SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES
Physics Area
Astroparticle Physics Curriculum

Framing the Dark: Theory and phenomenology of a non-minimally coupled dark matter fluid

Thesis submitted for the degree of
Doctor Philosophiæ

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September 2013

The hardest thing of all
is to find a black cat in a dark room,
especially if there is no cat.
– Confucius

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Chapter 1

Introduction

"Having its roots in philosophic speculation, cosmology evolved gradually into a physical science, but a science with so little observational basis that philosophical considerations still play a crucial if not dominant role." These 50 years old words, taken from a paper by R. H. Dicke [1], sound off-key today. The enormous wealth of observational data we have at our disposal and the variety of observables put this statement into the drawer of a faraway past. In fact, not only cosmology has entered the era of precision tests but also the picture emerging from the available data is coherently included in a standard model, the Lambda Cold Dark Matter (ΛCDM) model, which beautifully accounts for the observed Universe evolution from its very first stages, a few hundreds seconds after the Big Bang, up to present day, with only six free parameters. As a further occurrence we have experienced in the last year a boost in the experimental evidences: the first data release from the Planck survey agrees to an unprecedented precision with the predictions of the standard model of cosmology [2].

However, this success comes with a price. Despite working with unexpected accuracy, we know that the standard model of cosmology cannot be but an effective description of the Universe no matters how precise. And in fact, there are many observational evidences and theoretical issues that point towards some physics beyond ΛCDM model.

In order to fit the data, the matter content of the Universe must be equipped with two unknown components: dark energy (DE) and dark matter (DM). DE is a form of exotic matter/energy required to explain the observed accelerated expansion of the universe whose fundamental nature is far from being understood. The natural candidate within General Relativity (GR), a cosmological constant (CC) term, despite being able to fit observational data, is plagued by theoretical issues and it is very unlikely to represent the fundamental explanation for this component. Many alternatives have been proposed in which the DE role is played by extra dynamical degrees of freedom, like a scalar field with more or less complicated interactions, but up to now no smoking gun for any of these extensions has been found. If DE is indeed a CC its value will have to be explained most probably by Planck-scale/Quantum Gravity physics. If it is due to a field, then extra phenomenology with respect to a CC is expected, *e.g.,* dynamics and spatial fluctuations, and in particular a redshift dependence in the pressure and energy density of DE.

DM, instead, is thought to be made by a new, yet undiscovered, set of particles with at most weak interaction with standard model particles and its introduction is required in order to correctly form the large scales structures we observe. Contrarily to DE, DM seemed to be a settled issue with quite defined particle candidates in the context of Supersymmetric (SUSY) extensions of the standard model (SM) of particle physics and with a wealth of successes when theory is compared with observations. However, in the last years we have collected evidences that this picture may not be so definitive. In fact, despite the ability of this DM paradigm to provide a successful description of the Universe dynamics at cosmological scales, at smaller, galactic scales, it seems unable to reproduce the observed properties of structure formation. There is no general consensus on the origin of these discrepancies in the standard framework: they may be due to unaccounted baryons' feedbacks as well as to a modification of standard DM paradigm and this uncertainty is stressed by the large number of alternatives proposed in the last decade.

But the issue is possibly more dramatic: many observations seem to point towards correlations between luminous and dark components which are hard to explain in the standard DM scenario, thus suggesting a modified gravity explanation of the small scale dynamics. And in fact, there are phenomenological dark matter-less models, like MOdified Newtonian Dynamics (MOND), that are able to fit the data more accurately at small scales, through a modification of the gravitational laws. However, these models (and their relativistic generalizations) fail to be as good as the ΛCDM model at cosmological scales. This turns into an apparent dichotomy in our understanding of the Universe dynamics which seems to be fractioned into the successes in opposite regimes of two clashing models.

Furthermore, ΛCDM model is based on GR which, despite its successes, is probably not the ultimate theory of gravity. On the theoretical side, GR remains poorly understood in its foundations: we can construct very many alternative theories of gravitation but we do lack an axiomatic derivation of such theories and hence an authentic understanding of their reciprocal relation. Moreover, generalized theories of gravitation can be as well considered as different effective actions induced by physics beyond the Planck energy and as such their study as alternative models of gravitation could provide some insight on the long standing problem of building a quantum gravity theory. On the experimental side, we do lack severe experimental constraints on GR from galactic scales upwards. These issues, together with the fact that the 95% of the energy/matter content of the universe is of a yet unknown nature, have been among the most pressing motivations for the recent outburst of attention toward alternative theories.

Given the above picture of current understanding of DM and of gravitational dynamics, a natural direction of investigation seems to be towards the generalization of the interactions between matter content and curvature terms. The study of couplings between fields and gravity was started decades ago with the works of Brans and Dicke [3] and it is nowadays well structured into Scalar-Tensor Theories of gravity which have found their most successful application in the context of DE models. The topic was recently revitalized with the re-discovery of the most general scalar-tensor theory that gives second order field equations in four dimensions [4, 5], the so-called Horndeski action, which provides a coherent framework for extensions to the ΛCDM model.

In particular, the idea of non-minimally coupled DM was recently proposed [6] and indeed it proved to be an intriguing alternative to the standard paradigm as it is able to produce a mimicking of MONDian behavior in the context of DM theories, thus potentially being able to reconcile in a single scheme two apparently unrelated models.

In this thesis we will further explore this topic applying the techniques of Scalar-Tensor Theories to the DM sector by investigating both theoretical and phenomenological consequences of a model in which a DM fluid gets non-minimally coupled (NMC) to curvature terms. This phenomenological model shows to have relevant consequences on cosmological evolution, in particular on the process of structure formation, as the generalized couplings between DM and curvature terms lead, for example, to a modified Poisson equation in the non-relativistic limit of the theory. Furthermore, the investigation of this model brought us to the discovery that it is possible to find equivalent Einstein and Jordan frames which are connected by a generalization of the conformal transformation, the so called disformal transformation. This equivalence is a well known fact in the context of standard Scalar-Tensor Theories but is also quite new and unexpected in more general theories, as is the Horndeski one.

This point will quite naturally lead us to the question whether this invariance under generalized metric transformations could be more than a coincidence. Hence we shall investigate the relation between Horndeski action and disformal transformations discovering how these play a similar role to the one conformal metric transformations have for standard Scalar-Tensor Theories. We shall then be able to identify a class of metric transformations under which the Horndeski action is invariant, thus extending the class of Scalar-Tensor Theories that admits equivalent frames.

Far from being a mathematical curiosity, the invariance of the Horndeski action under a particular class of disformal transformations represents a first step in order to formalize some recently noticed relations between different theoretical models for DE, which may be seen as equivalent representations of the same fundamental theory.

The plan of the thesis is as follows. After briefly reviewing the ΛCDM model and its formalism together with its most relevant experimental evidences in chapter 2, we will discuss its main critical points and some of the proposed solution to them in chapter 3. Chapter 4 will be devoted to the introduction of our model and to the discussion of its characteristics, with particular emphasis on the weak field limit; in chapter 5 we will further extend the discussion of this model by investigating its cosmological consequences at the background and linear perturbations level. Chapter 6 will be devoted to a formal investigation of the transformation properties of the Horndeski action under disformal transformations, with particular attention on the equivalence between frames. Finally, in chapter 7 we will draw our conclusions.

This work is based on the following publications:

D. Bettoni, S. Liberati,
"*Disformal invariance of second order scalar tensor theories: framing the Horndeski action*",
arXiv:1306.6724 [gr-qc]

D. Bettoni, V. Pettorino, S. Liberati, C. Baccigalupi,
"*Non-minimally coupled dark matter: effective pressure and structure formation*",
JCAP 07(2012)027 [arXiv:1203.5735 [astro-ph.CO]]

D. Bettoni, S. Liberati, L. Sindoni,
"*Extended ΛCDM: generalized non-minimal coupling for dark matter fluids*",
JCAP 11(2011)007 [arXiv:1108.1728 [gr-qc]]

Chapter 2

The standard cosmological model

The ΛCDM model is based on GR which relates, in a beautiful and elegant way, the geometry of space-time to the matter content of the Universe, via the Einstein Field Equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2.1}$$

$G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor, constructed from the metric $g_{\mu\nu}$ and its first and second derivatives, Λ and G are the CC and the Newton constant respectively. The matter content is instead included in the Stress-Energy Tensor (SET) $T_{\mu\nu}$. These equations can be obtained through the variation with respect to the metric $g_{\mu\nu}$ of the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2\Lambda] + \int d^4x\sqrt{-g}\,\mathcal{L}_m(g,\psi_m)\,, \tag{2.2}$$

where $\mathcal{L}_m$ is the total matter Lagrangian and ψ_m collectively denotes matter fields.

The Einstein tensor is covariantly conserved as a consequence of the reduced Bianchi identities so that the companion set of equations for the evolution of matter fields can be obtained from the conservation of the total SET

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.3}$$

If there are no coupled species, the SET of the different matter components is individually conserved and the matter evolution can be split into equations for the single components. Alternatively, one can obtain the matter field equations via direct variation of the action with respect to the field variables obtaining the standard Euler–Lagrange equations.

Looking for a solution of this non-linear coupled system of equations may seem a tremendous task. However, at scales larger than about 100 Mpc, corresponding to the largest observed structures, the Universe is almost homogeneous and isotropic. The assumption of these two symmetries at large scales is confirmed by the high level of isotropy of the Cosmic Microwave Background (CMB) radiation and by the distribution of Large Scale Structures (LSS) and it is often referred to as the Copernican or Cosmological Principle.

The existence of these two symmetries dramatically simplifies the equations, reducing the number of functions required to describe the geometry and strongly constraining the form of the SET, and introduces a convenient reference frame, known as comoving frame. In fact, these symmetries are seen only by observers that are at rest with respect to the Universe expansion; otherwise a dipole anisotropy would be present for an observer moving with respect to this frame.

Of course, at smaller scales, deviations from isotropy and homogeneity are expected as a consequence of the collapse of matter into bounded objects. The process of structure formation is indeed allowed by the fact that the symmetries of space-time are not exact, small inhomogeneities being present throughout the whole history of the universe. However, these inhomogeneities can be considered as small perturbations around the homogenous background for most of the cosmological evolution and hence they can be mostly described via linear perturbation theory.

In the next two sections we will review the dynamics of a homogeneous and isotropic background Universe and the theory of linear perturbations.

2.1 The homogeneous and isotropic Universe

As said the isotropy and homogeneity of the Universe seen by a comoving observer at the largest scales highly simplify the equations of motion. These symmetries can be translated into the statement that the squared line element of the Universe is

$$ds^2 = -dt^2 + a(t)^2 d\sigma^2 \,, \tag{2.4}$$

where, using polar coordinates,

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \,, \tag{2.5}$$

where r is the comoving radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, while K is related to the spatial curvature and is commonly normalized in such a way to take the value -1 for an open Universe, 1 for a closed one and 0 for a flat one. As it is clear from equation (2.4) under the assumptions of the Cosmological Principle only one degree of freedom is required to fully describe the geometry of the Universe, the scale factor $a(t)$.

The structure of the metric derived from this line element forces the Einstein tensor to be diagonal and, for consistency, the SET can only take the perfect fluid form

$$T_{\mu\nu} = [\rho(t) + p(t)]\, u_\mu u_\nu + p(t) g_{\mu\nu}\,, \tag{2.6}$$

where $\rho(t)$ is the total energy density, $p(t)$ is the pressure and the four vector

$$u^\mu = (a^{-1}, 0, 0, 0)\,, \tag{2.7}$$

is the fluid four velocity whose normalization is such that $u_\mu u^\mu = -1$. This parametrization tells us about another important fact: when dealing with background cosmology, we can parametrize the matter content in a fluid limit with the consequence that only macroscopic thermodynamical quantities are relevant for the dynamics of the Universe.

When we plug the metric inferred from the line element (2.4) into the Einstein Field Equations and use the structure of the SET presented above we obtain the Friedman equations

$$H(t)^2 \equiv \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{K}{a(t)^2}\,, \tag{2.8}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\Big(\rho(t) + 3p(t)\Big)\,, \tag{2.9}$$

where we have introduced the Hubble parameter $H(t)$ whose present day value is

$$H_0 = 73.8 \pm 2.4\,\mathrm{km\ Mpc^{-1} s^{-1}}\,, \tag{2.10}$$

as obtained from the Hubble Space Telescope (HST) [8] through a direct measurement of the recession velocities of astrophysical objects around us. The Hubble parameter has been recently obtained also from CMB [2] which provided the value of $H_0^{\mathrm{CMB}} = 67.80 \pm 0.77\,\mathrm{km\ Mpc^{-1}s^{-1}}$. The reason of this difference is presently under investigation.

Equation (2.8) can be usefully described in terms of the density parameter

$$\Omega(t) = \frac{8\pi G \rho(t)}{3H(t)^2}\,. \tag{2.11}$$

thus taking the equivalent form

$$\Omega_{tot}(t) = 1 - \Omega_K(t)\,, \qquad \Omega_K(t) = -\frac{K}{(aH)^2}\,, \tag{2.12}$$

where Ω_K parametrizes deviations from spatial flatness. In what follows we will consider $K = 0$ as is suggested by CMB measurements, $\Omega_K = -0.037^{+0.043}_{-0.049}$ at the 95% limits [2].

A useful quantity to be defined at this point is the critical density ρ_c, the total matter density of the Universe in the absence of spatial curvature, whose present day value is

$$\rho_{c0} = \frac{3H_0^2}{8\pi G} = 1.88\, h^2 \times 10^{-29}\, g\, cm^{-3}\,, \tag{2.13}$$

where $h = H_0/100$ km s^{-1} Mpc^{-1}. The Friedman equations (2.8) and (2.9) can be combined together to give

$$\dot{\rho}(t) + 3H(t)\Big(\rho(t) + p(t)\Big) = 0\,, \tag{2.14}$$

which is an evolution equation for the total matter/energy component which reflects the conservation of the energy density. As said, if there is no interaction between the different matter species, the conservation equation can be split into equations for the single matter component

$$\dot{\rho}_i(t) + 3H(t)\Big(\rho_i(t) + p_i(t)\Big) = 0\,. \tag{2.15}$$

In the presence of interaction it is still possible to write separate equations but in this case they will be coupled, because of energy transfer from one species to the others. Under quite general assumptions the coupled equations can be written as

$$\dot{\rho}_i(t) + 3H(t)\Big(\rho_i(t) + p_i(t)\Big) = Q_i\,, \tag{2.16}$$

where Q_i encodes the effects of coupling of the i-th species with the others and is such that $\sum_i Q_i = 0$ in order to preserve the conservation of the total SET.

Of the three equations (2.8), (2.9) and (2.14) only two are independent so that we have a system of two equations with three unknowns. In order to close the system we need to provide an equation of state for matter which relates the pressure to the energy density. A standard choice is to consider barotropic fluids for which

$$p = p(\rho) = w(\rho)\rho \tag{2.17}$$

where we have introduced the equation of state parameter w which characterizes the different fluids' behavior. In the uncoupled case the continuity equation can be rewritten in terms of the equation of state parameter w and, if this is constant, the equation can be integrated to obtain the evolution of the particular matter species as a function of the scale factor, namely

$$\rho_i \propto a^{-3(1+w_i)}\,. \tag{2.18}$$

This clearly shows how matter species that differ in their pressure component will have a different scaling with the expansion of the Universe; hence, we can identify different eras in which a particular component is dominating the energy content of the Universe.

Before presenting the matter content that appears to compose the Universe we need to define a few more relevant quantities.

Due to the expansion of the Universe the wavelength of a light ray coming towards us from some distant object is stretched. It is then useful to introduce the concept of redshift which exactly measures this change, via

$$z = \frac{\lambda_{obs}}{\lambda_{emit}} - 1\,. \tag{2.19}$$

This quantity can be related to the scale factor a providing a fundamental relation for cosmology

$$1 + z = a^{-1}\,. \tag{2.20}$$

Present time corresponds to $z = 0$ as the scale factor is normalized in such a way to be unity today. This quantity is commonly used in cosmology as many equations looks simpler when expressed in terms of redshift and also because this quantity is more closely related to observables. For small redshifts we can expand the previous relation in powers of the lookback time $t - t_0$ to get

$$z = H_0(t_0 - t) + \frac{H_0^2}{2}\,(2 + q_0)\,(t_0 - t)^2 + \ldots \tag{2.21}$$

where we have defined the present day deceleration parameter

$$q_0 \equiv -\frac{\ddot{a}_0}{a_0 H_0^2}\,, \tag{2.22}$$

which is an observable quantity that tells about the acceleration of the Universe expansion. As we will see neither matter nor radiation are able to explain the currently observed value of $q_0 \sim -1/2$ which indicates an accelerated expansion of the Universe.

Finally, we introduce another time variable that will be extensively used in the following. We define the conformal time τ as

$$\tau = \int_0^t \frac{dt}{a(t)}\,, \tag{2.23}$$

which will be the natural time variable for the description of the evolution of perturbations. When expressed in terms of it, the FLRW metric reads

$$ds^2 = a(\tau)^2[-d\tau^2 + dx^2]\,, \tag{2.24}$$

thus making the line element conformally related to the Minkowski one. A useful relation is the following

$$\mathcal{H} \equiv \frac{1}{a(\tau)}\frac{da(\tau)}{d\tau} = a(t)H(t)\,, \tag{2.25}$$

so that the evolution equation for matter components is

$$\rho'(\tau) + 3\mathcal{H}(\tau)(\rho(\tau) + p(\tau)) = 0\,, \tag{2.26}$$

where the prime indicates differentiation with respect to the conformal time τ.

2.1.1 The ΛCDM model

Now that we have introduced the general framework for the evolution of an isotropic and homogeneous Universe, we can specify the different species that compose its energy/matter content. According to the most recent data from the Planck survey [2] 68.3% of the matter content is composed by DE, that in its simplest ΛCDM realization is encoded in the famous CC, an energy source constant in both space and time, required to explain the current accelerated expansion of the Universe. The remaining 30% is composed by pressureless matter divided in two species: baryonic matter and DM. The latter, of unknown origin, accounts for 26.8% of the cosmic energy density while only the 4.9% is made out of known particles.[1] A minimal fraction of the present day energy density is made of CMB and neutrinos while the contribution coming from curvature has been set to zero.

Due to the different scaling with the expansion of the various matter species we do not expect this relative abundances to be fixed, and in fact the Universes passed through different epochs before entering the current CC constant dominated era.

Radiation. With the generic term radiation we mean all particles that shows a relativistic behavior and whose equation of state parameter is $w = 1/3$. As a consequence the continuity equation (2.15) can be integrated to give

$$\rho_r(a) = \rho_{r0} a^{-4}\,, \tag{2.27}$$

where ρ_{r0} is the radiation energy density at present. In the ΛCDM model there are two relativistic components, photons and neutrinos, whose present day abundances are

$$\Omega_{\gamma 0} = 4.2032 \times 10^{-5} h^{-2}\,, \qquad \Omega_{\nu 0} = 3.2701 \times 10^{-5} h^{-2}\,, \tag{2.28}$$

representing subdominant contributions to the total energy content of the present day Universe. However due to their scaling, these components have dominated the energy density of the Universe at early times. During this era, radiation is the only

[1] These values are slightly different from those obtained by the previous CMB experiment WMAP [7] as a consequence of the lower value for today's Hubble parameter $H_0 = 67.4 \pm 1.4$ found by Planck.

energy component responsible for the expansion of the Universe and hence, using the scaling relation (2.27), the Friedman equation implies

$$H(t)^2 \propto a(t)^{-4} \,. \tag{2.29}$$

This equation can be integrated to give the evolution of the scale factor and hence of the density as a function of time

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} , \qquad \rho(t) = \frac{3}{32\pi G t^2} \,. \tag{2.30}$$

where $t_0 = 1/(2H_0)$ provides an estimate of the age of the Universe if it were to be always radiation dominated. The deceleration parameter during this era is $q_0 = 1$, which means that radiation cannot be the responsible for the accelerated expansion.

Matter. The matter content is made out of two contributions: baryons and DM. They are characterized by the absence of pressure, $w = 0$, and they are usually referred to as dust components. The continuity equation can be integrated to give

$$\rho_m(a) = \rho_{m0} a^{-3} \,, \tag{2.31}$$

where ρ_{m0} refers to the present day energy density for the two matter components. The measured present day abundance of these two components is

$$\Omega_{\mathrm{DM}0} = 0.12029\, h^{-2} \,, \qquad \Omega_{b0} = 0.022068\, h^{-2} \,. \tag{2.32}$$

Notice that due to the different scaling of matter and radiation at some point along the evolution of the Universe the two densities will be equal. If expressed in terms of redshift, this happens when

$$1 = \frac{\rho_m(z_{eq})}{\rho_r(z_{eq})} = \frac{\rho_{m0}}{\rho_{r0}}(1 + z_{eq})^{-1} \,, \tag{2.33}$$

which is realized at redshift $z \sim 3000$ thus defining the end of the radiation dominated era and the onset of matter domination.

During this era, matter, basically DM alone, is the dominant energy component responsible for the expansion of the Universe. Using the scaling relation (2.31), the Friedman equation reads

$$H(t)^2 \propto a(t)^{-3} \,. \tag{2.34}$$

This equation can be integrated to give the evolution of the scale factor and hence of the density as a function of time

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} , \qquad \rho(t) = \frac{1}{6\pi G t^2} \,. \tag{2.35}$$

where $t_0 = 2/(3H_0)$. Notice that the scaling of matter as a function of time is the same of that of radiation. This is a consequence of the fact that the scaling with time is independent of the equation of state parameter w. The deceleration parameter now is $q_0 = 1/2$, so, again, we see that the universe expansion is decelerated also during matter dominated era.

Cosmological constant. This component is related to the CC term appearing in the Einstein equations and is characterized by a constant density and by an equation of state parameter $w = -1$. Hence, we have

$$\rho_\Lambda = \frac{8\pi G\Lambda}{3}\,, \qquad p_\Lambda = -\rho_\Lambda\,, \tag{2.36}$$

whose present day abundance is

$$\Omega_\Lambda = 0.6825\,, \tag{2.37}$$

which makes this component the dominant contribution to the energy budget of present day Universe.

When the cosmological matter content is dominated by a CC the Hubble parameter is constant and the scale factor as a function of time is

$$a(t) = a_0 e^{H_\Lambda t}\,, \qquad H_\Lambda = \sqrt{\frac{8\pi G}{3}\rho_\Lambda}\,. \tag{2.38}$$

The deceleration parameter in this case gives $q_0 = -\frac{1}{2}$ meaning that this energy component provides an accelerated expansion when it dominates the energy content of the Universe. Given the present day abundance of this component we can quantify when the accelerated era begins

$$1 = \frac{\rho_m(z_{acc})}{\rho_\Lambda(z_{acc})} = \frac{\rho_{m0}}{\rho_\Lambda}(1+z_{acc})^{-3} \tag{2.39}$$

obtaining the result that accelerated expansion starts at $z_{acc} \sim 0.7$.

2.2 Growth of linear matter perturbations

If the Universe were perfectly isotropic and homogeneous the structure we see around us could have never formed. But in fact, as confirmed by the data from CMB and LSS, the Universe is quite but not completely homogeneous and isotropic. All the structure we see are the final result of the gravitational collapse of some small initial fluctuations. The fact that at the time at which the CMB radiation was emitted the degree of homogeneity was of 1 part over 10^5 allows us to use the tools of linear perturbation theory to study the dynamics of these perturbations. In fact only at recent times and at suitably small scales perturbations have become non-linear marking the breakdown of linear approximation. In any case linear perturbation theory can be used to evolve initial conditions to provide the starting point for a non-linear analysis of the gravitational collapse. In the next sections we review this topic providing the most relevant equations that will be used in the forthcoming chapters [9–12].

2.2.1 General theory and gauge freedom

Relaxing the assumption of a perfectly smooth Universe makes the dynamical variables depend also on spatial coordinates, not only on time. However, in linear theory this dependence shows up as small corrections to the smooth background and we can perturb the Einstein equations (2.1) and the energy conservation equation (2.3). Denoting the exact quantities with an overall bar one has

$$\bar{G}_{\mu\nu}(\tau, x) \Rightarrow \qquad G_{\mu\nu}(\tau) = 8\pi G T_{\mu\nu}(\tau)\,, \tag{2.40}$$

$$\delta G_{\mu\nu}(\tau, x) = \delta T_{\mu\nu}(\tau, x)\,, \tag{2.41}$$

$$\bar{\nabla}_\mu \bar{T}^{\mu\nu}(\tau, x) = 0 \Rightarrow \qquad \nabla_\mu T^{\mu\nu}(\tau) = 0\,, \tag{2.42}$$

$$\delta(\nabla_\mu T^{\mu\nu}(\tau, x)) = 0\,, \tag{2.43}$$

where we have separated the smooth background quantities, which only depends on (conformal) time, from the perturbations that depend also on the spatial coordinate. A great simplification to the analysis of the equations of motion comes directly from the structure of the chosen space-time metric. In fact the FLRW metric allows a splitting between spatial and time directions so that the perturbations can be classified accordingly to their transformation properties under coordinate transformations on the invariant spatial subspace. In particular we can define the following categories.

Scalar. Scalar quantities are defined by functions of position and time with no spatial indices so that the knowledge of their value at a point is enough to fully characterize their structure at that particular point in space.

Vector. Any vector quantity can be decomposed as the gradient of a scalar potential plus a divergence free vector

$$v^i = v^i_v + \nabla_i v\,, \qquad \Delta v = \nabla_i v^i\,, \qquad \nabla_i v^i_v = 0\,, \tag{2.44}$$

v is then the scalar part of v^i while v_v is the vector part.

Tensor. A similar decomposition can be performed for tensorial quantities of rank 2

$$t^{ij} = t_t^{ij} + (\nabla^j t_v^i + \nabla^i t_v^j) + \left(\nabla^i \nabla^j s - \frac{\gamma^{ij}}{3} \Delta s\right) + \frac{t}{3} \gamma^{ij} , \tag{2.45}$$

where

$$t = t_i^i , \qquad t_{t\ i}^i = 0 , \qquad \nabla_j t_t^{ij} = 0 , \tag{2.46}$$

which tells that t is the trace of the tensor perturbation and that t_t is traceless and transverse. Furthermore we have that t_v^i is the vector component of the tensorial perturbation while the scalar one is obtained by differentiating twice the scalar function s. Finally, γ_{ij} is the metric of the spatial hypersurface.

A great advantage of this decomposition is that in a FLRW space-time the equations for the different categories do not mix to linear order so that the analysis of linear perturbations can be separately studied for the single perturbations components [12].

To further simplify the analysis we can recall that in the linear regime a convenient description of perturbed quantities can be made in the Fourier space, given that different modes do not mix. We then define the scalar Fourier amplitude $A(\tau, k)$ as

$$A(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} A(\tau, \mathbf{k}) Y(\mathbf{x}, \mathbf{k}) , \tag{2.47}$$

where $Y(\mathbf{x}, \mathbf{k})$ is a complete set of harmonic functions. Analogous quantities can be defined for vectors and tensors. In particular, using the previous decomposition between scalar, vector and tensor quantities, we define the vector and tensor Fourier basis Y_i and Y_{ij} for the scalar component, $Y_i^{(1)}$ and $Y_{ij}^{(1)}$ for the vector one and $Y_{ij}^{(2)}$ for tensors [12].

Now that we have a Fourier basis for perturbed quantities we proceed with the definition of these. We will deal only with scalar perturbations so that from now on we will specialize to this component, referring to the cited works for a full treatment of linear perturbations theory in FLRW space-time.

Metric Perturbations. The perturbed metric can be expanded as $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$. They are defined as

$$\bar{g}_{00} = -a^2(1 + 2AY) , \quad \bar{g}^{00} = -a^{-2}(1 - 2AY) , \tag{2.48}$$

$$\bar{g}_{0i} = -a^2 B Y_i , \quad \bar{g}^{0i} - a^{-2} B Y^i , \tag{2.49}$$

$$\bar{g}_{ij} = a^2(\gamma_{ij}(1 + 2H_L Y) + 2H_T Y_{ij}) , \quad \bar{g}^{ij} = a^{-2}(\gamma^{ij}(1 - 2H_L Y) - 2H_T Y^{ij}) , \tag{2.50}$$

where Y_i and Y_{ij} are respectively the scalar parts of the vector and tensor Fourier basis introduced in (2.47).

SET Perturbations. The perturbed SET can be expanded in a similar way $\bar{T}^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + t^{\mu}{}_{\nu}$ where $t^{\mu}{}_{\nu} \ll 1$. The scalar part components are

$$\bar{T}^{0}{}_{0} = -\rho(1+\delta Y)\,, \qquad \bar{T}^{0}{}_{i} = (\rho+p)(v-B)Y_i\,, \tag{2.51}$$

$$\bar{T}^{j}{}_{0} = -(\rho+p)vY^j\,, \qquad \bar{T}^{i}{}_{j} = (\delta^{i}{}_{j}(p+\delta p) + \sigma Y^{i}_{j})\,, \tag{2.52}$$

where $\delta \equiv \delta\rho/\rho$ is the density contrast, v the perturbation to the four velocity, δp is the isotropic perturbation to the pressure and σ is the anisotropic stress of the perturbed fluid. From the normalization condition for the four velocity we have that the zero-zero component is not an independent perturbation variable as it can be written in terms of metric perturbations, $\bar{u}^0 = a^{-1}(1-AY)$, so that v is the only independent perturbation associated to the velocity field.

With the definition of the perturbed metric we can write the perturbations of the Einstein tensor and relate them to those in the matter SET. The full set of perturbed Einstein equations, divided in components, read

$$\frac{2}{a^2}\left[3\mathcal{H}^2 A - \mathcal{H}kB - 3\mathcal{H}H_L' - k^2\left(H_L + \frac{H_T}{3}\right)\right] = -8\pi G\rho\delta\,, \tag{2.53}$$

$$\frac{2}{a^2}\left[k\mathcal{H}A - kH_L' - k\frac{H_T'}{3}\right] = (\rho+p)(v-B)\,, \tag{2.54}$$

$$\frac{2}{a^2}\left[(\mathcal{H}'-\mathcal{H}^2)B - k\mathcal{H}A + kH_L' + k\frac{H_T'}{3}\right] = -8\pi G(\rho+p)v\,, \tag{2.55}$$

$$\frac{2}{a^2}\left[\left(2\frac{a''}{a} - \mathcal{H}^2\right)A + \mathcal{H}A' - k^2\frac{A}{3} - \frac{k}{3}(B' + 2\mathcal{H}B) + \right.$$
$$\left. - \frac{1}{a}(aH_L')' - \mathcal{H}H_L' - \frac{k^2}{3}\left(H_L + \frac{H_T}{3}\right)\right] = 8\pi G\delta p\,, \tag{2.56}$$

$$\frac{1}{a^2}\left[-k^2 A - k(B' + \mathcal{H}B) + \frac{1}{a}\left(aH_T'\right)' + \right.$$
$$\left. + \mathcal{H}(H_T' - kB) - k^2\left(H_L + \frac{H_T}{3}\right)\right] = 8\pi G\sigma\,, \tag{2.57}$$

which are respectively the $({}^{0}{}_{0})$, $({}^{0}{}_{i})$, $({}^{i}{}_{0})$ components and the trace and traceless part of $({}^{i}{}_{j})$ component of the perturbed Einstein equations. This is a set of 5 equations in eight variables that has to be added to the perturbed continuity equations, that read

$$\delta' - 3\mathcal{H}w\delta + (1+w)(vk + 3H_L') + 3\mathcal{H}\frac{\delta p}{\rho} = 0\,, \tag{2.58}$$

$$[h(v+B)]' + \frac{2}{3}k\sigma - h(kA + \mathcal{H}B) + 4\mathcal{H}h(v-B) - k\delta p = 0\,, \tag{2.59}$$

where we have defined $h = \rho + p$.

Before moving on with the analysis of these equations we have to discuss a somewhat subtle issue. When perturbations are introduced there is no unique way to define a co-moving frame. In fact, we may have an observer at rest with respect to perturbations that will see no deviation from the smooth background velocity field. Hence, he would conclude that there are only perturbations in the density of the cosmological fluid. However, this conclusion is only related to its particular frame choice and it is hence unphysical. In order not to draw erroneous conclusions we have to take into account this "gauge" freedom. At linear level this is formally described through coordinate transformation so that two different gauges are related by $(\tau, x) \to (\tilde{\tau}, \tilde{x})$, or

$$\tilde{\tau} = \tau + TY \, , \; \tilde{x}^i = x^i + LY^i \, . \tag{2.60}$$

Under this transformation the metric changes as

$$\tilde{\bar{g}}_{\mu\nu}(\tilde{\tau}, \tilde{x}) \sim \bar{g}_{\mu\nu}(\tau, x) + \bar{g}_{\mu\beta}\delta x^{\beta}_{,\nu} + \bar{g}_{\nu\beta}\delta x^{\beta}_{,\mu} - \bar{g}_{\mu\nu,\beta}\delta x^{\beta} \, , \tag{2.61}$$

so that

$$\tilde{A} = A - T' - \mathcal{H}T \, , \qquad \tilde{B} = B + L' + kT \, , \tag{2.62}$$

$$\tilde{H}_L = H_L - \frac{k}{3}L - \mathcal{H}T \, , \qquad \tilde{H}_T = H_T + kL \, , \tag{2.63}$$

where T and L are the Fourier amplitudes of the coordinate shift δx^0 and δx^i respectively. A similar reasoning holds for the SET of matter leading to

$$\tilde{\delta} = \delta + 3(1+w)\mathcal{H}T \, , \qquad \tilde{v} = v + L' \, , \tag{2.64}$$

$$\tilde{\delta p} = \delta p + 3\frac{c_s^2}{w}(1+w)\mathcal{H}T \, , \qquad \tilde{\sigma} = \sigma \, , \tag{2.65}$$

where we have defined the speed of sound $c_s^2 \equiv dp/d\rho$.

The gauge freedom tells us that of the eight perturbation variables only six are independent. We can thus construct combinations of these variables that are invariant under gauge transformations and write the evolution equations for those.

Another choice is instead to use the gauge freedom to fix two of the perturbation variables to a particular value. This gauge choice can be very useful as in some cases the equations are much easier to solve in some gauge or their physical meaning is clearer.

In the following we will adopt the second criterion and in particular we will use the so called Newtonian or longitudinal gauge in which off diagonal perturbations in the metric are taken to be zero.

2.2.2 Newtonian gauge and non-relativistic limit

The Newtonian gauge corresponds to the choice of no off-diagonal perturbations in the metric and is obtained by the choice $H_T = 0 = B$ and, by including the Bardeen potentials $A = \mathbf{\Psi}$, $H_L = \mathbf{\Phi}$, the metric is

$$ds^2 = -(1+\mathbf{\Psi})dt^2 + a^2(1+\mathbf{\Phi})\delta_{ij}dx^i dx^j \,. \tag{2.66}$$

This gauge choice not only simplifies the equations but also is free of any residual gauge modes(contrarily to the Synchronous gauge) for a flat Universe. Moreover, this gauge is particularly suited for late time and small scales cosmology as the gravitational potentials coincides with the gauge invariant ones. Finally, as it will be clear later, in case of no anisotropic stresses the two gravitational potentials are equal, with a further simplification of the equations.

The perturbed Einstein field equations in this gauge read

$$3\mathcal{H}(\tau)^2\mathbf{\Psi}(\tau,\mathbf{k}) - 3\mathcal{H}(\tau)\mathbf{\Phi}'(\tau,\mathbf{k}) - k^2\mathbf{\Phi}(\tau,\mathbf{k}) = -4\pi G a^2\delta\rho(\tau,\mathbf{k})\,, \tag{2.67}$$

$$k\mathcal{H}\mathbf{\Psi}(\tau,\mathbf{k}) - k\mathbf{\Phi}'(\tau,\mathbf{k}) = 4\pi G a^2 h(\tau) v(\tau,\mathbf{k})\,, \tag{2.68}$$

$$\begin{aligned}&\left(2a'' - \mathcal{H}(\tau)^2\right)\mathbf{\Psi}(\tau,\mathbf{k}) + \mathcal{H}(\tau)\mathbf{\Psi}'(\tau,\mathbf{k}) + \\ &\quad -\frac{1}{3}k^2\mathbf{\Psi}(\tau,\mathbf{k}) - \frac{1}{3}k^2\mathbf{\Phi}(\tau,\mathbf{k}) + -2\mathcal{H}(\tau)\mathbf{\Phi}'(\tau,\mathbf{k}) - \mathbf{\Phi}''(\tau,\mathbf{k}) = 4\pi G a^2\delta p(\tau,\mathbf{k})\,, \\ &\quad -k^2\left(\mathbf{\Psi}(\tau,\mathbf{k}) + \mathbf{\Phi}(\tau,\mathbf{k})\right) = 8\pi G a^2\sigma(\tau,\mathbf{k})\,,\end{aligned} \tag{2.69}$$

while the evolution equations for the fluid perturbations (2.59) become:

$$\delta' \quad +3\mathcal{H}\left(\frac{\delta p}{\delta\rho} - w\right)\delta + (1+w)(vk + 3\mathbf{\Phi}') = 0\,, \tag{2.70}$$

$$v' \quad +\mathcal{H}(1-3w)v - \frac{w'}{1+w}v + \frac{\delta p/\delta\rho}{1+w}k\delta + k\mathbf{\Psi} + \frac{2}{3}\frac{k\sigma}{(1+w)\rho} = 0\,, \tag{2.71}$$

where $\mathbf{\Psi}$ is the Newtonian potential. The system of equations (2.71) and (2.69) provides the evolution of matter and metric perturbation at all scales and times provided that linearity is a valid approximation. Further simplifications to the equations come if there are no anisotropic stresses, hence $\sigma = 0$, and if we consider barotropic fluids for which $p = p(\rho)$. The first request forces $\mathbf{\Phi} = -\mathbf{\Psi}$ as it can be seen from the last of the Einstein equations, in contrast with alternative theories of gravity for which the two potential are different. The second one, together with the assumption of adiabatic initial conditions, instead implies that

$$\delta p = c_s^2\delta\rho\,, \qquad c_s^2 = w + \rho\frac{dw}{d\rho}\,. \tag{2.72}$$

The dynamics of the previous system of equations is characterized by a single scale, known as the effective horizon, H^{-1}. As it may easily verified, both in the matter and the radiation dominated eras the horizon grows faster than physical distances. When a given scale is outside the horizon, then growing solutions exist for the density contrast, $\delta \propto a^p$ with $p > 0$. For perturbations inside the horizon, we can further simplify the equations. In fact, for scales such that $k\mathcal{H}^{-1} \gg 1$ we can neglect time derivatives with respect to terms proportional to the momentum k.

Putting together all the simplifications we obtain the following system of equations, namely the Poisson, the Continuity and the Euler linear equation in Fourier space,

$$k^2\mathbf{\Phi} = 4\pi G a^2 \rho \delta\,, \tag{2.73}$$

$$\delta' + 3\mathcal{H}(c_s^2 - w)\delta + (1+w)kv = 0\,, \tag{2.74}$$

$$v' + \mathcal{H}(1-3w)v - \frac{w'}{1+w}v + \frac{c_s^2 k}{1+w}\delta - k\mathbf{\Phi} = 0\,. \tag{2.75}$$

This set of equations can be cast into a single second order equation for the fluid overdensity δ. In particular if we set pressure to zero we obtain the equation for the evolution of the DM overdensity

$$\delta'' + \mathcal{H}\delta' - \frac{3}{2}\mathcal{H}^2\Omega_m\delta = 0\,. \tag{2.76}$$

The set of equations (2.73), (2.74) and (2.75) represents the basic equations that governs the formation of structure in the Universe when the perturbations are still small and will serve as initial conditions for subsequent non linear analysis.

2.3 Observational evidences for the ΛCDM model

We have today at our disposal an impressive amount of data from very different scales and epochs to which we can compare the predictions made by the ΛCDM model. It was just some months ago that we had the first release of the Planck data that showed to be in an incredible agreement with the prediction of the standard model of cosmology [13].

We will now present and discuss some of the evidences in support of the ΛCDM model as a standard framework for cosmology, with particular emphasis on DM.

2.3.1 Nucleosynthesis

Big Bang Nucleosynthesis (BBN) is the oldest observational evidence and it gives a picture of the Universe when it was 200 seconds old providing a remarkable way to test our cosmological models. When the temperature of the Universe is larger than ∼MeV there are no nuclei because the production of them is compensated with its destruction by an energetic photon. However, at energies around 0.1 MeV, the rate of weak interaction falls below a threshold allowing for a stabilization in the formation of light nuclei. After the freeze out of the weak interaction responsible for the thermal equilibrium, the amount of primordial light elements simply scales as a^{-3} to the present day value.

A particularly important quantity is the Helium abundance $Y \equiv 4n_{^4He}/n_b$. This is measured with precision to be

$$Y = 0.2477 \pm 0.0029\,, \tag{2.77}$$

$$Y = 0.250 \pm 0.004\,, \tag{2.78}$$

as reported in [14,15] respectively. These values are in good agreement with the theoretical prediction [16, 17]

$$Y = 0.2486 \pm 0.0002\,. \tag{2.79}$$

The main dependence on cosmological parameters is represented by the baryon abundance $\Omega_{b0}h^2$ and hence BBN can be used to cast constraints on the amount of baryons in the Universe. However, this can be nowadays inferred from other observations, like CMB anisotropies. Hence BBN is basically a parameter free process that can be used either as a consistency check for the ΛCDM model or as a test for alternatives. In particular, the existing limits tell us that dark, unobserved baryons, cannot play the role of DM, thus reinforcing the non-baryonic origin of the latter. Another interesting thing to note is that scalar-tensor theories of gravity generically induce a change in the Hubble parameter, as they add a time dependence to the bare gravitational constant hence producing a change in the redshift at which BBN onsets and hence a change in the present day abundance of the light elements [18].

2.3.2 CMB

The CMB is probably the most precise evidence for the ΛCDM model [2]. This relic radiation, whose present day temperature is measured with great accuracy to be $T_0 = 2.7255 \pm 0.0006K$ [19], is the living evidence for the phase of the Universe history when electrons and photons were in thermal equilibrium. Its existence represents a fundamental evidence for the expanding Universe and its high degree of homogeneity, $\Delta T/T \sim 10^{-5}$ is a great confirmation for the assumed initial homogeneity and isotropy.

However, the vast majority of the informations on the Universe is encoded in the small anisotropies in the CMB, $\Theta(\vec{x}, \hat{p}, \tau) \equiv \Delta T/T$. Given that we observe them from a single

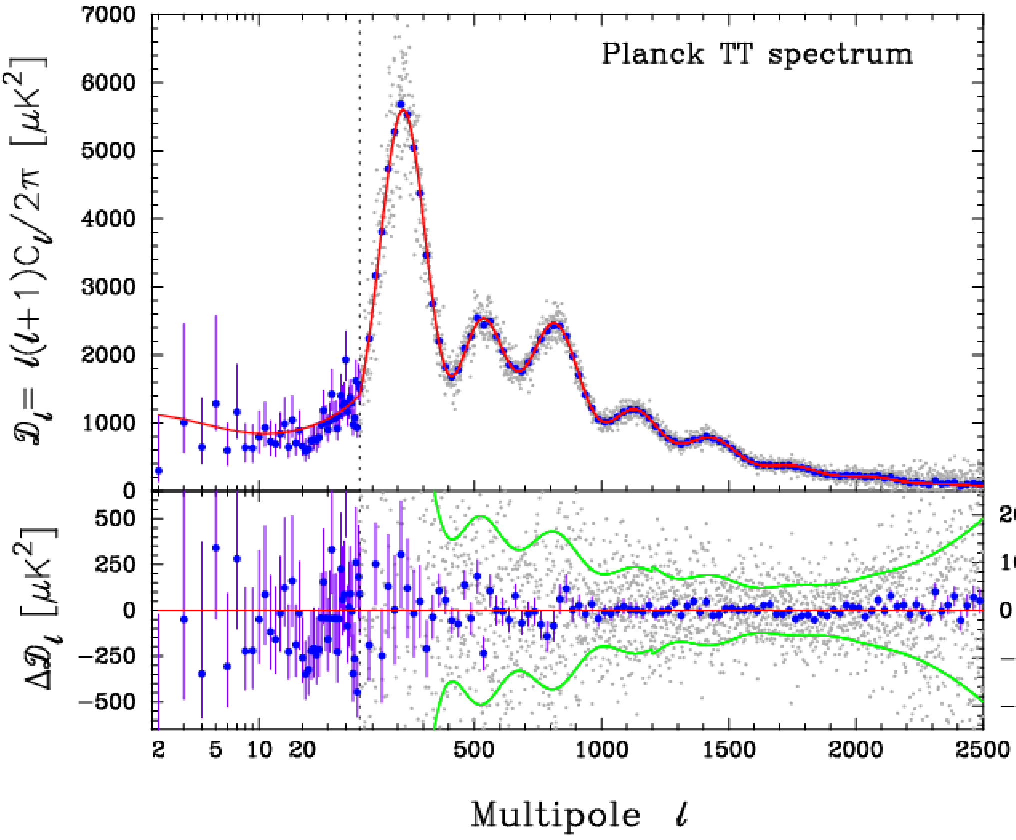

Figure 2.1: The Planck temperature power spectrum along with the Planck error bars in blue [2]. The red line shows the temperature spectrum for the best-fit base ΛCDM cosmology while the lower panel shows the power spectrum residuals with respect to this theoretical model. The green line represents a cumulative uncertainty made of instrument noise, sample (Gaussian) variance and angular resolution. See text for a detailed description of the CMB temperature pattern.

position it is very useful to perform an expansion in terms of spherical harmonics

$$\Theta(\vec{x},\hat{p},\tau) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\vec{x},\tau)Y_{\ell m}(\hat{p})\,. \tag{2.80}$$

The single coefficients of the expansion of temperature anisotropies are thought to obey a Gaussian statistics, as Planck has recently confirmed [2]. Therefore, what matters is the angular power spectrum (PS), *i.e.,* the variance C_l of the distribution at each angular scale l. The latter is the quantity which is used for direct comparison with the ΛCDM model and the derivation of the quoted constraints on cosmological parameters. Taking the ensemble average of the Gaussian distribution, one has:

$$\langle a_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}\mathcal{C}_\ell\,. \tag{2.81}$$

These are plotted in figure 2.1 in terms of the convenient normalization $\mathcal{D}_\ell = \ell(\ell+1)\mathcal{C}_\ell/2\pi$. We give now a brief description of the plotted red curve, representing the best fit. At low ℓs, which corresponds to large angles in the sky, the power is given by the spectrum of initial conditions, since no microphysics could have affected those large scales, corrected by the gravitational potential affecting photons, the so called Sachs–Wolfe effect. Then, at $\ell \geq 100$, corresponding to scales that crossed the horizon at the time of decoupling, the spectrum shows a series of acoustic oscillations. These are a picture, taken at the time of decoupling, of the oscillation pattern in the photon-baryon fluid, with the odd peaks corresponding to compression while the even ones to rarefaction of the fluid, under the effect of the potential wells provided by the DM overdensities. Both positions and amplitudes of the peaks are a manifestation of the underlying matter content of the Universe. In particular, the position of the first peak is associated to the sound horizon, the distance travelled by a sound wave, at decoupling. Hence, the fact that it is found at $\ell = 220$, corresponding to one degree in the sky, requires a particular balance between the amount of baryons and photons. Moreover, the differences in the amplitude of the spectrum at different scales, requires the presence of a non-baryonic matter component about 6 times more abundant than the baryonic one. Finally, at the largest multipoles, the power goes to zero as a consequence of the radiation diffusion that erases the anisotropies.

The CMB contains informations that are not related only to the time of decoupling. On the one hand, the knowledge of the primordial perturbations spectrum and of the dynamics that generates the CMB pattern, are at present one of the most powerful tools to constrain the cosmological parameters. On the other hand, we have to consider that the CMB radiation had to travel a long distance from the surface of last scattering up to

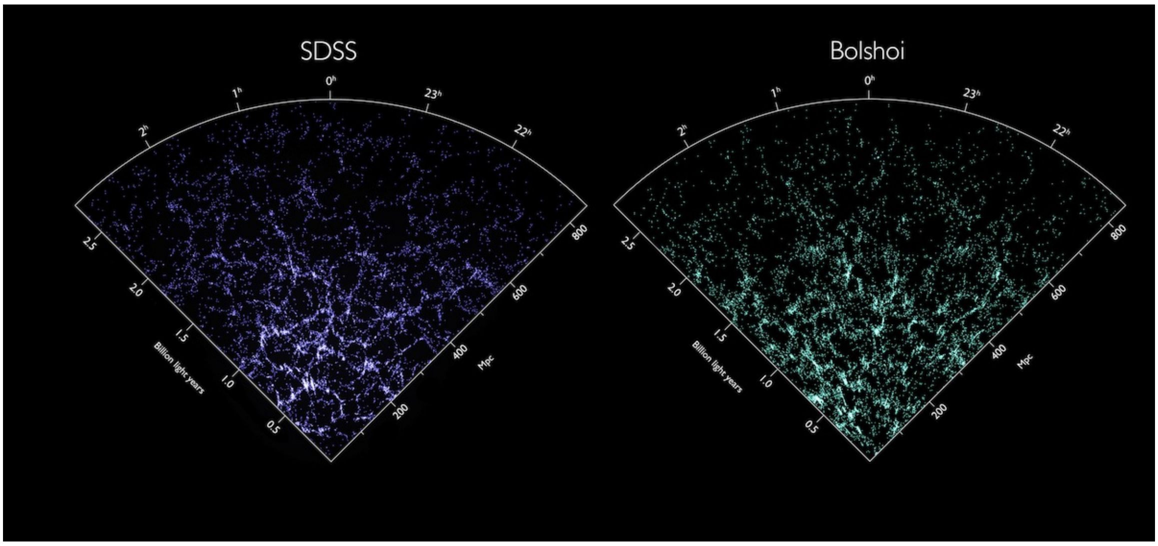

Figure 2.2: Galaxy distribution as obtained from the SDSS [20] (left) and the one extracted from the Bolshoi simulation [21] (right). The high degree of resemblance between simulated and real galaxy distribution is a clear evidence of the accuracy of the ΛCDM paradigm.

us, going through very different stages of the Universe evolution. In particular, DM perturbations act as lenses and are responsible for the CMB gravitational lensing effects and an Integrated Sachs-Wolfe (ISW) effect arises because of evolving gravitational potentials along the line of sight. Both effects have been detected and their relevance in cosmology is being studied (see [2] and references therein). In particular, the latter is can discriminate between GR and alternative theories of gravity.

Hence, fitting data coming from the CMB requires not only a precise knowledge of the physics at the time of decoupling but also a recipe for the initial stages of the Universe as well as a paradigm for structure formation. The fact that the ΛCDM fits so well the CMB spectrum represents a remarkable achievement of the corresponding modelization.

2.3.3 LSS and BAO

Another important source of cosmological information comes from the observation of the distribution of matter in the Universe. In fact, its isotropy at scales above 100 Mpc is a further confirmation of the assumptions on the isotropy of the distribution of fluctuations of the Universe. On the other hand, on smaller scales, gravitational dynamics tends to clump matter into more irregular structure, clusters and galaxies, which can be used to test our model of structure formation.

In the ΛCDM model the need for DM perturbations is required in order to evolve the initial $\sim 10^{-5}$ overdensities to the observed value for the density contrast of virialized objects at present time, roughly ~ 100. Perturbations in the baryonic component cannot be the sole actors as their growth only starts after they decouple from radiation at $z_{dec} \sim 1000$ as at earlier times they are in thermal equilibrium with the photon bath. DM instead can start to collapse much earlier, but not before z_{eq} when the gravitational effects of the pressure from relativistic species prevent DM to collapse.

During matter domination and at scales much smaller than the horizon the evolution of the perturbation in the DM component is almost scale invariant as far as linear theory is concerned and is ruled by the equation

$$\delta''_{\mathrm{DM}} + \mathcal{H}\delta'_{\mathrm{DM}} - \frac{3}{2}\mathcal{H}^2\Omega_{\mathrm{DM}}\delta_{\mathrm{DM}} = 0\,, \tag{2.82}$$

where primes denotes differentiation with respect to conformal time (2.23). During the first stages of matter growth the contribution from dark energy is negligible and the Universe is very close to an Einstein-de Sitter model with $\Omega_c \sim 1$. In this case perturbations in the DM component have a linear growing mode $\delta_{\mathrm{DM}} \propto a$.

Very similarly to what we did for the CMB, we can introduce the DM PS defined as the Fourier transform of the two point correlation function of the DM density contrast $\delta_{\mathrm{DM}}(t, \mathbf{x})$

$$\langle \delta_{\mathrm{DM}}(\mathbf{k})\delta_{\mathrm{DM}}(\mathbf{k}')\rangle = P_{\mathrm{DM}}(k)\delta_D(\mathbf{k} - \mathbf{k}')\,, \tag{2.83}$$

where the Dirac delta δ_D is introduced in order to conserve energy and the average is taken assuming a Gaussian and isotropic statistics.

The above expression is a definition given at a fixed time. If we want the time evolution of the PS we need a recipe on how to evolve overdensities through horizon crossing and matter-radiation equality since the evolution of perturbations in these regimes strongly depends on scale. This is done introducing the transfer function $T(k, a)$ so that

$$P_{\mathrm{DM}}(k, a) = T(k, a)^2 P_{\mathrm{DM,p}}(k)\,, \tag{2.84}$$

where p refers to the primordial spectrum and a is, as usual, the scale factor. As we describe below, the transfer function describes the super-horizon evolution on large, super-horizon scales, while for those re-entering the horizon before equivalence, a suppression is caused by acoustic oscillations.

Unfortunately we do not observe the DM power spectrum. What we observe is the galaxy power spectrum in redshift space. One needs to introduce baryons' physics in order to obtain the theoretical galaxy power spectrum from the DM one to be compared with the

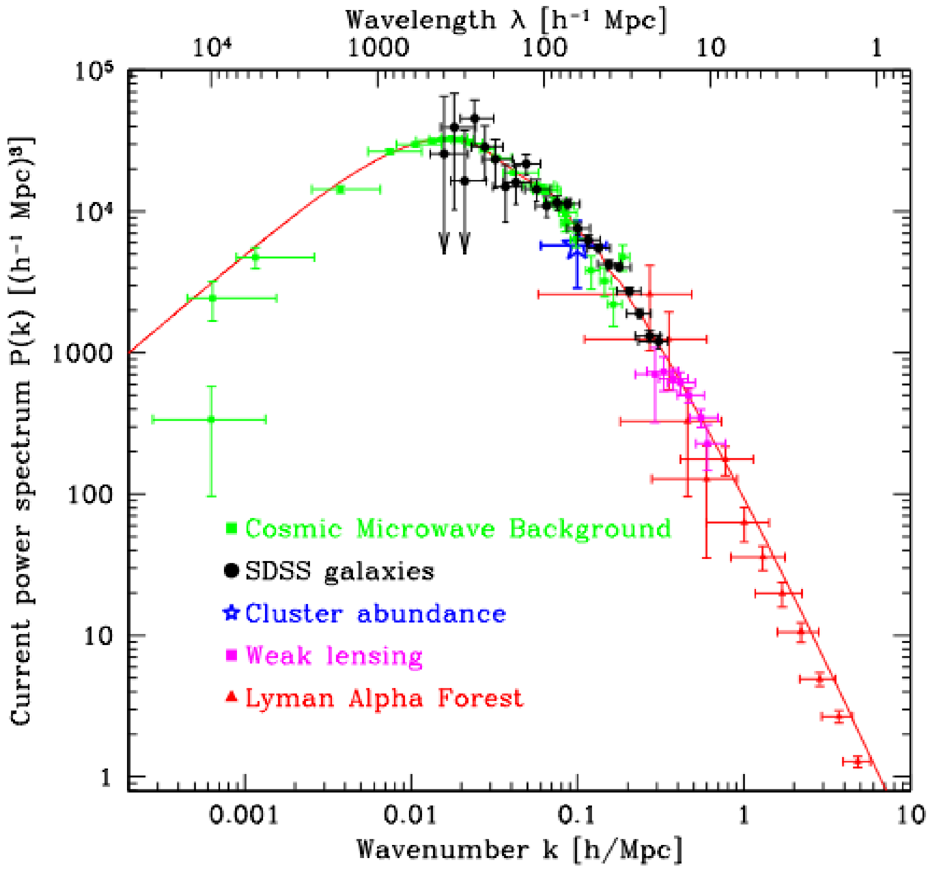

Figure 2.3: The galaxy PS as obtained in [24], compared with the PS obtained from other astrophysical and cosmological sources. The almost perfect overlap is a clear evidence of the common origin of the perturbation in different matter species and can be used to obtain informations on the evolution of the Universe. (See text for the details).

observed one. In figure 2.2 are reported the observed galaxy distribution [20] (left image) and the one obtained from the Bolshoi simulation [21] (right image). The concordance between the two is astonishing making indistinguishable by eye their different origin.

As it is what we actually observe, we now turn our attention to the distribution of visible matter. The distribution of baryonic matter has been measured with great accuracy by looking at different kinds of sources [20,22,23]. In figure 2.3 we report the matter power spectrum as obtained in [24]. This spectrum shows the typical Harrison–Zel'dovich scale invariant (almost proportional to k) growth up to ks corresponding to matter-radiation equality. These scales are those that entered the horizon during matter domination and hence started to grow in a way proportional to the scale factor a. For scales that entered the horizon before matter-radiation equality, k larger than $k_{\rm eq}$, the spectrum shows the usual suppression, due to the dominance of the relativistic pressure, that lasts until radiation era ends. In the latter regime, an oscillatory pattern have been measured by the surveys SDSS [24], 2dFGRS [25] and more recently by BOSS [26,27]. These are the counterpart of the acoustic oscillations present in the CMB spectrum, the well known Baryonic Acoustic Oscillations (BAO). The characteristic scales imprinted in these oscillations are the same as those of the CMB, only evolved in time. Hence, they can be used, together with CMB ones, as a standard ruler against which to test the cosmic evolution from last scattering to present day.

2.3.4 Hubble law

The distance between an observer and some faraway astrophysical object is not constant in time as a consequence of the predicted expansion of the Universe. We have in fact that the physical relative distance evolves according to

$$r = a(t)x\,, \tag{2.85}$$

where x is the comoving distance. From this we can compute the velocity of an object

$$\dot{r} = Hr + a\dot{x} = v_H + v_p\,, \tag{2.86}$$

where the first terms is the velocity due to the expansion of the Universe while the second is the peculiar velocity. The latter can be as large as thousands of km/s for galaxies inside clusters. This requires to look at scales larger than tenths of Mpc, corresponding to $z \gg 10^{-2}$, where peculiar motions are negligible. Moreover, we have to consider redshifts smaller than one, so that a perturbative expansion of the velocity distance relation around present day value is consistent. Under this constraints we obtain the Hubble law

$$v \sim H_0 r\,. \tag{2.87}$$

which states that distant galaxies should recede from us at a speed which is proportional to their distance. Given that it is very unlikely that we occupy a privileged position in the Universe the observation of such receding speed was the first indication of the cosmological expansion. In order to measure H we need to measure distances, which is generally a very hard task and requires the existence of some standard indicator. Two very important astrophysical objects of that kind are the Supernova Ia and the Cepheids, that have a luminosity that is quite independent on the details of the individual object. Using these standard candles it has been possible to measure the distance as a function

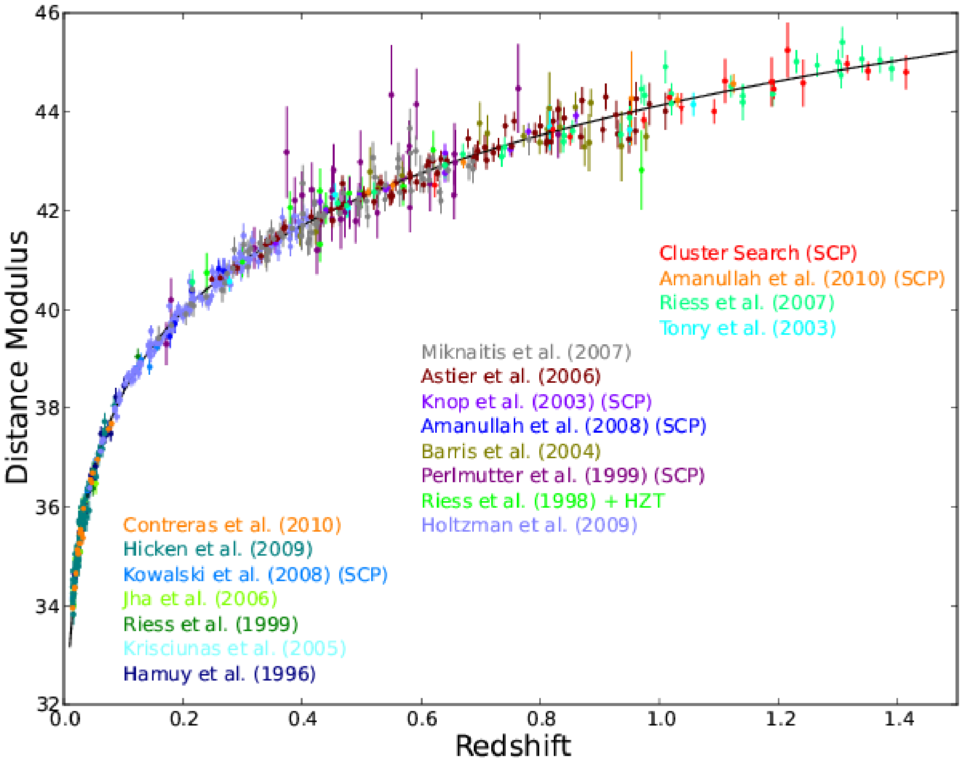

Figure 2.4: Hubble diagram for the Union 2.1 compilation. The solid line represents the best-fit cosmology for a flat ΛCDM Universe for supernovae alone [28].

of redshift and infer the value of H_0, producing a quite stable value for this fundamental parameter [7, 28]. The first measurements were limited to small redshift ($z < 0.1$) where cosmological evolution can be neglected. However, the discovery of standard candles at larger redshift opened up the possibility to test cosmological models with this parameter. These observations brought to the discovery of the accelerated expansion in 1998 [29–32] providing the first evidence of DE. In fact, as can be seen from figure 2.4, the predictions of ΛCDM model are in agreement with observations.

2.3.5 Dark Matter in galaxies

The first evidence of the need of a dark component to account for gravitational dynamics dates back to 1933 [33], when it was observed that the visible mass was not enough to explain the individual dispersion velocity of galaxies in the Coma cluster. This investigation was further extended during '70s and '80s [34–36] with the observations of the rotation curves of galaxies, providing further confirmations of the velocity anomaly.

All these astrophysical observations add up to the already mentioned evidences for DM coming from cosmological observations and altogether form a coherent framework in the context of ΛCDM. Moreover the astrophysical observations are possibly stronger than those coming from cosmology as their dependence on the assumptions on the cosmological model is somehow reduced.

Galaxy Rotation curves. Galaxies are bounded objects that have reached virial equilibrium and hence the velocity of stars at their interior is in equilibrium with the gravitational attraction. In particular, for spiral galaxies the gravitational mass of the galaxy can be obtained measuring the rotational velocity via Doppler shift. According to Newtonian gravity, if the mass of a galaxy is made mostly by its luminous components we have that the velocity field outside the matter distribution should show the Newtonian fall off $v \propto r^{-1/2}$ meaning that luminous objects trace well the matter distribution. However, as it is clear from figure 2.5, observations show an almost constant velocity profile well beyond the galaxy's core [37]. This behavior is described by adding the extra mass profile

$$m(r) \propto r\,. \tag{2.88}$$

which can be explained assuming that galaxies resides at the core of a DM halo, whose size extends far beyond the galaxy's one, providing another evidence for the presence of non-baryonic matter. The systematic investigation of these features brought to the discovery that the profile of the rotation velocity of stars in spiral galaxies has a universal profile leading to the so called Universal Rotation Curves (RC) paradigm [38, 39]. This is in agreement with the predictions coming from pure DM simulations since these predicts the existence of a universal profile for DM density which directly translates into a universal asymptotic rotation velocity for galaxies.

Radial Tully–Fisher Relations. In 1977 Tully and Fisher [40] proposed a new method for measuring the absolute luminosity of astrophysical objects based on the measurement of the width of the global neutral hydrogen line profile in galaxies. This led to the famous Tully–Fisher relation (TFR) between velocity and luminosity

$$M = a \log V_{\rm Max} + b\,, \tag{2.89}$$

where M is the absolute magnitude, $V_{\rm Max}$ is the maximum rotation velocity while a and b are the slope and the offset of the TFR. In particular, it was found that spiral

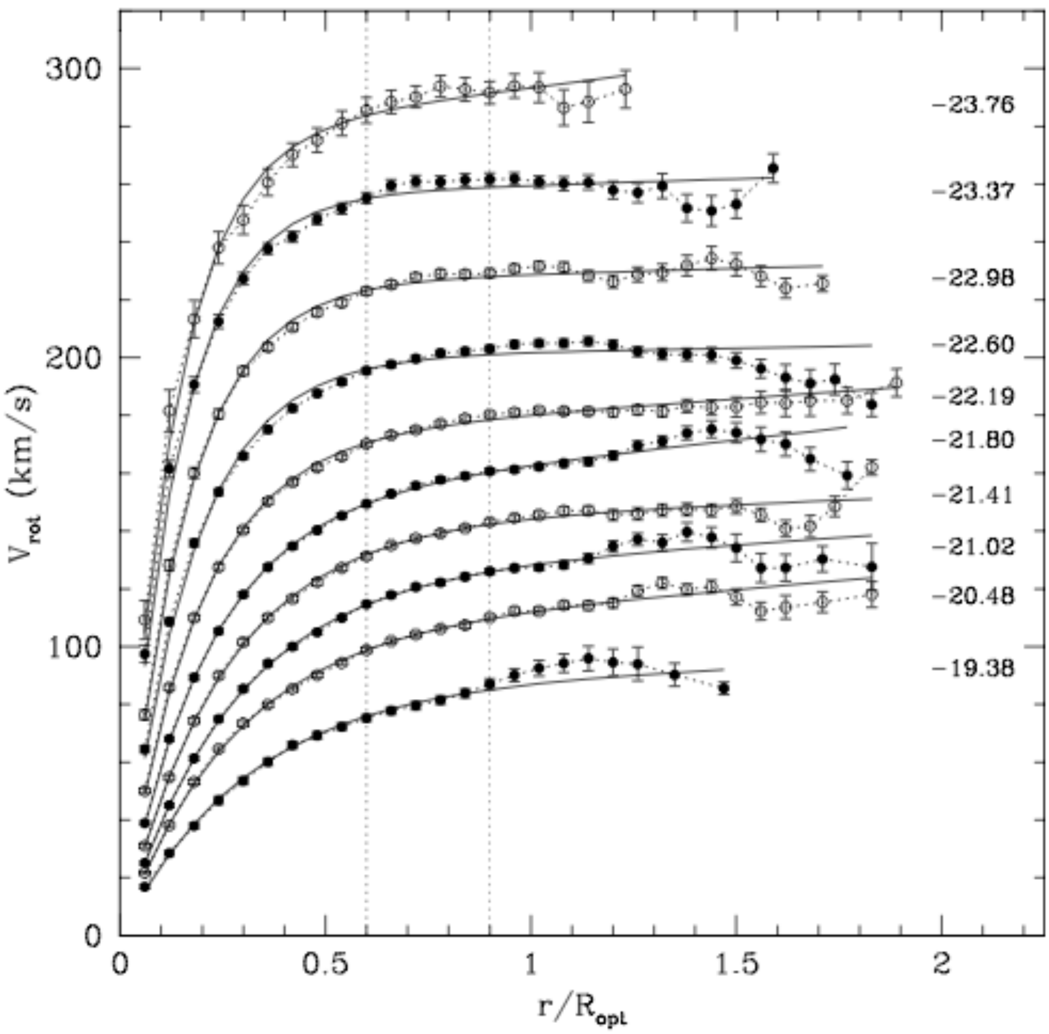

Figure 2.5: Galaxy RCs parameterized as functions of optical radii. This sample includes 2169 RCs extending beyond 0.6 $R_{\rm opt}$, $R_{\rm opt}$ being a characteristic scale related to the size of the luminous matter distribution. Notice the almost flat profile for large radii which contrasts with the expected Newtonian fall-off of the rotation velocity. The plot is taken from [37].

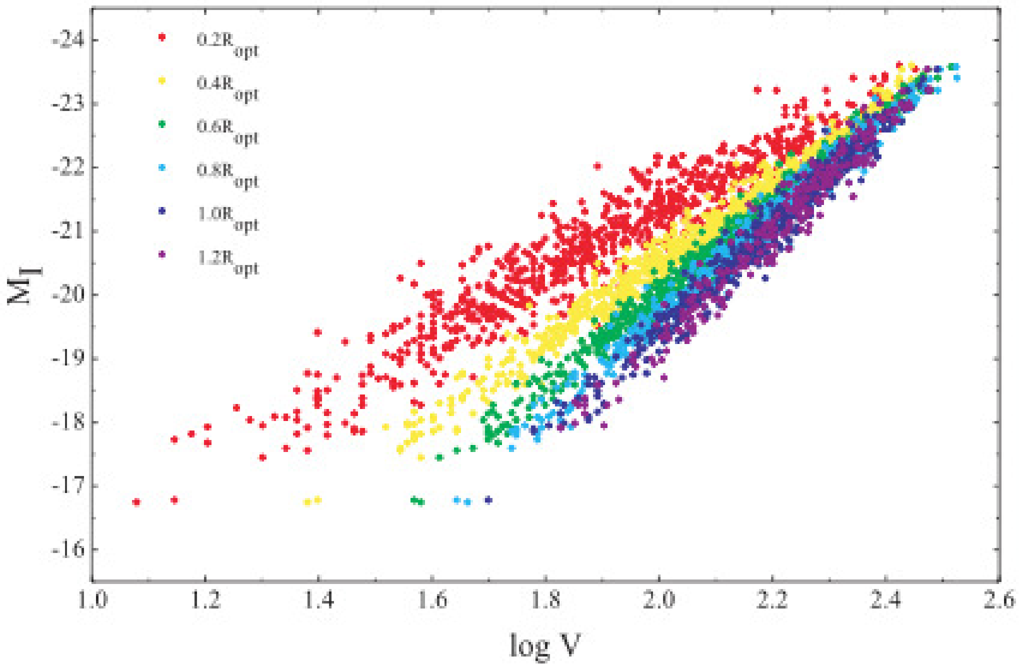

Figure 2.6: The RTF relations [41]. Each one of the RTF relations is indicated with different colors and it represents a TF-like relation at different radii.

galaxies share the common value for the slope parameter $a = 4$, which implies the proportionality law $M \propto v^4$. This relation is already intriguing per se as it links together two different quantities: luminous mass and gravitational mass. However, more recently it was found that spiral galaxies do possess what has been called a radial Tully–Fisher relation (RTF) [41]:

$$M = a_i \log V(R_i) + b_i \tag{2.90}$$

where the index i indicates different radii. This is a set of Tully–Fisher relations whose exponent changes alongside with the radial coordinate. These relations are plotted in figure (2.6) and it represents a very robust confirmation of the presence of a DM component. In fact, the particular change in the slope parameter a implies that light does not follow matter and that the dark component required to reproduce this RTF relation is more relevant at larger radii.

Bullet cluster. The discovery of a cluster collision from which it was possible to extract the mass center position of intracluster gas, DM and that of galaxies has been considered as a strong evidence of the DM paradigm, *e.g.,* against modified grav-

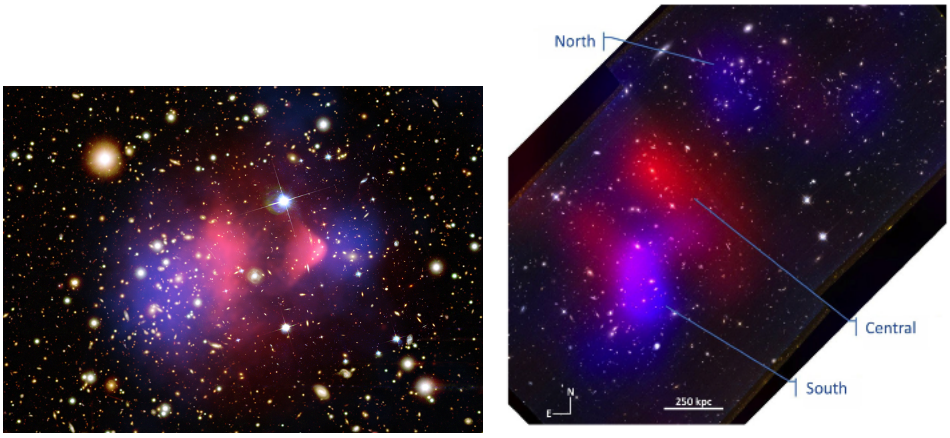

Figure 2.7: Left panel: the gas (violet), mass (blue) distribution in the bullet cluster. Right panel: same as left but for the Musket cluster.

ity predictions, and it has been used to put constraints on the interaction cross section-per mass of DM. In figure 2.7 two of such collisions are shown with in red the positions of the gas, as inferred from X-ray observations, and in blue the position of DM halos as inferred from gravitational lensing. The position of the galaxies that belong to the clusters are also inferred, even if it cannot be seen by eye from the figure. In both images it can be seen how the gas, due to its high friction, is still at the position of the clash. Galaxies are expected to behave as collisionless objects and to pass through the collision unaffected. Hence, the position of their center of mass can be used as a marker for the position of the DM halo.

The left image refers to the Bullet cluster [42, 43] for which the center of mass of DM and galaxies was found to coincide fact that the was considered an evidence of the collisionless nature of DM. Notice however that the high initial velocity required to explain the observed speed of the collision has been considered as a challenge for ΛCDM model [44] and various alternative explanations have been investigated in the context of both alternatives to DM [45] and DE [46]

Recently, a new cluster collision has been observed [47] which seems to point towards a different interpretation (right image in 2.7). In fact, this collision is found to be older than the previous one, and an offset in the position of galaxies and DM is found. A possible explanation of this being that DM may have a larger self-interaction, able to produce the friction needed to explain the observed mass configuration.

2.3.6 CDM candidates and their search

Despite its yet unknown particle origin, decades of observations have provided a certain amount of properties that a good DM candidate has to satisfy: it cannot be of baryonic origin, in order to satisfy BBN constraints, it must be cold, *i.e.,* non-relativistic, with very small velocity dispersion, massive enough as to match the observed bottom up structures' formation and it must interact very weakly with the particles of the standard model.

Here, we will briefly review some of the possible DM candidates and their searches. We refer to [48–50] and references therein for a thorough discussion.

WIMP Dark Matter

The non-baryonic nature of the CDM seemed to find a perfect candidate in the Super Symmetric (SUSY) extension of the SM. In fact, besides addressing many of the SM problems at the Electro-Weak (EW)scale, it provides a viable candidate for DM. The lightest supersymmetric particle (LSP) is stable, weakly interacting and it is able to provide the correct structure formation process and the right observed dark matter relic abundance. This Weakly Interacting Particle (WIMP) was so successful that the expression "WIMP Miracle" was coined. The most promising candidate is the neutralino, a mixture of four neutral fermions: the wino, the bino and the two Higgsinos, namely $\chi = \alpha\tilde{B} + \beta\tilde{W}^3 + \gamma\tilde{H}_1 + \delta\tilde{H}_2$, where α, β, γ and δ are parameters of the model [51].

Alternatives to SUSY have been proposed to explain physics at the EW scales which happens to contain viable DM candidates. In particular, models like Kaluza–Klein DM [52], little Higgs DM [53], Mirror DM [54] have new symmetries which provide stable particle states at the same time preventing interaction with standard model ones.

Non-WIMP DM

WIMP DM is fundamentally related to the EW scale and hence to weak interactions. However, one can think of particles that are disconnected from this scale and that have no restrictions on the smallness of the interactions strength with known SM particles. The discovery of the mass of neutrinos is the most relevant evidence for physics beyond the SM. In the context of effective field theory description, the mass matrix of neutrinos is $m_\nu = A_{\alpha\beta}v^2/\Lambda$, where $\alpha = e, \mu, \tau$, $v^2 = 174$GeV is the vacuum expectation value of the Higgs field and Λ is some high energy scale suppressing the operators. One possible explanation for the presence of these non-renormalizable terms can be associated to the existence of a new particle in the form of a Majorana fermion singlet under SM group.

From the cosmological point of view the most relevant fact is that sterile neutrinos have masses around the keV scale and hence are not CDM but rather Warm DM (WDM). These have major consequences on structure formation and this kind of DM may be a good candidate to solve some of the ΛCDM issues at small scales (see chapter 3).

Another DM candidate comes from an apparently unrelated framework. The strong Charge-Parity (CP) problem of the standard model of particle physics is associated to the presence of a CP violating term in the Quantum Chromo Dynamics (QCD) Lagrangian. Current constraints on CP violations force the parameter characterizing this extra contribution to the QCD Lagrangian to be of the order of 10^{-9}. In order to explain in a natural way this value, Peccei and Quinn [55, 56] introduced a global U(1) quasi-symmetry, *i.e.,* a symmetry at classical level which is broken by non-perturbative effects. The axion emerges as the quasi-Nambu–Goldstone boson associated to the spontaneous breaking of this symmetry. The axion is a CDM candidates [57] and its mass, given in terms of the axion decay constant f_a

$$m_a \sim 0.6eV \frac{10^7\mathrm{GeV}}{f_a}\,, \tag{2.91}$$

is constrained to be in the range $10^{-2} - 10^{-6}$eV from various particle physics experiments and astrophysical observations [58, 59].

Another line of research is that of minimal extensions of the SM like models with a single scalar field which is singlet under SM gauge group [60–62].

Present day status of direct and indirect dark matter searches

Astrophysical evidences in favor of the DM hypothesis and the large number of theoretical candidates have motivated the search for DM particles through other, more direct, channels. In particular, experiments that aim at a direct detection of DM are of extreme importance as such detection would be the final confirmation of the existence of DM and could be used to explore its properties with great consequences in the cosmological set up. In particular, if dark matter is weakly interacting a direct detection via interaction with standard model particles could be achieved while DM could be effectively created in particle colliders such as the Large Hadron Collider (LHC). Another way to detect DM particles is through the detection of the products of their annihilation in high density regions like the core of the Sun or DM dominated galaxies.

None of the ongoing experiments has been able to provide a definitive proof of DM detection, and at the moment only constraints on DM candidates properties are available.

LHC search. The LHC is a proton-proton collider whose purpose is to shed light onto the physics of the electroweak symmetry breaking. In the context of minimal supersymmetric model (MSSM) the lightest particle is stable and could provide a viable DM candidate whose detection may be at reach at LHC. Of course, a direct detection of a DM particle is not feasible as cosmology requires DM to be stable, but such particle could be detected as missing energy in the final products of some SM particle decay or via anomalous branching ratios in known decay channels. Despite the first LHC run showed no evidence for MSSM particles [63] still it was able to further reduce the parameter space of various MSSM in particular when combined with constraints coming from other experiments [64–69]. However, even in the case of no detection at LHC, a possibility often referred to as the "nightmare scenario", there still be room for MSSM DM, which can be explored through other experiments [70]. It is important to notice that the LHC DM search is very model-dependent, being related to MSSM particles. Hence no detection of viable DM candidates in this experiment would by no means imply the non existence of DM.

Direct detection. A WIMP dark matter candidate may be directly revealed exploiting the DM particle flux that flows through the Earth as a consequence of its motion inside the galactic halo. This kind of measurement is extremely difficult to achieve as a consequence of the large number of uncertainties in the DM astrophysical parameters (local density, DM velocity, Earth motion) and in the magnitude of the signal. However, several experiments are currently running in underground laboratories [71–74]. Among them only the DAMA-LIBRA experiment is claiming to have seen a clear evidence for DM: an annual modulation in the energy released in the detector which may be imputable to DM-nuclei collisions. Despite being a rather model independent test for the existence of DM there is not general consensus on this result, and at the moment only constraints on the DM-nucleus cross section for specific models are available as a function of particle mass (see fig 2.8).

Indirect detection. If DM annihilates or decays, it is then possible to detect the products of such interaction in the form of excesses of particle flux over the expected astrophysical backgrounds. In fact, several experiments (see [75] for a recent review) have observed an excess of the positron fraction [76, 77] which sounded like an evidence of DM. More recent data [78], shows how this excess is somehow reduced but that it is still present and its origin may be related to new physics even if it an explanation in terms of astrophysical sources is as well a possibility [79, 80]. Finally, if the excess is due to some exotic particle decay it should be accompanied by an analogous one in other annihilation channels and, for example, the observation

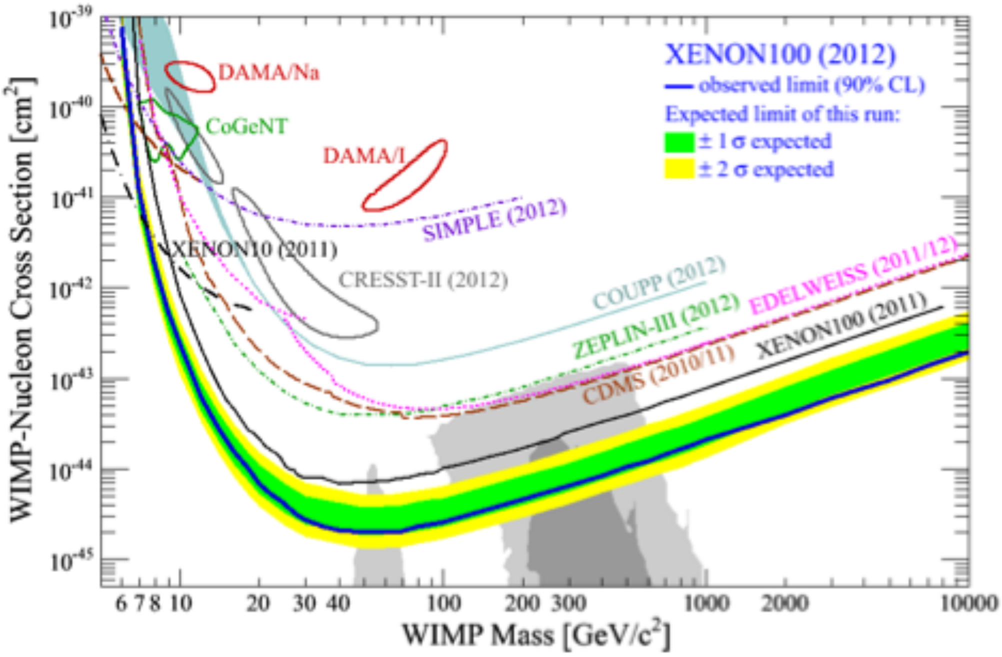

Figure 2.8: Current constraints on dark matter direct detection [74]. The best constraints come from the XENON 2012 data (blue line). Notice that the DAMA/LIBRA detection region (red contours) lies outside the region excluded by other experiments.

of antiproton flux may cast constraints on DM models [81]. Interestingly, Planck measured the emission from the Galactic "haze" at microwave wavelengths finding a morphology compatible with that of Fermi [82].

Chapter 3

Beyond ΛCDM?

In the previous chapter we have discussed some of the strongest observational evidences in support of the ΛCDM model. We have also seen that the recent Planck survey results do not show any significant deviation from it.

However, both theoretical and observational issues are challenging this picture, calling for a deeper understanding of the dark side of the Universe. The CC is plagued by the coincidence and fine tuning problems, while the standard CDM paradigm has problems on small, galactic scales, as anticipated in the Introduction. These issues may be considered as need for alternatives to CDM and to the CC.

3.1 The cosmological constant problem

The history of CC dates back from the first days of GR and was introduced by Einstein himself in the attempt to obtain a static Universe from his dynamical theory for space-time. After the realization that such Universe was unstable the CC disappeared from the scene for several decades. However, it was then realized that everything contributing to the vacuum energy produces a cosmological constant like term $\sim \lambda g_{\mu\nu}$. A simple and direct calculation of the approximate value of such a contribution leads to the incredibly high value for the energy density of the CC $\rho_\Lambda \sim 10^{71}\text{GeV}^4$ which is obtained by combining the Gravitational, Planck and speed of light constants. If a CC with such high value was present in the early universe no structure could ever been formed. In order to make such large value compatible with the measured value of the CC $\rho_\Lambda \sim 10^{-47}\text{GeV}^4$, one has to require that the CC appearing in the Einstein equations fine tunes to the vacuum contribution to about 120 orders of magnitude, unless some mechanism exists that sets to zero the vacuum contribution of quantum fields. It may be that a full theory of quantum gravity will set this issue but still the fine tuning problem remains to explain why the cosmological constant we observe is so small compared to all other energy scales of nature. The second problem with the CC is the so called coincidence problem. In fact it is hard to explain why the CC is comparable with that of DM the energy density at $z \sim 1$ despite the different scaling of the energy densities and the fact that the CC dominates for an infinite time in the future (see however [83] for an opposite point of view). The two scales are in principle completely unrelated and this second fine tuning problem can be rephrased as that of why the CC is small enough to allow the formation of the large scale structures we observe.

The DE models were introduced to solve these problems by providing a dynamic mechanism responsible for the observed accelerated expansion of the Universe (see [84] for a thorough review). It is possible to group DE models under two main categories: those that modify the right hand side of the Einstein equations, thus introducing a new matter field, and those that modify the gravity sector, namely the left hand side of the Einstein equations. We will see however how this distinction is not rigid as it is possible to show how some gravity models have an equivalent description in terms of dynamical matter degree(s) of freedom.

DE as a scalar field

The simplest idea is that of promoting the CC to a dynamical scalar field. In this case DE has a varying energy density which is not forced to stay fixed throughout the whole Universe evolution thus relieving the fine tuning problem.

One possibility is represented by the quintessence scenario (QE) [85–87] where the DE is described by a dynamical scalar field, whose properties are encoded in the scalar field potential. According to our previous discussion, at the background level, any matter component is described by its energy density and pressure from which one can construct the equation of state parameter. The condition that the scalar field is able to drive an accelerated expansion can then be written in the form[1]

$$w_{QE} = \frac{P_{QE}}{\rho_{QE}} = \frac{X - V(\phi)}{X + V(\phi)} < -\frac{1}{3} \Leftrightarrow X < V(\phi)\,, \tag{3.1}$$

which defines the different QE models. Note that the above condition is basically the requirement that the so called Strong Energy Condition is violated [88]. The above condition

[1] This is true only if DE is the only component in the Universe. When other matter species are included the condition for the onset of the accelerated expansion refers to the total equation of state parameter and hence is $w_{\rm tot} < -1/3$. Thus the constraint on DE equation of state must be consequently adapted to this more realistic situation.

tells us that the mechanism responsible for the acceleration is analogous to that driving inflation with a slowly varying potential. Remarkably, this model is able to alleviate the fine-tuning problem as it admits an attractor solution that reduces the dependence on the initial conditions.

However, the so called coincidence problem is still present as only a fine tuning in the parameters entering the potential can explain why the energy density of the QE field equals that of matter today.

The possibility of a DE-matter coupling has been widely investigated in the literature. Indeed, this coupling is expected to be generated unless some symmetry is present to prevent it. However, the coupling with baryons is strongly constrained by both local gravitational experiments and from precise measurement of the CMB and of galaxy distribution. On the contrary, the coupling with DM does not have to satisfy such local constraints and can be used as an appealing extension of QE models. Of course, the changes in the cosmological dynamics due to the DE-DM coupling are severely constrained at cosmological scales by the available data, *e.g.,* CMB, BBN or LSS. A general feature of these Coupled Quintessence (CQ) models [89] is that the single SET for DM and DE are not separately conserved

$$\nabla_\mu T^{\mu\nu}_{\rm DM} = -Q T_{\rm M} \partial^\nu \phi \,, \qquad \nabla_\mu T^{\mu\nu}_{DE} = Q T_{\rm DM} \partial^\nu \phi \,, \tag{3.2}$$

where ϕ is the CQ field, $T_{\rm M}$ is the trace of the matter SET and Q is the QE-DM coupling. The most relevant change with respect to uncoupled QE is that the CDM dominated epoch is substituted with a ϕDM era during which perturbations grow slower with respect to standard QE with observable effects on the CMB anisotropies spectrum. At background level we have that, during this era and for a constant coupling Q, the normalized Hubble parameter $E(z) \equiv H(z)/H_0$ evolves as

$$E(z)/E_0 \propto \left[\Omega_{m0}(1+z)^{3+2Q^2}\right]^{1/2} \,, \tag{3.3}$$

which reduces to the standard matter evolution for a vanishing coupling DE-DM.

DE as a modification of gravity

The second approach to DE is to modify the gravitational structure of the theory, leaving unaffected the matter content. There are several indications that GR, despite its successes in describing gravitational dynamics at many different scales, cannot be the ultimate theory of gravity. In this sense the idea that the anomalous expansion of the Universe may be due to a modification of GR at cosmological scales is appealing. Along this direction the first modification that one can look for is that in the form of the Lagrangian of the gravitational sector. The simplest extension of GR Lagrangian, namely the Ricci scalar R, is to use a generic function of it, $f(R)$ [90, 91]. Given that one wants to address late time dynamics, where the Ricci scalar is small, then one has to reduce to

$$f(R) = R - \frac{\alpha}{R^n} \tag{3.4}$$

where $\alpha, n > 0$ are new parameters. However, this is not enough to have a viable model for DE and indeed stringent limits on the freedom of the $f(R)$ function need to be imposed to match observations (see chap. 9 in [84]), thus reducing viable models [92–95]. These have been shown to be able to produce a ΛCDM-like background but they are expected to differ at perturbation level, thus making DE and $f(R)$ models distinguishable and testable with future tomographic surveys [96].

$f(R)$ theories have been shown to be equivalent to a sub-case of scalar-tensor theories of gravity [97]. These theories, which we will discuss in some detail in chapter 6, contain a new scalar degree of freedom for the gravitational interaction which is non-minimally coupled to curvature terms. These models have been recently investigated in the context of Covariant Galileons, [98, 99] and have been proven to be viable mechanisms to produce cosmic acceleration.

There is a strong connection between CQ and scalar-tensor theories. In fact, with some caveats, CQ can be seen as the Einstein frame version of scalar-tensor theories, thus explaining the origin of the scalar field-matter coupling. This link has been explored and extended recently, with the introduction of disformal transformations [100], showing that many models of DE can be connected by metric transformation [101–103]. This classification is consistent within the Horndeski theory [4, 5] since this is general enough to contain all single scalar field DE models as subcases. We will devote the last chapter to the investigation of the actual equivalence of these different theories exploring the symmetries of the Horndeski action under metric transformations.

Before closing this section we note that, up to present, the CC has not been ruled out by observations. However there is room for dynamical DE, and in fact in a recent paper [104] it has been shown how CQ models with a non-zero coupling are consistent with Planck data.

3.2 Challenging the CDM paradigm at small scales

If we were to know the DM particle, we probably wouldn't worry too much about the inconsistencies of ΛCDM at galaxy and cluster scales, referring to them as residual noise, or more than this, we could be able to provide an answer to them. However, the unknown nature of the main actor of structure formation lead naturally to think about the discrepancies between theoretical predictions and observations as hints towards the fundamental nature of DM.

At the above mentioned scales, the evolution of the perturbation can be described in terms of some (post-) Newtonian approximation, thus avoiding the complications of a full GR treatment. However, virialized objects have typical densities 200 times the background one, with the obvious consequence that linear theory cannot work anymore. Moreover, baryons cannot be considered as a pressureless fluid and the high level of complexity of the hydrodynamics of galaxy formation needs to be accounted for. For these reasons the largest source of information on the formation and evolution of structures at such small scales come mostly from large N-body numerical simulations, even though semi-analytical techniques, like spherical collapse, allow to grasp some relevant information about this regime.

We can classify the ΛCDM issues into two categories. On one side there are the discrepancies between observable quantities extracted from simulation and the ones effectively observed. On the other there exists sets of observational evidences that may be considered as hints for new physics in the DM sector.

The core-cusp problem. One of the most relevant quantities that can be extracted from simulations and directly compared with observations is the dark matter halos density profile. For a CDM model this turns out to be well reproduced by the Navarro–Frenk–White (NFW) profile [105]

$$\rho_{NFW}(r) = \frac{\rho_s}{r/r_s(1 + r/r_s)^2}\,, \tag{3.5}$$

where the two parameters ρ_s are r_s are a density and radius parameter respectively and are obtained from best fit of the DM distribution. At small radii this profile goes like r^{-1}. However when compared with the profile extracted from observations this appears to be too cuspy in its interior [106, 107] and in fact, a more suitable choice for the observed DM distribution seems to be the empirical Burkert profile [108]

$$\rho(r) = \frac{\rho_0 r_0^3}{(r + r_0)(r^2 + r_0^2)}\,, \tag{3.6}$$

where r_0 is the core radius and ρ_0 the central density. This profile shows the same behavior of NFW at large radii but produces a cored shape in the inner regions. This difference is at the basis of the so called core-cusp problem [109, 110].

It has to be stressed that the NFW profile emerges from pure DM simulations, in which baryons' effects are completely ignored and that the evidence for one or the other of the profiles seems to depend on the baryonic observable chosen [111]. We will discuss this in more detail in the next section.

Missing satellites. The missing satellites problem [112–115] is related to the discrepancy in the predicted number of satellites of the Milky Way (MW) from simulations and the actual observed number. This was considered to be a serious issue for the standard model of structure formation. However, with the increase of the sensitivity of the observational instruments it has been possible to discover ultrafaint galaxies thus reducing the discrepancy. Moreover, only a limited area of the sky has been covered by observations and suppression mechanisms may have inhibited star formation, hence making many dark halos not possible to be seen.

Too big to fail problem. Recent simulations [116, 117] have produced MW sub-halos that are too dense to host any of the luminous observed satellite galaxies. In fact, dwarf-spheroidal galaxies have typical maximal circular velocities less than 24 km s^{-1}, but in the simulations many sub-halos with velocities larger than this limit are generated, reaching up to 70 km s^{-1} [118–120]. The fact that we do not observe any luminous galaxy whose halo shows such velocities, is in contrast with the well known monotonic relation between luminosity and halo circular velocity (and hence halo mass). Hence, a priori these halos are too big to fail in producing galaxies but given that no galaxy with such properties is observed we need a new mechanism able either to avoid the formation of such halos or to suppress star formation in them.

Angular momentum problem. In the standard picture of structure formation it is supposed that the angular momentum of baryons is conserved during collapse. However, hydrodynamical simulations have shown that this is not the case as in the final state baryons have only $\sim 10\%$ of the initial angular momentum. This discrepancy is shown in figure 3.1, which clearly shows the offset between observed and simulated Tully–Fisher relation. The loss is imputed to the fast cooling of baryons in simulations that makes them to collapse to the center too fast, hence losing angular momentum. A possible solution to this problem was thought to be the injection of energy through supernovae feedbacks. However, more recent analysis [121] showed how this problem is mainly due to a too high concentration of DM haloes and it is hence unlikely that baryons' feedbacks can solve it, unless the effects of backreaction have been underestimated dramatically. A reduction in the concentration of

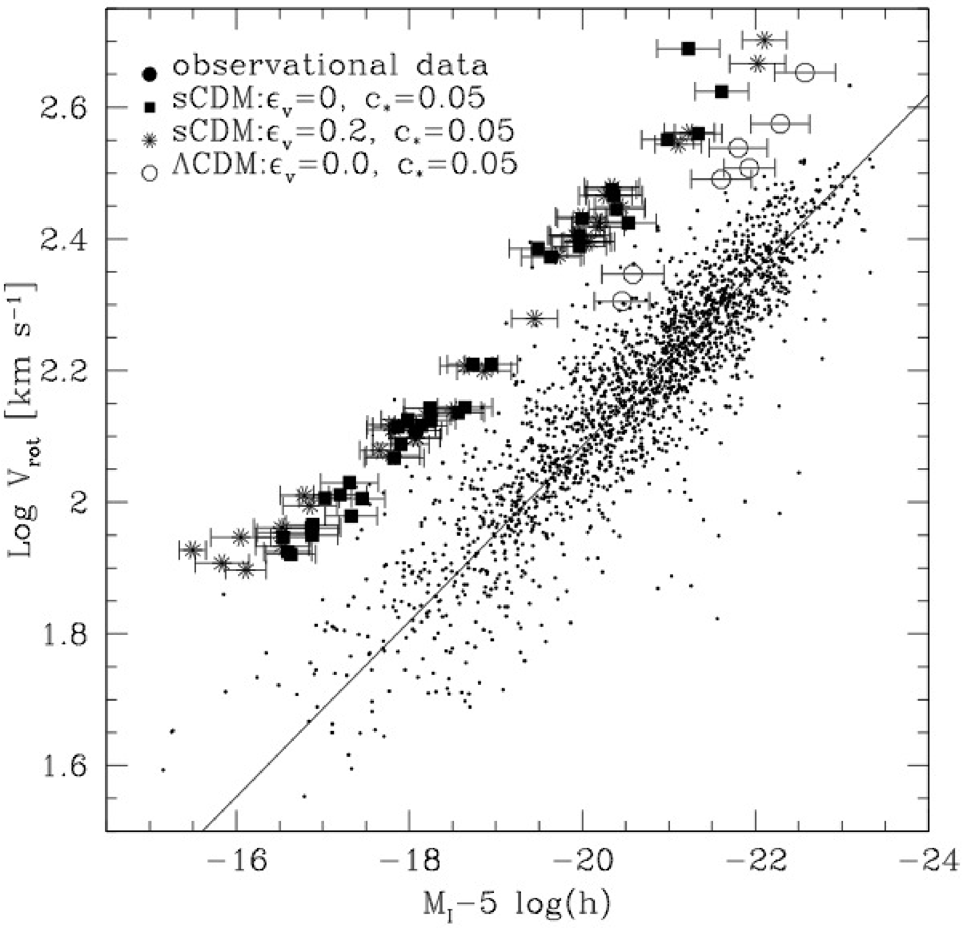

Figure 3.1: Simulated TFR compared with the results of the numerical simulations. Dots correspond to the observational samples while error bars to the simulated ones [121] . The offset indicates that the simulation generates a too high rotation velocity compared to the observed one.

CDM halo by a factor 3 or 5 may solve this issue and could be an indication of a modification of the CDM paradigm at small, galactic scales.

Correlations between DM and Baryonic parameters. Recent years have witnessed a boost in the number and quality of observations of galaxy and halos mass distribution, especially in their inner part. A remarkable outcome of these observations was the discovery of Universal properties of mass distribution. In particular, in [122] it was observed how the luminous surface density $\Sigma_0^{\rm bary}$ at the radius at which the DM density is almost constant, is constant in all the observed galaxies regardless of their luminosity as can be seen in figure 3.2. This is equivalent to say that the gravitational acceleration of baryons at the radius of the DM core is the same for all galaxies. The same relation was found for DM in [123] meaning that, despite the differences between halos, DM central distribution keeps constant the product between core radius and density. This intriguing results seems to tell that, even if in general the mass-to-light ratio is highly varying with luminosity, it is constant at the scale of the DM core. This can be interpreted as a correlation between dark and luminous matter in galaxies.

A second evidence is that of a common mass scale for Milky Way satellites. In fact, in [124] it was observed how all the velocities of galaxies in these satellites were compatible with a common mass of $10^7 M_\odot$ within 300 pc. This may be the evidence of a common scale for galaxy formation or a scale characteristic of DM.

The Baryonic Tully–Fisher relation (BTFR) [125] is an extension of the already discussed Tully–Fisher relation to include faint galaxies for which the TFR seemed to fail. However, taking into account also the gas contribution to the mass and not only the luminous one, the problem is solved thus ensuring that the TFR holds over a large range of magnitude making this relation an evidence of a universal property of galaxies. This relation is somewhat surprising as it has a very small scatter, which is a clear manifestation of a correlation between dark and luminous matter.

The main point of these observations is that, in order to be satisfied in the standard scenario of ΛCDM, they seem to require a high level of fine tuning, given the very different histories of merging and structure formation that each galaxy went through. A second interpretation can be that of considering these universal properties as hints of a change in the DM dynamics at galactic scales.

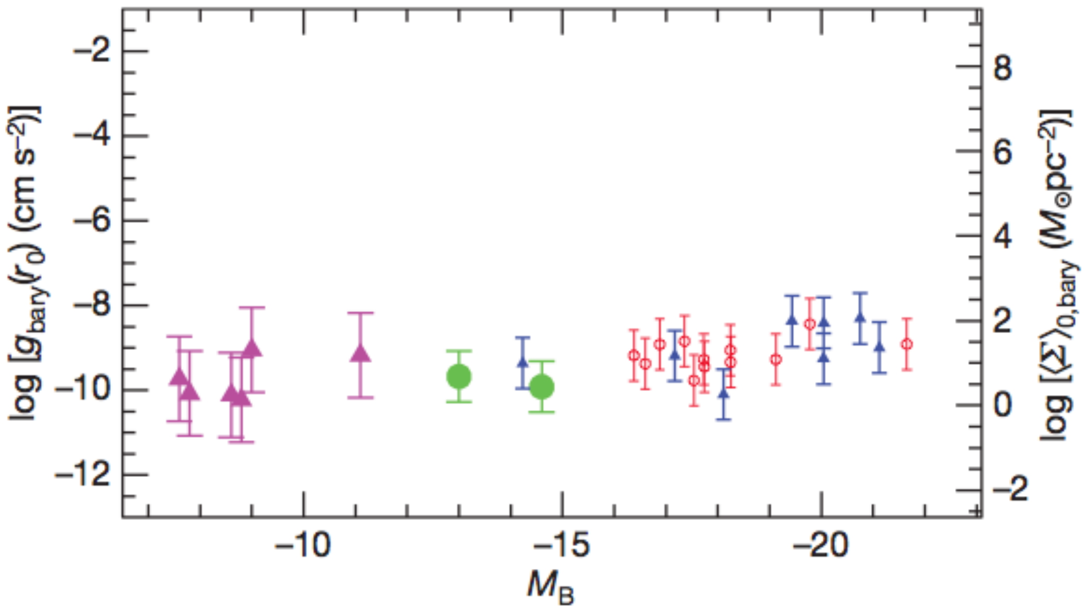

Figure 3.2: Universality of the average surface density and gravitational acceleration of baryons within the halo core radius as a function of galaxy luminosity [122]. From the plot it is clear how the gravitational acceleration for baryons, $g_{bary}(r_0)$, is basically constant over 16 orders of magnitude in luminosity, spanning over a set of galaxies with very different formation histories.

3.3 Proposed solution to the small scales issues of ΛCDM

Given the aforementioned issues, it is natural to question the CDM paradigm and its applicability at small scales by proposing alternatives or by putting more efforts to have better simulations (and a deeper understanding of their outcomes). In support of the first proposal many fundamental physics models may be introduced which, in the attempt to extend the SM, directly introduce new potential DM candidates in the cosmological context. On the other side, the ΛCDM is working so well that it is tempting to impute the present issues simply to a lack of precision in simulations. Actually, there is also a third point of view. As in the case of CC, the need of a DM component may just be the evidence of a modified theory of gravity, instead of a missing particle.

Baryons' feedbacks. Following the dynamics at galactic scales needs not only high performing computer simulations of DM structure formation but also a deep knowledge of the galaxy formation process [126], including the potential relevant baryons' feedbacks, like supernovae explosion. For example the core-cusp problem may be due to the non-inclusion of baryon's physics in the simulations [127–129] or, as said, may just be related to a bad choice of the visible tracer [111].

However, due to difficulties with including baryons physics into the simulations, today there is no a general consensus on the actual effects of baryon's feedback. In particular there is no prescription on how to address simultaneously all the above listed issues. In fact if, for example, in [130] it was shown how a proper treatment of baryonic feedbacks is able to erase many of the unwanted MW sub-haloes and in [131,132] a realistic MW-like galaxy was formed in the ΛCDM framework, in [133] it was claimed that these effects are not efficient in forming late time spiral galaxies.

Warm and Interacting Dark Matter. CDM has no relevant free streaming scale with the consequence that the collapse of matter keeps on going down to very small scales. A DM particle with a larger self interaction or with a non-negligible dispersion velocity would significantly suppress the growth of structure below the free streaming scale. This kind of DM, dubbed Warm DM (WDM), received a lot of attention in the last years as it seemed to be a good candidate for solving some of the issues the CDM is facing in the description of structure formation at small scale [134–137]. In fact, a thermal WDM of a mass of 1 keV has a free streaming length around the size of a galaxy so that it may be able to produce a cored profile for the DM distribution.

However, the effectiveness of these models is quite debated and recent works [138,139] showed that the WDM has to be quite tepid, almost reducing to CDM and hence not being able to address the core-cusp issue significantly.

In [140] it is instead discussed the case of interacting CDM. In particular it is discussed how a non-negligible interaction may resolve the aforementioned issues of standard CDM paradigm.

Ultralight DM. Recently, the possibility that DM may be made of ultralight scalar particle, with masses as small as 10^{-22} eV has been proposed. In this case the quantum properties of the scalar field cannot be neglected and a quantum pressure provides the required support to suppress the growth of structure below a certain scale [141–144]. Notice that, despite the smallness of the mass, this is a CDM candidate.

MOND and no DM matter models. A more radical point of view is represented by those models that try to explain galactic (and cosmological) dynamics without the need to introduce a new, yet unseen dark component, but rather via a modification of gravity. The most well known of such model is MOND [145, 146] which hinges on an empirical modification of Newton's law at galactic scales

$$\nabla \left(\mu \left(\frac{|\nabla\phi|}{a_0} \right) \nabla\phi \right) = 4\pi G\rho \,, \tag{3.7}$$

where $\mu(x)$ is an empirical function that reduces to 1 for large x while at small values of the argument $\mu(x) = x$. The constant a_0 is a new fundamental scale that divides the regime of applicability of GR from that of MOND.

Despite its phenomenological nature, this modification is able to properly reproduce many of the observed features of galaxy dynamics [147] better than what is achieved by CDM. In recent years some relativistic extensions of MOND have been proposed [148–151] with fairly good results, even if some problems in reproducing large scales data are still present [152]. Moreover DM is anyway needed in order to match observations [153–155].

Another approach recently explored that involves no DM particle is presented in [156–159]. In these models a direct coupling between the matter Lagrangian and curvature is present

$$S = \int d^4x \left[f_1(R) + (1 + \lambda f_2(R)\mathcal{L}_m) \right] \,, \tag{3.8}$$

where f_1 and f_2 are generic functions of the Ricci scalar and λ is the coupling strength. The model has been tested in many cosmological situations showing a good potential in reproducing some of the observations like the galaxy rotation curves [160].

Reconciling MOND with DM. A recent proposal, aiming at reconciling the merits of a MOND-like picture with the strengths of a CDM framework, has been suggested in [6,161]. There, it was shown how it is possible to reproduce a MONDian behavior at galactic scales in a standard CDM scenario by requiring DM to couple with baryons in a suitable way. In other words, the MONDian behavior would emerge as an effect of the specific interaction between DM and baryons. If this interaction can be built so as to be active at special scales and times, then one might be able to achieve the aforementioned marriage between competing models. In particular the interaction is such that

$$S_m[\psi, g_{\mu\nu}] + S_{Int}[\xi, \psi, g_{\mu\nu}] \sim S_m[\psi, g_{\mu\nu} + h_{\mu\nu}]\,. \tag{3.9}$$

$h_{\mu\nu}$ is a rank two symmetric tensor constructed with the metric $g_{\mu\nu}$ and with the scalar field χ and its derivatives.

This particular coupling would introduce a non-geodesic motion for matter at suitable scales and times and its origin can be envisaged in a geometric effect due to dark matter. In fact, this class of models is a generalization of standard scalar-tensor theories and indeed, for small $h_{\mu\nu}$, it is possible to derive such interaction from a non-minimal coupling of DM.

In the next chapters we will proceed along the latter point of view by extending the couplings that DM can have with gravity. In particular, we will consider a CDM fluid that gets non-minimally coupled to gravity at suitably late times and small scales and investigate both theory and phenomenology of this extended model for DM.

Chapter 4

Non-minimally coupled dark matter fluids

The picture emerging from the discussion of the previous chapter is that of a tension between the successes of the cosmological model at large scales and a difficulty in properly accounting for the observed properties of the DM distribution at galactic scales. In fact, even if the latter may be relaxed taking into account baryons physics in simulations, the evidence for correlations between dark and luminous matter suggests that this may not be the only reason for the mismatches between the prediction of the ΛCDM model and observations. In particular, the fact that models without DM, like MOND, are able to provide better fits to the data at small scales may be a hint of real differences with respect to the standard CDM paradigm. However, one cannot ignore the impressive successes of the latter at large scales. For this reason the possibility to reconcile the two schemes into a unitary picture is appealing. Moreover, the unknown nature of DM and the fact that gravity is poorly tested from astrophysical scales onwards [162, 163], seem to suggest that natural extensions of the CDM paradigm may be in the direction of generalizing the interaction between DM and gravity.

A further hint in this direction is provided by MOND itself. As discussed in the previous chapter, this model introduces a new fundamental constant, a_0, with the dimensions of an acceleration which enters in a modification of the Newton's gravitational law and as such breaking the equivalence of gravitational and inertial masses which is the Newtonian version of the weak equivalence principle. A similar non-geodesic motion for baryons can be achieved assuming that the metric g_{phys} along which propagate baryons is different from that defining the gravitational sector g_{grav}. If then we take DM to be responsible for this difference by postulating $g_{\rm phys} = g_{\rm phys}(g_{\rm grav}, \chi)$, where χ is the DM field, we may be able to mimic a MONDian behavior in the context of DM theories. This is indeed possible and in [6] it was shown how to construct such a theory.

In the same spirit, we propose and investigate a phenomenological model in which a generalized coupling between curvature terms and DM aims to address the small scale ΛCDM issues. In particular, we will consider a model in which a DM fluid, at suitably small scales and late times, gets non-minimally coupled to curvature terms. On quite general grounds this proposal is expected to have two main distinctive features: one is that the DM fluid is no longer perfect at the scales at which the non-minimal coupling is relevant; the other is that the metric along which baryons move gets redefined in a way that depends only on how DM and gravity are coupled. This leads to a modified effective dynamics for the fluid and for the gravitational field even in the Newtonian limit, as we shall see. While the first feature may address the problem of dark matter density in halos by assigning to the DM fluid an effective pressure, the second one may provide an explanation to the unexpected correlations between dark and luminous matter.

4.1 Non-minimally coupled dark matter fluid

We now proceed with the construction of such model. First we note that, being interested in astrophysical scales, we can describe the matter content of our theory in the fluid limit, in which only macroscopic quantities are relevant as all the quantum properties are negligible or averaged out at the scales of interest.[1] Hence, we introduce SM particles, basically baryons and photons, as non-interacting fluids, minimally coupled to the gravitational metric. Then, the equation of motion for these components will be given by the conservation of their individual SET, as in standard GR.

The original ingredient of our model comes from the new couplings between DM and curvature. Consider a system in which DM is described as a perfect fluid with a barotropic equation of state which couples non minimally to gravity. The easiest way to do this is to couple a scalar function of the DM variables to the Ricci scalar, adding to the Lagrangian a term like $f(R)\mathcal{L}_{dm}$, where $f(R)$ is a generic function of the Ricci scalar and $\mathcal{L}_{dm}$ would be the DM Lagrangian (for a thorough discussion of these models see, for instance, [157–159]).

However this particular scalar coupling would not affect the propagation of light rays, given that the Maxwell action is conformally invariant. Consequently it would not be enough for enhancing gravitational lensing as it seems necessary in order to account for the observed DM phenomenology [164–166].

[1] An exception to this statement could be realized in the presence of torsion or in Bose–Einstein Condensates (BEC), as we will discuss later.

Moreover, since we require the deviations from ΛCDM to be effective only at galactic scales (where densities are higher than the cosmological ones), the coupling has to be active only above a certain density threshold, at late times. This would fix a minimum scale at which deviations from ΛCDM are expected to be present. This is a subtle issue since we do not want to spoil the description of equally high density but spatially homogeneous early universe cosmology. We will come back to this point at the end of this chapter.

It is rather clear that, working with perfect fluids, there are not many possibilities to couple DM to gravity, given that we have at our disposal only scalars and the four vector field encoding the four velocity at each spacetime point. If we add the constraint that we want to keep the gravitational field equations of second order in the metric tensor, so that we do not introduce additional gravitational degrees of freedom, only the following five terms can be constructed

$$R_{\mu\nu\rho\sigma}u^\mu u^\nu u^\rho u^\sigma\,,\; R_{\mu\nu\rho\sigma}g^{\mu\nu}g^{\rho\sigma}\,,\; R_{\mu\nu\rho\sigma}g^{\mu\nu}u^\rho u^\sigma\,,\; R_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\,,\; R_{\mu\nu\rho\sigma}g^{\mu\sigma}u^\rho u^\nu\,. \tag{4.1}$$

The first three are zero because the antisymmetric indices of the Riemann tensor are contracted with a symmetric combination of the four velocity and the metric. Therefore, we end up with only the last two terms, namely

$$R\psi(\rho) \qquad \text{and} \qquad R_{\mu\nu}\xi(\rho)u^\mu u^\nu. \tag{4.2}$$

Furthermore, if we use perfect barotropic fluid, the residual information about the coupling can be parametrized completely with two arbitrary functions of the mass density (see again [167] for another example of density dependent couplings).

Before moving on, it is worth to recall that we are indicating as ρ the mass density, that $u_\mu u^\mu = -1$ and that for the rest we are following the treatment presented in [168].

4.1.1 Action and equations of motion

In the previous section we have discussed a set of requirements that have led us to propose the terms coupling curvature to fluid quantities. In this section we formalize the previous analysis into an action and derive the equation of motion from it. For the sake of simplicity and to enlighten the effects of the modified DM couplings, we will neglect other cosmological fluids. Their inclusion is in any case trivial and can be done in a second moment.

The action for our model is given by the standard GR one, plus the two terms introduced in the previous section, namely

$$S = \frac{c^3}{16\pi G_N}\int d^4x\,\sqrt{-g}\,\Psi(\rho)R + \frac{\alpha_{\rm R}c^3}{16\pi G_N}\int d^4x\,\sqrt{-g}\,R_{\mu\nu}\xi(\rho)u^\mu u^\nu + S_{\rm DM}[g,\rho]. \tag{4.3}$$

The DM fluid action is that for a perfect fluid [168]

$$S_{\rm DM} = -2c\int d^4x\,\left[\sqrt{-g}\,\rho(n,s) + J^\mu(\phi_{,\mu} + s\theta_{,\mu} + \beta_A\alpha^A_{,\mu})\right], \tag{4.4}$$

where n is the particle number density, s is the entropy per particle and the second term implements the constraints for the flow of perfect fluid. In particular, ϕ and θ serve as Lagrangian multipliers for the particle number conservation, while β_A is needed to restrict the fluid four-velocity to to be along the flow lines. The density vector J^μ is related to the fluid variables as

$$J^\mu = nu^\mu\sqrt{-g}\,. \tag{4.5}$$

The function $\Psi(\rho) = 1 + \alpha_{\rm S}\psi(\rho)$ controls the coupling of the dark fluid to the Ricci scalar, while the function $\xi(\rho)$ mediates the coupling to the Ricci tensor. Both these functions are dimensionless, and hence they must involve, for dimensional reasons, at least another density parameter ρ_* which sets the characteristic, phenomenological, scale of the model. Finally, the dimensionless constants $\alpha_{\rm S}, \alpha_{\rm R}$ control the strength of the non-minimal couplings. In fact, they could be reabsorbed in the functions ψ and ξ, without loss of generality. However, it is useful to keep them explicit since they can be used as dimensionless parameters for an expansion whenever the non-minimal coupling is expected to be a subdominant effect.

The equations obtained varying the action with respect to the metric are

$$\begin{aligned} G^{\mu\nu} = \frac{8\pi G_{\rm eff}(\rho)}{c^2}\Big[&T^{\mu\nu}_{\rm standard} + \alpha_{\rm S}\rho_*\ell^2\left(-\Box\tilde\psi g^{\mu\nu} + \nabla^\mu\nabla^\nu\tilde\psi\right) + \\ &-\frac{\alpha_{\rm R}}{2}\rho_*\ell^2\left(-\Box\tilde t^{\mu\nu} + \nabla_\rho\nabla^\mu\tilde t^{\rho\nu} + \nabla_\rho\nabla^\nu\tilde t^{\rho\mu} - g^{\mu\nu}\nabla_\alpha\nabla_\beta\tilde t^{\alpha\beta}\right) + \\ &+\frac{\rho_*\ell^2}{2}\left(\alpha_{\rm S}\tilde\psi'\rho R - \alpha_{\rm R}\left(\tilde\xi'\rho - \tilde\xi\right)R_{\alpha\beta}u^\alpha u^\beta\right)H^{\mu\nu}\Big]\,, \end{aligned} \tag{4.6}$$

where the prime indicates derivative with respect to the DM density and where we have introduced the notation $H^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ for the projector on the subspace orthogonal to u^μ and the tensor $\tilde t^{\alpha\beta} = \tilde\xi(\rho)u^\alpha u^\beta$ to slightly simplify the expressions. Moreover we have redefined the functions ψ and ξ as

$$\tilde\psi = \frac{c^2}{8\pi G_N\rho_*\ell^2}\psi \qquad \text{and} \qquad \tilde\xi = \frac{c^2}{8\pi G_N\rho_*\ell^2}\xi \tag{4.7}$$

so to make more explicit the structure of the terms. We have thus introduced two new constants, ρ_* and ℓ, that represents respectively the characteristic density and size of the system under consideration. Notice that both ψ and $\tilde\psi$ are dimensionless. This parametrization will become clearer when we will discuss the Newtonian limit in the next section.

The SET for the standard dark matter fluid is instead

$$T^{\mu\nu}_{\text{standard}} = [\rho + p(\rho)]\, u^\mu u^\nu + p(\rho) g^{\mu\nu} \,. \tag{4.8}$$

The first thing to notice in the modified Einstein equations (4.6) is that the non-minimal coupling affects Newton's constant G_N that now reads

$$G_{\text{eff}} = \frac{G_N}{1 + \alpha_{\text{S}} \psi(\rho)} \,, \tag{4.9}$$

which implies that now the value of the gravitational constant depends on the local density of DM. This is a universal effect in the sense that if we were to include other (minimally coupled) fluids they would all feel the same modified gravitational strength. Notice that only the conformal coupling contributes to this modification, while the one associated to the coupling with the Ricci tensor is not present. Then, as one can easily see, the Einstein equations do not contains higher derivatives of the metric tensor. However, in addition to the SET for a fluid made of dust, there are a certain number of terms that concur to define an effective SET, depending on higher derivatives of the fluid variables and on the curvature. This is indeed what should be expected, given that the basic idea of non-minimal coupling is that the field, or fluid, is able to probe geometry on a given length scale, not only point-wise as in the standard case.

The equations of motion for the fluid can be obtained either varying the action with respect to the various fluid fields, or by using the Bianchi identities on the modified Einstein equations. We will not discuss them in full generality here given that for our purpose we can limit ourselves to their Newtonian limit, which is the relevant regime to discuss galactic dynamics.

Indeed, from the contracted Bianchi identities for the system we have that

$$\nabla^\mu T^{\text{eff}}_{\mu\nu} = 0 \,, \tag{4.10}$$

where $T^{\text{eff}}_{\mu\nu}$ is the effective SET, which contains all the terms appearing on the right hand side of equation (4.6). In order to compute the covariant derivative of the SET notice that

$$\nabla^\mu H_{\mu\nu} = \vartheta u_\nu + u^\mu \nabla_\mu u_\nu \,, \tag{4.11}$$

where

$$\vartheta = \nabla_\mu u^\mu \,, \tag{4.12}$$

is not yet defined as the expansion of the bundle of geodesics (but rather as the expansion of the bundle of curves whose tangent vector field is u^μ).

Then using the commutation rules for the covariant derivatives:

$$\nabla_\alpha \nabla_\beta \nabla_\nu f = \nabla_\alpha \nabla_\nu \nabla_\beta f = \nabla_\nu \nabla_\alpha \nabla_\beta f - R^\rho{}_{\beta\alpha\nu} \nabla_\rho f \,, \tag{4.13}$$

and

$$\nabla_\rho \nabla_\nu t^{\rho\mu} = \nabla_\nu \nabla_\rho t^{\rho\mu} + R^\rho{}_{\sigma\rho\nu} t^{\sigma\mu} + R^\mu{}_{\sigma\rho\nu} t^{\sigma\rho} \,, \tag{4.14}$$

we have that the complete fluid equations reduce to the following expression:

$$\begin{aligned} \nabla^\mu T^{\text{standard}}_{\mu\nu} = & -\alpha_{\text{S}} \ell^2 \rho_* \left(\frac{R}{2} \nabla_\nu \tilde{\psi} - \frac{1}{2} H_{\mu\nu} \nabla^\mu (R\tilde{\psi}'\rho) - \frac{R\tilde{\psi}'\rho}{2} \vartheta u_\nu - H_{\nu\mu} \frac{R\tilde{\psi}'\rho}{2} u^\rho \nabla_\rho u^\mu \right) + \\ & - \alpha_{\text{R}} \ell^2 \rho_* \left[-g^{\alpha\beta} R^\rho{}_{\nu\alpha\mu} \nabla_\beta \tilde{t}^\mu{}_\rho + \nabla_\mu \left(R^\rho{}_{\sigma\rho\nu} \tilde{t}^{\sigma\mu} + R^\mu{}_{\sigma\rho\nu} \tilde{t}^{\sigma\rho} \right) \right. \\ & \left. + R_{\sigma\nu} \nabla_\mu \tilde{t}^{\mu\sigma} + H_{\mu\nu} \nabla^\mu W + W \vartheta u_\nu + H_{\mu\nu} W u^\rho \nabla_\rho u^\mu \right] \,. \end{aligned} \tag{4.15}$$

Notice that this expression contains the full Riemann tensor, and in particular the Weyl tensor. Therefore, this kind of non-minimally coupled matter can have nontrivial behavior even in Ricci-flat spacetimes (as, for instance, Schwarzschild spacetime).

It is probably worth to stress that even if in these equations higher derivatives of curvature terms appear, these can be traded for fluid derivatives for which higher order derivatives can find a natural interpretation in terms of effective viscosities. Hence, the theory is indeed second order in the metric.

4.1.2 Newtonian limit

To properly discuss the Newtonian limit, it is important to work out the weak field limit of equation (4.6). Indeed, it is important to understand how to establish a comparison between terms that have different physical dimensions. As we have discussed, ψ, ξ are dimensionless, as $\alpha_{\text{S}}, \alpha_{\text{R}}$ and we have taken the typical size of the system under consideration to be of order ℓ. This enables us to compare the different terms in a consistent way. The weak field limit is achieved whenever the curvature radius is much larger than the size of the system, *i.e.,*

$$|G_{\mu\nu}| \ll \ell^{-2}. \tag{4.16}$$

For consistency, then, ξ, ψ must be small quantities, as well as the properly normalized matter density. In short, the weak field limit is achieved when the following condition holds on the SET:

$$\psi \simeq \xi \simeq \frac{8\pi G_N}{c^2} \rho \ell^2 \ll 1 \,. \tag{4.17}$$

With this treatment, we can consistently extract the weak field limit of the equations Eq. (4.6), taking systematically into account the relative size of the various terms. If we define

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}; \qquad \bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma; \qquad \gamma = \eta^{\mu\nu}\gamma_{\mu\nu}\,, \tag{4.18}$$

the modified Einstein equations, in the weak field limit and in the transverse gauge, read

$$-\frac{1}{2}\Box\bar{\gamma}_{\mu\nu} = \frac{8\pi G_N}{c^2}\left\{-\rho_*\left[\alpha_{\mathrm{S}}\left(\eta_{\mu\nu}\Box\tilde{\psi} - \partial_\mu\partial_\nu\tilde{\psi}\right) - \frac{\alpha_{\mathrm{R}}}{2}\Omega_{\mu\nu}\right] + T^{\mathrm{standard}}_{\mu\nu}\right\}\,, \tag{4.19}$$

where

$$\Omega_{\mu\nu} = -\Box\tilde{t}_{\mu\nu} + \nabla_\rho\nabla_\nu\tilde{t}^\rho{}_\mu + \nabla_\rho\nabla_\mu\tilde{t}^\rho{}_\nu - g_{\mu\nu}\nabla_\alpha\nabla_\beta\tilde{t}^{\alpha\beta}\,. \tag{4.20}$$

with $\tilde{t}_{\mu\nu} = \tilde{\xi}(\rho)u_\mu u_\nu$. In the weak field limit we have that $u_\mu = \delta^0{}_\mu$, and hence

$$\Omega_{\mu\nu} = \delta^0_\mu\delta^0_\nu\Box\tilde{\xi}(\rho) - \delta^0_\nu\partial_0\partial_\mu\tilde{\xi}(\rho) - \delta^0_\mu\partial_0\partial_\nu\tilde{\xi}(\rho) + \eta_{\mu\nu}\partial_0\partial_0\tilde{\xi}(\rho)\,. \tag{4.21}$$

As one immediately sees, the effect of the non-minimal coupling is still present, even in the weak field limit, and the fluid is not behaving as a perfect one in Minkowski spacetime: the non-minimal coupling has generated a SET which contains additional terms, constructed out of the derivatives of the fluid variables. Notice that in this limit the effective gravitational constant has turned back to its original constant value.

Putting everything together, and considering the static, nonrelativistic limit (*i.e.,* the $c^2 \to \infty$ limit), we get the modified Poisson equation

$$\nabla^2\Phi_N = 4\pi G_N\left(\rho - \frac{\alpha_{\mathrm{R}}}{2}\rho_*\nabla^2\tilde{\xi}(\rho) + \alpha_{\mathrm{S}}\rho_*\nabla^2\tilde{\psi}(\rho)\right)\,. \tag{4.22}$$

We see that, in this modified scenario, the Newtonian potential is sourced not only by the mass density ρ, but also a certain number of derivative terms. Moreover, contrarily to what happens in GR, the Newtonian potential Φ_N is not the only potential concurring in defining the gravitational dynamics. In fact now we also have that the spatial gradients of the metric are present:

$$\nabla^2\gamma_{ij} = -\frac{8\pi G_N}{c^2}\left\{\left[\rho + \rho_*\left(2\alpha_{\mathrm{S}}\nabla^2\tilde{\psi}(\rho) - \frac{\alpha_{\mathrm{R}}}{2}\nabla^2\tilde{\xi}(\rho)\right)\right]\delta_{ij} - 2\rho_*\alpha_{\mathrm{S}}\partial_i\partial_j\tilde{\psi}(\rho)\right\}\,. \tag{4.23}$$

The gravitational potential associated to the spatial part of the Einstein equations is sourced by two terms: an isotropic contribution and an anisotropic one. Notice that while in the first case both NMCs contributes, in the second only the scalar (conformal) coupling matters. This is a novelty introduced by the NMC as in GR this potential has to satisfy the equation $\Box\bar{\gamma}_{ij} = 0$ whose only well behaved solution at infinity is $\bar{\gamma}_{ij} = 0$ (*e.g.,* see [169]).

Turning now the attention to the equations of motion for the DM fluid (4.15) we have that in the weak field limit, these will reduce to:

$$\nabla^{\mu} T^{\text{standard}}_{\mu\nu} = 0 \,. \tag{4.24}$$

However, in light of the fact that the definition of the dark matter mass density is given in terms of the density that enters the right hand side of the equation of motion for the gravitational field (either the Poisson equation or the general-relativistic version), one might define an effective mass density and effective stresses, that do not coincide with those that are defined out of the fluid action when the non-minimal coupling is absent. In particular notice that, from an observational point of view, we can reconstruct the total gravitational potential looking at baryons' motion and hence, what we actually measure is the right hand side of the Einstein equation. In terms of these effective quantities the equations of motion for the DM fluid will get modified also in the non relativistic limit. In fact, in the case of pressureless DM

$$\rho_{\text{eff}} = T^{\text{eff}}_{\mu\nu} u^{\mu} u^{\nu} = \rho + \alpha_{\text{S}} \rho_* \nabla^2 \tilde{\psi} - \frac{\alpha_{\text{R}}}{2} \rho_* \nabla^2 \xi \,, \tag{4.25}$$

$$3 p_{\text{eff}} = T^{\text{eff}}_{\mu\nu} H^{\mu\nu} = -2 \alpha_{\text{S}} \rho_* \nabla^2 \tilde{\psi} \,. \tag{4.26}$$

One should then re-express ρ, p in terms of $\rho_{\text{eff}}, p_{\text{eff}}$, with the consequence that the system in the new variables will show a standard Poisson equation but modified fluid equations.

Finally, notice from (4.26) the role of the NMC: even if we take a pressureless dust as DM candidate, it will end up to have an effective pressure related to the gradients of the DM distribution.

4.2 Phenomenological constraints

Now that we have defined our model and analyzed its Newtonian limit in which it can address some of the CDM issues, we are in the condition to discuss more accurately its predictions in different regimes and consequently use current observations to bound it (in particular by constraining the behavior of the functions $\psi(\rho)$, $\xi(\rho)$). There are two obvious regimes at which the model has to offer new phenomenology: galactic dynamics and cosmology. However, to be viable, any modified gravity model (in a broad sense) must be compatible with solar system constraints on gravitational phenomena. We shall hence start our discussion from here.

Solar System scales: Of course, our model must reduce to general relativity at these scales. In particular, if we impose that $\psi(0) = \xi(0) = 0$, we are sure that the dynamics of a purely baryonic system will be described by general relativity without corrections. Given that, at the level of solar system, it is safe to say that the density of baryonic matter is much larger that the density of dark matter, this condition ensures that the agreement with observational constraints will be achieved, provided that $\alpha_{\rm S}, \alpha_{\rm R}$ are not too large.

Galactic dynamics: We have shown that the Poisson equation gets modified by a term which depends on gradients of the density. This means that the more inhomogeneous a distribution of DM is, the stronger is the effect. As a consequence, structures may grow faster or slower than expected, according to the structure of the additional terms, and, ultimately, to the signs of the coupling constants $\alpha_{\rm S}, \alpha_{\rm R}$.

As we have mentioned, the NMC coupling also generates a pressure term, which is structured in two components. On the one hand, there is an isotropic pressure that again is related to gradients of the density. This is a key feature as pressure may stabilize halo's cores preventing the formation of cusps, given that its magnitude increases with the inhomogeneity.

On the other hand, there is an anisotropic pressure term which represents a distinguishing feature of our model. In standard CDM particles forming halos are collisionless and hence they have no global collective motion. This anisotropic pressure may generate a net overall rotation of DM halos which modifies the caustic structure of the infalling dark matter particles with respect to the irrotational flow. There is convincing evidence that such overall rotation can lead to a caustic structure closer to the observed one [59].

Given that the puzzles related to mass discrepancies are harder to address at the galactic scale, one needs the NMC terms to be larger in these regimes and consequently to SET $\rho_* \approx \rho_{gal}$; basically assuming that the functions ψ, ξ will attain their maxima in this density regime.

With this model we may be able to address some of the problems that ΛCDM is suffering, by reproducing, at suitable scales, a MOND-like behavior. However, we do not have yet established a one to one correspondence between our model and MOND in its traditional incarnation. Actually, this correspondence could be achieved only if baryons would end up tracking DM. Indeed, if we interpret the extra contribution emerging from the modified dynamics of MOND as DM, it would be nonetheless locked to the baryon density. To settle this point, the detailed analysis of the gravitational dynamics of a galaxy, within this model, is required. While the form of the Poisson equation gives the feeling that at least a slight tracking will be present, it is worth stressing that this model will generically show a richer phenomenology than MOND and could at most mimic it in some regimes.

We now show the connection between our model and the one proposed in [6]. Consider the action (4.3) in which we take $\psi = \xi$ and $\alpha_{\rm S} = \alpha_{\rm R}/2$. Then we have

$$S = \frac{c^3}{16\pi G_N}\int d^4x\,\sqrt{-g}R + \frac{\alpha_{\rm R}c^3}{16\pi G_N}\int d^4x\,\sqrt{-g}\xi(\rho)G_{\mu\nu}u^\mu u^\nu + S_{DM}[g,\rho]\,, \tag{4.27}$$

which is the same NMC coupling found in [6], between the Einstein tensor and the field/fluid variables. If we then define

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \alpha_{\rm R}\xi(\rho)u_\mu u_\nu \tag{4.28}$$

for small deviations from GR, *i.e.*,$\bar{g}_{\mu\nu} - g_{\mu\nu} \ll 1$, we can substitute this expression into the above action to find

$$S = S_{EH}[\bar{g}] + S_{DM}[\bar{g}_{\mu\nu} + \alpha_{\rm R}\xi(\rho)u_\mu u_\nu, \rho] + \mathcal{O}(\alpha_{\rm R}^2) \tag{4.29}$$

which shows in a more explicit way the connection between NMC and geometric effects on the propagation of matter fields.

Moreover, with this choice of the parameters and functions we see that the contributions to the Poisson equations (4.22) coming from NMC terms, exactly cancels. This is an interesting coincidence as it seems to indicate that at scales where gravity is weak and for non relativistic fluids this particular case of the model reduces exactly to GR. This can be seen as a particular screening that prevents modifications of gravity to be relevant at solar system scales where the tightest bound on gravitational interactions severely constrain alternatives to GR.

Cosmology: As pointed out earlier, a key feature of this model is the presence of spatial gradients of the density in the non-relativistic limit. However, in the full relativistic theory, not only spatial derivatives, but also time derivatives are relevant, and the additional terms might be active even in spatially homogeneous cases. The NMC may affect the cosmological evolution in a dramatic way that might lead to a sharp contrast with the observations, whenever the time derivatives become relevant, *i.e.,* at sufficiently early times in cosmology.

Consider the flat FLRW metric:

$$ds^2 = e^{2n(t)}dt^2 + e^{2a(t)}d\mathbf{x}^2. \tag{4.30}$$

We can compute the Lagrangian for our model inserting the metric defined by the above line element in the action 4.3. This gives the following effective Lagrangian density (where a boundary term has been discarded):

$$\mathcal{L}_{\rm grav} = \frac{e^{-n+3a}}{16\pi G}\left\{-6\left[\dot{a}^2 + (\alpha_{\rm S}\psi' + \alpha_{\rm R}\xi')\dot{a}\dot{\rho}\right] - 6\alpha_{\rm S}\psi\dot{a}^2 - 3\alpha_{\rm R}\xi\dot{a}^2\right\}, \tag{4.31}$$

to which the fluid Lagrangian density has to be added.

We now want to recover, at large scales and at early times, the ΛCDM model. Given that on large scales we can safely use spatially homogeneous configurations, we need only to take care of temporal gradients. To be sure that these are not effective in changing much the dynamics away from ΛCDM, we need to ask that the non-minimal coupling terms disappear for a sufficiently dense or hot fluid.

This requirement suggests that our functions ψ, ξ must be strongly peaked around ρ_*. Concretely, this means that as the density reaches the value ρ_*, then we get modified cosmological evolution, until ρ drops well below ρ_*. If we take today cosmological DM density to be of the order of $.24 \times 10^{-29}\text{g/cm}^3$ and the reference density to be $\rho_* \approx 10^{-21}\text{g/cm}^3$ – the typical value for dwarf spheroidal galaxies – we get that

$$1 + z_* = \left(\frac{\rho_*}{\rho_0^{dm}}\right)^{1/3} \sim 700\,. \tag{4.32}$$

This seems to indicate that our model may strongly affect the background evolution in a small redshift window in the matter dominated era, something for which there is no evidence. Nonetheless, it is not obvious that these modifications of the early universe dynamics could not be made compatible with current observations. We just notice here that the latter are normally able to cast strong constraints, for example via the CMB or BBN.

The above discussion holds only if the NMC is taken as fundamental so that its action is present all along the whole history of the universe. We have no reason to believe that this is true and we shall argue below reasons to expect the contrary.

4.3 Origin of the non-minimal coupling

Up to now we have not given any reason why only dark matter should couple non minimally to gravity. Furthermore, we have seen that a parametrization of the functions ξ, ψ with only densities might lead to discrepancies from the expected behavior starting at relatively large redshifts. Therefore, to address this tension we need to understand more of the possible mechanisms that can lead to the non-minimal coupling as a phenomenologically more accurate description of the dark matter fluid.

The fact that only DM couples to gravity in a non trivial way may be seen as a violation of the weak equivalence principle (WEP). However, here we are dealing with fluids, not elementary particles. Hence WEP is safe as long as single particles have the same coupling with gravity, while the WEP can be nonetheless violated at the level of the collective behavior of the fluid.

There are two main mechanisms that may produce a non-minimal coupling: either it appears through an averaging procedure that brings from particles to fluids or it can emerge from some collective behavior of the DM particles.

In the first case there is a scale, the averaging scale which depends on the number density of the DM particles. If these are heavy, the size of the averaging scale may be large enough to be comparable with the curvature radius of the galaxy and hence generate a non-minimal coupling, given that the minimal cell needed to define a fluid element is able to probe geometry in a nonlocal way, becoming explicitly sensitive to curvature. In this case, however, the reasoning applies to DM as well as to baryons, for which the non-minimal coupling does not seem so well motivated (see, however [158–160] for a proposal to explain dark matter as an effect of non-minimally coupled baryons).

The second picture is related to the possibility for DM particles to develop a macroscopic coherence length. A recent investigation in this sense is represented by BEC [141,143].[2] The condensate possesses a characteristic coherence length, the healing length, that controls the deviation of the fluid dynamics of the condensate from the one of an ordinary perfect fluid. The BEC option seems to be rather intriguing for our model, given that it would be able to reconcile the puzzle between the large density MONDian regime of galaxies and of ΛCDM in the early universe. Notice that, while BEC is a macroscopic quantum configuration of matter, it admits a rather standard hydrodynamical description, given by the Gross–Pitaevski equation for the classical condensate wave function.

The answer to the puzzle would be that the functions ψ, ξ, besides the density, depend on the *temperature* of the fluid itself: if the temperature of the fluid is smaller than the critical temperature, condensation sets in, and with it the non-minimal coupling (provided that the coherence length is large enough). On the contrary, if the fluid is too hot, the condensation is impossible, and it behaves like an ordinary one. Noticeably, in trapped BECs, the critical temperature increases with the depth of the potential well in which they

[2] See also [142] for a slightly different approach to the solution of the core-cusp problem, but also [144] for a counterexample to it.

are confined. Similarly, clumping of dark matter halos at the galactic scales might raise the critical temperature above the temperature of the DM fluid, triggering condensation. On the contrary, large density but too high temperatures, as in the high redshift universe, might make condensation impossible.

In this sense, it is intriguing the idea that, due to the formation of deep enough gravitational potential wells, a dark matter condensation can be triggered at suitable scales and times and that this phenomenon might be indeed considered as a candidate for the physical origin of the here generalized non-minimal coupling. While this is an exciting perspective worth exploring, we feel that some caution should be used, especially when applying our laboratory based intuition of BEC features to cosmology.

First of all, for this mechanism to take place and be effective in cosmology, a tight balance between the microscopic properties of the dark matter bosons and the various macroscopic parameters observed must be realized (*e.g.,* the required size of the healing length, needed to solve the cusp problem, is of the order of some parsecs).

Secondly, there is a big qualitative difference between the fluid dynamics of a standard BEC and the fluid dynamics of the NMC fluid that we have explored in this chapter. In fact, the pressure of the BEC gets corrected by the so-called quantum potential,

$$p_{BEC} = p_{\text{hydro}}(\rho) + V_Q; \qquad V_Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}}. \tag{4.33}$$

This gives rise to a dependence of the pressure of the fluid on the gradients of the density closely resembling what found in the Newtonian limit of our model. However, it is easy to see that no anisotropic stresses are present in this case while the NMC seems to lead generically to the appearance of off-diagonal terms in the SET. This issue probably requires a more accurate analysis possibly by considering more general theoretical settings for the condensation with respect to the standard one based on scalar fields.

In this direction, we are currently generalizing the BEC theory to a full GR regime, *i.e.,* we consider a relativistic BEC in a curved spacetime, including a NMC as well [170]. Our preliminary investigation confirms the nature of the corrections to the Poisson equation (4.22), even though we found that the BEC fluid representation and the actual fluid variables are related in a subtle way, thus leading to potential confusion when comparing normal to condensed fluids. Moreover, one can show that anisotropic stresses in the BEC are indeed present as a consequence of the quantum potential.

4.4 Summary

In this chapter we have laid down a new framework for DM fluid dynamics. In particular, we have shown how a NMC for a DM fluid can provide the latter with an effective pressure which can be relevant in solving the small scales issues of the CDM paradigm providing, for example, a characteristic scale for the collapse. We have also isolated some peculiar effects related to this model, like the presence of anisotropic stresses or the fact that the source of the gravitational potential depends on the way DM is distributed. We have also envisaged a possible candidate which might lead to such phenomenology: a scalar field which has undergone a phase transition. This seems to be a good choice as it implies a two phase dynamics. Before the transition the fluid behaves as standard CDM, albeit being slightly self-interacting as needed in order to induce the phase transition for suitably deep gravitational wells and DM densities. Such a regime would be well suited for reproducing the observed phenomenology of CMB and early universe. After the transition the fluid develops a macroscopic coherence length which is assumed to be the responsible for the onset of the NMC. In fact, we argued that if a fluid possesses a macroscopic characteristic length scale then it would be able to probe gravity in a non-local way, thus developing NMC of the sort discussed here.

Of course, further investigations are needed. On one side, we need to focus on extracting more detailed predictions from the model, for example by considering the issue of structure formation, *e.g.,* investigating linear perturbations or the effects of the NMC on galaxy rotation curves. On the other side, it is worth exploring the origin of the extended non-minimal coupling of dark matter both for its connection with ideas about the nature of dark matter (BEC) as well as for its implications with regard the particle physics nature of this evasive cosmological component.

In the next chapter we will stick to this program in two ways. First we will further investigate the formal properties of this fluid scalar-tensor theory by constructing its Einstein frame version and showing how it is related to the Jordan one by a generalization of the conformal metric transformation; secondly we will study the effects of the NMC on background cosmology and linear perturbations.

Chapter 5

Non-minimally coupled dark matter: Cosmology in the Einstein frame

In the previous chapter we addressed the small scale issues of the CDM paradigm through a generalization of the interactions between DM and gravity. This is implemented by considering a DM fluid that gets non-minimally coupled to curvature terms. We report for convenience the action

$$S = \kappa^2 \int d^4x \sqrt{-\bar{g}} \left[(1 + \alpha_{\mathrm{S}} \psi(\rho_{\mathrm{DM}})) \bar{R} + \alpha_{\mathrm{R}} \xi(\rho_{\mathrm{DM}}) \bar{R}_{\mu\nu} u^\mu u^\nu \right] + S_{\mathrm{DM}}[\bar{g}, \rho_{\mathrm{DM}}] + S_{\mathrm{SM}}[\bar{g}, \rho_{\mathrm{SM}}] \,, \tag{5.1}$$

where $\kappa^2 = \frac{c^3}{16\pi G_N}$ and S_{SM} refers to the action for standard model fluids. In the weak field limit, this model is able to provide several new welcome features for the gravitational dynamics of the DM fluid, for example a modified Poisson equation (4.22). The phenomenological idea that brought to the construction of the aforementioned model is that DM may develop, under suitable conditions, geometrical properties, thus making this matter component to directly enter the definition of the metric along which matter fields move (4.28). This is achieved assuming that DM undergoes a phase transition at suitably late times and small scales, thus developing a coherence length that forces the DM fluid to couple to the curvature.

Here we will further investigate this model and in particular its cosmological consequences on the background and linear perturbation evolution. To do so we exploit the invariance of Scalar-Tensor Theories under conformal transformation of the metric to map the Jordan frame action (5.1) into the equivalent representation of the theory provided by the Einstein frame, where the gravitational sector is described by the standard Einstein–Hilbert action and the NMC is translated into couplings between particles. This is standard in the context of Scalar-Tensor Theories of gravity [171] and, being our model an extension of these [6], a similar procedure is as well viable under certain approximations. When in the Jordan frame the NMC (5.1) is taken to be with the Einstein tensor, *i.e.*, $\alpha_{\rm S} = \alpha_{\rm R}/2 \equiv \epsilon$ and $\xi = \psi$, the change of frame can be achieved with the following metric transformation

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \,, \tag{5.2}$$

where $h_{\mu\nu} = \psi(\rho_{\rm DM})u_\mu u_\nu$, being ψ a generic function of the DM density and u_μ the DM four velocity; this is a particular case of the most general relation between metrics that respect both causality and the WEP and is called disformal transformation [100]. We stress that here $h_{\mu\nu}$ is a function of DM density itself and hence the metric $\bar{g}$ now depends on the DM field as well as on the gravitational metric g. If we re-express action (5.1) in terms of the metric $g_{\mu\nu}$ and then expand in powers of $h_{\mu\nu}$ up to order $\mathcal{O}(h^2_{\mu\nu})$ we obtain the following action

$$S = S_{\rm HE}[g] + S_{\rm DM}[g,\rho_{\rm DM}] + S_{\rm SM}[g,\rho_{\rm SM}] + S_{\rm int}[g,\rho_{\rm SM},\rho_{\rm DM}] \,, \tag{5.3}$$

where $S_{\rm HE}[g]$ is the standard Einstein–Hilbert gravitational action, $S_{\rm DM}[g,\rho_{\rm DM}]$ and $S_{\rm SM}[g,\rho_{\rm SM}]$ are the DM and standard model actions in the metric g, while $S_{\rm int}[g,\rho_{\rm DM},\rho_{\rm SM}]$ is

$$S_{\rm int} = -\frac{\epsilon}{2}\int d^4x\sqrt{-g}\,(T^{\mu\nu}_{\rm DM} + T^{\mu\nu}_{\rm SM})\,h_{\mu\nu}(g,\rho_{\rm DM}) \,, \tag{5.4}$$

and represents a new interaction term which in general involves the metric, DM and standard model particles. As usual the stress energy tensor $T_{\mu\nu}$ for the i-th component is given by

$$T^i_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_i}{\delta g_{\mu\nu}} \,. \tag{5.5}$$

In this new frame the effects of the NMC have been transferred into a coupling term for the stress energy tensors of DM and standard model fluids and $h_{\mu\nu}$ which, as stressed before, is itself a function of the metric as well as of DM fluid variables. This translates directly into a coupling between DM and standard model fluids and a self coupling for DM. Notice that those are not to be intended in the particle physics sense but they rather emerge from a geometrical coupling between DM and gravity. A manifestation of the geometric origin of the interaction is the universality of the coupling that in fact affects all matter species in the same way. This is indeed a relevant point as one does not have the freedom to suppress/enhance the strength of the interaction for a particular matter species leaving unchanged the others, as this would result in a violation of the WEP.

We further stress here that the action (5.3) is obtained from (4.3) through an expansion and hence the two actions are equivalent up to order $\mathcal{O}(h^2)$. In fact, consider the action

$$S = S_{\rm HE}[g] + S_{\rm DM}[g,\rho_{\rm DM}] + S_{\rm SM}[g,\rho_{\rm SM}] + \\ + \int d^4x\sqrt{-g}G^{\mu\nu}h^{(1)}_{\mu\nu}(g,\rho_{\rm DM}) - \frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}_{\rm M}h^{(2)}_{\mu\nu}(g,\rho_{\rm DM})\,, \quad (5.6)$$

where the subscript M refers to the total matter SET and where $h^{(i)}_{\mu\nu}(g,\rho_{DM})$ are generic functions of the metric and DM variables. Now consider the metric transformation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(3)}_{\mu\nu}(g,\rho_{\rm DM})\,. \quad (5.7)$$

If we take $h^{(i)}_{\mu\nu}(g,\rho_{\rm DM})$ to be small so that a perturbative approach is well defined then we can express the action in terms of the transformed metric and expand at linear order. We get:

$$S = S_{\rm HE}[g] + S_{\rm DM}[g,\rho_{\rm DM}] + S_{\rm SM}[g,\rho_{\rm SM}] + \\ + \int d^4x\sqrt{-g}G^{\mu\nu}\bar{h}^{(1)}_{\mu\nu}(g,\rho_{\rm DM}) - \frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}_{\rm M}\bar{h}^{(2)}_{\mu\nu}(g,\rho_{\rm DM})\,, \quad (5.8)$$

where

$$\bar{h}^{(1)}_{\mu\nu}(g,\rho_{\rm DM}) = h^{(1)}_{\mu\nu}(g,\rho_{\rm DM}) - h^{(3)}_{\mu\nu}(g,\rho_{\rm DM})\,, \quad (5.9)$$

$$\bar{h}^{(2)}_{\mu\nu}(g,\rho_{\rm DM}) = h^{(2)}_{\mu\nu}(g,\rho_{\rm DM}) - h^{(3)}_{\mu\nu}(g,\rho_{\rm DM})\,. \quad (5.10)$$

This shows how, at linear level, the action (5.6) is formally invariant under the metric transformation proposed in the same way as Scalar-Tensor Theories are. This means that we are free to choose the shape of the $h^{(3)}$ metric function such that the action has a NMC but no SET coupling (Jordan frame) or the vice versa (Einstein frame), being the two choices just different representations of the same theory. In chapter 6 we will discuss in depth the issue of the equivalence between frames in the context of single scalar-field theories, here we only comment that when dealing with fluids it is not obvious that the equivalence holds beyond the perturbative level we are using here. In any case, this is not crucial for the forthcoming analysis because we could have started directly from the action (5.3), considering the particular form of couplings as a phenomenological ansatz.

5.1 Dark matter with an effective pressure

Our model has been designed to modify the standard dynamics at late times. Hence, as a first investigation, we can neglect the contribution coming from baryons and photons as they are subdominant components during matter domination. Moreover we are interested in the growth of DM perturbations and hence we can consider the effects of the NMC only on this species. Of course, if we were to investigate the effects of the NMC on the CMB spectrum, we should have retained also photons and it is clear that a coupling between DM and baryons may have major consequences on the dynamics at galactic scales, which are worth investigating.

The action (5.3) is defined in terms of fluid variables, because, as we pointed out, these are the most appropriate quantities to describe cosmological matter fields. However, we found that the field language is more tractable when dealing with coupled matter fields in the Einstein frame. Hence we switch to field formalism using standard conventions [172]:

$$\rho = \rho(X, \varphi)\,, \qquad p = p(X, \varphi)\,, \qquad u_\mu = \nabla_\mu \varphi / \sqrt{2X}\,, \tag{5.11}$$

where now φ is the DM scalar field and $X = -g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi/2$ is its kinetic part. The field variables are not fundamental fields, but rather stand for a different representation of the fluid. In this sense we must ensure that in absence of a coupling we recover the equations for the standard DM fluid. In general a scalar field will not behave as a pressureless dust; actually, it does not reproduce any fluid with a time independent equation of state. In fact, the pressure for a scalar field is defined in terms of its potential $V(\varphi)$ and kinetic energy $X(\varphi, \partial\varphi)$ as $p = X - V^2(\varphi)$ which means

$$w \equiv \frac{p}{\rho} = \frac{X - V^2(\varphi)}{X + V^2(\varphi)}\,, \tag{5.12}$$

where the potential appears to the second power for later convenience. In order for this to be constant we need to specify a relation between the potential and the kinetic term which is *time independent*, otherwise we can at most find asymptotic limits, *e.g.,* a CC-like equation of state can be mimicked if the scalar field kinetic energy is subdominant in some regime.

In order to enforce a constant equation of state, actually the one for pressureless dust, we introduce a Lagrangian multiplier in the Lagrangian for the DM field

$$\mathcal{L} = \lambda(x^\mu)\left(X - \frac{1}{2}V^2(\varphi)\right)\,, \tag{5.13}$$

where $V(\varphi)$ is a generic potential so that when we take the variation of the action with respect to λ we get

$$\frac{\delta S_{DM}}{\delta\lambda} = 0 \Rightarrow X - \frac{1}{2}V^2(\varphi) = 0\,, \tag{5.14}$$

which exactly sets the pressure to zero [173, 174].

We have already discussed how in our model, in order for the interaction to be active, the temperature of the cosmological bath (*i.e.,* time or redshift) must be below a certain threshold. This is implemented through a time dependence of the coupling ϵ appearing in the interaction Lagrangian (5.4). However, even after the critical temperature has been reached, the density may be too low for the interaction to be efficient. Hence, one needs high densities in order to make the interaction relevant. In order to implement phenomenologically the desired scale and time dependences of the coupling, we fix the parametrization of the coupling function in the following way:

$$h_{\mu\nu}(X,\varphi) = \frac{F(X,\varphi)}{\rho_*}\nabla_\mu\varphi\nabla_\nu\varphi\,. \tag{5.15}$$

The function F gives the scale dependence of the interaction by suppressing it until the density is large enough to overcome a given threshold fixed by ρ_*.

The action (5.3) when only DM is present, can then be rewritten in terms of the field variables as:

$$S = S_{\rm HE}[g] + S_{\rm DM}[g,\varphi,\lambda] + \epsilon\int d^4x\sqrt{-g}\mathcal{L}^{\rm NMC}(X,\varphi)\,, \tag{5.16}$$

where now $S_{\rm DM}$ depends on $\{\varphi,\lambda\}$ rather than on $\rho_{\rm DM}$ and

$$\mathcal{L}^{\rm NMC} = -\frac{F(X,\varphi)}{\rho_*}X\left(X+\frac{1}{2}V^2(\varphi)\right)\,, \tag{5.17}$$

is the interaction term generated by the NMC. As said, the coupling ϵ is a switch that is chosen to be zero at $t < t_c$, t_c being the time when the coupling is activated; for $t > t_c$, ϵ becomes different from zero, reaching a constant value of about $\epsilon < 1$, as required by the expansion of $h_{\mu\nu}$ to order (ϵ^2). In other words, ϵ becoming different from zero indicates the onset of the NMC epoch. At the same time, for the purpose of the present analysis, which is limited to order $\mathcal{O}(\epsilon^2)$, its value must be small compared to one.[1] From a cosmological point of view the time dependence of the coupling is required because we want to study how the NMC modifies the ΛCDM behavior at small scales and late times only, as we stressed in the introduction and discussed in chapter 4. To be fully rigorous we should have given ϵ a spatial dependence as well. That would cause the activation of the NMC to happen at different times and in different regions. Such a spatial dependence would give rise to a most interesting phenomenology but strongly dependent on which powering mechanism is chosen for such a dependence on space. We therefore do not discuss it here leaving its analysis for further studies.

[1] ϵ is just a phenomenological parameter that should be given dynamically by the NMC generating mechanism. In this sense we are not introducing a non-dynamical field and background independence should be though of as preserved.

The variation of (5.16) with respect to φ and λ, derived in a detailed way in appendix A, gives the two combined equations of motion which together specify the fluid dynamics

$$\dot{\lambda} = V^{-2}(\varphi)\left[V(\varphi)\rho_{,\varphi} - (\rho + \epsilon p)\vartheta\right] , \quad (5.18)$$

$$\dot{\varphi} = -V(\varphi) , \quad (5.19)$$

where

$$\dot{()} \equiv u_\mu \nabla^\mu, \qquad \vartheta \equiv \nabla_\mu u^\mu , \quad (5.20)$$

and $\rho_{,\varphi}$ is the derivative of the density with respect to the field φ. The energy density ρ and pressure p are derived from the total DM SET

$$T_{\mu\nu} = \left(\lambda + \epsilon\mathcal{L}^{\rm NMC}_{,X}\right) V^2(\varphi) u_\mu u_\nu - \epsilon\mathcal{L}^{\rm NMC} g_{\mu\nu} . \quad (5.21)$$

By a direct comparison with the form of the perfect fluid SET we can identify the following thermodynamic quantities

$$\rho = \left(\lambda + \epsilon\mathcal{L}^{\rm NMC}_{,X}\right) V^2(\varphi) - \epsilon\mathcal{L}^{\rm NMC} , \quad (5.22)$$

$$p = \mathcal{L}^{\rm NMC} , \quad (5.23)$$

$$u^\mu = V^{-1}(\varphi)\nabla^\mu\varphi . \quad (5.24)$$

Notice that they are not the same appearing in action (5.3) because here the interaction term is directly involved in the definition of both density and pressure. In other terms in (5.3) ρ_{DM} is the DM density for a pressureless fluid which has some non trivial self interaction while here the interaction has been absorbed into the definition of the DM energy. ρ_{DM} in (5.3) is related to the $\{\lambda, \varphi\}$ variables by the relation $\rho_{DM} = \lambda V(\varphi)^2$. Since our knowledge of the DM distribution comes through its gravitational effects, the definition (5.22) gives the actual measured DM density.

The effect of the NMC is twofold: on the one side it modifies the DM energy density while on the other side it introduces a pressure term that would be absent in the standard ΛCDM scenario. We expect both these terms to have relevant cosmological consequences at the time and scales of interest, as it will be shown below. We anticipate that this model is able, with appropriate potential shapes, to reduce the source of the gravitational potential with the consequence of smoothing out the overdensities at small scales. The equations for a pressureless dust (5.18)-(5.19) in the limit of vanishing coupling are independent on the choice of the potential [173] that is to say the density scales like a^{-3} no matter what potential is chosen. This does not represent a problem since the interaction term, which in turns fixes the shape of the potential, is fixed once the mechanism that generates the

NMC is known. We will not investigate the ultimate nature of the NMC here but we will give some examples of potentials that could lead to an interesting phenomenology.

Finally, we notice that the pressure defined in (5.23) acts only along the direction defined by the fluid four velocity, as it can be seen from the fact that the four acceleration a_ν is identically zero. In fact,

$$a_\nu \equiv u^\mu \nabla_\mu u_\nu = V^{-1}\nabla^\mu\varphi\nabla_\mu(V^{-1}\nabla_\nu\varphi) = \frac{1}{V}\left(-\frac{dV}{d\varphi}\nabla_\nu\varphi + \nabla^\mu\varphi\nabla_\mu\nabla_\nu\varphi\right) = 0\,, \quad (5.25)$$

where we have used the constraint (5.14). As a consequence this model does not show any spatial pressure and we do not have anisotropic stresses that in more general situations may be present as discussed in chapter 4. This is due to the particular mechanism to implement the condition $p = 0$ for the scalar field when the NMC interaction is switched off.

5.2 Background and linear perturbations dynamics

We will now study the cosmological consequences of our model. We will assume a flat FRLW universe filled with the DM field plus a CC.[2] In terms of the conformal time (2.23) we have

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left[\left(\lambda + \epsilon\mathcal{L}^{\rm NMC}_{,X}\right)V^2(\varphi) - \epsilon\mathcal{L}^{\rm NMC} + \rho_\Lambda\right]\,, \quad (5.26)$$

$$\lambda' = -a(\tau)V^{-2}(\varphi)\left[V(\varphi)\rho_{,\varphi} + 3\mathcal{H}(\rho + \epsilon p)\right]\,, \quad (5.27)$$

$$\varphi' = a(\tau)V(\varphi)\,, \quad (5.28)$$

where primes indicate time derivatives and $\rho_\Lambda = 3\Lambda/(8\pi G)$. The equation for λ can be rewritten in terms of the more physical quantity ρ defined by equation (5.22) which is what appears on the right hand side of the Friedman equation (5.26) with no CC. In this respect we are defining the density of the DM fluid as the quantity which plays the role of gravitational source. With this consideration the previous system of equations is

$$\mathcal{H}^2 = \frac{8\pi G a^2}{3}\left[\rho + \rho_\Lambda\right]\,, \quad (5.29)$$

$$\rho' + 3\mathcal{H}(1 + \epsilon w(\varphi))\rho = 0\,, \quad (5.30)$$

$$\varphi' = a(\tau)V(\varphi)\,, \quad (5.31)$$

[2] Since here we are only interested in cluster/galaxy scales, we are not concerned about the nature of dark energy. We thus make the minimal choice of a CC.

where $w(\varphi) = p/\rho$. The continuity equation can be interpreted as that of a fluid with a field dependent equation of state which in turns means a time dependent equation of state.

We now present the equations for the linear perturbations in the Newtonian gauge for scales which are well inside the horizon following the notation of [12], introduced in chapter 2. The system of equations governing the evolution of the linear perturbations is given as usual by the continuity equation, the Euler equation and the Poisson equation; however, in this case, there are some differences due to the fact that we are expanding at linear order around the coupling ϵ, in order to be consistent with the linearization of the action (5.3). Moreover, the constraint (5.19) provides an extra perturbation equation

$$\delta\varphi' = a\left[V_{,\varphi}(\varphi)\delta\varphi + \mathbf{\Phi} V(\varphi)\right] \,, \tag{5.32}$$

and in the class of models under consideration extra relations exist that link together some of the variables. We have, in fact,

$$\begin{aligned} v &= V(\varphi)^{-1} k\delta\varphi \,, \\ \delta p &= p_{,\varphi}\delta\varphi \,. \end{aligned} \tag{5.33}$$

The last relation has an important consequence on the dynamics of the NMC DM fluid as it sets to zero the fluid speed of sound

$$c_s^2 \equiv \frac{dp}{d\rho} = p_{,\varphi}\frac{d\varphi}{d\rho} = 0 \,, \tag{5.34}$$

given that φ and ρ are two independent variables.

Hence, in the Newtonian gauge for scales much smaller than the Hubble scale, $\lambda \equiv \mathcal{H}/k \ll 1$, the system of equations for the evolution of linear perturbations is the following

$$\delta\varphi' = a\left(V_{\varphi}(\varphi)\delta\varphi + \mathbf{\Phi} V(\varphi)\right) \,, \tag{5.35}$$

$$\delta\rho' + hkv + 3\mathcal{H}\left(\delta\rho + \epsilon p_{\varphi}\delta\varphi\right) = 0 \,, \tag{5.36}$$

$$v' + \left(\mathcal{H} + \epsilon\frac{p'}{\rho}\right) v - k\mathbf{\Phi} = \epsilon k\frac{\delta p}{\rho} \,, \tag{5.37}$$

$$k^2\mathbf{\Phi} = 4\pi G a^2 \delta\rho \,. \tag{5.38}$$

Let's consider now the Euler equation (5.37). Using (5.33) the two terms proportional to ϵ cancel giving formally the same expression as in the standard, pressureless case

$$v'(k,\tau) + \mathcal{H}(\tau)v(k,\tau) - k\mathbf{\Phi}(k,\tau) = 0 \,. \tag{5.39}$$

However here $\mathcal{H}$ is the modified Hubble expansion rate obtained by solving equation (5.26). Eq. (5.39) is therefore actually non standard both in the friction term and in the gravitational potential that feeds it, which, from eq. (5.38), can be rewritten as

$$k^2\mathbf{\Phi}(k,\tau) = 4\pi Ga^2 Q(k,\tau)\rho(\tau)\delta(k,\tau)\,, \tag{5.40}$$

where the Q function measures deviation from the ΛCDM model which has $Q = 1$ and where δ is the dimensionless density contrast $\delta \equiv \delta\rho/\rho$. Explicit shapes for this function are given in section (5.3). Finally we have

$$\delta'(k,\tau) - 3\mathcal{H}(\tau)\epsilon w(\tau)\delta(k,\tau) + (1+\epsilon w(\tau))kv(k,\tau) \;=\; 0\,, \tag{5.41}$$

$$v'(k,\tau) + \mathcal{H}(\tau)v(k,\tau) + k\mathbf{\Phi}(k,\tau) \;=\; 0\,, \tag{5.42}$$

$$k^2\mathbf{\Phi}(k,\tau) = 4\pi Ga^2 Q(k,\tau)\rho_{mc}(\tau)\delta(k,\tau)\,, \tag{5.43}$$

where $\delta \equiv \delta\rho/\rho$ and ρ_{mc} is the minimally coupled background DM density. With this formalism the Q function takes the general form $Q(k,\tau) = (1+\mathcal{O}(\epsilon))$, where the corrections come from the modified background DM density.

From eq. (5.41) we can see that the continuity equation is modified in two ways: the last term can lead to a speed up or slow down of the growth of perturbations, depending on the sign of ϵ; the second term on the left hand side is a new genuine effect of this model that closely resembles a dilution term.

The Euler equation is modified in two parts: $\mathcal{H}(\tau)$ is modified as in the Friedman equation (5.26) and the gravitational potential is changed as from the Poisson equation (5.40). Notice also that despite the presence of an effective pressure, no Jeans length appears in the equation. This is a consequence of the time-like character of the pressure term, as noted above. Deviation from the ΛCDM model can be parametrized by two functions [84]: $\zeta = (\mathbf{\Psi}+\mathbf{\Phi})/\mathbf{\Phi}$, that characterizes the effects of anisotropic stresses, and Q related to the deviation from the standard Poisson equation. The ΛCDM model has $\zeta = 0$ and $Q = 1$. We have no anisotropic stresses and hence $\zeta = 0$ as well. However the Q function is in general different from unity and hence it is a measure of the departure from ΛCDM. As said, $Q = 1 + \epsilon f(\rho, p)$ in general. Explicit forms for the function f will be given below.

It is a standard procedure to take the derivative of (5.41) and using (5.42) and (5.43) to obtain one single equation describing the evolution of the linear perturbations. In our case this gives

$$\delta'' + \left[\mathcal{H} - \epsilon\left(\mathcal{H}w - w'\right)\right]\delta' + 3\mathcal{H}\epsilon\left[\mathcal{H}\left(\frac{1}{2}+w\right) - w'\right]\delta - \frac{3}{2}(1+\epsilon w)\mathcal{H}^2 Q\Omega_{mc}\delta \;=\; 0\,, \tag{5.44}$$

Potential	**Pressure**	**F(ϕ,X)**	**Effective coupling ($\epsilon_{\rm eff}$)**	**coupling redshift**
$V(\varphi)=\sqrt{\rho_0^{DM}}$	$p(z)=-(\rho_0^{DM})^2/\rho_*$	1	$(\epsilon\rho_0^{DM}/\rho_*)=1$	$z_c=5$
$V(\varphi)=\sqrt{\rho_0^{DM}}e^{-\kappa\varphi}$	$p(z)=-K\,\mathrm{arcsinh}\left[\left(\frac{\Omega_\Lambda/\Omega_{DM}}{(1+z)^3}\right)^{1/2}\right]^4$	1	$(\epsilon\rho_0^{DM}/\rho_*)=-5\times10^{-3}$	$z_c=5$

Table 5.1: Functions and parameters. In the first column the two potentials used in the paper are reported while in the second column the related pressures as a function of the redshift are given. The third column reports the used value for the scale function F. The value is set to a constant meaning that no scale dependence is present. The last two columns report the value for the effective coupling constant $\epsilon_{\rm eff}=\epsilon\rho_0^{DM}/\rho_*$ and the redshift at which the NMC is activated.

which reduces to the standard ΛCDM equation (2.76) in the limit $\epsilon\to0$. As already anticipated, notice that there is no scale dependence in this equation, despite the presence of a non-zero pressure; this has relevant consequences as it means that our model, as it stands, affects the DM dynamics at all scales. For completeness we give also the Euler equation in real space

$$v'(x,\tau)+\mathcal{H}(\tau)v(x,\tau)=-\nabla\mathbf{\Phi}(x,\tau)\,. \tag{5.45}$$

Note again that here both $H(\tau)$ and $\mathbf{\Phi}(x,\tau)$ are modified according to equations (5.26) and (5.43) respectively.

5.3 Results

In this section we present the results of the integration of equations (5.30)-(5.44) for different choices of the potential which directly translates in different time behaviors for the pressure. In the absence of a clear mechanism that can predict the form of the potential, we consider various examples and derive their possible cosmological implications. We will first explore the case in which the potential for the field φ is a constant and then we will consider a decaying exponential potential.

As a further work assumption we choose $F=1$. This means that the coupling is only time dependent and not also scale dependent so that as soon as the critical temperature is reached the coupling is active everywhere. This is a crude simplification which enables us to illustrate, in a first concrete example, the phenomenological impact that the DM

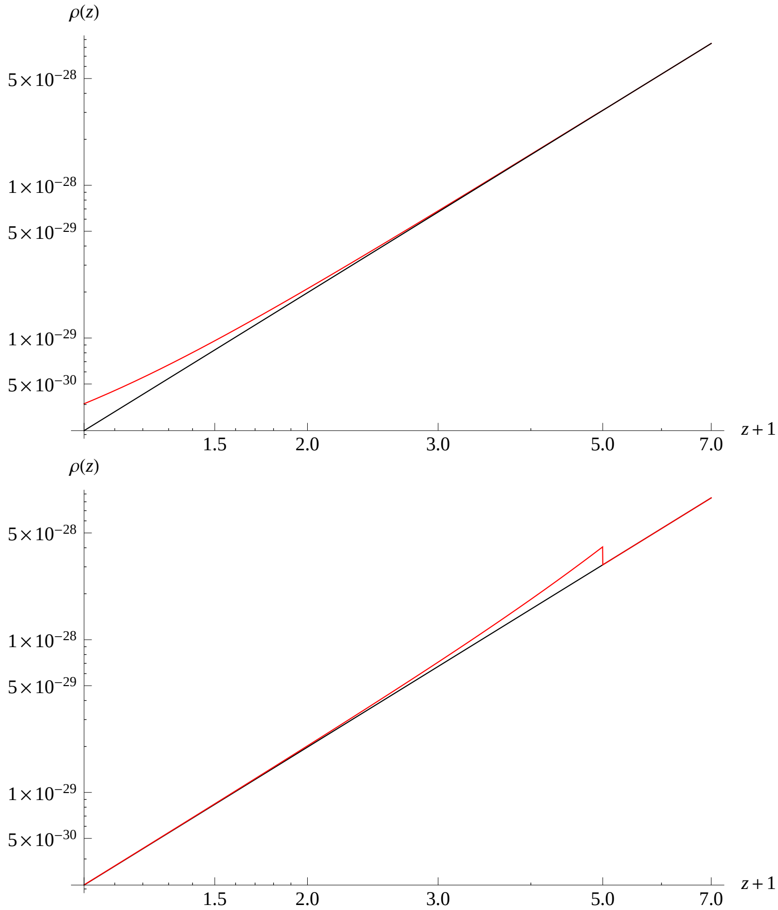

Figure 5.1: Background density plots for (a) the constant potential (top panel, red line) and (b) the exponential potential (low panel,red line), compared to standard pressureless DM (black dashed line). In both plots we fix $F(X, \varphi) = 1$ and $z_c = 5$. The density is in g/cm^3. The constant ϵ_{eff} in the case of the constant potential is chosen in order to give a clear idea of the effects of the geometrical interaction term. In particular, in this case, $\epsilon_{\text{eff}} = 1$.

NMC illustrated in section 5.1 can have. We leave more realistic scale dependent scenarios to future study. The set of functions and parameters used are reported in table 5.2. We also fix the cosmological parameters as follows: $\Omega_\Lambda = 0.76$, $\Omega_{DM} = 0.24$ and $H_0 = 74$ km/s/Mpc.

Constant potential

In the case of a constant potential the effects on the background density evolution are shown in figure 5.1a as compared to the evolution of standard CDM for different values of the coupling constant ϵ. Remarkably the effect of the coupling is to mimic a CC as can be seen from the background solution

$$\rho(z) = \rho^0_{\rm DM}(1+z)^3 + \frac{\epsilon_{\rm eff}}{2}\rho^0_{\rm DM}\,, \tag{5.46}$$

$$p(z) = -\frac{\epsilon_{\rm eff}}{2}\rho^0_{\rm DM}\,, \tag{5.47}$$

where $\epsilon_{\rm eff} = \epsilon\rho_{\rm DM}/\rho_*$ with $\rho^0_{\rm DM}$ the present day density of DM, and ρ_* a reference density characteristic of the scale under analysis. In this case the background DM fluid behaves as if it were composed by two fluids, a standard dust plus a fluid with a CC equation of state $p = -\rho$, as pointed out also in [173] in a different context, so that it may be possible to avoid the introduction of an extra cosmological term in eq (5.26).

The evolution for the density contrast is shown in figure 5.2a as compared to that of the density contrast for the ΛCDM model for the value of the coupling ϵ reported in table 5.2. The suppression is enhanced for illustrative purposes but it is clear that the growth of linear perturbations is suppressed with respect to the one in standard ΛCDM, in a way that mimics a ΛCDM model with a larger CC. This is also clear from figure 5.2b where the growth function $f \equiv -\log\delta/\log z$ for our model decreases faster compared to the ΛCDM function. Notice that the plots hold for all sub-horizon scales.

The Q function, defined in the Poisson equation and responsible for deviation in the gravitational potential has the following form in the case of a constant potential

$$Q(k,z) = \left(1 - \frac{\epsilon_{\rm eff}}{2(1+z)^3}\right)\,, \tag{5.48}$$

which implies that the effective gravitational constant that generates the gravitational potential is reduced at those scales at which the interaction is active.

To conclude, the effect of the gravitational self coupling with a constant DM potential is to add an extra contribution analogous to that given by a CC with the result that the growth of the density contrast is more suppressed than in the standard ΛCDM model.

Exponential potential

The exponential potential acts in a completely different way. The background DM density is shown in figure 5.1b as compared to the evolution of standard pressureless DM. The effect of the NMC is to slow down the dilution of the DM density as soon as the coupling is switched on. Notice that the sudden change in the evolution behavior is a mere consequence of the step function that switches on the coupling only for $t > t_c$. More realistically, a smoother crossing is expected. The effect of the coupling fades away with time and the model tends asymptotically to ΛCDM. This is a welcome feature as it could possibly boost the number of high redshift clusters observed around $z \sim 2$ without affecting present day halos [175, 176], similarly to the scenario pictured in [177, 178] but here relying only on the DM NMC to gravity.

In this case the explicit form of the pressure term, as obtained from eq. (5.23) is

$$p(z) = K \text{arcsinh}\left[\left(\frac{\Omega_\Lambda/\Omega_{DM}}{(1+z)^3}\right)^{1/2}\right]^4 , \tag{5.49}$$

where

$$K = \frac{1}{2}\left(\frac{3}{2}\right)^4 \rho_0 \left(\frac{3\rho_\Lambda}{8\pi G}\right)^2 . \tag{5.50}$$

In figure 5.3a the evolution of the density contrast as a function of redshift is plotted. In this case the Q function has a complicated expression, not reported here, due to the non-trivial relation between time and redshift for a ΛCDM model. We just comment that also in this case the function is always less than one, thus reducing the gravitational potential.

In figure 5.3b we plot the growth function f as a function of redshift. As in the case of the constant potential, here again we notice that the effect of the coupling is to reduce the growth of linear perturbations. In this case, however, the sign of the coupling constant is opposite. This is not surprising as ϵ is a phenomenological parameter and different potentials represent different theories: thus the sign of the coupling is not *a priori* determined. A mathematical explanation for this fact can be given in the limit of Einstein–de Sitter. In this case the equation to be solved for the density can be generally written as

$$\rho'(z) - \frac{3}{z}\left(\rho(z) + \epsilon\rho^0_{\rm DM} z^\alpha\right) = 0 . \tag{5.51}$$

This equation has the general solution

$$\rho(z) = \rho^0_{\rm DM} z^3 + 3Cz^\alpha , \tag{5.52}$$

where $C = -\epsilon/(-3+\alpha)$ with $\alpha \neq 3$.[3] For a constant potential $\alpha = 0$ and thus the particular solution to the differential equation is positive, hence the density is higher compared to ΛCDM. In the exponential case instead $\alpha = 6$ and hence the situation is reversed. Interestingly, for an exponential potential, this suppression is limited in time: perturbations are maximally suppressed around the time of the switching but asymptotically the model reduces to ΛCDM. Again, This can be interesting when trying to get a higher number of halos in the past only, without affecting present abundances.

5.4 Summary

In this chapter we have addressed the question of whether the DM fluid can behave differently at galactic scales rather than at cosmological ones, due to the presence of a time-dependent non-minimal interaction between DM and gravity. We have extended the analysis done in chapter 4, constructing the Einstein frame for the NMC DM model there introduced. We have then illustrated for the first time the cosmological consequences of such a scenario, both at the background level and within linear perturbation theory. In particular we have shown that a NMC DM fluid is able to produce two relevant effects: a pressure term for DM able to reduce the growth of structures at small scales, plus an effective interaction term between DM and baryons that can explain correlations between the two components of the cosmic fluid.

In this scenario we have considered the situation in which DM, at suitably late times, undergoes some sort of phase transition, analogous to the BEC discussed in the previous chapter, consequently developing a coherence length of a size comparable to that of the local curvature radius, thus becoming non-minimally coupled.

We have studied in details the DM pressure term, neglecting the roles of baryons in the present analysis. In particular, we have analyzed the system for two choices of the DM potential that generates the pressure term: a constant potential, resembling a CC contribution, and an exponential potential. These two choices are a good sample as all power law potentials have intermediate behaviors.

For a constant DM potential, the DM fluid behaves like the superposition of two fluids, one standard pressureless dust plus a fluid that behaves like a cosmological term, with a consequent suppression of the density contrast at small redshifts.

In the case of an exponential DM potential the effects are mostly relevant near the time of activation of the coupling. The background density is enhanced and the linear growth

[3] Notice that the case of $\alpha = 3$ would simply rescale the coefficient of the homogeneous solution.

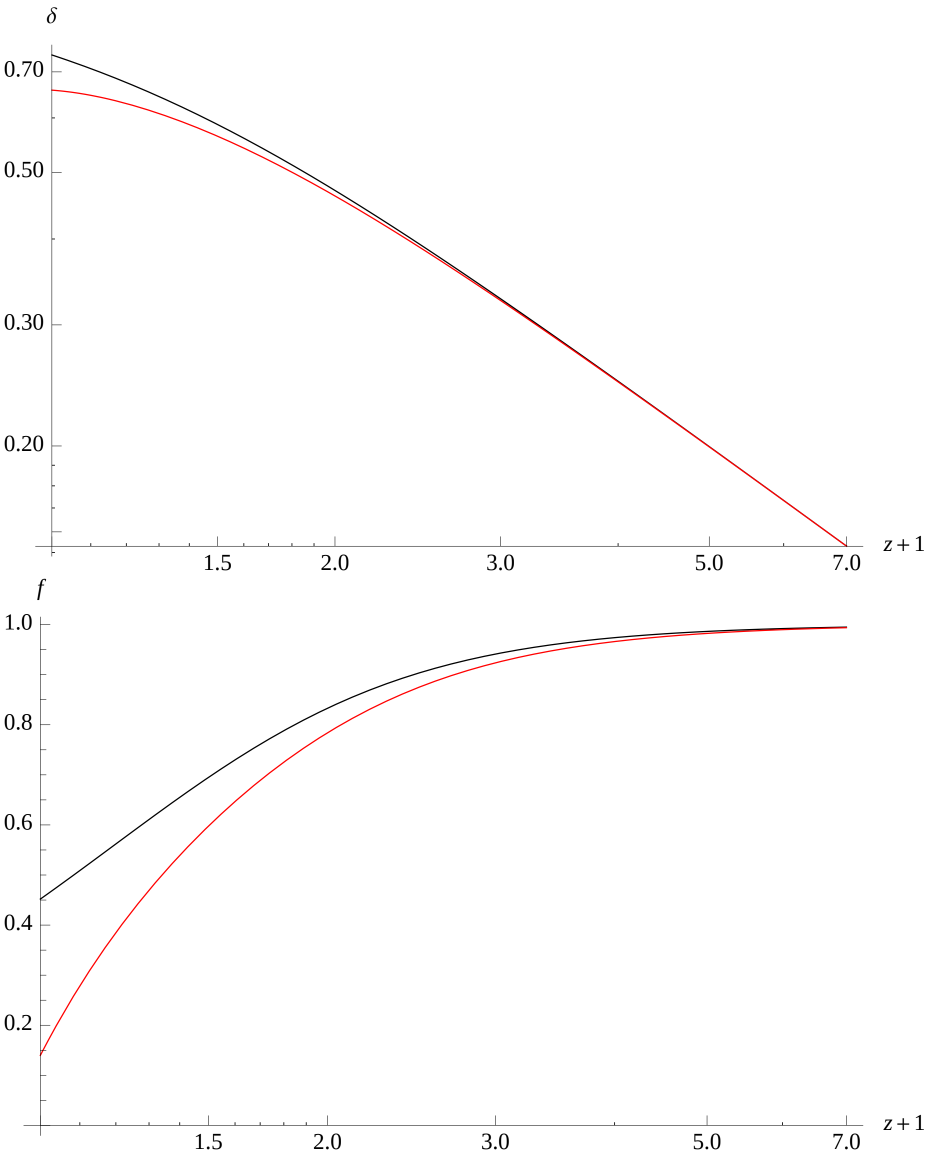

Figure 5.2: (a) Evolution of the density contrast versus redshift in the case of constant potential (top panel, red line) and (b) the growth function $f(z) = -d\log\delta/d\log z$ (low panel, red line), compared to the standard ΛCDM results (black dashed line), both with $F(X,\varphi) = 1$ and $z_c = 5$. The constant $\epsilon_{\rm eff}$ in the case of the constant potential is chosen in order to give a clear idea of the effects of the geometrical interaction term. In particular, in this case, $\epsilon_{eff} = 1$.

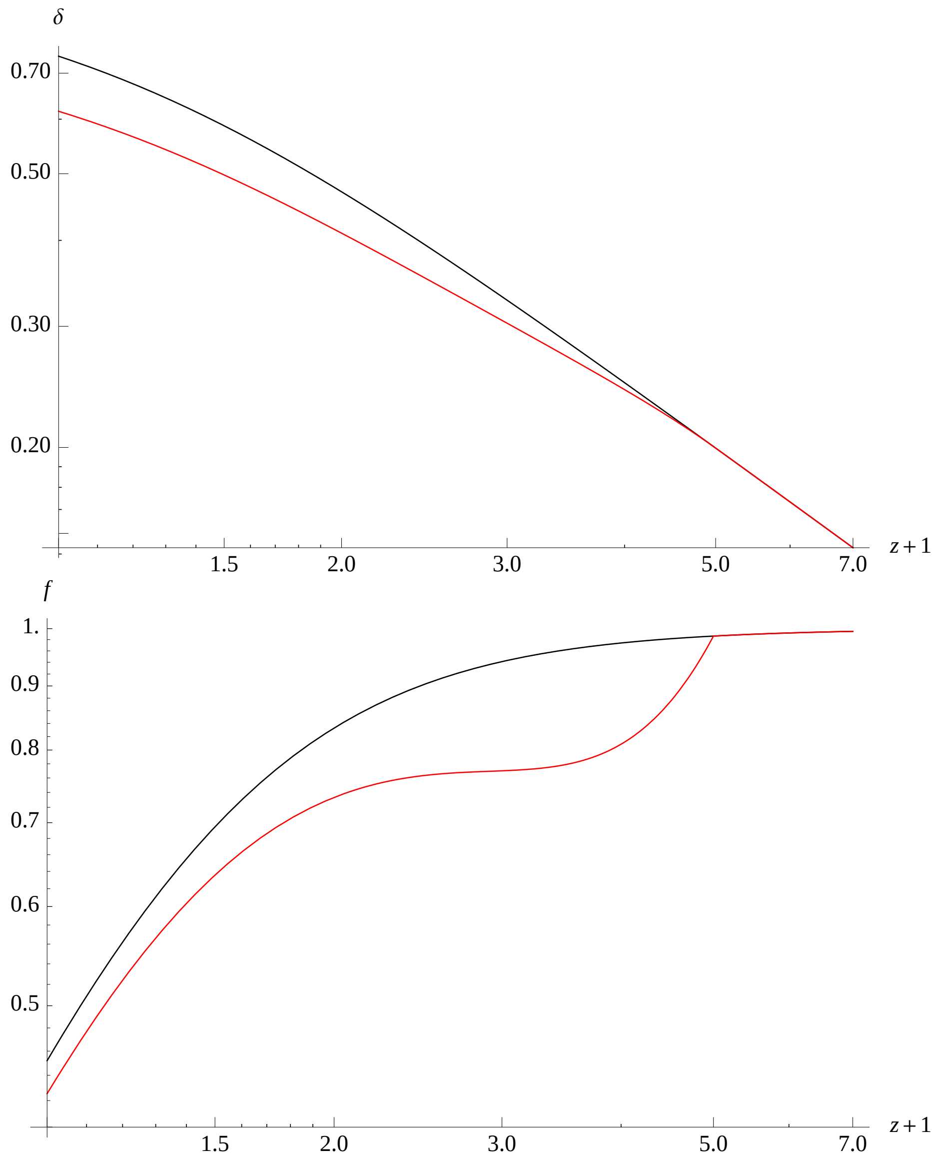

Figure 5.3: (a) Evolution of the density contrast versus redshift in the case of exponential potential (top panel red line) and (b) the growth function $f(z) = -d\log\delta/d\log z$ (low panel red line) compared with the standard ΛCDM results (black dashed line), both with $F(X,\varphi) = 1$ and $z_c = 5$.

is suppressed for a limited redshift interval. In fact the pressure decays with redshift so that standard ΛCDM evolution is recovered asymptotically.

We also provided the Euler and Poisson equations (5.42), (5.43) in a form convenient for N-body simulations for any choice of the potential V together with explicit expressions and predictions for the parameters Q and ζ, that characterize the deviations from ΛCDM. In particular we found that $\zeta = 0$, meaning that no anisotropic stresses are generated by our model, and $Q < 1$, after the coupling is switched on, thus reducing the gravitational potential. Constraints on these functions are rather weak and strongly model dependent [179–183] thus making necessary a direct confrontation between our model and observations in order to cast constraints on the deviations from ΛCDM.

For both choices of the potential we have obtained a suppression in the growth of linear perturbations as in the figures (5.2) and (5.3). This is a good indication that this class of models may be a viable possibility to solve some of the ΛCDM paradigm problems, like the core-cusp, the missing satellites, high-z clusters. Of course, a non-linear analysis is required to evaluate these effects.

We have derived the general perturbation equations valid for any $F(\varphi, X)$ and $V(\varphi)$ (see equations (5.29, 5.30, 5.41, 5.42, 5.43)), though we have limited our specific examples to $F(\varphi, X) = 1$, for simplicity. Baryons can be included, introducing a much welcome relation between DM properties and baryonic features as expected from observations [122].

On a more formal ground we have also showed how, at linear level in the metric function $h_{\mu\nu}$ appearing in (5.2), we can construct a scalar-tensor theory for fluids (5.6) which admits equivalent Jordan and Einstein frames, connected by a disformal transformation. This is interesting for at least two reasons: on one side the recent flourishing of modified theories of gravity for cosmological fluids calls for an analysis similar to the one made for standard Scalar-Tensor Theories while, on the other side, the coupling between the Einstein tensor and matter variables resembles closely the one that is included in the recently rediscovered Horndeski theory. Even if the connection between theories written in terms of fluid and those in terms of field is not completely clear, this may be a first hint in the construction of a Horndeski like model for fluids or, more generally, for vectors.

Indeed in the next chapter we will show, in a more rigorous way, how disformal transformation can be used to investigate the structure of the Horndeski action, in its standard scalar field formulation, with particular emphasis on the construction of equivalent frames.

Chapter 6

Generalized Scalar-Tensor Theories: the Horndeski action and disformal transformations

There is nowadays a rather broad set of evidences, both theoretical and observational, that points towards modifications of the standard paradigm for gravitational dynamics represented by GR. As argued in the previous chapters, modified theories of gravity seem to be very effective at addressing many of the problems of the ΛCDM model.

Among these, generalized Brans–Dicke (BD) Scalar-Tensor Theories have acquired, since their initial proposal more than half a century ago [3], a most relevant role as the standard alternative theories of gravitation. The investigation of formal aspects of these theories has played a fundamental role for several theoretical and observational issues in gravitation. In particular, Scalar-Tensor Theories have represented an ideal setting for understanding the thorny issue of the different representations of a given gravitational theory. For example, it has been realized that a whole class of higher curvature theories, $f(R)$ theories, can be recast as special cases of Scalar-Tensor Theories (with the number of scalars related to the order of the initial field equations). Even more interestingly, the invariance of the action of generalized Scalar-Tensor Theories under metric conformal transformations and redefinitions of the scalar field, can be used to relate several equivalent frames, for example trading off a space-time varying gravitational constant (*i.e.,* a non minimal coupling) for a GR-like gravitational sector (*i.e.,* minimally coupled) associated to a matter action with field-dependent masses and coupling constants.

It is worth stressing that such features are not only theoretically interesting, but are also relevant for the actual observational tests of the theory. So much so that the question of whether conformally related frames are physically distinguishable is still an open issue in the literature (see *e.g.,* [184]). Furthermore, this kind of investigations become even more important as one moves further away from GR into more general theories.

Further generalizations of the Scalar-Tensor Theories have been extensively investigated in the contexts of cosmology [171], Dark Energy [84, 98, 185], inflationary models [186, 187] and in the context of extended DM models, as discussed in the previous chapters, and have indeed provided very efficient frameworks for explaining (in an alternative ways w.r.t. GR) the observed properties of the universe.

An extension of the Scalar-Tensor framework that that has attracted a lot of interests is represented by the Horndeski action [4], recently rediscovered in the context of the Covariant Galileon theory [5, 188]. This action provides the most general Lagrangian for a metric and a scalar field that gives second order field equations and as such is a well motivated effective field theory. It has been extensively investigated since it includes, as sub-cases, basically all known models of DE and single scalar field inflation. However, this generality comes at a dear price. In fact, the physics derived from the full action is rather obscure and the theory has been investigated only in few regimes or for particular models, like the FLRW universe, so that a systematic investigation is still missing (see however [102,189] for a first attempt in this direction and [190,191] for a method to derive constraints in the context of DE models).

Given the above mentioned fruitful interplay between Scalar-Tensor Theories and conformal transformations and the intriguing connection between frames found in chapter 5, one may wonder whether a generalization along this line might help shedding some light on the properties and structure of the Horndeski theory. This is the main motivation of the present work. As we shall see in what follows, simple conformal transformations are not enough for this task, due to the more complicate structure of the Horndeski actions, and the use of generalized metric transformation will be required.

An example of such generalized metric transformation is given by disformally related metrics. These have been proposed in [100] and applied first in the context of relativistic extensions of MOND-like theories [192] in order to account for measured light deflection by galaxies. Later they found applications in varying speed of light models [193], DE [99,101, 102, 194], inflation [195] and modified DM models as discussed in the previous chapters. More recently, empirical tests of these ideas have been proposed in laboratory experiments [196] as well as in cosmological observations [197,198], highlighting the important role that disformal transformations are playing in contemporary cosmology and gravitation theory.

6.1 The Horndeski action and disformal transformations

The Horndeski Lagrangian [4] is the most general Lagrangian that involves a metric and a scalar field that gives second order field equations in both fields in four dimensions. Recently generalized to arbitrary dimensions by Deffayet et al. in [5], it is the natural extension of Scalar-Tensor *a là* Brans–Dicke.

Horndeski theory remained a sort of theoretical curiosity for more than thirty years but it was recently rediscovered as a powerful tool in cosmology. In fact, its generality (within the bound of second order field equations) made of it an ideal meta-theory for Scalar-Tensor models of DE and DM. However, up to now, no structural analysis analogous to the one carried out for standard Scalar-Tensor was performed. In particular there is no obvious extension of the concept of equivalent frames and no first principles to fix the shape of the free parameter functions. In order to address these, after briefly reviewing the Horndeski action and disformal transformations, we shall discuss here the behavior of this theory under such extended class of metric transformations.

6.1.1 Horndeski Lagrangian

The Horndeski action, rephrased in the modern language of Galileons [188][1] can be written as follows

$$\mathcal{L} = \sum_i \mathcal{L}_i \,, \tag{6.1}$$

where

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X)\,, && (6.2)\\
\mathcal{L}_3 &= G_3(\phi, X)\Box\phi\,, && (6.3)\\
\mathcal{L}_4 &= G_4(\phi, X)R - G_{4,X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]\,, && (6.4)\\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + && (6.5)\\
&+ \frac{G_{5,X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\nu\nabla_\mu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\,, && (6.6)
\end{aligned}$$

where

$$X = \nabla_\mu\phi\nabla^\mu\phi/2\,, \quad (\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\,, \quad (\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\nu\nabla_\mu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi\,, \tag{6.7}$$

while $G_{i,X} = \partial G_i/\partial X$, R is the Ricci scalar and $G_{\mu\nu}$ is the Einstein tensor. The coefficient function G_4 has the dimensions of a mass square and it plays the role of a varying

[1] Notice that we have a different sign convention w.r.t. [188] due to the different definition of the function $X \equiv \nabla_\mu\phi\nabla^\mu\phi/2$

Gravitation constant, while G_5 has those of a mass to the fourth power. The field ϕ is taken to have mass dimension 1. As said, this gravitational action is the most general one that can be built with a metric and a scalar field, providing second order field equations in four dimensions. We will not discuss here the equations of motion referring the interested reader to [5, 199] for a general analysis. Let us instead focus our attention on some important properties of this Lagrangian.

First of all notice that, beyond the usual conformal non-minimal coupling, there is another source which couples the Einstein tensor to second order derivatives of the field. This represents a novelty as, contrarily to what happens for the coupling to the Ricci scalar, in this case we have a direction dependent coupling. Secondly all the sub-Lagrangians give second order field equations independently so that one could in principle neglect some of them without spoiling the second order nature of the field equations. However, as is shown in appendix B.3, neglected terms can alway be eventually generated through redefinitions of the field variables. Finally, we notice that compared with the standard Scalar-Tensor action the NMC coefficients now depends also on the kinetic term.

Given that this model is a generalization of standard Scalar-Tensor Theory one may wonder whether suitable metric transformations can be introduced also in this case, leaving the action invariant and linking alternative frames. It is not hard to realize that simple conformal transformations have limited power in this sense. In standard Scalar-Tensor Theories these transformations allow to replace by constants some of the field dependent coefficients. However, the various terms appearing in the Horndeski action ($G_i(\phi, X)$) are also dependent on the kinetic term X and hence more general transformations are clearly needed.

The most natural extension of the conformal transformation in this sense would be $A(\phi) \to A(\phi, X)$. However, even if this can remove the non-minimal coupling in the $\mathcal{L}_4$, it is basically ineffective on the the non-minimally coupling provided by $\mathcal{L}_5$. Moreover, this generalized conformal transformation contains derivatives of the field and hence one must be careful that those do not end up introducing higher derivatives in the equations of motion. In this sense, the next natural candidate for a suitable set of metric transformations is then represented by the disformal ones.

6.1.2 Disformal transformations

Disformal transformations are defined by the following relation[2]

$$\bar{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_\mu\phi_\nu\,, \tag{6.8}$$

where the disformal functions A and B now depend on both the scalar field ϕ and its kinetic term X and where we have defined for convenience $\phi_\mu = \nabla_\mu\phi$. We can classify the properties of this generalization in two main categories: first the new functions do not simply depend on the local value of the field but also on the metric itself, hidden inside the definition of the kinetic term. Secondly we have a translation along the lines of variation of the field which means that the new metric will also depends on the way the field is changing through space-time.

When dealing with metric transformations one has to ensure that the new metric is still a good one. We can formally define the goodness of a metric transformation with a set of properties: it must preserve Lorentzian signature, it must be causal and it has to be invertible, with a non zero volume element. All these properties directly translate into constraints on the two free functions A and B which we are going to discuss one by one.

Lorentzian signature. Consider a frame in which $\phi_\mu \equiv \nabla_\mu\phi = (\nabla_0\phi, \vec{0})$. Then the Lorentzian requirement can be translated into

$$\bar{g}_{00} = A(\phi, X)g_{00} + B(\phi, X)\phi_0\phi_0 < 0\,. \tag{6.9}$$

This constraint must hold true for all values of the field and its derivative. Given that we cannot exclude that for some values of the field variables the function B can be zero, a first requirement is that $A > 0$. This is the usual requirement made also for standard Scalar-Tensor Theories. Then by multiplying equation (6.9) with g^{00} we found that the condition to be fulfilled for preventing $\bar{g}_{00}$ from sign inversion is:

$$A(\phi, X) + 2B(\phi, X)X > 0\,. \tag{6.10}$$

As a consequence, to have this relation to hold true for all values of X, it is necessary to have some kinetic dependence at least in one of the two disformal functions. This result was first derived in [151] (see also the original paper by Bekenstein [100]). However in [102] it was argued that the dynamics of the scalar field can be such that it is possible to keep the metric Lorentzian even with no X dependences in

[2] More general formulations may be possible, for example including higher derivatives of the scalar field or by adding vector fields [100].

the disformal functions A and B. For example this can happen when the scalar fields enters a slow-roll phase *e.g.,* when thought to be the field responsible for Dark Energy. However this subject is not yet fully understood and, being not mandatory for our purposes its investigation, in the following we will assume that both metrics are Lorentzian for all the values of the scalar field and its kinetic term.

Causal behavior. The disformal metric can have, depending on the sign of the B function, light cones wider or narrower than those of the metric g. This may lead to think that particles moving along one metric may show superluminal or a-causal behavior. However, the requirement of the invariance of the squared line element and recalling that physical particles satisfy $ds^2 < 0$ is enough to ensure causal behavior. This objection has been discussed in some detail in [200].

Invertible. We also must be sure that an inverse of the metric and the volume element are never singular. The inverse disformal metric is given by:

$$\bar{g}^{\mu\nu} = \frac{1}{A(\phi, X)} g^{\mu\nu} - \frac{B(\phi, X)/A(\phi, X)}{A(\phi, X) + 2B(\phi, X)X} \nabla^\mu \phi \nabla^\nu \phi \,, \tag{6.11}$$

while the volume element is given by $\sqrt{-\bar{g}} = A(\phi)^2 \left(1 + 2XB/A\right)^{1/2} \sqrt{-g}$. The constraint derived from these requirements are weaker than those already obtained hence there are no new potential issues.

From this analysis we learn that the extension of conformal transformations to disformal ones is well posed, even if all previous points deserve a deeper analysis which, in any case, is beyond the scopes of the present investigation and is left for further studies.

Disformal metrics seem to be good candidates for our purposes as they possess, beyond a purely conformal term, another one which is a deformation of the metric along the direction of variation of the field and indeed disformal transformations have for the Horndeski action a role very similar to that of conformal transformations for standard Scalar-Tensor Theories.

6.2 Invariance of the Horndeski Lagrangian under disformal transformations

The ability of the Horndeski action to give second order field equations resides in a fine cancellation between higher derivatives coming from NMC terms and those produced from derivative counterterms. This is the defining feature of the theory which fixes once and for all its structure. Consequently, it seems very natural to require that any transformation operated on the action must preserve the same structure (6.1), plus possibly inducing surface terms, in order to preserve this property. In summary, we are interested in those metric transformations which leave the structure of the Horndeski action invariant.

It is easy to see that this requirements already reduces the freedom in the disformal functions A and B. In fact, any kinetic dependence of these two terms would lead unavoidably to the breaking of the Horndeski structure, *i.e.,* to higher order equations of motion. We prove this through some examples in appendix B.1 while here we give a first principle argument why one should expect this to happen. The ability of the Horndeski action to give second order field equations lies on the antisymmetric structure of second derivatives terms, as has been made clear in [5]. Consider the $\mathcal{L}_4$ part of the Lagrangian. This can be rewritten in the following form

$$\mathcal{L}_4 = \left(g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)\left[G_4(\phi, X)R_{\mu\nu\alpha\beta} - G_{4,X}(\phi, X)\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi\right] , \tag{6.12}$$

where the antisymmetric structure is made clear. Given that we have to preserve this structure the only effects that can have a transformation is to renormalize the function G_4 and its derivative modulo a surface term. However, any kinetic dependence in the disformal functions will spoil this structure. In fact, consider the transformation property of the second derivatives of the scalar field under the conformal transformation $\hat{g}_{\mu\nu} = A(X)g_{\mu\nu}$

$$\nabla_\mu\nabla_\nu\phi \to \nabla_\mu\nabla_\nu\phi + \frac{A_{,X}}{A}\left[g_{\mu\nu}\phi^\alpha\phi^\beta\nabla_\alpha\nabla_\beta\phi - \phi_\mu\phi^\alpha\nabla_\alpha\nabla_\nu\phi - \phi_\nu\phi^\alpha\nabla_\alpha\nabla_\mu\phi\right] . \tag{6.13}$$

When inserted in (6.12), among other terms, the following one is generated

$$\sim 4G_{4,X}\left(\frac{A_{,X}}{A}\right)^2\phi^\mu\phi^\nu\phi^\alpha\phi^\beta\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi\,, \tag{6.14}$$

which is clearly symmetric in the four indices and hence will produce higher than second derivatives in the equations of motion. One may wonder whether there may be counterterms coming from curvature that eliminate this but, as shown in appendix B.1, this is not the case.

We hence conclude that in order to preserve second order field equations, we have to restrict our analysis to the following class of disformal transformations

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_\mu\phi_\nu\,. \tag{6.15}$$

The structural invariance of the action under the disformal transformation (6.15) translates into the statement that such transformations represent a symmetry of the Horndeski action so that all the functions are defined modulo a conformal and a disformal transformation. This reminds very closely the case of standard Scalar-Tensor Theories where invariance under conformal transformations is used to reduce the number of free functions that defines the theory. However, the generalization of this reasoning to the case of Horndeski theory is not straightforward. In fact, the subset of disformal transformations (6.15) does not allow for kinetic term dependent coefficients, consequently one cannot generically rescale the functions $G_i(\phi, X)$ characterizing the Horndeski action.

The effects of the disformal transformation on the generic coefficient function $G_i(\phi, X)$ can be schematically described as follows. After the application of (6.15) on the Horndeski action the renormalized coefficient will take the form

$$G_i(\phi, X) = \mathrm{f}(\phi, X; A, B) G_i(\phi, \bar{X}) + \mathrm{g}(\phi, X; G_j, A', B', A'', B'')\,, \tag{6.16}$$

where we have defined

$$\bar{X} = \frac{X/A}{1 + 2XB/A}\,. \tag{6.17}$$

We can identify two main effects from the above equation. A multiplicative factor has appeared in front of the original function which depends on the disformal functions A and B and a second contribution has appeared that depends explicitly on the coefficient functions themselves. This dependence enters the coefficients in a hierarchical way: only the coefficient functions with $j \geq i$ will contribute, as we shall see below. Hence, if any one of the pieces of the Horndeski Lagrangian is initially omitted, it will be generated by the disformal transformation, with the important exception of $\mathcal{L}_5$. Moreover, notice that the second piece depends on the derivatives of the disformal functions with respect to the scalar field. This fact implies that if we take them to be constant, there will be no mixing between different Lagrangian coefficient functions. We will see explicit examples of this in the next section where we will study the transformation properties of the Horndeski action under pure conformal and disformal transformations separately. In particular we will derive the sub-class of Horndeski theories that admits a representation in which all NMC terms are eliminated via a disformal transformation. We refer to appendix B.3 for the detailed transformation properties.

As a concluding remark, let us add that the Horndeski action is also invariant under the field rescaling $\phi \to s(\phi)\phi$ (this is explicitly discussed in appendix B.4 where we consider the effects of this transformation on the Horndeski coefficient functions). This property will play an important role later on in our discussion when we shall deal with the equivalence of disformal frames.

6.2.1 Purely conformal transformations

Let us first consider the effects of conformal transformations $\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu}$ on the Horndeski action (6.1), extending the well known results for these transformations in Scalar-Tensor Theories to this more general class of actions. The transformed Lagrangian coefficient functions read

$$\bar{G}_2(\phi, X) = A^2 G_2(\phi, X_C) + 2XG_3AA' + 3X\frac{G_4A'\left[1-2A\right]}{A} + \\ + \frac{6G_5X^2A'}{A}\left[\frac{A''}{A} - \frac{A'^2}{A^2}\right] - 2XH_{5,\phi} + \frac{2G_{5,X}X^3}{A^3}A'^2 , \quad (6.18)$$

$$\bar{G}_3(\phi, X) = AG_3(\phi, X_C) - 2G_{4,X}A' + X\left(-2H_{\Box,\phi} - \frac{G_5A'^2}{2A^2} + \frac{2G_5A''}{A} + \frac{G_{5,X}XA'^2}{A^2}\right) - H_5 , \quad (6.19)$$

$$\bar{G}_4(\phi, X) = A(\phi)G_4(\phi, X_C) , \qquad \bar{G}_5(\phi, X) = G_5(\phi, X_C) , \quad (6.20)$$

where

$$X_C = X/A(\phi) , \qquad H_\Box = G_5\frac{A'}{A} , \qquad H_5 = \int dX \left[H_{\Box,\phi} + \frac{G_5A''}{A} + \frac{5G_5}{2}\frac{A'^2}{A^2} + 2G_{5,X}\frac{A'}{A}\right] . \quad (6.21)$$

Here we can clearly see the hierarchical propagation of terms from higher derivatives Lagrangians towards lower ones together with the special case represented by the $\mathcal{L}_5$ Lagrangian that does not receive any contribution from the other parts of the Lagrangian. Then notice how the conformal NMC $G_4(\phi, X)$, is modified by a multiplicative factor while the NMC with the Einstein tensor is unaffected apart from a redefinition of the kinetic term inside $G_5(\phi, X)$.

Given that in general all the coefficient functions depend on both the scalar field and its kinetic term, it is clear that using only a conformal transformation we shall not be able to eliminate NMCs for any choice of the conformal factor $A(\phi)$. Even in the special case when the coefficient functions depend only on the field and not on its derivatives, we are able to set at most $G_4(\phi) = 1$ while retaining the generalized NMC between the Einstein tensor and the field derivatives (6.6), given that part of the Lagrangian is not affected by conformal transformations, see Eq. (6.20).

Notice that even if we were to take $G_5(\phi, X)$ to be a function of the scalar field only, we would not be able to eliminate it. In fact, we then have the following relation

$$G_5(\phi)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi = G_{5,\phi}XR - G_{5,\phi}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] + \\ - G_{5,\phi\phi}\left[2X\Box\phi - \phi^\mu\phi^\nu\nabla_\mu\nabla_\nu\phi\right] , \quad (6.22)$$

that shows how in this case $\mathcal{L}_5$ is a contribution to the $\mathcal{L}_4$ (as well as to $\mathcal{L}_3$ and $\mathcal{L}_2$) that depends explicitly on the kinetic term and cannot be eliminated by a simple conformal transformation.

6.2.2 Purely disformal transformations

We turn now our attention to the case of a pure disformal transformation, *i.e.,* when the conformal factor $A(\phi)$ is set to one while the disformal function $B(\phi)$ is left unspecified. Given that we are mainly interested on the effects of transformations on the NMC terms we will report here only the relevant coefficient functions. The remaining ones can be easily derived from the equations in appendix B.3.

In the case under consideration we have that the transformed NMC coefficient functions read

$$\bar{G}_4(\phi, X) = (1+2XB)^{1/2}\, G_4(\phi, X_D) + \frac{G_5(\phi, X_D)B'(\phi)X^2}{(1+2XB)^{3/2}} - H_{R,\phi}(\phi, X)X\,, \tag{6.23}$$

$$\bar{G}_5(\phi, X) = \frac{G_5(\phi, X_D)}{(1+2XB)^{1/2}} + H_R(\phi, X)\,, \tag{6.24}$$

where

$$X_D = X/(1+2BX)\,, \qquad H_R(\phi, X) = B\int dX \frac{G_5(\phi, X_D)}{(1+2XB)^{3/2}}\,. \tag{6.25}$$

Here we notice that the effects of the disformal transformation are richer than those of the conformal one. In fact, besides a conformal modification of G_4 we have other contributions to $\mathcal{L}_4$ and in this case G_5 is modified as well. In particular, the modified coefficient functions receive corrections that depend on the kinetic term but, as can be seen from equations (6.23) and (6.24), even in this case one cannot generically eliminate the NMC.

Let us focus on this last point and study which constraints can be imposed on the coefficient functions of the Hornedski action so to be able to eliminate all the NMC *i.e.,* to use the disformal transformation so to obtain $\bar{G}_4 = 1$ and $\bar{G}_5 = 0$. The latter condition is satisfied if

$$\frac{G_5(\phi, X_D)}{(1+2XB)^{1/2}} + B\int \frac{G_5(\phi, X_D)}{(1+2XB)^{3/2}}dX = 0 \Rightarrow \int dX\left[\frac{G_{5,X}(\phi, X_D)}{(1+2XB)^{1/2}}\right] = 0\,. \tag{6.26}$$

In general, if $G_5 = G_5(\phi)$ then the above constraint is automatically satisfied. We cannot exclude the existence of other solutions in which an X dependence is also allowed, for example if the integrand function is fast oscillating. However these will depend on the specific model chosen and would need to be investigated case by case. Finally, notice that this constraint is not influenced by the freedom in rescaling the scalar field.

In order to have no conformal coupling we have to impose

$$1 = (1 + 2XB)^{1/2}\, G_4(\phi, X_D) + \frac{G_{5,\phi}(\phi)X}{(1 + 2XB)^{1/2}} - \tilde{G}_5 X\,, \tag{6.27}$$

where

$$\tilde{G}_5(\phi, X) = \int dX \frac{G_{5,X}(\phi, X_D)}{(1 + 2BX)^{1/2}}\,, \tag{6.28}$$

that, when expressed in terms of the initial variable X, gives

$$G_4(\phi, X) = (1 - 2B(\phi)X)^{1/2} - G_{5,\phi}(\phi)X + \tilde{G}_{5,\phi}(\phi, X)X\,, \tag{6.29}$$

with

$$\tilde{G}_5(\phi, X) = \int dX (1 - 2BX)^{1/2} G_{5,X}(\phi, X)\,. \tag{6.30}$$

Given that we want both constraints to be satisfied at the same time, we have then

$$G_5 = G(\phi) \qquad \text{and} \qquad G_4(\phi, X) = (1 - 2B(\phi)X)^{1/2} - G_{5,\phi}(\phi)X\,, \tag{6.31}$$

which fixes once and for all the functional dependence of the $G_4(\phi, X)$ function on the kinetic term.

We conclude that the following Lagrangian

$$S_{NMC} = \int d^4x\sqrt{-g}\left[G_4(\phi, X)R - G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2\right] + G_5(\phi)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\right]\,, \tag{6.32}$$

where G_4 is given by (6.31), is the only one that admits a disformal map able to eliminate all the NMC terms in the context of Horndeski theory.

However, it is worth noticing that inserting equations (6.22) and (6.31) in (6.32) all the terms depending on $G_5(\phi)$ end up canceling. Hence, if the function G_5 depends only on the scalar field, we conclude that the existence of a disformal metric able to cancel all NMC *requires* the absence of $\mathcal{L}_5$.[3] We are hence left with the following action

$$S_{NMC} = \int d^4x\sqrt{-g}\left[G_E(\phi, X)R - G_{E,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]\right]\,, \tag{6.33}$$

where

$$G_E = (1 - 2B(\phi)X)^{1/2}\,. \tag{6.34}$$

[3] It may seem that a constant G_5 could be included without spoiling our request of no NMC. However in this case $\mathcal{L}_5$ reduces to a surface term and hence does not contribute to the dynamics.

While we have here considered a special subset of the ϕ-dependent disformal transformations (6.15) we can easily extend our conclusions to transformations including a conformal factor $A(\phi)$. Indeed, in this case the most general action allowing for a full elimination of the NMC would be the same as equation (6.33) modulo a conformal rescaling of the $G_E(\phi, X)$ function.

As a final remark, it is perhaps worth stressing that, as noted in [102], the non-relativistic limit of the action (6.33) corresponds to the quartic covariant term of the Galileon action with the appropriate non-minimal coupling to yield second order field equations [103].

6.3 Disformal frames

The invariance of an action under metric transformations implies the possibility to fix some of the free functions characterizing the theory, similarly to what is done when choosing a gauge. Consequently, the number of the independent functions is reduced. In our specific case the Horndeski action (6.6) is invariant under both purely conformal and disformal transformations. This freedom allows us to define an infinite set of equivalent frames defined by different fixings of two of the free functions in the action (see [201, 202] for a similar reasoning in standard Scalar-Tensor Theories).

Among all these equivalent representations of the theory two are most relevant as they correspond to somewhat opposite situations: the Einstein and Jordan frame. For the sake of clarity we provide here generalized definitions relevant for the Horndeski actions under consideration here.

Jordan Frame In the *Jordan Frame* the Lagrangian of the gravitational sector includes a non-minimally coupled scalar field meanwhile all the matter fields follow the geodesics of the gravitational metric (the stress energy tensor of the matter fields is covariantly conserved w.r.t. the gravitational metric).

Einstein Frame In the *Einstein Frame* the gravitational dynamics is described by the standard Einstein–Hilbert Lagrangian (plus possibly a cosmological constant). However, matter fields are coupled to the gravitational metric via some function of the scalar field and its derivatives. They hence move on geodesics that can be different from the one determined by the metric defining the Ricci scalar. Moreover, the gravitational equations in absence of matter do not reduce to $R = 0$, as in GR, but in general will retain the scalar field as a possible source.

We now proceed to recall the issue of frames and their equivalence in standard Scalar-Tensor and then extend this to the case of the Horndeski action.

6.3.1 Scalar-Tensor Theories and conformal transformations

A minimal prescription for generalizing GR is to promote the gravitational constant to a scalar field which must be provided with its own dynamics in order to preserve diffeomorphism invariance. Furthermore, the Einstein Equivalence Principle (EEP) allows such scalar field to also mediate the coupling of the matter to the metric (albeit in an universal way). This reasoning then leads to the following action:

$$S = \int d^4x \left[G(\phi)R - \frac{f(\phi)}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi) \right] + S_m[e^{2\alpha(\phi)}g, \psi] \,, \tag{6.35}$$

where the four functions $G(\phi)$, $f(\phi)$, $V(\phi)$ and $\alpha(\phi)$ are general functions of their argument. We will not enter into the details of the applications of this theory, referring to the above cited papers and to references therein for details, but we will focus on some more formal properties of this action.

First of all, the above mentioned free functions in the action are actually redundant [201,202]. Indeed, the invariance of action (6.35) under the conformal transformations $\bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$ and the scalar field redefinitions $\bar{\phi} = F(\phi)$ allows to freely choose two out of the four functions. Hence, implementations of (6.35) differing only for the fixing of two of the four coefficient functions are indeed just different representation of the same physical theory [201].

For this class of theories the Einstein frame is defined by the choice $G(\phi) = 1$ and $f(\phi) = 1$ so that gravity is described by the standard Einstein–Hilbert action, the scalar field has a canonical kinetic term while matter fields follows the geodesics of a physical metric conformally related to the gravitational one. The Jordan frame is instead obtained choosing $\alpha(\phi) = 0$ and $G(\phi) = \phi$. In this case we have that all fields follows the same metric but now the scalar field is non-minimally coupled to curvature and it may possess a non-standard kinetic term, *i.e.,* $f(\phi) \neq 1$. The fact that the above two frames are picked up from (6.35) by just fixing two of the four coefficient functions implies their physical equivalence (i.e. a varying gravitational coupling in the Jordan frame is translated into field dependent matter masses and couplings when the action is in the Einstein frame).

The lesson that we want to capture with this short introduction is that when dealing with generalized actions like (6.35) one has to pay attention to their symmetries in order to correctly identify the set of physically equivalent frames (*i.e.,* different representations of the same theory) which one can alternatively use for more conveniently dealing with different physical issues.

These considerations become even more important as further modifications of gravity are introduced and complicated terms are added. In what follows we shall first investigate the issue for that class of Horndeski actions admitting an Einstein frame (as this frame is often adopted for physical investigations). Later, we shall extend the discussion to more general actions.

6.3.2 Horndeski action and the Einstein frame

In section 6.2 we have derived the most general action in the Jordan frame for which all NMC can be eliminated via the disformal transformation (6.15). However, the discussion of the possible equivalence of frames requires us to include also the action for matter fields with possible generalized coupling to the metric. This leads to the following completion of (6.33):

$$S = \int d^4x\sqrt{-g}\Big[G(\phi,X)R - G_{,X}(\phi,X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2\right] + \\ + G_2(\phi,X) + G_3(\phi,X)\Box\phi\Big] + S_m[\tilde{g},\psi]\,, \quad (6.36)$$

where

$$G(\phi,X) = C(\phi)^2(1-2D(\phi)X)^{1/2}\,, \quad (6.37)$$

S_m is the total matter action defined in terms of the physical metric

$$\tilde{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu\,, \quad (6.38)$$

and ψ stands generically for matter fields.

Among the six free functions, four are related to the field-metric couplings, $C(\phi)$, $D(\phi)$, $\alpha(\phi)$ and $\beta(\phi)$ and two are defining the minimally coupled scalar field Lagrangian, $G_2(\phi,X)$ and $G_3(\phi,X)$. Thanks to the invariance under both conformal and disformal transformations we can fix two out of the four metric functions $C(\phi)$, $D(\phi)$, $\alpha(\phi)$ and $\beta(\phi)$ with appropriate choice of the functions $A(\phi)$ and $B(\phi)$ appearing in the disformal transformation (6.15). In principle, we could act on $G_2(\phi,X)$ and $G_3(\phi,X)$ but given their generic dependence on the kinetic term, (6.15) is not effective for fixing them. Hence, with a general disformal transformation (6.15), we can define a Jordan and an Einstein frame in the same sense as it can be done for standard Scalar-Tensor Theories.

However, we can also use the invariance of the Horndeski action under field rescaling to further constrain the number of independent functions (as in the case of action (6.35)). In fact, as shown in appendix B.4, we can always rescale the field ϕ by an arbitrary function. This amounts to say that we can fix one more of the free functions $\alpha(\phi)$, $\beta(\phi)$, $C(\phi)$ and $D(\phi)$ to arbitrary values so that the Einstein and Jordan frames defined above represents a class of equivalent theories that can be further fixed with a field redefinition. We conclude that implementations of (6.36) which differ only by the fixing of three out of six functions are nothing but equivalent representations of the same physical theory.

It is worth noticing that the invariance under two metric transformations allows the definition of more physically interesting equivalent frames, w.r.t. standard Scalar-Tensor Theories. In fact, we can actually define the following four equivalent frames, all obtained from the action (6.36) with different fixing of the free functions.

Jordan Frame. The Jordan frame is defined by the action

$$S_J = \int d^4x\sqrt{-g}\left[G_J(\phi, X)R - G_{J,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2\right] + \right. \\ \left. +G_2(\phi, X) + G_3(\phi, X)\Box\phi\right] + S_m[g, \psi]\,, \quad (6.39)$$

where we have fixed $\alpha = 1$ and $\beta = 1$ so that matter is minimally coupled to the metric that defines the curvature terms appearing in the action. As a consequence a conformal non-minimal coupling term, described by the presence of the function $G_J = C(\phi)^2(1 - 2D(\phi)X)^{1/2}$, is present and can be further constrained with a field redefinition.

Einstein Frame. The Einstein frame is given by the action

$$S_E = \int d^4x\sqrt{-g}\left[R + G_2(\phi, X) + G_3(\phi, X)\Box\phi\right] + S_m[\tilde{g}, \psi]\,, \quad (6.40)$$

where the NMC has been eliminated by the fixing $C(\phi) = 1$ and $D(\phi = 0)$ in the action (6.36) but now matter feels a physical metric related via a disformal transformation to that defining curvature terms, *i.e.*, $\tilde{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$. Again we can fix one of the two functions α and β via a field rescaling.

Galileon Frame. This frame is given by the action

$$S_G = \int d^4x\sqrt{-g}\left[G_G(\phi, X)R - G_{G,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\mu\phi)^2\right] + \right. \\ \left. +G_2(\phi, X) + G_3(\phi, X)\Box\phi\right] + S_m[\tilde{g}, \psi]\,, \quad (6.41)$$

where

$$G_G = (1 - 2D(\phi)X)^{1/2}; \qquad \tilde{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu}\,, \quad (6.42)$$

which corresponds to the choice $C(\phi) = 1$ and $\beta(\phi) = 1$. In this case we have both NMC and matter fields feeling a physical metric which is now conformally related to the gravitational one.

Disformal Frame. This frame is given by the action

$$S_D = \int d^4x\sqrt{-g}\,[G_G(\phi, X)R + G_2(\phi, X) + G_3(\phi, X)\Box\phi] + S_m[\tilde{g}, \psi]\,, \tag{6.43}$$

where

$$G_G = C(\phi)^2; \qquad \tilde{g}_{\mu\nu} = g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu\,, \tag{6.44}$$

which amounts to the choice $D = 0$ and $\alpha = 1$.

It is worth stressing here that the last two frames, which can be seen as some sort of intermediate frames between the Jordan and Einstein ones, can actually reduce to the latter for suitable choices of the rescaling of the field, as it can be seen from the last columns of table 6.3.2. This is a consequence of the fact that we have four free metric functions, $\alpha(\phi)$, $\beta(\phi)$, $C(\phi)$ and $D(\phi)$, three of which can be arbitrarily fixed.

As a final remark, while all these equivalent frames are connected by disformal transformations and field rescaling, one has also to be careful about accordingly rescale also the so far neglected functions $G_2(\phi, X)$ and $G_i(\phi, X)$ in order to preserve the equivalence of frames.

The above mentioned frames where first proposed in [102] and partially discussed in [103] where it was pointed out how disformal transformations relate them, albeit no discussion about their actual physical equivalence was provided. Here we have re-derived the same results in a different way and in addition we have proved the frames equivalence. This has relevant consequences, for example it implies that not only DBI Galileon models with a non-minimally coupled scalar field can be cast via a disformal transformation into the simpler Einstein frame, but also guarantees the equivalence of these representations. Furthermore, the equivalence of the frames allows us to claim the equivalence of many apparently unrelated models as those reported in [101] given that we can move from one to the other through appropriately chosen disformal transformations and field redefinitions.

6.3.3 More general disformal frames

We have seen in the previous section that the requirement of an Einstein frame strongly constrains the shape of the Horndeski Lagrangian with a specific form for $G_4(\phi, X)$ and

Frame	**Disformal transformation**		**Field rescaling**	
	Matter Metric	**NMC function**	**Matter Metric**	**NMC function**
Jordan Frame	$g_{\mu\nu}$	$C(\phi)^2(1+2D(\phi)X)^{1/2}$	$g_{\mu\nu}$ $g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$ $C(\phi)^2(1-2\Lambda X)^{1/2}$
Einstein Frame	$e^{\alpha(\phi)}g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	1	$g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$ $e^{\alpha(\phi)}g_{\mu\nu}+\Lambda\phi_\mu\phi_\nu$	1 1
Galileon Frame	$e^{\alpha(\phi)}g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$	$g_{\mu\nu}$ $e^{\alpha(\phi)}g_{\mu\nu}$	$(1-2D(\phi)X)^{1/2}$ $(1-2\Lambda X)^{1/2}$
Disformal Frame	$g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	$C(\phi)^2$	$g_{\mu\nu}+\Lambda\phi_\mu\phi_\nu$ $g_{\mu\nu}+\beta(\phi)\phi_\mu\phi_\nu$	$C(\phi)^2$ 1

Table 6.1: Disformal frames obtained for different fixing of the Horndeski coefficient functions of (6.36). The first two columns show the results of the fixing after a disformal transformation while the last two show the effects of the further freedom associated to the invariance under field rescaling (there are two possibilities in each slot in this case as one can alternatively rescale the metric or the field ϕ derivative terms). Λ is a dimensional constant introduced to keep track of the dimensions of the coefficient functions.

forcing $G_5(\phi, X) = 0$. However, there is no real physical need to have an Einstein frame so that one may wonder about the existence of more general Lagrangians that do not possess an Einstein frame but that show in any case interesting properties under disformal transformation. We list and analyze here some examples.

Disformal matter When we add the matter Lagrangian to the full Horndeski action, the Einstein Equivalence Principle allows matter fields to be coupled to a metric which is disformally related to the one defining the Horndeski action

$$S = S_H[g, \phi] + S_m[\bar{g}, \psi]\,, \tag{6.45}$$

where S_H is the full Horndeski action (6.1), ψ collectively defines matter fields and where $\bar{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$. Thanks to the invariance of the full Horndeski action under disformal transformations and field rescaling we are free to fix both $\alpha(\phi)$ and $\beta(\phi)$ in such a way that, after the transformation, matter propagates along the geodesics defined by the metric $g_{\mu\nu}$ that appears in the Horndeski action. These transformations will of course affect the Horndeski Lagrangian, but only in the shape if its coefficient functions, not in its structure. Hence, a Horndeski theory in which matter propagates on the metric $\bar{g}_{\mu\nu} = e^{\alpha(\phi)}g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$ is equivalent to another Horndeski theory, with redefined coefficient functions, in which matter propagates along the same metric $g_{\mu\nu}$ that enters the Horndeski action.

This fact is not particularly surprising but it is nonetheless interesting as it shows how, without any assumption on the shape of the Horndeski action, we can see that apparently different matter behaviours are in fact different representations of the same theory.

Einstein coupling Another possible extension is to include the $\mathcal{L}_5$ Lagrangian while keeping the requirement of having a frame with no conformal coupling. Using the relations derived in appendix (B.3) we see that this requirement translates into a condition on the initial shape of the $G_4(\phi, X)$ function

$$G_4(\phi, X) = (1 - 2B(\phi)X)^{1/2} - G_{5,\phi}(\phi, X)X + \tilde{G}_{5,\phi}(\phi, X)X \,, \tag{6.46}$$

where

$$\tilde{G}_5(\phi, X) = \int dX (1 - 2BX)^{1/2} G_{5,X}(\phi, X) \,. \tag{6.47}$$

With this requirement we can consider the following action

$$\begin{aligned} S = \int d^4x \sqrt{-g} \Big[& G_4(\phi, X)R - G_{4,X}(\phi, X) \left[(\Box\phi)^2 - (\nabla_\mu \nabla_\mu \phi)^2\right] + \\ & + G_2(\phi, X) + G_3(\phi, X)\Box\phi \Big] + \int d^4x \sqrt{-g} \Big[G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi + \\ & - \frac{1}{6} \left((\Box\phi)^3 - 3\Box\phi(\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right) \Big] + S_m[\tilde{g}, \psi] \,, \end{aligned} \tag{6.48}$$

where $G_4(\phi, X)$ is given by the previous expressions while $G_5(\phi, X)$ is left totally free. With the disformal transformation (6.15) we can eliminate the conformal coupling leaving only a NMC via the Einstein tensor and matter fields propagating along disformal geodesics.

We conclude this section recalling that the invariance of the Horndeski action under disformal transformations and field rescaling holds true for the full Horndeski theory (6.1). Possible restrictions on the shape and functional dependencies of the free functions of the theory are to be ascribed only to physical motivations, *e.g.,* the requirement of an Einstein frame, or to classification aims, *e.g.,* identify equivalent models, but not to constraints imposed by the invariance itself.

6.4 Summary

The gravitational interaction has been the first one studied in a systematic way and its modern formulation is encoded in the theory of General Relativity. Despite its successes, GR is nowadays challenged both at the theoretical and experimental level, leading to several proposals for alternatives theories of gravity. However the lack of an axiomatic procedure for the construction of such theories and the limited regime for which we have highly constraining observational data, make hard to reduce the number of alternative theories and to find their mutual relations.

A major tool in physics is represented by symmetries. This is a clean and precise way to order models, find their simplest formulations and identify the minimal set of degrees of freedom required to fully define a theory. In the context of standard Scalar-Tensor Theories this has been systematically investigated and the discovery of the invariance of such theories under conformal metric transformations and field rescaling has made possible to identify the minimal number of functions required to describe the theory and showed the mutual relations between apparently different representations.

Along this line of reasoning, we have investigated the symmetries of the Horndeski action and found that it is invariant under a more general metric transformation than conformal, the so called disformal transformation, as well as under field rescalings. These transformations contain free functions and hence can in principle be used to constrain the coefficient functions that define the Horndeski action. However, we have shown that the most general disformal transformation (6.8) cannot be used to this purpose as the Horndeski action is not invariant under transformations induced by it. We have hence circumscribed our investigation to a subset of disformal transformations, (6.15) where the two free functions needed to define it only depends on the scalar field. We have shown that the Horndeski action is actually invariant under such class of disformal transformations albeit the generality of the Horndeski action does not allow for an efficient fixing of the coefficient functions.

For this reason, we looked to the constraints that one has to impose on the Horndeski coefficient functions in order to have a theory that admits an Einstein frame. We discovered that this is a quite constraining request as in fact the full Horndeski action is reduced to the action (6.36) where only a conformal non-minimal coupling is present. This allowed us to investigate the existence of equivalent frames, in an analogous way to what is done for standard Scalar-Tensor Theories. We found that apart from the well known Einstein and Jordan frames, the invariance under disformal transformations allows for the definition of two more equivalent frames: the so called Galileon and Disformal frames. We further extend our analysis to frames that do not admit an Einstein frame and showed that even without this requirement one can find physically relevant frames connected by disformal transformations.

In conclusion, we have found a new class of Scalar-Tensor Theories of gravity that admits physically equivalent frames, which are related by disformal transformation and field rescaling, thus generalizing the previous results obtained in the context of standard Scalar-Tensor Theories. This may have important consequences in cosmological context, in particular for DE models, as may allow to identify a large class of models into different representations of the same theory.

Chapter 7

Discussion and conclusions

In this thesis we have explored an alternative to the standard CDM paradigm in which a DM fluid gets non-minimally coupled to curvature terms, investigating both its theoretical and phenomenological consequences.

In first place we have shown how to construct such a theory, providing the most general NMC that can be obtained from a fluid if one wants to preserve second order field equations for gravity. This turned out to produce a very rich phenomenology which may be able to capture many of the currently unexplained properties of structure formation. Of particular interest is the presence of anisotropic stresses and the change in the way DM sources the gravitational potential. In fact the latter, in the weak field limit, depends not only on the local matter density but also on how it changes in space. A very welcome feature indeed when addressing the small scales issues of the CDM paradigm, as this effect can contrast the gravitational infall and consequently alleviate longstanding problems such as the core-cusp one. More than this, the fact that the gravitational dynamics is modified due to DM NMC directly influences baryons' motion thus potentially explaining the observed correlations between dark and luminous matter.

This is even more clear when one moves to the Einstein frame of the theory where the NMC is translated into couplings between DM and the other matter components. Here the geometric nature of the coupling is manifest in the universal way in which it affects all matter species, including DM itself. In fact, the couplings are fixed once and for all when the form of the NMC is chosen. The investigation in the Einstein frame showed how the NMC coupling acts as an effective pressure term for DM thus affecting significantly the clustering properties. In particular, we have shown how the general effect of the pressure is to reduce the growth of linear perturbation for a quite broad class of potentials.

We saw that the effects of the NMC are manifold and one has to be careful that they do not affect the dynamics in regimes where the standard model is a good description of the observations. Indeed, our model is designed to modify standard cosmology only at suitable scales and times. The NMC has not to be intended as a fundamental coupling but rather as an effective one emerging as a collective behavior of DM particles.

A particularly intriguing driving mechanism might be provided by the condensation of DM particles. In fact, in this case some special conditions need to be realized in order to catalyze the NMC of DM to gravity. In primis the temperature must be below a certain critical value, but also the particle density must be high enough. This in turns would protect our model from affecting CMB and early universe physics (high densities but temperature above the critical one) while making its phenomenology relevant at galactic scales (low temperature and high density). Moreover, a condensation will provide DM with a characteristic length scale that would make it able to probe gravity on a non-local scales, thus activating the NMC. We hence consider this dynamical realization of a NMC for DM worth further investigations which is currently being carried on through the construction of a coherent framework for NMC BEC in curved spacetimes [170].

We also noticed that the proposed model belongs to a generalized class of Scalar Tensor Theories. An intriguing result, evidenced by our analysis, was that the Einstein and Jordan frame are not related, as in standard Scalar-Tensor Theories, via conformal metric transformations, but rather through a more general one, the so called disformal transformation.

Of course the NMC DM model here considered is in an early stage of development and much work has still to be done. On the theoretical side, a better understanding on the mechanism that triggers the NMC is needed in order to reduce the functional freedom and to make more "observations-friendly" the model. On the phenomenological side, a detailed investigation on the effects on gravitational dynamics is required to see to which extent this model can relax CDM issues without spoiling large scale dynamics. Of particular interest would be the analysis of the effects on CMB secondary indicators (ISW, weak lensing), lensing and on the galaxy rotation curves. In particular the latter, being in weak field limit, can be used to constrain the functional freedom of our model.

On general grounds the possibility that DM might have non trivial interactions with gravity has been little explored despite its high potentialities. We believe that further studies in this direction are worthy as they can shed light on cosmological dynamics and on the nature of the gravitational interactions. In particular, the fact that when coupled non-minimally to gravity DM develops an effective pressure may be tested against a large number of observables. In the non-linear regime of structure formation the critical density required to form virialized objects will be changed and hence observables effects are expected on halo properties and formation history. Moreover we can compute the pressure effects on the statistical properties of matter distributions and on the CMB anisotropies spectrum.

Finally, also inspired by the previous results, we investigated the effects of disformal transformations on the most general Scalar-Tensor Theory we have at our disposal: the Horndeski action. We found that, even if in its most general form this metric transformation spoils the second order character of the equations of motion, a sub-case is available that leaves the Horndeski action invariant. This led us to ask, in an analogous way to what is done for standard scalar-tensor theories, if this invariance can be used to fix some of the free functions of the theory. We found that this is in general not possible. We then turned our attention to which is the most general scalar-tensor theory that admits an Einstein frame under the reduced disformal transformation. The result is interesting for at least two reasons. First, such a theory exists and it does not reduce to the standard scalar-tensor theory, showing a more general conformal coupling. Secondly, we found that, besides the well known Jordan and Einstein frame, two more physically interesting frames are available: the Galileon and the Disformal ones. These were investigated in previous works that aimed at showing how some DE models can be related by such disformal transformation. With our result we prove that they are indeed equivalent representations.

To conclude, in this thesis, we have adopted two different strategies to face current cosmological puzzles. On one side we have chosen a phenomenological approach and we have built a model that aims to address small scales DM issues. On the other side we have conducted a formal investigation of the Horndeski action that led to the identification of a theory that can accommodate, as different representations, apparently unrelated models for DE. We believe that both routes are much needed in order to move forward our understanding and hope that further investigations in these directions will provide a deeper insight into the fundamental properties of the Universe, especially when done in connection with observations. In particular, the Euclid mission [204] will represents a fundamental mean to test couplings between matter and gravity, possibly including the one presented in this work.

Appendix A

Derivation of the background equations in the Einstein frame

In this appendix we give full details of the derivation of the equations reported in section (5.1). The starting action is

$$S = S_{HE}[g] + S_{DM}[g,\varphi] + \frac{\epsilon}{\rho_*}\int d^4x\sqrt{-g}XF(X,\varphi)\left(X+\frac{1}{2}V^2(\varphi)\right). \tag{A.1}$$

The variation with respect to the independent variables φ, λ and the metric $g_{\mu\nu}$ respectively gives:

$$\begin{aligned}
&\lambda\Box\varphi + \nabla_\mu\lambda\nabla^\mu\varphi - \lambda V(\varphi)V_{,\varphi}(\varphi)+\\
&\quad\frac{\epsilon}{\rho_*}\left[\Box\varphi\left(F(X,\varphi)\left(2X+\frac{1}{2}V(\varphi)^2\right)+F_{,\varphi}(X,\varphi)X\left(X+\frac{1}{2}V^2(\varphi)\right)\right)+\right.\\
&\quad+\nabla_\mu\varphi\nabla^\mu\left(F(X,\varphi)\left(2X+\frac{1}{2}V(\varphi)^2\right)+F_{,X}(X,\varphi)X\left(X+\frac{1}{2}V^2(\varphi)\right)\right)+\\
&\quad\left.+F(X,\varphi)XV(\varphi)V_{,\varphi}(\varphi)+F_{,X}(X,\varphi)X\left(X+\frac{1}{2}V^2(\varphi)\right)\right]=0\,,
\end{aligned} \tag{A.2}$$

$$X - \frac{1}{2}V(\varphi)^2 = 0\,, \tag{A.3}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + 8\pi G T^{\rm int}_{\mu\nu}\,, \tag{A.4}$$

where

$$\begin{aligned}
T^{\rm int}_{\mu\nu} = \frac{2\epsilon}{\rho_*}&\left\{\nabla_\mu\varphi\nabla_\nu\varphi\left[F(X,\varphi)\left(2X+\frac{1}{2}V(\varphi)^2\right)+\right.\right.\\
&\left.\left.F_{,X}(X,\varphi)X\left(X+\frac{1}{2}V^2(\varphi)\right)\right]+g_{\mu\nu}F(X,\varphi)X\left(X+\frac{1}{2}V(\varphi)^2\right)\right\}\,,
\end{aligned} \tag{A.5}$$

In order to simplify the notation we define the following quantity:

$$\mathcal{L}^{\rm NMC}(X,\varphi) \equiv \frac{1}{2}F(X,\varphi)T^{\mu\nu}\frac{\nabla_\mu\varphi\nabla_\nu\varphi}{\rho_*} = \frac{F(X,\varphi)}{\rho_*}X\left(X+\frac{1}{2}V(\varphi)^2\right)\,, \tag{A.6}$$

so that the stress energy tensor can be rewritten as

$$T_{\mu\nu} = \left(\lambda + \epsilon\mathcal{L}^{\rm NMC}_X\right)\nabla_\mu\varphi\nabla_\nu\varphi + \epsilon\mathcal{L}^{\rm NMC}g_{\mu\nu}\,. \tag{A.7}$$

By a direct confrontation with the shape of the perfect fluid stress energy tensor we can identify the pressure of the field as

$$p = \mathcal{L}^{\rm NMC}(X,\varphi)\,. \tag{A.8}$$

Then, using the constraint equation (A.3) we get:

$$\rho = \left(\lambda + \epsilon\mathcal{L}^{\rm NMC}_X\right)V(\varphi)^2 - \epsilon\mathcal{L}^{\rm NMC}\,, \tag{A.9}$$

$$p = \mathcal{L}^{\rm NMC}, \tag{A.10}$$

$$u^\mu = V(\varphi)^{-1}\nabla^\mu\varphi\,. \tag{A.11}$$

The constraint (A.3) can be rewritten as:

$$\dot{\varphi} = -V(\varphi), \qquad \dot{} \equiv u_\mu\nabla^\mu\,, \tag{A.12}$$

while we have that

$$\vartheta \equiv \nabla_\mu u^\mu = V(\varphi)^{-1}\Box\varphi + V_\varphi(\varphi)\,, \tag{A.13}$$

with this set of definitions, equations (A.2) and (A.3) can be rewritten as

$$\dot{\lambda} = V^{-2}\left[V(\varphi)\rho_{,\varphi} - (\rho+\epsilon p)\,\vartheta\right]\,, \tag{A.14}$$

$$\dot{\varphi} = -V(\varphi), \tag{A.15}$$

where we notice that the minus sign in equation (A.14) has appeared coming from the definition of the derivative (A.12) and where

$$\rho_{,\varphi} = 2\lambda V V_{,\varphi} + \epsilon\left[\mathcal{L}^{\rm NMC}_{,XX}V^3V_{,\varphi} + \mathcal{L}^{\rm NMC}_{,X\varphi}V^2 + \mathcal{L}^{\rm NMC}_{,X}VV_{,\varphi} - \mathcal{L}^{\rm NMC}_{,\varphi}\right]\,. \tag{A.16}$$

Here we have defined the various terms in the Lagrangian as follows:

$$\mathcal{L}^{\rm NMC} = \frac{1}{2}F(X,\varphi)\frac{V(\varphi)^4}{\rho_*},\ \mathcal{L}^{\rm NMC}_{,X} = \frac{3}{2}F(X,\varphi)\frac{V(\varphi)^2}{\rho_*} + \frac{1}{2}F_{,X}(X,\varphi)\frac{V(\varphi)^4}{\rho_*}\,, \tag{A.17}$$

$$\mathcal{L}^{\rm NMC}_{,XX} = 2\frac{F(X,\varphi)}{\rho_*} + \frac{1}{2}F_{,XX}(\varphi,X)\frac{V(\varphi)^4}{\rho_*} + \frac{1}{2}F_{,X}(X,\varphi)\frac{V(\varphi)^2}{\rho_*}\,, \tag{A.18}$$

$$\begin{aligned}\mathcal{L}^{\rm NMC}_{,X\varphi} &= F(X,\varphi)\frac{V(\varphi)}{\rho_*}V_{,\varphi}(\varphi) + \frac{1}{2}F_{,\varphi X}(\varphi,X)\frac{V(\varphi)^4}{\rho_*} + \\ &+ \frac{1}{2}F(X,\varphi)_{,X}V_{,\varphi}(\varphi)\frac{V(\varphi)^3}{\rho_*} + \frac{3}{2}F_{,\varphi}(X,\varphi)\frac{V(\varphi)^2}{\rho_*}\,,\end{aligned} \tag{A.19}$$

$$\mathcal{L}^{\rm NMC}_{,\varphi} = \frac{1}{2}F(X,\varphi)\frac{V(\varphi)^3}{\rho_*}V_{,\varphi}(\varphi) + \frac{1}{2}F_{,\varphi}(X,\varphi)\frac{V(\varphi)^4}{\rho_*}\,. \tag{A.20}$$

Notice that the Klein-Gordon equation for the field φ is now an evolution equation for the Lagrangian multiplier λ equation (A.14).

Appendix B

Disformal transformation and the Horndeski action

In this appendix we provide in detail the results of disformal transformations on the Horndeski action. We will first show how the most general disformal transformation (6.8) spoils the second order nature of the equations of motion derived from the Horndeski action and then, after providing the transformation rules for geometrical quantities, we will discuss the invariance of the theory under the reduced disformal transformation (6.15).

B.1 Keeping second order field equations

In this section we show how a metric transformation induced by the general disformal relation (6.8) spoils the property of the Horndeski action of producing second order field equations.[1]

Our proof consists of direct calculation of the modifications that the disformal transformation has onto a particular term of the full Lagrangian, namely $\mathcal{L}_4$, when the disformal functions depends only on the kinetic term of the scalar field ϕ. Despite this does not represent a formal proof of our statement it is nonetheless general enough to discard any kinetic term dependence in the disformal transformation if second order field equations are to be preserved. We leave the formal proof of this for further work, but we stress that the result obtained here holds in general. Our calculations make use of [5], where a general procedure on how to build actions for a metric and a scalar field that keeps the equation

[1] We want to stress that this result holds on both curved background as well as on flat backgrounds with the exception that on flat space times there exist subcases that give second order field equations even after a disformal transformation.

of motion second order was put forward. We will shortly review it for what concerns us, referring the interested reader to the original paper.[2]

In flat space times consider the following Lagrangian:

$$\mathcal{L} = \mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\nabla_{\mu_1}\nabla_{\nu_1}\phi\ldots\nabla_{\mu_n}\nabla_{\nu_n}\phi\,, \tag{B.1}$$

where

$$\mathcal{T} = \mathcal{T}(\phi,\phi_\alpha), \qquad \mathcal{L} = \mathcal{L}(\phi,\phi_\mu,\nabla_\mu\nabla_\nu\phi)\,, \tag{B.2}$$

then the following lemma holds:

Lemma 1 *A sufficient condition for the field equations derived from the Lagrangian B.1 to remain second order or less is that* $\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdot\nu_n}$ *is totally antisymmetric in its first indices* μ_i *as well as (separately) in its last indices* ν_i.

Notice that this is a sufficient conditions. However the opposite statement has been proven and a uniqueness condition exists so that the condition is both necessary and sufficient.

When one moves to curved space times and covariantizes promoting partial derivatives to covariant derivatives, third order derivatives of the metric are produced. It has been shown that adding a suitable finite number of non-minimally coupled terms to the Lagrangian is enough to eliminate the higher than second derivatives from the equations of motion in both the scalar field and in the metric. As a final result the authors of [5] gave the form of the Lagrangian that preserves the second order equations:

$$\mathcal{L}_n\{f\} = \sum_{p=0}^{\lfloor n/2\rfloor}\mathcal{C}_{n,p}\mathcal{L}_{n,p}\{f\}\,, \tag{B.3}$$

where $\lfloor n/2\rfloor$ indicates the integer part while the curly bracket indicates that $\mathcal{L}$ is a functional of f, which is in general different for any n, and where

$$\mathcal{L}_{n,p}\{f\} = P_{(p)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\nabla_{\mu_1}R_{(p)}S_{(q\equiv n-2p)}\,, \tag{B.4}$$

$$R_{(p)} = \prod_{i=1}^{p}R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}}\,, \qquad S_{(q\equiv n-2p)} = \prod_{i=0}^{q-1}\nabla_{\mu_{n-i}}\nabla_{\nu_{n-i}}\phi\,, \tag{B.5}$$

while

$$P_{(p)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n} = \int_{X_0}^{X}dX_1\cdots\int_{X_0}^{X_{p-1}}dX_p\mathcal{T}_{(2n)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}(\phi,X_1)\,, \tag{B.6}$$

[2] Notice that in our work we have the following correspondences: $\pi\to\phi$, $\pi_\mu\to\phi_\mu$, $\pi_{\mu\nu}\to\nabla_\mu\nabla_\nu\phi$ and $X\to 2X$.

while the coefficients are given by:

$$\mathcal{C}_{n,p} = \left(-\frac{1}{8}\right)^p \frac{n!}{(n-2p)!p!} \,. \tag{B.7}$$

Using the Lagrangian B.1 and the rules reported above it is possible to construct all covariant theories that gives second order field equations and in particular in four dimensions we have that the Horndeski action is a linear combination of the following terms:

$$\begin{aligned}
\mathcal{L}_{0,0} &= X f_0(\phi, X)\,, \qquad \mathcal{L}_{1,0} = X f_1(\phi, X) A_2^{\mu\nu} \nabla_\mu \nabla_\nu \phi\,, && \text{(B.8)}\\
\mathcal{L}_{2,0} &= X f_2(\phi, X) A_4^{\mu_1\mu_2\nu_1\nu_2} \nabla_{\mu_1}\nabla_{\nu_2}\phi \nabla_{\mu_2}\nabla_{\nu_2}\phi\,, && \text{(B.9)}\\
\mathcal{L}_{3,0} &= X f_3(\phi, X) A_6^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} \nabla_{\mu_1}\nabla_{\nu_2}\phi \nabla_{\mu_2}\nabla_{\nu_2}\phi \nabla_{\mu_3}\nabla_{\nu_3}\phi\,, && \text{(B.10)}\\
\mathcal{L}_{2,1} &= P_{(1)}^{\mu_1\mu_2\nu_1\nu_2} R_{\mu_1\mu_2\nu_1\nu_2}\,, \qquad P_{(1)} = \int dX_1 \mathcal{A}_4^{\mu_1\mu_2\nu_1\nu_2} X_1 f_{(2)}(\phi, X_1)\,, && \text{(B.11)}\\
\mathcal{L}_{3,1} &= P_{(1)}^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} R_{\mu_1\mu_2\nu_1\nu_2} \nabla_{\mu_3}\nabla_{\nu_3}\phi\,, && \text{(B.12)}\\
P_{(1)} &= \int dX_1 \mathcal{A}_6^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} X_1 f_{(3)}(\phi, X_1)\,, && \text{(B.13)}
\end{aligned}$$

where the coefficients $\mathcal{C}$ are

$$\mathcal{C}_{0,0} = 1\,, \quad \mathcal{C}_{1,0} = 1\,, \quad \mathcal{C}_{2,0} = 1\,, \quad \mathcal{C}_{3,0} = 1\,, \quad \mathcal{C}_{2,1} = -\frac{1}{4}\,, \quad \mathcal{C}_{3,1} = -\frac{3}{4}\,, \tag{B.14}$$

where we have redefined the form function $\mathcal{T}_{2n}(\phi, X) = X f_n(\phi, 2X)\mathcal{A}_{2n}$ in such a way to separate the field dependences (ϕ, X) from the structure term $\mathcal{A}(g_{\alpha\beta}, \phi_\alpha)$. Notice that the terms $(2,0)$ and $(2,1)$ as well as $(3,0)$ and $(3,1)$ are coupled terms whose joint presence is required in order to cancel the unwanted higher order derivatives. The Horndeski action can be rephrased in these terms with the following identifications:[3]

$$G_2(\phi, X) = X f_{(0)}(\phi, X)\,, \qquad G_3(\phi, X) = X f_{(1)}(\phi, X)\,, \qquad \mathcal{A}_{(2)}^{\mu\nu} = g^{\mu\nu}\,, \tag{B.15}$$

$$G_4(\phi, X) = \int \left[X_1 f_{(2)}(\phi, X_1) dX_1\right]; \qquad \mathcal{A}_{(4)}^{\mu\alpha\nu\beta} = g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}\,, \tag{B.16}$$

$$\begin{aligned}
G_5(\phi, X) &= \int \left[X_1 f_{(3)}(\phi, X_1) dX_1\right] && \text{(B.17)}\\
\mathcal{A}_{(6)}^{\mu\sigma\alpha\nu\rho\beta} &= g^{\alpha\nu}\left[g^{\beta\mu} g^{\sigma\rho} - g^{\beta\rho} g^{\sigma\mu}\right] && \text{(B.18)}\\
&+ g^{\alpha\sigma}\left[-g^{\beta\mu} g^{\nu\rho} + g^{\beta\rho} g^{\nu\mu}\right] + g^{\alpha\beta}\left[g^{\sigma\mu} g^{\nu\rho} - g^{\sigma\rho} g^{\nu\mu}\right]. && \text{(B.19)}
\end{aligned}$$

[3]Notice that compared with the convention used in the definition of the kinetic term X in [5] there are factors 1/2 that have been reabsorbed into the definition of the function $f_{(n)}$.

In order to prove our statement we need to show how a kinetic dependent metric transformation spoils the antisymmetric structure of the model. Before entering the calculations we note that the effects of a disformal metric transformation on the coefficient functions $f_{(n)}$ only redefines its functional dependence, while the structure functions $\mathcal{A}$ are again only redefined with no modifications on their antisymmetric structure. Hence, in order to check the breaking of the antisymmetric structure, we only need to compute the effects of metric transformations on second covariant derivatives of the field and on the Riemann tensor. In order to do this in a simple way, we will look at the effects of the kinetic dependence of the disformal functions $A(X)$ and $B(X)$ by applying separately a conformal transformation and a purely disformal one on the terms corresponding to $\mathcal{L}_4$ in the rephrased Horndeski action (B.13).

B.1.1 Conformal transformation

Consider a conformal transformation of the kind

$$\bar{g}_{\mu\nu} = A(X) g_{\mu\nu}. \tag{B.20}$$

After the conformal transformation B.20 is performed the original $\mathcal{L}_4$ Lagrangian is mapped into:

$$\begin{aligned}\mathcal{L}_4 = A^2 G_4 R - A^2 G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] - 6AA'G_4(\nabla_\mu\nabla_\nu\phi)^2 \\ - 6AA'G_4\phi^\alpha\Box\nabla_\alpha\phi - 2AA'G_{4,X}\left[\Box\phi + \frac{A'}{A}\phi^\mu\phi^\nu\nabla_\mu\nabla_\nu\phi\right]\phi^\alpha\phi^\beta\nabla_\alpha\nabla_\beta\phi \\ + \phi^\mu\phi^\nu\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla^\alpha\phi\left[-4AA'G_{4,X} + AA'^2X - 6A''AG_4\right].\end{aligned} \tag{B.21}$$

From this expression it is clear that the first two terms are not dangerous as they have the same structure as those in the original Lagrangian. In order to better understand the others we proceed in rewriting them in the form $\mathcal{A}^{\mu\alpha\nu\beta}\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi$. Any antisymmetry violating term will then directly lead to higher derivatives in the equation of motions. After some manipulation we arrive at the expression:

$$\begin{aligned}\sim \Big[-6AA'G_4(g^{\alpha\mu}g^{\beta\nu}) + (-2G_{4,X}AA' + 6A'^2G_4 + AA''G_4 + AA'G_{4,X})g^{\mu\nu}\phi^\alpha\phi^\beta \\ +(-4G_{4,X}AA' + 4G_{4,X}A'^2X - 6AA''G_4)g^{\nu\beta}\phi^\alpha\phi^\mu - 2G_{4,X}A'^2\phi^\mu\phi^\nu\phi^\alpha\phi^\beta\Big]\nabla_\mu\nabla_\nu\phi\nabla_\alpha\nabla_\beta\phi,\end{aligned} \tag{B.22}$$

where the symbol $\sim$ indicates that only the dangerous terms have been considered and notice that we have added a surface term to rewrite the third order derivative. As can be easily seen antisymmetry breaking terms have appeared in the Lagrangian. We can then conclude that the generalized conformal transformation B.20 spoils the antisymmetric structure of the Horndeski action and hence gives equation of motion for the fields that are higher than second order.

B.1.2 Disformal transformations

Consider now a metric transformation of the form

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + B(X)\phi_\mu\phi_\nu \,. \tag{B.23}$$

Using the same procedure of the previous section we can write the transformed $\mathcal{L}_4$ part of the Lagrangian and see whether or not it is possible to recover the antisymmetric structure. The dangerous terms of the transformed Lagrangian read

$$\begin{aligned}
\sim \Bigg[g^{\mu\nu}\phi^\alpha\phi^\beta \Bigg(& \frac{2G_{4,X}(B'X+B)}{(1+2XB/A)^{1/2}} - \frac{2G_4(B^2 - B'(1+BX))}{(1+2XB/A)^{3/2}} \\
& -2\frac{G_4}{(1+2XB/A)^{1/2}} R_{\alpha\mu\beta\nu} - \frac{G_4B'}{(1+2XB/A)^{1/2}} + \frac{2B'G_4}{(1+2XB/A)^{1/2}} \\
& + \frac{2XB''G_4}{(1+2XB/A)^{1/2}} + \frac{2B'G_{4,X}}{(1+2XB/A)^{1/2}} - \frac{2B'G_4(B'X+B)}{(1+2XB/A)^{3/2}} \Bigg) \Bigg] \\
g^{\mu\alpha}\phi^\beta\phi^\nu \Bigg[& -\frac{G_{4,X}}{(1+2XB/A)^{1/2}} \Bigg(B - 2XB'(-1X^2B') \Bigg) + \frac{G_4}{(1+2XB/A)^{3/2}} (B^2 - B' + B'^2X^2 \\
& -XB''(1+2XB/A)\big) + \frac{G_4B'}{(1+2XB/A)^{1/2}} - 2\frac{G_{4,X}B'X}{(1+2XB/A)^{1/2}} - 2\frac{G_4B'}{(1+2XB/A)^{1/2}} \Bigg] \\
\phi^\mu\phi^\nu\phi^\alpha\phi^\beta \Bigg[& \frac{G_{4,X}B'(1-2X^2B')}{(1+2XB/A)^{1/2}} + \frac{G_4(-XB'^2 + B''(1+2XB/A) - BB')}{(1+2XB/A)^{3/2}} \\
& - \frac{G_{4,X}B'}{(1+2XB/A)^{1/2}} - \frac{G_4B''}{(1+2XB/A)^{1/2}} + \frac{G_4B'(B'X+B)}{(1+2XB/A)^{3/2}} \Bigg] \nabla_\mu\nabla_\nu\nabla_\alpha\phi\nabla_\beta\phi \,,
\end{aligned} \tag{B.24}$$

which, again, contains terms which are not antisymmetric in the couples (α, β) and (μ, ν) hence giving rise to higher derivatives in the equations of motion.

In conclusion, even if a formal proof of this result would be desirable, our result clearly states that if one wants to preserve second order field equations, then the most general disformal transformation that can be used is the one reported in equation (6.15) where the disformal functions A and B only depends on the scalar field ϕ.

B.2 Transformation properties of geometrical quantities.

We provide here the transformation rules for geometric quantities when the metric undergoes a disformal transformation of the kind

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi\mu\phi\nu\,, \tag{B.25}$$

where both metric g and $\bar{g}$ are well defined metrics that can be equally be used to rise and lower indices. The transformed inverse is:

$$\bar{g}^{\mu\nu} = \frac{1}{A(\phi)}g^{\mu\nu} - \frac{B(\phi)}{A(\phi)^2\,(1+2XB/A)}\phi^\mu\phi^\nu\,, \tag{B.26}$$

while the volume element changes (see appendix C of [203]) as

$$\sqrt{-\bar{g}} = A(\phi)^2\,(1+2XB/A)^{1/2}\,\sqrt{-g}\,. \tag{B.27}$$

From this definitions one can express all the barred curvature quantities in function of the unbarred metric and the scalar field ϕ. We list these below.

Connection coefficient

$$\begin{aligned}\bar{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} &+ \frac{B}{A\,(1+2XB/A)}\phi^\mu\nabla_\alpha\nabla_\beta\phi + \frac{A'}{2A}\left(\delta^\mu_\alpha\phi_\beta + \delta^\mu_\beta\phi_\alpha\right) \\ &+ \frac{1}{2}\frac{\phi^\mu}{A^2\,(1+2XB/A)}\left(-AA'g_{\alpha\beta} + (AB'-2A'B)\phi_\alpha\phi_\beta\right)\,. \end{aligned}\tag{B.28}$$

Ricci Tensor

$$\begin{aligned}\bar{R}_{\alpha\beta} = R_{\alpha\beta} &+ \left[\frac{AB\,(1+2XB/A)\,\Box\phi - B^2\phi^\mu\nabla_\mu X - AA'\,(1+2XB/A) + (AB'-A'B)X}{A^2\,(1+2XB/A)^2}\right]\nabla_\alpha\nabla_\beta\phi \\ &+ \left[\frac{-A^2A'\,(1+2XB/A)\,\Box\phi + AA'B\phi^\mu\nabla_\mu X - 2A'X^2(A'B-AB') - 2A^2A''X\,(1+2XB/A)}{2A^3\,(1+2XB/A)^2}\right]g_{\alpha\beta} \\ &+ \left[\frac{\Box\phi(A^3B' - 4AA'B'X - 2A^2B(A'-B'X)) - 2A''(A^3 + 6AB^2X^2 + 5A^2BX)}{2A^4\,(1+2XB/A)^2}\right. \\ &+ \left.\frac{AB\phi^\mu\nabla_\mu X(2A'B - AB') + 6A'^2B^2X + 2AA'BX(5A' + 3B'X) + 3A'A^2B'X}{2A^4\,(1+2XB/A)^2}\right]\phi_\alpha\phi_\beta \\ &+ \left[\frac{-2AB\,(1+2XB/A)\,R_{\alpha\mu\beta\nu}\phi^\mu\phi^\nu - 2AB\,(1+2XB/A)\,\nabla_\alpha\nabla_\lambda\phi\nabla_\beta\nabla^\lambda\phi + 2B^2\nabla_\alpha X\nabla^\alpha X}{2A^2\,(1+2XB/A)^2}\right. \\ &+ \left.\frac{(A'B-AB')(\phi_\alpha\nabla_\beta X + \phi_\beta\nabla_\alpha X) - \phi_\alpha\phi_\beta\phi^\mu\phi\nabla_\mu X}{2A^2\,(1+2XB/A)^2}\right]\,. \end{aligned}\tag{B.29}$$

Ricci Scalar

$$\begin{aligned}\bar{R} &= R - \frac{2B}{A^2\,(1+2XB/A)} R_{\alpha\beta}\phi^\alpha\phi^\beta + \frac{B}{A^2\,(1+2XB/A)}\left[(\Box\phi)^2 - (\nabla_\alpha\nabla_\beta\phi)^2\right] \\ &+ \frac{2B}{A^3\,(1+2XB/A)^2}\left[\nabla^\alpha X \nabla_\alpha X - \phi^\alpha\nabla_\alpha X\Box\phi\right] - \frac{8A'BX + A(3A'-2B'X)}{A^3\,(1+2XB/A)^2}\Box\phi \\ &+ \frac{4A'B - AB'}{A^3\,(1+2XB/A)^2}\phi^\alpha\nabla_\alpha X + \frac{3A'X(A'+2B'X)}{A^2\,(1+2XB/A)^2} - \frac{6A''X}{A^2\,(1+2XB/A)}\,. \end{aligned} \tag{B.30}$$

Notice that both functions A and B are to be intended as general functions of the scalar field ϕ.

B.3 Transformation properties of the Horndeski action under disformal transformations

We explored the consequences on the Horndeski action when the metric is transformed via a disformal transformation

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_\mu\phi_\nu\,. \tag{B.31}$$

through a direct calculation. Our results show that after this transformation is performed the new action can be recast into the same initial Horndeski form given that all the effect of the transformation are absorbed into the rescaling of the free coefficient functions. As a consequence we can say that the Horndeski action is formally invariant under this class of disformal transformation. We report below the transformations properties of the Horndeski Lagrangian coefficient functions. The new Lagrangian is

$$\bar{\mathcal{L}} = \sum_i \bar{\mathcal{L}}_i\,, \tag{B.32}$$

where

$$\bar{\mathcal{L}}_2 = \bar{G}_2(\phi, X)\,, \tag{B.33}$$

$$\bar{\mathcal{L}}_3 = \bar{G}_3(\phi, X)\Box\phi\,, \tag{B.34}$$

$$\bar{\mathcal{L}}_4 = \bar{G}_4(\phi, X)R - \bar{G}_{4,X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]\,, \tag{B.35}$$

$$\bar{\mathcal{L}}_5 = \bar{G}_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi + \tag{B.36}$$

$$+ \frac{\bar{G}_{5,X}(\phi, X)}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\nu\nabla_\mu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\,, \tag{B.37}$$

where

$$\bar{G}_2(\phi,X) = (1+2XB/A)^{1/2}\,G_2(\phi,X_D) + 2X\left[\frac{G_3(\phi,X_D)AA'}{(1+2XB/A)^{1/2}} + \frac{G_3(\phi,X_D)(A'B)X}{(1+2XB/A)^{3/2}} + H_{3,\phi}(\phi,X)\right]$$
$$+3X\frac{G_4(\phi,X_D)\left[A'+2A'B'X-2AA'-4A'BX\right]}{A\,(1+2XB/A)^{3/2}} + 12X\frac{G_{4,X}(\phi,X_D)X\left[A'^2BX-AA'B'X\right]}{A^2\,(1+2XB/A)^{1/2}}$$
$$-2XH_{4,\phi}(\phi,X) + \frac{3G_5(\phi,X_D)X^2A'}{A^4\,(1+2XB/A)^{5/2}}\left[-A'^2BX+2A^2A''\,(1+2XB/A)-A(2A'^2+3A'B'X)\right]$$
$$-2XH_{5,\phi}(\phi,X) + \frac{2G_{5,X}(\phi,X_D)X^3}{A^4\,(1+2XB/A)^{3/2}}\left(A'^3BX+AA'(A'+3B'X)\right)\,, \quad \text{(B.38)}$$

$$\bar{G}_3(\phi,X) = \left[\frac{AG_3(\phi,X_D)}{(1+2XB/A)^{1/2}} + H_3(\phi,X)\right]$$
$$+\left[\frac{G_4(\phi,X_D)\left(4AA'B+ABB'X+A'B^2X\right)}{A^2\,(1+2XB/A)^{3/2}} + \frac{BG_{4,\phi}(\phi,X_D)}{(1+2XB/A)^{1/2}}\right.$$
$$\left.+\frac{G_{4,X}\left(AA'BX-2A^2A'+2A^2B'X\right)}{A^2\,(1+2XB/A)^{1/2}} - H_4(\phi,X)\right]$$
$$+\left[X\left(-2(H_{\Box,\phi}(\phi,X)-H_{R,\phi\phi}(\phi,X)) + \frac{G_5(\phi,X_D)}{A^3\,(1+2XB/A)^{5/2}}\left(5A'^2BX-A\left(\frac{A'^2}{2}+6A'B'X\right)\right)\right.\right.$$
$$\left.\left.+\frac{2G_5(\phi,X_D)}{A\,(1+2XB/A)^{3/2}}A'' + \frac{G_{5,X}XA'}{A^3\,(1+2XB/A)^{3/2}}\left(AA'-2A'BX+4AB'X\right)\right) - H_5(\phi,X)\right]\,, \quad \text{(B.39)}$$

$$\bar{G}_4(\phi,X) = A\,(1+2XB/A)^{1/2}\,G_4(\phi,X_D) - \left(\frac{G_5(\phi,X_D)X^2}{A^2\,(1+2XB/A)^{3/2}}(A'B-AB') + H_{R,\phi}(\phi,X)X\right)\,, \quad \text{(B.40)}$$

$$\bar{G}_5(\phi,X) = \frac{G_5(\phi,X_D)}{(1+2XB/A)^{1/2}} + H_R(\phi,X)\,, \quad \text{(B.41)}$$

where the explicit form of the functions H_i are

$$H_4(\phi,X) = \int dX\left[\frac{G_4(\phi,X_D)\left(4AA'B+ABB'X+A'B^2X\right)}{A^2\,(1+2XB/A)^{3/2}}\right]\,, \quad \text{(B.42)}$$

$$H_3(\phi,X) = B\int dX\frac{G_3(\phi,X_D)}{(1+2XB/A)^{3/2}}\,, \quad \text{(B.43)}$$

$$H_5(\phi,X) = \int dX \Big[H_{\Box,\phi}(\phi,X) - H_{R,\phi\phi}(\phi,X) + \frac{G_5(\phi,X_D)}{2A^3\,(1+2XB/A)^{5/2}} \big(-5A'BX - 2A^2A''\,(1+2XB/A) + A(5A'^2 + 6A'B'X)\big) + \frac{G_{5,X}(\phi,X_D)}{A^2\,(1+2XB/A)^{3/2}} \big(-A'BX + 2A(A'+B'X)\big) \Big] , \quad \text{(B.44)}$$

$$H_\Box(\phi,X) = G_5(\phi,X_D) \frac{AA' + (AB' - A'B)X}{A^2\,(1+2XB/A)^{3/2}} , \quad \text{(B.45)}$$

$$H_R(\phi,X) = \frac{B}{A} \int dX \frac{G_5(\phi,X_D)}{(1+2XB/A)^{3/2}} , \quad \text{(B.46)}$$

while

$$X_D = \frac{X/A}{1+2B/AX} , \quad \text{(B.47)}$$

and, again, the functions A and B depend on the scalar field ϕ. The most relevant conclusion is that the effect of the disformal transformation on the Horndeski action can be recast into renormalisation of the coefficient functions, exactly as in the case of conformal transformations for standard scalar-tensor theories, which, we stress, are a subcase of our result. Then notice that, if one starts with a only a subset of the Lagrangians, a disformal transformation will in general produce contributions at all sub-Lagrangians in a hierarchical way. Said in other words, the corrections propagate from higher derivatives down to lower derivatives terms.

B.4 Invariance under field rescaling

Besides the previously analysed invariance under disformal transformation it can be proved that the Horndeski action is also invariant under the rescaling of the scalar field

$$\phi = s(\psi)\psi . \quad \text{(B.48)}$$

In fact, the effects of this transformation can be again reabsorbed into redefinitions of the Horndeski coefficient functions which become

$$\bar{G}_2(\psi,\bar{X}) = G_2(\psi,\bar{X}) + 2YG_3(\psi,\bar{X})(2s' + \psi s'') - 2YH_{4,\psi}(\psi,\bar{X}) + 2YH_{\Box,\psi} , \quad \text{(B.49)}$$

$$\bar{G}_3(\psi,\bar{X}) = (s'\psi + s)G_3(\psi,\bar{X}) - \big(4YG_{4,Y}(\psi,\bar{X}) - 2G_4(\psi,\bar{X})\big) \frac{2s' + s''\psi}{s + \psi s'} + 2YH_{5,\psi} - H_\Box , \quad \text{(B.50)}$$

$$\bar{G}_4(\psi,\bar{X}) = G_4(\psi,\bar{X}) - Y(2s' + \psi s'')G_5(\psi,\bar{X}) , \qquad \bar{G}_5(\psi,\bar{X}) = (2s' + s''\psi)G_5(\psi,\bar{X}) , \quad \text{(B.51)}$$

where

$$H_4(\psi,\bar{X}) = G_4(\psi,\bar{X})\frac{2s' + s''\psi}{s+\psi s'}\,, \qquad H_5(\psi,\bar{X}) = (2s' + \psi s'')G_5\frac{2s' + s''\psi}{s+\psi s'}\,, \tag{B.52}$$

$$H_\Box(\psi,\bar{X}) = \int d\bar{X}\, H_{5,\psi}(\psi,\bar{X})\,, \tag{B.53}$$

where $\bar{X} = (s'(\psi)\psi + s(\psi))^2 Y$, being $Y = \psi^\mu\psi_\mu/2$, and where a prime denotes the derivative w.r.t. ψ.

The field transformation is in principle arbitrary. However, as can be seen from, *e.g.,* the $\bar{G}_3$ coefficient, infinities may be generated if $s + \psi s' = 0$. This amounts to say that the solution $s(\psi) = \psi^{-1}$ is excluded from the set of admissible rescaling. This fact is in some sense obvious because it is equivalent to the limit of having no scalar field. A second remark concerns the possibility of eliminating the NMC with the Einstein tensor with a field redefinition. In fact the transformed G_5 coefficient is proportional to $2s'(\psi)+\psi s''(\psi)$. This equation can be integrated once giving $s(\psi) = -\psi s'(\psi)$, whose solution is excluded by the previous requirement. We conclude that it is not possible to eliminate the NMC with the Einstein tensor with a field redefinition.

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Acknowledgements to...

Stefano “Capo” Liberati for his contagious passion and for being so supportive. I’ll always remember our never ending discussions after which I felt so recharged (or was the lack of sugars?). Carlo Baccigalupi for enthusiastically greeting any little progress I made, making my work looking prettier. You both have thought me so much (another thing is what I actually got..) and I have really enjoyed working with you and I hope to keep on doing it! The people I have collaborated with, Valeria Pettorino and Lorenzo Sindoni, for their invaluable help. Valeria’s thousands of tips on how to convince people that what I am doing is interesting (seriously they have been really precious). My officemates, Aurora and Giulio: if I managed to get here it’s also thanks to you. My big bro Vincenzo, for taking me under his protecting wing. studentreps and student-council and all the people that have been/are silently working behind them, for all the discussions and efforts to make SISSA an even nicer place and for keeping alive that feeling of adolescence rebellion.

Monte Cengio B&G, my second family. Without you guys my life here in Trieste would have been healthier but for sure with much less fun... “Jason” Santoro and his electric saw. The “muli triestini” which, despite the roughness of their language, have a marshmallow heart. Those that know what “Onna no Kappa” is. The long nights at the barachin de barcola, the local Bermuda triangle Neverinn-Tetris-Grip; All the “grande bouffe” had and those who showed me that behind a brain we have a larger stomach. Trieste’s crags for allowing me to experimentally test the effects of gravity. Paklenica, the steering-wheel of the Kangoo and the Russian tale (I still don’t remember how it ends), Li Sánti Marío de la Mar; the afterwork swims, Istanbul and everything that came afterwards.

Four years are maybe not such a long period of time. But four Ph.D. years are more like an entire life and it has been a pleasure to share them with all of you!

Finally, huge thanks to my family for being there, always, with their unconditioned support and their example.