Scuola Internazionale Superiore di Studi Avanzati – Trieste
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Cascading theories in gauge/gravity duality

Stefano Cremonesi

Supervisor: Prof. Matteo Bertolini

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Foreword

This dissertation summarizes the main line of research that I have followed during my PhD course. It is based on some of the papers that I have co-written in the past three years [1–3], as well as some yet unpublished material. The main topic is the study of cascading gauge theories by means of type IIB supergravity dual solutions. This thesis is structured as follows.

The first four chapters contain review material on the large N limit of gauge theories and its relation with string theory, beautifully realized in the celebrated AdS/CFT duality and in its nonconformal extensions. Chapter 1 reviews the arguments suggesting that string theories might arise in the large N limit of gauge theories. In chapter 2 we introduce the AdS/CFT correspondence, which has provided the first explicit implementation of that long-standing expectation. In chapter 3 we recall the application of AdS/CFT duality, along with its nonconformal extensions, to the arena of the conifold; we introduce and review in detail the prototypical example of a cascading gauge theory, the famous Klebanov-Strassler theory, which at low energies describes the minimally supersymmetric Yang-Mills theory.

The subsequent three chapters follow the tangential line of research of holographic dualities for gauge theories with dynamical flavor fields. This field is relevant *per se* because it allows the application of gauge/gravity duality to a realm of phenomenologically more appealing theories, closer to real world QCD. Therefore we have included in chapter 4 an introduction to large N expansions in gauge theories with fundamental flavor fields and to the extension of AdS/CFT duality to account for flavor fields, with a critical presentation of the different approaches that have been developed in the literature. This chapter concludes the review material of the thesis.

With the aim of finding a gravity dual of large N $\mathcal{N} = 1$ Super-QCD, we consider the problem of adding dynamical flavors to the Klebanov-Strassler theory. As a necessary intermediate step, we first study in chapter 5 the addition of dynamical flavors to the conformal Klebanov-Witten theory. After that preliminary work, in chapter 6 we find a flavored version of the Klebanov-Strassler solution, learning, by means of novel methods that we introduce, that a different cascade describes the RG flow of the dual field theory.

In the last two chapters of the thesis we apply our new methods to the study of cascading gauge theories living on regular and fractional D3 branes at conical singularities. In chapter 7 we find a gravity solution for fractional branes at an orbifold of the conifold. That is the first instance of a supergravity solution describing fractional branes which trigger different low energy dynamics; we study its RG flow, discovering that the interplay of various fractional branes considerably enriches the pattern of the cascade with respect to previously known cases.

Stimulated by those results, in the final chapter we reconsider the well known solution describing fractional branes on an $\mathcal{N} = 2$ orbifold: we propose a novel dual interpretation of that solution in terms of a cascading RG flow at a very specific point of the moduli space of the gauge theory. We also find novel vacua, for which we identify the dual type IIB backgrounds, and in this process we gain an understanding of some of the surprising aspects of cascading gauge theories that we uncovered in the previous chapters.

Finally, we collect in five appendices our conventions and additional technical material which may help the comprehension of the main text.

Contents

Part I

The large N limit of gauge theories and string theory

Chapter 1

Introduction

String theory was born in the sixties as an attempt to model strong interactions, motivating the approximately linear Regge trajectories that hadron spectra fit. Veneziano's dual model [4] (with respect to s-channel and t-channel exchanges of virtual particles) was soon interpreted as stemming from a theory of open strings [5]. The idea was that a meson could be modeled by an open string, representing a strong interaction flux tube, linking a quark and an antiquark at its endpoints. Mesons with different spins arise in this picture as different oscillation modes of the same string, and their interactions are interpreted as splitting and joining interactions of open strings. Unfortunately, it was soon understood that quantum consistency of Veneziano's dual model in four dimensions requires that the spin 0 state of the spectrum is a tachyon and the spin 1 state is massless. But in real world hadronic phenomenology, there are no tachyons nor massless vector mesons, therefore string theory started to look problematic as a quantum theory of strong interactions.

In the same years, Quantum Chromodynamics (QCD) was proposed as another candidate for describing strong interactions. In the early seventies, the discovery of asymptotic freedom [6] singled out QCD as the correct quantum field theory describing strong interactions between quarks and gluons. The high energy behavior of the theory has overcome innumerable tests at accelerator facilities, starting from the deep inelastic scattering experiments which originally led to the formulation of the phenomenological parton model in the late sixties, up to more modern experiments at LEP and Tevatron where the whole Standard Model of elementary particles and fundamental interactions, including QCD, has been extensively tested. The theory is well formulated and tested in the ultraviolet in terms of the interactions between quarks and gluons, but its large distance properties still escape a complete understanding. We know from observations that quarks and gluons are confined inside mesons and baryons, but we are still far from being able to understand and prove confinement from first principles, since for energies of the order of the dimensional transmutation scale of QCD – about 200 MeV – the theory is strongly coupled and perturbative computations are of no use. A low energy effective description of QCD is available – it goes under the name of Chiral Perturbation Theory – but the spectrum of hadrons and the coupling constants of their interactions cannot be obtained from the underlying microscopic theory.

Several brilliant methods have been developed through the years in order to overcome these difficulties. One of the most powerful approaches is lattice QCD [7], which apart from technical difficulties allows in principle the computation of all the low energy features of QCD; unfortunately, even when it is very successful in estimating quantities, this method fails in providing a qualitatively deeper insight of the phenomena it exhibits.

Another approach which looked very promising was the large N expansion of gauge theories [8]. 't Hooft rearranged the Feynman diagrams of QCD in a topological expansion which resembles that of a theory of open and closed strings, with coupling proportional to $1/N$, thus reconnecting gauge theories with string theory. Despite the simplification, nobody has been able to sum all the planar diagrams and solve $SU(N)$ QCD in the large N limit and propose an elementary string theory dual. Nevertheless, many distinctive phenomenological features of QCD have a natural explanation in the large N limit, suggesting that this be a very good approximation to real world $SU(3)$ QCD.

Although QCD is the exact theory of strong interactions, well describing the approximately Coulombic interactions at short distances, at larger distances, of the order of 1 fm, the quark-antiquark interaction seems to be well approximated by a linear potential, indicating the presence of a flux tube. String theory may therefore be viewed as an effective low energy description of chromoelectric flux tubes, reliable when the length of the flux tube makes its thickness negligible. Semiclassical quantization of long Nambu-Goto strings was considered by Lüscher, who predicted a quark-antiquark potential of the form $V(r) = Tr + \mu + \frac{\gamma}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$ at large distances [9]. The coefficient γ of the so-called Lüscher term is proportional to the zero-point energy of the string in four spacetime dimensions. Remarkably, lattice calculations are in good agreement with Lüscher's formula for $r > 0.7$ fm [10].

During the nineties, the discovery of D branes [11] in string theory and the following formulation of the AdS/CFT duality conjecture [12] and its nonconformal extensions, brought together string theory and gauge theory once again, providing the first explicit realization of 't Hooft's idea for a four-dimensional gauge theory, and allowing computations in strongly coupled large N gauge theories by means of weakly coupled and weakly curved string duals. Surprisingly, from that viewpoint the effective thick YM string emerges holographically from the string of a fundamental theory.

1.1 't Hooft's large N_c expansion for pure Yang-Mills theories

More than forty years after its formulation, and thirty-five years after the discovery of asymptotic freedom that established it as the theory of strong interactions between quarks and gluons, Quantum Chromodynamics (QCD) is still a challenge for theoretical physicists. The perturbative approach is not useful to describe long distance properties of QCD, because of dimensional transmutation: if the number of flavors is not too large, gauge dynamics runs towards strong coupling along the renormalization group flow, and dynamically generates a mass scale Λ_{QCD} that fixes the order of magnitude of nonperturbative observables such as the spectrum of hadrons and the thickness and the energy density of chromoelectric flux tubes which confine quarks inside hadrons. At energies of the order of $\Lambda_{QCD} \sim 200$ MeV, the gauge coupling becomes strong and the perturbative expansion which describes very well ultraviolet (UV) phenomena such as deep inelastic scatterings is no longer applicable. The best available quantitative method to tackle infrared (IR) properties of QCD is that of lattice gauge theories. However, getting some analytic theoretical handle on the theory would be clearly of great help.

Already in the 1970's, considerable progress arose from the insight of 't Hooft [8], who proposed to generalize the $SU(3)$ color group of real world QCD to an $SU(N_c)$ group, and showed that taking a suitable large N_c limit the diagrammatics of the gauge theory greatly simplifies: Feynman diagrams of the gauge theory rearrange in a topological expansion, weighted by powers of $1/N_c$, which closely resembles that of a string theory.

In addition to the original masterpiece [8], we refer the reader to the first part of [13] for a nice introduction to the subject and to [14] for a more comprehensive review.

Let us review here the argument for the pure Yang-Mills (YM) theory. In section 4.1.1 we will generalize to a Yang-Mills theory with flavor fields in the fundamental and antifundamental representation. For the sake of simplicity we restrict our attention to the $U(N_c)$ pure Yang-Mills theory, including only gluons in the adjoint representation of the gauge group.[1] The difference between $U(N_c)$ and $SU(N_c)$ can be traced back, and affects only subleading diagrams in the expansions. We will follow the convention according to which gauge fields are canonically normalized in the action

$$S_{YM} = -\frac{1}{2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\ , \tag{1.1.1}$$

which must be supplemented with a gauge fixing and a ghost term for quantization. In our conventions

$$\mathrm{Tr}(T_A T_B) = \frac{1}{2}\delta_{AB} \tag{1.1.2}$$

is the normalization of generators T_A in the fundamental representation of $U(N_c)$. The covariant derivative is

$$D_\mu \equiv \partial_\mu + igA_\mu\ , \tag{1.1.3}$$

and the nonabelian field strength is

$$F_{\mu\nu} \equiv -\frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\ . \tag{1.1.4}$$

For the sake of simplicity we will neglect ghosts in the treatment; since they also transform in the adjoint representation, they can be included safely without changing the qualitative results that are laid out in the following.

In a large N_c limit, it is important to keep track of traces over color indices arising from loops of virtual gluons, since each trace carries a factor of N_c. In view of this, 't Hooft introduced the so called 'double line notation'. Since $N_c \times \overline{N_c} = \mathrm{Adj}$, as far as the color structure is concerned we can associate to a gluon a quark-like index in the fundamental representation and an antiquark-like index in the antifundamental representation of the gauge group. A quark propagator and an antiquark propagator are usually denoted in Feynman rules as a single line, with an arrow pointing in one direction or the opposite one. One can therefore naturally represent a gluon propagator as a double line, made out of two parallel single lines with opposite orientations, as in Fig. 1.1.

The color structure in Feynman rules is simplified in this notation. Color conservation, or gauge invariance of the vertices, implies that each line entering a vertex (cubic or quartic) also exits the vertex as a neighboring line (there is no crossing). Furthermore, since the fundamental representation of a $U(N_c)$ group is complex, lines have an arrow on top and vertices preserve line orientations. The cubic and quartic gluon vertices in double line notation are presented in Fig. 1.2.

Let us now consider the one loop correction to the gluon propagator which arises from inserting two cubic vertices, depicted in Fig. 1.3. Each cubic vertex carries a g factor, and the sum over color indices k brings in an additional N_c factor; all in all, the diagram is proportional to $g^2 N_c$. This power counting analysis clarifies that a naive $N_c \to \infty$ limit would make radiative corrections

[1]The same argument goes through for a theory of $N \times N$ matrices, with a global $U(N)$ symmetry.

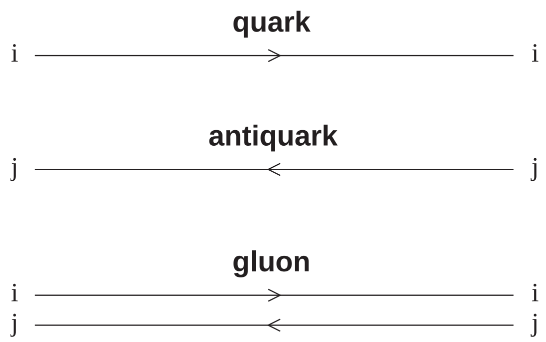


Figure 1.1: Propagators in double line notation.

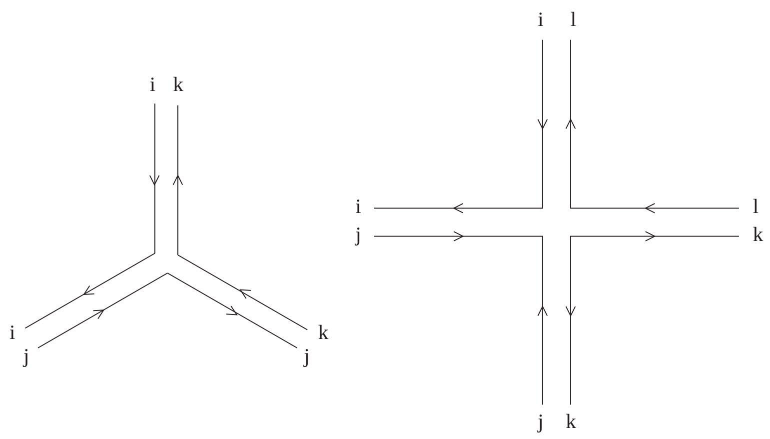


Figure 1.2: Gluon vertices in double line notation.

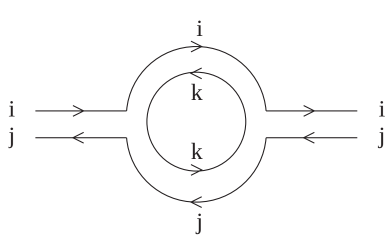


Figure 1.3: Radiative correction to the gluon propagator.

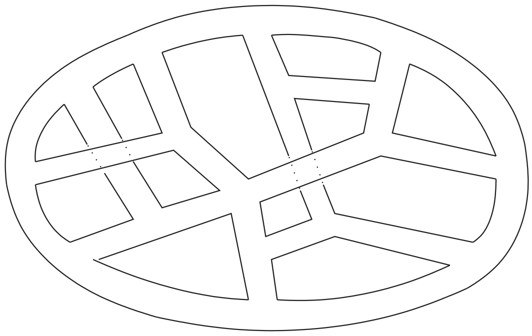


Figure 1.4: Instance of a connected vacuum diagram. This Feynman diagram is not planar since there are two overpasses which add two handles to the plane.

diverge because of multiplicities in loops. In order for this radiative correction to be finite, the limit which has to be considered is rather a double scaling limit:

$$\begin{cases} N_c \to \infty \\ g \to 0 \end{cases} \qquad \text{keeping} \qquad \lambda \equiv g^2 N_c \quad \textit{fixed} \, , \tag{1.1.5}$$

so that the vanishing of the gauge coupling cancels the divergence of the combinatoric factor. λ is usually called 't Hooft coupling, and is the coupling that naturally comes out of loop computations in gauge theories, once combinatoric factors are taken into account. In this limit, the diagram of Fig. 1.3 scales as λ, and has the same N_c^0 dependence as the tree level gluon propagator.

An alternative way to understand why we should consider the large N_c limit (1.1.5) is to look at the Yang-Mills one-loop β-function,

$$\mu \frac{dg}{d\mu} = -\frac{11}{3} N_c \frac{g^3}{16\pi^2} \, . \tag{1.1.6}$$

This equation does not have a sensible large N_c limit if we keep the Yang-Mills coupling fixed, since the right-hand side diverges with N_c. If instead we introduce the 't Hooft coupling λ, its one-loop β-function is

$$\mu \frac{d\lambda}{d\mu} = -\frac{22}{3} \frac{\lambda^2}{16\pi^2} \, , \tag{1.1.7}$$

therefore we can sensibly keep λ fixed as we send $N_c \to \infty$.

In the large N_c limit (1.1.5), the Yang-Mills coupling g, which weights vertices in Feynman rules, goes to zero. In order for a diagram to survive the large N_c limit, its combinatoric factors must be large enough to compensate the vanishing of the gauge coupling. It turns out that only 'planar diagrams', those which can be drawn on a plane without lines crossing out of a vertex, survive this limit. More generally, Feynman diagrams nicely rearrange in a $1/N_c$ expansion where the powers of the expansion parameter depend on the topology of the diagram. Let us review the argument. We will first restrict our attention to connected vacuum diagrams, which at lowest order in perturbation theory scale like N_c^2. By attaching little surfaces (polygons) to each color loop, we get a two-dimensional surface: it is characterized by having as many vertices V as the interaction vertices of the perturbative expansion, as many faces F as the number of color loops, and as many edges E as the number of propagators. We further denote by V_3 and V_4 the number of cubic and quartic vertices respectively, so that $V = V_3 + V_4$. A generic example is drawn in figure 1.4. We invite the reader to count the number of vertices, edges and faces of the diagram. Because Feynman rules assign weight g to the cubic vertex and g^2 to the quartic vertex, the diagram will scale like

$$r = g^{V_3+2V_4} N_c^F \ . \tag{1.1.8}$$

Making use of the relation[2]

$$2E = 3V_3 + 4V_4 = 2V + (V_3 + 2V_4) \tag{1.1.9}$$

between the number of propagators and the numbers of cubic and quartic vertices, we find that

$$r = g^{2(E-V)} N_c^F \ , \tag{1.1.10}$$

and using the definition of the Euler character of the surface $\chi \equiv F - E + V = 2 - 2h$ (h is the number of handles) we get that the diagram scales like

$$r = \lambda^{\frac{1}{2}V_3+V_4} N_c^{2-2h} \ . \tag{1.1.11}$$

The result is that the N_c dependence of the diagram is fixed by the Euler character of the minimal genus Riemann surface over which the diagram can be drawn. Planar diagrams can be drawn on a 2-sphere ($h = 0$),[3] and all scale as N_c^2, like the lowest order diagram. The first subleading nonplanar diagrams are those which can be drawn on a 2-torus ($h = 1$), and are subleading by a factor of $1/N_c^2$. Diagrams with genus h are subleading by a factor of N_c^{-2h}. At each topology, an infinite number of diagrams must be summed, constructing a series expansion in λ. At fixed genus, the power of λ in this expansion grows as the number of loops F. We can be more specific: the exponent of λ in (1.1.11) is $\frac{1}{2}V_3 + V_4 = E - V = l - 1$, where $l = E - V + 1 = F - (2 - 2h) + 1$ is the number of unconstrained loop momenta that we have to integrate over after imposing energy-momentum conservation.

The sum of all connected vacuum amplitudes is given by the double expansion

$$\mathcal{A} = \sum_{h=0}^{\infty} N_c^{2-2h} \sum_{F=2-2h}^{\infty} c_{h,F} \lambda^{F-(2-2h)} \ . \tag{1.1.12}$$

Formula (1.1.11), which extends also to theories with matter in the adjoint representation, looks like the topological expansion of a closed oriented string theory where the closed string coupling is proportional to $1/N_c$. The series expansion in λ, namely the loop expansion in the field theory at fixed genus of the diagram, should be related to an expansion in the worldsheet coupling constant. Formula (1.1.12) suggests that any gauge theory might be reformulated as a suitable string theory.[4]

[2] We remark here a subtlety: this formula fails to hold for the simplest vacuum diagrams having a single propagator going back to itself. In order to avoid this caveat, one should add at least one ‘quadratic’ vertex, which is actually an intermediate point for the propagation, that has to be integrated over and carries no gauge coupling dependence. This insertion of a quadratic vertex has the effect of adding one edge to all diagrams except the ‘pathological’ one. The more correct formulae are then $V = V_2 + V_3 + V_4$ and $2E = 2V_2 + 3V_3 + 4V_4$.

[3] We add the point at infinity so as to compactify the surface.

[4] In the case of $SU(N_c)$ gauge group, the lack of the last $U(1)$ factor brings in $1/N_c^2$ corrections. The result is that each Feynman diagram contributes in the expansion for any Riemann surface over which it can be drawn. Similar large N_c expansions can be formulated for orthogonal and symplectic groups; since the fundamental representations are real in that case, the outcome is that the topological expansion is in term of both oriented and nonoriented Riemann surfaces.

This derivation can be easily generalized to the case when Green functions of gauge invariant glueball operators are present. In the case of single trace operators of the form $\mathrm{Tr}(F^M)$, the only change is that these insertions appear as punctures in the Riemann surface; they add one (external) vertex and one edge.

The power of 't Hooft's expansion and the simplifications of the large N_c limit has led to very important qualitative understandings of YM and QCD phenomenology. For the pure YM theory, assuming the existence of the large N_c limit and confinement, one can prove that glueballs for large N_c are free, stable and not interacting. The amplitude for a glueball to decay into two glueballs scales like $1/N_c$, whereas the amplitudes for glueball-glueball elastic scatterings are of order $1/N_c^2$. Glueballs are in infinite number, and their masses have smooth large N_c limits. In the large N_c expansion, they are the weakly coupled degrees of freedom appearing in the spectrum of Yang-Mills theory. In the strict limit, the theory of glueballs is free. The $1/N_c$ expansion can be rephrased as a loop expansion in this effective theory of glueballs.

Unfortunately, despite the simplifications, the resummation of the planar diagrams of a four-dimensional interacting quantum field theory has not been achieved by field theory technique, nor it has been possible to understand which kind of closed orientable string theory should be dual to large N_c Yang-Mills theory in four dimensions. Along this last route, Polyakov [15] has suggested that the dual string theory should live in a world with an additional dimension. More recently, just before the advent of the AdS/CFT conjecture, he has proposed that this fifth dimension, which has to be related to the energy scale of the field theory, should be warped [16]. Right after, the AdS/CFT duality proposal formulated by Maldacena in [12] provided the first explicit and quantitative realization of 't Hooft's and Polyakov's ideas, by conjecturing that the maximally supersymmetric four-dimensional field theory is equivalent to type IIB superstring theory on a manifold $AdS_5 \times S^5$.

Chapter 2

AdS/CFT duality

In this chapter, we give a brief introduction to the subject of AdS/CFT duality, as the paradigm of more general gauge/string dualities.

After recalling the complementary descriptions of D branes in section 2.1, in section 2.2 we consider the decoupling limit of D3 branes that led Maldacena to conjecture the duality between $\mathcal{N} = 4$ $SU(N)$ SYM and type IIB string theory on an $AdS_5 \times S^5$ manifold, that we formulate more precisely in section 2.3. In section 2.4 we review the dictionary between observables in the two theories, as well as the prescription for computing gauge theory correlators from the knowledge of the dual string background. For a more complete exposition of the subject, we refer the reader to some of the many reviews on the subject [17]. Finally, in section 2.5, we briefly mention some of the generalizations of AdS/CFT duality to nonconformal and less supersymmetric settings.

2.1 D branes

The existence of Ramond-Ramond (RR) potentials (a 1-form C_1 and a 3-form C_3 in type IIA string theory; an axial scalar C_0, a 2-form C_2 and a 4-form C_4 with self-dual field strength in type IIB string theory) has been known since the birth of superstring theory. Such potentials arise at the massless level in the RR sector of the perturbative spectrum, but it is easy to see that the corresponding vertex operators do not involve them, but only their gauge invariant field strengths (and their Hodge duals). The fundamental string is not electrically charge under any of these potentials (not even C_2, which has the correct rank to couple to a string), but only feels their field strengths, in an analogous way to the dipole interactions of electrodynamics. Until the discovery of D branes by Polchinski in 1995 [11], it was not clear how to describe perturbatively the states charged electrically and magnetically under such potentials, although string dualities required their existence. By minimal coupling, it is clear that a $(p+1)$-dimensional extended object can couple electrically to a $p+1$-form potential C_{p+1} via a term $\mu_p \int \hat{C}_{p+1}$, where μ_p is the charge density carried by the source and the hat means pullback.

2.1.1 Black p branes

The low energy limits of type IIA and IIB superstring theory are type IIA and type IIB supergravity. Horowitz and Strominger [18] were able to find classes of solitonic solutions of the equations of motion of those theories with the topology of extended $(p+1)$-dimensional objects surrounded by a horizon, and carrying electric charge with respect to the RR potential C_{p+1}. These solutions were named 'black p branes'. They are subject to a BPS bound, stating that their mass (per unit volume) is bounded from below by the value of the charge (per unit volume). In these classes of solutions, the extremal objects saturate the BPS bound and preserve half of the original supersymmetries. At least in the BPS case, one can naturally expect that such solutions extend to solitonic solutions of the full-fledged string theory. For an overview of solitons in 10- and 11-dimensional supergravities, we refer the reader to the classical review [19], which is also historically instructive for young readers since it was written before the advent of D branes.

We review here the black p branes solutions found by Horowitz and Strominger [18] in the type IIA and IIB supergravities. The result will be expressed in string frame.

The metric of the black p brane solution is

$$ds^2 = Z_p(r)^{-1/2}\left[-K(r)dt^2 + \sum_{i=1}^{p} dx_i^2\right] + Z_p(r)^{1/2}\left[\frac{dr^2}{K(r)} + r^2 ds^2_{S^{8-p}}\right] , \tag{2.1.1}$$

where t is the timelike coordinate, x^i $(i = 1, \ldots, p)$ are the spacelike coordinates parallel to the brane, r is a radial coordinate in the directions transverse to the brane, and $ds^2_{S^{8-p}}$ is the metric of a unit round $(8-p)$-sphere;

$$Z_p(r) = 1 + \alpha_p \left(\frac{r_p}{r}\right)^{7-p} \tag{2.1.2}$$

$$K(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p} \tag{2.1.3}$$

are the radial functions appearing in the metric (2.1.1), and we have defined the constants

$$r_p^{7-p} = d_p (2\pi)^{p-2} g_s N \alpha'^{\frac{7-p}{2}} \tag{2.1.4}$$

$$d_p = 2^{7-2p}\pi^{\frac{9-3p}{2}}\Gamma\left(\frac{7-p}{2}\right) \tag{2.1.5}$$

$$\alpha_p = \sqrt{1 + \left(\frac{r_H^{7-p}}{2r_p^{7-p}}\right)^2} - \frac{r_H^{7-p}}{2r_p^{7-p}} . \tag{2.1.6}$$

These solutions have a horizon at $r = r_H$, covering a singularity at $r = 0$. In addition, there are nontrivial profiles for the dilaton (except for $p = 3$) and the $(p+1)$-form RR potential:

$$e^{\Phi} = g_s Z_p(r)^{\frac{3-p}{4}} \tag{2.1.7}$$

$$C_{p+1} = g_s^{-1} Z_p(r)^{-1}\, dx^0 \wedge \cdots \wedge dx^p . \tag{2.1.8}$$

This solution therefore carries an electric charge with respect to the RR field strengths F_{p+2} and a mass. As for the Reissner-Nordström black hole in General Relativity, the event horizon exists only if the ratio between mass and charge satisfies a certain lower bound, which in this case is a BPS bound.

These solutions simplify in the extremal limit $r_H = 0$, where the bound is saturated. The corresponding solutions, which preserve half the supersymmetries of the asymptotic background, are:[1]

$$ds^2 = H_p(r)^{-1/2} dx_{1,p}^2 + H_p(r)^{1/2} \left[dr^2 + r^2 ds_{S^{8-p}}^2\right] \tag{2.1.9}$$

$$e^{\Phi} = g_s H_p(r)^{\frac{3-p}{4}} \tag{2.1.10}$$

$$C_{p+1} = g_s^{-1} H_p(r)^{-1}\, dx^0 \wedge \cdots \wedge dx^p \ , \tag{2.1.11}$$

where the warp factor

$$H_p = 1 + \left(\frac{r_p}{r}\right)^{7-p} \tag{2.1.12}$$

is a spherically symmetric harmonic function in the transverse space. The solution can easily be generalized to a multicentered solution by choosing as the warp factor a generic harmonic function

$$H_p = 1 + \frac{1}{N} \sum_{i=1}^{N} \left(\frac{r_p}{|\vec{r} - \vec{r}_i|}\right)^{7-p} \ . \tag{2.1.13}$$

Let us make some remarks. The solution for seven-branes in type IIB supergravity, which is a codimension 2 object, can be obtained by putting a harmonic function $H_7(r) \propto \log \frac{r}{r_p}$. Eight-branes in type IIA (massive) supergravity are codimension 1 and form a domain wall. Finally, the derivation of the solution for three-branes in type IIB supergravity is subtle since they are both electric and magnetic sources of the self-dual RR field strength F_5. Nevertheless, the punchline is that the solution falls in the class described above, and can be obtained by simply fixing $p = 3$ in the formulae written above.

2.1.2 Dp branes

Few years later, Polchinski [11] reconsidered the addition of open strings with mixed Neumann-Dirichlet boundary conditions to type II superstring theories. If Neumann boundary conditions are imposed on $p+1$ coordinates and Dirichlet boundary conditions are imposed on the remaining $9-p$ coordinates, the endpoints of open strings are constrained to live on a hyperplane, called D(irichlet) brane, with p spatial dimensions and one timelike dimension. Consistency requires p to be even in type IIA string theory and odd in type IIB string theory. Mixed Neumann-Dirichlet boundary conditions break one half of the supersymmetries of the background. The D brane gravitationally interacts with bulk fields via open-closed string interactions, therefore it is not a rigid object, but can fluctuate. The tension of the D brane arises from a disk diagram, therefore scales as g_s^{-1} in string frame.

Massless open string excitations describe fluctuations of the D brane: a vector boson describes fluctuations parallel to the brane, $9-p$ scalars are the Goldstone bosons of the spontaneously broken translation symmetry in the transverse directions, and their fermionic superpartners are the Goldstinos of the spontaneously broken supersymmetries. The starting point and the endpoint

[1] We have not been careful here in the sign of the RR field, which will determine if this solution will correspond to an extremal black p brane or an extremal black anti-p brane.

of an open string need not be on the same D3 brane: both the Chan-Paton factors of open strings range from 1 to N, and all the excitations transform in the adjoint representation of $U(N)$, where N is the number of D branes. These massless fields and the action describing the interactions among them, which is that of maximally supersymmetric $U(N)$ YM theory in $p+1$ dimensions, can be obtained by dimensional reduction of $\mathcal{N}=1$ $U(N)$ Super-Yang-Mills (SYM) theory in ten dimensions to $p+1$ dimensions.

The fundamental observation made by Polchinski is that not only Dp branes carry tension, but they also carry electric charge with respect to the RR potential C_{p+1}. The computation of a cylinder/annulus amplitude shows that the gravitational attraction is exactly balanced by the repulsion due to the RR field sourced by the D branes. There is no net force between two parallel Dp branes, as should be expected since the objects are mutually BPS: their tension is equal to their charge density. Dirac quantization, together with string dualities, fixes unambiguously the value of the tension of a Dp brane. We refer the reader to appendix A.2.1 for details on the worldvolume action of D branes.

Once these properties were understood, it was natural to connect the open string perspective of a Dp brane as spacetime defects where open strings can end with the closed string perspective of an extremal black p brane as a solitonic solution of closed string theory. Those are two complementary descriptions of one and the same object.

2.2 The decoupling limit of D3 branes

In the previous section we have seen that Dp branes have two complementary descriptions. From the perturbative/open string point of view, they represent the locus of the endpoints of open strings, whose massless fluctuations are described by a maximally supersymmetric Yang-Mills theory. These open strings interact not only among themselves but also with the closed string modes propagating and interacting in the bulk, whose massless fluctuations are described by a maximally supersymmetric supergravity theory (type IIA if p is even, or type IIB if p is odd). On the other hand, from the closed string point of view, they arise as nonperturbative solutions of the equations of motion (of the low energy supergravities), carrying tension and charge so that they curve spacetime.

More than ten years ago now, in a beautiful paper [12] Maldacena studied the decoupling limit of extremal nondilatonic branes: D3 branes in type IIB string theory, and M2 and M5 branes in M-theory. This analysis led him to the AdS/CFT duality conjecture. We will review here the argument for D3 branes.

From a perturbative point of view, the string theory description of a stack of N D3 branes in flat space goes as follows (see Figure 2.1). Open strings are attached to the D3 branes, and their Chan-Paton factors are in the fundamental times antifundamental representation of the gauge group $U(N)$. The massless level of the open string sector is described by a four-dimensional $\mathcal{N}=4$ $U(N)$ gauge theory, with Yang-Mills coupling constant $g^2_{YM} \propto g_s$. In addition, there is a tower of massive open string modes, with masses of the order of the inverse string length. In the bulk ten-dimensional Minkowski space, closed strings propagate. The massless level of the closed string spectrum is described by type IIB supergravity in flat ten-dimensional spacetime, with Newton constant $G_N \propto \kappa^2 \propto g_s^2 \alpha'^4$, where $\alpha' = l_s^2$, with l_s the string length. In addition, there is a tower of massive closed string modes, with masses of the order of l_s^{-1}. Open string and closed string modes interact. In particular, at the massless level the Yang-Mills theory hosted by the D3 branes is coupled to supergravity. In order for this perturbative picture to be valid, we need $g_s N \ll 1$,

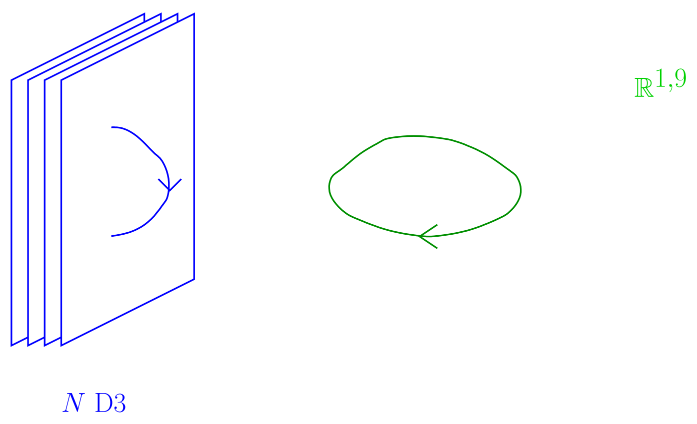


Figure 2.1: Perturbative picture ($g_s N \ll 1$): type IIB superstring theory with N D3 branes in flat spacetime.

since loop diagrams of open strings (higher genera) are weighted by powers of $g_s N$, whereas loop diagrams of closed strings are weighted by g_s.

In a low energy limit ($E \ll \alpha'^{-1/2}$), all massive string modes, open and closed, decouple. Furthermore, because Newton's constant in ten dimensions has negative mass dimension, (super)gravity decouples too. This is nothing but the well known statement that gravity is IR-free. In this low energy limit, the $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory hosted by the stack of N D3 branes remains interacting, since its gauge coupling is dimensionless. Actually, the diagonal $U(1)$ is free and decoupled.

As usual in effective field theories, we can also define the low energy limit formally as a parametric limit in which the string mass scale $\alpha'^{-1/2}$ is sent to infinity and the typical energies relevant to the gauge theory are kept fixed. A typical energy scale is the mass of a W-boson in a vacuum belonging to the Coulomb branch. For simplicity, we can think of displacing a single D3 brane from the stack of the other $N-1$ D3 branes by a distance r, so that we spontaneously break the gauge group $U(N) \to U(N-1) \times U(1)$. The massive W-boson corresponds to a string stretched between the single displaced brane and the stack of the remaining ones, which has mass $r/(2\pi\alpha')$. The decoupling limit will therefore be

$$\alpha' \to 0 \,, \qquad U \equiv \frac{r}{\alpha'} \quad fixed\,. \tag{2.2.1}$$

In this decoupling limit we bring the branes together, but the expectation value of the Higgs fields remains fixed.

Summarizing, what survives the low energy limit is the four-dimensional $\mathcal{N} = 4$ $U(N)$ gauge theory, together with a decoupled free gravity in ten flat noncompact spacetime dimensions.

From the dual closed string point of view, the stack of N D3 branes is represented by a warped RR background in type IIB superstring theory, where closed strings propagate. In the string frame, the background has a constant dilaton, a warped metric and N units of 5-form flux through the sphere parameterizing the angles of the transverse $\mathbb{R}^6$ (plus the Hodge dual term):

$$ds^2 = h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \left[dr^2 + r^2 ds_{S^5}^2\right] \tag{2.2.2}$$

$$g_s F_5 = (1 + *)\, d^4x \wedge dh(r)^{-1} \tag{2.2.3}$$

$$h(r) = 1 + \frac{L^4}{r^4} \tag{2.2.4}$$

$$e^\Phi = g_s \tag{2.2.5}$$

where $ds_{S^5}^2$ is the metric of a round 5-sphere, and the integration constant $L^4 = 4\pi\alpha'^2 g_s N$ is fixed by requiring that the D3 brane charge of the solution be N. See appendix A.2.1 for the definition of brane charges.

The D3 brane solution preserves 16 supercharges, and the metric asymptotes to that of ten-dimensional flat spacetime far from the branes. Close to the branes there is a throat: the metric asymptotes to the near-horizon geometry,[2] which can be obtained by dropping the 1 in the warp factor and is $AdS_5 \times S^5$,[3]

$$ds^2 = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2}{r^2} dr^2 + L^2 ds_{S^5}^2 \tag{2.2.6}$$

with constant dilaton and N units of 5-form flux piercing the sphere. The curvature radii of AdS^5 and S^5 are the same: $L = (4\pi\alpha'^2 g_s N)^{1/4}$. The near-horizon solution preserves all the 32 supersymmetries of type IIB supergravity.

In order for this supergravity picture to be valid, we need $g_s N \gg 1$ so that the curvature is everywhere small (in string units), and $g_s \ll 1$ to suppress interactions between closed strings. The D3 brane geometry is depicted in Figure 2.2.

We now want to take the same low energy limit $E \ll \alpha'^{-1/2}$ considered before. Here the energy E refers to the one measured by an observer in the asymptotically flat Minkowski region.[4] What survives the low energy limit in the asymptotically flat region is a ten-dimensional free supergravity multiplet, for the same reason expounded before. These modes also decouple from the ones in the throat region because, having very long wavelength, they cannot sense the throat. On the other hand, the whole tower of massive string modes localized in the deep throat region survives the limit. Because of the warp factor, they have to climb an infinite gravitational barrier to exit the throat. Any closed string state with arbitrarily high proper energy is redshifted to arbitrarily low energy as measured by an observer in the asymptotic region, provided that the string is located deep enough inside the throat.

Summarizing, what survives the low energy limit in this closed string perspective is the direct sum of free gravity in flat ten-dimensional spacetime and the full interacting type IIB superstring theory on the near-horizon manifold.

As in the dual perspective, we can equivalently view the low energy limit as the parametric limit $\alpha' \to 0$, keeping $U = \frac{r}{\alpha'}$ fixed (2.2.1). This means that we are concentrating our attention

[2] We adhere to the usual nomenclature in the literature. However, one should keep in mind that in the extremal 3-brane solution (2.2.2) there is no horizon nor brane-source singularity at all: the metric is geodesically complete and nonsingular.

[3] Actually what we see here is only the Poincaré patch of the full AdS_5 geometry.

[4] The fact that the field theory energy (the conjugate momentum to t) of a string stretched between the 'horizon' and a probe brane displaced at r is proportional to r/α' also in this dual picture comes from the cancellation of the $\sqrt{g_{rr}}$ factor in the proper distance element with the redshift factor $\sqrt{g_{tt}}$.

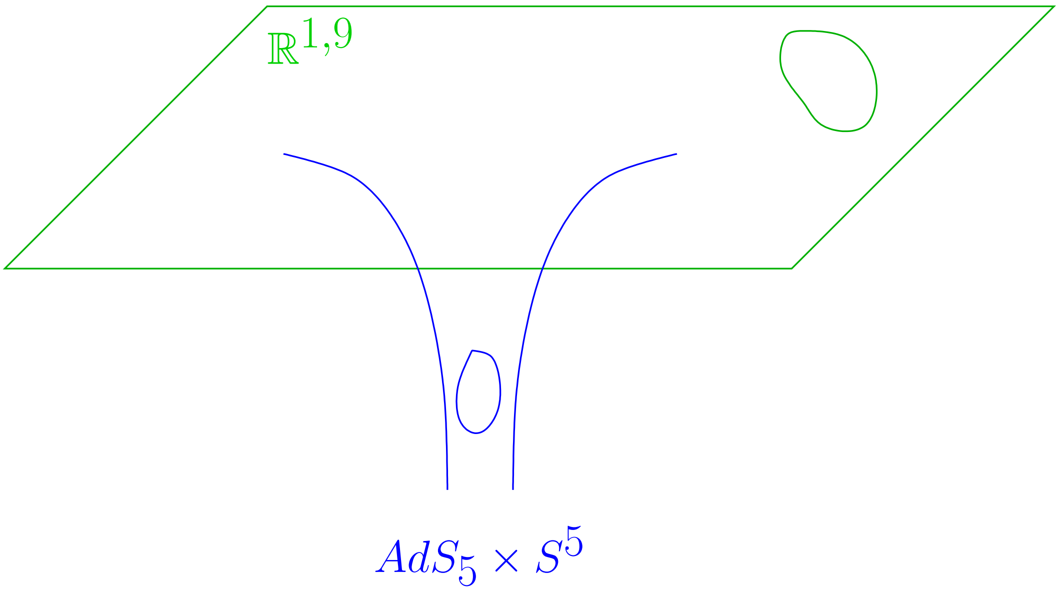


Figure 2.2: Gravitational picture ($g_s N \gg 1$): type IIB closed strings propagating in the geometry generated by a stack of D3 branes in flat spacetime.

on substringy distances from the D3 branes, since $\frac{r}{\sqrt{\alpha'}} = \frac{r}{\alpha'}\sqrt{\alpha'} \to 0$. We will only see their near-horizon geometry. In this limit the warp factor (2.2.4) loses the additive 1, so that the geometry does not asymptote flat spacetime anymore, and becomes

$$h(U) = \frac{R^4}{\alpha'^2 U^4} \ , \tag{2.2.7}$$

where we conveniently introduced $R^4 = L^4/\alpha'^2 = 4\pi g_s N$. Plugging this into the metric (2.2.2), we find the near-horizon $AdS^5 \times S^5$ geometry, expressed in terms of the U coordinate which remains fixed in the decoupling limit:

$$ds^2 = \alpha'^2 \left[\frac{U^2}{R^2} dx_{1,3}^2 + \frac{R^2}{U^2} dU^2 + R^2 ds^2(S^5) \right] \ . \tag{2.2.8}$$

The previous analysis concerns the same low energy limit applied to two complementary pictures of the same system, a stack of N coincident D3 branes in flat ten-dimensional space. Therefore the two pictures should lead to the same result. In the perturbative string theory picture what survives the low energy limit is $\mathcal{N} = 4$ $U(N)$ SYM theory in four dimensions, decoupled from free type IIB supergravity in Minkowski space. In the complementary picture, what survives the low energy limit is the full interacting type IIB string theory on $AdS_5 \times S^5$, decoupled from free type IIB supergravity in Minkowski space. This analysis led Maldacena to conjecture that $\mathcal{N} = 4$ $U(N)$ SYM in four dimensions is equivalent (dual) to type IIB superstring theory on $AdS_5 \times S^5$ with N units of RR 5-form flux.

2.3 The different formulations of AdS/CFT duality

Let us start by formulating the AdS/CFT duality conjecture (for D3 branes) in its full glory. In the strong form, it states the equivalence between the following *a priori* very different theories:

- $\mathcal{N} = 4$ SYM theory in 4-dimensions, with gauge group $SU(N)$ and complexified coupling

$$\tau_{YM} \equiv \frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g_{YM}^2} \ . \tag{2.3.1}$$

- Type IIB superstring theory on the background $AdS_5 \times S^5$, where AdS_5 and S^5 have the same curvature radius $L = (4\pi g_s N \alpha'^2)^{1/4}$, with N units of 5-form flux

$$N = -\frac{1}{(4\pi^2\alpha')^2}\int_{S^5} F_5 \tag{2.3.2}$$

  and with axio-dilaton (complexified string coupling)

$$\tau = C_0 + ie^{-\Phi} \ . \tag{2.3.3}$$

Some comments are in order at this point.

First of all, the duality is conjectured to hold for the full quantum theories. In particular it should apply to all the vacua of the two theories. Until now we have discussed the origin of the moduli space of $\mathcal{N} = 4$ SYM, known to preserve the full conformal symmetry, which corresponds to having N coincident D3 branes in the string theory realization. This vacuum is dual to the unexcited $AdS_5 \times S^5$ background 2.2.6 with constant axio-dilaton $\tau = C_0 + \frac{i}{g_s}$ of type IIB superstring theory, which is the near-horizon geometry of the background generated by N coincident D3 branes. But the duality statement is much more powerful than this: it allows us to relate any states and marginal/relevant deformations of the theory to asymptotically $AdS_5 \times S^5$ geometry perturbed by normalizable and nonnormalizable deformations respectively. We will elaborate on this issue in section 2.4.

Secondly, we have to comment on the gauge group of the $\mathcal{N} = 4$ SYM theory. In section 2.1 we have reviewed that a stack of N D3 branes hosts a $U(N)$ gauge theory. The diagonal $U(1)$ factor is decoupled and free from the remaining nonabelian $SU(N)$ dynamics. One could wonder whether or not this free abelian factor is described by type IIB string theory on the AdS background. It turns out that it is not. A first evidence for this will be clearer after we review the operator/field correspondence in section 2.4. For the time being, let us only state that protected operators (short multiplets) of the gauge theory are dual to supergravity modes in the $AdS_5 \times S^5$ background. It turns out that the classification of S^5 KK modes of type IIB supergravity on $AdS_5 \times S^5$ performed in the eighties exhausts all the field theory short multiplets, except for the one having as a bosonic component the trace of a single adjoint complex scalar field. This operator would not exist in an $SU(N)$ gauge theory, whereas it exists in a $U(N)$ gauge theory. A second evidence comes from the existence of the 'baryonic vertex': a D5 brane wrapped on S^5 represents a baryonic vertex on which external quarks, in the form of strings starting from the boundary of AdS_5, end. This configuration represents in the field theory a coupling of the form $\epsilon_{a_1 \ldots a_N} Q^{a_1} \ldots Q^{a_n}$, where Q's are external quarks. Had the gauge group been $U(N)$, this coupling would have been forbidden by gauge invariance. These two facts show that the gauge group of the dual field theory captured by type IIB string theory on the $AdS_5 \times S^5$ background is $SU(N)$ rather than $U(N)$.

Let us now return to the statement of the AdS_5/CFT_4 duality and concentrate on the superconformal phase of the gauge theory and the $AdS_5 \times S^5$ vacuum of type IIB string theory. In order to define the duality properly, we have to map parameters of the field theory into parameters of the string theory. This map follows from the way the correspondence was "derived" by Maldacena [12]. The low energy physics of a stack of N D3 branes, is described by $\mathcal{N} = 4$ SYM with a complexified gauge coupling τ_{YM} (2.3.1), which gets identified with the axio-dilaton τ (2.3.3) of the type IIB background, as results from expanding the Dirac-Born-Infeld action (A.2.28) and the Wess-Zumino action (A.2.29) of D3 branes in powers of α':

$$\frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g_{YM}^2} \equiv \tau_{YM} = \tau \equiv C_0 + \frac{i}{g_s} \ . \qquad (2.3.4)$$

This last equality allows us to map the complex modulus τ of type IIB string theory on $AdS_5 \times S^5$ with the complexified coupling τ_{YM} of the gauge theory, which parameterizes the maximally symmetric submanifold of conformal fixed points of the field theory. The number of colors N of the gauge theory is mapped into the number of units of 5-form flux on the 5-sphere.

We can rephrase the map in terms of the string coupling and the string tension:

$$g_s = \frac{1}{4\pi} g_{YM}^2 = \frac{1}{4\pi}\sqrt{\frac{\lambda}{N}} \qquad \frac{L^2}{\alpha'} = \sqrt{4\pi g_s N} = \sqrt{\lambda} \ , \qquad (2.3.5)$$

where L^2/α', the square of the curvature radius of AdS_5 and S^5 in string units, is the inverse of the dimensionless coupling appearing in the nonlinear sigma model on $AdS_5 \times S^5$, and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling of the gauge theory.

The strong form of the conjecture states the equivalence between the two theories for any values of the rank N and of the 't Hooft coupling of the field theory, or equivalently for any values of the string coupling and of the sigma model coupling of the string theory. In this form, it is very difficult to test the duality because quantization of string theory on a curved RR background is still a challenge.[5]

For this reason, it has proven useful to consider more tractable but still very interesting limits of the duality. These limits are related to the two possible expansions in string theory, the genus expansion and the worldsheet perturbative expansion.

The first limit that we may consider consists in keeping the size of AdS_5 and S_5 fixed and sending the string coupling $g_s \to 0$, thus suppressing higher genus worldsheet topologies with respect to the sphere, at fixed worldsheet coupling. In the field theory, it corresponds to keeping the 't Hooft coupling λ fixed and sending the number of colors N to infinity. It is exactly the large N limit considered by 't Hooft, reviewed in section 1.1, which suppresses nonplanar diagrams in the double line notation.

We remark that considering this limit provides the first precise formulation of the duality between a string theory and a four-dimensional gauge theory in the large N limit, that was suggested by 't Hooft more than two decades earlier in [8]. The strict $N \to \infty$ limit relates the planar gauge

[5]Very nontrivial tests of the correspondence at the quantum level in a curved background have been performed by considering the duality between type IIB string theory on the maximally supersymmetric PP-wave and the BMN subsector of $\mathcal{N} = 4$ SYM [20].

Quantum type IIB string theory on $AdS_5 \times S^5$ $\forall g_s\ ,\ \ \forall \frac{\alpha'}{L^2}$	Full quantum $\mathcal{N}=4$ SYM $\forall N\ ,\ \ \forall \lambda$
Classical type IIB string theory on $AdS_5 \times S^5$ $g_s \to 0\ ,\ \ \forall \frac{\alpha'}{L^2}$	Planar $\mathcal{N}=4$ SYM $N \to \infty\ ,\ \ \forall \lambda$
Classical type IIB supergravity on $AdS_5 \times S^5$ $g_s \to 0\ ,\ \ \frac{\alpha'}{L^2} \ll 1$	Strongly coupled planar $\mathcal{N}=4$ SYM $N \to \infty\ ,\ \ \lambda \gg 1$

Figure 2.3: The three formulations of the AdS/CFT duality conjectures, from the strongest to the weakest.

theory to classical string theory on $AdS_5 \times S^5$, and the $1/N$ expansion is mapped into the genus expansion on worldsheets of different topologies.

Notice however that whereas the gauge theory is weakly coupled when $\lambda \ll 1$, the dual sigma model on the sphere is weakly curved in the opposite region of parameter space, $\lambda \gg 1$.

We can simplify even more the correspondence, by taking further the large λ limit of the previous weak form of the conjecture. In this way we suppress nonlinearities in the worldsheet sigma model and reduce to type IIB supergravity on an $AdS_5 \times S^5$ with large curvature radii for the two factors. Stringy α' corrections to supergravity are mapped into $\lambda^{-1/2}$ corrections in the strongly coupled planar dual $\mathcal{N}=4$ field theory.

We conclude this section with a discussion of the global symmetries of the two theories which are proposed to be dual. $\mathcal{N}=4$ SYM is a superconformal theory, whose global symmetries form the supergroup $PSU(2,2|4)$. The bosonic part of this supergroup consists of the product of the $SO(2,4) \approx SU(2,2)$ group of conformal symmetries in four dimensions and the $SO(6)_R \approx SU(4)_R$ group of R-symmetries, under which the vector boson of the $\mathcal{N}=4$ vector multiplet does not transform, the 4 Weyl fermions transform in the $\underline{4}$ representation, and the 6 real scalars transform in the $\underline{6}$. In addition to the bosonic charges of the $SO(2,4) \times SO(6)$ algebra, there are 32 fermionic supercharges: 16 of them generate $\mathcal{N}=4$ Poincaré supersymmetry, while the additional 16 are conformal partners of the ordinary supercharges and can be obtained by commuting ordinary supercharges with conformal boosts.

If the maximal conformal supergroup $PSU(2,2|4)$ is not anomalous, these symmetries must show up also on the other side of the duality. They do. The $SO(2,4)$ conformal group arises as the isometry group of AdS_5, whereas the $SO(6)$ R-symmetry group arises as the isometry group of S^5. The $AdS_5 \times S^5$ background is a maximally supersymmetric background, which preserves 32 supercharges: 16 of them are the supercharges preserved by the D3 brane background, acting as ordinary supercharges on the 4d Minkowski part; the extra 16 supercharges arise only in the near-horizon limit, and may be obtained again by commuting ordinary supercharges with conformal boosts.

We see that global symmetries of the field theory are translated into large gauge transformations leaving the background invariant. Symmetries on the two sides of the duality match.

Finally, $\mathcal{N} = 4$ is expected to have an $SL(2,\mathbb{Z})$ Montonen-Olive duality symmetry, acting on the complexified gauge coupling as

$$\tau_{YM} \longmapsto \frac{a\,\tau_{YM} + b}{c\,\tau_{YM} + d}\,, \tag{2.3.6}$$

with $\{a,b,c,d\} \in \mathbb{Z}$ and $ad - bc = 1$. This duality symmetry maps into the type IIB string theory S-duality symmetry, which acts on the axio-dilaton τ in the same way as in (2.3.6), leaving the metric background and the 5-form flux invariant.

2.4 The operator-field correspondence

In order for the equivalence to be well defined, we still have to provide a map between the observables in the dual theories and a recipe for computing gauge theory correlators from the dual string theory. This result was achieved in two independent papers by Gubser, Klebanov and Polyakov [21] and by Witten [22], few months after the conjecture was proposed.

$\mathcal{N} = 4$ SYM at the origin of its moduli space is a scale-invariant theory. As such, it is not sensible to define particles as asymptotic states and to compute an S-matrix. The observables of this conformal field theory are the correlators of gauge invariant operators. We will discuss local operators, although nonlocal operators such as the Wilson loop may be included. These local operators can be single trace operators or multiple trace operators.[6] Multiple trace operators can be defined through the Operator Product Expansion (OPE) of single trace operators, but a renormalization is required to take care of UV divergences. In the large N limit, knowledge of the correlation functions of single trace operators fixes the correlation functions of all multiple trace operators to leading order in the $1/N$ expansion.

Local gauge invariant operators are labeled by their quantum numbers under the $PSU(2,2|4)$ global symmetry group of the field theory. The short multiplets of the supersymmetry algebra are: 1/2 BPS multiplets, spanning 2 units of helicity, 1/4 BPS multiplets, spanning 3 units of helicity, and 1/8 BPS multiplets, spanning 7/2 units of helicity. Long multiplets span 4 units of helicity.

On the other side of the duality, there are the excitations of type IIB string theory on the $AdS_5 \times S^5$ background. These fields are also labeled by their quantum numbers under the $PSU(2,2|4)$ symmetries of the background, and belong to $\mathcal{N} = 8$ multiplets. A long $\mathcal{N} = 8$ multiplet spans 4 units of spin. Therefore modes arising from KK compactification on S^5 of the supergravity fields, having maximum spin 2, must fill 1/2 BPS multiplets.

Maldacena's conjecture is refined by declaring a map between local gauge invariant operators $\mathcal{O}(x)$ of $\mathcal{N} = 4$ SYM and the quanta $|\phi\rangle$ of type IIB string theory on $AdS_5 \times S^5$. All the quantum numbers have to match; in particular, we will see that there is a dictionary between the conformal dimension Δ of $\mathcal{O}(x)$ and the 'mass' m of the string theory quantum $|\phi\rangle$.[7]

In order to understand the prescription for computing gauge theory observables by means of type IIB string theory on $AdS_5 \times S^5$ [21,22], we first recall that according to the argument which led

[6] The notion of single-traceness is only well defined in the large N limit. At finite N, only traces of less than N operators are independent. Traces with more operators can be expressed as linear combinations of lower multiple trace operators.

[7] The AdS_5 'mass squared' we are referring to here is defined as the eigenvalue of $-P_M P_M$, in analogy with the case of the Poincaré group. We remark that, unlike the Poincaré case, this is not a Casimir of the AdS_5 isometry supergroup nor of the $SO(2,4)$ bosonic group.

Maldacena to conjecture the duality, the rescaled coordinate r/α' in the metric (2.2.6) represents an energy scale in the dual field theory. The metric of AdS_5 (2.2.6) in Poincaré coordinates has $\mathbb{R}^{1,3}$ Minkowski slices at fixed r. On the other hand, a quantum field theory is usually defined by a bare Lagrangian in the UV. Therefore one is tempted to associate the conformal boundary of AdS_5 living at $r=\infty$ with the spacetime where the field theory lives.

The argument that we have presented here for the sake of simplicity, although physically intuitive, is not very firm. We refer the reader to the paper by Witten [22] for a more precise discussion of global issues. Suffices to say that the boundary of the conformally compactified AdS_5 is identical to the conformal compactification of the 4-dimensional Minkowski space, over which the conformal group acts. Therefore, the identification of this boundary with the spacetime where the conformal field theory lives naturally arises. Furthermore, the boundary of AdS_5 extends in the timelike direction; consequently, boundary conditions have to be specified there in order for the Cauchy problem to be well posed.

We then recall that we have already identified the complexified gauge coupling τ_{YM} of $\mathcal{N}=4$ SYM with the expectation value of the axio-dilaton field τ in the string background. The expectation value is fixed by the boundary condition at $r=\infty$. Changing the gauge theory coupling constant amounts to changing the boundary value of the dilaton. In the gauge theory, τ_{YM} can be viewed as a source for the $\mathcal{N}=1$ field strength vector multiplet $\mathrm{Tr}W^\alpha W_\alpha$, because of the term

$$\begin{aligned}\mathcal{L}_{SYM} &= \frac{1}{16\pi}\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}W^\alpha W_\alpha\right)+c.c.=\\ &=\frac{1}{g_{YM}^2}\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-2i\lambda\sigma^\mu D_\mu\overline{\lambda}+D^2\right)+\frac{\theta_{YM}}{16\pi^2}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}\end{aligned} \tag{2.4.1}$$

appearing in the Lagrangian density. In particular, we see that we can rewrite this term as a boundary term in AdS_5, having compactified on the transverse S^5:

$$S_{bdry}\supset\frac{1}{16\pi}\mathrm{Im}\left(\tau\int d^4x\int d^2\theta\,\mathrm{Tr}W^\alpha W_\alpha\right)+c.c\,. \tag{2.4.2}$$

The boundary behavior of the axio-dilaton field (reduced to AdS_5), that we have to specify in order to well pose the Cauchy problem, acts as a (constant) source for a gauge invariant operator of the field theory.

We need to specify the boundary conditions of all fields in the noncritical string theory on AdS_5 which is obtained upon S^5 compactification of type IIB superstring theory on $AdS_5\times S^5$. In analogy to the previous observation, it is then natural to associate to each $\phi(x,r)$ of these fields a gauge invariant local operator $\mathcal{O}(x)$ in the gauge theory, in such a way that its asymptotic behavior $\phi_0(x)$ at the boundary $r\to\infty$ (after factorization of the r dependence: $\phi(x,r)\sim r^{\Delta-4}\phi_0(x)$) acts as a source in the dual field theory:

$$Z_{CFT}[\phi_0(x)]\equiv\left\langle e^{\int d^4x\,\phi_0(x)\mathcal{O}(x)}\right\rangle_{CFT}=Z_{string}\left[\phi(x,r)\Big|_{r\to\infty}\simeq\left(\frac{r}{L}\right)^{\Delta-4}\phi_0(x)\right]\,. \tag{2.4.3}$$

Z_{CFT} is the generating functional of connected Green's functions of gauge invariant local operators $\mathcal{O}(x)$ in the conformal field theory, and ϕ_0 are the associated external sources; on the right hand side there is the (unknown) partition function of the S^5 compactification of string theory on $AdS_5\times S^5$ with boundary conditions $\phi_0(x)$ for the bulk fields $\phi(x,r)$.

In the supergravity limit of the correspondence ($N \to \infty$, $\lambda \gg 1$), the unknown string theory partition function reduces to its saddle point value

$$Z_{string}\left[\phi(x,r)\Big|_{r\to\infty} \simeq \left(\frac{r}{L}\right)^{\Delta-4}\phi_0(x)\right] \approx \exp\left[-S_{sugra+KK}\right] \ , \tag{2.4.4}$$

where $S_{sugra+KK}$ is the *on shell* action of the 5-dimensional supergravity obtained by KK compactification, with the prescribed boundary conditions for the supergravity fields.

Summarizing, in order to compute correlators of gauge invariant local operators in the planar $\lambda \gg 1$ limit of $\mathcal{N}=4$ $SU(N)$ SYM, one has to: specify the boundary conditions for the fields of KK compactification of $AdS_5 \times S^5$ type IIB supergravity; solve the equations of motion for those fields subject to the boundary conditions; plug them back into the supergravity action in the right-hand side of (2.4.4); finally, take functional derivatives with respect to the sources/boundary conditions and set to zero the sources not appearing in the Lagrangian.

The computation may also be rephrased in a more intuitive fashion by using Witten's formalism [22], which mimics the usual Feynman diagrams technique by means of boundary-to-bulk and bulk-to-bulk propagators, together with bulk interaction vertices that can be read from the five-dimensional action.

We are now in the position to understand the relation between the conformal dimension Δ of a gauge invariant operator $\mathcal{O}$ and the mass m of the dual string or supergravity field ϕ. Let us consider for simplicity a free scalar field ϕ with mass m propagating in AdS_5.[8] The argument generalizes to tensors and spinors, with simple modifications of the mass-conformal dimension relation. Its equation of motion $(\Box_5 - m^2)\phi(x,r) = 0$ has two independent solutions close to the boundary $r \to \infty$: the asymptotic general solution is

$$\phi(x,r) \simeq \left(\frac{r}{L}\right)^{-\Delta_+}\phi_+(x) + \left(\frac{r}{L}\right)^{-\Delta_-}\phi_-(x) \ , \tag{2.4.5}$$

where

$$m^2L^2 = \Delta(\Delta-4) \ , \tag{2.4.6}$$

and we define $\Delta_\pm$ to be the roots of this equation:

$$\Delta_\pm = 2 \pm \sqrt{4+m^2L^2} \ . \tag{2.4.7}$$

In AdS_5, all scalar modes with a value of m^2 such that the argument of the square root in (2.4.7) is nonnegative,

$$m^2L^2 \geq -4 \ , \tag{2.4.8}$$

are stable. This is the famous Breitenlohner-Freedman bound [23].

Since $\Delta_+ > \Delta_-$, the leading asymptotics is

$$\phi(x,r) \simeq \left(\frac{r}{L}\right)^{-\Delta_-}\phi_-(x) \ , \tag{2.4.9}$$

where ϕ_- is what was earlier called ϕ_0.

Consider a scale transformation

$$(x^\mu, r) \longmapsto (\alpha\, x^\mu, \alpha^{-1}\, r) \ , \tag{2.4.10}$$

[8] It is assumed that far from the bulk interaction region, the bulk field is asymptotically free.

which leaves the AdS_5 metric, and therefore the scalar field $\phi(x,r)$, invariant. Relation (2.4.9) implies that under the scale transformation $\phi_-(x) = \alpha^{\Delta_-}\,\phi_-(\alpha x)$. Finally, scale invariance of the source term $\int d^4x\,\phi_-(x)\mathcal{O}(x)$ in the CFT partition function implies that

$$\mathcal{O}(\alpha x) = \alpha^{-\Delta_+}\mathcal{O}(x)\ , \tag{2.4.11}$$

meaning that

$$\Delta \equiv \Delta_+ = 2 + \sqrt{4 + m^2L^2} \tag{2.4.12}$$

is the scaling dimension of the gauge invariant operator $\mathcal{O}$.

We remark here that the (2.4.8) bound would imply $\Delta \geq 2$. This relation follows from unitarity of the CFT in most AdS/CFT pairs. In $\mathcal{N} = 4$ SYM, the gauge invariant operator with lowest conformal dimension is the trace of a product of two adjoint scalars, which has quantum dimension 2. In very special cases, like the Klebanov-Witten theory that we will review in the next Chapter, gauge invariant operators with conformal dimension less than 2 exist. In such cases, the conformal dimension is Δ_- instead of Δ_+ [24]. Secondly, we read immediately from (2.4.12) that irrelevant operators correspond to positive m^2 fields, marginal operators correspond to vanishing m^2 fields, and relevant operators correspond to negative m^2 field above the Breitenlohner-Freedman bound.

We can now revisit the general asymptotic solution (2.4.5) for a scalar field with mass parameter m, that we rewrite as:

$$\phi(x,r) \simeq \left(\frac{r}{L}\right)^{-\Delta}\phi_1(x) + \left(\frac{r}{L}\right)^{4-\Delta}\phi_0(x)\ . \tag{2.4.13}$$

The leading term that specifies the boundary behavior is the second one, which is not normalizable. $\phi_0(x)$ is not interpreted as a state in the dual CFT, but rather as a source deforming the theory on the boundary and coupling to a gauge invariant local operator $\mathcal{O}$ with conformal dimension Δ, as in (2.4.3). The reason is clear: since the mode is not normalizable, the divergence it would lead to must be subtracted by a new boundary term. In some case, it occurs that the second term in (2.4.13) vanishes, so that the asymptotics is driven by the first term in (2.4.13), which is normalizable. In such circumstances, no boundary (counter)term is needed, and $\phi_1(x)$ is interpreted as the vacuum expectation value of the dual operator $\mathcal{O}(x)$ [25].

Let us conclude this section by commenting on some tests and a peculiar prediction of the field/operator correspondence. We have mentioned at the beginning of this section that among $\mathcal{N} = 8$ multiplets on $AdS_5 \times S^5$, the supergravity modes, which include supergravity fields on AdS_5 as well as their KK replicas from the S^5 compactification, must fill 1/2 BPS multiplets otherwise they would exceed spin 2. These modes were classified in the eighties [26]. On the gauge theory side, the corresponding 1/2 BPS multiplets are obtained by applying the supersymmetry charges to operators $\mathrm{Tr}\Phi_{\{i_1}\ldots\Phi_{i_k\}}-$traces, of dimension k, where Φ_i is one of the three $\mathcal{N} = 1$ adjoint chiral superfields [22, 27]. These operators are protected against acquiring anomalous dimension by the representation being short; consequently, the dimension of these operators must remain the classical one even in the strong coupling $\lambda \gg 1$ regime of the planar gauge theory which was conjectured to be dual to type IIB supergravity on $AdS_5 \times S^5$. There is an exact matching not only between the $SO(2,4) \times SO(6)$ quantum numbers of the field/operator pairs, but very remarkably also between the masses of the supergravity fields and the anomalous dimensions of the dual operator, which fulfil formula (2.4.6). Supergravity and KK modes exhaust all the gauge invariant single trace chiral operators of the CFT.

Stringy states correspond instead to longer multiplets, and they are dual to operators which acquire large anomalous dimensions. Let us consider a massive string state at excitation level n:

its mass squared will be $m^2 \propto n/\alpha'$. Plugging this into the mass-anomalous dimension formula (2.4.6), and recalling (2.3.4), we easily obtain a scaling [21]

$$\Delta \propto n^{1/2}\lambda^{1/4} . \tag{2.4.14}$$

This scaling by a fractional power of the 't Hooft coupling λ for large value of λ is completely unexpected in field theory, and was one of the first nontrivial prediction of AdS/CFT. The simple instance of an operator dual to a stringy state is $\mathrm{Tr}(\Phi_i\Phi_i)$. More recently, massive employment of integrability techniques in planar $\mathcal{N}=4$ SYM, together with symmetry requirements, has allowed the computation of the scaling dimension of some of these operators for any values of the 't Hooft coupling, interpolating between the perturbative field theory region and the strongly coupled $\lambda \gg 1$ region where supergravity is useful and the behavior (2.4.14) holds.

2.5 Other AdS/CFT dualities and non-AdS/non-CFT dualities

Already in the original paper [12] where the duality between $\mathcal{N}=4$ SYM and type IIB string theory on $AdS_5 \times S^5$ was conjectured, Maldacena proposed other AdS/CFT duality pairs for field theories in different dimensions, by considering similar decoupling limits to the one (2.2.1) suitable for D3 branes. Two instances of these are the dualities between the conformal theories describing the low energy dynamics of stacks of coincident M2 and M5 branes and M theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively.

In the realm of four-dimensional field theories, AdS/CFT duality was extended to orbifolds by Kachru and Silverstein [28], who considered the decoupling limit of D3 branes probing orbifolds of flat $\mathbb{R}^6$. In this way the correspondence was extended to systems with $\mathcal{N}=2$, 1 and even 0 supersymmetry. The duality is between the planar CFT obtained by orbifolding the $\mathcal{N}=4$ gauge theory by a discrete group Γ and type IIB supergravity on $AdS_5 \times S^5/\Gamma$. Subsequently, Klebanov and Witten considered the decoupling limit of D3 branes at a conifold singularity [29], which was conjectured to be dual to a nontrivial IR fixed point of a particular $\mathcal{N}=1$ gauge theory. We will discuss this dual pair in the next chapter. This paper opened the way for AdS/CFT duality to be applied to the low energy dynamics of branes at conical singularities [30–32].

So far we have mentioned less supersymmetric AdS/CFT dualities. Another very important step towards building phenomenologically more attractive theories goes into the direction of breaking conformal symmetry. The first breaking of scale invariance that was achieved in the literature is the one due to thermal effects. Black holes in asymptotically AdS spaces were considered by Maldacena [12] and by Witten [33] as dual to the finite temperature versions of conformal field theories, with a temperature equal to the Hawking temperature of the black hole. This line of research has become very relevant in recent years because it provides the only available method for computations of thermodynamic and hydrodynamic properties of strongly coupled gauge theories which are at least qualitatively similar to QCD, whose strongly coupled gluon plasma is under study in heavy ion collision experiments (RHIC) in Brookhaven.

As regards breaking conformal invariance in the zero temperature field theory, three paths have been followed in the literature. The first method consists in perturbing a conformal field theory by relevant operators or by vacuum expectation values: this triggers an RG flow which leads the theory either to a different IR fixed point or to nontrivial IR dynamics such as confinement and that can be investigated holographically [34]. In the dual supergravity description this implies some different dependence on the radial coordinate, breaking the scale isometry of the $AdS_5 \times S^5$ solution. As reviewed in the previous section, relevant deformations are dual to nonnormalizable deformations of the $AdS_5 \times S^5$ background, whereas vacuum expectation values correspond to normalizable deformations.

A second approach, that has proven very powerful, was initiated by Klebanov and collaborators [35–37], who studied the addition to the duality of so-called fractional branes, that we will introduce in the next chapter. The effect is an unbalance of the ranks of gauge groups, which breaks conformality in a very nontrivial way. The common lore is that the renormalization group flow is described by a cascade of Seiberg dualities; the low energy dynamics may be very interesting, such as in the celebrated Klebanov-Strassler background [37] which displays confinement and R-symmetry breaking in the dual field theory. The original papers in this line of research will be reviewed in detail in the next chapter. Qualitatively similar results were obtained in the case of $\mathcal{N} = 2$ theories realized by placing regular and fractional D3 branes on a $\mathbb{Z}_2$ orbifold of flat space [38,39] and in the case of $\mathcal{N} = 1$ theories from branes on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of flat space [40].

Since the main subject of this Ph.D. thesis.is the study of cascading gauge theories via dual supergravity (plus branes) solutions, in the next chapter we will review in detail the gauge/gravity dualities obtained by placing regular and fractional D3 branes at the conifold singularity.

A third alternative and complementary approach takes advantage of wrapped branes. Gauge/string (or M theory) dualities for other than D3 branes were pioneered by the beautiful paper [41], where subtleties related to the decoupling problem and the validity regimes of different approaches in studying the field theory limit of the dynamics of coincident Dp branes ($p \leq 6$) were uncovered. Two years later, Maldacena and Núñez [42] studied the gravity backgrounds generated by higher dimensional branes wrapped on nontrivial cycles, that host a four-dimensional theory at low energies. The study of $\mathcal{N} = 2$ field theories, further investigated in [43, 44], as well of an $\mathcal{N} = 1$ field theory possessing a low energy dynamics very similar to the one of the Klebanov-Strassler solution[9], was pursued by considering D5 branes wrapping two-cycles.

We refer the reader who intends to delve into this very interesting subject to some of the very good reviews on non-AdS/non-CFT dualities in the literature [46–48].

We would like to conclude this chapter mentioning a common problem which arises in different forms when trying to describe interesting, maybe confining, low energy dynamics in these three different methods. The problem lies in the fact that the ultraviolet physics (relevant deformation of a conformal theory, infinite cascade, higher dimensional physics) which completes the interesting low energy dynamics in such a way that the supergravity approximation is reliable, actually shows up at an energy scale of the same order of the strong coupling scale of the IR field theory. The reason is very intuitive in the case where masses are given to the chiral adjoint superfields of $\mathcal{N} = 4$ SYM so as to get $\mathcal{N} = 2, 1, 0$ supersymmetry at low energies (these theories are called $\mathcal{N} = 2^*, 1^*, 0^*$). The dynamically generated scale of the low energy physics will be qualitatively of the form $\Lambda \sim Me^{-c/\lambda(M)}$, where M is the mass scale of the deformation, c an order 1 factor proportional to the coefficient of the β-function, and $\lambda(M)$ the value of the running 't Hooft coupling at scale M. In order to decouple the UV completion from the low energy physics, we should take a limit $M \to \infty$ keeping Λ fixed: this implies that $\lambda(M)$ should go to zero, invalidating the supergravity approximation. The full string theory is needed to decouple the UV completion. Similar arguments exist for gauge/gravity dualities from fractional branes and from wrapped branes.

[9]Maldacena and Núñez gave a gauge/gravity interpretation to a solution previously found by Chamseddine and Volkov [45], as the dual of a confining gauge theory with IR dynamics very similar to $\mathcal{N} = 1$ pure SYM.

Another qualitative argument pointing to the necessity of the full string theory for describing a confining gauge theory without any unwanted unconventional ultraviolet completion applies generally and is the following: in any realistic confining theory, glueballs of spin larger than two are expected to exist, whereas at the level of supergravity the only gauge invariant operators that can be described in the dual theory have at most spin two. The full string theory is needed to describe the higher spin glueballs.

Chapter 3

(non-)AdS/(non-)CFT dualities on the conifold

In this chapter, we review the extensions of AdS/CFT duality that have been achieved by considering D3 branes at the conifold singularity. For a review of these gauge/gravity dualities, we refer the reader to [49]. After an introduction, in section 3.2 we review the Klebanov-Witten gauge/gravity pair. In section 3.3 we review the Klebanov-Tseytlin background, dual to a nonconformal theory; in section 3.4 we review its desingularization in the Klebanov-Strassler background and discuss the dual field theory interpretation.

3.1 Introduction

A very interesting generalization of the original AdS/CFT duality goes in the direction of lowering the number of (super)symmetries of the four-dimensional field theory, without spoiling conformal invariance. One can start by considering a stack of N coincident D3 branes probing a different manifold than $\mathbb{R}^6$, and, following Maldacena's argument, construct a duality between type IIB string theory on the 'near-horizon' geometry generated by the stack of D3 branes, and the conformal field theory controlling the low energy dynamics of the stack of D3 branes (in the conformal phase) [22]. The low energy gauge theory on the D3 branes depends on the transverse space the branes can probe.

The transverse manifold should be noncompact to allow decoupling of gravity. The requirement of conformal invariance of the gauge theory imposes that this transverse manifold be a 6-dimensional real cone, namely its metric can be written as

$$ds_6^2 = dr^2 + r^2 ds_{X_5}^2 \ . \tag{3.1.1}$$

The space is a cone because there is a group of diffeomeorphisms $r \mapsto tr$ ($t \in \mathbb{R}_+^*$) which rescales the metric. X_5 is a 5-dimensional compact submanifold called the 'base' of the cone. Except for

$\mathbb{R}^6$, whose base is the round S^5, any cone has a (conical) singularity at $r = 0$. Examples of conical singularities are orbifolds of flat space and conifolds, of which string theory can make sense.

The additional requirement that $\mathcal{N} = 1$ supersymmetry in four dimensions (4 supercharges) be preserved imposes that the cone is a Calabi-Yau (CY) manifold. Ricci-flatness of the cone implies that X_5 is an Einstein manifold [29], which satisfies

$$R_{ij} = 4\, g_{ij} \ , \tag{3.1.2}$$

whereas the $SU(3)$ holonomy condition imposes that X_5 is a Sasaki-Einstein space [31]. For our purposes, we can take as a definition of Sasaki-Einstein space the property that the real cone over it is Calabi-Yau.

Any D3 brane can be displaced at any point in the transverse Calabi-Yau manifold without spending energy; these possibilities must be recovered in the moduli space of the gauge theory. This observation puts in one-to-one correspondence the moduli space of the gauge theory living on the D3 branes with the singularity structure of the transverse manifold, strongly constraining the gauge theory. General techniques have been developed to determine the superconformal field theory of D3 branes at toric singularities [32, 50].

The curved background produced by the stack of N D3 branes at the tip of the cone (3.1.1) is:

$$ds^2 = h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \left[dr^2 + r^2 ds_{X_5}^2\right] \tag{3.1.3}$$

$$g_s F_5 = (1 + *)\, d^4x \wedge dh(r)^{-1} \tag{3.1.4}$$

$$h(r) = 1 + \frac{L^4}{r^4} \tag{3.1.5}$$

$$e^{\Phi} = g_s \tag{3.1.6}$$

where the integration constant

$$L^4 = 4\pi\alpha'^2 g_s N \frac{\pi^3}{\mathrm{Vol}(X_5)} \tag{3.1.7}$$

is fixed by requiring the D3 brane charge of the solution be N. See appendix A.2.1 for the definition of brane charges. Taking Maldacena's decoupling limit (2.2.1) leaves the near-horizon metric

$$ds^2 = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2}{r^2} dr^2 + L^2 ds_{X_5}^2 \ . \tag{3.1.8}$$

Type IIB string theory on the $AdS_5 \times X_5$ manifold (3.1.8) with N units of F_5-flux is conjectured to be dual to the CFT describing the IR fixed point of the D3 branes theory. The different X_5, the different the gauge theory. In the vary large class of D3 branes at conical toric Calabi-Yau singularities, the geometry can be obtained by partial resolutions of orbifold singularities of the $\mathbb{C}^3/(\mathbb{Z}_p \times \mathbb{Z}_q)$, with suitable integer p and q. The field theory moduli space is in one-to-one correspondence with the transverse space that can be probed by the D3 branes, modded out by identifications between indistinguishable branes. Partial resolutions on the geometric sides correspond to Higgsings in the gauge theory living on D3 branes. Since the gauge theory living on D3 branes at a $\mathbb{C}^3/(\mathbb{Z}_p \times \mathbb{Z}_q)$ orbifold can be derived by perturbative open string methods, this correspondence allows us to find the gauge theory living on a stack of D3 branes living at a generic conical toric singularity: it is obtained by orbifolding and Higgsing the $\mathcal{N} = 4$ field theory. This gauge theory exhibits at least a fixed point, which is dual to type IIB string theory on $AdS_5 \times X_5$, with X_5 the 5-dimensional Sasaki-Einstein space the real cone over which is the toric CY.

The first instance of such an AdS/CFT duality that cannot be obtained by orbifold methods was found by Klebanov and Witten [29], who studied D3 branes on the conifold, which is the real cone over a specific Sasaki-Einstein manifold known as $T^{1,1}$. We are now going to illustrate this duality.

3.2 D3 branes at the conifold singularity and the Klebanov-Witten CFT

In this section we review the AdS/CFT duality between type IIB string theory on $AdS_5 \times T^{1,1}$ and the $\mathcal{N} = 1$ gauge theory on D3 branes at the conifold singularity. We refer the reader to the original paper [29] for a more complete analysis and to [49] for a nice review. The reader who is not familiar with such a manifold is advised to read appendix B.1, where some generalities on the conifold geometry, as well as our conventions, that differ slightly from those common in the literature, are collected. In that appendix, the Kähler structure and the complex structure of this Calabi-Yau are also explicitly provided. For later purposes, in appendix C.1 we have rederived the solution for D3 branes on the conifold and its near-horizon Klebanov-Witten solution by means of supersymmetry methods. It falls in the general class of equations (3.1.8) and (3.1.4), with $X_5 = T^{1,1}$.

Here we recall that the conifold can be described algebraically as the affine variety

$$\det Z = \begin{vmatrix} z_1 & z_4 \\ z_3 & z_2 \end{vmatrix} = z_1 z_2 - z_3 z_4 = 0 \tag{3.2.1}$$

in $\mathbb{C}_4$. It possesses a manifest $\frac{SU(2)\times SU(2)}{\mathbb{Z}_2} \times \mathbb{C}^*$ symmetry, where the two $SU(2)$ factors act on the rows and columns of the Z matrix, and the $\mathbb{C}^* = \mathbb{R}^*_+ \times U(1)$ acts as an overall complex rescaling. The base of the conifold is the Sasaki-Einstein 5-manifold called $T^{1,1}$, described by the intersection of (B.1.1) with the unit sphere

$$\sum_{i=1}^{4} |z_i|^2 = \sum_{i=1}^{4} |w_i|^2 = 1 \tag{3.2.2}$$

in $\mathbb{R}^8$. It has $S^3 \times S^2$ topology, and is actually a coset manifold $\frac{SU(2)\times SU(2)}{U(1)}$. Since the eighties, type IIB string theory has been known to admit a consistent $AdS_5 \times S^5$ compactification with 5-form fluxes [52]. Six real coordinates on the conifold can be introduced as in (B.1.5,B.1.5) so to solve the defining equation (3.2.1). The Calabi-Yau metric of [51] is of conical type (3.1.1), with the following Einstein metric for the $T^{1,1}$ base:

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\varphi_i\right)^2 + \frac{1}{9}\left(d\psi - \sum_{i=1}^{2}\cos\theta_i d\varphi_i\right) . \tag{3.2.3}$$

(θ_i, φ_i), $i = 1, 2$, parameterize the Kähler-Einstein space $S^2 \times S^2$, whereas $\psi \in [0, 4\pi[$ parameterizes a $U(1)$, which is fibered over the Kähler-Einstein base. The following angular periodicities hold:

$$\begin{pmatrix}\psi\\ \varphi_1\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi + 4\pi\\ \varphi_1\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi + 2\pi\\ \varphi_1 + 2\pi\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi + 2\pi\\ \varphi_1\\ \varphi_2 + 2\pi\end{pmatrix} . \tag{3.2.4}$$

The volume of $T^{1,1}$ is $\frac{16}{27}\pi^3$.

Using these coordinates, the symmetries of the space are manifest. As shown in Appendices B.1 and C.1, the $U(1)$ isometry along the ψ circle acts nontrivially on the holomorphic top form of the Calabi-Yau and therefore on the Killing spinor. By the general arguments of section 2.4, this isometry is mapped to the R-symmetry of the superconformal algebra of the dual CFT. Instead, the $SU(2) \times SU(2)$ isometries acting on the two 2-spheres are mapped to global non-R-symmetries of the dual CFT. Finally, the existence of a nontrivial 3-cycle in the $T^{1,1}$ base leads to an additional gauge symmetry on the gravity side, whose gauge potential is the reduction of the RR C_4 potential on this 3-cycle. This gauge symmetry will be mapped to the baryonic symmetry of the CFT.

To construct the field theory describing D3 branes at a conifold singularities, Klebanov and Witten used the description of the conifold as a quotient. The defining equation (3.2.1) can be solved by introducing four complex variables A_1, A_2, B_1, B_2 and letting

$$Z = \begin{pmatrix} z_1 & z_4 \\ z_3 & z_2 \end{pmatrix} \equiv \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix} . \tag{3.2.5}$$

In such a way, under the $SU(2) \times SU(2)$ symmetry acting on the columns and rows A_i and B_j $(i, j = 1, 2)$ transform in the isospin $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations respectively. Under the overall complex rescaling symmetry $z_i \mapsto \beta z_i$, they can be chosen to transform as $(A, B) \mapsto \beta^{1/2}(A, B)$. Actually, we have to mod out by the $\mathbb{C}^*$ rescaling

$$A_j \to \lambda A_j \ , \qquad\qquad B_k \to \lambda^{-1} B_k \tag{3.2.6}$$

which leaves the z coordinates invariant. Away from the singular point, the noncompact part of this redundancy can be fixed by

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 \ , \tag{3.2.7}$$

and the conifold is finally obtained after modding out by the remaining $U(1)$ redundancy

$$A_j \to e^{i\alpha} A_j \ , \qquad\qquad B_k \to e^{-i\alpha} B_k \ . \tag{3.2.8}$$

The conifold must also be recovered as the moduli space of the supersymmetric gauge theory on a D3 brane at the conifold singularity. Recalling that the action of the gauge group on the charged superfields in a supersymmetric gauge theory gets complexified, the $\mathbb{C}^*$ redundancy can be viewed as a complexified $U(1)$ gauge symmetry under which four chiral superfields A_j, B_k, $j, k = 1, 2$ have charge 1 and -1 respectively. In such a gauge theory, the D-term equation, in the absence of a Fayet-Iliopoulos term, fixes the $\mathbb{R}^*_+$ redundancy as in (3.2.7).[1] When A and B acquire a VEV, the $U(1)$ symmetry previously discussed is Higgsed. This happens when the D3 brane exits the singularity and goes to a smooth point. In such a case, an unbroken $U(1)$ group should survive, hence a second unbroken gauge group must be included, under which A and B are neutral. The gauge theory living on a single D3 brane at the conifold singularity can therefore be summarized as a $U(1) \times U(1)$ $\mathcal{N} = 1$ gauge theory with bifundamental fields A_j with charges $(1, -1)$ and B_k with charges $(-1, 1)$, possessing a global $SU(2) \times SU(2) \times U(1)_R$ symmetry.

[1] A nonzero Fayet-Iliopoulos term appears in the presence of a small resolution, which substitutes the conifold singularity with a $\mathbb{P}^1$. there are actually two possible resolutions, depending on the sign of the FI parameter, which are related by a flop transition.

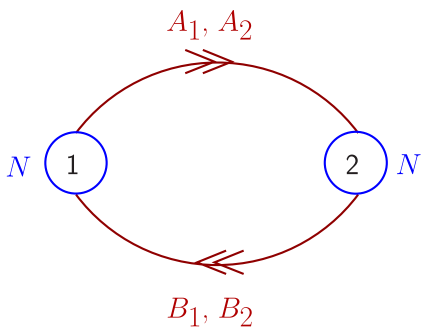


Figure 3.1: Quiver diagram for the Klebanov-Witten theory. Circles represent gauge groups (and vector superfields), arrows represent bifundamental chiral superfields.

In generalizing to the nonabelian case of N D3 branes, the bifundamental fields A_j and B_k are promoted to matrices transforming under a $U(N) \times U(N)$ gauge group. The diagonal $U(1)$ acting as (3.2.8) is IR free and becomes a global $U(1)_B$ baryonic symmetry. The other diagonal axial $U(1)$, apart from being IR free, is anomalous and therefore massive, and its D-term equation should not be imposed at low energies [53]. In order for the moduli space of this theory to reproduce correctly the conifold (3.2.1) modulo identifications, a superpotential enforcing the conifold equation (3.2.1) on the eigenvalues has to be introduced. Before introducing the superpotential, a nonanomalous R-symmetry under which the gluini have charge 1 and the superfields A_j and B_k have charge 1/2 can be found. The only superpotential consistent with the symmetries is

$$W = h\, \epsilon^{ij} \epsilon^{kl} \mathrm{Tr}(A_i B_k A_j B_l) \,, \tag{3.2.9}$$

whose F-term equations correctly remove the unwanted flat directions and leave the conifold (modulo permutations S_N) as the moduli space of the theory of N D3 branes.

We summarize the representations/quantum numbers of the chiral superfields under the gauge and global symmetry groups in Table 3.1. On top of these, there are also two discrete symmetries (exchange of the gauge groups and charge conjugation), which are discussed in detail in [29]. The

	$[SU(N) \times SU(N)]$	$SU(2)_A$	$SU(2)_B$	$U(1)_R$	$U(1)_B$
A	$(N, \overline{N})$	2	1	1/2	1
B	$(\overline{N}, N)$	1	2	1/2	-1

Table 3.1: Field content and symmetries of the KW field theory with massless flavors.

field content is usually summarized in the ‘quiver diagram’ of Figure 3.1, where circles represent gauge groups and arrows represent bifundamental fields (fundamental under the gauge group the arrow is exiting from, antifundamental under the gauge group the arrow is entering).

An analysis along the lines of [54] shows that this gauge theory has a superconformal fixed point where the R-symmetry under which the chiral superfields have charge $R[A_j] = R[B_k] = 1/2$ and the gluini have charge $R[\lambda^{(1)}] = R[\lambda^{(2)}] = 1$ becomes a generator of the superconformal algebra.

The existence of such a superconformal point is equivalent to the existence of an exact $U(1)_R$ symmetry in the superconformal algebra, because of the relation $\Delta = \frac{3}{2}R$ between the R-charge and the quantum dimension of a gauge invariant operator. The quantum dimension of the chiral superfields is 3/4, indicating that the theory is at a strongly coupled fixed point. A beautiful description of the manifold of conformal fixed points preserving maximal symmetry can be found in the review [55].

We should mention here that in addition to the mesonic branch of the moduli space, which is the conifold modulo S_N, there exists also a baryonic branch where some of the dibaryonic operators [56] of the schematic form A^N and B^N (indices are suitably contracted by antisymmetric tensors) acquire a VEV, spontaneously breaking the baryonic $U(1)_B$ symmetry. In that case the baryonic current gets a VEV, and the effect mirrors the presence of a D-term in the abelian theory: the moduli space experienced by mesonic operators (D3 branes) becomes the resolved conifold.

The Klebanov-Witten conformal field theory, dual to type IIB string theory on $AdS_5 \times T^{1,1}$, has an interesting relationship with a conformal field theory conjectured by Kachru and Silverstein to be dual to type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_2$ [28]. The latter field theory is an $\mathcal{N}=2$ gauge theory describing the low energy dynamics of D3 branes at a $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity, where the nontrivial element of the $\mathbb{Z}_2$ orbifold group acts on the three natural complex coordinates (z_1, z_2, z_3) of the covering space as $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, z_3)$. Along the fixed line $z_1 = z_2 = 0$ parameterized by z_3, there is an exceptional 2-cycle. The near-horizon geometry is $AdS_5 \times S^5/\mathbb{Z}_2$, where the angular part has an $S^1/\mathbb{Z}_2$ fixed locus of the $\mathbb{Z}_2$ action. The gauge group arising from the $\mathbb{Z}_2$ action on the Chan-Paton factors is $U(N) \times U(N)$ in the conformal case. There are two hypermultiplets, one in the bifundamental representation $(N, \overline{N})$, the other in the conjugate representation $(\overline{N}, N)$. Decomposed in $\mathcal{N}=1$ language, they appear as four chiral superfields A_1, A_2, B_1, B_2 as those appearing in the Klebanov-Witten theory. Because of $\mathcal{N}=2$ supersymmetry, to each gauge group is associated an $\mathcal{N}=1$ adjoint chiral superfield belonging the $\mathcal{N}=2$ vector multiplet. We will call these adjoint superfields Φ and $\tilde{\Phi}$. The superpotential is determined by $\mathcal{N}=2$ supersymmetry to be

$$W = g\mathrm{Tr}\left[\Phi(A_1B_1 - A_2B_2)\right] + g\mathrm{Tr}\left[\tilde{\Phi}(B_1A_1 - B_2A_2)\right] , \qquad (3.2.10)$$

where g is the gauge coupling of the two groups, that we assume to be the same for simplicity. This is the case when the integral of the NSNS 2-form potential on the exceptional 2-cycle is $\frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 = \frac{1}{2}$. If we now add a relevant term [29]

$$\Delta W = m\mathrm{Tr}\Phi^2 - m\mathrm{Tr}\tilde{\Phi}^2 , \qquad (3.2.11)$$

at energies below m the massive adjoint superfields should be integrated out by imposing their F-terms. The resulting low energy superpotential is exactly the Klebanov-Witten superpotential (3.2.9) with coupling $h = g^2/m$. This indicates that the $\mathcal{N}=2$ $\mathbb{Z}_2$ orbifold theory flows to the Klebanov-Witten upon mass deformation (3.2.11). Although naively the superpotential coupling g^2/m seems to disappear in the IR (which is the same as the parametric limit $m \to \infty$), it actually does not and it becomes marginal due to large anomalous dimensions.

On the string theory side, the relevant deformation (3.2.11) appears as a twisted sector blowup mode that smoothens the orbifold singularity $S^5/\mathbb{Z}_2$ into $T^{1,1}$. Clearly, a conical singularity survives in the IR. In this $\mathcal{N}=2$ case, the blowup mode is in the same $SU(2)_R$ multiplet as the complex structure deformation modes. It is easier to see its effect by rotating to the description of a complex structure deformation. We start from an A_1 singularity described by $w_1^2 + w_2^2 + w_3^2 = 0$, times a complex plane parameterized by x (related to the VEV of Φ and $\tilde{\Phi}$). The geometric picture of the mass deformation corresponds to fibering the x plane over the A_1 singularity as in $w_1^2 + w_2^2 + w_3^2 + m^2x^2 = 0$. The exceptional 2-cycle is blown up, except at the conifold singularity. The resulting geometry is the conifold geometry.

Remarkably, an interpolating Calabi-Yau solution between the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold and the conifold and the holographic Klebanov-Witten RG flow between the two corresponding conformal field theories has been shown to exist in [57].

We conclude this section by specifying the holographic dictionary that maps parameters in type IIB string theory on $AdS_5 \times T^{1,1}$ into parameters of the Klebanov-Witten $\mathcal{N} = 1$ CFT, with matter content as in Figure (3.1) and a quartic superpotential (3.2.9) [29, 55, 61].[2] The number N of units of RR 5-form flux through $T^{1,1}$ is mapped to the number of colors of the two gauge groups in the dual gauge theory, as follows from the brane construction.

More interesting is the mapping between the moduli of type IIB string theory and the couplings of the dual gauge theory. We will restrict our analysis to the space of deformations that preserve the $SU(2) \times SU(2) \times U(1)_R$ global symmetry of the theory at the Klebanov-Witten point and of string theory on the $AdS_5 \times T^{1,1}$ geometry. On the string side, we are free to change the value of the type IIB complexified coupling, the axio-dilaton $\tau \equiv C_0 + ie^{-\Phi} = C_0 + \frac{i}{g_s}$, provided that we keep it constant as a function of the coordinates. We are also free to change the value of the complexified 2-form potential $C_2 + \tau B_2$ integrated over the nontrivial 2-cycle (B.1.21) of the conifold geometry, provided that this remains constant, or in other terms that the complexified 2-form potential remains closed.[3]

Changing these moduli does not change the $AdS_5 \times T^{1,1}$ structure of the solution, therefore they correspond to motions in the maximally symmetric subsurface of fixed points of the Klebanov-Witten field theory. These string moduli are mapped to renormalization group invariant quantities in the gauge theory. If we define the holomorphic dynamically generated scales of the two gauge groups,[4]

$$\Lambda_j^N = \mu^N \, e^{-\frac{8\pi^2}{g_j^2} + i\theta_j} \equiv \mu^N \, e^{2\pi i \tau_j} \tag{3.2.12}$$

two RG invariant dimensionless quantities can be constructed out of them and the superpotential coupling h, consistently with the global symmetries. They are $L_1 = h^N \Lambda_1^N$ and $L_2 = h^N \Lambda_2^N$.[5] Comparison with the transformation properties under exchange of the two gauge groups and analogy with the dictionary of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ $\mathcal{N} = 2$ orbifold gauge theory clarifies that the correct dictionary

[2] It is perhaps worth commenting that in taking the large N 't Hooft limit, also the quartic superpotential coupling should be suitably scaled. However, since the coupling is superficially nonrenormalizable and the theory is at a strongly coupled point, it is difficult to understand which scaling would fit.

[3] Making a quick jump ahead, we can perhaps mention to the educated reader that these two moduli are related to the sum and the difference of the actions of the two fractional D-instantons (fractional D(-1) branes) representing the supersymmetric gauge theory instantons of the D3 brane theory.

[4] As explained in [58], in the presence of matter fields these holomorphic dynamically generated scales are not RG invariant if renormalization is done in the usual way, namely including the matter fields kinetic term. They are renormalized by powers of the wave function renormalizations of the matter fields along the RG flow, consistently with the exact Novikov-Shifman-Vainshtein-Zakharov β-function [59, 60].

[5] The h^N prefactors make the quantity dimensionless and cancel the renormalization by wave functions of the matter fields.

is

$$L_1 L_2 = \exp\left[2\pi i\tau\right] \tag{3.2.13}$$

$$L_1/L_2 = \exp\left[\frac{i}{\pi\alpha'}\int_{S^2}(C_2 + \tau\tilde{B}_2)\right] . \tag{3.2.14}$$

Here $\tilde{B}_2$ represents the fluctuation of B_2 from the 'orbifold point' value in the $\mathcal{N} = 2$ theory that, as discussed above, can provide a UV completion to the Klebanov-Witten theory. In that theory, the 'orbifold point' is the one for which $\int_{S^2} C_2 = 0$ and $\int_{S^2} B_2 = 2\pi^2\alpha'$: in such a case, we are able to quantize string theory and check that the two gauge couplings are indeed equal [62]. Finally, the holographic flow from the orbifold theory to the conifold theory does not change the value of B_2.

This dictionary can be rewritten in terms of the complexified gauge couplings $\tau_j = \frac{\theta_j}{2\pi} + i\frac{4\pi}{g_j^2}$ of the two groups and of the superpotential coupling h as

$$C_0 + \frac{i}{g_s} \equiv \tau = \tau_1 + \tau_2 + \frac{2N}{2\pi i}\log(h\mu) \tag{3.2.15}$$

$$\frac{1}{2\pi^2\alpha'}\int_{S^2}\left(C_2 + \tau\tilde{B}_2\right) = \tau_1 - \tau_2 . \tag{3.2.16}$$

Notice the presence of the additional $\log h$ term in (3.2.15) with respect to more conventional orbifold theories. This term should be regarded as a genuine effect of the nontrivial $\mathcal{N} = 1$ dynamics of the Klebanov-Witten theory (or, in other words, of the conifold singularity).

3.3 Fractional D3 branes and the Klebanov-Tseytlin solution

The scale invariance of the Klebanov-Witten field theory, describing the low energy dynamics of N D3 branes at the conifold singularity, can be broken explicitly, in quite a peculiar way that we will explain in this and in next section, by adding to the stack of N D3 branes a number M of fractional D3. These fractional D3 branes can be viewed as D5 branes wrapped on the collapsed 2-cycle at the conifold singularity.

This fruitful idea arose from a study of D branes wrapped over cycles of $T^{1,1}$ [56]. $T^{1,1}$ has $S^2 \times S^3$ topology, so that branes can be wrapped either on the 2-cycle or on the 3-cycle. In [63], Witten studied the effect of adding a D3 brane in $AdS_5 \times S^5$, which is a domain wall in AdS_5: the flux of the RR 5-form field strength F_5 jumps by one unit when the D3 brane wall is crossed. Since this flux is related to the number of colors of the dual gauge theory, the effect in field theory is to decrease the gauge group from $SU(N+1)$ (outside) to $SU(N)$ (inside). Indeed, we know that there is a moduli space for the positions of D3 branes, and that the effect of the Higgs mechanism consists precisely in reducing the rank of the gauge group. Analogously, adding a D3 brane at some radial position r_0 in $AdS_5 \times T^{1,1}$ has the effect of increasing the gauge group from $SU(N)\times SU(N)$ in the IR ($r < r_0$) to $SU(N+1) \times SU(N+1)$ in the UV ($r > r_0$). Gubser and Klebanov [56] argued that a D5 brane wrapped on the two cycle of $T^{1,1}$ has instead the effect of increasing the gauge group from $SU(N) \times SU(N)$ to $SU(N) \times SU(N+1)$, leaving the matter content in the same bifundamental representations as before, but now with respect to the new gauge group with unbalanced ranks: the A_j superfields transform in $(\mathbf{N}, \overline{\mathbf{N+1}})$ and the B_k superfields transform in $(\overline{\mathbf{N}}, \mathbf{N+1})$. Notice that such a wrapped D5 brane is not a stable object, since the two-cycle minimizes its volume at the tip of the cone, where it shrinks. There is no moduli space for the wrapped D5 brane position: they are stuck at the singularity. The D5 brane does not disappear provided a nontrivial B_2 flux exists through the collapsed cycle. However, one can think of holding the wrapped D5 brane fixed with an external source of energy and studying its effect, or of starting with a wrapped D5 at very large r, where the energy of the system is very large, and then letting the system relax to the vacuum and seeing what happens. The argument for the change in the ranks relies only on charges and is firm.

There are different ways of arguing that the change in the ranks is the one proposed by Gubser and Klebanov. First of all, one can make an analogy with the $\mathcal{N} = 2$ theory of D3 branes on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. In that case, it is possible to add fractional D3 branes (which are D5 branes wrapped on the exceptional 2-cycle living at the singularity line), so that the gauge groups become $SU(N) \times SU(N+1)$, with bifundamental fields (and adjoint fields) remaining in the same representations as before. This can be checked by explicit quantization of open strings. Giving mass to the adjoint fields of this theory and integrating them out, we end up with an $SU(N) \times SU(N+1)$ gauge theory with the same matter content as the Klebanov-Witten theory.[6] Notice that the $SU(N) \times SU(N+1)$ gauge theory is not conformal invariant anymore.

An alternative argument was used by Gubser and Klebanov, who studied the behavior of a D3 brane wrapped on S^3 as it crosses the D5 brane wrapped on S^2 wall. In the conformal theory, a D3 brane wrapped on S^3 is dual to a dibaryonic operator of the form $\epsilon_{a_1 \ldots a_N} \epsilon^{b_1 \ldots b_N} A^{a_1}_{b_1} \ldots A^{a_N}_{b_N}$ (or one of the similar operators built out of B's), where a and b are color indices, running from 1 to N. In [56], a precise matching of the quantum numbers of the dibaryonic operators with those of these string states was provided. In the $SU(N) \times SU(N+1)$ theory, the operators which are more similar to the dibaryonic operators of the conformal theory are $\epsilon_{a_1 \ldots a_N} \epsilon^{b_1 \ldots b_N b_{N+1}} A^{a_1}_{b_1} \ldots A^{a_N}_{b_N}$, transforming in the fundamental of $SU(N+1)$, and $\epsilon_{a_1 \ldots a_N} \epsilon^{b_1 \ldots b_N b_{N+1}} A^{a_1}_{b_1} \ldots A^{a_N}_{b_N} A^{a_{N+1}}_{b_{N+1}}$, transforming in the antifundamental of $SU(N+1)$. The dual interpretation is simple: the wrapped D5 brane has a string attached to it. The wrapped D5 brane generates one unit of F_3 flux on the 3-sphere on one side of it. When the wrapped D3 brane crosses it, it must reemerge with a string attached, because the RR 3-form flux induces minus one unit of charge for the gauge field on the brane. Since the brane is wrapped on a compact manifold, the total charge on this manifold has to vanish. The endpoint of the string provides the necessary extra charge.[7]

As already mentioned, the wrapped D5 brane domain wall is not stable, and it actually falls at $r = 0$, behind the horizon, being replaced only by its RR 3-form flux. The supergravity solution corresponding to N regular D3 branes and M fractional D3 branes replaced by their fluxes should therefore provide a dual of the $SU(N) \times SU(N+M)$ nonconformal version of the Klebanov-Witten field theory, that from now on we will call Klebanov-Tseytlin (KT) or Klebanov-Strassler (KS)

[6]This argument is not as firm as it may look, since the two theories have a nontrivial RG flow. However, if the mass of the adjoints is far from the dynamically generated scales of the two gauge groups, it is very reasonable that the result should be this one, without any subtleties.

[7]To be more precise, the other endpoints of the strings attached to the wrapped D3 brane should also end somewhere, because of charge conservation. This issue is related to gauge invariance of the field theory operator. The configuration dual to a gauge invariant dibaryonic operator, of the schematic forms $(A^N)^{N+1}$ (contractions with epsilon tensors are understood in these powers), is made of $N+1$ wrapped D3 branes, each one with a string attached, and with the other $N+1$ endpoints on a single D5 brane wrapped over $T^{1,1}$, which acts as a baryon vertex for $SU(N+1)$ [64].

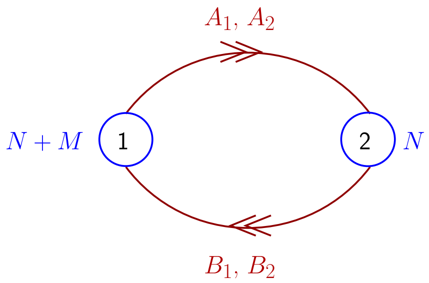


Figure 3.2: Quiver diagram for the Klebanov-Tseytlin/Strassler theory.

theory, and whose quiver diagram we reproduce in Figure 3.2. The superpotential remains (3.2.9):

$$W = h\,\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_i B_k A_j B_l)\ . \tag{3.3.1}$$

In [35], the supergravity equations of the system of N regular and M fractional D3 branes at the tip of the conifold were solved to leading order in M/N, and the logarithmic running of the difference of the inverse squared gauge couplings $g_1^{-2} - g_2^{-2}$ was first pointed out. In [36], that solution was completed to all orders: it is the famous Klebanov-Tseytlin solution, with 3-form fluxes and running 5-form flux on the singular conifold; the warp factor has logarithmic corrections to its AdS_5 expression, the relative gauge coupling runs logarithmically at all scales, and the 5-form flux decreases logarithmically as well. However, since the singular conifold has no preferred scale, the logarithms are not cut off at small r and at some point the D3 brane charge becomes negative and the metric develops a naked singularity. We review the Klebanov-Tseytlin solution in the remainder of this section.

In appendix C.2, we have rederived such a solution by requiring supersymmetry and fulfilment of Bianchi identities. That derivation shows how that solution can be found by simply adding an $SU(2)\times SU(2)\times U(1)_R$ invariant closed primitive and imaginary-selfdual $(2,1)$-form $G_3 = F_3 + ie^{-\Phi}H_3$, with the normalization fixed by the conserved charge of the system, and then determining the warp factor by imposing the Bianchi identity for F_5. The result is the following. The solution, that we will lay out in string frame, has a constant axio-dilaton $\tau = C_0 + ie^{-\Phi} = C_0 + \frac{i}{g_s}$, a complexified 3-form field strength proportional to the $SU(2)\times SU(2)$ invariant closed primitive and imaginary-selfdual $(2,1)$-form $\omega_{CF}^{(2,1)}$ defined in (B.1.29).

$$G_3 = F_3 + \frac{i}{g_s}H_3 = -\frac{M\,\alpha'}{2}\left(\zeta - 3\,i\,\frac{dr}{r}\right)\wedge\omega_2^{CF}\ , \tag{3.3.2}$$

where $\zeta = d\psi - \sum_{i=1}^{2}\cos\theta_i d\varphi_i$ is the 1-form of the $U(1)_R$ fiber, and ω_2^{CF} is the 2-cocycle of $T^{1,1}$ defined in (B.1.23) of appendix B.1. The prefactor is fixed by the number of fractional D3 branes, namely the D5 brane charge (A.2.35), being M:

$$M = -\frac{1}{4\pi^2\alpha'}\int_{S^3} F_3\ , \tag{3.3.3}$$

where S^3 is the 3-cycle of $T^{1,1}$, whose representative can be chosen as in (B.1.24). Because of the right-hand side in the Bianchi identity $dF_5 = -H_3\wedge F_3$, the flux of F_5 is not a conserved charge.

This signals the fact that the number of D3 branes N or the 'background value' N of the ranks is not really a conserved quantity.[8] The warp factor can be easily found to be:

$$h(r) = \frac{27}{4}\,\pi\alpha'^2\,\frac{1}{r^4}\left[\frac{3}{2\pi}(g_sM)^2\,\log\frac{r}{r_s}\right]\ , \tag{3.3.4}$$

where r_c is an integration constant. There is a single integration constant r_s. Its value can be related to the D3 brane Maxwell charge (A.2.34) at some reference scale, because $g_sF_5 = (1+*)d^4x\wedge dh^{-1}$. Maxwell charges are the usual brane charges defined as fluxes of gauge invariant RR field strengths, see appendix A.2.1. Following Klebanov and Tseytlin [36], we will name N the D3 brane Maxwell charge at some other reference scale r_0, so that the warp factor (3.3.4) is rewritten as:

$$h(r) = \frac{27}{4}\,\pi\alpha'^2\,\frac{1}{r^4}\left[g_sN + \frac{3}{2\pi}(g_sM)^2\,\left(\log\frac{r}{r_0}+\frac{1}{4}\right)\right]\ . \tag{3.3.5}$$

We stress again that N and r_0 are not independent: there is really a single integration constant r_s. The D3 brane Maxwell charge (A.2.34), namely the flux of F_5, is not quantized and acquires a radial dependence because of the 3-fluxes:

$$N_{eff}(r) \equiv -\frac{1}{(4\pi^2\alpha')^2}\int_{\mathcal{C}_5} F_5 = N + \frac{3}{2\pi}g_sM^2\log\frac{r}{r_0}\ . \tag{3.3.6}$$

Klebanov and Tseytlin noticed that the RG flow described by this dual background seems to enjoy a cascade where $N_{eff}(r)\mapsto N_{eff}(r') = N_{eff}(r) - M$ as $r\mapsto r' = e^{-\frac{2\pi}{3g_sM}}r$: the 'effective number of colors' $N_{eff}(r)$ drops by M units as $r\mapsto r' = e^{-\frac{2\pi}{3g_sM}}r$. This indicates that the dual gauge theory changes from $SU(N+M)\times SU(N)$ to $SU(N)\times SU(N-M)$ as the logarithm of the energy scale $\mu\sim r/\alpha'$[9] drops by a constant factor $\log\mu\mapsto\log\mu-\frac{2\pi}{3g_sM}$.

The solution has a naked singularity of repulson type, located at the position r_s where the warp factor vanishes. It is a bad IR singularity according to the criterion of [65]: this solution cannot be used to extract IR properties of the dual gauge theory. At the radial position where the effective number of colors vanishes (namely the derivative of the warp factor changes sign), gravity becomes repulsive. Curvature invariants are small in the UV region. Even if g_sM were very small, the curvature is small provided $g_sN_{eff}(r)\gg 1$. In that regime, field theory quantities can be reliably computed.

Furthermore, running gauge couplings were computed by using the holographic relations valid in the $\mathcal{N}=2$ orbifold theory:[10]

$$\begin{aligned}\frac{8\pi^2}{g_1^2}+\frac{8\pi^2}{g_2^2} &= \frac{2\pi}{g_s}\\ \frac{8\pi^2}{g_1^2}-\frac{8\pi^2}{g_2^2} &= \frac{2}{g_s}\left[\frac{1}{2\pi\alpha'}\int_{S^2}B_2-\pi\pmod{2\pi}\right]\ ,\end{aligned} \tag{3.3.7}$$

[8] We have been a little sloppy here, but we will discuss this point in detail in the following.

[9] This radius-energy relation is obtained using the stretched-string prescription. In nonconformal gauge/gravity duals, it is known that the radius/energy relation usually depends on the process and the quantity under study.

[10] The use of orbifold formulae may look naive. However, *a posteriori* it turns out [55] that the corrections due to the quartic superpotential can be neglected in the UV, where the Klebanov-Tseytlin solution gives a faithful picture of the dual theory.

using the NSNS potential

$$B_2 = \frac{3g_s M\alpha'}{2} \log \frac{r}{r_0}\, \omega_2^{CF} \;, \tag{3.3.8}$$

which is obtained by integrating H_3.[11] The result exactly agrees with the field theory computation of the β-functions of the gauge couplings, provided that the anomalous dimensions of the bifundamental fields are the same ($\gamma = -\frac{1}{2}$) as in the conformal Klebanov-Witten theory:[12]

$$\beta_{\frac{8\pi^2}{g_1^2}} = 3(N+M) - 2N(1-\gamma) = 3M \tag{3.3.9}$$

$$\beta_{\frac{8\pi^2}{g_2^2}} = 3N - 2(N+M)(1-\gamma) = -3M \;, \tag{3.3.10}$$

where g_1 is the coupling of the $SU(N+M)$ group and g_2 of $SU(N)$. The gauge coupling g_1 runs towards strong coupling, and diverges at some energy scale. If we start at an energy not much larger than the one dual to r_0, we see that the gauge coupling g_1 diverges ar $r = r_0$, where B_2 vanishes. From the holographic relations and the validity of the solution at least in the UV region, Klebanov and Tseytlin argued that the field theory can be continued past this seeming strong coupling singularity provided that N is substituted by $N - M$. In the IR regime where this jumping N becomes of the same order of M, the solution cannot be trusted and must be desingularized.

We end this section by mentioning that not only β-functions, but also the anomaly of the $U(1)_R$ R-symmetry of the field theory is succesfully reproduced by this dual background, as was shown in [66]. The Klebanov-Tseytlin geometry is symmetric under a $\psi \mapsto \psi + 2\epsilon$ shift of the coordinate of the $U(1)$ fiber in $T^{1,1}$, which is dual to a $U(1)_R$ transformation parameterized by ϵ (under which gluini transform as $\lambda \mapsto e^{i\epsilon}\lambda$). RR field strengths are also invariant under this shift, but the C_2 potential is not: $C_2 \mapsto C_2 - M\alpha'\epsilon\,\omega_2^{CF}$. On the other hand, we know that a Euclidean D1 brane wrapped on the 2-cycle S^2 (Euclidean fractional D(-1) brane) is an instanton, and its WZ action $\frac{1}{2\pi\alpha'}\int_{S^2} C_2$ is defined modulo 2π. Therefore $\psi \mapsto \psi + 2\epsilon$ is a symmetry when $\epsilon \in \frac{\pi}{M}\mathbb{Z}$. Since $\epsilon \in [0, 2\pi]$, the result is that a $\mathbb{Z}_{2M}$ subgroup of the $U(1)_R$ classical R-symmetry group of the gauge theory is not anomalous. In [66], it was also shown that the vector boson of the dual gauge symmetry in the supergravity background acquires a mass eating the field $\int_{S^2} C_2$ dual to the difference of the theta angles.

3.4 The Klebanov-Strassler solution

In the beautiful paper [37], Klebanov-Strassler achieved two main results: they were able to provide a field theory interpretation of the renormalization group cascade displayed by the Klebanov-Tseytlin solution, which reliably describes the ultraviolet regime of the gauge theory, and they desingularized the Klebanov-Tseytlin solution by replacing the singular conifold with the deformed conifold, which has a natural cutoff scale. The infrared regime of the dual gauge theory can be sensibly investigated by means of their warped deformed conifold solution: up to subtleties related to the non-decoupling of the UV completion, the IR dynamics is that of pure $\mathcal{N} = 1$ $SU(M)$ SYM,

[11] Here we have picked for latter purposes a convenient choice of the additive integration. In the Klebanov-Tseytlin solution, which is singular at small r, there is really no physical way of fixing this integration constant.

[12] In [37], it was shown that in the UV, where $M/N \ll 1$, because of conformal invariance of the Klebanov-Witten theory and of a $\mathbb{Z}_2$ symmetry, anomalous dimensions are $\gamma = -\frac{1}{2} + \mathcal{O}((M/N)^2)$.

which displays confinement and chiral (R-) symmetry breaking. The strong coupling scale of the IR gauge theory is related to the complex deformation parameter in the geometry.

3.4.1 UV: a cascade of Seiberg dualities

In the previous section, we have mentioned that Klebanov and Tseytlin argued that the $SU(N+M)\times SU(N)$ gauge theory dual to their background (in some energy range) might be continued past the energy scale at which the $SU(N+M)$ gauge coupling diverges, provided that $N \to N-M$. We can be more specific. Continuity of gauge couplings is achieved provided the $SU(N)$ gauge group is untouched and the $SU(N+M)$ gauge group becomes $SU(N-M)$, and the approximate holographic relations (3.3.7) are modified into:

$$\begin{aligned} \frac{8\pi^2}{g_l^2} + \frac{8\pi^2}{g_s^2} &= 2\pi e^{-\Phi} \\ \frac{8\pi^2}{g_l^2} - \frac{8\pi^2}{g_s^2} &= 2e^{-\Phi}\left[\frac{1}{2\pi\alpha'}\int_{S^2} B_2 - \pi \pmod{2\pi}\right] , \end{aligned} \tag{3.4.1}$$

where g_l is always the coupling of the larger rank gauge group and g_s the coupling of the smaller rank gauge group; on the right-hand side we have written e^{Φ} for the string coupling, in order not to ingenerate confusion between the string coupling and the gauge coupling g_s appearing in the left-hand side. This reshuffling precisely matches what occurs in the dual field theory, where the continuation past strong coupling was understood in [37] to be described by a Seiberg duality [67]. When the $SU(N+M)$ gauge group becomes infinitely coupled, we are able to describe the dynamics of the gauge theory at lower energies by resorting to a Seiberg dual description, where the $SU(N+M)$ group is substituted by an $SU(N-M)$ gauge group, and with a superpotential coupling the magnetic gauge singlet dual to the electric mesons $M_{ij} = B_j A_i$ to the $N_f = 2N$ fundamental and antifundamental chiral superfields b_j and a_i of the magnetic gauge group. The dual superpotential is:

$$\begin{aligned} W_{mag} = h\,\mathrm{Tr}(M_{12}M_{21} - M_{11}M_{22}) + \frac{1}{\hat\Lambda}\,\mathrm{Tr}(b_1 M_{11} a_1) + \frac{1}{\hat\Lambda}\,\mathrm{Tr}(b_1 M_{12} a_2) + \\ + \frac{1}{\hat\Lambda}\,\mathrm{Tr}(b_2 M_{21} a_1) + \frac{1}{\hat\Lambda}\,\mathrm{Tr}(b_2 M_{22} a_2) , \end{aligned} \tag{3.4.2}$$

where $\hat\Lambda$ is the mass scale that has to be inserted for dimensional reasons and enters the relation between the electric and magnetic holomorphic dynamical scales. The M_{ij} $SU(N-M)$ gauge singlets are massive and can be integrated out. The resulting superpotential for the light fields is

$$W' = \frac{1}{h\hat\Lambda^2}\,\mathrm{Tr}(a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1) , \tag{3.4.3}$$

a quartic superpotential of the same form as the electric one 3.2.9, with the only difference that $h \to \frac{1}{h\hat\Lambda^2}$. The magnetic dual theory (after an innocuous charge conjugation) has the same matter content and superpotential of the electric theory, differing from that only in the rank of the gauge group: the $SU(N+M)\times SU(N)$ gauge group in the electric description valid at high energies becomes the $SU(N-M)\times SU(N)$ gauge group of the magnetic description valid at low energies. This peculiar property of SQCD-like theories with a quartic superpotential is called *self-similarity*. Continuity of

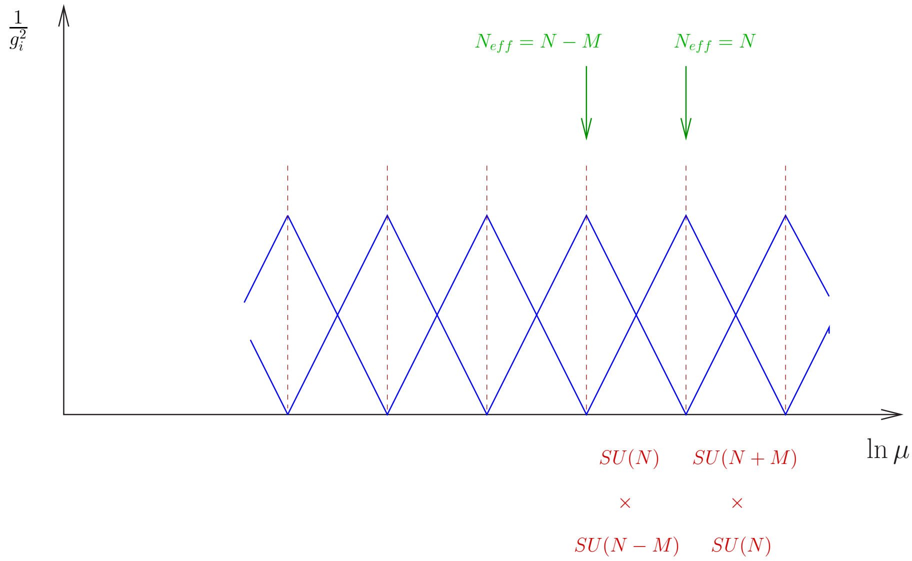


Figure 3.3: Renormalization group flow of the Klebanov-Tseytlin/Strassler theory in the UV regime. At its infrared strong coupling scale, each gauge group is substituted by its magnetic dual, which subsequently runs towards weak coupling.

the gauge couplings, as required by the dual supergravity solution with the holographic relations (3.4.1), implies that the strong coupling scales of the electric and magnetic groups are the same and equal $\hat{\Lambda}$. The RG flow of the Klebanov-Strassler field theory is depicted in Figure 3.3. In the UV regime, using the fact that the anomalous dimensions of the bifundamental superfields do not acquire corrections at linear order in M/N, we conclude that the β-function of the dimensionless quartic coupling $\tilde{\lambda}(\mu) = \mu h(\mu)$ is subleading with respect to the gauge coupling β-functions. To leading order, $\tilde{\lambda}$ does not run. Under Seiberg duality, $\tilde{\lambda} \to 1/\tilde{\lambda}$. Continuity of this quartic coupling under Seiberg duality imposes that $\tilde{\lambda} \simeq 1$ in the UV region. A remarkably detailed analysis of the Seiberg duality cascade in the Klebanov-Strassler theory can be found in the lecture notes [55].

We end this subsection by proposing corrected holographic relations for the UV of the Klebanov-Strassler theory, obtained by completing an analysis of [61]. They are a generalization of formulae (3.2.13)-(3.2.14) which hold in the Klebanov-Witten theory. On the gauge theory side, we can build two renormalization group invariant quantities out of the quartic coupling h and the holomorphic dynamically generated scales Λ_1 and Λ_2 of the $SU(N+M)$ and $SU(N)$ gauge groups, consistently with the symmetries: they are $L_1 = h^N \Lambda_1^{N+3M}$, with engineering dimension $3M$, and $L_2 = h^{N+M} \Lambda_1^{N-2M}$, with engineering dimension $-3M$. Their product, which is invariant under interchange of the gauge groups, should be matched with the amplitude $e^{2\pi i \tau}$ of a type IIB D-instanton, which is the sum of the two fractional D-instantons on the conifold which are the instantons of the two gauge groups. The ratio of L_1 and L_2 should instead be proportional to the amplitude of the difference of the two fractional D-instantons. The proportionality factor should carry the right engineering dimension and be consistent with the RG flow properties of the two sides. We propose the following relations:

$$L_1 L_2 = \exp\left[2\pi i \tau\right] \tag{3.4.4}$$

$$h^{6M}\frac{L_1}{L_2} = \exp\left[\frac{i}{\pi\alpha'}\int_{S^2}(C_2+\tau\tilde{B}_2)\right] . \tag{3.4.5}$$

Again, here $\tilde{B}_2$ represents the fluctuation of B_2 from the 'orbifold point' value in the $\mathcal{N}=2$ theory that can provide a UV completion. The β-functions and the $U(1)_R$ properties of the two sides exactly match in the UV, using the stretched string radius-energy relation and the $U(1)_R$ infinitesimal transformation $\psi \to \psi + 2\epsilon$. The symmetry properties and engineering dimensions of both sides of formulae (3.4.5) are the same.

This dictionary can be rewritten in terms of the complexified gauge couplings $\tau_j = \frac{\theta_j}{2\pi} + i\frac{4\pi}{g_j^2}$ of the two groups and of the dimensionless superpotential coupling $h\mu$ as

$$C_0 + \frac{i}{g_s} \equiv \tau = \tau_l + \tau_s + \frac{2N+M}{2\pi i}\log(h\mu) \tag{3.4.6}$$

$$\frac{1}{2\pi^2\alpha'}\int_{S^2}\left(C_2+\tau\tilde{B}_2\right) = \tau_l - \tau_s + \frac{5M}{2\pi i}\log(h\mu) . \tag{3.4.7}$$

We stress again that the additional $\log(h\mu)$ terms with respect to the orbifold formulae can be neglected in the UV regime, where to leading order $h\mu$ does not run.

3.4.2 IR: confinement and chiral symmetry breaking

The infinite cascade of Seiberg dualities explained in the previous subsection goes on until the IR, where at some point different dynamics occurs and resolves the singularity of the Klebanov-Tseytlin solution. If we start at some UV scale from $N = kM$, where k is some large integer, and if corrections to anomalous dimensions do not become too large in the IR, the cascade goes on until the gauge group is reduced to $SU(2M)\times SU(M)$. At that point, $SU(2M)$ flows towards strong coupling but we cannot move to a Seiberg dual description because $N_f = N_c = 2M$. Instead, the moduli space is deformed quantum mechanically [68]. The classical constraint $\det M - B\tilde{B} = 0$, where M is the meson matrix and B, $\tilde{B}$ the baryon and antibaryon superfields, is modified at quantum level to $\det M - B\tilde{B} = \Lambda_{2M}^{4M}$. In the Klebanov-Strassler theory there is also a quartic superpotential. The full exact superpotential is

$$W = h\,\mathrm{Tr}(M_{11}M_{22} - M_{12}M_{21}) + \alpha(\det M - B\tilde{B} - \Lambda_{2M}^{4M}) , \tag{3.4.8}$$

where α is the Lagrange multiplier enforcing the quantum constraint. There are two disjoint branches of the moduli space of this gauge theory. In the mesonic branch the baryon VEV's vanish and the eigenvalues m_{ij} of the meson matrices M_{ij} lie on a deformed conifold $m_{11}m_{22} - m_{12}m_{21} = \Lambda_{2M}^4$. This branch has a dual description in terms of mobile D3 branes on the deformed conifold. A similar analysis at a generic step of the cascade can be found in [61] and leads to the same result: the transverse space probed by D3 brane should be not the singular, but the deformed conifold. In the baryonic branch, the mesons vanish and $B\tilde{B} = -\Lambda_{2M}^{4M}$. The solution to this equation is $B = \frac{i}{\zeta}\Lambda_{2M}^2$, $\tilde{B} = i\zeta\Lambda_{2M}^2$. At the $\mathbb{Z}_2$ symmetric point $\zeta = 1$ of the baryonic branch and below the scale Λ_{2M}, what survives is the pure $SU(M)$ gauge theory of the other group, which finally flows towards strong coupling at a scale Λ_M. This point has a dual description in terms of fluxes on the smooth warped deformed conifold geometry, which was proposed by Klebanov and Strassler and we are to review now.

The previous analysis led Klebanov and Strassler to consider the smooth deformed conifold as the internal Calabi-Yau space of the solution. Some generalities on the deformed conifold can be found in appendix B.2. The Calabi-Yau metric on the deformed conifold is [51]:

$$ds_6^2 = \frac{1}{2}\epsilon^{4/3}K(\tau)\Bigg[\frac{1}{3K^3(\tau)}\left(d\tau^2+\zeta^2\right)+\cosh^2\frac{\tau}{2}\,\left[(g^3)^2+(g^4)^2\right]+ + \sinh^2\frac{\tau}{2}\,\left[(g^1)^2+(g^2)^2\right]\Bigg]\,, \tag{3.4.9}$$

where

$$K(\tau)=\frac{(\frac{1}{2}\sinh(2\tau)-\tau)^{1/3}}{\sinh\tau} \tag{3.4.10}$$

and the 1-forms g^1, g^2, g^3, g^4 and ζ are defined in Appendices B.1 and B.2 in terms of the angular coordinates. For large r and τ , using the asymptotic change of coordinate

$$r^3 \simeq \frac{3^{3/2}}{2^{5/2}}\,\epsilon^2\,e^{\tau}\,, \tag{3.4.11}$$

the metric (B.2.7) reduces to that of the singular conifold $dr^2+r^2ds^2_{T^{1,1}}$. The deformed conifold metric approaches that of $\mathbb{R}^3\times S^3$ as $\tau\to 0$:

$$ds_6^2 \simeq 6^{-1/3}\,\epsilon^{4/3}\,\left\{\frac{1}{2}\left[d\tau^2+\tau^2\frac{1}{2}\left((g^1)^2+(g^2)^2\right)\right]+\left(\frac{1}{2}\zeta^2+(g^3)^2+(g^4)^2\right)\right\}\,. \tag{3.4.12}$$

The ansatz for the RR 3-form field strength is

$$F_3=\frac{M\alpha'}{2}\left\{\zeta\wedge\left[(1-F(\tau))g^3\wedge g^4+F(\tau)g^1\wedge g^2\right]+F'(\tau)d\tau\wedge(g^1\wedge g^3+g^2\wedge g^4)\right\}\,, \tag{3.4.13}$$

where $F(0)=0$ in order to ensure regularity at the tip of the deformed conifold and $F(\infty)=\frac{1}{2}$ to ensure the Klebanov-Tseytlin limit. This form is automatically closed. The ansatz for the NSNS 2-form potential is

$$B_2=-\frac{g_sM\alpha'}{2}\left(f(\tau)g^1\wedge g^2+k(\tau)g^3\wedge g^4\right)\,, \tag{3.4.14}$$

where $f(0)=k(0)=0$ so as to ensure regularity at the tip of the deformed conifold and $f(\tau)\approx k(\tau)\approx\tau/2$ as $\tau\to\infty$ so as to approximate the Klebanov-Tseytlin solution.

Supersymmetry of the 3-form fluxes imposes the following system of first order differential equations for the functions $F(\tau)$, $f(\tau)$ and $k(\tau)$:

$$\begin{aligned} f' &= (1-F)\tanh^2\frac{\tau}{2} \\ k' &= F\coth^2\frac{\tau}{2} \\ F' &= \frac{1}{2}(k-f)\,. \end{aligned} \tag{3.4.15}$$

which guarantees that $G_3 = F_3 + \frac{i}{g_s} H_3$ is $(2,1)$, primitive and imaginary-selfdual ($*_6 G_3 = iG_3$). Finally, the Bianchi identity $dF_5 = -H_3 \wedge F_3$ for $F_5 = \frac{1}{g_s}(1+*)d^4x \wedge dh^{-1}$ implies

$$h' = -\alpha \frac{f(1-F) + kF}{K^2 \sinh^2 \tau} \ , \tag{3.4.16}$$

where

$$\alpha = 4(g_s M \alpha')^2 \epsilon^{-8/3} \ . \tag{3.4.17}$$

The BPS system for the flux factors can be solved first, and yields

$$F(\tau) = \frac{\sinh \tau - \tau}{2 \sinh \tau} \tag{3.4.18}$$

$$f(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1) \tag{3.4.19}$$

$$k(\tau) = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1) \ . \tag{3.4.20}$$

Finally, requiring that the warp factor vanishes as $\tau \to \infty$, one obtains

$$h(\tau) = \alpha 2^{-4/3} I(\tau) = (g_s M \alpha')^2 2^{2/3} \epsilon^{-8/3} I(\tau) \ , \tag{3.4.21}$$

where

$$I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} [\sinh(2x) - 2x]^{1/3} \ . \tag{3.4.22}$$

For small τ, $I(\tau) = a_0 + \mathcal{O}(\tau^2)$, with $a_0 \approx 0.71805$: the warp factor approaches a constant. Consequently, in the neighborhood of the tip of the deformed conifold $\tau = 0$, the 10-dimensional geometry is metrically the product of 4-dimensional Minkowski space and of $\mathbb{R}^3 \times S^3$. Corrections to this asymptotics are $\mathcal{O}(\tau^2)$, so that the metric is smooth. Notice that the scale r_s, necessary for the definition of the logarithms in the Klebanov-Tseytlin solution (3.3.4), has dimensionally transmuted into the deformation parameter ϵ of the deformed conifold: $r_s \sim \epsilon^{2/3}$.

Curvature invariants are small everywhere, also close to the tip, provided that $g_s M \gg 1$. This implies that the cascade steps are very close one to the other, so that we cannot decouple the UV completion (duality cascade) from the IR dynamics ($\mathcal{N} = 1$ pure SYM) at the level of supergravity.

Let us now make some comments about interesting checks of the duality between the Klebanov-Strassler solution and the proposed cascading gauge theory, which reduces in the IR to pure $\mathcal{N} = 1$ $SU(M)$ SYM.

Since the Klebanov-Strassler geometry ends smoothly at $\tau = 0$, it is dual to a confining gauge theory, as $SU(M)$ pure SYM is. Indeed, the Wilson loop in the fundamental representation and in the q-antisymmetric representations obey the area law $-\log\langle W_q(\mathcal{C})\rangle = T_q A(\mathcal{C})$ in the limit of large area A of the loop $\mathcal{C}$ [7]. The Wilson loop in the fundamental representation is holographically realized as a fundamental string whose worldsheet extends in the bulk geometry and has the loop as a boundary at $\tau = \infty$ [69].[13] In the Klebanov-Strassler geometry, for sufficiently large separation between the external quark an antiquark pair that can be thought to bound a rectangular Wilson loop, the string profile in the τ direction takes a 'U-shape'. The bottom of this U is an almost flat tensionful string at the tip of the geometry, representing the chromoelectric flux tube in the dual

[13] See [70] for a nice review of the geometric translation of confinement in gauge/gravity duality.

gauge theory; it is connected to the boundary by the vertical parts of the U, which are approximately fundamental strings stretched radially from the boundary to the end of the geometry, and represent the infinite masses of the external quark and antiquark, that have to be subtracted if we want to compute the potential between quark and antiquark. The result is a linear confining potential, with a tension of the confining string [37]

$$T_1 \sim \frac{\epsilon^{4/3}}{g_s M \alpha'^2} , \tag{3.4.23}$$

where we have omitted order 1 numerical factors.

A similar analysis can be performed for the tensions of the q-strings, which have a dual description in terms of q coincident fundamental strings [73]. In the S-dual description, it is possible to see that because of the NSNS 3-flux through the 3-sphere at the tip of the geometry, the q coincident D strings blow up into a D3 brane wrapping an S^2 inside the S^3, with q units of worldvolume flux. The tension of these confining q-strings can be computed, and give to a very good approximation the sine law

$$\frac{T_q}{T_{q'}} = \frac{\sin \frac{\pi q}{M}}{\sin \frac{\pi q}{M}} \tag{3.4.24}$$

that was found in softly broken $\mathcal{N} = 2$ SYM [71] and in $\mathcal{N} = 1$ MQCD [72], and ensures stability of a q-string against decay into strings with smaller q.

Glueball masses can also be evaluated by studying fluctuations of supergravity modes. The result is that the masses of glueballs are of order $\epsilon^{2/3}/(g_s M\alpha')$, of the same order of the masses of Kaluza-Klein modes from the S^3. This is another face of the decoupling problem. The supergravity solution is mildly curved provided $g_s M \gg 1$; but then the UV completion of the IR theory, the cascade, shows up at energies of the same order as the strong coupling scale of the IR theory, and there is a mixing between states in the low energy theory and states in the completion. In order to decouple the UV completion with all its undesired stuff, we should let $g_s M$ be small; consequently, the curvature radius becomes small in string units close to the tip of the deformed conifold. The full string theory therefore necessary for a dual description of $\mathcal{N} = 1$ pure SYM decoupled from its UV completion.

Another remarkable check concerns the field-theoretic expectation that $SU(M)$ pure SYM possesses M confining vacua parameterized by the gluino condensate $\mathrm{Tr}\lambda\lambda \sim \Lambda^3 e^{i\frac{2\pi k}{M}}$, $k = 0, 1, \ldots, M-1$, which break spontaneously the $\mathbb{Z}_{2M}$ nonanomalous R-symmetry to $\mathbb{Z}_2$. This phenomenon is manifest in the dual picture. The $\mathbb{Z}_{2M}$ R-symmetry is the R-symmetry of the UV background, which is approximately the Klebanov-Tseytlin background of the previous section. However, the full Klebanov-Strassler solution preserves only a $\mathbb{Z}_2$ subgroup: already the defining equation of the deformed conifold (B.2.2) preserves only a $\mathbb{Z}_2$ R-symmetry which maps $z_i \to -z_i$. Fluxes and potentials of the Klebanov-Strassler solution preserve this $\mathbb{Z}_2$ too. It is also possible to compute holographically the gluino condensate [74], with the result $\mathrm{Tr}\lambda\lambda \sim M\frac{\epsilon^2}{\alpha'^3}$. Furthermore, the domain wall separating two of these M vacua, with k and $k' = k + l$, have a holographic description as l BPS D3 branes wrapping the S^3 at $\tau = 0$, with the remaining directions along Minkowski space. As the domain wall is crossed, for large τ the RR 2-form potential is shifted by $\Delta C_2 = \pi\alpha' l \omega_2^{CF}$, reproducing the broken $\mathbb{Z}_{2M}$ transformation that relates the two vacua. The tension of the domain wall is found to be $T_{wall} \sim \frac{\epsilon^2}{g_s \alpha'^3}$.

We end this section by mentioning that other solutions describing different points of the moduli space of the Klebanov-Strassler cascading gauge theory exist. A thorough analysis of the moduli space of this theory was done in [61]. In [75], a family of $SU(3)$ structure explicit solutions parameterizing the full baryonic branch $B = \frac{i}{\zeta}\Lambda_{2M}^2$, $\tilde{B} = i\zeta\Lambda_{2M}^2$ of the Klebanov-Strassler theory and so extending the work of [76], was found. Interestingly, in a limit the background reduces to the one proposed by Maldacena and Núñez as another dual of $\mathcal{N} = 1$ $SU(N)$ pure SYM [42]. More recently, a partially numeric solution describing the mesonic branch of the Klebanov-Strassler theory appeared in [77].

Part II

Unquenched flavors in gauge/gravity duality

Chapter 4

Flavors in AdS/CFT duality

4.1 Flavors in large N_c expansions

4.1.1 Flavors in 't Hooft's limit

In section 1.1 we have introduced the large N_c expansion for a pure $SU(N_c)$ Yang-Mills (YM) theory, or a YM theory with matter in the adjoint representation. We have seen that Feynman diagrams rearrange in a way that mimics the topological expansion of a theory of closed oriented strings. In the original paper [8], 't Hooft studied the large N_c expansion of QCD. He started from a YM theory with gauge group $U(N_c)$[1] and N_f flavors of vector-like quarks, and considered the limit (1.1.5), keeping the number of flavors N_f fixed. Let us review here 't Hooft's argument.

The action we use is

$$S_{YM} = -\frac{1}{2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \int d^4x \sum_{a=1}^{N_f} \bar{\psi}^a(i\not{D} - m_a)\psi_a\ , \qquad (4.1.1)$$

where the Dirac spinor field ψ_a includes the quark and antiquark degrees of freedom, a is a flavor index, and the covariant derivative D_μ was defined in (1.1.3). Hidden inside the covariant derivative term in (4.1.1), there is an interaction vertex for emission of a gluon by a quark, which is proportional to the YM coupling g and in double line notation is shown in Fig. 4.1. Quark lines are single lines and enter and exit the vertex of Fig. 4.1; therefore quark loops introduce boundaries in the Riemann surface which is obtained by gluing faces to edges (propagators).[2]

Let us first consider connected vacuum diagrams.[3] The only difference with respect to the pure

[1] Also in this case, the adaptation to $SU(N_c)$ can be worked out easily.

[2] In order to keep track of the dependence on the number of flavors, Veneziano [78] has later introduced a double line notation for quarks, where the second line is wavy and follows the flavor index. Although Veneziano's diagrammatic notation is very convenient, we will keep using 't Hooft's double and single line notation, paying attention to N_f factors.

[3] The same subtlety as in the pure YM case arises for the simplest vacuum diagrams with a single quark propagator going back to itself. This pathology can be dealt with in the same way as in the unflavored theory, see section 1.1 around equation (1.1.9).

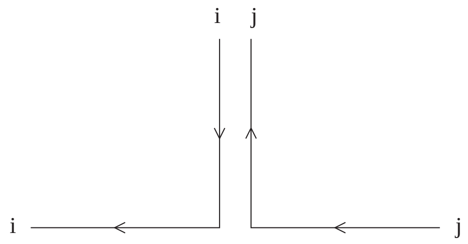


Figure 4.1: Quark-gluon vertex in double line notation.

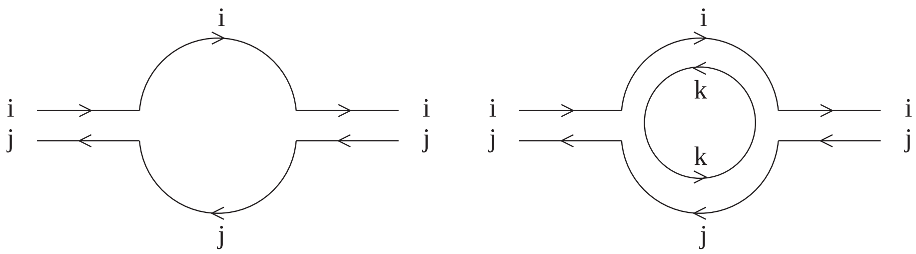


Figure 4.2: The quark loop contribution to the gluon propagator (on the left) scales like N_f/N_c with respect to the gluon loop contribution (on the right).

YM case arises when there are internal quark loops: any internal gluon loop which touches only trivalent vertices can be replaced with a quark loop, introducing a boundary and substituting one power of N_c with one power of N_f. See Fig. 4.2 for an instance of this.

The final result is

$$\mathcal{A} = \sum_{\chi}^{\infty} N_c^{\chi} N_f^{b} \sum_{F=\chi}^{\infty} c_{\chi,F} \lambda^{F-\chi} , \tag{4.1.2}$$

with b the number of boundaries. The exponent $F-\chi$ of λ can be rewritten as $F-\chi = E-V = l-1$, where l is the number of unconstrained loop momenta we have to integrate over.

This formula looks like a topological expansion for a theory of orientable closed and open strings, where the open string sector is related to the quarks and the closed string sector is related to the gluons. This is a strong suggestion that Yang-Mills theories with fundamental flavors may be reformulated as suitable theories of closed and open strings.

One can also consider generic connected diagrams, where also insertions of gauge invariant mesonic-like sources, like for instance the pseudoscalar $\bar{\psi}\gamma_5\psi$ or the vector current $\bar{\psi}\gamma_\mu\psi$, can appear. These sources will generically have definite flavor quantum numbers. What is important here is that the source is gauge invariant, implying that color lines run through it undisturbed. The previous argument goes through also for this case, with the only difference that the insertion of mesonic-like sources in a quark loop fixes the flavor index running along the loop, hence lowering the power dependence on the number of flavors N_f.

In the large N_c limit (1.1.5) with N_f kept fixed, the dependence of a diagram on the number of colors is again N_c^{χ}, where now the Euler character is

$$\chi = F - E + V = 2 - 2h - b . \tag{4.1.3}$$

The leading order diagrams in 't Hooft's expansion are planar diagrams with a minimum number of quark loops. In other words, quark loops appear only as external loops touching mesonic source vertices, in a way analogous to the 'quenched approximation' in lattice gauge theories, where the fermionic determinant is approximated by 1. There are two classes of subleading diagrams: diagrams with internal quark loops, which are suppressed by powers of N_f/N_c, and nonplanar diagrams, which are suppressed by $1/N_c^2$.

A nice phenomenology can be derived in this large N_c expansion, in addition to the pure glue phenomenology mentioned in section 1.1 [13]. As consequences of the previously illustrated scaling and of the assumptions of existence of this large N_c limit and of confinement, one can show the following properties. In this large N_c limit mesons for large N_c are free, stable and not interacting. Meson decay amplitudes are of order $1/\sqrt{N_c}$, amplitudes for meson-meson elastic scatterings are of order $1/N_c$ and are given by a sum of tree diagrams involving the exchange of physical mesons with local vertices. There are infinitely many mesons, and their masses have smooth large N_c limits. The amplitude for a glueball to mix with a meson is of order $1/\sqrt{N_c}$, the decay amplitude of a glueball into two mesons is of order $1/N_c$, and an interaction vertex with k meson legs and l glueball legs scales like $N_c^{-l-\frac{1}{2}k+1}$. Mesons and glueballs are the weakly coupled degrees of freedom (with masses not diverging like N_c) appearing in the spectrum of Yang-Mills theory. The $1/N_c$ expansion can be rephrased as a loop expansion in an effective theory of mesons and glueballs. Furthermore, in this limit the Okubo-Zweig-Iizuka (OZI) selection rule is exact, the mixing between flavor singlet and octet mesons is suppressed, and mesons are pure $q\bar{q}$ states. Because of the N_f/N_c factors arising from internal quark loops, all the phenomena which are related to the quark-antiquark sea are suppressed.

Real world QCD has three colors and two (or three) light flavors, so that N_c does not look very large (but actually without resumming planar diagrams we cannot know what is the effective coupling) and N_f is of the order of N_c. Despite this fact, the qualitative similarity of 't Hooft's large N_c limit phenomenology with real QCD phenomenology seems to indicate that this limit provides a good approximation scheme. Quantitatively, in cases where $1/N_c$ corrections vanish and the first subleading corrections are at $1/N_c^2$, this limit allows to make predictions at the 10% level.

4.1.2 Veneziano's large N_c expansion

Few years after 't Hooft's paper, Veneziano reconsidered large N_c expansions in [78]. He pointed out that 't Hooft's large N_c expansion, although very promising in tackling the issue of confinement, had the phenomenological disadvantage that its higher orders mix together planar corrections and nonplanar loop corrections. Nonplanar corrections can be shown to modify qualitatively the physics of the leading planar term: they introduce violations of the OZI rule, as well as absorptive corrections (such as cuts and long-range correlations) to the pole dominated leading terms of the $1/N_c$ expansion. On the contrary, planar corrections coming from internal quark loops are expected to modify quantitatively, but not qualitatively, the leading term of the expansion. They do not introduce violations of the Zweig rule nor absorptive corrections. Their effect is that of renormalizing the quantities relevant to the leading term in 't Hooft's expansion, for instance by giving a finite width to the mesons, and of enforcing a set of unitarity-like constraints that the leading order expansion obviously misses. Phenomenological observations give ground to a validity of the OZI rule, to a good approximation, and also indicate that short range interactions between mesons are dominant. Therefore it seemed phenomenologically desirable to include quark-induced planar corrections in a leading order expansion.

For these reasons, Veneziano decided to study a different scaling limit than 't Hooft's one. He considered the scaling limit $N_c \to \infty$ with $\lambda = g^2 N_c$ fixed, together with $x = N_f/N_c$ fixed. Feynman diagrams do not change, therefore the expansion is the same as (4.1.2), together with (4.1.3). It still looks like a topological expansion for a theory of closed and open strings.[4] What changes in this limit is that since x is kept fixed, planar diagrams with internal quark loops are not suppressed with respect to planar diagrams without internal quark loops. Similarly, the two diagrams of Fig. 4.2 contribute both to the leading order in Veneziano's expansion.

What the leading order of Veneziano's expansion captures, in addition to the physics of 't Hooft's large N_c limit, is the dynamics due to the quark-antiquark sea. One can then expect that phenomena like screening of color charges and breaking of confining chromoelectric flux tube can be studied already at the leading order of this expansion. Furthermore, since quarks are allowed to run inside loops, the way the coupling runs, or equivalently the magnitude of the dynamically generated scale, is affected by the dynamics of fundamental fields. This is clearly seen in the expression of the 1-loop β-function for the 't Hooft coupling:

$$\mu \frac{d\lambda}{d\mu} = -\frac{2}{3}\left(11 - 2\frac{N_f}{N_c}\right)\frac{\lambda^2}{16\pi^2} \ . \tag{4.1.4}$$

One nice consequence for our modern taste is that Veneziano's large N_c and N_f expansion looks also as a promising arena where one can study phases of (non)supersymmetric gauge theories with fundamental flavors, that the power of holomorphy and duality has revealed so beautifully in supersymmetric gauge theories. All these effects are washed out at the leading order of 't Hooft's expansion, but are kept in Veneziano's expansion.

4.2 Adding flavors to the AdS/CFT correspondence

The original version of AdS/CFT duality proposed by Maldacena and developed also by many other authors involves gauge theories with multiple gauge groups and fields in the adjoint or bifundamental representation of the gauge groups [32]. In order to make contact with phenomenologically more interesting QCD-like theories, the addition of fields transforming in the fundamental or antifundamental representation of the gauge groups, as well as in the fundamental or antifundamental representation of a global flavor group is of obvious importance. Large N expansions in gauge theories, reviewed in the previous section, strongly suggest that the addition of flavor degrees of freedom amounts to introducing an open string sector to the previously considered closed string dual. This expectation is beautifully realized in the AdS/CFT correspondence.

In this section we will first review the argument of Karch and Katz [79], who considered the addition of AdS_5 filling (noncompact) probe D7 branes to Maldacena's setting of type IIB string theory on $AdS_5 \times S^5$, as a way of including fundamental flavors in the correspondence. Then we will elaborate on their argument, extending it to more general situations where the D7 branes cannot be treated as probe.

[4] Actually, the resemblance can be even pushed further: it is possible to see for instance that correlation functions of n mesonic currents are exactly the same as in a theory of oriented closed and open strings, where the string mass scale is related to the dynamically generated scale of the gauge theory, and the string coupling is (in principle) a calculable constant divided by $\sqrt{N_c}$.

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–						
D7	–	–	–	–	–	–	–	–		

Table 4.1: D3/D7 system in ten-dimensional Minkowski spacetime.

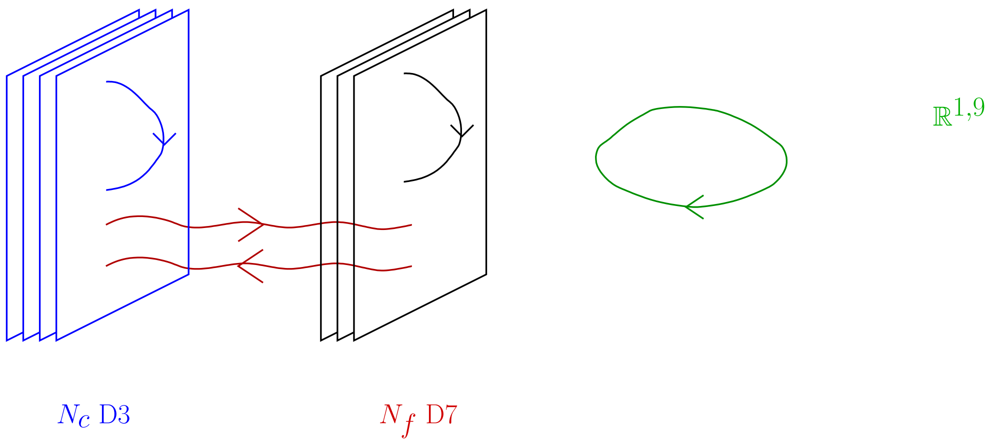


Figure 4.3: Perturbative picture ($g_sN_c, g_sN_f \ll 1$): type IIB superstring theory with N_c D3 branes and N_f D7 branes in flat spacetime.

Let us consider a system of N_c parallel D3 branes and N_f D7 branes in ten-dimensional Minkowski spacetime, in the perturbative description. The D3 branes are placed along the directions 0123, whereas the D7 branes are placed along 01234567, as in Table 4.1. This system preserves 8 supercharges. D3 branes can either be placed inside the stack of D7 branes and form a defect, or they can be separated in the 89 complex plane from the stack of D7 branes. The sectors of string theory in this picture are those depicted in Figure 4.3:

- closed strings in flat $\mathbb{R}^{1,9}$
- open 3-3 strings
- open 7-7 strings
- open 3-7 and 7-3 strings.

The dynamics of closed strings is described at low energies by type IIB supergravity in flat ten-dimensional spacetime, with Newton's constant proportional to $g_s^2\alpha'^4$. The dynamics of open 3-3 strings alone would be described at low energies by an $\mathcal{N} = 4$ $U(N_c)$ four-dimensional SYM theory with dimensionless gauge coupling $g^2_{(4)} \sim g_s$. The dynamics of open 7-7 strings alone would be described at low energies by a nonrenormalizable $U(N_f)$ 8-dimensional SYM gauge theory preserving 16 supercharges, with dimensionful gauge coupling $g^2_{(8)} \sim g_s\alpha'^2$. The 3-7 and 7-3 open string sectors introduce N_f $\mathcal{N} = 2$ hypermultiplets in the four-dimensional gauge theory on D3 branes, and a defect sector in the eight-dimensional gauge theory on D7 branes.

The $SO(4) \times SO(2)$ symmetry of the system maps to the $SU(2)_\Phi \times SU(2)_R \times U(1)_R$ global symmetries of the four-dimensional $\mathcal{N} = 2$ theory obtained adding N_f hypermultiplets coupled to an $\mathcal{N} = 2$ vector multiplet inside the $\mathcal{N} = 4$ vector multiplet. The hypermultiplets are massless or massive depending on the D3 branes being on top of or displaced from the D7 branes. If they are massive, the $U(1)_R$ R-symmetry corresponding to the $SO(2)$ rotational symmetry in the 89 complex plane is explicitly broken in the gauge theory.

We can trust this perturbative description provided that $g_s N_c \ll 1$ and $g_s N_f \ll 1$.

Let us now consider the decoupling limit (2.2.1) of D3 branes from bulk physics. If the D3 branes and the D7 branes are separated, we rescale their separations so as to keep four-dimensional gauge theory quantities fixed. The result is the following. Closed string dynamics reduces to free ten-dimensional supergravity in $\mathbb{R}^{1,9}$, decoupled from everything else, because Newton's constant has negative mass dimension. For the same dimensional analysis reason, gauge dynamics on the D7 branes disappears, and the $U(N_f)$ group becomes global. On the other hand, dynamics on the D3 branes reduces to a four-dimensional $\mathcal{N} = 2$ $U(N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation of a $U(N_f)$ flavor group. If there are massless flavors, the diagonal $U(1)$ factor in the gauge group is IR free and decouples at low energies.

Summarizing, after the decoupling limit we are left with an interacting four-dimensional $\mathcal{N} = 2$ $SU(N_c)$ SYM theory with two vector multiplets, one of which is coupled to N_f flavors, plus free supergravity in ten flat spacetime dimensions and free eight-dimensional maximal SYM on the D7 branes.

We now consider the dual picture. If one wants to work in full generality, one should start from the supergravity solution generated by a stack of N_c D3 branes and a stack of N_f D7 branes, and then take the decoupling limit. Being D7 branes higher dimensional objects, for generic values of N_c and N_f the backreaction of D7 branes would be the leading effect. D7 branes are subtle, because as codimension two objects they can introduce a deficit angle in the transverse space or even 'eat it up' and close it [80]. Far from the branes, the dilaton becomes large and the full string theory is needed to understand their physics. We will comment more on these subtleties in subsection 4.2.2. Differently from the D3 brane solution, which is geodesically complete, the D7 brane source is at finite distance from any point in the bulk geometry. This means that a source action for the D7 branes, therefore an open string sector, has to be introduced in the equations of motion.

4.2.1 The probe brane approximation

In order to avoid subtleties related to the leading backreaction of D7 branes, in their original paper Karch and Katz restricted their attention to the probe approximation for the flavor D7 branes. In the limit of very large N_c with $g_s N_c \gg 1$ and fixed, if N_f is kept finite we see that the backreaction of D7 branes, proportional to $g_s N_f$ is small. The background is therefore generated solely by the stack of D3 branes; D7 branes have to minimize their action in the D3 brane background. The situation is illustrated in Figure 4.4.

In the decoupling limit, the asymptotically flat region decouples, with its ten dimensional free supergravity and eight-dimensional free SYM hosted by the asymptotic part of the D7 brane worldvolume. We are left with interacting type IIB supergravity in the near-horizon $AdS_5 \times S^5$ geometry of the stack of N_c D3 branes, together with the open string theory on the piece of the D7 brane worldvolume which lives in this near-horizon region of the D3 branes, which is described by a Dirac-Born-Infeld plus Wess-Zumino effective action. The reason, analogous to the one explained

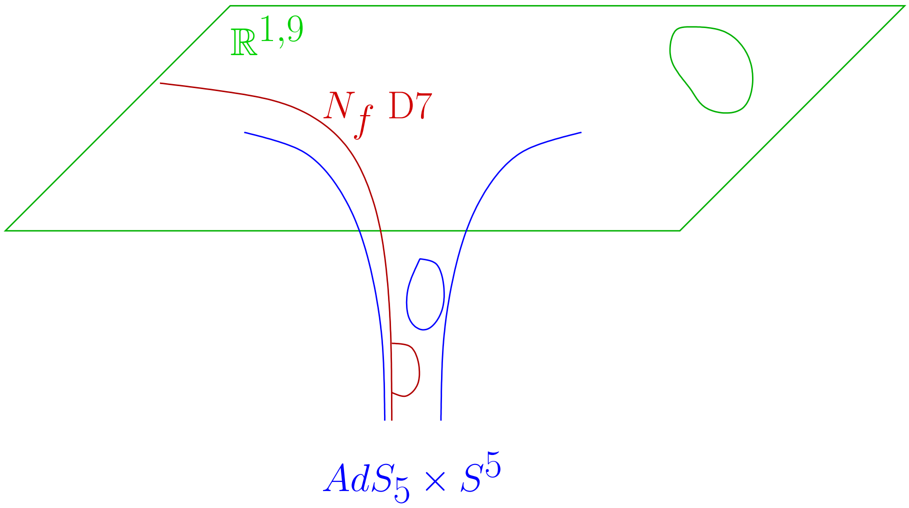


Figure 4.4: N_f probe D7 branes (in red) in the N_c D3 brane background.

in section 2.2, is that for a field theory observer out of the throat all massive closed and open string excitations inside the throat have arbitrarily low energies, provided that they propagate deep enough in the throat. For the same reason, although naive dimensional analysis would suggest that all gravity and eight-dimensional gauge interactions disappear in a low energy limit, they actually survive in the throat.

This analysis led Karch and Katz to extend Maldacena's AdS/CFT duality to field theories with fundamental flavors. In the maximally supersymmetric setting, and in the probe brane approximation, the conjecture states the equivalence between:

- the 't Hooft limit ($N_c \to \infty$, with λ and N_f fixed) of the $\mathcal{N} = 2$ $SU(N)$ SYM theory obtained by coupling N_f fundamental hypermultiplets to $\mathcal{N} = 4$ SYM, in the strongly coupled $\lambda \gg 1$ regime;

- Type IIB supergravity on $AdS_5 \times S_5$, with N_c units of RR 5-form flux, plus N_f probe D7 branes.

An important remark is in order at this point. The reader should notice that the usual open/closed string duality which is at the heart of Maldacena's AdS/CFT has been enlarged with an open/open string duality. The Dirac-Born-Infeld action for the D7 brane on the 'string side' describes the dynamics of 7-7 open strings which are *not* the 7-7 open string on the perturbative ('field theory') side, which decouple; rather, it describes the dynamics of the 3-7 and 7-3 strings of the 'field theory' side. Boundaries of the string worldsheet on the D7 branes represent mesonic operators in the gauge theory.

A related comment concerns the fact that in AdS/CFT duality (see section 2.3) a global symmetry on the field theory side is mapped into a gauge symmetry on the string side. This still holds in our setup: here the global flavor symmetry of the gauge theory, under which mesonic operators transform in the adjoint representation, is mapped into the gauge symmetry hosted by the flavor D7 branes on the string side.

Meson spectra and correlators in this limit can be obtained by computing fluctuations of the flavor branes [79, 81, 82]. We refer the reader to the review [83] for an overview of the subject and a complete list of references.

This is the first explicit realization of 't Hooft's idea reviewed in section 4.1.1: a gauge theory at large N_c and N_f fixed is translated into a classical closed and open string theory. 'Quarks' (fundamentals) are only external objects, which do not propagate in loops, where only 'gluons' (adjoints) run. This translates in a geometry generated by closed strings, plus an open string sector (hosted by flavor branes) which does not backreact on the geometry but can fluctuate in the geometry.

4.2.2 Backreacting flavor branes

In the previous subsection we have discussed the addition of flavors to the AdS/CFT duality in the probe brane approximation and we have pointed out that such an approximation amounts to considering the leading order of 't Hooft's large N_c expansion, in which the 't Hooft coupling λ and the number of flavors N_f are kept fixed. However, 't Hooft's field theoretical diagrammatic arguments which led to 4.1.2, suggesting a dual closed and open string theory interpretation of a large N gauge theory with flavors, hold irrespective of the ratio between the number of flavors and colors. In particular, we have seen in section 4.1.2 that a different scaling limit can be considered in which N_f/N_c is kept fixed instead of N_f: in that limit the leading order includes diagrams with flavor fields running in loops. It should be clear from the previous discussion that if we want to capture the leading order of Veneziano's expansion we should let the flavor branes backreact [84, 85]. Some papers which studied backreacting flavor branes even before the understanding of the decoupling limit by Karch and Katz are [86–88].

Let us revisit in more generality the decoupling limit of D3 branes in the presence of D7 branes. As a starting point, we recall the duality between the open string and the closed string pictures of D branes. Conceptually, the D3 brane geometry could be obtained by resumming an infinite number of diagrams with boundaries for the worldsheet of a closed string propagating in the presence of the D3 branes. The decoupling limit further corresponds to the near-horizon limit of such a geometry. The conjecture in its strongest forms states that quantum type IIB string theory on the near-horizon geometry of a stack of D3 branes is dual to the field theory limit of the D3 brane dynamics.

In the presence of D7 branes we still consider the field theory/decoupling limit of D3 branes. Doing this should be dual to considering propagation of closed strings in the near-horizon geometry of the D3 branes, which is a consistent string background. Taking the decoupling limit of D3 branes on this side, as we have already reviewed in the previous subsection, not only the full closed string theory but also a full open string sector hosted by D7 branes (dual to mesonic excitations in the gauge theory) in such a throat background survives, because of the redshift. We are therefore led to the strongest form of the flavored extension of AdS/CFT duality, which states the equivalence of:

- the full quantum four-dimensional $\mathcal{N} = 2$ theory obtained by coupling N_f fundamental hypermultiplets to $\mathcal{N} = 4$ $SU(N)$ SYM theory;

- the full quantum type IIB string theory on $AdS_5 \times S_5$, with N_c units of RR 5-form flux, endowed with an open string sector hosted by N_f D7 branes in such a background.

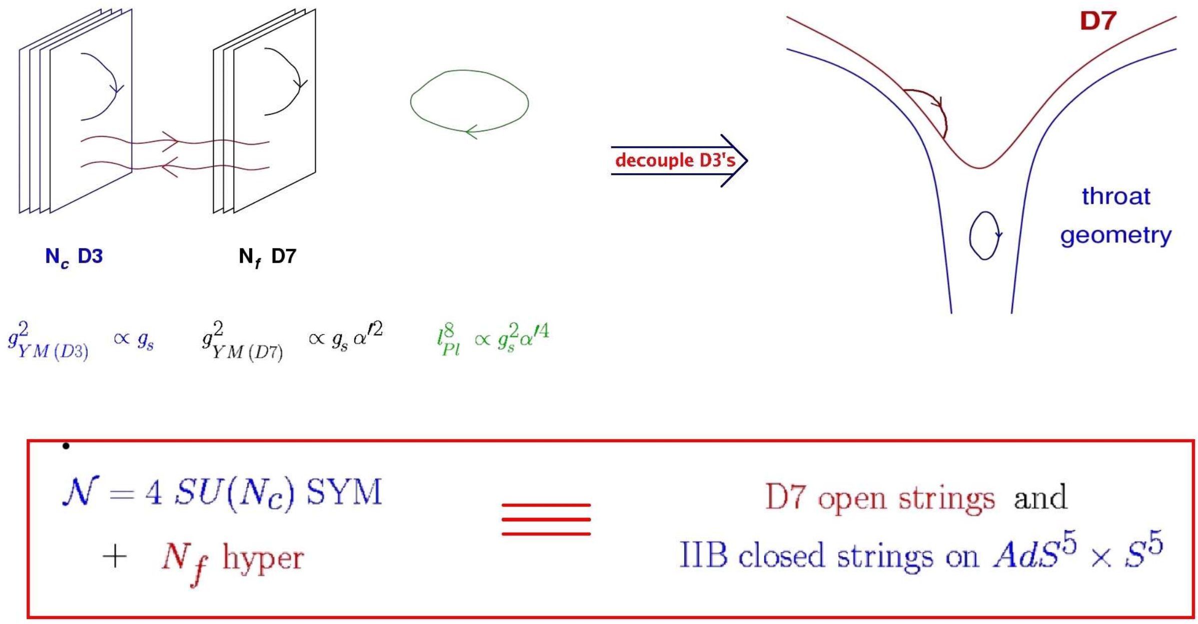


Figure 4.5: AdS/CFT correspondence with flavors.

In order to obtain the string dual of the field theory with arbitrary λ, N_c, N_f, we should be able to quantize both closed and open strings in a curved RR $AdS_5 \times S_5$ background. This achievement is out of our current reach, but we can reduce to more controllable regimes.

If we take the usual supergravity limit $g_s \ll 1$, $g_s N_c \gg 1$, the D3 brane sector can be easily dealt with. Depending on N_f/N_c, the D7 brane sector has to be dealt with differently. For general values of N_f/N_c, the full backreaction of D7 branes in the near-horizon background generated by D3 branes should be computed. This introduces an open string sector: string worldsheets (without handles) can now acquire any number of boundaries. This means that in the dual field theory fundamental flavor fields are allowed to run in loops. We are at the leading order of Veneziano's expansion, where amplitudes contain planar diagrams with any number of empty quark loops.

If we further take the limit in which $\frac{N_f}{N_c} \to 0$ ('t Hooft's limit), the backreaction of D7 branes goes to zero, and the open string sector remains as external. In the jargon of lattice gauge theorists, flavors are 'quenched' and cannot be pair-produced. We only find worldsheets with the topology of a sphere, or with the topology of a disc if there are open string vertices insertions at the boundary.

Therefore, the advantages of the backreacting flavor branes approach that we will follow over the probe branes approach are the same advantages of Veneziano's expansion over 't Hooft's expansion, discussed briefly in section 4.1.2. New hadronic physics is captured, and nontrivial dynamics arising from flavor fields quantum effects is visible in this approach.

At first sight, there are however some disadvantages, beyond the increased degree of technical complexity. The first issue is related to the geometry generated by D7 branes. The authors of [80] have shown that the geometry generated by a stack of localized N_f D7 branes in the transverse two-dimensional space is asymptotically conical with a deficit angle of $2\pi\frac{N_f}{12}$ if $N_f < 12$, asymptotically cylindrical if $N_f = 12$, whereas it closes and is singular in all other cases, except for the regular case $N_f = 24$. Therefore there is no known nonsingular global solution for more than 24 D7 branes. Furthermore, close to the D7 branes the effective string coupling e^{Φ} goes to zero, bringing an additional curvature singularity in string frame. This facts could worry the reader, since in the next part of this thesis we are going to discuss solutions for backreacting D7 branes in the context of gauge/gravity dualities.

We can anticipate here that, luckily, in this framework such singularities have a physical meaning and are actually expected to be there. The IR singularity is related to a coupling in the dual field theory flowing to zero. It is common lore in gauge/gravity dualities that when the field theory becomes weakly coupled, the dual supergravity description develops large curvatures. This happens for the maximally supersymmetric theory we have discussed so far [89], and will also happen for the flavored Klebanov-Witten theory that we are going to discuss in the next chapter [1]. In that case we give arguments why the coupling that is flowing to zero is not the gauge coupling but rather the quartic superpotential coupling of bifundamental fields.

The UV singularity has a dual interpretation too, as a Landau pole in the dual gauge theory that is obtained by adding flavors to a conformal field theory [1,89]. Such a behavior is expected on field theoretic grounds for any nonvanishing value of N_f/N_c in the large N_c limit. The conclusion is that, as a gravity dual of a field theory with a Landau pole, the UV singularity of a D3-D7 brane geometry is not only acceptable but also expected.[5] The backgrounds are sensible for any number of flavors. It would be of course nice to find a globally nonsingular string theory solution extending in the UV for $N_f > 12$ (24), but for what concerns us that solution would only shed light on the way string theory UV-completes the field theory with a Landau pole that we are studying. The backgrounds that are found at the level of supergravity can be trusted in some energy range, thus providing a tool for studying the physics of the dual gauge theory.

[5]In the (orbifold of) flat space case, it is possible to show that massless open string states (related to the field theory picture) precisely map into massless closed string states (related to the supergravity pictures) without any mixing with massive string states [89, 90]. As a consequence, gauge/gravity duality works even in the absence of a true boundary corresponding to an infinite energy 'cutoff'.

Chapter 5

Unquenched flavors in the Klebanov-Witten theory

In this chapter, we study the addition of backreacting noncompact D7 branes to the Klebanov-Witten geometry. The resulting background is dual to a flavored version of the Klebanov-Witten field theory, to leading order in Veneziano's large N_c expansion. This is a first step towards flavoring the cascading Klebanov-Strassler field theory, the result of which is hoped to give a gravity dual of an $\mathcal{N} = 1$ SQCD-like theory in Veneziano's large N_c limit. That will be the content of the next chapter.

This chapter is mostly based on reference [1], written by the present author in collaboration with Francesco Benini, Felipe Canoura, Carlos Núñez and Alfonso V. Ramallo.

5.1 Introduction

Our starting point is the type IIB $AdS_5 \times T^{1,1}$ solution dual to the Klebanov-Witten field theory [29] earlier introduced in section 3.2. Our aim is to add a large number of flavors, comparable to the number of colors, to each of the gauge groups. Our procedure can be generalized to cases describing different duals to $\mathcal{N} = 1$ SCFT's constructed from D3 branes placed at conical Calabi-Yau singularities, as described in [1].

Let us briefly describe the procedure we will follow, inspired mostly by the papers [84, 85, 91] and more recently [92, 93] and introduced in subsection 4.2.2. In those papers (dealing with the addition of many fundamentals in the noncritical string and type IIB string respectively), flavors were added to the dynamics of the dual background via the introduction of N_f spacetime filling flavor branes, whose dynamics is given by a Dirac-Born-Infeld and a Wess-Zumino action. This dynamics is intertwined with the usual Einstein-like action of type IIB supergravity and a new solution is found, up to technical subtleties described below.

5.1.1 Generalities of the procedure

To illustrate the way flavor branes will be added, let us first recall the background of type IIB supergravity that is dual to the Klebanov-Witten field theory. The dual type IIB background in string frame reads[1]

$$\begin{aligned} ds^2 &= h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \Big\{ dr^2 + \frac{r^2}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2) + \frac{r^2}{9} (d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i)^2 \Big\} \\ g_s F_5 &= (1+*)\, d^4x \wedge dh(r)^{-1} \\ h(r) &= \frac{27\pi g_s N_c \alpha'^2}{4r^4} \end{aligned} \tag{5.1.1}$$

with constant dilaton $e^\Phi = g_s$ and all the other fields in type IIB supergravity vanishing. The set of coordinates that will be used in the rest of the chapter is $x^M = \{x^0, x^1, x^2, x^3, r, \psi, \theta_1, \varphi_1, \theta_2, \varphi_2\}$. We recall that a review of the conifold geometry can be found in appendix B.1.

We will add N_f noncompact D7 branes to this geometry, in a way that preserves four supercharges. This problem was studied in [79,94,95] for the conformal case and in [96] for the cascading theory.[2] These authors found calibrated embeddings of D7 branes which preserve four conserved supercharges of the background. We will choose to put two sets of D7 branes on the surfaces parameterized by

$$\begin{aligned} \xi_1^\alpha &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_2, \varphi_2\} \qquad \theta_1 = \text{const.} \qquad \varphi_1 = \text{const.}\ , \\ \xi_2^\alpha &= \{x^0, x^1, x^2, x^3, r, \psi, \theta_1, \varphi_1\} \qquad \theta_2 = \text{const.} \qquad \varphi_2 = \text{const.}\ . \end{aligned} \tag{5.1.2}$$

These two configurations are mutually supersymmetric with the background. The two sets of flavor branes introduce a chiral $U(N_f) \times U(N_f)$ flavor symmetry,[3] the maximal flavor symmetry for massless flavors. The configuration with two sets (two branches) can be deformed to a single set, shifted from the origin, that represents massive flavors, and realizes the explicit breaking of the flavor symmetry to the diagonal vector-like $U(N_f)$. Our configuration (eq. (5.1.2)) for probes is nothing else than the $z_1 = 0$ holomorphic embedding of [95], having two branches, $z_3 = 0$ and $z_4 = 0$. Notice also that placing stacks of localized D7 branes along the embeddings (5.1.2) breaks the $SU(2) \times SU(2)$ symmetry of the background and of the dual gauge theory to $U(1) \times U(1)$.

We will then write an action for a system consisting of type IIB supergravity[4] plus D7 branes described by their Dirac-Born-Infeld and Wess-Zumino actions:

$$\begin{aligned} S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \Big[R - \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} e^{2\Phi} |F_1|^2 - \frac{1}{4} |F_5|^2 \Big] + \\ - \mu_7 \sum^{N_f} \int d^8\xi \, e^\Phi \Big[\sqrt{-\hat{g}_8^{(1)}} + \sqrt{-\hat{g}_8^{(2)}} \Big] + \mu_7 \sum^{N_f} \int \hat{C}_8\ , \end{aligned} \tag{5.1.3}$$

[1] As in appendix B.1, we have flipped for convenience a sign in the fibration of the ψ angles with respect to the most widespread conventions in the literature [49].

[2] A stable configuration of flavor branes that breaks supersymmetry was previously considered in [97].

[3] The diagonal axial $U(1)$ is anomalous, and its anomaly can be seen at leading order in Veneziano's large N_c limit [98].

[4] The problems with writing an action for type IIB that includes the self-duality condition are well known. Here, as in appendix A.2, we just loosely mean a Lagrangian from which the equations of motion of type IIB supergravity are derived. The self-duality condition is imposed on shell.

where we have chosen the normalization $|F_p|^2 = \frac{1}{p!}F_pF_p(g^{-1})^p$ (see appendix A.1). The action, the derivation of the equations of motion, as well as the definition of the Einstein frame can be found in Appendices A.2 and A.2.1. Here we use the Einstein frame defined by (A.2.5), where we rescale the string frame metric by the whole dilaton.

Notice that we did not excite the worldvolume gauge fields, but this is a freedom of the approach we adopted. Otherwise one may need to find new suitable κ-symmetric embeddings.

These two sets of D7 branes are localized in their two transverse directions, hence the equations of motion derived from (5.1.3) will be quite complicated to solve, due to the presence of localized source terms in the form of Dirac delta functions. But we can take some advantage of the fact that we are adding lots of flavors, and that D7 branes along embeddings related by the broken subgroup of $SU(2) \times SU(2)$ symmetry are mutually BPS. Since we will have many ($N_f \to \infty$) flavor branes, we might think about distributing them in a homogeneous way on their respective transverse directions, thus restoring the $SU(2) \times SU(2)$ symmetry of the unflavored Klebanov-Witten background. This 'smearing procedure' boils down to approximating

$$\begin{aligned} \mu_7 \sum^{N_f} \int d^8\xi \, e^{\Phi} \sqrt{-\hat{g}_8^{(i)}} \quad &\to \quad \frac{\mu_7 N_f}{4\pi} \int d^{10}x \, e^{\Phi} \, \sin\theta_i \sqrt{-\hat{g}_8^{(i)}} \\ \mu_7 \sum^{N_f} \int \hat{C}_8 \quad &\to \quad \frac{\mu_7 N_f}{4\pi} \int \Big[d\text{vol}(Y_1) + d\text{vol}(Y_2) \Big] \wedge C_8 \; , \end{aligned} \tag{5.1.4}$$

with $d\text{vol}(Y_i) = \sin\theta_i \, d\theta_i \wedge d\varphi_i$ the volume forms of the S^2's.

Notice that this smearing differs conceptually from the one that is encountered for instance when T-dualizing at the level of supergravity brane solutions in transverse directions. Here we are really considering $N_f \to \infty$ flavor D7 branes, that we are free to place where we want in the transverse directions, because these configurations are mutually BPS. This is a legitimate choice: the only approximation we are doing consists in taking the continuum limit substituting an infinite sum with an integral. This smearing effectively generates a ten-dimensional action

$$\begin{aligned} S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \Big[R - \frac{1}{2}\partial_M\Phi\partial^M\Phi - \frac{1}{2}e^{2\Phi}|F_1|^2 - \frac{1}{4}|F_5|^2 \Big] + \\ - \frac{\mu_7 N_f}{4\pi} \int d^{10}x \, e^{\Phi} \sum_{i=1,2} \sin\theta_i \sqrt{-\hat{g}_8^{(i)}} + \frac{\mu_7 N_f}{4\pi} \int \Big[d\text{Vol}(Y_1) + d\text{Vol}(Y_2) \Big] \wedge C_8 \; . \end{aligned} \tag{5.1.5}$$

From this action, we can derive the following equations of motion:

$$\begin{aligned} R_{MN} - \frac{1}{2}g_{MN}R &= \frac{1}{2}\Big(\partial_M\Phi\partial_N\Phi - \frac{1}{2}g_{MN}\partial_P\Phi\partial^P\Phi \Big) + \frac{1}{2}e^{2\Phi}\Big(F_MF_N - \frac{1}{2}g_{MN}|F_1|^2 \Big) + \\ &\quad + \frac{1}{96}F_{MPQRS}F_N{}^{PQRS} + 2\kappa_{10}^2 T_{MN} \\ D^M\partial_M\Phi &= e^{2\Phi}|F_1|^2 + \frac{2\kappa_{10}^2\mu_7}{\sqrt{-g}}\frac{N_f}{4\pi}e^{\Phi} \sum_{i=1,2} \sin\theta_i \sqrt{-\hat{g}_8^{(i)}} \\ d\Big(e^{2\Phi} * F_1 \Big) &= 0 \\ dF_1 &= 2\kappa_{10}^2\mu_7\frac{N_f}{4\pi}\Big[d\text{vol}(Y_1) + d\text{vol}(Y_2) \Big] \\ dF_5 &= 0 \; . \end{aligned} \tag{5.1.6}$$

The modified Bianchi identity for F_1 is obtained through $F_1 = e^{-2\Phi} * F_9$, and comes from the WZ part of the action (5.1.5), see appendix A.2.1. The contribution to the stress-energy tensor coming from the two sets of N_f D7 flavor branes is given by

$$T^{MN} = \frac{1}{\sqrt{-g}}\frac{\delta S^{flavor}}{\delta g_{MN}} = -\mu_7 \frac{N_f}{4\pi}\frac{e^{\Phi}}{\sqrt{-g}}\sum_{i=1,2}\sin\theta_i \frac{1}{2}\sqrt{-\hat{g}_8^{(i)}}\,\hat{g}_8^{(i)\alpha\beta}\delta_\alpha^M\delta_\beta^N\ , \tag{5.1.7}$$

where α, β are coordinate indices on the D7 branes worldvolumes. We will solve the equations of motion (5.1.6)-(5.1.7) and propose that the resulting type IIB background is dual to the Klebanov-Witten field theory when two sets of N_f flavors are added for each gauge group. We will actually find BPS equations for the purely bosonic background, by imposing that the variations of the dilatino and gravitino vanish and the Bianchi identities.

Let us add some remarks on some important points about the resolution of the system of equations of motion and Bianchi identities. First of all, it is clear from the Bianchi identity of F_1 in (5.1.6) that we will not be able to define the axion field C_0 on open subsets. Regarding the solution of the equations of motion, we will proceed by proposing a *deformed background* ansatz of the form

$$\begin{aligned} ds^2 &= h^{-1/2}dx_{1,3}^2 + h^{1/2}\Big\{dr^2 + \frac{e^{2g}}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i\,d\varphi_i^2) + \frac{e^{2f}}{9}(d\psi - \sum_{i=1,2}\cos\theta_i\,d\varphi_i)^2\Big\} \\ F_5 &= (1+*)\,d^4x\wedge dh^{-1} \\ F_1 &= \frac{N_f}{4\pi}(d\psi - \sum_{i=1,2}\cos\theta_i\,d\varphi_i)\ . \end{aligned} \tag{5.1.8}$$

Thanks to the smearing procedure, all the unknown function h, f, g, K and the dilaton Φ only depend on the radial coordinate r. We will impose the Bianchi identity for the 5-form field strength $dF_5 = 0$ (the one for F_1 is automatically solved by our ansatz) and we will obtain solutions to (5.1.6) by imposing that the BPS equations derived from the vanishing of the gravitino and gaugino variations and the Bianchi identities are satisfied. These will produce ordinary first-order equations for $f(r)$, $g(r)$, $h(r)$, $\Phi(r)$. It is also possible to derive these BPS equations from a superpotential in the reduction of type IIB supergravity, as shown in [1]. The explicit solution of the system of BPS equations for the flavored Klebanov-Witten theory will be given in the next section.

We will study in detail the field theory dual to the supergravity solutions mentioned above, making a number of matchings and predictions. Let us anticipate the main results. The field theory turns out to have positive β-function along the flow, exhibiting a Landau pole in the UV. We will propose that the IR is also described by a strongly coupled conformal field theory. All these results can be generalized to the interesting case of a large class of different $\mathcal{N} = 1$ SCFT's on D3 branes at conical Calabi-Yau singularities, deformed by the addition of flavors. In particular, using the same method it is possible to add flavors to every gauge theory whose dual is $AdS_5 \times M_5$, where M_5 is a five-dimensional Sasaki-Einstein manifold. Finally, a possible way of handling the massive flavor case is undertaken.

We have explained the strategy we adopt to add flavors, so this is perhaps a good place to discuss some interesting issues.

First of all, it is important to stress again that we are considering the backreaction of the flavor branes along the lines discussed in subsection 4.2.2, rather than treating the flavor branes as probes, as in the approach exposed in subsection 4.2.1. This is necessary if N_f/N_c does not

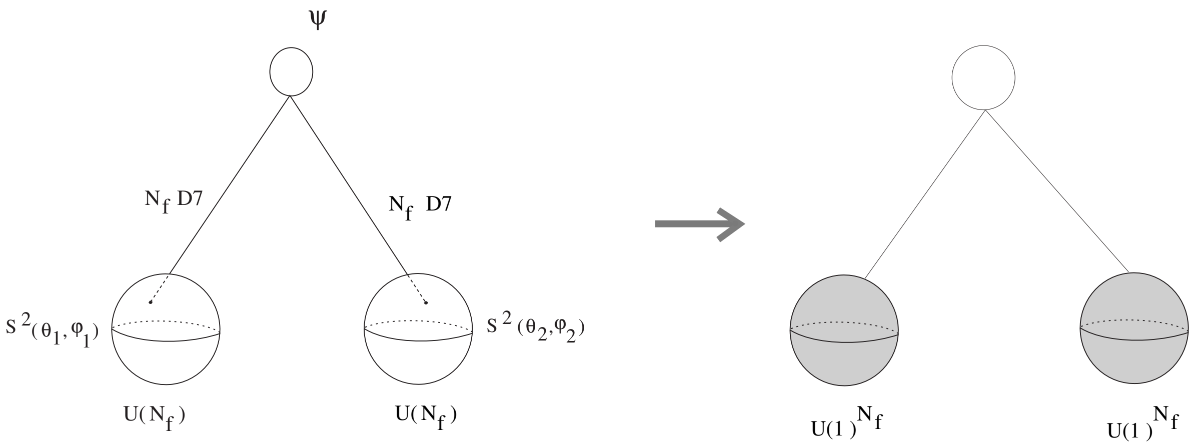


Figure 5.1: On the left side, the two stacks of N_f flavor branes are localized on each of their respective S^2's (they wrap the other S^2). The flavor group of the fundamental fields is clearly $U(N_f) \times U(N_f)$. In addition, an unbroken $U(1) \times U(1)$ subgroup of the original $SU(2) \times SU(2)$ symmetry group (plus the R-symmetry) is acting also on bifundamental fields. After the smearing, on the right side of the figure, the global flavor symmetry is broken to $U(1)^{N_f-1} \times U(1)^{N_f-1} \times U(1)_B \times U(1)_A$, and the $U(1) \times U(1)$ group is enhanced to the original $SU(2) \times SU(2)$ of the KW theory.

vanish in the large N_c limit, and allows the study of the influence of flavor fields on color dynamics. Our supergravity plus branes setting will describe the leading order of Veneziano's expansion (see subsection 4.1.2) of the dual gauge theory, rather than the leading order of 't Hooft's large N_c expansion (see subsection 4.1.1).

Secondly, the reader might be wondering about the 'smearing procedure' discussed above, what is its significance and effect on the dual gauge theory. We make clear that we smear the flavor branes just to be able to write a 10-dimensional action that will produce ordinary (in contrast to partial) differential equations without Dirac delta functions source terms. Let us pause for a while on the global symmetries introduced by flavor fields, and let us go back to the weak coupling ($g_s N_c \ll 1$) limit, in which we have branes living on a spacetime that is the product of four Minkowski directions and the conifold. When all the flavor branes of the two separate stacks (5.1.2) are on top of each other, the gauge symmetry on the D7 branes worldvolume is given by the product $U(N_f) \times U(N_f)$. When we take the decoupling limit for the D3 branes $\alpha' \to 0$, with fixed $g_s N_c \gg 1$ and keeping constant the energies of the excitations on the branes, we are left with a solution of type IIB supergravity that we propose is dual to the Klebanov-Witten field theory with N_f flavors for both gauge groups [95], following the general discussion of subsection 4.2.2. In this setup the flavor symmetry is $U(N_f) \times U(N_f)$, where the axial $U(1)$ is anomalous. As we have anticipated, this background would be for sure very involved because of the low degree of symmetry: it would depend on the coordinates (r, θ_1, θ_2), if the embeddings of the two stacks of D7 branes are $\theta_1 = 0$ and $\theta_2 = 0$ respectively. When we smear the N_f D7 branes, we are recovering the full $SU(2) \times SU(2)$ symmetry related to the internal space. At the same time, we are breaking each flavor group $U(N_f) \to U(1)^{N_f}$ (see figure 5.1). We will need to compare our way of introducing holographic flavors, which is dual to a field theory with maximal global symmetry acting on the bifundamental fields, with the more conventional flavoring obtained by placing stacks of coincident D7 branes, dual to a field theory where the superpotential coupling flavor fields to bifundamental fields explicitly breaks some of the global symmetries of the unflavored theory (this coupling will be presented in the next section). The results we will lay out and the experience obtained in [85, 92] show that many properties of the more conventionally flavored field theory are still well captured by the solutions obtained following the procedure described above. It is not completely clear what important phenomena in the conventionally flavored gauge theory we are losing in smearing, but see the next sections for subtleties and discussions.

Another issue that is worth revisiting at this point is the objection on the limit on the number of D7 branes that can be added. Indeed, since a D7 brane is a codimension two object (like a vortex in $2+1$ dimensions) its gravity solution will generate a deficit angle; having many seven branes, will basically "eat-up" the transverse space. This led to the conclusion that solutions that can be globally extended cannot have more than a maximum number of twelve D7 branes [80] (and exactly twenty-four in compact spaces). In this paper we are adding a number $N_f \to \infty$ of D7 branes, certainly larger that the bound mentioned above. We refer the reader to the end of subsection 4.2.2 for a thorough discussion of the solution of this issue. We add here that the smearing procedure distributes the D7 branes all over a 2-dimensional space, in such a way that the equation for the axio-dilaton is not the one in the vacuum at any point. Therefore, strictly speaking, the analysis of [80] does not apply to our case. Nevertheless, we will solve the equations of motion and discuss the results, that are qualitatively similar to those of [80]. However, as already discussed in 4.2.2, in the context of gauge/gravity duality having a solution which cannot be globally extended at large r is not a problem. In our setup, it is precisely the dual of the field theory having a Landau pole.

5.2 Adding flavors to the Klebanov-Witten field theory

5.2.1 What to expect from field theory considerations

Before presenting the solution for backreacting D7 branes in the Klebanov-Witten background, in this subsection we would like to have a look at the dual field theory, and sketch which are the features we expect.

The addition of fundamental and antifundamental flavors to the Klebanov-Witten theory can be addressed by including probe D7 branes into the geometry. This was done in a sequence of papers [79, 94–97], where different embeddings were studied. We will follow here reference [95], where the embedding of the flavor branes we are considering in this work and the corresponding superpotential for the fundamental and antifundamental superfields were found. The D7 branes have four Minkowski directions parallel to the stack of D3 branes transverse to the conifold, whereas the other four directions are embedded holomorphically in the conifold. In particular, D7 branes describing massless flavors can be introduced by considering the holomorphic noncompact embedding $z_1 = 0$, in the complex coordinates of appendix B.1 where the conifold equation is $z_1 z_2 - z_3 z_4 = 0$. Flavors from 3-7 and 7-3 strings are massless because the D7 branes intersect the D3 branes placed at the tip of the cone. It is clear from the embedding equation that the D7 branes are made of two branches, described by $z_1 = z_3 = 0$ and $z_1 = z_4 = 0$, to each one of which a stack will be associated. The presence of two branches is required by RR tadpole cancellation, as will be explicitly shown in the next subsection: in the field theory this amounts to adding flavors in vector-like representations

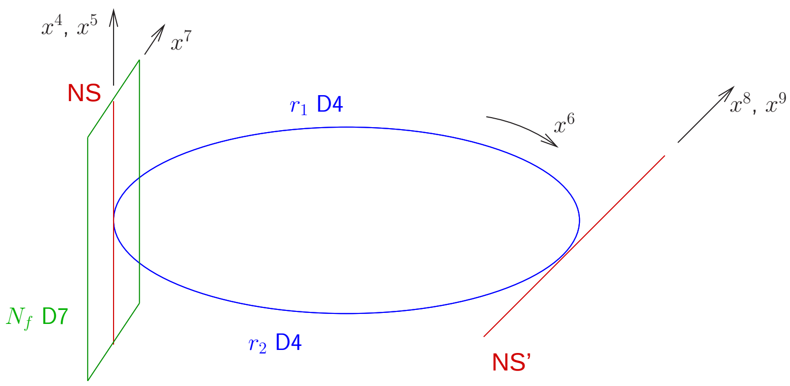


Figure 5.2: Type IIA Hanany-Witten brane engineering of the conifold theory with chiral flavors. The flavored Klebanov-Witten theory is obtained by setting $r_1 = r_2 = N_c$.

to each gauge group, hence preventing gauge anomalies.

This is clear also from the classical type IIA T-dual picture discussed in [95]. The T-dual Hanany-Witten brane engineering of the conifold theory [100] involves two stacks of D4 branes suspended between orthogonal NS5 branes, along a circle direction. Chiral flavors are provided by a D6 brane coincident with one of the two NS5 branes, in such a way that the NS5 brane splits the flavor D6 brane in two pieces. Each of these two pieces provides a chiral flavor group for both the gauge groups [99], see figure 5.2. A single half D6 brane cannot consistently end on an NS5 because of RR charge conservation: another half D6 brane must end on the NS5 brane, in such a way that the two halves do not form an angle. In the gauge theory, RR charge conservation maps into the absence of gauge anomalies: a single chiral flavor group would lead to gauge anomalies, but two chiral flavor groups with the same rank do not, since they build vector-like representations of the gauge groups.

The fundamental and antifundamental chiral superfields of the two gauge groups will be denoted as q, $\tilde{q}$ and Q, $\tilde{Q}$ respectively. The matter content of the flavored KW theory we are studying is summarized in figure 5.3. The gauge invariant and flavor invariant superpotential proposed in [95] is

$$W = W_{KW} + W_f \ , \tag{5.2.1}$$

where

$$W_{KW} = h \,\mathrm{Tr}(A_i B_k A_j B_l)\, \epsilon^{ij} \epsilon^{kl} \tag{5.2.2}$$

is the $SU(2) \times SU(2)$ invariant Klebanov-Witten superpotential for the bifundamental fields; as far as W_f is concerned, given a stack of N_f flavor branes localized in two branches at given points on the two 2-spheres (say the north poles), we can conventionally take the coupling between bifundamentals and quarks to be

$$W_f = h_1\, \tilde{q}^a A_1 Q_a + h_2\, \tilde{Q}^a B_1 q_a \ . \tag{5.2.3}$$

This coupling between bifundamental fields and the fundamental and antifundamental flavors is dictated by the D7 embedding $z_1 = 0$. The explicit indices are flavor indices. This superpotential, as well as the holomorphic embedding $z_1 = 0$, explicitly breaks the $SU(2) \times SU(2)$ global symmetry. This global symmetry will be recovered after the smearing. In the type IIA brane picture, $h_1 = -h_2$.

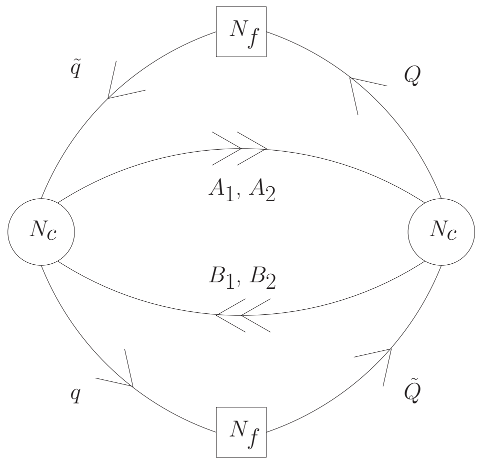


Figure 5.3: Quiver diagram of the Klebanov-Witten gauge theory with flavors.

The gauge and flavor symmetries of the theory are summarized in Table 5.1. The $U(1)_R$ R-

	$[SU(N_c)^2]$	$SU(N_f)^2$	$SU(2)^2$	$U(1)_R$	$U(1)_B$	$U(1)_{B'}$
A	$(N_c, \overline{N_c})$	$(1,1)$	$(2,1)$	$1/2$	0	1
B	$(\overline{N_c}, N_c)$	$(1,1)$	$(1,2)$	$1/2$	0	-1
q	$(N_c, 1)$	$(\overline{N_f}, 1)$	$(1,1)$	$3/4$	1	1
$\tilde{q}$	$(\overline{N_c}, 1)$	$(1, N_f)$	$(1,1)$	$3/4$	-1	-1
Q	$(1, N_c)$	$(1, \overline{N_f})$	$(1,1)$	$3/4$	1	0
$\tilde{Q}$	$(1, \overline{N_c})$	$(N_f, 1)$	$(1,1)$	$3/4$	-1	0

Table 5.1: Field content and symmetries of the KW field theory with massless flavors.

symmetry is preserved at the classical level by the inclusion of D7 branes embedded in such a way to describe massless chiral flavors, as can be seen from the fact that the equation $z_1 = 0$ is invariant under the rotation $z_i \to e^{i\alpha} z_i$ and the D7 wrap the R-symmetry circle. However, the $U(1)_R$ turns out to be anomalous after the addition of flavors, due to the nontrivial C_0 gauge potential sourced by the D7. The baryonic symmetry $U(1)_B$ inside the flavor group is anomaly free, being vector-like.

The theory including D7 brane probes is also invariant under a real rescaling $z_i \to \beta z_i$, therefore the field theory is scale invariant in the quenched approximation. In this limit the scaling dimension of the bifundamental fields is $3/4$ and the one of the flavor fields is $9/8$, as required by power counting in the superpotential. In that case, the exact beta function for the gauge couplings[5] is

$$\beta_{\frac{8\pi^2}{g_i^2}} = -\frac{16\pi^2}{g_i^3}\beta_{g_i} = -\frac{3}{4}N_f \qquad\qquad \beta_{\lambda_i} = \frac{1}{(4\pi)^2}\frac{3N_f}{2N_c}\lambda_i^2 \, , \tag{5.2.4}$$

with $\lambda_i = g_i^2 N_c$ the 't Hooft couplings of the two gauge groups. At leading order in 't Hooft's limit,

[5]We choose the normalization where the inverse square gauge coupling appears in front of the YM action, as is natural on the D brane side.

the field theory has a fixed point specified by the aforementioned choice of scaling dimensions, because the beta functions of the superpotential couplings and the 't Hooft couplings vanish. As soon as N_f/N_c corrections cannot be neglected, like in the leading order of Veneziano's limit, the previous choice of anomalous dimensions does not lead to a fixed point anymore. Rather, if a regime of the RG flow exists where these anomalous dimensions are correct, in that regime the theory flows close to what we will call a "near conformal point" with vanishing β- functions for the superpotential couplings but nonvanishing β-functions for the 't Hooft couplings, with very slow running if N_f/N_c is small. This flow was considered in [95], where the leading order backreaction of flavor branes was computed. Let us remark that in a N_f/N_c expansion, formula (5.2.4) holds at order N_f/N_c *if* the anomalous dimensions of the bifundamental fields A_j and B_j do not get corrections at this order. A priori it is not obvious that we should expect such a behavior from string theory, since the energy-momentum tensor of the flavor branes will induce backreaction effects on the geometry already at linear order in N_f/N_c, unlike the fluxes, which will backreact at order $(N_f/N_c)^2$. Moreover, since we are adding flavors to a conformal theory, we expect a Landau pole to appear in the UV. Since the UV behavior changes drastically after flavors are added, there is a priori no reason why the gauge theory should run through this "near conformal point" where the anomalous dimensions of the flavor fields are those valid in the probe/'t Hooft limit. In a following section we will propose an interpretation of the RG flow described by our supergravity solutions: we will see that provided N_f/N_c is small enough, an intermediate regime of the flow where the anomalous dimensions are those previously discussed exists. The smaller N_f/N_c, the longer this regime will be.

We make here a remark which will turn out to be very important in that section for the discussion of the IR dynamics of our solution. Notice that if the anomalous dimensions of the fields become at some energy $\gamma_A = \gamma_B = -1/2$, $\gamma_Q = \gamma_q = 1$, the gauge couplings and the bifundamental quartic superpotential coupling stop running, whereas the superpotential coupling bifundamentals to flavor fields is irrelevant and runs to zero in the IR. The flavored theory under study possesses in principle a strongly coupled IR fixed point. We will argue in a following section that the RG flow described by our solution actually ends in such a point.

5.2.2 The setup and the general solutions

The starting point for adding backreacting branes to a given background is the identification of the supersymmetric embeddings in that background, that is the analysis of probe branes. In [94], by imposing κ-symmetry on the brane world-volume, the following supersymmetric embeddings for D7 branes on the Klebanov-Witten background were found:

$$\begin{aligned} &\xi_1^\alpha = \{x^0, x^1, x^2, x^3, r, \psi, \theta_2, \varphi_2\} \qquad \theta_1 = \text{const.} \qquad \varphi_1 = \text{const.} \\ &\xi_2^\alpha = \{x^0, x^1, x^2, x^3, r, \psi, \theta_1, \varphi_1\} \qquad \theta_2 = \text{const.} \qquad \varphi_2 = \text{const.} \end{aligned} \tag{5.2.5}$$

They are precisely ($SU(2)\times SU(2)$ rotations of) the two branches of the supersymmetric embedding $z_1 = 0$ first proposed in [95]. Each branch realizes a $U(N_f)$ symmetry group, giving the total flavor symmetry group $U(N_f) \times U(N_f)$ of massless flavors (a diagonal axial $U(1)_A$ is anomalous in field theory, which is dual to the corresponding gauge field getting massive in string theory through Green-Schwarz mechanism). We choose these embeddings because of the following properties: they reach the tip of the cone and intersect the color D3 branes; they wrap the $U(1)_R$ circle corresponding to rotations $\psi \to \psi + \alpha$; they are invariant under radial rescalings. So they realize in field theory massless flavors, without breaking explicitly the $U(1)_R$ and the conformal symmetry. Actually, these symmetries are both broken by quantum effects. Moreover the configuration does not break the $\mathbb{Z}_2$ symmetry of the conifold solution which corresponds to exchanging the two gauge groups.

As already mentioned, the fact that we must include both the branches is due to D7 charge tadpole cancellation, which is dual to the absence of gauge anomalies in field theory. We can see this explicitly at this point. The integral of the charge density on any 2-cycle in the geometry must vanish. The only nontrivial 2-cycle in the conifold has the topology of a 2-sphere and can be represented as $S^2 : \ \{\theta_1 = \theta_2,\ \varphi_1 = 2\pi - \varphi_2,\ \psi = \text{const},\ r = \text{const}\}$, as reviewed in appendix B.1. The charge distributions of the two branches are

$$\omega^{(1)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_1, \varphi_1)\, d\theta_1 \wedge d\varphi_1 \qquad\qquad \omega^{(2)} = \sum\nolimits_{N_f} \delta^{(2)}(\theta_2, \varphi_2)\, d\theta_2 \wedge d\varphi_2 \ , \tag{5.2.6}$$

where the sum is over the various D7 branes, possibly localized at different points, and a correctly normalized scalar delta function (localized on an 8-submanifold) is $\delta^{(2)}(x)\sqrt{-\hat{g}_8}/\sqrt{-g}$. Integrating the two D7-charges on the 2-submanifold we get:

$$\int_{S^2} \omega^{(1)} = -N_f \qquad\qquad \int_{S^2} \omega^{(2)} = N_f \ . \tag{5.2.7}$$

Thus, whilst the two branches have separately nonvanishing tadpole, putting an equal number of them on the two sides the total D7-charge cancels.

The embeddings can be deformed into a single D7 brane embedding that only reaches a minimum radius and realizes a merging of the two branches. This corresponds to giving mass to flavors and explicitly breaking the flavor symmetry to $SU(N_f)$ and the R-symmetry completely.

Each embedding preserves the same four supercharges, irrespectively to where the branes are located on the two 2-spheres parameterized by (θ_1, φ_1) and (θ_2, φ_2). Thus we can smear the distribution and still preserve the same amount of supersymmetry. The 2-form charge distribution is readily obtained to be the same as the volume forms on the two 2-spheres in the geometry, and through the modified Bianchi identity it sources the flux F_1. Because of the symmetries, we look for a solution where all the functions have only radial dependence. Moreover we were careful in never breaking the $\mathbb{Z}_2$ symmetry that exchanges the two spheres. The natural ansatz is:

$$\begin{aligned} ds^2 = h(r)^{-1/2} dx_{1,3}^2 + h(r)^{1/2} \Big\{ dr^2 + \frac{e^{2g(r)}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) + \\ + \frac{e^{2f(r)}}{9} (d\psi - \sum_{i=1,2} \cos\theta_i\, d\varphi_i)^2 \Big\} \end{aligned} \tag{5.2.8}$$

$$\Phi = \Phi(r) \tag{5.2.9}$$

$$F_5 = \frac{h'(r)}{h(r)^{5/4}} \Big(- E^{0123r} + E^{\psi\theta_1\varphi_1\theta_2\varphi_2} \Big) \tag{5.2.10}$$

$$F_1 = \frac{N_f}{4\pi} \big(d\psi - \cos\theta_1\, d\varphi_1 - \cos\theta_2\, d\varphi_2 \big) = \frac{3N_f}{4\pi}\, h(r)^{-1/4} e^{-f(r)}\, E^\psi \tag{5.2.11}$$

$$dF_1 = \frac{N_f}{4\pi} \big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \big) \ , \tag{5.2.12}$$

where the unknown functions are $h(r)$, $g(r)$, $f(r)$ and $\Phi(r)$. The vielbein is:

$$\begin{aligned} E^\mu &= h^{-1/4}\, dx^\mu & E^{\theta_i} &= h^{1/4}\frac{e^g}{\sqrt{6}}\, d\theta_i \\ E^r &= h^{1/4}\, dr & & \\ E^\psi &= h^{1/4}\frac{e^f}{3}\,(d\psi - \sum_{i=1,2}\cos\theta_i\, d\varphi_i) & E^{\varphi_i} &= h^{1/4}\frac{e^g}{\sqrt{6}}\,\sin\theta_i\, d\varphi_i \end{aligned} \tag{5.2.13}$$

In appendix C.3 we have collected the derivation of the solution, in a way that makes the comparison with the unflavored Klebanov-Witten solution simple. Our technique is to enforce Bianchi identities and require that the solution preserve the same four ordinary supercharges preserved by the Klebanov-Witten solution and by the probe D7 brane embedding. We end up with a BPS system of first order differential equations, that we are able to solve.

We have explicitly checked that the Einstein, Maxwell and dilaton equations are solved. This can be done even before finding actual solutions of the BPS system. The first-order system (Bianchi identities plus vanishing variations of dilatino and gravitini) in fact implies the second order Einstein, Maxwell and dilaton differential equations. An analytic proof is given in [1]. In checking Einstein equations, we have used the following expression for the stress-energy tensor (5.1.7) of the distribution of flavor branes (in the coordinate basis):

$$\begin{aligned} 2\kappa_{10}^2\, T_{\mu\nu} &= -\frac{3N_f}{2\pi}h^{-1}e^{\Phi-2g}\,\eta_{\mu\nu} & 2\kappa_{10}^2\, T_{\theta_i\theta_i} &= -\frac{N_f}{8\pi}e^{\Phi} \\ 2\kappa_{10}^2\, T_{rr} &= -\frac{3N_f}{2\pi}e^{\Phi-2g} & 2\kappa_{10}^2\, T_{\varphi_i\varphi_i} &= -\frac{N_f}{24\pi}e^{\Phi-2g}\Big[4e^{2f}\cos^2\theta_i + 3e^{2g}\sin^2\theta_i\Big] \\ 2\kappa_{10}^2\, T_{\psi\psi} &= -\frac{N_f}{6\pi}e^{\Phi+2f-2g} & 2\kappa_{10}^2\, T_{\varphi_1\varphi_2} &= -\frac{N_f}{6\pi}e^{\Phi+2f-2g}\cos\theta_1\cos\theta_2\ . \\ 2\kappa_{10}^2\, T_{\psi\varphi_i} &= \frac{N_f}{6\pi}e^{\Phi+2f-2g}\cos\theta_i & & \end{aligned} \tag{5.2.14}$$

It is correctly linear in N_f. The Dirac-Born-Infeld equations for the D7 brane distribution are solved because of κ-symmetry (supersymmetry) on their world-volume.

The general solution, obtained after a change of radial coordinate $dr = e^f\, d\rho$, is:

$$\begin{aligned} e^\Phi &= \frac{4\pi}{3N_f}\frac{1}{\rho_{max}-\rho} \\ e^g &= C\,\Big[(1-6(\rho-\rho_{max}))\,e^{6(\rho-\rho_{max})} + c_1\Big]^{1/6} \\ e^f &= C\,\Big[-6(\rho-\rho_{max})e^{6(\rho-\rho_{max})}\Big]^{1/2}\Big[(1-6(\rho-\rho_{max}))\,e^{6(\rho-\rho_{max})} + c_1\Big]^{-1/3} \\ h &= -27\pi\alpha'^2 N_c\int_{\rho_{max}}^{\rho} d\rho'\, e^{-4g(\rho')} + c_2\ . \end{aligned} \tag{5.2.15}$$

In these new coordinates, the metric is

$$ds^2 = h^{-1/2}dx_{1,3}^2 + h^{1/2}\,e^{2f}\left\{d\rho^2 + \frac{1}{9}(d\psi - \sum_{i=1,2}\cos\theta_i\, d\varphi_i)^2 + \frac{e^{-2u}}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2)\right\}\ , \tag{5.2.16}$$

where

$$e^u \equiv e^{2(f-g)} = \frac{-6(\rho-\rho_{max})e^{6(\rho-\rho_{max})}}{(1-6(\rho-\rho_{max}))\, e^{6(\rho-\rho_{max})}+c_1} \; . \qquad (5.2.17)$$

The new radial coordinate ρ ranges from $-\infty$ (at most), corresponding to the IR of the dual gauge theory, to a maximal value ρ_{max} where the dilaton diverges; this will be later interpreted as dual to the UV Landau pole of the flavored gauge theory.

5.2.3 The unflavored limit

Naively, one could think about reabsorbing the integration constants ρ_{max} and C by a shift of the radial coordinate and a rescaling of the Minkowski coordinates respectively, so as to simplify the notation. However, this turns out not to be a good choice if we are interested in making contact with the unflavored Klebanov-Witten solution.

First of all, the Klebanov-Witten solution (5.1.1) has a constant dilaton, whereas in our solution the dilaton runs according to (5.2.15). In order to reduce our solution to the Klebanov-Witten background with constant dilaton $e^{\Phi} = g_s$, we conclude that the unflavored limit must involve a scaling limit $N_f \to 0$ with $N_f \rho_{max} \to \frac{4\pi}{3g_s}$. The value of ρ (dual to some energy scale) remains finite in this limit.[6] We will see in a moment that C has to scale properly too.

Secondly, continuity with the Klebanov-Witten solution requires also that c_1 and c_2 vanish. $c_1 = 0$ ensures that in this limit $e^u \to 0$, whereas $c_2 \to 0$ ensures that the warp factor vanish at infinity.

Finally, inserting $c_1 = 0$ we get the following behaviors for the squash factors f and g as $\rho_{max} \to \infty$:

$$e^f \approx e^g \approx C(6\rho_{max})^{1/6} e^{-\rho_{max}} e^{\rho}; \, . \qquad (5.2.18)$$

This suggests that the correct scaling for C in the unflavored limit is $C \approx (6\rho_{max})^{-1/6} e^{\rho_{max}}$.

Summarizing, the unflavored scaling limit is

$$N_f \to 0 \, , \qquad \text{with} \qquad N_f \rho_{max} \to \frac{4\pi}{3g_s} \quad \text{and} \quad C(6\rho_{max})^{1/6} e^{-\rho_{max}} \to r_0 \, , \qquad (5.2.19)$$

where r_0 is a reference scale with the dimension of a length.

After this limit is taken, we find $e^f = e^g = r_0 e^{\rho} = r$, and the warp factor $h(r) = \frac{27}{4}\pi\alpha'^2 N_c \frac{1}{r^4}$. The Klebanov-Witten solution has been recovered.

Keeping the previous subtlety in mind, from now on we will reabsorb ρ_{max} and C and set

[6] Adding massive flavors, it is possible to stop the running and get a conformal Klebanov-Witten solution in the IR, at energy scales below the mass. In that case, in the IR conformal range the conformal radius-energy relation can be applied, so that, up to small threshold effects, a fixed mass m of the flavors is dual to a fixed value ρ_0 of the radial coordinate, where the transition to the Klebanov-Witten behavior occurs. This value ρ_0 does not depend on the number of flavors. This reasoning shows that ρ must be kept fixed in the unflavored scaling limit.

$c_1 = c_2 = 0$, so that:

$$\begin{aligned}
e^{\Phi} &= -\frac{4\pi}{3N_f}\frac{1}{\rho} \\
e^{g} &= \left[(1-6\rho)\,e^{6\rho}\right]^{1/6} \\
e^{f} &= \left(-6\rho e^{6\rho}\right)^{1/2}\left[(1-6\rho)\,e^{6\rho}\right]^{-1/3} \\
e^{u} &= \frac{-6\rho e^{6\rho}}{(1-6\rho)\,e^{6\rho}} \\
h &= -27\pi\alpha'^2 N_c \int_0^{\rho} d\rho'\, e^{-4g(\rho')}\ .
\end{aligned} \tag{5.2.20}$$

In the original paper [1], a discussion of the holographic interpretation of c_1 and c_2 as vacuum expectation values of certain dimension 6 and 8 operators was provided. That interpretation is consistent with the analysis of the unflavored limit. The IR behavior of the solution dramatically changes if a nonvanishing value of c_1 is inserted. The behavior and the sign of the warp factor in the UV ($\rho \to 0^-$) instead depends on c_2. An analysis of such behaviors can be found in [1] as well. Here we will discuss only the solution having $c_1 = c_2 = 0$, which has a continuous unflavored limit to the Klebanov-Witten solution.

As a final remark, we notice that in the IR ($\rho \to -\infty$) the string coupling goes to zero. Note however that the solution could stop at a finite negative $\rho_{\min}$ due to integration constants or, for example, more dynamically, due to the presence of massive flavors.

5.2.4 Solution with general couplings

We can generalize our set of solutions by switching on nonvanishing VEV's for the bulk gauge potentials C_2 and B_2. This result can be achieved without modifying the previous set of equations, and the two parameters are present for every solution of them. The condition is that the gauge potentials are flat, that is with vanishing field strength. They correspond thus to (higher rank) Wilson lines for the corresponding bundles.

Let us switch on the following fields:

$$C_2 = c\,\omega_2^{CF} \qquad\qquad B_2 = b\,\omega_2^{CF}\ , \tag{5.2.21}$$

where the 2-form ω_2 (B.1.20) defined in appendix B.1 is proportional to the Poincaré dual of the 2-cycle S^2:

$$S^2:\ \{\theta_1 = \theta_2,\ \varphi_1 = 2\pi - \varphi_2,\ \psi = \text{const},\ r = \text{const}\} \tag{5.2.22}$$

$$\omega_2^{CF} = \frac{1}{2}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big)\ , \qquad\qquad \int_{S^2}\omega_2^{CF} = 4\pi\ . \tag{5.2.23}$$

We see that $F_3 = 0$ and $H_3 = 0$. So the supersymmetry variations are not modified, neither are the gauge invariant field strengths. In particular the BPS system does not change.

Consider the effects on the action (the argument is valid both for localized and smeared branes). It can be written as a bulk term plus the D7 brane terms:

$$S = S_{bulk} - \mu_7 \int d^8\xi\, e^{\Phi}\sqrt{-\det(\hat{g}_8 + \mathcal{F})} + \mu_7 \int \Big[\sum\nolimits_q \hat{C}_q \wedge e^{\mathcal{F}}\Big]_8\ , \tag{5.2.24}$$

with $\mathcal{F} = \hat{B}_2 + 2\pi\alpha'\, F_2$ is the gauge invariant field strength on the D7 brane worldvolume (see appendix A.2.1 for our conventions). To get solutions of the κ-symmetry conditions and of the equations of motion, we must take F_2 such that

$$\mathcal{F} = \hat{B}_2 + 2\pi\alpha'\, F_2 = 0\ . \tag{5.2.25}$$

Notice that there is a solution for F because B_2 is flat: $d\hat{B}_2 = \widehat{dB_2} = 0$. With this choice κ-symmetry is preserved as before, since it depends on the combination $\mathcal{F}$. The dilaton equation is fulfilled. The Bianchi identities and the bulk field strength equations of motion are not modified, since the WZ term only sources C_8. The energy momentum tensor is not modified, so the Einstein equations are fulfilled. The last steps are the equations of B_2 and A_1 (the gauge potential on the D7). For this notice that they can be written as:

$$d\frac{\delta S}{\delta F_2} = 2\pi\alpha'\, d\frac{\delta S_{brane}}{\delta \mathcal{F}} = 0 \tag{5.2.26}$$

$$\frac{\delta S}{\delta B_2} = \frac{\delta S_{bulk}}{\delta B_2} + \frac{\delta S_{brane}}{\delta \mathcal{F}} = 0\ . \tag{5.2.27}$$

The first is solved by $\mathcal{F} = 0$ since in the equation all the terms are linear or higher order in $\mathcal{F}$. This is because the brane action does not contain terms linear in $\mathcal{F}$, and this is true provided $C_6 = 0$ (which in turn is possible only if C_2 is flat). The second equation then reduces to $\frac{\delta S_{bulk}}{\delta B_2} = 0$, which amounts to $d(e^{-\phi} * H_3) = 0$ and is solved.

As we will see in section 5.2.6, being able to switch on arbitrary constant values c and b for the (flat) gauge potentials, we can freely tune the two gauge couplings (or better the two dynamically generated scales Λ's)[7] and the two theta angles, according to the dictionary (3.2.15-3.2.16). This turns out to break the $\mathbb{Z}_2$ symmetry that exchanges the two gauge groups, even if the breaking is mild and only affects C_2 and B_2, while the metric, the dilaton and all the field strengths continue to have that symmetry. The dual effect is an unbalance between the two gauge couplings, such that the relative gauge coupling does not run.

5.2.5 Analysis of the solution: asymptotics and singularities

We perform here a systematic analysis of the possible solutions of the BPS system, and study the asymptotics in the IR and in the UV. In this section we will make use of the following formulae for the Ricci scalar curvature, which can be obtained for solutions of the BPS system. In string frame

$$R^{(s)} = -2\,\frac{3N_f}{4\pi}\, h^{-1/2} e^{-2g+\frac{1}{2}\Phi} \left[7 + 4\frac{3N_f}{4\pi} e^{2g-2f+\Phi}\right]\ , \tag{5.2.28}$$

whereas in Einstein frame

$$R^{(E)} = \frac{3N_f}{4\pi}\, h^{-1/2} e^{-2f+\Phi} \left[4e^{2f-2g} + \frac{3N_f}{4\pi} e^{\Phi}\right]\ . \tag{5.2.29}$$

Although the warp factor $h(\rho)$ cannot be analytically for generic values of the integration constant c_1, it can be if $c_1 = 0$, as in the case we are interested in. Indeed, introducing the

[7] Actually, two combinations of the dynamically generated scales and the superpotential coupling.

incomplete gamma function, defined as follows:

$$\Gamma[a,x] \equiv \int_x^\infty t^{a-1}e^{-t}dt \xrightarrow[x\to-\infty]{} e^{i2\pi a}e^{-x}\left(\frac{1}{x}\right)^{1-a}\left\{1+\mathcal{O}\Big(\frac{1}{x}\Big)\right\}, \tag{5.2.30}$$

we can integrate

$$\begin{aligned} h(\rho) &= -27\pi N_c\alpha'^2\int d\rho\,\frac{e^{-4\rho}}{(1-6\rho)^{2/3}} = \\ &= \frac{9}{2}\pi N_c\alpha'^2(\frac{3}{2e^2})^{1/3}\Gamma[\frac{1}{3},-\frac{2}{3}+4\rho] \simeq \\ &\simeq \frac{27}{4}\pi N_c(-6\rho)^{-2/3}e^{-4\rho} \;\text{ for }\; \rho\to-\infty\,. \end{aligned} \tag{5.2.31}$$

The warp factor diverges for $\rho\to-\infty$. If we integrate the proper line element ds from a finite point to $\rho=-\infty$, we see that the throat has an *infinite invariant length.* This holds both in Einstein and in string frame.

The function $r(\rho)$ cannot be given as an analytic integral, but using the asymptotic behavior of e^f for $\rho\to-\infty$ we can approximately integrate it:

$$r(\rho)\simeq 6^{1/6}\Big[(-\rho)^{1/6}e^{\rho}+\frac{1}{6}\Gamma[\frac{1}{6},-\rho]\Big]+c_3 \tag{5.2.32}$$

in the IR. Fixing $r\to0$ when $\rho\to-\infty$ we set $c_3=0$. We approximate further on

$$r(\rho)\simeq(-6\rho)^{1/6}e^{\rho}\,. \tag{5.2.33}$$

Substituting r in the asymptotic behavior of the functions appearing in the metric, we find that up to logarithmic corrections of relative order $1/|\log(r)|$:

$$\begin{aligned} e^{g(r)} &\simeq e^{f(r)} \simeq r \\ h(r) &\simeq \frac{27\pi N_c}{4}\frac{1}{r^4}\,. \end{aligned} \tag{5.2.34}$$

Therefore the geometry in Einstein frame approaches $AdS_5\times T^{1,1}$ with logarithmic corrections in the IR limit $\rho\to-\infty$.

UV limit

The solution with backreacting flavors has a 'Landau pole' in the ultraviolet ($\rho\to0^-$), since the dilaton diverges (see (5.2.20)). The asymptotic behaviors of the functions appearing in the metric are:

$$e^{2g}\simeq 1-6\rho^2+\mathcal{O}(\rho^3) \tag{5.2.35}$$

$$e^{2f}\simeq -6\rho\Big[1+6\rho+\mathcal{O}(\rho^2)\Big] \tag{5.2.36}$$

$$h\simeq 27\pi N_c\alpha'^2\Big[-\rho-4\rho^3+\mathcal{O}(\rho^4)\Big]\,. \tag{5.2.37}$$

$h(\rho)$ is monotonically decreasing with ρ and goes to zero at the pole $\rho\to0^-$.

The curvature invariants, evaluated in string frame, diverge when $\rho\to0^-$, indicating that the supergravity description cannot be trusted in the UV. For instance the Ricci scalar is $R^{(s)}\simeq \frac{2}{9\pi}\left(\frac{N_f}{N_c}\right)^{1/2}(-\rho)^{-3}$. The scalar curvature in Einstein frame diverges too.

IR limit

The IR ($\rho \to -\infty$) limit of the geometry of the flavored solutions in Einstein frame is independent of the number of flavors, if we neglect logarithmic corrections to the leading term. Indeed, at the leading order, flavors decouple from the theory in the IR (see the discussion below eq. (5.2.4)). The counterpart in our supergravity plus branes solution is evident when we look at the BPS system (C.3.7),(C.3.15),(C.3.16),(C.3.10): when $\rho \to -\infty$ the e^{Φ} term disappears from the system, together with all the backreaction effects of the D7 branes on the geometry (see subsection 5.2.6 for a detailed analysis of this phenomena), therefore the system reduces to the unflavored one.

The asymptotics of the functions appearing in the metric in the IR limit $\rho \to -\infty$ are:

$$e^g \simeq e^f \simeq (-6\rho)^{1/6} e^{\rho} \tag{5.2.38}$$

$$h \simeq \frac{27}{4} \pi \alpha'^2 N_c (-6\rho)^{-2/3} e^{-4\rho} \; . \tag{5.2.39}$$

Formula (5.2.28) implies that the scalar curvature in string frame vanishes in the IR limit: $R^{(s)} \simeq -\frac{14}{3\pi}\left(\frac{N_f}{N_c}\right)^{1/2}(-\rho)^{-1/2} \to 0$. An analogous but lengthier formula for the square of the Ricci tensor gives

$$R^{(s)}_{MN} R^{(s)\,MN} \simeq \frac{160}{9\pi^2} \frac{N_f}{N_c} (-\rho) + \mathcal{O}(1) \;\to\; \infty \; , \tag{5.2.40}$$

thus the supergravity description presents a singularity and some care is needed when computing observables from it. The same quantities in Einstein frame have limiting behavior $R^{(E)} \simeq 8(27\pi\alpha'^2 N_c)/(-\rho) \to 0$ and $R^{(E)}_{MN} R^{(E)\,MN} \to 640/(27\pi N_c)$.

Using the criterion in [65] in its strong form, that proposes the IR singularity to be physically acceptable for the dual gauge theory if g_{tt} does not increase as we approach the IR problematic point, we observe that these singular geometries are all acceptable. Gauge theory physics can be read from these supergravity backgrounds. We call them "good singularities".

We remark here that it is possible to show that a region where all the curvature invariants in string units and the effective string coupling are small exists if N_f/N_c is sufficiently smaller than 1. For instance, if we want the curvature invariants to be smaller than 1/4 string units, N_f/N_c should be approximately 1/20 or less. This range becomes larger and larger as we reduce N_f/N_c.

5.2.6 Detailed study of the dual field theory

In this section we are going to undertake a detailed analysis of the dual gauge theory features, reproduced by the supergravity solution. The first issue we want to address is what is the effect of the smearing on the gauge theory dual.

As we wrote above, the addition to the supergravity solution of one stack of localized non-compact D7 branes at $z_1 = 0$ put in the field theory flavors coupled through a superpotential term

$$W = h\,\mathrm{Tr}(A_i B_k A_j B_l)\,\epsilon^{ij}\epsilon^{kl} + h_1\,\tilde{q}^a A_1 Q_a + h_2\,\tilde{Q}^a B_1 q_a \; , \tag{5.2.41}$$

where we explicitly wrote the flavor indices a. For this particular embedding the two branches are localized at $\theta_1 = 0$ and $\theta_2 = 0$ respectively on the two spheres. One can exhibit a lot of features in common with the supergravity plus D7 branes solution:

- the theory has $U(N_f) \times U(N_f)$ flavor symmetry (the diagonal axial $U(1)_A$ is anomalous), each group corresponding to one branch of D7's;
- putting only one branch there are gauge anomalies in QFT and a tadpole in SUGRA, while for two branches they cancel;
- adding a mass term for the fundamentals the flavor symmetry is broken to the diagonal $U(N_f)$, while in SUGRA there are embeddings moved away from the origin for which the two branches merge.

The $SU(2) \times SU(2)$ part of the isometry group of the background without D7 branes is broken by the presence of localized branes. It amounts to separate rotations of the two S^2 in the geometry and shifts the location of the branches. Its action is realized through the superpotential, and exploiting its action we can obtain the superpotential for D7 branes localized in other places. The two bifundamental doublets A_j and B_j transform as spinors of the respective $SU(2)$. So the flavor superpotential term for a configuration in which the two branches are located at x and y on the two spheres can be obtained by identifying two rotations that bring the north pole to x and y. There is of course a $U(1) \times U(1)$ ambiguity in this. Then we have to act with the corresponding $SU(2)$ matrices U_x and U_y on the vectors (A_1, A_2) and (B_1, B_2) (which transform in the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representations) respectively, and select the first vector component. In summary we can write[8]

$$W_f = h_1\, \tilde{q}^x \Big[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \Big]_1 Q_x + h_2\, \tilde{Q}^y \Big[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \Big]_1 q_y \,, \tag{5.2.42}$$

where the notation $\tilde{q}^x$, Q_x stands for the flavors coming from a first D7 branch being at x, and the same for a second D7 branch at y.

To understand the fate of the two phase ambiguities in the couplings h_1 and h_2, we appeal to symmetries. The $U(1)$ action which gives $(q, \tilde{q}, Q, \tilde{Q})$ charges $(1, -1, -1, 1)$ is a symmetry explicitly broken by the flavor superpotential. The freedom of redefining the flavor fields acting with this $U(1)$ can be exploited to reduce to the case in which the phase of the two holomorphic couplings is the same. The $U(1)$ action with charges $(1, 1, 1, 1)$ is anomalous with equal anomalies for both the gauge groups, and it can be used to absorb the phase ambiguity into a shift of the sum of Yang-Mills theta angles $\theta_1^{YM} + \theta_2^{YM}$ (while the difference holds steady). This is what happens for D7 branes on flat spacetime. The ambiguity we mentioned amounts to rotations of the transverse $\mathbb{R}^2$ space, whose only effect is a shift of C_0. As we show in the next section, the value of C_0 is our way of measuring the sum of theta angles through probe D(-1) branes. Notice that if we put in our setup many separate stacks of D7 branes, all their superpotential $U(1)$ ambiguities can be reabsorbed in a single shift of C_0.

From a physical point of view, the smearing corresponds to place the D7 branes at different points on the two spheres, distributing each branch on one of the 2-spheres. This is done homogeneously so that there is one D7 at every point of S^2. The nonanomalous flavor symmetry is broken from $U(1)_B \times SU(N_f)_R \times SU(N_f)_L$ (localized configuration) to $U(1)_B \times U(1)_V^{N_f - 1} \times U(1)_A^{N_f - 1}$ (smeared configuration).[9]

[8]In case the two gauge couplings and theta angles are equal, we could appeal to the $\mathbb{Z}_2$ symmetry that exchanges them to argue $|h_1| = |h_2|$, but no more because of the ambiguities.

[9]The axial $U(1)$ which gives charges $(1, 1, -1, -1)$ to one set of fields $(q_x, \tilde{q}^x, Q_x, \tilde{Q}^x)$ coming from a single D7 is

Let us introduce a pair of flavor indices (x, y) that naturally live on $S^2 \times S^2$ and specify the D7. The superpotential for the whole system of smeared D7 branes is just the sum (actually an integral) over the indices (x, y) of the previous contributions:

$$W = h\,\mathrm{Tr}(A_i B_k A_j B_l)\,\epsilon^{ij}\epsilon^{kl} + h_1 \int_{S^2} d^2x\, \tilde{q}^x \Big[U_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\Big]_1 Q_x + h_2 \int_{S^2} d^2y\, \tilde{Q}^y \Big[U_y \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\Big]_1 q_y\ . \quad (5.2.43)$$

Again, all the $U(1)$ ambiguities have been reabsorbed in field redefinitions and a global shift of $\theta_1^{YM} + \theta_2^{YM}$.

In this expression the $SU(2)_A \times SU(2)_B$ symmetry is manifest: rotations of the bulk fields A_j, B_j leave the superpotential invariant because they can be reabsorbed in rotations of the dummy indices (x, y). In fact, roughly speaking we can think of the action of $SU(2)_A \times SU(2)_B$ on the flavors as a subgroup of the broken $U(N_f) \times U(N_f)$ flavor symmetry. In the smeared configuration, there is a D7 brane at each point of the spheres and the group $SU(2)^2$ rotates all the D7 branes in a rigid way, moving each D7 where another was. So it is a flavor transformation contained in $U(N_f)^2$. By combining this action with a rotation of A_i and B_i, we get precisely the claimed symmetry.

Even if written in an involved fashion, the superpotential (5.2.43) does not spoil the main features of the gauge theory. In particular, the addition of a flavor mass term still would give rise to the symmetry breaking pattern

$$U(1)_B \times U(1)_V^{N_f-1} \times U(1)_A^{N_f-1} \quad \rightarrow \quad U(1)_V^{N_f}\ .$$

Gauge couplings and β-functions

In order to extract information on the gauge theory from the supergravity solution, we need to know the holographic relations between the gauge couplings, the theta angles and the supergravity fields. In general, finding these relations is a very difficult task.

These formulae can be properly derived only in orbifold theories, when string theory can be quantized, by considering fractional D3 branes placed at the singularity. For instance, in the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, this can be done honestly only for the conformal field theory with equal couplings. Symmetry arguments allow to extend this mapping also out of the so called 'orbifold point' where we are able to quantize string theory [62], so that the moduli space of marginal deformations of the field theory can be matched successfully with the subspace of the moduli space of string theory that leaves the form of the metric background invariant.

In the case of the conformal Klebanov-Witten theory, similar relations can be obtained although we are not able to quantize string theory on the conifold. The relations were guessed in [29] and later specified and motivated better in [61]. In reference [55], Strassler has shown what the orbifold relations, usually assumed to hold in the literature also for the conifold, actually miss in the conifold theory. The argument was explained at the end of section 3.2. The result (3.2.15)-(3.2.16) differs from its $\mathcal{N} = 2$ orbifold counterpart by corrections depending on the quartic superpotential coupling.

The extension of the conformal formulae to nonconformal cases, obtained either by adding fractional D3 branes or D7 branes, is problematic. A smart attempt for the Klebanov-Tseytlin/Strassler

an anomalous symmetry. For each D7 brane we consider, the anomaly amounts to a shift of the two theta angles of the gauge theory. We can combine this $U(1)$ with an axial rotation of all the flavor fields and get an anomaly free symmetry. Altogether, from N_f D7 branes we can find $N_f - 1$ such anomaly free axial $U(1)$ symmetries.

field theory can be found in [61]. Their attempt can be completed successfully, as exposed at the end of section 3.4.1, equations (3.4.6)-(3.4.7), in such a way that the dictionary holds in the UV region, mapping quantities with the same dimensions, charges and renormalization group flow properties. The holographic formulae differ from the conformal ones, and reduce to them only when the number M of fractional D3 branes added to the regular D3 branes vanish. Knowledge of the anomalous dimensions in the UV limit of the field theory is necessary to guess how to complete the proposal of [61] in such a way that the map is consistent with the RG flow properties.

An analogous extension from the conformal Klebanov-Witten theory to its flavored version that we are considering here could help our analysis. Unfortunately, such an extension is not known and is very difficult to guess. It is not difficult to build two RG invariant quantities out of the holomorphic dynamically generated scales of the gauge groups Λ_j and the superpotential couplings h and h_j, analogously to the L_1 and L_2 of section 3.2. The problem instead arises when trying to match the product and the ratio of these quantities with the exponential of the axio-dilaton and of the integrals of $C_2+\tau B_2$ on the 2-cycle, since the axio-dilaton is not constant anymore. Furthermore, since the UV of the flavor field theory involves a Landau pole, we lack field theory arguments to pinpoint the anomalous dimensions of the fields in such a regime and also an asymptotic radius-energy relation. Therefore we are not able to propose a dictionary that complies with the RG flow properties.

The best that we are able to do is to use the holographic relations (3.2.15-3.2.16) valid in the conformal Klebanov-Witten theory. As these relations change when fractional D3 branes are added to the setting, they could also change when D7 branes are added. However, we do not feel so uncomfortable: after all, all the analogous results in the Klebanov-Strassler cascading theory, matching to a very good qualitative level field theory expectations, were derived in the literature using the dictionary holding in the conformal $\mathcal{N}=2$ orbifold theory (see [49] for a review), which already in the Klebanov-Witten theory acquires corrections. Relations (3.2.15-3.2.16), valid in the Klebanov-Witten theory, can be rewritten as follows:

$$-2N\log|h| + \frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} = 2\pi e^{-\Phi} \tag{5.2.44}$$

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 2e^{-\Phi}\left[\frac{1}{2\pi\alpha'}\int_{S^2} B_2 - \pi \pmod{2\pi}\right] \tag{5.2.45}$$

$$2N\arg(h) + \theta_1^{YM} + \theta_2^{YM} = 2\pi C_0 \tag{5.2.46}$$

$$\theta_1^{YM} - \theta_2^{YM} = 2\frac{1}{2\pi\alpha'}\int_{S^2}(C_2 + C_0 B_2) - 2\pi C_0\ . \tag{5.2.47}$$

The ambiguity in (5.2.45) is the 2π periodicity of $\frac{1}{2\pi\alpha'}\int_{S_2} B_2$ which comes from the quantization condition on H_3. A shift of 2π amounts to move to a dual description of the gauge theory.

Let us now make contact with our supergravity solution. In the smeared solution, since $dF_1 \neq 0$ at every point, it is not possible to define a scalar potential C_0 such that $F_1 = dC_0$. We by-pass this problem by restricting our attention to the noncompact 4-cycle defined by $\{\rho, \psi, \theta_1 = \theta_2, \varphi_1 = 2\pi - \varphi_2\}$ (note that it wraps the R-symmetry direction ψ), so that we can pull-back on it and write

$$F_1^{eff} = \frac{N_f}{4\pi}\, d\psi \tag{5.2.48}$$

and therefore

$$C_0^{eff} = \frac{N_f}{4\pi}(\psi - \psi_0) \ . \tag{5.2.49}$$

Now we can identify:

$$\frac{8\pi^2}{g^2} - N \log|h| = \pi\, e^{-\phi} = -\frac{3N_f}{4}\rho \tag{5.2.50}$$

$$\theta_1^{YM} + \theta_2^{YM} + 2N \arg(h) = \frac{N_f}{2}(\psi - \psi_0) \ , \tag{5.2.51}$$

where we suppose for simplicity the two gauge couplings to be equal ($g_1 = g_2 \equiv g$). The generalization to an arbitrary constant B_2 is straightforward since the difference of the inverse squared gauge couplings does not run. Although, as discussed above, one cannot be sure of the validity of (5.2.50), we can try to extract some information.

Let us first compute the β-function of the gauge couplings. The identification (5.2.44) allows us to define a "radial" β-function that we can directly compute from supergravity [101]

$$\beta^{(\rho)}_{\frac{8\pi^2}{g^2}} \equiv \frac{\partial}{\partial\rho}\frac{8\pi^2}{g^2} = \pi\frac{\partial e^{-\phi}}{\partial\rho} = -\frac{3N_f}{4} \ , \tag{5.2.52}$$

which gives the same numerical result of the gauge theory computation eq. (5.2.4) for the β-function. The physical β-function defined in the field theory is of course:

$$\beta_{\frac{8\pi^2}{g^2}} \equiv \frac{\partial}{\partial \log\frac{\mu}{\Lambda}}\frac{8\pi^2}{g^2} \ , \tag{5.2.53}$$

where μ is the subtraction scale and Λ is a renormalization group invariant scale. In order to get the precise field theory β-function from the supergravity computation one needs the *energy-radius* relation $\rho = \rho\big(\frac{\mu}{\Lambda}\big)$, from which $\beta = \beta^{(\rho)}\,\partial\rho/\partial\log\frac{\mu}{\Lambda}$. In general, for nonconformal duals, the radius-energy relation depends on the phenomenon one is interested in and accounts for the scheme-dependence in the field theory. The coincidence between (5.2.52) and (5.2.4) shows that if a regime where the anomalous dimensions are those of the probe limit exists, then the radius-energy relation in that regime is $\rho = \log\frac{\mu}{\Lambda} + const.$.

Even without knowing the precise radius-energy relation all along the flow, there is some physical information that we can extract from the radial β-function (5.2.52). In particular, being the energy-radius relation $\rho = \rho\big(\frac{\mu}{\Lambda}\big)$ monotonically increasing, the signs of the two beta functions, the field theory one and the radial one, coincide.

In the section where we studied the IR asymptotics of our solution, we have found that the Einstein frame metric approaches the $AdS_5 \times T^{1,1}$ metric, with $r = r(\rho)$ of equation (5.2.33) playing the rôle of the usual radial coordinate of AdS_5. In that case, using equation (5.2.33) and the stretched string radius-energy relation $r = \mu/\Lambda$, one gets the previous radius-energy relation $\rho = \log\frac{\mu}{\Lambda} + const.$, up to logarithmic corrections.

R-symmetry anomaly and vacua

Now we move to the computation of the $U(1)_R$ anomaly. On the field theory side we follow the convention that the R-charge of the superspace Grassmann coordinates is $R[\vartheta] = 1$. This fixes the

R-charge of the gauginos $R[\lambda] = 1$. Let us consider an infinitesimal R-symmetry transformation and calculate the $U(1)_R - SU(N_c) - SU(N_c)$ triangle anomaly. The anomaly coefficient in front of the instanton density of a gauge group is $\sum_f R_f T[\mathcal{R}^{(f)}]$, where the sum runs over the fermions f, R_f is the R-charge of the fermion and $T[\mathcal{R}^{(f)}]$ is the Dynkin index of the gauge group representation $\mathcal{R}^{(f)}$ the fermion belongs to, normalized as $T[\mathcal{R}^{(fund.)}] = 1$ and $T[\mathcal{R}^{(adj.)}] = 2N_c$. Consequently the anomaly relation in our theory is the following:

$$\partial_\mu J^\mu_R = -\frac{N_f}{2} \frac{1}{32\pi^2} \left(F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a + G^a_{\mu\nu} \tilde{G}^{\mu\nu}_a \right) , \tag{5.2.54}$$

or in other words, the effect of a $U(1)_R$ transformation of parameter ε on the Lagrangian is equivalent to shifting the theta angles of both gauge groups as[10]

$$\theta^{YM}_i \to \theta^{YM}_i + \frac{N_f}{2} \varepsilon . \tag{5.2.55}$$

On the string/gravity side a $U(1)_R$ transformation of parameter ε is realized by the shift $\psi \to \psi + 2\varepsilon$. This can be derived from the transformation of the complex variables (B.1.1), which under a $U(1)_R$ rotation transform as $z_i \to e^{i\varepsilon} z_i$, or directly by the decomposition of the 10d spinor ϵ into 4d and 6d factors provided in appendices C.1 and C.3. By means of the dictionary (5.2.51) we obtain for the sum of the theta angles

$$\theta^{YM}_1 + \theta^{YM}_2 \to \theta^{YM}_1 + \theta^{YM}_2 + 2\, \frac{N_f}{2} \varepsilon , \tag{5.2.56}$$

in agreement with (5.2.55).

The $U(1)_R$ anomaly is responsible for the breaking of the symmetry group, but usually a discrete subgroup survives. Disjoint physically equivalent vacua, not connected by other continuous symmetries, can be distinguished thanks to the formation of domain walls among them, whose tension could also be measured. We want to read the discrete symmetry subgroup of $U(1)_R$ and the number of vacua both from field theory and supergravity. In field theory the $U(1)_R$ action has an extended periodicity (range of inequivalent parameters) $\varepsilon \in [0, 8\pi)$ instead of the usual 2π periodicity, because the minimal charge is $1/4$. Let us remark however that when ε is a multiple of 2π the transformation is not really an R-symmetry, since it commutes with supersymmetry. The global symmetry group contains the baryonic symmetry $U(1)_B$ as well, whose parameter we call $\alpha \in [0, 2\pi)$, and the two actions $U(1)_R$ and $U(1)_B$ satisfy the following relation: $\mathcal{U}_R(4\pi) = \mathcal{U}_B(\pi)$. Therefore the group manifold $U(1)_R \times U(1)_B$ is parameterized by $\varepsilon \in [0, 4\pi)$, $\alpha \in [0, 2\pi)$ (this parametrization realizes a nontrivial torus) and $U(1)_B$ is a true symmetry of the theory. The theta angle shift (5.2.55) allows us to conclude that the $U(1)_R$ anomaly breaks the symmetry according to $U(1)_R \times U(1)_B \to \mathbb{Z}_{N_f} \times U(1)_B$, where the latter is given by $\varepsilon = 4n\pi/N_f$ ($n = 0, 1, \dots, N_f - 1$), $\alpha \in [0, 2\pi)$.

Coming to the string side, the solution for the metric, the dilaton and the field strengths is invariant under arbitrary shifts of ψ. But the nontrivial profile of C_0, which can be probed for instance by D(-1) branes, breaks this symmetry. The presence of DBI actions in the functional integral tells us that the RR potentials are quantized, in particular C_0 is defined modulo integers.

[10] It is a common manipulation, following Seiberg, to assign inverse transformations properties to the theta angles, so that the Lagrangian remains invariant. In the case at hand, this amounts to setting $R[\Lambda_j^{N_c - N_f}] = -\frac{N_f}{2}$.

Taking formula (5.2.49) and using the periodicity 4π of ψ, we conclude that the true invariance of the solution is indeed $\mathbb{Z}_{N_f}$.

We end this analysis with some comments on the difference between the two theta angles (5.2.47). On the field theory side, the R-anomalies of the two gauge groups are the same. On the string side, a rotation in the $U(1)$ fiber parameterized by ψ only shifts C_0. Using formula (5.2.47), we obtain the expected result only in the special case $\frac{1}{2\pi\alpha'}\int_{S^2} B_2 = \pi$. The same problem arises also in the only other D3-D7 case in the literature where the anomaly has been computed, namely the $\mathcal{N} = 2$ $\mathbb{C} \times \mathbb{C}_2/\mathbb{Z}_2$ orbifold [88]. To our knowledge, the solution of this puzzle is not known. Even when conformality is broken by adding fractional D3 branes [36], similar subtleties related to the $C_0 B_2$ term in (5.2.47) appear (compare with [66]): if one adds a flat C_0, the strange phenomenon that theta angles as defined holographically change as we change the value of the radial coordinate r shows up. Since we know in field theory that anomalies do not renormalize, this could indicate that there are additional terms correcting the holographic relations and solving the puzzle (which, at least in that instance, seems not to be the case), or that we are forced to change the origin of the reference frame of the theta angles as we change r, or more probably we have to appeal to something else. We suggest that the solution of the puzzle could perhaps be achieved by substituting in (5.2.47) a quantized integral, related to the potentials that are used to compute Page charges.

Proposal for the RG flow

The supergravity solution allows us to extract the renormalization group flow of the KW field theory with massless flavors.

Let us first analyze the UV regime of the theory. It is dominated by flavors, whose addition to the conformal theory makes the gauge couplings increase with the energy. At a finite energy scale dual to ρ_{max}, that we conventionally fixed to $\rho = 0$, the gauge theory develops a Landau pole, as confirmed by the string coupling that diverges at that particular radius. This energy scale is finite, because $\rho = 0$ is at finite proper distance from the bulk points $\rho < 0$.

At the Landau pole radius the supergravity description breaks down for many reasons: the string coupling diverges as well as the curvature invariants (both in Einstein and string frame), and the ψ circle shrinks. It is conceivable that string theory may provide a UV completion for this field theory, and finding it is an interesting problem. One could think about obtaining a new description in terms of supergravity plus branes through various dualities. In particular, T-duality could map our solution to a system of NS5, D4 and D6 branes, which could then be uplifted to M theory. Anyway, T-duality has to be applied with care because of the presence of D branes on a nontrivial background, and we actually do not know how to T-dualize the Dirac-Born-Infeld action. Alternative, one could look instead for an F theory lift of the solution. This is an interesting open problem.

We stress again that in our setting the UV singularity is not a problem at all: it is actually expected and needed to describe the Landau pole of the gauge theory. Gauge/gravity duality holds also for theories with a Landau pole, as discussed in subsection 4.2.2. If string theory provides a UV completion of such a field theory, then a globally valid solution may be found. At any rate, although very interesting, this would only shed light on the UV completion that removes the field theory singularity.

We now want to propose an interpretation of the whole RG flow described by our flavored

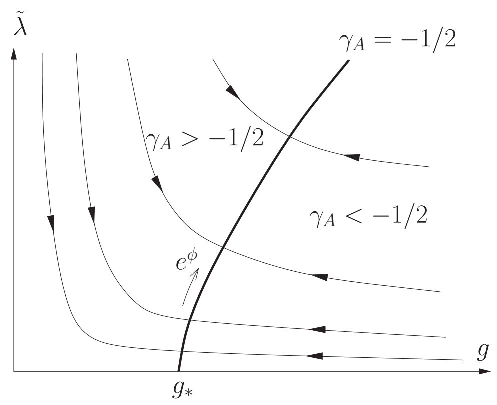


Figure 5.4: RG flow phase space for the Klebanov-Witten model.

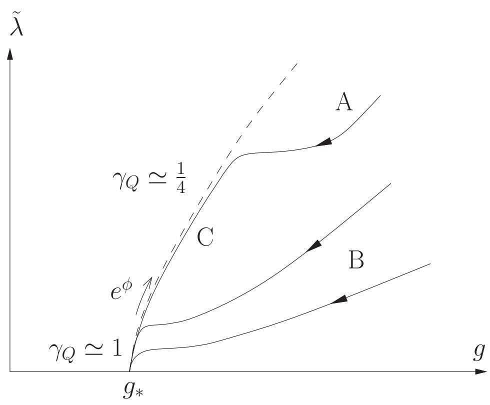


Figure 5.5: Klebanov-Witten model with flavors. The A-C flow has backreacting D7 branes in the A piece and then follows the KW line in the C piece; it corresponds to $N_f \ll N_c$. The B flow is always far from the KW line, and corresponds to $N_f \gtrsim N_c$.

solution, and in particular of its IR dynamics. It mainly arises from the analysis of the Klebanov-Witten model at small values of the string coupling, and it is based on the nonvalidity of the orbifold relations for all values of the parameters in the KW model, that was extensively pointed out in [55]. The correct formulae that we will use for the conifold theory are (5.2.44)-(5.2.45), which include corrections due to the quartic superpotential h with respect to the $\mathcal{N} = 2$ theory. In the following analysis, we will consider for the sake of clarity only the case of equal gauge couplings $g_1 = g_2 \equiv g$.

The curve of conformal points in the Klebanov-Witten model is obtained by requiring the anomalous dimension of the fields A, B to be $\gamma_A(g, \tilde{\lambda}) = -1/2$, which assures $\beta_g = \beta_{\tilde{\lambda}} = 0$, where $\tilde{\lambda} = h\mu$ is the dimensionless coupling constant of the quartic superpotential (μ is the energy scale). The qualitative shape of the curve is depicted in Figure 5.4, as well as some possible RG flows. The important feature is that there is a minimum value $g_* > 0$ that fixed points can have (due to the perturbative β_g being negative, so that $g = 0$ is an unstable IR point). One way to determine this curve of fixed points is to apply the a-maximization procedure originally spelled in [102] by using Lagrange multipliers enforcing the marginality constraints [103], and then express the Lagrange multipliers in terms of the gauge and superpotential couplings. This computation for the Klebanov-Witten model was done in [104]. One can show that the curve of fixed points does not pass through the origin of the space of Lagrange multipliers, which is mapped into the origin of the space of couplings (free theory). In a particular scheme the curve of fixed points is an arc of hyperbola with the major axis along $\tilde{\lambda} = 0$. The exact shape of the curve is scheme-dependent, due to scheme-dependence of the relation between Lagrange multipliers and couplings: we choose a scheme in which the Lagrange multipliers are quadratic in the couplings. This choice fixes a conical section, and it is such a hyperbola because the one-loop anomalous dimensions of the chiral superfields get a negative contribution from gauge interactions and a positive contribution from superpotential interactions. The conclusion that the curve of conformal points does not pass through the origin of the space of coupling constants is physical.

The family of Klebanov-Witten supergravity solutions describes the fixed curve. It is parameterized by e^Φ that can take arbitrary values. For sufficiently large values of it, we can neglect the dependence on the quartic coupling and trust the orbifold formula:

$$\frac{g^2}{8\pi} = e^\Phi \qquad \text{for} \qquad e^\Phi N_c \gtrsim 1 \, . \tag{5.2.57}$$

The 't Hooft coupling $g^2 N_c$ is large (at least of order 1, so the theory is strongly coupled and the anomalous dimensions are of order 1) and the string frame curvature $R_S \sim 1/(e^\Phi N_c)$ is small. For smaller values $e^\Phi N_c \lesssim 1$, (5.2.57) cannot be correct: it would give small 't Hooft coupling while the gauge dynamics is instead strongly coupled. The answer is in (5.2.44): the quartic coupling is very small, whereas the gauge coupling are not. The bottom end of the line corresponds to:

$$\{e^\Phi \to 0\} \qquad \leftrightarrow \qquad \{g = g_*, \tilde{\lambda} = 0\} \, , \tag{5.2.58}$$

and the curvature of the supergravity solution is large even if the field theory is still strongly coupled. Anyway some quantities, for instance the quantum dimension of A, B, are protected and do not depend on the coupling, so they can be computed in supergravity even for small values of $e^\Phi N_c$. The duality between the Klebanov-Witten CFT and the $AdS_5 \times T^{1,1}$ is still conjectured to hold, at the level of string theory.

The supergravity solution of our system with D7 branes is in the IR quite similar to the KW geometry: the IR asymptotic background is $AdS_5 \times T^{1,1}$ (with corrections), but with running dilaton. The gravitational coupling of the D7 branes goes to zero and the flavor branes tend to decouple. The signature of this is in equation (C.3.16) of the BPS system: the quantity $e^\Phi N_f$ can be thought of as the effective size of the flavor backreaction, which indeed vanishes in the far IR. The upshot is that flavors can be considered as an irrelevant deformation of the $AdS_5 \times T^{1,1}$ geometry (but not of the dilaton).

The field theory is thus deduced to be close to KW fixed line, but running along it as $e^\Phi \to 0$ in the IR. As soon as $e^\Phi N_f \lesssim 1$, flavor branes begin to behave as probes. In this regime, we expect the quantities computable from the background to be equal to the KW model ones: in particular $\gamma_A = -1/2$.

We can distinguish different regimes, starting from the UV to the IR. Depending on the values of N_c and N_f they can be either well separated or not present at all. A section of the space of couplings and some RG flows are drawn in Figure 5.5, but one should include the third orthogonal direction h_i which is not plotted.

- For $1 < e^\Phi$ we are in the Landau pole regime, and the dilaton (string coupling e^Φ) is large.
- For $\frac{1}{N_f} < e^\Phi < 1$ we are in a complicated piece of the flow, quite far from the KW fixed line, as in the type A-B flows of Figure 5.5. In particular the D7 branes are largely backreacting. In this regime our supergravity solution is perfectly behaved (as long as $\frac{1}{N_c} < e^\Phi$).
- For $\frac{1}{N_c} < e^\Phi < \frac{1}{N_f}$ (this regime exists for $N_f < N_c$) we are in a region with almost probe D7 branes,[11] so we are close to the KW line, but with large 't Hooft coupling, so that we can

[11] The dual in field theory of the D7 branes being probes is that Feynman graphs with flavors in the loops are suppressed with respect to gauge fields in the loops, since $N_f < N_c$.

trust (5.2.57). Furthermore, in this regime (if it exists) all the curvature invariants are small in string units, so that we can trust our solution. We can expect the energy/radius relation to be quite similar to the conformal ones, thus we can compute the gauge β-function and deduce the flavor anomalous dimensions γ_Q. Apart from corrections, we get:

$$\gamma_A \simeq -\frac{1}{2} \qquad R_A \simeq \frac{1}{2} \qquad\qquad \gamma_Q \simeq \frac{1}{4} \qquad R_Q \simeq \frac{3}{4}\,. \tag{5.2.59}$$

The R-symmetry is classically preserved but anomalous as in supergravity. The various β-functions are computed to be

$$\beta_g = \frac{3}{4} N_f \frac{g^3}{16\pi^2} \qquad\qquad \beta_{\tilde{\lambda}} \simeq 0 \qquad\qquad \beta_h \simeq 0\,. \tag{5.2.60}$$

We want to stress that this regime is *not* conformal, and in fact the theory flows along the KW fixed line, as in the type C flow of Figure 5.5. The smaller is N_f/N_c, the longer is this piece of the flow. For $N_f \gtrsim N_c$ this regime does not exist, and the theory probably follows the type B flows of Figure 5.5, although we cannot really trust our solution anywhere because curvatures are always large.

- For $e^\phi < \min(\frac{1}{N_c}, \frac{1}{N_f})$ we are close to the end of the KW fixed line, and the gauge coupling is close to g_*. Again the D7 branes are almost probes. The string frame curvature is large, as in the KW model at small $g_s N_c$. Since the gauge coupling cannot go below g_*, its β-function vanishes even if the string coupling continues flowing to zero. We get in field theory the following anomalous and quantum dimensions for the fundamental and bifundamental fields:

$$\gamma_A \simeq -\frac{1}{2} \qquad \Delta[A] =\simeq \frac{3}{4} \qquad\qquad \gamma_Q \simeq 1 \qquad \Delta[Q] \simeq \frac{3}{2} \tag{5.2.61}$$

$$\beta_g \simeq 0 \qquad\qquad \beta_{\tilde{\lambda}} \simeq 0 \qquad\qquad \beta_{h_i} = \frac{3}{4} h_i\,, \tag{5.2.62}$$

using the following exact formulae for the β-functions:

$$\beta_{\frac{8\pi^2}{g_i^2}} = 3N_c - 2N_c(1-\gamma_A) - N_f(1-\gamma_Q) \tag{5.2.63}$$

$$\beta_{\tilde{\lambda}} = (1+2\gamma_A)\tilde{\lambda} \tag{5.2.64}$$

$$\beta_{h_i} = \frac{1}{2}(\gamma_A + 2\gamma_Q) h_i\,. \tag{5.2.65}$$

γ_Q are the anomalous dimensions of the (anti)fundamental flavor chiral superfields, whereas γ_A are the anomalous dimensions of the bifundamental chiral superfields.

All the flows accumulate at the point $\{g = g_*, \tilde{\lambda} = 0\}$ of Figure 5.5, but the theory is *not* conformal. In fact the coupling h_i always flows to smaller values, and the theory moves "orthogonal" to the figure.

- The end of the flow is the superconformal point with $h_i = 0$ (and $g = g_*$), which should correspond to $e^\Phi = 0$ and cannot be described by supergravity. Without the cubic superpotential one can construct a new anomaly free R-symmetry with $R_Q = 1$, by combining the previous one ($R_A = 1/2$, $R_Q = 3/4$), which is natural in the probe limit, with the anomalous axial

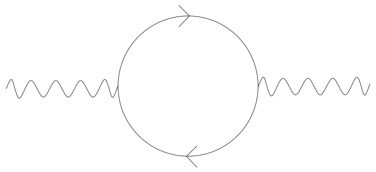


Figure 5.6: Flavor 1-loop correction to the gauge propagator.

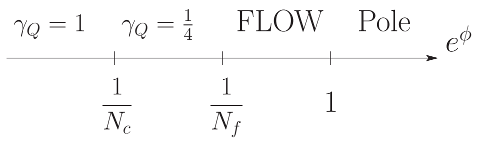


Figure 5.7: Regimes of KW with flavors for $N_f < N_c$.

symmetry assigning charge 1/4 to each flavor superfield. This $U(1)$ R-symmetry is the one that enters the conserved current multiplet of the superconformal algebra. Moreover, the fact that $h_i \to 0$ in the far infrared realizes in field theory the incapability of resolving the D7 brane separation at small energies, and the flavor symmetry $S(U(N_f) \times U(N_f))$ is restored.

Note that when $N_f \gtrsim N_c$ and the D7 branes are probes (this is the regime $e^{\Phi} < \frac{1}{N_f} < \frac{1}{N_c}$ and $g = g_*$) one could think hard to see in field theory a suppression of graphs with flavors in the loops, with respect to gauge fields in the loops. Consider the gauge propagator at 1-loop with flavors (Figure 5.6). It is of order $g_*^2 N_f$, not suppressed with respect to the graph with gauge fields in the loop of order $g_*^2 N_c$. But if we sum all the loops with flavors, we must obtain the flavor contribution to the β-function, which for $g \simeq g_*$ and so $\gamma_Q \simeq 1$ is indeed very small.

A summary of the phase space for $N_f < N_c$ is in Figure 5.7. The computation of the β-functions performed in [95] using the anomalous dimensions for the conformal theory in the quenched limit is valid in the region $\frac{1}{N_c} < e^{\Phi} < \frac{1}{N_f}$ of the phase space, where the flavor branes can actually be treated as small perturbations of the conformal Klebanov-Witten background.

5.3 Generalizations

5.3.1 Kuperstein's embeddings

Until now, as in the original paper [1], we have considered smearing Ouyang's embeddings [95], which are of the kind $z_1 = 0$. With the aim of understanding some general features of the smearing procedure and for later convenience, we believe that it is useful at this point to discuss the smearing of different supersymmetric embeddings, such as those studied by Kuperstein in [96]. The embeddings considered by Kuperstein are the holomorphic embeddings of the type $w_i = 0$ for some w_i, where w's are the coordinates in terms of which the conifold equation in $\mathbb{C}^4$ is written as

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \, . \tag{5.3.1}$$

They are related to the coordinates previously used as $Z = \frac{i}{\sqrt{2}} \sigma^m w_m$, with $(\sigma^m) = (\vec{\sigma}, i\mathbb{1})$ (see appendix B.1). If we concentrate for instance on the embedding $w_2 = 0$, it can be rewritten as $z_3 - z_4 = 0$. An embedding of this kind preserves supersymmetry because of holomorphy of the equation. Indeed, κ-symmetry on the D7 brane worldvolume is realized with a flat gauge connection [96]. This embedding breaks the $SO(4)$ symmetry of the conifold to an $SO(3)$ subgroup, and preserves $U(1)_R$ symmetry.

In terms of the angular coordinates introduced in (B.1.5-B.1.8), this embedding is defined as: $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\forall\psi, \forall\tau$. We can obtain other embeddings with the same properties by acting on it with the broken generators. The charge distribution obtained by homogeneously spreading the D7 branes in this class is the same as for the smeared Ouyang's embeddings:

$$\Omega = \frac{N_f}{4\pi}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big) \ , \tag{5.3.2}$$

where N_f is the total number of D7 branes. This form is fixed by symmetries.

The kind of flavor fields that are introduced by these embeddings and their interactions among themselves and with the bifundamental fields of the Klebanov-Witten theory will be discussed in detail in the next chapter, where we consider the addition of this kind of flavors to the Klebanov-Strassler theory.[12] For the time being, suffices to say that there are actually two kinds of D7 branes along these embeddings, the difference between them being whether or not there is a worldvolume flux on the exceptional 2-cycle living at the conifold singularity, which these D7 branes wrap. These are really fractional D7 branes. N_f is the sum of the fractional D7 branes of one kind, which couple to one gauge group, and of the fractional D7 brane of the other kind, which couple to the other gauge group. They provide nonchiral flavors for both gauge groups, coupled to the bifundamental fields through a quartic superpotential and also self-interacting through a quartic superpotential.

What matters for the present discussion is to underline that the field content and interactions are very different from those arising from Ouyang's embedding, that we have considered so far. Despite these differences, the effect of smearing is that of restoring all the symmetries of the unflavored theory, at least at the classical level, and this fixes completely the charge distribution of the D7 branes. Therefore, the background that is generated by this distribution is the same as the one generated by the smeared distribution of Ouyang's D7 branes. We are led to conclude that the effect of the two distributions of flavor branes (Ouyang's D7 branes and Kuperstein's D7 branes) on the gauge dynamics is the same.[13]

This points to some strange universality of the behavior of gauge theories with 'smeared' flavors in Veneziano's limit and at strong coupling. Consideration of localized flavor brane embeddings preserving different symmetries would *a priori* lead to different results. But we cannot exclude that the gauge dynamics could be qualitatively similar with the insertion of different flavors. It would be very interesting to obtain a clear-cut understanding of whether the bulk of this universality lies in the large 't Hooft coupling and large N Veneziano's limit or it is only a consequence of the smearing procedure. An analogous analysis to that of the previous subsection can be worked out as plausibly for the Klebanov-Witten theory with Kuperstein's flavors, with the same results of the RG flows of the right graph in Figure 5.5: a UV Landau pole, followed by a complicated flow, then by a flow almost along the surface of fixed points of the unflavored Klebanov-Witten theory, until its bottom is reached and finally the gauge theory flows to a strongly coupled fixed point with vanishing superpotential. This analysis seems to hint that the bulk of the universality is in fact due to the large 't Hooft coupling and large N Veneziano's limit.

Notice that, on the contrary, the impact of gauge dynamics over the flavor fields is different for distinct kinds of flavors. For instance, it is fairly clear that the spectrum of mesons, which

[12] Adding Ouyang's flavors to the Klebanov-Strassler theory is more complicated because, since the pullback of the NSNS field strength H_3 on the D7 branes worldvolumes does not vanish, a nontrivial gauge field has to be added on the worldvolume so as to preserve supersymmetry.

[13] Here by gauge dynamics we actually mean the dynamics of the fields of the conformal theory, dual to closed strings.

can be obtained by computing fluctuations of the D7 brane embeddings, will be quite different for Ouyang's and Kuperstein's flavors.

5.3.2 Massive flavors

In the ansatz we have been using up to now we have assumed that the density of RR charge of the D7 branes is independent of the holographic coordinate. This is, of course, what occurs for a flavor brane configuration which corresponds to massless quarks. On the contrary, in the massive quark case, a supersymmetric D7 brane has a nontrivial profile in the radial direction [94] and, in particular ends at some nonzero value of the radial coordinate. These massive embeddings have free parameters which could be used to smear the D7 branes in the transverse directions. It is natural to think that the corresponding charge and mass distribution of the smeared flavor branes will depend on the radial coordinate in a nontrivial way.

It turns out that there is a simple modification of our ansatz for F_1 which gives rise to a charge and mass distribution with the characteristics required to represent smeared flavor branes with massive quarks. Indeed, let us simply substitute in the expression for F_1 in (5.1.8) the constant N_f by a function $N_f(r)$. In this case:

$$\begin{aligned} F_1 &= \frac{3N_f(r)}{4\pi}(d\psi - \sum_{i=1,2}\cos\theta_i d\varphi_i) \\ dF_1 &= \frac{3N_f(r)}{4\pi}\sum_{i=1,2}\sin\theta_i d\theta_i \wedge d\varphi_i + \frac{3N_f'(r)}{4\pi}dr \wedge (d\psi - \sum_{i=1,2}\cos\theta_i d\varphi_i)\ . \end{aligned} \tag{5.3.3}$$

Notice that the supersymmetry analysis of appendix C.3 remains unchanged since only F_1, and not its derivative, appears in the supersymmetric variations of the dilatino and gravitino. The final result is just the same system of first order BPS equations (C.3.7,C.3.10,C.3.15,C.3.16), where now one has to understand that $N_f(r)$ is a prescribed function of r, which encodes the nontrivial profile of the D7 brane. Notice that $N_f(r)$ determines the running of the dilaton which, in turn, affects the other functions of the ansatz.

It is not hard to prove that the modified BPS system solves automatically the equation of motion for the dilaton and the Einstein equations, provided that the functions $N_f(r)$ and $N_f'(r)$ are nonnegative. These two very weak conditions are certainly met by any smeared configuration of D7 branes introducing massive flavor branes. The reader interested in the technicalities can find them in [1].

The function $N_f(r)$ is not really so arbitrary. Given a choice of masses, it can (in principle) be computed. However, we have a lot of freedom, because an arbitrary distribution of flavor masses can be chosen consistently. Any such arbitrary distribution will meet the requirements $N_f(r), N_f'(r) \geq 0$.

Finally, in order to get a qualitative understanding of the physics when all the flavors have a fixed mass m, let us consider the approximation where the function $N_f(r)$ has a Heaviside shape, starting to be nonzero at some finite value ρ_m of the radial coordinate. In that case the BPS equations and solutions will be the ones of massless flavors discussed in this chapter for values of the radial coordinate larger than ρ_m. Below that radial value, the solution will be the one of Klebanov-Witten, with a constant dilaton. Aside from decoupling in the field theory, this is clearly indicating that the addition of massive flavors should "resolve" the IR singularity: the curvature in the IR should be small provided that the value of the running dilaton at the threshold scale $g_s^{IR} = e^{\Phi(\rho_m)} = -\frac{4\pi}{3N_f}\frac{1}{\rho_m}$ is such that $g_s^{IR} N_c = -\frac{4\pi}{3\rho_m}\frac{N_c}{N_f} \gg 1$. Physically this behavior is expected and makes these massive flavors interesting. A study of the solution dual to backreacting massive flavors has been recently pursued in [105], confirming and refining the qualitative results of the Heaviside approximation. We finally comment on the validity of this approximation. It is very good as soon as the logarithm of the energy is sufficiently far from the logarithm of the mass. Close to the mass scale, there are threshold effects that this approximation misses, and the energy range where these effects are relevant scale logarithmically as we change the mass. One could naively think that the approximation is better and better as the mass of the flavors goes to zero (in which case $N_f(r)$ becomes a constant). This is true and false at the same time. It is true if we compare energies in a linear scale, but it is false if, perhaps more appropriately, we compare them in a logarithmic scale. Indeed, the IR field theory is scale invariant, so that its cutoff m (together with its threshold effects) can be scaled by scaling invariance without affecting the qualitative picture. The range of the threshold effects in logarithmic scale does not change as m is changed.

5.3.3 Generalization to Sasaki-Einstein spaces

The results discussed in this chapter can be extended to the much wider class of gauge theories living on D3 branes at conical Calabi-Yau singularities, as explained at length in [1]. The interested reader can find in that reference all the technical details. Here we only explain the rationale.

In deriving the solution for backreacting symmetrically distributed flavor D7 branes in $AdS_5 \times T^{1,1}$, we have really used only the structure of $T^{1,1}$ as a 5-dimensional Sasaki-Einstein space, namely a 4-dimensional Kähler-Einstein base (in the $T^{1,1}$ case it is $S^2 \times S^2$), endowed with a local $U(1)$ fibration over it.[14] The connection of the fiber in Sasaki-Einstein spaces obeys the property that its curvature (field strength) is proportional to the Kähler form of the Kähler-Einstein space. Let us consider the addition of massless flavors, that classically preserve the $U(1)_R$ symmetry, to the D3 brane gauge theory. This implies that the embedding of the flavor D7 branes extends along the whole radial direction and the $U(1)_R$ ψ-direction. Supersymmetric embeddings for the D7 branes (without gauge field flux on the worldvolume) may be found either by imposing holomorphy of the embedding equations or κ-symmetry. In general, they will span the four Minkowski directions, the radial direction and the ψ direction, and finally a 2-dimensional submanifold of the 4-dimensional Kähler-Einstein space, breaking the global non-R-symmetries of the gauge theory (except for the baryonic symmetries). Finding a solution in the presence of localized backreacting D7 branes is a formidable task, which has been pursued only in some very special cases, like flat space or orbifolds thereof, and often in quite an implicit way. If instead we take advantage of the symmetries of the unflavored theory and of the number of flavor branes going to infinity, we can again consider a special distribution of flavor branes that in the $N_f \to \infty$ (and continuum) limit classically preserves all the global symmetries of the unflavored theory. In that case, the derivation that we worked out for $T^{1,1}$ goes through for a general Sasaki-Einstein space. The supersymmetry projections on the Killing spinors have the same form as those imposed in the conifold, but are written in the complex vielbein basis suitable for a generic conical Calabi-Yau. Two projectors are related to the Kähler-Einstein base, and one is related to the $\mathbb{C}^*$ fiber. The magnetic charge distribution of the

[14] We have written 'local' meaning that the $U(1)$ fibration need not be globally defined. In many cases it is really an $\mathbb{R}$ fibration. Since the corresponding isometry is dual to the R-symmetry of the gauge theory, the fibration is $\mathbb{R}$ when the R-charges of the fields are incommensurable.

flavor branes is proportional to the Kähler form of the Kähler-Einstein base, for symmetry reasons. This fixes F_1 to be proportional to the $U(1)_R$ fiber $d\psi - A$, where A is the connection, defined on the Kähler-Einstein space. The proportionality factor is fixed by the number of flavor branes. The dilaton is therefore fixed to have the same dependence we obtained in the flavored Klebanov-Witten solution, because of holomorphy/supersymmetry. The five-form RR field strength (or the warp factor) is obtained by requiring fulfilment of its Bianchi identity. Finally, the metric ansatz has again the same structure considered in the conifold case, with two squash factors, one in front of the Kähler-Einstein part and one in front of the $U(1)_R$ fiber. The supersymmetry conditions and the Bianchi identities lead to a BPS system of the same form as the one considered in the case of the conifold, and can be solved analogously.

The only subtlety arises when one wants to check that the equations of motions are solved. For the Einstein equations and the dilaton equations, one needs the action of the smeared distribution of flavor branes. Whilst in the Wess-Zumino action only the volume form of the Kähler-Einstein base appears as the effect of smearing, what appears in the Dirac-Born-Infeld part is actually a sum of the moduli of the decomposable pieces in this volume form.[15] In [1], we provided a coordinate invariant definition of the action of the smeared Dirac-Born-Infeld action of the flavored D7 branes, and showed that it is the correct action by exploiting supersymmetry, which relates the mass and charge distributions.

We end this short section by commenting on the surprising result of this analysis: the same structure of BPS equations repeats for all $AdS_5 \times X_5$ manifolds, provided X_5 is Sasaki-Einstein. This clearly points to some universality of the behavior of 4-dimensional $\mathcal{N} = 1$ superconformal gauge theories with flavors at strong coupling. The effect of flavors on the gauge dynamics has the universal property of introducing an ultraviolet Landau pole in the gauge theory. The IR seems to be described by a strongly coupled conformal fixed point, where the flavor fields acquire large anomalous dimensions that make their superpotential coupling to bifundamental fields irrelevant and cancels their contributions to the running of gauge couplings. This proposal for the IR has to be taken with care however, since the curvature of the dual supergravity background is very large and the correct treatment would involve the full string theory. Understanding in detail what is the universality that produces the same dynamics for a large class of $\mathcal{N} = 1$ gauge theories with flavors would be very interesting.

[15] A form is decomposable if it can be written as the exterior product of one-forms.

Chapter 6

Unquenched flavors in the Klebanov-Strassler theory

In this chapter, we study the addition of backreacting noncompact D7 branes to the Klebanov-Tseytlin and Klebanov-Strassler geometries. The resulting backgrounds are dual to flavored versions of the cascading Klebanov-Strassler field theory to leading order of Veneziano's large N_c expansion. Even after the addition of dynamical flavors, the renormalization group flow is described in the ultraviolet by a cascade of Seiberg dualities. We will see that flavors change the ultraviolet behavior of the theory and the way ranks decrease in the cascade. The two backgrounds we will exhibit are dual to RG flows with two different IR behaviors, which are selected according to the initial conditions (in the ultraviolet) for the ranks. The flavored Klebanov-Strassler theory is an important step toward finding a gravity dual of $\mathcal{N}=1$ Super-QCD in Veneziano's large N_c limit.

This chapter is a refinement of [2], written by the present author in collaboration with Francesco Benini, Felipe Canoura, Carlos Núñez and Alfonso V. Ramallo.

6.1 Introduction and summary

Early ideas of 't Hooft [8] and the experimental evidence for stringy behavior in hadronic physics (see chapter 1 of this thesis) suggested that aspects of the strong interaction can be described, predicted, and understood using a (not yet known) string theory. These ideas started to materialize when the Maldacena conjecture about AdS/CFT duality (see chapter 2) was formulated. A four-dimensional field theory was shown to contain strings that captured nonperturbative and perturbative physics. The downside was that the field theory in question ($\mathcal{N}=4$ SYM) was not of immediate relevance to hadronic physics. The necessity of finding extensions of these ideas to phenomenologically more interesting field theories was then well motivated. In chapter 3, we have reviewed the very fruitful extension of the AdS/CFT correspondence which stemmed from studying branes at conical singularities, of which the case of D3 branes on conifolds is a particular example presenting especially rich dynamics.

In this chapter, endowed with the knowledge the flavored Klebanov-Witten theory, we will tackle the problem of adding flavor degrees of freedom to the Klebanov-Tseytlin (KT) and Klebanov-Strassler (KS) solutions of sections 3.3 and 3.4. According to the proposal of Karch and Katz, these new excitations will be incorporated in the form of noncompact D7 flavor branes, corresponding to fundamental matter in the dual field theory. Unlike the color branes, which disappear in the geometric transition and are substituted by closed string fluxes, the flavor branes appear explicitly in our solutions: they correspond to the open strings which are suggested by the topological expansion of large N gauge theories and that appearing in the leading order of Veneziano's expansion. The addition of flavors to these field theories was first considered in [79,95,96]. We will follow the general idea of chapter 4, but instead of working in the probe brane approximation of section 4.2.1 we will consider the case in which the number of fundamental fields scales in the same way as the number of color fields in the large N limit, that is $N_f \sim N_c$. This means that the new (strongly coupled) dynamics of the field theory is captured by a background that includes the backreaction of the flavor branes. In order to find the new solutions, we follow the ideas and techniques of [1, 85, 92, 107], exploiting in particular the smearing procedure explained in the previous chapter.

We will present analytic solutions for the equations of motion of type IIB supergravity coupled to the DBI+WZ action of the flavor D7 branes that preserve minimal supersymmetry in four dimensions; we show how to reduce these solutions to those found by Klebanov-Tseytlin/Strassler when the number of flavors is taken to zero. With these solutions at hand, we make a precise matching between the field theory cascade (that, enriched by the presence of the fundamentals, is still self-similar) and the string predictions. We will also match anomalies and beta functions by using our new supergravity background.

The organization of this chapter goes as follows. In section 6.2 we present the setup, the ansatz and the strategy to find supersymmetric solutions of the Bianchi identities and equations of motion. We also introduce the notion of the so-called Page charges and we compute their values in our particular ansatz. In sections 6.3 and 6.4 we present two main solutions, which reflect the addition of flavors to the Klebanov-Strassler and Klebanov-Tseytlin backgrounds respectively. In section 6.5 we present the dual field theory and propose that its RG flow can be understood in terms of a cascade of Seiberg dualities. In section 6.6 we show that the duality cascade is encoded in our supergravity solutions, by comparing ranks of the groups in field theory with effective charges in supergravity, and matching R-anomalies and β-functions of gauge couplings on both sides of the gauge/gravity duality. The behavior of the background in the UV of the gauge theory suggests that the field theory generates a 'duality wall'. We also provide a nice translation of the effect of Seiberg duality in the cascading gauge theory as the effect of a large gauge transformation on the supergravity background. In section 6.7, we add a final remark on the flavor groups of our cascading solutions. We close the chapter illustrating the relations between RG flows of different gauge theories living on branes at a conifold singularity in section 6.8.

6.2 The setup and the ansatz

We are interested in adding to the KT/KS cascading gauge theory a number of flavors comparable with the number of colors, by means of N_f D7 branes representing the flavor mesonic degrees of freedom of the dual gauge theory. The dynamics of these branes is governed by the corresponding Dirac-Born-Infeld and Wess-Zumino actions. The solution we are after, which encodes the back-reaction of the flavor branes, will have a nontrivial metric and dilaton Φ and, as in any cascading background, nonvanishing RR 3- and 5-forms F_3 and F_5, as well as a nontrivial NSNS 3-form H_3. In addition, D7 branes act as a magnetic source for the RR 1-form F_1 through the WZ coupling:

$$S_{WZ}^{D7} = \mu_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 + \cdots , \tag{6.2.1}$$

which induces a violation of the Bianchi identity $dF_1 = 0$. Therefore our configuration will also necessarily have a nonvanishing value of F_1, as the flavored Klebanov-Witten solution of the previous chapter. The ansatz we shall adopt for the Einstein frame metric[1] is the following:

$$\begin{aligned} ds^2 = h(r)^{-\frac{1}{2}}\, dx_{1,3}^2 + h(r)^{\frac{1}{2}} \Big[dr^2 + e^{2G_1(r)} \, \left(u_1^2 + u_2^2\right) + \\ + e^{2G_2(r)} \Big(\left(w_1 + g(r) u_1\right)^2 + \left(w_2 + g(r) u_2\right)^2 \Big) + \frac{e^{2G_3(r)}}{9} \, \zeta^2 \Big] , \end{aligned} \tag{6.2.2}$$

where $dx_{1,3}^2$ denotes the four-dimensional Minkowski metric, u_i and w_i ($i = 1, 2$) are real 1-forms that can be written in terms of the angular coordinates or of the 1-forms (B.1.12) as follows

$$u_1 + i u_2 = d\theta_1 - i \sin\theta_1 d\varphi_1 = e^{i\psi/2} \left(\sigma_1 - i\sigma_2\right) \tag{6.2.3}$$

$$w_1 + i w_2 = e^{i\psi} (d\theta_2 + i \sin\theta_2 d\varphi_2) = e^{i\psi/2} \left(\Sigma_1 + i\Sigma_2\right) , \tag{6.2.4}$$

and finally $\zeta = d\psi - \sum_{i=1,2} \cos\theta_i d\varphi_i$. Notice that our metric ansatz (6.2.2) depends on five unknown radial functions $G_i(r)$ ($i = 1, 2, 3$), $g(r)$ and $h(r)$. The ansatz for F_5 has the standard form required by supersymmetry, namely

$$F_5 = (1 + *) \, d^4x \wedge dh^{-1}(r) . \tag{6.2.5}$$

The flavor D7 branes will be extended along the four Minkowski coordinates as well as an internal noncompact four-dimensional manifold. The κ-symmetric embedding of the D7 branes we start from will be discussed in section 6.5. In order to simplify the computations, following the approach of [1], we will smear the D7 branes in their two transverse directions in such a way that the symmetries of the unflavored background are recovered. As explained in [1], this smearing amounts to the following generalization of the WZ term of the D7 brane action:

$$S_{WZ}^{D7} = \mu_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 + \cdots \qquad \rightarrow \qquad \mu_7 \int_{\mathcal{M}_{10}} \Omega_2 \wedge C_8 + \cdots , \tag{6.2.6}$$

where Ω is a 2-form which determines the distribution of the RR charge of the D7 brane and $\mathcal{M}_{10}$ is the full ten-dimensional manifold. Notice that Ω_2 acts as a magnetic charge source for F_1 which generates a violation of its Bianchi identity. Actually, from the equation of motion of C_8 one gets:

$$dF_1 = \Omega_2 . \tag{6.2.7}$$

In what follows we will assume that the flavors introduced by the D7 brane are massless, which is equivalent to require that the flavor brane worldvolume reaches the origin in the holographic

[1] As in the previous chapter, we will use the Einstein frame defined in (A.2.5), which makes sense in the presence of D7 branes.

direction. Under this condition the D7 brane charge density is radial coordinate independent. Moreover, the D7 brane embeddings that we will smear imply that Ω_2 is symmetric under the exchange of the two S^2's parameterized by (θ_1, φ_1) and (θ_2, φ_2), and independent of ψ (see section 6.5). The smeared charge density distribution is the one already adopted in the previous chapter, namely:

$$dF_1 = \frac{N_f}{4\pi} (\sin\theta_1 \, d\theta_1 \wedge d\varphi_1 + \sin\theta_2 \, d\theta_2 \wedge d\varphi_2) = \frac{N_f}{4\pi} (-u_1 \wedge u_2 + w_1 \wedge w_2) \, , \tag{6.2.8}$$

where the coefficient $N_f/(4\pi)$ is determined by normalization. With this ansatz for Ω_2 the modified Bianchi identity (6.2.7) determines the value of F_1, namely:

$$F_1 = \frac{N_f}{4\pi} \zeta \, . \tag{6.2.9}$$

The ansatz for the RR and NSNS 3-forms that we propose is an extension of the one given by Klebanov and Strassler and it is simply:

$$\begin{aligned}
B_2 &= -\frac{M\alpha'}{2} \Big[f \, g^1 \wedge g^2 + k \, g^3 \wedge g^4 \Big] \\
H_3 = dB_2 &= -\frac{M\alpha'}{2} \Big[dr \wedge (f' \, g^1 \wedge g^2 + k' \, g^3 \wedge g^4) + \frac{1}{2}(k - f) \, \zeta \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \Big] \\
F_3 &= \frac{M\alpha'}{2} \Big\{ \zeta \wedge \Big[\big(F + \frac{N_f}{4\pi} f\big) g^1 \wedge g^2 + \big(1 - F + \frac{N_f}{4\pi} k\big) g^3 \wedge g^4 \Big] + F' dr \wedge \big(g^1 \wedge g^3 + g^2 \wedge g^4\big) \Big\}
\end{aligned} \tag{6.2.10}$$

where M is a constant, $f(r)$, $k(r)$ and $F(r)$ are functions of the radial coordinate, and the g^i's are the set of 1-forms defined in (B.2.9), that we rewrite here in the form

$$\begin{aligned}
g^1 &= \frac{-u_2 + w_2}{\sqrt{2}} & \qquad g^3 &= -\frac{u_2 + w_2}{\sqrt{2}} \\
g^2 &= \frac{u_1 - w_1}{\sqrt{2}} & \qquad g^4 &= \frac{u_1 + w_1}{\sqrt{2}} \, .
\end{aligned} \tag{6.2.11}$$

The forms F_3, H_3 and F_5 must satisfy the following set of Bianchi identities:

$$dF_3 = -H_3 \wedge F_1 \, , \qquad dH_3 = 0 \, , \qquad dF_5 = -H_3 \wedge F_3 \, . \tag{6.2.12}$$

Notice that the equations for F_3 and H_3 are automatically satisfied by our ansatz (6.2.10). Instead, the Bianchi identity for F_5 gives rise to the following differential equation:

$$\frac{d}{dr} \Big[h' \, e^{2G_1 + 2G_2 + G_3} \Big] = -\frac{3}{4} M^2 \alpha'^2 \Big[(1 - F + \frac{N_f}{4\pi} k) f' + (F + \frac{N_f}{\pi} f) k' + (k - f) F' \Big] \, , \tag{6.2.13}$$

which can be integrated, with the result:

$$h' \, e^{2G_1 + 2G_2 + G_3} = -\frac{3}{4} M^2 \alpha'^2 \Big[f - (f - k) F + \frac{N_f}{4\pi} f k \Big] + \text{constant} \, . \tag{6.2.14}$$

Let us now parameterize F_5 as

$$F_5 = \frac{\pi}{4} \alpha'^2 \, N_{eff}(r) \, \zeta \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 + \text{Hodge dual} \, , \tag{6.2.15}$$

and let us define the five-manifold $\mathcal{M}_5$ as the one that is obtained by taking the Minkowski coordinates and r fixed to a constant value.[2] As $\int_{\mathcal{M}_5} F_5 = -(4\pi^2\alpha')^2\, N_{eff}(r)$, according to (A.2.34), it follows that $N_{eff}(r)$ can be interpreted as an effective D3 brane charge at the value r of the holographic coordinate. From our ansatz (6.2.5), it follows that:

$$N_{eff}(r) \,=\, -\frac{4}{3\pi\alpha'^2}\, h'\, e^{2G_1+2G_2+G_3}\;, \tag{6.2.16}$$

and taking into account (6.2.14), we can write

$$N_{eff}(r) \equiv -\frac{1}{(4\pi^2\alpha')^2}\int_{\mathcal{M}_5} F_5 = N_0 + \frac{M^2}{\pi}\left[\, f - (f-k)F + \frac{N_f}{4\pi} fk\right]\,, \tag{6.2.17}$$

where N_0 is a constant. It follows from (6.2.17) that the RR five-form F_5 is determined once the radial functions F, f and k that parameterize the three-forms are known. Moreover, eq. (6.2.14) allows to compute the warp factor once the functions G_i and the three-forms are determined. Notice also that the effective D5 brane charge is obtained, according to (A.2.35), by integrating the gauge invariant field strength F_3 over the 3-cycle S^3: $\theta_2 = \text{const.}$, $\varphi_2 = \text{const.}$. The result is:

$$M_{eff}(r) \equiv -\frac{1}{4\pi^2\alpha'}\int_{S^3} F_3 = M\Big[1 + \frac{N_f}{4\pi}(f+k)\Big]\,. \tag{6.2.18}$$

The strategy to proceed further is to look at the conditions imposed by supersymmetry. We will smear, as in the previous chapter, κ-symmetric D7 brane embeddings. Therefore, the supersymmetry requirement is equivalent to the vanishing of the variations of the dilatino and gravitino of type IIB supergravity under supersymmetry transformations. These conditions give rise to a large number of BPS first order ordinary differential equations for the dilaton and the different functions that parameterize the metric and the forms. In the end, one can check that the first order differential equations imposed by supersymmetry and Bianchi identities imply the second order differential equations of motion. In particular, from the variation of the dilatino we get the following differential equation for the dilaton:

$$\phi' \,=\, \frac{3N_f}{4\pi}\, e^{\phi-G_3}\;. \tag{6.2.19}$$

A detailed analysis of the conditions imposed by supersymmetry shows that the fibering function g in formula (6.2.2) is subject to the following algebraic constraint:

$$g\Big[g^2 - 1 + e^{2(G_1-G_2)}\Big] = 0\;, \tag{6.2.20}$$

which has obviously two solutions. The first of these solutions is $g = 0$ and, as it is clear from our metric ansatz (6.2.2), it corresponds to the cases of the flavored version of the warped singular and resolved conifolds. In the second solution g is such that the term in brackets on the right-hand side of (6.2.20) vanishes. This solution gives rise to the flavored version of the warped deformed conifold. The flavored KT solution will be presented in section 6.4, whereas the flavored KS solution will be analyzed in section 6.3.

[2] In order to define the Hodge dual, we have to specify an ordering. We choose an ordering such that the orientation of the frame (as a $\mathbb{Z}_2$ valued object) is the same as the one previously considered in the conical case. The orientation of $\mathcal{M}_5$ is therefore chosen to be the same that we picked for $T^{1,1}$. We remark that $\zeta\wedge\gamma^1\wedge g^2\wedge g^3\wedge g^4 = -d\psi\wedge d\theta_1\wedge\sin\theta_1 d\varphi_1\wedge d\theta_1\wedge\sin\theta_1 d\varphi_1 = -108\, d\mathrm{vol}_{T^{1,1}}$.

6.2.1 Maxwell and Page charges

Before presenting the explicit solutions for the metric and the forms of the supergravity equations, let us discuss the different charges carried out by our solutions. In theories, like type IIB supergravity, that have Chern-Simons terms in the action (which give rise to modified Bianchi identities), it is possible to define more than one notion of charge associated with a given gauge field. Let us discuss here, following the presentation of reference [108], two particular definitions of this quantity, namely the so called Maxwell and Page charges [109], whose definitions can be found also in appendix A.2.1. Given a gauge invariant field strength F_{8-p}, up to a p dependent sign the (magnetic) Maxwell current associated to it is defined through the following relation:

$$d\,F_{8-p} \,=\, \star j_{D_p}^{Maxwell}\ , \tag{6.2.21}$$

or equivalently, the Maxwell charge in a volume V_{9-p} is given by:

$$Q_{D_p}^{Maxwell} \sim \int_{V_{9-p}} \star j_{D_p}^{Maxwell}\ , \tag{6.2.22}$$

with a suitable normalization. Taking $\partial V_{9-p} = M_{8-p}$ and using (6.2.21) and Stokes theorem, we can rewrite the previous expression as:

$$Q_{D_p}^{Maxwell} \sim \int_{M_{8-p}} F_{8-p}\ . \tag{6.2.23}$$

This notion of current is gauge invariant and conserved and it has other properties that are discussed in [108]. In particular, it is not "localized" in the sense that for a solution of pure supergravity (for which $d\,F_{8-p} = -H_3 \wedge F_{6-p}$) this current does not vanish. These are the kind of charges we have calculated so far (6.2.17)-(6.2.18), namely:

$$\begin{aligned} Q_{D5}^{Maxwell} &= M_{eff} = -\frac{1}{4\pi^2\alpha'}\int F_3\ ,\\ Q_{D3}^{Maxwell} &= N_{eff} = -\frac{1}{(4\pi^2\alpha')^2}\int F_5\ . \end{aligned} \tag{6.2.24}$$

An important issue regarding these charges is that, in general, they are not quantized. Indeed, we have seen explicitly in (6.2.18) and (6.2.17) that $Q_{D5}^{Maxwell} = M_{eff}$ and $Q_{D3}^{Maxwell} = N_{eff}$ vary continuously with the holographic variable r.

Let us move on to the notion of Page charge. The idea is first to write the Bianchi identities for F_3 and F_5 as the vanishing of the exterior derivatives of some differential form, which in general will not be gauge invariant; then we define it to be the Page current. Using the polyform notation of Appendices A.2 and A.2.1, we can define (up to signs), the polyform $*j^{Page}$ (the formal sum of $*j_{Dp}^{Page}$ components of definite ranks, where here $*$ is the Hodge star) as $d(e^{B_2}\wedge F) = *j^{Page}$. In our case, we can define the following (magnetic) Page currents:

$$\begin{aligned} &d(F_3 + B_2\wedge F_1) = -\star j_{D5}^{Page}\\ &d(F_5 + B_2\wedge F_3 + \frac{1}{2}B_2\wedge B_2\wedge F_1) = -\star j_{D3}^{Page}\ . \end{aligned} \tag{6.2.25}$$

The Page charges Q_{D5}^{Page} and Q_{D3}^{Page} are just defined as the integrals of $\star j_{D5}^{Page}$ and $\star j_{D3}^{Page}$ with the appropriate normalization, *i.e.*:

$$\begin{aligned} Q_{D5}^{Page} &= -\frac{1}{4\pi^2\alpha'}\int_{V_4}\star j_{D5}^{Page} \\ Q_{D3}^{Page} &= -\frac{1}{(4\pi^2\alpha')^2}\int_{V_6}\star j_{D3}^{Page} \ , \end{aligned} \tag{6.2.26}$$

where V_4 and V_6 are submanifolds in the transverse space to the D5 and D3 branes respectively, which enclose the branes. Actually, we can use Page charges also when branes have transmuted to fluxes via a geometric transition. In that case we define V_4 and V_6 by their boundaries, in such a way that the D branes were enclosed before transmuting into fluxes. In this case, Page charges are carried by fluxes. By using the expressions of the currents $\star j_{D5}^{Page}$ and $\star j_{D3}^{Page}$ given in (6.2.25), and by applying Stokes theorem, we get for our geometries:

$$\begin{aligned} Q_{D5}^{Page} &= -\frac{1}{4\pi^2\alpha'}\int_{S^3}\Big(F_3 + B_2\wedge F_1\Big) \\ Q_{D3}^{Page} &= -\frac{1}{(4\pi^2\alpha')^2}\int_{\mathcal{M}_5}\Big(F_5 + B_2\wedge F_3 + \frac{1}{2}B_2\wedge B_2\wedge F_1\Big) \ , \end{aligned} \tag{6.2.27}$$

where S^3 and $\mathcal{M}_5$ are the same manifolds used to compute the Maxwell charges in equations. (6.2.18) and (6.2.17). It is not difficult to establish the topological nature of these Page charges. Outside sources, Hodge duals of magnetic Page charges are closed. In compact notation: $d(*j^{Page}) = d\left(e^{B_2}\wedge F\right) = H_3\wedge e^{B_2}\wedge F + e^{B_2}\wedge(-H_3\wedge F_3) = 0$, where we have used the Bianchi identities or equations of motion of RR field strengths collected in appendix (A.2) Locally we can write $e^{B_2}\wedge F = d\left(e^{B_2}\wedge C\right)$, where C is the polyform of RR potentials. Therefore, if the RR potentials C were globally defined on the integration manifolds, Page charges would vanish identically as a consequence of Stokes theorem. But they do not if the corresponding components of $e^{B_2}\wedge C$ of definite rank are instead topologically nontrivial and need to be patched, as happens for the monopole number.

Due to the topological nature of the Page charges defined above, one naturally expects that they are quantized and, as we shall shortly verify, they are independent of the holographic coordinate. This shows that they are the natural objects to compare with the numbers of branes that create the geometry in backgrounds with varying flux. However, as it is manifest from the fact that Q_{D5}^{Page} and Q_{D3}^{Page} are given in (6.2.27) in terms of the B_2 field and not in terms of its field strength H_3, Page charges are not gauge invariant under large gauge transformations. They are instead invariant under small gauge transformations. Consider a small gauge transformation under which $\delta B_2 = d\Lambda_1$ and RR improved field strengths F are invariant: outside sources the gauge variation $\delta(e_2^B\wedge F) = d\Lambda_1\wedge e^{B_2}\wedge F = d\left(\Lambda_1\wedge e^{B_2}\wedge F\right)$ is exact (provided there are no NS5 branes making B_2 not globally defined), so that Page charges do not change under small gauge transformations. In subsection 6.6.2 we will relate this noninvariance to the Seiberg duality of the field theory dual.

Let us now calculate the associated Page charges for our ansatz (6.2.10) . We shall start by computing the D5 brane Page charge for the 3-sphere S^3 defined by $\theta_2, \varphi_2 =$ constant. We already know the value of the D5 brane Maxwell charge (6.2.18), which gives precisely M_{eff}. Taking into account that

$$\int_{S^3}\zeta\wedge g^1\wedge g^2 = \int_{S^3}\zeta\wedge g^3\wedge g^4 = -\int_{S^3}\omega_3^{CF} = -8\pi^2 \ , \tag{6.2.28}$$

we readily get:

$$-\frac{1}{4\pi^2}\int_{S^3} B_2\wedge F_1 = -\frac{MN_f}{4\pi}(f+k)\;, \tag{6.2.29}$$

and therefore:

$$Q_{D5}^{Page} = M_{eff} - \frac{MN_f}{4\pi}(f+k)\;. \tag{6.2.30}$$

Using the expression of M_{eff} given in (6.2.18), we obtain:

$$Q_{D5}^{Page} = M\;, \tag{6.2.31}$$

which is certainly quantized and independent of the radial coordinate.

Let us now look at the D3 brane Page charge, which can be computed as an integral over the angular manifold M_5. Taking into account that

$$\int_{\mathcal{M}_5} g^1\wedge g^2\wedge g^3\wedge g^4\wedge g^5 = -(4\pi)^3\;, \tag{6.2.32}$$

we get that, for our ansatz (6.2.10):

$$\begin{aligned} &-\frac{1}{(4\pi^2\alpha')^2}\int_{\mathcal{M}_5} B_2\wedge F_3 = -\frac{M^2}{\pi}\left[f-(f-k)F+\frac{N_f}{2\pi}fk\right] \\ &-\frac{1}{(4\pi^2\alpha')^2}\int_{\mathcal{M}_5}\frac{1}{2}B_2\wedge B_2\wedge F_1 = \frac{M^2}{\pi}\frac{N_f}{4\pi}fk\;, \end{aligned} \tag{6.2.33}$$

and thus

$$Q_{D3}^{Page} = N_{eff} - \frac{M^2}{\pi}\left[f-(f-k)F+\frac{N_f}{4\pi}fk\right]\;, \tag{6.2.34}$$

and, using the expression of N_{eff} (6.2.17), we obtain

$$Q_{D3}^{Page} \;=\; N_0\;, \tag{6.2.35}$$

which is again independent of the holographic coordinate. Recall that these Page charges are not gauge invariant and we will study in subsection 6.6.2 how they change under a large gauge transformation.

We now proceed to present the solutions to the BPS equations of motion.

6.3 Flavored warped deformed conifold

In this section we consider the following solution of the algebraic constraint (6.2.20):

$$g^2 = 1 - e^{2(G_1-G_2)}\;. \tag{6.3.1}$$

In order to write the equations for the metric and dilaton in this case, we perform the following change of coordinate:

$$3\,e^{-G_3}\,dr = d\tau\;. \tag{6.3.2}$$

In terms of this new variable, the differential equation for the dilaton is simply:

$$\dot{\Phi} = \frac{N_f}{4\pi}\, e^{\Phi} \,, \tag{6.3.3}$$

where the dot means derivative with respect to τ. This equation can be straightforwardly integrated, namely:

$$\frac{N_f}{4\pi}\, e^{\Phi} = \frac{1}{\tau_0 - \tau} \,, \qquad\qquad 0 \le \tau \le \tau_0 \,, \tag{6.3.4}$$

where τ_0 is an integration constant. The equations imposed by supersymmetry on the metric functions G_1, G_2 and G_3 are:

$$\begin{aligned} \dot{G}_1 \,-\, \frac{1}{18} e^{2G_3 - G_1 - G_2} \,-\, \frac{1}{2} e^{G_2 - G_1} \,+\, \frac{1}{2} e^{G_1 - G_2} \,&=\, 0 \\ \dot{G}_2 \,-\, \frac{1}{18} e^{2G_3 - G_1 - G_2} \,+\, \frac{1}{2} e^{G_2 - G_1} \,-\, \frac{1}{2} e^{G_1 - G_2} \,&=\, 0 \\ \dot{G}_3 \,+\, \frac{1}{9} e^{2G_3 - G_1 - G_2} \,-\, e^{G_2 - G_1} \,+\, \frac{N_f}{8\pi} e^{\Phi} \,&=\, 0 \,. \end{aligned} \tag{6.3.5}$$

In order to write the solution of this system of first order equations, we define the following function

$$\Lambda(\tau) \equiv \frac{\Big[2(\tau - \tau_0)(\tau - \sinh 2\tau) + \cosh(2\tau) - 2\tau\tau_0 - 1 \Big]^{\frac{1}{3}}}{\sinh\tau} \,. \tag{6.3.6}$$

Then, the metric functions G_i are given by:

$$\begin{aligned} e^{2G_1} &= \frac{1}{4}\, \mu^{\frac{4}{3}}\, \frac{\sinh^2\tau}{\cosh\tau}\, \Lambda(\tau) \\ e^{2G_2} &= \frac{1}{4}\, \mu^{\frac{4}{3}}\, \cosh\tau\, \Lambda(\tau) \\ e^{2G_3} &= 6\, \mu^{\frac{4}{3}}\, \frac{\tau_0 - \tau}{\big[\Lambda(\tau)\big]^2} \,, \end{aligned} \tag{6.3.7}$$

where μ is an integration constant. Notice that the range of the τ variable chosen in (6.3.4) is the one that makes the dilaton and the metric functions real. Moreover, for the solution we have found, the fibering function g is given by

$$g = \frac{1}{\cosh\tau} \,, \tag{6.3.8}$$

as in the deformed conifold. Using this result, we can write the ten-dimensional metric as:

$$ds^2 = h(\tau)^{-\frac{1}{2}}\, dx_{1,3}^2 + h(\tau)^{\frac{1}{2}}\, ds_6^2 \,, \tag{6.3.9}$$

where ds_6^2 is the metric of the 'flavored' deformed conifold, namely

$$\begin{aligned} ds_6^2 = \frac{1}{2}\, \mu^{\frac{4}{3}}\, \Lambda(\tau) \Bigg[& \frac{4(\tau_0 - \tau)}{3\Lambda^3(\tau)} \left(d\tau^2 + (g^5)^2 \right) + \cosh^2\left(\frac{\tau}{2}\right) \Big((g^3)^2 + (g^4)^2 \Big) + \\ & + \sinh^2\left(\frac{\tau}{2}\right) \Big((g^1)^2 + (g^2)^2 \Big) \Bigg] \,. \end{aligned} \tag{6.3.10}$$

Notice the similarity between the metric (6.3.10) and the one corresponding to the 'unflavored' deformed conifold (3.4.9). To further analyze this similarity, let us study the $N_f \to 0$ limit of our solution. By looking at the expression of the dilaton in (6.3.4), one realizes that this limit is only sensible if one also sends $\tau_0 \to +\infty$ with $N_f\tau_0$ fixed.[3] Indeed, by performing this scaling and neglecting τ versus τ_0, one gets a constant value for the dilaton. Moreover, the function $\Lambda(\tau)$ reduces in this limit to $\Lambda(\tau) \approx (4\tau_0)^{\frac{1}{3}}\, K(\tau)$, where $K(\tau)$ is the function appearing in the metric of the deformed conifold, namely:

$$K(\tau) = \frac{\left[\sinh 2\tau - 2\tau\right]^{\frac{1}{3}}}{2^{\frac{1}{3}}\sinh\tau} \,. \tag{6.3.11}$$

Using this result it is straightforward to verify that the metric (6.3.10) reduces to the metric of the deformed conifold (3.4.9) with $\epsilon = (4\tau_0)^{1/4}\mu$. In the unflavored limit, μ has to scale so that $\epsilon = (4\tau_0)^{1/4}\mu$ stays finite.

The requirement of supersymmetry imposes the following differential equations for the functions k, f and F appearing in the fluxes of our ansatz:

$$\begin{aligned} \dot{k} &= e^{\Phi}\left(F + \frac{N_f}{4\pi} f\right)\coth^2\frac{\tau}{2} \\ \dot{f} &= e^{\Phi}\left(1 - F + \frac{N_f}{4\pi} k\right)\tanh^2\frac{\tau}{2} \\ \dot{F} &= \frac{1}{2} e^{-\Phi}\,(k - f)\,. \end{aligned} \tag{6.3.12}$$

Notice that in the unflavored limit the system (6.3.12) reduces to the one (3.4.15) found in [37].[4] Moreover, for $N_f \neq 0$ this system can be solved as:

$$\begin{aligned} e^{-\phi} f &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1) \\ e^{-\phi} k &= \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1) \\ F &= \frac{\sinh\tau - \tau}{2\sinh\tau}\,, \end{aligned} \tag{6.3.13}$$

where e^{Φ} is given in eq. (6.3.4). By using the solution given by (6.3.7) and (6.3.13) in the general eq. (6.2.14) we can immediately obtain the expression of the warp factor $h(\tau)$. Actually, if we require that h is regular at $\tau = 0$, the integration constant N_0 in (6.2.17) must be chosen to be zero. In this case, we get:

$$h(\tau) = -\frac{\pi M^2\alpha'^2}{4\mu^{8/3}N_f}\int^{\tau} dx\, \frac{x\coth x - 1}{(x-\tau_0)^2\sinh^2 x}\, \frac{-\cosh 2x + 4x^2 - 4x\tau_0 + 1 - (x - 2\tau_0)\sinh 2x}{(\cosh 2x + 2x^2 - 4x\tau_0 - 1 - 2(x-\tau_0)\sinh 2x)^{2/3}}\,. \tag{6.3.14}$$

The integration constant can be fixed by requiring that the analytic continuation of $h(\tau)$ goes to zero as $\tau \to +\infty$, to connect with the Klebanov-Strassler solution in the unflavored (scaling) limit. Then, close to the tip of the geometry, $h(\tau) \approx h_0 - \mathcal{O}(\tau^2)$.

[3] Other rescalings are also necessary in taking this unflavored limit, similarly to what we did in detail in subsection 5.2.3 in the context of the Klebanov-Witten theory.

[4] The overall g_s factor is compensated by the slightly different ansatz we chose.

We should emphasize an important point: even though at first sight this solution may look smooth in the IR limit ($\tau \to 0^+$), where all the components of our metric (as well as the 3- and 5-form fluxes, the NSNS 2-form potential and the dilaton) approach the same limit as those of the KS solution (up to a suitable redefinition of parameters), there is actually a curvature singularity. For instance, the curvature scalar diverges as $1/\tau$. The reason is that in this flavored solution corrections to the leading behavior are $\mathcal{O}(\tau)$ instead of $\mathcal{O}(\tau^2)$, which is the case in the Klebanov-Strassler solution. The fibration of S^3 over $\mathbb{R}^3$ becomes singular.[5] This singularity of course disappears when taking the unflavored limit, using the scaling described above.

This is a good singularity according to the criterion of [65], therefore low energy properties of the dual gauge theory can be extracted from our solution. There is a physically intuitive reason why such an 'infrared' singularity should occur. Were not for this curvature singularity, the geometry would have ended smoothly, and linear confinement (infinitely long QCD strings) would have been found by studying the behavior of rectangular Wilson loops. In Veneziano's large N limit, pair production of flavor fields is allowed already at the leading order, that we are studying by means of a dual background. Pair production of quarks leads to charge screening and chromoelectric flux tube breaking. Hence it is natural to expect a dual solution with a curvature singularity in the IR region. We have checked numerically, by means of the holographic method for computing the Wilson loop [69], that exactly the $\mathcal{O}(\tau)$ corrections which are responsible for the curvature singularity are also responsible for the existence of a maximal distance up to which a pair of external quark and antiquark can be separated before the flux tube in between breaks.

The solution presented above is naturally interpreted as the addition of fundamentals to the KS background [37]. In the next section, we will present a solution that can be understood as the addition of flavors to the KT background [36].

6.4 Flavored warped singular conifold

In this section we consider the solutions with $g = 0$. An alternative derivation is collected in appendix C.4, where we start directly from the singular conifold and exploit the Kähler and complex structure of the ('flavored') singular conifold; reading that appendix is instructive because it highlights the links with the unflavored Klebanov-Tseytlin solution and the flavored Klebanov-Witten solution.

First of all, let us change the radial variable from r to ρ, where the latter is defined by the relation $dr = e^{G_3}\, d\rho$. The equation for the dilaton can be integrated trivially:

$$e^{\Phi} = -\frac{4\pi}{3N_f}\,\frac{1}{\rho}\,, \qquad \rho < 0\,. \tag{6.4.1}$$

Here we have already absorbed the integration constant that would have been useful if we wanted to take the unflavored limit. It can be reinstated trivially by redefining the radial coordinate ρ by a shift. The unflavored limit is exactly the same as the one we analyzed for the flavored Klebanov-Witten solution (5.2.19).

Requiring supersymmetry imposes that the metric functions G_i satisfy in this case the following

[5] The simplest example of this kind of singularity appears at $r = 0$ in a 2-dimensional manifold whose metric is $ds^2 = dr^2 + r^2(1 + r)d\varphi^2$.

system of differential equations:

$$\begin{aligned}\dot{G}_i &= \frac{1}{6}\, e^{2G_3-2G_i} \qquad\qquad\qquad (i=1,2)\\ \dot{G}_3 &= 3-\frac{1}{6}\, e^{2G_3-2G_1}-\frac{1}{6}\, e^{2G_3-2G_2}-\frac{3N_f}{8\pi}\, e^{\Phi}\;,\end{aligned} \tag{6.4.2}$$

where now the dot refers to the derivative with respect to ρ. This system is equivalent to the one analyzed in [1] for the Klebanov-Witten model with flavors. In what follows we will restrict ourselves to the particular solution with $G_1 = G_2$ given by:

$$\begin{aligned}e^{2G_1} &= e^{2G_2} = \frac{1}{6}\,(1-6\rho)^{\frac{1}{3}}\, e^{2\rho}\\ e^{2G_3} &= -6\rho\,(1-6\rho)^{-\frac{2}{3}}\, e^{2\rho}\;.\end{aligned} \tag{6.4.3}$$

Notice that, as for the flavored Klebanov-Witten solution of the previous chapter, the range of values of ρ for which the metric is well defined is $-\infty < \rho < 0$. The equations for the flux functions f, k and F are now:

$$\begin{aligned}\dot{f}-\dot{k} &= 2e^{\Phi}\dot{F}\\ \dot{f}+\dot{k} &= 3e^{\Phi}\Big[1+\frac{N_f}{4\pi}(f+k)\Big]\\ F &= \frac{1}{2}\Big[1+\Big(e^{-\Phi}-\frac{N_f}{4\pi}\Big)(f-k)\Big]\;.\end{aligned} \tag{6.4.4}$$

We will focus on the particular solution of this system such that $f = k$ and F is constant, namely:

$$\begin{aligned}F &= \frac{1}{2}\\ f &= k = -\frac{2\pi}{N_f}\left(1-\frac{\Gamma}{\rho}\right)\;,\end{aligned} \tag{6.4.5}$$

where Γ is an integration constant. These are the solutions that reduce to the Klebanov-Tseytlin solution in the unflavored limit. By substituting these values of F, f and k in our ansatz (6.2.10) we obtain the form of F_3 and H_3. Notice that the constants M and Γ only appear in the combination $M\Gamma$. Accordingly, let us define $\mathcal{M}$ as $\mathcal{M} \equiv M\Gamma$. We will write the result in terms of the function

$$M_{eff}(\rho) \equiv \frac{\mathcal{M}}{\rho}\;, \tag{6.4.6}$$

which is the D5 brane Maxwell charge of the solution at a value ρ of the holographic coordinate. We find:

$$\begin{aligned}F_3 &= \frac{\alpha'}{4}\, M_{eff}(\rho)\, \zeta\wedge\big(g^1\wedge g^2+g^3\wedge g^4\big)\\ H_3 &= \frac{\pi\alpha'}{N_f}\,\frac{M_{eff}(\rho)}{\rho}\, d\rho\wedge\big(g^1\wedge g^2+g^3\wedge g^4\big)\;.\end{aligned} \tag{6.4.7}$$

Moreover, the RR five-form F_5 can be written as in (6.2.15) in terms of the effective D3 brane charge defined in (6.2.17). For the solution (6.4.5) one gets:

$$N_{eff}(\rho) = N+\frac{\mathcal{M}^2}{N_f}\frac{1}{\rho^2} = N+\frac{M_{eff}^2(\rho)}{N_f}\;, \tag{6.4.8}$$

where $N \equiv N_0 - \frac{M^2}{N_f}$. Integration of equation (6.2.14) gives the warp factor:

$$h(\rho) = -27\pi\alpha'^2 \int_0^\rho dx \left[N + \frac{\mathcal{M}^2}{N_f} \frac{1}{x^2} \right] \frac{e^{-4x}}{(1-6x)^{\frac{2}{3}}} \,. \tag{6.4.9}$$

We have chosen the integration constant so as to recover the Klebanov-Tseytlin solution in the scaling unflavored limit, which can be taken after redefining ρ by a shift by suitably scaling the maximal value of ρ to infinity.

To interpret the solution just presented, it is interesting to study it in the deep IR region $\rho \to -\infty$. Notice that in this limit the three-forms F_3 and H_3 vanish. Actually, it is easy to verify that for $\rho \to -\infty$ the solution obtained here reduces to the one studied in [1], corresponding to the Klebanov-Witten [29] model with flavors. Indeed, in this IR region it is convenient to go back to our original radial variable r. The relation between r and ρ for $\rho \to -\infty$ is $r \approx (-6\rho)^{\frac{1}{6}}\, e^{\rho}$. Moreover, one can prove that for $\rho \to -\infty$ (or equivalently $r \to 0$), the warp factor h and the metric functions G_i become:

$$h(r) \approx \frac{27\pi\alpha'^2 N}{4} \frac{1}{r^4} \,, \qquad e^{2G_1} = e^{2G_2} \approx \frac{r^2}{6} \,, \qquad e^{2G_3} \approx r^2 \,, \tag{6.4.10}$$

which implies that the IR Einstein frame metric is $AdS_5 \times T^{1,1}$ plus logarithmic corrections, exactly as the flavored Klebanov-Witten solution found in [1]. The interpretation of the RG flow of the field theory dual to this solution will be explained in sections 6.5 and 6.6.

Finally, let us stress that the UV behavior of this solution (coincident with that of the solution presented in section 6.3) presents a divergent dilaton at the point $\rho = 0$ (or $\tau = \tau_0$ for the flavored warped deformed conifold). Hence the supergravity approximation fails at some value of the radial coordinate that we will associate in section 6.6 with the presence of a duality wall [111] in the cascading field theory.

6.5 The field theory with flavors: a cascade of Seiberg dualities

The field theories dual to our supergravity solutions can be engineered by placing stacks of two kinds of fractional D3 branes (color branes) and two kinds of fractional D7 branes (flavor branes) on the singular conifold. In section 3.3, we have introduced fractional D3 branes on the conifold. The conifold has a single nontrivial 2-cycle at the singularity, over which it is possible to wrap D5 branes or anti-D5 branes with a suitable worldvolume gauge flux, still preserving the same supersymmetries as the conventional (regular) D3 branes. Since the geometric volume of the 2-cycle vanishes, these objects are really D3 branes, but they carry noninteger D3 charge, in particular they carry half a unit of D3 brane charge in a background where $\int_{S^2} B_2 = 2\pi^2\alpha'$. A regular D3 brane is a bound state of the two kinds of fractional branes that unwraps the cycle and can move off the singular point. We have also reviewed the arguments explaining why each kind of fractional D3 branes increases the rank of one or the other gauge group of the quiver gauge theory.

Similarly to the fractional D3 branes, two kinds of fractional D7 branes, providing flavors for either one or the other gauge group can be introduced. These fractional D7 branes are not wrapped D9 or anti-D9 branes, which would have no transverse directions, but true D7 branes. They are fractional because the 4-dimensional manifold they are both embedded along inside the conifold, that we will define momentarily, touches the two-cycle at the singularity, and actually wraps it,

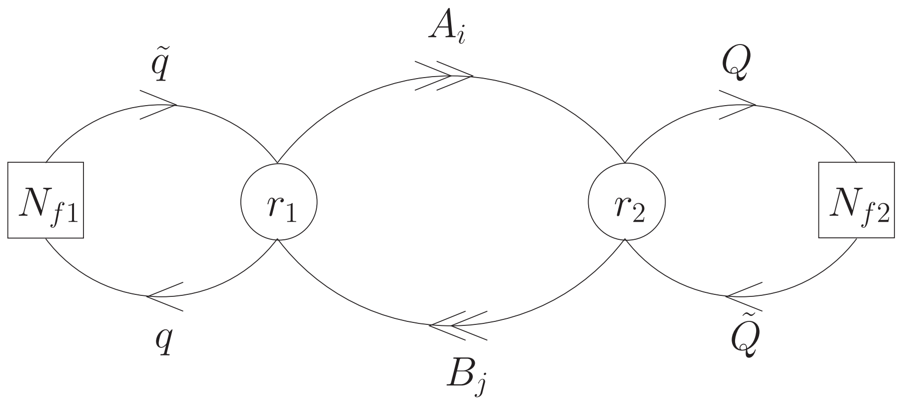


Figure 6.1: The quiver diagram of the gauge theory. Circles are gauge groups, squares are flavor groups, and arrows are bifundamental chiral superfields. N_{f1} and N_{f2} sum up to N_f.

although leaving four other directions to be spanned by the branes outside the singularity. There are two kinds of fractional D7 branes, because the can carry either zero or minus one unit of worldvolume gauge flux on the exceptional cycle, still preserving the same supercharges as the regular D3 branes. The worldvolume action of these fractional branes, which clarifies their properties, will be defined and discussed in detail in the next section.

The supersymmetric embedding for these fractional D7 branes on the conifold was considered by Kuperstein [96]:

$$z_3 - z_4 = 0 \,, \tag{6.5.1}$$

which in the w coordinates defined in appendix B.1 is $w_2 = 0$. We immediately see that the embedding submanifold in the singular conifold, $z_1 z_2 = z_3^2$ or $w_1^2 + w_3^2 + w_4^2 = 0$, is algebraically the orbifold $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, having an exceptional 2-cycle at the singularity $z_1 = z_2 = z_3 = z_4 = 0$, which coincides with the conifold singularity. In order to convince the reader that this embedding actually wraps the singular cycle, it is enough to consider either the small resolution of the conifold or the deformation: in both cases the embedding $z_3 = z_4$ acquires a blown-up two-cycle (either as a Kähler or a complex structure deformation). Furthermore, an easy way of understanding why fractional D7 branes along the embedding (6.5.1) provide flavors coupling either to one or the other gauge group is to recall that fractional D7 branes on the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, where explicit quantization of open strings is possible, precisely provide nonchiral flavors for both the color groups of the corresponding $\mathcal{N} = 2$ quiver gauge theory [88]. Upon mass deformation, that $\mathcal{N} = 2$ gauge theory with two gauge groups and two flavor groups reduces to the one under consideration. Making contact with this possible UV completion, it is also possible to understand which kind of fractional D7 brane (flavor group) couples to which kind of fractional D3 brane (color group). It turns out that D7 branes with flux flavor the gauge group related to wrapped D5 branes without flux, and D7 branes without flux flavor the gauge group related to wrapped anti-D5 branes with flux.

Summarizing, by placing at the conifold singularity arbitrary numbers of the two kinds of fractional D3 branes and placing arbitrary numbers of the two kinds of fractional D7 branes along the embedding (6.5.1), we can engineer a flavored version of the Klebanov-Strassler theory whose matter content is depicted in the quiver diagram of figure 6.1.

We have not yet found the superpotential of the flavored gauge theory. Before facing this issue, let us preliminarily observe that the embedding (6.5.1) is invariant under $U(1)_R$ and a diagonal $SU(2)_D$ subgroup of the $SU(2) \times SU(2) \times U(1)_R$ symmetry of the conifold (and a $\mathbb{Z}_2$ which exchanges $z_3 \leftrightarrow z_4$). It could be useful to write it in the angular coordinates of the previous section, using (B.2.5)-(B.2.6): $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\forall\psi, \forall\tau$. We can obtain other embeddings with the same properties by acting on it with the broken generators. One can show that the charge distribution obtained by homogeneously distributing the D7 branes in this class of mutually BPS embedding is (6.2.8):

$$\Omega_2 = \frac{N_f}{4\pi}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big)\ , \tag{6.5.2}$$

where N_f is the total number of D7 branes.[6]

Different techniques developed in the literature allow us to identify the field theory dual to our type IIB plus D7 branes background. First of all, we have already mentioned that one possibility consists in starting from an $\mathcal{N} = 2$ quiver gauge theory realized via a generic configuration of fractional D3 and D7 branes on the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold, giving masses to the adjoint chiral superfields in the $\mathcal{N} = 2$ vector multiplets, and integrating them out to find our theory as a low energy effective description. Another equivalent technique, that we found very transparent, is that of performing a T-duality along the isometry $(z_1, z_2) \to (e^{i\alpha} z_1, e^{-i\alpha} z_2)$. The system is mapped into type IIA string theory: neglecting the common spacetime directions, there is an NS5 brane along $x^{4,5}$, another orthogonal NS5 brane along $x^{8,9}$, r_1 D4 branes along x^6 (which is a compact direction) connecting them on one side, other r_2 D4 branes connecting them on the other side, N_{f1} D6 branes along x^7 and at a $\frac{\pi}{4}$ angle between $x^{4,5}$ and $x^{8,9}$, touching the stack of r_1 D4 branes, and N_{f2} D6 branes along x^7 and at a $\frac{\pi}{4}$ angle between $x^{8,9}$ and $x^{4,5}$, touching the stack of r_2 D4 branes. NS5 branes are T-dual to the conifold singularity, the two kinds of suspended D4 branes are T-dual to the two kinds of fractional D3 branes, and finally the two kinds of rotated D6 branes are T-dual to the two kinds of fractional D7 branes. The type IIA brane system is depicted in Figure 6.2. The spectrum is directly read off, and the superpotential comes from the analysis of the moduli space [112]:[7]

$$\begin{aligned} W = h\,(A_1B_1A_2B_2 - A_1B_2A_2B_1) + h_1\,\tilde{q}(A_2B_1 - A_1B_2)q + h_2\,\tilde{Q}(B_1A_2 - B_2A_1)Q + \\ + \alpha\,\tilde{q}q\tilde{q}q + \beta\,\tilde{Q}Q\tilde{Q}Q\ . \end{aligned} \tag{6.5.3}$$

The factors $A_2B_1 - A_1B_2$ directly descend from the embedding equation (6.5.1), while the quartic term in the fundamental fields is derived from the type IIA Hanany-Witten brane setup or from the $\mathcal{N} = 2$ completion. This superpotential explicitly breaks the $SU(2)_A \times SU(2)_B$ global symmetry of the unflavored theory to a diagonal subgroup $SU(2)_D$, but this global symmetry is recovered after smearing the flavors. It is worth here stressing that the smearing procedure does not influence at all either the duality cascade, which is the main feature of our solutions that we want to address here, nor (presumably) the infrared dynamics.

Notice that we have engineered a field theory with four independent ranks. In particular, the number of flavors for one or the other gauge group depends on the number of D7 branes with or without worldvolume gauge flux on the exceptional cycle. Since this flux is stuck at the tip

[6] Notice that one could have considered the more general embedding: $z_3 - z_4 = m$, where m corresponds in field theory to a mass term for quarks. These embeddings and their corresponding supergravity solutions are not worked out here.

[7] Sums over gauge and flavor indices are understood.

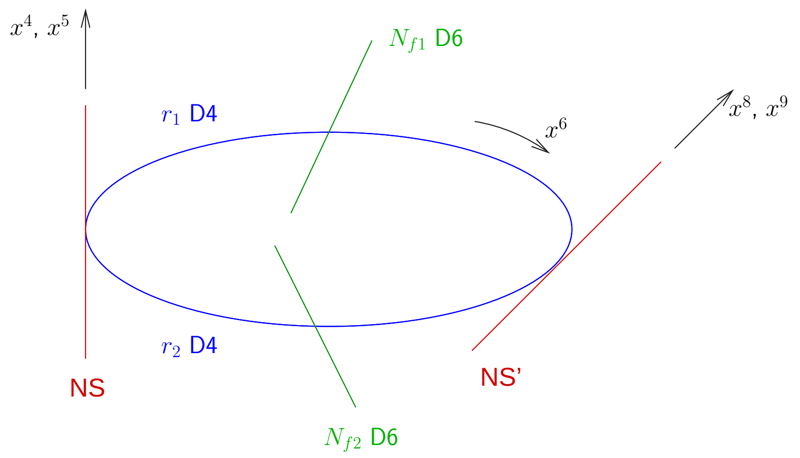


Figure 6.2: Type IIA Hanany-Witten brane engineering of the conifold theory with nonchiral Kuperstein flavors.

of the cone, outside the singularity we can only measure the D3, D5 and D7-charges produced. Unfortunately three charges are not enough to fix four ranks. This curious ambiguity will show up again in section 6.6. We will return more deeply on this issue (and solve it for the flavored warped deformed conifold solution, where the previous statement is not correct) in section 6.7.

6.5.1 The cascade

Now that we know the field theory we are dealing with, we can propose a plausible pattern for its RG flow, that will finally be confirmed by the dual supergravity backgrounds we have found.

Let us consider our gauge theory at an ultraviolet cutoff scale, below the Landau pole. It looks natural to conceive that, as in the unflavored theory, the β-functions of the two gauge couplings have opposite sign. If we follow the RG flow, at some energy scale the gauge coupling of the gauge group with larger rank will diverge; at that point we can resort to a Seiberg-dual description [67]. Remarkably, a straightforward computation shows that the quartic superpotential makes the field theory *self-similar*: the field theory in the dual description is a quiver gauge theory with exactly the same field content and superpotential; the only quantities that change are the ranks of the groups.[8]

From now on, we define the gauge theory at any energy scale by specifying the ranks of the color and flavor groups: the gauge group will be $SU(r_l) \times SU(r_s)$, where l always stands for the larger gauge group and s for the smaller ($r_l > r_s$), and the flavor groups $SU(N_{fl})$ and $SU(N_{fs})$ will be associated to the gauge groups $SU(r_l)$ and $SU(r_s)$ respectively. We start at the ultraviolet cutoff scale, conventionally, with ranks $r_l = r_1$, $r_s = r_2$, $N_{fl} = N_{f1}$ and $N_{fs} = N_{f2}$, and follow the flow until the gauge coupling of $SU(r_1)$ diverges. At that point we need to Seiberg-dualize that gauge

[8]This is not the case for the chirally flavored version of Klebanov-Strassler's theory proposed by Ouyang [95], and for the flavored version of nonconformal theories obtained by putting branes at conical Calabi-Yau singularities [113]. In those realizations the superpotential is cubic, and the theory is not self-similar under Seiberg duality: new gauge singlet fields appear or disappear after a Seiberg duality, making the cascade subtler. This setup has been investigated later in [110].

group: as a result, the field theory becomes $SU(2r_2 - r_1 + N_{f1}) \times SU(r_2)$, with again N_{f1} and N_{f2} flavors respectively. In identifying which gauge group is now the larger and which is the smaller, we have to exchange the labelling of the groups, so that we get $r'_l = r_2$, $r'_s = 2r_2 - r_1 + N_{f1}$, $N'_{fl} = N_{f2}$ and $N'_{fs} = N_{f1}$. As long as our assumption about opposite signs for the gauge coupling β-functions is still met, the flow goes on in a similar way: the new larger gauge group will run to strong coupling, at that point we need to dualize it, and the former smaller gauge group becomes the larger gauge group, then flowing to strong coupling, and so on. The assumption leads to an RG flow which is described by a self-similar cascade of Seiberg dualities, analogously to [36,37]. By inspection of the transformation properties of ranks under a Seiberg duality that we have considered above, we can see that in the UV the ranks of the gauge groups are much larger than their difference, which in turn is much larger than the number of flavors. Hence the assumption of having β-functions with opposite sign is for sure justified in the ultraviolet regime of the RG flow.

The previous expectation is confirmed by our backgrounds. We will extract the field theory cascade from our supergravity solution. We can anticipate that the 'flavored warped deformed conifold' background of section 6.3 is dual to a quiver gauge theory where the cascade goes on until the deep IR, with nonperturbative dynamics dual to the complex structure deformation occurring at the end, like in the Klebanov-Strassler solution.

In the 'flavored warped singular conifold' background of section 6.4 instead, the cascade does not take place anymore below some value of the radial coordinate, and the background asymptotes to the flavored Klebanov-Witten solution [1] illustrated in the previous chapter. In the field theory, this reflects the possibility that, because of a suitable initial choice of ranks at a UV cutoff scale, at some point along the cascade we are driven to a field theory where the β-functions of both gauge couplings are positive. In this situation, the infrared dynamics is the one discussed in the previous chapter, but with a quartic superpotential for the flavors as in subsection 5.3.1.

Unlike the unflavored case, where the Klebanov-Tseytlin background is only an approximation of the correct Klebanov-Strassler background, here the 'flavored warped singular conifold' and the 'flavored warped deformed conifold' solutions are two morally distinct and equally legitimate solutions, describing RG flows with different IR dynamics.

The description of the duality cascade in our solutions and its interesting ultraviolet behavior will be the content of the next section.

6.6 The cascade: supergravity side

We claim that our supergravity solutions are dual to the class of quiver gauge theories with backreacting fundamental flavors introduced in the previous section. Indeed we will show that the effective brane charges, the R-anomalies and the beta functions of the gauge couplings that we can read from the supergravity solutions precisely match the picture of a cascade of Seiberg dualities that we expect to describe the RG flow of the field theories, generalizing the results of [36, 37] to gauge theories which include dynamical flavors.

6.6.1 Effective brane charges and ranks

By integrating fluxes over suitable compact cycles, we can compute three effective D brane charges in our solutions, which are useful to pinpoint the changes in the ranks of gauge groups when the field theory undergoes a Seiberg duality along the cascade: one of them (D7) is dual to a quantity which is constant along the RG flow, whereas two of them (D3, D5) are not independent of the holographic coordinate and are dual to the nontrivial part of the RG flow. The (Maxwell) charges of D3 and D5 brane (N_{eff} and M_{eff}) for our ansatz were already calculated in section 6.2 (see equations. (6.2.17) and (6.2.18)). Let us now compute the D7 brane charge, integrating (6.2.8) on a 2-manifold with boundary which is intersected once by all the smeared D7 branes (e.g. $\mathcal{D}_2$: $\theta_2 = \text{const.}$, $\varphi_2 = \text{const.}$, $\psi = \text{const.}$). This charge is conserved along the RG flow because no fluxes appear on the right hand side of (6.2.8). The D7 brane charge, which we interpret as the total number of flavors added to the Klebanov-Strassler gauge theory, is indeed:

$$N_{flav} \equiv \int_{\mathcal{D}_2} dF_1 = N_f \; . \tag{6.6.1}$$

Another important quantity is the integral of B_2 over the nontrivial 2-cycle, which has S^2 topology and can be represented by $\theta_1 = \theta_2 \equiv \theta$, $\varphi_1 = 2\pi - \varphi_2 \equiv \varphi$, $\psi = \text{const.}$:

$$b(\tau) \equiv \frac{1}{4\pi^2\alpha'} \int_{S^2} B_2 = \frac{M}{\pi}\Big(f \sin^2\frac{\psi}{2} + k\cos^2\frac{\psi}{2}\Big) \; . \tag{6.6.2}$$

This quantity is important because in the absence of fractional branes, string theory is invariant as it undergoes a shift of 1. For instance, in the KW background it amounts to move to a Seiberg dual description. In a cascading background, the effect of changing the radial coordinate so that b is shifted by 1 corresponds to moving to the following step of the cascades, as is clear from the approximate formulae (3.4.1). So we will shift this last quantity by one unit, identify a shift in the radial variable τ that realizes the same effect, and see what happens to M_{eff} and N_{eff}. Actually, the cascade matching is not exact along the whole flow down to the IR but only in the UV asymptotic (below the UV cut-off τ_0 obviously). The same happens for the unflavored solutions of [36] and [37]: in the KT solution one perfectly matches the cascade in field theory and supergravity, while in the KS solution close to the tip of the warped deformed conifold the matching is not so clean. On the other hand, this is expected, since the last step of the cascade is not a Seiberg duality. Thus we will not be worried and compute the cascade only in the UV asymptotic for large τ which also requires $\tau_0 \gg 1$ (we neglect $\mathcal{O}(e^{-\tau})$): in that regime the functions f and k become equal, and b is ψ-independent.
Actually, we will not compute the explicit shift in τ but rather the shift in the functions f and k. We have:

$$b(\tau) \to b(\tau') = b(\tau) - 1 \quad \Longrightarrow \quad \begin{array}{l} f(\tau) \to f(\tau') = f(\tau) - \dfrac{\pi}{M} \\ k(\tau) \to k(\tau') = k(\tau) - \dfrac{\pi}{M} \end{array} \tag{6.6.3}$$

Correspondingly, after a Seiberg duality step from τ to $\tau' < \tau$, that is going toward the IR, we have:

$$N_f \to N_f \tag{6.6.4}$$

$$M_{eff}(\tau) \to M_{eff}(\tau') = M_{eff}(\tau) - \frac{N_f}{2} \tag{6.6.5}$$

$$N_{eff}(\tau) \to N_{eff}(\tau') = N_{eff}(\tau) - M_{eff}(\tau) + \frac{N_f}{4} \tag{6.6.6}$$

This result is valid for all our solutions.

We would like to compare this result with the action of Seiberg duality in field theory, as computed in section 6.5. We need an identification between the brane charges computed in supergravity and the ranks of the gauge and flavor groups in the field theory.

The field theory of interest for us, with gauge groups $SU(r_l) \times SU(r_s)$ ($r_l > r_s$), and flavor groups $SU(N_{fl})$ and $SU(N_{fs})$ for the gauge groups $SU(r_l)$ and $SU(r_s)$ respectively, is engineered, at least effectively at some radial distance, by the following objects: r_l fractional D3 branes of one kind (D5 branes wrapped on the shrinking 2-cycle), r_s fractional D3 branes of the other kind ($\overline{\text{D5}}$ branes wrapped on the shrinking cycle, supplied with -1 quantum of gauge field flux on the 2-cycle), N_{fs} fractional D7 branes without gauge field strength on the 2-cycle, and N_{fl} fractional D7 branes with -1 unit of gauge field flux on the shrinking 2-cycle. This description is valid for $b \in [0,1]$.

This construction can be checked explicitly in the case of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold [39, 88], where one is able to quantize the open and closed string system for the case $b = \frac{1}{2}$ that leads to a free CFT [114]. That is the $\mathcal{N} = 2$ field theory which flows to the field theory we are considering, when equal and opposite masses are given to the adjoint chiral superfields (the geometric description of this relevant deformation is a blowup of the orbifold singularity) [29, 32]. Fractional branes are those branes which couple to the twisted closed string sector.[9]

Here we will consider a general background value for B_2. In order to compute the charges, we will follow quite closely the computations in [87].

We will compute the gauge invariant charges[10] of D7 branes and wrapped D5 branes on the singular conifold, described by $z_1 z_2 - z_3 z_4 = 0$. The D5 brane Wess-Zumino action is

$$S_{D5} = \mu_5 \int_{M^4 \times S^2} \Big\{ C_6 + (2\pi\alpha' F_2 + B_2) \wedge C_4 \Big\} \,, \tag{6.6.7}$$

where S^2 is the only 2-cycle in the conifold, vanishing at the tip, that the D5 brane is wrapping. We write also a world-volume gauge field F_2 on S^2. All forms except F_2 are meant to be pulled back. We suppress hats to avoid cluttered formulae. Then we expand:

$$B_2 = 2\pi\alpha' \, \theta_B \, \omega_2 \qquad\qquad \theta_B = 2\pi \, b \qquad\qquad F_2 = \Phi \, \omega_2 \,, \tag{6.6.8}$$

where ω_2 is the 2-form on the 2-cycle, which satisfies $\int_{S^2} \omega_2 = 1$.[11] In this conventions, b has period 1, and Φ is quantized in $2\pi\,\mathbb{Z}$. We obtain (using $\mu_p(4\pi^2\alpha') = \mu_{p-2}$):

$$S_{D5} = \mu_5 \int_{M^4 \times S^2} C_6 + \frac{\mu_3}{2\pi} \int_{M^4} (\Phi + \theta_B) \, C_4 \,. \tag{6.6.9}$$

The first fractional D3 brane [38] is obtained with $\Phi = 0$ and has D3-charge b, D5-charge 1. The second fractional D3 brane is obtained either as the difference with a D3 brane, or as an anti D5

[9] Notice that one can build, out of a fractional D3 of one kind and a fractional D3 of the other kind, a regular D3 brane (*i.e.* not coupled to the twisted sector) that can move outside the orbifold singularity; on the contrary, there is no regular D7 brane: the two kinds of fractional D7 branes we are considering, extending entirely along the orbifold, cannot bind into a regular D7 brane that does not touch the orbifold fixed locus and is not coupled to the twisted sector [88].

[10] These charges, that we read naively from the Wess-Zumino couplings of D branes, were called 'brane source charges' in [108].

[11] We see that $\omega_2 = \frac{1}{4\pi}\omega_2^{CF}$, in terms of the 2-form defined in (B.2.17).

brane (global $-$ sign in front) with $\Phi = -2\pi$, and has D3-charge $1-b$, D5-charge -1. These charges are summarized in Table 6.1.

Now consider a D7 brane along the surface $z_3 = z_4$. It describes a $z_1 z_2 = z_3^2$ inside the conifold, which is a copy of $\mathbb{C}^2/\mathbb{Z}_2$. The D7 brane Wess-Zumino action is (up to a curvature term considered below)

$$S_{D7} = \mu_7 \int_{M^4\times\Sigma} \Big\{ C_8 + (2\pi\alpha' F_2 + B_2)\wedge C_6 + \frac{1}{2}(2\pi\alpha' F_2 + B_2)\wedge(2\pi\alpha' F_2 + B_2)\wedge C_4 \Big\} \,. \quad (6.6.10)$$

The surface $\Sigma = \mathbb{C}^2/\mathbb{Z}_2$ has a vanishing 2-cycle at the origin. Since the conifold has only one 2-cycle, these two must be one and the same and we can expand on Σ using ω_2 again. Moreover, being ω_2 the Poincaré dual to the 2-cycle on Σ,

$$\int_\Sigma \omega_2 \wedge \alpha_2 = \frac{1}{2}\int_{S^2} \alpha_2 \quad (6.6.11)$$

holds for any closed 2-form α_2, and $\frac{1}{2}$ arises from the self-intersection number of the S^2. There is another contribution of induced D3-charge coming from the curvature coupling [115]:

$$\frac{\mu_7}{96}(2\pi)^2 \int_{M^4\times\Sigma} C_4 \wedge \mathrm{Tr}\,\mathcal{R}_2\wedge\mathcal{R}_2 = -\mu_3 \int_{M^4\times\Sigma} C_4 \wedge \frac{p_1(\mathcal{R})}{48} \,. \quad (6.6.12)$$

This can be computed in the following way. On K3 $p_1(\mathcal{R}) = 48$ and the induced D3-charge is -1. In the orbifold limit K3 becomes $T^4/\mathbb{Z}_2$ which has 16 orbifold singularities, thus on $\mathbb{C}^2/\mathbb{Z}_2$ the induced D3-charge is $-1/16$.[12]

Collecting everything together we get:

$$S_{D7} = \mu_7 \int_{M^4\times\Sigma} C_8 + \frac{\mu_5}{4\pi}\int_{M^4\times S^2} (\Phi+\theta_B)\, C_6 + \frac{\mu_3}{16\pi^2}\int_{M^4} \Big[(\Phi+\theta_B)^2 - \pi^2\Big]\, C_4 \,. \quad (6.6.13)$$

The second fractional D7 brane (the one that couples to the second gauge group) is obtained with $\Phi = 0$ and has D7-charge 1, D5-charge $\frac{b}{2}$ and D3-charge $(4b^2-1)/16$. With $\Phi = 2\pi$ we get a nonsupersymmetric or nonminimal object (see [38] for some discussion of this). The first fractional D7 brane (coupled to the first gauge group) has $\Phi = -2\pi$ and has D7-charge 1, D5-charge $\frac{b-1}{2}$ and D3-charge $(4(b-1)^2-1)/16$. This is summarized in Table 6.1. Which fractional D7 brane provides flavors for the gauge group of which fractional D3 brane can be determined from the orbifold case with $b = \frac{1}{2}$ (compare with [88]).

Given these charges, we can compare with the field theory cascade. First of all we construct the dictionary:

$$N_f = N_{fl} + N_{fs} \quad (6.6.14)$$

$$M_{eff} = r_l - r_s + \frac{b-1}{2}N_{fl} + \frac{b}{2}N_{fs} \quad (6.6.15)$$

$$N_{eff} = b\, r_l + (1-b)\, r_s + \frac{4(1-b)^2-1}{16}N_{fl} + \frac{4b^2-1}{16}N_{fs} \quad (6.6.16)$$

[12] To be fair, the previous computation is a bit too naive. Because the 4-dimensional manifold spanned by D7 branes inside the Calabi-Yau is noncompact, some additional care is needed in computing this topological quantity: as the asymptotics changes, this topological quantity can change too. However, changing these curvature couplings does not affect the matchings and the arguments that are laid out in the following, since it amounts simply to a constant shift, proportional to the total number of flavors N_f, in the D3 brane charge. Therefore we will neglect this subtlety, which is only relevant for a detailed analysis of the deep IR regime.

Object	frac D3 (1)	frac D3 (2)	frac D7 (1)	frac D7 (2)
D3-charge	b	$1-b$	$\frac{4(b-1)^2-1}{16}$	$\frac{4b^2-1}{16}$
D5-charge	1	-1	$\frac{b-1}{2}$	$\frac{b}{2}$
D7-charge	0	0	1	1
Number of objects	r_l	r_s	N_{fl}	N_{fs}

Table 6.1: Charges of fractional branes on the conifold

To derive this, we have only used that the brane configuration that engineers the field theory we consider consists of r_l fractional D3 branes of the 1^{st} kind, r_s fractional D3 branes of the 2^{nd} kind, N_{fl} fractional D7 branes of the 1^{st} kind, and N_{fs} fractional D7 branes of the 2^{nd} kind. Recall that, by convention, $r_l > r_s$ and N_{fl} (N_{fs}) are the flavors for $SU(r_l)$ ($SU(r_s)$).

We remark that this method is a clever trick. These D branes of which we are computing the charges do not actually exist in the background. The whole idea is that, in the presence of a fixed background value of b, we could engineer the field theory under study by placing a particular configuration of branes (at the singularity). The gauge invariant 'brane source charges' of this configuration can be computed and depend on the value of b. On the other hand, in the dual background D branes are replaced by their fluxes, and in a cascading solution b varies as we move in the radial direction; but at any value τ of the radial coordinate, we can compute the gauge invariant Maxwell charges of the solution. Being both gauge invariant, these Maxwell charges must equal by construction the brane source charges, in a background for b equal to $b(\tau)$, of the hypothetical brane configuration that would provide the gauge and flavor groups describing the gauge theory at the dual energy scale.

It is important to remember that in the holographic formulae (3.4.1) b is defined modulo 1, and when b dynamically crosses an integer value a gauge coupling diverges and we go to a Seiberg dual description in the field theory. At any given energy scale in the cascading gauge theory, there are infinitely many Seiberg dual descriptions of the field theory, because Seiberg duality is exact along the RG flow [55]. Among these different pictures, there is a single description that we can really make sense of within our perturbative intuition, having positive squared gauge couplings: it is the one where b has been redefined, by means of a large gauge transformation, so that $b \in [0, 1]$ (see subsection 6.6.2). This is the description that we will use when we effectively engineer the field theory in terms of branes in some range of the RG flow that lies between two adjacent Seiberg dualities.

In field theory, as before, we start with gauge group $SU(r_1) \times SU(r_2)$ and N_{f1} flavors for $SU(r_1)$, N_{f2} flavors for $SU(r_2)$, with $r_1 > r_2$. The gauge group $SU(r_1)$ flows toward strong coupling, and when its gauge coupling diverges we turn to a Seiberg dual description. After the Seiberg duality on the larger gauge group, we get $SU(2r_2 - r_1 + N_{f1}) \times SU(r_2)$, and the flavor groups are left untouched.

The effective D5 and D3 brane charges of a brane configuration that engineers this field theory *before* the duality are:

$$\begin{aligned} M_{eff} &= r_1 - r_2 + \frac{b-1}{2} N_{f1} + \frac{b}{2} N_{f2} \,, \\ N_{eff} &= b r_1 + (1-b) r_2 + \frac{4(1-b)^2 - 1}{16} N_{f1} + \frac{4b^2-1}{16} N_{f2} \,. \end{aligned} \tag{6.6.17}$$

After the duality they become:

$$\begin{aligned} M'_{eff} &= -r_2 + r_1 - N_{f1} + \frac{b-1}{2} N_{f2} + \frac{b}{2} N_{f1} = M_{eff} - \frac{N_f}{2} \,, \\ N'_{eff} &= b r_2 + (1-b)(2r_2 - r_1 + N_{f1}) + \frac{4(1-b)^2-1}{16} N_{f2} + \frac{4b^2-1}{16} N_{f1} = \\ &= N_{eff} - M_{eff} + \frac{N_f}{4} \,. \end{aligned} \tag{6.6.18}$$

They *exactly* reproduce the SUGRA behavior (6.6.4)-(6.6.6). Notice that the matching of the cascade between supergravity and field theory is there, irrespective of how we distribute the flavors between the two gauge groups; so, from the three charges and the cascade we are not able to determine how the flavors are distributed, but only their total number. Notice also that the fact that M_{eff} shifts by $N_f/2$ instead of N_f confirms that the flavored version of the Klebanov-Strassler theory we are describing has nonchiral flavors (with a quartic superpotential) rather than chiral flavors (with a cubic superpotential) like in [95], where the shift goes with units of N_f.

6.6.2 Page charges and Seiberg duality as a large gauge transformation

Even though the effective brane charges (Maxwell charges) computed in supergravity in the previous subsection run and take integer values only at some values of the holographic coordinate, the ranks of gauge and flavor groups computed from them are constant and integer (for suitable choice of the integration constants) in the whole range of radial coordinate dual to the energy range where we use a specific field theory description. This range of scales corresponds to $b \in [k, k+1]$ (with integer k) in the gravity dual. We can compute ranks of the groups at any energy scale inside each of these ranges. At the boundaries of each energy range, in field theory we perform a Seiberg duality and move to a new sensible description with real gauge couplings. In particular, if ranks were integer before the duality, they still are after it. On the dual supergravity side, using the dictionary (6.6.15) between ranks and Maxwell charges (via b) we can compute ranks at any energy and see that ranks remain constant in each energy range where we use a specific field theory description, because the extra running of Maxwell charges is compensated by b; furthermore, when we change description in the field theory, the ranks we extract from the supergravity solution change by integers, exactly matching the field theory description, so that relations (6.6.15) between Maxwell charges and ranks via b still hold.

Although correct, the previous procedure is quite cumbersome. In this subsection, we provide a different prescription which directly give ranks of color and flavor groups in terms of quantized charges of the background.

In view of that, let us try to understand better what happens on the dual gravity side at the threshold where b crosses an integer value, say 0.[13] The auxiliary system of D branes that we

[13] As we will stress momentarily, this can always be achieved by a large gauge transformation.

used to compute the charges before crossing the threshold is no longer BPS, because some of the constituents become tensionless at the threshold and then develop negative charges. At this point, new minimal BPS auxiliary branes need to be considered in order to engineer the system [38]. They are different with respect to those previously considered because they have different worldvolume fluxes when they are compared at the same background value of b. However, they look the same if we insist to remain in a gauge where $b \in [0, 1]$. We can achieve this by performing a large gauge transformation, as we discuss below. For instance, the fractional D3 brane that is associated to the larger gauge group still looks as a wrapped D5 brane with no flux of the (gauge noninvariant) worldvolume field strength F_2 in this 'preferred' gauge. But in the previous range, it would have had one unit of worldvolume flux, therefore one unit more of gauge invariant D3 brane charge. We would have called it more naturally a bound state of a regular D3 brane and a D5 brane.

Notice that there is nothing so strange in this. The situation is somehow analogous to monopoles and dyons in $\mathcal{N} = 2$ gauge theories: it does not make sense to say that some object is a monopole instead of a dyon, because this statement depends on the duality frame one may choose, and different duality frames are equivalent; it is sensible instead to compare two objects, and seeing whether the charges of the two objects are different or not. In our situation, for our convenience we prefer to change 'frame' by means of a large gauge transformation each time a threshold is crossed, so that we still call 'wrapped D5' and 'wrapped anti-D5 brane with minus one unit of (gauge noninvariant) worldvolume flux' the minimal BPS objects. But when we compare the 'wrapped D5' above the threshold with the 'wrapped D5' below the threshold, we see that their gauge invariant fluxes are different, therefore the two objects are different.

At the threshold (the scale where we must perform a Seiberg duality in order to remain in a sensible field theory description), Maxwell charges are continuous. Consistency therefore requires that the rearrangement of the auxiliary brane system does not induce a discontinuity in the gauge invariant charges of the system. This is exactly what happens: remarkably, this rearrangement ensures continuity of the gauge couplings and exactly mimics the effect of Seiberg duality on the ranks of the groups. We will delve into this subtle point in the last chapters of this thesis.

For a given value of the holographic coordinate τ, we choose a gauge where $b(\tau) \in [0, 1]$. Therefore we can use the approximate holographic formulae (3.4.1) forgetting the 'mod'. When we hit the value τ_t for which $b(\tau_t) = 0$, we have seen that there is a rearrangement in the auxiliary brane system, which mimics Seiberg duality and ensures continuity of the gauge coupling. The fact that the gauge theory is self-similar along the cascade after a single Seiberg duality translates in the fact that the new BPS and minimal objects look always the same if we insist in remaining in the preferred 'frame' such that $b(\tau) \in [0, 1]$. This can be always achieved by performing a large gauge transformation whenever we hit a threshold. The integral of the NSNS B_2 potential is not a gauge invariant quantity and can be changed under a large gauge transformation. Indeed, if we denote by ω_2^{CF} the following two-cocycle:

$$\omega_2^{CF} = -\frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) \ , \tag{6.6.19}$$

we can perform a large gauge transformation which changes B_2 as follows:

$$B_2 \to B_2 + \Delta B_2 \ , \qquad\qquad \Delta B_2 = \pi\alpha' \, \omega_2^{CF} \ . \tag{6.6.20}$$

As ω_2^{CF} is closed, the field strength H_3 does not change and our transformation is a gauge transformation. The RR field strengths F_1, F_3 and F_5 are gauge invariant under this transformation.

Being ω_2^{CF} not exact, it is a large gauge transformation. Therefore, the flux of B_2 does change as:

$$\int_{S^2} B_2 \to \int_{S^2} B_2 + 4\pi^2\alpha' \ , \tag{6.6.21}$$

or equivalently $b \to b+1$.

Since RR field strengths F's are invariant under gauge transformations, we see that the combination $F_3 + B_2 \wedge F_1$ that enters the definition of the D5 brane Page charge changes as follows under (6.6.20):

$$\Delta(F_3 + B_2 \wedge F_1) = \Delta B_2 \wedge F_1 = \pi\alpha'\omega_2^{CF} \wedge \frac{N_f}{4\pi}\,\zeta = \frac{N_f}{4}\,\alpha'\,\omega_3^{CF} \ . \tag{6.6.22}$$

For the holographic computation of $U(1)_R$ anomalies, we will need $C_2 + C_0 B_2$. Locally, we can write $F_3 + B_2 \wedge F_1 = d\,(C_2 + C_0 B_2)$, but the quantity inside parentheses is not globally defined. Comparing with the variation (6.6.22), we get

$$\Delta(C_2 + C_0 B_2) = \frac{N_f}{8}\,\alpha' \Big[(\psi - \psi_*)\,(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 - \sin\theta_2\, d\theta_2 \wedge d\varphi_2) + \cos\theta_1 \cos\theta_2\, d\varphi_1 \wedge d\varphi_2\Big] \ , \tag{6.6.23}$$

where ψ_* is a constant. In the study of the R-symmetry anomaly of the next subsection we will actually need the change of the restriction of $C_2 + C_0 B_2$ to the submanifold $\theta_1 = \theta_2 \equiv \theta$, $\varphi_1 = 2\pi - \varphi_2 \equiv \varphi$. Denoting by $(C_2 + C_0 B_2)^{eff}$ the pullback of $C_2 + C_0 B_2$ to this submanifold, we get from (6.6.23) that:

$$\Delta(C_2 + C_0 B_2)^{eff} = \frac{N_f}{4}\,\alpha'\,(\psi - \psi_*)\,\sin\theta\, d\theta \wedge d\varphi \ . \tag{6.6.24}$$

Let us now study how Page charges change under these large gauge transformations. From the expressions written in (6.2.27), we obtain:

$$\begin{aligned} \Delta Q_{D5}^{Page} &= -\frac{1}{4\pi^2\alpha'}\int_{S^3} \Delta B_2 \wedge F_1 \\ \Delta Q_{D3}^{Page} &= -\frac{1}{(4\pi^2\alpha')^2}\int_{M_5} \Big(\Delta B_2 \wedge F_3 + \Delta B_2 \wedge B_2 \wedge F_1 + \frac{1}{2}\,\Delta B_2 \wedge \Delta B_2 \wedge F_1\Big) \ . \end{aligned} \tag{6.6.25}$$

Plugging in (6.6.25) our ansatz for F_3 and B_2 (6.2.10), together with the expression of ΔB_2 given in (6.6.20) as well as the relations (6.2.28) and (6.2.32), one readily gets:

$$\begin{aligned} \Delta Q_{D5}^{Page} &= -\frac{N_f}{2} \\ \Delta Q_{D3}^{Page} &= -M - \frac{N_f}{4} \ . \end{aligned} \tag{6.6.26}$$

Recall that for our solutions $Q_{D5}^{Page} = M$ (6.2.31) and $Q_{D3}^{Page} = N_0$ (6.2.35), in the gauge we have chosen initially. Formula (6.6.26) gives how these constants change under a large gauge transformation. At a given holographic scale τ we should perform as many large transformations as needed to have $b \in [0,1]$. Suppose that we start at some radius/energy scale in the preferred gauge frame in which $b \in [0,1]$. Flowing toward the infrared, at some point b will become negative (a gauge coupling becomes imaginary): at that point, we have to perform the large gauge transformation that we have considered (a Seiberg duality in field theory) so as to go back to the preferred gauge. Accordingly, Page charges change as in (6.6.26).

We can get an explicit expression of Q_{D5}^{Page} and Q_{D3}^{Page} in terms of the ranks r_l and r_s and the number of flavors N_{fl} and N_{fs}. In order to verify this fact, let us suppose that we are in a region of the holographic coordinate such that the two functions f and k of our ansatz are equal. Notice that for the flavored KS solution this happens in the UV, while for the flavored KT this condition holds for all values of the radial coordinate. If $f = k$ the normalized flux b in (6.6.2) can be written as:

$$b(\tau) = \frac{M}{\pi} f(\tau) \, . \tag{6.6.27}$$

Using this expression we can write the D5 brane Page charge (6.2.30) as:

$$Q_{D5}^{Page} = M_{eff} - \frac{N_f}{2} b \, . \tag{6.6.28}$$

Notice also that the supergravity expression (6.2.18) of M_{eff} can be written when $f = k$ as:

$$M_{eff} = M + \frac{N_f}{2} b \, . \tag{6.6.29}$$

Let us next assume that we have chosen our gauge such that, at the given holographic scale, $b \in [0, 1]$. In that case we can use the value of M_{eff} obtained by the field theory calculation of subsection 6.6.1 to evaluate the Page charge Q_{D5}^{Page}. Actually, by plugging the value of M_{eff} given in (6.6.15) on the right-hand side of (6.6.28) we readily get the following relation between Q_{D5}^{Page} and the field theory data:

$$Q_{D5}^{Page} = r_l - r_s - \frac{N_{fl}}{2} \, . \tag{6.6.30}$$

Similarly, for $f = k$, one can express the D3 brane Page charge (6.2.34) as:

$$Q_{D3}^{Page} = N_{eff} - bM - \frac{b^2}{4} N_f \, , \tag{6.6.31}$$

which, after using the relation (6.6.29), can be written in terms of M_{eff} as:

$$Q_{D3}^{Page} = N_{eff} - bM_{eff} + \frac{N_f}{4} b^2 \, . \tag{6.6.32}$$

Again, if we assume that $b \in [0, 1]$ and use the field theory expressions (6.6.16) and (6.6.15) of N_{eff} and M_{eff}, we get:

$$Q_{D3}^{Page} = r_s + \frac{3N_{fl} - N_{fs}}{16} \, . \tag{6.6.33}$$

Notice that, as it should, the expressions (6.6.30) and (6.6.33) of Q_{D5}^{Page} and Q_{D3}^{Page} that we have just found are independent of b, as far as $b \in [0, 1]$.

To make things simpler, a much more practical way of relating Page charges of the background in the preferred gauge to ranks in the dual gauge theory consists in assigning to the auxiliary branes that would engineer the gauge theory charges which change under large gauge transformations, and that can be obtained by taking the brane source charges of the previous subsection and setting $b = 0$. With this shortcut, we find directly (6.6.30) and (6.6.33). Such total source charges of the auxiliary brane system is mapped into the Page charges of the background in the preferred frame. One can straightforwardly verify that under a field theory Seiberg duality the right-hand sides of (6.6.30) and (6.6.33) transform as the left-hand sides do under a large gauge transformation (6.6.20) of supergravity.

Finally, let us remark that in this approach we follow the RG flow pointwise not only as far as gauge couplings are concerned, but also as ranks are concerned. Seiberg duality is performed at a fixed energy scale where a gauge coupling diverges (b becomes integer); M_{eff} and N_{eff} are left invariant (recall that Maxwell charges are gauge invariant), or better they keep changing continuously. From formulae (6.6.30) and (6.6.33) it is clear instead that Page charges provide a clean way to extract ranks and number of flavors of the corresponding good field theory (the one with real gauge couplings) dual at a given energy scale. The ranks of this good field theory description change as step-like functions along the RG flow, due to the fact that b varies continuously and needs to suffer a large gauge transformation every time that, flowing toward the IR, it reaches the value $b = 0$ in the good gauge. This large gauge transformation changes Q_{D5}^{Page} and Q_{D3}^{Page} in the way described above, which realizes in supergravity the change of the ranks under a Seiberg duality in field theory.

Let us now focus on a different way of matching the behavior of the field theory and our solutions.

6.6.3 R-symmetry anomalies and β-functions

We can compute the β-functions (up to the energy-radius relation) and the R-symmetry anomalies for the two gauge groups both in supergravity and in field theory in the spirit of [66, 88, 116]. In the UV, where the cascade takes place, they nicely match. For the comparison we make use of the following holographic formulae, which can be derived in the $\mathcal{N} = 2$ orbifold case by looking at the Lagrangian of the low energy field theory living on probe (fractional) D3 branes:[14]

$$\begin{aligned}
\frac{8\pi^2}{g_l^2} + \frac{8\pi^2}{g_s^2} &= 2\pi\, e^{-\Phi} \\
\frac{8\pi^2}{g_l^2} - \frac{8\pi^2}{g_s^2} &= \frac{e^{-\Phi}}{\pi}\Big[\frac{1}{\alpha'}\int_{S^2} B_2 - 2\pi^2\Big] \\
\theta_l + \theta_s &= 2\pi\, C_0 \\
\theta_l - \theta_s &= \frac{1}{\pi\alpha'}\int_{S^2}(C_2 + C_0 B_2) - 2\pi C_0\ .
\end{aligned} \tag{6.6.34}$$

We have written the holographic relations directly in the preferred gauge where $b = \frac{1}{4\pi^2\alpha'}\int_{S^2} B_2 \in [0,1]$, that we reach by performing large gauge transformations.

We have adapted the indices in (6.6.34) to the previous convention for the gauge group with the larger (the smaller) rank. Let us restrict our attention to an energy range, between two subsequent Seiberg dualities, where a specific field theory description in terms of fixed ranks holds. In this energy range the gauge coupling g_l of the gauge group with larger rank flows toward strong coupling, while the gauge coupling g_s of the gauge group with smaller rank flows toward weak

[14]These formulas provide a very good approximation when the quartic (dimensionless) couplings are almost constant. This is what occurs in the Klebanov-Strassler cascade, and we will assume it occurs for our solutions as well.

coupling. Indeed, as formulae (6.6.34) confirm, the coupling g_l was not touched by the previous Seiberg duality, starts different from zero and flows to infinity at the end of this range, where a Seiberg duality on its gauge group is needed. The coupling g_s of the gauge group with smaller rank is the one which starts very large (actually divergent) after the previous Seiberg duality on its gauge group, and then flows toward weak coupling.

In supergravity, due to the presence of magnetic sources for F_1, we cannot globally define a potential C_0.[15] Therefore we project our fluxes on the submanifold S^2: $\theta_1 = \theta_2 \equiv \theta$, $\varphi_1 = 2\pi - \varphi_2 \equiv \varphi$, $\forall\, \psi, \tau$ before integrating them. Recalling that locally we can write $F_3 + B_2 \wedge F_1 = d(C_2 + C_0 B_2)$, what we get from (6.2.9)-(6.2.10) (in the UV limit) are the effective potentials

$$\begin{aligned} C_0^{eff} &= \frac{N_f}{4\pi}\,(\psi - \psi_{0*}) \\ (C_2 + C_0 B_2)^{eff} &= -\alpha' \left(\frac{M}{2} + n\frac{N_f}{4} \right) (\psi - \psi_*)\, \sin\theta\, d\theta \wedge d\varphi\ . \end{aligned} \tag{6.6.35}$$

The integer n in $(C_2 + C_0 B_2)^{eff}$ arises from n inverse large gauge transformations (6.6.20) on B_2, that are needed to shift $b(\tau) \in [n, n+1]$ by n units, so that the gauge transformed $\tilde{b}(\tau) = b(\tau) - n$ lies between 0 and 1.

The field theory possesses an anomalous R-symmetry which assigns charge $\frac{1}{2}$ to all chiral superfields.[16] The field theory R-anomalies are easily computed. Continuing to use r_l (r_s) for the larger (smaller) group rank and N_{fl} (N_{fs}) for the corresponding flavors (see Figure 6.1), the anomalies under a $U(1)_R$ rotation of parameter ε are:

$$\text{Field theory:} \qquad \begin{aligned} \delta_\varepsilon \theta_l &= [2(-r_l + r_s) + N_{fl}]\,\varepsilon \\ \delta_\varepsilon \theta_s &= [2(r_l - r_s) + N_{fs}]\,\varepsilon\ . \end{aligned} \tag{6.6.36}$$

Along the cascade of Seiberg dualities, the coefficients of the anomalies for the two gauge groups change when we change the effective description; what does not change is the unbroken subgroup of the R-symmetry group. Because we want to match them with the supergravity computations, it will be convenient to rewrite the field theory anomalies in the following form:

$$\text{Field theory:} \qquad \begin{aligned} \delta_\varepsilon(\theta_l + \theta_s) &= N_f\,\varepsilon \\ \delta_\varepsilon(\theta_l - \theta_s) &= [-4(r_l - r_s) + N_{fl} - N_{fs}]\,\varepsilon\ . \end{aligned} \tag{6.6.37}$$

An infinitesimal $U(1)_R$ rotation parameterized by ε in field theory corresponds to a shift $\psi \to \psi + 2\varepsilon$ in the geometry. Therefore, making use of (6.6.35), we find on the supergravity side:

$$\text{SUGRA:} \qquad \begin{aligned} \delta_\varepsilon(\theta_l + \theta_s) &= N_f\,\varepsilon \\ \delta_\varepsilon(\theta_l - \theta_s) &= -[4M + (2n+1)N_f]\,\varepsilon\ . \end{aligned} \tag{6.6.38}$$

These formulae exactly agree with those computed in the field theory. In order to see that this agreement holds not only for the sum but also for the difference of the theta angles, we have to translate M into a linear combination of ranks in the dual gauge theory. In the preferred gauge we

[15] To be precise, since we have smeared the magnetic D7 brane sources, we cannot even define it locally.

[16] Although the R-charges of the chiral superfields are half-integer, an R-rotation of parameter $\varepsilon = 2\pi$ coincides with a baryonic rotation of parameter $\alpha = \pi$. It follows that $U(1)_R \times U(1)_B$ is parameterized by $\varepsilon \in [0, 2\pi]$, $\alpha \in [0, 2\pi]$.

are working, the D5 brane Page charge, which is related to the ranks of the field theory by (6.6.30), is

$$Q_{D5}^{Page} = M + n\frac{N_f}{2} \ . \tag{6.6.39}$$

Again, the second term arises because of the $-n$ large gauge transformations (6.6.20) necessary to move to the preferred gauge where (6.6.34) hold. If we trade M for the ranks, using (6.6.39) and (6.6.30), we see that in the right-hand side of (6.6.38) $-[4M+(2n+1)N_f] = -4(r_l-r_s)+N_{fl}-N_{fs}$ precisely gives the same result of the field theory computation in (6.6.37).

It is very easy to check also that these anomalies change under a Seiberg duality (large gauge transformations) in the way predicted by the field theory. This should be obvious by now, because we have already seen that the variation of Page charges under a large gauge transformation agrees with the variation of ranks under Seiberg duality.

The dictionary (6.6.34) allows us also to compute the β-functions of the two gauge couplings and check further the picture of the duality cascade.

Since we will be concerned in the cascade, we will make use of the flavored Klebanov-Tseytlin solution of section 6.4, to which the flavored Klebanov-Strassler solution of section 6.3 reduces in the UV limit.

We shall keep in mind that, at a fixed value of the radial coordinate, we want to shift $b = \frac{1}{4\pi^2\alpha'}\int_{S^2} B_2$ by means of a large gauge transformation in supergravity in such a way that its gauge transformed $\tilde{b} = b - [b] \equiv b - n$[17] belongs to $[0,1]$: in doing so, we are guaranteed to be using the good description in terms of a field theory with positive squared gauge couplings.

Recall that

$$e^{-\Phi} = \frac{3N_f}{4\pi}(-\rho) \tag{6.6.40}$$

$$b = \frac{2M}{N_f}\left(\frac{\Gamma}{\rho} - 1\right) \ , \tag{6.6.41}$$

and the dictionary (6.6.34), that we rewrite in our preferred gauge as:

$$\frac{8\pi^2}{g_+^2} \equiv \frac{8\pi^2}{g_l^2} + \frac{8\pi^2}{g_s^2} = 2\pi e^{-\Phi} \tag{6.6.42}$$

$$\frac{8\pi^2}{g_-^2} \equiv \frac{8\pi^2}{g_l^2} - \frac{8\pi^2}{g_s^2} = 2\pi e^{-\Phi}(2\tilde{b} - 1) \ . \tag{6.6.43}$$

Then we can compute the following 'radial' β-functions from the gravity dual

$$\beta_+^{(\rho)} \equiv \beta^{(\rho)}_{\frac{8\pi^2}{g_+^2}} \equiv \frac{d}{d\rho}\frac{8\pi^2}{g_+^2} \tag{6.6.44}$$

$$\beta_-^{(\rho)} \equiv \beta^{(\rho)}_{\frac{8\pi^2}{g_-^2}} \equiv \frac{d}{d\rho}\frac{8\pi^2}{g_-^2} \ , \tag{6.6.45}$$

[17] $n(\rho)$ here should be thought as a step function of the radial coordinate.

and we would like to match these with the field theory computations.
Using the expressions (6.6.42)-(6.6.43), we can conclude that

$$\beta_+^{(\rho)} = -3\frac{N_f}{2} \tag{6.6.46}$$

$$\beta_-^{(\rho)} = 3\left(\frac{N_f}{2} + Q\right) , \tag{6.6.47}$$

where $Q = N_f[b(\rho)] + 2M = N_f n(\rho) + 2M$ is a quantity which undergoes a change $Q \to Q - N_f$ as $b(\rho) \to b(\rho') = b(\rho) - 1$ (one Seiberg duality step along the cascade toward the IR), or equivalently $n(\rho) \to n(\rho') = n(\rho) - 1$. Notice that we can rewrite these two 'radial' β-functions as

$$\begin{aligned} \beta_+^{(\rho)} &= -\frac{3}{2} N_f \\ \beta_-^{(\rho)} &= \frac{3}{2}\left[4M + (2n+1)N_f\right] = -\frac{3}{2}\left[-4(r_l - r_s) + N_{fl} - N_{fs}\right] , \end{aligned} \tag{6.6.48}$$

which, up to an overall factor of $-3/2$, are the same quantities appearing in the R-anomalies in (6.6.38).

The field theory computations of the β-functions give:

$$\beta_l \equiv \beta_{\frac{8\pi^2}{g_l^2}} = 3r_l - 2r_s(1-\gamma_A) - N_{fl}(1-\gamma_q) \tag{6.6.49}$$

$$\beta_s \equiv \beta_{\frac{8\pi^2}{g_s^2}} = 3r_s - 2r_l(1-\gamma_A) - N_{fs}(1-\gamma_q) , \tag{6.6.50}$$

with the usual conventions. Hence

$$\beta_+ \equiv \beta_l + \beta_s = (r_l + r_s)(1+2\gamma_A) - N_f(1-\gamma_q) \tag{6.6.51}$$

$$\beta_- \equiv \beta_l - \beta_s = (5-2\gamma_A)(r_l - r_s) + (N_{fs} - N_{fl})(1-\gamma_q) . \tag{6.6.52}$$

In order to match the above quantities with the gravity computations (6.6.46)-(6.6.47), an energy-radius relation is required. This is something we lack here. Although it is not really needed to extract from our supergravity solutions the qualitative information on the running of the gauge couplings, we are going initially to make an assumption, which can be viewed as an instructive simplification. Let us assume that the radius-energy relation is $\rho = \log \frac{\mu}{E_{UV}}$, where E_{UV} is the scale of the UV cutoff dual to the maximal value of the radial coordinate $\rho = 0$. Matching β_+ and β_- implies $\gamma_A = \gamma_q = -\frac{1}{2}$.

Actually, the qualitative picture of the RG flow in the UV can be extracted from our supergravity solution even without knowing the precise radius-energy relation, but simply recalling that the radius must be a monotonic function of the energy scale.

It is interesting to notice the following phenomenon: as we flow up in energy and approach the far UV $\rho \to 0^-$ in (6.6.41), a large number of Seiberg dualities is needed to keep b varying in the interval $[0,1]$. The Seiberg dualities pile up the more we approach the UV cut-off E_{UV}. Meanwhile, formula (6.6.47) reveals that, when going toward the UV cutoff E_{UV}, the 'slope' in the plots of $\frac{1}{g_i^2}$ versus the energy scale becomes larger and larger, and (6.6.46) reveals that the sum of the inverse squared gauge coupling goes to zero at this UV cutoff. At the energy scale E_{UV} the effective number of degrees of freedom needed for a sensible field theory description becomes

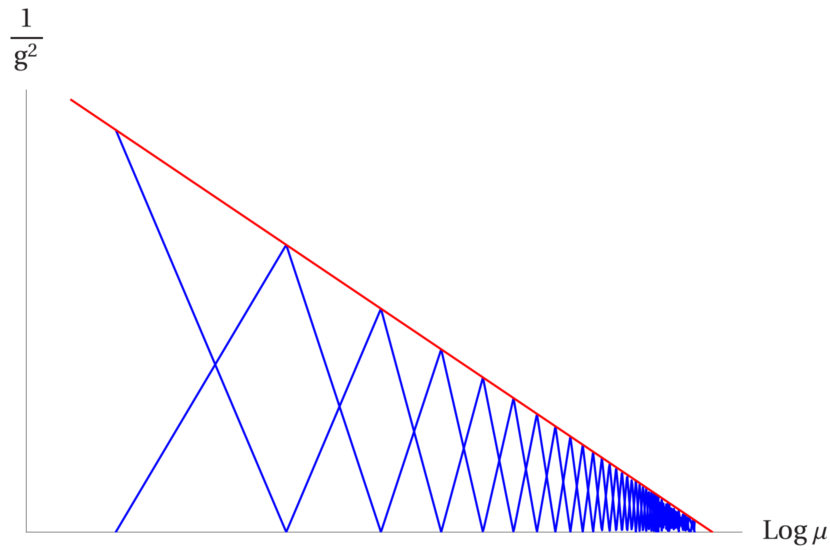


Figure 6.3: Qualitative plot of the running gauge couplings as functions of the logarithm of the energy scale in our cascading gauge theory. The blue lines are the inverse squared gauge couplings, while the red line is their sum.

infinite. Since $\rho = 0$ is at finite proper radial distance from any point placed in the interior $\rho < 0$, E_{UV} is a finite energy scale.

The picture which stems from our flavored Klebanov-Tseytlin/Strassler solution is that E_{UV} is a so-called "Duality Wall", namely an accumulation point of energy scales at which Seiberg dualities are required in order to have a sensible gauge theory description [111]. Above the duality wall, Seiberg duality does not proceed, see figure 6.3. If a UV completion exists, it is not field theoretical. However, it could be defined holographically if a string theory solution for the D3-D7 brane system which can be extended globally is found.

Duality walls have been studied in the context of quiver gauge theories first in [117] and later in [118,119]. That analysis of this phenomenon was in the framework of quiver gauge theories with only bifundamental chiral superfields, and was restricted to the field theory. In [119] a gravity dual was constructed, but it did not described a field theory with a duality wall.

To our knowledge, our solutions are the first explicit realizations of this exotic ultraviolet phenomenon on the supergravity side of the gauge/gravity correspondence.

We remark that our flavored Klebanov-Strassler flow not only displays a duality wall in the ultraviolet, but also a Landau pole for g_+ at the same energy scale. These are two possibly independent phenomena. A duality wall can exist independently of the fate of the coupling g_+ related to the dilaton. Examples of field theories where a duality wall is expected, but the positive linear combination of inverse squared gauge coupling dual related to the dilaton stays constant were studied in [118].

6.7 A comment on the partition of flavors and the IR

So far, we have seen that from the three brane charges that we measure and the cascade pattern that we have analyzed, we cannot conclude anything about the partition of the total numbers N_f into N_{fl} and N_{fs} flavors for one or the other color group respectively. We have mentioned the difference between the two kinds of D7 branes that provide flavors for one group or the other: they lie on the same embedding, which has $\mathbb{C}^2/\mathbb{Z}_2$ topology, but they carry either zero or minus one unit of flux of the flat worldvolume gauge field on the exceptional two-cycle of the embedding, sitting at the conifold singularity. Kuperstein's D7 branes are then fractional branes, in the sense that they unavoidably wrap the exceptional two-cycle.

The remnant of this flux stuck at the singularity is a Wilson line of the gauge field at infinity, along the noncontractible cycle of the embedding [113, 120]. For toric Calabi-Yau cones, to any holomorphic D7 brane embedding along a noncompact (toric) divisor is associated a 3-cycle, the radial section of the divisor, over which a D3 brane can be wrapped supersymmetrically. Wrapped D3 branes are dual to dibaryonic operator in the conformal theory, and there exist as many baryonic $U(1)$'s as 3-cycles in the Sasaki-Einstein base. This correspondence ensures that flavor fields from D3-D7 strings are coupled to the bifundamental field (elementary or composite) which is the building block of the dibaryon related to the 3-cycle. For toric CY cones, these 3-cycles are generically Lens spaces with $S^3/\mathbb{Z}_m$ topology, having fundamental group $\pi_1(S^3/\mathbb{Z}_m) = \mathbb{Z}_m$. This means that there exist m possible values that the Wilson line of the gauge field living on the D brane worldvolume can acquire.

In the case of Kuperstein's embedding, the divisor is algebraically $\mathbb{C}^2/\mathbb{Z}_2$, whose base is $S^3/\mathbb{Z}_2$. Consequently, the holonomy at infinity of the D7 brane gauge field is $e^{i\oint_C A} = \pm 1$. If the D7 brane has an odd number of units of flux of the gauge field strength through the exceptional 2-cycle, then the Wilson line along the noncontractible 1-cycle is -1, otherwise it is 1.

Let us come back to our solutions, and let us start by considering the one on the singular conifold (section 6.4). As we have explained, in this case the two-cycle supporting the gauge field flux on the D7 branes is at the conifold singularity, that we cannot access in our analysis of charges. The gauge field on the flavor branes is flat outside the singularity, for both kinds of fractional D7 branes. Therefore our supergravity background (color dynamics) should be valid for any partition of the N_f flavor branes into branes with trivial and nontrivial Wilson line. What will change, depending on the number of D7 branes of one or the other kind, is the dynamics of flavor fields.

Things are different when the manifold is the deformed conifold, as in section 6.3. The effect of the deformation is to give support to a 3-cycle at $\tau = 0$. Correspondingly, the (formerly exceptional) 2-cycle the D7 branes were wrapping develops a nonvanishing size even at $\tau = 0$: the embedding of the D7 branes has now the topology of the blown up $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$. Having the singularity on the embedding disappeared, we are able to distinguish the two kinds of flavor branes. Indeed, the D7 brane configuration with a nonvanishing Wilson line at infinity must now have a nonvanishing worldvolume gauge flux along the 2-cycle at any value of τ, because the gauge field is not flat anymore, but only approaches the flat configuration asymptotically. On the contrary, the D7 brane which was lacking worldvolume flux on the exceptional 2-cycle still has vanishing flux on the blown up 2-cycle for any value of τ, since in that case $F_2 = 0$ everywhere. Since in our solution we did not put any worldvolume gauge flux, we are forced to conclude that our solution describes the gauge theory with flavors coupled to a single gauge group. The fact that the flavored warped deformed conifold solution we have presented has vanishing D3 brane charge in the IR indicates that a single

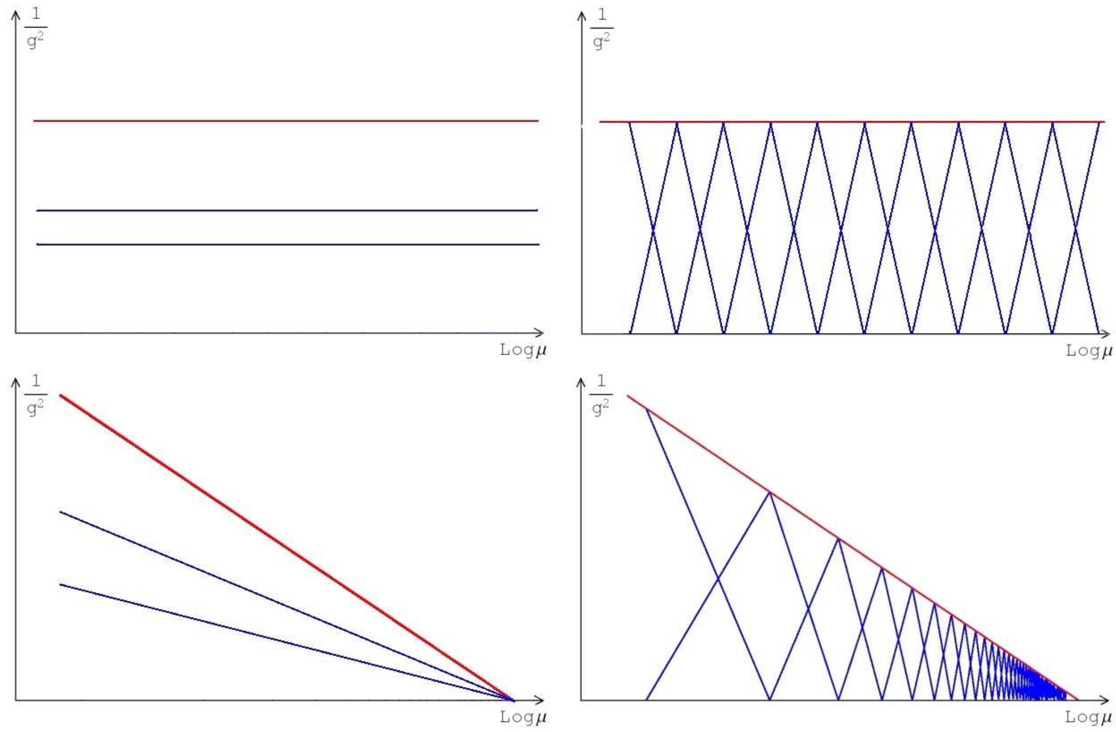


Figure 6.4: Ultraviolet RG flows of KW (top left), KT/KS (top right), flavored KW (bottom left), flavored KT/KS (bottom right) theories.

gauge group survives in the deep IR, after the "last step" of the cascade of Seiberg duality (which is actually not a Seiberg duality). The analysis of the matter content associated to the different fractional branes of section 6.6.1 tells us that the flavor fractional brane with trivial Wilson line, that our solution is accounting, is coupled to the color group which disappears in the IR. Therefore we conclude that at the very last step of the cascade we end up with a gauge theory without flavors.

Finding the general solution for Kuperstein flavors on the deformed conifold would provide the dual of a cascading gauge theory reducing in the deep IR to $\mathcal{N} = 1$ SQCD with a quartic superpotential, a theory with rich dynamics and of great interest. This requires the addition of a worldvolume gauge field on some of the D7 branes. We leave this very important problem to future work.

6.8 Remarks on RG flows of conifold theories

Finally, it is very instructive to compare the different (ultraviolet) RG flows of the KW solution, KT solution, flavored KW solution and flavored KT (or KS) solution, which are summarized in fig. 6.4. In view of this comparison, we collected the derivation of these solutions by supersymmetry methods in appendices C.1-C.4. The derivation highlights the similarities and differences between these four solutions, which are mirrored in the RG flow analysis.

Adding fractional D3 branes to the conformal KW solution induces the difference of the inverse squared gauge coupling to run and creates an infinite Seiberg duality cascade ($b \to \infty$ when τ approaches its maximal value ∞). The KT solution is obtained by adding $(2,1)$ primitive ISD closed 3-fluxes G_3 to the KW solution. Adding flavor D7 branes to the conformal KW solution induces the sum of the inverse squared gauge coupling to run, and creates a Landau pole at a finite UV energy scale. This flavored KW solution is obtained by adding a supersymmetric $(1,0)$ $d\tau$ to the KW solution. Pictorially, the flavored KW RG flow can be obtained from the trivial KW one by pinching the point at infinity and pulling it to a finite value.

The flavored KT/KS solution can be obtained by adding supersymmetric $(2,1)$ primitive ISD 3-fluxes G_3 to the flavored KW solution, but also adding a supersymmetric $(1,0)$ $d\tau$ to the unflavored KT/KS solution. In the first case, because of the running 5-flux $\int F_5$, an RG flow for g_- is induced on top of the running which drives g_+ to a Landau pole in the UV. In the second case, because of the running of the dilaton caused by D7 branes, an RG flow for g_+ which drives it to a Landau pole in the UV is induced on top of the flow of g_- leading to the cascade of Seiberg dualities. The inverse unflavored limits have been discussed in this chapter for the flavored KT/KS solutions and in the previous one for the flavored KW solution: it involves sending the Landau pole scale to infinity. We stress that the Landau pole scale (the scale where the dilaton diverges) and the duality wall scale (the scale where b diverges) are the same in the flavored KT/KS solution. Again, pictorially the flavored KT/KS RG flow can be obtained from the KT/KS one by pinching the point at infinity (where b diverges) and pulling to a finite value.

Part III

Surprises in cascading gauge theories

Chapter 7

Fractional branes interplay in cascading theories

In this chapter, we provide the gravity dual describing the UV regime (and in some cases also the IR regime) of cascading gauge theories realized by a general configuration of fractional D3 branes on a $\mathbb{Z}_2$ orbifold of the conifold. Several matchings and predictions for the cascade of the dual gauge theories are easily extracted by means of an algorithm that we develop. We find hints of a generalization of the familiar cascade of Seiberg dualities due to a nontrivial interplay between the different types of fractional branes.

This chapter is based on [3], written in collaboration with Riccardo Argurio, Francesco Benini, Matteo Bertolini and Cyril Closset.

7.1 Introduction

The correspondence between gauge theories with nontrivial low-energy dynamics and string theory backgrounds has an enormous potential. The string theory setup is usually established drawing uniquely on the holomorphic data of a supersymmetric gauge theory, including a specific choice of vacuum. Then, solving the classical equations of motion of supergravity one can in principle obtain, through the warp factor, all the dynamical information on the gauge theory low-energy dynamics, that would instead usually imply precise knowledge of the Kähler sector. The limitation of this procedure to supergravity and not to full string theory corresponds in the gauge theory to taking some large N and strong 't Hooft coupling limit.

A fruitful arena where to address these issues has proven to be that of D3 branes at Calabi-Yau (CY) singularities. In this context, the most celebrated example where such a program has been successfully completed is the warped deformed conifold [37], which describes a theory with confinement and chiral symmetry breaking.

It is of obvious interest to apply the above program to gauge theories with a varied low-energy behavior. D3 branes at CY singularities typically give rise to $\mathcal{N} = 1$ quiver gauge theories, which are supersymmetric theories characterized by product gauge groups, matter in the bifundamental representation and a tree level superpotential, all such data being dictated by the structure of the singularity. Most quiver gauge theories can have several different IR behaviors, depending on which branch of the moduli space one is sitting on. Already in the simple conifold theory, one has a baryonic branch displaying confinement and a mass gap in the gauge sector, and mesonic branches with a dynamics which is $\mathcal{N}=4$ to a good approximation. In more general quivers, other kinds of low-energy behaviors are possible. Some quivers will actually have no vacua and display a runaway behavior [53, 121–124], but this leaves little hope of finding a regular gravity dual. Other quivers will on the other hand contain branches of the moduli space where the dynamics is approximately the one on the Coulomb branch of an $\mathcal{N}=2$ theory. The latter can also be thought of as mesonic branches, albeit of complex dimension one instead of three as in the (generic) $\mathcal{N}=4$ case.

As has been shown in [125, 126], theories with both baryonic and $\mathcal{N}=2$ mesonic branches can be very interesting because they are likely to possess, besides the supersymmetric vacua, also metastable supersymmetry breaking vacua. The latter arise precisely because there is a tension between the conditions for realizing baryonic or mesonic vacua among the various nodes of the quiver. On the gravity/string side, the metastable vacua are associated to the presence of anti-D3 branes. They are only metastable because they can decay through an instanton that shifts the flux in such a way that their charge is cancelled. Of course, a full gravity solution of such a supersymmetry breaking vacuum would be a wonderful arena for studying quantitatively the low energy dynamics of such theories.

In this chapter, we take a first step towards this goal. We construct the gravity dual of the most generic gauge theory one can engineer using D3 branes at the tip of a $\mathbb{Z}_k$ nonchiral orbifold of the conifold [100], focusing for simplicity, but with little loss of generality, on the case $k=2$. This singularity admits different kinds of fractional branes, triggering confinement or enjoying an $\mathcal{N}=2$ mesonic branch and known as deformation or $\mathcal{N}=2$ fractional branes, respectively. We aim at describing the backreaction of the most general D3 brane bound state. The difficulty in doing so stems from the fact that the UV completion which corresponds to the supergravity solution is qualitatively different in the two cases. For deformation branes, the common lore is that the renormalization group (RG) flow is described in terms of a cascade of Seiberg dualities which increases the overall rank of the quiver nodes towards the UV. For $\mathcal{N}=2$ branes, the RG flow (which is indeed present and also increases the ranks towards the UV [38, 39]) seems to be better represented by some form of Higgsing [64]. We will study this phenomenon in detail in the last chapter of this thesis.

It should be clear that whenever there are $\mathcal{N}=2$ branes around the IR of the gravity dual is bound to contain some singularity. This is because their open string degrees of freedom cannot completely transmute into flux. Indeed, on the Coulomb branch we still have by definition some surviving abelian gauge group, which cannot be described in terms of closed string degrees of freedom. This situation is similar to the situation where one aims at describing theories with flavors. There too, flavor degrees of freedom must be described by open strings, and hence flavor branes must be present in the gravity dual as physical sources [79], as we have seen in the previous chapters. Thus in our set up we expect to have physical sources corresponding to $\mathcal{N}=2$ fractional branes. The main difference with respect to the case of flavor branes is that $\mathcal{N}=2$ fractional branes are not infinitely extended but rather localized in the Calabi-Yau.

The main results of our analysis can be summarized as follows. We find an explicit analytical supergravity solution corresponding to the UV regime of a generic distribution of fractional branes, both of the deformation and $\mathcal{N}=2$ kind, on the orbifolded conifold. It describes holographically an RG flow which exactly matches the beta functions that one can compute in the dual field theory and the expected reduction of degrees of freedom towards the IR, which occurs through a cascade. We develop an algorithm to follow the RG flow of each gauge coupling from the supergravity solution.

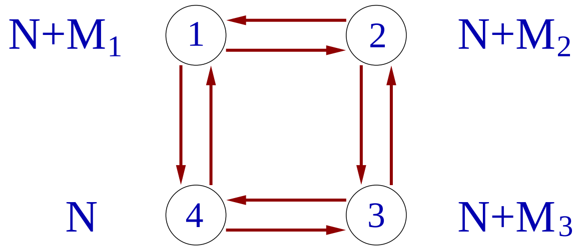


Figure 7.1: The quiver diagram of the gauge theory, for the most generic choice of ranks. For later purposes we have parameterized the four independent ranks in terms of a common N.

An interesting feature is that in this general setting there are cascade steps that do not always have a simple interpretation in terms of Seiberg dualities. This is due to the presence of $\mathcal{N} = 2$ fractional branes, or more generally to the presence of twisted fluxes. Nevertheless, supergravity considerations and field theory expectations (based on the nonholomorphic beta function) exactly match.

As far as the IR regime of the gauge theory is concerned, in [3] we also performed a nontrivial consistency check matching the field theory effective superpotential with that predicted from the geometric background. We also provided the solution for the 3-form fluxes and discussed the pattern of singularities resolution, which depends on the number and the species of fractional branes placed at the singularity, while we only set the stage for computing the exact warp factor. The discussion of the IR regime has its own interest, especially in view of the proposed existence of metastable supersymmetry breaking vacua obtained by adding anti-branes to the supersymmetric background, but has not been included here.

This chapter is structured as follows. In section 7.2 we explain our setup and introduce the minimal geometrical data that is needed in the following. In section 7.3 we present the supergravity solution which is expected to reproduce the UV behavior of our quiver gauge theory. We take the CY base to be the orbifold of the singular conifold, but we take into account all the fluxes sourced by the fractional branes and compute their backreaction on the warp factor. We then check that the result is indeed compatible with the expected RG flow and perform a number of nontrivial gauge/gravity duality checks and predictions. We briefly summarize our results in a final section. We have added appendix D, containing many technical data which might help in better understanding the form of the supergravity ansatz that we solve in the main text and the geometric structure of the orbifolded conifold CY singularity we consider.

7.2 The orbifolded conifold

We consider in what follows an orbifolded avatar of the familiar conifold quiver. We focus on a nonchiral $\mathbb{Z}_2$ orbifold of the conifold and consider the corresponding $\mathcal{N} = 1$ supersymmetric quiver gauge theory obtained by placing a bound state of regular and fractional D3 branes at its tip. This theory has been analyzed at great length in [125], to which we refer for more details.

The quiver gauge theory is shown in Figure 7.1. The gauge theory has four gauge factors and a tree level superpotential for the bifundamental fields

$$W = h\left(X_{12}X_{21}X_{14}X_{41} - X_{23}X_{32}X_{21}X_{12} + X_{34}X_{43}X_{32}X_{23} - X_{41}X_{14}X_{43}X_{34}\right) , \qquad (7.2.1)$$

where X_{ij} is a chiral superfield in the fundamental representation of the i-th gauge group and antifundamental representation of the j-th gauge group, and traces on the gauge degrees of freedom are understood.

We are interested in the dynamics of the gauge theory with the most generic rank assignment, as in Figure 7.1. Depending on the values of the M_i's, various kinds of IR dynamics can occur: confinement, runaway behavior or a (locally $\mathcal{N} = 2$) quantum moduli space.

There is a relation between the ranks of the various gauge groups in the quiver and the number of fractional branes wrapping the different 2-cycles in the geometry. In turn, the fractional branes source the RR 3-form flux which is an important ingredient in order to determine the supergravity solution. In the following of this section we provide the link between these three sets of data (ranks, branes wrapping cycles, fluxes). For a more detailed discussion we refer to appendix D.

7.2.1 Regular and fractional branes

The superconformal theory ($N \neq 0\,,\ M_i = 0$) can be engineered by placing N regular D3 branes at the tip of the cone. Unbalanced ranks in the quiver of Figure 7.1 correspond instead to the presence of fractional D3 branes and the corresponding breaking of conformal invariance. From the gauge theory viewpoint, fractional branes correspond to independent anomaly free rank assignments in the quiver (modulo the superconformal one). Hence, in the present case, we have three types of fractional branes to play with.

In general, fractional branes can be classified in terms of the IR dynamics they trigger [122].

A first class of fractional branes are those associated to a single node in the quiver, or to several decoupled nodes, or else to several contiguous nodes whose corresponding closed loop operator appears in the tree level superpotential. This subsector of the quiver gauge theory undergoes confinement. The dual effect in string theory is a geometric transition, which means that the branes induce a complex structure deformation. Hence the name deformation fractional branes. Examples of this kind in our theory correspond to rank assignments $(1,0,0,0)$, $(1,0,1,0)$ or $(1,1,1,0)$ and cyclic permutations.

Another class of fractional branes are those associated to closed loops in the quiver whose corresponding operator does not appear in the superpotential. Such a subquiver has a mesonic moduli space which corresponds to the Coulomb branch of an effective $\mathcal{N} = 2$ SYM theory. Hence the name $\mathcal{N} = 2$ fractional branes. Geometrically, $\mathcal{N} = 2$ fractional branes are located at nonisolated codimension four singularities in the CY threefold. Such singularities locally look like $\mathbb{C} \times \mathbb{C}^2/\Gamma$ (where $\Gamma = \mathbb{Z}_2$ in our case), where the $\mathbb{C}$ complex line corresponds to the Coulomb branch of the effective $\mathcal{N} = 2$ gauge theory. In the gauge theory a $U(1)^{N-1}$ gauge group survives. In this case the branes cannot undergo a geometric transition, because there exists no local complex deformation of such a nonisolated singularity. Hence the supergravity dual background is expected to display some leftover singularity. Rank assignments corresponding to this class of branes in our quiver are for instance $(1,1,0,0)$ and cyclic permutations.

Finally, fractional branes of any other class (which is the most generic case, in fact) lead to ADS-like superpotential and runaway behavior and as such are called DSB (dynamical supersymmetry breaking) branes. Geometrically, they are associated with geometries where the complex structure deformation is obstructed, this tension being the geometric counterpart of the runaway. In this case the occupied nodes have unbalanced ranks.

Obviously, combining different fractional branes of a given class, one can obtain fractional branes of another class. Hence one can choose different fractional brane bases to describe the gauge theory. In our present case, we will be able to choose a basis composed only of deformation and $\mathcal{N}=2$ fractional branes. We have just seen to which rank assignments the various branes should correspond, now we have to review which 2-cycles they are associated to.

7.2.2 Geometry, cycles and quiver ranks

There is a well established relation between quiver configurations, the primitive topologically non-trivial shrinking 2-cycles of a given CY singularity, and the possible existing fractional D3 branes, since the latter can be geometrically viewed as D5 branes wrapped on such cycles. Let us review such relation for our CY singularity (see appendix D for a full analysis).

The conifold is a noncompact CY three-fold described by the following equation in $\mathbb{C}^4$: $z_1 z_2 - z_3 z_4 = 0$. We consider a $\mathbb{Z}_2$ orbifold of such singularity defined by the symmetry

$$\Theta : \quad (z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, -z_3, -z_4) \ . \tag{7.2.2}$$

The resulting orbifolded geometry is described by the following equation in $\mathbb{C}^4$

$$(z_1 z_2)^2 - xy = 0 \ , \tag{7.2.3}$$

where $x = z_3^2$ and $y = z_4^2$. There is a singular locus in this variety which consists of two complex lines, that we call the p and q lines, respectively. They meet at the tip $\{z_1 = z_2 = x = y = 0\}$ and correspond to the fixed point locus of the orbifold action Θ.

One can as well describe the variety as a real manifold. The coordinates we use are defined in appendix B.1. From this point of view the conifold is a real cone over $T^{1,1}$, which in turn is a $U(1)$ bundle over $S^2 \times S^2$. The orbifold action (7.2.2) reads in this case

$$\Theta : \quad (\varphi_1, \varphi_2) \rightarrow (\varphi_1 - \pi, \varphi_2 + \pi) \ . \tag{7.2.4}$$

The two complex lines are defined, in complex and real coordinates respectively, as

$$\begin{aligned} p &= \{z_1 = x = y = 0, \, \forall z_2\} = \{\theta_1 = \theta_2 = 0, \, \forall r, \psi'\} \\ q &= \{z_2 = x = y = 0, \, \forall z_1\} = \{\theta_1 = \theta_2 = \pi, \, \forall r, \psi''\} \ , \end{aligned} \tag{7.2.5}$$

where $\psi' = \psi - \varphi_1 - \varphi_2$ and $\psi'' = \psi + \varphi_1 + \varphi_2$ are (well defined) angular coordinates along the singularity lines. In a neighborhood of the singular lines (and outside the tip) the geometry looks locally like the A_1-singularity $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. The fixed point curve p sits at the north poles of both S^2's while the curve q sits at the south poles. A sketch of the conifold geometry in these real coordinates and of the fixed points of Θ is given in Figure 7.2.

Our CY cone has three vanishing 2-cycles. Two of these three 2-cycles arise due to the orbifold action. Such exceptional 2-cycles are located all along the $\mathbb{C}^2/\mathbb{Z}_2$ singular lines p and q, and we call them $\mathcal{C}_2$ and $\mathcal{C}_4$, respectively. The third relevant 2-cycle descends from the 2-cycle of the parent conifold geometry, whose base $T^{1,1}$ is topologically $S^2 \times S^3$. Correspondingly, we will have a basis consisting of three fractional branes.

In appendix D we construct different fractional brane bases. However, the basis we will favor here is the one arising most naturally when viewing our singularity as a $\mathbb{Z}_2$ projection of the conifold, which as anticipated is given in terms of the two $\mathcal{N}=2$ 2-cycles $\mathcal{C}_2$ and $\mathcal{C}_4$ and a deformation 2-cycle, $\mathcal{C}_\beta$. This basis of 2-cycles corresponds to a particular resolution of the singularity, which is

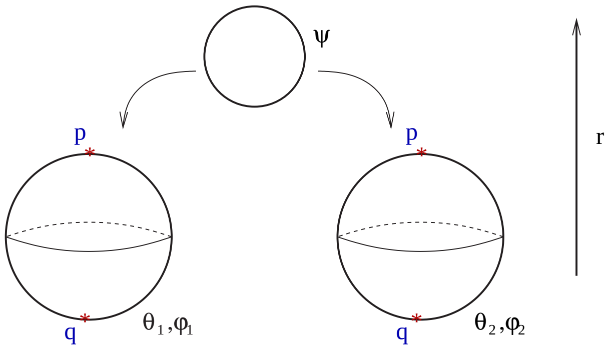


Figure 7.2: The singular conifold in real angular coordinates: it is a real cone in r over $T^{1,1}$, which in turn is a $U(1)$ fibration in ψ over the Kähler-Einstein space $\mathbb{P}^1 \times \mathbb{P}^1$ parameterized by θ_i and φ_i. The fixed point locus of the orbifold action Θ is given by two lines p and q, localized at antipodal points on the two S^2's. At the tip the spheres shrink and p and q meet.

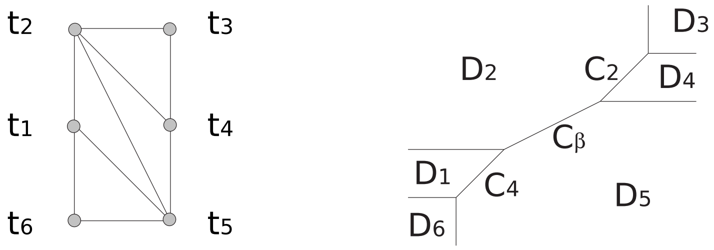


Figure 7.3: The (p,q)-web (right) associated to the specific triangulation (which corresponds to a specific resolution) of the toric diagram of the orbifolded conifold (left).

encoded in the triangulation of the toric diagram (and the associated (p,q)-web) reported in Figure 7.3.

We now mention some results derived in appendix D. First, a linear combination of the three cycles above, $\mathcal{C}_{CF} \equiv 2\,\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$, has a vanishing intersection with the exceptional 2-cycles $\mathcal{C}_2$ and $\mathcal{C}_4$ and it corresponds to the 2-cycle of the double covering conifold geometry. Hence, a brane wrapping it does not couple to closed string twisted sectors, which are those associated to exceptional cycles, and it gives rise to the orbifold of the configuration of a fractional brane at the singular conifold [36]. It thus corresponds to a quiver rank assignment $(1,0,1,0)$. Given the obvious rank assignments $(0,1,1,0)$ and $(1,1,0,0)$ for branes wrapped on $\mathcal{C}_2$ and $\mathcal{C}_4$ respectively, it follows that the rank associated to a D5 brane wrapped on $\mathcal{C}_\beta$ is $(0,-1,0,0)$. We will find it more convenient to use a D5 brane wrapped on $-\mathcal{C}_\beta \equiv \mathcal{C}_\alpha$, corresponding to the quiver $(0,1,0,0)$.

Eventually, one needs to compute the RR 3-form fluxes sourced by each fractional brane. Our findings, which are derived in appendix D, are summarized in the Table below:

	$-\int_{A_2} F_3$	$-\int_{A_4} F_3$	$-\int_{A_{CF}} F_3$	gauge theory
D5 on $\mathcal{C}_2$	2	0	0	(0,1,1,0)
D5 on $\mathcal{C}_4$	0	2	0	(1,1,0,0)
D5 on $\mathcal{C}_\alpha$	1	1	-1	(0,1,0,0)

(7.2.6)

where fluxes are understood in units of $4\pi^2\alpha' g_s$. The 3-cycle A_2 corresponds to the product of the exceptional 2-cycle $\mathcal{C}_2$ transverse to the p-line with the S^1 on p. Similarly, A_4 is the product of the exceptional $\mathcal{C}_4$ with the S^1 in the q-line. Finally, A_{CF} is the image of the compact 3-cycle of the double covering conifold under the orbifold projection.

The table above is all we need to translate directly a quiver with generic rank assignment to a supergravity solution with the corresponding 3-form flux.

7.3 Supergravity background for the UV regime

In this section we present the supergravity solution describing the most general D3 brane system one can consider on the orbifolded conifold. The solution is expected to be dual to the previously discussed gauge theory with the most general rank assignment: $(N+M_1, N+M_2, N+M_3, N)$.[1]

Fractional branes are magnetic sources for the RR 3-form flux. This typically results in some singularity of the backreacted supergravity solution. In some cases, namely when there are only deformation branes around, the singularity is smoothed out by the complex structure deformation the branes induce. One gets back a singularity-free solution where branes are replaced by fluxes [37,127]. In more general situations it is more difficult to find a regular solution. As already noticed, in the case of $\mathcal{N}=2$ fractional branes this is in fact not even expected to be possible, because there should always be some remaining open string modes corresponding to the left over $U(1)^{N-1}$ gauge degrees of freedom on the Coulomb branch. Hence, (a remnant of) the brane sources remains in the gravity dual.

This said, in order to take the leading effect of any such kind of fractional brane into account, it is enough to make an educated ansatz for the supergravity fields and to impose suitable boundary conditions on the system of differential equations. Therefore, in what follows, we will only consider the type IIB bulk action S_{IIB}, see appendix A.2, and implement the effects of each brane source by properly chosen boundary conditions.

7.3.1 Running fluxes and singularity lines

The general solution we are looking for has constant axio-dilaton $\tau = C_0 + ie^{-\phi} = i$, but nontrivial RR and NSNS 3-form fluxes (which are usually organized in a complex 3-form $G_3 = F_3 + ie^{-\phi}H_3 = F_3 + iH_3$), RR 5-form field strength F_5 and warp factor. The ansatz reads

$$
\begin{aligned}
ds^2_{10} &= h^{-1/2}dx^2_{3,1} + h^{1/2}(dr^2 + r^2 ds^2_{T^{1,1}}) \\
F_5 &= (1+*)\, dh^{-1} \wedge d\mathrm{vol}_{3,1} \\
G_3 &= G_3^U + G_3^T
\end{aligned}
\tag{7.3.1}
$$

where the orbifold $\mathbb{Z}_2$ identification (7.2.2) acting on the internal coordinates is understood, h is the warp factor, while the superscripts U and T on the 3-form flux stand for untwisted and twisted sector fluxes, respectively. The above ansatz is the one of a warped singular cone. Any deformation of the singular geometry will still asymptote to this cone for large values of the radial coordinate,

[1] Our conventions for type IIB supergravity and D brane actions, together with the equations of motion for the bulk fields, can be found in appendix A.2-A.2.1. Throughout this chapter, we work in the Einstein frame defined in (A.2.3), where we rescale the metric by the fluctuating part of the dilaton, and the RR fields by the string coupling.

and it is in this sense that we will think of the solution as representing (at least) the UV regime of the dual gauge theory.

Recall that for the solution to be supersymmetric, the closed complex 3-form G_3 should be $(2,1)$, primitive and imaginary-self-dual [87]

$$*_6 G_3 = i\, G_3 \ , \tag{7.3.2}$$

where $*_6$ is constructed with the unwarped metric. We will see that the warp factor depends on the radial coordinate as well as some of the angular coordinates, as typical for solutions with $\mathcal{N}=2$ branes around [39].

The equations of motion we have to solve can be worked out from appendix A.2, substituting $\Phi \to \phi$ and $\kappa_{10} \to \kappa$ in equations (A.2.7-A.2.15). The warp factor equation is given by the Bianchi identity for F_5. The Einstein equations are then automatically satisfied by our ansatz (7.3.1).

It is easy to check that, given all the geometrical data discussed in the previous section, and taking for simplicity all fractional branes sitting at the tip, the complex 3-form G_3 reads[2]

$$\begin{aligned} G_3 = & -\frac{\alpha'}{2} g_s \left(M_1 - M_2 + M_3\right) \left[\omega_3^{CF} - 3i\frac{dr}{r} \wedge \omega_2^{CF}\right] + \\ & + 2i\pi\alpha' g_s \left(-M_1 + M_2 + M_3\right) \frac{dz_2}{z_2} \wedge \omega_2^{(p)} + 2i\pi\alpha' g_s \left(M_1 + M_2 - M_3\right) \frac{dz_1}{z_1} \wedge \omega_2^{(q)} \\ = & -\frac{\alpha'}{2} g_s \left(M_1 - M_2 + M_3\right) \left[\omega_3^{CF} - 3i\frac{dr}{r} \wedge \omega_2^{CF}\right] + \\ & + i\pi\alpha' g_s \left(-M_1 + M_2 + M_3\right) \left(3\frac{dr}{r} + i\, d\psi'\right) \wedge \omega_2^{(p)} + \\ & + i\pi\alpha' g_s \left(M_1 + M_2 - M_3\right) \left(3\frac{dr}{r} + i\, d\psi''\right) \wedge \omega_2^{(q)} \ , \end{aligned} \tag{7.3.3}$$

where ω_3^{CF} and ω_2^{CF} are defined in appendix B.1, and $\omega_2^{(p)}$ and $\omega_2^{(q)}$ are the two normalized exceptional 2-cocycles defined by the integrals below.

For the present purposes it suffices to recall that

$$\int_{\mathcal{C}_{CF}} \omega_2^{CF} = 4\pi \ , \qquad \int_{\mathcal{C}_2} \omega_2^{(p)} = \int_{\mathcal{C}_4} \omega_2^{(q)} = 1 \ , \qquad \text{and} \qquad \int_{A_{CF}} \omega_3^{CF} = 8\pi^2 \ , \tag{7.3.4}$$

where A_{CF} is the image under the orbifold projection of the 3-sphere on the double covering conifold. The second equality in (7.3.3) can be easily obtained by using equations. (B.1.5-B.1.8). It is then easy to check that the RR 3-form fluxes on the A-cycles are

$$-\frac{1}{4\pi^2\alpha' g_s}\int_{A_{CF}} F_3 = M_1 - M_2 + M_3 \tag{7.3.5}$$

$$-\frac{1}{4\pi^2\alpha' g_s}\int_{A_2} F_3 = -M_1 + M_2 + M_3 \tag{7.3.6}$$

$$-\frac{1}{4\pi^2\alpha' g_s}\int_{A_4} F_3 = M_1 + M_2 - M_3 \ . \tag{7.3.7}$$

[2]The vielbein we use for the singular conifold can be found in (B.1.13). Alternatively, one can use (B.1.25). Appendix B.1 contains a review of the singular conifold geometry.

It is important to stress at this point that the above equations are really the input (i.e. the asymptotic conditions) in solving the equations. They are in one-to-one correspondence with a choice of ranks in the quiver. The real part of G_3, that is F_3, is thus essentially determined in this way. Then the imaginary self-dual condition (7.3.2) fixes also H_3, the imaginary part of G_3. The latter is thus the output of solving the supergravity equations. As we will see in the next subsection, this is a nontrivial output in the sense that it will contain information about the running of the gauge couplings. Further dynamical data on the dual gauge theory is contained in the warp factor.

From the ansatz (7.3.1), one sees that the warp factor should satisfy the following equation in the unwarped internal manifold

$$*_6\, d *_6 dh \equiv \Delta h = - *_6 (H_3 \wedge F_3)\ , \tag{7.3.8}$$

with boundary conditions dictated by the D brane sources. To compute $H_3 \wedge F_3$ from (7.3.3) and to solve for the warp factor h in (7.3.8), the first issue is whether there are mixed terms between twisted and untwisted sectors in the expansion of such 6-form in the cocycle basis. Let us consider a closed 2-form ω_2, that represents the Poincaré dual of an exceptional cycle $\mathcal{C}$ in any submanifold transverse to the singularity line, and α_2 a smooth 2-form with vanishing flux on the exceptional cycle. The 4-form $\omega_2 \wedge \alpha_2$, which would give mixed terms, vanishes at any point but the singular one. One can then write $\omega_2 \wedge \alpha_2 = C\, \delta_4$ and compute C as

$$C = \int \omega_2 \wedge \alpha_2 = \int_{\mathcal{C}} \alpha_2 = 0\ . \tag{7.3.9}$$

This implies that there are no mixed terms between the twisted sector and the untwisted one. Then the 6-form $H_3 \wedge F_3$ is easily computed. From (7.3.3) for the 3-form fluxes, using

$$\begin{aligned} \frac{dr}{r} \wedge \omega_2^{CF} \wedge \omega_3^{CF} &= -\frac{54}{r} dr \wedge d\mathrm{vol}_{T^{1,1}} \\ \omega_2^{(p)} \wedge \omega_2^{(p)} &= -\frac{1}{4\pi^2}\, \delta^{(2)}(1-\cos\theta_1, 1-\cos\theta_2)\, \sin\theta_1\, d\theta_1 \wedge d\varphi_1 \wedge \sin\theta_2\, d\theta_2 \wedge d\varphi_2 \\ \omega_2^{(q)} \wedge \omega_2^{(q)} &= -\frac{1}{4\pi^2}\, \delta^{(2)}(1+\cos\theta_1, 1+\cos\theta_2)\, \sin\theta_1\, d\theta_1 \wedge d\varphi_1 \wedge \sin\theta_2\, d\theta_2 \wedge d\varphi_2\ , \end{aligned} \tag{7.3.10}$$

we get

$$\begin{aligned} H_3 \wedge F_3 = 81\, \alpha'^2 g_s^2\, \frac{1}{r^6} \Big\{ &\frac{1}{2}(M_1 - M_2 + M_3)^2 + (M_1 - M_2 - M_3)^2\, \delta^{(2)}(1-\cos\theta_1, 1-\cos\theta_2) + \\ &+ (M_1 + M_2 - M_3)^2\, \delta^{(2)}(1+\cos\theta_1, 1+\cos\theta_2) \Big\}\, dr \wedge r^5\, d\mathrm{vol}_{T^{1,1}}\ . \end{aligned} \tag{7.3.11}$$

The equation we have to solve for the warp factor is then

$$\begin{aligned} \Delta h = -81\, \alpha'^2 g_s^2 \frac{1}{r^6} \Big\{ &\frac{1}{2}(M_1 - M_2 + M_3)^2 + (M_1 - M_2 - M_3)^2\, \delta^{(2)}(1-\cos\theta_1, 1-\cos\theta_2) \\ &+ (M_1 + M_2 - M_3)^2\, \delta^{(2)}(1+\cos\theta_1, 1+\cos\theta_2) \Big\}\ . \end{aligned} \tag{7.3.12}$$

Defining the angular function

$$f(x,y) = \frac{1}{24} \sum_{(n,m)\neq(0,0)}^{\infty} \frac{(2n+1)(2m+1)}{n(n+1)+m(m+1)} P_n(x)P_m(y) \ , \tag{7.3.13}$$

where $P_n(t)$ are Legendre polynomials, and which satisfies the differential equation

$$\Delta_{ang} f(\cos\theta_1, \cos\theta_2) = -\delta^{(2)}(1-\cos\theta_1, 1-\cos\theta_2) + \frac{1}{4} \ , \tag{7.3.14}$$

the solution finally reads (see appendix D of [126] for details)

$$\begin{aligned} h = \frac{27\pi\alpha'^2}{2}\frac{1}{r^4}\Big\{ & g_s N + \frac{3g_s^2}{4\pi}\Big[(M_1-M_2+M_3)^2+(M_1-M_3)^2+M_2^2\Big]\Big(\log\frac{r}{r_0}+\frac{1}{4}\Big)+ \\ & + \frac{6g_s^2}{\pi}\Big[(M_1-M_2+M_3)^2 f(\cos\theta_1,\cos\theta_2)+(M_1+M_2-M_3)^2 f(-\cos\theta_1,-\cos\theta_2)\Big]\Big\} \ . \end{aligned} \tag{7.3.15}$$

The constant terms inside the $\{\dots\}$ in eq. (7.3.15) have been fixed in such a way that the effective D3-charge at $r=r_0$ is N. This is a choice for the physical meaning one wants to give to r_0, as any such constant term can be absorbed into a redefinition of r_0.

The above solution is not smooth, as the warp factor displays singularities at small r. Moreover, as already anticipated, we expect an enhançon behavior to be at work whenever there are $\mathcal{N}=2$ branes in the original bound state. Similarly to [38,39], the enhançon radius can be defined by the minimal surface below which the effective D3-charge changes sign. The resolution of the singularities has to do with the IR dynamics of the dual gauge theory. The structure of the vacua, as well as the phases the gauge theory can enjoy, depend crucially on the classes of fractional branes present and on the hierarchy of the scales Λ_i associated to each quiver node. Hence, the way the singularity is dealt with will change accordingly. These issues are discussed in detail in [3]. Here we just want to stress that no matter the hierarchy between the dynamically generated scales Λ_i and the specific fractional branes content, the above solution is a good description of the UV regime of the dual gauge theory. In the following we will then present a number of nontrivial checks of the duality which apply in this regime.

7.3.2 Checks of the duality: β-functions and Maxwell charges

In this subsection we perform some nontrivial checks of the proposed gauge/gravity duality: we discuss the computation of gauge coupling β-functions and analyze the RG flow of our solutions using standard techniques. In the following subsection we adopt the new perspective explained in the previous chapter, which is based on Page charges and enables us to get stronger predictions from supergravity.

Typically, given a supergravity background dual to a quiver gauge theory, the knowledge of the various brane charges at any value of the radial coordinate r allows one, in principle, to extract the gauge ranks of the dual theory at the scale μ holographically dual to r. Furthermore, from the value of closed string fields, one can learn about parameters and running couplings appearing in the dual field theory. In theories like IIB supergravity, whose action contains Chern-Simons terms leading to modified Bianchi identities for the gauge invariant field strengths, different notions of charges carried by the same fields may be introduced [108]. Following standard techniques, we will start using Maxwell charges, which are integrals of gauge invariant RR field strengths.

In order to specify the dictionary between the string and the gauge sides, one needs to understand the details of the microscopic D brane configuration that realizes the field theory. As explained in [38] and in the previous chapter, the idea is to match the brane charges of the supergravity solution at some value of r with the charges of an auxiliary system of fractional branes that, in the presence of the same closed string fields as those of the supergravity solution, engineers the field theory: in this way one reads the effective theory at the scale μ. A complication arises because the meaningful brane configuration changes along the radial direction: when certain radial thresholds are crossed the D3-charge of one of the effective constituents of the system changes sign, and the auxiliary system is no longer BPS. One has then to rearrange the charges into different BPS constituents. The field theory counterpart is that, when one of the gauge couplings diverges, one has to resort to a different description.

When the theory admits only deformation fractional branes, the link between different field theory descriptions is established by Seiberg duality. This was originally proposed and checked in the conifold theory [37], then applied to other singularities [119, 128] and even to theories with noncompact D7 branes [2, 110] like the one of the previous chapter. In $\mathcal{N}=2$ solutions like the one of [39] the procedure works also well [38]. In this latter case, however, one expects the cascade not to be triggered by subsequent Seiberg dualities: the correct interpretation is more along the lines of a Higgsing phenomenon [64]. We will elaborate on this phenomenon in the next and final chapter of this thesis.

The supergravity solution presented in section 7.3.1 is the first example of a solution describing the backreaction of a bound state containing both deformation and $\mathcal{N}=2$ fractional branes, and hence represents an excellent opportunity to study their interplay. One expects $\mathcal{N}=2$ fractional branes to behave as their cousins in pure $\mathcal{N}=2$ setups, and we will find good evidence that this is the case. The novelty is that even deformation fractional branes, when probing a geometry admitting $\mathcal{N}=2$ branes, may have that kind of behavior, sometimes.

Let us first compare the gauge theory β-functions with the supergravity prediction. The anomalous dimensions of matter fields in the UV are to leading order the same as in the conformal theory, $\gamma=-1/2$. Defining $\chi_a = 8\pi^2/g_a^2$, the four one-loop β-functions $b_a \equiv \partial/\partial(\log\mu)\,\chi_a$ are then

$$\begin{aligned} b_1 &= \frac{3}{2}\,(2M_1 - M_2) \qquad & b_2 &= \frac{3}{2}\,(-M_1 + 2M_2 - M_3) \\ b_4 &= \frac{3}{2}\,(-M_1 - M_3) \qquad & b_3 &= \frac{3}{2}\,(-M_2 + 2M_3)\ . \end{aligned} \tag{7.3.16}$$

On the other hand, inspection of the action of probe fractional D3 branes allows one to find the dictionary between the gauge couplings and the integrals of B_2 on the corresponding shrinking 2-cycles [28, 29, 32].[3] With the conventions laid out in appendix A.2.1, the dictionary is easily found to be

$$\begin{aligned} \chi_2 + \chi_3 &= \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_2} B_2 \qquad & \chi_1 + \chi_3 &= \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_{CF}} B_2 \\ \chi_1 + \chi_2 &= \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_4} B_2 \qquad & \chi_1+\chi_2+\chi_3+\chi_4 &= \frac{2\pi}{g_s}\ , \end{aligned} \tag{7.3.17}$$

[3]We recall that such formulæ are derived in $\mathcal{N}=2$ orbifolds. It is well known [55] that they get corrected by superpotential couplings in cases where the geometry is not an orbifold of flat space, as we have discussed in chapter 3. Nevertheless, the correction due to the quartic superpotential coupling is negligible in the UV regime.

with a radius-energy relation in the UV region $r/\alpha' = \mu$, like in the conformal case. Recall that $\mathcal{C}_{CF} = \mathcal{C}_2 + \mathcal{C}_4 - 2\mathcal{C}_\alpha$.

Integrating the NSNS 3-form given in eq. (7.3.3) one gets for the B_2 field

$$B_2 = \frac{3}{2}\,\alpha' g_s \log\frac{r}{r_0}\Big[(M_1 - M_2 + M_3)\,\omega_2^{CF} + 2\pi(-M_1 + M_2 + M_3)\,\omega_2^{(p)} + 2\pi(M_1 + M_2 - M_3)\,\omega_2^{(q)}\Big] + \pi\alpha'\Big[a_{CF}\,\omega_2^{CF} + 4\pi(a_2\,\omega_2^{(p)} + a_4\,\omega_4^{(p)})\Big] \ , \quad (7.3.18)$$

where a_{CF}, a_2, a_4 are integration constants. This implies that

$$\begin{aligned} \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_{CF}} B_2 &= 3\,(M_1 - M_2 + M_3)\log\frac{r}{r_0} + \frac{2\pi}{g_s}\,a_{CF} \\ \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_2} B_2 &= \frac{3}{2}\,(-M_1 + M_2 + M_3)\log\frac{r}{r_0} + \frac{2\pi}{g_s}\,a_2 \\ \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}_4} B_2 &= \frac{3}{2}\,(M_1 + M_2 - M_3)\log\frac{r}{r_0} + \frac{2\pi}{g_s}\,a_4 \ . \end{aligned} \quad (7.3.19)$$

The three integration constants a_{CF}, a_2, a_4 correspond to the periods of B_2 at $r = r_0$, the latter having being chosen to be the value of the holographic coordinate where the effective D3 brane charge is N, see the discussion after eq. (7.3.14). We can think of it as a UV cut-off for the dual gauge theory, i.e. the scale where the dual UV bare Lagrangian is defined. Then the integration constants fix, through equations. (7.3.17), the bare couplings of the dual nonconformal gauge theory. It is easy to check that the logarithmic derivatives of (7.3.19) give exactly the same β-functions as the field theory computation in (7.3.16).

As generically happens in supergravity solutions dual to nonconformal theories, the Maxwell D3-charge runs. It is easily computed from eq. (7.3.15) to be in our case

$$Q_{D3}(r) = -\frac{1}{(4\pi^2\alpha')^2 g_s}\int_{T^{1,1}/\mathbb{Z}_2} F_5 = N + \frac{3g_s}{2\pi}\left[M_1^2 + M_2^2 + M_3^2 - M_1M_2 - M_2M_3\right]\,\log\frac{r}{r_0} \ . \quad (7.3.20)$$

As in [37], the periods of B_2 are no more periodic variables in the nonconformal supergravity solutions. One should then investigate what the shift in $Q_{D3}(r)$ is once we move in the radial direction from r down to r', where $\Delta r = r - r' > 0$ is the minimal radius shift for which all the periods of B_2 on $\mathcal{C}_\alpha$, $\mathcal{C}_2$, C_4 change by an integer (in units of $4\pi^2\alpha'$). The shift in $Q_{D3}(r)$ should then be compared against the gauge theory expectation for the decrease of the ranks under a specific sequence of cascade steps. What changes after such a sequence are the ranks of the gauge groups, all decreasing by the same integer number, the theory being otherwise self-similar, and with the initial values of the couplings. Sometimes a cyclic permutation of the gauge group factors is also needed, as in [37]. We will call such a sequence of cascade steps a quasi-period.

We are now ready to check the supergravity predictions against the field theory cascade in some simple cases with deformation fractional branes only, where the RG flow can be followed by performing successive Seiberg dualities.

1. $(N + P, N, N + P, N)$

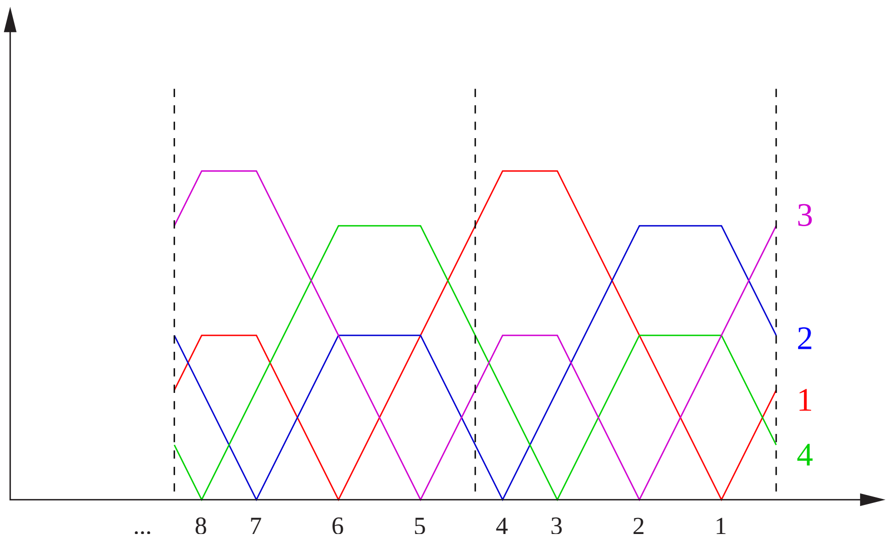


Figure 7.4: Example of the pattern of the cascade of Seiberg dualities for ranks $(N+P, N, N+P, N)$ as derived from the field theory. Black numbers indicate Seiberg dualities, performed on gauge groups with diverging couplings. Inverse squared gauge couplings are plotted versus the logarithm of the energy scale.

This theory is the daughter of the duality cascade discussed in [37]. There are P deformation branes of type $(1,0,1,0)$ (corresponding to D5 branes wrapped over $\mathcal{C}_{CF}$). We get for the charge and the periods

$$\begin{gathered} Q_{D3}(r) = N + \frac{3g_s}{4\pi}\, 4P^2 \log\frac{r}{r_0} \\ b_{\mathcal{C}_\alpha} = -\frac{3g_s}{4\pi}\, 2P\, \log\frac{r}{r_0} + a_\alpha\ , \qquad b_{\mathcal{C}_2} = a_2\ , \qquad b_{\mathcal{C}_4} = a_4\ , \end{gathered} \tag{7.3.21}$$

where $a_{CF} = a_2 + a_4 - 2a_\alpha$ and $b_{\mathcal{C}_i}$ are the periods of B_2 along the cycle $\mathcal{C}_i$ in units of $4\pi^2\alpha'$. From the above equation we see that $r' = r\,\exp[-4\pi/(6g_sP)]$, and under this radial shift $Q_{D3}(r') = Q_{D3}(r) - 2P$. This matches with the gauge theory expectations since the theory is quasi-periodic with a shift $N \to N - 2P$, which is obtained after four subsequent Seiberg dualities on the different gauge groups. See Figure 7.4 for an explicit example of the RG flow computed in field theory, for some values of the bare couplings. Obviously, for any cyclic permutation of the above rank assignment we have the same story.

2. $(N+P, N, N, N)$

$$\begin{gathered} Q_{D3}(r) = N + \frac{3g_s}{4\pi}\, 2P^2 \log\frac{r}{r_0} \\ b_{\mathcal{C}_\alpha} = -\frac{3g_s}{4\pi}\, P\log\frac{r}{r_0} + a_\alpha\ , \quad b_{\mathcal{C}_2} = -\frac{3g_s}{4\pi}\, P\log\frac{r}{r_0} + a_2\ , \quad b_{\mathcal{C}_4} = \frac{3g_s}{4\pi}\, P\log\frac{r}{r_0} + a_4\ . \end{gathered} \tag{7.3.22}$$

From the above equation we see that $r' = r\,\exp[-4\pi/(3g_sP)]$ and consequently $Q_{D3}(r') = Q_{D3}(r) - 2P$. This matches again with gauge theory expectations. Although the quiver looks self-similar after four Seiberg dualities, the theory is not: the gauge couplings return to their original values

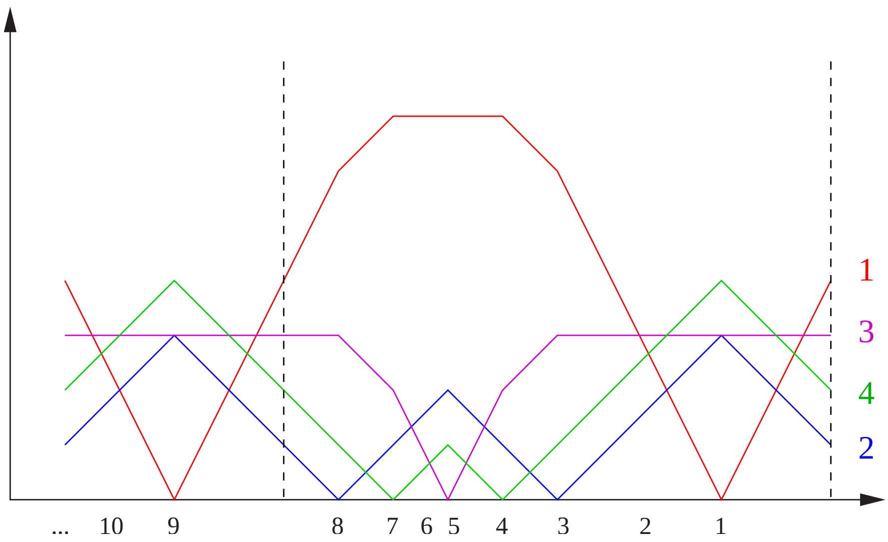


Figure 7.5: Example of the pattern of the cascade of Seiberg dualities for ranks $(N+P, N, N, N)$ as derived from the field theory.

only after eight Seiberg dualities, as shown in Figure 7.5. Hence in this case a quasi-period needs eight dualities and the shift in the ranks is indeed $N \to N - 2P$. Again, similar conclusions hold for any cyclic permutations of the above rank assignment.

3. $(N+Q, N+Q, N+Q, N)$

$$\begin{gathered} Q_{D_3}(r) = N + \frac{3g_s}{4\pi}\, 2Q^2 \log\frac{r}{r_0} \\ b_{\mathcal{C}_\alpha} = a_\alpha\,, \qquad b_{\mathcal{C}_2} = \frac{3g_s}{4\pi}\, Q\log\frac{r}{r_0} + a_2\,, \qquad b_{\mathcal{C}_4} = \frac{3g_s}{4\pi}\, Q\log\frac{r}{r_0} + a_4\,. \end{gathered} \tag{7.3.23}$$

Here, $r' = r\ \exp[-4\pi/(3g_sQ]$ and $Q_{D3}(r') = Q_{D3}(r) - 2Q$. A quasi-period requires eight Seiberg dualities and again agrement with field theory expectations is found. Notice that this theory appears along the RG flow of the theory $(N', N', N', N'+Q)$.

7.3.3 Page charges and the RG flow from supergravity

There is another way of matching our running supergravity solutions (and more generally type IIB solutions constructed from fractional branes at conical singularities) with cascading field theories. The method was originally proposed in [2], on which the previous chapter is based, working on ideas in [108]. Instead of using Maxwell charges, which are conserved and gauge invariant but not quantized nor localized, the method is based on Page charges [109] which are conserved and quantized, and therefore more suitable to be identified with gauge ranks, even though they shift under large gauge transformations.

We recall that Maxwell and Page charges are defined, again up to signs, as as

$$\text{Maxwell:}\quad Q_p = \frac{1}{2\kappa^2\tau_p}\int_{S^{8-p}} F_{8-p} \qquad\qquad \text{Page:}\quad Q_p^{Page} = \frac{1}{2\kappa^2\tau_p}\int_{S^{8-p}} e^{B_2}\wedge F\Big|_{8-p}\,. \tag{7.3.24}$$

We will need the D3 and D5 brane Page charges, which are both defined with a minus sign in front, see appendix A.2.1.

We recall here that the general idea is that it is possible to read the field theory RG flow from supergravity pointwise. At fixed radial coordinate r dual to some scale μ, standard formulæ allow us to compute the gauge couplings from the dilaton and the integrals of B_2. Such formulæ do not give real couplings in general, but need particular integer shifts of B_2, which are large gauge transformations. Consequently, Page charges get shifted by some integer values. Having at hand a dictionary, they are readily mapped to the ranks of the gauge theory at that scale.

Flowing in the holographic coordinate, at some specific radii, in order to keep the couplings real, one has to perform a further large gauge transformation, shifting B_2 and therefore ending up with different ranks. These points connect different steps of the cascade and can usually be interpreted in the field theory as Seiberg dualities [37] or Higgsings [64]. In particular, ranks are not continuously varying functions but rather integer discontinuous ones. This is not the end of the story: in general the shifts of B_2 are not enough to save us from imaginary couplings, and one is forced to introduce multiple dictionaries. We will see how everything beautifully merges.

Let us make the point clear using a popular example, the Klebanov-Strassler cascade [36, 37] of section 3.4.1. The first step is to identify a dictionary between the field theory ranks and Page charges. An $SU(N+M)\times SU(N)$ theory is microscopically engineered with N regular and M fractional D3 branes at the tip of the conifold, and in the suitable gauge we get $Q_3^{Page}=N$, $Q_5^{Page}=M$. The formulæ for the gauge couplings are

$$\chi_1=\frac{2\pi}{g_s}\,b \qquad\qquad \chi_2=\frac{2\pi}{g_s}\,(1-b)\;, \tag{7.3.25}$$

where $\chi_a=8\pi^2/g_a^2$ and $a=1$ refers to the larger group, while $4\pi^2\alpha'\,b=\int_{S^2}B_2$. From the actual UV solution [36], we have (for B_2 in some gauge)

$$b=\frac{1}{4\pi^2\alpha'}\int_{S^2}B_2=\frac{3g_sM}{2\pi}\log\frac{r}{r_0} \qquad\qquad Q_3=-\frac{1}{2\kappa^2\tau_3}\int_{T^{1,1}}F_5=N+\frac{3g_sM^2}{2\pi}\log\frac{r}{r_0}\;. \tag{7.3.26}$$

At any radius/energy scale $x\equiv\log r/r_0$ one should perform a large gauge transformation and shift b by some integer Δb such that $\chi_a\geq 0$, compute the Page charges in such a gauge, and finally use the dictionary to evaluate the ranks at that scale.

It is easy to evaluate Δb and Q_3^{Page} in this example. They read

$$\Delta b=-\Big[\frac{3g_sM}{2\pi}x\Big]_- \qquad\qquad Q_3^{Page}=N-\Delta b\,M=N+\Big[\frac{3g_sM}{2\pi}x\Big]_-\,M\;, \tag{7.3.27}$$

where the floor function $[y]_-$ is the greatest integer less than or equal to y. Applying the algorithm at any x, we can plot the RG flow of the gauge couplings and the ranks along it. The result (the famous KS cascade) is depicted in Figure 7.6. Notice that we never imposed continuity of the gauge couplings (even though it is a well motivated physical requirement), nevertheless the supergravity solution predicts it. Moreover it also suggests a reduction in the gauge group ranks without explaining the corresponding field theory mechanism. It turns out that in this case Seiberg duality can beautifully account for it [37, 55].

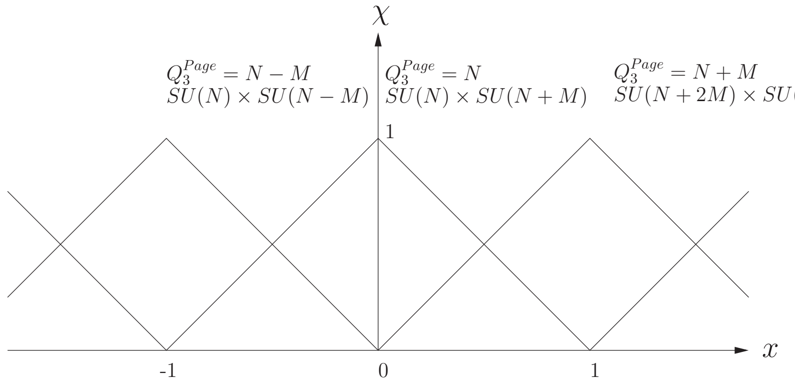


Figure 7.6: Flow in the KS theory as computed with the algorithm. x is in units of $2\pi/3g_sM$ while χ in units of $2\pi/g_s$. At integer values of x a large gauge transformation is required. At each step the Page D3-charge and the field theory is indicated.

We want to apply the same procedure to our class of solutions. In order to do that, however, we need some more machinery. Given a basis of 2-cycles $\mathcal{C}_i$ and 3-cycles A_j on radial sections, one defines an intersection matrix

$$C_i \cdot A_j = \mathcal{I}_{ij} \qquad\qquad i,j = 1 \dots p \ , \tag{7.3.28}$$

where p is the number of fractional branes. Let $(n_I) = (\#D5_i, \#D3)$, $I = 1 \dots p+1$ be the occupation vector, that is the numbers of D5 branes wrapped on $\mathcal{C}_i$ and of D3 branes. A *dictionary* $F_{(m)}$ relates this system to the ranks r_a, $a = 1 \dots P$ of the dual gauge theory

$$r_a = [F_{(m)}]_{aI}\, n_I \ . \tag{7.3.29}$$

In general $P \geq p+1$, but for our nonchiral theory $P = p+1$ and $F_{(m)}$ is invertible. In the following $i,j = 1 \dots p$ while $I, J, a, b = 1 \dots p+1$. Let (Q_I) be the vector of Page charges

$$(Q_I) = \Big(-\frac{1}{2\kappa^2 \tau_5} \int_{A_j} F_3 \, , \, -\frac{1}{2\kappa^2\tau_3} \int F_5^{Page} \Big) \ , \tag{7.3.30}$$

then the definition of Page charges (7.3.24) (with a minus sign for both the D3 and D5 brane Page charges) implies that $Q_j = -\mathcal{I}^t{}_{ji}\, n_i$. Introducing the matrix $\tilde{\mathcal{I}} = \mathrm{diag}(-\mathcal{I}^t, 1)$ we can write: $Q_I = \tilde{\mathcal{I}}_{IJ}\, n_J$. It follows that (suppressing indices)

$$r = \Big(F_{(m)}\, \tilde{\mathcal{I}}^{-1} \Big)\ Q \ . \tag{7.3.31}$$

The formulæ relating the gauge couplings to the supergravity solution can be derived by considering the worldvolume action of probe D3 and wrapped D5 branes [119]. Let $\chi_a = 8\pi^2/g_a^2$ as before. Considering D3 branes one concludes that $\sum \chi_a = 2\pi/g_s$; then the integral of B_2 on some 2-cycle $\mathcal{C}_j$ is related to the gauge coupling on the probe D5 brane, which is itself related to the sum of the χ's corresponding to the ranks increased by the D5 brane, as in (7.3.17). Defining the vector

$$(B_I) = \Big(\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_i} B_2 \, , \, 1 \Big) \tag{7.3.32}$$

one can summarize the relations by

$$\frac{2\pi}{g_s} B = F_{(m)}^t \chi \qquad \Rightarrow \qquad \chi = \frac{2\pi}{g_s} F_{(m)}^{-1t} B \ . \tag{7.3.33}$$

Under large gauge transformations the integrals of B_2 change by integer amounts, thus the first p components of the vector B undergo a particular shift $B_i \to B_i + Z_i$, for some $Z_i \in \mathbb{Z}$. As a result the Page D3-charge is shifted by

$$\Delta Q_3^{Page} = -\frac{1}{2\kappa^2 \tau_3} \int \Delta B_2 \wedge F_3 = Q_j \, (\mathcal{I}^{-1})_{jk} \, Z_k \ , \tag{7.3.34}$$

while the inferred gauge couplings change according to eq. (7.3.33).

We now apply the algorithm to our solutions (7.3.3), where the integrals of B_2 are (7.3.19), for some values of the charges (equivalently for some M_i's). Using the basis $\{\mathcal{C}_2, \mathcal{C}_4, \mathcal{C}_\alpha\}$ for the 2-cycles and $\{A_2, A_4, A_{CF}\}$ for the 3-cycles, the intersection matrix $\mathcal{I}_{ij}$ is given by

$$\mathcal{I}_{ij} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{pmatrix} \tag{7.3.35}$$

as in (D.2.4), while the dictionary $[F_{(1)}]_{aI}$ derived in section 7.2.2 (see Table (7.2.6)), referring to the central quiver in Figure 7.7, is reported in Figure 7.8. One quickly discovers that, for generic values of the integration constants a_i and of the radial coordinate r, there is no gauge transformation that produces positive χ_a in eq. (7.3.33).

One is led to the conclusion that *multiple dictionaries* are needed. This had to be expected since performing any Seiberg duality on the central quiver in Figure 7.7 one obtains the lateral quivers (depending on the node chosen), which are substantially different and cannot be described by the same dictionary, even up to reshuffling of the nodes.

It turns out that even two dictionaries are not enough in our case. We provide a set of six dictionaries such that, at any energy, for one and only one dictionary there is one large gauge transformation that gives nonnegative χ_a, see Figure 7.8.

The dictionaries besides $F_{(1)}$ are obtained from it through formal Seiberg dualities. Consider a system with occupation vector $n = (n_1, n_2, n_3, N)$. Start with the central quiver where the ranks are given by eq. (7.3.29) using $F_{(1)}$. Then a formal Seiberg duality on one node gives a new quiver with new ranks (and superpotential), from which a new dictionary $F_{(m)}$ is directly read. Actually there is an ambiguity because the number of D3 branes N could have changed in the process (but not the other charges) and then one is free to add lines of 1's to any of the first three columns. One can show that the physical result, that is the gauge couplings and ranks in the correct gauge of B_2, is not affected. In our case, a Seiberg duality on node 1 gives $F_{(4)}$, on node 2 $F_{(6)}$, on node 3 $F_{(3)}$, on node 4 $F_{(5)}$ and on two opposite nodes $F_{(2)}$.

We can finally apply the algorithm at any radius $x \equiv \log r/r_0$, that is:

- find a dictionary in the set $\{F_{(m)}\}$ and a large gauge transformation $B_i(x) \to B_i(x) + Z_i$ such that, according to eq. (7.3.33), $\chi_I \geq 0 \quad \forall I$. It turns out that there is always one and only one solution;[4]

[4] To be precise, when one of the χ_I vanishes there are two dictionaries (with their gauges) that do the job, one which is valid for larger r and another for smaller r. At these radii there is the transition between the validity domains of two different field theory duals.

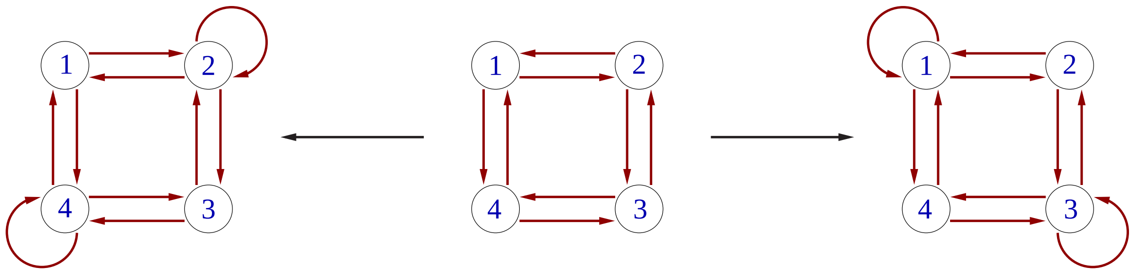


Figure 7.7: Seiberg dual quivers. The central quiver is the most extensively discussed one in the paper. The left quiver is obtained with a Seiberg duality on node 1 or 3, while the right one on node 2 or 4.

$$F_{(3)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{(1)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{(5)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$F_{(4)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{(2)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad F_{(6)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 7.8: A set of six dictionaries for the orbifolded conifold theory. $F_{(3)}$, $F_{(4)}$ refer to the left quiver, with adjoints on nodes 2-4; $F_{(1)}$, $F_{(2)}$ refer to the central quiver, without adjoints; $F_{(5)}$, $F_{(6)}$ refer to the right quiver, with adjoints on nodes 1-3. The four columns represent the nodes activated by a D5 brane on $\mathcal{C}_2$, $\mathcal{C}_4$, $\mathcal{C}_\alpha$ and a D3 brane respectively.

- compute the D3 brane Page charge in this gauge, using eq. (7.3.34) (D5 brane charges are invariant);
- use the dictionary and the charges in eq. (7.3.31) to evaluate the ranks at that scale in the corresponding quiver.

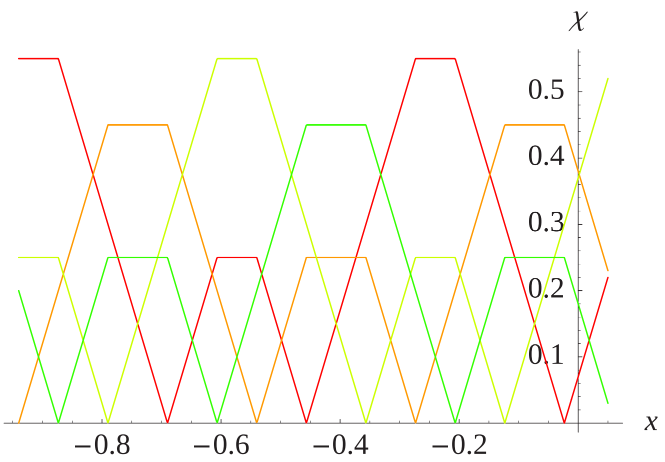
As a result, one can plot the gauge couplings along the flow and keep track of the various field theory descriptions.

It is clear that the transition radii between two different descriptions (dictionaries) occur when one of the χ_I vanishes. But in principle there is no reason why one should expect, from the procedure above, continuous couplings at the transition points. Surprisingly enough, it turns out that the resulting coupling are indeed continuous. Some plots with explanation are in Figures 7.9, 7.10, 7.11, 7.12 (obtained via a mathematica code). In the following, we comment on interesting examples.

1. $(N+P, N, N+P, N)$

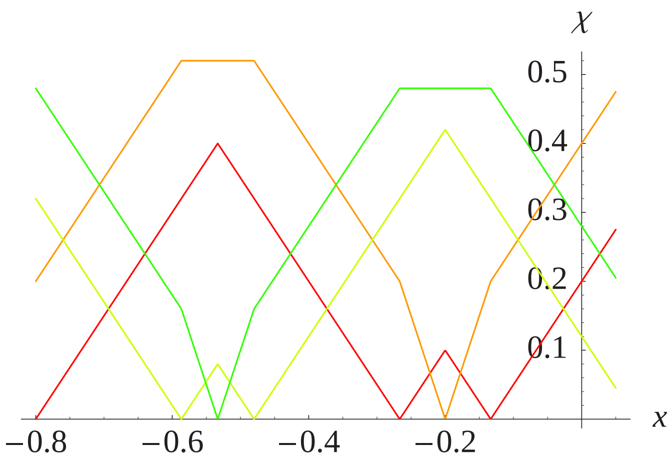
The RG flow, as computed from supergravity with the algorithm above, is plotted in Figure 7.9 (for $P = 1$ and some typical choice of the integration constants a_2, a_4, a_α and the starting radius $x = \log r/r_0$). It precisely matches with the field theory expectations, with respect to both gauge couplings and ranks at any step. All transition points can be interpreted by means of a single Seiberg duality, as the prototypical example in [37]. Notice that the integral of B_2 on $\mathcal{C}_2$ and $\mathcal{C}_4$ is

Figures: the following figures represent the RG flow as computed from SUGRA with the algorithm, for typical values of the integration constants a_2, a_4, a_α and the initial radius $x = \log r/r_0$. The gauge couplings are in units of $2\pi/g_s$. On the right side we report, for each step, the dictionary used and the ranks in the quiver; the addition of N is understood. Underlined ranks signal an adjoint chiral superfield at the corresponding node. The red line represents the first group, the orange the second one, the light green the third one, the dark green the fourth one.



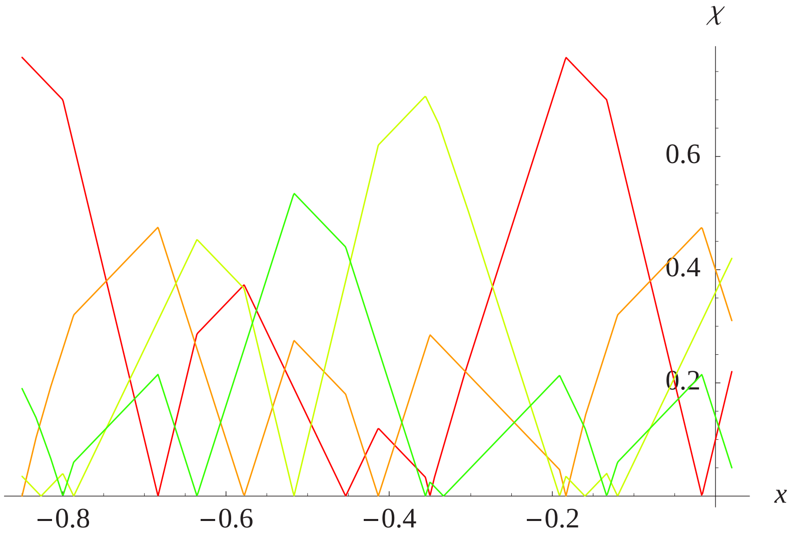
Dic.	Ranks
1	(1 , 0 , 1 , 0)
4	(-1 , $\underline{0}$, 1 , $\underline{0}$)
2	(-1 , 0 , -1 , 0)
5	($\underline{-1}$, 0 , $\underline{-1}$, -2)
1	(-1 , -2 , -1 , -2)
	⋮

Figure 7.9: RG flow for the $(N+1, N, N+1, N)$ theory from supergravity.



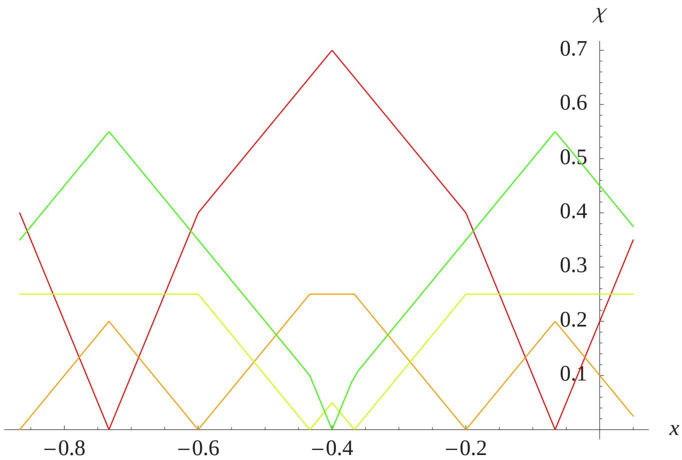
Dic.	Ranks
1	(1 , 1 , 0 , 0)
4	(0 , $\underline{1}$, 0 , $\underline{0}$)
5	(0 , $\underline{-1}$, 0 , $\underline{0}$)
2	(-1 , -1 , 0 , 0)
6	(-1 , $\underline{-1}$, -1 , $\underline{0}$)
3	(-1 , $\underline{-1}$, -1 , $\underline{-2}$)
1	(-1 , -1 , -2 , -2)
	⋮

Figure 7.10: RG flow for the $(N+1, N+1, N, N)$ theory from supergravity.



Dic.	Ranks
1	(2 , 0 , 1 , 0)
4	(-2 , 0 , 1 , 0)
2	(-2 , 0 , -1 , 0)
5	(-2 , 0 , -1 , -3)
3	(-2 , 0 , -2 , -3)
2	(-2 , -4 , -2 , -3)
6	(-2 , -4 , -5 , -3)
3	(-2 , -4 , -5 , -4)
2	(-6 , -4 , -5 , -4)
5	(-6 , -4 , -5 , -7)
1	(-6 , -7 , -5 , -7)
3	(-8 , -7 , -5 , -7)
2	(-8 , -7 , -9 , -7)
5	(-8 ,-10, -9 , -7)
1	(-8 ,-10, -9 ,-10)
	⋮

Figure 7.11: RG flow for the $(N+2, N, N+1, N)$ theory from supergravity.



Dic.	Ranks
1	(1 , 0 , 0 , 0)
4	(-1 , 0 , 0 , 0)
5	(-1 , -1 , 0 , 0)
1	(-1 , -1 , -1 , 0)
4	(-1 , -1 , -1 , -2)
6	(-1 , -1 , -2 , -2)
1	(-1 , -2 , -2 , -2)
	⋮

Figure 7.12: RG flow for the $(N+1, N, N, N)$ theory from supergravity.

constant and generically not integer.

2. $(N+P, N+P, N, N)$

The supergravity RG flow is shown in Figure 7.10 (for $P = 1$ and typical integration constants). This theory is realized with $\mathcal{N} = 2$ fractional branes only, and one expects a behavior quite similar to the $\mathcal{N} = 2$ setup of [39]. The algorithm confirms that there are steps of the cascade where the node with divergent coupling has an adjoint chiral field and $\mathcal{N} = 2$ superpotential. In the example of Figure 7.10, after a Seiberg duality on node 1, one is left with the left hand side quiver of Figure 7.7, and superpotential

$$W = -X_{12}X_{21}X_{14}X_{41} + M_{22}(X_{21}X_{12} - X_{23}X_{32}) + X_{32}X_{23}X_{34}X_{43} - M_{44}(X_{43}X_{34} - X_{41}X_{14}) \ . \tag{7.3.36}$$

The next node with diverging coupling is node 2. Notice that if one neglects the gauge dynamics on the other nodes and possible subtleties related to a nontrivial Kähler potential and anomalous dimensions of node 2, the theory is effectively $\mathcal{N}=2$ massless SQCD with $N+P$ colors and $2N$ flavors. One is tempted to think that this piece of the RG flow can be interpreted as in the $\mathcal{N}=2$ theory of [38, 39, 64] (see also [129]).

It is beyond the scope of the present work to fully understand the field theory dynamics. We just want to observe that on the gravity side this step in the cascade, possibly understandable as Higgsings, precisely occurs when

$$\frac{1}{4\pi^2\alpha'}\int_{\mathcal{C}_2} B_2 \in \mathbb{Z} \qquad \text{or} \qquad \frac{1}{4\pi^2\alpha'}\int_{\mathcal{C}_4} B_2 \in \mathbb{Z} \tag{7.3.37}$$

(in this case only $\mathcal{C}_4$). Since $\mathcal{C}_2$ and $\mathcal{C}_4$ are shrunk 2-cycles along the $\mathcal{N}=2$ singularity lines, at these radii (called generalized enhançons in [64]) there are extra massless fields and tensionless objects in supergravity. We will discuss this phenomenon in the next chapter.

3. $(N+P, N, N+Q, N)$

The supergravity RG flow for the case $(N+2, N, N+1, N)$ is shown in Figure 7.11. This theory is realized with deformation fractional branes only. Nevertheless, the fact that the geometry admits $\mathcal{N}=2$ fractional branes causes that, at some steps, there is a reduction of rank in a node with adjoint; as before, this cannot be interpreted as a Seiberg duality and some other mechanism, such as Higgsing, should be invoked. Shells where such transitions occur are again precisely at radii where one of the periods of B_2 on $\mathcal{C}_2$ or $\mathcal{C}_4$ vanishes.

This rather intriguing fact can be understood by noticing that in some intermediate steps, i.e. when there are nodes with adjoints, the relevant dictionary forces us to reinterpret the configuration as if it were composed of deformation fractional branes together with a number of $\mathcal{N}=2$ fractional branes.

For generic P and Q things can be analyzed in a similar way. Notice that for P and Q large and coprime, the flow becomes quickly very complicated.

4. $(N+P, N, N, N)$

The supergravity RG flow for the case $(N+1, N, N, N)$ is shown in Figure 7.12. As in the previous examples, when one of the periods of B_2 on $\mathcal{C}_2$ or $\mathcal{C}_4$ vanishes supergravity predicts some transition that cannot be interpreted as a Seiberg duality in the FT. This flow is anyway peculiar because performing a Seiberg duality on a conformal node it is possible to provide a dual FT interpretation of the RG flow using only Seiberg dualities, as was done in the previous subsection. However, supergravity seems to predict a different pattern of dualities which nevertheless leads to the same evolution of the gauge couplings.

7.4 Summary and conclusions

Let us summarize what we found. There exists a well-defined algorithm that, given a minimal set of dictionaries, allows one to derive the field theory RG flow from a supergravity solution. For toric singularities, as the one we are describing, the dictionaries can be derived using standard techniques (see for instance [130]) and, given the first, the other ones follow applying formal Seiberg dualities. It is not clear to us how to determine the minimal number of dictionaries, and we have obtained them by hand. Moreover, it would be interesting to understand how to extend the algorithm to supergravity solutions dual to chiral gauge theories, as those in [128].

Our geometry admits both deformation and $\mathcal{N} = 2$ fractional branes. We saw examples of cascades from deformation branes that can be interpreted in term of Seiberg dualities only, examples with $\mathcal{N} = 2$ branes that are very close to pure $\mathcal{N} = 2$ theories and whose interpretation should be similar to the Higgsing proposed of [64], but also examples which one would say are realized with deformation branes only that require something like a Higgsing, at some steps. This field theory interpretation will be further explored, in an $\mathcal{N} = 2$ setting, in the next chapter.

Chapter 8

The $\mathcal{N} = 2$ cascade revisited and the enhançon bearings

This final chapter is based on some work in progress done in collaboration with Riccardo Argurio, Francesco Benini, Matteo Bertolini and Cyril Closset.

8.1 Introduction

In the previous chapter we have studied a generic cascading gauge theory by means of its holographic dual. We have made use, as in the Klebanov-Strassler cascade, of an auxiliary brane system which enabled us to extract ranks in the gauge theory from brane charges in supergravity. At special positions along the holographic coordinate, some of the constituents of this auxiliary brane system were no longer BPS, but they could always rearrange in a different auxiliary brane system which is BPS in the new energy range. In generic theories like the one studied in the last chapter, the new brane system can be quite different from the previous one: this translates to the fact that the field theory need not be self-similar after a single strong coupling transition, that is interpreted as a Seiberg duality or a peculiar kind of Higgsing, depending on the kind of fractional brane becoming tensionless at the transition. As such a generic situation is encountered, a new dictionary is needed in order to extract field theory observables from dual string theory fields. This is a signal that the rearrangement of the BPS objects in the auxiliary brane system has a very nontrivial meaning.

In this chapter, we would like to understand better what occurs at the strong coupling transition scales. We will restrict our attention to a more controllable $\mathcal{N} = 2$ cascade, where the Seiberg-Witten technology allows us to extract exact results. We will provide a cascading interpretation of the $\mathcal{N} = 2$ solution from fractional branes at the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold [39], identifying a cascading vacuum in the dual field theory, whose RG flow coincides with the one that can be extracted holographically from the type IIB supergravity solution. In particular, we will understand that the strong coupling transition is driven by a highly nonperturbative Higgsing, which to leading order approximation is the one occurring at the baryonic root of the SQCD theory effectively describing the dynamics of the strongly coupled group as the remaining gauge dynamics is neglected.

In the analysis of the Coulomb branch of the field theory, we will also encounter a particular reincarnation of the enhançon ring, that we will call enhançon bearing and that will allow us to gain a clearer understanding of what happens to brane probes, which in this $\mathcal{N} = 2$ situation are the branes of the auxiliary system, at the strong coupling transition scales.

This chapter is structured as follows. In section 8.2 we review the supergravity solution for fractional D3 branes at a $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold. In section 8.3 we review some different field theory interpretations that have been proposed for that solution in the literature. In section 8.4, after a review of the Coulomb branch of $\mathcal{N}=2$ SQCD, we will propose our interpretation of the dual field theory at a special vacuum and make some matchings between the two sides of the duality. In section 8.5 we will study different vacua of the same theory, displaying enhançon bearings, and we will see how they are connected to the cascading vacuum. In section 8.6 we summarize our results and add some final comments. Finally, we have gathered in appendix E a short review of the derivation of Seiberg-Witten curves in the context of M theory.

8.2 Review of the $\mathcal{N}=2$ cascading solution

In this section we review the $\mathcal{N}=2$ solution describing fractional D3 branes at a $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold [39].

We parameterize the $\mathbb{C}$ plane by a complex coordinate $z=\rho\, e^{i\varphi}=y^1+iy^2$; the covering space of $\mathbb{C}^2/\mathbb{Z}_2$ is parameterized by two complex coordinates $(z_1,z_2)=(y^3+iy^4,y^5+iy^6)$, on which the generator g of $\mathbb{Z}_2$ acts as $g:\ (z_1,z_2)\mapsto(-z_1,-z_2)$, leaving z invariant. The fixed point locus is the z plane at $z_1=z_2=0$. The orbifold group is contained in $SU(2)$, therefore string theory on this orbifold has sixteen supersymmetries. An exceptional 2-cycle $\mathcal{C}$, with S^2 topology, lives at the orbifold plane $z_1=z_2=0$. As in the previous chapter, we will associate to this exceptional 2-cycle a closed anti-selfdual (1,1)-form ω_2 with delta-function support at the orbifold plane, normalized as follows:

$$\int_{\mathcal{C}}\omega_2=1\ ,\qquad \int_{\mathbb{C}^2/\mathbb{Z}_2}\omega_2\wedge\omega_2=-\frac{1}{2}\ . \tag{8.2.1}$$

The supergravity solution for fractional D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold preserves eight supercharges and enjoys logarithmically varying twisted fluxes, that can be thought of as varying 3-form fluxes on the (exceptional) 3-cycles of the geometry, analogously to the Klebanov-Tseytlin/Strassler solution for fractional D3 branes at the conifold. We will see that also in the orbifold case the supergravity solution naturally suggests that the dual field theory has a cascading RG flow, where the number of degrees of freedom decreases along the flow.

The authors of [39] considered the simplest configuration, with M fractional D3 branes placed at the origin $z=z_1=z_2=0$. The metric is a warped product of the 4-dimensional Minkowski metric and the 6-dimensional Euclidean metric on the orbifold:

$$ds^2=h^{-1/2}dx_{1,3}^2+h^{1/2}dy_6^2\ , \tag{8.2.2}$$

where the warp factor has the dependence $h=h(\rho^2,r^2)$, with $r^2=\sum_{i=1}^6(y^i)^2$ the square of the overall radius in the $\mathbb{C}^3$ covering space of the orbifold and ρ the radius in the complex $\mathbb{C}$ plane. The RR 5-form field strength is related to the warp factor in the usual way, $F_5=(1+*)d^4x\wedge dh^{-1}$, by supersymmetry. The axio-dilaton $\tau=C_0+ie^{-\phi}=C_0+i$ is taken to be constant[1] because fractional D3 branes do not couple to it. Being D5 branes wrapped on the exceptional cycle, fractional D3

[1] We work here in the Einstein frame defined by (A.2.3), as in the previous chapter.

branes source magnetically the twisted scalar c, living at the orbifold plane, which can be thought of as the reduction of the RR 2-form potential on this cycle

$$c = \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} C_2 \ , \tag{8.2.3}$$

and therefore by supersymmetry they also source its NSNS partner

$$b = \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} B_2 \ . \tag{8.2.4}$$

These two twisted scalars are normalized with periodicities 1. They can be summarized in the complex twisted scalar

$$\gamma = c + \tau b = \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} (C_2 + \tau B_2) \ , \tag{8.2.5}$$

which is in general a function $\gamma = \gamma(z, \bar{z})$. By (8.2.1), we can write

$$C_2 + \tau B_2 = 4\pi^2\alpha' \, \gamma \, \omega_2 \ , \tag{8.2.6}$$

whose exterior differential is the complexified 3-form field strength

$$G_3 = 4\pi^2\alpha' \, d\gamma \wedge \omega_2 \ . \tag{8.2.7}$$

The complex twisted scalar γ is subject to a two-dimensional Laplace equation with sources at the positions of fractional D3 branes. The supersymmetry condition that G_3 be closed, primitive, ISD and $(2,1)$ translates into the requirement that $\gamma = \gamma(z)$ be a meromorphic function. Fractional D3 brane sources introduce poles in the meromorphic form $d\gamma(z)$, with residues proportional to the number of branes, by Gauss' law. For the configuration with M fractional D3 branes at $z = z_1 = z_2 = 0$, the solution is

$$\gamma(z) = i\frac{g_s M}{\pi} \log \frac{z}{z_e} \ , \tag{8.2.8}$$

where the integration constant z_e formally acts as a regulating scale.

Finally, the warp factor is obtained by integrating the Bianchi identity $dF_5 = -H_3 \wedge F_3$, with the result [39]:

$$h = \frac{4\pi g_s M \alpha'^2}{r^4} \left\{ 1 + 8\pi g_s M \left[\log \frac{r^4}{\rho_e^2 (r^2 - \rho^2)} - 1 + \frac{\rho^2}{r^2 - \rho^2} \right] \right\} \ , \tag{8.2.9}$$

where $\rho_e = |z_e|$. The resulting geometry has an unphysical repulsive region near the origin $r = 0$,[2] which is expected to be resolved by string theory by an enhançon mechanism along the lines of [134].

8.3 Available interpretations of the cascading solutions

Several different dual field theory interpretations of the previous solution have been proposed in the literature. We review them in some detail in this section.

[2]See appendix A of [133] for an analytic study of the warp factor.

8.3.1 The pure YM theory: the enhançon and the Seiberg-Witten curve

The authors of [39] found the previous solution starting from quantization of open strings on the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold, and computing the coupling of fractional D3 branes to the closed string sector by means of the boundary state formalism. It was therefore natural for them to take the perspective that the solution presented above is defined in the z plane from a radial cutoff scale $\rho_{UV}=\rho_e\, e^{\frac{\pi}{2g_sM}}$, where the twisted scalar b takes its 'orbifold point' value $b=\frac{1}{2}$ which makes the orbifold sigma model free [114], down to the interior. They also computed the moduli space metric for a probe fractional D3 brane in the geometry, that by open-closed string duality was expected to map to the moduli space metric on the Coulomb branch of the dual $\mathcal{N}=2$ $SU(M)$ pure SYM theory (related by extended supersymmetry to the running gauge coupling and the theta angle). They found precise agreement with the perturbative field theory computation. In particular, the moduli space metric takes the form

$$g_{ab}=\frac{2M}{2\pi}\log\frac{\rho}{\rho_e}\;\delta_{ab} \tag{8.3.1}$$

in a basis of canonically normalized real scalar fields, and becomes singular at $\rho=\rho_e$, the so-called 'enhançon' radius. At that radial position fractional D3 branes become tensionless, so that they cannot enter the interior region. As in [134], the stringy interpretation is that the M fractional D3 branes generating the geometry are not actually allowed to stay coincident at the origin: the correct microscopic description involves rather M tensionless fractional D3 branes melted in a thin ring at the enhançon radius, which excises the repulsive region. The metric in the interior is flat, and the twisted scalar γ is constant there. The source introduced at the excision surface between the exterior warped geometry with fluxes and the interior flat and fluxless geometry behaves exactly as a thin ring of wrapped D5 branes, which become tensionless at the enhançon radius, providing support for the enhançon proposal [135].

Let us also mention that the background displays a $\mathbb{Z}_{2M}$ symmetry, realized in the geometry as discrete rotations $\varphi\mapsto\varphi+\frac{2k\pi}{2M}$ $(k=0,1,\ldots,2M-1)$; this maps to the nonanomalous subgroup of the $U(1)_R$ symmetry nontrivially acting on the Coulomb branch of the dual $\mathcal{N}=2$ $SU(M)$ pure SYM theory, that we mentioned earlier.

The enhançon mechanism is nothing but the large M manifestation of the Seiberg-Witten curve at the origin of the moduli space of the dual $SU(M)$ Yang-Mills theory; as suggested by (8.3.1), the enhançon radius is mapped to the holomorphic dynamically generated scale Λ of the $\mathcal{N}=2$ SYM theory [136]. Let us review the argument here.

The result for the tree-level plus one-loop prepotential of the gauge theory

$$\mathcal{F}_{pert}=\frac{i}{2\pi}\mathrm{Tr}\left(\Phi^2\log\frac{\Phi^2}{\Lambda_M^2}\right) \tag{8.3.2}$$

implies that the perturbative moduli space develops, at a scale of order Λ_M, $2M$ singularities in the plane of the adjoint field Φ VEV's; such singularities are related by $\mathbb{Z}_{2M}$ rotations, the subgroup of the nonanomalous abelian R-symmetry which acts nontrivially on this moduli space, and which is unbroken in this vacuum. Along the lines of the Seiberg-Witten analysis of the $SU(2)$ theory, it is possible to show that instantonic corrections, whose size becomes important at energies comparable to Λ_M, modify this picture and drastically change the IR behavior. The quantum moduli space has only M singularities, corresponding to monopoles or dyons becoming massless. These particular vacua break the $\mathbb{Z}_{2M}$ R-symmetry to $\mathbb{Z}_2$.

Instead, the $\mathbb{Z}_{2M}$-symmetric origin of the Coulomb branch is a smooth point, but has very interesting physics, as can be seen from the Seiberg-Witten curve for the $\mathcal{N} = 2$ pure $SU(M)$ Yang-Mills theory [137]:

$$y^2 = \prod_{a=1}^{M}(x - \phi_a)^2 - 4\Lambda_M^{2M} , \tag{8.3.3}$$

where ϕ_a are the eigenvalues of the scalar adjoint field in the vector multiplet, which parameterize the Coulomb moduli space up to Weyl group (gauge) identifications.[3]

Let us pause for a moment for an important comment. Classically ($\Lambda_M = 0$), the eigenvalues ϕ_a coincide with the branch points of (8.3.3) and also correspond to the positions of fractional D3 branes in the z plane [134]. At the quantum level, the eigenvalues of the adjoint scalar field still parameterize the whole moduli space, but the physical information resides in the branch points of the Seiberg-Witten curve (8.3.3), whose knowledge allows us to compute for instance the masses of BPS states. In the perturbative semiclassical regime of the moduli space ($|\phi_a| \gg |\Lambda_M|$), branch points of the Seiberg-Witten curve appear in pairs close to locations ϕ_a, with small separations, so that we can still approximately associate fractional D3 brane positions with VEV's. As soon as the VEV's get closer to the nonperturbative region (at energy scales comparable to Λ_M), the separations between branch points become large, so that it does not make much sense to talk about fractional D3 brane positions (almost double branch points) anymore. This point becomes clearer in the M theory derivation of the Seiberg-Witten curve, that we have reviewed in appendix E.1.

Let us now go back to the origin of the moduli space $\phi_a = 0\ \forall a = 1, \ldots, M$. This is a smooth point: the hyperelliptic curve (8.3.3) becomes $y^2 = x^{2M} + 4\Lambda_M^{2M}$, which has $2M$ separate branch points at the $2M$-th roots of $(2\Lambda_M^{2M})^{2M}$. This is a highly nonperturbative configuration, since branch points are equally separated. In the large M limit, one can think that these branch points densely fill the ring of radius $\Lambda_e \approx \Lambda_M$. This is the field theory manifestation of the enhançon mechanism: although classically fractional D3 branes can be placed at any points in the complex plane, and in particular they can stay at the origin, at the quantum level some regions are not allowed; the origin of the moduli space corresponds to fractional D3 branes melted at the enhançon ring.

It is also possible to see explicitly from the Seiberg-Witten curve (8.3.3) that if we add to the previous configuration a fractional D3 brane probe (in field theory terms, an additional VEV ϕ such that $|\phi| \gg |\Lambda_M|$), it can freely move in the semiclassical region. The corresponding two branch points are placed at about ϕ, with a small separation of order $\Lambda_M\,(\Lambda_M/\phi)^M$. However, as soon as ϕ becomes comparable to Λ_M, its two branch points split considerably and finally melt in the enhançon ring. The branch points cannot enter the interior, as the fractional D3 brane cannot. In the large M limit, the splitting occurs abruptly at the enhançon scale, mimicking very accurately the behavior of a fractional D3 brane probe in the solution of the previous section.

This analysis strongly suggests that the IR limit of the solution for fractional D3 branes on the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold actually describes $\mathcal{N} = 2$ pure $SU(M)$ SYM.

[3] Here we have to assume a basic knowledge of Seiberg-Witten theory. We refer the reader to the original papers [136, 138] and to the review [139] for a detailed pedagogical introduction. In appendix E we briefly review the derivation of Seiberg-Witten curves in the context of M theory.

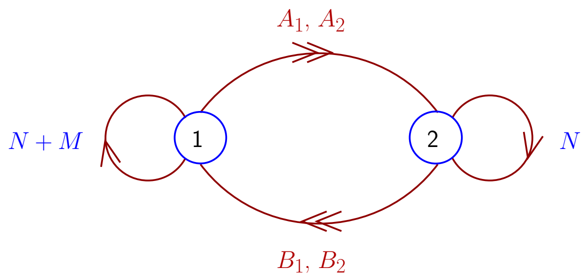


Figure 8.1: Quiver diagram of the theory on regular and fractional D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold, in an $\mathcal{N}=1$ notation where arrows represent bifundamental $\mathcal{N}=1$ chiral superfields. Arrows from one node to itself are adjoint chiral superfields.

8.3.2 An $\mathcal{N}=2$ Seiberg duality cascade

A broader point of view was taken by Polchinski [38]. He noticed that the logarithmically varying fluxes imply that the D3 brane Maxwell charge, the flux of F_5 through the base $S^5/\mathbb{Z}_2$ at radius r, grows indefinitely towards $r=\infty$, in a way that is completely analogous to the Klebanov-Strassler solution.[4] In both these supergravity solutions of fractional D3 branes, an effective D3 brane charge, growing toward large r, is generated by requiring the supergravity limit. By the same kind of analysis, based on Maxwell charges, that we followed in chapter 6, Polchinski concluded that at higher energies the full $\mathcal{N}=2$ quiver of the theory of regular and fractional D3 branes, see figure 8.1, is supported.

We repeat here the argument for later convenience, although the reader should know it quite well by now. The auxiliary brane system used to match the charges and extract the ranks of the gauge group is now made, at some radius r dual to an energy scale μ, of N+M D5 branes wrapped on the exceptional 2-cycle and N anti-D5 branes wrapped on the same cycle, with the suitable unit of worldvolume gauge flux that makes it BPS and minimal with respect to the wrapped D5. For the sake of convenience, we use the preferred gauge that we introduced in chapter 6: by a large gauge transformation, we transform $b(\rho)\mapsto\tilde{b}(\rho)\in[0,1]$.[5] The holographic relations between running gauge couplings and closed string fields are found expanding the DBI action of probe fractional D3 branes: they are equations (6.6.42)-(6.6.43), where $e^{\Phi}=g_s$, and l refers again to the larger and s to the smaller gauge group. Wrapped D5 branes are always associated to the larger gauge group, and wrapped anti-D5 with flux to the smaller gauge group, since the D5 brane charge of the solution is positive. As we flow toward the interior (the IR in the gauge theory), the larger gauge group flows towards strong coupling and the smaller towards weak coupling, as in the Klebanov-Strassler cascade, with the difference that here the β-functions of the gauge couplings are proportional to $\pm 2M$ instead of $\pm 3M$. At a first threshold scale ρ_1, the gauge coupling of the larger group diverges. On the supergravity side, the wrapped D5 branes of the auxiliary brane system become tensionless and then their D3 brane charge changes sign, making the system no longer BPS. At this point a

[4]Actually, in the $\mathcal{N}=2$ case under consideration, a different UV completion in terms of a conformal theory, stopping the cascade in the UV, can be achieved easily, as we will discuss below.

[5]Would not we have chosen that gauge, the previous objects would have had different units of worldvolume gauge noninvariant flux on the exceptional 2-cycle, that can be easily worked out.

transition occurs.[6] The constituents rearrange into a new BPS system in the adjacent interval, made of objects which look like N wrapped D5 and $N-M$ wrapped anti-D5 (with worldvolume flux) after having chosen the good preferred gauge. In the old gauge we would have called them a wrapped D5 with one unit of flux and a wrapped anti-D5 without flux, respectively. We stress once again that a very nontrivial rearrangement has occurred: these two constituents are not the same as those which were BPS in the previous interval, they only look the same in the preferred gauge we chose. In particular, the $N+M$ wrapped D5 branes of the initial interval do not become the N wrapped D5 branes of the new one, nor the N initial wrapped anti-D5 branes become the $N-M$ new wrapped anti-D5 branes. On the contrary, as in the Klebanov-Strassler cascade of section 3.4, continuity of the gauge couplings at this seeming strong coupling singularity teaches us that the gauge group $SU(N)$ behaves as a spectator, whereas $SU(N+M)$ becomes $SU(N-M)$. The previous analysis can be simplified by exploiting our method based on Page charges. The result of course confirms the previous analysis done with Maxwell charges.

This pattern is the smoking gun of a self-similar cascading RG flow of the dual field theory. Polchinski proposed that the field theory interpretation should go along the lines of an unknown $\mathcal{N}=2$ Seiberg duality.

It was also noticed that a Higgsing $SU(N+M) \to SU(N-M)$ (times a group of rank $2M$) by the adjoint scalar field in the vector multiplet of the broken gauge group, leaving the $SU(M)$ group as a spectator, could fit the numerology. However, he discarded this possibility because $SU(N+M)$ initially acted on the D5 brane Chan-Paton factors, whereas $SU(N-M)$ finally acts on the anti-D5 brane Chan-Paton factors. Actually, the last argument can be questioned since the transition occurs in a strongly coupled nonperturbative regime where no Chan-Paton description can be applied. The nonperturbative nature of the field theory dynamics responsible for the transitions driving the cascade is reflected in the highly nontrivial rearrangement of the auxiliary brane system. We will return on this possibility in a following section.

8.3.3 A Higgsing interpretation

A different, more mundane interpretation of the cascade was proposed by Aharony [64], who pointed out that the electric-magnetic duality suggested by Polchinski, at least in its simplest form, could not ensure matching of the moduli spaces of the proposed dual pair. He instead provided pieces of evidence for a diagonal Higgsing of $SU(N+M) \times SU(N)$ to a $SU(N) \times SU(N-M)$ times a subgroup of $U(M) \times U(M)$, by M eigenvalues for the adjoint scalars of the two gauge groups.

Aharony's point was that the source for F_5 could be thought of as coming from an actual distribution of regular or fractional D3 branes, dual to field theory VEV's of the adjoint scalars, rather than from the 3-form fluxes, which have no direct analogue in the field theory. In order to match the decrease of 5-form flux, he proposed that M eigenvalues of the two adjoint scalars should lie in each band between a radius where the D3 brane Maxwell charge is N and a radius where the D3 brane Maxwell charge is $N-M$, for any N modulo M. And in order not to source the complex twisted scalar γ, he proposed that these sources are regular D3 branes (equal sets of eigenvalues for the two adjoint scalars in the band) rather than fractional D3 branes.

[6] Remark that the wrapped D5 which ceases being BPS at ρ_1 is actually BPS for any $\rho > \rho_1$. Similarly, the wrapped anti-D5 with flux that is BPS in the same interval $\rho \in [\rho_1, \rho_0]$, with $\rho_0 = \rho_1\, e^{\frac{2\pi}{2g_s M}}$, is actually BPS for any $\rho < \rho_0$. In the $[\rho_1, \rho_0]$ interval, the D5 and wrapped anti-D5 with flux are the minimal BPS objects, in the sense that any BPS object can be built as a linear combination thereof, with nonnegative integer coefficients.

As noticed also by Polchinski, the unbalance between the ranks of the two groups can be achieved starting from a number of regular D3 branes, splitting M of them into wrapped D5 and wrapped anti-D5 brane, and placing these M anti-D5 branes at a UV cutoff scale by a perturbative Higgsing which triggers the RG flow. This provides a UV cutoff to the cascade in terms of a conformal field theory completion. Eventually, these cutoff branes can be sent to infinity so that the cascade becomes infinite. Remark that this is not possible in the $\mathcal{N}=1$ Klebanov-Strassler cascade, which lacks such a moduli space.

The main evidence for a Higgsing interpretation rather than a duality interpretation arose from a holographic computation of the adjoint scalar VEV's by means of the AdS/CFT machinery of section 2.4, which is made possible thanks to the conformal field theory UV completion in the cutoff cascade configuration previously mentioned. An operator dual to an untwisted field, of the schematic form $\mathrm{Tr}(\Phi_1^\dagger\Phi_1)+\mathrm{Tr}(\Phi_2^\dagger\Phi_2)+$ hypermultiplets, was shown to acquire a VEV, inconsistently with the duality interpretation but consistently with the Higgsing interpretation for the cascade.

Aharony himself singled out some problems with its Higgsing interpretation. The first one is that the Higgsing he proposed has actually a huge moduli space of possibilities, that is not clear how to see in the dual background. But the main problem comes from the observation, already presented in the previous subsection, that the field theory computation of the running gauge couplings shows that the coupling of the larger gauge group diverges (at least using the perturbative result) at some scale. This behavior is not affected by the regular D3 brane Higgsing, therefore something else has to be invoked to explain what happens there. The holographic computation of the gauge couplings shows how this gauge coupling goes past this perturbative singularity, in a way which is more consistent with an electric-magnetic duality interpretation, although such a duality is not known.

Let us end this section by remarking another related problem for the perturbative Higgsing interpretation, that is clearer to us now that the method based on Page charges is available. Associating D3 brane sources to the continuous variation of the D3 brane Maxwell charge turns out to be incorrect: it is an artifact of Chern-Simons terms in the type IIB action. Taking the solution of the previous section in the gauge we have written it, we can compute the D3 brane Page charge and see easily that it is constant as we vary r, showing that there are no actual D3 brane sources distributed in the geometry.[7] This is also consistent with the continuity of the gauge couplings which is displayed by the holographic formulae (6.6.42)-(6.6.43), that are obtained assuming that the minimal BPS constituents in each energy range are always the wrapped D5 without flux and the wrapped anti-D5 with flux in the preferred gauge.

Remark, however, that the above comment does not spoil the important observation that some Higgsing is occurring in the dual field theory, as the aforementioned holographic computation confirms. In the following sections, we will propose and provide evidence for an alternative interpretation of the cascade, which reconnects Polchinski's and Aharony's results.

Before doing that, let us mention another interesting proposal for the gauge/gravity pair under consideration, whose upshot is somehow closer to the perspective illustrated in subsection 8.3.1.

[7] The reader should not be confused by the fact that here we do not change gauge at the transitions. That is done when we want to extract the gauge theory ranks by matching them with charges of the auxiliary brane system. Here instead we simply want to see whether there are D3 branes in the geometry. There are not, neither in-between transition scales nor at transition scales.

8.3.4 The Seiberg-Witten curve analysis

An analysis of the Coulomb branch of the moduli space of the $SU(N+M) \times SU(N+M)$ orbifold quiver gauge theory, spontaneously broken to $SU(N+M) \times SU(N) \times U(1)^M$ in a perturbative regime by adjoint scalar VEV's, was performed using the Seiberg-Witten curve for the conformal gauge theory in [129].

In appendix E we briefly recall the derivation of Seiberg-Witten curves in the M theory lift of type IIA brane engineering models for the $\mathcal{N}=2$ pure YM and SQCD theories (appendix E.1) and for the elliptic model under study (appendix E.2).

Petrini, Russo and Zaffaroni [129] studied the RG flow and the dynamics of the origin of the moduli space of the $SU(N+M) \times SU(N)$ theory. That nonconformal theory is obtained via a perturbative Higgsing of the conformal $SU(N+M) \times SU(N+M)$ theory by M eigenvalues of the adjoint scalar $\tilde{\Phi}$ of the second gauge group, which we have already mentioned in the previous subsections as a possible conventional UV completion of the field theory vacuum dual to the supergravity solution. This completion is particularly useful because it allows to exploit the Seiberg-Witten technology.

We recall here that the Seiberg-Witten curve (E.2.13), found in M theory as a holomorphic M5 brane embedding [140], may be written as

$$\frac{S(v)+R(v)}{S(v)-R(v)} = f(u|\tau) \equiv \frac{\theta_3(u|\tau/2)}{\theta_4(u|\tau/2)} = \frac{\theta_3(2u|2\tau)+\theta_2(2u|2\tau)}{\theta_3(2u|2\tau)-\theta_2(2u|2\tau)} \tag{8.3.4}$$

in terms of quasi-modular Jacobi θ-functions defined in appendix E.3, by suitably choosing the UV values of the gauge couplings of the two $SU(N+M)$ gauge groups to be the same. More details on its derivation can be found in appendix E.2. $R(v)=\prod_{a=1}^{N+M}(v-\phi_a)$ and $S(v)=\prod_{a=1}^{N+M}(v-\tilde{\phi}_a)$ are degree $N+M$ polynomials whose zeroes ϕ_a and $\tilde{\phi}_a$ are the eigenvalues of the VEV's of the adjoint scalars of the two gauge groups. $u=i(x^6+ix^{10})/(2\pi R)$ parameterizes an M theory 2-torus defined by the identifications $u \sim u+1 \sim u+\tau$, where the complex structure τ is nothing but the type IIB axio-dilaton of section 8.2, but expressed in the string frame.

Let us briefly review the analysis of [129]. The VEV's were chosen for simplicity to be $\mathbb{Z}_M$-invariant, and the other VEV's were chosen to be at the origin, so that the polynomials R and S are:

$$R(v) = v^{N+M} \qquad\qquad S(v) = v^N(v^M - z_\infty^M)\ . \tag{8.3.5}$$

By means of the Seiberg-Witten curve (8.3.4), it is possible to extract both the branch points and the RG flow of the gauge theory at the chosen point of the moduli space. As already stressed in appendix E.2, the N common zeros of R and S, corresponding in the type IIB picture to N regular D3 branes at the origin, factor out of the curve, without affecting the RG flow and the rest of the dynamics. Therefore we can actually use

$$R(v) = v^M \qquad\qquad S(v) = v^M - z_\infty^M\ , \tag{8.3.6}$$

so that the curve is actually the same as that of an $SU(M) \times SU(M)$ gauge theory, spontaneously broken to $SU(M)$ at a scale z_∞. If the IR dynamics is not much affected by the UV Higgsing, as it is natural to expect, the low energy physics should be similar to the one of the enhançon, but with N leftover regular D3 branes. This is indeed the upshot of the analysis.

Thanks to the relation between the Seiberg-Witten curve and the brane embedding explained in appendix E, the RG flow can be extracted from the roots of the curve as v varies. In the case at

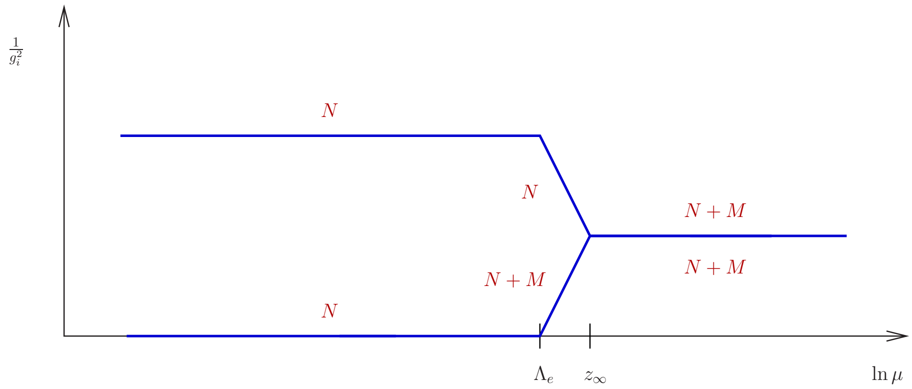


Figure 8.2: RG flow of the theory at the enhançon vacuum (origin of the moduli space).

hand, this is done by approximating the exact curve in different regimes, according to the value of the ratio

$$\frac{S(v) + R(v)}{S(v) - R(v)} = 1 - 2\left(\frac{v}{z_\infty}\right)^M , \tag{8.3.7}$$

which has to be equated to $f(u|\tau)$. In the UV regime $|v| > |z_\infty|$,[8] the ratio is very large and the theory is conformal with equal gauge couplings, because of the initial choice. On the contrary, when $|v| < |z_\infty|$ the ratio is almost equal to one. For $|\Lambda| < |v| < |z_\infty|$, with $\Lambda = z_\infty q^{\frac{1}{4M}}$ the dynamically generated IR strong coupling scale of the unbroken $SU(N + M)$ group, the two gauge couplings start to run in opposite directions. In the previous relations, $q = e^{2\pi i\tau}$, whose modulus is related to the type IIB dilaton by $|q| = e^{-2\pi/g_s}$ and is very small in the supergravity approximation, allowing a series expansion in its positive powers. It is possible to use the expansion $f = 1+2q^{1/4}(t+t^{-1})+\mathcal{O}(q^{1/2})$ to get the approximate roots of the curve. The result is the perturbative running of the gauge couplings, with β-functions equal to $\pm 2M$ for the two gauge groups respectively; the gauge coupling of the unbroken $SU(N + M)$ group diverges at a scale Λ. This running agrees with the type IIB formulae according to which

$$\begin{aligned} \frac{8\pi^2}{g_l^2} &= \frac{2\pi}{g_s} b \\ \frac{8\pi^2}{g_s^2} &= \frac{2\pi}{g_s}(1 - b) . \end{aligned} \tag{8.3.8}$$

Analogously, for $|v| < |\Lambda|$ it is possible to see that the couplings remain constant: there is an infinitely coupled $SU(N)$ gauge group, and another $SU(N)$ group at its minimal value for the coupling, as if in (8.3.8) b remained constant and equal to 0, see figure 8.2. This is precisely what would occur if an enhançon mechanism were at work. The previous analysis is also confirmed by the positions of the branch points of the Seiberg-Witten curve, that correspond to the double points of $f(u|\tau)$, at this point of the moduli space. It turns out to be simpler to use the alternative meromorphic function $g(u|\tau)$ defined in (E.2.11), related to f by (E.2.12), and in terms of which

[8]As usual in the large M limit, transitions occur abruptly at the threshold scales, allowing us to use $>$ instead of $\gg$.

the curve is $R/S = g$. These double points come in two sets: those placed at $u = 0, 1/2$, where $g \approx \pm 2q^1/4$ is small in the supergravity limit, become zeros of the polynomial R in the classical limit; instead those placed at $u = \tau/2, (1+\tau)/2$, where $g \approx \pm(2q^1/4)^{-1}$ is large in the supergravity limit, can be associated to the polynomial S. The first set of branch points turns out to be placed at

$$u = 0, \frac{1}{2}: \quad v \simeq v_k = 2^{\frac{1}{M}} \Lambda e^{\frac{2\pi i k}{2M}} , \qquad k = 0, 1, \ldots, 2M - 1 . \tag{8.3.9}$$

They correspond to M fractional D3 branes melted at an enhançon ring at a scale Λ. The second set of branch points are almost double branch points

$$u = \frac{\tau}{2}, \frac{1+\tau}{2}: \quad v \simeq v_h^{(\pm)} = z_\infty \left[1 \pm \frac{2}{M} \left(\frac{\Lambda}{z_\infty} \right)^M \right] e^{\frac{2\pi i h}{M}} , \qquad h = 0, 1, \ldots, M - 1 . \tag{8.3.10}$$

They correspond to M semiclassical fractional D3 branes of the opposite kind, located about the VEV's $z_\infty e^{\frac{2\pi i h}{M}}$ of the adjoint scalar $\tilde{\Phi}$.

Moreover, as in the pure YM case, it turns out to be possible to understand what happens to fractional D3 brane probes by means of the Seiberg-Witten curve. It is enough to add one eigenvalue to Φ and one to $\tilde{\Phi}$. If the two eigenvalues are added at an intermediate scale between z_∞ and Λ, it is easy to see that the eigenvalue of Φ (corresponding to a wrapped D5 brane) can freely move outside the enhançon ring, but, as it approaches it, its two branch points split and melt at the enhançon, which then consists of $M + 1$ wrapped D5 branes. On the contrary, the eigenvalue of $\tilde{\Phi}$ (corresponding to a wrapped anti-D5 brane with flux) is free to move at $|v| < |z_\infty|$ (that it cannot cross), and can easily penetrate the enhançon ring: when this happens, it unchains two branch points from the enhançon ring, that follow the two anti-D5 brane branch points inside the ring. The interpretation of this phenomenon is that a wrapped anti-D5 brane with flux captures a melted D5 brane from the ring, forming a D3 brane that is free to move everywhere.

An important remark is in order at this point. The one previously considered is not exactly the same enhançon of section 8.3.1, but actually a generalized enhançon, in the language of [64]. The difference is that the gauge group whose coupling diverges is $SU(N + M)$ rather than $SU(M)$, leaving N units of D3 brane charge inside the. It reduces to the usual enhançon only when $N = 0$.

This analysis led the authors of [129] to conclude that actually there is no cascade: as soon as the first generalized enhançon (the first threshold scale) is met, the gauge theory remains infinitely coupled, and the fractional D3 brane melt in a ring.

This is certainly true for the vacuum at the origin of the moduli space of the $SU(N+M) \times SU(N)$ gauge theory, for any value of N: its large N and M correct string dual is described by a solution with constant twisted fields from infinity down to a cutoff radius where M cutoff wrapped anti-D5 branes with flux are placed; at that radius the b field starts to run as in the solution of section 8.2; at the radius where b vanishes, there are actually M wrapped D5 branes melted in an enhançon ring; in the interior $b = 0$ and the metric is flat.

In the following sections, we would like to propose that there exists a different vacuum which instead displays a cascading behavior like the one of the supergravity solution of section 8.2. We will see that it can be obtained as a particular limiting case of an infinite class of vacua, including also the previous one, whose string dual can be easily identified.

Before presenting our proposal for the cascading vacuum, it is convenient to make a detour on the moduli space of $\mathcal{N} = 2$ SQCD.

8.4 The cascading vacuum in the dual field theory

8.4.1 The baryonic root of $\mathcal{N}=2$ SQCD and the cascade proposal

In chapter 3, we have seen that in the $\mathcal{N}=1$ Klebanov-Strassler cascade, the nonperturbative dynamics which resolves the seeming perturbative strong coupling singularity and triggers transitions along the cascade, Seiberg's electric-magnetic duality, could be understood by neglecting the dynamics of the finitely coupled gauge group, which acts as a spectator, and concentrating solely on the gauge group flowing to infinite coupling.

It turns out that such an approach can be also successfully applied to the $\mathcal{N}=2$ cascade from fractional D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity, with gauge group $SU(N+M)\times SU(N)$. If we treat, to first approximation, the finitely coupled $SU(N)$ gauge group as a global flavor group, the relevant field theory is $\mathcal{N}=2$ SQCD with $n_c=N+M$ colors and $n_f=2N$ massless flavors.

The moduli space of vacua of $\mathcal{N}=2$ SQCD has been explored and characterized in a series of papers. The Coulomb branch of the moduli space was studied in [141], where the Seiberg-Witten curve for this theory was found. Later, the full quantum moduli space of $\mathcal{N}=2$ SQCD was analyzed by Argyres, Plesser and Seiberg [142]. The result of that analysis is the following.[9] The Coulomb branch is parameterized by the vacuum expectation value of the adjoint scalar field Φ in the $\mathcal{N}=2$ vector multiplet, that after a suitable gauge rotation may be written in the diagonal form $\Phi=\mathrm{diag}(\phi_1,\phi_2,\ldots,\phi_{n_c})$; the eigenvalues are subject to the tracelessness constraint $\sum_{a=1}^{n_c}\phi_a=0$ and are identified under permutations, which are gauge symmetries. The VEV's generically breaks spontaneously the nonabelian $SU(n_c)$ gauge group to its Cartan subgroup $U(1)^{n_c-1}$. However, at special submanifolds, where the Higgs branch touches the Coulomb branch, a nonabelian gauge symmetry survives. Since the Higgs branch can be divided into nonbaryonic branches and a baryonic branch, the corresponding intersections with the Coulomb branch were named nonbaryonic roots and baryonic root respectively. nonbaryonic roots are true submanifolds, whereas the baryonic root is a single point. Classically, the nonbaryonic roots and the baryonic root intersect, but at quantum level they are separated.

nonbaryonic branches are labeled by an integer $r\leq[n_f/2]$. The low energy effective theory at the roots are the IR free or finite $SU(r)\times U(1)^{n_c-r}$ SQCD with n_f hypermultiplets in the fundamental representation and charged under one of the $U(1)$ factors. At special points along these submanifolds, the Seiberg-Witten curve (E.1.5) shows that n_c-r-1 additional massless singlet hypermultiplets arise, each one charged under one of the remaining $U(1)$ factors. It is important for us to remark that there are $2n_c-n_f$ such vacua, related by the broken $\mathbb{Z}_{2n_c-n_f}$ nonanomalous R-symmetry acting on the Coulomb branch.

Instead the baryonic root is a single point, invariant under the $\mathbb{Z}_{2n_c-n_f}$ R-symmetry. This implies that its coordinates on the Coulomb branch are of the form

$$\Phi_b=\alpha(\omega,\omega^2,\ldots,\omega^{2n_c-n_f},0,\ldots,0)\,, \tag{8.4.1}$$

where $\omega=e^{\frac{2\pi i}{2n_c-n_f}}$, for some constant α. The gauge group is thus broken to $SU(n_f-n_c)\times U(1)^{2n_c-n_f}$, which is IR-free. The requirement that a Higgs branch originates from this root implies the presence of $2n_c-n_f$ massless singlet hypermultiplets charged under the $U(1)$ factors. The charges of the massless matter fields under the gauge symmetries can be found in [142]. The

[9] We will restrict our attention to the window $n_c<n_f\leq 2n_c-2$, where the nonconformal orbifold theory under consideration lies.

presence of additional massless hypermultiplets allows the identification of the constant α in (8.4.1) by means of the Seiberg-Witten curve (E.1.5). The result is that $\alpha = (-1)^{1/(2n_c-n_f)}\Lambda$, so that the Seiberg-Witten curve takes the singular form:

$$y^2 = x^{2(n_f-n_c)}\left(x^{2n_c-n_f} - \Lambda^{2n_c-n_f}\right)^2 \ . \tag{8.4.2}$$

The $x^{2(n_f-n_c)}$ factor, corresponding to $2(n_f-n_c)$ coincident branch points at the origin, signals the presence of the massless gluons of the nonabelian $SU(n_f-n_c)$ group. The remaining $2(2n_c-n_f)$ branch points show up in coincident pairs, placed at $x_k = \Lambda\omega^k$, with $k = 0, 1, \ldots, 2n_c-n_f-1$: this is the right singularity structure to account for $2n_c-n_f$ mutually local massless hypermultiplets.

The reader has perhaps already got the reason of this detour: the nonperturbative Higgsing at the baryonic root preserves the same $\mathbb{Z}_{2n_c-n_f}$ R-symmetry of the supergravity solution of section 8.2, and its low energy effective theory possesses a nonabelian $SU(n_f-n_c)$ gauge symmetry precisely matching the numerology of the cascade interpretation of section 8.3.2! It is therefore natural for us to interpret the solution of fractional D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold as dual to the cascading $SU(N+M)\times SU(N)$ quiver gauge theory at subsequent baryonic roots of the strongly coupled groups.[10] This proposal reconnects the viewpoints of sections 8.3.2 and 8.3.3. Highly nonperturbative dynamics occurs at energy scales where one of the couplings of the gauge theory diverges, reducing the nonabelian group from $SU(N+M)$ to $SU(N-M)$ (modulo M). This phenomenon is not interpreted as an $\mathcal{N}=2$ Seiberg duality, but as a Higgsing, as suggested also by the holographic field/operator computation mentioned in section 8.3.3. However, this Higgsing is driven by the adjoint scalar of the strongly coupled group, and stems out of nonperturbative dynamics.

In the next subsection, we will study the Seiberg-Witten curve of the gauge theory in this vacuum, showing that it actually mirrors exactly the type IIB supergravity solution. In the following sections, we will study in more detail the moduli space of vacua of the $SU(N+M)\times SU(N)$ theory, showing how it is possible to construct the cascading vacuum starting from the enhançon vacuum at the origin of the moduli space illustrated in section 8.3.4.

Before proceeding, a couple of important remarks are in order. The first observation concerns the $2M$ massless photons of the $U(1)^{2M}$ abelian groups and the $2M$ massless hypermultiplets at the baryonic root. Their presence is clear from the Seiberg-Witten curve (8.4.2), but these massless states are more elusive in the type IIB picture. In the latter picture, they should be related to subtle tensionless string phases of the background, arising when the complex twisted scalar $\gamma = c+\tau b$ vanishes up to periodicities: $\gamma\in\mathbb{Z}+\tau\mathbb{Z}$. This becomes clearer following the duality map between the type IIB picture and the M theory picture (which is nothing but the Seiberg-Witten curve). We will see that points in the complex plane where γ vanishes (up to its periodicities) correspond to the double branch points of the massless hypermultiplets in the cascading Seiberg-Witten curve.

Another important remark regards the relation to Seiberg duality in $\mathcal{N}=1$ theories pointed out in [142]. Upon mass deformation, the $\mathcal{N}=2$ SQCD theory reduces at low energies to $\mathcal{N}=1$ SQCD with a quartic superpotential for the quarks, if the mass parameter is larger than the strong coupling scale. Even in the deep IR (or in other terms, sending the mass deformation to infinity), the quartic superpotential survives, as it is marginal for $n_f = 2n_c$ and relevant for $n_f < 2n_c$ at

[10] We should mention at this point that a proposal for an $\mathcal{N}=2$ cascade at the baryonic root has actually been proposed in [143] in the context of the M theory realization of this elliptic model, following the same basic SQCD reasoning explained above, without mentioning its relation to the type IIB solution of [39].

the quantum level.[11] When the mass parameter is much smaller than the strong coupling scale of the $\mathcal{N}=2$ theory, the use of the effective theory at the roots is justified. $\mathcal{N}=1$ supersymmetry prevents phase transitions as the mass parameter varies, so that the results should be the same as those in the previous microscopic picture. As soon as the mass deformation is turned on, the Coulomb branch is lifted, except for the baryonic root and the special points in the nonbaryonic root where there is a maximal number of massless hypermultiplets. As the mass parameter is sent to infinity (or in other words, in the IR), these points merge into a single one. Moreover, new branches appear [144]. It is possible to show that this macroscopic description of the mass deformation leads precisely to the magnetic dual theory – $\mathcal{N}=1$ $SU(n_f-n_c)$ SQCD with n_f (magnetic) quark superfields and a quartic superpotential– of the microscopic (electric) theory – $\mathcal{N}=1$ $SU(n_c)$ SQCD with n_f (electric) quark superfields and a quartic superpotential –. This is practically a proof of Seiberg duality in SQCD with quartic superpotentials.

Given this relation, it is natural to expect that the cascading vacuum of the $\mathcal{N}=2$ quiver gauge theory of fractional branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold should reduce upon mass deformation to the Klebanov-Strassler cascading vacuum of the $\mathcal{N}=1$ quiver gauge theory of fractional branes at the conifold singularity. We will return to this point in the next subsection.

8.4.2 The cascading vacuum in the quiver gauge theory

In this subsection we consider the cascading vacuum at subsequent baryonic roots that we have suggested in the previous subsection; we study it by means of the Seiberg-Witten curve for the quiver gauge theory, introduced in section 8.3.4 and explained in appendix E.2. We will see that the analysis of the curve at this special point indeed leads to a cascading RG flow for the field theory which mimics precisely the running gauge couplings and the decrease of ranks that can be extracted from the supergravity background of section 8.2, along the lines of the holographic analysis of section 8.3.2.

Let us consider the elliptic model of section 8.3.4, starting from an $SU((2h+1)M)\times SU((2h+1)M)$ conformal theory in the ultraviolet and then breaking the gauge group to $SU((2h+1)M)\times SU(2hM)$ by giving VEV's of order z_∞ to some components of the adjoint scalar of the second gauge group, so as to trigger an RG flow: the vacuum we choose is given by subsequent alternating baryonic root VEV's for the two gauge groups. The Seiberg-Witten curve is

$$\frac{R(v)}{S(v)}=g(u|\tau)\ , \tag{8.4.3}$$

where the meromorphic function $g(u|\tau)$ is defined in (E.2.11) in terms of quasi-modular functions on the u torus of complex structure τ. The polynomials $R(v)$ and $S(v)$, of degree $(2h+1)M$, can be written as:

$$\begin{aligned}R(v)&=v^M\prod_{j=0}^{h-1}(v^{2M}+q^{\frac{1}{2}+2j}z_\infty^{2M})\\ S(v)&=(v^M-z_\infty^M)\prod_{j=0}^{h-1}(v^{2M}+q^{\frac{3}{2}+2j}z_\infty^{2M})\ .\end{aligned}\tag{8.4.4}$$

[11] We do not consider $n_f>2n_c$ since the theory is not UV free and needs a completion.

The polynomial $R(v)$ is related to the $SU((2h+1)M)$ group that starts flowing toward strong coupling at the cutoff scale, whereas the polynomial $S(v)$ is related to the $SU((2h+1)M)$ group which is spontaneously broken to $SU(2hM)$ at the scale z_∞. The eigenvalues of the two adjoint scalar fields are put, in an alternating manner, at energies corresponding to their (subsequent) strong coupling scales along the cascade: in the limit in which we neglect the gauge dynamics of the finitely coupled gauge group at those strong coupling scales, the vacuum expectation values are placed exactly at the baryonic roots of the relevant SQCD theories at those energy scales. In agreement with the cascading RG flow of the supergravity solution, the hierarchy of the strong coupling scales is controlled by $q = e^{2\pi i\tau}$, so that the strong coupling scales are equally spaced in a logarithmic plot.

It is clear that the RG flow of the quiver gauge theory on this vacuum coincides by construction with the RG flow that can be extracted holographically, as in section 8.3.2, from the type IIB supergravity solution of fractional branes at the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold. Indeed, because of the large M limit, the Seiberg-Witten curve shows that out of the strong coupling scales of the gauge group the running is led by the perturbative β-functions. This can be explicitly checked either numerically using the exact curve, or analytically by expanding the polynomials energy range by energy range, in an effective field theory approach that makes the previous statement obvious.

We now move to the study of the branch points of the curve. At $t = \pm 1$ ($u = 0$ and $u = 1/2$ respectively) we find that the branch points of the Seiberg-Witten curve (points in the v plane where NS5 branes touch) associated to the polynomial $R(v)$ lie at $v^M = \mp q^{\frac{1}{4}+n} z_\infty^M$ ($n = 0, 1, \ldots, h-1$) and $v^M = \mp 2 q^{\frac{1}{4}+h} z_\infty^M$ respectively. The former class of points consists of h sets of baryonic root VEV's of the first gauge group, whereas the latter branch points are those of the true enhançon of the low energy $SU(M)$ theory.

At $t = \pm q^{1/2}$ ($u = \tau/2$ and $u = (\tau+1)/2$ respectively), the branch points of the Seiberg-Witten curve associated to the polynomial $S(v)$ lie at $v^M = (1 \pm 2q^{\frac{1}{4}}) z_\infty^M$ and $v^M = \mp q^{\frac{3}{4}+n} z_\infty^M$ ($n = 0, 1, \ldots, h-1$) respectively. The first set of points are the almost coincident branch points associated to semiclassical fractional D3 branes at the cutoff scale z_∞, whereas the other ones are h sets of baryonic root VEV's of the second gauge group.

From the number of solutions found in the previous analysis, we immediately have an indication that, to leading order in q, all the branch points are double except those related to the enhançon, which are well split, and those related to the cutoff, whose relative splitting is of order $q^{1/4}$; the separations between close branch points at the cutoff are larger than those at the baryonic root points, which are at least of order q, as we will see below.

We would like to estimate to which extent the 'baryonic root' branch points may be considered double in the curve for the quiver gauge theory; in other words, we would like to see how much this curve is similar, close to each transition scale, to the curve at the baryonic root of the related SQCD theory. In particular, an important property of the baryonic root curve of SQCD is that it has genus 0, having all double branch points: this is what allows this point of the moduli space to survive the mass deformation to $\mathcal{N} = 1$ [144, 145]. It would be very nice to find a cascading $\mathcal{N} = 2$ vacuum with genus zero Seiberg-Witten curve: such a vacuum would survive a mass deformation, which could be used to provide an exact interpolation between an $\mathcal{N} = 2$ cascade and the $\mathcal{N} = 1$ Klebanov-Strassler cascade. Unfortunately, we have not succeeded in this achievement yet.

We will see explicitly that the branch points of the Seiberg-Witten curve of the cutoff cascading vacuum (8.4.4), to order $\mathcal{O}(q)$,[12] are all double except those at the cutoff and at the enhançon. Clearly, any vacuum with a conformal UV completion does not survive the mass deformation, since the adjoint VEV's which Higgs the conformal group to a nonconformal one are not allowed anymore. But as soon as we send the cutoff to infinity, hence constructing an infinite cascade, the approximation to which the branch points are double becomes better and better in the UV.

Let us see this explicitly. In order to study the infinite cascade limit, where we keep the scales of strong coupling fixed as we send the cutoff to infinity, it is convenient to rewrite the two polynomials in terms of the IR enhançon scale Λ_e:

$$\begin{aligned} R_h(v) &= v^M \prod_{j=1}^{h} (v^{2M} + q^{-2j}\Lambda_e^{2M}) \\ S_h(v) &= (v^M - q^{-\frac{1}{4}-h}\Lambda_e^M) \prod_{j=1}^{h} (v^{2M} + q^{1-2j}\Lambda_e^{2M}) \; . \end{aligned} \tag{8.4.5}$$

The limit of infinite cascade is simply $h \to \infty$. We then define $\tilde{v} = v/\Lambda_e$, and introduce dimensionless polynomials $\tilde{R}_h(\tilde{v}) = \Lambda_e^{(1+2h)M} R_h(v)$ and $\tilde{S}_h(\tilde{v}) = \Lambda_e^{(1+2h)M} S_h(v)$. Further definition of $x = \tilde{v}^M$ allows us to write

$$\begin{aligned} \tilde{R}_h(x) &= x \prod_{j=1}^{h} (x^2 + q^{-2j}) \\ \tilde{S}_h(x) &= (x - q^{-\frac{1}{4}-h}) \prod_{j=1}^{h} (x^2 + q^{1-2j}) \; . \end{aligned} \tag{8.4.6}$$

Let us also define

$$T_h(x) \equiv \frac{R_h(x)}{S_h(x)} = \frac{x}{x - q^{-\frac{1}{4}-h}} \, \frac{\prod_{j=1}^{h}(x^2 + q^{-2j})}{\prod_{j=1}^{h}(x^2 + q^{1-2j})} \tag{8.4.7}$$

and

$$T(x) \equiv \lim_{h\to\infty} T_h(x) \; , \tag{8.4.8}$$

so that the exact Seiberg-Witten curve (8.4.3) for the infinite cascade we are studying becomes formally

$$T(x) = g(u|\tau) \; . \tag{8.4.9}$$

We would like to see to which level of approximation the branch points of this curve, at $u = 0, \frac{1}{2}$ and $u = \frac{\tau}{2}, \frac{1+\tau}{2}$ may be considered double in an expansion in positive powers of the small parameter q. We will make use of the following property of g at its double points: $g(0|\tau) = -g(\frac{1}{2}|\tau) = 1/g(\frac{\tau}{2}|\tau) = -1/g(\frac{1+\tau}{2}|\tau)$. Let us start from the branch points at $u = 0, \frac{1}{2}$. It is easy to show that

$$T_h(-sq^{-n}) = \frac{2sq^{\frac{1}{4}}}{1 + sq^{\frac{1}{4}+h-n}} \, \frac{\prod_{i=1}^{n-1}(1 + q^{2i})^2}{\prod_{i=1}^{n}(1 + q^{2i-1})^2} \, \frac{\prod_{i=n}^{h-n}(1 + q^{2i})}{\prod_{i=n+1}^{h-n}(1 + q^{2i-1})} \; , \tag{8.4.10}$$

[12] Recall that in the supergravity limit $q \to 0$.

where $s = \pm 1$. Therefore, corrections due to the cutoff h enter $q^{-\frac{1}{4}} T_h(-sq^{-n})$ at orders $q^{h-n+\frac{1}{4}}$ and $q^{2(h-n)+1}$. We should then compare $s\,T(-sq^{-n})$ with

$$g_0(q) \equiv g(0|\tau) = \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)} = 2q^{\frac{1}{4}} \prod_{j=1}^{\infty} \frac{(1+q^{2j})^2}{(1+q^{2j-1})^2} \ . \qquad (8.4.11)$$

The comparison yields

$$\frac{s\,T(-sq^{-n})}{g_0(q)} = \frac{\prod_{i=n+1}^{\infty}(1+q^{2i-1})}{\prod_{i=n}^{\infty}(1+q^{2i})} = 1 + O(q^{2n}) \ . \qquad (8.4.12)$$

Similarly, for the branch points at $u = \frac{\tau}{2}, \frac{1+\tau}{2}$,

$$T_h(-sq^{-n+\frac{1}{2}}) = \frac{1}{2sq^{\frac{1}{4}}\,(1+sq^{\frac{3}{4}+h-n})} \frac{\prod_{i=1}^{n-1}(1+q^{2i-1})^2}{\prod_{i=1}^{n-1}(1+q^{2i})^2} \frac{\prod_{i=n}^{h-n+1}(1+q^{2i-1})}{\prod_{i=n}^{h-n}(1+q^{2i})} \ . \qquad (8.4.13)$$

Corrections due to the finite cutoff h enter $q^{\frac{1}{4}} T_h(-sq^{-n+\frac{1}{2}})$ at orders $q^{h-n+\frac{3}{4}}$ and $q^{2(h-n+1)}$. We should now compare $s/T(-sq^{-n+\frac{1}{2}})$ with $g_0(q)$:

$$\frac{s\,T(-sq^{-n+\frac{1}{2}})^{-1}}{g_0(q)} = \frac{\prod_{i=n}^{\infty}(1+q^{2i-1})}{\prod_{i=n}^{\infty}(1+q^{2i})} = 1 + O(q^{2n-1}) \ . \qquad (8.4.14)$$

We therefore conclude that, except at the true IR enhançon scale Λ_e, the branch points of the curve are double in the $q \to 0$ limit. The approximation to which the branch points can be considered double (or the extra hypermultiplets have small masses) is better and better as we go to the ultraviolet (increasing n), and is instead poor in the IR. This had to be expected, because the IR physics is given by $SU(M)$ SYM, and, having chosen the origin of the Coulomb branch, we find a true enhançon at $x^2 = 4$. Hence we can conclude that this curve is not 'rotatable', in the sense that the parallel NS5 branes of the corresponding type IIA setup cannot be rotated to an $\mathcal{N} = 1$ supersymmetric configuration (the tangent of the rotation angle is proportional to the mass parameter): it has not genus 0, because its far IR part carries genus $M - 1$.

This means that the cascading vacuum defined by (8.4.5) in the $h \to \infty$ limit is not exactly related to the $\mathcal{N} = 1$ Klebanov-Strassler vacuum. This should have been expected too, since the IR regime of the Klebanov-Strassler cascade is pure $\mathcal{N} = 1$ SYM. The M vacua of this theory, breaking the nonanomalous R-symmetry $\mathbb{Z}_{2M}$ to $\mathbb{Z}_2$, can be obtained by mass deformation of M vacua of the $\mathcal{N} = 2$ SYM theory, also breaking the nonanomalous R-symmetry acting on the Coulomb branch, $\mathbb{Z}_{2M}$, to $\mathbb{Z}_2$. These latter vacua, whose Seiberg-Witten curve has genus 0, differ from the vacuum at the origin of the moduli space. We conclude that an exact genus 0 Seiberg-Witten curve for an infinite $\mathcal{N} = 2$ cascade, which would be related by mass deformation to the Klebanov-Strassler cascade, necessarily involves breaking of the $\mathbb{Z}_{2M}$ symmetry, which the background of section 8.2 instead preserves.

Let us end this section summarizing the result and making a remark. The $\mathbb{Z}_{2M}$-symmetric type IIB background of section 8.2 is dual to a cascading $\mathcal{N} = 2$ gauge theory on the vacuum described by the polynomials (8.4.5) in the $h \to \infty$ limit, up to possible subleading corrections in q which are not visible in the supergravity approximation. It possibly approximates a $\mathbb{Z}_2$-symmetric background whose Seiberg-Witten curve has genus zero[13] and which upon massive deformation flows to the Klebanov-Strassler cascade. This may be viewed as an $\mathcal{N}=2$ analogue of how the Klebanov-Tseytlin solution approximates the Klebanov-Strassler solution, but with an important difference: unlike its $\mathcal{N}=1$ counterpart, the $\mathcal{N}=2$ $\mathbb{Z}_{2M}$-symmetric cascading solution describes a perfectly legitimate vacuum, whose IR dynamics, described by an enhançon vacuum, is very different from the one of a genus 0 point in the moduli space, with additional massless monopoles.

In the next section we will study in more detail the moduli space of the quiver gauge theory, and show how it is possible to construct the cascading vacuum of this section starting from the enhançon vacuum of section (8.3.4).

8.5 The enhançon bearings

In this section we study a class of vacua of the $SU(N+M)\times SU(N)$ quiver gauge theory which preserve the same $\mathbb{Z}_{2M}$ R-symmetry as the supergravity cascading solution. We will start from the enhançon vacuum of section 8.3.4 and gradually construct the cascading vacuum by pulling VEV's out of the origin. In this process, we will observe new nontrivial vacua, for which we will propose novel type IIB dual backgrounds.

Let us consider the following polynomials for the Seiberg-Witten curve of the $SU(N+M)\times SU(N+M)$ gauge theory:

$$\begin{aligned} R(v) &= v^{N-M}(v^{2M}-\phi^{2M}) \\ S(v) &= v^{N}(v^{M}-z_\infty^{M}) \ . \end{aligned} \tag{8.5.1}$$

An overall v^{N-M} (interpreted as $N-M$ D3 branes at the origin) factor in R and S decouples from the Seiberg-Witten curve (8.4.3), so that we will effectively reduce to the $SU(2M)\times SU(2M)$ case, with

$$\begin{aligned} R(v) &= v^{2M}-\phi^{2M} \\ S(v) &= v^{M}(v^{M}-z_\infty^{M}) \ . \end{aligned} \tag{8.5.2}$$

When $\phi=0$ we are at the enhançon vacuum at the origin of the moduli space. We want to study the branch points of the Seiberg-Witten curve as we vary ϕ continuously. We work in the supergravity approximation of small q, so that $g_0(q)=2q^{1/4}+\mathcal{O}(q^{5/4})$. We will use the shorthand notation $\xi=v^M$; let us also define the enhançon scale as $\Lambda_e=(2q^{1/4}z_\infty^M)^{1/M}$.

Let us first consider the branch points at $u=0,1/2$, related to the R polynomial. According to the value of $|\phi|$, we can use different approximations. The branch points are the following:[14]

- $|\phi^M|<|q^{1/4}z_\infty^M|$:
$$\xi\simeq\pm\Lambda_e^M\ , \qquad \xi\simeq\pm\phi^M\left(\frac{\phi}{\Lambda_e}\right)^M\ , \tag{8.5.3}$$
namely $2M$ equally separated branch points at the enhançon ring and $2M$ equally spaced branch points at a ring of radius $|\phi^2/\Lambda_e|$.

[13] It would be very interesting to prove the existence of such a vacuum by finding its genus 0 Seiberg-Witten curve.
[14] We write the first corrections only when they are necessary to split double branch points.

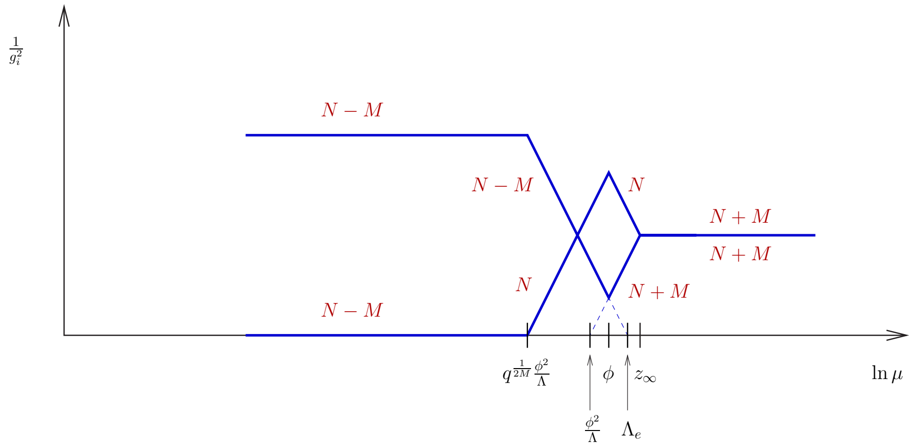


Figure 8.3: RG flow of the theory at a vacuum with a perturbative Higgsing.

- $|\phi^M| > |q^{1/4} z_\infty^M|$:
$$\xi \simeq \pm(1+\epsilon q^{1/4})\phi^M \ , \qquad \epsilon = \pm 1 \ , \tag{8.5.4}$$
namely $2M$ almost double branch points on a circle of radius $|\phi|$.

As far as the branch points at $u = \tau/2, (1+\tau)/2$, related to the S polynomial, are concerned, as long as $|\phi^M| < |q^{-1/4} z_\infty^M|$, which will always be the case if $|\phi| < |z_\infty|$, the branch points are

$$\xi \simeq (1 \pm 2q^{1/4}) z_\infty^M \ , \qquad \xi \simeq \pm 4 q^{1/2} \phi^M \left(\frac{\phi}{\Lambda_e}\right)^M \ , \tag{8.5.5}$$

namely $2M$ almost double branch points along a circle of radius the cutoff scale $|z_\infty|$ and $2M$ equally spaced branch points on a ring of radius $4^{1/M} q^{1/(2M)} |\phi^2/\Lambda_e|$.

The picture which stems from the branch points of the curve and from the study of the RG flow, which can be found immediately using an effective field theory point of view, is very interesting. In the case $|\Lambda_e| < |\phi| < |z_\infty|$, whose RG flow is depicted in figure 8.3, the theory is conformal in the UV, up to z_∞,[15] where M perturbative eigenvalues of one adjoint scalar field Higgs the gauge group to $SU(N+M) \times SU(N) \times U(1)^M$, triggering the RG flow. These eigenvalues correspond to M semiclassical wrapped anti-D5 branes with flux in the type IIB picture. At a scale ϕ, there are $2M$ almost double branch points at the positions of the $2M$ VEV's of the other adjoint field, which break $SU(N+M)$ to $SU(N-M) \times U(1)^{2M}$ and invert the RG flow. They correspond to $2M$ semiclassical wrapped D5 branes in the geometry, which invert the twisted fluxes: in particular, b starts to grow as the radius decreases. At a lower energy scale $q^{1/(2M)}\phi^2/\Lambda_e$ the $SU(N)$ gauge coupling diverges and there are $2M$ branch points equally spaced along a ring; in the interior the gauge couplings do not run, and we are left with an $SU(N-M) \times SU(N-M)$ gauge theory with a divergent coupling. In the type IIB picture b reaches the value 1, and there are M tensionless anti-D5 branes with one unit of flux smeared at this anti-enhançon. It is possible to see by adding an anti-D5 brane probe that it cannot penetrate the interior, whereas a D5 can penetrate it, unchaining an anti-D5 from the anti-enhançon. So far everything was expected.

[15] For the sake of brevity, from now on we omit the modulus when we discuss of energy scales.

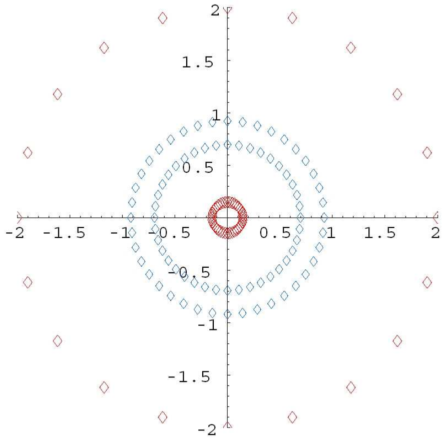


Figure 8.4: Branch points at a vacuum with one enhançon bearing, a nonperturbative region between two enhançon rings. Red circles denote branch points related to the S polynomial, whereas blue circles denote branch points related to the R polynomial.

The behavior is more surprising in the case $|\phi| < |\Lambda_e|$. If $\phi = 0$ we are at the enhançon vacuum of section 8.3.4. Instead, if ϕ does not vanish, the branch points follow the pattern of figure 8.4, whereas the RG flow is the one depicted in figure 8.5. There is a cutoff scale where M wrapped anti-D5 branes are placed, then a flow with decreasing b toward smaller radii, and an enhançon ring with $2M$ equally spaced branch points at Λ, where b reaches 0, M tensionless D5 branes melt on the ring and the related gauge coupling diverges. At lower energies the couplings remain constant as $b = 0$ in the dual supergravity solution, because of M D5 branes at the enhançon shell. One could have expected that a new flow started at a scale ϕ because of the VEV's, but it does not: it actually starts only at a lower scale ϕ^2/Λ_e, where there are $2M$ additional equally spaced branch points; below this energy scale, the gauge group which was at infinite coupling starts to run toward weak coupling, whereas the other one toward strong coupling; we enter a new perturbative regime,

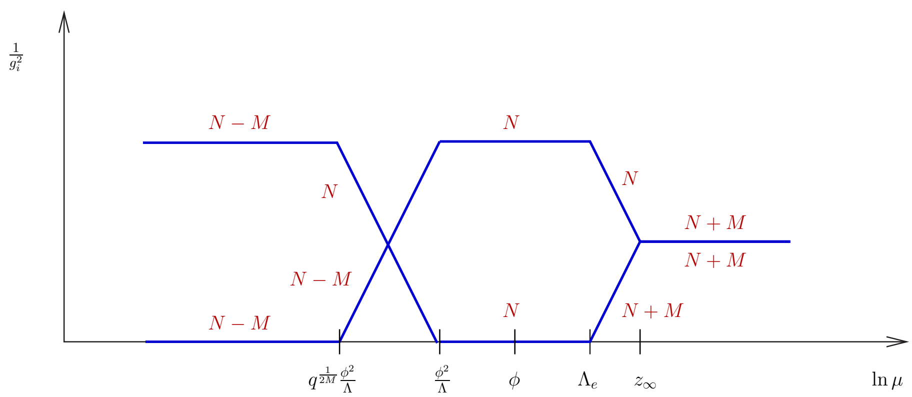


Figure 8.5: RG flow of the theory at a vacuum with one enhançon bearing.

which ends with a final ring of equally spaced branch points at a scale of order $q^{1/(2M)}|\phi^2/\Lambda_e|$ where one gauge coupling diverges; in the interior the couplings remain constant, one infinitely and the other finitely coupled, down to the IR.

It turns out that there is an interesting ambiguity in the type IIB interpretation of the flow: there is an $\mathcal{N}=2$-inspired point of view, which by continuity with the perturbative Higgsing regime discussed before interprets the ring of branch points at ϕ^2/Λ_e as an enhançon made of M tensionless melted D5 branes (like the one at Λ_e), which therefore forces b to grow as the radius decreases, so that it remains bounded by 0 and 1. Finally, the innermost ring, placed where b reaches 1, is an anti-enahnçon ring made of smeared tensionless wrapped anti-D5 branes with flux. In this picture, wrapped D5 branes are always associated to the first gauge group, and wrapped anti-D5 branes with flux are always associated to the second gauge group.

But there is also an $\mathcal{N}=1$-inspired point of view, which works by analogy with the Klebanov-Strassler cascade and interprets the ring of branch points at ϕ^2/Λ_e as an anti-enhançon made of M tensionless melted wrapped anti-D5 branes with 0 units of flux, so that b becomes negative at smaller radii.[16] Then b is monotonic, and the innermost ring at $b=-1$ would be interpreted again as an enhançon ring made of M tensionless wrapped D5 branes with one unit of flux.[17] This is the picture that was followed by Polchinski: b is monotonic in the solution of section 8.2, and the auxiliary brane system changes as b crosses integer values; the association between fractional branes and gauge groups is such that wrapped D5 branes always correspond to the larger group and wrapped anti-D5 branes to the smaller group.

Notice that for cascading transitions triggered by fractional D3 branes not of $\mathcal{N}=2$ kind, namely branes wrapped on isolated cycles, that do not possess an $\mathcal{N}=2$-like moduli space,[18] the $\mathcal{N}=1$ inspired interpretation is the only available choice, since no supersymmetric brane sources can be placed out of the isolated singularity to explain the reduction of degrees of freedom as a Higgsing.

Let us also remark that from the field theory Seiberg-Witten analysis, any of these two interpretations is equivalent: they only give different names to a unique field theory physics. What is physically meaningful in the field theory is to see whether pairs of branch points starting in a perturbative region split and melt as they hit a ring of equally spaced branch points (which occurs if they correspond to VEV's of the same gauge group as those related to the ring) or rather they cross it without any problems, unchaining a pair of branch points from the ring (which happens if they are VEV's for the other adjoint field). The probe brane analysis in the two interpretations gives the same result.

This ambiguity is perhaps only a matter of definition: after all, the corresponding type IIB supergravity solutions can be obtained by excising and gluing pieces of the solution of section 8.2 (possibly generated by one or the other kind of fractional D3 branes) and of a flat space fluxless solution, with suitable sources accounting for the discontinuities at the glued surfaces, along the lines of [135]. In each piece of solution we could act with the discrete S-duality transformation which changes sign to the twisted fluxes, and so switch from one picture to the other.

Let us finally notice that we can keep playing the same game of adding suitable VEV's, explained so far in this section, to the newly found solutions, so as to generate longer RG flows with more

[16] In the preferred gauge that we have used many times, they would have looked as wrapped anti-D5 branes with flux.

[17] In the preferred gauge, they would have looked as wrapped D5 branes without flux.

[18] The reader may keep in mind the Klebanov-Strassler cascade as an example.

and more transitions and reduction of degrees of freedom.

Let us summarize what we have found. Conventional enhançon rings are barriers for a kind of fractional D3 brane probes, containing a disk in the complex plane, where there are no fluxes and the metric is flat. Here we have found that in different vacua of the same $\mathcal{N}=2$ quiver gauge theory, continuously connected to the usual enhançon vacuum, there exist *enhançon bearings*, regions bounded by two enhançon rings (or an enhançon and an anti-enhançon ring), which also represent barriers for (one or two kinds of) fractional D3 brane probes and inside which there are no fluxes and the metric is flat, which separate different regions of the plane. To any of these configurations, at least in the $\mathbb{Z}_{2M}$-symmetric case, we are able to associate a corresponding type IIB solution obtained by gluing pieces of the fractional D3 brane solution and pieces of the flat space solution (the enhançon bearings), up to the aforementioned ambiguity. Finally, we are free to rotate as we like the pieces of brane solutions, effectively shifting the c axion without affecting the b field, since the enhançon bearings we have found act as domain walls between different pieces of the fractional brane solution. In the dual field theory, this is achieved by changing the phase of the ϕ VEV's.

8.5.1 Reconstructing the cascading vacuum at the baryonic roots

Let us conclude this section be connecting these enhançon bearing vacua to the cascading vacuum at the baryonic roots of the previous section. Such a cascading vacuum is characterized by the property that all the complex strong coupling scales along the cascade are related by the same hierarchy $q^{1/(2M)}$. That ensures that, in the supergravity approximation, the branch points at the strong coupling scales along the cascade (except for the last IR one) pair up in double branch points.

We can reconstruct such a vacuum from a vacuum with enhançon bearings as follows. We start from a vacuum with an enhançon bearing and send the thickness of the bearing to zero sending $|\phi| \to |\Lambda_e|$ for the relevant strong coupling scale Λ_e. So doing, we end up with a single circle, at a scale Λ_e, over which $4M$ branch points lie, $2M$ coming from inside and $2M$ coming from outside. For generic phases of ϕ, these branch points do not pair up, and on the type IIB side we end up with a source term at the glued surface. But if the phase of ϕ is suitably tuned, branch points coming from the outer boundary and branch points coming from the inner boundary of the bearing collide, hence forming double branch points. If we start from an infinite RG flow with enhançon bearings, and if we fine-tune all the ϕ's, we can obtain the cascading vacuum along the baryonic roots.

On the type IIB side, as we reduce the bearing to zero thickness we make the two smeared sources at the inner and outer boundaries of the bearing coincide. From the $\mathcal{N}=2$-inspired point of view, we end up with a source for $2M$ smeared tensionless wrapped D5 branes, say. There might be some additional subtleties at the special points where $\gamma \in \mathbb{Z}+\tau\mathbb{Z}$, where tensionless strings appear: these points are nothing but the locations of the double branch points. From the $\mathcal{N}=1$-inspired point of view, the two smeared sources along the rings correspond to tensionless wrapped D5 branes and tensionless wrapped anti-D5 branes, both with no worldvolume gauge flux. If the branch points of the inner and outer rings coincide as we shrink the bearing, the two smeared sources annihilate leaving a continuous solution, otherwise a source remains accounting for the discontinuity of c: very roughly speaking, it is tempting to think that it would be made of smeared dipoles of wrapped D5 anti-D5.

It would be very nice to study in more detail at the level of type IIB string theory the action of these smeared sources, in such a way to account for possible shifts of c.

Let us make here a final remark on the ambiguity that we have mentioned previously. We have recalled in appendix E how to reinterpret the Seiberg-Witten curve as the embedding of an M5 brane. Branch points of the curve correspond to NS5 branes touching or intersecting in the type IIA picture which arises upon compactification. In the cascading configuration we have studied, by continuously moving along the Coulomb branch it is possible to see that the vortex numbers on the NS5 branes are such that the branch points can be interpreted either to NS5 branes touching or to NS5 branes crossing. This is the T-dual of the type IIB ambiguity: NS5 branes touching correspond to the first, $\mathcal{N}=2$-inspired, picture where $b \in [0,1]$; instead, NS5 branes crossing correspond to the $\mathcal{N}=1$-inspired picture where b is monotonic.[19]

An important remark is in order at this point, concerning the behavior of probes and its relation to the rearrangement of the auxiliary brane system that is used to extract ranks and gauge couplings from the cascading supergravity solution. It will prove useful to start again from a vacuum with an enhançon bearing, see what happens and finally shrink the bearing and let it disappear. Let us add a probe VEV for the adjoint scalars of both gauge groups, in the perturbative regime at energies higher than the scale of the outer enhançon of the bearing. We can then move the VEV for the adjoint field of the gauge group related to the branch points of the outer ring, keeping the additional VEV for the other adjoint field fixed. As we decrease this probe VEV towards the outer enhançon scale, the two branch points which are semiclassically associated to the probe VEV start to separate and finally end on the outer enhançon ring. When this VEV becomes smaller than the scale of the inner ring of the bearing, two branch points escape from this ring, get closer, and then continue their motion as almost double branch points. In the $\mathcal{N}=1$-inspired point of view, this is interpreted as a wrapped D5 brane which melts at the enhançon, and then comes out of the inner anti-enhançon as a wrapped anti-D5 brane. Similarly, we can move the probe VEV of the other gauge group. The result is that its two branch points cross the outer enhançon ring, unchaining two of its branch points. When they reach the inner ring, they leave two branch points there. In the $\mathcal{N}=1$-inspired point of view, where b is monotonic, this is interpreted as a wrapped anti-D5 brane with flux, capturing a wrapped D5 brane at the enhançon and becoming a D3 brane, which then leaves a wrapped anti-D5 with no flux at the anti-enhançon, becoming a wrapped D5 brane with one unit of worldvolume flux when $b \in [-1,0]$. Let us finally shrink the enhançon bearing: if the positions of the branch points on the two rings are tuned (as occurs at the baryonic root), the sources at the boundaries of the bearing annihilate, and we are left with the fractional D3 brane solution of [39], with monotonic b. As a result, the interpretation of the behavior of probe VEV's as they cross the transition scales precisely accounts for the nontrivial rearrangement in the auxiliary brane system.

It is worth noticing that the previous discussion can be rephrased consistently in the $\mathcal{N}=2$ perspective where $b \in [0,1]$ takes a saw-shape: the language is simpler, and there is apparently no rearrangement of branes. However, it is important to keep in mind that highly nonperturbative dynamics occurs at the strong coupling scale; at that energy it does not make sense to think about fractional D3 branes using our perturbative intuition: pairs of branch points from probe VEV's split and then rejoin at the surface where the source is, showing that some rearrangement has

[19]The latter is the only picture surviving rotation of the NS5 branes (mass deformation): in this $\mathcal{N}=1$ case NS5 branes do not cross anymore, but it remains true that their coordinates along the T-duality direction keep crossing on the circle.

actually occurred.

To conclude this section, let us see a posteriori how the cascading vacuum at subsequent baryonic roots naturally arises as the dual of the supergravity solution of section (8.2). Requiring rotational isometry in the $\mathbb{C}$ plane implies that the background has a $\mathbb{Z}_{2M}$ symmetry. Moreover, no sources were placed in the geometry (except those at the enhançon scale which resolve the repulson singularity), so that the background for the twisted fields has everywhere continuous derivatives, except at the enhançon. On the field theory side, vacua which preserve the same $\mathbb{Z}_{2M}$ symmetry and that do not display perturbative Higgsing in the large N limit are those where the scales of the ϕ VEV's occur at strong coupling scales. Among these, only the cascading vacuum at the baryonic root avoids seeming discontinuities (which would actually be resolved by instanton effects) of the theta angles at the strong coupling scales: this maps into the continuity of c on the supergravity side. Continuity of the derivatives is finally achieved in the $\mathcal{N}=1$-inspired picture.

8.6 Summary and conclusions

In this chapter, we have proposed a field theory vacuum dual to the supergravity solution of M fractional D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold. It enjoys a cascading RG flow similar to the Klebanov-Strassler flow, where the number of degrees of freedom repeatedly drops by M units at the strong coupling scales of one or the other gauge group; this RG flow can be compared faithfully with the properties of the type IIB background from the UV down to the IR, which is described by an enhançon mechanism. Unlike its $\mathcal{N}=1$ counterpart, the decrease of the number of degrees of freedom is not described by an electric-magnetic duality but is due to a highly nonperturbative Higgsing. This nonperturbative Higgsing, occurring at the roots of the baryonic branches of the strongly coupled gauge group was shown in [142] to reduce to Seiberg duality upon mass deformation. It would be very interesting to find an interpolating supergravity solution between the $\mathcal{N}=2$ cascading background of [39] and the $\mathcal{N}=1$ Klebanov-Tseytlin background of [36], and even more interesting to find a genus 0 Seiberg-Witten curve on an $\mathcal{N}=2$ cascading vacuum, reducing to the exact $\mathcal{N}=1$ MQCD curve for the Klebanov-Strassler cascade upon mass deformation.

Analyzing $\mathbb{Z}_{2M}$-symmetric vacua of the moduli space of the (cutoff) $SU(N+M)\times SU(N)$ $\mathcal{N}=2$ quiver gauge theory, we have also discovered a new incarnation of enhançon rings as boundaries of enhançon bearings which separate disconnected regions. These vacua, for which we have been able to propose type IIB duals, allow us to interpolate between the enhançon vacuum of section 8.3.4 and the cascading vacuum of section 8.4. In this process, we have also understood the origin of the (perhaps mysterious) rearrangement of the constituents of Polchinski's auxiliary brane system as strong coupling scales are crossed.

In the background of the previous chapter, we have seen that even in $\mathcal{N}=1$ setups, at scales where an $\mathcal{N}=2$ fractional D3 brane (of the auxiliary brane system) becomes tensionless a gauge group with an adjoint field reduces its ranks and crosses a seeming strong coupling singularity. When the gauge coupling diverges, the holomorphic part of the dynamics of that node of the quiver is $\mathcal{N}=2$, whereas we lack information about the Kähler structure. However, from the vantage point of string theory it is natural to expect that the nonperturbative dynamics responsible for the transition is the same as that considered in this chapter: nonperturbative Higgsing at the root of the baryonic branch. This issue deserves further investigation.

APPENDICES

Appendix A

Conventions for type IIB supergravity

A.1 Forms and Hodge duality

We define a p-form ω on an n-dimensional Riemannian manifold X as

$$\omega \equiv \frac{1}{p!}\,\omega_{\mu_1\dots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\ . \tag{A.1.1}$$

Hodge duality is an operation that maps p-forms into $(n-p)$-forms. The Hodge star is defined as follows:

$$*dx^{\mu_1}\wedge\dots dx^{\mu_p} \equiv \frac{\sqrt{sg}}{(n-p)!}\,g^{\mu_1\rho_1}\dots g^{\mu_p\rho_p}\,\epsilon_{\rho_1\dots\rho_p\nu_1\dots\nu_{n-p}}\,dx^{\nu_1}\wedge\dots dx^{\nu_{n-p}}\ , \tag{A.1.2}$$

where $s=\pm1$ depending on X having Euclidean or Minkowskian signature respectively, and $\epsilon_{\mu_1\dots\mu_n}$ is the Levi-Civita symbol, which is a completely antisymmetric density and equals $+1$ (-1) when its indices lie in the same (opposite) order as the ordered vielbein. For a generic p-form ω, the Hodge dual form is then

$$\begin{aligned} *\omega &\equiv \frac{1}{(n-p)!}\,(*\omega)_{\rho_1\dots\rho_{n-p}}\,dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_{n-p}} = \\ &= \frac{1}{(n-p)!}\left(\frac{\sqrt{sg}}{p!}\,\epsilon_{\nu_1\dots\nu_p\rho_1\dots\rho_{n-p}}\,\omega^{\nu_1\dots\nu_p}\right)dx^{\rho_1}\wedge\cdots\wedge dx^{\rho_{n-p}}\ , \end{aligned} \tag{A.1.3}$$

so that

$$\begin{aligned} \omega\wedge*\omega &= \frac{1}{p!}\sqrt{sg}\,\omega_{\mu_1\dots\mu_p}\omega^{\mu_1\dots\mu_p}\,d^mx = \frac{1}{p!}\,\omega_{\mu_1\dots\mu_p}\omega^{\mu_1\dots\mu_p}\,d\mathrm{vol}_X \equiv |\omega|^2\,d\mathrm{vol}_X \\ **\,\omega &= s(-1)^{p(n-p)}\,\omega\ . \end{aligned} \tag{A.1.4}$$

A.2 Type IIB supergravity: action and equations of motion

We follow conventions in which the action of type IIB supergravity in string frame reads

$$\begin{aligned} S_{IIB}^{string} = \frac{1}{2\kappa_{10}^2}\Big\{ \int e^{-2\Phi} \left[R * 1 + 4\, d\Phi \wedge *d\Phi - \frac{1}{2} H_3 \wedge *H_3\right] + \\ - \frac{1}{2} \int \left[F_1 \wedge *F_1 + F_3 \wedge *F_3 + \frac{1}{2} F_5 \wedge *F_5 - C_4 \wedge H_3 \wedge F_3\right] \Big\} , \end{aligned} \tag{A.2.1}$$

where $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$, and we define the gauge invariant field strengths

$$F_1 = dC_0 \ , \qquad F_3 = dC_2 + C_0\, H_3 \ , \qquad F_5 = dC_4 + C_2 \wedge H_3 \ , \qquad H_3 = dB_2 \ . \tag{A.2.2}$$

Self-duality of F_5 has to be imposed on shell. The $\frac{1}{2} F_5 \wedge *F_5$ term in (A.2.1) is a common abuse of notation, since a simple Poincaré invariant action principle for the self-dual field does not exist.[1] We also introduce the gauge invariant field strength polyform $F = dC + C \wedge H_3$,[2] satisfying the Bianchi identity $dF + H_3 \wedge F = 0$. F_7 and F_9 are proportional to the Hodge duals of F_3 and F_1 respectively; their precise definition will be derived below by consistency of the Bianchi identities of higher rank field strengths with the equations of motion of lower rank field strengths.

When the dilaton is a constant or asymptotes to a constant, we can switch to Einstein frame by rescaling the metric by the fluctuating part of the dilaton and the RR potentials by the string coupling $g_s \equiv e^{\Phi_0}$:

$$g^s_{MN} = e^{\phi/2}\, g^E_{MN} \ , \qquad \phi \equiv \Phi - \Phi_0 \ , \qquad C^s_p \equiv \frac{C^E_p}{g_s} \ . \tag{A.2.3}$$

The action of type IIB supergravity (A.2.1) in this Einstein frame reads

$$\begin{aligned} S_{IIB}^{Einstein} = \frac{1}{2\kappa^2}\Big\{ \int R * 1 - \frac{1}{2} \int \Big[d\phi \wedge *d\phi + e^{2\phi}\, F_1 \wedge *F_1 + e^{\phi}\, F_3 \wedge *F_3 + \frac{1}{2} F_5 \wedge *F_5 + \\ + e^{-\phi}\, H_3 \wedge *H_3 - C_4 \wedge H_3 \wedge F_3 \Big] \Big\} , \end{aligned} \tag{A.2.4}$$

where $2\kappa^2 = 2\kappa_{10}^2\, g_s^2 = (2\pi)^7\, g_s^2\, \alpha'^4$ is the ten-dimensional Newton constant for an observer in the asymptotic region.

When dealing with D3-D7 backgrounds, where the dilaton does not asymptote a fixed value, we prefer instead to rescale only the metric by the whole dilaton field:

$$g^s_{MN} = e^{\Phi/2}\, g^E_{MN} \ . \tag{A.2.5}$$

After all, the true string coupling e^{Φ} is not a constant, but rather the vacuum expectation value of a field, which may vary in the manifold. The action of type IIB supergravity in Einstein frame in

[1] See however [146] for recent progresses in the subject. A related subtlety that we have to bear in mind is that when varying with respect to C_4 the D3 brane action, we have to split by hand the $\int C_4$ Wess-Zumino term into electric and magnetic parts, and vary only with respect to the electric part. In other terms, we have to put a factor of 1/2 in front of that term.

[2] A polyform is a formal sum of forms of different degrees. In the present context $C = C_0 + C_2 + C_4 + C_6 + C_8$ and $F = F_1 + F_3 + F_5 + F_7 + F_9$.

this case reads

$$
\begin{aligned}
S_{IIB}^{Einstein} = \frac{1}{2\kappa_{10}^2}\Big\{ \int R*1 - \frac{1}{2}\int \Big[& d\Phi\wedge *d\Phi + e^{2\Phi}\, F_1\wedge *F_1 + e^{\Phi}\, F_3\wedge *F_3 + \frac{1}{2}\, F_5\wedge *F_5 + \\
& + e^{-\Phi}\, H_3\wedge *H_3 - C_4\wedge H_3\wedge F_3 \Big]\Big\} \,.
\end{aligned}
\tag{A.2.6}
$$

In the conventions of (A.2.6), the equations of motion and Bianchi identities are:

$$
\begin{aligned}
R_{MN} - \frac{1}{2}g_{MN}\, R = & \frac{1}{2}\left(\partial_M\Phi\,\partial_N\Phi - \frac{1}{2}g_{MN}|d\Phi|^2\right) + \frac{1}{2}\, e^{2\Phi}\left(F_M\, F_N - \frac{1}{2}g_{MN}|F_1|^2\right) + \\
& + \frac{1}{4}\, e^{\Phi}\left(F_{MPQ}F_N{}^{PQ} - g_{MN}|F_3|^2\right) + \frac{1}{4}\, e^{-\Phi}\left(H_{MPQ}H_N{}^{PQ} - g_{MN}|H_3|^2\right) + \\
& + \frac{1}{96}\, F_{MPQRS}F_N{}^{PQRS}
\end{aligned}
\tag{A.2.7}
$$

$$d*d\Phi = e^{2\Phi}\, F_1\wedge *F_1 + \frac{1}{2}\, e^{\Phi}\left(F_3\wedge *F_3 - e^{-2\Phi}\, H_3\wedge *H_3\right) \tag{A.2.8}$$

$$d\left(e^{2\Phi}*F_1\right) = e^{\Phi}\, H_3\wedge *F_3 \tag{A.2.9}$$

$$d\left(e^{\Phi}*F_3\right) = H_3\wedge *F_5 \tag{A.2.10}$$

$$dF_5 = -H_3\wedge F_3 \tag{A.2.11}$$

$$dF_3 = -H_3\wedge F_1 \tag{A.2.12}$$

$$dF_1 = 0 \tag{A.2.13}$$

$$d\left(e^{-\Phi}*H_3\right) = F_5\wedge F_3 - e^{\Phi}\, F_1\wedge *F_3 \tag{A.2.14}$$

$$dH_3 = 0\,, \tag{A.2.15}$$

where we have consistently imposed $*F_5 = F_5$. Consistency of the equations of motion for F_1, F_3 with the Bianchi identities $dF_9 + H_3\wedge F_7 = 0$ and $dF_7 + H_3\wedge F_5 = 0$ for the dual field strengths imposes the identifications

$$F_7 = -e^{\Phi}*F_3 \qquad\qquad F_9 = e^{2\Phi}*F_1\,. \tag{A.2.16}$$

Given these identifications, equations (A.2.9)-(A.2.13) are elegantly summarized in the Bianchi identity $dF + H_3\wedge F = 0$ for the polyform F.

In the presence of D brane source terms S_{sou} in the action, equations of motions and Bianchi identities become:

$$
\begin{aligned}
R_{MN} - \frac{1}{2} g_{MN}\, R = \frac{1}{2}\left(\partial_M \Phi\, \partial_N \Phi - \frac{1}{2} g_{MN} |d\Phi|^2\right) + \frac{1}{2}\, e^{2\Phi}\left(F_M\, F_N - \frac{1}{2} g_{MN} |F_1|^2\right) + \\
+ \frac{1}{4}\, e^{\Phi}\left(F_{MPQ} F_N{}^{PQ} - g_{MN}|F_3|^2\right) + \frac{1}{4}\, e^{-\Phi}\left(H_{MPQ} H_N{}^{PQ} - g_{MN}|H_3|^2\right) + \\
+ \frac{1}{96}\, F_{MPQRS} F_N{}^{PQRS} + 2\kappa_{10}^2\, T_{MN}^{sou}
\end{aligned}
\tag{A.2.17}
$$

$$
d * d\Phi = e^{2\Phi}\, F_1 \wedge *F_1 + \frac{1}{2}\, e^{\Phi}\left(F_3 \wedge *F_3 - e^{-2\Phi}\, H_3 \wedge *H_3\right) - 2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta \Phi} \tag{A.2.18}
$$

$$
dF_9 + H_3 \wedge F_7 = -2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta C_0} \tag{A.2.19}
$$

$$
dF_7 + H_3 \wedge F_5 = +2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta C_2} \tag{A.2.20}
$$

$$
dF_5 + H_3 \wedge F_3 = -2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta C_4} \tag{A.2.21}
$$

$$
dF_3 + H_3 \wedge F_1 = -2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta C_6} \tag{A.2.22}
$$

$$
dF_1 = +2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta C_8} \tag{A.2.23}
$$

$$
d\left(e^{-\Phi} * H_3\right) = F_5 \wedge F_3 - e^{\Phi}\, F_1 \wedge *F_3 - 2\kappa_{10}^2 \frac{\delta S_{sou}}{\delta B_2} \tag{A.2.24}
$$

$$
dH_3 = 0\,, \tag{A.2.25}
$$

where

$$
T_{MN} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta S_{sou}}{\delta g^{MN}} \tag{A.2.26}
$$

is the stress-energy tensor of the sources, and we still use the definitions (A.2.16) and $*F_5 = F_5$.

A.2.1 D brane actions and charges

The worldvolume action of a localized Dp brane in string frame is the sum

$$
S^{Dp} = S^{Dp}_{DBI} + S^{Dp}_{WZ} \tag{A.2.27}
$$

of a Dirac-Born-Infeld (DBI) and a Wess-Zumino (WZ) term:

$$
S^{Dp}_{DBI} = -\mu_p \int_{Dp} d^{p+1}\xi\, e^{-\Phi}\, \sqrt{-\det(\hat{g} + \mathcal{F})} \tag{A.2.28}
$$

$$
S^{Dp}_{WZ} = \mu_p \int C \wedge e^{\mathcal{F}} \wedge \Omega_{9-p}\,. \tag{A.2.29}
$$

ξ^α, $\alpha = 0, \ldots, p$, are coordinates on the worldvolume, and a hat denotes pullback. $\mathcal{F} = \hat{B}_2 + 2\pi\alpha'\, F_2$ is the gauge invariant field strength on the brane worldvolume, $\mu_p = [(2\pi)^p \alpha'^{\frac{p+1}{2}}]^{-1}$ is the brane tension, C is the RR potential polyform, and Ω_{9-p} is a form localized on the Dp brane worldvolume (the Poincaré dual to the cycle) and closed. The magnitude of the backreaction of a Dp brane on the geometry and the RR field strength is measured by the combination $2\kappa_{10}^2\mu_p = (4\pi^2\alpha')^{\frac{7-p}{2}}$.

When moving to Einstein frame according to the field redefinitions (A.2.3), that we use in absence of D7 branes, the DBI and WZ action become

$$S_{DBI}^{Dp} = -\tau_p \int_{Dp} d^{p+1}\xi \, e^{-\frac{p-3}{4}\phi} \sqrt{-\det(\hat{g} + e^{-\frac{\phi}{2}}\mathcal{F})} \tag{A.2.30}$$

$$S_{WZ}^{Dp} = \tau_p \int C \wedge e^{\mathcal{F}} \wedge \Omega_{9-p} \, , \tag{A.2.31}$$

where $\tau_p = \mu_p/g_s = [(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s]^{-1}$. The relevant parameter measuring the backreaction of a Dp brane is $2\kappa^2\tau_p = 2\kappa_{10}^2\mu_p\, g_s = (4\pi^2\alpha')^{\frac{7-p}{2}}\, g_s$.

In the presence of D7 branes we define instead the Einstein frame by the field redefinitions (A.2.5). The DBI and WZ worldvolume actions are

$$S_{DBI}^{Dp} = -\mu_p \int_{Dp} d^{p+1}\xi \, e^{-\frac{p-3}{4}\Phi} \sqrt{-\det(\hat{g} + e^{-\frac{\Phi}{2}}\mathcal{F})} \tag{A.2.32}$$

$$S_{WZ}^{Dp} = \mu_p \int C \wedge e^{\mathcal{F}} \wedge \Omega_{9-p} \, . \tag{A.2.33}$$

The combination $2\kappa_{10}^2\mu_p = (4\pi^2\alpha')^{\frac{7-p}{2}}$ will appear in Bianchi identities and equations of motion.

We are now in the position to define the RR charges of D branes as fluxes of RR field strengths. We do that by comparing Dp brane actions and Bianchi identities/EOM modified by the presence of sources. We will be interested in D3, D5 and D7 brane charges, therefore we compare (A.2.21-A.2.23) with (A.2.33). We carry out this computation in the Einstein frame defined in (A.2.5). Following the nomenclature of [108], we define 'Maxwell charges' as fluxes of gauge invariant improved RR field strengths on all the possible cycles in the odd homology classes:

$$Q_{D3}^{Maxwell} \equiv -\frac{1}{(4\pi^2\alpha')^2} \int_{\mathcal{C}_5} F_5 \tag{A.2.34}$$

$$Q_{D5}^{Maxwell} \equiv -\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_3} F_3 \tag{A.2.35}$$

$$Q_{D7}^{Maxwell} \equiv \int_{\mathcal{C}_1} F_1 \, , \tag{A.2.36}$$

where $\mathcal{C}_n$ is an n-cycle. Maxwell (magnetic) currents $j^{Maxwell}$ are defined by $dF = *j^{Maxwell}$, up to signs. The use of Maxwell charges in the literature is widespread. The most famous example is [37]. They are gauge invariant by construction; they are carried by fluxes, hence they are not localized; because of the Chern-Simons term in the action (A.2.6), even when the fluxes are computed out of sources, these charges are not conserved (as the manifold changes continuously) nor quantized if there are lower RR field strengths. Indeed Bianchi identities in absence of sources tell us that $dF = -H_3 \wedge F \neq 0$, so that Maxwell charges do depend on the representative chosen in a homology class.

In the alternative Einstein frame defined by (A.2.3), Maxwell charges are:

$$Q_{D3}^{Maxwell} \equiv -\frac{1}{(4\pi^2\alpha')^2\, g_s} \int_{\mathcal{C}_5} F_5 \tag{A.2.37}$$

$$Q_{D5}^{Maxwell} \equiv -\frac{1}{4\pi^2\alpha'\, g_s} \int_{\mathcal{C}_3} F_3 \tag{A.2.38}$$

$$Q_{D7}^{Maxwell} \equiv \frac{1}{g_s} \int_{\mathcal{C}_1} F_1 \;. \tag{A.2.39}$$

Long ago, Page introduced a different notion of charge that obeys quantization [109]. Page (magnetic) currents are defined by $*j^{Page} = d(e^{B_2} \wedge F)$, and Page charges are obtained by integration of $*j^{Page}$. In the Einstein frame defined in (A.2.5), 'Page charges' are

$$Q_{D3}^{Page} \equiv -\frac{1}{(4\pi^2\alpha')^2} \int_{\mathcal{C}_5} (F_5 + B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1) \tag{A.2.40}$$

$$Q_{D5}^{Page} \equiv -\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_3} (F_3 + B_2 \wedge F_1) \tag{A.2.41}$$

$$Q_{D7}^{Page} \equiv \int_{\mathcal{C}_1} F_1 \;. \tag{A.2.42}$$

Like Maxwell charges, Page charges are carried by fluxes, therefore they are not localized. But unlike Maxwell currents, out of sources Page currents are closed, then they are locally exact. Their integrals do not vanish, because they are differentials of forms which are not globally defined and need to be patched. In other terms, like monopole numbers, Page charges are topological invariants which are quantized. They are invariant under small but not under large gauge transformations.[3]

In the alternative Einstein frame defined by (A.2.3), Page charges are:

$$Q_{D3}^{Page} \equiv -\frac{1}{(4\pi^2\alpha')^2\, g_s} \int_{\mathcal{C}_5} (F_5 + B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1) \tag{A.2.43}$$

$$Q_{D5}^{Page} \equiv -\frac{1}{4\pi^2\alpha'\, g_s} \int_{\mathcal{C}_3} (F_3 + B_2 \wedge F_1) \tag{A.2.44}$$

$$Q_{D7}^{Page} \equiv \frac{1}{g_s} \int_{\mathcal{C}_1} F_1 \;. \tag{A.2.45}$$

Notice that the D7 brane Maxwell charge is equal to the D7 brane Page charge, hence it is quantized.

[3] Here we refer to the small gauge transformation under which $\delta B_2 = d\Lambda_1$, whereas the RR field strengths F's are invariant.

A.3 Type IIB SUSY transformations in string and Einstein frame

The supersymmetry transformations of type IIB supergravity were found long ago in ref. [147]. Here we will follow the conventions of the appendix A of3 [148], where they are written in string frame. Let us recall them:

$$\begin{aligned}\delta_\epsilon\lambda^{(s)} &= \frac{1}{2}\Big(\Gamma^{(s)\,M}\partial_M\Phi\,+\,\frac{1}{2\cdot 3!}H_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_3\Big)\epsilon^{(s)}\,-\,\frac{1}{2}e^{\Phi}\Big(F_M\,\Gamma^{(s)\,M}\,(i\sigma_2)+\\&\quad+\,\frac{1}{2\cdot 3!}F_{MNP}\,\Gamma^{(s)\,MNP}\,\sigma_1\Big)\epsilon^{(s)},\\ \delta_\epsilon\psi^{(s)}_M &= \nabla^{(s)}_M\epsilon^{(s)}\,+\,\frac{1}{4}\frac{1}{2!}H_{MNP}\,\Gamma^{(s)\,NP}\,\sigma_3\epsilon^{(s)}\,+\,\frac{1}{8}e^{\Phi}\Big(F_N\,\Gamma^{(s)\,N}\,(i\sigma_2)+\\&\quad+\,\frac{1}{3!}F_{NPQ}\,\Gamma^{(s)\,NPQ}\,\sigma_1\,+\,\frac{1}{2\cdot 5!}F_{NPQRT}\,\Gamma^{(s)\,NPQRT}\,(i\sigma_2)\Big)\Gamma^{(s)}_M\epsilon^{(s)}\ ,\end{aligned} \tag{A.3.1}$$

where the superscript s refers to the string frame, σ_i $(i = 1, 2, 3)$ are Pauli matrices, H is the NSNS three-form and F's are the RR field strengths of obvious ranks. In (A.3.1)

$$\epsilon = \begin{pmatrix}\epsilon^1\\ \epsilon^2\end{pmatrix} \tag{A.3.2}$$

is a doublet of Majorana-Weyl spinors of negative chirality:

$$\Gamma_{(11)}\epsilon^i = -\epsilon^i\ , \tag{A.3.3}$$

where $\Gamma_{(11)}$ is the chirality matrix in ten dimensions, expressed in the vielbein basis as

$$\Gamma_{(11)} \equiv \Gamma^{012\ldots 9} = -\Gamma_{012\ldots 9}\ . \tag{A.3.4}$$

We also define the chirality matrix in the 4 Minkowksi dimensions

$$\Gamma_{(5)} = -i\,\Gamma_{0123} \tag{A.3.5}$$

and the chirality matrix in the 6 internal dimensions

$$\Gamma_{(7)} = -i\,\Gamma_{456789}\ , \tag{A.3.6}$$

so that

$$\Gamma_{(11)} = \Gamma_{(5)}\Gamma_{(7)}\ . \tag{A.3.7}$$

We are using conventions in which the ten-dimensional Γ matrices are real.

We can study how these equations change under a rescaling of the metric

$$g^{(s)}_{MN}\,=\,e^{\Phi/2}\,g_{MN} \tag{A.3.8}$$

which moves us from string to Einstein frame.[4] In doing that it is useful to follow section 2 of [149]. Under the above change for the metric, there are some quantities which also change:

$$\begin{aligned}\Gamma^{(s)}_M &= e^{\Phi/4}\,\Gamma_M\ ,\\ \epsilon^{(s)} &= e^{\Phi/8}\,\epsilon\ ,\\ \lambda^{(s)} &= e^{-\Phi/8}\,\lambda\ ,\\ \psi_M &= e^{-\Phi/8}\Big(\psi^{(s)}_M\,-\,\frac{1}{4}\Gamma^{(s)}_M\,\lambda^{(s)}\Big)\ .\end{aligned} \tag{A.3.9}$$

[4]Here we use the rescaling by the whole dilaton, which is suitable for solutions including D7 branes.

The equation for the dilatino in the new frame can be easily obtained, whereas in doing the same for the gravitino equation we will use

$$\nabla^{(s)}_M \epsilon^{(s)} \,=\, e^{\Phi/8}\Big[\nabla_M \epsilon \,+\, \frac{1}{8}\Gamma_M^{\ N}(\nabla_N \Phi) \,+\, \frac{1}{8}(\nabla_M \Phi)\Big]\ . \tag{A.3.10}$$

After some algebra with Γ matrices, the supersymmetry transformations in Einstein frame we obtain are the following ones:

$$\begin{aligned}
\delta_\epsilon \lambda &= \frac{1}{2}\Gamma^M\big(\partial_M\Phi - e^{\Phi}F_M(i\sigma_2)\big)\,\epsilon \,+\, \frac{1}{4\cdot 3!}\Gamma^{MNP}\big(e^{-\Phi/2}H_{MNP}\,\sigma_3 - e^{\Phi/2}F_{MNP}\,\sigma_1\big)\,\epsilon,\\
\delta_\epsilon \psi_M &= \nabla_M\epsilon + \frac{1}{4}e^{\Phi}F_M(i\sigma_2)\epsilon - \frac{1}{96}\big(e^{-\Phi/2}H_{NPQ}\,\sigma_3 + e^{\Phi/2}F_{NPQ}\,\sigma_1\big)\big(\Gamma_M{}^{NPQ} - 9\,\delta^N_M\Gamma^{PQ}\big)\,\epsilon +\\
&\quad + \frac{1}{16\cdot 5!}F_{NPQRT}\Gamma^{NPQRT}(i\sigma_2)\Gamma_M\epsilon.
\end{aligned} \tag{A.3.11}$$

In order to write the expression of the supersymmetry transformations, it is convenient to change the notation used for the spinor. Up to now we have considered the double spinor notation, namely the two Majorana-Weyl spinors ϵ^1 and ϵ^2 form a two-dimensional vector (A.3.2). We can rewrite the double spinor in complex notation as

$$\epsilon = \begin{pmatrix}\epsilon^1\\ \epsilon^2\end{pmatrix} \longleftrightarrow \epsilon = \epsilon^1 \,+ i\,\epsilon^2\ . \tag{A.3.12}$$

It is then straightforward to find the following rules to pass from real doublets of spinors to complex spinors:

$$-i\,\sigma_2\,\epsilon = \begin{pmatrix}-\epsilon^2\\ \epsilon^1\end{pmatrix} \longleftrightarrow i\,\epsilon \qquad\qquad \sigma_3\,\epsilon = \begin{pmatrix}\epsilon^1\\ -\epsilon^2\end{pmatrix} \longleftrightarrow \epsilon^* \qquad\qquad \sigma_1\,\epsilon = \begin{pmatrix}\epsilon^2\\ \epsilon^1\end{pmatrix} \longleftrightarrow i\,\epsilon^*\ . \tag{A.3.13}$$

The supersymmetry variations of the dilatino and the gravitini in Einstein frame become:

$$\delta_\epsilon \lambda = \frac{1}{2}\,\Gamma^M\,\big(\partial_M\Phi + i\,e^{\Phi}F_M\big)\,\epsilon - \frac{i}{4\cdot 3!}\,e^{\Phi/2}\,\Gamma^{MNP}\,\big(F_{MNP} + i\,e^{-\Phi}H_{MNP}\big)\,\epsilon^* \tag{A.3.14}$$

$$\begin{aligned}
\delta_\epsilon \psi_M &= \nabla_M\epsilon - \frac{i}{4}\,e^{\Phi}F_M\,\epsilon - \frac{i}{96}\,e^{\Phi/2}\,\big(F_{NPQ} - i\,e^{-\Phi}H_{NPQ}\big)\,\big(\Gamma_M{}^{NPQ} - 9\,\delta^N_M\,\Gamma^{PQ}\big)\,\epsilon^* +\\
&\quad - \frac{i}{16\cdot 5!}\,F_{NPQRT}\,\Gamma^{NPQRT}\Gamma_M\,\epsilon\ .
\end{aligned} \tag{A.3.15}$$

Appendix B

Generalities on the conifold geometry

B.1 The singular conifold

The singular conifold C_0 can be defined as an affine variety in $\mathbb{C}^4 \cong \{z_1, z_2, z_3, z_4\}$,

$$z_1 z_2 - z_3 z_4 = 0 \ . \tag{B.1.1}$$

It is convenient to introduce the matrix notation

$$Z = \begin{pmatrix} z_1 & z_4 \\ z_3 & z_2 \end{pmatrix} \ , \tag{B.1.2}$$

in terms of which the defining equation (B.1.1) becomes $\det Z = 0$. This notation makes manifest an $SU(2) \times SU(2)$ symmetry acting on the rows and the columns respectively. There are also a $\mathbb{C}^*$ symmetry acting as a complex rescaling $Z \mapsto \alpha Z$, $\alpha \in \mathbb{C}^*$. The symmetry is $\frac{SU(2)\times SU(2)}{\mathbb{Z}_2} \times \mathbb{C}^*$. By a linear change of coordinates $Z = \frac{i}{\sqrt{2}} \sigma^m w_m$, where $(\sigma^m) = (\vec{\sigma}, i\mathbb{1})$, σ^a being Pauli matrices, equation (B.1.1) can also be written as:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \ , \tag{B.1.3}$$

which has manifest $SO(4) \times \mathbb{C}^*$ symmetry. This manifold can be given a Calabi-Yau metric preserving the symmetries previously discussed [51]. The base manifold, described by the intersection of (B.1.1) with the unit sphere

$$\sum_{i=1}^{4} |z_i|^2 = \sum_{i=1}^{4} |w_i|^2 = 1 \ , \tag{B.1.4}$$

is called $T^{1,1}$ [52]. In terms of real coordinates $(x_m, y_m) \in \mathbb{R}^8$, such that $w_m \equiv x_m + i y_m$, $T^{1,1}$ is described by the constraints $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = \frac{1}{2}$, $\vec{x} \cdot \vec{y} = 0$: it is an S^2 fibration over S^3. However such a fibration is trivial,[1] so that topologically $T^{1,1} \cong S^2 \times S^3$. The following coordinate system on the

[1] We can cover S^3 with two patches, intersecting at the equator. The bundle is constructed by specifying a transition function on this equator (itself an S^2), which is a map from S^2 to $SO(3)$, the structure group of the fiber. Such maps are always trivial ($\pi_2(SO(3)) = 0$), so the bundle is trivial.

cone will be useful[2]

$$z_1 = r^{3/2}\, e^{\frac{i}{2}(\psi+\varphi_1+\varphi_2)} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\ , \tag{B.1.5}$$

$$z_2 = r^{3/2}\, e^{\frac{i}{2}(\psi-\varphi_1-\varphi_2)} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2}\ , \tag{B.1.6}$$

$$z_3 = r^{3/2}\, e^{\frac{i}{2}(\psi-\varphi_1+\varphi_2)} \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2}\ , \tag{B.1.7}$$

$$z_4 = r^{3/2}\, e^{\frac{i}{2}(\psi+\varphi_1-\varphi_2)} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2}\ . \tag{B.1.8}$$

Here, $0 \le \psi \le 4\pi$, $0 \le \varphi_i \le 2\pi$, $0 \le \theta_i \le \pi$, and we have the following angular periodicities

$$\begin{pmatrix}\psi\\ \varphi_1\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi+4\pi\\ \varphi_1\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi+2\pi\\ \varphi_1+2\pi\\ \varphi_2\end{pmatrix} \simeq \begin{pmatrix}\psi+2\pi\\ \varphi_1\\ \varphi_2+2\pi\end{pmatrix}\ . \tag{B.1.9}$$

In these coordinates, the Calabi-Yau metric reads

$$ds^2_{C_0} = \mathrm{d}r^2 + r^2\, ds^2_{T^{1,1}}\ ,, \tag{B.1.10}$$

with the Sasaki-Einstein metric of $T^{1,1}$

$$ds^2_{T^{1,1}} = \sum_{i=1,2} \frac{1}{6}\big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\big) + \frac{1}{9}\big(d\psi - \sum_{i=1,1} \cos\theta_i\, d\varphi_i\big)^2\ . \tag{B.1.11}$$

It describes a circle bundle, where the circle ψ is fibered over $S^2 \times S^2$. In terms of the natural vielbein for the two 2-spheres, it is useful to define rotated vielbein for the 2-spheres [150]

$$\begin{aligned}\sigma_1 + i\sigma_2 &= e^{i\psi/2}\,(d\theta_1 + i\sin\theta_1\, d\varphi_1)\ ,\\ \Sigma_1 + i\Sigma_2 &= e^{i\psi/2}\,(d\theta_2 + i\sin\theta_2\, d\varphi_2)\ ,\end{aligned} \tag{B.1.12}$$

where σ_i and Σ_i are real by definition. Let us also define $\zeta = d\psi - \sum_{i=1,2}\cos\theta_i\, d\varphi_i$. For the singular conifold, we will use the following ordered vielbein

$$\Big\{e^r = dr\ ,\ e^\psi = \frac{r}{3}\zeta\ ,\ e^1 = \frac{r}{\sqrt{6}}\sigma_1\ ,\ e^2 = \frac{r}{\sqrt{6}}\sigma_2\ ,\ e^3 = \frac{r}{\sqrt{6}}\Sigma_1\ ,\ e^4 = \frac{r}{\sqrt{6}}\Sigma_2\Big\}\ . \tag{B.1.13}$$

The metric of the conifold then reads $ds^2_{C_0} = \sum_{n=1}^6 (e^n)^2$, and the volume form is

$$d\mathrm{vol}_{C_0} = e^r\wedge e^\psi \wedge e^1\wedge e^2\wedge e^3\wedge e^4 = \frac{1}{108}\, r^5\, dr\wedge d\psi\wedge d\theta_1\wedge \sin\theta_1\, d\varphi_1\wedge d\theta_2\wedge\sin\theta_2\, d\varphi_2\ . \tag{B.1.14}$$

A complex vielbein can be defined as

$$\Big\{E^1 = e^1 + ie^2\ ,\ E^2 = e^3 + ie^4\ ,\ E^3 = e^r + ie^\psi\Big\}\ . \tag{B.1.15}$$

[2]Remark that we differ from the conventions of [37] by a flip in the orientation of the angles ϕ_i.

In terms of this complex structure, the Kähler form is

$$\begin{aligned} J &\equiv \frac{i}{2}\left(E^1\wedge\overline{E^1}+E^2\wedge\overline{E^2}+E^3\wedge\overline{E^3}\right) = d\left(\frac{r^2}{6}\,\zeta\right) = \\ &= \frac{r}{3}\,dr\wedge(d\psi-\sum_{i=1,2}\cos\theta_i\,d\varphi_i)+\frac{r^2}{6}\sum_{i=1,2}\sin\theta_i\,d\theta_i\wedge d\varphi_i\ , \end{aligned} \tag{B.1.16}$$

which is $(1,1)$, closed, coclosed (because $*J=\frac{1}{2}J\wedge J$), and satisfies $J\wedge J\wedge J=6\,d\mathrm{vol}_{C_0}$. It is not only closed but also exact, since we are at the zero resolution point in Kähler moduli space where the cohomology class of J is trivial. The holomorphic top form is

$$\begin{aligned} \Omega^{(3,0)} &\equiv E^1\wedge E^2\wedge E^3 = \\ &= e^{i\psi}\,\frac{r^2}{6}\left[dr+i\,\frac{r}{3}\left(d\psi-\sum_{i=1,2}\cos\theta_i\,d\varphi_i\right)\right]\wedge(d\theta_1+i\sin\theta_1\,d\varphi_1)\wedge(d\theta_2+i\sin\theta_1\,d\varphi_2) = \\ &= -\frac{4}{9}\,\frac{dz_1\wedge dz_2\wedge dz_3}{z_3}\ , \end{aligned} \tag{B.1.17}$$

which is closed and coclosed (because $*\Omega=-i\,\Omega$). Finally, $d\mathrm{vol}_{C_0}=\frac{1}{3!}\,J\wedge J\wedge J=\frac{i}{8}\,\Omega\wedge\overline{\Omega}$.

The last equality in (B.1.17) provides a first check that the coordinates z_i introduced in (B.1.5-B.1.8) are holomorphic with respect to the complex structure defined in (B.1.15). More explicitly, you can define the complex structure as a map from the tangent space to itself that squares to minus the identity. Given the Kähler metric $g_{\mu\nu}$ (B.1.10,B.1.11) and the Kähler form $J_{\mu\nu}$ (B.1.16), the complex structure is $\mathcal{J}_\mu{}^\nu=J_{\mu\rho}\,g^{\rho\nu}$. In the coordinate basis

$$\mathcal{J}_\mu{}^\nu = \begin{pmatrix} 0 & \frac{3}{r} & 0 & 0 & 0 & 0 \\ -\frac{r}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \cot\theta_1 & 0 & \csc\theta_1 & 0 & 0 \\ \frac{r}{3}\cos\theta_1 & 0 & -\sin\theta_1 & 0 & 0 & 0 \\ 0 & \cot\theta_2 & 0 & 0 & 0 & \csc\theta_2 \\ \frac{r}{3}\cos\theta_2 & 0 & 0 & 0 & -\sin\theta_2 & 0 \end{pmatrix}\ , \tag{B.1.18}$$

where μ runs through rows and ν runs through columns. The holomorphic and antiholomorphic projectors

$$P=\frac{\mathbb{1}-i\,\mathcal{J}}{2} \qquad\qquad \overline{P}=\frac{\mathbb{1}+i\,\mathcal{J}}{2} \tag{B.1.19}$$

allow us to construct holomorphic and antiholomorphic exterior differentials $\partial=P\,d$ and $\overline{\partial}=\overline{P}\,d$. It is then straightforward to explicitly check that $\overline{P}\,dz_i=0$ $(i=1,\ldots,4)$ as well as $\overline{P}\,E^l=0$ $(l=1,2,3)$.

Let us now review 2- and 3-(co)cycles for the conifold. We have the closed (1,1)-form

$$\begin{aligned} \omega_2^{CF} &\equiv \frac{3i}{2r^2}\left(E^1\wedge\overline{E^1}-E^2\wedge\overline{E^2}\right)=\frac{1}{2}(\sigma_1\wedge\sigma_2-\Sigma_1\wedge\Sigma_2) = \\ &= \frac{1}{2}\left(\sin\theta_1\,d\theta_1\wedge d\varphi_1-\sin\theta_2\,d\theta_2\wedge d\varphi_2\right)\ . \end{aligned} \tag{B.1.20}$$

The 2-cycle in $T^{1,1}$ is topologically a 2-sphere S^2. It can be represented by

$$S^2 : \qquad \theta_1 = \theta_2 \equiv \theta \ , \quad \varphi_1 = 2\pi - \varphi_2 \equiv \varphi \ , \quad \psi = 0 \ , \qquad \varphi \in [0, 2\pi) \ , \ \theta \in (0, \pi) \ . \tag{B.1.21}$$

It turns out that $\int_{S^2} \omega_2^{CF} = 4\pi$. In addition, one usually defines the real closed 3-form

$$\omega_3^{CF} \equiv \zeta \wedge \omega_2^{CF} \ , \tag{B.1.22}$$

which is the real part of the imaginary-self-dual (ISD) closed primitive (2,1)-form

$$\omega^{(2,1)} \equiv \frac{9}{2r^3} E^3 \wedge \left(E^1 \wedge \overline{E^1} - E^2 \wedge \overline{E^2} \right) = \left(\zeta - 3i \frac{dr}{r} \right) \wedge \omega_2^{CF} \ , \tag{B.1.23}$$

defined on the whole conifold. Imaginary self-duality means that $*_6 \, \omega^{(2,1)} = i \, \omega^{(2,1)}$. The 3-cycle in $T^{1,1}$ has the topology of a 3-sphere. We call it simply S^3. It can be represented by

$$S^3 : \qquad \theta_2 = \varphi_2 = 0 \ . \tag{B.1.24}$$

Its orientation is such that $\int_{S^3} \omega_3^{CF} = 8\pi^2$.

B.1.1 Simplified orthonormal basis

The frame basis introduced in (B.1.13) and (B.1.15) has the nice feature that the holomorphic top form can be expressed as the unit $(3,0)$-form $E^1 \wedge E^2 \wedge E^3$, without any function in front. However, the rotation by $\psi/2$ introduced in (B.1.12) complicates the expression of the vielbein in terms of the coordinates ψ, θ_1, φ_1, θ_2, φ_2. Consequently, the spin connection is more involved. In view of this, we introduce an alternative frame basis where the $\psi/2$ rotation is undone and a prefactor $e^{i\psi}$ appears in the holomorphic top form of the conifold. In this choice, the ordered real vielbein is

$$\left\{ e^r = dr \ , \ e^\psi = \frac{r}{3} \zeta \ , \ e^{\theta_1} = \frac{r}{\sqrt{6}} d\theta_1 \ , \ e^{\varphi_1} = \frac{r}{\sqrt{6}} \sin\theta_1 \, d\varphi_1 \ , \ e^{\theta_2} = \frac{r}{\sqrt{6}} d\theta_2 \ , \ e^{\varphi_2} = \frac{r}{\sqrt{6}} \sin\theta_2 \, d\varphi_2 \right\} \ , \tag{B.1.25}$$

the ordered complex vielbein is

$$\left\{ E^1 = e^{\theta_1} + i e^{\varphi_1} \ , \ E^2 = e^{\theta_2} + i e^{\varphi_2} \ , \ E^3 = e^r + i e^\psi \right\} \ , \tag{B.1.26}$$

the Kaḧler form is

$$\begin{aligned} J &\equiv \frac{i}{2} \left(E^1 \wedge \overline{E^1} + E^2 \wedge \overline{E^2} + E^3 \wedge \overline{E^3} \right) = d \left(\frac{r^2}{6} \zeta \right) = \\ &= \frac{r}{3} dr \wedge (d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i) + \frac{r^2}{6} \sum_{i=1,2} \sin\theta_i \, d\theta_i \wedge d\varphi_i \ , \end{aligned} \tag{B.1.27}$$

the holomorphic top form is

$$\begin{aligned} \Omega^{(3,0)} &\equiv e^{i\psi} \, E^1 \wedge E^2 \wedge E^3 = \\ &= e^{i\psi} \frac{r^2}{6} \left[dr + i \, \frac{r}{3} \left(d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i \right) \right] \wedge (d\theta_1 + i \sin\theta_1 \, d\varphi_1) \wedge (d\theta_2 + i \sin\theta_1 \, d\varphi_2) = \\ &= -\frac{4}{9} \, \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_3} \ , \end{aligned} \tag{B.1.28}$$

and the ISD primitive (2,1)-form is

$$\omega^{(2,1)} \equiv \frac{9}{2r^3} E^3 \wedge \left(E^1 \wedge \overline{E^1} - E^2 \wedge \overline{E^2} \right) = \left(\zeta - 3i \frac{dr}{r} \right) \wedge \omega_2^{CF} \ , \tag{B.1.29}$$

where

$$\omega_2^{CF} \equiv \frac{3i}{2r^2} \left(E^1 \wedge \overline{E^1} - E^2 \wedge \overline{E^2} \right) = \frac{1}{2} \left(\sin\theta_1 \, d\theta_1 \wedge d\varphi_1 - \sin\theta_2 \, d\theta_2 \wedge d\varphi_2 \right) \ . \tag{B.1.30}$$

B.2 The deformed conifold

The deformed conifold is defined as an affine variety in $\mathbb{C}^4$ by the following equation:

$$\sum_{i=1}^{4} w_i^2 = \left(\frac{3}{2}\right)^{3/2} \epsilon^2 \tag{B.2.1}$$

or

$$z_1 z_2 - z_3 z_4 = -\frac{1}{2}\left(\frac{3}{2}\right)^{3/2} \epsilon^2 \,, \tag{B.2.2}$$

if z's and v's are defined as in appendix B.1. The ugly normalization has been put for consistency with following formulae. The following assignments

$$z_1 = 2^{-\frac{1}{2}}\left(\frac{3}{2}\right)^{\frac{3}{4}} \epsilon\, e^{\frac{i}{2}(\varphi_1+\varphi_2)}\left[\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{\frac{1}{2}(\tau+i\psi)} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{-\frac{1}{2}(\tau+i\psi)}\right] \tag{B.2.3}$$

$$z_2 = 2^{-\frac{1}{2}}\left(\frac{3}{2}\right)^{\frac{3}{4}} \epsilon\, e^{-\frac{i}{2}(\varphi_1+\varphi_2)}\left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{\frac{1}{2}(\tau+i\psi)} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{-\frac{1}{2}(\tau+i\psi)}\right] \tag{B.2.4}$$

$$z_3 = 2^{-\frac{1}{2}}\left(\frac{3}{2}\right)^{\frac{3}{4}} \epsilon\, e^{\frac{i}{2}(-\varphi_1+\varphi_2)}\left[\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{\frac{1}{2}(\tau+i\psi)} + \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{-\frac{1}{2}(\tau+i\psi)}\right] \tag{B.2.5}$$

$$z_4 = 2^{-\frac{1}{2}}\left(\frac{3}{2}\right)^{\frac{3}{4}} \epsilon\, e^{\frac{i}{2}(\varphi_1-\varphi_2)}\left[\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{\frac{1}{2}(\tau+i\psi)} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{-\frac{1}{2}(\tau+i\psi)}\right] \tag{B.2.6}$$

solve the defining equation (B.2.2).

The deformed conifold can be given a Calabi-Yau metric which asymptotes the one of the singular conifold for large values of $\sum_i |w_i|^2 = \sum_i |w_i|^2 = r^3$ [51]:

$$\begin{aligned} ds_6^2 = \frac{1}{2}\epsilon^{4/3}K(\tau)\Bigg[&\frac{1}{3K^3(\tau)}\left(d\tau^2+\zeta^2\right) + \cosh^2\frac{\tau}{2}\left[(g^3)^2+(g^4)^2\right] + \\ &+ \sinh^2\frac{\tau}{2}\left[(g^1)^2+(g^2)^2\right]\Bigg] \,, \end{aligned} \tag{B.2.7}$$

where

$$K(\tau) = \frac{(\frac{1}{2}\sinh(2\tau)-\tau)^{1/3}}{\sinh\tau} \tag{B.2.8}$$

and the 1-forms g^1, g^2, g^3, g^4 are defined in terms of (B.1.12) as follows:

$$\begin{aligned} g^3 + ig^4 &= \frac{i}{\sqrt{2}}e^{i\psi/2}\left[(\Sigma_1+i\Sigma_2)+(\sigma_1-i\sigma_2)\right] \\ g^1 + ig^2 &= \frac{i}{\sqrt{2}}e^{i\psi/2}\left[-(\Sigma_1+i\Sigma_2)+(\sigma_1-i\sigma_2)\right] \,. \end{aligned} \tag{B.2.9}$$

The change of coordinates between τ and r is

$$r^3 = \left(\frac{3}{2}\right)^{3/2} \epsilon^2 \cosh\tau \,, \tag{B.2.10}$$

which for large r and τ can be approximated as

$$r^3 \simeq \frac{3^{3/2}}{2^{5/2}}\,\epsilon^2\,e^\tau\;. \tag{B.2.11}$$

Using this asymptotic change of radial coordinate, the metric (B.2.7) reduces to that of the singular conifold $dr^2 + r^2 ds^2_{T^{1,1}}$. The deformed conifold metric approaches that of $\mathbb{R}^3 \times S^3$ as $\tau \to 0$:

$$ds_6^2 \simeq 6^{-1/3}\,\epsilon^{4/3}\,\left\{\frac{1}{2}\left[d\tau^2 + \tau^2\frac{1}{2}\left((g^1)^2 + (g^2)^2\right)\right] + \left(\frac{1}{2}\zeta^2 + (g^3)^2 + (g^4)^2\right)\right\}\;. \tag{B.2.12}$$

Some useful formulae are:

$$d\zeta = -(g^1 \wedge g^4 + g^3 \wedge g^2) \tag{B.2.13}$$

$$d(g^3 \wedge g^4) = -d(g^1 \wedge g^2) = \frac{1}{2}\zeta \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \tag{B.2.14}$$

$$d(g^1 \wedge g^3 + g^2 \wedge g^4) = \zeta \wedge (g^1 \wedge g^2 - g^3 \wedge g^4) \tag{B.2.15}$$

$$d(\zeta \wedge g^1 \wedge g^2) = d(\zeta \wedge g^3 \wedge g^4) = 0\;. \tag{B.2.16}$$

Notice also that

$$\omega_2^{CF} = -\frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) \tag{B.2.17}$$

$$g^4 \wedge g^1 + g^2 \wedge g^3 = \sin\theta_1 d\theta_1 \wedge d\varphi_1 + \sin\theta_2 d\theta_2 \wedge d\varphi_2\;. \tag{B.2.18}$$

Finally,

$$\omega_3^{CF} \equiv \zeta \wedge \omega_2^{CF} = -\frac{1}{2}\zeta \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) \tag{B.2.19}$$

is the 3-cocycle such that $\int_{S^3} \omega_3^{CF} = 8\pi^2$.

Appendix C

Supersymmetric type IIB solutions on the conifold

In this appendix we derive supersymmetric solutions of type IIB supergravity (plus branes) on the singular conifold, by imposing Bianchi identities and vanishing supersymmetry variations of dilatino and gravitini, which together lead to a BPS system of first order differential equations. It is possible to prove that this requirements imply fulfilment of the equations of motion. We have checked it in all the cases under consideration in this thesis.

We start in subsection C.1 by deriving the solution for D3 branes at the tip of the conifold and its near horizon limit, the Klebanov-Witten (KW) solution [29]. In subsection C.2 we derive the singular Klebanov-Tseytlin (KT) solution [36] for regular and fractional D3 branes on the singular conifold. In subsection C.3 we derive the solution which takes into account the backreaction of an $SU(2) \times SU(2)$ symmetric distribution of noncompact D7 branes in the KW background [1]. Finally, in subsection C.4 we derive the solution for an $SU(2) \times SU(2)$ symmetric distribution of backreacting noncompact D7 branes in the KT background [2].

We have collected all these solutions one after the other in order to clarify the connections and the differences among them.

In this appendix we will use the vielbein introduced in subsection B.1.1 for the singular conifold.

C.1 The Klebanov-Witten solution

In this subsection we derive by means of supersymmetry methods the solution for a stack of N_c D3 branes placed at the singularity of the conifold. The D3 branes span four-dimensional Minkowski space and are a point in the conifold. In the absence of D3 branes, the product of $\mathbb{R}^{1,3}$ and the conifold is a solution of type IIB string theory preserving 8 supercharges. The addition of D3 branes breaks half supersymmetry down to 4 supercharges, because of the RR 5-form flux sourced by the branes.

Symmetry considerations imply that he solution for D3 branes will be a warped product of $\mathbb{R}^{1,3}$ and the conifold

$$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2}\left\{dr^2 + r^2\left[\frac{1}{9}(d\psi - \sum_{i=1,2}\cos\theta_i\, d\varphi_i)^2 + \frac{1}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2)\right]\right\}\,, \quad \text{(C.1.1)}$$

plus a self-dual 5-form flux along the $T^{1,1}$ angles and along Minkowski and r. The warp factor h is a function of the radial coordinate r only. Supersymmetry or Einstein equations relates the warp factor and the 5-form flux. Here we will directly impose the ansatz

$$F_5 = (1 + *)\, dh^{-1} \wedge d^4x \tag{C.1.2}$$

which solves this constraint.[1] Having done so, we will only have to impose the Bianchi identity for F_5, which determines the warp factor, and check that the supersymmetry variations of the dilatino and the gravitini vanish.

The ordered vielbein related to the metric (C.1.1) is $\{E^0, E^1, E^2, E^3, E^r, E^\psi, E^{\theta_1}, E^{\varphi_1}, E^{\theta_2}, E^{\varphi_2}\}$, where

$$\begin{aligned} E^\mu &= h^{-1/4}\, dx^\mu \\ E^r &= h^{1/4}\, e^r = h^{1/4}\, dr \\ E^\psi &= h^{1/4}\, e^\psi = h^{1/4}\, \frac{r}{3}\big(d\psi - \sum_{i=1,2} \cos\theta_i\, d\varphi_i\big) \\ E^{\theta_i} &= h^{1/4}\, e^{\theta_i} = h^{1/4}\, \frac{r}{\sqrt{6}}\, d\theta_i \\ E^{\varphi_i} &= h^{1/4}\, e^{\varphi_i} = h^{1/4}\, \frac{r}{\sqrt{6}}\, \sin\theta_i\, d\varphi_i\ . \end{aligned} \tag{C.1.3}$$

In the vielbein basis, the RR 5-form reads

$$F_5 = (1 + *)\, dh^{-1} \wedge d^4x = \frac{h'}{h^{5/4}}(-E^{0123r} + E^{\psi\theta_1\varphi_1\theta_2\varphi_2})\ , \tag{C.1.4}$$

where $E^{MNPQR} \equiv E^M \wedge E^N \wedge E^P \wedge E^Q \wedge E^R$.

First of all, we have to impose the Bianchi identity for F_5, which in the absence of 3-form fluxes is $dF_5 = 0$. Since in the coordinate basis

$$F_5 = -\frac{h'}{h^2}\, dr \wedge d^4x + h'\, r^5\, d\mathrm{vol}(T^{1,1})\ , \tag{C.1.5}$$

the Bianchi identity becomes

$$d\,(h'\, r^5) = 0\ , \tag{C.1.6}$$

which has two solutions which are not diffeomorphic to each other:

$$h(r) = 1 + \frac{L^4}{r^4} \tag{C.1.7}$$

and

$$h(r) = \frac{L^4}{r^4}\ . \tag{C.1.8}$$

[1] This holds for both the Einstein frames we have defined, (A.2.3) and (A.2.5). If instead one prefers to work in string frame, the ansatz is $g_s F_5 = (1 + *)\, dh^{-1} \wedge d^4x$.

The first one is the D3 brane solution, which has asymptotically flat metric.[2] The second one is its near-horizon limit, where the metric takes the $AdS_5 \times T^{1,1}$ form

$$ds^2 = \frac{r^2}{L^2}\, dx_{1,3}^2 + \frac{L^2}{r^2}\, dr^2 + L^2\, ds^2_{T^{1,1}} \;. \tag{C.1.9}$$

The integration constant L^4 is fixed by imposing that the D3 brane charge of the solution be N_c. In the Einstein frame which is commonly used when the dilaton is constant (A.2.3), the condition is

$$N_c = -\frac{1}{(4\pi^2\alpha')^2\, g_s} \int_{T^{1,1}} F_5 = \frac{L^4}{4\pi\alpha'\, g_s} \frac{\text{Vol}(T^{1,1})}{\pi^3} \;, \tag{C.1.10}$$

and since $\text{Vol}(T^{1,1}) = \frac{16}{27}\,\pi^3$, this fixes

$$L^4 = 4\pi\alpha'\, g_s N_c \cdot \frac{\pi^3}{\text{Vol}(T^{1,1})} = \frac{27}{4}\pi\alpha'\, g_s N_c \;. \tag{C.1.11}$$

The same result holds in the string frame. Instead, in the alternative Einstein frame (A.2.5) that we use in presence of D7 branes,

$$L^4 = 4\pi\alpha'\, N_c \cdot \frac{\pi^3}{\text{Vol}(T^{1,1})} = \frac{27}{4}\pi\alpha'\, N_c \;. \tag{C.1.12}$$

The number of supercharges preserved by the D3 brane background is 4, which is one eighth of the supercharges conserved by the IIB action. Therefore we have to impose 3 commuting projections on the Killing spinor. We will work with the complex notation for the ten-dimensional spinors (A.3.12). Recall that according to our conventions the type IIB complexified supersymmetry parameter has negative chirality in 10 dimensions, see (A.3.4):

$$\Gamma_{0123r\psi\theta_1\varphi_1\theta_2\varphi_2}\, \epsilon = \epsilon \;, \tag{C.1.13}$$

working in the vielbein basis. Consistently with the complex and Kähler structures of the conifold which are summarized in the complex orthonormal basis (B.1.26), we will impose the projections

$$\Gamma_{r\psi}\, \epsilon = \Gamma_{\theta_1\varphi_1}\, \epsilon = \Gamma_{\theta_2\varphi_2}\, \epsilon = i\, \epsilon \;, \tag{C.1.14}$$

which imply

$$\Gamma_{r\psi\theta_1\varphi_1\theta_2\varphi_2}\, \epsilon = -i\, \epsilon \;, \qquad\qquad \Gamma_{0123}\, \epsilon = i\epsilon \;. \tag{C.1.15}$$

In particular, the Killing spinor of negative ten-dimensional chirality can be decomposed into a tensor product of a 4d spinor of positive chirality and a 6d spinor of negative chirality, according to the definitions (A.3.5) and (A.3.6).

[2]The additive constant is fixed to 1 in string frame and in the Einstein frame (A.2.3) by requiring that the metric become the unit flat metric when the flux is switched off. When going to the Einstein frame (A.2.5), a suitable rescaling of the radial and Minkowski coordinate like $x^\mu = g_s^{-1/4}\, x^\mu_{(s)}$ and $r = g_s^{-1/4}\, r_{(s)}$ may be chosen so to fix the additive constant to 1 again, so that in these rescaled coordinates the metric approaches the unit flat metric as the flux is switched off. Obviously there is no physical content in this rescaling, that we choose here only for the sake of convenience.

In order to check supersymmetry and to find the Killing spinor of the D3 brane background, we need the spin connection, which in the vielbein basis (C.1.3) is:

$$\begin{aligned} \omega^{\mu r} &= -\frac{h'}{4h^{5/4}}\, E^{\mu} \\ \omega^{r\psi} &= -\frac{4h+rh'}{4h^{5/4}}\, E^{\psi} \\ \omega^{r\theta_i} &= -\frac{4h+rh'}{4h^{5/4}}\, E^{\theta_i} \\ \omega^{r\varphi_i} &= -\frac{4h+rh'}{4h^{5/4}}\, E^{\varphi_i} \end{aligned} \qquad \begin{aligned} \omega^{\psi\theta_i} &= \frac{1}{rh^{1/4}}\, E^{\varphi_i} \\ \omega^{\psi\varphi_i} &= -\frac{1}{rh^{1/4}}\, E^{\theta_i} \\ \omega^{\theta_i\varphi_i} &= -\frac{1}{rh^{1/4}}\, E^{\psi} - \frac{\sqrt{6}\,\cot\theta_i}{rh^{1/4}}\, E^{\varphi_i} \end{aligned} \tag{C.1.16}$$

The variation of the dilatino (A.3.14) vanishes trivially. Imposing vanishing variation of the gravitini (A.3.15) in this F_5 background reduces to requiring:

$$0 = \partial_A \epsilon + \frac{1}{4}\,\omega_A^{BC}\,\Gamma_{BC}\,\epsilon - \frac{i}{16\cdot 5!}\,F_{BCDEF}\,\Gamma^{BCDEF}\Gamma_A\,\epsilon\ . \tag{C.1.17}$$

Because of the symmetries, we look for a Killing spinor which depends only on r and ψ. In the AdS case, this dependence is related to dilatation and $U(1)_R$ symmetry transformations of the 4-dimensional spinor in the dual field theory. ψ dependence arises because that angular direction is fibered over the other four angles, which parameterize a 4-dimensional Kähler-Einstein space. We will work in the vielbein basis, with capital Latin indices at the beginning of the alphabet. The partial derivative has to be translated to this basis:

$$\partial_A \epsilon = \delta_A^r\,\frac{1}{h^{1/4}}\,\partial_r \epsilon + \delta_A^{\psi}\,\frac{3}{rh^{1/4}}\,\partial_\psi \epsilon + \delta_A^{\varphi_i}\,\frac{\sqrt{6}\,\cot\theta_i}{rh^{1/4}}\,\partial_\psi \epsilon\ . \tag{C.1.18}$$

The variation of the gravitino in Minkowski components vanishes because of the chiralities (C.1.15). Indeed, the condition is

$$0 = \frac{1}{2}\left(-\frac{h'}{4h^{5/4}}\right)\Gamma_{1r}\,\epsilon - \frac{i}{16}\frac{h'}{h^{5/4}}\,\left(\Gamma_{0123r} + \Gamma_{\psi\theta_1\varphi_1\theta_2\varphi_2}\right)\Gamma_1\,\epsilon\ , \tag{C.1.19}$$

which after factorization of Γ_{1r} becomes

$$2\epsilon - i\,\left(-\Gamma_{0123} + \Gamma_{r\psi\theta_1\varphi_1\theta_2\varphi_2}\right)\epsilon = 0\ , \tag{C.1.20}$$

that is fulfilled once we impose the chiralities (C.1.15). Notice that it is the condition that the variation of the Minkowski components of the gravitino vanish that fixes the relation (C.1.4) between the RR 5-form and the warp factor, that we have chosen to impose from the beginning for the sake of brevity.

Vanishing variation of the r and ψ components fix the dependence of the Killing spinor on those coordinates. The 10-dimensional Killing spinor is

$$\epsilon = h^{-1/8}\,e^{i\psi/2}\,\epsilon_4^+ \otimes \epsilon_6^-\ , \tag{C.1.21}$$

where ϵ_4^+ is a 4-dimensional constant spinor of positive chirality, and ϵ_6^- is a constant spinor on the conifold satisfying the projections

$$\Gamma_{r\psi}\,\epsilon_6^- = \Gamma_{\theta_1\varphi_1}\,\epsilon_6^- = \Gamma_{\theta_2\varphi_2}\,\epsilon_6^- = i\,\epsilon_6^-\ . \tag{C.1.22}$$

To see this explicitly, we first consider the condition that the supersymmetry variation of ψ_r vanish:

$$0 = \frac{1}{h^{1/4}}\,\partial_r\epsilon - \frac{i}{16}\frac{h'}{h^{5/4}}\,(\Gamma_{0123r} + \Gamma_{\psi\theta_1\varphi_1\theta_2\varphi_2})\,\Gamma_r\,\epsilon\,, \tag{C.1.23}$$

where the derivative is the usual one in the coordinate basis. After imposing the chiralities (C.1.15), this boils down to

$$\partial_r\epsilon = -\frac{1}{8}\,\partial_r(\log h)\,\epsilon\,, \tag{C.1.24}$$

which tells us that $\epsilon \propto h^{-1/8}$. Then we consider the condition $\delta_\epsilon\,\psi_\psi = 0$:

$$\begin{aligned} 0 =& \frac{3h}{rh^{5/4}}\,\partial_\psi\epsilon - \frac{1}{2}\,\frac{4h+rh'}{4rh^{5/4}}\,\Gamma_{r\psi}\,\epsilon - \frac{1}{2}\,\frac{4h}{4rh^{5/4}}\,(\Gamma_{\theta_1\varphi_1} + \Gamma_{\theta_2\varphi_2})\,\epsilon + \\ &- \frac{i}{16}\frac{h'}{h^{5/4}}\,(\Gamma_{0123r} + \Gamma_{\psi\theta_1\varphi_1\theta_2\varphi_2})\,\Gamma_\psi\,\epsilon\,. \end{aligned} \tag{C.1.25}$$

After imposing the projections (C.1.14), it simplifies to

$$\partial_\psi\epsilon = \frac{i}{2}\,\epsilon\,, \tag{C.1.26}$$

which tells us that $\epsilon \propto e^{i\psi/2}$.

Similarly, the variations of the θ_i and φ_i components of the gravitino vanish because of (C.1.14) and (C.1.21).

Incidentally, let us mention that the near-horizon solution $AdS_5 \times T^{1,1}$ preserves four additional supercharges, which are the superconformal partners of the ones discussed here; they are related to additional solutions of the Killing spinor equations. The additional Killing spinor can also be found by using the superconformal algebra, since the commutator of a special conformal transformation and an ordinary supercharge gives this 'conformal' supercharge.

C.2 The Klebanov-Tseytlin solution

Klebanov and Tseytlin (KT) considered the addition of M fractional D3 branes to the Klebanov-Witten background [36], as a way of breaking the conformal symmetry in the dual field theory. Fractional D3 branes can be thought of as D5 branes wrapping the exceptional rigid 2-cycle living at the conical singularity, and as such they source 3-form fluxes: they source F_3, whose flux through the nontrivial 3-cycle in the $T^{1,1}$ base counts the number of fractional D3 branes, and by supersymmetry they also source H_3. Being D3 branes (the 2-cycle they wrap, when they are thought of as D5 branes, has vanishing volume), they do not couple to the dilaton. The solution, that we will rederive here by means of supersymmetry methods, will therefore involve nontrivial 3- and 5-form fluxes.

Inspection of the Einstein equations (A.2.7) reveals that the Ricci scalar vanishes as long as the complexified 3-form flux $G_3 = F_3 + i\,e^{-\Phi}\,H_3$ is imaginary-self-dual (ISD) $*_6 G_3 = i\,G_3$ or anti-imaginary-self-dual (AISD) $*_6 G_3 = -i\,G_3$, since those conditions imply that $|F_3|^2 = e^{-2\Phi}\,|H_3|^2$. Supersymmetry will require G_3 to be ISD. Therefore, the metric ansatz will be the same as for the Klebanov-Witten solution. The relation between the 5-form and the warp factor will not be spoilt too. This is clear from the requirement that the variation of a μ component of the gravitino vanish, or from Einstein's equation along Minkowski directions.

We are after a solution with a constant axio-dilaton, a nontrivial 5-form flux, and a closed complexified 3-form flux G_3 (from the Bianchi identities), which preserves the same four ordinary supersymmetries that we have exhibited for the Klebanov-Witten solution; therefore the Killing spinor will be again of the form (C.1.21), subject to the same projections. Requiring the same supersymmetries preserved by D3 branes imposes conditions on G_3: not only it has to be closed, but also primitive, ISD, and of $(2,1)$ kind. After the global symmetries are imposed, it is easy to see that primitivity of G_3, namely $J \wedge G_3 = 0$, ensures vanishing dilatino variation and vanishing gravitino variation along r and ψ, and together with the $(2,1)$ condition ensures vanishing gravitino variations along the other angles. Imaginary-self-duality comes out as a consequence of the other requirements.

3-form fluxes break the four additional (conformal) supercharges; indeed we have to fulfil the Bianchi identity $dF_5 = -H_3 \wedge F_3$, and because of relation (C.1.5) the warp factor h will develop a more complicated r dependence than the simple L^4/r^4 one, so that the background will not display an AdS_5 factor anymore.

Let us now determine the 3-form fluxes and the warp factor, and show more explicitly why the solution preserves the same four ordinary supersymmetries of the KW background. In appendix B.1 we have found the $SU(2) \times SU(2) \times U(1)_R$ invariant closed, primitive, ISD $(2,1)$-form (B.1.29), whose real part has nonvanishing integral on the 3-cycle (B.1.24) of the conifold. The complexified 3-form G_3 we are after has to be proportional to this (B.1.29). We find that

$$G_3 = F_3 + \frac{i}{g_s} H_3 = -\frac{M\,\alpha'}{2} \left(\zeta - 3\, i\, \frac{dr}{r} \right) \wedge \omega_2^{CF} \;, \tag{C.2.1}$$

where the prefactor is fixed by the number of fractional D3 branes, namely the D5 brane charge (A.2.35), being M:[3]

$$M = -\frac{1}{4\pi^2 \alpha'} \int_{S^3} F_3 \;. \tag{C.2.2}$$

Therefore

$$F_3 = -\frac{M\alpha'}{2}\, \zeta \wedge \omega_2^{CF} = -M\,\alpha' \frac{9\, h^{-3/4}}{2\, r^3}\, E^{\psi} \wedge \left(E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2} \right) \tag{C.2.3}$$

and

$$H_3 = g_s\, \frac{3M\,\alpha'}{2}\, \frac{dr}{r} \wedge \omega_2^{CF} = g_s\, M\,\alpha' \frac{9\, h^{-3/4}}{2\, r^3}\, E^{r} \wedge \left(E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2} \right) \;, \tag{C.2.4}$$

so that

$$F_3 \wedge H_3 = -g_s\, \frac{81}{2}\, M^2\, \alpha'^2 \frac{h^{-3/2}}{r^6}\, E^{r\psi\theta_1\varphi_1\theta_2\varphi_2} = -g_s\, \frac{81}{2}\, M^2\, \alpha'^2 \frac{1}{r}\, dr \wedge \mathrm{dvol}(T^{1,1}) \;. \tag{C.2.5}$$

The Bianchi identity $dF_5 = F_3 \wedge H_3$ becomes

$$(r^5\, h')' = -g_s\, \frac{81}{2}\, M^2\, \alpha'^2\, \frac{1}{r} \;, \tag{C.2.6}$$

whose solution is

$$h(r) = \frac{27}{4}\, \pi\alpha'^2\, \frac{1}{r^4} \left[\frac{3}{2\pi}\, g_s\, M^2\, \log \frac{r}{r_c} \right] \;, \tag{C.2.7}$$

[3]Here we are using the natural Einstein frame (A.2.5) that extends to the situation where D7 branes are present, rather than the most common ones (A.2.3) or string frame.

where r_c is an integration constant. There is a single integration constant r_c. Its value can be related to the D3 brane Maxwell charge (A.2.34) at some reference scale.

We can also write the solution for the RR B_2 potential:

$$B_2 = g_s \frac{3M\,\alpha'}{2} \log\frac{r}{r_0}\,\omega_2^{CF} \ . \tag{C.2.8}$$

What is relevant for supersymmetry is the structure of the added 3-form flux which follows from the requirement that it be primitive, $(2,1)$ and ISD, as well as $SU(2)\times SU(2)\times U(1)_R$ invariant:

$$G_3 \propto (E^r + i\,E^\psi)\wedge(E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2}) \ . \tag{C.2.9}$$

The additional term due to the 3-flux in the supersymmetry variation of the dilatino (A.3.14) is proportional to

$$(G_3)_{ABC}\,\Gamma^{ABC}\,\epsilon^* \ , \tag{C.2.10}$$

which vanishes because $\Gamma^{\theta_1\varphi_1}\,\epsilon^* = \Gamma^{\theta_2\varphi_2}\,\epsilon^*$. This is ensured by $SU(2)\times SU(2)$ invariance and the primitivity condition. As far as the supersymmetry variations of the gravitino (A.3.15) are concerned, the additional term is proportional to the complex conjugate of

$$(G_3)_{BCD}\,({\Gamma_A}^{BCD} - 9\,\delta_A^B\,\Gamma^{CD})\,\epsilon \ . \tag{C.2.11}$$

The first term vanishes for any A for the same reason as before, as well as the second term when $A = r,\,\psi$. When A is one of the four other angles, the second term vanish because the $E^r + i\,E^\psi$ part in (C.2.9) brings $\Gamma^r + i\,\Gamma^\psi$, which annihilates ϵ because of the projection $\Gamma^{r\psi}\,\epsilon = i\,\epsilon$. This is the $(2,1)$ condition, which together with the previous conditions fixes G_3 (C.2.9) to be ISD.

Finally, we can check that the equations of motion for F_3 (A.2.10) and H_3 (A.2.14) are automatically satisfied. To see this, we rephrase the imaginary-self-duality condition on G_3 as $*_6\,F_3 = -e^{-\Phi}H_3$ and $*_6\,H_3 = e^{\Phi}F_3$, and make use of the identity

$$*\,\omega_3 = h^{-1}\,d^4x\wedge *_6\,\omega_3 \ , \tag{C.2.12}$$

which holds for any 3-form ω_3 with all legs in the internal space. The equation of motion for F_3 (A.2.10) is solved, because the left hand side (LHS) is $-d(h^{-1}\,d^4x\wedge H_3) = -F_5\wedge H_3$, which is equal to the right hand side (RHS). The equation of motion for H_3 (A.2.14) is also solved, because the LHS is $d(h^{-1}\,d^4x\wedge F_3) = F_5\wedge F_3$, which is equal to the RHS.[4]

C.3 Backreacting D7 branes in the Klebanov-Witten background

In this subsection we derive the solution which takes into account the backreaction of an $SU(2)\times SU(2)$ symmetric distribution of noncompact D7 branes in the KW background [1]. D7 branes source the dilaton and (magnetically) the RR field strength F_1.

$$F_1 = \frac{N_f}{4\pi}\big(d\psi - \sum_{i=1,2}\cos\theta_i\,d\varphi_i\big) = \frac{N_f}{4\pi}\,\zeta \tag{C.3.1}$$

[4] We have used that in this solution $F_1 = 0$, but the equation is satisfied also with a nonzero F_1.

satisfies the modified Bianchi identity for an $SU(2) \times SU(2)$ invariant D7 brane charge distribution with the correct normalization for a total number N_f of D7 branes:

$$N_f = \int_{\mathcal{D}_2} dF_1 \ , \tag{C.3.2}$$

where

$$\mathcal{D}: \qquad \psi = \theta_2 = \varphi_2 = const. \tag{C.3.3}$$

is a 2-chain with boundary which is intersected once by every D7 brane. A metric ansatz that respects the symmetry of the problem is

$$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2} \left\{ dr^2 + \frac{e^{2f}}{9} (d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i)^2 + \frac{e^{2g}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i \, d\varphi_i^2) \right\} \ , \tag{C.3.4}$$

where the warp factor h and the squash factors f and g are functions of the radial coordinate only. The ordered vielbein related to the metric (C.3.4) is $\{E^0, E^1, E^2, E^3, E^r, E^\psi, E^{\theta_1}, E^{\varphi_1}, E^{\theta_2}, E^{\varphi_2}\}$, where

$$\begin{aligned} E^\mu &= h^{-1/4} \, dx^\mu \\ E^r &= h^{1/4} \, e^r = h^{1/4} \, dr \\ E^\psi &= h^{1/4} \, \frac{e^f}{3} \, \zeta = h^{1/4} \, \frac{e^f}{3} \big(d\psi - \sum_{i=1,2} \cos\theta_i \, d\varphi_i \big) \\ E^{\theta_i} &= h^{1/4} \, \frac{e^g}{\sqrt{6}} \, d\theta_i \\ E^{\varphi_i} &= h^{1/4} \, \frac{e^g}{\sqrt{6}} \, \sin\theta_i \, d\varphi_i \ . \end{aligned} \tag{C.3.5}$$

Relation (C.1.2) between the self-dual 5-form and the warp factor still holds, since it follows from requiring vanishing variation of a Minkowski component of the gravitino. The RR 5-form reads

$$F_5 = (1+*) \, dh^{-1} \wedge d^4x = -\frac{h'}{h^2} \, d^4x \wedge dr + h' \, e^{4g+f} \, \mathrm{dvol}(T^{1,1}) = \frac{h'}{h^{5/4}} (-E^{0123r} + E^{\psi\theta_1\varphi_1\theta_2\varphi_2}) \ , \tag{C.3.6}$$

so that the Bianchi identity $dF_5 = 0$ becomes

$$h' \, e^{4g+f} = -27\pi \, \alpha'^2 \, N_c \ , \tag{C.3.7}$$

where the right hand side is fixed by the condition that the D3 brane charge (A.2.34) be N_c.

We now want to impose supersymmetry. We will again impose the projections (C.1.14), which are fixed by holomorphy. The condition that the supersymmetry variation of the dilatino vanishes is

$$0 = \delta_\epsilon \lambda = \frac{1}{2} \Gamma^A \left(\partial_A \Phi + i \, e^\Phi F_A \right) \epsilon \ . \tag{C.3.8}$$

This equation will determine the dilaton. Let us remark here that

$$\partial_A \Phi + i \, e^\Phi F_A = i \, \epsilon^\Phi \left(F_1 + i \, de^{-\Phi} \right)_A \ . \tag{C.3.9}$$

In this solution with distributed D7 branes, C_0 (and then the axio-dilaton $\tau = C_0 + i\, e^{-\Phi}$) is not well defined, but its differential (and then $d\tau = F_1 + i\, de^{-\Phi}$) is. With the supersymmetry projections of the D3 brane solution (C.1.14), the supersymmetry variation of the dilatino vanishes if and only if the differential of the axio-dilaton is a $(1,0)$-form. The equation gives

$$\left(e^{-\Phi}\right)' = -\frac{3N_f}{4\pi} e^{-f} \ . \tag{C.3.10}$$

We can see immediately that

$$F_1 + i\, de^{-\Phi} = -i\, \frac{3N_f}{4\pi}\, h^{-1/4}\, e^{-f}\, (E^r + i\, E^\psi) \tag{C.3.11}$$

is a $(1,0)$-form.

The conditions that the supersymmetry variations of the gravitino vanish are

$$0 = \delta_\epsilon \psi_A = \nabla_A\, \epsilon - \frac{i}{4}\, e^\Phi F_A\, \epsilon \ . \tag{C.3.12}$$

In the basis (C.3.5), the spin connection is:

$$\begin{aligned}
\omega^{\mu r} &= -\frac{h'}{4h^{5/4}}\, E^\mu & \omega^{\psi\theta_i} &= \frac{e^{f-2g}}{h^{1/4}}\, E^{\varphi_i} \\
\omega^{r\psi} &= -\frac{4hf' + h'}{4h^{5/4}}\, E^\psi & \omega^{\psi\varphi_i} &= -\frac{e^{f-2g}}{h^{1/4}}\, E^{\theta_i} \\
\omega^{r\theta_i} &= -\frac{4hg' + h'}{4h^{5/4}}\, E^{\theta_i} & \omega^{\theta_i\varphi_i} &= -\frac{e^{f-2g}}{h^{1/4}}\, E^\psi - \frac{\sqrt{6}\, e^{-g}\, \cot\theta_i}{h^{1/4}}\, E^{\varphi_i} \\
\omega^{r\varphi_i} &= -\frac{4hg' + h'}{4h^{5/4}}\, E^{\varphi_i} & &
\end{aligned} \tag{C.3.13}$$

Expressed in the vielbein basis, the partial derivative is

$$\begin{aligned}
\partial_A = E^M{}_A \partial_M = \delta^\mu_A\, h^{1/4}\, \partial_\mu + \delta^r_A\, h^{-1/4}\, \partial_r + \delta^\psi_A\, \frac{3\, e^{-f}}{h^{1/4}}\, \partial_\psi + \delta^{\theta_i}_A\, \frac{\sqrt{6}\, e^{-g}}{h^{1/4}}\, \partial_{\theta_i} + \\
+ \delta^{\varphi_i}_A\, \frac{\sqrt{6}\, e^{-g}}{h^{1/4}} (\cot\theta_i\, \partial_\psi + \csc\theta_i\, \partial_{\varphi_i}) \ .
\end{aligned} \tag{C.3.14}$$

Exactly as in the Klebanov-Witten case, the μ component of (C.3.12) is (C.1.19), which is satisfied because of (C.1.2), and the r component is (C.1.23), which implies that $\epsilon \propto h^{-1/8}$. Requiring vanishing variation along the θ_i component leads to the equation

$$g' = e^{f-2g} \ . \tag{C.3.15}$$

Once inserted in the conditions for vanishing variation along φ_i and ψ, it gives a system of two first order differential equations which can be disentangled and imply the dependence $\epsilon \propto e^{\frac{i}{2}\psi}$, as well as the final differential equation

$$e^f\, (f' + 2\, e^{f-2g}) + \frac{1}{2}\, \frac{3N_f}{4\pi}\, e^\Phi - 3 = 0 \ . \tag{C.3.16}$$

The equation of motion for F_1 (A.2.9) is automatically satisfied: the LHS vanishes because the dilaton is a radial function, and the RHS is equal to $-h^{-1}\, d^4x \wedge H_3 \wedge H_3 = 0$.

We can now solve the first order differential equations (C.3.10), (C.3.15), and (C.3.16). The dilaton can be found after a change of radial coordinate from r to ρ, such that $\frac{d}{d\rho} = e^f \frac{d}{dr}$. From now on, we will denote with a dot derivatives with respect to ρ. Equation (C.3.10) is rewritten as

$$\dot{(e^{-\Phi})} = -\frac{3N_f}{4\pi} \; , \tag{C.3.17}$$

whose solution is

$$e^{\Phi} = \frac{4\pi}{3N_f} \frac{1}{\rho_{max} - \rho} \; . \tag{C.3.18}$$

The radial coordinate ρ ranges from $-\infty$ to ρ_{max}. At any nonzero fixed value of N_f, ρ_{max} can be absorbed by redefining ρ. However, the integration constant ρ_{max} is useful in taking the unflavored limit: if we want to recover the Klebanov-Witten solution [29], which has a constant dilaton, we must take a double scaling limit $N_f \to 0$ with $N_f \, \rho_{max}$ fixed. ρ remains finite in this scaling limit. We will keep in mind this point, but from now on we will reabsorb ρ_{max} so as not to clutter formulae. We will therefore use

$$e^{\Phi} = -\frac{4\pi}{3N_f} \frac{1}{\rho} \; . \tag{C.3.19}$$

We can now rewrite the two remaining equations (C.3.15) and (C.3.16) as

$$\dot{g} = e^{2f-2g} \tag{C.3.20}$$

$$\dot{f} = 3 - 2\dot{g} - \frac{1}{2}\dot{\Phi} \; . \tag{C.3.21}$$

It proves convenient to introduce the new function $u \equiv 2f - 2g$, subject to the first order equation

$$\dot{u} = 6\,(1 - e^{u}) + \frac{1}{\rho} \; , \tag{C.3.22}$$

whose solution is

$$e^{u} = \frac{-6\rho\, e^{6\rho}}{(1-6\rho)\, e^{6\rho} + c_1} \; . \tag{C.3.23}$$

The integration constant c_1 cannot be reabsorbed. We can then integrate (C.3.20) to obtain

$$e^{g} = C\, \left[(1-6\rho)\, e^{6\rho} + c_1\right]^{1/6} \; , \tag{C.3.24}$$

and use the definition of u to get

$$e^{f} = C\,(-6\rho\, e^{6\rho})^{1/2}\, \left[(1-6\rho)\, e^{6\rho} + c_1\right]^{-1/3} \; . \tag{C.3.25}$$

Finally, we can integrate the warp factor in (C.3.7):

$$h = -27\pi\, \alpha'\, N_c \int d\rho\, e^{-4g} \; . \tag{C.3.26}$$

The metric (C.3.4) in terms of the coordinate ρ is

$$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2}\, e^{2f} \left\{ d\rho^2 + \frac{1}{9}(d\psi - \sum_{i=1,2} \cos\theta_i\, d\varphi_i)^2 + \frac{e^{-2u}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2) \right\} . \tag{C.3.27}$$

The integration constant C in (C.3.24) and (C.3.25) can be reabsorbed by rescaling $x^\mu \to x^\mu / C$.

C.4 Backreacting D7 branes in the Klebanov-Tseytlin background

In this subsection we derive the solution which takes into account the backreaction of an $SU(2) \times SU(2)$ symmetric distribution of noncompact D7 branes in the KT background [2]. The RR 1-form field strength which is sourced by the flavor brane sources is still (C.3.1). Our strategy to find the solution should be clear at this point. We will keep the same metric as in the case without 3-fluxes (C.3.4), with the same squash factors (C.3.24) and (C.3.25) (with $C = 1$), as well as the same dilaton (C.3.19); we will add to that solution a primitive, $(2,1)$ and ISD complexified 3-form $G_3 = F_3 + i\,e^{-\Phi}\,H_3$, so that the supersymmetry arguments of the solution without 3-form fluxes go through. Finally we will impose Bianchi identities $dF_3 = -H_3 \wedge F_1$ and $dF_5 = -H_3 \wedge F_3$ so as to fix the 3-forms and the warp factor.

The complexified 3-form that meets the requirements is

$$\begin{aligned} G_3 = F_3 + i\,e^{-\Phi}\,H_3 &= -\frac{M\alpha'}{2}\,l\,(\zeta - 3i\,e^{-f}\,dr) \wedge \omega_2^{CF} = \\ &= -M\alpha'\,\frac{9e^{-f-2g}}{2h^{3/4}}\,l\,(E^\psi - i\,E^r) \wedge (E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2})\,, \end{aligned} \tag{C.4.1}$$

where $l = l(r)$ is a function that will be determined by imposing the Bianchi identity $dF_3 = -H_3 \wedge F_1$. Inserting

$$dF_3 = -M\alpha'\,\frac{9e^{-f-2g}}{2h}\,l'\,E^{r\psi} \wedge (E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2})\,, \tag{C.4.2}$$

$$H_3 = e^{\Phi}\,M\alpha'\,\frac{9e^{-f-2g}}{2h^{3/4}}\,l\,E^r \wedge (E^{\theta_1\varphi_1} - E^{\theta_2\varphi_2})\,, \tag{C.4.3}$$

$$F_1 = \frac{3N_f}{4\pi}\,h^{-1/4}\,e^{-f}\,E^\psi \tag{C.4.4}$$

and switching to the radial coordinate ρ, in terms of which the dilaton is (C.3.19), we find that the radial function l appearing in G_3 is

$$l = \frac{\Gamma}{\rho}\,, \tag{C.4.5}$$

where Γ is an integration constant. As in [2], we define $M_{eff}(\rho)$ as the D5 brane Maxwell charge (A.2.35) of the solution at a radial position ρ:

$$M_{eff}(\rho) \equiv -\frac{1}{4\pi^2\alpha'}\int_{S^3(\rho)} F_3 = M\,l(\rho) = \frac{M\Gamma}{\rho} \equiv \frac{\mathcal{M}}{\rho}\,. \tag{C.4.6}$$

the solution for G_3 is therefore

$$G_3 = F_3 + i\,e^{-\Phi}\,H_3 = -\frac{\alpha'}{4}\,\frac{\mathcal{M}}{\rho}\,(\zeta - 3i\,d\rho) \wedge (\sin\theta_1\,d\theta_1 \wedge d\varphi_1 - \sin\theta_2\,d\theta_2 \wedge d\varphi_2)\,. \tag{C.4.7}$$

Finally, we have to solve the last Bianchi identity $dF_5 = -H_3 \wedge F_3$. We will use

$$dF_5 = (e^{4g+f}\,h')'\,dr \wedge \mathrm{dvol}(T^{1,1}) = \frac{d}{d\rho}(e^{4g}\,\dot h)\,d\rho \wedge \mathrm{dvol}(T^{1,1}) \tag{C.4.8}$$

$$-H_3 \wedge F_3 = 54\,\alpha'^2\,\frac{\pi}{N_f}\,\frac{\mathcal{M}^2}{\rho^3}\,d\rho \wedge \mathrm{dvol}(T^{1,1})\,. \tag{C.4.9}$$

We first integrate

$$e^{4g}\,\dot{h} = -27\pi\alpha' \left(N + \frac{1}{N_f}\,\frac{\mathcal{M}^2}{\rho^2} \right) \equiv -27\pi\alpha'\,N_{eff}(\rho)\;, \tag{C.4.10}$$

where N is an integration constant and we have defined $N_{eff}(\rho)$ as the D3 brane Maxwell charge (A.2.34) of the solution at a radial position ρ:

$$N_{eff}(\rho) = \frac{1}{(4\pi^2\alpha')^2}\int_{T^{1,1}} F_5 = -\frac{1}{27\pi\alpha'^2}\,e^{4g}\,\dot{h} = N + \frac{1}{N_f}\,\frac{\mathcal{M}^2}{\rho^2}\;. \tag{C.4.11}$$

Finally, the warp factor is

$$h = -27\pi\,\alpha' \int d\rho\, e^{-4g}\,N_{eff}\;. \tag{C.4.12}$$

The equations of motion for F_1, F_3 and H_3 are (A.2.9), (A.2.10), (A.2.14), since there are no electric sources for these field strengths. We are about to show that they are automatically solved. Equation (A.2.9) is trivially solved because the dilaton is a radial function and F_1 is not along E^r. Equation (A.2.10) is solved for the same reason as in the KT case: the LHS can be rewritten as $-d(h^{-1}\,d^4x \wedge H_3 = -F_5 \wedge H_3$. Finally, equation A.2.14 is also solved: the LHS can be rewritten as $d(h^{-1}\,d^4x \wedge F_3) = F_5 \wedge F_3 - h^{-1}\,d^4x \wedge H_3 \wedge F_1$, whereas the RHS can be rewritten as $F_5 \wedge F_3 + F_1 \wedge h^{-1}\,d^4x \wedge H_3$.

It is possible to check that also the equations of motion for the dilaton and the metric are satisfied, provided that we use the BPS system of first order differential equations for the radial functions appearing in the ansatz.

Appendix D

The orbifolded conifold and its fractional branes

In this appendix, we derive the results presented in section 7.2 concerning the relation between the ranks in the quiver, the cycles wrapped by the different fractional branes, and the fluxes present in the supergravity solution. In order to do this, we need first to discuss in detail the compact 2-cycles of the geometry, on which the branes can wrap. Then we discuss the compact 3-cycles of the geometry, which support the RR fluxes sourced by the branes, and their intersections with the 2-cycles (in the base of the singular cone). This will allow us to write the 3-form fluxes directly in terms of the ranks of the gauge groups in the quiver.

The CY singularity on which our gauge theory is engineered is a nonchiral $\mathbb{Z}_2$ orbifold of the conifold (B.1.1), obtained considering the following action on the coordinates z_i in $\mathbb{C}^4$

$$\Theta : \quad (z_1, z_2, z_3, z_4) \;\rightarrow\; (z_1, z_2, -z_3, -z_4) \; . \tag{D.0.1}$$

The orbifold geometry is still an algebraic variety. To describe it one can introduce a complete set of invariants: $x \equiv z_3^2$, $y \equiv z_4^2$ and $t \equiv z_3 z_4$, which satisfy the constraint $xy = t^2$. The conifold equation is rewritten as $t = z_1 z_2$ so that t can be eliminated and we are left with

$$f = (z_1 z_2)^2 - xy = 0 \; . \tag{D.0.2}$$

The singular locus $f = df = 0$ consists of two complex lines that meet at the tip of the geometry $\{z_1 = z_2 = x = y = 0\}$, and corresponds to the fixed point locus of the orbifold action Θ.

One can use real coordinates as well, those already defined in appendix B.1. The orbifold action (D.0.1), which is an identification in the covering space, where we will work, reads

$$\Theta : \quad (\varphi_1, \varphi_2) \;\rightarrow\; (\varphi_1 - \pi, \varphi_2 + \pi) \; . \tag{D.0.3}$$

The two complex lines, that we call the p and q line respectively, are defined, in complex and real coordinates, as

$$\begin{aligned} p &= \{z_1 = x = y = 0, \,\forall z_2\} = \{\theta_1 = \theta_2 = 0, \,\forall r, \psi'\} \\ q &= \{z_2 = x = y = 0, \,\forall z_1\} = \{\theta_1 = \theta_2 = \pi, \,\forall r, \psi''\} \; , \end{aligned} \tag{D.0.4}$$

where $\psi' = \psi - \varphi_1 - \varphi_2$ and $\psi'' = \psi + \varphi_1 + \varphi_2$ are (well defined) angular coordinates along the singularity lines. In a neighborhood of the singular lines (and outside the tip) the geometry looks locally like the A_1-singularity $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. The fixed point curve p sits at the north poles of both S^2's while the curve q sits at the south poles.

D.1 2-cycles and resolutions

From the above analysis it follows that the singular geometry has three vanishing 2-cycles. Two of these three cycles arise due to the orbifold action; such exceptional 2-cycles are located all along the $\mathbb{C}^2/\mathbb{Z}_2$ singular lines p and q (D.0.4), and we call them $\mathcal{C}_2$ and $\mathcal{C}_4$, respectively. Locally, one could resolve the space into an ALE space fibered over $\mathbb{C}^*$. The third relevant 2-cycle descends from the 2-cycle of the double covering conifold geometry, whose base $T^{1,1}$ is topologically $S^2 \times S^3$.

Our goal in what follows is to pinpoint the precise map between vanishing 2-cycles, wrapped D5 branes, 3-form RR fluxes and quiver rank assignments. To this end, it will prove useful to take advantage of our CY cone being a toric variety[1], since in this case one can use standard techniques to understand the structure of 2-cycles and their intersections. Let us sketch how this comes about.

A toric variety can be described as the moduli space of an associated supersymmetric gauged linear σ-model (GLSM). Consider n chiral superfields t_i, $i = 1 \dots n$ charged under a product of abelian gauge groups $U(1)^s$, with charges ${Q_a}^i$, $a = 1 \dots s$. In the absence of a superpotential, the potential for the scalar components is

$$V(t_i) = \sum_{a=1}^{s} \Big(\sum_{i=1}^{n} {Q_a}^i \, |t_i|^2 - \xi_a \Big)^2 \, . \tag{D.1.1}$$

where ξ_a are Fayet-Iliopoulos parameters (FI). The moduli space of vacua $\mathcal{M}$ is given by the D-flatness equations modulo $U(1)^s$ gauge transformations

$$\mathcal{M} = \Big\{ t_i \in \mathbb{C}^n \Big| \sum_{i=1}^{n} {Q_a}^i \, |t_i|^2 = \xi_a \quad \forall a = 1, \dots, s \Big\} \Big/ U(1)^s \, , \tag{D.1.2}$$

where $U(1)^s$ acts as $t_i \to e^{i\,{Q_a}^i\,\phi^a} t_i$. When the FI's are such that $\dim \mathcal{M} = n - s$, $\mathcal{M}$ is the desired toric variety (and $n - s = r$ is just the number of isometry abelian factors). Putting the FI's to zero the variety, if admissible, is scale invariant: this corresponds to a cone. As the FI's change, the Kähler moduli of $\mathcal{M}$ also change and one gets resolutions or blow-ups. Generically, different regions in the parameter space of the FI parameters correspond to different resolutions, delimited by flop transition curves.

In our case the GLSM has six fields t_i whose charges ${Q_a}^i$ are reported in the table below

t_1	t_2	t_3	t_4	t_5	t_6	
0	0	1	−2	1	0	ξ_2
1	−1	0	1	−1	0	ξ_β
−2	1	0	0	0	1	ξ_4

(D.1.3)

We can parameterize the toric variety with the gauge invariants

$$t_3 t_4 t_5 = z_1 \qquad t_1 t_2 t_6 = z_2 \qquad t_1 t_2^2 t_3^2 t_4 = x \qquad t_1 t_4 t_5^2 t_6^2 = y \tag{D.1.4}$$

which, consistently, satisfy the defining equation (D.0.2). We can also give a parametrization for the so-called toric divisors, which are the four-dimensional hypersurfaces in the toric CY defined

[1] A toric manifold is a manifold of complex dimension r which admits an isometry group (at least as big as) $U(1)^r$. A toric CY threefold is then a CY threefold whose isometry group is at least $U(1)^3$. For a recent introduction, see e.g. [131].

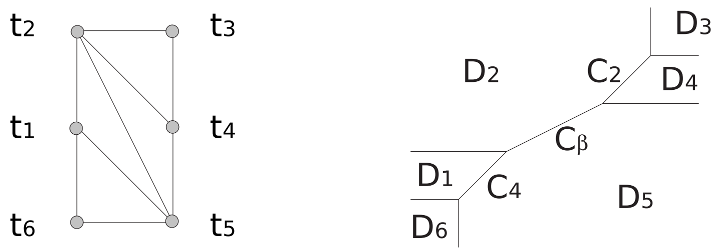


Figure D.1: The toric diagram and the dual (p, q)-web. The specific toric diagram triangulation is the one related to having all $\xi_a > 0$ in the associated GLSM.

by $D_i = \{t_i = 0\}$. We recognize $D_4 = \{z_1 = x = y = 0\}$ as the p line and $D_1 = \{z_2 = x = y = 0\}$ as the q-line.

The toric diagram and the related (p, q)-web corresponding to choosing all $\xi_a > 0$ (which amounts to a given triangulation of the toric diagram) are depicted in Figure D.1. For the particular resolution corresponding to $\xi_2, \xi_\beta, \xi_4 > 0$ the three holomorphic 2-cycles can be directly read from the (p, q)-web. They can be explicitly constructed as intersections of toric divisors

$$\mathcal{C}_2 = D_2 \cdot D_4 \qquad \mathcal{C}_\beta = D_2 \cdot D_5 \qquad \mathcal{C}_4 = D_1 \cdot D_5 \ . \tag{D.1.5}$$

This can be explicitly checked using D-term equations, which for the intersections of interest are

$$\begin{array}{llll} D_2 D_4 : & |t_3|^2 + |t_5|^2 = \xi_2 & |t_6|^2 = 2|t_1|^2 + \xi_4 & |t_1|^2 = |t_5|^2 + \xi_\beta \\ D_2 D_5 : & |t_4|^2 + |t_1|^2 = \xi_\beta & |t_3|^2 = 2|t_4|^2 + \xi_2 & |t_6|^2 = 2|t_1|^2 + \xi_4 \\ D_1 D_5 : & |t_2|^2 + |t_6|^2 = \xi_4 & |t_3|^2 = 2|t_4|^2 + \xi_2 & |t_4|^2 = |t_2|^2 + \xi_\beta \ . \end{array} \tag{D.1.6}$$

As one can see, each $\mathcal{C}_i$ topologically is a $\mathbb{P}^1$ (parameterized by the first two variables in each row) of volume ξ_i.

Let us consider also another basis of 2-cycles, which arises in a different resolution of the singular conical geometry (corresponding to a different triangulation of the toric diagram). Consider the region in the space of FI parameters where $\xi_\beta < 0$ with $\xi_2 + \xi_\beta > 0$ and $\xi_4 + \xi_\beta > 0$. We can introduce

$$\xi_1 = \xi_4 + \xi_\beta > 0 \qquad \xi_3 = \xi_2 + \xi_\beta > 0 \qquad \xi_\alpha = -\xi_\beta > 0 \ . \tag{D.1.7}$$

This new resolution can be obtained from the one in Figure D.1 with a flop transition on $\mathcal{C}_\beta \leftrightarrow \mathcal{C}_\alpha$. The toric diagram triangulation and the corresponding dual (p, q)-web for the new geometry are sketched in Figure D.2. In order to have a nice presentation of the GLSM charges in terms of the new positive FI's, we can linearly reshuffle Table (D.1.3) getting

t_1	t_2	t_3	t_4	t_5	t_6	
-1	0	0	1	-1	1	ξ_1
-1	1	0	-1	1	0	ξ_α
1	-1	1	-1	0	0	ξ_3

(D.1.8)

Repeating the same analysis as before one finds the holomorphic[2] 2-cycles in this new resolution

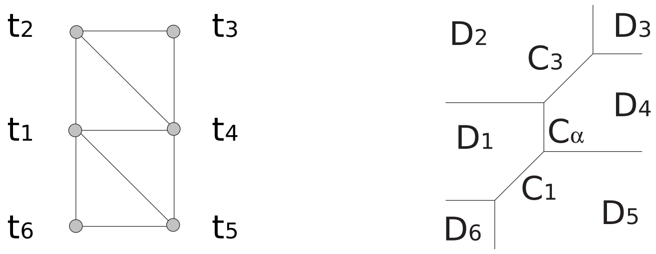


Figure D.2: The toric diagram and the dual (p, q)-web in the region of the FI parameter space where $\xi_\beta < 0$.

in terms of toric divisors

$$\mathcal{C}_3 = D_2 \cdot D_4 \qquad \mathcal{C}_\alpha = D_1 \cdot D_4 \qquad \mathcal{C}_1 = D_1 \cdot D_5 \ . \tag{D.1.9}$$

Again the FI parameters are the positive volumes of the corresponding 2-cycles $\mathcal{C}_i$. From the relations among FI parameters we read the relations

$$\mathcal{C}_1 = \mathcal{C}_4 + \mathcal{C}_\beta \qquad \mathcal{C}_3 = \mathcal{C}_2 + \mathcal{C}_\beta \ , \tag{D.1.10}$$

which can be thought of as relations in homology between vanishing cycles.

A comment is in order at this point. In this nonchiral case, vanishing 2-cycles are in one-to-one correspondence with possible fractional branes. All the divisors are non compact 4-cycles. This implies that all dual 2-cycles support nonanomalous fractional branes. This does not hold in general, as only 2-cycles dual to noncompact 4-cycles give anomaly-free fractional branes, their number being equal to the number of 3-cycles in the real base of the CY cone (which in turn corresponds to the number of baryonic charges). This is the geometric counterpart of the dual gauge theory being nonchiral. Conversely, chiral theories are related to CY cones where there are compact 4-cycles around. The latter put constraints on the allowed fractional D3 branes configurations, because of the RR tadpole cancellation condition.

Once we wrap a D5 brane on a 2-cycle, it will thus source a 3-form RR flux. We turn to consider the compact 3-cycles of the geometry which can support this flux, and their dual noncompact 3-cycles.

D.2 3-cycles and deformations

The study of compact and noncompact 3-cycles is best performed in a regular geometry obtained by complex deformation of the singular space, rather than by resolution (which is a Kähler deformation).

The algebraic variety (D.0.2) admits two normalizable complex deformations parameterized by ϵ_1 and ϵ_3 [125]

$$f = (z_1 z_2 - \epsilon_1)(z_1 z_2 - \epsilon_3) - xy = 0 \ . \tag{D.2.1}$$

[2]Notice that generically if a homology class $\mathcal{C}$ has a holomorphic representative, $-\mathcal{C}$ has not because the representative becomes antiholomorphic and one should look for a different one. In particular, in different resolutions the rôle of homology classes with a holomorphic representative is exchanged.

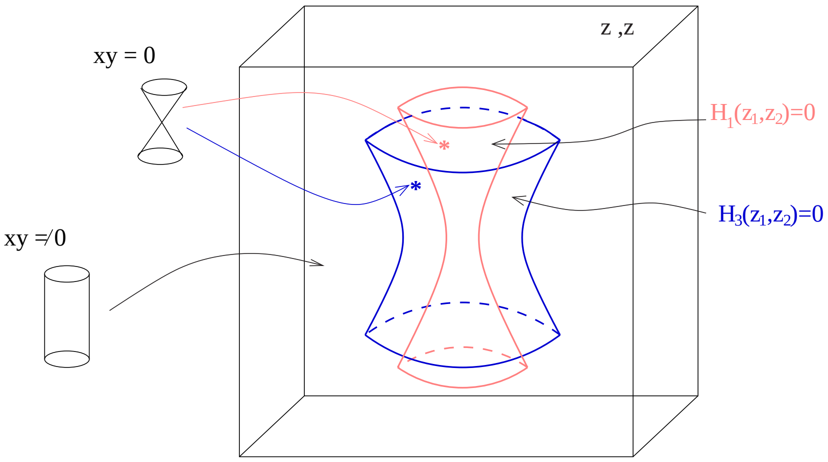


Figure D.3: The 6-dimensional manifold seen as a singular $\mathbb{C}^*$ fibration over the (z_1, z_2) space. The surfaces $H_k(z_1, z_2) = z_1 z_2 - \epsilon_k = 0$, $k = 1, 3$, are the loci where the $\mathbb{C}^*$ fiber degenerates to a cone $xy = 0$ and a nontrivial S^1 shrinks.

The deformed geometry is regular for $\epsilon_1 \neq \epsilon_3$, provided $\epsilon_1\epsilon_3 \neq 0$. For $\epsilon_1 = \epsilon_3 \neq 0$ it still has a $\mathbb{C}^*$ line of A_1 singularities (locally $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$) and corresponds to a $\mathbb{Z}_2$ orbifold of the deformed conifold. For $\epsilon_3 = 0$ it has a conifold singularity at the tip.

A convenient way to visualize the geometry is to regard (D.2.1) as a singular $\mathbb{C}^*$ fibration over $\mathbb{C}^2 \simeq (z_1, z_2)$

$$xy = H_1(z_1, z_2)\, H_3(z_1, z_2) \qquad \text{with} \qquad H_k(z_1, z_2) = z_1 z_2 - \epsilon_k \; . \tag{D.2.2}$$

At any point (z_1, z_2) where $H_1(z_1, z_2)H_3(z_1, z_2) \neq 0$ the fiber has equation $xy = c \neq 0$ and is a copy of $\mathbb{C}^*$. On each surface $H_k(z_1, z_2) = 0$ the fiber degenerates to a cone $xy = 0$ and an S^1 shrinks. On the other hand, each surface $H_k(z_1, z_2) = 0$ is an hyperboloid in $\mathbb{C}^2$ and has the topology of $\mathbb{C}^*$. For a general deformation, $\epsilon_1 \neq \epsilon_3$, they are disjoint and never touch. When $\epsilon_1 = \epsilon_3$ they degenerate one on top of the other, while when one deformation parameter vanishes the corresponding hyperboloid degenerates into a cone. See Figure D.3 for a picture of the geometry.

Figure D.3 is very useful to visualize compact and noncompact 3-cycles as well as 2-cycles in the deformed geometry. Any line segment of real dimension one in the $\mathbb{C}^2$ space (z_1, z_2) which begins and ends on the locus $xy = 0$ represents a closed submanifold of real dimension two, obtained by fibering on that segment an S^1 which lives in the $\mathbb{C}^*_{x,y}$ cylinder and shrinks to zero at the endpoints. When the line segment is noncontractible (keeping the endpoints on the $xy = 0$ locus), it represents a nontrivial element in the homology group $H_2(\mathcal{M}, \mathbb{Z})$. In the same way, a real dimension two surface with boundary on the $xy = 0$ locus gives rise to a closed dimension three submanifold after the S^1 has been fibered on it. When the surface is noncontractible (keeping the boundary on the $xy = 0$ locus), it gives rise to a nontrivial 3-cycle. Compact 3-cycles A_i arise from compact surfaces while noncompact 3-cycles B_i arise from noncompact surfaces.

In Figure D.4 we depicted the various 2-cycles $\mathcal{C}_i$ and compact 3-cycles A_i for the deformed orbifolded conifold. We have used the basis which is most natural when complex deformations are concerned. noncompact 3-cycles B_i are easily obtained as well: the real dimension two base

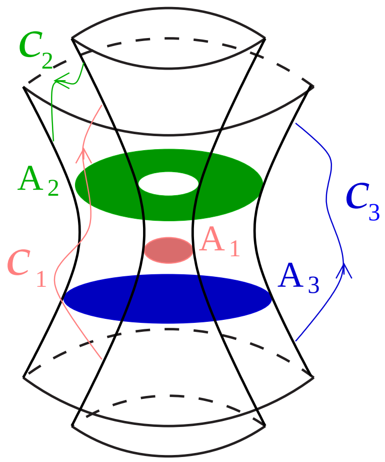


Figure D.4: The projection of the A and $\mathcal{C}$ cycles in the (x, y) space. The noncompact B-cycles are obtained as $\mathcal{C}$-cycles fibers over r.

surfaces are noncompact "vertical" foils with one or two boundaries on the degeneration loci, and are related to the line segment supporting the 2-cycles $\mathcal{C}_i$.

In the regular deformed geometry, a canonical symplectic basis for the third homology group $H_3(\mathcal{M}, \mathbb{Z})$ is given by $\{A_1, A_3, B_1, B_3\}$ with intersection numbers $A_i \cdot B_j = \delta_{ij}$. A_1 and A_3 have topology S^3 while B_1 and B_3 have topology $\mathbb{R}^3$. One can also consider a linear combination of them, $A_2 = A_1 - A_3$ (see Figure D.4) and its dual $B_2 = -B_1 + B_3$: they have intersection number $A_2 \cdot B_2 = -2$.

The asymptotic behavior of supergravity solutions based on these spaces is fixed, among other parameters, by the D5 charges at infinity. These are constructed by integrating suitable currents on the 3-cycles in radial sections of the asymptotically conical geometry. This is equivalent to considering any radial section in the singular conical geometry ($\epsilon_1 = \epsilon_3 = 0$). The latter perspective is useful because from any 3-cycle in a radial section we can construct a noncompact conical 4-cycle having the 3-cycle as its radial section: this allows us to introduce a concept of holomorphy and to use toric divisors instead of 3-cycles in radial sections.

From the GLSM description we know that the number of compact 3-cycles in radial sections (which equals the number of baryonic charges and the number of nonanomalous fractional branes) is three. For concreteness we choose the following basis: A_2, A_4 and A_{CF}. A_2 is the radial section of the toric divisor D_4, and corresponds to the product of the exceptional 2-cycle $\mathcal{C}_2$ along the p-line (which is $\cong \mathbb{C}^*$) with S^1 in the latter; in the same way, A_4 is the radial section of the toric divisor D_1, and is the product of the exceptional $\mathcal{C}_4$ along the q-line times S^1. A_{CF} is the compact 3-cycle of the covering space conifold[3]: under the orbifold action it has an image, and no fixed points. In particular, the representative 3-cycle at $\theta_2 = \pi/2$ and $\varphi_2 = 0$ is mapped to the divisor $\{x = z_1^2, y = z_2^2\}$ which has the GLSM description $t_1 t_2^2 = t_4 t_5^2$. Comparing the charges we find that A_{CF} corresponds to the toric divisor $D_1 + 2D_2 = D_4 + 2D5$. Summarizing, our basis of 3-cycles and the corresponding toric divisors are

$$A_2 \simeq D_4 \qquad A_4 \simeq D_1 \qquad A_{CF} \simeq D_1 + 2D_2 = D_4 + 2D_5 \ . \tag{D.2.3}$$

[3] Actually $A_{CF} = A_1 + A_3$.

Notice that in the deformed geometry $A_2 = -A_4$ in homology. Nevertheless they can give rise to different charges when explicit sources are present in the geometry and this is in fact the case of $\mathcal{N} = 2$ branes which do not undergo complete geometric transition.

In order to compute the 3-form fluxes generated by D5 branes wrapped on 2-cycles, we will need the intersection matrix between divisors and 2-cycles. In our basis we find

	$A_2 \simeq D_4$	$A_4 \simeq D_1$	A_{CF}
$\mathcal{C}_2$	-2	0	0
$\mathcal{C}_4$	0	-2	0
$\mathcal{C}_\beta$	1	1	-1

(D.2.4)

This table is computed from the charges in Table (D.1.3): in the GLSM construction each gauge field gives rise to an element $\mathcal{C}_a$ of the homology group $H_2(\mathcal{M}, \mathbb{Z})$, and the intersection between it and a toric divisor D_i is the charge ${Q_a}^i$.

D.3 The fractional branes/ranks correspondence

We have now all the ingredients to finally figure out the precise correspondence between fractional branes (that is wrapped D5 branes) and quiver rank assignments.

Consider a D5 brane wrapped on a 2-cycle $\mathcal{C}_i$ of our CY$_3$. The Bianchi identity for F_3 is violated by the source

$$dF_3 = -2\kappa^2 \tau_5 \, \Omega_4 \ , \tag{D.3.1}$$

where Ω_4 is a 4-form with δ-function support on the D5 world-volume, see (A.2.22). We are interested in the flux generated on a 3-cycle A_j in the radial section. First we have to resolve the geometry, switching on the FI parameters of the associated GLSM. This does not change the holomorphic data nor the quantized charges. Then we identify a noncompact divisor D_j which has A_j as radial section. Being the geometry smooth, A_j turns out to be (plus or minus) the boundary of D_j, therefore

$$\int_{A_j} F_3 = -\int_{D_j} dF_3 = 2\kappa^2 \tau_5 \int_{D_j} \Omega_4 = 2\kappa^2 \tau_5 \, (D_j \, , \, \mathcal{C}_i) \ , \tag{D.3.2}$$

where $(D_j \, , \, \mathcal{C}_i)$ is the intersection number as in Table (D.2.4), and we have fixed the orientation ambiguity requiring consistency with known cases, such as the conifold and the $\mathbb{Z}_2$ orbifold. If there is a holomorphic representative for $\mathcal{C}_i$, we can then directly compute the intersection from the GLSM data.

The last thing to determine are the quiver rank assignments corresponding to each fractional brane. A D5 brane wrapped on the exceptional 2-cycles $\mathcal{C}_2$ and $\mathcal{C}_4$ along the $\mathbb{C}^2/\mathbb{Z}_2$ lines p and q gives rise to an $\mathcal{N} = 2$ fractional brane, and we conventionally choose the rank assignments to be, respectively, $(0, 1, 1, 0)$ and $(1, 1, 0, 0)$. The rank assignment for a D5 brane wrapped on $\mathcal{C}_\beta$ can be defined by observing that the combination $\mathcal{C}_{CF} = 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ does not couple to twisted fields and gives rise to the orbifold of the Klebanov-Tseytlin theory [36], see Table (D.2.4). This implies that the corresponding gauge theory is the orbifold of the KT theory. We can say that the ranks for one D5 on $\mathcal{C}_\beta$ are (a, b, c, d). Requiring that $2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ is in the class $(N+1, N, N+1, N)$ or $(N, N+1, N, N+1)$, which do correspond to the orbifold of the KT theory, singles out two possibilities for $\mathcal{C}_\beta$: either $(1, 0, 1, 1)$ or $(0, 0, 0, 1)$. To select the correct option we should consider the induced D3-charge on the fractional D3 probe.

The induced D3-charge is proportional to the integral of B_2 (or more generally of $\mathcal{F} = B_2 + 2\pi\alpha' F_2$) on the corresponding 2-cycle $\mathcal{C}$:

$$Q_3 \; = \; \tau_5 \int_{\mathcal{C}} \mathcal{F} \; = \; \tau_3 \, \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} \left(B_2 + 2\pi\alpha' F\right) \; . \tag{D.3.3}$$

The actual value depends on the background value of B_2. This is arbitrary at this level (and it is related to the UV cut-off values of the gauge couplings in the dual gauge theory). We only require these background values to be positive (so as to describe mutually BPS objects) and less than one (in order to describe noncomposite, that is elementary, objects). Along the p and q lines the physics is locally $\mathbb{C}^2/\mathbb{Z}_2$, thus we can naturally set [132]: $\int_{\mathcal{C}_2} B_2 = \int_{\mathcal{C}_4} B_2 = (4\pi^2\alpha')/2$. If we consider the KT theory and set also [36] $\int_{\mathcal{C}_{CF}} B_2 = (4\pi^2\alpha')/2$, then using the previous relation $\mathcal{C}_{CF} = 2C_\beta + C_2 + C_4$, we get $\int_{C_\beta} B_2 = -(4\pi^2\alpha')/4$.

This implies that while the $\mathcal{N} = 2$ branes have positive D3-charge, a D5 brane wrapped on $\mathcal{C}_\beta$ has negative D3 brane charge and it is not mutually BPS. Putting one unit of worldvolume flux on the wrapped D5 we get positive D3-charge: $3/4$. The total D3-charge for $\mathcal{C}_{CF} = 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ (with two units of flux on $\mathcal{C}_\beta$) is $5/2$. This is exactly the D3-charge of the configuration $(3, 2, 3, 2)$, which implies that one D5 brane wrapped on $\mathcal{C}_\beta$ with one unit of worldvolume flux gives rise to the theory $(1, 0, 1, 1)$. A similar analysis shows that a D5 brane wrapped on $\mathcal{C}_\alpha = -\mathcal{C}_\beta$ (with no background world-volume flux) corresponds to a rank assignment $(0, 1, 0, 0)$. Finally, direct application of Table (D.2.4) tells us what the fluxes sourced by D5 branes wrapped on any 2-cycles are.

Our findings are summarized in the Table below

	$-\int_{A_2} F_3$	$-\int_{A_4} F_3$	$-\int_{A_{CF}} F_3$	D3-charge	gauge theory
D5 on $\mathcal{C}_2$	2	0	0	1/2	(0,1,1,0)
D5 on $\mathcal{C}_4$	0	2	0	1/2	(1,1,0,0)
D5 on $\mathcal{C}_\beta$	−1	−1	1	3/4	(1,0,1,1)
D5 on $\mathcal{C}_\alpha$	1	1	−1	1/4	(0,1,0,0)

(D.3.4)

where fluxes are in units of $4\pi^2\alpha' g_s$.

As anticipated, we will use D5 branes wrapped on $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_\alpha = -\mathcal{C}_\beta$ without worldvolume flux as a basis for fractional branes to discuss our gauge/gravity duality. This is the most natural basis for discussing rank assignments parametrized as in Figure 7.1, where fractional branes modify the ranks of the first three quiver nodes only.

Appendix E

Introduction to Seiberg-Witten curves from M theory

E.1 $\mathcal{N} = 2$ pure YM and SQCD theories

$\mathcal{N} = 2$ unitary YM theories can be realized in type IIA string theory by means of systems of D4 branes suspended between parallel NS5 branes.[1] Let us first recall, as an instructive example, the construction for the $SU(M)$ pure gauge theory, which may be used to derive (8.3.3).

Let us then consider a system of two parallel NS5 branes in ten-dimensional spacetime, spanning the $0, 1, 2, 3, 4, 5$ directions and separated along the 6 direction by a distance Δx^6. Between the NS5 branes, let M D4 branes be suspended in the 6 direction and span the $0, 1, 2, 3$ directions. The system preserves eight supercharges and at high energies hosts a 5-dimensional $U(M)$ Yang-Mills theory on the D4 branes worldvolume, which at low energies (compared to the inverse distance of the five-branes in string units) reduces by compactification to a 4-dimensional $\mathcal{N} = 2$ $U(M)$ SYM theory on the $0, 1, 2, 3$ directions of the D4 branes worldvolume. The distance between the NS5 branes Δx^6 in string units is proportional to the classical inverse squared gauge coupling of the 4-dimensional field theory, by compactification from 5 to 4 dimensions. At the classical level, each of the M suspended D4 branes is free to move along the $4, 5$ directions, provided it remains attached to the NS5 branes, matching the classical moduli space of the $U(M)$ gauge theory.

At the quantum level, the presence of M suspended D4 branes bends the NS5 branes embeddings logarithmically in the 6 direction as we move along the $4, 5$ plane, because brane tensions have to balance at the intersections. This phenomenon is easily seen in the M theory lift that we will discuss momentarily. The distance between the NS5 branes becomes a function of the coordinates along the $4, 5$ plane: this logarithmic asymptotic bending is interpreted as the logarithmic running of gauge coupling of the quantum theory, because that is what results from probing the RG flow by moving in the Coulomb moduli space. Special points in the $4, 5$ plane where the two NS5 branes touch or intersect correspond to energies in the dual field theory where the running gauge coupling diverges. It is also possible to see that the diagonal $U(1)$ subgroup freezes, otherwise the NS5 branes kinetic energy would diverge when the ends of D4 branes fluctuate. Finally, as in the field theory the inverse squared gauge coupling is accompanied by the theta angle, with which it forms

[1] A very detailed review of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ four-dimensional gauge theories realized on systems of branes in type IIA string theory can be found in [151], to which we refer the interested reader.

the complexified gauge coupling $\tau_{YM} = \frac{\theta}{2\pi} + i\frac{8\pi^2}{g^2}$, the field x^6 of each NS5 brane embedding has as a partner an axionic scalar field propagating on the five-brane. Each D4 brane ending on the NS5 brane forms a vortex (or an antivortex) for the field strength of this scalar.

This can be understood easily by lifting the type IIA brane configuration to M theory compactified on a circle, parameterized by a coordinate $x^{10} \sim x^{10} + 2\pi R$ [140]. The NS5 and D4 brane configuration in type IIA string theory maps to a single (but multi-covering) generically smooth M5 brane configuration in M theory,[2] and the NS5 brane worldvolume scalar becomes the x^{10} coordinate of the M5 brane embedding. The M5 brane worldvolume spans the Minkowski $0, 1, 2, 3$ directions, is localized at $x^7 = x^8 = x^9 = 0$, and finally spans a 2-dimensional complex Riemann surface in the remaining directions, in the complex structure in which $x^4 + ix^5$ and $x^6 + ix^{10}$ are holomorphic. This Riemann surface, which is a multi-cover of the (x^6, x^{10}) strip, is nothing but the auxiliary Riemann surface of the Seiberg-Witten theory. Finally, it is possible to take a scaling limit where the radius of the M theory circle goes to infinity, keeping proportions in the M5 brane embedding fixed. This limit does not affect relevant and marginal field theory quantities, but allows a semiclassical treatment of the M5 brane configuration. The abstract auxiliary Riemann surface associated to any point of the Coulomb branch of the $\mathcal{N} = 2$ gauge theory, describing the quantum dynamics of the theory at that point of the moduli space, has translated into the embedding of a physical M5 brane, which makes the visualization of gauge theory quantities easier, thanks to the semiclassical type IIA intuition. The M5 brane embedding is best described by introducing the following complex coordinates

$$t \equiv e^{-s} , \qquad s \equiv \frac{x^6 + ix^{10}}{R} . \tag{E.1.1}$$

Since there are 2 NS5 branes and M D4 branes, the embedding equation for the M5 brane is written as a polynomial equation which is quadratic in t and of degree M in v, the complex dimensionless coordinate along $4, 5$ which measures lengths in 11-dimensional Planck units. Moreover, the D4 branes are suspended between the NS5 branes, that therefore bend outwards so as to equilibrate their tension, and go to $t = 0, \infty$ only at $v = \infty$. Therefore there are no poles or zeros of $t(v)$ at finite v. The equation for the M5 brane embedding takes then the form

$$\begin{aligned} &t^2 + B_M(v)t + 1 = 0 , \\ &B_M(v) = \prod_{i=1}^{M}(v - v_i) . \end{aligned} \tag{E.1.2}$$

It can be rewritten as $\tilde{t}^2 = \frac{B(v)^2}{4} - 1$ in terms of $\tilde{t} = t + \frac{B(v)}{2}$; finally, we can reinstate dimensions by introducing $y = 2\Lambda_M^M \tilde{t}$, $x = \Lambda_M v$, $\phi_i = \Lambda_M v_i$, so as to obtain the Seiberg-Witten curve in its conventional field theory look (8.3.3).

We see that the roots of the Seiberg-Witten curve, as functions of v, get translated into the positions of the NS5 branes in the $6, 10$ strip, as functions of v, up to an exponential map. The running complexified gauge coupling of the gauge theory (2.3.1) at a scale v (in units of Λ_M) is readily extracted from the roots $t_{1,2}(v)$ as follows:

$$\tau_{YM}(v) = i\left(s_2(v) - s_1(v)\right) , \tag{E.1.3}$$

[2] We recall that an M5 brane wrapped on the M theory circle becomes a D4 brane in type IIA string theory, whereas an M5 brane which does not wrap the circle becomes a type IIA NS5 brane.

where $s_2(v) > s_1(v)$ and we have used (E.1.1).

Notice that branch points of the curve (as a double cover of the complex plane) are translated in the M theory pictures into special points v_* in the $4,5$ plane where the two roots $t_{1,2}(v)$ of the (E.1.2) or (E.1.4) embedding equation coincide: $t_1(v_*) = t_2(v_*)$. In the IIA picture this occurs when not only the two NS5 branes touch or intersect, but at the same time the worldvolume axions (related to the 10 direction in M theory) take on the two five-branes the same value, modulo the periodicity. In general, these are not singular points, since the curve is smooth. Singularities arise whenever at least two of these branch points collide.

Let us also remark that in the M theory lift of the type IIA brane configuration, each single D4 brane is thickened into an M5 brane and this results into two branch points in the v plane, which lie close to one another in a semiclassical regime but need not in a nonperturbative regime where the corresponding M5 brane is fat. Again, singularities occur when two of these branch points, in general coming from different semiclassical branes, collide.

Finally, it is worth mentioning a straightforward generalization to SQCD theories, that will be useful in the following: flavor hypermultiplets can be included in the type IIA picture by attaching semi-infinite D4 branes along the 6 direction, on the opposite side with respect the finite D4 branes (either on the left or on the right, or also on both sides), and spanning the $0,1,2,3$ directions as well. The position of the endpoints of these semi-infinite D4 branes are related to the mass parameters of the quark hypermultiplets in the gauge theory, which are provided by strings stretching form the semi-infinite to the finite D4 branes. From the M theory viewpoint, the embedding equation is simply obtained by promoting (E.1.2) to

$$A_{N_1}(v)t^2 + B_M(v)t + C_{N_2}(v) = 0 \,, \tag{E.1.4}$$

where $A_{N_1}(v)$ and $C_{N_2}(v)$ are degree N_1 and N_2 polynomials, and $N_1 + N_2 = N_f$ is the total number of flavors. By suitable redefinitions of variables, the embedding equation (E.1.4) can be rewritten in the standard form [141] for $SU(n_c)$ SQCD with n_f flavors:

$$y^2 = \prod_{a=1}^{n_c}(x-\phi_a)^2 - 4\Lambda^{2n_c-n_f}\prod_{i=1}^{n_f}(x+m_i) \,, \tag{E.1.5}$$

valid for $n_f \leq n_c - 2$, where here $n_c = M$ and $n_f = N_f$.

E.2 The Seiberg-Witten curve for the elliptic model

In [140], among other things Witten also found the Seiberg-Witten curve for the conformal $\mathcal{N}=2$ $SU(n)\times SU(n)\times U(1)$ quiver gauge theory arising on n regular D3 branes at the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold. Before introducing the curve, following the presentation of [129], let us mention the relation between D3 branes in type IIB string theory on the $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$ orbifold and the M5 brane configuration whose holomorphic embedding gives the desired hyperelliptic curve.

First of all, we can map the type IIB system to a type IIA string theory Hanany-Witten-like setup, by T-dualizing along the $U(1)$ symmetry $(x,y)\mapsto(e^{i\alpha}x, e^{-i\alpha}y)$ of the A_1 singularity $xy = z^2$ (the algebraic description of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold). If we call x^6 the compact T-duality direction, the singularity is mapped into two parallel NS5 branes, spanning the $0,1,2,3$ Minkowski directions and the $4,5$ directions, separated along the 6 direction (a circle of radius L) by a distance $2\pi bL$, proportional to the B_2 flux through the exceptional cycle. The two kinds of type IIB fractional D3 branes become D4 branes suspended between the two NS5 branes, on one side or the other: wrapped D5 branes become D4 branes extended along the $2\pi bL$ long arc of the circle, wrapped anti-D5 branes with flux become D4 branes extended on the other arc of the circle, of length $2\pi(1-b)L$. Again, these lengths are nothing but the gauge couplings of the two gauge groups. Each of the two kinds of suspended D4 branes has a classical moduli space matching the type IIB one. If there is an equal number of suspended D4 branes on both sides of each NS5 brane, the field theory is conformal in the UV and the NS5 branes are asymptotically parallel even at the quantum level.

The lift to M theory proceeds by adding the circle in the eleventh dimension parameterized by x^{10}, which joins x^6 in a complex variable $x^6 + ix^{10}$ parameterizing a two-torus E:

$$(x^6, x^{10}) \sim (x^6, x^{10} + 2\pi R) \sim (x^6 + 2\pi L, x^{10} - \theta R) \ , \tag{E.2.1}$$

whose complex structure τ turns out to be the same as the type IIB axio-dilaton in string frame. We will use the complex coordinate

$$u \equiv i\frac{x^6 + ix^{10}}{2\pi R} \ , \tag{E.2.2}$$

so that the two-torus E is defined by the identifications

$$\begin{aligned} &u \sim u + 1 \sim u + \tau \ , \\ &\tau = \frac{\theta}{2\pi} + i\frac{L}{R} \ . \end{aligned} \tag{E.2.3}$$

The Seiberg-Witten curve for the $SU(n) \times SU(n) \times U(1)$ quiver gauge theory[3] is obtained as an n-sheeted cover of the torus E, where the n sheets correspond to n suspended D4 branes between each ordered pair of NS5 branes. The 2-torus E can be defined by the following equation in $\mathbb{C}^2$ (parameterized by a and y):

$$y^2 = x(x-1)(x-\lambda) \ , \tag{E.2.4}$$

where $\lambda = -\theta_2^4(\tau)/\theta_4^4(\tau)$ in terms of the Jacobi theta functions defined in appendix E.3. The Seiberg-Witten curve is then defined by the equation $F(x, y, v) = 0$, where $F(x, y, v)$ is a polynomial of degree n in v:

$$F(x, y, v) = v^n + f_1(x, y)v^{n-1} + \cdots + f_n(x, y) \ . \tag{E.2.5}$$

$f_i(x, y)$ are meromorphic functions on the torus, with simple poles at the positions of NS5 branes: all the f_i's have simple poles at the same two points, and depend on two additional complex parameters each. All in all, the Seiberg-Witten curve (E.2.5) depends on the asymptotic positions of the NS5 branes (the poles of f_i) plus $2n$ parameters which are one mass parameter (related to how the v plane is fibered over E) and the $2n - 1$ moduli of the Coulomb branch. The f_i's cover the complex plane twice, with four double points. They can be chosen as follows,

$$f_i(x, y) = c_i + d_i \frac{y + y_B}{x - x_B} \ , \tag{E.2.6}$$

placing one pole at the point at infinity P_∞ and the other one at $P_B = (x_B, y_B)$. A straightforward manipulation brings the curve to the form

$$(R + S) + (R - S)\frac{y + y_B}{x - x_B} \ , \tag{E.2.7}$$

[3]The additional $U(1)$ which is not frozen is free and decoupled, therefore we will not bother much about it.

where

$$
\begin{aligned}
R(v) &= \prod_{a=1}^{n} (v - \phi_a) \\
S(v) &= \prod_{a=1}^{n} (v - \tilde{\phi}_a) \; ,
\end{aligned}
\tag{E.2.8}
$$

ϕ_a and $\tilde{\phi}_a$ being the eigenvalues of the VEV of the adjoint scalars of the two gauge groups. Notice that whenever the two adjoints have a common eigenvalue, this factorizes from the Seiberg-Witten curve. It corresponds to two suspended D4 branes reconnecting into a complete D4 brane, that can move off the NS5 branes. It is the T-dual of a regular D3 brane in type IIB string theory. Instead noncommon eigenvalues force the NS5 branes to bend.

It proves useful to map the problem to the u parallelogram (E.2.3). Choosing, with no loss of information, the 'orbifold point' asymptotic values $c^{(0)} = 0$ and $b^{(0)} = \frac{1}{2}$, namely $\gamma^{(0)} = \frac{\tau}{2}$, for the type IIB twisted fields, and fixing the asymptotic positions of the NS5 branes at $u = \frac{1}{4}\tau$ and $u = \frac{3}{4}\tau$, the meromorphic form $f = \frac{y+y_B}{x-x_B}$ can be rewritten as

$$
f = \frac{\theta_3(u|\tau/2)}{\theta_4(u|\tau/2)} = \frac{\theta_3(2u|2\tau) + \theta_2(2u|2\tau)}{\theta_3(2u|2\tau) - \theta_2(2u|2\tau)}
\tag{E.2.9}
$$

in terms of the elliptic θ-functions defined in appendix E.3. The Seiberg-Witten curve (E.2.7) is then rewritten as

$$
R(v)\, \theta_3(2u|2\tau) = S(v)\, \theta_2(2u|2\tau) \; .
\tag{E.2.10}
$$

It is an infinite series in $t = e^{2\pi i u}$. It is also common to introduce $q = e^{2\pi i \tau}$, whose modulus is related to the type IIB dilaton by $|q| = e^{-\frac{2\pi}{g_s}}$. We find it useful to introduce the meromorphic function

$$
g(u|\tau) \equiv \frac{\theta_2(2u|2\tau)}{\theta_3(2u|2\tau)} \; ,
\tag{E.2.11}
$$

which obeys $g(u + \frac{1}{2}|\tau) = -g(u)$ and $g(u + \frac{\tau}{2}|\tau) = 1/g(u|\tau)$. It is related to f (E.2.9) by

$$
f = \frac{1+g}{1-g} \qquad\qquad g = \frac{f-1}{f+1} \; .
\tag{E.2.12}
$$

The Seiberg-Witten curve becomes then

$$
\frac{R(v)}{S(v)} = g(u|\tau) \; .
\tag{E.2.13}
$$

$g(u|\tau)$ has four double points in the u parallelogram at the points $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}\}$, which have the physical interpretation of points where the two NS5 branes touch, like the branch points discussed in the previous section.

The complexified gauge couplings $\tau_{1,2}$ of the two gauge groups, defined as in (2.3.1), in the type IIB picture are given by the formulae

$$
\begin{aligned}
\tau_1 &= b\tau + c = \gamma \\
\tau_2 &= (1-b)\tau - c = \tau - \gamma \; ,
\end{aligned}
\tag{E.2.14}
$$

where again the axio-dilaton is evaluated in string frame, and in the M theory picture are translated into

$$\begin{aligned}\tau_1 &= u_2(v) - u_1(v)\\ \tau_2 &= u_1(v) + \tau - u_2(v)\ ,\end{aligned} \tag{E.2.15}$$

assuming that $\mathrm{Im}(u_2) > \mathrm{Im}(u_2)$.

E.3 Elliptic θ-functions

In this appendix we collect our conventions for Jacobi's elliptic θ-functions. They are quasi-modular functions on a two-torus E with complex structure τ. We parameterize the torus as in the previous section by means of a complex coordinate u with the identifications $u \sim u+1 \sim u+\tau$. The four θ-functions are defined as follows:

$$\begin{aligned}\theta_1(u|\tau) &= i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau\left(n-\frac{1}{2}\right)^2} e^{i2\pi u\left(n-\frac{1}{2}\right)}\\ \theta_2(u|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau\left(n-\frac{1}{2}\right)^2} e^{i2\pi u\left(n-\frac{1}{2}\right)}\\ \theta_3(u|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{i2\pi u n}\\ \theta_4(u|\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau n^2} e^{i2\pi u n}\ .\end{aligned} \tag{E.3.1}$$

They are semi-periodic in the $\mathbb{Z}+\tau\mathbb{Z}$ lattice:

	$\theta_i(u+1\|\tau)/\theta_i(u\|\tau)$	$\theta_i(u+\tau\|\tau)/\theta_i(u\|\tau)$
θ_1	-1	$-N$
θ_2	-1	N
θ_3	1	N
θ_4	1	$-N$

(E.3.2)

where $N = e^{-i\pi\tau} e^{-i2\pi u}$. They have no poles, and have zeros at:

	zeroes
$\theta_1(u\|\tau)$	$\mathbb{Z}+\tau\mathbb{Z}$
$\theta_2(u\|\tau)$	$\left(\mathbb{Z}+\frac{1}{2}\right)+\tau\mathbb{Z}$
$\theta_3(u\|\tau)$	$\left(\mathbb{Z}+\frac{1}{2}\right)+\tau\left(\mathbb{Z}+\frac{1}{2}\right)$
$\theta_4(u\|\tau)$	$\mathbb{Z}+\tau\left(\mathbb{Z}+\frac{1}{2}\right)$

(E.3.3)

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