)} u^2 \right)^{1-\tau} .$$

Now since $B_{\rho_0}(y_0) \subset B_{3\rho_0}(y_1)$ and by (2.18), we have that

$$\int_{B_{\rho_0}(y_0)} u^2 \leq C \left(\int_{B_{3\rho_0}(y_1)} u^2 \right)^{\tau} \cdot E^{(1-\tau)} .$$

An iterated application of the three spheres inequality leads to

$$\int_{B_{\rho_0}(y_0)} u^2 \leq C \left(\int_{B_{\rho_0}(y_s)} u^2 \right)^{\tau^s} \cdot E^{(1-\tau^s)} .$$

Finally observing that $B_{\rho_0}(y_s) \subset B_{3\rho_0}(x_0)$ the theorem follows. □

2.2.3 Stability estimate up to the boundary

In this subsection we give the proof of the stability estimate up to the portion of the boundary Γ for the solution u to the problem (2.1) in terms of the Cauchy data.
In order to obtain such an estimate we need to make use of some a priori bounds on a weak solution $u \in H^1(\Omega)$ to the Cauchy problem (2.1), as well as a further regularity assumption on the portion Γ.
Let us require the following.

A regularity assumption on Γ

Given $\alpha,\ 0 < \alpha \leq 1$, we shall require that the portion of the boundary

$$\Gamma \text{ is } C^{1,\alpha} \text{ smooth with constants } r_0, M. \tag{2.37}$$

A priori bound on the $C^{1,\alpha}$ regularity at the boundary

Given $\alpha,\ 0 < \alpha \leq 1$, we shall assume that, for every $\rho \in (0, r_0)$, $u \in C^{1,\alpha}(U^{\Gamma}_{\rho})$ and that there exists a constant C_ρ depending on ρ, such that

$$\|u\|_{C^{1,\alpha}(U^{\Gamma}_{\rho})} \leq C_\rho. \tag{2.38}$$

Let us stress that in the treatment of the inverse corrosion problem and of the inverse scattering one we will not need to *a priori* require a bound of the type (2.38). Indeed, in Theorem (3.4) and in Theorem (5.3), we will prove a property of this sort by making use of the boundary condition and of the *a priori* bounds on the unknown boundary terms.

Theorem 2.7 (Stability for the Cauchy problem). *Let Ω, Σ and Γ be such that (2.11),(2.12),(2.13) and (2.37) are satisfied. Let (2.14)-(2.17) be satisfied. Let $u_i \in H^1(\Omega)$, $i = 1, 2$ be weak solutions to the Cauchy problem (2.1) with $\psi = \psi_i$ and $g = g_i$ respectively, such that (2.18) and (2.38) hold for each u_i. Suppose that*

$$\|\psi_1 - \psi_2\|_{L^2(\Sigma)} \leq \varepsilon, \tag{2.39}$$

$$\|g_1 - g_2\|_{L^2(\Sigma)} \leq \varepsilon, \tag{2.40}$$

then, for every $\rho \in (0, r_0)$ there exists a constant $c_\rho > 0$ depending on the a priori data *and on ρ only, such that*

$$\|u_1 - u_2\|_{C^1(\Gamma^\rho)} \leq c_\rho |\log(\varepsilon)|^{-\theta}, \tag{2.41}$$

where $\theta, 0 < \theta < 1$ is a constant depending on a priori data *only.*

Proof. Since the boundary of Ω is of Lipschitz class, then it satisfies the cone property. More precisely, if Q is a point of $\partial\Omega$, then there exists a rigid transformation of coordinates under which we have $Q = 0$. Moreover, considering the finite cone

$$\mathcal{C} = \left\{ x : |x| < r_0, \ \frac{x \cdot \xi}{|x|} > \cos\theta \right\}$$

with axis in the direction ξ and width 2θ, where $\theta = \arctan \frac{1}{M}$, we have that $\mathcal{C} \subset \Omega$. Let us consider now a point $Q \in \Gamma$ and let Q_0 be a point lying on the axis ξ of the cone with vertex in $Q = 0$ such that $d_0 = \mathrm{dist}(Q_0, 0) < \frac{r_0}{2}$. Let us define $u = u_1 - u_2$.

Using the notation introduced in the Proposition 2.4, we define the point $P = P_0 - \frac{1}{2\sqrt{1+M^2}} r_0 \cdot \nu$, $\rho_0 = \min\{\frac{1}{128M\sqrt{1+M^2}} r_0, \frac{r_0}{4}\sin\theta\}$. By Theorem 2.6 with $x_0 = P$ and $y_0 = Q_0$ and by (2.18), we have that

$$\int_{B_{\rho_0}(Q_0)} u^2 \leq C \left(\int_{B_{3\rho_0}(P)} u^2 \right)^{\tau^s} \cdot E^{(1-\tau^s)} \ . \tag{2.42}$$

Moreover, since $B_{3\rho_0}(P) \subset B_{\frac{3M}{4\sqrt{1+M^2}} r_0}(P_0) \cap U^\Sigma_{2r_0}$, then by Proposition 2.4, (2.18) and the bounds on the error (2.39) and (2.40), we can infer that

$$\int_{B_{\rho_0}(Q_0)} u^2 \leq C \left\{ (\varepsilon + E)^{1-\delta} \cdot (\varepsilon)^\delta \right\}^{\tau^s} \ .$$

We shall construct a chain of balls $B_{\rho_k}(Q_k)$ centered on the axis of the cone, pairwise tangent to each other and all contained in the cone

$$\mathcal{C}' = \left\{ x : |x| < r_0, \ \frac{x \cdot \xi}{|x|} > \cos\theta' \right\} ,$$

where $\theta' = \arcsin\left(\frac{\rho_0}{d_0}\right)$. Let $B_{\rho_0}(Q_0)$ be the first of them, the following are defined by induction in such a way

$$\begin{aligned} Q_{k+1} &= Q_k - (1+\tilde{\mu})\rho_k \xi \ , \\ \rho_{k+1} &= \tilde{\mu}\rho_k \ , \\ d_{k+1} &= \tilde{\mu} d_k \ , \end{aligned}$$

with

$$\tilde{\mu} = \frac{1-\sin\theta'}{1+\sin\theta'} \ .$$

Hence, with this choice, we have $\rho_k = \tilde{\mu}^k \rho_0$ and $B_{\rho_{k+1}}(Q_{k+1}) \subset B_{3\rho_k}(Q_k)$. Arguing with analogous arguments to those developed in Theorem (2.6), we have that

$$\begin{aligned} \|u\|_{L^2(B_{\rho_k}(Q_k))} &\leq \|u\|_{L^2(B_{3\rho_{k-1}}(Q_{k-1}))} \leq \\ &\leq \|u\|^{\tau}_{L^2(B_{\rho_{k-1}}(Q_{k-1}))} \|u\|^{1-\tau}_{L^2(B_{4\rho_{l-1}}(Q_{k-1}))} \\ &\leq C\|u\|^{\tau^k}_{L^2(B_{\rho_0}(Q_0))} \leq C\Big\{\left[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^{\delta}\right]^{\tau^s}\Big\}^{\tau^k} \end{aligned} \tag{2.43}$$

For every r, $0 < r < d_0$, let $k(r)$ be the smallest positive integer such that $d_k \leq r$ then, since $d_k = \tilde{\mu}^k d_0$, it follows

$$\frac{|\log(\frac{r}{d_0})|}{\log\tilde{\mu}} \leq k(r) \leq \frac{|\log(\frac{r}{d_0})|}{\log\tilde{\mu}} + 1 \ , \tag{2.44}$$

and by (2.43) we deduce

$$\|u\|_{L^2(B_{\rho_{k(r)}}(Q_{k(r)}))} \leq C\Big\{\left[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^{\delta}\right]^{\tau^s}\Big\}^{\tau^{k(r)}} \ . \tag{2.45}$$

Let $\bar{x} \in \Gamma^{\frac{\rho}{2}}$ with $\rho \in (0, r_0)$ and let $x \in B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$. By the a priori assumption (2.38) we have, in particular, that $u \in C^{1,\alpha}(U^{\Gamma}_{\frac{\rho}{4}})$ with

$$\|u\|_{C^{1,\alpha}(U^{\Gamma}_{\frac{\rho}{4}})} \leq C_\rho. \tag{2.46}$$

Then (2.46) yields to

$$|u(\bar{x})| \leq |u(x)| + C_\rho |x-\bar{x}|^\alpha \leq |u(x)| + C_\rho\left(\frac{2}{\tilde{\mu}}r\right)^\alpha \ .$$

Integrating this inequality over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, we have that

$$|u(\bar{x})|^2 \leq \frac{2}{\omega_n\left(\frac{\rho_{k-1}}{2}\right)^n}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})} |u(x)|^2 \mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\tilde{\mu}^2}\right)^\alpha . \tag{2.47}$$

Being k the smallest integer such that $d_k \leq r$, then $d_{k-1} > r$ and thus (2.47) yields to

$$|u(\bar{x})|^2 \leq \frac{C}{\left(r\sin\theta'\right)^n} \int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} |u(x)|^2 \mathrm{d}x + C_\rho r^{2\alpha} \ .$$

By (2.45) we have that

$$|u(\bar{x})|^2 \leq \frac{C}{r^n}\left\{\left[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^\delta\right]^{\tau^s}\right\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} \ . \tag{2.48}$$

By the bound (2.46) we deduce also that

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right| \leq \left|\frac{\partial u(x)}{\partial\nu}\right| + C_\rho\left(\frac{2}{\tilde{\mu}}r\right)^\alpha \ .$$

Integrating over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$ we obtain that

$$\begin{aligned}\left|\frac{\partial u(\bar{x})}{\partial\nu}\right|^2 &\leq \frac{2}{\omega_n\left(\frac{\rho_{k-1}}{2}\right)^n}\int_{B_{\frac{\rho_{k(r)-1}}{2}}\left(Q_{k(r)-1}\right)}\left|\frac{\partial u(x)}{\partial\nu}\right|^2\mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\tilde{\mu}^2}\right)^\alpha \leq \\ &\leq \frac{2}{\omega_n\left(\frac{\rho_{k-1}}{2}\right)^n}\int_{B_{\frac{\rho_{k(r)-1}}{2}}\left(Q_{k(r)-1}\right)}|\nabla u(x)|^2\mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\tilde{\mu}^2}\right)^\alpha \ .\end{aligned}$$

Applying the Caccioppoli inequality, we have

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right|^2 \leq \frac{C}{\left(\rho_{k-1}\right)^{n+2}}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} u(x)^2\mathrm{d}x + C_\rho r^{2\alpha} \ .$$

Rephrasing the arguments that have led to (2.48), we obtain that

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right|^2 \leq \frac{C}{r^{n+2}}\left\{\left[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^\delta\right]^{\tau^s}\right\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} \ . \tag{2.49}$$

The choice in (2.44) guarantees that

$$\tau^{k(r)-1} \geq \left(\frac{r}{d_0}\right)^\nu,$$

where $\nu = -\log\left(\frac{1}{\tilde{\mu}}\right)\log\tau$. Thus, by (2.48) and by (2.49), it follows that

$$|u(\bar{x})| \leq C_\rho\left\{r^{-\frac{n}{2}}\left[\left((\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^\delta\right)^{\tau^s}\right]^{\frac{r^\nu}{2}} + r^\alpha\right\} , \tag{2.50}$$

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right| \leq C_\rho\left\{r^{-\frac{n}{2}}\left[\left((\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^\delta\right)^{\tau^s}\right]^{\frac{r^\nu}{2}} + r^\alpha\right\} . \tag{2.51}$$

Minimizing the right hand sides of the above inequalities with respect to r, with $r \in (0, \frac{r_0}{4})$, we deduce

$$|u(\bar{x})| \leq C_\rho |\log(\varepsilon)|^{-\frac{2\alpha}{\nu+2}}\ , \tag{2.52}$$

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right| \leq C_\rho |\log(\varepsilon)|^{-\frac{2\alpha}{\nu+2}}, \tag{2.53}$$

where $C_\rho > 0$ is a constant depending on the *a priori* data and on ρ only. Thus, since $\bar{x}$ is an arbitrary point in $\Gamma^{\frac{\rho}{2}}$, by (2.52) and (2.53) we have that

$$\|u(\bar{x})\|_{L^\infty(\Gamma^{\frac{\rho}{2}})} \leq C_\rho |\log(\varepsilon)|^{-\frac{2\alpha}{\nu+2}}\ , \tag{2.54}$$

$$\left\|\frac{\partial u(\bar{x})}{\partial \nu}\right\|_{L^\infty(\Gamma^{\frac{\rho}{2}})} \leq C_\rho |\log(\varepsilon)|^{-\frac{2\alpha}{\nu+2}}. \tag{2.55}$$

By an interpolation inequality we have

$$\|\nabla_t(u)\|_{L^\infty(\Gamma^\rho)} \leq c_\rho \|u\|^\beta_{L^\infty(\Gamma^{\frac{\rho}{2}})} \|u\|_{C^{1,\alpha}(\Gamma^\rho)}{}^{1-\beta}\ ,$$

where $\beta = \frac{\alpha}{\alpha+1}$ and $c_\rho > 0$ depends on the *a priori data* and on ρ only. Thus, by (2.46), we obtain

$$\|\nabla_t(u)\|_{L^\infty(\Gamma^\rho)} \leq c_\rho \|u\|^\beta_{L^\infty(\Gamma^{\frac{\rho}{2}})} C_\rho{}^{1-\beta}\ .$$

It follows that for every $\varepsilon < \varepsilon_0$, with ε_0 depending only on the *a priori data*,

$$\begin{aligned} \|\nabla(u)\|_{L^\infty(\Gamma^\rho)} &\leq \left\|\frac{\partial u}{\partial \nu}\right\|_{L^\infty(\Gamma^\rho)} + \|\nabla_t(u)\|_{L^\infty(\Gamma^\rho)} \leq \\ &\leq c_\rho |\log(\varepsilon)|^{-\frac{2\alpha\beta}{\nu+2}}, \end{aligned} \tag{2.56}$$

where $c_\rho > 0$ depends on the *a priori* data and on ρ only. Hence, by a possible replacing of ε_0 with a smaller one depending on the *a priori* data only, we have that

$$\|u_1 - u_2\|_{C^1(\Gamma^\rho)} \leq c_\rho |\log(\varepsilon)|^{-\frac{2\alpha\beta}{\nu+2}} \quad \text{for every } \varepsilon, 0 < \varepsilon < \varepsilon_0. \tag{2.57}$$

Thus the thesis follows with $\theta = \frac{2\alpha\beta}{\nu+2}$. □

2.3 Doubling inequalities

In this section we list two versions of doubling inequalities. The first one is the following doubling inequality in the interior.

Proposition 2.8 (Doubling inequality in the interior). *Let the conductivity σ satisfies* (2.16), (2.17). *Let $u \in H^1(\Omega)$ be a weak solution to the equation* (2.33). *For every $\bar{r} > 0$ and for every $x_0 \in \Omega_{\bar{r}}$,*

$$\int_{B_{\beta r}(x_0)} u^2 \leq C\beta^{\tilde{K}} \int_{B_r(x_0)} u^2 \tag{2.58}$$

for every r, β such that $1 \leq \beta$ and $0 < \beta r \leq \bar{r}$, where C only depends on μ and K, whereas $\tilde{K}$ only depends on μ, K and increasingly on

$$N(\bar{r}) = \bar{r}^2 \frac{\int_{B_{\bar{r}}(x_0)} |\nabla u|^2}{\int_{B_{\bar{r}}(x_0)} |u|^2}. \tag{2.59}$$

Proof. For the proof we refer to Garofalo and Lin [38]. See also, for a more recent proof, Kukavica [51]. □

We state below the following doubling inequality at the boundary.

Proposition 2.9 (Doubling inequality at the boundary). *Let Ω be a domain satisfying* (2.12) *and let $x_0 \in \partial\Omega$. Let v be a solution to*

$$div(\sigma' \nabla v) = 0 \qquad in\ \Omega \cap B_{R_0}(x_0) \tag{2.60}$$

$$\sigma' \nabla v \cdot \nu = 0 \qquad in\ \partial\Omega \cap B_{R_0}(x_0), \tag{2.61}$$

for some $R_0 > 0$, where σ' is a function from $\mathbb{R}^n$ with values in an $n \times n$ symmetric matrix $\sigma'(x) = (\sigma'_{ij}(x))_{i,j=1}^n$ satisfying the following assumptions, for given positive constants μ_0, α and C,

i)

$${\mu_0}^{-1}|\xi|^2 \leq \sum_{i,j=1}^n \sigma'_{ij}(x)\xi_i\xi_j \leq \mu_0|\xi|^2, \quad for\ all\ \ x \in \Omega \ \ and\ \ \xi \in \mathbb{R}^n, \tag{2.62}$$

ii)

$$\sigma'(0) = Id, \tag{2.63}$$

iii)

$$\sigma'(x)x \cdot \nu = 0, \quad for\ a.e.\ x \in\ \partial\Omega \cap B_{R_0}(x_0), \tag{2.64}$$

iv)

$$|\nabla \sigma'(x)| \leq \frac{C}{{r_0}^\alpha}|x|^{\alpha-1},\ |\sigma'(x) - \sigma'(x_0)| \leq \frac{C}{{r_0}^\alpha}|x|^\alpha,\ for\ every\ x \in B_{R_0}(x_0) \tag{2.65}$$

Then there exists R, $0 < R < R_0$, *depending on* μ_0, α *and* C *only, such that*

$$\int_{\Omega\cap B_{\beta r}(x_0)} u^2 \leq c\beta^{\tilde{K}} \int_{\Omega\cap B_r(x_0)} u^2 \tag{2.66}$$

for every r, β *such that* $1 \leq \beta$ *and* $0 < \beta r \leq R$, *where* $c > 0$ *only depends on* μ_0, α, C, *whereas* $\tilde{K}$ *only depends on* μ_0, α, C *and increasingly on*

$$N(R_0) = {R_0}^2 \frac{\int_{\Omega\cap B_{R_0}(x_0)} |\nabla v|^2}{\int_{\partial B_{R_0}(x_0)\cap\Omega} |v|^2}. \tag{2.67}$$

Proof. For the proof we refer to [2, Theorem 1.3]. □

Chapter 3

Stability for the inverse corrosion problem

In this chapter we shall discuss the stability issue for the determination of the nonlinear term f in the boundary value problem (1.5). Before stating the main results of this chapter let us formulate the main hypothesis on the data of the problem and on the *a priori* assumptions on the unknown nonlinear term under which we shall prove the stability estimate.

Assumptions on the domain

Given positive constants D, r_0, M, we assume throughout this chapter that the assumptions (2.11) and (2.12) are satisfied.
We suppose that Γ_1, Γ_2 are two mutually disjoint, nonempty, connected, open subsets of $\partial\Omega$ and

$$\Gamma_D = \partial\Omega \setminus (\overline{\Gamma_1 \cup \Gamma_2}) \quad \text{and} \quad \overline{\Gamma}_1 \cap \overline{\Gamma}_D \neq \emptyset. \tag{3.1}$$

Moreover, given $0 < \alpha \leq 1$, we assume that the portions of the boundary Γ_i are contained respectively into surfaces S_i, $i = 1, 2$ which are $C^{1,\alpha}$ smooth with constants r_0, M.
More precisely, for any $x_0 \in S_i$, $i = 1, 2$, we have that up to a rigid change of coordinates,

$$S_i \cap B_{r_0}(x_0) = \{(x', x_n) : x_n = \varphi_i(x')\} \ , \tag{3.2}$$

with φ_i $i = 1, 2$ satisfying (2.4)-(2.6) with $\varphi = \varphi_i$ and $k = 1$.
In particular it follows that if

$$x_0 \in \Gamma_i \quad \text{and} \quad \text{dist}(x_0, \Gamma_D) > r_0 \ ,$$

then

$$\Omega \cap B_{r_0}(x_0) = \{(x', x_n) \in B_{r_0}(x_0) : x_n > \varphi_i(x')\} \ , \tag{3.3}$$

where φ_i is the Lipschitz function whose graph locally represents $\partial\Omega$. Moreover, since $\Omega \cap B_{r_0}(x_0) \cap \Gamma_D = \emptyset$, φ_i must also be the $C^{1,\alpha}$ function whose graph locally represents S_i. We also suppose that the boundary of Γ_i, within S_i, is of $C^{1,\alpha}$ class with constants r_0, M, namely, for any $x_0 \in \partial\Gamma_i$, there exists a rigid transformation of coordinates under which

$$\partial\Gamma_i \cap B_{r_0}(x_0) = \{(x', x_n) \in B_{r_0}(x_0) : \quad x_n = \varphi_i(x'),\ x_{n-1} = \psi_i(x'')\} \tag{3.4}$$

and

$$\psi_i : B''_{r_0}(x_0) \subset \mathbb{R}^{n-2} \longrightarrow \mathbb{R} \tag{3.5}$$

satisfying $\psi_i(0) = |\nabla\psi_i(0)| = 0$ and $\|\psi_i\|_{C^{1,\alpha}(B''_{r_0}(x_0))} \leq M$.

Assumptions on the boundary data

Given G, m positive constants, we assume that the current flux g is a prescribed function such that

$$\|g\|_{C^{0,\alpha}(\Gamma_2)} \leq G \ , \tag{3.6}$$

and furthermore

$$\|g\|_{L^\infty(\Gamma_2^{2r_0})} \geq m > 0 \ . \tag{3.7}$$

A priori bound on the energy

Given $E > 0$, we assume that the voltage potential u satisfies the a priori bound (2.18).

A priori information on the nonlinear term

Given $L > 0$ given positive constant, we assume that the function f belongs to $C^{0,1}(\mathbb{R}, \mathbb{R})$ and, in particular,

$$f(0) = 0 \quad \text{and} \quad |f(u) - f(v)| \leq L|u - v| \quad \text{for every } u, v \in \mathbb{R} \ . \tag{3.8}$$

Let us recall that a weak solution to the problem (1.5) is a function $u \in H^1_0(\Omega, \Gamma_D)$, such that

$$\int_\Omega \nabla u \cdot \nabla\rho = \int_{\Gamma_2} g\rho + \int_{\Gamma_1} f(u)\rho \quad \text{for all } \rho \in H^1_0(\Omega, \Gamma_D). \tag{3.9}$$

We shall refer in the sequel to the *a priori data* as to the set of quantities $r_0, M, \alpha, L, G, E, D$.

Before stating the main theorems of this chapter let us recall that we shall denote with $\eta(t)$ and $\omega(t)$, two positive increasing functions defined on $(0, +\infty)$, that satisfy

$$\eta(t) \geq \exp\left[-\left(\frac{t}{c}\right)^{-\gamma}\right], \quad \text{for every} \quad 0 < t \leq G \quad , \tag{3.10}$$

$$\omega(t) \leq C\,|\log(t)|^{-\vartheta}\ , \quad \text{for every} \ \ 0 < t < 1 \ \ , \tag{3.11}$$

where $c > 0$, $C > 0$, $\gamma > 1$, $0 < \theta < 1$ are constants depending on the *a priori data* only.
The statements of the main results are the following.

Theorem 3.1 (Lower bound for the oscillation). *Let* Ω, g *satisfying the a priori assumptions. Let* u *be a weak solution of* (1.5) *satisfying the a priori bound* (2.18) *then*

$$\operatorname*{osc}_{\Gamma_1} u \geq \eta(\|g\|_{L^\infty(\Gamma_2^{2r_0})})$$

where η *satisfies* (3.10).

Theorem 3.2 (Stability for the nonlinear term f). *Let* $u_i \ \in H^1_0(\Omega, \Gamma_D)$, $i = 1, 2$ *be weak solutions of the problem* (1.5)*, with* $f = f_i$ *and* $g = g_i$ *respectively and such that* (2.18) *holds for each* u_i*. Let us also assume that, for some positive number* m*, the following holds*

$$\|g_1\|_{L^\infty(\Gamma_2^{2r_0})} \geq m > 0\ . \tag{3.12}$$

Moreover, let $\psi_i = u_i\big|_{\Gamma_2}$*,* $i = 1, 2$*. There exist* $C > 0$*,* $\varepsilon_0 > 0$ *only depending on the* a priori data *and on* m *such that, if, for some* ε*,* $0 < \varepsilon < \varepsilon_0$*, we have*

$$\|\psi_1 - \psi_2\|_{L^2(\Gamma_2)} \leq \varepsilon\ ,$$
$$\|g_1 - g_2\|_{L^2(\Gamma_2)} \leq \varepsilon\ ,$$

then

$$\|f_1 - f_2\|_{L^\infty(V)} \leq \omega(\varepsilon)\ ,$$

where

$$V = (\alpha, \beta) \subseteq [-CE, CE]\ ,$$

is such that

$$\beta - \alpha > \frac{\eta(m)}{2}$$

and η, ω *satisfy* (3.10)*,* (3.11) *respectively.*

3.1 The regularity results

Lemma 3.3 (Hölder regularity at the boundary). *Let* u *be a solution to* (1.5)*, satisfying the a priori bound* (2.18) *then there exists a constant* $C > 0$*, depending on the* a priori data *only, such that*

$$\|u\|_{L^\infty(B_{\frac{r_0}{4}}(z_0)\cap\Omega)} \leq CE, \quad \textit{for every} \ \ z_0 \in \Gamma_1 \tag{3.13}$$

and

$$\|u\|_{C^{0,\alpha}(\Gamma_1)} \leq CE \tag{3.14}$$

where $0 < \alpha < 1$ *is a constant depending on* r_0, M, n *only.*

Proof. For any $z_0 \in \Gamma_1$ and for any $\rho > 0$, we shall denote

$$\Gamma_\rho(z_0) = \Omega \cap B_\rho(z_0)\ , \tag{3.15}$$

$$\Delta_\rho(z_0) = \overline{\Gamma}_\rho(z_0) \cap \partial\Omega. \tag{3.16}$$

Let $0 < \rho_1 < \rho_2 \leq r_0$ and let us consider a test function $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that

i) $0 \leq \varphi \leq 1$;

ii) $\varphi = 1$ in $\Gamma_{\rho_1}(z_0)$ and $\varphi = 0$ in $\Omega \setminus \Gamma_{\rho_2}(z_0)$;

iii) $|\nabla\varphi| \leq \dfrac{2}{\rho_2 - \rho_1}$.

For any integer $s \geq 2$, let us define the function $\psi = |u|^{s-2}u\varphi^2$. Hence, choosing ψ as test function in the weak formulation of the problem (3.9) we have that

$$\int_{\Gamma_{\rho_2}(z_0)} (s-1)|\nabla u|^2|u|^{s-2}\varphi^2 + \int_{\Gamma_{\rho_2}(z_0)} 2\nabla u \cdot \nabla\varphi|u|^{s-2}u\varphi = \int_{\Delta_{\rho_2}(z_0)} f(u)|u|^{s-2}u\varphi^2. \tag{3.17}$$

Hence,

$$\int_{\Gamma_{\rho_2}(z_0)} (s-1)|\nabla u|^2|u|^{s-2}\varphi^2 \leq \Big|\int_{\Gamma_{\rho_2}(z_0)} 2\nabla u \cdot \nabla\varphi|u|^{s-2}u\varphi\Big| + \tag{3.18}$$
$$+ \Big|\int_{\Delta_{\rho_2}(z_0)} f(u)|u|^{s-2}u\varphi^2\Big|.$$

By applying the Hölder inequality to the first term on the right hand side of (3.18), we obtain

$$\Big|\int_{\Gamma_{\rho_2}(z_0)} 2\nabla u \cdot \nabla\varphi|u|^{s-2}u\varphi\Big| \leq \frac{4}{\rho_2 - \rho_1}\left(\int_{\Gamma_{\rho_2}(z_0)} |\nabla u|^2|u|^{s-2}\varphi^2\right)^{\frac{1}{2}}\left(\int_{\Gamma_{\rho_2}(z_0)} |u|^s\right)^{\frac{1}{2}}$$

By the Schwartz inequality, it then follows that for every $\varepsilon > 0$

$$\Big|\int_{\Gamma_{\rho_2}(z_0)} 2\nabla u \cdot \nabla\varphi|u|^{s-2}u\varphi\Big| \leq \tag{3.19}$$
$$\leq \varepsilon\left(\int_{\Gamma_{\rho_2}(z_0)} |\nabla u|^2|u|^{s-2}\varphi^2\right) + \frac{16}{(\rho_2 - \rho_1)^2\varepsilon}\left(\int_{\Gamma_{\rho_2}(z_0)} |u|^s\right)$$

Let us now consider the second term on the right hand side of (3.18). The assumption (3.8) yields

$$\Big|\int_{\Delta_{\rho_2}(z_0)} f(u)|u|^{s-2}u\varphi^2\Big| \leq L\int_{\Delta_{\rho_2}(z_0)} |u|^s\varphi^2. \tag{3.20}$$

Furthermore by a trace inequality, see for instance [1, Theorem 5.22], we infer that

$$\Big|\int_{\Delta_{\rho_2}(z_0)} f(u)|u|^{s-2}u\varphi^2\Big| \leq CL\int_{\Gamma_{\rho_2}(z_0)} |\nabla(|u|^s\varphi^2)| \tag{3.21}$$

where $C>0$ is a constant depending on the *a priori data* only. Hence by the Schwartz inequality, it then follows that for every $\varepsilon>0$

$$\begin{aligned}&\Big|\int_{\Delta_{\rho_2}(z_0)} f(u)|u|^{s-2}u\varphi^2\Big| \leq \\ &\leq \varepsilon\int_{\Gamma_{\rho_2}(z_0)} |u|^{s-2}|\nabla u|^2\varphi^2 + \frac{s^2C^2L^2}{\varepsilon}\int_{\Gamma_{\rho_2}(z_0)}|u|^s + \frac{4CL}{\rho_2-\rho_1}\int_{\Gamma_{\rho_2}(z_0)}|u|^s\end{aligned} \tag{3.22}$$

Inserting (3.19) and (3.22) in (3.18), we obtain

$$\begin{aligned}&(1-2\varepsilon)\left(\int_{\Gamma_{\rho_2}(z_0)}|u|^{s-2}|\nabla u|^2\varphi^2\right) \leq \\ \leq\ &\left(\frac{16}{(\rho_2-\rho_1)^2\varepsilon} + \frac{L^2s^2C^2}{\varepsilon} + \frac{4CL}{\rho_2-\rho_1}\right)\left(\int_{\Gamma_{\rho_2}(z_0)}|u|^s\right)\end{aligned}$$

Choosing $\varepsilon = \frac{1}{4}$ in the above inequality we have that

$$\int_{\Gamma_{\rho_1}(z_0)}|u|^{s-2}|\nabla u|^2 \leq \left(\frac{32}{(\rho_2-\rho_1)^2\varepsilon} + \frac{2L^2s^2C^2}{\varepsilon} + \frac{2CL}{\rho_2-\rho_1}\right)\left(\int_{\Gamma_{\rho_2}(z_0)}|u|^s\right)$$

By the Sobolev inequality, see for instance [1, Chap. 5], we have that

$$\left(\int_{\Gamma_{\rho_1}(z_0)}|u|^{\frac{\hat{n}s}{\hat{n}-2}}\right)^{\frac{\hat{n}-2}{\hat{n}s}} \leq \left(\frac{C(1+s)}{\rho_2-\rho_1}\right)^{\frac{2}{s}}\left(\int_{\Gamma_{\rho_2}(z_0)}|u|^s\right)^{\frac{1}{s}},$$

where $\hat{n}=n$ for $n>2$, $\hat{2}>2$ and $C>0$ is a constant depending on the *a priori data* only.
Now, dealing as in [39, Chap. 8], we observe that the above inequality can be iterated. Indeed, setting $s=s_m=2\left(\frac{\hat{n}}{\hat{n}-2}\right)^m$ and $\rho_m=\frac{r_0}{4}+2^{-m}\frac{r_0}{4}$, $m=0,1,\ldots$, by (3.23) it follows

$$\|u\|_{L^{s_m}\left(\Gamma_{\frac{r_0}{4}}(z_0)\right)} \leq \left(C\frac{\hat{n}}{\hat{n}-2}\right)^{\sum 4m\left(\frac{\hat{n}}{\hat{n}-2}\right)^{-m}}\|u\|_{L^2\left(\Gamma_{\frac{r_0}{2}}(z_0)\right)}. \tag{3.23}$$

Letting m tends to ∞ in (3.23), we can infer that

$$\|u\|_{L^\infty\left(\Gamma_{\frac{r_0}{4}}(z_0)\right)} \leq C\|u\|_{L^2\left(\Gamma_{\frac{r_0}{2}}(z_0)\right)}, \tag{3.24}$$

where $C>0$ is a constant depending on the *a priori data* only.
Hence combining (2.18) and (3.24) the inequality (3.13) follows.
Let us now prove the inequality (3.14).
Let $0<r_1<r_2\leq\frac{r_0}{4}$ and let us consider a test function $\eta\in C^1(\Omega)$ such that

i) $0 \le \eta \le 1$;

ii) $\eta = 1$ in $\Gamma_{r_1}(z_0)$ and $\eta = 0$ in $\Omega \setminus \Gamma_{r_2}(z_0)$;

iii) $|\nabla \eta| \le \dfrac{2}{r_2 - r_1}$.

By (3.13), we have that

$$M_2 = \sup_{x \in \Gamma_{r_2}(z_0)} u(x) < +\infty. \tag{3.25}$$

Let us define the following non-negative function

$$v(x) = M_2 - u(x), \quad \text{for every } x \in \Gamma_{r_2}(z_0). \tag{3.26}$$

Let us introduce the following quantities.
For every $\rho \in (0, \frac{r_0}{4})$, let

i) $b = 2LC$,

ii) $h = bM_2$;

iii) $k = k(\rho) = \rho^\delta h$,

iv) $\bar{b} = b^2 + k^{-2}h^2$;

v) $\bar{v} = v + k$.

where $C > 0$ is the constant appearing in the inequality (3.21) and δ is such that $0 < \delta < 1$.
Let us define, for $\beta \in \mathbb{R} \setminus \{0\}$ the function $\chi = \eta^2 \bar{v}^\beta$. Hence choosing χ as test function in the weak formulation (3.9), it follows that

$$\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2 \bar{v}^{\beta-1}\eta^2 + \frac{2}{\beta}\int_{\Gamma_{r_2}(z_0)} \nabla v \cdot \nabla \eta \eta \bar{v}^\beta = -\frac{1}{\beta}\int_{\Delta_{r_2}(z_0)} f(M_2 - v)\eta^2\bar{v}^\beta. \tag{3.27}$$

By the hypothesis (3.8) and by (3.27), we can infer that

$$\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2 \bar{v}^{\beta-1}\eta^2 + \frac{2}{\beta}\int_{\Gamma_{r_2}(z_0)} \nabla v \cdot \nabla \eta \eta \bar{v}^\beta \le \frac{1}{|\beta|}\int_{\Delta_{r_2}(z_0)} L|M_2 - v|\eta^2\bar{v}^\beta. \tag{3.28}$$

Furthermore by the trace inequality used in (3.21), we have that

$$\int_{\Gamma_{r_2}} |\nabla v|^2 \bar{v}^{\beta-1}\eta^2 + \frac{2}{\beta}\int_{\Gamma_{r_2}} \nabla v \cdot \nabla \eta \eta \bar{v}^\beta \le \frac{LC}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla[(M_2 - v)\eta^2\bar{v}^\beta]|.$$

After straightforward calculations, we have that

$$\begin{aligned}
&\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2 \bar{v}^{\beta-1}\eta^2 - LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla v|\bar{v}^{\beta-1}\eta^2 \le \\
\le \; & \frac{2}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla v||\nabla \eta|\eta\bar{v}^\beta + \frac{2LC}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla\eta|\eta\bar{v}^\beta + \\
+ \; & \frac{LC}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla v|\eta^2\bar{v}^\beta.
\end{aligned} \tag{3.29}$$

By the Schwartz inequality it follows that for every $\varepsilon > 0$

$$LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla v|\bar{v}^{\beta-1}\eta^2 \leq \varepsilon\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 + \frac{L^2C^2}{\varepsilon}\int_{\Gamma_{r_2}(z_0)} |M_2 - v|^2\bar{v}^{\beta-1}\eta^2. \tag{3.30}$$

Hence choosing $\varepsilon = \frac{1}{2}$ in (3.30), we obtain

$$\begin{aligned}
&\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla v|\bar{v}^{\beta-1}\eta^2 \geq \\
\geq\ & \frac{1}{2}\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - 2L^2C^2\int_{\Gamma_{r_2}(z_0)} |M_2 - v|^2\bar{v}^{\beta-1}\eta^2 \geq \\
\geq\ & \frac{1}{2}\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - b^2\int_{\Gamma_{r_2}(z_0)} v^2\bar{v}^{\beta-1}\eta^2 - h^2\int_{\Gamma_{r_2}(z_0)} \bar{v}^{\beta-1}\eta^2.
\end{aligned} \tag{3.31}$$

Moreover, observing that $b^2v^2 + h^2 \leq \bar{b}\bar{v}^2$, by (3.31) we can infer that

$$\begin{aligned}
&\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla v|\bar{v}^{\beta-1}\eta^2 \geq \\
\geq\ & \frac{1}{2}\left(\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - 2\bar{b}\int_{\Gamma_{r_2}(z_0)} \bar{v}^{\beta+1}\eta^2\right)
\end{aligned} \tag{3.32}$$

On the other hand we have also that

$$\begin{aligned}
&\int_{\Gamma_{r_2}(z_0)} |\nabla v||\nabla\eta|\eta\bar{v}^{\beta} + LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla\eta|\eta\bar{v}^{\beta} = \\
\leq\ & \int_{\Gamma_{r_2}(z_0)} \frac{1}{2}\left(2|\nabla v| + bv + h\right)|\nabla\eta|\eta\bar{v}^{\beta}.
\end{aligned} \tag{3.33}$$

Noticing that $bv + h \leq 2\sqrt{\bar{b}}\bar{v}$, we have that (3.33) yields

$$\begin{aligned}
&\int_{\Gamma_{r_2}(z_0)} |\nabla v||\nabla\eta|\eta\bar{v}^{\beta} + LC\int_{\Gamma_{r_2}(z_0)} |M_2 - v||\nabla\eta|\eta\bar{v}^{\beta} \leq \\
\leq\ & \int_{\Gamma_{r_2}(z_0)} \left(|\nabla v| + \sqrt{\bar{b}}\bar{v}\right)|\nabla\eta|\eta\bar{v}^{\beta}.
\end{aligned} \tag{3.34}$$

Hence inserting (3.32) and (3.34) in (3.29) we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 - 2\bar{b}\int_{\Gamma_{r_2}(z_0)} \bar{v}^{\beta+1}\eta^2\right) \leq \\
\leq\ & \frac{1}{|\beta|}\int_{\Gamma_{r_2}(z_0)} \eta\bar{v}^{\beta}|\nabla\eta||\nabla v| + \frac{1}{|\beta|}\int_{\Gamma_{r_2}(z_0)} \eta\bar{v}^{\beta+1}\sqrt{\bar{b}}|\nabla\eta| + \\
+\ & \frac{LC}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla v|\eta^2\bar{v}^{\beta}.
\end{aligned} \tag{3.35}$$

Moreover, by the Schwartz inequality and by (3.35) we obtain that for every $\varepsilon > 0$

$$\frac{1}{2}\left(\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2 \bar{v}^{\beta-1}\eta^2 - 2\bar{b}\int_{\Gamma_{r_2}(z_0)} \bar{v}^{\beta+1}\eta^2\right) \leq \tag{3.36}$$

$$\leq \frac{\varepsilon}{|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\eta^2\bar{v}^{\beta-1} + \frac{1}{\varepsilon|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla\eta|^2\bar{v}^{\beta+1} +$$

$$+ \frac{1}{2|\beta|}\int_{\Gamma_{r_2}(z_0)} |\nabla\eta|^2\bar{v}^{\beta+1} + \frac{\bar{b}}{2|\beta|}\int_{\Gamma_{r_2}(z_0)} \eta^2\bar{v}^{\beta+1} +$$

$$+ \varepsilon\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 + \frac{L^2C^2}{\varepsilon|\beta|^2}\int_{\Gamma_{r_2}(z_0)} \eta^2\bar{v}^{\beta+1}.$$

From the above inequality it follows that

$$\left(\frac{1}{2} - \frac{\varepsilon}{|\beta|} - \varepsilon\right)\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 \leq$$

$$\leq \int_{\Gamma_{r_2}(z_0)} \left(2\bar{b} + \frac{\bar{b}}{2|\beta|} + \frac{L^2C^2}{\varepsilon|\beta|}\right)\eta^2\bar{v}^{\beta+1} + \int_{\Gamma_{r_2}(z_0)} \left(\frac{1}{2|\beta|} + \frac{1}{\varepsilon|\beta|}\right)|\nabla\eta|^2\bar{v}^{\beta+1}. \tag{3.37}$$

Thus, choosing $\varepsilon = \min\{\frac{1}{8}, \frac{|\beta|}{8}\}$, we have that

$$\int_{\Gamma_{r_2}(z_0)} |\nabla v|^2\bar{v}^{\beta-1}\eta^2 \leq \hat{C}\int_{\Gamma_{r_2}(z_0)} \left(\eta^2 + |\nabla\eta|^2\right)\bar{v}^{\beta+1}\ , \tag{3.38}$$

where $\hat{C}$ is a positive constant depending on $|\beta|, L, C, M_2, \rho, \delta$.
Let w be a function defined as follows

$$w = \begin{cases} \bar{v}^{\frac{\beta+1}{2}}, & \text{if } \beta \neq -1, \\ \log\bar{v}, & \text{if } \beta = -1. \end{cases}$$

Hence we can reformulate (3.38) as follows

$$\int_{\Gamma_{r_2}(z_0)} |\eta\nabla w|^2 \leq \begin{cases} (\beta+1)^2\hat{C}\int_{\Gamma_{r_2}(z_0)} \left[\eta^2 + |\nabla\eta|^2\right]w^2, & \text{if } \beta \neq -1\ , \\ \hat{C}\int_{\Gamma_{r_2}(z_0)} \left[\eta^2 + |\nabla\eta|^2\right], & \text{if } \beta = -1. \end{cases} \tag{3.39}$$

By the Sobolev inequality, see for instance [1, Chap. 5], we have that

$$\|\eta w\|^2_{L^{\frac{2\hat{n}}{\hat{n}-2}}(\Gamma_{r_2}(z_0))} \leq C\int_{\Gamma_{r_2}(z_0)} \left(|\eta\nabla w|^2 + |w\nabla\eta|^2\right) \tag{3.40}$$

where $\hat{n} = n$ for $n > 2$, $\hat{2} > 2$ and $C > 0$ is a constant depending on the *a priori data* only. Combining (3.39) and (3.40) we obtain

$$\|\eta w\|^2_{L^{\frac{2\hat{n}}{\hat{n}-2}}(\Gamma_{r_2}(z_0))} \leq c(\beta+1)^2\int_{\Gamma_{r_2}(z_0)} (\eta^2 + |\nabla\eta|^2)w^2\ , \tag{3.41}$$

where $c > 0$, depending on the *a priori data*, on ρ, on $|\beta|$ and on δ only, is bounded when $|\beta|$ is bounded away from zero.
Hence from (3.41) we obtain

$$\|w\|^2_{L^{\frac{2\hat{n}}{\hat{n}-2}}(\Gamma_{r_1}(z_0))} \leq c' \frac{(|\beta+1|+1)}{r_2-r_1}\|w\|_{L^2(\Gamma_{r_2}(z_0))} \tag{3.42}$$

where $c > 0$, depending on the *a priori data*, on ρ and on δ only.
At this stage arguing as in [39, Theorem 8.18], we obtain the following weak Harnack inequality for the function v.
For every $0 < \rho < \frac{r_0}{16}$, we have that

$$\rho^{-n}\|v\|_{L^1(\Gamma_{2\rho}(z_0))} \leq C\left(\inf_{\Gamma_\rho(z_0)} v + \rho^\delta |M_2|\right) , \tag{3.43}$$

where $C > 0$ is a constant only depending on the *a priori* data .
On the other hand by (3.13) we have also that,

$$m_2 = \inf_{x\in\Gamma_{r_2}(z_0)} u(x) < +\infty. \tag{3.44}$$

Then, we define the following non-negative function

$$z(x) = u(x) - m_2 \ \text{ for every } x \in \Gamma_{r_2}(z_0). \tag{3.45}$$

Hence, by analogous arguments to those developed for the function v, we find also the following weak Harnack inequality for the function z.
For every $0 < \rho < \frac{r_0}{16}$, we have that

$$\rho^{-n}\|z\|_{L^1(\Gamma_{2\rho}(z_0))} \leq C\left(\inf_{\Gamma_\rho(z_0)} z + \rho^\delta |m_2|\right) , \tag{3.46}$$

where $C > 0$ is a constant only depending on the *a priori* data .
For every $\rho \in (0, \frac{r_0}{16})$, let us denote

$$M(\rho) = \sup_{\Gamma_\rho(z_0)} u , \tag{3.47}$$

$$m(\rho) = \inf_{\Gamma_\rho(z_0)} u. \tag{3.48}$$

By (3.13),(3.43) and (3.46), we have that there exists a constant $K > 0$ depending on the *a priori* data only, such that

$$\rho^{-n}\int_{\Gamma_{2\rho}(z_0)}(M_2 - u) \leq K\left(M_2 - M + \rho^\delta\right) , \tag{3.49}$$

$$\rho^{-n}\int_{\Gamma_{2\rho}(z_0)}(u - m_2) \leq K\left(m - m_2 + \rho^\delta\right). \tag{3.50}$$

Moreover, let us observe that being the boundary $\partial\Omega$ of Lipschitz class, we have that there exists a constant $c_1 > 0$, depending on r_0, M only, such that for every $\rho \in (0, \frac{r_0}{16})$

$$\rho^{-n}|\Gamma_{2\rho}(z_0)| \geq c_1. \tag{3.51}$$

Hence adding (3.49) and (3.50), we obtain

$$M - m \leq \Big(1 - \frac{c_1}{K}\Big)(M_2 - m_2) + 2K\rho^{\delta}. \tag{3.52}$$

Denoting by $\omega(\rho) = \underset{\Gamma_\rho(z_0)}{\operatorname{osc}} u$, we have that by (3.52) it follows

$$\omega(\rho) \leq \gamma\omega(4\rho) + c_2\rho^{\delta} \ , \tag{3.53}$$

where $c_2 = 2K$ and $\gamma = 1 - \frac{c_1}{K}$.
By the arguments in [39, Lemma 8.23], it follows that for any $\mu \in (0,1)$ and any $0 < \rho \leq \rho_0 \leq \frac{r_0}{16}$

$$\omega(\rho) \leq C\left(\left(\frac{\rho}{\rho_0}\right)^{\alpha}\omega(\rho_0) + c_2\rho^{\mu\delta}{\rho_0}^{(1-\mu)\delta}\right) \ , \tag{3.54}$$

where C is a constant depending on the *a priori* data only, whereas α is such that $\alpha = (1-\mu)(\frac{\log(\gamma)}{\log(\frac{1}{4})})$. Hence choosing μ such that $(1-\mu)\frac{\log(\gamma)}{\log(\frac{1}{4})} < \mu\delta$, we have that (3.54) leads to

$$\frac{\omega(\rho)}{\rho^{\alpha}} \quad \leq \quad c\left({\rho_0}^{-\alpha}\omega(\rho_0) + \rho^{\beta}\right) \ , \tag{3.55}$$

where c is a constant depending on the *a priori data* only and β is such that $\beta = \mu(\delta - 1) - \alpha + 1 > 0$. Furthermore, we have that the above inequality and (3.13) lead to

$$\frac{\omega(\rho)}{\rho^{\alpha}} \quad \leq \quad c\left({\rho_0}^{-\alpha}2CE + \rho^{\beta}\right) \ , \tag{3.56}$$

where C is a constant depending on the *a priori* data only.
Hence we can infer that for any $z_0 \in \Gamma_1$

$$\|u\|_{C^{0,\alpha}\left(\Gamma_{\frac{r_0}{16}}(z_0)\right)} \leq CE. \tag{3.57}$$

where $C > 0, 0 < \alpha < 1$ are constants depending on the *a priori* data only.
Thus the lemma follows. □

Theorem 3.4 ($C^{1,\alpha}$ regularity at the boundary). *Let u be a solution of* (1.5)*, satisfying the a priori bound* (2.18)*, then for any $\rho \in (0, r_0)$, $u \in C^{1,\alpha}(U_\rho^{\Gamma_1})$ and there exists a constant $C_\rho > 0$, depending on the* a priori data *and on ρ only, such that the following estimate holds*

$$\|u\|_{C^{1,\alpha}(U_\rho^{\Gamma_1})} \leq C_\rho E. \tag{3.58}$$

Proof. Since, by Lemma 3.3, we know that $u \in C^{0,\alpha}(\Gamma_1)$, by the Lipschitz regularity of f we have that

$$\frac{\partial u}{\partial \nu}(x) = f(u(x)) \quad \in C^{0,\alpha}(\Gamma_1) \ .$$

By well-known regularity bounds for the Neumann problem (see for instance [5, p.667]) it follows that $u \in C^{1,\alpha}(U_\rho^{\Gamma_1})$ and the following estimate holds

$$\begin{aligned} \|u\|_{C^{1,\alpha}(U_\rho^1)} &\leq C\left(\|u\|_{C^{0,\alpha}(\Gamma_1^{\frac{\rho}{2}})} + \left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_1^{\frac{\rho}{2}})} + \|\nabla u\|_{L^2(\Omega)}\right) \leq \\ &\leq C\left(\left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_1^{\frac{\rho}{2}})} + E\right) \end{aligned} \tag{3.59}$$

where $C > 0$ depends on the *a priori data* and on ρ only. Moreover, we can estimate the $C^{0,\alpha}$ norm of $\dfrac{\partial u}{\partial \nu}$ in terms of E, in fact

$$\begin{aligned} \left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_1^{\frac{\rho}{2}})} &= \sup_{x \in \Gamma_1^{\frac{\rho}{2}}} \left|\frac{\partial u(x)}{\partial \nu}\right| + \left(\frac{\rho}{2}\right)^\alpha \sup_{x,y \in \Gamma_1^{\frac{\rho}{2}}} \frac{\left|\frac{\partial u(x)}{\partial \nu} - \frac{\partial u(y)}{\partial \nu}\right|}{|x-y|^\alpha} = \\ &= \sup_{x \in \Gamma_1^{\frac{\rho}{2}}} |f(u(x))| + \left(\frac{\rho}{2}\right)^\alpha \sup_{x,y \in \Gamma_1^{\frac{\rho}{2}}} \frac{|f(u(x)) - f(u(y))|}{|x-y|^\alpha} \ . \end{aligned}$$

By the Lipschitz bound (3.8) on f and by Lemma 3.3 we obtain

$$\begin{aligned} \left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_1^{\frac{\rho}{2}})} &\leq L \sup_{x \in \Gamma_1^{\frac{\rho}{2}}} |u(x)| + L\left(\frac{\rho}{2}\right)^\alpha \sup_{x,y \in \Gamma_1^{\frac{\rho}{2}}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \\ &\leq L\|u\|_{C^{0,\alpha}(\Gamma_1)} \leq CE \ . \end{aligned} \tag{3.60}$$

So inserting this estimate in (3.59) we have the thesis. □

Corollary 3.5. *Let u be as above, then, for every $\rho > 0$, the function $\frac{\partial u}{\partial \nu}$ belongs to $C^{0,1}(\Gamma_1^\rho)$, with Lipschitz constant $\tilde{L}$ depending on the a* a priori data *and on ρ only.*

Proof. Let x and y be two points in Γ_1^ρ then, by the assumption (3.8) and by Theorem 3.4, it follows that

$$\begin{aligned} \left|\frac{\partial u(x)}{\partial \nu} - \frac{\partial u(y)}{\partial \nu}\right| &= |f(u(x)) - f(u(y))| \leq L|u(x) - u(y)| \leq \\ &\leq LC_\rho E|x-y| \ . \end{aligned}$$

The thesis follows with $\tilde{L} = LC_\rho E$. □

3.2 The lower bound for the oscillation

Proposition 3.6 (Stability near the boundary). *Let Ω satisfies the a priori assumptions and let $v \in H^1(\Omega)$ be a solution of the following Cauchy problem*

$$\begin{cases} \Delta v = 0 \ , & in\ \Omega, \\ v = \varphi \ , & on\ \Gamma_1, \\ \dfrac{\partial v}{\partial \nu} = h, & on\ \Gamma_1, \end{cases} \tag{3.61}$$

where $\varphi, h \in L^2(\Gamma_1)$ and the boundary conditions are considered in the weak sense.
Then, for every $P_1 \in \Gamma_1^{2r_0}$, v satisfies the following estimate

$$\begin{aligned} \|v\|_{L^2(B_\rho(P_0)\cap U^1_{2r_0})} \quad \leq \quad & C\left(\|\varphi\|_{L^2(\Gamma_1^\rho)} + \ \|h\|_{L^2(\Gamma_1^\rho)} \ + \|v\|_{H^1(\Omega)}\right)^{1-\delta} \cdot \\ & \cdot \left(\|\varphi\|_{L^2(\Gamma_1^\rho)} + \ \|h\|_{L^2(\Gamma_1^\rho)}\right)^{\delta} \end{aligned}$$

where $\rho \in \left(\frac{M}{4\sqrt{1+M^2}} r_0, \frac{3M}{4\sqrt{1+M^2}} r_0\right)$, $P_0 = P_1 + \frac{M}{4\sqrt{1+M^2}} r_0 \cdot \nu$, ν is the outer unit normal to Ω at P_1 and $C > 0$, $0 < \delta < 1$ are constants depending on ρ, r_0, n, M only.

Proof. The proposition follows by applying the same arguments introduced in Theorem 2.4 with $\sigma = Id$ and $\Sigma = \Gamma_1$. □

Proof of Theorem 3.1. Let $\varepsilon = \operatorname*{osc}_{\Gamma_1} u > 0$, since $u = 0$ on Γ_D we have that

$$0 \in [\min_{\Gamma_1} u, \max_{\Gamma_1} u] \tag{3.62}$$

and hence $\|u\|_{L^\infty(\Gamma_1)} \leq \varepsilon$ and also

$$\|u\|_{L^2(\Gamma_1^{r_0})} \leq C_1 \varepsilon \tag{3.63}$$

where C_1 is a positive constant depending on the *a priori data* only. By the a priori assumption (3.8) on f, we have that $|f(u)| \leq L|u|$, moreover, since

$$\left|\frac{\partial u(x)}{\partial \nu}\right| = |f(u(x))| \ \text{ on } \Gamma_1,$$

then

$$\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\Gamma_1^{r_0})} \leq C_1 L \varepsilon \ . \tag{3.64}$$

By Proposition 3.6, it follows

$$\|u\|_{L^2(B_\rho(P_0)\cap U^1_{2r_0})} \leq C(\varepsilon + E)^{1-\delta} \cdot \varepsilon^\delta \tag{3.65}$$

where C is a constant depending on the *a priori data* only. Since the boundary of Ω is of Lipschitz class, then it satisfies the cone property. More precisely, if Q is a point of $\partial\Omega$, then there exists a rigid transformation of coordinates under which we have $Q = 0$. Moreover, considering the finite cone

$$\mathcal{C} = \left\{ x : |x| < r_0,\ \frac{x \cdot \xi}{|x|} > \cos\theta \right\}$$

with axis in the direction ξ and width 2θ, where $\theta = \arctan \frac{1}{M}$, we have that $\mathcal{C} \subset \Omega$. Let us consider now a point $Q \in \Gamma_2^{r_0}$ and let Q_0 be a point lying on the axis ξ of the cone with vertex in $Q = 0$ such that $d_0 = \mathrm{dist}(Q_0, 0) < \frac{r_0}{2}$. Following Lieberman [57], we introduce a regularized distance $\tilde{d}$ from the boundary of Ω. We have that there exists $\tilde{d}$ such that $\tilde{d} \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$, satisfying the following properties

i) $\gamma_0 \leq \dfrac{\mathrm{dist}(x, \partial\Omega)}{\tilde{d}(x)} \leq \gamma_1,$

ii) $|\nabla \tilde{d}(x)| \geq c_1,\quad$ for every x such that $\mathrm{dist}(x, \partial\Omega) \leq b r_0,$

iii) $\|\tilde{d}\|_{C^{0,1}} \leq c_2 r_0,$

where $\gamma_0, \gamma_1, c_1, c_2, b$ are positive constants depending on M only, (see also [8, Lemma 5.2]).
Let us define for every $\rho > 0$

$$\tilde{\Omega}_\rho = \{x \ \in \Omega \quad : \quad \tilde{d}(x) > \rho\} \ .$$

It follows that, there exists a, $0 < a \leq 1$, only depending on M such that for every ρ, $0 < \rho \leq a r_0$, $\tilde{\Omega}_\rho$ is connected with boundary of class C^1 and

$$\tilde{c}_1 \rho \leq \mathrm{dist}(x, \partial\Omega) \leq \tilde{c}_2 \rho \qquad \text{for every} \ \ x \ \in \ \partial\tilde{\Omega}_\rho \cap \Omega \tag{3.66}$$

where $\tilde{c}_1, \tilde{c}_2$ are positive constants depending on M, α only. By (3.66) it follows that

$$\Omega_{\tilde{c}_2 \rho} \subset \tilde{\Omega}_\rho \subset \Omega_{\tilde{c}_1 \rho} \ .$$

Using the notation introduced in the Proposition 3.6, we define the point $P = P_0 - \frac{1}{4\sqrt{1+M^2}} r_0 \cdot \nu$ and $\rho_0 = \min\{\frac{1}{32M\sqrt{1+M^2}} r_0, \frac{r_0}{4} \sin\theta\}$. Moreover, let γ be a path in $\tilde{\Omega}_{\frac{\rho_0}{\tilde{c}_1}}$ joining P to Q_0 and let us define $\{y_i\}$, $i = 0, \dots, s$ as follows $y_0 = Q_0$, $y_{i+1} = \gamma(t_i)$, where $t_i = \max\{t : |\gamma(t) - y_i| = 2\rho_0\}$ if $|P - y_i| > 2\rho_0$ otherwise let $i = s$ and stop the process.
Now, we will use the three spheres inequality for harmonic functions (see for instance [48] or [10, Appendix E]) that is

$$\int_{B_{3\rho_0}(y_0)} u^2 \leq \left(\int_{B_{\rho_0}(y_0)} u^2 \right)^\tau \cdot \left(\int_{B_{4\rho_0}(y_0)} u^2 \right)^{1-\tau}$$

where $0 < \tau < 1$ is an absolute constant. Now since $B_{\rho_0}(y_0) \subset B_{3\rho_0}(y_1)$ and since, by hypothesis $\|u\|_{H^1(\Omega)} \leq E$, then we have

$$\int_{B_{\rho_0}(y_0)} u^2 \leq \left(\int_{B_{3\rho_0}(y_1)} u^2\right)^{\tau} \cdot E^{1-\tau} .$$

An iterated application of the three spheres inequality leads to

$$\int_{B_{\rho_0}(y_0)} u^2 \leq \left(\int_{B_{\rho_0}(y_s)} u^2\right)^{\tau^s} \cdot E^{1-\tau^s} .$$

Finally, since we have $B_{\rho_0}(y_s) \subset B_{\frac{3M}{4\sqrt{1+M^2}}r_0}(P_0) \cap U^{\Gamma_1}_{2r_0}$, then by the Proposition 3.6 it follows

$$\int_{B_{\rho_0}(y_0)} u^2 \leq C\left\{(\varepsilon + E)^{1-\delta} \cdot (\varepsilon)^{\delta}\right\}^{\tau^s} .$$

We shall construct a chain of balls $B_{\rho_k}(Q_k)$ centered on the axis of the cone, pairwise tangent to each other and all contained in the cone

$$\mathcal{C}' = \left\{x : |x| < r_0,\ \frac{x \cdot \xi}{|x|} > \cos\theta'\right\}$$

where $\theta' = \arcsin\left(\frac{\rho_0}{d_0}\right)$. Let $B_{\rho_0}(Q_0)$ be the first of them, the following are defined by induction in such a way

$$\begin{aligned} Q_{k+1} &= Q_k - (1+\mu)\rho_k \xi \ , \\ \rho_{k+1} &= \mu \rho_k \ , \\ d_{k+1} &= \mu d_k \ , \end{aligned}$$

with

$$\mu = \frac{1 - \sin\theta'}{1 + \sin\theta'} .$$

Hence, with this choice, we have $\rho_k = \mu^k \rho_0$ and $B_{\rho_{k+1}}(Q_{k+1}) \subset B_{3\rho_k}(Q_k)$.
Let us now consider the following estimate obtained by a repeated application of the three spheres inequality

$$\begin{aligned} \|u\|_{L^2(B_{\rho_k}(Q_k))} &\leq \|u\|_{L^2(B_{3\rho_{k-1}}(Q_{k-1}))} \leq \\ &\leq \|u\|^{\tau}_{L^2(B_{\rho_{k-1}}(Q_{k-1}))} \|u\|^{1-\tau}_{L^2(B_{4\rho_{l-1}}(Q_{k-1}))} \\ &\leq C\|u\|^{\tau^k}_{L^2(B_{\rho_0}(Q_0))} \leq \\ &\leq C\left\{\left[(\varepsilon+E)^{1-\delta} \cdot (\varepsilon)^{\delta}\right]^{\tau^s}\right\}^{\tau^k} . \end{aligned} \tag{3.67}$$

For every r, $0 < r < d_0$, let $k(r)$ be the smallest positive integer such that $d_k \leq r$, then since $d_k = \mu^k d_0$, it follows

$$\frac{|\log(\frac{r}{d_0})|}{\log \mu} \leq k(r) \leq \frac{|\log(\frac{r}{d_0})|}{\log \mu} + 1 \tag{3.68}$$

and by (3.67), we have

$$\|u\|_{L^2(B_{\rho_{k(r)}}(Q_{k(r)}))} \leq C\Big\{\big[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^{\delta}\big]^{\tau^s}\Big\}^{\tau^{k(r)}} . \tag{3.69}$$

Since, by hypothesis, Γ_2 is contained in a $C^{1,\alpha}$ surface and by the regularity assumption (3.6) on g, it follows, by the same argument used in Theorem 3.4, that $u \in C^{1,\alpha}(U^{\Gamma_2}_{2r_0})$.
Let $\bar{x} \in \Gamma_2^{2r_0}$, $x \in B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, since $u \in C^{1,\alpha}(U^{\Gamma_2}_{2r_0})$ we have

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right| \leq \left|\frac{\partial u(x)}{\partial \nu}\right| + C|x-\bar{x}|^{\alpha} \leq \left|\frac{\partial u(x)}{\partial \nu}\right| + C\left(\frac{2}{\mu}r\right)^{\alpha} .$$

Integrating over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, we deduce that

$$\begin{aligned}\left|\frac{\partial u(\bar{x})}{\partial \nu}\right|^2 &\leq \frac{2}{\omega_n(\frac{\rho_{k-1}}{2})^n}\int_{B_{\frac{\rho_{k(r)-1}}{2}}\left(Q_{k(r)-1}\right)}\left|\frac{\partial u(x)}{\partial \nu}\right|^2 \mathrm{d}x + 2C^2\left(\frac{4r^2}{\mu^2}\right)^{\alpha} \leq \\ &\leq \frac{2}{\omega_n(\frac{\rho_{k-1}}{2})^n}\int_{B_{\frac{\rho_{k(r)-1}}{2}}\left(Q_{k(r)-1}\right)}|\nabla u(x)|^2 \mathrm{d}x + 2C^2\left(\frac{4r^2}{\mu^2}\right)^{\alpha} .\end{aligned}$$

Applying the Caccioppoli inequality, we have

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{\big(\rho_{k-1}\big)^{n+2}}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} u(x)^2 \mathrm{d}x + Cr^{2\alpha}$$

and since k is the smallest integer such that $d_k \leq r$, then $d_{k-1} > r$, it follows

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{\big(r\sin\theta'\big)^{n+2}}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} u(x)^2 \mathrm{d}x + Cr^{2\alpha} .$$

From (3.69), we deduce

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{r^{n+2}}\Big\{\big[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^{\delta}\big]^{\tau^s}\Big\}^{\tau^{k(r)-1}} + Cr^{2\alpha} .$$

Let us define

$$\sigma(\varepsilon) = \big[(\varepsilon+E)^{1-\delta}\cdot(\varepsilon)^{\delta}\big]^{\tau^s},$$

thus the previous inequality becomes

$$\left|\frac{\partial u(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{r^{n+2}}\big\{\sigma(\varepsilon)\big\}^{\tau^{k(r)-1}} + Cr^{2\alpha} .$$

Now, using (3.68), we have

$$\tau^{k(r)-1} \geq \left(\frac{r}{d_0}\right)^{\nu}$$

where $\nu = -\log\left(\frac{1}{\mu}\right)\log\tau$. We have

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right| \leq C\left\{r^{-\frac{n+2}{2}}\left[\sigma(\varepsilon)\right]^{\frac{r^{\nu}}{2}} + r^{\alpha}\right\} .$$

Now minimizing the function on the right hand side, with respect to r, with $r \in (0, \frac{r_0}{4})$, we deduce

$$\left|\frac{\partial u(\bar{x})}{\partial\nu}\right| \leq C\left(\log\frac{1}{\sigma(\varepsilon)}\right)^{-\frac{2\alpha}{\nu+2}} .$$

Since this estimate holds for every $\bar{x} \in \Gamma_2^{2r_0}$, we infer

$$\left\|\frac{\partial u}{\partial\nu}\right\|_{L^{\infty}(\Gamma_2^{2r_0})} \leq C\left(\log\frac{1}{\sigma(\varepsilon)}\right)^{-\frac{2\alpha}{\nu+2}}$$

where C is a constant depending on the *a priori data* only. Hence, solving for ε, we can compute

$$\varepsilon \geq C\exp\left\{-\left\|\frac{\partial u}{\partial\nu}\right\|_{L^{\infty}(\Gamma_2^{2r_0})}^{-\frac{\nu+2}{2\alpha}}\right\} .$$

Note that, recalling the a priori bound (3.6), and choosing $c = 2(1-\log CG^{\gamma})$ and $\gamma = \frac{\nu+2}{2\alpha}$ one trivially obtains

$$\varepsilon \geq \exp\left[-\left(\frac{t}{c}\right)^{-\gamma}\right], \quad \text{for every} \quad t \in (0, G] .$$

□

3.3 The stability result

Theorem 3.7 (Stability for a Cauchy problem). *Let Ω, f_i $i = 1, 2$ and g_i satisfy the a priori assumptions described above. Let $u_i \in H_0^1(\Omega, \Gamma_D)$, $i = 1, 2$ be weak solutions of the problem* (1.5)*, with $f = f_i$ and $g = g_i$ respectively and such that* (2.18) *holds for each u_i.*
Moreover, let $\psi_i = u_i\big|_{\Gamma_2}$, $i = 1, 2$. Suppose that

$$\|\psi_1 - \psi_2\|_{L^2(\Gamma_2)} \leq \varepsilon ,$$
$$\|g_1 - g_2\|_{L^2(\Gamma_2)} \leq \varepsilon ,$$

then, for every $\rho \in (0, r_0)$

$$\|u_1 - u_2\|_{C^1(\Gamma_1^{\rho})} \leq \omega(\varepsilon) \tag{3.70}$$

where ω is given by (3.11) *with a constant $C > 0$ which depends on the* a priori data *and on ρ only.*

Proof. The proof follows by considering the procedure developed in Theorem 2.7 with $\sigma = Id$ and $\Sigma = \Gamma_2$. □

Proposition 3.8 (Local monotonicity). *Let u be a solution of* (1.5) *satisfying* (2.18)*, then there exist a point $\bar{x} \in \Gamma_1^{\tau}$ and a direction $\xi \in \mathbb{R}^{n-1}$, $|\xi| = 1$ such that, in the representation* (2.3) *of Γ_1 near $\bar{x}$, the following holds*

$$|\nabla_{x'} u(x', \varphi_1(x')) \cdot \xi| \geq \eta\left(\|g\|_{L^\infty(\Gamma_2^{2r_0})}\right), \quad x' \in U_{\bar{x}'} = \{x' = t \cdot \xi + \bar{x}', \ |t| \leq \tau\} \tag{3.71}$$

with

$$\tau = \min\left\{\frac{r_0}{4}, \frac{a\tilde{c_1} r_0}{4}, \eta(\|g\|_{L^\infty(\Gamma_2^{2r_0})})\right\} \tag{3.72}$$

where $0 < a < 1, \tilde{c_1} > 0$ are constants depending on the a priori data *only and η satisfies* (3.10)*.*

Proof. Arguing as in Theorem 3.1, we can introduce a regularized distance, in the sense of Lieberman, on S_1 from the boundary of Γ_1 and consequently construct connected sets $\tilde{\Gamma}_1^{\rho}$ for every ρ, $0 < \rho \leq ar_0$, which satisfy

$$\Gamma_1^{\tilde{c}_2 h} \subset \tilde{\Gamma}_1^{h} \subset \Gamma_1^{\tilde{c}_1 h} \tag{3.73}$$

where $0 < a < 1, \tilde{c}_2 > \tilde{c}_1 > 0$ are constants depending on M, α only.
Since, by Lemma 3.3, $u \in C^{0,\alpha}(\Gamma_1)$, we have that by (3.73) it follows

$$\operatorname*{osc}_{\tilde{\Gamma}_1^{\frac{\rho}{\tilde{c}_1}}} u \geq \operatorname*{osc}_{\Gamma_1^{\frac{\tilde{c}_2\rho}{\tilde{c}_1}}} u \geq \operatorname*{osc}_{\Gamma_1} u - 2CE\left(\frac{\rho}{\tilde{c_1}}\right)^{\alpha} \tilde{c_2}^{\alpha} \ .$$

Moreover by Theorem 3.1, we infer that

$$\operatorname*{osc}_{\tilde{\Gamma}_1^{\frac{\rho}{\tilde{c}_1}}} u \geq \eta(\|g\|_{L^\infty(\Gamma_2^{2r_0})}) - 2CE\left(\frac{\rho}{\tilde{c_1}}\right)^{\alpha} \tilde{c_2}^{\alpha} \ .$$

Possibly replacing c by a larger constant in (3.10) and taking

$$r_1 = \min\left\{\eta(\|g\|_{L^\infty(\Gamma_2^{2r_0})}), a\tilde{c_1} r_0, r_0\right\}$$

we have that

$$\operatorname*{osc}_{\tilde{\Gamma}_1^{\frac{r_1}{\tilde{c}_1}}} u \geq \eta(\|g\|_{L^\infty(\Gamma_2^{2r_0})}) \ . \tag{3.74}$$

Let us set, for simplicity, $\eta = \eta\left(\|g\|_{L^\infty(\Gamma_2^{2r_0})}\right)$. Since in the a priori assumptions we have assumed that the portion Γ_1 of the boundary is of $C^{1,\alpha}$ class, then we can locally represent the restriction of u (the solution to (1.5)) to Γ_1, as a function of $n-1$ variables, more precisely, for every $x_0 \in \Gamma_1$, up to a rigid change of coordinates, we denote

$$w(x') = u(x', \varphi_1(x')) \quad \text{for all} \quad x \in \Gamma_1 \cap B_{r_0}(x_0). \tag{3.75}$$

By (3.74), it follows that exist two points x and y in $\tilde{\Gamma}_1^{\frac{r_1}{c_1}}$, such that

$$\eta \leq u(x) - u(y) \ . \tag{3.76}$$

Let us consider a continuous path $\sigma \subset \tilde{\Gamma}_1^{\frac{r_1}{c_1}}$ joining x to y and let us define a sequence $\{x_i\}_{i=0,\dots,l}$ as follows $x_0 = x,\ x_i = \sigma(s_i)$ where $s_i = \max\{s\ , |\sigma(s) - x_i| = \frac{r_1}{4}\}$ if $|y - x_i| > \frac{r_1}{4}$ otherwise let $i = l$ and otherwise stop the process.
The number l of balls is bounded from above by $CM\left(\frac{D}{r_1}\right)^{n-1}$, where $C > 0$ is a constant depending on n only.
Let us define

$$M_i = \max_{B_{\frac{r_1}{4}}(x_i)\cap\Gamma_1} |\nabla_t u(x)|$$

where ∇_t denotes the tangential gradient on Γ_1. Let $\bar{M}$, $\bar{\imath}$, $\bar{x}$ be such that $\bar{x} \in B_{\frac{r_1}{4}}(x_{\bar{\imath}}) \cap \Gamma_1$ and

$$\bar{M} = \max_{i=1,\dots,l} \{M_i\} = |\nabla_t u(\bar{x})|. \tag{3.77}$$

By (3.76) and the mean value Theorem, it follows that

$$\begin{aligned} \eta \quad &\leq |u(x) - u(x_1)| + \cdots + |u(x_l) - u(y)| \leq \\ &\leq \sum_{i=1,\dots,l} M_i \frac{r_1}{4} \leq \bar{M} C_1 \end{aligned}$$

where $C_1 > 0$ is a constant depending on the *a priori data* only. Thus we have

$$\bar{M} \geq \frac{\eta}{C_1} > 0 \ . \tag{3.78}$$

Now we use the local representation of u as a function of $n-1$ variables (3.75), within $\Gamma_1 \cap B_{\frac{r_1}{4}}(x_{\bar{\imath}})$. Let us define the direction $\xi = \frac{\nabla_{x'} w}{|\nabla_{x'} w|}(\bar{x}')$. We shall further restrict the function w to the segment $t \cdot \xi + \bar{x}'$, with

$$v(t) = w(t \cdot \xi + \bar{x}') \ .$$

Now, we look for a neighborhood U_0 of $t = 0$ such that

$$|v'(t)| \geq \frac{\eta}{2C_1} \quad \text{for every} \quad t \in U_0 \ . \tag{3.79}$$

It follows that for every $|t| < \frac{r_1}{4}$

$$|v'(0) - v'(t)| \leq C_2 |t|^{\alpha}$$

where $C_2 > 0$ is a constant depending on the *a priori data* only.
Thus we have

$$\bar{M} = |v'(0)| \leq |v'(t)| + C_2|t|^\alpha \ .$$

Hence by (3.78),

$$\frac{\eta}{C_1} - C_2|t|^\alpha \leq |v'(t)| \ .$$

Let us choose t in such a way

$$C_2|t|^\alpha \leq \frac{\eta}{2C_1} \ .$$

Hence (3.79) holds with $U_0 = [-\tau, \tau]$, where $\tau = \min\left\{\frac{r_1}{4}, \left(\frac{\eta}{2C_1C_2}\right)^{\frac{1}{\alpha}}\right\}$. The thesis follows, observing that $v'(t) = \frac{\partial w(x')}{\partial \xi} = \nabla_{x'} u(x', \varphi(x')) \cdot \xi$ and, possibly, by a further adjustment of the constant c in (3.10). □

Proof of Theorem 3.2. Let $\bar{x} \in \Gamma_1^{\tau_1}, \tau_1, \xi \in \mathbb{R}^{n-1}$ be the point, the length and the direction introduced in Proposition 3.8, with u replaced with u_1. Up to a change of coordinates, we assume $\xi = e_1$. Let

$$v_i(t) = u_i(t \cdot \xi + \bar{x}', \varphi_1(t \cdot \xi + \bar{x}')) \ , \quad i = 1, 2 \ ,$$

where $x = (x', \varphi_1(x'))$ is the local representation of Γ_1 near $\bar{x}$.
By Proposition 3.8 and assumption (3.12), we have that

$$|v_1'(t)| \geq \eta(m) \ , \qquad \text{for every} \ \ t \in U_0 = [-\tau_1, \tau_1] \ . \tag{3.80}$$

We shall denote by $\eta_1 = \eta(\|g_1\|_{L^\infty(\Gamma_2^{2r_0})})$. By the stability estimate (3.70) of Theorem 3.7, we have that

$$v_2'(t) \geq \eta_1 - \omega(\varepsilon) \ , \ \text{for every } t \in U_0 \ .$$

Thus choosing ε_0 such that

$$\omega(\varepsilon_0) \leq \frac{\eta_1}{2}$$

we have

$$|v_2'(t)| \geq \frac{\eta_1}{2} \ , \ \ \text{for every } t \in U_0 \ . \tag{3.81}$$

Thus the functions v_i are invertible on U_0, let us denote by V_i their respective images and by

$$s^i : V_i \to U_0 \ , \quad i = 1, 2 \ , \tag{3.82}$$

their inverse functions. Let us observe that the intervals V_1 and V_2 overlap on a sufficiently large interval V. In fact, by (3.80) and (3.81) it follows that v_i are monotone.Without loss of generality, let us assume they are both increasing. We have that, taken

$$a = -\frac{\tau_1}{2} \ , \quad b = \frac{\tau_1}{2} \ ,$$

the following hold

$$v_i(a) < v_i(t) < v_i(b) \ , \quad \text{for every } t \in (a,b) \ , \quad i = 1,2 \ .$$

Moreover, since by the Theorem 3.7 we have

$$\|u_1 - u_2\|_{L^\infty(\Gamma_{1,\frac{\tau_1}{2}})} \leq \omega(\varepsilon)$$

then, it follows that, for $\varepsilon < \varepsilon_0$, setting $V = (v_1(a) + 2\omega(\varepsilon), v_1(b) - 2\omega(\varepsilon))$, for every $u \in V$, there exists $t \in (a,b)$ such that $v_2(t) = u$.
Let us estimate from below the length of the interval V. By the mean value Theorem, (3.80) and (3.72), it follows that

$$|v_1(a) - v_1(b)| = |v_1'(\xi)||b - a| \geq \eta_1 \tau_1 \ .$$

Thus the length $\mathcal{L}$ of V is bounded from below by

$$\mathcal{L} \geq \tau_1 \eta_1 - \omega(\varepsilon) \ .$$

Hence, possibly adjusting the constant c in the definition (3.10) of η, we have that

$$\mathcal{L} \geq \eta(m) - \omega(\varepsilon_0) \geq \frac{1}{2}\eta(m) > 0 \ .$$

Let us consider any value $u \in V$, then using the inverse function s^i, we have

$$u = v_1(s^1(u)) = v_2(s^2(u)) \ .$$

Let us estimate

$$\begin{aligned}
&|f_1(u) - f_2(u)| = \\
&\left| \frac{\partial u_1}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) - \frac{\partial u_2}{\partial \nu}(s^2(u)e_1, \varphi_1(s^2(u)e_1)) \right| \leq \\
&\left| \frac{\partial u_1}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) - \frac{\partial u_2}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) \right| + \\
&\left| \frac{\partial u_2}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) - \frac{\partial u_2}{\partial \nu}(s^2(u)e_1, \varphi_1(s^2(u)e_1)) \right|
\end{aligned}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{n-1}$. By Theorem 3.7 it follows that, for all $u \in V$,

$$\left| \frac{\partial u_1}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) - \frac{\partial u_2}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) \right| \leq \omega(\varepsilon) \ . \tag{3.83}$$

By Corollary 3.5, we infer that

$$\begin{aligned}&\left|\frac{\partial u_2}{\partial \nu}(s^1(u)e_1, \varphi_1(s^1(u)e_1)) - \frac{\partial u_2}{\partial \nu}(s^2(u)e_1, \varphi_1(s^2(u)e_1))\right| \leq \\ &\tilde{L}\left(|s^1(u) - s^2(u)| + |\varphi_1(s^1(u)e_1) - \varphi_1(s^2(u)e_1)|\right) \leq \\ &\tilde{L}(1+M)|s^1(u) - s^2(u)| \ .\end{aligned}$$

By the mean value Theorem, we find

$$v_2(s^2(u)) = v_2(s^1(u)) + v_2'(\bar{s})(s^2(u) - s^1(u))$$

where $\bar{s}$ is a point between $s^2(u)$ and $s^1(u)$. Since

$$v_2(s^2(u)) = v_1(s^1(u)) \ ,$$

by (3.81) and by Theorem 3.7, it follows that

$$\begin{aligned}|s^1(u) - s^2(u)| &\leq \frac{2}{\eta_1}|v_2(s^1(u)) - v_1(s^1(u))| \leq \\ &\leq \frac{2}{\eta_1}\omega(\varepsilon) \ , \qquad \text{for every} \quad u \in V \ .\end{aligned}$$

Finally, we infer that

$$|f_1(u) - f_2(u)| \leq \omega(\varepsilon) \ , \qquad \text{for every} \quad u \in \ V \ ,$$

possibly by a further adjustment of the constant C in (3.11). □

Chapter 4

Resolution of elliptic Cauchy problems and reconstruction of the nonlinear corrosion

In this chapter we shall study the issue of solving the Cauchy problem for elliptic equations in divergence form (2.1) as well as the reconstruction issue for the nonlinearity f in the boundary value problem (1.5). First we shall solve the Cauchy problem by means of regularization techniques, then we shall propose a reconstruction procedure for the identification of the nonlinear corrosion under some additional *a priori* assumptions on the solution of the problem.
Before discussing the main results of this chapter, let us introduce the notion of regularization strategy and collect some reconstruction techniques, that we shall apply in the course of the exposition.

4.1 Regularization theory for compact operators

A lot of inverse problems can be formulated as operator equations of the form

$$Kx = y \ , \tag{4.1}$$

where K is a linear compact operator between Hilbert spaces X and Y.
For a sake of simplicity let us assume that the compact operator K is injective.
Let us now introduce the notion of *regularization strategy.*

Definition 4.1. *A* regularization strategy *is a family of linear and bounded operators*

$$R_\alpha : X \ \rightarrow \ Y, \quad \alpha > 0 \tag{4.2}$$

such that

$$\lim_{\alpha \to 0} R_\alpha K x = x, \text{ for every } x \in X \ , \tag{4.3}$$

i.e. the operators $R_\alpha K$ *converge* pointwise *to the identity.*

As a consequence of the compactness of the operator K, we state the following theorem.

Theorem 4.2. *Let* R_α *be a regularization strategy for* (4.1)*, where dim*$X = \infty$*. Then we have*

i) *The operators* R_α *are not uniformly bounded.*

ii) *There is no convergence* $R_\alpha K$ *to the identity* I *in the operator norm.*

Proof. See [46, Chap.2]. □

Let us observe that the definition of a regularization strategy is based on unperturbed data. Indeed, let us assume that there exists a solution $x \in X$ of the unperturbed equation (4.1). However, in practice, the right hand-side of (4.1), will be affected by errors and thus it is never known exactly, but only up to an error $\varepsilon > 0$. Hence, let us assume to know the measured data y_ε with

$$\|y - y_\varepsilon\|_Y \leq \varepsilon. \tag{4.4}$$

Let us define

$$x_{\alpha,\varepsilon} = R_\alpha y_\varepsilon. \tag{4.5}$$

Thus, $x_{\alpha,\varepsilon}$ can be thought as an approximate solution of the exact one x.
By a trivial application of the triangle inequality, we can split the error in two parts, as follows.

$$\|x - x_{\alpha,\varepsilon}\|_X \leq \|R_\alpha y_\varepsilon - R_\alpha y\|_X + \|R_\alpha y - x\|_X \leq \tag{4.6}$$

$$\leq \|R_\alpha\| \|y_\varepsilon - y\|_Y + \|R_\alpha K x - x\|_X. \tag{4.7}$$

Hence by (4.4), we have

$$\|x - x_{\alpha,\varepsilon}\|_X \leq \varepsilon \|R_\alpha\| + \|R_\alpha K x - x\|_X. \tag{4.8}$$

Our aim now is to choose the regularization parameter α, dependent upon ε, so that the approximate solutions $x_{\alpha,\varepsilon}$ actually converge to the exact solution x. In this respect, let us observe that the first term in the right-hand side of (4.8) might diverge as α tends to zero, whereas the second term tends to zero as α tends to zero. Hence we have to balance these two behaviors by minimizing (4.8) with respect to α.
We introduce the following notion.

Definition 4.3. *A regularization strategy* $\alpha = \alpha(\varepsilon)$ *is called* admissible *if*

$$\alpha(\varepsilon) \to 0 \quad \text{and} \quad \sup\{\|R_{\alpha\varepsilon y_\varepsilon - x}\| \ : \ \|Kx - y_\varepsilon\| \leq \varepsilon\} \to 0, \quad \varepsilon \to 0, \tag{4.9}$$

for every $x \in X$.

Before introducing a regularization strategy for the problem (4.1), let us recall the following property of the compact operators.

Proposition 4.4. *Let* $K : X \to Y$ *be a compact operator between Hilbert spaces* X *and* Y*. Then there exists a triple* $\{\sigma_j, x_j, y_j\}_{j=1}^{\infty}$ *called* singular value decomposition, *such that* $\{\sigma_j\}_{j=1}^{\infty}$ *is a non increasing infinitesimal sequence of nonnegative numbers,* $\{x_j\}_{j=1}^{\infty}$,$\{y_j\}_{j=1}^{\infty}$ *are orthonormal bases for* X *and* Y *respectively, such that*

$$Ky_j = \sigma_j x_j \ , \quad \text{for every } j = 1, 2 \dots \ , \tag{4.10}$$

$$K^* x_j = \sigma_j y_j \ , \quad \text{for every } j = 1, 2 \dots \ , \tag{4.11}$$

where K^* *denotes the adjoint operator to* K.

Proof. See [46, Appendix A]. □

Let us now state the following regularization theorem.

Theorem 4.5. *Let* $K : X \to Y$ *be a compact operator with singular value decomposition* $\{\sigma_j, x_j, y_j\}_{j=1}^{\infty}$ *and*

$$q : (0, \infty) \times (0, \|K\|] \to \mathbb{R}$$

be a function with the following properties

i) $|q(\alpha, \sigma)| \leq 1$ *for every* $\alpha > 0$ *and* $0 < \sigma \leq \|K\|$;

ii) *for every* $\alpha > 0$ *there exists* $c(\alpha)$ *such that*

$$|q(\alpha, \sigma)| \leq c(\alpha)\sigma \text{ for every } 0 < \sigma \leq \|K\|;$$

iii) $\lim_{\alpha \to 0} q(\alpha, \sigma) = 1$ *for every* $0 < \sigma \leq \|K\|$.

Then the operator $R_\alpha : Y \to X$, $\alpha > 0$ *defined by*

$$R_\alpha y = \sum_{j=1}^{\infty} \frac{q(\alpha, \sigma_j)}{\sigma_j} (y, y_j) x_j, \quad y \in Y$$

is a regularization strategy with $\|R_\alpha\| \leq c(\alpha)$.
A choice $\alpha = \alpha(\varepsilon)$ *is admissible if* $\alpha(\varepsilon) \to 0$ *and* $\varepsilon c(\alpha(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$. *The function* q *is called a* regularizing filter *for* K.

Proof. See [46, Chap. 2]. □

Let us note that the cut-off function q defined as follows

$$q(\alpha,\sigma)=\begin{cases} 1 & if\ \sigma^2\geq\alpha\ , \\ 0 & if\ \sigma^2<\alpha\ , \end{cases} \tag{4.12}$$

is a regularization filter, since it satisfies the properties i),ii),iii) of Theorem 4.5.

Corollary 4.6. *The operator $R_\alpha : Y \to X$, $\alpha > 0$ defined by*

$$R_\alpha y = \sum_{\sigma_j\geq\alpha} \frac{1}{\sigma_j}(y, y_j)x_j, \quad y \in Y$$

is a regularization strategy.
Moreover, every choice

$$\alpha(\varepsilon) = \varepsilon^{2(1-\gamma)} \tag{4.13}$$

for the regularization parameter, with γ, $0 < \gamma < 1$, is admissible.

In the course of the present chapter we shall recall, when needed, some quantitative formulations of the a priori assumptions made in Chapter 2 and in Chapter 3. Hence, shall refer to the *a priori data* as the set of quantities $r_0, M, \alpha, L, G, E, D, m, \mu, K$, previously introduced in Chapter 2 and in Chapter 3.

4.2 Solving the Cauchy problem

In this section we return to the study of the Cauchy problem (2.1) for variable coefficients elliptic equations, started in Chapter 2 with the stability analysis. Now we are concerned with the reconstruction issue for the same problem.
Before discussing the reconstruction techniques developed in this chapter, let us recall the main assumptions and briefly outline the trace space setting needed in this context.
We shall assume that the hypothesis (2.11)-(2.18) are satisfied.
We introduce the trace spaces $H^{\frac{1}{2}}(\Sigma)$, $H^{\frac{1}{2}}_{00}(\Sigma)$ as the interpolation spaces $[H^1(\Sigma), L^2(\Sigma)]_{\frac{1}{2}}$, $[H^1_0(\Sigma), L^2(\Sigma)]_{\frac{1}{2}}$ respectively, see [58, Chap. 1] for details. We shall denote the corresponding dual spaces by $H^{\frac{1}{2}}(\Sigma)^*$, $H^{\frac{1}{2}}_{00}(\Sigma)^*$, respectively.
We recall that there exists a linear extension operator

$$\begin{aligned} &\mathcal{E} : H^{\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\partial\Omega)\ , \quad \text{such that} \quad \mathcal{E}(\psi) = \psi \quad \text{on } \Sigma \quad \text{and} \\ &\|\mathcal{E}(\psi)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|\psi\|_{H^{\frac{1}{2}}(\Sigma)} \text{ for every } \psi \in H^{\frac{1}{2}}(\Sigma), \end{aligned} \tag{4.14}$$

where $C > 0$ is a constant depending on the *a priori data* only, see for instance [1, Lemma 7.45]. Also we recall that the operator $\mathcal{E}_0$ of continuation to zero outside Σ,

$$\mathcal{E}_0(\varphi) = \begin{cases} \varphi, & \text{in } \Sigma, \\ 0, & \text{in } \partial\Omega \setminus \Sigma, \end{cases} \tag{4.15}$$

is bounded from $H^{\frac{1}{2}}_{00}(\Sigma)$ into $H^{\frac{1}{2}}(\partial\Omega)$. Note that, by such an extension, $H^{\frac{1}{2}}_{00}(\Sigma)$ can be identified with the closed subspace of $H^{\frac{1}{2}}(\partial\Omega)$ of functions supported in $\overline{\Sigma} \subset \partial\Omega$. More precisely, recalling the notations (2.10) and (2.20) we can identify $H^{\frac{1}{2}}_{00}(\Sigma)$ with the trace space of $H^1_0(\Omega, \Gamma)$ on $\partial\Omega$. See [58, Chap. 1] and also, for more details, [68].

Given $\psi \in H^{\frac{1}{2}}(\Sigma)$ and $g \in H^{\frac{1}{2}}_{00}(\Sigma)^*$ we shall say that $u \in H^1(\Omega)$ is a weak solution to (2.1) if $u|_\Sigma = \psi$ in the trace sense and also

$$\int_\Omega \sigma \nabla u \cdot \nabla \eta =< g, \eta|_\Sigma > \tag{4.16}$$

for every $\eta \in H^1_0(\Omega, \Gamma)$. Here $< \cdot, \cdot >$ denotes the pairing between $H^{\frac{1}{2}}_{00}(\Sigma)^*$ and $H^{\frac{1}{2}}_{00}(\Sigma)$ based on the $L^2(\Sigma)$ scalar product. Our first step in the solution of the Cauchy problem (2.1) is the reduction to the case when $\psi = 0$. To this purpose we consider the weak solution $W \in H^1(\Omega)$ to the well-posed Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla W) = 0 & \text{in } \Omega, \\ W = \mathcal{E}\psi & \text{on } \partial\Omega. \end{cases} \tag{4.17}$$

Setting $U = u - W$ and $G = g - \sigma \nabla W \cdot \nu|_{H^{\frac{1}{2}}_{00}(\Sigma)} \in H^{\frac{1}{2}}_{00}(\Sigma)^*$, we have that U is a weak solution to the Cauchy problem

$$\begin{cases} \operatorname{div}(\sigma \nabla U) = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Sigma, \\ \sigma \nabla U \cdot \nu = G & \text{on } \Sigma. \end{cases} \tag{4.18}$$

For every $h \in H^{\frac{1}{2}}_{00}(\Gamma)^*$ let us consider the mixed boundary value problem

$$\begin{cases} \operatorname{div}(\sigma \nabla v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \\ \sigma \nabla v \cdot \nu = h & \text{on } \Gamma\ . \end{cases} \tag{4.19}$$

A function $v \in H^1_0(\Omega, \Sigma)$ is said to be a weak solution to (4.19) if

$$\int_\Omega \sigma \nabla v \cdot \nabla \eta =< h, \eta|_\Gamma > \quad \text{for every } \eta \in H^1_0(\Omega, \Sigma). \tag{4.20}$$

It is readily seen, by the Lax-Milgram Theorem, that such mixed boundary value problem (4.19) is well-posed. It is also evident that, finding the appropriate $h \in H^{\frac{1}{2}}_{00}(\Gamma)^*$ such that $\sigma \nabla v \cdot \nu|_{H^{\frac{1}{2}}_{00}(\Gamma)} = G$, would imply that $v = U$ and provide

us with the solution to (4.18). We note however, that given $\rho_0 > 0$ such that Σ^{ρ_0} has nonempty interior, it would suffice to check that for some $\rho, 0 < \rho < \rho_0$, $\sigma\nabla v \cdot \nu = G$ when both functionals are restricted to $H_{00}^{\frac{1}{2}}(\Sigma^\rho)$. In fact, this is a consequence of the uniqueness of the solution of the Cauchy problem when the Cauchy data are prescribed on Σ^ρ (instead than on all of Σ). Thus, having fixed $\rho,\ 0 < \rho < \rho_0$, the solution of the Cauchy problem (4.18) amounts to find $h \in H_{00}^{\frac{1}{2}}(\Gamma)^*$ such that $\sigma\nabla v \cdot \nu = G$ on $H_{00}^{\frac{1}{2}}(\Sigma^\rho)$.
We prove the following.

Theorem 4.7. *For any $\rho, 0 < \rho < \rho_0$, let T_ρ be the operator*

$$\begin{array}{rcl} T_\rho : H_{00}^{\frac{1}{2}}(\Gamma)^* & \rightarrow & H_{00}^{\frac{1}{2}}(\Sigma^\rho)^* \\ h & \mapsto & \sigma\nabla v \cdot \nu|_{\Sigma^\rho} \end{array} \tag{4.21}$$

where $v \in H_0^1(\Omega, \Sigma)$ solves the mixed problem (4.19)*. The operator T_ρ is compact.*

Proof. By the well posedness of the mixed boundary value problem (4.19), the linear operator

$$\begin{array}{rcl} S : H_{00}^{\frac{1}{2}}(\Gamma)^* & \rightarrow & H_0^1(\Omega, \Sigma) \\ h & \mapsto & v \end{array}$$

is bounded.
Moreover, by a standard result of regularity at the boundary, it follows that for every $\rho > 0,\ v \in C^{1,\alpha}(U_\rho^\Sigma)$ and there exists a constant $C_\rho > 0$ depending on the *a priori data* and on ρ only, such that

$$\|v\|_{C^{1,\alpha}(\Sigma^\rho)} \leq C_\rho \|v\|_{H_0^1(\Omega)}.$$

Thus the operator

$$\begin{array}{rcl} D_\rho : H^1(\Omega) & \rightarrow & C^{0,\alpha}(\Sigma^\rho) \\ v & \mapsto & \sigma\nabla v \cdot \nu|_{\Sigma^\rho} \end{array}$$

is bounded. Finally, since the inclusion

$$i_\rho :\ C^{0,\alpha}(\Sigma^\rho) \hookrightarrow\ H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*$$

is compact and T_ρ can be factored as $T_\rho = i_\rho \circ D_\rho \circ S$, the thesis follows. □

Being T_ρ a compact operator between Hilbert spaces, then it admits a *singular value decomposition* $\{\sigma_j^\rho, h_j, g_j^\rho\}_{j=1}^\infty$
By Corollary 4.6, we have that, denoting with $(\cdot,\cdot)_{H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*}$ the scalar product for the Hilbert space $H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*$, the family of operators $R_\alpha,\ \alpha > 0$

$$\begin{array}{rcl} R_\alpha : H_{00}^{\frac{1}{2}}(\Sigma^\rho)^* & \rightarrow & H_{00}^{\frac{1}{2}}(\Gamma)^* \\ g & \mapsto & \sum_{\sigma_k^\rho \geq \alpha} \frac{1}{\sigma_k^\rho} (g, g_k^\rho)_{H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*} h_k \end{array} \tag{4.22}$$

is a regularization strategy for T_ρ, namely

$$\lim_{\alpha\to 0} R_\alpha T_\rho h = h \ , \ \text{for every } h \in H_{00}^{\frac{1}{2}}(\Gamma)^*. \tag{4.23}$$

Moreover, we recall that the choice (4.13) where γ is a fixed number, $0 < \gamma < 1$, is an admissible one, this means that if given, for every $\varepsilon > 0$, $g, g_\varepsilon \in H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*$ and $h \in H_{00}^{\frac{1}{2}}(\Gamma)^*$ such that

$$g = T_\rho h \quad \text{and} \quad \|g - g_\varepsilon\|_{H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*} \leq \varepsilon \ , \tag{4.24}$$

then it follows that

$$\lim_{\varepsilon\to 0} \|R_{\alpha(\varepsilon)} g_\varepsilon - h\|_{H_{00}^{\frac{1}{2}}(\Gamma)^*} = 0 \ . \tag{4.25}$$

We can return now to the Cauchy problem (2.1), when ψ is arbitrary in $H^{\frac{1}{2}}(\Sigma)$. Let us suppose that, for every $\varepsilon > 0$, $\psi_\varepsilon \in H^{\frac{1}{2}}(\Sigma)$, $g_\varepsilon \in H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*$, and let $W_\varepsilon \in H^1(\Omega)$ be the weak solution of (4.17), with $\psi = \psi_\varepsilon$. Let us denote by $R^\varepsilon = R_{\alpha(\varepsilon)}(g_\varepsilon - \sigma\nabla W_\varepsilon \cdot \nu|_{\Sigma^\rho}) + \sigma\nabla W_\varepsilon \cdot \nu|_\Gamma \in H_{00}^{\frac{1}{2}}(\Gamma)^*$, where R_α and $\alpha(\varepsilon)$ are the regularization strategy and the regularization parameter introduced in (4.22) and (4.13), respectively. We propose as approximate regularized solution to the problem (2.1) the function $u_\varepsilon \in H^1(\Omega)$ which is a weak solution of the mixed boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ u_\varepsilon = \psi_\varepsilon & \text{on } \Sigma, \\ \sigma\nabla u_\varepsilon \cdot \nu = R^\varepsilon & \text{on } \Gamma. \end{cases} \tag{4.26}$$

In analogy to (4.19) and (4.20), we shall call weak solution of the problem (4.26), a function $u_\varepsilon \in H^1(\Omega)$ such that $u_\varepsilon|_\Sigma = \psi_\varepsilon$ in the trace sense and such that

$$\int_\Omega \sigma\nabla u_\varepsilon \cdot \nabla\eta = < R^\varepsilon, \eta|_\Gamma > \quad \text{for every} \quad \eta \in H_0^1(\Omega, \Sigma). \tag{4.27}$$

The well-posedness of problem (4.26) is again a consequence of the Lax-Milgram Theorem. The following Theorem provides a convergence results for the procedure of regularized inversion of the Cauchy problem (2.1) that we have just outlined, when we start with approximate Cauchy data $\psi_\varepsilon, g_\varepsilon$ close to the exact Cauchy data ψ, g.

Theorem 4.8. *Let $\psi \in H^{\frac{1}{2}}(\Sigma)$ and $g \in H_{00}^{\frac{1}{2}}(\Sigma)^*$ be such that there exists $u \in H^1(\Omega)$, which is a weak solution to the Cauchy problem* (2.1)*. If, given $\varepsilon > 0$, we have that $\psi_\varepsilon \in H^{\frac{1}{2}}(\Sigma)$ and $g_\varepsilon \in H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*$*

$$\|\psi - \psi_\varepsilon\|_{H^{\frac{1}{2}}(\Sigma)} \leq \varepsilon \ , \tag{4.28}$$

$$\|g - g_\varepsilon\|_{H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*} \leq \varepsilon \ , \tag{4.29}$$

then

$$\lim_{\varepsilon\to 0} u_\varepsilon|_\Gamma = u|_\Gamma \quad in \quad H^{\frac{1}{2}}(\Gamma) \quad , \tag{4.30}$$

$$\lim_{\varepsilon\to 0} \sigma\nabla u_\varepsilon \cdot \nu|_\Gamma = \sigma\nabla u \cdot \nu|_\Gamma \quad in \quad H_{00}^{\frac{1}{2}}(\Gamma)^* \quad . \tag{4.31}$$

Proof. Let us observe that given S any open and connected portion of $\partial\Omega$, the following holds

$$\|\sigma\nabla W_\varepsilon \cdot \nu|_S - \sigma\nabla W \cdot \nu|_S\|_{H_{00}^{\frac{1}{2}}(S)^*} \leq c_1\|W - W_\varepsilon\|_{H^1(\Omega)} \leq c_2\|\mathcal{E}\psi_\varepsilon - \mathcal{E}\psi\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

then replacing in (4.14) ψ with $\psi_\varepsilon - \psi$, we have by (4.28) that

$$\|\sigma\nabla W_\varepsilon \cdot \nu|_S - \sigma\nabla W \cdot \nu|_S\|_{H_{00}^{\frac{1}{2}}(S)^*} \leq c_3\varepsilon \ , \tag{4.32}$$

where $c_1, c_2, c_3 > 0$ are constants depending on the *a priori data* and on S only. Thus by (4.32), with $S = \Sigma^\rho$, and by (4.29), we have that

$$\lim_{\varepsilon\to 0} \|g - g_\varepsilon + \sigma\nabla W_\varepsilon \cdot \nu|_{\Sigma^\rho} - \sigma\nabla W \cdot \nu|_{\Sigma^\rho}\|_{H_{00}^{\frac{1}{2}}(\Sigma^\rho)^*} = 0 \tag{4.33}$$

Moreover, we have that (4.31) follows by applying (4.32) with $S = \Gamma$, (4.25) with g_ε replaced with $g_\varepsilon - \sigma\nabla W_\varepsilon \cdot \nu|_{\Sigma^\rho}$ and (4.33). Indeed, we have

$$\begin{aligned}
&\|\sigma\nabla u \cdot \nu|_\Gamma - \sigma\nabla u_\varepsilon \cdot \nu|_\Gamma\|_{H_{00}^{\frac{1}{2}}(\Gamma)^*} \leq \\
&\leq \left\|R_{\alpha(\varepsilon)}\left(g_\varepsilon - \sigma\nabla W_\varepsilon \cdot \nu|_{\Sigma^\rho}\right) + \sigma\nabla W \cdot \nu|_\Gamma - \sigma\nabla u \cdot \nu|_\Gamma\right\|_{H_{00}^{\frac{1}{2}}(\Gamma)^*} + \\
&\quad + \|\sigma\nabla W \cdot \nu|_\Gamma - \sigma\nabla W_\varepsilon \cdot \nu|_\Gamma\|_{H_{00}^{\frac{1}{2}}(\Gamma)^*} .
\end{aligned}$$

Finally, by a standard trace inequality

$$\begin{aligned}
&\|u|_\Gamma - u_\varepsilon|_\Gamma\|_{H^{\frac{1}{2}}(\Gamma)} \leq c_4\|u - u_\varepsilon\|_{H^1(\Omega)} \leq \\
&\leq c_5\left(\|\sigma\nabla u \cdot \nu|_\Gamma - \sigma\nabla u_\varepsilon \cdot \nu|_\Gamma\|_{H_{00}^{\frac{1}{2}}(\Gamma)^*} + \|\psi - \psi_\varepsilon\|_{H^{\frac{1}{2}}(\Sigma)}\right)
\end{aligned} \tag{4.34}$$

where $c_4, c_5 > 0$ are constants depending on the *a priori data* only, then (4.30) follows by recalling (4.31) and from (4.28). □

4.3 A special case

The aim of this section is to specialize the approach of the previous section to the Laplace equation in a domain with a singular geometry, which might be well-suited to a reference conductor specimen, and to the model of electrochemical corrosion.

Let D be a bounded domain in $\mathbb{R}^{n-1}$, with Lipschitz boundary ∂D with constants r_0, M. From now on we shall consider this special choice of Ω

$$\Omega = D \times (0,1)\ ,\ \Gamma_2 = D \times \{0\}\ ,\ \Gamma_1 = D \times \{1\}\ ,\ \Gamma_D = \partial D \times (0,1).$$

In the following we will denote by $\lambda_k, \varphi_k,\ k = 1,2,\dots,$ the Dirichlet eigenvalues and eigenfunctions of $-\Delta$ on D, namely

$$\begin{cases} -\Delta\varphi_k = \lambda_k \varphi_k & \text{in } D, \\ \varphi_k \in H_0^1(D)\ . \end{cases} \tag{4.35}$$

We recall that the family $\{\varphi_k\}_{k=1}^{\infty}$ is an orthogonal basis in $L^2(D)$ and also in $H_0^1(D)$. In the following we shall refer to the $\{\varphi_k\}_{k=1}^{\infty}$ as the basis normalized in the $L^2(D)$ norm. We have that $\psi \in H_{00}^{\frac{1}{2}}(D)$ if and only if its Fourier coefficients

$$\psi_k = \int_D \psi\varphi_k \tag{4.36}$$

satisfy

$$\sum_{k=1}^{\infty} {\lambda_k}^{\frac{1}{2}} {\psi_k}^2 < \infty \tag{4.37}$$

and that, as a norm on $H_{00}^{\frac{1}{2}}(D)$ we can choose

$$\|\psi\|_{H_{00}^{\frac{1}{2}}(D)} = \left(\sum_{k=1}^{\infty} {\lambda_k}^{\frac{1}{2}} {\psi_k}^2 \right)^{\frac{1}{2}}\ . \tag{4.38}$$

Moreover, $h \in H_{00}^{\frac{1}{2}}(D)^*$ if and only if, its Fourier coefficients

$$h_k = < h, \varphi_k >\ , \tag{4.39}$$

satisfy

$$\sum_{k=1}^{\infty} {\lambda_k}^{-\frac{1}{2}} {h_k}^2 < \infty \tag{4.40}$$

and the norm on $H_{00}^{\frac{1}{2}}(D)^*$ turns out to be

$$\|h\|_{H_{00}^{\frac{1}{2}}(D)^*} = \left(\sum_{k=1}^{\infty} {\lambda_k}^{-\frac{1}{2}} {h_k}^2 \right)^{\frac{1}{2}}. \tag{4.41}$$

Here $< \cdot, \cdot >$ denotes the pairing between $H_{00}^{\frac{1}{2}}(D)^*$ and $H_{00}^{\frac{1}{2}}(D)$ based on the $L^2(D)$ scalar product. Note also that $\{{\lambda_k}^{-\frac{1}{4}}\varphi_k\}$ and $\{{\lambda_k}^{\frac{1}{4}}\varphi_k\}$ constitute orthonormal bases for $H_{00}^{\frac{1}{2}}(D)$ and $H_{00}^{\frac{1}{2}}(D)^*$ respectively.

Due to the cylindrical geometry of Ω, we remark that we can identify the spaces $H_{00}^{\frac{1}{2}}(\Gamma_i)$, $H_{00}^{\frac{1}{2}}(\Gamma_i)^*$, $i=1,2$, with $H_{00}^{\frac{1}{2}}(D)$, $H_{00}^{\frac{1}{2}}(D)^*$ respectively. Furthermore, as noted in Section 4, we can identify $H_{00}^{\frac{1}{2}}(\Gamma_1)$ with the trace space on $\partial\Omega$ of $H_0^1(\Omega,\Gamma)$ when $\Gamma = \overset{\circ}{\overline{(\Gamma_2 \cup \Gamma_D)}}$, and the same holds when the roles of Γ_1 and Γ_2 are exchanged.
Let $\psi \in H_{00}^{\frac{1}{2}}(\Gamma_2)$, $g \in H_{00}^{\frac{1}{2}}(\Gamma_2)^*$ and let us consider the following Cauchy problem with auxiliary homogeneous condition on Γ_D

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \Gamma_2, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma_2, \\ u = 0 & \text{on } \Gamma_D. \end{cases} \tag{4.42}$$

We shall say that u is a weak solution to the problem (4.42) if $u|_{\overset{\circ}{\overline{(\Gamma_2 \cup \Gamma_D)}}} = \mathcal{E}_0(\psi)$ in the trace sense and if

$$\int_\Omega \nabla u \cdot \nabla \eta = < g, \eta|_{\Gamma_2} > \quad \text{for every } \eta \in H_0^1(\Omega, \overset{\circ}{\overline{(\Gamma_1 \cup \Gamma_D)}}).$$

Here $\mathcal{E}_0(\psi)$ denotes the extension of ψ by zero outside Γ_2 and $< \cdot, \cdot >$ denotes the pairing between $H_{00}^{\frac{1}{2}}(\Gamma_2)^*$ and $H_{00}^{\frac{1}{2}}(\Gamma_2)$ based on the $L^2(\Gamma_2)$ scalar product.
We shall use a strategy similar to the one discussed in Section 4, but with some slight variations, suggested by the presence of the portion Γ_D of the boundary where $u=0$. As before, we reduce the problem (4.42) to the special case when $\psi = 0$ and introduce the well-posed Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = \xi & \text{on } \Gamma_1, \\ v = 0 & \text{on } \overset{\circ}{\overline{(\Gamma_2 \cup \Gamma_D)}}, \end{cases} \tag{4.43}$$

where ξ is a prescribed function in $H_{00}^{\frac{1}{2}}(\Gamma_1)$. To this purpose, in analogy with (4.17), we consider $W \in H^1(\mathbb{R}^n \setminus D)$ as the weak solution to the Dirichlet problem

$$\begin{cases} \Delta W = 0 & \text{in } \Omega, \\ W = \psi & \text{on } \Gamma_2, \\ W = 0 & \text{on } \overset{\circ}{\overline{(\Gamma_1 \cup \Gamma_D)}}\ . \end{cases} \tag{4.44}$$

The difference $U = u - W$ shall satisfy (4.42) with $\psi = 0$ and g replaced with $G = g - \frac{\partial W}{\partial \nu}|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}$.
Note that the well posed boundary value problem (4.43), will take the place of (4.19). We intend to invert the map

$$T : \xi \to \left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_2} \tag{4.45}$$

in order to solve the Cauchy problem. It is convenient at this stage to recall the identification of the trace spaces on $\Gamma_i,\ i=1,2$ with the corresponding ones on D.

Lemma 4.9. *Let T be the operator*

$$T : H_{00}^{\frac{1}{2}}(D) \quad \to \quad H_{00}^{\frac{1}{2}}(D)^* \tag{4.46}$$

$$\xi \quad \mapsto \quad \left.\frac{\partial v}{\partial \nu}\right|_{\Gamma_2} \tag{4.47}$$

where v is the weak solution of the problem (4.43)*. Then T extends to a compact and self-adjoint operator on $L^2(D)$, such that $\left\{-\lambda_k^{\frac{1}{2}}(\sinh(\lambda_k^{\frac{1}{2}}))^{-1}, \varphi_k\right\}_{k=1}^{\infty}$ are its eigenvalues and eigenfunctions respectively. The singular value decomposition of $T : H_{00}^{\frac{1}{2}}(D) \to H_{00}^{\frac{1}{2}}(D)^*$ is given by*

$$\{-(\sinh(\lambda_k^{\frac{1}{2}}))^{-1}, \lambda_k^{-\frac{1}{4}}\varphi_k, \lambda_k^{\frac{1}{4}}\varphi_k\}_{k=1}^{\infty} \ . \tag{4.48}$$

Proof. Let us first observe that the operator T is well defined since the problem (4.43) is well-posed. In this special setting we can represent the solution v of (4.43) by separation of variables, namely

$$v(x', x_n) = \sum_{k=1}^{\infty} \frac{\xi_k}{\sinh(\lambda_k^{\frac{1}{2}})} \sinh(\lambda_k^{\frac{1}{2}} x_n)\varphi_k(x') \tag{4.49}$$

where $\{\xi_k\}_{k=1}^{\infty}$ are the Fourier coefficients of ξ with respect to the $L^2(D)$ basis $\{\varphi_k\}_{k=1}^{\infty}$. After straightforward calculations we have that

$$T\left(\sum_{k=1}^{\infty} \xi_k\varphi_k\right) = \sum_{k=1}^{\infty}\left(-\frac{\xi_k\lambda_k^{\frac{1}{2}}}{\sinh(\lambda_k^{\frac{1}{2}})}\right)\varphi_k \tag{4.50}$$

thus the operator extends to a self-adjoint operator on $L^2(D)$ and since the eigenvalues are infinitesimal we conclude that T is compact as an operator from $L^2(D)$ into $L^2(D)$. Moreover, since $H_{00}^{\frac{1}{2}}(D)$ is continuously embedded in $L^2(D)$ and $L^2(D)$ is continuously embedded in $H_{00}^{\frac{1}{2}}(D)^*$, also $T : H_{00}^{\frac{1}{2}}(D) \to H_{00}^{\frac{1}{2}}(D)^*$ is compact and its SVD turns out to be (4.48). □

As a consequence of the above Lemma 4.9, we obtain that the family of operators

$$R_\alpha : H_{00}^{\frac{1}{2}}(D)^* \longrightarrow H_{00}^{\frac{1}{2}}(D), \quad \text{such that}$$

$$R_\alpha(G) = \sum_{\mu_k \geq \alpha} (-\sinh(\lambda_k^{\frac{1}{2}}))(G, \varphi_k)_{H_{00}^{\frac{1}{2}}(D)^*}\varphi_k \tag{4.51}$$

where $\mu_k = (\sinh(\lambda_k^{\frac{1}{2}}))^{-1}$, is a regularization strategy for T and the choice (4.13) for the parameter α is still admissible. We are in the position now to present the regularized approximate solution for the following special case of the problem (4.42). That is, given $G \in H_{00}^{\frac{1}{2}}(\Gamma_2)$,

$$\begin{cases} \Delta U = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial U}{\partial \nu} = G & \text{on } \Gamma_2, \\ U = 0 & \text{on } \Gamma_D. \end{cases} \tag{4.52}$$

In this section we shall denote by $[r]$ the integral part of the real number r.

Theorem 4.10. *For every $\varepsilon > 0$, let $G_\varepsilon \in H_{00}^{\frac{1}{2}}(\Gamma_2)^*$ and let $G \in H_{00}^{\frac{1}{2}}(\Gamma_2)^*$ be such that there exists $U \in H^1(\Omega)$, which is a weak solution of the problem* (4.52)*. If we have*

$$\|G_\varepsilon - G\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} \leq \varepsilon$$

then for every choice of $\gamma, 0 < \gamma < 1$, the function

$$U_\varepsilon(x', x_n) = \sum_{k=1}^{[\log(\varepsilon^{\gamma-1})]^{n-1}} (-\lambda_k^{-\frac{1}{2}} G_{k,\varepsilon}) \sinh(\lambda_k^{\frac{1}{2}} x_n) \varphi(x') \tag{4.53}$$

where $\{G_{k,\varepsilon}\}_{k=1}^{\infty}$ are the $L^2(D)$ Fourier coefficients of G_ε (according to the formula (4.39)*), satisfies*

$$\lim_{\varepsilon \mapsto 0} U_\varepsilon|_{\Gamma_1} = U|_{\Gamma_1} \quad \text{in} \quad H_{00}^{\frac{1}{2}}(\Gamma_1) \ . \tag{4.54}$$

Proof. Since the one defined in (4.51) is a family of regularizing operators and since the choice (4.13) is admissible, we have that

$$\lim_{\varepsilon \to 0} \|R_{\alpha(\varepsilon)}(G_\varepsilon) - U|_{\Gamma_1}\|_{H_{00}^{\frac{1}{2}}(D)} = 0 \ . \tag{4.55}$$

By the asymptotic bounds of the eigenvalues of the Laplace operator (see for instance [26, Chap. 12]) we have that there exist constants $c, C > 0$ depending on the *a priori data* only, such that

$$ck^{\frac{2}{n-1}} \leq \lambda_k \leq Ck^{\frac{2}{n-1}}, \quad k = 1, 2, \dots \quad .$$

Thus it follows that the integer k such that $\mu_k \geq \alpha(\varepsilon)$ is of the order $[\log(\varepsilon^{\gamma-1})]^{n-1}$. Moreover, since

$$(G_\varepsilon, \varphi_k)_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} = G_{k,\varepsilon} \lambda_k^{-\frac{1}{2}},$$

the thesis follows immediately by (4.55). □

The following Corollary 4.11 provides us with the approximate regularized solution to the Cauchy problem (4.42).

Corollary 4.11. *For every* $\varepsilon > 0$, *let* $\psi_\varepsilon \in H_{00}^{\frac{1}{2}}(\Gamma_2), g_\varepsilon \in H_{00}^{\frac{1}{2}}(\Gamma_2)^*$ *and suppose that there exists* $u \in H^1(\Omega)$ *which is a weak solution of the problem* (4.42), *with exact Cauchy data* $\psi \in H_{00}^{\frac{1}{2}}(\Gamma_2), g \in H_{00}^{\frac{1}{2}}(\Gamma_2)^*$. *If we have*

$$\|\psi_\varepsilon - \psi\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \leq \varepsilon \tag{4.56}$$

$$\|g_\varepsilon - g\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} \leq \varepsilon \tag{4.57}$$

then for every choice of γ, $0 < \gamma < 1$, *the function*

$$u_\varepsilon(x', x_n) = \sum_{k=1}^{[\log(\varepsilon^{\gamma-1})]^{n-1}} (-{\lambda_k}^{-\frac{1}{2}} G_{k,\varepsilon}) \sinh({\lambda_k}^{\frac{1}{2}} x_n)\varphi_k(x') + \tag{4.58}$$
$$+ \sum_{k=1}^{\infty} \psi_{k,\varepsilon} \frac{\sinh({\lambda_k}^{\frac{1}{2}}(1-x_n))}{\sinh({\lambda_k}^{\frac{1}{2}})} \varphi_k(x'),$$

where

$$G_{k,\varepsilon} = g_{k,\varepsilon} - \psi_{k,\varepsilon}{\lambda_k}^{\frac{1}{2}} \coth({\lambda_k}^{\frac{1}{2}}), \quad k = 1, 2, \ldots \tag{4.59}$$

$\{\psi_{k,\varepsilon}\}_{k=1}^{\infty}$, $\{g_{k,\varepsilon}\}_{k=1}^{\infty}$ *are the* $L^2(D)$*-Fourier coefficients of* ψ_ε *and* g_ε *respectively, is an approximate regularized solution of* (4.42). *Moreover, we have*

$$\lim_{\varepsilon \mapsto 0} u_\varepsilon|_{\Gamma_1} = u|_{\Gamma_1} \quad in \quad H_{00}^{\frac{1}{2}}(\Gamma_1)\ , \tag{4.60}$$

$$\lim_{\varepsilon \mapsto 0} \left.\frac{\partial u_\varepsilon}{\partial \nu}\right|_{\Gamma_1} = \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1} \quad in \quad H_{00}^{\frac{1}{2}}(\Gamma_1)^*. \tag{4.61}$$

Proof. Let W_ε be the solution of (4.44) with $\psi = \psi_\varepsilon$, respectively. Thus we can decompose $u = U + W$ where U is the solution of (4.52) with $G = g - \frac{\partial W}{\partial \nu}|_{\Gamma_2}$. Moreover, by (4.56) we have

$$\begin{aligned} \left\| \frac{\partial W_\varepsilon}{\partial \nu} - \frac{\partial W}{\partial \nu} \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} &\leq C_1 \|W_\varepsilon - W\|_{H^1(\Omega)} \leq C_2 \|\mathcal{E}_0 \psi_\varepsilon - \mathcal{E}_0 \psi\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \\ &\leq C_3 \|\psi_\varepsilon - \psi\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \leq C_3 \varepsilon\ , \end{aligned} \tag{4.62}$$

where $C_i > 0,\ i = 1, 2, 3$, are constants depending on the *a priori data* only. Thus denoting with $G_\varepsilon = g_\varepsilon - \frac{\partial W_\varepsilon}{\partial \nu}|_{\Gamma_2}$, (4.57) and (4.62) leads to

$$\|G_\varepsilon - G\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} \leq \|g_\varepsilon - g\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} + \left\| \frac{\partial W_\varepsilon}{\partial \nu} - \frac{\partial W}{\partial \nu} \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)^*} \leq C\varepsilon$$

where $C > 0$ is a constant depending on the *a priori data* only. By (4.55) in the proof of Theorem 4.10 and recalling that $W = 0$ on Γ_1, we have

$$\lim_{\epsilon \to 0} \|R_{\alpha(\epsilon)}(G_\varepsilon) - u|_{\Gamma_1}\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} = 0\ . \tag{4.63}$$

Finally, let us consider the following Dirichlet problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega, \\ u_\varepsilon = R_{\alpha(\epsilon)}(G_\varepsilon) & \text{on } \Gamma_1, \\ u_\varepsilon = \psi_\varepsilon & \text{on } \Gamma_2, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \end{cases} \tag{4.64}$$

we have that

$$\begin{aligned} \left\| \frac{\partial u_\varepsilon}{\partial \nu} - \frac{\partial u}{\partial \nu} \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)^*} &\leq C_4 \| u_\varepsilon - u \|_{H^1(\Omega)} \leq \\ &\leq C_5 \big(\| R_{\alpha(\epsilon)}(G_\varepsilon) - u|_{\Gamma_1} \|_{H_{00}^{\frac{1}{2}}(\Gamma_1)} + \| \psi_\varepsilon - \psi \|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \big) \end{aligned}$$

where $C_4, C_5 > 0$ are constants depending on the *a priori data* only, thus by (4.63) and by (4.56)

$$\lim_{\varepsilon \to 0} \left\| \frac{\partial u_\varepsilon}{\partial \nu} - \frac{\partial u}{\partial \nu} \right\|_{H_{00}^{\frac{1}{2}}(\Gamma_1)^*} = 0.$$

After straightforward calculations, (4.60) and (4.61) follow. □

Thus, for a given error level $\varepsilon > 0$, the regularized solution of the Cauchy problem (4.42) is given by (4.58) and in particular we obtain the following formulas for the Cauchy data on Γ_1 as follows

$$u_\varepsilon|_{\Gamma_1} = \sum_{k=1}^{[\log(\varepsilon^{\gamma-1})]^{n-1}} (\lambda_k{}^{-\frac{1}{2}} \psi_{k,\varepsilon} \coth(\lambda_k{}^{-\frac{1}{2}}) - g_{k,\varepsilon}) \lambda_k{}^{-\frac{1}{2}} \sinh(\lambda_k{}^{\frac{1}{2}}) \varphi(x') \tag{4.65}$$

$$\begin{aligned} \left. \frac{\partial u_\varepsilon}{\partial \nu} \right|_{\Gamma_1} &= \sum_{k=1}^{[\log(\varepsilon^{\gamma-1})]^{n-1}} (\lambda_k{}^{-\frac{1}{2}} \psi_{k,\varepsilon} \coth(\lambda_k{}^{-\frac{1}{2}}) - g_{k,\varepsilon}) \cosh(\lambda_k{}^{\frac{1}{2}}) \varphi(x') + \\ &+ \sum_{k=1}^{\infty} \left(- \frac{\psi_{k,\varepsilon} \lambda_k{}^{\frac{1}{2}}}{\sinh(\lambda_k{}^{\frac{1}{2}})} \right) \varphi_k(x') \end{aligned} \tag{4.66}$$

where the coefficients $\psi_{k,\varepsilon}$ and $g_{k,\varepsilon}$, with $k = 1, 2, \dots$, are the Fourier coefficients of ψ_ε and g_ε, with respect to the $L^2(D)$ basis $\{\varphi_k\}_{k=1}^\infty$.

4.4 A procedure for reconstruction

In this section we briefly discuss a procedure for the determination of the nonlinearity f in (1.5) when the measurement $u|_{\Gamma_2} = \psi$ is available for a given Neumann data g. First, we use the methods described in Section (4.2) and in Section (4.3). We shall assume that the assumptions (2.11),(2.12),(2.18),(3.1)-(3.8) are satisfied. In Subsection 4.4.1, we outline the adaptations to the method of Section (4.2) needed for our corrosion problem. In Subsection 4.4.2 we propose a method for the identification of the nonlinearity f from approximate values of $u|_{\Gamma_1}$, $\frac{\partial u}{\partial \nu}|_{\Gamma_1}$.

4.4.1 Solving the Cauchy problem

- We need to solve a Cauchy problem of the form

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \Gamma_2, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma_2, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{4.67}$$

where $u \in H^1(\Omega)$, and where in this special setting we choose $\psi \in H_{00}^{\frac{1}{2}}(\Gamma_2)$ and we have $g \in L^2(\Gamma_2) \subset H_{00}^{\frac{1}{2}}(\Gamma_2)^*$. The procedure introduced in Section 4 can be applied by considering $\sigma = Id$, $\Sigma = \Gamma_2$, $\Gamma = \overset{\circ}{\overline{(\Gamma_1 \cup \Gamma_D)}}$. Note that in this case, we have $\psi \in H_{00}^{\frac{1}{2}}(\Gamma_2)$. Therefore, it is convenient, in the formulation of the Dirichlet problem (4.17), to replace the Dirichlet data $\mathcal{E}(\psi)$ with $\mathcal{E}_0(\psi)$. We consider W as the solution to (4.17) with such modified Dirichlet data, that is

$$\begin{cases} \Delta W = 0 & \text{in } \Omega, \\ W = \mathcal{E}_0(\psi) & \text{on } \partial\Omega. \end{cases} \tag{4.68}$$

Performing as before the decomposition $u = U + W$, we obtain that U is the solution to the following variant of the Cauchy problem (4.18)

$$\begin{cases} \Delta U = 0 & \text{in } \Omega, \\ U = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial U}{\partial \nu} = g - \dfrac{\partial W}{\partial \nu}\Big|_{\Gamma_2} & \text{on } \Gamma_2, \\ U = 0 & \text{on } \Gamma_D\ . \end{cases} \tag{4.69}$$

- We can use the regularization strategy used in (4.22). Note that here $\Sigma^\rho = \Gamma_2^\rho$ and v turns out to be the solution of the following problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial v}{\partial \nu} = h & \text{on } \Gamma_1, \\ v = 0 & \text{on } \Gamma_D. \end{cases} \tag{4.70}$$

According to (4.22), we obtain a regularized inversion procedure for T_ρ.

- We obtain an approximate regularized solution to (4.67) by solving the analogue of the mixed boundary value problem (4.26), which in detail, takes the form

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega, \\ u_\varepsilon = \psi_\varepsilon & \text{on } \Gamma_2, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = R_{\alpha(\varepsilon)}(g_\varepsilon - \frac{\partial W_\varepsilon}{\partial \nu}\big|_{\Gamma_2^\rho}) + \frac{\partial W_\varepsilon}{\partial \nu}\big|_{\Gamma_1} & \text{on } \Gamma_1, \\ u_\varepsilon = 0 & \text{on } \Gamma_D, \end{cases} \tag{4.71}$$

where $\psi_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2)$, $g_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2^\rho)^*$ are the approximate Cauchy data and where $W_\varepsilon \in H^1(\Omega)$ is the weak solution of (4.17), with $\sigma(x) = Id$ and with $\mathcal{E}(\psi)$ replaced by $\mathcal{E}_0(\psi_\varepsilon)$. Having solved (4.71) we can determine the approximate regularized values of $u|_{\Gamma_1}$, $\frac{\partial u}{\partial \nu}|_{\Gamma_1}$ according to Theorem 4.8.

We observe that if the conducting specimen has the special geometry introduced in Section (4.3), that is $\Omega = D \times (0,1)$, then the above described scheme simplifies to the formulas (4.65) and (4.66).

4.4.2 Solving the algebraic equation $f(u) = \frac{\partial u}{\partial \nu}$

We cannot expect that, for the regularized solution u_ε, the Neumann data $\frac{\partial u_\varepsilon}{\partial \nu}$ on Γ_1 is precisely constant on each level set of $u_\varepsilon|_{\Gamma_1}$, as it should happen for the exact solution u to (1.5). Therefore, it is necessary to extract an approximate expression of the nonlinearity $f = f(u)$ when $u_\varepsilon|_{\Gamma_1}$ and $\frac{\partial u_\varepsilon}{\partial \nu}|_{\Gamma_1}$ may have different level sets. We propose to obtain such approximate nonlinear term by minimizing the *best fit* functional defined as follows,

$$F_\varepsilon[f] = \int_{\Gamma_1} \left(f(u_\varepsilon) - \frac{\partial u_\varepsilon}{\partial \nu} \right)^2 \mathrm{d}\sigma_{n-1}. \tag{4.72}$$

By the Coarea formula, (see for instance [34, Chap.3]), we have that we can express $F_\varepsilon[f]$ as follows

$$F_\varepsilon[f] = \int_{\mathbb{R}} \mathrm{d}t \int_{u_\varepsilon = t} \frac{(f(t) - \frac{\partial u_\varepsilon}{\partial \nu})^2}{|\nabla_{x'} u_\varepsilon|} \mathrm{d}\sigma_{n-2},$$

here, by σ_{n-2} we denote the $(n-2)$-dimensional Hausdorff measure. Thus, by formal differentiation it follows that

$$DF_\varepsilon[f](g) = \frac{\mathrm{d}}{\mathrm{d}s} F_\varepsilon[f + sg] \Big|_{s=0} = \int_{\mathbb{R}} g(t) \mathrm{d}t \int_{u_\varepsilon = t} 2 \frac{(f(t) - \frac{\partial u_\varepsilon}{\partial \nu})}{|\nabla_{x'} u_\varepsilon|} \mathrm{d}\sigma_{n-2}.$$

Hence a candidate minimizer for F_ε is given by the following weighted average of $\frac{\partial u_\varepsilon}{\partial \nu}|_{\Gamma_1}$ on the level sets of $u_\varepsilon|_{\Gamma_1}$, that is

$$f_\varepsilon(t) = \frac{1}{\displaystyle\int_{u_\varepsilon = t} \frac{1}{|\nabla_{x'} u_\varepsilon|} \mathrm{d}\sigma_{n-2}} \int_{u_\varepsilon = t} \frac{\frac{\partial u_\varepsilon}{\partial \nu}}{|\nabla_{x'} u_\varepsilon|} \mathrm{d}\sigma_{n-2}. \tag{4.73}$$

We note the consistency of this formula in the limiting case when u_ε is replaced by the exact solution u. In fact, in this case, the above formula leads to the correct values of f for every regular value t of $u|_{\Gamma_1}$.

4.5 Reconstruction of the nonlinear corrosion

In the present section we shall obtain a reconstruction result for the nonlinearity f under suitable *a priori* assumptions.

Indeed in order to recover the nonlinearity f we shall require a further regularity assumption on the smoothness of the portion Γ_1, namely we shall assume that given $\alpha, 0 < \alpha \leq 1$

$$\Gamma_1 \text{ is of class } C^{m+\frac{1}{2},\alpha} \text{ with constants } r_0, M, \tag{4.74}$$

with

$$m = \left[\frac{n}{2} + 2\right] + \frac{1}{2}. \tag{4.75}$$

In the sequel we shall make use of fractional order spaces, in this respect let us introduce the trace space $H^m_{00}(\Gamma_1)$, with m given by (4.75), as the interpolation space $[H^{2m}_0(\Gamma_1), L^2(\Gamma_1)]_{\frac{1}{2}}$. Moreover we shall denote with $H^m_{00}(\Gamma_1)^*$ its dual space.

We now outline a procedure, based on a slight modification of the arguments developed in Section (4.2), to obtain a convergence result for the solution to the Cauchy problem (2.1). The new feature consists in an improvement of such a convergence due to the stronger assumption (4.74) made on the portion Γ_1, as well as a further *a priori* assumption on the solution u to (2.1), namely we suppose that

$$u|_{\Gamma_1} \in H^m_{00}(\Gamma_1). \tag{4.76}$$

Remark 4.12. *Let us observe that the assumption* (4.76) *can be achieved by imposing a stronger regularity assumption on the nonlinearity f and by limiting ourselves to a particular geometry, for instance to a cylinder one or considering a geometry such that Γ_1 is a connected component of the boundary $\partial\Omega$.*

For every $\xi \in H^{\frac{1}{2}}_{00}(\Gamma_1)$ let us consider the following Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = \mathcal{E}_0(\xi) & \text{on } \partial\Omega, \end{cases} \tag{4.77}$$

where $\mathcal{E}_0$ is the operator of continuation to zero defined in (4.15) with Σ replaced by Γ_1.

By the Lax-Milgram theorem, it follows that the above Dirichlet problem is well posed.

Theorem 4.13. *For any $\rho, 0 < \rho < \rho_0$, let $\tilde{T}_\rho$ be the operator*

$$\begin{array}{rcl} \tilde{T}_\rho : H^m_{00}(\Gamma_1) & \rightarrow & H^{\frac{1}{2}}_{00}(\Gamma_2^\rho)^* \\ \xi & \mapsto & \left.\dfrac{\partial v}{\partial \nu}\right|_{\Gamma_2^\rho} \end{array} \tag{4.78}$$

where $v \in H^1_0(\Omega, \overset{\circ}{\overline{\Gamma_2 \cup \Gamma_D}})$ solves the mixed problem (4.77)*. The operator $\tilde{T}_\rho$ is compact.*

Proof. Noticing that the space $H^m_{00}(\Gamma_1)$ is continuously embedded into $H^{\frac{1}{2}}_{00}(\Gamma_1)$ and recalling that the problem (4.77) is well-posed, we have that the linear operator

$$\begin{array}{rcl} \tilde{S}: H^m_{00}(\Gamma_1) & \to & H^1_0(\Omega, \overline{\Gamma_2 \cup \Gamma_D}^{\circ}) \\ \xi & \mapsto & v \end{array}$$

is bounded.
At this stage the proof follows using analogous arguments to those developed in Theorem 4.7. □

Let us denote with $\{\tilde{\sigma}^\rho_j, \tilde{\xi}_j, \tilde{g}^\rho_j\}_{j=1}^\infty$ the singular value decomposition admitted by the the compact operator $\tilde{T}_\rho$.
By Theorem 4.13 we can conclude that the family of operators

$$\begin{array}{rcl} \tilde{R}_\alpha : H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^* & \to & H^m_{00}(\Gamma_1) \\ g & \mapsto & \sum_{\tilde{\sigma}^\rho_k \geq \alpha} \frac{1}{\tilde{\sigma}^\rho_k} (g, \tilde{g}^\rho_k)_{H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^*} \tilde{\xi}_k \end{array} \tag{4.79}$$

is a regularization strategy for $\tilde{T}_\rho$ and the choice (4.13) for the parameter α is still admissible.
Let us suppose that for every $\varepsilon > 0, \psi_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2)$ and $g_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^*$ and let $W_\varepsilon \in H^1_0(\Omega, \overline{\Gamma_1 \cup \Gamma_D}^{\circ})$ be the solution to the problem (4.68) with ψ replaced by ψ_ε.
For every $\varepsilon > 0$, let $u_\varepsilon \in H^1(\Omega, \Gamma_D)$ be the weak solution to the problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega, \\ u_\varepsilon = \psi_\varepsilon & \text{on } \Gamma_2, \\ u_\varepsilon = \tilde{R}_{\alpha(\varepsilon)}(g_\varepsilon - \frac{\partial W_\varepsilon}{\partial \nu}\big|_{\Gamma^\rho_2}) & \text{on } \Gamma_1, \\ u_\varepsilon = 0 & \text{on } \Gamma_D. \end{cases} \tag{4.80}$$

We are now in position to state the following convergence theorem.

Theorem 4.14. *Let $\psi \in H^{\frac{1}{2}}_{00}(\Gamma_2)$ and $g \in H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^*$ be such that there exists $u \in H^1(\Omega)$, which is a weak solution to the Cauchy problem* (2.1) *and let the assumption* (4.76) *be satisfied. If, given $\varepsilon > 0$, we have that $\psi_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma_2)$ and $g_\varepsilon \in H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^*$*

$$\|\psi - \psi_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\Gamma_2)} \leq \varepsilon \ , \tag{4.81}$$

$$\|g - g_\varepsilon\|_{H^{\frac{1}{2}}_{00}(\Gamma^\rho_2)^*} \leq \varepsilon \ , \tag{4.82}$$

then

$$\|u_\varepsilon - u\|_{C^1(\overline{\Gamma}_1)} \to 0, \ \text{as } \varepsilon \to 0, \tag{4.83}$$

$$\|u_\varepsilon - u\|_{H^1_0(\Omega)} \to 0, \ \text{as } \varepsilon \to 0, \tag{4.84}$$

where for every $\varepsilon > 0$, u_ε is the solution to the Dirichlet problem (4.80).

Proof. Let us observe that dealing with an analogous procedure to the one introduced in Section (4.2) and in Section (4.3), it follows that

$$\|u_\varepsilon - u\|_{H^m_{00}(\Gamma_1)} \to 0, \text{ as } \varepsilon \to 0 \tag{4.85}$$

$$\|u_\varepsilon - u\|_{H^1_0(\Omega)} \to 0, \text{ as } \varepsilon \to 0. \tag{4.86}$$

Moreover, we conclude that the convergence (4.83) holds noticing that $H^m_{00}(\Gamma_1)$ is continuously embedded into $C^1(\overline{\Gamma}_1)$, (see for instance [58, Chap.1]). □

Let τ be the length introduced in Proposition 3.8 and let for every $\varepsilon > 0$, $u_\varepsilon \in H^1(\Omega)$ be the solution to the problem (4.80). Let us now propose the following function

$$f^\tau_\varepsilon(t) = \frac{1}{\displaystyle\int_{\{x\in\Gamma_1^{\frac{\tau}{2}}:u_\varepsilon=t\}} |\nabla_{x'}u_\varepsilon|^{-1}\mathrm{d}\sigma_{n-2}} \int_{\{x\in\Gamma_1^{\frac{\tau}{2}}:u_\varepsilon=t\}} \frac{\partial u_\varepsilon}{\partial\nu}|\nabla_{x'}u_\varepsilon|^{-1}\mathrm{d}\sigma_{n-2} \ , \tag{4.87}$$

as an approximation of the exact nonlinearity f.
In the following theorem we will show that the sequence $\{f^\tau_\varepsilon\}_{\varepsilon>0}$ introduced in (4.87) actually converges to the nonlinearity f. Before stating the convergence result, let us recall that for every $k > 0$ we shall denote with $f^\tau_{\frac{1}{k}}$ the function introduced in (4.87) with $\varepsilon = \frac{1}{k}$.

Theorem 4.15. *Let the hypothesis of Theorem 4.14 be satisfied. Then there exist an interval V and an integer $k_0 > 0$ depending on the* a priori data *only such for a.e. $t \in V$*

$$f^\tau_{\frac{1}{k}}(t) \to f(t) \quad as \quad k \to \infty \ , \tag{4.88}$$

where $k \geq k_0$.

Proof. Let $\bar{x} \in \Gamma^\tau_1, \tau, \xi \in \mathbb{R}^{n-1}$ be the point, the length and the direction introduced in Proposition 3.8. For every $\varepsilon > 0$, let

$$v(s) = u(s\cdot\xi + \bar{x}', \varphi_1(s\cdot\xi + \bar{x}')) \ , \tag{4.89}$$

$$v_\varepsilon(s) = u_\varepsilon(s\cdot\xi + \bar{x}', \varphi_1(s\cdot\xi + \bar{x}')) \ , \tag{4.90}$$

where $x = (x', \varphi_1(x'))$ is the local representation of Γ_1 near $\bar{x}$.
By Proposition 3.8 and assumption (3.12), we have that

$$|v'(s)| \geq \eta(m) \ , \qquad \text{for every} \ \ s \in U_0 = [-\tau, \tau] \ . \tag{4.91}$$

By the convergence result (4.83) achieved in Theorem 4.14, we have that there exists an $\varepsilon_0 > 0$ only depending on the *a priori* data, such that for every ε, $0 < \varepsilon < \varepsilon_0$, we have

$$|v'_\varepsilon(s)| \geq \frac{\eta(m)}{2} \ , \qquad \text{for every} \ \ s \in U_0 = [-\tau, \tau] \ . \tag{4.92}$$

Let

$$V_0 = \{t \in \mathbb{R} \ : \exists \ s \in U_0 : v(s) = t\}, \tag{4.93}$$

then arguing as in the proof of Theorem 3.2, we can infer that, by a possible replacement of ε_0, there exists an interval $V \subset V_0$, such that for every $t \in V$ and for every $\varepsilon, 0 < \varepsilon < \varepsilon_0$, there exist $s_0, s_\varepsilon \in \left(-\frac{\tau}{2}, \frac{\tau}{2}\right)$ such that $v(s_0) = t$ and $v_\varepsilon(s_\varepsilon) = t$.
In other words we have found an interval V of common values of u and $\{u_\varepsilon\}_{\varepsilon>0}$.
By a consequence of the Coarea formula we have that

$$\sigma_{n-2}(\{x \in \Gamma_1^{\frac{\tau}{2}} : |\nabla_{x'} u_\varepsilon(x)| = 0\} \cap \{x \in \Gamma_1^{\frac{\tau}{2}} : u_\varepsilon(x) = t\}) = 0 \ , \tag{4.94}$$

for every $t \in \mathbb{R} \setminus A_\varepsilon$, with $\mathcal{L}^1(A_\varepsilon) = 0$. And analogously

$$\sigma_{n-2}(\{x \in \Gamma_1^{\frac{\tau}{2}} : |\nabla_{x'} u(x)| = 0\} \cap \{x \in \Gamma_1^{\frac{\tau}{2}} : u(x) = t\}) = 0 \ , \tag{4.95}$$

for every $t \in \mathbb{R} \setminus A_0$, with $\mathcal{L}^1(A_0) = 0$.
Let us set $\varepsilon = \frac{1}{k}$ and define the set of measure zero $A = \bigcup_{k=1}^{\infty} A_{\frac{1}{k}} \cup A_0$.
Let

$$x_0 \ \in \ \{x \in \Gamma_1^{\frac{\tau}{2}} : u(x) = t\} \ , \tag{4.96}$$

where t is a value in $V \setminus A$.
We consider now the following local representations of u and u_ε near x_0

$$w(x') = u(x', \varphi_1(x')), \quad \text{for all} \ \ x' \in B'_{r_1}(x_0{}'), \tag{4.97}$$
$$w_\varepsilon(x') = u_\varepsilon(x', \varphi_1(x')), \quad \text{for all} \ \ x' \in B'_{r_1}(x_0{}'), \tag{4.98}$$

where $r_1 = r_0(\sqrt{1+M^2}\ )^{-1}$.
Let U be the function defined as follows

$$U : B'_{r_1}(x_0{}') \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \ , \tag{4.99}$$

such that for every $\varepsilon, 0 < \varepsilon < \varepsilon_0$

$$U(x', 0) = w(x') - t \ , \tag{4.100}$$
$$U(x', \varepsilon) = U(x', -\varepsilon) = w_\varepsilon(x') - t \ . \tag{4.101}$$

By the choice (4.96), we have that

$$U(x_0{}', 0) = 0 \ , \tag{4.102}$$

and furthermore, being t a regular value of u, up to a change of coordinates, we have that

$$\frac{\mathrm{d}U}{\mathrm{d}x_{n-1}}(x_0', 0) \neq 0. \tag{4.103}$$

Hence by the Implicit Function Theorem it follows that there exist $\delta_0, \tilde{\varepsilon_0}, \eta_0 > 0$ and a function Ψ

$$\Psi : B''_{\delta_0}(x_0{}'') \times (-\tilde{\varepsilon_0}, \tilde{\varepsilon_0}) \to (x_{0_{n-1}} - \eta_0, x_{0_{n-1}} + \eta_0) \ , \tag{4.104}$$

such that for every $(x'', \varepsilon) \in B''_{\delta_0}(x_0{}'') \times (-\tilde{\varepsilon_0}, \tilde{\varepsilon_0})$

$$U(x'', \Psi(x'', \varepsilon), \varepsilon) = 0 \ . \tag{4.105}$$

Moreover, the function Ψ is continuous with respect to (x'', ε), differentiable with respect to x'' with partial derivatives

$$\frac{\partial \Psi}{\partial x_i}(x'', \varepsilon) = -\frac{U_{x_i}(x'', \Psi(x'', \varepsilon), \varepsilon)}{U_{x_{n-1}}(x'', \Psi(x'', \varepsilon), \varepsilon)} \ , \ i = 1, \dots n-2 \tag{4.106}$$

continuous with respect to (x'', ε). And furthermore, we have that

$$\Psi(x_0{}'', 0) = x_{0_{n-1}} \ . \tag{4.107}$$

Let us define for every $\varepsilon, 0 < \varepsilon < \tilde{\varepsilon_0}$ the functions

$$\psi_\varepsilon, \psi : B''_{\delta_0}(x_0{}'') \to \ (x_{0_{n-1}} - \eta_0, x_{0_{n-1}} + \eta_0) \ , \tag{4.108}$$

as

$$\psi_\varepsilon(x'') = \Psi(x'', \varepsilon) \ , \tag{4.109}$$

$$\psi(x'') = \Psi(x'', 0). \tag{4.110}$$

Hence we have that

$$\|\psi_\varepsilon - \psi\|_{C^1(B''_{\frac{\delta_0}{2}}(x_0{}''))} \to 0 \ , \ \text{as } \varepsilon \to 0 \ , \tag{4.111}$$

and moreover for every $\varepsilon, 0 < \varepsilon < \tilde{\varepsilon_0}$, (4.105) yields

$$w(x'', \psi(x'')) = t \ , \ \text{for every } x'' \in B''_{\delta_0}(x_0{}'') \ , \tag{4.112}$$

$$w_\varepsilon(x'', \psi_\varepsilon(x'')) = t \ , \ \text{for every } x'' \in B''_{\delta_0}(x_0{}''). \tag{4.113}$$

Repeating the arguments introduced above for every point

$$x_0 \in \{x \in \Gamma_1^{\frac{\tau}{2}} : u(x) = t\} \tag{4.114}$$

we can extract a finite covering

$$\{B''_{\frac{\delta_j}{2}}(x_j{}'') \times (x_{j_{n-1}} - \eta_j, x_{j_{n-1}} + \eta_j)\}_{j=1}^J \ , \tag{4.115}$$

of the sets $\{x \in \Gamma_1^{\frac{\tau}{2}} : u(x) = t\}$,$\{x \in \Gamma_1^{\frac{\tau}{2}} : u_\varepsilon(x) = t\}$ with $0 < \varepsilon < \tilde{\varepsilon_0}$ and a finite numbers of functions

$$\psi_\varepsilon^j, \psi^j : B''_{\delta_j}(x_j{}'') \to (x_{j_{n-1}} - \eta_j, x_{j_{n-1}} + \eta_j) \ \ j = 1 \dots, J \ , \tag{4.116}$$

verifying (4.111),(4.112) and (4.113) with $x_0 = x_j$, $\psi = \psi^j$, $\psi_\varepsilon = \psi^j_\varepsilon$, $\delta_0 = \delta_j$ and $\eta_0 = \eta_j$.
Denoting for every $j = 1, \dots, J$

$$\mathcal{U}_j = B''_{\frac{\delta_j}{2}}(x_j'') \times (x_{j\,n-1} - \eta_j, x_{j\,n-1} + \eta_j) \ , \tag{4.117}$$

let $\{\alpha_j\}_{j=1}^J$ be a smooth partition of unity subordinate to the open sets $\{\mathcal{U}_j\}_{j=1}^J$, namely suppose that

i) $0 \leq \alpha_j \leq 1$, $\alpha_j \in C^\infty_c(\mathcal{U}_j)$;

ii) $\sum_{j=1}^J \alpha_j = 1$, on $\cup_{j=1}^J \{\mathcal{U}_j\}$.

Let us consider the sequence $\{u_{\frac{1}{k}}\}$ obtained by setting $\varepsilon = \frac{1}{k}$ for every integer $k > 0$.
By the change of variables formula we have that for any function $h \in L^1(\Gamma_1^{\frac{\tau}{2}})$ the following holds

$$\int_{\{u_{\frac{1}{k}}=t\}} h(x')|\nabla_{x'} u_{\frac{1}{k}}(x')|^{-1} \mathrm{d}\sigma_{n-2} = \sum_{j=1}^J \int_{\{u_{\frac{1}{k}}=t\}} \alpha_j(x')\ h(x')|\nabla_{x'} u_{\frac{1}{k}}(x')|^{-1} \mathrm{d}\sigma_{n-2} =$$

$$\sum_{j=1}^J \int_{B''_{\frac{\delta_j}{2}}(x_j'')} \alpha_j(x'', \psi^j_{\frac{1}{k}}(x''))h(x'', \psi^j_{\frac{1}{k}}(x''))|\nabla_{x'} u_{\frac{1}{k}}(x'', \psi^j_{\frac{1}{k}}(x''))|^{-1} \sqrt{1 + |\nabla_{x''}\psi_{\frac{1}{k}}(x'')|^2} \mathrm{d}x''$$

Letting k tends to ∞ we obtain by (4.111) with $\varepsilon = \frac{1}{k}$ that

$$\left(\int_{\{u_{\frac{1}{k}}=t\}} h(x')|\nabla_{x'} u_{\frac{1}{k}}(x')|^{-1} \mathrm{d}\sigma_{n-2} - \int_{\{u=t\}} h(x')|\nabla_{x'} u_{\frac{1}{k}}(x')|^{-1} \mathrm{d}\sigma_{n-2} \right) \rightarrow 0 \ . \tag{4.118}$$

In particular, the above convergence implies that there exists a constant $c_0 > 0$ and an integer $k_0 > 0$ depending on the *a priori* data only such that for every $k \geq k_0$

$$\left| \int_{\{u_{\frac{1}{k}}=t\}} |\nabla_{x'} u_{\frac{1}{k}}(x')|^{-1} \mathrm{d}\sigma_{n-2} \right| \geq c_0 \ . \tag{4.119}$$

Let us notice that by the arguments in [12, Proposition 5.1], we have that there exists a constant $C > 0$ depending on the *a priori* data only, such that

$$\int_{\Gamma_1^{\frac{\tau}{2}}} \left| \frac{\partial u_{\frac{1}{k}}}{\partial \nu} - \frac{\partial u}{\partial \nu} \right| \mathrm{d}\sigma_{n-1} \leq C \left(\int_{\Gamma_1^{\frac{\tau}{4}}} |\nabla_{x'} u_{\frac{1}{k}} - \nabla_{x'} u|^2 \mathrm{d}\sigma_{n-1} + \int_\Omega |\nabla u_{\frac{1}{k}} - \nabla u|^2 \mathrm{d}x \right).$$

Hence by (4.83) and (4.84), it follows that

$$\int_{\Gamma_1^{\frac{\tau}{2}}} \left| \frac{\partial u_{\frac{1}{k}}}{\partial \nu} - \frac{\partial u}{\partial \nu} \right| \mathrm{d}\sigma_{n-1} \rightarrow 0, \text{ as } k \rightarrow \infty. \tag{4.120}$$

By a further application of the Coarea formula we have that for every $k \geq k_0$

$$\int_{\Gamma_1^{\frac{\tau}{2}}} \left| \frac{\partial u_{\frac{1}{k}}}{\partial \nu} - \frac{\partial u}{\partial \nu} \right| \mathrm{d}\sigma_{n-1} = \int_{\mathbb{R}} \mathrm{d}t \int_{\{u_{\frac{1}{k}}=t\}} \left| \frac{\partial u_{\frac{1}{k}}}{\partial \nu} - \frac{\partial u}{\partial \nu} \right| |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \mathrm{d}\sigma_{n-2}. \quad (4.121)$$

Hence by (4.121) and by (4.120) we have that, up to extract a subsequence

$$\int_{\{u_{\frac{1}{k}}=t\}} \left| \frac{\partial u_{\frac{1}{k}}}{\partial \nu} - \frac{\partial u}{\partial \nu} \right| |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \mathrm{d}\sigma_{n-2} \to 0 \ , \ \text{as } k \to \infty \ , \quad (4.122)$$

for a.e. $t \in V$.
Let $f^{\tau}_{\frac{1}{k}}$ be defined as in (4.87), then we have that for a.e. $t \in V$ the following holds

$$|f^{\tau}_{\frac{1}{k}}(t) - f(t)| \leq \quad (4.123)$$

$$\left| \left(\int_{u_{\frac{1}{k}}=t} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \right)^{-1} \left(\int_{u_{\frac{1}{k}}=t} \frac{\partial u_{\frac{1}{k}}}{\partial \nu} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \mathrm{d}\sigma_{n-2} - \int_{u_{\frac{1}{k}}=t} \frac{\partial u}{\partial \nu} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \mathrm{d}\sigma_{n-2} \right) \right| +$$

$$\left| \left(\int_{u_{\frac{1}{k}}=t} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \right)^{-1} \left(\int_{u_{\frac{1}{k}}=t} \frac{\partial u}{\partial \nu} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \mathrm{d}\sigma_{n-2} - \int_{u=t} \frac{\partial u}{\partial \nu} |\nabla_{x'} u|^{-1} \mathrm{d}\sigma_{n-2} \right) \right| +$$

$$\left| \left(\left(\int_{u_{\frac{1}{k}}=t} |\nabla_{x'} u_{\frac{1}{k}}|^{-1} \right)^{-1} - \left(\int_{u=t} |\nabla_{x'} u|^{-1} \right)^{-1} \right) \int_{u=t} \frac{\partial u}{\partial \nu} |\nabla_{x'} u|^{-1} \mathrm{d}\sigma_{n-2} \right| .$$

Let us observe that, by (4.119) and (4.122), the first term on the right hand side of the above inequality tends to zero as k tends to ∞. On the other hand, by (4.119) and (4.118) with $h = \frac{\partial u}{\partial \nu}$ the second term on the right hand side of (4.123) tends to zero as k tends to ∞. Finally, we have that by (4.118) with $h = 1$ and by (3.58), the third term on the right hand side of (4.123) tends to zero as well. Hence the theorem follows.

□

Chapter 5

Stability for the inverse scattering problem

In this chapter we shall treat the stability issue for the determination of the surface impedance λ in the boundary value problem (1.8). As usual, let us start the discussion by stating the main assumptions on the data and the *a priori* conditions on the unknown impedance term.

Assumptions on the obstacle

Given positive constants D, r_0, M, we assume throughout this chapter that the obstacle D is a bounded domain satisfying the assumptions (2.11) and (2.12) with Ω replaced by D.
We suppose that Γ_I, Γ_D are two disjoint, nonempty, connected, open subsets of ∂D such that

$$\partial D = \overline{\Gamma}_I \cup \overline{\Gamma}_D. \tag{5.1}$$

Moreover, we assume that the portion of the boundary

$$\Gamma_I \text{ is of class } C^{1,1} \text{ with constants } r_0, M. \tag{5.2}$$

We recall that by the above assumption it follows that there exists a function φ_I, satisfying (2.3)-(2.6) with $\varphi = \varphi_I$ and $S = \Gamma_I$.

A priori informations on the impedance term

We assume that the impedance coefficient λ belongs to $C^{0,1}(\Gamma_I, \mathbb{R})$ and is such that

$$\lambda(x) \geq \lambda_0 > 0 \tag{5.3}$$

for every $x \in \Gamma_I$.

Moreover we assume that, for a given constant $\Lambda > 0$, we have that

$$\|\lambda\|_{C^{0,1}(\Gamma_I)} \leq \Lambda. \tag{5.4}$$

Let us introduce some notations that we shall use in the course of the present chapter.
For a sake of simplicity we shall assume that $0 \in D$.
Fixed $R > d$, $\rho \in (0, r_0)$ and $x_0 \in \Gamma_I$, let us define the following sets

$$D^+ = \mathbb{R}^3 \setminus \overline{D}, \tag{5.5}$$
$$D_R^+ = B_R(0) \cap D^+, \tag{5.6}$$
$$D_{R,\rho}^+ = \{x \in \overline{D_R^+} \ : \mathrm{dist}(x, \Gamma_D) > \rho\}, \tag{5.7}$$
$$\Gamma_I^\rho = \partial D_{R,\rho}^+ \cap \Gamma_I, \tag{5.8}$$
$$\Gamma_{I,\rho}(x_0) = B_\rho(x_0) \setminus \overline{D}, \tag{5.9}$$
$$\Delta_{I,\rho}(x_0) = \overline{\Gamma_{I,\rho}(x_0)} \cap \partial D. \tag{5.10}$$
$$H^1_{\mathrm{loc}}(D^+) = \{v \in D^*(D^+) : \ v|_{D_R^+} \in H^1(D_R^+), \ \forall \ R > 0 \text{ s.t. } \overline{D} \subset B_R(0)\} \tag{5.11}$$

where $D^*(D^+)$ is the space of distribution on D^+. (5.12)

Let us present the statement of the main result.

Theorem 5.1 (Stability for λ). *Let $u_i, \ i = 1, 2$, be the weak solutions to the problem (1.8) with $\lambda = \lambda_i$ respectively and let $u_{i,\infty}$ be their respectively far field patterns. There exists $\varepsilon_0 > 0$ constant only depending on the* a priori data, *such that, if for some $\varepsilon, \ 0 < \varepsilon < \varepsilon_0$, we have*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leq \varepsilon, \tag{5.13}$$

then

$$\|\lambda_1 - \lambda_2\|_{L^\infty(\Gamma_I^{r_0})} \leq \omega(\varepsilon), \tag{5.14}$$

where ω is given by (3.11).

5.1 The direct scattering problem

A weak solution to the problem (1.8) is a function $u = \exp(ik\omega \cdot x) + u^s$, where $u^s \in H^1_{\mathrm{loc}}(D^+)$ is a weak solution to the problem

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & \text{in } D^+, \\ u^s = -\exp(ik\omega \cdot x), & \text{on } \Gamma_D, \\ \dfrac{\partial u^s}{\partial \nu} + i\lambda(x) u^s = -\dfrac{\partial}{\partial \nu} \exp(ik\omega \cdot x) - i\lambda(x)\exp(ik\omega \cdot x), & \text{on } \Gamma_I, \\ \lim_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial r}(r\hat{x}) - iku^s(r\hat{x})\right) = 0, & \text{uniformly in } \hat{x}. \end{cases} \tag{5.15}$$

Let us recall that a weak solution of (5.15) is a function $u^s \in H^1_{\text{loc}}(D^+)$, with $u^s|_{\Gamma_D} = -\exp(ik\omega\cdot x)$ in the trace sense, such that, for all test functions $\eta \in H^1(D^+)$ with compact support in $\mathbb{R}^3$ and $\eta|_{\Gamma_D} = 0$, the following holds

$$\begin{aligned}\int_{D^+} \nabla u^s \cdot \nabla\overline{\eta} - k^2 \int_{D^+} u^s\overline{\eta} \;&=\; \int_{\Gamma_I} \left(\frac{\partial}{\partial\nu}\exp(ik\omega\cdot x) + i\lambda(x)\exp(ik\omega\cdot x)\right)\overline{\eta} + \\ &\quad + \int_{\Gamma_I} ik\lambda u^s\overline{\eta}\ . \qquad (5.16)\end{aligned}$$

Furthermore, u^s satisfies the asymptotic condition (1.9).

Lemma 5.2 (Well-posedness). *The problem* (5.15) *has one and only one weak solution* u^s. *Moreover, for every* $R > d$, *there exists a constant* $C_R > 0$ *depending on the* a priori data *and on* R *only, such that the following holds*

$$\|u^s\|_{H^1(D_R^+)} \leq C_R\ . \qquad (5.17)$$

Proof. For the proof we refer to [21, Theorem 2.5], in which the authors, among various results, show that the exterior mixed boundary value problem (5.15) can be reformulated as a 2×2 system of boundary integral equations. In [21], Theorem 2.5 has been proved in two dimensions for a constant λ, however it can be verified that the same techniques can be carried over in three dimensions and with $\lambda = \lambda(x) \in C^{0,1}(\Gamma_I)$. □

Theorem 5.3 ($C^{1,\alpha}$ regularity at the boundary). *Let* u *be the weak solution to* (1.8), *then there exists a constant* α, $0 < \alpha < 1$, *such that for every* $R > d$ *and* $\rho \in (0, r_0)$, $u \in C^{1,\alpha}(D^+_{R,\rho})$. *Moreover, there exists a constant* $C_{R,\rho} > 0$ *depending on the* a priori data, *on* R *and on* ρ *only, such that*

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \leq C_{R,\rho}\ . \qquad (5.18)$$

Proof. From the weak formulation (5.16), it follows that the total field u satisfies

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u\cdot\nabla\bar{\eta} - k^2\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u\bar{\eta} = -i\int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x)u\bar{\eta}\ ,$$

where $x_0 \in \Gamma_I$ and η is any test function such that $\text{supp}\eta \subset \overline{\Gamma}_{I,\frac{r_0}{2}}(x_0)$.
By (5.4) we have that

$$\left|\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u\cdot\nabla\bar{\eta}\right| \leq k^2\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |u\bar{\eta}| + \Lambda\int_{\Delta_{I,\frac{r_0}{2}}(x_0)} |u\bar{\eta}| \qquad (5.19)$$

and by a trace inequality (see [1, p.114]) it follows that

$$\left|\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u\cdot\nabla\bar{\eta}\right| \leq k^2\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |u\bar{\eta}| + C\Lambda\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |\nabla(u\bar{\eta})|\ , \qquad (5.20)$$

where $C>0$ is a constant depending on the *a priori data* only.
By the standard iteration techniques due to Moser (see for instance [39]), we obtain the following local bound for u

$$\|u\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} \le C\|u\|_{H^1\left(\Gamma_{I,\frac{r_0}{2}}(x_0)\right)} \ , \tag{5.21}$$

where $C>0$ is a constant depending on the *a priori data* only.
Let us denote by u_1 and u_2 the real and the imaginary part of u respectively. Thus by the elliptic equations in weak form satisfied by u_1 and u_2, it follows that

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u_1 \cdot \nabla \eta - k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u_1\eta = \int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x) u_2 \eta \ , \tag{5.22}$$

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u_2 \cdot \nabla \eta - k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u_2\eta = -\int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x) u_1 \eta \ , \tag{5.23}$$

where η is any real valued test function such that supp$\eta \subset \overline{\Gamma}_{I,\frac{r_0}{2}}(x_0)$.
By applying again the Moser method to the weak formulations (5.22) and (5.23), we obtain the following bounds of the Hölder continuity of u_1 and u_2, namely

$$\|u_1\|_{C^{0,\alpha}(\Gamma_{I,\frac{r_0}{8}}(x_0))} \le C(\|u_1\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} + \|u_2\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))}) \ , \tag{5.24}$$

$$\|u_2\|_{C^{0,\alpha}(\Gamma_{I,\frac{r_0}{8}}(x_0))} \le C(\|u_2\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} + \|u_1\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))}) \ , \tag{5.25}$$

where $\alpha, 0<\alpha<1, C>0$ are constants depending on the *a priori data* only.
Combining the two last inequalities with (5.21), we obtain

$$\|u\|_{C^{0,\alpha}(\Gamma_I)} \le C\|u\|_{H^1(D_R^+)} \ , \tag{5.26}$$

where $C>0$ are constants depending on the *a priori data* only and $R=d+r_0$.
By (5.17) we have that

$$\|u^s\|_{H^1(D_R^+)} \le C, \tag{5.27}$$

where C is a constant depending on the *a priori data* only. Moreover, since $u=\exp(ik\omega\cdot x)+u^s$, by (5.26) and (5.27), we have that

$$\|u\|_{C^{0,\alpha}(\Gamma_I)} \le C, \tag{5.28}$$

where C is a constant depending on the *a priori data* only. By (5.28) and by (5.4), we have that

$$\frac{\partial u}{\partial \nu}(x) = -i\lambda(x)u(x) \ \in \ C^{0,\alpha}(\Gamma_I). \tag{5.29}$$

By well-known regularity bounds for the Neumann problem (see for instance [5, p.667]) it follows that, for every $R>d, \rho\in(0,r_0), u\in C^{1,\alpha}(D^+_{R,\rho})$ and the following estimate holds

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \le C_{R,\rho}\left(\|u\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} + \left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} + \|u\|_{H^1(D^+_{2R})}\right) \ , \tag{5.30}$$

where $C_{R,\rho} > 0$ is a constant depending on the *a priori data*, on R and on ρ only. We shall estimate the $C^{0,\alpha}$ norm of $\frac{\partial u}{\partial \nu}$ in terms of the *a priori data*, indeed

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}}))} &= \sup_{x\in\Gamma_I^{\frac{\rho}{2}}}\left|\frac{\partial u(x)}{\partial \nu}\right| + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{\left|\frac{\partial u(x)}{\partial \nu}-\frac{\partial u(y)}{\partial \nu}\right|}{|x-y|^{\alpha}} = \\ &\le \sup_{x\in\Gamma_I^{\frac{\rho}{2}}}|\lambda(x)u(x)| + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|\lambda(x)||u(x)-u(y)|}{|x-y|^{\alpha}} + \\ &\quad + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|u(y)||\lambda(x)-\lambda(y)|}{|x-y|^{\alpha}} \ .\end{aligned}$$

Combining (5.4) and (5.28) we obtain

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} &\le \Lambda\sup_{x\in\Gamma_I^{\frac{\rho}{2}}}|u(x)| + \Lambda\left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} + \\ &\quad + \left(\frac{\rho}{2}\right)^{\alpha}|\Gamma_I|^{1-\alpha}\|u\|_{C^{0,\alpha}(\Gamma_I)}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|\lambda(x)-\lambda(y)|}{|x-y|} \le \\ &\quad \le \bar{C}_{\rho}\end{aligned}$$

where $\bar{C}_{\rho} > 0$ is a constant depending on the *a priori data* and on ρ only. Moreover, since $u = \exp(ik\omega\cdot x) + u^s$, we have that (5.17) yields to

$$\|u\|_{H^1(D^+_{2R})} \le C_R, \tag{5.31}$$

where $C_R > 0$ is a constant depending on the *a priori data* and on R only.
Thus, inserting (5.28), (5.31) and (5.31) in (5.30), we obtain that

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \le C_{R,\rho}, \tag{5.32}$$

where $C_{R,\rho} > 0$ is a constant depending on the *a priori data*, on R and on ρ only. □

Corollary 5.4 (Lower bound). *Let u be the weak solution to* (1.8)*, then there exists a radius $R_0 > 0$ depending on the* a priori data *only, such that*

$$|u(x)| > \frac{1}{2} \quad \textit{for every } x, \ |x| > R_0 \ . \tag{5.33}$$

Proof. Let us choose $R = 4d + 4r_0$. By Theorem 5.3 it follows that there exists a constant $C > 0$ depending on the *a priori data* only, such that

$$\|u\|_{C^{1,\alpha}\left(D^+_{2R,\frac{r_0}{2}}\right)} \le C \ . \tag{5.34}$$

In particular, by (5.34), it follows that

$$|u^s| \leq C_1 \ , \ \left|\frac{\partial u^s}{\partial \nu}\right| \leq C_1 \ \text{ on } \ \partial B_R(0), \tag{5.35}$$

where $C_1 > 0$ is a constant depending on the *a priori data* only.
By the Green's formula for the scattered wave u^s (see for instance [32, p.18]), we have that

$$u^s(x) = \int_{\partial B_R(0)} \left(u^s(y)\frac{\partial \phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu(y)}\phi(x,y)\right) \mathrm{d}s(y), \qquad |x| > R, \tag{5.36}$$

where

$$\phi(x,y) = \frac{1}{4\pi}\frac{\exp\left(ik|x-y|\right)}{|x-y|}, \quad x \neq y \ ,$$

is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$.
Thus, by (5.36) and by (5.35) it follows that

$$|u^s(x)| \leq C_1 \int_{\partial B_R(0)} \left|\frac{\partial \phi(x,y)}{\partial \nu(y)}\right| + |\phi(x,y)|\mathrm{d}s(y) \leq \tag{5.37}$$

$$\leq C_1 R^2 \left(\frac{kR}{||x|-R|^2} + \frac{R}{||x|-R|^3} + \frac{1}{||x|-R|}\right) . \tag{5.38}$$

Straightforward calculations show that

$$|u^s| < \frac{1}{2} \ , \text{ for every } x, |x| > R_0, \tag{5.39}$$

where $R_0 = (k+1)8R^3C_1 + 2R$.
The thesis follows observing that $|u| \geq 1 - |u^s|$. □

5.2 The inverse scattering problem

Lemma 5.5 (From the far field to the near field). *Let $u_i, u_{i,\infty}, \ i = 1,2,$ be as in Theorem 5.1. Suppose that, for some ε, $0 < \varepsilon < 1$, (5.13) holds, then there exist a radius $R_1 > 0$ and a constant $C > 0$, depending on the* a priori data *only, such that*

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))} \leq C\varepsilon^{\alpha(\varepsilon)}, \tag{5.40}$$

where the function $\alpha(\varepsilon)$ is defined as follows

$$\alpha(\varepsilon) = \frac{1}{1 + \log(\log(\varepsilon^{-1}) + e)} \ . \tag{5.41}$$

Proof. Let us choose $R = 4d + 4r_0$ and let us denote by $u_i^s,\ i = 1, 2$, the scattered wave of the problem (1.8) with $\lambda = \lambda_i$ respectively. By (5.35) it follows that

$$\|u_1^s - u_2^s\|_{L^2(\partial B_R(0))} \leq C\ , \tag{5.42}$$

where $C > 0$ is a constant depending on the *a priori data* only.
By the argument in [43] (see also [19]), it follows that there exists a constant $C > 0$ depending on the *a priori data* only, such that, for every $r \in (4R, 4R+1)$, the following holds

$$\|u_1^s - u_2^s\|_{L^2(\partial B_r(0))} \leq C\varepsilon^{\alpha(\varepsilon)}. \tag{5.43}$$

Integrating (5.43) with respect to r over $(4R, 4R+1)$, we obtain that

$$\|u_1^s - u_2^s\|_{L^2(B_{4R+1}(0)\setminus B_{4R}(0))} \leq C\varepsilon^{\alpha(\varepsilon)}\ , \tag{5.44}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Thus the thesis follows with $R_1 = 16d + 16r_0$ and by observing that $u_1^s - u_2^s = u_1 - u_2$.
Let us stress, that Hölder stability doesn't hold, indeed, in [19, Section 4], it has been proved that it is not possible to choose α independently on ε. $\square$

Theorem 5.6 (Stability at the boundary). *Let $u_i, u_{i,\infty},\ i = 1, 2$, be as in Theorem 5.1. We have that there exists $\varepsilon_0 > 0$ depending on the* a priori data *only, such that, if for some ε, $0 < \varepsilon < \varepsilon_0$, (5.13) holds, then for every $\rho \in (0, r_0)$ we have*

$$\|u_1 - u_2\|_{C^1(\Gamma_I^\rho)} \leq \omega(\varepsilon)\ , \tag{5.45}$$

where ω is given by (3.11)*, with a constant $C > 0$ depending on the* a priori data *and on ρ only.*

Proof. By the Lipschitz regularity of the boundary ∂D, it follows that the cone property holds. Namely, for every point $Q \in \partial D$, there exists a rigid transformation of coordinates under which we have $Q = 0$ and the finite cone

$$\mathcal{C} = \left\{ x : |x| < r_0,\ \frac{x \cdot \xi}{|x|} > \cos\theta \right\}$$

with axis in the direction ξ and width 2θ, where $\theta = \arctan\frac{1}{M}$, is such that $\mathcal{C} \subset D^+$.
Let Q be a point such that $Q \in \Gamma_I^{r_0}$ and let Q_0 be a point lying on the axis ξ of the cone with vertex in $Q = 0$ such that $d_0 = \mathrm{dist}(Q_0, 0) < \frac{r_0}{2}$.
Let us define $R_2 = 2R_1+2$, where R_1 is the radius introduced in the statement of Lemma 5.5. Dealing as in Lieberman [57], we consider a regularized distance $\tilde{d}$ from the boundary of ∂D such that, $\tilde{d} \in C^2(D^+_{R_2}) \cap C^{0,1}(\overline{D^+_{R_2}})$ and furthermore the following properties hold

- $\gamma_0 \leq \dfrac{\operatorname{dist}(x, \partial D)}{\tilde{d}(x)} \leq \gamma_1$,
- $|\nabla \tilde{d}(x)| \geq c_1$, for every x such that $\operatorname{dist}(x, \partial D) \leq b r_0$,
- $\|\tilde{d}\|_{C^{0,1}} \leq c_2 r_0$,

where $\gamma_0, \gamma_1, c_1, c_2, b$ are positive constants depending on M only, (see also [8, Lemma 5.2]).
Let us define for every $\rho > 0$

$$D^\rho = \{x \ \in D^+_{R_2} : \operatorname{dist}(x, \partial D) > \rho\} \ , \tag{5.46}$$

$$\tilde{D}^\rho = \{x \ \in D^+_{R_2} : \tilde{d}(x) > \rho\} \ . \tag{5.47}$$

It follows that there exists a, $0 < a \leq 1$, only depending on M such that for every ρ, $0 < \rho \leq a r_0$, $\tilde{D}^\rho$ is connected with boundary of class C^1 and

$$\tilde{c}_1 \rho \leq \operatorname{dist}(x, \partial D) \leq \tilde{c}_2 \rho \quad \text{ for every } \ x \ \in \ \partial \tilde{D}^\rho, \tag{5.48}$$

where $\tilde{c}_1, \tilde{c}_2$, are positive constants depending on M only. By(5.48) we deduce that

$$D^{\tilde{c}_2 \rho} \subset \tilde{D}^\rho \subset D^{\tilde{c}_1 \rho} \ .$$

Let us now define $\rho_0 = \min\{\frac{1}{16}, \frac{r_0}{4} \sin\theta\}$ and let P be a point in the annulus $B_{R_1+1}(0) \setminus B_{R_1}(0))$, such that $B_{4\rho_0}(P) \subset B_{R_1+1}(0) \setminus B_{R_1}(0))$. Furthermore, let γ be a path in $\tilde{D}^{\frac{\rho_0}{\tilde{c}_1}}$ joining P to Q_0 and let us define $\{y_i\}$, $i = 0, \dots, s$ as follows $y_0 = Q_0$, $y_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \text{ s.t. } |\gamma(t) - y_i| = 2\rho_0\}$ if $|P - y_i| > 2\rho_0$, otherwise let $i = s$ and stop the process.
Let us introduce the function $U \in H^1_{\text{loc}}(D^+)$ defined as follows

$$U(x) = u_1(x) - u_2(x). \tag{5.49}$$

We shall denote with U_1 and U_2 the real and the imaginary part of U respectively. Namely

$$U(x) = U_1(x) + iU_2(x).$$

It immediately follows that U_1, U_2, are both real valued solutions to the Helmholtz equation in D^+.
Thus, by the three spheres inequalities for elliptic system with Laplacian principal part, (see [11, Theorem 3.1]), we have that for every $\beta_1, \beta_2, \ 1 < \beta_1 < \beta_2$, there exist $\bar{r} > 0, \tau, \ 0 < \tau < 1$ and $C > 0$ depending on the *a priori data* and on β_1, β_2 only, such that for every $x \in D^{\beta_2 \rho}$ the following holds

$$\int_{B_{\beta_1 \rho}(x)} |U|^2 \leq C \left(\int_{B_\rho(x)} |U|^2 \right)^\tau \cdot \left(\int_{B_{\beta_2 \rho}(x)} |U|^2 \right)^{1-\tau} \tag{5.50}$$

for every $\rho \in (0, \bar{r})$. By a possible replacement of ρ_0 with $\bar{r}$ if $\rho_0 > \bar{r}$ and choosing in (5.50) $\beta_1 = 3, \ \beta_2 = 4, \rho = \rho_0, \ x = y_0$, we infer that

$$\int_{B_{3\rho_0}(y_0)} |U|^2 \leq C \left(\int_{B_{\rho_0}(y_0)} |U|^2 \right)^\tau \cdot \left(\int_{B_{4\rho_0}(y_0)} |U|^2 \right)^{1-\tau} . \tag{5.51}$$

As a consequence of Lemma 5.2, we have that

$$\|U\|_{H^1(D^+_{R_2})} \leq C, \tag{5.52}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us observe that $B_{4\rho_0}(y_0) \subset D^+_{R_2}$ and $B_{\rho_0}(y_0) \subset B_{3\rho_0}(y_1)$. Thus by (5.51) and (5.52) we deduce that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leq C \left(\int_{B_{3\rho_0}(y_1)} |U|^2 \right)^{\tau} \cdot C^{1-\tau} \ .$$

An iterated application of the three spheres inequality leads to

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leq \left(\int_{B_{\rho_0}(y_s)} |U|^2 \right)^{\tau^s} \cdot C^{1-\tau^s} \ .$$

Finally, since $B_\rho(y_s) \subset B_{R_1+1}(0) \setminus B_{R_1}(0))$, by (5.40) we obtain that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leq C \big\{ \varepsilon^{\alpha(\varepsilon)} \big\}^{\tau^s} \ .$$

We shall construct a chain of balls $B_{\rho_k}(Q_k)$ centered on the axis of the cone, pairwise tangent to each other and all contained in the cone

$$\mathcal{C}' = \left\{ x : |x| < r_0, \ \frac{x \cdot \xi}{|x|} > \cos\theta' \right\} ,$$

where $\theta' = \arcsin\left(\frac{\rho_0}{d_0}\right)$. Let $B_{\rho_0}(Q_0)$ be the first of them, the following are defined by induction in such a way

$$\begin{aligned} Q_{k+1} &= Q_k - (1+\mu)\rho_k \xi \ , \\ \rho_{k+1} &= \mu \rho_k \ , \\ d_{k+1} &= \mu d_k \ , \end{aligned}$$

with

$$\mu = \frac{1 - \sin\theta'}{1 + \sin\theta'} \ .$$

Hence, with this choice, we have $\rho_k = \mu^k \rho_0$ and $B_{\rho_{k+1}}(Q_{k+1}) \subset B_{3\rho_k}(Q_k)$.
Considering the following estimate obtained by a repeated application of the three spheres inequality, we have that

$$\begin{aligned} \|U\|_{L^2(B_{\rho_k}(Q_k))} &\leq \|U\|_{L^2(B_{3\rho_{k-1}}(Q_{k-1}))} \leq \\ &\leq \|U\|^{\tau}_{L^2(B_{\rho_{k-1}}(Q_{k-1}))} \|U\|^{1-\tau}_{L^2(B_{4\rho_{l-1}}(Q_{k-1}))} \\ &\leq C \|U\|^{\tau^k}_{L^2(B_{\rho_0}(Q_0))} \leq C \Big\{ \big[\varepsilon^{\alpha(\varepsilon)} \big]^{\tau^s} \Big\}^{\tau^k} \ . \end{aligned} \tag{5.53}$$

For every r, $0 < r < d_0$, let $k(r)$ be the smallest positive integer such that $d_k \leq r$ then, since $d_k = \mu^k d_0$, it follows

$$\frac{|\log(\frac{r}{d_0})|}{\log\mu} \leq k(r) \leq \frac{|\log(\frac{r}{d_0})|}{\log\mu} + 1 \ , \tag{5.54}$$

and by (5.53) we deduce

$$\|U\|_{L^2(B_{\rho_{k(r)}}(Q_{k(r)}))} \leq C\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)}} \ . \tag{5.55}$$

Let $\bar{x} \in \Gamma_I^{\frac{\rho}{2}}$ with $\rho \in (0, r_0)$ and let $x \in B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$. By Theorem 5.3, in particular, it follows that $U \in C^{1,\alpha}(D^+_{R_2,\frac{\rho}{4}})$ with

$$\|U\|_{C^{1,\alpha}(D^+_{R_2,\frac{\rho}{4}})} \leq C_\rho, \tag{5.56}$$

where $C_\rho > 0$ is a constant depending on the *a priori data* and on ρ only. Then (5.56) yields to

$$|U(\bar{x})| \leq |U(x)| + C_\rho|x-\bar{x}|^\alpha \leq |U(x)| + C_\rho\left(\frac{2}{\mu}r\right)^\alpha \ .$$

Integrating this inequality over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, we have that

$$|U(\bar{x})|^2 \quad\leq\quad \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})}|U(x)|^2\mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\mu^2}\right)^\alpha . \tag{5.57}$$

Being k the smallest integer such that $d_k \leq r$, then $d_{k-1} > r$ and thus (5.57) yields to

$$|U(\bar{x})|^2 \leq \frac{C}{(r\sin\theta')^3}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})}|U(x)|^2\mathrm{d}x + C_\rho r^{2\alpha} \ .$$

By (5.55) we deduce that

$$|U(\bar{x})|^2 \leq \frac{C}{r^3}\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} \ . \tag{5.58}$$

The estimate (5.56) also provides us that

$$\left|\frac{\partial U(\bar{x})}{\partial\nu}\right| \leq \left|\frac{\partial U(x)}{\partial\nu}\right| + C_\rho\left(\frac{2}{\mu}r\right)^\alpha \ .$$

Integrating over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$ we deduce that

$$\begin{aligned}\left|\frac{\partial U(\bar{x})}{\partial\nu}\right|^2 \quad&\leq\quad \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})}\left|\frac{\partial U(x)}{\partial\nu}\right|^2\mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\mu^2}\right)^\alpha \leq \\ &\leq\quad \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})}|\nabla U(x)|^2\mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\mu^2}\right)^\alpha .\end{aligned}$$

Applying the Caccioppoli inequality, we have

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{\left(\rho_{k-1}\right)^5} \int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} U(x)^2 \mathrm{d}x + C_\rho r^{2\alpha} .$$

Dealing with the same arguments that lead to (5.58), we obtain that

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right|^2 \leq \frac{C}{r^5}\left\{\left[\varepsilon^{\alpha(\varepsilon)}\right]^{\tau^s}\right\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} . \tag{5.59}$$

The choice in (5.54) guarantees that

$$\tau^{k(r)-1} \geq \left(\frac{r}{d_0}\right)^{\nu},$$

where $\nu = -\log\left(\frac{1}{\mu}\right)\log\tau$. Thus, by (5.58) and by (5.59), it follows that

$$|U(\bar{x})| \leq C_\rho \left\{ r^{-\frac{3}{2}} \left[\left(\varepsilon^{\alpha(\varepsilon)}\right)^{\tau^s}\right]^{\frac{r^\nu}{2}} + r^\alpha \right\} , \tag{5.60}$$

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right| \leq C_\rho \left\{ r^{-\frac{5}{2}} \left[\left(\varepsilon^{\alpha(\varepsilon)}\right)^{\tau^s}\right]^{\frac{r^\nu}{2}} + r^\alpha \right\} . \tag{5.61}$$

Minimizing the right hand sides of the above inequalities with respect to r, with $r \in (0, \frac{r_0}{4})$, we deduce

$$|U(\bar{x})| \leq C_\rho \big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}} , \tag{5.62}$$

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right| \leq C_\rho \big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}}, \tag{5.63}$$

where $C_\rho > 0$ is a constant depending on the *a priori data* and on ρ only. Thus, since $\bar{x}$ is an arbitrary point in $\Gamma_I^{\frac{\rho}{2}}$, by (5.62) and (5.63) we have that

$$\|U(\bar{x})\|_{L^\infty(\Gamma_I^{\frac{\rho}{2}})} \leq C_\rho \big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}} , \tag{5.64}$$

$$\left\|\frac{\partial U(\bar{x})}{\partial \nu}\right\|_{L^\infty(\Gamma_I^{\frac{\rho}{2}})} \leq C_\rho \big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}}. \tag{5.65}$$

By an interpolation inequality we have

$$\|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leq c_\rho \|U\|^\beta_{L^\infty(\Gamma_{1,\frac{\rho}{2}})} \|U\|_{C^{1,\alpha}(\Gamma_{1,\rho})}{}^{1-\beta} ,$$

where $\beta = \frac{\alpha}{\alpha+1}$ and $c_\rho > 0$ depends on the *a priori data* and on ρ only. Thus, by (5.56), we obtain

$$\|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leq c_\rho \|U\|^\beta_{L^\infty(\Gamma_{1,\frac{\rho}{2}})} C_\rho{}^{1-\beta} .$$

It follows that for every $\varepsilon < \varepsilon_0$, with ε_0 depending only on the *a priori data*,

$$\begin{aligned} \|\nabla(U)\|_{L^\infty(\Gamma_{1,\rho})} &\leq \left\|\frac{\partial U}{\partial \nu}\right\|_{L^\infty(\Gamma_{1,\rho})} + \|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leq \\ &\leq C_\rho\big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha\beta}{\nu+2}}, \end{aligned} \tag{5.66}$$

where $C_\rho > 0$ depends on the *a priori data* and on ρ only.
After straightforward calculations and by a possible replacing of ε_0 with a smaller one depending on the *a priori data* only we have that

$$\|u_1 - u_2\|_{C^1(\Gamma_{1,\rho})} \leq C_\rho\big(|\log(\varepsilon)|\big)^{-\frac{\alpha\beta}{\nu+2}} \quad \text{for every } \varepsilon, 0 < \varepsilon < \varepsilon_0. \tag{5.67}$$

Thus the thesis follows replacing in (3.11) C with C_ρ and θ with $\frac{\alpha\beta}{\nu+2}$. □

Proposition 5.7. *There exists a radius $r_1 > 0$ depending on the* a priori data *only such that, for every $x_0 \in \Gamma_I^{r_0}$, the problem*

$$\begin{cases} \Delta\psi + k^2\psi = 0, & \text{in } \Gamma_{I,r_1}(x_0), \\ \dfrac{\partial\psi}{\partial\nu} + i\lambda(x)\psi = 0, & \text{on } \Delta_{I,r_1}(x_0), \end{cases} \tag{5.68}$$

admits a solution $\psi \in H^1(\Gamma_{I,r_1}(x_0))$ satisfying

$$|\psi(x)| \geq 1 \text{ for every } x \in \Gamma_{I,r_1}(x_0). \tag{5.69}$$

Moreover, there exists a constant $\bar{\psi} > 0$ depending on the a priori data *only, such that for every $x_0 \in \Gamma_I^{r_0}$*

$$\|\psi\|_{C^1(\Gamma_{I,r_1}(x_0))} \leq \bar{\psi}. \tag{5.70}$$

Proof. Let us consider a point $x_0 \in \Gamma_I^{r_0}$. After a translation we may assume that $x_0 = 0$ and, fixing local coordinates, we can represent the boundary as a graph of a $C^{1,1}$ function. Namely, we have that

$$D^+ \cap B_{r_0}(0) = \{(x', x_3) \in B_{r_0}(0) : x_3 < \varphi_I(x')\}\ , \tag{5.71}$$

where φ_I is the $C^{1,1}$ function satisfying (2.4)-(2.6) with $\varphi = \varphi_i$ and $k = \alpha = 1$. Let $\Phi \in C^{1,1}(B_{\frac{r_0}{4M}}, \mathbb{R}^3)$ be the map defined as follows

$$\Phi(y', y_3) = (y', y_3 + \varphi_I(y'))\ . \tag{5.72}$$

We have that there exist $\theta_1, \theta_2, \theta_1 > 1 > \theta_2 > 0$, constants depending on M and r_0 only, such that, for every $r \in (0, \frac{r_0}{4M})$, it follows that

$$\Gamma_{I,\theta_2 r}(0) \subset \Phi(B_r^-(0)) \subset \Gamma_{I,\theta_1 r}(0)\ , \tag{5.73}$$

where $B_r^-(0) = \{y \in \mathbb{R}^3 : |y| < r,\ y_3 < 0\}$ and furthermore we have

$$|\det\ D\Phi\ | = 1\ . \tag{5.74}$$

The inverse map $\Phi^{-1} \in C^{1,1}(\Gamma_{I,r_0}(0), \mathbb{R}^3)$ and is defined by

$$\Phi^{-1}(x', x_3) = (x', x_3 - \varphi_I(x'))\ . \tag{5.75}$$

Denoting by

$$\sigma(y) = (\sigma_{i,j}(y))_{i,j=1}^3 = (D\Phi^{-1})(\Phi(y)) \cdot (D\Phi^{-1})^T(\Phi(y))\ , \tag{5.76}$$

$$\lambda'(y) = \lambda(\Phi(y))\ , \tag{5.77}$$

$$\lambda_0{}' = \lambda'(0)\ , \tag{5.78}$$

it follows that

$$\sigma(0) = \mathrm{I}, \tag{5.79}$$

$$\|\sigma_{i,j}\|_{C^{0,1}(\Gamma_{I,r_0})} \leq \Sigma, \quad \text{for} \quad i,j = 1,2,3, \tag{5.80}$$

$$\frac{1}{2}|\xi|^2 \leq \sigma(y)\xi \cdot \xi \leq C_1|\xi|^2, \text{ for every } y \in B^-_{(\frac{r_0}{4M})}(0) \text{ and every } \xi \in \mathbb{R}^3, \tag{5.81}$$

$$\|\lambda'\|_{C^{0,1}(B'_{\frac{r_0}{4M}}(0))} \leq \Lambda'\ , \tag{5.82}$$

where $\Sigma > 0, C_1 > 0, \Lambda' > 0$ are constants depending on M, r_0, Λ only.

Claim 5.8. *There exists a radius r_2, $0 < r_2 < \frac{r_0}{4M}$ and a solution $\psi' \in H^1(B_{r_2}^-(0))$ to the problem*

$$\begin{cases} div(\sigma\nabla\psi') + k^2\psi' = 0\ , & in \quad B_{r_2}^-(0)\ , \\ \sigma\nabla\psi' \cdot \nu' + i\lambda'\psi' = 0\ , & on \quad B'_{r_2}(0), \end{cases} \tag{5.83}$$

where $\nu' = (0,0,1)$ such that

$$|\psi'| \geq 1 \ in\ B_{r_2}^-(0).$$

Proof. of Claim 5.8.
We look for a radius $r_2 > 0$ and for a solution of the form $\psi' = \psi_0 - s$ such that, $\psi_0 \in H^1(B_{r_2}^-(0))$ is a weak solution to the problem

$$\begin{cases} \Delta\psi_0 + k^2\psi_0 = 0\ , & \text{in} \quad B_{r_2}^-(0), \\ \dfrac{\partial\psi_0}{\partial\nu} + i\lambda_0{}'\psi_0 = 0\ , & \text{on} \quad B'_{r_2}(0), \end{cases} \tag{5.84}$$

satisfying $|\psi_0| \geq 2$ in $B^-_{r_2}(0)$.
Whereas $s \in H^1(B^-_{r_2}(0))$ is a weak solution to the problem

$$\begin{cases} \mathrm{div}(\sigma\nabla s) + k^2 s = \mathrm{div}((\sigma - I)\nabla\psi_0) \ , & \text{in} \quad B^-_{r_2}(0), \\ \sigma\nabla s\cdot\nu + i\lambda' s = (\sigma - I)\nabla\psi_0\cdot\nu + i(\lambda' - {\lambda_0}')\psi_0 \ , & \text{on} \quad B'_{r_2}(0), \\ s = 0 \ , & \text{on} \quad |y| = r_2, \end{cases} \tag{5.85}$$

such that $s(y) = O(|y|^2)$ near the origin.
We can construct ψ_0 explicitly as follows

$$\psi_0(y_1, y_2, y_3) = 8\cosh\left(|{\lambda_0}'^2 - k^2|^{\frac{1}{2}} y_1\right)\left[\sin\left({\lambda_0}' y_3\right) + i\cos\left({\lambda_0}' y_3\right)\right], \text{ if } k^2 < {\lambda_0}'^2,$$
$$\psi_0(y_1, y_2, y_3) = 8\cos\left(|k^2 - {\lambda_0}'^2|^{\frac{1}{2}} y_1\right)\left[\sin\left({\lambda_0}' y_3\right) + i\cos\left({\lambda_0}' y_3\right)\right], \text{ if } k^2 > {\lambda_0}'^2,$$
$$\psi_0(y_1, y_2, y_3) = 8\sin\left({\lambda_0}' y_3\right) + i8\cos\left({\lambda_0}' y_3\right), \text{ if } k^2 = {\lambda_0}'^2.$$

Denoting by

$$\tilde{r} = \frac{\pi}{4}\min\left\{\frac{1}{\sqrt{|k^2 - {\lambda_0}'^2|}}, \frac{1}{{\lambda_0}'}\right\} \ , \tag{5.86}$$

it follows, by straightforward calculations, that $\psi_0 \in H^1(B^-_{\tilde{r}}(0))$ is a weak solution of (5.84) with $r_2 = \tilde{r}$ and $|\psi_0| \geq 2$ in $B^-_{\tilde{r}}(0)$.
Let us now look for a solution s to the problem (5.85).
Fixed $r \in (0, \frac{r_0}{8M})$, let us define the space

$$H^1_{0^-}(B^-_r(0)) = \{\eta \in H^1(B^-_r(0)) \text{ such that } \eta(y) = 0 \text{ on } \ |y| = r\}, \tag{5.87}$$

endowed with the usual $\|\cdot\|_{H^1_0(B^-_r(0))}$ norm. Thus the weak formulation of the problem (5.85) reads in this way: find $s \in H^1_{0^-}(B^-_r(0))$ such that, for every $\eta \in H^1_{0^-}(B^-_r(0))$, the following holds

$$\int_{B^-_r(0)} \sigma\nabla s\cdot\nabla\bar{\eta} - \int_{B^-_r(0)} k^2 s\bar{\eta} - \int_{B'_r(0)} i\lambda' s\bar{\eta} = \int_{B^-_r(0)} (\sigma - I)\nabla\psi_0\cdot\nabla\bar{\eta} + i\int_{B'_r(0)} (\lambda' - {\lambda_0}')\psi_0\bar{\eta}. \tag{5.88}$$

Let us introduce the following bilinear form

$$A \quad : \quad H^1_{0^-}(B^-_r(0)) \times H^1_{0^-}(B^-_r(0)) \to \mathbb{C} \tag{5.89}$$

such that

$$A(\eta_1, \eta_2) = \int_{B^-_r(0)} \sigma\nabla\eta_1\cdot\nabla\bar{\eta_2} - \int_{B^-_r(0)} k^2\eta_1\bar{\eta_2} - \int_{B'_r(0)} i\lambda'\eta_1\bar{\eta_2} \tag{5.90}$$

and the following functional

$$F : H^1_{0^-}(B^-_r(0)) \to \mathbb{C} \tag{5.91}$$

such that

$$F(\eta) = \int_{B_r^-(0)} (\sigma - I)\nabla\psi_0 \cdot \nabla\bar{\eta} + i\int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0\bar{\eta}\ . \tag{5.92}$$

It immediately follows that A and F are continuous on $H^1_{0^-}(B_r^-(0))$ as bilinear form and as a functional respectively.
Moreover, dealing as in [39, Lemma 8.4], we have that, by the Hölder inequality, it follows that for every $\eta \in H^1_{0^-}(B_r^-(0))$

$$\int_{B_r^-(0)} |\eta|^2 \leq \tilde{c_1} r^2 \Big(\int_{B_r^-(0)} |\eta|^6\Big)^{\frac{1}{3}}, \tag{5.93}$$

where $\tilde{c_1} > 0$ is a constant depending on the *a priori data* only. Hence by the Sobolev Embedding Theorem, (see [1, Chap.4]), and by (5.93), we have that

$$\int_{B_r^-(0)} |\eta|^2 \leq c_1 r^2 \int_{B_r^-(0)} |\nabla\eta|^2, \tag{5.94}$$

where $c_1 > 0$ is a constant depending on the *a priori data* only.
Analogously, by the Hölder inequality on the boundary, it follows that

$$\int_{B_r'(0)} |\eta|^2 \leq \tilde{c_2} r \Big(\int_{B_r'(0)} |\eta|^4\Big)^{\frac{1}{2}}, \tag{5.95}$$

where $\tilde{c_2} > 0$ is a constant depending on the *a priori data* only. By a trace inequality (see for instance [1], Chap.5), it follows that

$$\int_{B_r'(0)} |\eta|^2 \leq c_2 r \int_{B_r^-(0)} |\nabla\eta|^2, \tag{5.96}$$

where $c_2 > 0$ is a constant depending on the *a priori data* only.
Thus, by (5.81),(5.94) and (5.96), we deduce that

$$|A(\eta,\eta)| \geq (\frac{1}{2} - c_1 r^2 k^2 - c_2 r\Lambda') \int_{B_r^-(0)} |\nabla\eta|^2.$$

Denoting by

$$r_3 = \min\left\{1, \frac{1}{8}(c_1 k^2 + c_2\Lambda), \frac{r_0}{8M}\right\}, \tag{5.97}$$

we have that for every $r \in (0, r_3)$

$$|A(\eta,\eta)| \geq \frac{1}{4}\int_{B_r^-(0)} |\nabla\eta|^2. \tag{5.98}$$

Thus it follows that, for every $r \in (0, r_3)$, the bilinear form A is coercive on $H^1_{0^-}(B_r^-(0))$. Hence by the Lax-Milgram theorem we can infer that, for every $r \in (0, r_3)$, there exists a unique solution $s \in H^1_{0^-}(B_r^-(0))$ to the problem (5.85).

Fixing $r \in (0, r_3)$ and choosing $\eta = s$ as test function in the weak formulation (5.88), we obtain

$$\int_{B_r^-(0)} \sigma \nabla s \cdot \nabla \bar{s} - \int_{B_r^-(0)} k^2 |s|^2 - \int_{B_r'(0)} i\lambda' |s|^2 = \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} + + i \int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s}. \quad (5.99)$$

By (5.98), we have that

$$\frac{1}{4} \int_{B_r^-(0)} |\nabla s|^2 \leq \Big| \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} \Big| + \Big| \int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big|. \quad (5.100)$$

By the Schwartz inequality, by (5.79) and by (5.80) we have that

$$\Big| \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} \Big| \leq 16\Sigma r^2 \int_{B_r^-(0)} |\nabla \psi_0|^2 + \frac{1}{16} \int_{B_r^-(0)} |\nabla s|^2 \ . \quad (5.101)$$

Analogously, we have that, by the Schwartz inequality, by (5.78) and by (5.82) it follows that

$$\Big| \int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big| \leq 16 c_2 \Lambda' r^2 \int_{B_r'(0)} |\psi_0|^2 + \frac{1}{16 c_2} \int_{B_r'(0)} |s|^2 \ . \quad (5.102)$$

Moreover, by the inequality (5.96) and by (5.102) we deduce

$$\Big| \int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big| \leq c_2^2 r^4 16\Lambda' \int_{B_r^-(0)} |\nabla \psi_0|^2 + \frac{1}{16} r \int_{B_r^-(0)} |\nabla s|^2 \ . \quad (5.103)$$

Hence inserting (5.101) and (5.103) in (5.100) we obtain that

$$\frac{1}{8} \int_{B_r^-(0)} |\nabla s|^2 \leq (16\Sigma + c_2^2 16\Lambda') r^2 \int_{B_r^-(0)} |\nabla \psi_0|^2 \ . \quad (5.104)$$

Denoting by

$$Q = \sup_{B^-_{\frac{r_0}{8M}}(0)} |\nabla \psi_0|^2,$$

we have that

$$\frac{1}{8} \int_{B_r^-(0)} |\nabla s|^2 \leq \frac{4}{3}\pi (16\Sigma + c_1^2 16\Lambda') r^5 Q \ . \quad (5.105)$$

By standard estimates for solutions of elliptic equations (see for instance [39], Chap.8) and observing that $Q > 0$ depends on the *a priori data* only, we can infer that for every $r \in (0, \frac{r_3}{2})$

$$\|s\|_{L^\infty(B_r^-(0))} \leq c_4 r^2,$$

where $c_4 > 0$ is a constant depending on the *a priori data* only.
Hence the Claim follows choosing $r_2 = \min\{\tilde{r}, \frac{r_3}{2}, \frac{1}{\sqrt{c_4}}\}$ and observing that

$$|\psi'| \geq |\psi_0| - |s| \geq 1 \quad \text{in } B^-_{r_2}(0) \ .$$

□

Let us notice that choosing $r_1 = \theta_2 r_2$ and $\psi(x', x_3) = \psi'(\Phi^{-1}(x', x_3))$, we have that $\psi \in H^1(\Gamma_{I,r_1}(0))$ is a weak solution to the problem (5.68) and is such that $|\psi| \geq 1$ in $\Gamma_{I,r_1}(0)$.
Finally, we conclude the proof of Proposition 5.7 observing that (5.70) follows dealing with the same argument used in the proof of Theorem 5.3. □

Lemma 5.9 (Volume doubling inequality). *Let u be the solution to the problem* (1.8)*, then there exists a radius $\bar{r} > 0$ such that for every $x_0 \in \Gamma_I^{r_0}$ the following holds*

$$\int_{\Gamma_{I,\beta r}} |u|^2 \leq C\beta^K \int_{\Gamma_{I,r}} |u|^2 \tag{5.106}$$

for every r, β such that $\beta > 1$ and $0 < \beta r < \bar{r}$, where $C > 0, K > 0$ are constants depending on the a priori data *only.*

Proof. Let $x_0 \in \Gamma_I^{r_0}$ and let r_1 and ψ be, respectively, the radius and the function, introduced in Proposition 5.7. Denoting by

$$z = \frac{u}{\psi}, \tag{5.107}$$

it follows that $z \in H^1(\Gamma_{I,r_1}(x_0))$ is a weak solution to the problem

$$\begin{cases} \Delta z + 2\dfrac{\nabla\psi}{\psi} \cdot \nabla z = 0, & \text{in } \Gamma_{I,r_1}(x_0), \\ \dfrac{\partial z}{\partial \nu} = 0, & \text{on } \Delta_{I,r_1}(x_0). \end{cases} \tag{5.108}$$

Dealing as in Proposition 5.7, we may assume that, up to a rigid transformation of coordinates, $x_0 = 0$ and, by local coordinates, we can locally represent the boundary as a graph of a $C^{1,1}$ function as in (5.71).
Following [2, Theorem 0.8], (see also [8, Proposition 3.5]), we have that there exists a map $\Psi \in C^{1,1}(B_{\rho_2}(0), \mathbb{R}^3)$ such that

$$\Psi(B_{\rho_2}(0)) \subset B_{\rho_1}(0), \tag{5.109}$$

$$\Psi(y', 0) = (y', \varphi_I(y')), \qquad \text{for every } y' \in B'_{\rho_2}(0), \tag{5.110}$$

$$\Gamma_{I,\frac{\rho}{2}} \subset \Psi(B^-_\rho(0)) \subset \Gamma_{I,c_1\rho}, \quad \text{for every } \rho \in (0, \rho_2), \tag{5.111}$$

$$\frac{1}{8} \leq |\det D\Psi| \leq c_2, \tag{5.112}$$

where $\rho_1, 0 < \rho_1 < r_0, \rho_2 > 0, c_1 > 0, c_2 > 0$ are constants depending on r_0, M, Λ only. Denoting by

$$A(y) = |\det D\Psi(y)|(D\Psi^{-1})(\Psi(y))(D\Psi^{-1})^T(\Psi(y)), \tag{5.113}$$

$$B(y) = 2|\det D\Psi(y)|(D\Psi^{-1})(\Psi(y))\frac{\nabla\psi(\Psi(y))}{\psi(\Psi(y))}, \tag{5.114}$$

$$v(y) = z(\Psi(y)), \tag{5.115}$$

it follows that

$$A(0) = I \ , \tag{5.116}$$

$$A(y',0)(y',0)\cdot e_3 = 0, \text{ for every } \ y', |y'| \leq \rho_2, \tag{5.117}$$

$$c_3|\xi|^2 \leq A(y)\xi\cdot\xi \leq c_4|\xi|^2, \text{for every } \ y \in B^-_{\rho_2}(0) \text{ and for every } \xi \in \mathbb{R}^3, \tag{5.118}$$

$$|A(y_1) - A(y_2)| \leq c_5|y_1 - y_2|, \quad \text{for every } \ y_1, y_2 \in B^-_{\rho_2}(0), \tag{5.119}$$

$$|B(y)| \leq c_6, \text{ for every } \ y \in B^-_{\rho_2}(0), \tag{5.120}$$

where $c_4 > 0, c_5 > 0, c_6 > 0$ are constants depending on r_0, M, Λ only.
Let us observe that $v \in H^1(B^-_{\rho_2}(0))$ is a weak solution to the problem

$$\begin{cases} \operatorname{div}(A\nabla v) + B\nabla v = 0, & \text{in } B^-_{\rho_2}(0), \\ A(y',0)\nabla v\cdot\nu' = 0, & \text{on } B'_{\rho_2}(0). \end{cases} \tag{5.121}$$

Hence we are under the assumptions of Theorem 1.3 in [2] and thus we can infer that there exists a radius $\rho_3, 0 < \rho_3 < \rho_2$, depending on the *a priori data* only, such that

$$\int_{B^-_{\beta\rho}(0)} |v|^2 \leq c\beta^K \int_{B^-_{\rho}(0)} |v|^2 \ , \tag{5.122}$$

for every ρ, β such that $\beta > 1$ and $0 < \beta\rho \leq \rho_3$, where $c > 0$ is constant depending on the *a priori data* only, and $K > 0$ depends on the *a priori data* and increasingly on

$$N(\rho_3) = \rho_3 \frac{\int_{B^-_{\rho_3}(0)} A\nabla v\cdot\nabla\bar{v} + Re(\bar{v}\ \operatorname{div}(A\nabla v))}{\int_{\partial B^-_{\rho_3}(0)\setminus B'_{\rho_3(0)}} \mu|v|^2} \ , \tag{5.123}$$

where we denote

$$\mu(x)=\frac{A(x)x\cdot x}{|x|^2},\quad \text{for every } x\in B^-_{\rho_2}(0). \tag{5.124}$$

By (5.118) it follows that

$$c_3\le \mu(x)\le c_4,\quad \text{for every } x\in B^-_{\rho_2}(0). \tag{5.125}$$

Let us observe that the proof of Theorem 1.3 in [2] needs, in this context, a slight modification due to the fact that we deal with complex valued functions. We omit the details.
Denoting by

$$\tilde N(\rho_3)=\frac{\int_{B^-_{\rho_3}(0)}\rho_3^2|\nabla v|^2+|v|^2}{\int_{B^-_{\rho_3}(0)}|v|^2}\ , \tag{5.126}$$

we notice, following the arguments in [11, Lemma 3.3], that

$$N(\rho_3)\le C\tilde N(\rho_3), \tag{5.127}$$

where $C>0$ is a constant depending on the *a priori data* only.
By (5.111), it follows, that for every r and $\beta>1$ such that $0<r<\beta r<\frac{\rho_3}{2}$

$$\int_{\Gamma_{I,\beta r}(0)}|z|^2\le C\int_{B^-_{2\beta r}(0)}|v|^2\ , \tag{5.128}$$

where $C>0$ is a constant depending on r_0,M,Λ only. Moreover, by (5.122) and by (5.111) we have that

$$\int_{B^-_{2\beta r}(0)}|v|^2\le C(2\beta c_1)^K\int_{B^-_{\frac{r}{c_1}}(0)}|v|^2\le C(2\beta c_1)^K\int_{\Gamma_{I,r}(0)}|z|^2, \tag{5.129}$$

where $C>0$ is a constant depending on r_0,M,Λ only.
Combining (5.128) and (5.129), we have that

$$\int_{\Gamma_{I,\beta r}}|z|^2\le C(2\beta c_1)^K\int_{\Gamma_{I,r}(0)}|z|^2\ . \tag{5.130}$$

Finally the last inequality, (5.69),(5.70) imply that

$$\int_{\Gamma_{I,\beta r}}|u|^2\le C(\beta)^K\int_{\Gamma_{I,r}(0)}|u|^2\ , \tag{5.131}$$

where $C>0,K>0$ are constants depending on *a priori data* and on $\tilde N(\rho_3)$ only. Thus the Lemma follows with

$$\bar r=\frac{\rho_3}{2}. \tag{5.132}$$

It only remains to majorize the quantity (5.126) by a constant depending on the *a priori data* only. Let us observe that by (5.111), by (5.69) and by (5.70), we have that

$$\int_{B^-_{\rho_3}(0)} |\nabla v|^2 + |v|^2 \leq C \int_{\Gamma_{I,\rho_3 c_1}(0)} |\nabla u|^2 + |u|^2, \tag{5.133}$$

where $C > 0$ is a constant depending on the *a priori data* only. Moreover, by the above inequality and by (5.18), we can conclude that

$$\int_{B^-_{\rho_3}(0)} |\nabla v|^2 + |v|^2 \leq C, \tag{5.134}$$

where $C > 0$ is a constant depending on *a priori data* only.
On the other hand, we have that choosing $P_0 = \frac{M}{8\sqrt{1+M^2}\rho_3}\nu$ and $\rho_4 = \frac{1}{32}\frac{M}{\sqrt{1+M^2}}\rho_3$, where ν is the outer unit normal to D at 0, it follows that $B_{\rho_4}(P_0) \subset \Gamma_{I,\frac{\rho_3}{2}}(0)$. Thus, by (5.111) and by (5.70) it follows that

$$\int_{B^-_{\rho_3}(0)} |v|^2 \geq C \int_{\Gamma_{I,\frac{\rho_3}{2}}(0)} |u|^2 \geq C \int_{B_{\rho_4}(P_0)} |u|^2 \ , \tag{5.135}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us consider a point $Q \in \mathbb{R}^3 \setminus D^+_{2R_0}$ such that

$$B_{4\rho_4}(Q) \subset \mathbb{R}^3 \setminus \overline{D}^+_{2R_0}, \tag{5.136}$$

where R_0 is the radius introduced in Corollary 5.4. Dealing as in the proof of Theorem 5.6, we cover a path joining P_0 to Q by a chain of balls of radius ρ_4 pairwise tangent to each other. Hence, by an iterated use of the three spheres inequality, we have that the following holds

$$\|u\|_{L^2(B_{\frac{\rho_4}{4}(Q)})} \leq C\|u\|^{\tau^s}_{L^2(B_{\rho_4(P_0)})} \ , \tag{5.137}$$

where $C > 0, s > 0$ and $\tau, 0 < \tau < 1$ are constants depending on the *a priori data* only. By the last inequality, by (5.136) and by (5.33), we can infer that

$$\|u\|_{L^2(B_{\rho_4(P_0)})} \geq \left(\frac{\pi\rho_4^3}{C48}\right)^{\frac{1}{\tau^s}} . \tag{5.138}$$

Hence, by (5.138) and by (5.135), we have that

$$\int_{B^-_{\rho_3}(0)} |v|^2 \geq C, \tag{5.139}$$

where $C > 0$ is a constant depending on *a priori data* only. Hence, by (5.134) and by (5.139), we can majorize $\tilde{N}(\rho_3)$ by a constant depending on the *a priori data* only and thus the Lemma follows. □

Theorem 5.10 (Surface doubling inequality). *Let u be the solution to the problem* (1.8)*, then there exists a constant $C>0$ depending on the* a priori data *only such that, for every $x_0\in\Gamma_I^{r_0}$ and for every $r\in(0,\frac{\bar{r}}{4})$, the following holds*

$$\int_{\Delta_{I,2r}(x_0)}|u|^2\,d\sigma\leq C\int_{\Delta_{I,r}(x_0)}|u|^2\,d\sigma\ . \tag{5.140}$$

Proof. Let $x_0\in\Gamma_I^{r_0}$ and let $z\in H^1(\Gamma_{I,r_1}(x_0))$ and $\bar{r}$ be, respectively, the solution to the problem (5.108) defined by (5.107) and the radius introduced in (5.132). By a regularity estimate at the boundary, (see for instance [8, p.777]) we have that, for any $r\in(0,\frac{\bar{r}}{4})$, the following holds

$$\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2\leq C\left(\frac{1}{r}\int_{\Gamma_{I,2r}(x_0)}|\nabla z|^2\right)^{1-\gamma}\left(\frac{1}{r^2}\int_{\Delta_{I,r}(x_0)}|z|^2\right)^{\gamma}, \tag{5.141}$$

where $C>0$ and $0<\gamma<1$ are constants depending on the *a priori data* only and where $\nabla_{\mathrm{t}}z$ represents the tangential gradient.
Thus, by the Young inequality we have that for every $\varepsilon>0$ the following holds

$$\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2\leq\frac{C\varepsilon^{\frac{1}{1-\gamma}}}{r}\int_{\Gamma_{I,2r}(x_0)}|\nabla z|^2+\frac{C}{\varepsilon^{\frac{1}{\gamma}}r^2}\int_{\Delta_{I,r}(x_0)}|z|^2, \tag{5.142}$$

where $C>0$ is a constant depending on the *a priori data* only.
Moreover, by a well-known estimate of stability for the Cauchy problem (see for instance [69]), we have that

$$\begin{aligned}\int_{\Gamma_{I,\frac{r}{2}}(x_0)}|z|^2\quad&\leq Cr\left(\int_{\Delta_{I,r}(x_0)}|z|^2+r^2\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2\right)^{1-\delta}\cdot \\ &\cdot\left(\int_{\Delta_{I,r}(x_0)}|z|^2+r^2\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2+r\int_{\Gamma_{I,r}(x_0)}|\nabla z|^2\right)^{\delta},\end{aligned} \tag{5.143}$$

where $C>0$ and $0<\delta<1$ are constants depending on the *a priori data* only.
Hence, by (5.143) and by the Young inequality, we have that for every $\beta>0$ the following holds

$$\begin{aligned}\int_{\Gamma_{I,\frac{r}{2}}(x_0)}|z|^2\leq\frac{C}{\varepsilon^{\frac{\beta}{1-\delta}}}\left(r\int_{\Delta_{I,r}(x_0)}|z|^2+r^3\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2\right)+ \\ +C\varepsilon^{\frac{\beta}{\delta}}\left(r\int_{\Delta_{I,r}(x_0)}|z|^2+r^3\int_{\Delta_{I,r}(x_0)}|\nabla_t z|^2+r^2\int_{\Gamma_{I,r}(x_0)}|\nabla z|^2\right),\end{aligned} \tag{5.144}$$

where $C>0$ is a constant depending on the *a priori data* only.
Choosing β in (5.144) such that $\beta=\frac{1-\delta}{1-\gamma}\gamma$ and inserting (5.142) in (5.144), we obtain

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)}|z|^2\leq\frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}}\int_{\Delta_{I,r}(x_0)}|z|^2+C\varepsilon r^2\int_{\Gamma_{I,2r}(x_0)}|\nabla z|^2,$$

where $C > 0$ is a constant depending on the *a priori data* only. By the Caccioppoli inequality we have that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |z|^2 \leq \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}} \int_{\Delta_{I,r}(x_0)} |z|^2 + C\varepsilon \int_{\Gamma_{I,4r}(x_0)} |z|^2,$$

where $C > 0$ is a constant depending on the *a priori data* only.
Thus by (5.69) and (5.70) we can infer that

$$\int_{\Gamma_{I,r}(x_0)} |u|^2 \leq \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}} \int_{\Delta_{I,2r}(x_0)} |u|^2 + C\varepsilon \int_{\Gamma_{I,8r}(x_0)} |u|^2,$$

where $C > 0$ is a constant depending on the *a priori data* only.
By (5.106) it follows that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2 \leq \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}} \int_{\Delta_{I,r}(x_0)} |u|^2 + C(8)^K\varepsilon \int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2, \qquad (5.145)$$

where $C > 0$ is a constant depending on the *a priori data* only.
Hence, choosing ε in (5.145) such that $\varepsilon = \frac{1}{2C(8)^K}$, we obtain that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2 \leq Cr \int_{\Delta_{I,r}(x_0)} |u|^2, \qquad (5.146)$$

where $C > 0$ is a constant depending on the *a priori data* only.
By applying again (5.106) on the left hand side of (5.146), we obtain that

$$\int_{\Gamma_{I,2r}(x_0)} |u|^2 \leq Cr \int_{\Delta_{I,r}(x_0)} |u|^2, \qquad (5.147)$$

where $C > 0$ is a constant depending on the *a priori data* only.
Moreover, by a standard Dirichlet trace inequality, we have that

$$\int_{\Delta_{I,2r}(x_0)} |u|^2 \leq C \int_{\Delta_{I,r}(x_0)} |u|^2, \qquad (5.148)$$

where $C > 0$ is a constant depending on the *a priori data* only.

$\square$

Corollary 5.11 (A_p property on the boundary). *Let u be the solution to the problem* (1.8)*, then there exist $p > 1, A > 0$ constants depending on the* a priori data *only, such that, for every $x_0 \in \Gamma_I^{r_0}$ and every $r \in (0, \frac{\bar{r}}{4})$, the following holds*

$$\left(\frac{1}{|\Delta_{I,r}(x_0)|} \int_{\Delta_{I,r}(x_0)} |u|^2 d\sigma\right) \left(\frac{1}{|\Delta_{I,r}(x_0)|} \int_{\Delta_{I,r}(x_0)} |u|^{-\frac{2}{p-1}} d\sigma\right)^{p-1} \leq A. \qquad (5.149)$$

Proof. Let $x_0 \in \Gamma_I^{r_0}$ and let $r \in (0, \frac{\bar{r}}{4})$, then by a trace inequality, (see for instance [1], Chap. 5), it follows that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leq C\|u\|_{H^1(\Gamma_{I,r}(x_0))}, \tag{5.150}$$

where $C > 0$ is a constant depending on the *a priori data* only. By the Caccioppoli inequality we deduce that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leq \frac{C}{r}\|u\|_{L^2(\Gamma_{I,2r}(x_0))}. \tag{5.151}$$

Applying the Doubling inequality (5.106) on the right hand side of (5.151), we obtain that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leq \frac{C}{r}\|u\|_{L^2(\Gamma_{I,r}(x_0))}, \tag{5.152}$$

where $C > 0$ is a constant depending on the *a priori data* only. Combining (5.146) and (5.152) we have that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leq \frac{C}{\sqrt{r}}\|u\|_{L^2(\Delta_{I,2r}(x_0))}, \tag{5.153}$$

where $C > 0$ is a constant depending on the *a priori data* only. Thus by the doubling inequality (5.140) we have

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leq \frac{C}{\sqrt{r}}\|u\|_{L^2(\Delta_{I,r}(x_0))}. \tag{5.154}$$

Hence, we infer that for every $r \in (0, \frac{\bar{r}}{4})$ and for every $x_0 \in \Gamma_I^{r_0}$, the following holds

$$\left(\frac{1}{r^2}\int_{\Delta_{I,r}} |u|^4\right)^{\frac{1}{4}} \leq \left(\frac{C}{r^2}\int_{\Delta_{I,r}} |u|^2\right)^{\frac{1}{2}},$$

obtaining a reverse Hölder inequality.
The result in [28] assures the existence of some $p > 1$ and $A > 0$ depending on the *a priori data* only such that (5.149) holds. □

Proof of Theorem 5.1. Let x_0 be a point in $\Gamma_I^{r_0}$. Let us pick $r = \frac{\bar{r}}{8}$, thus by (5.146) with $u = u_2$ it follows that

$$\int_{\Delta_{I,\frac{\bar{r}}{8}}(x_0)} |u_2|^2 \mathrm{d}\sigma \geq C\int_{\Gamma_{I,\frac{\bar{r}}{16}}(x_0)} |u_2|^2 \mathrm{d}x, \tag{5.155}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let P_0 and $\rho_4 > 0$ be, respectively a point and a radius, such that $B_{\rho_4}(P_0) \subset \Gamma_{I,\frac{\bar{r}}{16}}(x_0)$. By rephrasing the argument leading to (5.138) we deduce by (5.155) that

$$\int_{\Delta_{I,\frac{\bar{r}}{8}}(x_0)} |u_2|^2 \mathrm{d}\sigma \geq C, \tag{5.156}$$

where $C>0$ is a constant depending on the *a priori data* only.
Combining (5.149) and (5.156), we have that for every $x_0 \in \Gamma_I^{r_0}$ the following holds

$$\left(\int_{\Delta_{I,\frac{\bar r}{8}}(x_0)} |u_2|^{-\frac{2}{p-1}} d\sigma\right)^{p-1} \leq C, \tag{5.157}$$

where $C>0$ is a constant depending on the *a priori data* only.
Let us now consider $x \in \Delta_{I,\frac{\bar r}{8}}(x_0)$, then it follows that

$$\begin{aligned} |\lambda_1(x)-\lambda_2(x)| &= \left|-\lambda_1(x)\frac{u_1(x)-u_2(x)}{u_2(x)} + \frac{1}{iu_2(x)}\left(\frac{\partial u_2(x)}{\partial\nu} - \frac{\partial u_1(x)}{\partial\nu}\right)\right| \leq \\ &\leq |\lambda_1(x)|\frac{|u_1(x)-u_2(x)|}{|u_2(x)|} + \frac{1}{|u_2(x)|}\left|\frac{\partial u_2(x)}{\partial\nu} - \frac{\partial u_1(x)}{\partial\nu}\right| . \end{aligned}$$

Then by Theorem 5.6 and by (5.4) we have that, if $0<\varepsilon<\varepsilon_0$, then

$$|\lambda_1(x)-\lambda_2(x)| \leq (\Lambda+1)\omega(\varepsilon)\frac{1}{|u_2(x)|}. \tag{5.158}$$

Hence denoting by $\delta = \frac{2}{p-1}$, (5.158) yields to

$$\left(\int_{\Delta_{I,\frac{\bar r}{8}}(x_0)} |\lambda_1(x)-\lambda_2(x)|^\delta\right)^{\frac{1}{\delta}} \leq (\Lambda+1)\omega(\varepsilon)\left(\int_{\Delta_{I,\frac{\bar r}{8}}(x_0)} \frac{1}{|u_2(x)|^\delta}\right)^{\frac{1}{\delta}} . \tag{5.159}$$

By (5.157) and by a possible replacement of the constant C in (3.11), we have that

$$\left(\int_{\Delta_{I,\frac{\bar r}{8}}(x_0)} |\lambda_1(x)-\lambda_2(x)|^\delta\right)^{\frac{1}{\delta}} \leq \omega(\varepsilon). \tag{5.160}$$

By the a priori bound (5.4), we can infer that

$$|\lambda_1(x)-\lambda_2(x)| \leq |\lambda_1(x)-\lambda_2(x)|^{\frac{\delta}{2}}(2\Lambda)^{1-\frac{\delta}{2}} . \tag{5.161}$$

Integrating the above inequality with respect to x over $\Delta_{I,\frac{\bar r}{8}(x_0)}$ we have

$$\|\lambda_1(x)-\lambda_2(x)\|_{L^2(\Delta_{I,\frac{\bar r}{8}(x_0)})} \leq (2\Lambda)^{1-\frac{\delta}{2}}\left(\int_{\Delta_{I,\frac{\bar r}{8}(x_0)}} |\lambda_1(x)-\lambda_2(x)|^\delta\right)^{\frac{1}{2}} . \tag{5.162}$$

Hence, by a possible further replacement of the constants C,θ in (3.11), we can infer that the last inequality and (5.160) yield to

$$\|\lambda_1(x)-\lambda_2(x)\|_{L^2(\Delta_{I,\frac{\bar r}{8}(x_0)})} \leq \omega(\varepsilon) . \tag{5.163}$$

By an interpolation inequality, see for instance [8, p.777], we have that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar r}{8}(x_0)})}\le C\|\lambda_1-\lambda_2\|^{\frac12}_{L^2(\Delta_{I,\frac{\bar r}{8}(x_0)})}\|\lambda_1-\lambda_2\|^{\frac12}_{C^{0,1}(\Delta_{I,\frac{\bar r}{8}(x_0)})}, \tag{5.164}$$

where $C>0$ is a constant depending on the *a priori data* only. Hence by (5.4), it follows that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar r}{8}(x_0)})}\le C(2\Lambda)^{\frac12}\|\lambda_1-\lambda_2\|^{\frac12}_{L^2(\Delta_{I,\frac{\bar r}{8}(x_0)})}. \tag{5.165}$$

Combining (5.163) with (5.165) we obtain, by a possible further replacement of the constants C,θ in (3.11), that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar r}{8}(x_0)})}\le\omega(\varepsilon). \tag{5.166}$$

Let us cover $\Gamma_I^{r_0}$ with the sets $\Delta_{I,\frac{\bar r}{8}}(x_j),\ j=1,\dots,J$, with $x_j\in\Gamma_I^{r_0}$.
Let i be an index such that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar r}{8}(x_i)})}=\|\lambda_1-\lambda_2\|_{L^\infty(\Gamma_I^{r_0})}. \tag{5.167}$$

Thus, by a further possible replacement of the constant C,θ in (3.11), we deduce (5.14) by combining (5.167) and (5.166) with $x_0=x_i$. $\square$

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