Quantisation of Gauge Systems:

Application to Minisuperspace Models

in Canonical Quantum Gravity

Thesis Submitted for the Degree of

"Doctor Philosophiae"

Candidate: Marco Cavaglià.

Supervisors: Prof. Vittorio de Alfaro.
Prof. Dennis W. Sciama.

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SISSA–ISAS

INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

ASTROPHYSICS SECTOR

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"We had the sky, up there, all speckled with stars, and we used to lay on our backs and look up at them, and discuss whether they was made, or only just happened..."

Mark Twain, "Huckleberry Finn"

Preface

I started working on the subject of this PhD thesis in the spring of 1992. At that time, I was studying for my finals at the University of Torino. The final examination of the Physics course represents a "symmetry breaking" in the academic life of a student. Indeed, the candidate must choose a well-defined area of research where probably to spend a great deal of his academic career. I chose to work on quantum gravity. The title of my thesis was "Anisotropic Electromagnetic Wormholes". Prof. Vittorio de Alfaro played a very important role in this choice.

When I was accepted at SISSA in the astrophysics group, I planned to continue my work in quantum gravity, depending on the interests of the SISSA staff. My tutor and supervisor at SISSA, Prof. Dennis William Sciama, agreed to that. Of course I was aware that I belonged to the astrophysics group. Hence, I decided to work on what Isham call the "phenomenological theory of quantum gravity". It is well-known that the energy scale of the ultimate theory of quantum gravity is 10^{19} GeV. This energy scale is so far beyond the range of any experiment based on human devices that perhaps there is no need for a quantum theory of gravity at all. No experimental tests can be planned in order to obtain hints for the quantisation of the gravitational field. However, two kind of "astrophysics systems" may provide important interesting areas for the study of quantum gravity: black holes and the post big-bang era of the universe. If the situation concerning experimental predictions from quantum gravity is not completely hopeless, it is there that we should look for suggestions about the ultimate theory of everything.

Hence, I decided to investigate the early universe and quantum black holes from the point of view of canonical quantum gravity, as the physicists of the beginning of the 20th century investigated the hydrogen atom during the first stages of the development of quantum mechanics. Indeed, we can consider the Friedmann-Robertson-Walker homogeneous isotropic universe, or the spherically symmetric Schwarzschild black hole, as the analogue in quantum gravity of the hydrogen atom in quantum mechanics. So in this thesis I shall not deal with the ultimate theory of quantum gravity, superstrings, or topological gravity, or something else. I shall discuss quantum gravity from a phenomenological level. For instance, I shall investigate the quantum theory relevant to the initial stages of the universe when an underlying field structure (superstrings, or any sort of quantum field treatment) is not yet predominant. Even though the scientific literature had plenty of articles dealing with quantum cosmological models and quantum black holes when I started out my research, many problems were unsolved or not fully discussed. In this thesis I complete the exact quantisation for all models considered, essentially the Friedmann-Robertson-Walker, de Sitter, and Kantowski-Sachs Universes filled with several kinds of matter, the Schwarzschild and Reissner-Nordström black holes, and wormholes generated by the electromagnetic field. This programme is performed in the usual Einstein gravity framework, as well in the string framework. For these models I am able to solve all conceptual problems connected to the search for a coherent theory of quantum gravity (probability, unitarity, interpretation, problem of time...) and complete the quantisation procedure. This provides a solid background that can be used for further investigations and represents the main result of this thesis. For instance, if the anisotropies in the microwave background radiation originate from quantum fluctuations of the wave function of the universe, as Halliwell and Hawking have suggested, only the complete quantisation of the Robertson-Walker universe, together with the definition of the Hilbert space of states, of the observables, of the operator ordering, of the time evolution, and of the equivalence between different gauges can allow a meaningful discussion of this claim.

The outline of the thesis is the following. In the next two chapters I shall present briefly the historical background from which my research developed, its motivations, and the mathematical techniques that I used. The first chapter is an overview of the different approaches to quantum gravity. The second chapter deals with the theory and the mathematics of constrained gauge systems with a finite number of degrees of freedom; in particular attention is focused on canonical quantum gravity and minisuperspace models, i.e. gravitational systems with a finite number of degrees of freedom. I shall illustrate the importance that minisuperspace models have as simplified models in the context of the attempts to construct a mathematically coherent, self-consistent quantum theory of the gravitational field, as well as their importance as models describing intriguing physical processes (wormholes, birth of the universe, black holes...) at (Planck) energy scales, where the classical theory breaks down. In this framework the major part of my research has been developed. Of course, the content of these chapters is quite general and not exhaustive. The only aim is to give the reader a discursive overview of the argument and the minimum background to understand the following material. Throughout, references to more complete and technical reviews are given.

The following chapters contain the original part of my work. Chapt. 3 is devoted to minisuperspace models in Euclidean Gravity, i.e. wormholes. In order to illustrate wormholes, I present and discuss in depth the solutions generated by the coupling of gravity to the electromagnetic field for Kantowski-Sachs and Bianchi I models. In chapt. 4 I deal with cosmological minisuperspace models discussed and analysed in my research articles. In particular I focus attention on the Friedmann-Robertson-Walker universe filled with radiation and matter fields and quantum string cosmology. In chapt. 5 I discuss the Hamiltonian formalism for Schwarzschild and Reissner-Nordström black holes and their quantisation.

For all these models I complete the quantisation along the lines discussed in chapt. 2. This is performed both by the reduced and Dirac methods and I prove that the two quantisation procedures lead to identical results. I obtain the Hilbert space of states for these models, the prerequisite for solving some of the deepest issues in quantum gravity, for instance the entropy of black holes and the evolution of the pre-inflationary universe. As stressed above these models can be interpreted as useful reduced models for understanding conceptual issues, for instance the problem of time or the equivalence between different gauges, as well as self-consistent models representing physical processes at small scales due to the quantum nature of the gravitational field. In the last chapter I present my conclusion.

I would like to conclude this short introduction with the acknowledgements to all my colleagues and friends who assisted me during these years. First of all I would like to mention Vittorio de Alfaro. During these years he has become a good friend of mine and has given me an important and indispensable human and scientific support in a difficult and complicated field like quantum gravity; I know that without him my work would not have been possible. I dedicate this thesis to him. I am indebted to my tutor and supervisor Dennis Sciama. He agreed that I continued my activity in quantum gravity at SISSA, even though this argument was a little outside the customary research in the astrophysics group. His suggestions and support allowed me to spend three wonderful years at SISSA. I would like to thank my colleagues who have collaborated with me during my research: Sasha Filippov, who taught me the techniques of constrained systems, a fundamental step in my research; Fernando de Felice, who taught me General Relativity and made important contributions to my publications; Mariano Cadoni, who taught me string theory and allowed me to generalise my results to this theory. I am glad to thank all my colleagues who helped me and gave me hints, even though we did not publish, in particular Orfeu Bertolami for valuable discussions when he was in Torino; all researchers and students of the astrophysics group at SISSA; the members of the organising committee of the two meetings on quantum gravity that I helped to organise; the chairmen of the XI Italian Meeting of General Relativity and Gravitational Theory, and in particular Mauro Carfora and Pietro Fré, who gave me the opportunity of being the editor of the proceedings. I thank Vittorio de Alfaro, Dennis Sciama, Jeanette Nelson, Hugo Morales-Técotl, and Mauro Carfora for important suggestions and criticisms about the contents of this thesis and Marisa Seren who drew the illustrations. Finally, I express my gratitude to my parents and to Marisa, who always assisted me in hard times, and to all my colleagues and friends who have taught me something or, simply, made my life pleasant.

Marco Cavaglià

Trieste, August 16th, 1996.

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1

Quantum Gravity: an Overview

In this chapter I shall give a short overview of the present status of research in quantum gravity. In the first section I shall illustrate why both elementary particle physicists and general relativists are interested in the quantisation of the gravitational field. In the following section I shall briefly discuss the four presently most popular approaches used to build a coherent theory of quantum gravity and what a future theory of quantum gravity might look like. Finally, in the last section a brief history of the research in the subject is sketched.

The content of the current chapter is nor a complete nor an exhaustive review of the present research in quantum gravity. For a full discussion of the subject the reader is referred to the excellent recent reviews of Isham (1992, 1993a, 1995), Kuchař (1992, 1993), Alvarez (1989) and Ashtekar (1991). Other excellent introductions to the argument can also be found in Esposito (1992), Williams and Tuckey (1992), and Ambjorn (1994).

1.1 Why Quantise Gravity?.

A great number of elementary particle physicists, field theory physicists, and general relativists are currently involved in the search for a quantum theory of the gravitational field. What are their motivations?

It is well-known that no physical tests nor observations are presently suggesting that the gravitational field must be quantised. The reason for the lack of experimental data is due to the energy scale at which quantum effects of gravity become non-negligible (around 10^{19} GeV). This value is far beyond any possibility of laboratory tests (see for instance Misner, Thorne, and Wheeler, 1973): presently there are no laboratory-based experiments whose data could be inequivocally interpreted as due to quantum effects of the gravitational field. However, even though the situation concerning experimental predictions from quantum gravity seems today to be completely hopeless, there are physical "observable" systems for which the classical Einstein theory appears clearly to be incomplete. The birth of the universe and the black hole evolution are the most remarkable examples. They may provide potential "non laboratory-based experiments" whose effects can be compared, at least in principle, to the predictions of a quantum theory of the gravitational field. The investigation of these systems allows to discuss quantum gravity from a "phenomenological level" (Isham, 1995). In this framework we can try and solve some of the conceptual problems connected to the formulation of a coherent theory of the quantum gravitational field as well as investigate possible quantum effects for suitable gravitational entities. In order to discuss the motivations for studying a quantum theory of gravity, a good starting point is thus to look for the physics at the boundary of the theory of general relativity, namely singularities, birth of the universe, and black hole evolution.

a) Singularities.

The most important disease of the Einstein theory of gravitation seems to be the presence of singularities. It is well-known that already one of the simplest exact solutions of the Einstein theory, the spherically symmetric vacuum solution, is affected by the presence of a singularity. Is this singularity due to the classical nature of general relativity? Let us discuss this point drawing the analogy between the Schwarzschild black hole and the hydrogen atom. In a range of energy appropriate, both of them can be seen as "fundamental" entities. In classical mechanics the electromagnetic potential generated by a single charge at rest with respect to a given reference frame with spherical coordinates (r, θ, φ) has the spherically symmetric form

$$V(r) = \frac{q}{r}, \tag{1.1.1}$$

where q is the charge. Obviously the potential is singular at $r = 0$. As a consequence, the solutions of the equations of motion are singular at $r = 0$. In quantum mechanics the picture changes completely because the quantum theory "smears" the singularity at $r = 0$. Thus the s-wave of a electron in a hydrogen atom is well-behaved at $r = 0$. The analogy to the Schwarzschild black hole is straightforward. We will see in chapt. 5 that the quantum canonical treatment of the Schwarzschild black hole removes the singularity of classical solutions; matrix elements describing quantum black holes are well-behaved also at the points where classical solutions are singular.

b) Birth of the Universe.

A similar analogy can be drawn for cosmological solutions. Since the universe is on large scales homogeneous and isotropic, and its deviations from homogeneity and isotropy are believed to have grown during its termal evolution, it can be described with good approximation by a model with a finite small number of gravitational and matter degrees of freedom, at least during its initial stages

(Misner, 1969, 1972). Hence, if we assume that the Friedmann-Robertson-Walker metric gives an appropriate description of the primordial universe, we meet in cosmology the same problem just found for the Schwarzschild black hole: the presence of a singularity. This signals that the birth of the universe must be investigated in the framework of a quantum theory of gravity. Finally, why the spacetime is four-dimensional and what is the origin of the inflationary era, are questions strictly related to the birth of the universe that perhaps will be solved only in the framework of a self-consistent and well-defined theory of quantum gravity.

c) Black Holes Evolution.

The final state of black holes is another important physical system in which the effects of a quantum theory of gravity may be relevant. The thermal radiation of black holes is one of the most striking discoveries in gravitation in the last twenty years (Hawking, 1974, 1975 – see later). Classically a black hole is stable and does not radiate. The situation changes radically when the interaction with a matter field is taken into account: black holes radiate. This phenomenon can be imagined as the production of particle-antiparticle pairs by the gravitational field near the horizon via the polarisation of the vacuum. This process seems to last up to the complete evaporation of the black hole and raises interesting questions related to the non-unitarity of the process, to the loss of coherence (see for instance Hawking, 1987; Coleman, 1988a), and to black hole remnants. The end of the black hole evolution remains an unsolved puzzle whose solution can be found only going beyond the classical treatment of the gravitational field (see the discussion in chapt. 5). The main conclusion following from the Hawking result is that the gravitational field must be connected to thermodynamical quantities, as the entropy, the link being possibly given by the quantum theory of gravity.

Isham defines the motivations presented above "motivations from the perspective of a general relativist" (Isham, 1995) since the existence of singularities, the birth of the universe, the end state of a gravitationally collapsing matter are typical unsolved puzzles in the framework of Einstein gravity. The formulation of a self-consistent theory able to explain these points is the main goal of researchers in quantum gravity with a "classical" background coming from general relativity.

There are of course different motivations for studying quantum gravity, coming from a different "classical" background. Isham call them "motivations from the perspective of an elementary particle physicist". From the point of view of elementary particle physics and quantum field theory the main reasons to formulate a quantum theory of gravity are essentially two: the non-renormalisability of the Einstein theory and the desire for the unification of all the fundamental forces.

d) Non-renormalisability of Einstein Gravity.

The non-renormalisability of the perturbative theory derived from Einstein general relativity (see Goroff and Sagnotti, 1985; 1986) strongly motivates different approaches to a quantum theory of the gravitational field. Indeed, even though quantum gravity is negligible at the energy scales of particle physics, the non-renormalisability of the perturbative theory prevents the calculation of possible quantum corrections (Isham, 1993b). Further, a theory in which the gravitational field is not quantised cannot be obviously physically self-consistent since elementary particles interact with the gravitational field and their nature is quantised.

e) Theory of Everything.

The main goal of a great number of theorists is the unification of all fundamental forces in a unique theory, the "theory of everything" (see for instance Albrecht, 1994). The history of the modern physics is traced by attempts, sometime successful, towards the completion of this programme. Presently, the weak and electromagnetic forces have been unified in a unique theory, the Glashow-Weinberg-Salam electroweak model, and there are strong hints that favour a great unification with the strong nuclear force, to be expected around 10^{16} GeV, where the running coupling constants of the electroweak and strong fields seem to coincide.[1] It is then natural to conjecture that the ultimate theory of quantum gravity must include all fundamental forces in a unique framework. This idea can be carried out starting from two complementary points of view (Isham, 1995):

i) general relativity is an unavoidable ingredient of a self-consistent, mathematically coherent theory of everything;

ii) the ultimate theory of quantum gravity will include by itself the other fundamental forces.

Obviously, these two points of view come from different approaches to the quantum gravity theory: the first one is preferred from people who start from the classical theory of Einstein and then apply to it some quantisation algorithm; the second one is preferred by people who try to build quantum gravity starting *ab initio* with a new theory.

These are the main motivations for studying quantum gravity. Let us see now what are the most promising canditates in building the ultimate theory of the quantum gravitational field.

1.2 Candidates for Quantum Gravity Theories.

Let us discuss briefly the current research programmes in quantum gravity. The candidates for the ultimate theory of the gravitational field can be roughly divided in two main alternative categories:

[1] Supersymmetry favours it (see for instance Ellis, 1986).

i) theories in which a given quantisation procedure is applied to the classical theory of gravitation (Einstein general relativity or slight modification of it);

ii) theories which start *ab initio* with a new quantum theory; classical gravity is recovered through a low-energy limit or a semiclassical reduction.

Canonical quantisation of gravity (see Isham, 1992), Euclidean quantum gravity (see Esposito, 1992), the Ashtekar non-perturbative programme (Ashtekar, 1991) belong to the first class; supergravity (see van Nieuwenhuizen, 1981), superstrings (Green and Schwarz, 1984), topological quantum gravity (Crane, 1993, 1995; Atiyah, 1995), non-commutative geometry (Connes, 1994, 1995), simplicial quantum gravity (Williams and Tuckey, 1992; Williams, 1995), dynamical triangulations (Ambjorn et al., 1995) belong to the second one.

Following Isham (1995) the subdivision in different categories may be made more precise. We have four distinctive approaches:

a) theories in which people try to quantise general relativity by quantising the metric tensor $g_{\mu\nu}$ defined on a four-dimensional Riemannian or hyperbolic manifold (spacetime approach): Euclidean quantum gravity, path-integral approaches;

b) theories in which the operator fields are defined on a three-surface and the canonical analysis is applied: canonical quantum gravity, Ashtekar formalism.

The two categories above are called by Isham "quantise general relativity". This is the point of view that I will adopt in the original part of the thesis. Note that in this context the dimensionally reduced theories of gravity play a very important role as simplified models useful to clarify conceptual problems arising in the quantisation procedure (for a review on lower-dimensional gravity see for instance Moncrief, 1990; Carlip, 1995).

c) Quantisation of matter fields in a fixed background spacetime endowed with a metric: quantum mechanics on curved spacetimes, semiclassical quantum gravity.

This approach is called by Isham "general-relativise quantum theory";

d) general relativity as low-limit of a more complete quantum field theory: strings, superstrings;

e) new theories that do not include *a priori* the notion of spacetime manifold (spacetime continuum): topological quantum gravity, simplicial quantum gravity, dynamical triangulations.

Canonical and Euclidean approaches to quantum gravity will be briefly discussed in sect. [2.5] and sect. [3.1] respectively. For a complete discussion of these and other approaches the reader is referred to Isham (1992) and Esposito (1992).

What do we expect from the quantisation of the gravitational field? Following Isham (1995) we can divide the new possible features of the ultimate theory of quantum gravity in two main categories:

i) the properties at basic level;

ii) the "phenomenological" properties.

The difference underlying this classification is essentially based on the range of energy at which we want to observe the quantum effects of the theory. The fundamental properties at basic level concern mainly the gravitational field at energies beyond the Planck energy.[2] In this regime we expect that the ultimate description of the quantised gravitational field may involve mathematical tools and concepts very different from those of the usual quantum field theory. For instance, it is commonly believed that beyond the Planck energy, the use of the continuum and/or the concept of spacetime point is no longer appropriate. Further, the usual (Copenhagen) interpretation of quantum mechanics should probably be abandoned (see for instance Hartle, 1995). At energy scales just below the Planck energy, where the effects mentioned above are not yet predominant, "phenomenological" effects of quantum gravity may take place. In this range of energy a quantum theory of gravity might use mathematical tools and concepts more similar to the usual quantum field theory or quantum mechanics. For instance, the notions of continuum and spacetime manifold should emerge from the basic structure of the theory in its semiclassical limit. The formalism of the theory might then use the standard quantum mechanics based on Hilbert spaces and the usual Copenhagen interpretation. In this framework one may obtain a correct "phenomenological" description of well-defined "quantum gravity entities", for instance quantum black holes, or the primordial universe. For this sort of quantum gravitational systems, what Isham calls the "phenomenological quantum gravity" emerges from the underlying theory (Isham, 1995). In this framework I have developed the major part of my research work.

1.3 A Short History of the Research in Quantum Gravity.

What is the origin of the attempts to formulate a quantum treatment of the gravitational field? What have been the most important discoveries? To conclude this short review, let us summarize briefly the research in quantum gravity over the last thirty-five years.

[2] See for instance Morales-Técotl (1994).

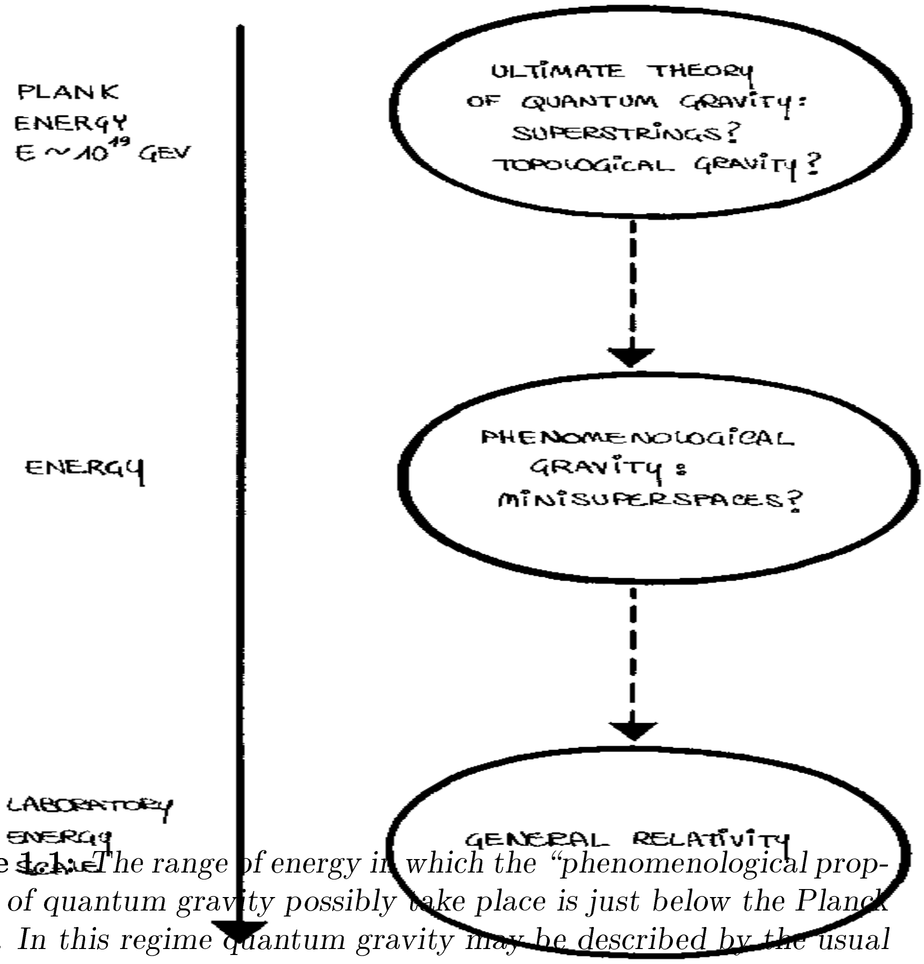

Figure 1.1: *The range of energy in which the "phenomenological properties" of quantum gravity possibly take place is just below the Planck energy. In this regime quantum gravity may be described by the usual formalism of quantum mechanics. By a process of "coarse graining" the appropriate description of the gravitational field moves from the ultimate theory of quantum gravity to general relativity.*

1960-1970's: The Origin of Quantum Gravity Studies.

The origin of quantum gravity studies dates back to the pioneristic works of Dirac and Bergmann in the late 50's (Dirac, 1958; Bergmann and Goldberg, 1955). Indeed, it was Dirac who first investigated the canonical analysis of classical general relativity in order to apply to the gravitational field the theory of quantisation of constrained systems (see Dirac, 1964) he was developing (quantisation *à la* Dirac – see sect. [2.3]). This line of research led to the Hamiltonian formulation of the Einstein theory by Arnowitt, Deser, and Misner (1962) in the early sixties and to the application of Dirac quantisation to the ADM formalism by Wheeler (1968) and DeWitt (1967) (Wheeler-DeWitt equation, see sect. [2.5]). The first attempt in quantising gravity moves thus from the canonical formalism. The first works on reduced models (*minisuperspace models*, *quantum cosmology* – see [2.5] and chapt. 4) date back to the end of sixties (DeWitt, 1967; Misner, 1969; 1972). The conceptual problems arising in the quantum canonical formulation of gravity *à la* Dirac, stimulated the discussion about the correct interpretation of quantum mechanics, definition of time, etc. The first studies on renormalisability of the gravitational field start in this period.

More or less at the time when the canonical quantum gravity is developing, Regge (1961) starts to investigate what has come to be called the simplicial quantum gravity (Regge calculus). The first application of Regge calculus in quantum gravity dates back in 1968 to Ponzano and Regge (1968). The twistor approach is introduced by Penrose (1967).

1970-1980's: The Middle Ages of Quantum Gravity.

In early 70's the definitive proof that the Einstein theory is not renormalisable (see Isham, Penrose, and Sciama, 1975; 1981) is given. In this period all standard methods of quantum field theory seem to fail when applied to gravity. However, the proof of the non-renormalisability of general relativity is a great achievement in the research in quantum gravity because it stimulates work on non-perturbative methods of quantisation. More or less in the same time, Hawking shows that black holes radiate (Hawking, 1974; 1975). This result has a natural precursor in the formulation of the area theorem (Hawking, 1971).[3] In a few years it is realised that the Hawking discovery allows to relate the black hole parameters to thermodynamical quantities as the entropy (Bekenstein, 1973; 1974a). The results on the black hole thermodynamics are then summarised in the *laws of the black hole mechanics* analogous to the laws of thermodynamics (Bardeen, Carter, and Hawking, 1973; Bekenstein, 1974a). Finally, the exact connection between thermodynamics and mechanics of black holes is established by Hawking (1974). The work of Hawking and Bekenstein leads to a great interest in the quantum field theory in curved spacetime. The so-called *Euclidean quantum gravity* (see sect. [3.1]) begins to develop. In this context, around the mid 70's, supergravity was born (for a review, see for instance van Nieuwenhuizen, 1981).

1980-1990's: The Renaissance.

In parallel to the research in black holes, a great deal of activity is devoted to the applications of Regge calculus to quantum gravity. In early eighties we find the first works on four-dimensional quantum Regge calculus by Williams (1986a,b) and her collaborators (Roček and Williams, 1981; 1982; 1984). In 1985 Hartle proposes and develops the concept of *simplicial minisuperspace* in quantum cosmology (Hartle, 1985). Physicists realise that topological quantum field theories (Witten, 1988; Atiyah, 1989) may be relevant for quantum gravity. Studies on dynamical triangulations and conformal theories begin to develop (see for instance Ambjorn, 1994) and the Regge calculus becomes the basis for numerical quantum gravity.

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[3] According to this theorem, the area of a black hole always increases.

The development of the superstring theory dates back to mid 80's (Green and Schwarz, 1984) and becomes one of the most promising candidates for the ultimate theory of quantum gravity. Superstrings incorporate the old hope that quantum gravity is unified with the other fundamental interactions. More or less in the same time Ashtekar finds a set of new canonical variables that simplify the structure of the constraints (Ashetkar, 1986). This development produces a new intriguing programme in the non-perturbative approach to quantum gravity leading to the interesting issue of the use of gauge-invariant loop variables (Rovelli and Smolin, 1990). The research in low-dimensional quantum gravity is also developing in this period (for a review on 2+1 gravity, see for instance Carlip, 1995).

1990's-today: The Modern Epoch.

All the lines of research that have been developed in the last thirty-five years are currently active. Presently, a great amount of activity is devoted to the Ashtekar programme and loop representation of quantum gravity (see Ashtekar, 1994; Rovelli and Smolin, 1990) also in connection with black hole physics (Barreira, Carfora, and Rovelli, 1996; Rovelli, 1996), superstrings (see for instance Amati, 1995), lower-dimensional models (see Carlip, 1995; Nelson and Regge, 1994), topological gravity (see Crane, 1993; 1995), simplicial quantum gravity and dynamical triangulations (see Williams, 1995; David, 1992; Ambjorn et al., 1995), Euclidean quantum gravity (see Esposito, 1992) and conceptual problems of quantisation (*Copenhagen interpretation and decoherence*, see Hartle, 1995; *semiclassical approximation*, see Embacher, 1996; *problem of time*, see Isham 1992, 1993a; Kuchař 1992; Barbour, 1994a, 1994b; *structural issues*, see Isham 1993b, 1995) the two last approaches being the framework in which the original part of my work has been developed. As far as the "phenomenological point of view" is concerned, during the last years black hole and cosmological solutions have been widely investigated in the framework of low-energy effective string field theory (see for instance Witten, 1991), also in connection with the scale-factor duality (Veneziano, 1991). The general idea underlying these investigations is that the short-distance modifications of general relativity due to string theory could be crucial in order to understand long-standing problems of quantum gravity such as the loss of information in the black hole evaporation process or the nature of singularities in the Einstein theory.

2

Constrained Hamiltonian Systems

In this chapter we shall deal with constrained Hamiltonian systems. The investigation of constrained systems is of particular relevance in physics. Indeed, gauge systems are always endowed with constraints. General relativity, Maxwell's theory of electromagnetism, and the massive relativistic free particle are the most remarkable examples. Since the Hamiltonian formulation of a system is fundamental to obtain quantum mechanics, the analysis of the Hamiltonian formalism for constrained systems is an essential tool for the formulation of the quantum theory of gauge systems.

The next section is devoted to the essential properties of classical constrained Hamiltonian systems. It consists of a short review of the well-known techniques developed first by Dirac (1958) and Bergmann (see Bergmann and Goldberg, 1955) to discuss the formal aspects of finite dimensional systems endowed with constraints. Sect. [2.2] deals with finite-dimensional systems invariant under time reparametrisation. Sect. [2.3] is devoted to the discussion of the various approaches to the quantisation of these systems. In sect. [2.4] I shall apply the techniques discussed in the previous sections to an academic case: the relativistic free massive particle. Finally, in the remaining two sections I shall introduce the Hamiltonian formalism for general relativity. Sect. [2.5] is devoted to the formal aspects of the theory and to canonical quantum gravity. In the last section I shall discuss in detail minisuperspace models, i.e. models in which only a finite number of gravitational and matter degrees of freedom is retained.[1]

The purpose of this chapter is to present some basic ideas on classical and quantum constrained systems, in particular general relativity. Of course, this is not meant to be a complete discussion of the subject. For a more complete

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[1] The content of sect. [2.6] and part of sect. [2.4] is original work done in collaboration with Prof. de Alfaro (Università di Torino) and Prof. Alexandre T. Filippov (JINR, Dubna) – see Cavaglià, de Alfaro, and Filippov (1995a; 1995b; 1995c; 1996).

introduction to constrained Hamiltonian systems the reader is referred to the seminal papers by Dirac (1964), Hanson, Regge, and Teitelboim (1976), and to the recent excellent book by Henneaux and Teitelboim (1992).

2.1 Finite-dimensional Constrained Systems.

Our starting point is the action

$$S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i)\,, \tag{2.1.1}$$

where L is the Lagrangian, q_i ($i = 0, ...D-1$) are the Lagrangian coordinates of the D-dimensional system, and $\dot{q}_i$ are the velocities. The action (2.1.1) is obviously taken to be stationary under variations $\delta q_i(t)$ vanishing at the endpoints t_1 and t_2. The ensuing equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,. \tag{2.1.2}$$

From (2.1.1) it is straightforward to pass to the Hamiltonian formalism via the usual Legendre transformation. Define the canonical momenta as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}\,. \tag{2.1.3}$$

The action (2.1.1) can be written in the form

$$S = \int_{t_1}^{t_2} dt\, \left\{\dot{q}_i p_i - H\right\}\,, \tag{2.1.4}$$

where H is the Hamiltonian

$$H \equiv \dot{q}_i p_i - L(q_i, \dot{q}_i)\,. \tag{2.1.5}$$

H does not depend on the velocities $\dot{q}_i$ and is only a function of the coordinates and momenta. The Hamiltonian (2.1.5) formally generates the equations of motion

$$\dot{q}_i = [q_i, H]_P = \frac{\partial H}{\partial p_i}\,, \tag{2.1.6a}$$

$$\dot{p}_i = [p_i, H]_P = -\frac{\partial H}{\partial q_i}\,, \tag{2.1.6b}$$

where the Poisson bracket $[\,,\,]_P$ is defined as

$$[A, B]_P = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\,. \tag{2.1.7}$$

Eqs. (2.1.3) can in principle be inverted in order to give the canonical velocities $\dot{q}_i$ in function of the canonical coordinates q_i and momenta p_i: $\dot{q}_i = \dot{q}_i(q_j, p_j)$. If

$$\det \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0\,, \tag{2.1.8}$$

the solution $\dot{q}_i = \dot{q}_i(q_j, p_j)$ does not exist. This means that the canonical momenta (2.1.3) are not all independent. Thus we have some relations between coordinates and momenta

$$\phi_n(q_i, p_i) \approx 0\,, \tag{2.1.9}$$

where $n = 1, ...N$. Eqs. (2.1.9) are called *primary constraints* of the system. Since the formal quantities ϕ_n usually have non-vanishing Poisson brackets with the Hamiltonian H (even though their numerical values are zero) following Dirac (1964) we have used the symbol "$\approx$"[2] to stress that the canonical expressions $\phi_n(q_i, p_i)$ are numerically restricted to vanish but are not identically null in the entire phase space.[3]

Eqs. (2.1.9) imply that the Hamiltonian is not unique. Indeed, since in the phase space the Hamiltonian is defined only on the hypersurface determined by primary constraints, it can be "extended" off that hypersurface, i.e. we can always substitute H with the *total Hamiltonian*

$$H_T = H + u_n \phi_n(q_i, p_i)\,, \tag{2.1.10}$$

where u_n are arbitrary coefficients depending from canonical coordinates, $u_n \equiv u_n(q_j, p_j)$. It is easy to verify that the equations of motion (2.1.6) become

$$\dot{q}_i \approx [q_i, H_T]_P = \frac{\partial H}{\partial p_i} + u_n \frac{\partial \phi_n}{\partial p_i}\,, \tag{2.1.11a}$$

$$\dot{p}_i \approx [p_i, H_T]_P = -\frac{\partial H}{\partial q_i} - u_n \frac{\partial \phi_n}{\partial q_i}\,. \tag{2.1.11b}$$

Since constraints must be preserved in time, the time derivatives of (2.1.9) have to (weakly) vanish, i.e. we must impose the n consistency equations

$$\dot{\phi}_n \approx [\phi_n, H_T]_P = [\phi_n, H]_P + u_m\,[\phi_n, \phi_m]_P \approx 0\,. \tag{2.1.12}$$

Eqs. (2.1.12) can be interpreted as n non-homogeneous linear equations in the unknown u_m with coefficients depending on the canonical variables. Let us look at (2.1.12). They can be divided in four classes:

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[2] It reads "weakly equal to".

[3] This implies that one must not make use of weakly vanishing quantities before one works out Poisson brackets. In the following chapters we will often substitute $\approx$ with $=$ for simplicity.

1) equations that have no solutions, i.e. leading to inconsistencies;

2) equations identically solved (using the constraints);

3) equations reducing to relations independent from the unknown u_m;

4) equations that give u_m in function of the phase space variables.

Let us discuss the above cases. In the first one we find that our system is inconsistent, i.e. the original Lagrangian is incompatible with constraints. In the second case the equation does not give any further information about the system; the constraint is automatically preserved in time due to the other constraints and we can neglect the equation. The remaining cases are more subtle.

An equation of the third class implies the relation

$$\chi(q_i, p_i) \approx 0 \, . \tag{2.1.13}$$

Eq. (2.1.13) means that we have a new constraint involving the phase space variables. We will call (2.1.13) a *secondary constraint* since it does not follow directly from the original Lagrangian but is a consequence of the consistency equations (2.1.12). In this case we have to require the vanishing of the time derivative of (2.1.13) as done for the primary constraints. So we write

$$\dot{\chi} \approx [\chi, H_T]_P = [\chi, H]_P + u_n \, [\chi, \phi_n]_P \approx 0 \, , \tag{2.1.14}$$

and repeat the procedure until we are into the case four. At the end of the procedure we will find K additional secondary constraints as (2.1.13). Thus all constraints of the system can be written

$$\phi_\mu(q_i, p_i) \approx 0 \, , \tag{2.1.15}$$

where $\mu = 1, ...N + K = \bar{N}$.

Finally, let us discuss the fourth case. Rewrite (2.1.12) as

$$\dot{\phi}_\mu \approx [\phi_\mu, H_T]_P = [\phi_\mu, H]_P + u_m \, [\phi_\mu, \phi_m]_P \approx 0 \, , \tag{2.1.16}$$

to take into account all primary and secondary constraints. These equations can be seen as a set of $\bar{N}$ non-homogeneus linear equations in the N unknown u_m ($N \leq \bar{N}$). The most general solution of this system of equations is given by the sum of a particular solution plus the general solution of the homogeneus associated system. Since the latter is given by a linear combination of all independent solutions of the homogeneus equations associated to (2.1.16) we can write

$$u_n = \tilde{u}_n(q_i, p_i) + v_a v_{an}(q_i, p_i) \, , \qquad a = 1, ...A \, , \tag{2.1.17}$$

where $\tilde{u}_n$ is a particular solution of the system, v_{an} are the independent solutions of the homogeneus equations associated to (2.1.16), and v_a are arbitrary coefficients.

Introducing

$$H' = H + \tilde{u}_n(q_i, p_i)\phi_n(q_i, p_i)\,, \tag{2.1.18}$$

and recalling (2.1.17), the total Hamiltonian (2.1.10) can be rewritten

$$H_T = H' + v_a\phi_a(q_i, p_i)\,, \tag{2.1.19}$$

where $\phi_a(q_i, p_i) = v_{an}(q_i, p_i)\phi_n(q_i, p_i)$. Note that the separation (2.1.19) is not unique because it depends on the choice of the particular solution in (2.1.17).

At this point it is useful to introduce a classification of constraints. We define *first-class* a function of the phase space variables whose Poisson brackets with all the constraints are weakly zero, i.e.

$$\text{If: } \forall\mu \qquad [F, \phi_\mu]_P \approx 0 \quad \rightarrow \quad F : \text{First-class.} \tag{2.1.20}$$

A non-first-class function will be said to be *second-class*, i.e.

$$\text{If: } \exists\mu \;\; / \qquad [F, \phi_\mu]_P \not\approx 0 \quad \rightarrow \quad F : \text{Second-class.} \tag{2.1.21}$$

Using this classification it is easy to see that H' is the first-class replacement of the original non-extended Hamiltonian H (*first-class Hamiltonian*). Thus the total Hamiltonian H_T can be interpreted as the sum of a first-class Hamiltonian plus a linear generic combination of first-class primary constraints ϕ_a. Finally, the total Hamiltonian can further be extended. Let us see this point in detail.

We define the *extended Hamiltonian*

$$H_E = H' + u_a\gamma_a(q_i, p_i)\,, \tag{2.1.22}$$

where u_a are arbitrary coefficients and γ_a are all first-class constraints (primary + secondary). Of course, H_E differs from H_T for a linear combination of first-class secondary constraints. One can prove that the equations of motion derived from (2.1.22) are equivalent to the ones derived by (2.1.18), thus to the equations of motion following from the original Hamiltonian, since the constraints are preserved.[4]

The importance of the extended Hamiltonian lies on the relation between first-class constraints and gauge transformations of the system. Since the Hamiltonian of the system is not unique, not all the canonical variables are observable. Indeed,

[4] See Henneaux and Teitelboim (1992).

the presence of arbitrary quantities in the total Hamiltonian implies that the solution of the equations of motion depends on unknown arbitrary functions of time. Since a physical quantity must be unambigously defined, only those particular quantities that do not depend on the unknown arbitrary functions are observable.[5] This property is necessary in order to determine completely the physical state of the system at any time. Indeed, suppose that the state of the system is given at a certain time t_0 by the set of initial conditions $\{q_i(t_0), p_i(t_0)\}$. Since the solutions of the equations of motion depend on arbitrary functions of time, the state at the time $t > t_0$ will not be uniquely determined. As a consequence, there are several final states, with the same physical content, corresponding to different choices of initial conditions. The situation is analogous to gauge fields. We call *gauge transformations* those transformations that do not alter physical states of the system defined by observables, (*gauge-invariant quantities*). Let us see how gauge transformations are defined.

Let us return back for a moment to the total Hamiltonian and consider a generic canonical quantity $X(q_i, p_i)$. Let $X_0 \equiv X(q_i(t_0), p_i(t_0))$ the value of X at a given initial time t_0. After an infinitesimal time interval δt, X will be

$$X = X_0 + \big([X, H']_P + v_a\,[X, \phi_a]_P\big)\delta t\,. \tag{2.1.23}$$

Since the quantities v_a are completely arbitrary, we can rewrite (2.1.23) for a different choice of v_a. The variation induced on X is

$$\delta X = \big(v_a^{(1)} - v_a^{(2)}\big)\,[X, \phi_a]_P\,\delta t\,, \tag{2.1.24}$$

or, in another form

$$\delta X = [X, \phi_a]_P\,\varepsilon_a\,, \tag{2.1.25}$$

where $\varepsilon_a = \big(v_a^{(1)} - v_a^{(2)}\big)\delta t$. The transformation (2.1.25) is equivalent to the infinitesimal contact transformation of X generated by ϕ_a. Of course the observables must be invariant under (2.1.25). First-class primary constraints can thus be seen as "generating functions of infinitesimal contact transformations that lead to changes in canonical variables but do not affect the physical state" (Dirac, 1964). Secondary first-class constraints may also generate gauge transformations. Indeed, a Poisson bracket of two first-class primary constraints, and the Poisson bracket of any first-class primary constraint with the Hamiltonian H' (first-class), are generating functions of gauge transformations and in general linear combinations of (primary and secondary) first-class constraints. It has not been proved that every first-class secondary constraint is a generating function of a gauge transformation (*Dirac conjecture*, see Dirac (1964)). However, the known counterexamples[6] involve pathological Lagrangians, so usually one assumes that

[5] They are indeed called the *observables* of the system.

[6] For a counterexample see for instance Henneaux and Teitelboim (1992).

all first-class constraints generate gauge transformations. Since gauge transformations do not affect the physical observables of the system we are led to use the extended Hamiltonian (2.1.22). The latter describes the most general motion compatible with gauge transformations and constraints.

To conclude this section, let us count the number of degrees of freedom of a system endowed with first- and second-class constraints (see for instance Henneaux and Teiteboim, 1992). The number $N_{\rm phys}$ of physical canonical degrees of freedom[7] is

$$N_{\rm phys} = N_{\rm can} - N_{\rm fc} - N_{\rm sc}/2\,, \tag{2.1.26}$$

where $N_{\rm can}$ is the original (unconstrained) Lagrangian number of degrees of freedom[8], $N_{\rm fc}$ is the number of first-class constraints, and $N_{\rm sc}$ is the number of second-class constraints. From (2.1.26) we can see that the number of second-class constraints is always even (for the proof, see Henneaux and Teitelboim, 1992).

2.2 Generally Covariant Systems.

In the previous section we have seen how constrained Hamiltonian systems correspond essentially to gauge systems. A first-class constraint can be interpreted as the infinitesimal generating function of a gauge invariance of the system. From now on we will consider a special class of finite-dimensional gauge systems, invariant under reparametrisation of time, i.e. the *generally covariant systems.*[9]

We define "generally covariant" a finite-dimensional system whose action (2.1.4) remains invariant under the time redefinition

$$t \to \bar{t} = f(t)\,. \tag{2.2.1}$$

It is straightforward[10] to see that if the canonical variables transform as scalars under reparametrisation, the form of the action must be

$$S = \int_{t_1}^{t_2} dt\ \{\dot{q}_i p_i - H_E\}\ , \tag{2.2.2}$$

where the extended Hamiltonian is simply a linear combination of first- and second-class constraints

$$H_E = u_a \gamma_a(q_i, p_i)\,, \tag{2.2.3}$$

i.e. the first-class Hamiltonian in (2.1.22) must be identically (weakly) zero.

[7] We define canonical degree of freedom each pair $\{q_i, p_i\}$.

[8] $N_{\rm can}=D$, where D is the dimensionality of the configuration space, see (2.1.1).

[9] One of the most remarkable examples among generally covariant systems is the massive relativistic free particle that will be discussed in sect. [2.4].

[10] See Henneaux and Teitelboim (1992).

The system described by (2.2.2,3) is invariant under the (gauge) transformation

$$\delta q_i = \varepsilon(t)\dot{q}_i \approx \varepsilon\big[q_i, H_E\big]_P\,, \tag{2.2.4a}$$

$$\delta p_i = \varepsilon(t)\dot{p}_i \approx \varepsilon\big[p_i, H_E\big]_P\,, \tag{2.2.4b}$$

$$\delta u_a = \big(u_a\varepsilon(t)\big)^{\cdot}, \tag{2.2.4c}$$

with $\varepsilon(t_1) = \varepsilon(t_2) = 0$. In the case of a single first-class constraint and no second-class constraints[11], i.e. $H_E = uH$, where u is the Lagrange multiplier (non-dynamical variable), eqs. (2.2.4) can be written in the more compact form

$$\delta q_i = \epsilon\frac{\partial H}{\partial p_i} = \epsilon\big[q_i, H\big]_P\,, \tag{2.2.5a}$$

$$\delta p_i = -\epsilon\frac{\partial H}{\partial q_i} = \epsilon\big[p_i, H\big]_P\,, \tag{2.2.5b}$$

$$\delta u = \frac{d\epsilon}{dt}\,, \tag{2.2.5c}$$

where $\epsilon = u\varepsilon$ and $\epsilon(t_1) = \epsilon(t_2) = 0$. From eqs. (2.2.5) we see that the transformation (2.2.5) corresponds to the gauge transformation generated by H. We will denote it as $\mathcal{H}$. Note that the finite form of the gauge transformation (2.2.5) has the same content of the solution of the equations of motion.

To conclude this section, let us spend a few words about the restrictions to be imposed on the Lagrange multiplier. We require u to be positive, at least on the constraint surface. Indeed, with this choice the evolution parameter

$$\tau(t) = \int_0^t u(p_i(t), q_i(t); t)dt \tag{2.2.6}$$

is a monotonic increasing function in t on any trajectory. In quantum mechanics one never gets the Feynman propagator without this positivity restriction (see e.g. Teitelboim, 1991; 1983; Banks and O'Loughlin, 1991). Possibly, some simple zeroes may be harmless but one cannot make any general statement about this. As a consequence, in the redefinition of the Lagrange multiplier $u' = f(q_i, p_i)u$ one has to impose $f \geq 0$.

[11] I.e. there are no gauge generators except the generator of t-reparametrisation. Minisuperspace models that will be discussed in the following chapters are systems of this sort.

2.3 Quantisation of Discrete Gauge Systems.

This section is devoted to the quantisation of the generally covariant constrained Hamiltonian systems discussed in the previous section. Attention will be focused on the so-called *discrete gauge theories*, namely systems endowed only with first-class constraints whose Poisson brackets form a Lie algebra. Minisuperspace models that will be investigated in the following chapters are of this kind.[12]

There are essentially two standard operator approaches to the quantisation of gauge systems. The first is the Dirac method in which the constraints of the theory are promoted to quantum operators (*quantise before constraining*). This leads in gravity to the Wheeler-DeWitt equation (see sect. [2.5]) whose solutions need gauge fixing before being interpreted, in order to eliminate the unphysical degrees of freedom and define the inner product. The method suffers from factor ordering ambiguities and the related problem of the representation of the operators. This difficulty is usually overcome by the definition of an invariant measure in the off-gauge shell phase space together with the implementation of the Faddeev-Popov procedure. The second approach is the reduced canonical method leading to a classical reduced gauge-fixed phase space where quantisation can be carried out as usual (Schrödinger equation) and wave functions have the customary interpretation (*constrain before quantising*).

In our treatment of the quantisation of discrete gauge systems we will discuss both methods. Further, in the particular examples that will be examined in the following chapters we will be able to show that the two methods lead to the same results for correct gauge fixing conditions, thus proving the equivalence of the two approaches when viable.

We will start by discussing the gauge fixing and then we will investigate the Dirac and reduced methods.

2.3.a Gauge Fixing.

We have seen in sect. [2.1] that the number of physical degrees of freedom for a system endowed with only first-class constraints is equal to the number of original degrees of freedom of the system minus the number of first-class constraints, eq. (2.1.26). Since a canonical degree of freedom is represented by a pair $\{q_i, p_i\}$, in order to reduce the system to the physical degrees of freedom we have to impose a number of extra conditions (*gauge identities*) equal to the number of first-class constraints.[13] By means of these extra conditions the system is reduced to a complete set of physical observables.

[12] Examples of these systems are also the free relativistic particle, the theory of confined bound states of relativistic particles, etc.. See for instance Filippov (1996a).

[13] Here and throughout the section we will consider constraints at least bilinear in the momenta.

In field theory, essentially four different kinds of gauge fixing have been discussed in the literature (see Filippov, 1996a). We retain here that nomenclature, according to the type of gauge fixing condition: *unitary gauges, Lorentz gauges, axial gauges*, and *stochastic quantisation.* Let us discuss briefly the first three approaches.[14]

a) *Unitary Gauges.* Remarkable examples of unitary gauge fixings are the Coulomb gauge in electromagnetism and the light-cone gauge in string theory. The gauge fixing identities are of the form

$$F_a(q_i, p_i; t) = 0 \,. \tag{2.3.1}$$

Since the gauge fixing must be preserved in time, we require that the total time derivatives of (2.3.1) vanish. Further, we impose the Poisson brackets of F with the constraints to be different (weakly) from zero. These requirements are necessary and sufficient conditions leading to the reduction of the system to physical degrees of freedom and to the determination of the Lagrange multipliers. Note that these conditions imply the system to become second-class. After gauge fixing there are no first-class constraints left. In field theory gauge fixings of this sort destroy Lorentz invariance.

b) *Lorentz Gauges.* Lorentz gauges are defined by the equations

$$\dot{u}_a = F_a(q_i, p_i; u_b) \,, \tag{2.3.2}$$

even though one usually sets simply $\dot{u}_a = 0$. This sort of gauge fixing is suited to the Faddeev-Popov (FP) – Batalin-Fradkin-Vilkovisky (BFV) – Becchi-Rouet-Stora-Tyutin (BRST) approach to quantisation. The gauge fixing (2.3.2) requires the introduction of additional (ghost) degrees of freedom. Gauge fixings used in the path-integral approach are of this sort. In electromagnetism the Lorentz condition $\partial_\mu A^\mu = 0$ is of the form (2.3.2).

c) *Axial Gauges.* In this case the gauge fixing identities are

$$F_a(q_i, p_i; u_b) = 0 \,. \tag{2.3.3}$$

This sort of gauge corresponds to fix directly the value of the Lagrange multipliers. The analogue case in electromagnetism is $n_\mu A^\mu = 0$. Analogously to (2.3.1) we require the system to become second-class.

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[14] The stochastic quantisation is a *sui generis* gauge fixing not suitable for minisuperspace models in which we are interested here.

To conclude the discussion on gauge fixing, let us stress that gauge fixing conditions are local relations that might not globally hold in phase space. This phenomenon is known as *Gribov obstruction* (see Henneaux and Teitelboim, 1992) and in the case of t-reparametrisation invariance is strictly related to the non-existence of a *global time* (see Hájíček, 1986). Usually, an integrable system invariant under t-reparametrisation does not suffer from the Gribov obstruction, since it can be reduced to the Shanmugadhasan form (Shanmugadhasan, 1973) by a canonical transformation[15]. In that case the conjugate variable to the Hamiltonian defines a global time parameter and there is no Gribov obstruction. Geometrically, the existence of a global time parameter implies that the gauge conditions intersect the gauge orbits on the constraint surface once and only once.

2.3.b Dirac Method.

We have seen that in order to quantise a discrete gauge system one has to reduce its degrees of freedom to the physical ones. This can be performed after the application of the quantisation algorithm or before. In the former case the quantisation procedure is called *Dirac method*, in the latter it is called *canonical reduced method*. Let us first discuss the Dirac method.

The starting point of the Dirac method is to set the commutation relations[16]

$$\left[\hat{q}_i, \hat{p}_j\right] = i\delta_{ij} \,. \tag{2.3.4}$$

So far no gauge conditions have been imposed. Thus eqs. (2.3.4) hold in the entire (unconstrained) phase space and the system carries unphysical information. Using (2.3.4) the quantisation of the system can be achieved imposing that the constraints be enforced by operator identities on the wave functions. We have

$$\hat{\gamma}_a \Psi = 0 \,. \tag{2.3.5}$$

The relations (2.3.5) enforce the gauge invariance of the theory. Indeed, all physical states remain unchanged if one performs the general finite gauge transformations generated by constraints.

Once one has imposed the quantum relations (2.3.5), the gauge must be fixed. This can be performed using the Faddeev-Popov procedure to define the inner product. The scalar inner product of solutions of (2.3.5) is defined as (see Henneaux and Teitelboim, 1992)

$$(\Psi_2, \Psi_1) = \int d[\alpha] \prod_a \Psi_2^*(\alpha)\delta(F_a)\Delta_{FP}\Psi_1(\alpha) \,, \tag{2.3.6}$$

where $F_a(\alpha_i) = 0$ are the gauge fixing identities, $\alpha_i \equiv \alpha_i(q_j, p_j)$ are the polarisation coordinates which appear in the wave functions, $d[\alpha]$ is the measure

[15] See later [2.3.c].

[16] We represent operators with the symbol "^".

defined in the unconstrained phase space, and finally Δ_{FP} is the Faddeev-Popov determinant

$$\Delta_{FP}^{-1} = \int d[h]\, \delta\big(F(h)\big)\,, \tag{2.3.7}$$

where h_i are the coordinates on the (compact) group manifold and $d[h]$ is the product over t of the invariant measure of the group (see Itzykson and Zuber, 1980; Henneaux and Teitelboim, 1992).

Let us see how the method works in the case of (one) simple constraint $p_0 = 0$. Just to fix ideas, let us use

$$d[\alpha] = dq_0 dq_1, ...dq_{D-1}\,, \tag{2.3.8}$$

and thus the operator representation

$$\hat{q}_i \to q_i\,, \qquad \hat{p}_i \to -i\frac{\partial}{\partial q_i}\,. \tag{2.3.9}$$

The single constraint equation (2.3.5) reads

$$-i\frac{\partial}{\partial q_0}\Psi(q_i) = 0\,. \tag{2.3.10}$$

Eq. (2.3.10) implies that the physical wave functions do not depend on q_0, i.e. the degree of freedom $\{q_0, p_0\}$ is unphysical. A suitable gauge fixing condition is

$$F \equiv q_0 - k = 0\,, \tag{2.3.11}$$

where k is a parameter. Using (2.3.11) in the definition of the Faddeev-Popov determinant we find $\Delta_{FP} = 1$. Finally, the inner product (2.3.6) is

$$(\Psi_2, \Psi_1) \;=\; \int dq_2...dq_n\, \Psi_2^*(q_2, ...q_n)\Psi_1(q_2, ...q_n)\,. \tag{2.3.12}$$

In practice, the main problems in order to implement the Dirac procedure are the choice of the factor ordering to be used in (2.3.5) and the choice of the measure in superspace (and, as a consequence, the choice of the variables to be used for the wave functions). Indeed, in order to represent the formal commutation relations (2.3.4) as differential operators one must first choose a pair of commuting variables as coordinates and establish the form of the (non gauge-fixed) measure $d[\alpha]$. For instance, the measure $d[\alpha]$ can be determined by the requirement that it be invariant under the symmetry transformations of the system (rigid and gauge transformations).[17]

[17] See Cavaglià, de Alfaro, and Filippov (1996).

2.3.c Reduced Method.

In this case the quantisation of the system is implemented by first reducing the system to the physical degrees of freedom by gauge fixing and then applying the quantisation algorithm. Let us see the procedure in detail.

The starting point to eliminate the gauge redundancy and reduce the canonical degrees of freedom is the introduction (at the classical level) of a number of unitary gauge conditions (2.3.1) equal to the number of first-class constraints. The difficulty of the choice of gauge identities lies in that the local conditions (2.3.1) must be globally compatible with the equations of motion. This can be checked using the finite gauge transformations (2.2.4). The completion of the procedure depends thus on the integrability properties of the system. For instance, the gauge fixing can be successfully implemented if the system is integrable[18] or separable in one degree of freedom since one can pass to action-angle variables (Arnold, 1978). Let us see this point in the case of a single first-class constraint $H = 0$ – thus eqs. (2.2.5). To implement the reduction procedure one needs the general solution of the equations

$$\frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}\,; \qquad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i}\,, \tag{2.3.13}$$

where $d\tau = udt$. The solution of these equations, i.e.

$$q_i(\tau) = f_i(q,p;\tau)\,, \qquad p_i(\tau) = g_i(q,p;\tau) \tag{2.3.14}$$

satisfying the initial conditions

$$f_i = q_i\,, \qquad g_i = p_i \qquad \text{for} \quad \tau = 0\,, \tag{2.3.15}$$

define the finite gauge transformation. We can then eliminate the unphysical canonical degree of freedom as function of the remaining ones and time (see below); time is connected to the original set of canonical variables and the problem is reduced to the gauge shell with $2(N_{\rm can} - 1)$ canonical coordinates (analogous to the unitary gauge in field theories). Naturally the Poisson bracket $[F, H]_P$ must not vanish even weakly, thus the set $\{F, H\}$ is second-class. The gauge condition (2.3.1) determines the Lagrange multipliers $u(t)$ as a function of time and in general of canonical variables. Finally, the consistency of the procedure can be verified using (2.3.14) and substituting the explicit form of u in eq. (2.2.6). The gauge-fixing function F must depend explicitly upon time since we want to define time as a function of the phase space variables.

The reduction of the system is usually implemented by a canonical transformation. For instance, if the gauge fixing function F is of the form $F = p_0 - f(q_i, t)$

[18] An Hamiltonian system with D canonical degrees of freedom is called integrable (*à la* Liouville) if D independent first integrals in involution are known. See for instance Arnold (1978).

one can choose the canonical transformation $\{q_0, p_0\} \to \{Q_0, P_0\}$ so that $P_0 = F$, $Q_0 = q_0$. Then the gauge shell is obtained by imposing $P_0 = 0$ and fixing Q_0 from the constraint $H = 0$ (see sect. [2.6]). One canonical degree of freedom is thus eliminated and the motion is reduced to $N_{\rm can} - 1$ canonical degrees of freedom with an effective Hamiltonian $H_{\rm eff}$ (that can be time-dependent) generating canonical motion on the gauge shell.[19] The gauge fixing identity is the fundamental gauge fixing equation that states how time is connected to canonical variables.

In order to have a sensible quantum mechanics, usually one requires that the effective Hamiltonian be local, Hermitian and unique. If these conditions are satisfied, the system can be quantised and the Hilbert space recovered in the effective reduced canonical space (gauge shell) by writing a time-dependent Schrödinger equation[20]

$$i\frac{\partial}{\partial t}\psi = \hat{H}_{\rm eff}\psi\,, \tag{2.3.16}$$

where ψ is the on-gauge shell wave function defined in the Hilbert space. In a given gauge, time is now connected to the canonical degree of freedom that has been eliminated by the constraint and gauge fixing.

2.3.d Equivalence between Different Approaches.

For a generic system it is not proved that the two different approaches to the quantisation (Dirac vs. reduced) lead to the same Hilbert space. However, the equivalence between the two methods can be proved if we are able to pass, via a canonical transformation, to the *Shanmugadhasan representation*, namely to the maximal set of gauge-invariant canonical variables (Shanmugadhasan, 1973). Let us see this point in detail. Consider again for simplicity the case of one single constraint H. The canonical variables in the Shanmugadhasan representation are defined as

$$\begin{aligned} &X_i \equiv X_i(q,p)\,, \quad P_j \equiv P_j(q,p)\,, \quad H\,, \quad T \equiv T(q,p)\,;\\ &i,j = 1,...N_{\rm can} - 1\,, \end{aligned} \tag{2.3.17}$$

where $\{X_i,\, P_i\}$ and $\{H,\, T\}$ are canonical coordinates and momenta:

$$\big[X_i, P_j\big]_P = \delta_{ij}\,, \qquad \big[T, H\big]_P = 1\,. \tag{2.3.18}$$

(The remaining Poisson brackets are zero.) Consequently $\{X_i,\, P_i\}$ are gauge-invariant quantities and T transforms linearly for the gauge transformation generated by H. The quantities $\{X_i, P_j, H\}$ form a complete set of gauge-invariant quantities (observables) that are in a one-to-one correspondence with the initial

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[19] For the procedure see for instance Sec. [2.4] or Sec. [2.6].

[20] Or, what has the same content, the Heisenberg equations.

conditions.[21] The classical integrability of the system is thus a necessary condition to pass to the Shanmugadhasan representation. The explicit form of X_i and P_j is obtained using the solutions of the equations of motion to express the initial conditions as functions of the phase space coordinates. Finally, T can be found using the relations (2.3.18).

Now the quantity T can be used to fix the gauge since its transformation properties for the gauge transformation imply that time defined by this variable covers once and only once the symplectic manifold, i.e. time defined by T is a global time (see Hájíček, 1986). Using the Shanmugadhasan representation the quantisation procedure becomes trivial and the Dirac and reduced approaches lead to the same Hilbert space.

To conclude this section let us stress that quantisations of classical systems in different representations usually lead to different quantum theories. This remains true here of course (see figure 2.1).

2.4 An Example: the Relativistic Free Massive Particle.

As a remarkable example of the methods illustrated above, let us discuss the case of the relativistic free massive particle. Its action is

$$S = -m \int_1^2 ds \equiv -m \int_1^2 (-dx_\mu dx^\mu)^{1/2} , \tag{2.4.1}$$

where m is the mass of the particle and ds is the infinitesimal path lenght. Defining an arbitrary monotonic parameter t that labels the world line position of the particle the action (2.4.1) can be written

$$S = -m \int_{t_1}^{t_2} dt \sqrt{-\dot{x}_\mu \dot{x}^\mu} , \tag{2.4.2}$$

where $\dot{x}_\mu = dx_\mu/dt$. One can verify that eq. (2.1.8) holds, so the system described by (2.4.2) is constrained. Indeed the Lagrangian in (2.4.2) is invariant under the transformation (2.2.1). One can show that (2.4.2) is equivalent to

$$S = \frac{1}{2} \int dt \left[\frac{1}{u} \dot{x}_\mu \dot{x}^\mu - u m^2 \right] , \tag{2.4.3}$$

where u is a non-dynamical variable because no derivatives of u appear in the action.[22] From (2.4.3) we can see that the relativistic free massive particle can be interpreted as the one-dimensional gravity described by the line element $ds^2 = -u^2(t) dt^2$ coupled to the matter fields $x^\mu(t)$.

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[21] These are indeed $2N_{\rm can}-1$ because of the invariance under reparametrisation.

[22] The equivalence between (2.4.2) and (2.4.3) can be easily proved substituting in (2.4.3) the equation of motion for the non-dynamical variable.

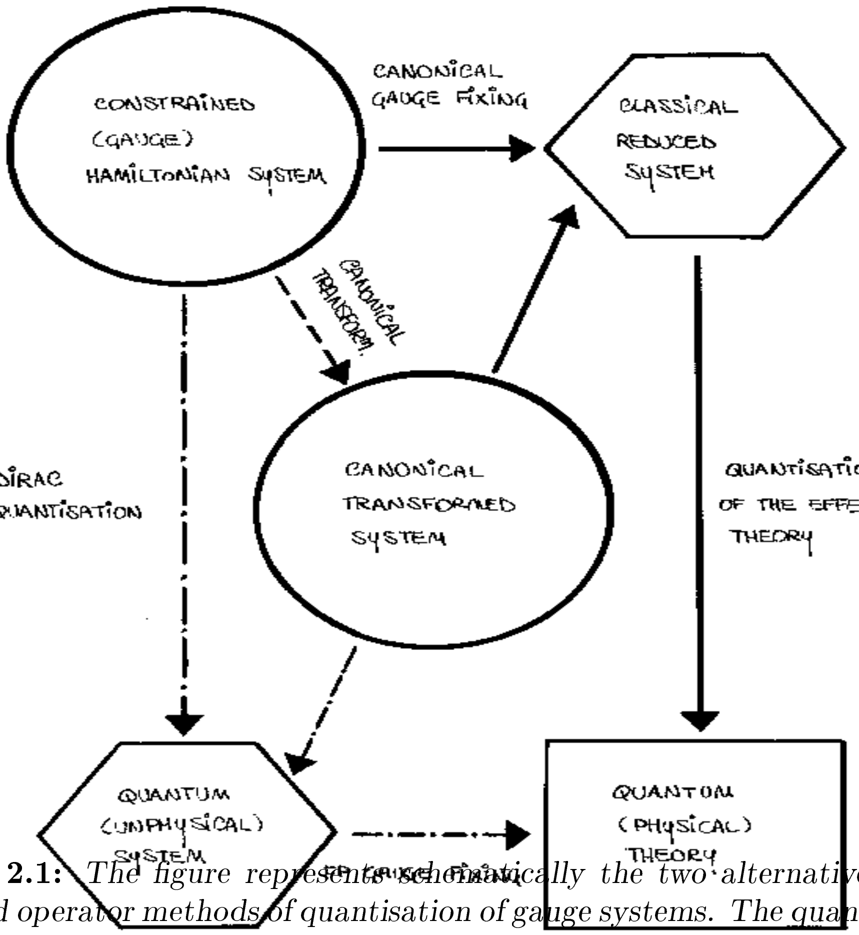

Figure 2.1: *The figure represents schematically the two alternative standard operator methods of quantisation of gauge systems. The quantum physical theory (bottom right corner) is obtained through the application of the quantisation algorithm (vertical lines) and of the gauge fixing (horizontal lines) to the classical constrained system (upper left corner). The equivalence of all possible paths in the figure represents the problem of the equivalence between different quantisation approaches.*

Defining as usual the conjugate momenta, the action can be cast in the Hamiltonian form

$$S = \int dt \left(-p_0 \dot{x}_0 + \mathbf{p}\dot{\mathbf{x}} - uH\right), \tag{2.4.4}$$

where $H_E \equiv uH$ is the extended Hamiltonian[23] and $u \equiv u(t)$ is the Lagrange multiplier (analogous to the gauge potential in standard gauge theories) that

[23] Note that the first-class Hamiltonian is zero, as expected for generally covariant systems when coordinates transform as scalars under time-reparametrisation. See sect. [2.2].

enforces the constraint

$$H = \frac{1}{2}(\mathbf{p}^2 - p_0^2 + m^2) = 0\,. \tag{2.4.5}$$

Finally, the equations of motion are:

$$\dot{\mathbf{x}} = u\mathbf{p}\,, \qquad \dot{\mathbf{p}} = 0\,, \qquad \dot{x}_0 = up_0\,, \qquad \dot{p}_0 = 0\,. \tag{2.4.6}$$

Let us discuss first the reduced method. The gauge can be fixed via the canonical method performing the following canonical transformation

$$X_0 = x_0 - t\, p_0\,, \tag{2.4.7a}$$
$$P_0 = p_0\,. \tag{2.4.7b}$$

and imposing the gauge-fixing condition

$$F(X_0, P_0) \equiv X_0 = 0. \tag{2.4.8}$$

Note that the Poisson bracket $[X_0, H]_P$ is not (weakly) vanishing (it is not a linear combination of X_0 and H). As a consequence, the set of constraints $\{F, H\}$ becomes second-class and the system can be reduced. The use of the consistency condition in the canonical equations of motion generated by the Hamiltonian fixes the Lagrange multiplier as

$$u(t) = 1\,. \tag{2.4.9}$$

Using the gauge-fixing (2.4.8) and the constraint equation (2.4.5) the constrained on-gauge shell action is

$$S_c = \int dt\, \left(\mathbf{p}\cdot\mathbf{x} - \frac{1}{2}P_0^2\right) \tag{2.4.10}$$

with P_0^2 given by (2.4.5) and (2.4.8). Thus, with this choice of gauge, in the sector $\{\mathbf{x},\, \mathbf{p}\}$ the motion is generated by the Hamiltonian

$$H_{\text{eff}} = \frac{1}{2}P_0^2 = \frac{1}{2}(\mathbf{p}^2 + m^2). \tag{2.4.11}$$

Note that the Hamiltonian (2.4.11) generates the motion in the physical three-dimensional canonical space $\{\mathbf{x},\, \mathbf{p}\}$; this is equivalent to the motion in the sector $\{\mathbf{x},\, \mathbf{p}\}$ generated by the action (2.4.1-3) with the constraints (2.4.5) and (2.4.8). Let us note that the canonical transformation (2.4.7) must involve the time parameter explicitly and that (2.4.8) can be thought of as the definition of time in terms of canonical variables. It is easy to verify that t defined in (2.4.7) is a so-called *global time* (see Hájíček, 1986).

Now the system can be quantised imposing the formal commutators

$$\left[\hat{x}_i, \hat{p}_j\right] = i\delta_{ij}\,. \tag{2.4.12}$$

In the Schrödinger picture the algebra is realised by

$$\hat{x}_i \to x_i\,, \qquad \hat{p}_j \to -i\partial_j\,. \tag{2.4.13}$$

Using (2.4.13) we obtain the Schrödinger equation (2.3.16)

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \frac{1}{2}\left[-\nabla^2 + m^2\right]\psi(\mathbf{x},t)\,, \tag{2.4.14}$$

whose stationary solutions are

$$\psi(\mathbf{k};\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}e^{-i(Et-\mathbf{kx})}\,, \tag{2.4.15}$$

where $E = \frac{1}{2}(\mathbf{k}^2 + m^2)$. This is obviously a positive definite Hilbert space since the Hamiltonian is positive-definite and Hermitian. Usual quantum mechanics applies.

It is interesting to compare this result with a different gauge-fixing condition. Let us choose as time the zeroth component of x. To do that, eq. (2.4.7a) has to be substituted by

$$X_0 = x_0 - t. \tag{2.4.16}$$

Now, imposing the identity (2.4.8) we find

$$H_{\text{eff}} = p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.4.17}$$

and the consistency equation gives the Lagrangian multiplier

$$u = 1/p_0\,. \tag{2.4.18}$$

With this gauge fixing choice, in order to have a positive Lagrange multiplier (see sect. [2.2]), and a sensible quantum mechanics of a single particle, we have to choose $p_0 > 0$ in (2.4.17,18) and thus the positive sign in (2.4.17).[24] So we end with the reduced space Hamiltonian

$$H_{\text{eff}} = \sqrt{\mathbf{p}^2 + m^2}\,. \tag{2.4.19}$$

This is the gauge-fixed Hamiltonian of the relativistic particle. The choice of the $-$ sign in (2.4.17) would be wrong in quantum mechanics of a single particle (see e.g. Bjorken and Drell, 1964). The (non-local) Schrödinger equation is ($+$ stands for $u > 0$)

$$i\frac{\partial}{\partial t}\psi_+(x,t) \;= \hat{H}_{\text{eff}}\;\psi_+\,, \tag{2.4.20}$$

[24] Thus using (2.4.16) a positive-defined Hamiltonian does not follow automatically from the theory.

and the eigenfunctions of $\hat{p}$ are

$$\psi_+(\mathbf{k};\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} e^{-i(Et-\mathbf{kx})}\,, \tag{2.4.21}$$

where of course $E = \sqrt{\mathbf{k}^2 + m^2}$. As before this is obviously a positive-definite Hilbert space. Also with this choice of time we can easily verify that t satisfies the Hájíček conditions of global time (see Hájíček, 1986). We will see later that this result follows directly from the method of gauge-fixing via canonical transformation and it is not a property of the relativistic particle.

Now let us see how the Dirac method works for the gauge (2.4.7). In order to implement the Dirac quantisation procedure it is suitable to perform a canonical transformation in the sector $\{x_0, p_0\}$. We write

$$X_0 = \frac{x_0}{p_0}\,, \qquad P_0 = \frac{1}{2}p_0^2\,. \tag{2.4.22}$$

In the new variables the constraint (2.4.5) reads

$$H = -P_0 + \frac{1}{2}(\mathbf{p}^2 + m^2) = 0\,. \tag{2.4.23}$$

Promoting as usual the canonical coordinates $\{X_0, P_0, x_i, p_i\}$ to the operators

$$\begin{aligned} \hat{X}_0 &\to X_0\,, & \hat{x}_i &\to x_i\,, \\ \hat{P}_0 &\to i\partial_{X_0}\,, & \hat{p}_i &\to -i\partial_i\,, \end{aligned} \tag{2.4.24}$$

the quantum constraint becomes

$$\left[-i\partial_{X_0} + \frac{1}{2}\left(-\nabla^2 + m^2\right)\right]\Psi(X_0,\mathbf{x}) = 0\,. \tag{2.4.25}$$

A basis of solutions of (2.4.25) is

$$\Psi(\mathbf{k};\mathbf{x},X_0) = \frac{1}{(2\pi)^{3/2}} e^{-i(EX_0-\mathbf{kx})}\,, \tag{2.4.26}$$

where E is defined as in (2.4.15). Imposing the gauge fixing identity $F \equiv X_0 - t = 0$ (where now t is a parameter) we obtain $\Delta_{FP} = 1$ and the gauge-fixed inner product becomes

$$(\psi_2,\psi_1) = \int d^3x\,\psi^*(\mathbf{k}_2,\mathbf{x})\,\psi(\mathbf{k}_1,\mathbf{x})\,, \tag{2.4.27}$$

where the wave functions are those in (2.4.26) with $X_0 = t$. The Hilbert space coincides with that obtained by the reduced method.

The Dirac method can be also implemented for the gauge (2.4.16). In this case using (2.4.16) the Faddeev-Popov determinant becomes $\Delta_{FP} = -p_0$ and the inner product (2.3.6) becomes

$$(\psi_2, \psi_1) = \int d^3x \, [\psi_2^* \partial_{x_0} \Psi_1 - \psi_1 \partial_{x_0} \Psi_2^*]_{x_0=t} \, . \qquad (2.4.28)$$

The inner product (2.4.28) is not positive-defined (see note 24). In the gauge $x_0 = t$ the wave function must be re-interpreted as a second-quantised field.

2.5 Canonical Quantum Gravity.

Having discussed the theory of Hamiltonian constrained systems, we are now able to investigate the canonical approach to quantum gravity that forms the background of this thesis and a great deal of research articles in quantum gravity. In this section we will deal first with the classical formalism, devoting the last part of the section to the quantum theory. We will see how to implement both the Dirac approach, that leads to the Wheeler-DeWitt equation, and the reduced method. Throughout the section we will adopt the notations of Isham (1992).

The basis of the canonical approach to quantum gravity is the Arnowitt-Deser-Misner (ADM) formalism for general relativity, or *geometrodynamics* (Arnowitt, Deser, and Misner, 1962; see also Wheeler, 1964). In geometrodynamics the starting point is a four-dimensional hyperbolic manifold $\mathcal{M}$ endowed with a metric $g_{\mu\nu}$ with signature $(-,+,+,+)$, and a three-dimensional manifold Σ.[25] Σ plays essentially the role of the three-space. In this section we will consider for simplicity Σ compact. We will also take the signature of Σ to be positive-definite, according to the spirit of geometrodynamics. However, both conditions are not strictly necessary; we will relax them in chapt. 3 and chapt. 5. In the ADM formalism the topology of $\mathcal{M}$ is assumed to be $\mathbb{R} \times \Sigma$. Thus it is possible to define the map (a diffeomorphism of $\mathbb{R} \times \Sigma$ with $\mathcal{M}$)

$$\mathcal{F} : \mathbb{R} \times \Sigma \to \mathcal{M} \, , \qquad (t, x) \mapsto \mathcal{F}(t, x) \, , \qquad (2.5.1)$$

where t and x are respectively the coordinates of a point in $\mathbb{R}$ and Σ in a given reference frame. As a consequence, $\mathcal{M}$ can be foliated by a one-parameter family of embeddings

$$\mathcal{F}_t : \Sigma \to \mathcal{M} \, , \qquad x \mapsto \mathcal{F}_t(x) \equiv \mathcal{F}(t, x) \, . \qquad (2.5.2)$$

Since $\mathcal{F}$ is a diffeomorphism of $\mathbb{R} \times \Sigma$ with $\mathcal{M}$, for any point of $\mathcal{M}$ we can define the inverse map $\mathcal{F}^{-1}$

$$\tau : \mathcal{M} \to \mathbb{R} \, , \qquad \tau(\mathcal{F}_t(x)) = t \, , \qquad (2.5.3a)$$

$$\sigma : \mathcal{M} \to \Sigma \, , \qquad \sigma(\mathcal{F}_x(t)) = x \, . \qquad (2.5.3b)$$

[25] The generalisation to Riemannian manifolds is straightforward and we will omit it.

The map (2.5.3a) is usually called the *global time function* and τ the *natural parameter* of the foliation. Finally, for any point of Σ the map

$$\mathcal{F}_x : \mathbb{R} \to \mathcal{M}\,, \qquad t \mapsto \mathcal{F}_x(t) \equiv \mathcal{F}(t,x)\,, \tag{2.5.4}$$

represents a curve in $\mathcal{M}$ and it is possible to define a one-parameter vector field (*deformation vector*) given by the tangent vectors on $\mathcal{M}$. The deformation vector $\dot{\mathcal{F}}_x(t)$ can be decomposed in two components, one lying in the hypersurface Σ and one along the normal vector. In components we have

$$\dot{\mathcal{F}}^\mu_x(t) = N(t,x) g^{\mu\nu} \mathcal{F}(t,x)\, n_\nu(t,x) + N^i(t,x)\partial_i \mathcal{F}^\mu(t,x)\,, \tag{2.5.5}$$

where n_ν is the normal vector on the hypersurface Σ, the greek indices refer to $\mathcal{M}$ and the latin indices to Σ. N and N^i are usually called the *lapse function* and the *shift vector* associated to the foliation.

The lapse function and the shift vector depend on both the foliation and the geometry of $\mathcal{M}$. From the pull-back of g by the map $\mathcal{F}$ in the coordinates $\{t,x\}$ we obtain

$$\begin{aligned}
(\mathcal{F}^* g)_{00}(t,x) &= N^i(t,x) N^j(t,x) h_{ij}(t,x) - N^2(t,x)\,,\\
(\mathcal{F}^* g)_{0j}(t,x) &= N^i(t,x) h_{ij}(t,x)\,,\\
(\mathcal{F}^* g)_{ij}(t,x) &= h_{ij}(t,x)\,,
\end{aligned} \tag{2.5.6}$$

where

$$h_{ij} = g_{\mu\nu}(\mathcal{F}) \partial_i \mathcal{F}^\mu(t,x) \partial_j \mathcal{F}^\nu(t,x) \tag{2.5.7}$$

is the metric on Σ. h_{ij} is often called the *first fundamental form* of the hypersurface. The line element of $\mathcal{M}$ can thus be locally cast in the form

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = (N^i N^j h_{ij} - N^2) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j\,. \tag{2.5.8}$$

The inverse metric is

$$g^{00} = -\frac{1}{N^2}\,, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2}\,, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}\,. \tag{2.5.9}$$

From (2.5.8) we can easily see that the lapse function and the shift vector are related respectively to the 00 and $0i$ components of the spacetime metric $g_{\mu\nu}$. Given a foliation of the spacetime manifold, the lapse function relates the hypersurface with parameter t (Σ_t) to the hypersurface with parameter $t+\delta t$ ($\Sigma_{t+\delta t}$), i.e. it measures the proper distance between the two hypersurfaces. The shift vector N^i represents the distorsion of the hypersurface as the parameter t increases, i.e. measures for any point of Σ how a point x in $\Sigma_{t+\delta t}$ is displaced with respect to

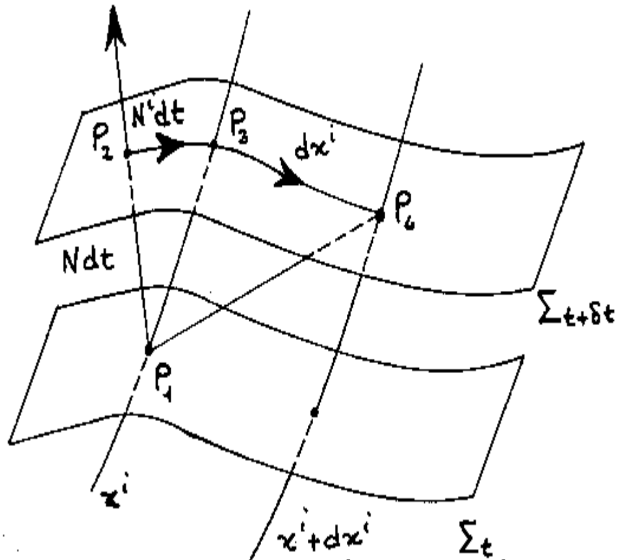

Figure 2.2: *The geometrical meanings of the lapse function and shift vector are illustrated. The lapse function (the vector from P_1 to P_2) measures the proper distance between the hypersurfaces at constant time; the shift vector (from P_2 to P_3) measures the distorsion of the hypersurface as it evolves in time.*

the intersection of the hypersurface with the normal drawn in Σ_t from the point x (see figure 2.2).

Having chosen a foliation, the canonical analysis of the gravitational field proceeds as follows. The Einstein-Hilbert action is (we consider for simplicity vacuum gravity)

$$S_{EH}[g_{\mu\nu}] = \int_{\mathcal{M}} d^4x \sqrt{-g}\, [R[g_{\mu\nu}] - 2\Lambda] \ , \qquad (2.5.10)$$

where $g_{\mu\nu}$ is the metric on $\mathcal{M}$, $g \equiv \det g_{\mu\nu}$, and Λ is the cosmological constant.[26] The problem is now to find a set of canonical variables that determine the evolution of the metric of $\mathcal{M}$. In particular we are interested in the evolution of the foliation (2.5.2), i.e. we want to determine how the hypersurface Σ evolves in function of the parameter t. This represents a well-posed Cauchy problem. The components of the metric on Σ play the role of the configuration variables. Their conjugate momenta represent the rate of change of $g_{\mu\nu}$ in function of the evolution parameter. We define

$$K_{ij} = \frac{1}{2N(t,x)}[-\dot{h}_{ij}(t,x) + L_{\vec{N}} h_{ij}(t,x)] \, , \qquad (2.5.11)$$

[26] In the literature the action (2.5.10) in the Euclidean regime is usually defined with an overall minus sign. In chapt. 3 we will also follow this convention.

where $L_{\vec{N}} h_{ij} = N_{i|j} + N_{j|i}$ is the Lie derivative of h_{ij} along the shift vector on the hypersurface. K_{ij} is usually called the *extrinsic curvature* or the *second fundamental form* of the hypersurface with respect to the embeddings (2.5.2). Now we can pull-back the Lagrangian density in (2.5.10) by the foliation. The action (2.5.10) can be thus expressed as a function of the metric of the hypersurface h_{ij}, the lapse function N, the shift vector N^i, and the extrinsic curvature (2.5.11). The result is

$$S_{EH}[N, N^i, h_{ij}] = \int dt \int_\Sigma d^3x\, N\sqrt{h}\big(K_{ij}K^{ij} - K^2 + {}^{(3)}R[h_{ij}] - 2\Lambda\big) + \ldots$$
$$\ldots + \text{surface terms}\,, \tag{2.5.12}$$

where K is the trace of the extrinsic curvature ($K = K_i{}^i$) and ${}^{(3)}R$ is the scalar curvature of the hypersurface Σ. Note that in (2.5.12) we have neglected a surface term arising from the integration by parts in (2.5.10).[27] This is perfectly legitimate in order to obtain a genuine action principle. However, surface terms play a very important role in the calculation of S_{EH} (for example when we compute the probability amplitude for a baby universe formation or when we deal with asymptotic quantities).[28] For instance, for asymptotically flat metrics (2.5.10) must be substituted by

$$S_{\rm GR}[g_{\mu\nu}] = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left[R[g_{\mu\nu}] - 2\Lambda\right] + 2\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,(K - K_0)\,. \tag{2.5.13}$$

The first contribution to the surface integral makes the action stationary under arbitrary variations of the metric when the variation of the latter is fixed on the boundary $\partial\mathcal{M}$. K_0 is the second fundamental form of the asymptotic hypersurface embedded in flat space and must be introduced to remove the infinite contribution coming from the asymptotic region.[29]

The canonical analysis of (2.5.12) now proceeds as follows. Define via a Legendre transformation the conjugate momenta of variables in the action. Note that in (2.5.12) time derivatives of N and N^i do not appear. So the conjugate momenta vanish identically

$$\pi_N = 0\,, \qquad \vec{\pi}_{\vec{N}} = 0\,. \tag{2.5.14}$$

Hence, the lapse function and the shift vector are non-dynamical variables. The conjugate momenta to h_{ij} are

$$\pi^{ij} = -\sqrt{h}\left(K^{ij} - h^{ij}K\right)\,. \tag{2.5.15}$$

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[27] See for instance York (1972) and Gibbons and Hawking (1977).

[28] For a deeper discussion on the role of surface terms, see Teitelboim and Regge (1974).

[29] See for instance Hawking (1979).

As a consequence of the non-dynamical nature of the lapse function and shift vector, the Legendre-transformed action becomes

$$S_{EH} = \int dt \int_\Sigma d^3x(\pi^{ij}\dot{h}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i)\,, \tag{2.5.16}$$

where $\mathcal{H}$ and $\mathcal{H}_i$ are respectively the super-Hamiltonian and the super-momentum generators

$$\mathcal{H} = \mathcal{H}_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\big({}^{(3)}R - 2\Lambda\big)\,, \tag{2.5.17a}$$

$$\mathcal{H}_i = -2\pi_i{}^j{}_{|j}\,. \tag{2.5.17b}$$

Here

$$\mathcal{H}_{ijkl} = \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \tag{2.5.18}$$

is the metric on the space of all three-metrics (*DeWitt supermetric*).

From (2.5.16) we see that N and N^i are Lagrange multipliers. They enforce the constraints

$$\mathcal{H} = 0\,, \tag{2.5.19a}$$

$$\mathcal{H}_i = 0\,, \tag{2.5.19b}$$

that are called the *super-Hamiltonian* constraint and the *super-momentum* constraint respectively. Finally, we have the Hamiltonian equations of motion for h_{ij} and π^{ij}

$$\dot{h}_{ij} = \frac{\delta H_E}{\delta \pi^{ij}}\,, \tag{2.5.20a}$$

$$\dot{\pi}_{ij} = -\frac{\delta H_E}{\delta h^{ij}}\,, \tag{2.5.20b}$$

where

$$H_E = \int_\Sigma d^3x\,\big(N\mathcal{H} + N^i\mathcal{H}_i\big) \tag{2.5.21}$$

is the extended Hamiltonian of the system. The equation (2.5.20a) can be inverted and coincides with the definition of the extrinsic curvature (2.5.11,15). The equations (2.5.20b) together with the super-Hamiltonian and the super-momentum constraints are completely equivalent to the Einstein equations when the space-time manifold $\mathcal{M}$ is compatible with the foliation (2.5.2).[30]

The classical canonical algebra is given by the Poisson brackets

$$\begin{aligned}
&[h_{ij}(x), h_{kl}(x')]_P = 0\,,\\
&[\pi^{ij}(x), \pi^{kl}(x')]_P = 0\,,\\
&[h_{ij}(x), \pi^{kl}(x')]_P = \frac{\delta(x,x')}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\,.
\end{aligned} \tag{2.5.22}$$

[30] For details see Isham (1992); Fischer and Marsden (1979).

However, not all the phase space variables of the system are independent, because of the constraints (2.5.19). The set of constraints is first-class, i.e. the algebra of Poisson brackets of the constraints is closed. Indeed

$$\begin{aligned}
&[\mathcal{H}(x),\mathcal{H}(x')]_P = \big(h^{ij}(x)\mathcal{H}_i(x)\partial_j^{(x')} - h^{ij}(x')\mathcal{H}_i(x')\partial_j^{(x)}\big)\delta(x,x')\,,\\
&[\mathcal{H}_i(x),\mathcal{H}(x')]_P = \mathcal{H}(x)\partial_i^{(x)}\delta(x,x')\,,\\
&[\mathcal{H}_i(x),\mathcal{H}_j(x')]_P = \big(\mathcal{H}_i(x')\partial_j^{(x)} - \mathcal{H}_j(x)\partial_i^{(x')}\big)\delta(x,x')\,.
\end{aligned} \tag{2.5.23}$$

From (2.5.23) it is straightforward to show (see for instance Isham, 1992) that the first term of the Hamiltonian (2.5.21) can be interpreted as the generator of the deformations of the hypersurface Σ normal to itself and the second term as the generator of spatial diffeomorphisms. This is a consequence of the space and time reparametrisation invariances of general relativity.

The canonical structure that we have developed up to now is the starting point of the canonical quantisation of the gravitational field. We have seen that the canonical treatment of general relativity leads to a theory endowed with gauge invariances and first-class constraints. So the quantisation of theories of this kind requires the careful analysis illustrated in the previous sections. Indeed, the non-dynamical variables of the system must be removed and the physical degrees of freedom isolated. The completion of this program is perhaps an impossible task in the general case, at least in the standard canonical formalism. The possibility to simplify the canonical structure of the gravitational field is at the basis of the Ashtekar programme (see Ashtekar, 1994).

2.5.a Dirac Quantisation of Gravity.

Even though the quantisation of the gravitational field seems far to be achieved, gravity can be formally quantised using both the Dirac and reduced approaches. So let us discuss how canonical gravity can be formally quantised. In this subsection we will deal with the Dirac method. The next subsection is devoted to the reduced method.

We have seen in sect. [2.3] that the first step in the Dirac quantisation procedure is the imposition of formal Heisenberg commutators in the (unconstrained) phase space. By the usual procedure from (2.5.22) we write

$$\begin{aligned}
&[\hat{h}_{ij}(x),\ \hat{h}_{kl}(x')] = 0\,,\\
&[\hat{\pi}^{ij}(x),\ \hat{\pi}^{kl}(x')] = 0\,,\\
&[\hat{h}_{ij}(x),\ \hat{\pi}^{kl}(x')] = i\frac{\delta(x,x')}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\,.
\end{aligned} \tag{2.5.24}$$

The operators in (2.5.24) are obviously defined on the three-dimensional manifold Σ. In order to implement the quantisation algorithm the choice (2.5.24) is not unique. For instance the commutation relations (2.5.24) can be slightly changed in order to take into account at the quantum level that the classical three-metric h_{ij} is positive defined.[31] However, in our applications we will consider only the choice (2.5.24).

In order to implement the Dirac procedure, the first problem we meet is the choice of the representation for the quantum operators in (2.5.24). The choice of the measure in superspace and the choice of the variables to be used for the wave functions are related problems. We will see in chapt. 5 that the invariance under gauge and rigid transformations in superspace may suggest a recipe to solve these problems, at least for simple reduced models, as those investigated there.[32]

The Dirac quantisation can be implemented promoting the constraints to functional operators acting on wave functions defined in a suitable space. Identifying the classical three-metric h_{ij} and the classical momenta π^{ij} defined in (2.5.15) with the functional operators[33]

$$\hat{h}_{ij}(x) \to h_{ij}(x)\,, \tag{2.5.25a}$$

$$\hat{\pi}^{ij}(x) \to -i\frac{\delta}{\delta h_{ij}(x)}\,, \tag{2.5.25b}$$

the super-Hamiltonian constraint (2.5.19a) becomes the Wheeler-DeWitt equation (Wheeler, 1968; DeWitt, 1967)

$$\left[\mathcal{H}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \sqrt{h}\left({}^{(3)}R - 2\Lambda\right)\right]\Psi(h_{ij}) = 0\,, \tag{2.5.26}$$

where we have suppressed the functional dependence on the space variables x. In (2.5.26) Ψ represents the off-gauge shell wave function of the system. As stressed above, eq. (2.5.26) is defined apart from factor ordering ambiguities. From the super-momentum constraints (2.5.19b) we obtain the super-momentum constraint equations

$$-2\left[\frac{\delta}{\delta h_{ij}}\right]_{|j}\Psi(h_{ij}) = 0\,. \tag{2.5.27}$$

From the classical meaning of (2.5.19b) we see that eqs. (2.5.27) enforce the invariance of wave functions under spatial diffeomorphisms. This means that Ψ is only a functional of the three-geometry and not of the particular three-metric h_{ij}.

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[31] See for instance Isham (1992).

[32] Other problems that arise in the Dirac approach concern the validity of the algebra (2.5.23) at quantum level and the self-adjointness of the super-Hamiltonian and super-momentum constraints. For the discussion, see Isham (1992).

[33] For the moment we neglect factor ordering and representation problems.

At this point one could in principle recover the Hilbert space of states by fixing the gauge and reducing the system to the physical degrees of freedom.[34] The difficulty in the implementation of the gauge fixing lies in the identification of the gauge fixing identities and is usually surmounted by reducing the original system to a finite number of degrees of freedom (minisuperspace model). We would like to stress that the gauge fixing implementation is a necessary step in order to recover the dynamical evolution of the system from the Wheeler-DeWitt equation because time evolution is "hidden" in eq. (2.5.26).

2.5.b Reduced Quantisation of Gravity.

Let us now briefly discuss how the reduction procedure of quantisation described in sect. [2.3] can be implemented, at least formally, for the gravitational field.[35]

In the reduced approach to quantisation, the physical degrees of freedom must be identified before the application of the quantisation algorithm. This can be performed by a canonical transformation. So the first step is to define the canonical transformation

$$(h_{ij}, \pi^{ij}) \to (W_a, Z_b; w_r, z_s)\,, \tag{2.5.28}$$

where $a, b = 1, ...4$, $r, s = 1, 2$, and the only non-zero Poisson brackets are

$$[W_a(x), Z_b(x')]_P = \delta_{ab}\delta(x, x')\,, \qquad [w_r(x), z_s(x')]_P = \delta_{rs}\delta(x, x')\,. \tag{2.5.29}$$

In (2.5.28,29) the quantities $\{W_a,\ Z_b\}$ and $\{w_r,\ z_s\}$ identify the four canonical unphysical degrees of freedom (per space point) and the two physical ones respectively. The system can then be reduced using the procedure illustrated in sect. [2.3] imposing for instance the four gauge identities

$$F_a(h_{ij}, \pi^{ij}; x, t) \equiv W_a(h_{ij}, \pi^{ij}; x) - f_a(x, t) = 0\,, \tag{2.5.30}$$

where f_a are four given functions of the spacetime coordinates. The system is then reduced on the gauge shell and one obtains an effective Hamiltonian that is function only of the physical variables. The system can be quantised on the gauge shell imposing the formal commutator relations

$$[\hat{w}_r(x), \hat{z}_s(x')] = i\delta_{rs}\delta(x, x')\,, \tag{2.5.31}$$

or writing the Schrödinger equation analogue to (2.3.16).

As stressed above, the main problem in the reduced quantisation procedure for the gravitational field is that one does not know how to solve the super-Hamiltonian and the super-momentum constraints in a closed form, or, in other

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[34] Indeed the quantum system has only two physical degrees of freedom (per space point), corresponding to the dimension of the phase space (six) minus the number of first-class constraints (four).

[35] For a deep discussion, see Hanson, Regge, and Teitelboim (1976).

words, how to write explicitly the global canonical transformation (2.5.28). Further, since the gauge fixing identities are highly non-linear, quantisation in different gauges may produce different quantum theories. We will see that for the simple (minisuperspace) models discussed in the following chapters, these problems can be solved.

2.6 Minisuperspace Models and Definition of Time.

We have seen in the previous section that the canonical quantisation of the gravitational field is far from being completed. However, something can be said if one imposes by hand some additional constraints to the theory. For instance, one can consider gravity in lower dimensions (see for instance Carlip, 1995; Moncrief, 1990; Menotti, 1996), or restrict oneself to manifolds endowed with a suitable group of symmetries. In the latter case, the implementation of additional spacetime symmetries can be formally obtained by reducing the degrees of freedom of the theory. As a consequence, by retaining only a finite number of degrees of freedom the quantum field theory reduces to quantum mechanics and within this framework typical problems due to the field nature of the system, for example anomalies, disappear. Further, other conceptual problems (definition of time, Hilbert space of states,...) can be successful investigated, at least for simple models.

In this section attention is focused on minisuperspace models. In particular we will see how the problem of time can be addressed in this framework.[36] Since the identification of time can be naturally discussed in the reduced method, this section will be essentially devoted to the classical reduction of minisuperspace models using Coulomb-like gauges (see sect. [2.3]).[37]

The dynamics of a minisuperpace model is expressed by the action

$$S = \int_{t_1}^{t_2} dt \ \{p_i \dot{q}_i - H_E(q_i, p_i)\} \ , \qquad i = 0, ...N_{\rm can} - 1 \, . \tag{2.6.1}$$

where the phase space variables $\{q_i, p_i\}$ represent matter and gravitational degrees of freedom. The action (2.6.1) has been obtained from (2.5.10) by imposing

[36] The origin of the so-called problem of time follows from the fact that the first-class Hamiltonian for general relativity is identically zero, due to general covariance. The evolution of the system is then "hidden" entirely in the constraints: in other words, "constraints generate dynamics". The problem of time is the problem of recovering a suitable parameter to describe evolution. See for instance Kuchař (1992); Barbour (1994a, 1994b).

[37] Not all minisuperspace models admit a global time (see Hájíček, 1986). The existence of a global parameter to describe evolution depends on the geometry of the symplectic space and thus is strictly related to the model under consideration.

some suitable ansatz for the metric tensor $g_{\mu\nu}$.[38] The usual type of truncation for the metric is of the form

$$ds^2 = -N^2(t)dt^2 + h_{ab}(x^a; q_i(t))dx^a dx^b \,, \tag{2.6.2}$$

where the lapse function and the set of parameters q_i depend only on time t. The set of parameters q_i labels the reduced space (minisuperspace). Inserting (2.6.2) in (2.5.10) and integrating over surfaces at constant t one obtains (2.6.1).

The Lagrangian (2.6.1) is of the generally covariant type (see sect. [2.2]) and thus is invariant (up to a total t-derivative) under the infinitesimal gauge transformation (2.2.5). We have stressed in sect. [2.2] that the gauge invariance of the canonical equations corresponds to the invariance of the Lagrange equations under t-reparametrisation. The Lagrange multiplier u is analogous to the gauge potential and enforces the vanishing of the Hamiltonian. So the extended Hamiltonian for minisuperspace models will contain only the constraint (see sect. [2.2])

$$H_{\mathrm{E}} \equiv u(t)H(q_i, p_i)\,, \qquad H = 0\,. \tag{2.6.3}$$

Hence, the models described by (2.6.1) have close similarity to the dynamics of the free relativistic particle (see sect. [2.4]) or to the dynamics of the relativistic harmonic oscillator. The system can thus be quantised along the lines of sect. [2.3]. In this section we will restrict our discussion to the (classical) issue of the definition of time in these models, in view of the application to quantum cosmology that will be investigated in chapt. 4.

We can eliminate redundancy introducing the dynamical way of fixing the gauge discussed in sect. [2.3], i.e. a relationship between canonical variables and time

$$F(q, p; t) = 0 \tag{2.6.4}$$

(Coulomb-like gauge). This allows us to determine time as a function of canonical variables. Let us see this procedure in detail. Define a canonical transformation

$$P_i = p_i - f_i(q_j; t)\,, \tag{2.6.5a}$$

$$Q_i = q_i\,. \tag{2.6.5b}$$

(the roles of the p's and q's may be interchanged of course). From the Poisson identity $[P_i, P_j]_P = 0$ follows, at least locally, that $f_i = \partial/\partial q_i f(q_j, t)$. In this section we will denote partial differentiations by commas. Then we have

$$p_i \dot{q}_i = P_i \dot{Q}_i - f_{,t}(Q_j; t) + \frac{d}{dt} f(Q_j; t)\,. \tag{2.6.6}$$

[38] ...and obviously a similar ansatz for matter degrees of freedom. In this case we have to consider also additional matter contributions in the action (2.5.10).

Now let us introduce the gauge-fixing condition (2.3.1, 2.6.4) in the form

$$P_0 = 0\,. \tag{2.6.7}$$

Note that the Poisson bracket of the gauge-fixing function and the constraint, $[P_0, H]_P$, must not be (weakly) zero. The function f must depend explicitly on t in order that the procedure works (and completely determines the Lagrange multiplier). Now the system has two constraints of second-class, $P_0 = 0$ and $H = 0$. The variable Q_0 conjugate to P_0 is fixed by the constraint: $Q_0 = Q_c(Q_\alpha, P_\alpha; t)$, where $\alpha = 1, ...N_{\text{can}} - 1$ and Q_c is obtained from

$$\left[H\big(Q_0, Q_\alpha, p_0 = f_{,Q_0}(Q_0, Q_\alpha; t), P_\alpha + f_{,Q_\alpha}\big)\right]_{Q_0=Q_c} = 0\,. \tag{2.6.8}$$

The system is now on the gauge shell, the $2(N_{\text{can}}-1)$-hypersurface in the $2N_{\text{can}}$-phase space (Q_i, P_i) defined by

$$P_0 = 0, \qquad Q_0 = Q_c(Q_\alpha, P_\alpha; t)\,. \tag{2.6.9}$$

The effective Lagrangian in the (Q_α, P_α) sector is obtained from (2.6.1,3) using (2.6.6), (2.6.7), and (2.6.9). Neglecting a total derivative we have

$$L_{\text{eff}} = L[P_0 = 0, Q_0 = Q_c] = P_\alpha \dot{Q}_\alpha - H_{\text{eff}}(Q_\alpha, P_\alpha; t)\,, \tag{2.6.10}$$

where the effective Hamiltonian on the gauge shell is

$$H_{\text{eff}}(Q_\alpha, P_\alpha; t) = f_{,t}\big(Q_c, Q_\alpha; t\big)\,. \tag{2.6.11}$$

The effective Hamiltonian is in general time-dependent. One can check that the canonical equations

$$\dot{Q}_\alpha = \frac{\partial H_{\text{eff}}}{\partial P_\alpha}\,, \tag{2.6.12a}$$

$$\dot{P}_\alpha = -\frac{\partial H_{\text{eff}}}{\partial Q_\alpha}\,, \tag{2.6.12b}$$

are equivalent to the original canonical system using (2.6.8) and (2.6.14) (see below).

Finally, for consistency with the gauge-fixing condition (2.6.7) we require also

$$\dot{P}_0 = 0\,. \tag{2.6.13}$$

This determines the expression for the Lagrange multiplier:

$$u(t) = -\left[\frac{f_{,tq_0}}{H_{,q_0} + f_{,q_0 q_i} H_{,p_i}}\right]_{\substack{q_0=Q_c\\ p_i=f_i}}\,. \tag{2.6.14}$$

In general the expression (2.6.14) can be complicated, however this does not necessarily concern us: the important fact is that the gauge-fixing of the form (2.6.7) determines $u(t)$. The problem of the motion is thus reduced to the task of determining the gauge function $f(q_i; t)$.

Time is thus determined by the gauge-fixing condition to be a function of the variables q_i and p_i from the condition $P_0 = 0$. For the relativistic particle we have seen that often t turns out to be a function only of a couple of canonical variables p_0, q_0, thus a single degree of freedom defines the time in that gauge. This is an interesting situation. A particular situation occurs when t depends only on q_0 or p_0. This allows identification of time in that gauge with a canonical coordinate of immediate physical relevance. Obviously this case is not in general possible. Further, the effective Hamiltonian can become highly not polynomial and after the identification of time with a degree of freedom H_{eff} is in general time-dependent.

When t depends only on q_0 or p_0 it is straightforward to show that time determined by the gauge-fixing is a global time. Suppose for simplicity that the gauge-fixing is of the form (2.4.16) ($x_0 \to q_0$); in this case it is straightforward to verify that the Hamiltonian vector field is (using the notations of Hájíček, 1986):

$$H^A = u(H_{,p_0}, H_{,p_\alpha}, -H_{,q_0}, -H_{,q_\alpha}) \,. \tag{2.6.15}$$

Then, considering the projection of (2.6.15) on the configuration space we obtain that t is a global time if and only if the condition

$$uH_{,p_0} > 0 \tag{2.6.16}$$

holds. But (2.6.16) is certainly true because u is fixed by the gauge:

$$u = \frac{1}{H_{,p_0}} \,. \tag{2.6.17}$$

When t does not depend only on q_0 or p_0, one can perform in principle a canonical transformation reducing to the previous case.[39] Clearly, u determines the range in which canonical variables are defined.

To conclude this discussion, let us spend some words on the usually called *intrinsic* and *extrinsic* times. In the literature, the intrinsic time is defined as a function only of the three-metric h_{ij}, i.e. in our minisuperspace models

$$t \equiv t(q_i) \,. \tag{2.6.18}$$

[39] If this is not possible a global time cannot be defined. Thus the existence of a global time is related to the integrability properties of the model (see [2.3.c]).

Conversely, the extrinsic time depends not only on the first fundamental form h_{ij} but also on the extrinsic curvature. In minisuperspace models we write

$$t \equiv t(q_i, p_i)\,. \tag{2.6.19}$$

In our general approach the above distinction disappears. From the general theory of gauge fixing (see sect. [2.3]) we know that the difference between (2.6.18) and (2.6.19) is purely philosophical. Note that time can also be defined using matter (non-gravitational) degrees of freedom.

3

Minisuperspaces in Euclidean Quantum Gravity: Wormholes

In the last section of the previous chapter we have introduced minisuperspace models essentially as a crude approximation of general relativity. In this chapter and in the following ones we will see how minisuperspace models can be successfully used to obtain classical and quantum descriptions of well-defined gravitational systems, as wormholes, black holes, and the early universe.

The current chapter is devoted to minisuperspace models in Euclidean quantum gravity. In the first section I shall present briefly the Euclidean quantum gravity approach and introduce wormholes. Wormholes are indeed the most striking example of minisuperspace models in Euclidean gravity. In the second section I shall illustrate the considerations of sect. [3.1] by a model of wormhole generated by the electromagnetic field. Sect. [3.3] deals with the generalisation of the model to string theory. Finally, the quantisation of the model is completed in sect. [3.4]. We will see from this particular example that minisuperspace models allow to obtain interesting results about tunnelling processes in quantum gravity and open the way to interesting speculations on the nature of singularities in general relativity.

The content of Sects. [3.2-4] is original work done by the author in collaboration with Prof. Vittorio de Alfaro (Università di Torino), Prof. Fernando de Felice (Università di Padova), and Dr. Mariano Cadoni (Università di Cagliari) (see Cavaglià, de Alfaro, and de Felice, 1994; Cavaglià, 1994a, 1994b; Cadoni and Cavaglià, 1995a, 1995b).

3.1 Euclidean Gravity, Path-integral Approach and Wormholes.

The starting point of the Euclidean quantum gravity approach is the observation that the Hawking result on black hole radiance can be derived using thermal Green functions (Gibbons and Perry, 1987). In the usual quantum field theory, thermal Green functions are related to the complexification of time; the partition function of a thermodynamical system is formally obtained by the Wick rotation $t \to it$ in the path-integral.

Let us see briefly this point. Consider a quantum mechanical system with a single degree of freedom. Let $\{q, p\}$ the phase space coordinates and H the Hamiltonian. At any time the system is described by the spectrum of H. Let $q_1 \equiv q(t_1)$ and $q_2 \equiv q(t_2)$, where $t_2 > t_1$. The amplitude to go from the configuration q_1 to the configuration q_2 is given by (see for instance Ramond, 1989; Feynman and Hibbs, 1965)

$$< q_2|q_1 >=< q_2| \exp(-iH(t_2 - t_1))|q_1 >= \int D[q]D[p]e^{iS[p,q]}\,, \tag{3.1.1}$$

where

$$S = \int_{t_1}^{t_2} dt\,[p\dot{q} - H(q,p)] \tag{3.1.2}$$

is the action of the system. The functional integration in (3.1.1) is over all possible paths in phase space with fixed values of the coordinate q at the end points $q(t_1) = q_1$ and $q(t_2) = q_2$.

Comparing (3.1.1) with the expression of the partition function for the same system at temperature T

$$Z = \mathrm{tr}\; e^{-H/T} = \sum_q < q| \exp(-H/T)|q >\,, \tag{3.1.3}$$

it is straightforward to see that (3.1.1) and (3.1.3) coincide if the time separation in (3.1.1) is taken to be pure imaginary (Wick rotation: $i(t_2 - t_1) = 1/T$) and the configurations q_1 and q_2 coincide. Thus the partition function of the system at temperature T is given by a path-integral where the functional integration is over the space of periodic functions with period T^{-1}.

The Hawking result that black holes produce a thermal emission with a temperature $T = 1/8\pi M$ (Hawking, 1975) can be easily derived by noticing that the Euclidean-Schwarzschild black hole solution is periodic with period $8\pi M$. This analogy suggests to define in gravitational theory the functional (see Hawking, 1979)

$$\Psi(\mathrm{boundary}) = \int_{\mathcal{C}} D[g_{\mu\nu}]D[\varphi]e^{-S_E[g_{\mu\nu},\varphi]}\,, \tag{3.1.4}$$

where $g_{\mu\nu}$ is the positive-definite metric tensor defined on Riemannian manifolds, $\mathcal{C}$, φ represents the matter fields, and S_E is the Euclidean action for gravity plus matter fields. Eq. (3.1.4) is the basis of the Euclidean quantum gravity approach.[1]

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[1] The generalisation of the construction developed in sect. [2.5] to the Euclidean case is straightforward.

Starting from (3.1.4) one can prove that the wave functional Ψ satisfies, at least formally, the Wheeler-DeWitt equation (2.5.26).[2]

One of the main problems of this approach is that the functional measure in (3.1.4) is not mathematically well-defined, so one does not know how to evaluate the path-integral. A second important difficulty in giving a physical meaning to (3.1.4) is that the Euclidean gravitational action S_E is not positive-definite. Further, it is not obvious how to relate the Euclidean and the Lorentzian regimes since in general a real hyperbolic metric cannot be made Riemannian (and real) by a complexification procedure or viceversa. In other words, there is not a one-to-one relation between Riemannian and hyperbolic metrics: a given real four-dimensional hyperbolic metric has not in general a real and positive-definite section in the complexified spacetime (and viceversa). Finally, it is not clear what boundary conditions have to be imposed in (3.1.4).[3]

In spite of these mathemathical problems, the path-integral approach has stimulated a great deal of work. A great amount of attention has been devoted to approximations of (3.1.4): saddle-point and one-loop approximations (see for instance Esposito, 1993). The saddle-point approximation has been widely used for the study of gravitational istantons. Further, the functional integral (3.1.4) is the basis for the so-called *wave function of the universe* of Hartle and Hawking (1983).[4] We will see below that wormholes in semiclassical approximation are described by gravitational instantons whose probability of formation can be calculated from (3.1.4).

The Euclidean quantum gravity is the arena for the investigation of wormholes. Semiclassically, wormholes in dilute approximation[5] are classical Euclidean solutions for the gravitational field coupled to matter or gauge fields, that asymptotically connect two four-dimensional manifolds (see for instance Hawking and Page, 1990; Carlini, 1991); they are interpreted as tunnelling processes between the two asymptotic configurations, namely as instantons[6] joining two flat (or asymptotically flat) regions defined by $\tau \to \pm\infty$, where τ is the Euclidean time. Hence, they can be seen as the saddle point approximations of (3.1.4) with particular boundary conditions that identify the asymptotically flat behaviour. The exponential of the action integrated on the solution gives the semiclassical probability amplitude for the wormhole formation per Planck volume and per Planck time.

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[2] Of course matter contributions must be included in (2.5.26). See for instance Halliwell and Hartle (1991).

[3] For a deeper discussion of the meaning of the functional approach to quantum gravity, see Esposito (1992, 1993), Ashtekar (1991), and Hawking (1979).

[4] See chapt. 4.

[5] I.e. we neglect in first approximation interactions between wormholes.

[6] See Coleman, 1977.

If a wormhole can be continued into a hyperbolic universe whose spatial three-dimensional hypersurface is compact the Euclidean solution can be interpreted as nucleating a baby universe from an asymptotic region and gives the semiclassical amplitude for creating a disconnected closed universe in that space.[7] The baby universe then evolves according to its equations of motion (Hawking, 1990a; 1990b).

In the last years, a large amount of attention has been devoted to explicit wormhole solutions.[8] However, in order to calculate the effective interactions caused by wormholes, one needs to go beyond the semiclassical description. In the quantum framework wormholes are defined as non-singular solutions of the Wheeler-DeWitt equation with boundary conditions which identify the asymptotically flat space (Hawking and Page, 1990). So let us conclude this introduction defining boundary conditions in (3.1.4) that determine wormhole wave functions.

Following Hawking and Page (1990) we define quantum wormholes as non-singular solutions of the Wheeler-DeWitt equation and constraints (see sect. [2.5]) that for large three-metrics reduce to the vacuum wave function. The latter is defined by a path-integral (3.1.4) over all asymptotically Euclidean metrics $\mathcal{C}$ (not necessarily $\mathbb{R}^4$, depending on topology of the space) and matter fields vanishing at infinity. Now $\mathcal{C}$ in (3.1.4) represents the class of three-metrics and matter fields which satisfy the conditions

$$h_{ij}(x,\tau) = \tilde{h}_{ij}, \qquad \phi(x,\tau) = \tilde{\phi}, \tag{3.1.5a}$$

$$\phi(x,\infty) = 0, \qquad h_{ij}(x,\infty) = h_\infty. \tag{3.1.5b}$$

In (3.1.5b) h_∞ is the asymptotic three-metric corresponding to the space with minimal gravitational excitation (Garay, 1991). Evaluating the path-integral in the asymptotic limit, we obtain the asymptotic behaviour of the wormhole wave function (see for instance Garay, 1991; Cavaglià, 1994b)

$$\Psi(\tilde{h}_{ij}) \sim \exp\left[\int d^3x \pi^{ij} h_{ij}\right]_\tau , \tag{3.1.6}$$

Finally, (3.1.5) and (3.1.6) are the boundary conditions which identify the wormhole wave functions.

---

[7] The analitic continuation is performed at the extremal configurations of the solution, namely where the extrinsic curvature of the three-surface at constant Euclidean time is vanishing.

[8] See for instance Giddings and Strominger (1988); Myers (1988); Halliwell and Laflamme (1989); Coule and Maeda (1990); Hawking (1987); Hosoya and Ogura (1989); Keay and Laflamme (1989); and Carlini (1991).

3.2 An Example: Kantowski-Sachs Electromagnetic Wormholes.

As a remarkable example of the ideas illustrated in the previous section, let us discuss a wormhole generated by the coupling of gravity to the electromagnetic field. [9]

3.2.a Euclidean Solution as Tunnelling in Time.

Let us consider the Euclidean action for gravity minimally coupled to the electromagnetic field:[10]

$$S_E = \int_{\mathcal{M}} d^4x\,\sqrt{g}\,[-R + 2\Lambda + \varepsilon F_{\mu\nu}F^{\mu\nu}] \;-\; 2\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}\,(K - K_0)\,. \qquad (3.2.1)$$

where $\mathcal{M}$ is now a Riemannian four-dimensional manifold, $F_{\mu\nu}$ is the usual electromagnetic field tensor, and $\varepsilon = \pm 1$.[11]

Now, we look for a solution of the form[12]

$$ds^2 = N^2(\tau)d\tau^2 + a^2(\tau)d\chi^2 + b^2(\tau)d\Omega_2^2\,, \qquad (3.2.2)$$

where χ is the coordinate of the one-sphere, $0 \leq \chi < 2\pi$ and $d\Omega_2^2$ represents the line element of the two-sphere. The line element (3.2.2) is known as the Euclidean Kantowski-Sachs type (Kompaneets and Chernov, 1964; Kantowski and Sachs, 1966) and describes a $\mathbb{R} \times \Sigma$ space, where Σ is a three-dimensional homogeneus and non-isotropic hypersurface with topology $S^1 \times S^2$. Since we are interested in the computation of the classical solution, let us choose for the moment $N(\tau) = 1$. This corresponds to break the τ-reparametrisation invariance of the action.[13]

Let us first discuss the case of vanishing cosmological constant. The ansatz for the electromagnetic field must be chosen according to the symmetry of (3.2.2). We choose the potential one-form

$$\mathbf{A} = A(\tau)\,d\chi\,. \qquad (3.2.3)$$

[9] Recently wormholes generated by coupling of gravity to abelian and non-abelian gauge fields have raised a great interest. See for instance Dzhunushaliev (1996) and references therein.

[10] The overall change of sign with respect to (2.5.13) is due to usual conventions for Riemannian spaces.

[11] The sign of ε is fixed by the request that a real electromagnetic field in the hyperbolic space remains real when rotated in the Euclidean space. See for instance Cadoni and Cavaglià (1995a). The discussion about the Euclidean formulation of the Maxwell theory can also be found in Brill (1992).

[12] Here and throughout this chapter t represents the Lorentzian time and τ the Euclidean time.

[13] Note the difference with sect. [3.4] where we shall deal with the Hamiltonian formalism and τ-reparametrisation invariant actions.

The only non-vanishing component of the electromagnetic field is thus along the χ direction. Note that the equations of motion for the electromagnetic field are

$$\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0\,. \tag{3.2.4}$$

Since the right-hand side of (3.2.4) is zero, there are no physical charges in the space. This will be useful later. Solving the equations of motion for the ansatz (3.2.2) and (3.2.3) one recovers after some algebra the general solution (Cavaglià, de Alfaro, and de Felice, 1994)

$$ds^2 = d\tau^2 + \bar{Q}^2 \frac{\tau^2}{Q^2+\tau^2} d\chi^2 + (Q^2+\tau^2)d\Omega_2^2\,, \tag{3.2.5a}$$

$$A(\tau) = -\frac{\bar{Q}Q}{\sqrt{Q^2+\tau^2}}\,. \tag{3.2.5b}$$

where Q and $\bar{Q}$ are integration constants with dimension of length. In the following we will set $\bar{Q} = Q$.

Let us now study the asymptotic behaviour of (3.2.5a) to discuss the wormhole interpretation of the solution. We have seen in the previous section that the line element must become asymptotically flat in the Euclidean time τ in order to be interpreted as a wormhole. When $\tau \to \pm\infty$, the metric (3.2.5a) becomes

$$ds^2 = d\tau^2 + Q^2 d\chi^2 + \tau^2 d\Omega_2^2\,. \tag{3.2.6}$$

Clearly the space becomes flat with topology $\mathbb{R}^3 \times S^1$. At $\tau = 0$ the metric is singular. However, this singularity is due to the choice of the coordinates that cover only half of the manifold. Indeed, in the neighbourhood of $\tau = 0$, (3.2.5a) becomes

$$ds^2 = d\tau^2 + \tau^2 d\chi^2 + Q^2 d\Omega_2^2\,. \tag{3.2.7}$$

The singularity at $\tau = 0$ can thus be removed going to Cartesian coordinates in the $(t,\, \chi)$ plane. This particular case has been classified by Gibbons and Hawking (1979) as a "bolt" singularity. In this case in the neighbourhood of $\tau = 0$ the topology is locally $\mathbb{R}^2 \times S^2$ with $\mathbb{R}^2$ contracting to zero as $\tau \to 0$. The coordinate nature of the singularity can be proved also by finding a new set of variables such that the whole Euclidean space is represented by a single chart. Let us define

$$\tau = Q \tan\frac{\xi}{2}\,, \tag{3.2.8}$$

where ξ is defined in the interval $(-\pi,\, \pi)$. Introduce the new coordinates

$$u = \frac{1-\cos\frac{\xi}{2}}{\sin\frac{\xi}{2}} e^{1/\cos\frac{\xi}{2}} \cos\chi\,, \tag{3.2.9a}$$

$$v = \frac{1-\cos\frac{\xi}{2}}{\sin\frac{\xi}{2}} e^{1/\cos\frac{\xi}{2}} \sin\chi\,. \tag{3.2.9b}$$

Using u and v, the line element becomes

$$ds^2 = Q^2\left(1+\frac{Q}{\sqrt{\tau^2+Q^2}}\right)^2 e^{-2\sqrt{\tau^2+Q^2}/Q}(du^2+dv^2)+(Q^2+\tau^2)d\Omega_2^2\,. \quad (3.2.10)$$

Since

$$u^2+v^2 = \frac{1-\cos\frac{\xi}{2}}{1+\cos\frac{\xi}{2}}e^{2/\cos\frac{\xi}{2}}\,, \quad (3.2.11a)$$

$$\frac{v}{u} = \tan\chi\,, \quad (3.2.11b)$$

the geodesics at constant χ are the straight lines passing through the origin while the geodesics at fixed τ are circles of radius

$$r = \sqrt{\frac{1-\cos\frac{\xi}{2}}{1+\cos\frac{\xi}{2}}e^{1/\cos\frac{\xi}{2}}}\,. \quad (3.2.12)$$

Let us now discuss the interpretation of the solution. The Euclidean instanton (3.2.5) can be joined to a real, Lorentzian-signature universe that is the "bounce" solution of the gravitational tunnelling. To find the hyperbolic metric describing the tunnelled spacetime, we have to investigate hyperbolic solutions of the coupled gravity and electromagnetic fields with the same symmetry as just discussed in the Euclidean case. We find

$$ds^2 = -dt^2+\bar{Q}^2\sin^2\left(\frac{t}{Q}\right)d\chi^2+Q^2 d\Omega_2^2\,, \quad (3.2.13a)$$

$$A(t) = -\bar{Q}\cos\left(\frac{t}{Q}\right)\,. \quad (3.2.13b)$$

Here, as before, the topology is $\mathbb{R}\times\Sigma$, $t\in\mathbb{R}$, and we will set $\bar{Q}=Q$. A simple analysis shows that the line element (3.2.13a) describes a non-isotropic universe with a constant two-sphere radius and a periodic one-sphere scale factor taking values in the interval $[0,Q]$. In the neighbourhood of $t=0$ the line element (3.2.13a) reduces to the form

$$ds^2 = -dt^2+t^2 d\chi^2+Q^2 d\Omega_2^2\,. \quad (3.2.14)$$

The electromagnetic field is well-defined for all values of the time. Analogously to the Euclidean case (3.2.5a), the singularity at $t=0$ can be removed; the curvature tensor is regular there even if at $t=0$ the physical size in χ is zero. Indeed, in the neighbourhood of $t=0$ the topology is locally $M^2\times S^2$ and the three-dimensional spatial hypersurface Σ of (3.2.13a), becomes homotopic to S^2 and a point. The metric (3.2.13a) describes a universe that periodically reproduces itself with period πQ.

The interpretation of the Euclidean instanton is related to the joining of (3.2.5) to the hyperbolic solution (3.2.13). The joining must obviously take place at $\tau = 0$, $t = 0$ where both the Euclidean and the hyperbolic solutions become extremal. Let us see this point in detail. Solution (3.2.5) with $\tau \in \mathbb{R}^-$ and (3.2.13) with $t \in \mathbb{R}^+$ satisfy the Darmois conditions for change of signature at $\tau = 0$, $t = 0$ (see for instance Ellis and Piotrkowska, 1994) since

i) the first and second fundamental forms of the three-dimensional hypersurfaces Σ in (3.2.5a) and (3.2.13a) are well-defined and coincide smoothly for $\tau \to 0^-$, $t \to 0^+$;

ii) the electromagnetic field is continuous with its derivative on the hypersurface $\tau = 0$, $t = 0$, where the change of signature occurs.

The electromagnetic field is well-behaved on the matching hypersurface $\tau = 0$, $t = 0$ because both the Euclidean and the hyperbolic manifolds are well-defined there. The regularity of the solution (3.2.5) and its asymptotic behaviour for $\tau \to \pm\infty$ (where the electromagnetic field vanishes) allow us to interpret that solution as describing a tunnelling between a flat vacuum Lorentzian spacetime and the spacetime described by the solution (3.2.13). Hence, the Euclidean solution (3.2.5) describes the nucleation of a non-isotropic baby universe starting from an original flat spacetime. The baby universe is nucleated in the phase of maximum shrinkage of the spatial three-metric.

To complete the discussion we must compute the probability amplitude for the formation of the baby universe. This can be performed at the semiclassical level evaluating the Euclidean path-integral of the sect. [3.1] on the saddle point. In the semiclassical approximation, the probability amplitude in a Planck volume and in a Planck time is given by (Vilenkin, 1984, 1988; Rubakov, 1984)

$$\Gamma_{\rm BU} = e^{-|\bar{S}_E|}\,, \tag{3.2.15}$$

where $\bar{S}_E$ represents the Euclidean action evaluated on the solutions of the classical equations of motion. Integrating the Euclidean action (3.2.1), we obtain

$$\bar{S}_E = \pi M_{\rm Pl}^2 Q^2\,, \tag{3.2.16}$$

where we have restored the Planck mass for clarity. The probability of formation of a baby universe in a Planck volume and in a Planck time is thus given by

$$\Gamma_{\rm BU} = \exp\left(-\pi M_{\rm Pl}^2 Q^2\right). \tag{3.2.17}$$

In order to have a probability of the order of unity, the constant Q appearing in the solution must satisfy the condition $Q^2 \approx M_{\rm Pl}^{-2}$. The nucleation probability is thus maximal for baby universes with dimensions of the order of the Planck lenght.

The above results can be easily generalised to the case of non-vanishing cosmological constant (Cavaglià, de Alfaro, and de Felice, 1994). Let us review briefly the formulae. Introducing for convenience a new Euclidean time coordinate $b \equiv b(\tau)$, the general solution is given by

$$ds^2 = \frac{b^2}{\lambda b^4 + b^2 - Q^2} db^2 + \bar{Q}^2 \frac{\lambda b^4 + b^2 - Q^2}{b^2} d\chi^2 + b^2 d\Omega_2^2 \,, \qquad (3.2.18a)$$

$$A(b) = -\bar{Q}\frac{Q}{b} \,, \qquad (3.2.18b)$$

where $\lambda = -\Lambda/3$. The solution (3.2.18) reduces to (3.2.5) when $\lambda = 0$ and $b^2 = Q^2 + \tau^2$.

The cases $\lambda > 0$ and $\lambda < 0$ must be discussed separately. In the first case the line element (3.2.18a) is defined for

$$b^2 > Q_0^2 \equiv \frac{\sqrt{1 + 4\lambda Q^2} - 1}{2\lambda} \,. \qquad (3.2.19)$$

With the transformation $b^2 = Q_0^2 + \tau^2$ the metric (3.2.18a) takes the form

$$\begin{aligned} ds^2 = &\frac{\tau^2}{\lambda(Q_0^2 + \tau^2)^2 + Q_0^2 + \tau^2 - Q^2} d\tau^2 + \\ &+ \bar{Q}^2 \frac{\lambda(Q_0^2 + \tau^2)^2 + Q_0^2 + \tau^2 - Q^2}{Q_0^2 + \tau^2} d\chi^2 + (Q_0^2 + \tau^2) d\Omega_2^2 \,, \end{aligned} \qquad (3.2.20)$$

where now $\tau \in \mathbb{R}$.

The asymptotic form of (3.2.20) for $\tau^2 \to \infty$ is

$$ds^2 = \frac{1}{\lambda \tau^2} d\tau^2 + \lambda \bar{Q}^2 \tau^2 d\chi^2 + \tau^2 d\Omega_2^2 \,. \qquad (3.2.21)$$

Contrary to the $\lambda = 0$ case, this is not a flat Euclidean space. Let us redefine the Euclidean time by

$$\tau = \pm \exp\left(\sqrt{\lambda \tau'^2}\right) \qquad (3.2.22)$$

so that the asymptotic form (3.2.21) becomes

$$ds^2 = d\tau'^2 + e^{2\sqrt{\lambda \tau'^2}} \left(\lambda \bar{Q}^2 d\chi^2 + d\Omega_2^2\right) . \qquad (3.2.23)$$

This line element defines an anisotropic universe whose scale factors expand exponentially; their ratio is fixed by the cosmological constant.

For $\lambda < 0$ (3.2.18a) is defined when

$$Q_-^2 < b^2 < Q_+^2 \,, \qquad (3.2.24)$$

where

$$Q_{\pm}^2 = \frac{1 \pm \sqrt{1-4|\lambda|Q^2}}{2|\lambda|} . \tag{3.2.25}$$

In this case, we can introduce the Euclidean proper time

$$\tau = \frac{1}{2\sqrt{|\lambda|}} \arcsin \left[\frac{2|\lambda|b^2 - 1}{\sqrt{1-4|\lambda|Q^2}} \right]. \tag{3.2.26}$$

We then have

$$\begin{aligned} ds^2 = d\tau^2 &+ \frac{\bar{Q}^2}{2} \frac{[1-4|\lambda|Q^2]\cos^2[2\sqrt{|\lambda|}\tau]}{1+\sqrt{1-4|\lambda|Q^2}\sin[2\sqrt{|\lambda|}\tau]} d\chi^2 \\ &+ \frac{1}{2|\lambda|}\left[1+\sqrt{1-4|\lambda|Q^2}\sin[2\sqrt{|\lambda|}\tau]\right] d\Omega_2^2 , \end{aligned} \tag{3.2.27a}$$

$$A(\tau) = -\frac{\bar{Q}Q\sqrt{2|\lambda|}}{\left[1+\sqrt{1-4|\lambda|Q^2}\sin[2\sqrt{|\lambda|}\tau]\right]^{1/2}} . \tag{3.2.27b}$$

The important feature of (3.2.27a), which will be relevant in the forthcoming discussion, is that of being a periodic solution in the Euclidean proper time τ. Analogously to the case of vanishing cosmological constant, the Euclidean solution (3.2.18) describes a tunnelling between two Lorentzian-signature regions. For instance, in the case $\lambda < 0$ the instanton (3.2.18) describes a tunnelling between the hyperbolic universes

$$ds^2 = -dt^2 + \sin^2\left[\omega_-\left(t + \frac{\pi}{4\sqrt{|\lambda|}}\right)\right] d\chi^2 + Q_- d\Omega_2^2 , \tag{3.2.28a}$$

and

$$ds^2 = -dt^2 + \sinh^2\left[\omega_+\left(t - \frac{\pi}{4\sqrt{|\lambda|}}\right)\right] d\chi^2 + Q_+ d\Omega_2^2 , \tag{3.2.28b}$$

where

$$\omega_{\pm}^2 = \sqrt{1-4|\lambda|Q^2}/Q_{\pm}^2 . \tag{3.2.29}$$

Here the tunnelling occurs when the one-sphere radius is zero; they notably describe an oscillating baby universe in (3.2.28a) and an ever expanding universe for $t > \pi/4\sqrt{|\lambda|}$ in (3.2.28b).

3.2.b Euclidean Solution as Tunnelling in Space.

The solution (3.2.5) can also be interpreted as an Euclidean wormhole joining two isometric, asymptotically flat spacetimes, described by Reissner-Nordström type of solutions.[14]

[14] Recently a great deal of attention has been devoted to Reissner-Nordstöm wormholes. See for instance Schein and Aichelburg (1996).

To see this, let us make a change of coordinates in (3.2.2) by substituting $\chi \to iT$ with T having the dimension of a length; for clarity we shall put $\tau \equiv r$. In the hyperbolic regime we find (Cavaglià, de Alfaro, and de Felice, 1994)

$$ds^2 = -\frac{r^2}{r^2 - Q^2} dT^2 + dr^2 + (r^2 - Q^2) d\Omega_2^2\,, \tag{3.2.30a}$$

$$A(r) = -\frac{Q^2}{\sqrt{r^2 - Q^2}}\,, \tag{3.2.30b}$$

where Q is a constant. The solution (3.2.30) is defined for $r^2 > Q^2$; at $|r| = |Q|$ there is a curvature singularity.

In the region $|r| < |Q|$ we have no hyperbolic solution. However, the solution (3.2.5) with $\tau \equiv r$, reduces to (3.2.30) if we Wick rotate the coordinate χ as $\bar{Q}\chi = iT$ and impose that the electric field remains real. Now, since (3.2.5) with $\tau \equiv r$ is well-behaved for $r \in \mathbb{R}$, we can interpret the two branches of solution (3.2.30) as joined by the Euclidean solution (3.2.5) at some $r^2 = Q^2 + \epsilon^2$ with ϵ arbitrary, via the complexification of χ as stated. With this procedure, setting $\epsilon \to 0^\pm$ we obtain an Euclidean wormhole joining the two branches of the solution (3.2.30). The interpretation of the Euclidean solution as a static wormhole differs from the customary one (see for instance eq. (3.2.5)). This is due to the particular nature of the electromagnetic field and to the coordinate used for the complexification. This fact can be at first sight quite surprising. However, we will see in sect. [3.4] that the quantum picture agrees with the above interpretation.

The parameter Q entering (3.2.30) can be given a particular meaning. Denoting $R^2 \equiv r^2 - Q^2$, we obtain

$$ds^2 = -\left(1 + \frac{Q^2}{R^2}\right) dT^2 + \left(1 + \frac{Q^2}{R^2}\right)^{-1} dR^2 + R^2 d\Omega_2^2\,, \tag{3.2.31}$$

where the radial coordinate R ranges in $\mathbb{R}^+$.

The line element in the form (3.2.31) can be regarded as of a Reissner-Nordström solution with effective gravitational mass equal to $-Q^2/2R$. However, we can deduce from eq. (3.2.4) that the constant Q is not a real charge since there are no physical charges in the field, but is a measure of the electric field flux through the wormhole throat at $R = 0$. Indeed, since the electric field is radial in R, its integral flux through a sphere containing the origin $R = 0$ is equal to

$$\Phi = 4\pi Q\,. \tag{3.2.32}$$

Therefore, the constant Q only determines the amount of flux that we want through any given surface containing the origin, similar to what is done by Giddings and Strominger (1988) for the axionic field. Thus, the electric field extends beyond the wormhole throat to the asymptotic infinities of the isometric spacetimes, generating in both cases an "apparent" charge Q.

In order to complete the discussion we might prove that the wormhole is traversable. This can be done studying the equation of motion for a test particle, having an electric charge per unit mass q , total specific energy E, and specific angular momentum L with respect to the flat infinity, that approaches $R = 0$. This is relevant since the particle can cross the wormhole throat only if it gets to $R = 0$ (classically or via quantum tunnelling). We omit here for brevity the discussion of this point. The reader may find the complete discussion in Cavaglià, de Alfaro, and de Felice (1994).

3.3 Generalisation to String Theory.

In this section we generalise the model of the previous section to string theory. We will see that the main results obtained for the Einstein gravity hold also for the low-energy effective string theory. In addition we will obtain new interesting results.

Our starting point is a generalised four-dimensional low-energy string effective action that takes into account, apart from the dilaton and the electromagnetic field, a modulus field which acquires non-minimal couplings to the gauge fields owing to string one-loop effects (see Cadoni and Mignemi, 1993, 1994 and references therein). The theory can be identified as a Jordan-Brans-Dicke gravity theory with Brans-Dicke parameter ω taking values in $[-1,\infty[$ and contains, as particular cases, both the dilaton-gravity theory of Garfinkle, Horowitz, and Strominger (1991), and Gibbons and Maeda (1988), as well the Einstein-Maxwell theory studied in the previous section.

Even though the solutions we are going to discuss do not seem to correspond, at least in the general case, to exact conformal string backgrounds, however they are interesting because the dimensional reduction of the four-dimensional theory on the background defined by the magnetically charged solutions produces a two-dimensional effective theory whose hyperbolic solutions have all the features of two-dimensional string cosmological solutions.

3.3.a Euclidean Solutions as Spacetime Tunnelling.

We start from the Euclidean action

$$S_E = \int_{\mathcal{M}} d^4x\,\sqrt{g}\,e^{-2\phi}\left[-R-4(\nabla\phi)^2+\frac{2}{3}(\nabla\psi)^2+\varepsilon F^2+\varepsilon e^{2\phi-(2/3)q\psi}F^2\right]+ \\ -2\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}\,e^{-2\phi}(K-K_0)\,, \tag{3.3.1}$$

where ϕ is the dilaton field, ψ a modulus field, and q is a coupling constant.

Following Cadoni and Mignemi (1993, 1994) we choose for the modulus field the ansatz

$$e^{-(2/3)q\psi} = \frac{3}{q^2} e^{-2\phi} . \tag{3.3.2}$$

Using (3.3.2), the action (3.3.1) reduces to the form

$$\begin{aligned} S_E = \int_{\mathcal{M}} d^4x \sqrt{g}\, e^{-2\phi} \left[-R + \tfrac{8k}{1-k} (\nabla\phi)^2 + \tfrac{3+k}{1-k} \varepsilon F^2 \right] \\ - 2 \int_{\partial\mathcal{M}} d^3x \sqrt{h} e^{-2\phi} (K - K_0) , \end{aligned} \tag{3.3.3}$$

where

$$k = \frac{3 - 2q^2}{3 + 2q^2}, \qquad -1 \leq k \leq 1 . \tag{3.3.4}$$

We have several interesting cases according to the value of k. For $k = -1$ (i.e. $q \to \infty$) the action reduces to the usual low-energy string action when the modulus ψ is not taken into account; (3.3.3) describes then the (Euclidean) four-dimensional dilaton-gravity theory considered by Garfinkle, Horowitz, and Strominger (1991), and by Gibbons and Maeda (1988). For $k = 0$ we have a four-dimensional action whose two-dimensional reduction gives the (Euclidean) Jackiw-Teitelboim theory (see for instance Cadoni and Mignemi, 1994). The case $k = 1$ looks singular. However, inserting $q = 0$ in the action (3.3.1) and using the equations of motion that enforce the dilaton to be constant, we recover the usual (Euclidean) Einstein-Maxwell theory. Using the ansatz (3.3.2) exact solutions can be obtained for any value of the coupling constant $k \in [-1, 1]$.

One can easily realise that the action (3.3.3) describes a Brans-Dicke theory coupled to the electromagnetic field. Indeed, the redefinition $\Phi = \exp(-2\phi)$ brings the action (3.3.3) in the Jordan-Brans-Dicke form with a Brans-Dicke parameter $\omega = 2k/1 - k$. As expected we recover general relativity ($\omega = \infty$) for $k = 1$.

Let us first consider the purely magnetic ansatz. A suitable configuration compatible with the topology of the spacetime is given by the magnetic monopole on the two-sphere:

$$\mathbf{F} = Q_m \sin\theta d\theta \wedge d\varphi , \tag{3.3.5}$$

where Q_m is the magnetic charge. Eq. (3.3.5) describes a purely magnetic field. Later on we shall consider a purely electric field with the only non-vanishing component along the χ direction as we have done in sect. [3.2]. We shall see that the solutions corresponding to the two choices (3.3.5) and (3.2.3) are related by a duality transformation.

The Euclidean solution of the equations of motion is (Cadoni and Cavaglià, 1995a)

$$ds^2 = d\tau^2 + 2^{(1-k)}Q^2 \frac{\tau^2}{\tau^2+Q^2}\left(1+\frac{Q}{\sqrt{\tau^2+Q^2}}\right)^{k-1} d\chi^2 + \\ + (\tau^2+Q^2)d\Omega_2^2, \qquad (3.3.6a)$$

$$\mathbf{F} = \frac{\sqrt{1-k}}{2} Q \sin\theta d\theta \wedge d\varphi\,, \qquad (3.3.6b)$$

$$e^{2(\phi-\phi_0)} = 2^{(1-k)/2}\left(1+\frac{Q}{\sqrt{\tau^2+Q^2}}\right)^{(k-1)/2}. \qquad (3.3.6c)$$

where we have redefined the magnetic charge Q_m through

$$Q_m = \frac{1}{2}\sqrt{1-k}\,Q\,. \qquad (3.3.7)$$

The solution (3.3.6) exists and is well-defined for any $-1 \leq k < 1$. For $k=1$ the redefinition of the magnetic charge (3.3.7) becomes singular. This is not surprising because the ansatz (3.3.2) is singular for $k=1$ (i.e. $q=0$); so, the solution for this particular case must be determined by starting directly from the action (3.3.1) with $q=0$. One can easily verify that the corresponding solution is described by (3.3.5) and (3.3.6a) where now $k=1$ and $Q_m = Q$ and coincide with the solution of sect. [3.2].[15]

As done in the previous section, let us study the properties of the solution. In the asymptotic regions $\tau \to \pm\infty$, the line element becomes

$$ds^2 = d\tau^2 + 2^{(1-k)}Q^2 d\chi^2 + \tau^2 d\Omega_2^2\,. \qquad (3.3.8)$$

Thus the asymptotic Riemannian space is flat with topology $R^3 \times S^1$. At $\tau = 0$, $\forall k \in [-1,1]$, the metric is singular. Analogously to sect. [3.2], the singularity is due to a bad choice of coordinates and it is possible to find a new chart that covers the whole space. One can easily verify this, observing that in the neighbourhood of $\tau = 0$, (3.3.6a) reduces to (3.2.7).

The asymptotic behaviour of the line element (3.3.6a) and its regularity allow to interpret the instanton (3.3.6) as a wormhole that connects two asymptotic flat regions. Again, the instanton can be joined at $\tau = 0$ with a cosmological solution. Hence, eq. (3.3.6) describes the nucleation of a baby universe starting

[15] The Euclidean solution (3.3.6) has been recently generalised to the case of multi-black hole instantons by Demelio and Mignemi (1996).

from an original flat region. The case of sect. [3.2] is of course a particular case of (3.3.6).

It is straightforward (see Cadoni and Cavaglià, 1995a) to find the magnetic charged solution of the Lorentzian-signature equations of motion:

$$ds^2 = -dt^2 + Q^2 \sin^2(t/Q) \left[\frac{1+\cos(t/Q)}{2}\right]^{k-1} d\chi^2 + Q^2 d\Omega_2^2\,, \quad (3.3.9a)$$

$$e^{2(\phi-\phi_0)} = \left[\frac{1+\cos(t/Q)}{2}\right]^{(k-1)/2}. \quad (3.3.9b)$$

It is interesting to study the properties of the solution (3.3.9). The line element (3.3.9a) describes a universe whose spatial sections are compact with topology $S^1 \times S^2$. The scale factor of the two-sphere is constant, while the radius of the one-sphere is periodic in time. The behaviour of the line element (3.3.9a) depends on k, so it is convenient to study separately the following cases:

a) $k = 1$, the Einstein-Maxwell theory. In this case the line element reduces to the one found in the previous section. The radius a of the one-sphere takes values in the range $[0, Q]$ and the line element is singular at $t = n\pi Q$, $n = 0, \pm 1, \pm 2, ...$ Thus eq. (3.3.9a) represents a universe which periodically reproduces itself with period πQ. The dilaton is constant.

b) $0 < k < 1$. Contrary to the previous case, when k takes values in the interval $]0, 1[$, the metric has a curvature singularity for $t = (2n + 1)\pi Q$ where the dilaton diverges and the theory becomes strong-coupled. At $t = 2n\pi Q$ there is a coordinate singularity analogous to the case a). The radius of the one-sphere vanishes for $t = n\pi Q$ and has a maximum for $\cos(t/Q) = (k-1)/(k+1)$. In this case (3.3.9a) describes a universe whose two-sphere scale factor remains constant, whereas the radius of the one-sphere vanishes at $t = 0$, grows till a maximum value and becomes again zero after a time $t = \pi Q$.

c) $k = 0$. In this case (3.3.9a) describes a periodic universe with period $2\pi Q$. The scale factor a vanishes for $t = 2n\pi Q$, where there is a coordinate singularity analogous to the case a), and takes its maximum value $a_{max} = 2Q$ when $t = (2n + 1)\pi Q$. Note that even though there are no curvature singularities, the dilaton diverges and the theory becomes strong-coupled for $t = (2n+1)\pi Q$.

d) $-1 \leq k < 0$. The scale factor a vanishes for $t = 2n\pi Q$, where the metric shows a coordinate singularity, and goes to infinity for $t = (2n + 1)\pi Q$ where the line element has a curvature singularity. Hence, the radius of the one-sphere starts with zero at $t = 0$ and grows to infinity at $t = \pi Q$. Note that $k = -1$ corresponds to the usual dilaton-gravity theory.

One can easily see that $\forall k \in [-1, 1]$ the solution (3.3.9) can be joined at $t = 0$ with the Euclidean instanton (3.3.6) and analogously to the previous section can be interpreted as the line element of a baby universe nucleated starting from an asymptotically flat region. Of course, the dilaton is continuous with its derivative on the hypersurface $t = \tau = 0$, where the change of signature occurs.

As far as the probability amplitude of nucleation of the baby universe (3.2.15) is concerned, after a straightforward calculation and taking into account the boundary terms to cancel the divergent contribution coming from the asymptotic region, one finds (we restore for a moment the Planck mass)

$$\Gamma_{\rm BU} = \exp\left[-\pi M_{\rm Pl}^2 e^{-2\phi_0} Q^2 (k+1)/2\right]. \tag{3.3.10}$$

For $k \neq -1$, in order to have a probability of the order of unity, the charge Q appearing in the solution must be of the order of the Planck lenght, so the nucleation probability is maximal for baby universes with dimensions of the order of the Planck length. Conversely, for the usual dilaton-gravity theory ($k = -1$), the semiclassical probability amplitude (3.3.10) does not determine the dimension of the baby universe, because one obtains $\Gamma_{\rm BU} = 1$ for any value of the charge Q. In this case, in order to fix the probability amplitude one must consider higher order contributions in the string tension α' to the low-energy string effective action.

We can use the solution (3.3.6) to compute the most probable value of the effective Brans-Dicke parameter. This can be done following the Coleman's argument for the vanishing of the cosmological constant (Coleman, 1988b), as in (Garay and Garcia-Bellido, 1993). Using a Jordan-Brans-Dicke theory with a cosmological constant Garay and Garcia-Bellido argue that the most probable value of the effective Brans-Dicke parameter is $\omega = \infty$, i.e. general relativity is the low-energy effective theory of gravity. Let us prove that the result of Garay and Garcia-Bellido holds also for the theory defined by the action (3.3.3). Using a dilute wormhole approximation (see sect. [3.1]) and taking into account that the main contribution to the path-integral comes from the classical Euclidean action evaluated at its saddle point, the Coleman's mechanism gives for the probability distribution $\mathcal{Z}(k)$ of the parameter k

$$\ln \mathcal{Z}(k) = \exp\left[32\pi^2 e^{-2\phi_0} Q_m^2 \frac{1+k}{1-k}\right], \tag{3.3.11}$$

where we have used eq. (3.3.7) to reinstate the full dependence of the action on the parameter k. The probability distribution for the coupling constant k is strongly peaked at $k = 1$, i.e the most probable value for the effective Brans-Dicke parameter is $\omega = \infty$, according to the results of Garay and Garcia-Bellido (1993).

To conclude this subsection, let us briefly discuss the solution obtained in presence of the purely electric field (3.2.3). We find

$$ds^2 = e^{4\phi_0}\left(1 + \frac{Q}{\sqrt{\tau^2 + Q^2}}\right)^{1-k} \Big[d\tau^2 + 2^{(1-k)} Q^2.$$

$$\left. \cdot \frac{\tau^2}{\tau^2+Q^2}\left(1+\frac{Q}{\sqrt{\tau^2+Q^2}}\right)^{k-1} d\chi^2 + (\tau^2+Q^2)d\Omega_2^2 \right] , \quad (3.3.12a)$$

$$\mathbf{F} = \frac{1}{2}\sqrt{1-k}Q^2 e^{2\phi_0} \frac{\tau}{\left(\tau^2+Q^2\right)^{3/2}} d\tau \wedge d\chi \, , \quad (3.3.12b)$$

$$e^{2(\phi-\phi_0)} = 2^{(k-1)/2}\left(1+\frac{Q}{\sqrt{\tau^2+Q^2}}\right)^{(1-k)/2} . \quad (3.3.12c)$$

Analogously to the purely magnetic configuration, one can easily verify that the solution (3.3.12) can be joined at $\tau = 0$ to the electric cosmological solution dual to (3.3.9)

$$ds^2 = e^{4\phi_0}\left[\frac{1+\cos(t/Q)}{2}\right]^{1-k}\left\{-dt^2 + Q^2\sin^2(t/Q)\cdot \right.$$

$$\left. \cdot \left[\frac{1+\cos(t/Q)}{2}\right]^{k-1} d\chi^2 + Q^2 d\Omega_2^2 \right\} , \quad (3.3.13a)$$

$$\mathbf{F} = \frac{1}{2}\sqrt{1-k}\; e^{2\phi_0}\sin(t/Q)dt\wedge d\chi \, , \quad (3.3.13b)$$

$$e^{2(\phi-\phi_0)} = \left[\frac{1+\cos(t/Q)}{2}\right]^{(1-k)/2} , \quad (3.3.13c)$$

at $t = 0$. Note that for $k = 1$ (3.3.9a) and (3.3.13a) coincide after taking $\phi_0 = 0$; indeed, in this case the dilaton is constant and the duality invariance holds also in the string frame. The solution (3.3.13) has properties analogous to the solution (3.3.9). The line elements differ only for a conformal factor because of the duality relation. The most striking difference between the two solutions resides in the fact that differently from (3.3.9) the scale factor for the two-sphere in (3.3.13a) is not constant. We shall see below that the two-dimensional section of the magnetic solution (3.3.9) can be described in terms of an effective two-dimensional theory obtained by dimensional reduction of the action (3.3.3). This is of course not possible for the electric solution. The probability amplitude for the nucleation of the baby universe coincides obviously with (3.3.10).

3.3.b Digression: Two-dimensional Reduction and Scale-Factor Duality.

The line element of the magnetic solution (3.3.9) has the form of a direct product of a two-dimensional solution and a two-sphere of constant radius. Thus, it is interesting to study the two-dimensional effective theory obtained by retaining only the time-dependent components of the four-dimensional metric. This two-dimensional theory is expected to describe the essential four-dimensional physics for perturbations around the background solution (3.3.9). The hyperbolic version of the action (3.3.3) can be dimensionally reduced by taking the angular coordinates to span a two-sphere of constant radius Q. The resulting two-dimensional action (in the hyperbolic regime) is

$$S = \int d^2x\sqrt{-g}e^{-2\phi}\left[R - \frac{8k}{1-k}(\nabla\phi)^2 + \lambda^2\right], \tag{3.3.14}$$

where $\lambda^2 = (1-k)/2Q^2$. This two-dimensional action has been studied in connection with its black hole solutions and its duality invariances by Cadoni and Mignemi (1994, 1995). In this section we shall study (3.3.14) from the cosmological point of view. As shown by Cadoni and Mignemi (1995) considering space-dependent field configurations, the action (3.3.14) possesses a duality symmetry. It is easy to see that this duality invariance also holds for time-dependent configurations. Let us consider the metric and the dilaton field of the form

$$ds^2 = -dt^2 + e^{2\rho(t)}dx^2, \qquad \phi = \phi(t)\,, \tag{3.3.15}$$

where $x \in \mathbb{R}^+ \cup \{0\}$. The action becomes

$$S = \int dt\, e^{-2\phi+\rho}\left[2(\ddot{\rho} + \dot{\rho}^2) + \frac{8k}{1-k}\dot{\phi}^2 + \lambda^2\right], \tag{3.3.16}$$

where dots represent time-derivatives. One can check that the transformation

$$\rho \to k\rho - 2(k+1)\phi, \qquad \phi \to \frac{k-1}{2}\rho - k\phi \tag{3.3.17}$$

leaves the action invariant modulo a total derivative. The duality transformation (3.3.17) is the generalisation for the action (3.3.1) of the scale-factor duality symmetry of string theory (Veneziano, 1991; Gasperini and Veneziano, 1992; Tseytlin and Vafa, 1992). Indeed, for $k = -1$ we get the standard scale-factor duality transformation $\rho \to -\rho, \quad \phi \to \phi - \rho$ which exchanges the radius of the two-dimensional universe with its inverse.[16]

Let us now discuss the cosmological solutions of the two-dimensional theory and their behaviour under the duality transformation (3.3.17). The time-dependent solution of the action (3.3.14) is

$$ds^2 = -dt^2 + \sin^2(t/2Q)\left[\cos^2(t/2Q)\right]^k dx^2\,, \tag{3.3.18a}$$

$$e^{2(\phi-\phi_0)} = \left[\cos^2(t/2Q)\right]^{(k-1)/2}. \tag{3.3.18b}$$

[16] See sect. [4.4]. This result has been recently generalised by Lidsey (1995) and in 2+1 dimensions by Cadoni (1996).

Considering a periodic space, i.e. setting $x = 2Q\chi$, $0 \leq \chi < 2\pi$, the solution (3.3.18) coincides with the two-dimensional section of solution (3.3.9). The effect of the duality transformation (3.3.17) on the solution (3.3.18) is to exchange the sine and the cosine everywhere:

$$ds^2 = -dt^2 + \cos^2(t/2Q)\big[\sin^2(t/2Q)\big]^k dx^2\,, \tag{3.3.19a}$$

$$e^{2(\phi-\phi_0)} = \big[\sin^2(t/2Q)\big]^{(k-1)/2}\,. \tag{3.3.19b}$$

The dual solution corresponds of course to a solution of the four-dimensional theory. Moreover, for $k = 1$ (3.3.18) and (3.3.19) are the same, i.e the solution is self-dual. Comparing eq. (3.3.18) with eq. (3.3.19), one realise that the effect of the duality transformation (3.3.17) on the solutions with $k \neq 1, 0$ is to exchange the coordinate singularities at $t = 2n\pi Q$ with the curvature singularities at $t = (2n+1)\pi Q$. For $k = 0$ there are no curvature singularities and the duality transformation simply exchanges strong string couplings with weak ones.

The previous cosmological solutions are further examples of the two-dimensional string cosmologies studied by Veneziano (1991), Gasperini and Veneziano (1992), and Tseytlin and Vafa (1992). They exhibit all the peculiar properties of string cosmological solutions such as the above-discussed duality invariance. In particular for $k = -1$ the solutions (3.3.18) and (3.3.19) correspond to well-known $D = 2$ cosmological conformal string backgrounds (Veneziano, 1991; Gasperini and Veneziano, 1992; Tseytlin and Vafa, 1992). However, for generic k we do not know if the interpretation of (3.3.18) and (3.3.19) as conformal string backgrounds can be maintained. In this context the case $k = 0$ seems very interesting. We have seen that the cosmological solution describes an universe which periodically reproduces itself without encountering a singularity, thus avoiding the singularity-problem which affects the models with $k \neq 0, 1$.

To conclude this digression, let us discuss the relationship between the two-dimensional cosmological solutions and the corresponding two-dimensional black hole geometries. For the particular case $k = -1$, it has been already shown that the cosmological solution (3.3.18) describes the region between the horizon and the singularity of the black hole geometry derived from (3.3.14) (Tseytlin and Vafa, 1992)

$$ds^2 = -4Q^2\tanh^2(x/2Q)d\tau^2 + dx^2\,, \tag{3.3.20a}$$

$$e^{2(\phi-\phi_0)} = \big[\cosh(x/2Q)\big]^{-2}\,. \tag{3.3.20b}$$

This construction can be generalised for arbitrary k. Consider the metric (3.3.18a) expressed in terms of the periodic coordinate $\chi = x/2Q$ and choose the new coordinates

$$u = e^{t^*+\chi}\,, \qquad v = e^{t^*-\chi}\,,$$

$$t^* = \frac{1}{2Q}\int \frac{dt}{\sin(t/2Q)\cos^k(t/2Q)} = \frac{y^{(1-k)/2}}{k-1}F\left(\frac{1-k}{2},1,\frac{3-k}{2},y\right), \tag{3.3.21}$$

where F is the hypergeometric function and $y = \cos^2(t/2Q)$. The line element (3.3.18a) becomes

$$ds^2 = -4Q^2\frac{(1-y)y^k}{uv}dudv\,. \tag{3.3.22}$$

An identical form for the metric can be obtained starting from the black hole solution of the action (3.3.14) (Cadoni and Mignemi, 1994)

$$ds^2 = -4Q^2\sinh^2(x/2Q)\big[\cosh(x/2Q)\big]^{2k}d\tau^2 + dx^2\,, \tag{3.3.23a}$$

$$e^{2(\phi-\phi_0)} = \big[\cosh(x/2Q)\big]^{k-1}\,, \tag{3.3.23b}$$

and introducing the coordinates

$$u = e^{x^*+\tau}\,, \qquad v = e^{x^*-\tau}\,,$$

$$x^* = \frac{1}{2Q}\int \frac{dx}{\sinh(x/2Q)\cosh^k(x/2Q)} = \frac{y^{(1-k)/2}}{k-1}F\left(\frac{1-k}{2},1,\frac{3-k}{2},y\right), \tag{3.3.24}$$

but now with $y = \cosh^2(x/2Q)$. In (3.3.21) y takes values in the interval $[0,1]$, whereas in (3.3.24) $1 \le y < \infty$. Hence, the solution (3.3.22) describes the region between the horizon and the singularity of the black hole solution (3.3.23).[17]

As shown by Tseytlin and Vafa (1992) for the case $k = -1$, one can continue the time past the singularity at $t = \pi Q$ to get an identical copy of the interior of the black hole where the universe now starts at the singularity and evolves till it reaches zero size at $t = 2\pi Q$. By continuing this procedure, i.e. not identifying $t \to t + 2\pi Q$, we end up with an universe which undergoes infinitely many oscillations. However, this construction cannot be taken too seriously because near the singularity, where the size of the universe becomes infinitely large, one cannot trust anymore the low-energy string effective action (3.3.14) and one should consider the exact theory. We will not discuss this point further, we just note

[17] Strictly speaking, because we are working with χ periodic, this correspondence holds for a wedge in the region between the horizon and the singularity (see Tseytlin and Vafa, 1992).

that our model with $k = 0$ avoids the singularity problem. Indeed, for this value of k the scalar curvature stays everywhere finite and only the dilaton diverges at $t = (2n + 1)\pi Q$ indicating that the theory becomes strong-coupled.

3.3.c Euclidean Solutions as Instability of the Vacuum.

The instanton (3.3.6) can also describe a different physical process taking place in the theory. Indeed, using a different analytical continuation to the hyperbolic space, the solution (3.3.6) can be interpreted as a semiclassical decay process of the ground state (vacuum) of the low-energy effective string theory (Cadoni and Cavaglià, 1995b).

The existence of a process of semiclassical decay is important since it may lead to the instability of the vacuum of the theory. Furthermore, a careful analysis of the geometric and topological features of the instanton will enable us to identify (3.3.6) also as a Hawking-type wormhole (Hawking, 1988) connecting two asymptotic regions with topology $\mathbb{R}^3 \times S^1$.

Here we shall follow an approach similar to the one used by Witten (1982) to prove the semiclassical instability of the Kaluza-Klein vacuum in five dimensions. Even though the theory considered here has little to do with the Kaluza-Klein theory in five dimensions, both instantons have common geometrical and topological features and consequently most of the mathematical techniques used by Witten can be implemented in this case.

Let us consider the solution (3.3.6) written in the form ($\chi \to 2^{(k-1)/2}\chi$, $\phi_0 \to \phi_0 + (k-1)\ln(2)/4$)

$$ds^2 = \left(1 - \frac{Q^2}{r^2}\right)^{-1} dr^2 + Q^2 \left(1 + \frac{Q}{r}\right)^{k-1} \left(1 - \frac{Q^2}{r^2}\right) d\chi^2 + + r^2 d\Omega^2 \,, \quad (3.3.25a)$$

$$\mathbf{F} = \frac{\sqrt{1-k}}{2} Q \sin\theta d\theta \wedge d\varphi \,, \quad (3.3.25b)$$

$$e^{2(\phi-\phi_0)} = \left(1 + \frac{Q}{r}\right)^{(k-1)/2} . \quad (3.3.25c)$$

The crucial point for the identification of (3.3.25) with a vacuum decay process is the analytical continuation of the line element to the hyperbolic space. Hence, let us discuss the geometric and topological properties of the Euclidean manifold described by (3.3.25a). Since the latter has by definition signature $(+, +, +, +)$, r can take values only in the range $[Q, \infty[$. For $r \to \infty$ the space is asymptotically flat with topology $\mathbb{R}^3 \times S^1$. For $r = Q$ the metric tensor is singular. However, in $r = Q$ the manifold is smooth, as one can see putting $r = \sqrt{Q^2 + \tau^2}$ ($\tau \in \mathbb{R}$) and defining χ as a periodic variable with period $2\pi \cdot 2^{(1-k)/2}$ (see the previous section). This conclusion seems to indicate that the coordinate system $(r, \chi, \theta, \varphi)$ does not cover the whole manifold. In order to obtain the maximal extension of the Euclidean metric (3.3.25a) we have to perform the coordinate transformation

$$r = \frac{(x^2 + \tau^2) + Q^2}{2\sqrt{x^2 + \tau^2}}, \qquad \tan\theta = \frac{x}{\tau}, \tag{3.3.26}$$

whose inverse is given by

$$x = f(r)\sin\theta, \qquad \tau = f(r)\cos\theta, \tag{3.3.27}$$

where

$$f(r) = \sqrt{x^2 + \tau^2} = Q\exp[\text{arccosh }(r/Q)]. \tag{3.3.28}$$

The coordinate transformation (3.3.27,28) is never singular. Using (3.3.26) the Euclidean solution (3.3.25) reads

$$\begin{aligned} ds^2 = \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 \left[d\tau^2 + dx^2 + x^2 d\varphi^2\right] + \\ + Q^2\left(1 - \frac{2Q^2}{f^2 + Q^2}\right)^2 \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{k-1} d\chi^2, \end{aligned} \tag{3.3.29a}$$

$$\mathbf{F} = \frac{1}{2}\sqrt{1-k}\, Q\frac{x}{f^3}\left[x\ d\tau \wedge d\varphi \ - \ \tau\ dx \wedge d\varphi\right], \tag{3.3.29b}$$

$$e^{2(\phi - \phi_0)} = \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{(k-1)/2}. \tag{3.3.29c}$$

Eq. (3.3.29a) represents the maximal extension of (3.3.25a). As before, when $x, \tau \to \infty$ the manifold is asymptotically flat with topology $\mathbb{R}^3 \times S^1$. The critical surfaces are $x^2 + \tau^2 = Q^2$ and $x^2 + \tau^2 = 0$. Using the coordinate transformation (3.3.26-28) it is easy to verify that the first critical surface corresponds to $r = Q$ and the second one to $r = \infty$. Eq. (3.3.25a) describes thus two asymptotically flat regions smoothly joined through the surface $r = Q$. This strange structure is related to the existence of a conformal equivalence between the region inside $x^2 + \tau^2 = Q^2$ and the region outside. Indeed, the solution (3.3.25) is invariant under the transformation

$$y^\mu \to \frac{Q^2}{y^2} O^\mu{}_\nu y^\nu, \qquad \mu, \nu = 1, 2, 3. \tag{3.3.30}$$

Here y^μ are Cartesian coordinates of the three-dimensional space $y^1 = t$, $y^2 = x\cos\varphi$, $y^3 = x\sin\varphi$, and $O^\mu{}_\nu$ is a 3×3 rotation matrix. As a consequence, the line element (3.3.29a) represents a Hawking-type wormhole (Hawking, 1988) that connects two asymptotically flat spaces with topology $\mathbb{R}^3 \times S^1$. The minimum radius of the wormhole is equal to Q.

In order to interpret (3.3.25) as a vacuum decay process, we have to continue analytically the Euclidean solution to the hyperbolic spacetime. In the previous sections the analytical continuation has been performed first by defining $\tau = \sqrt{r^2 - Q^2}$ thereafter by the complexification of τ, $\tau \to i\tau$. Finally, the resulting hyperbolic manifold has been interpreted as a baby universe of spatial topology $S^2 \times S^1$ nucleated at $\tau = 0$. However, we can perform different analytic continuations. For instance, we can complexify the θ coordinate of the two-sphere S^2. In this case, since $\theta = 0$ is a coordinate singularity of the metric, it is convenient to choose as symmetry plane the surface $\theta = \pi/2$ and to set

$$\theta \to \frac{\pi}{2} + i\xi\,. \tag{3.3.31}$$

Using (3.3.31) we obtain the hyperbolic solution

$$ds^2 = \left(1 - \frac{Q^2}{r^2}\right)^{-1} dr^2 + Q^2 \left(1 + \frac{Q}{r}\right)^{k-1} \left(1 - \frac{Q^2}{r^2}\right) d\chi^2 + \\ - r^2 d\xi^2 + r^2 \cosh^2 \xi d\varphi\,, \tag{3.3.32a}$$

$$\mathbf{F} = Q_m \cosh \xi d\xi \wedge d\varphi\,, \tag{3.3.32b}$$

$$e^{2(\phi - \phi_0)} = \left(1 + \frac{Q}{r}\right)^{(k-1)/2}\,. \tag{3.3.32c}$$

The electromagnetic field is real, and for $r \geq Q$ the spacetime is nonsingular, the coordinate singularity at $r = Q$ being harmless as it is for the Euclidean space (3.3.25). The line element for $r \geq Q$ represents the spacetime in which the $\mathbb{R}^3 \times S^1$ vacuum decays. The topology of the initial $\xi = 0$ hypersurface is $\mathbb{R}^2 \times S^1$. Note that the analytic continuation to the hyperbolic space of the previous section, even though it has been obtained from the same Euclidean instanton, has instead spatial topology $S^2 \times S^1$.

The topology of the analytic continuation to the hyperbolic space depends thus on the coordinate chosen to complexify. A better understanding of the features of this space can be achieved starting from the hyperbolic line element that covers only the region $r \geq Q$. Using the coordinate transformation

$$x = f(r) \cosh \xi\,, \qquad t = f(r) \sinh \xi\,, \tag{3.3.33}$$

where $f(r) = \sqrt{x^2 - t^2}$ is defined as function of r as in eq. (3.3.28), we obtain

$$ds^2 = \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 \left[-dt^2 + dx^2 + x^2 d\varphi^2\right] + \\ + Q^2\left(1 - \frac{2Q^2}{f^2 + Q^2}\right)^2 \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{k-1} d\chi^2 \,, \quad (3.3.34a)$$

$$\mathbf{F} = \frac{1}{2}\sqrt{1-k}Q\frac{x}{f^3}\left[x\ dt \wedge d\varphi \ - \ t\ dx \wedge d\varphi\right] \,, \quad (3.3.34b)$$

$$e^{2(\phi - \phi_0)} = \left(1 + \frac{2Qf}{f^2 + Q^2}\right)^{(k-1)/2} \,. \quad (3.3.34c)$$

Since $-1 \leq t/x \leq 1$, the new coordinates (x, t) do not cover the whole plane. They cover only the region outside to the light cone $x \pm t = 0$, corresponding to the physical region. Analogously to the Euclidean case, the critical surfaces are two: $x^2 - t^2 = Q^2$ corresponding to $r = Q$, and $x^2 - t^2 = 0$ representing the infinity (see figure 3.1). Of course, the manifold described by (3.3.34a) is geodesically complete and its topology is $\mathbb{R}^3 \times S^1$. Regions I and II in figure 3.1 are analogous to the Euclidean ones and their conformal equivalence can be proved using a coordinate transformation similar to (3.3.30).

The region II is the starting point for the vacuum decay interpretation of the Euclidean instanton. One can easily verify that the origin of the Euclidean plane (x, t) – coinciding with an asymptotically flat infinity – is not the only surface one can use to pass to the hyperbolic regime. At $\tau = 0$ we can join the Euclidean manifold described by (3.3.29a) with a hyperbolic spacetime, namely the region $x^2 - t^2 > Q^2$ of the spacetime (3.3.34a) (region II in figure 3.1). Indeed, at $\tau = 0$ the metric, the dilaton and the electromagnetic field assume a minimal configuration, so the extrinsic curvature vanishes and the joining is possible. The hyperbolic spacetime in which the vacuum decays is the region II in figure 3.1. Let us explore in detail its properties. Due to the maximal analytic extension, the regions on the left and on the right of the plane (x, t) are identical, so attention is focused on one of them. Choosing for simplicity hypersurfaces at constant χ, the line element (3.3.34a) becomes conformally equivalent to a $\mathbb{R}^3$ flat Lorentzian spacetime. Of course the manifold is not geodesically complete, since there exist geodesics crossing the boundary $x^2 - t^2 = Q^2$. The meaning of the boundary can be understood following its time evolution. Starting at $t = 0$, as t becomes larger and larger, the coordinate x of the boundary grows according to $x = \sqrt{Q^2 + t^2}$ (right region). Since the coordinate x corresponds to a radius in the cylindrical system of coordinates (t, x, φ), the boundary can be interpreted as a hole in space starting with radius Q at $t = 0$ and growing up for $t > 0$. At $t = 0$ the electromagnetic field is a purely electric field in the φ direction $E_\varphi = Q_m/x$; as the time t flows and E_φ changes in intensity, the latter generates a magnetic field

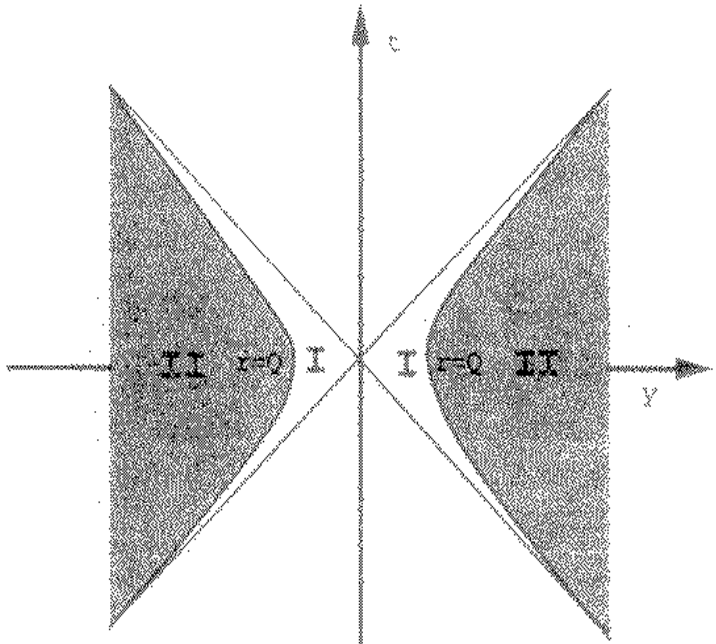

Figure 3.1: *The two-dimensional section of the hyperbolic space described by the line element (3.3.34a). The physical region corresponds to the shaded part (region II) of the picture enclosed by the hyperbola* $x^2 - t^2 = Q^2$.

in the perpendicular χ direction. Finally, when $x, t \to \infty$, the electromagnetic field vanishes, as expected because the spacetime is asymptotically flat. The Euclidean line element (3.3.29a) represents thus the decay process of the flat spacetime of topology $\mathbb{R}^3 \times S^1$ in a spacetime with a growing hole.

In conclusion, the Euclidean instanton (3.3.29) represents either a wormhole or a vacuum decay process according to the null-extrinsic curvature hypersurface used for the analytic continuation to the hyperbolic spacetime.

The previous results can be straightforwardly extended to the purely electric electromagnetic field configuration (3.2.3). Indeed, we have seen in the previous section that in this case the line element differs from (3.3.25) only by the conformal factor $e^{4\phi_0}(1 - Q/r)^{1-k}$, so the conclusion remains unchanged.

At this stage we can ask ourselves if the semiclassical vacuum decay process is consistent with energy conservation. Since the $\mathbb{R}^3 \times S^1$ vacuum has zero energy, the space (3.3.32a) in which it decays must also have zero energy. Using the ADM formula generalised to dilaton-gravity theories, the total energy of (3.3.32) can be calculated as usual by means of a surface integral depending on the asymptotic behaviour of the gravitational and dilaton fields. The line element (3.3.34a) is not static with respect to t, so the integral must be evaluated at the initial $t = 0$ surface, corresponding in (3.3.32a) to $\xi = 0$. The result of the integration is zero.

Indeed, the terms of the gravitational and dilaton fields which contribute to the total energy of the solution are those of order $1/r$. However, in our case these terms give a null contribution to the energy, owing to the $\mathbb{R}^2 \times S^1$ topology of the $\xi = 0$ surface. The spacetime described by (3.3.32) has thus zero energy. This feature makes the $\mathbb{R}^3 \times S^1$ vacuum not stable for the theory defined by (3.3.1), since there exists a solution with zero energy and the same asymptotic behaviour as the $\mathbb{R}^3 \times S^1$ vacuum. As a consequence, the positive energy theorem (Schoen and Yau, 1979) does not hold for the theory (3.3.1) if one considers vacua with topology $\mathbb{R}^3 \times S^1$.[18] The failure of the positive energy theorem seems related to the presence of the electromagnetic field: in the $\mathbb{R}^3 \times S^1$ vacuum there exist excitations of the electromagnetic field for which the positive energy theorem does not hold.

The interpretation of the Euclidean solution (3.3.25) as an instability process of the vacuum has been established using the analytical continuation (3.3.31). Considering a further analytical continuation to the hyperbolic spacetime, we have also seen that the instanton can be interpreted as a Hawking-type wormhole. The latter has an intrinsically three-dimensional nature because its topology is $\mathbb{R}^3 \times S^1$ and the radius of S^1 is equal to Q in the two asymptotic regions $f = \infty$, $f = 0$ and shrinks to zero for $r = Q$. Hence, the most natural interpretation of this solution can be found in the context of a $3+1$ Kaluza-Klein theory.

Starting from the action (3.3.1), setting to zero the components of the electromagnetic field along the χ direction and splitting the four-dimensional line element as

$$ds^{(4)} = ds^{(3)} + Q^2 e^{-2\psi} d\chi^2 \,, \tag{3.3.35}$$

after some manipulations we obtain the three-dimensional action:

$$S_E = \int_{\mathcal{M}} d^3x \sqrt{g^{(3)}} e^{-2\sigma} \left[-R^{(3)} + \tfrac{k-1}{2} [4(\nabla\sigma)^2 - (\nabla\eta)^2] - \tfrac{3+k}{1-k} F^2 \right] , \tag{3.3.36}$$

where $\sigma = \phi + \psi/2$, $\eta = \psi + 2\phi(k+1)/(k-1)$, and we have dropped the boundary terms.

A solution of the ensuing equations of motion is

$$\begin{aligned} ds^2 &= \frac{1}{4}\left(1 + \frac{Q^2}{f^2}\right)^2 \left[d\tau^2 + dx^2 + x^2 d\varphi^2\right] , \\ e^{2(\sigma-\sigma_0)} &= \frac{f^2+Q^2}{f^2-Q^2} , \qquad e^{2(\eta-\eta_0)} = \left(\frac{f+Q}{f-Q}\right)^2 , \end{aligned} \tag{3.3.37}$$

[18] The positive energy theorem states that every non-flat, asymptotically Lorentzian solution of the Einstein equations has positive energy. However, its validity for spacetimes with arbitrary topology and theories as (3.3.1) has not been yet proved.

where $f = \sqrt{x^2+\tau^2}$ and we have chosen the electromagnetic tensor $\mathbf{F}$ as in (3.3.29b). The solution of the three-dimensional theory is thus a Hawking-type wormhole connecting two asymptotic regions of topology $\mathbb{R}^3$.

Finally, let us compute the decay rate of the vacuum. Evaluating the action (3.3.1) on the Euclidean solution (3.3.25) as done for (3.2.17) and (3.3.10) we have (as usual we reintroduce the Planck mass)

$$S_E = \pi M_{\rm Pl}^2 e^{-2\phi_0} Q^2 (k+1)/4 \,. \tag{3.3.38}$$

This result has been obtained by integrating r and θ in the range $Q \le r < \infty$, $0 \le \theta \le \pi/2$, the appropriate one for the vacuum decay process. The rate of decay of the $\mathbb{R}^3 \times S^1$ vacuum is

$$\Gamma_{VD} = \exp\left[-\pi M_{\rm Pl}^2 e^{-2\phi_0} Q^2 (k+1)/4\right] . \tag{3.3.39}$$

The vacuum is long-lived for values of Q much greater than the Planck length and becomes unstable when Q is of the same order of magnitude of the Planck lenght. Finally, it is interesting to compare the vacuum decay rate Γ_{VD} with the probability for the nucleation of a baby universe $\Gamma_{\rm BU}$ in (3.3.10)

$$\Gamma_{\rm BU} = (\Gamma_{\rm VD})^2 \,. \tag{3.3.40}$$

The probability of nucleation of a baby universe is thus smaller than the probability of the vacuum decay.

3.4 Beyond the Semiclassical Interpretation.

The final stage in the discussion of our minisuperspace model is its quantisation. We have mentioned in the first section that the path-integral (3.1.1) satisfies formally the Wheeler-De Witt equation of the system (Halliwell and Hartle, 1991). So in order to go beyond the semiclassical approximation of sect. [3.2] and sect. [3.3] we have to investigate the solutions of the Wheeler-DeWitt equation.[19] We shall see later that the study of the Wheeler-DeWitt equation opens the way to a new interpretation of the solutions found in previous sections. At the end of the section we shall see that an analogous interpretation holds also for the Bianchi I spacetime.

Our starting point is again the action (3.2.1), the line element (3.2.2) and the ansatz (3.2.3).[20] Substituting (3.2.2) and (3.2.3) in (3.2.1) and neglecting surface terms[21] the action density in the minisuperspace becomes

$$S_E = 2\int d\tau \left[-\frac{a\dot{b}^2}{N} - 2\frac{\dot{a}b\dot{b}}{N} - aN + \frac{b^2}{aN}\dot{A}^2\right] , \tag{3.4.1}$$

[19] In this section we shall deal with the quantisation *à la* Dirac as usual in the literature.

[20] We shall discuss only the Einstein gravity theory. The generalisation to the dilaton case is straightforward (see Cavaglià, 1994a).

[21] We may do that, as we are only interested in the solutions of the equations of motion.

where dots represent derivatives with respect to τ. From (3.4.1) the Einstein equations can be recovered considering formally $a(\tau)$, $N(\tau)$, $b(\tau)$, and $A(\tau)$ as Lagrangian coordinates evolving in τ. We have seen in chapt. 2 that $N(\tau)$ acts as a Lagrange multiplier and thus imposes the Hamiltonian constraint $H = 0$. Of course, in the derivation of the classical solutions we have the freedom to choose N. The Euclidean instanton (3.2.5) coincides with the solution derived from (3.4.1) by setting $N = 1$.

The first step towards the Dirac quantisation of the system is to find the classical expression of the Hamiltonian constraint. The latter can be cast in a quadratic form in the canonical momenta using the change of variables (canonical transformation) $\{a, b\} \to \{x, y\}$

$$a = f \equiv f(x)\,, \tag{3.4.2a}$$

$$b = \frac{h(y)}{f(x)}\,, \tag{3.4.2b}$$

where $f(x)$ and $h(y)$ are arbitrary functions of x and y respectively. We obtain the extended Hamiltonian

$$H_E = u\left[-\frac{1}{(h_{,y}/h)^2}p_y^2 + \frac{1}{(f_{,x}/f)^2}p_x^2 + 16h^2 + f^2p_A^2\right] = 0\,, \tag{3.4.3}$$

where we have defined the Lagrange multiplier $u = Nf/8h^2$. p_x, p_y, and p_A are the canonical conjugate momenta of x, y, and A respectively.

Now we can quantise the system *à la* Dirac substituting quantum operators for classical quantities (see chapt. 2). Starting from the classical extended Hamiltonian (3.4.3) written in the form

$$H_E = u\left[p_\xi^2 - p_\eta^2 + e^{2\xi}p_A^2 + 16e^{2\eta}\right]\,, \tag{3.4.4}$$

where $f(x) = \exp[\xi(x)]$ and $h(y) = \exp[\eta(y)]$, and choosing the Laplace-Beltrami ordering for the kinetic part of the Hamiltonian,[22] the Wheeler-DeWitt equation becomes separable

$$\left[-\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2} + 16e^{2\eta} + e^{2\xi}\frac{\partial^2}{\partial A^2}\right]\Psi(\xi, \eta, A) = 0\,. \tag{3.4.5}$$

Now it is straighforward to write the solutions of (3.4.5) that are eigenstates of the operator $\hat{p}_A$

$$\Psi_{(\nu,\omega)}(f, h, A) = C(\nu, \omega)K_{i\nu}(\omega f)K_{i\nu}(4h)e^{i\omega A} \tag{3.4.6}$$

[22] In chapt. 5 we shall see that this choice corresponds essentially to choose a measure invariant for the gauge and rigid transformations of the system.

or, in terms of the scale factors a and b:

$$\Psi_{(\nu,\omega)}(a,b,A) = C(\nu,\omega)K_{i\nu}(\omega a)K_{i\nu}(4ab)e^{i\omega A}\,, \tag{3.4.7}$$

where ν and ω are real parameters and $K_{i\nu}$ is the modified Bessel function of index $i\nu$ (see Bateman, 1953).

Before dealing with gauge fixing and complete the quantisation of the model, let us discuss the physical meaning of the solution (3.4.7) and focus attention on the behaviour in the variable b. The wave functions (3.4.7) are singular at the origin, where they oscillate an infinite number of times since Ψ goes like $b^{\pm i\nu}$ as one approaches $b = 0$. The wave functions oscillate for $4ab < |\nu|$ (Lorentzian region), and are exponentially damped for $4ab > |\nu|$ (Euclidean region). The oscillating region of (3.4.7) can be interpreted as the quantum analogue of (3.2.13) with an initial singularity at $b = 0$ and maximum dimension $4ab \approx \nu$.

Let us now discuss the asymptotic behaviour of the wave functions (3.4.7) for large values of the scale factor b. From (3.4.7) we have

$$\Psi \sim \exp(-4ab)\,. \tag{3.4.8}$$

This is the asymptotic behaviour corresponding to the flat $\mathbb{R}^3\times S^1$ space.[23] Hence, for any value of ν the wave functions (3.4.7) represent quantum wormholes joining two asymptotically flat regions with topology $\mathbb{R}^3 \times S^1$. The size of the throat is defined by the condition $4ab = |\nu|$ and does not depend on the scale factor a nor on the parameter ω. Since $\omega \in \mathbb{R}$ there is a real flux of the electric field through any closed hypersurface due to the electric field along χ. The physical meaning of this flux will be clear in a moment.

Considering a linear superposition of the wave functions (3.4.7)

$$\Psi' = \int d\nu f(\nu)\Psi_\nu\,, \tag{3.4.9}$$

we can find regular wormhole wave functions. For instance, choosing the Kontorovich-Lebedev transform (see Bateman, 1954)

$$f(\nu) = \nu \tanh(\pi\nu)\,, \tag{3.4.10}$$

we obtain the wave function

$$\Psi'_\omega(a,b,A) = C'(\omega)\frac{\sqrt{b\omega}}{\omega+4b}e^{-a(\omega+4b)}e^{i\omega A}\,, \qquad \omega > 0\,. \tag{3.4.11}$$

We can easily verify that (3.4.11) is regular and its asymptotic behaviour for $b \to \infty$ coincides with (3.4.8). Again, the wave function (3.4.11) can be interpreted

[23] See eqs. (3.1.5,6) and Cavaglià (1994a, 1994b).

as a wormhole generated by the electromagnetic field joining two flat $\mathbb{R}^3 \times S^1$ regions. Analogously to (3.4.7), the solution (3.4.11) is an eigenfunction of the operator $\hat{p}_A$, so there is a real flux through any closed hypersurface . The physical interpretation of (3.4.11) must take some care since the regularity at small three-geometries means that the space "closes regularly" there, so the electromagnetic flux cannot go through the wormhole throat.

This property makes (3.4.11) very different from any other quantum wormhole solutions known in the literature. We have seen that there is a real flux both for singular and regular wave functions. For the solutions (3.4.7) this is not surprising, because one can imagine the flux coming out or going into the singularity at the origin. However, for (3.4.11) where can the flux go?

To answer to this question and shed light on the physical interpretation of the solutions, we have to consider the structure of the Euclidean electromagnetic field (3.2.3). The ansatz (3.2.3) represents a purely electric field along the χ direction, i.e. an electromagnetic field whose dynamics is confined in the one-sphere with radius a. The asymptotic behaviour of the wave function for large three-geometries and the radius of the throat depend on b. Hence, the dynamics of the electromagnetic field is decoupled from the dynamics of the wormhole and the flux of the electric field must coincide for regular and non-regular wave functions.

Even though there are no physical charges in the field equations[24], the observer in the asymptotically flat region measures a real finite flux and sees an apparent charge in the origin. Thus the geometry must be non trivial. Otherwise, since there are no physical sources for the electromagnetic field, the Gauss law should imply a vanishing flux through any closed surface around the origin. Further, the solution (3.4.11) describes an asymptotically flat space, so the electromagnetic field is confined in a finite region. We conclude that (3.4.11) describes the quantum analogue of an electromagnetic geon because the charge can be seen as an electric field trapped in a finite region of space, without any source.

The interpretation sketched above agrees with the classical picture of sect. [3.2] – see eq. (3.2.31). As shown there, a macroscopic observer measures an apparent electric charge Q even though physical charges are absent. Since the spacetime is non-trivial, the electric field extends beyond an Euclidean region joined to the isometric Reissner-Nordström type spacetimes via a classical change of signature occurring in the naked singularity at $R = 0$. Indeed, the latter can be "expanded" and continued analitically in the Euclidean space, where the solution is regular. This Euclidean region describes the wormhole. The wave function (3.4.11) is the quantum analogue of this picture. In the full quantum treatment we avoid the problems of the classical case, namely the *ad hoc* change of signature: the solution (3.4.11) is regular everywhere and thus describes a geometry that closes regularly.

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[24] See eq. (3.2.4)

In the classical theory, the lines of force of the electric field are convergent at $R = 0$, where the Lorentzian spacetime becomes singular. So we need a little trick, namely the transition from a Lorentzian to an Euclidean region. In the quantum theory, the solution is regular everywhere and no tricks are necessary.

These results depend of course on the ansatz (3.2.3) chosen for the electromagnetic field but seem not to depend on the Kantowski-Sachs geometry. Indeed, we will see later that identical conclusions can be drawn for the Bianchi I model.

Up to now the gauge has not been fixed and we have no Hilbert space. In order to complete the quantisation of the model, let us discuss how the Hilbert space can be recovered in the Dirac method. Since the signature of the minisuperspace metric is positive-defined in the sector $\{\xi, A\}$, the sector $\{\eta, p_\eta\}$ is the best candidate for gauge fixing. Analogously to the relativistic free massive particle of sect. [2.4] in order to implement the gauge fixing with the Dirac method it is convenient to perform a canonical transformation in the sector $\{\eta, p_\eta\}$ and pass to action-angle variables (see Arnold, 1978). We write

$$\theta = \frac{1}{2H_\eta}\text{arcsinh}\left[\frac{H_\eta}{4}e^{-\eta}\right], \qquad p_\theta = -H_\eta^2, \tag{3.4.12}$$

where $H_\eta^2 = p_\eta^2 - 16e^{2\eta}$. Using the new variables, the extended Hamiltonian (3.4.4) becomes

$$H_E = u\left[p_\xi^2 + e^{2\xi}p_A^2 + p_\theta\right]. \tag{3.4.13}$$

The solutions (3.4.6,7) of the Wheeler-DeWitt equation that are eigenstates of $\hat{p}_A$ read in the new variables

$$\Psi_{(\nu,\omega)}(a,\theta,A) = C(\nu,\omega)K_{i\nu}(\omega a)e^{-\nu^2\theta + i\omega A} \tag{3.4.14}$$

Using in (2.3.6) the quantum measure (see chapt. 5)

$$d[\alpha] = \frac{da}{a}\,dA\,d\theta = d\xi\,dA\,d\theta, \tag{3.4.15}$$

and imposing the gauge

$$F \equiv \theta - \tau = 0, \tag{3.4.16}$$

where τ is a parameter, we obtain $\Delta_{FP} = 1$ and the scalar inner product (2.3.6) can be written

$$(\psi_2, \psi_1) = \int \frac{da}{a}dA\,\psi_2^*\,\psi_1, \tag{3.4.17}$$

One can easily check that the solutions (3.4.14) of the Wheeler-DeWitt equation with a suitable normalisation factor form an orthonormal basis in the Hilbert space defined by (3.4.17)

$$(\psi_2, \psi_1) = \int_0^\infty \frac{da}{a}\,dA\,\psi^*_{(\nu_2,\omega_2)}(a,A)\psi_{(\nu_1,\omega_1)}(a,A) = \delta(\nu_1 - \nu_2)\delta(\omega_1 - \omega_2). \tag{3.4.18}$$

To conclude this section let us discuss briefly the Bianchi I model. In this case the line element reads

$$ds^2 = N^2(\tau)d\tau^2 + a^2(\tau)d\chi^2 + b^2(\tau)d\theta^2 + c^2(\tau)d\varphi^2\,, \qquad (3.4.19)$$

where χ, θ and φ are defined in the interval $[0, 2\pi[$. We choose the electric field along the χ direction, analogously to what done for the Kantowski-Sachs case.[25]

The asymptotic behaviour of the wormhole wave functions in terms of the scale factors is (Cavaglià, 1994b)

$$\Psi(a, b, c) \sim \exp\left[-\frac{2}{N}\frac{d}{d\tau}(abc)\right]_\tau. \qquad (3.4.20)$$

Note that the asymptotic behaviour of the wormhole wave functions does not depend on the structure constants of the three-dimensional isometry group G generating the homogeneous hypersurface (see for instance Jantzen, 1984; Ryan and Shepley, 1975), so (3.4.20) holds for the all Bianchi and Kantowski-Sachs models. Using the classical Euclidean equations of motion and the Hamiltonian constraint we find for the Bianchi I space (Cavaglià, 1994b)

$$\Psi \sim e^{-\omega a (a^\rho b^\sigma c^\lambda)/2}, \qquad \rho = \sigma + \lambda \quad \text{or} \quad \rho = 2(\sigma + \lambda) - 1, \qquad (3.4.21)$$

where σ and λ are two arbitrary positive parameters. The wormhole wave functions behave for large three-geometries essentially in two different ways according to the asymptotic three-metric. This occurs because the asymptotic region with minimal gravitational excitation is not unique and does not have the topology $\mathbb{R}^3 \times S^1$ as it happens, for instance, to the Kantowski-Sachs model. Hence, the wormhole wave functions that for large three-geometries behave as

$$\Psi \sim e^{-\omega a^2\sqrt{bc}/2} \qquad (3.4.22)$$

represent Riemannian spaces asymptotically of the form

$$a = (\alpha\tau)^{-1/2}, \qquad b = c = (\alpha\tau)^{1/2}, \qquad (3.4.23a)$$

and

$$a = (\alpha\tau)^{-1/3}, \qquad b = c = (\alpha\tau)^{2/3}. \qquad (3.4.23b)$$

Wave functions with behaviour (3.4.22) correspond thus to wormholes joining two asymptotically flat regions with metric (3.4.23a,b) respectively.

[25] Clearly, identical results can be obtained choosing θ or φ directions.

Now we are able to find wormhole solutions for the Bianchi I model. Choosing the Hawking-Page prescription for the factor ordering (Hawking and Page, 1990), the Wheeler-DeWitt equation can be cast in the form

$$\Delta\Psi(a,b,c,A) = 0, \tag{3.4.24}$$

where Δ is the Laplace-Beltrami covariant operator in the minisuperspace:

$$\begin{aligned}\Delta = \frac{2}{c}\partial_a\partial_b + \frac{2}{b}\partial_a\partial_c + \frac{2}{a}\partial_b\partial_c - \frac{a}{bc}\partial_a^2 - \frac{b}{ac}\partial_b^2 - \frac{c}{ab}\partial_c^2 + \\ + \frac{1}{bc}\partial_a - \frac{1}{ac}\partial_b - \frac{1}{ab}\partial_c - \frac{a}{bc}\partial_A^2.\end{aligned} \tag{3.4.25}$$

A set of solutions of (3.4.24) is

$$\Psi(a,b,c,A;p,k,\omega) = \frac{1}{\sqrt{bc}} a^{i(p+k)/2} b^{ip/2} c^{ik/2} K_{i\sqrt{pk}}(\omega a) e^{i\omega A}, \tag{3.4.26}$$

where p, k and ω are real constants and K is the modified Bessel function. As in the Kantowski-Sachs case, the wave functions (3.4.28) oscillate an infinite number of times approaching the origin. They are damped for large values of a but not for large b and c because they oscillate in the (b,c) plane. This feature is not surprising, since we have chosen a electromagnetic field that "lives" in the one-sphere with radius a, so its dynamics does not depend on b and c. Using the variables $\log b$ and $\log c$, we see that the oscillating factors $b^{ip/2}$ and $c^{ik/2}$ look formally as $e^{i\omega A}$. So, as far as the dynamics of the wormhole is concerned, the scale factors b and c behave essentially as matter fields and together with A determine the extent of the wormhole mouth. Note that for real values of ω there is a real flux of the electric field through any closed $T^2(\theta,\phi)$ surface due to the electric field along χ. The factor $1/\sqrt{bc}$ is eliminated by the covariant measure in the integral when we deal with matrix elements.

Analogously to the Kantowski-Sachs case, we can find regular wormhole wave functions. Let us put $p = \xi^2 k$, where ξ is a real number, and take the Fourier transform. We obtain (see for instance Bateman, 1954)

$$\begin{aligned}\Psi(a,b,c,A;\mu,\xi,\omega) &= \int dk e^{i\mu k}\Psi(a,b,c,A;k,\xi,\omega) \\ &= \frac{1}{\sqrt{bc}} e^{i\omega A} e^{-\omega a \cosh[\log(a^{\sigma+\lambda} b^{\sigma} c^{\lambda})+\mu]},\end{aligned} \tag{3.4.27}$$

where $\sigma = \xi/2$ and $\lambda = 1/2\xi$. The wave functions (3.4.27) represent Riemannian spaces asymptotically of the form (3.4.23a) and are again eigenfunctions of the operator $\hat{p}_A$, so there is a real flux through any closed $T^2(\theta,\phi)$ surface analogously to (3.4.26).

Choosing $\xi = 1$ we obtain

$$\Psi(a,b,c,\mu,\omega) = \frac{1}{\sqrt{bc}} e^{i\omega A} e^{-\omega a \cosh\left[\log(a\sqrt{bc})+\mu\right]}. \tag{3.4.28}$$

Now, let us put for simplicity $\mu = 0$. The solution (3.4.28) can be cast in the form

$$\Psi(a,b,c,A;\omega) = \frac{1}{\sqrt{bc}} e^{-\omega a^2\sqrt{bc}/2} e^{-\omega/2\sqrt{bc}} e^{i\omega A}, \tag{3.4.29}$$

which coincides with a Kontorovich-Lebedev transform (see Bateman, 1954) of (3.4.26) with respect to the index $k = p$. Using a different type of Kontorovich-Lebedev transform we can find a further solution

$$\Psi(a,b,c,A;\omega) = \frac{1}{\sqrt{bc}}\left(a^2\sqrt{bc} - \frac{1}{\sqrt{bc}}\right) e^{-\omega a^2\sqrt{bc}/2} e^{-\omega/2\sqrt{bc}} e^{i\omega A}. \tag{3.4.30}$$

The asymptotic behaviour of (3.4.29) and (3.4.30) suggests to use in the Wheeler-DeWitt equation the new variables

$$\eta^2 = \omega a^2\sqrt{bc}/2, \qquad \xi^2 = \omega/2\sqrt{bc}. \tag{3.4.31}$$

Recalling (3.4.24) we obtain

$$\Psi_n(\eta,\xi,A;\omega) = \psi_n(\eta+\xi)\psi_n(\eta-\xi)\xi^2 e^{i\omega A}, \tag{3.4.32}$$

where $\psi_n(x)$ is the harmonic wave function of order n:

$$\psi_n(x) = \frac{1}{\left(2^n n!\sqrt{\pi}\right)^{1/2}} H_n(x) e^{-x^2/2}. \tag{3.4.33}$$

Solutions (3.4.29) and (3.4.30) correspond (apart from normalisation factors) to Ψ_0 and Ψ_1. As in the Kantowski-Sachs case, we have a non-zero flux even if the wave functions are regular for small three-geometries.

The physical interpretation of these solutions as geons and the gauge fixing implementation follow the lines of the Kantowski-Sachs case (Cavaglià, 1994b).

3.5 Results.

Let us summarise the results that we have obtained in this chapter. We have investigated an Euclidean minisuperspace model with symmetry $\mathbb{R} \times S^1 \times S^2$. The gravitational field is coupled to the electromagnetic field. The study has been performed both for Einstein gravity (classical and quantum) and in the low-energy string theory regime. The Euclidean solutions of the classical equations of motion and the solutions of the Wheeler-DeWitt equation have been interpreted

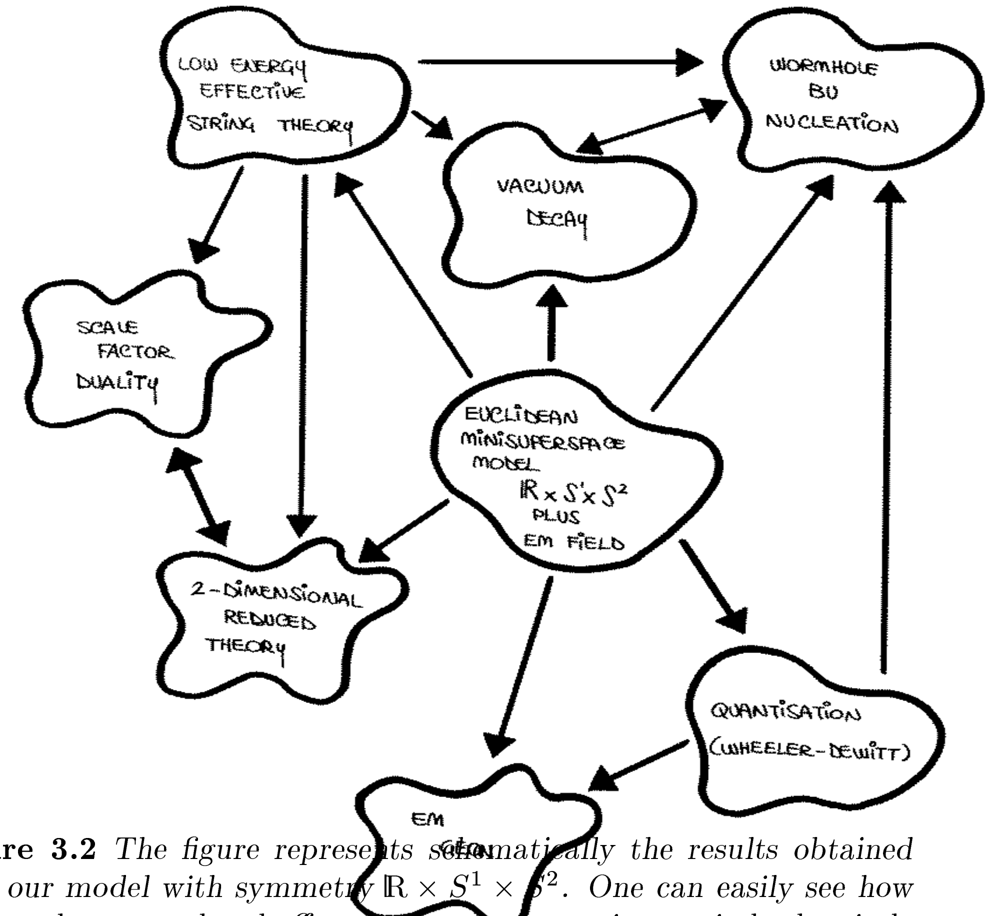

Figure 3.2 *The figure represents schematically the results obtained from our model with symmetry $\mathbb{R} \times S^1 \times S^2$. One can easily see how apparently uncorrelated effects in quantum gravity are indeed strictly related in the phenomenological regime.*

as wormholes. Further, the model leads to different interpretations according to the framework in which the analysis is completed.

The Euclidean instanton found in sect. [3.2] can be interpreted in several ways, depending on the Wick-rotated coordinate. This is an important property that remains true also in the low-energy string effective theory (sect. [3.3]). The fact that different complexifications of a given Euclidean solution may lead to different interpretations in the Lorentzian regime is an unexpected and intriguing result. This property opens the way to interesting speculations about the relations between typical Planck-scale phenomena (topology changes, singularities, decay of the vacuum...) and/or low-energy phenomena (effective theory of gravity at low-energy scales, renormalisation of coupling constants...) – see figure 3.1. For instance, we have seen in sect. [3.3] that in the Jordan-Brans-Dicke regime the Euclidean instanton can be interpreted either as a tunnelling process leading to the nucleation of a baby universe starting from a flat vacuum region, or as describing a semiclassical decay process of the ground state of the low-energy string effective theory. Further, the solution is characterized by the Brans-Dicke parameter and its behaviour depends crucially on this parameter. This fact has been used to show that the probability distribution for the effective coupling constant is strongly peaked at the value of the effective Brans-Dicke parameter corresponding to general relativity (Coleman's mechanism). The above result connects quantum gravity effects at Planck-scales with low-energy properties of the gravitational field (the value of coupling constants). In this sense we can say that the model gives "experimental predictions".

We have also seen that the theory can be dimensionally reduced to a two-dimensional theory that exhibits a scale-factor duality symmetry which is a generalisation of the duality symmetries found for exact cosmological conformal string backgrounds. For a particular value of the Brans-Dicke parameter the two-dimensional solution reduces to a well-known string conformal background (Veneziano, 1991; Gasperini and Veneziano, 1992; Müller, 1990; Tseytlin and Vafa, 1992). These two-dimensional time-dependent solutions describe the region between the horizon and the singularity of the black hole solution of the two-dimensional theory and lead to possible speculations about the relation between black holes and wormholes (see chapt. 5).

Finally, in sect. [3.4] we have discussed the model from the point of view of canonical quantisation. We have seen that the solutions can also be interpreted as the quantum corresponding of an Einstein-Rosen-Misner-Wheeler electromagnetic geon since the electric charge can be viewed as electric lines of force trapped in a finite region of spacetime. This interpretation makes more transparent and mathematically coherent the semiclassical suggestion of sect. [3.2] and perhaps can shed light on the meaning of classical singularities in general relativity.

In conclusion, these results give a striking example of the physical information that can be extracted from minisuperspace models. Even though we do not expect our model really to explain the meaning of singularities, the vacuum decay, or other processes possibly taking place in quantum gravity, however it represents an intriguing "phenomenological" picture useful to obtain some insight on these issues and so it is worth to be explored.

4

Minisuperspaces in Quantum Cosmology: Early Universe

This chapter is devoted to minisuperspace models in cosmology. The idea of investigating the birth of the universe from the quantum point of view (quantum cosmology) dates back to the beginning of the research in quantum gravity (see chapt. 1). Indeed, the present description of the hot big bang universe is classical; quantisation of the metric and matter fields seems appropriate to the investigation of the very early stages. Hence, in the past two decades the hypotesis of the birth of the universe as a quantum effect has raised a great interest.

We have stressed in chapt. 2 that the investigation of quantum cosmological models is very appealing for two reasons: *i)* there is not a complete quantum theory of gravitation yet, in spite of the efforts and much progress achieved; *ii)* in the general present picture of the theoretical foundations of physics, general relativity is believed to be the proper theory at low-energy scales and a more fundamental theoretical structure is needed in order to represent physics at the initial scales, when the universe gets very near to the singularity (see chapt. 1). It is then plausible that for intermediate scales, of order of few Planck lenghts, a reduced quantum representation is effective, namely a universe with a finite number of degrees of freedom, similarly to what happens in electrodynamics: at small energies the quantum mechanics of a proton and an electron is all we need to understand the hydrogen atom and $p - e^-$ scattering. In turn this quantum representation of the universe as a finite-dimensional system in phase space gives rise to a region of large scale factors where the semiclassical approximation holds and the transition to a classical universe is thus achieved. Finally, the formulation of a consistent quantum theory for these gravitational models with a finite number of degrees of freedom is by itself a very interesting problem.

The outline of the chapter is the following. After a few general considerations, in sect. [4.2] I shall present and discuss the quantisation procedure for models describing the primordial universe filled with different fields. In the next section I shall illustrate in detail the model of a FRW universe filled with radiation fields derived from Einstein gravity. Sections [4.4-5] deal with a model for the birth of the universe derived from string theory: sect. [4.4] is devoted to illustrate the issue of the formation of the universe evolving from the perturbative vacuum of string theory; the quantisation of the model and the *graceful exit* (see Gasperini et al., 1996; Gasperini and Veneziano, 1996) will be the subject of sect. [4.5]. Finally, in the last section I shall draw conclusions and introduce briefly the issue of third quantisation.

The content of this chapter (except sect. [4.4]) is original work done by the author in collaboration with Prof. Vittorio de Alfaro (Università di Torino) and Prof. Alexandre T. Filippov (JINR, Dubna) (see Cavaglià and de Alfaro, 1994; 1996b; Cavaglià, de Alfaro, and Filippov, 1995a; 1995b; 1995c; Cavaglià, 1994c).

4.1 The Wave Function of the Universe.

The idea of investigating the early universe using finite-dimensional reduced models dates back to the end of sixties (Misner, 1969) (see chapt. 1). In the last years quantum cosmology has undergone a renewed interest since the definition of the so-called *wave function of the universe* by Hartle and Hawking (1983) and Vilenkin (1984).

The importance of minisuperspace models in cosmology (see sect. [2.6]) is due to the fact that these models are characterized by a finite number of gravitational and matter degrees of freedom. This property allows to discuss at a simpler level the conceptual problems of quantum gravity as well as give a physical description – in first approximation – of the possible quantum effects that took place during the first stages of the evolution of the universe (see for instance Anini, 1991). Indeed, the standard (classical) cosmological model cannot be arbitrarily extended backward in time without running into inconsistencies. This implies that the universe was in a non-classical regime at a given time of its history. In this regime a quantum treatment of a few degrees of freedom could make sense (Misner 1969, 1972).

The usual starting point for quantum cosmology is the Friedmann-Robertson-Walker (FRW) ansatz for the line element, even though in the last years more complicated models, like Kantowski-Sachs and Bianchi models, have been discussed. The use of the FRW ansatz is based on the observation that the universe is today essentially homogeneus and isotropic and that the deviation from homogeneity and isotropy has been growing during its thermal history. The FRW ansatz is then appropriate to discuss the early stages of the universe if we neglect typical field effects as spacetime foam, topology changes, etc...

The goal of the quantum description of the universe is to obtain the wave function of the universe to investigate quantum mechanically the birth of the universe (*tunnelling from nothing*).[1] The first main problem in completing this task lies in

[1] See Vilenkin (1984, 1988); Hartle and Hawking (1983).

the fact that one deals with a closed system (the entire universe). This is strictly related to the problem of the determination of time evolution (see sect. [2.6]) and has an important consequence on the definition of the observables. Indeed, there are no “external observers” to the universe able to perform measurements. A re-interpretation of the usual quantum mechanics (the Copenhagen interpretation) is then necessary. We have seen in sect. [2.6] that a careful canonical analysis of the reparametrisation properties of minisuperspace models allows to overcome these problems.

The wave function of the universe can be obtained essentially through two complementary procedures: the path-integral formalism and the standard operator quantisation. In the first case the wave function is defined starting from a path-integral similar to the one introduced in sect. [3.1] to describe wormholes.[2] In the last years a great debate has been developing about boundary conditions to be imposed on the path-integral to obtain the correct wave function of the universe – see Hartle and Hawking (1983) and Vilenkin (1984, 1995). The main problem is the interpretation of the wave function so defined. The naive interpretation claims that the wave function simply gives the probability amplitude for a given configuration. This interpretation suffers from the problem that a inner product is not defined. Indeed, in quantum cosmological models there is a residual invariance under the reparametrisation of time, similar to what we find for the free massive particle in special relativity, so in the path-integral approach their canonical treatment requires an analysis of the gauge invariance and the reduction of the redundant degrees of freedom by the BRST formalism. In the standard approach the quantisation of the system can be completed along the lines discussed in chapt. 2. Solving the Wheeler-DeWitt equation is not sufficient to determine the Hilbert space. Indeed, the Wheeler-DeWitt equation, fundamental as it is, contains well-known ambiguities about which much has been written: absence of time, absence of conserved current, choice of boundary conditions, interpretation of the wave functions and normalisation, and so on. The gauge fixing procedure allows to overcome these ambiguities. We shall see in the following that the reduction method is the best suited procedure for quantum cosmological models.

4.2 Cosmological Models Coupled to Different Fields.

We shall apply the ideas exposed in sect. [2.6] to some simple general cases of relevance to the discussion of FRW or de Sitter models coupled to zero-modes of different fields (Cavaglià, de Alfaro, and Filippov, 1995a; 1995b; 1995c). In the next section we shall apply the techniques discussed here to models derived from Einstein general relativity; sect. [4.5] will be devoted to the low-energy effective string theory.

[2] Of course the signature is now Lorentzian.

For the models under consideration the constraint (2.6.3) has the form

$$H \equiv \frac{1}{2}p_0^2 + V(q_0) - H_1(q_i, p_\alpha) = 0\,, \tag{4.2.1}$$

where we may identify q_0 as the gravitational degree of freedom, even though this is not always needed (see next section).

In the relevant models the constraint turns out to be separable. Let us assume that the zeroth degree of freedom has the form of a harmonic oscillator: we shall see in the next section that this happens, for instance, when we deal with a conformal scalar field or a FRW closed metric.[3] So we have

$$V(q_0) = \frac{1}{2}q_0^2\,, \tag{4.2.2a}$$

$$H_1 \equiv H_1(q_\alpha, p_\alpha)\,. \tag{4.2.2b}$$

We choose the canonical transformation (2.6.5) as

$$P_0 = p_0 + q_0 \mathrm{ctg}\ t\,, \tag{4.2.3a}$$

$$Q_0 = q_0\,, \tag{4.2.3b}$$

and the gauge-fixing condition to be of the form (see chapt. 2)[4]

$$F \equiv p_0 + q_0\ \mathrm{ctg}\ t = 0\,. \tag{4.2.4}$$

By the method exposed in sect. [2.6] one obtains a very interesting result for the effective Hamiltonian on the gauge shell:

$$H_{\mathrm{eff}} = \ H_1\,. \tag{4.2.5}$$

The simplicity of (4.2.5) shows the interest of the gauge-fixing (4.2.4). Note that the gauge identity (4.2.4) fixes the Lagrange multiplier as

$$u(t) = -1\,. \tag{4.2.6}$$

Thus $u(t)$ is t-independent. Finally, the quantisation can be completed writing the Schrödinger equation as in (2.3.16). In this problem the natural choice of the global time is thus arctg q_0/p_0 as one sees from (4.2.4).

Intuitively it is appealing to identify q_0 (or even p_0) with time. Let us see what happens if one assumes gauge-fixing conditions of the form

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[3] See also Cavaglià, de Alfaro, and Filippov (1995a, 1995b, 1995c).

[4] Gauge-fixing conditions of this kind were introduced by Filippov (1989) in the context of a gauge approach to systems of many relativistic particles.

$$p_0 = \sqrt{2}\, t\,, \qquad \text{or} \qquad q_0 = -\sqrt{2}\, t\,. \tag{4.2.7}$$

We have respectively

$$u(t) = -\frac{\sqrt{2}}{q_0}, \qquad u(t) = -\frac{\sqrt{2}}{p_0}. \tag{4.2.8}$$

With this choice the effective (non-local) Hamiltonian is time-dependent:

$$H_{\text{eff}} = \pm 2\sqrt{H_1 - t^2}. \tag{4.2.9}$$

The positive-definiteness of the operator under square root implies that the support of q_0 or p_0 (and thus of the time) is restricted, in agreement with the general properties of the oscillatory motion in q_0, p_0.[5] One sees the advantage of the gauge choice (4.2.4) since in that case H_{eff} is independent of time and local.[6]

Using the general approach outlined in sect. [2.6] it is easy to find the gauge transformation relating both gauges[7]. Let us distinguish the variables in the two gauges by the superscript (i), where $i = 1$ for the gauge (4.2.4) and $i = 2$ for the gauge $q_0 = -\sqrt{2}t$. Then $u^{(1)} = u^{(2)} + \dot{\tau}$ and

$$q_0^{(1)} = q_0^{(2)} \cos(\tau) + p_0^{(2)} \sin(\tau)\,, \;\; p_0^{(1)} = -q_0^{(2)} \sin(\tau) + p_0^{(2)} \cos(\tau)\,, \tag{4.2.10}$$

where

$$\tau \equiv \tau(t) = -t + \arcsin\left(\frac{t}{H_1}\right). \tag{4.2.11}$$

Note that since

$$H_1(q_\alpha, p_\alpha) = \frac{1}{2}(p_0^2 + q_0^2)\,, \tag{4.2.12}$$

is a gauge-invariant function of the variables, q_α, p_α need not be transformed.[8]

In the next section we shall encounter a quartic potential when we shall discuss the zero-mode of the Yang-Mills field and the case of the cosmological constant. So let us discuss the gauge-fixing when $V(q_0)$ of (4.2.1) has the form

$$V(q_0) = kq_0^2 - \lambda q_0^4\,, \tag{4.2.13}$$

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5 Indeed we have $H_1 - t^2 > 0$ that implies $p_0^2 < 2H_1$ or $q_0^2 < 2H_1$ for (4.2.7) respectively.

6 Thus the gauge fixing conditions (4.2.7) are not suited for the quantisation of the system.

7 It is not so easy for the more complicated Lagrangians considered below because the equations for the gauge transformations are non-linear.

8 Of course a non-compact oscillator requires hyperbolic functions in place of circular.

and (4.2.2b) holds. In this case it is convenient to choose as new canonical coordinates (2.6.5)

$$P_0 = p_0 - \sqrt{2\lambda}\big(q_0^2 + g(t)\big)\,, \tag{4.2.14a}$$
$$Q_0 = q_0\,, \tag{4.2.14b}$$

where $g(t) = t^2 - k/2\lambda$. The gauge-fixing condition is

$$p_0 = \sqrt{2\lambda}\big(q_0^2 + g(t)\big)\,, \tag{4.2.15}$$

which together with (4.2.1) allows to fix the Lagrange multiplier $u(t)$ as

$$u(t) = -\frac{1}{\sqrt{2\lambda}tq_0}\,. \tag{4.2.16}$$

The effective Hamiltonian in the physical degrees of freedom becomes

$$H_{\rm eff} \;= \sqrt{2\lambda}\;\dot{g}(t)Q_c\,. \tag{4.2.17}$$

Now Q_c must be obtained from the constraint as in (2.6.8). We have

$$Q_c^2 = \frac{1}{k + 2\lambda g(t)}(H_1 - \lambda g^2(t))\,, \tag{4.2.18}$$

and thus

$$H_{\rm eff} = \pm 2\sqrt{H_1 - \lambda(t^2 - k/2\lambda)^2}\,. \tag{4.2.19}$$

It is evident that the final form of $H_{\rm eff}$ is not simple and is time-dependent.[9] Time is essentially fixed by (4.2.15) as a function of p_0 and q_0.

Now let us give some hints about the case of a general potential in (4.2.1).[10]

Eq. (2.6.8) that defines $Q_c(q_\alpha, p_\alpha; t)$ is now

$$\left[\frac{1}{2}\big(f_{,Q_0}(Q_0, Q_\alpha; t)\big)^2 + V(Q_0) - H_1(Q_0, Q_\alpha, P_\alpha + f_{,Q_\alpha})\right]_{Q_0 = Q_c} = 0\,. \tag{4.2.20}$$

From (4.2.20) we see that a good choice for f is $f \equiv f(Q_0; t)$; now one needs to connect $f_{,Q_0}$ to $f_{,t}$. We may proceed for instance as follows. Set

$$\frac{1}{2}\big(f_{,q}(q;t)\big)^2 + V(q) \;=\; g(t)\;F(q)\,, \tag{4.2.21}$$

[9] The particular case k=0, λ=0 has been recently discussed by Lemos (1996). In this case the effective Hamiltonian does not depend on time and a sensible quantum mechanics can be developed.

[10] Often this procedure is not the best one to follow as it happens, for instance, in the case just discussed.

where $g(t)$ and $F(q)$ are arbitrary functions of t and q respectively. Then

$$f(Q_0;t) \;=\; \sqrt{2}\int^{Q_0} dq\ \left[H_1(q,Q_\alpha,P_\alpha)-V(q)\right]^{1/2}, \tag{4.2.22}$$

and the effective Hamiltonian has the expression

$$H_{\text{eff}} = \frac{\dot g}{\sqrt{2}}\int^{Q_c} dq\sqrt{F(q)}\,\left(g(t)-\frac{V(q)}{F(q)}\right)^{-1/2}. \tag{4.2.23}$$

We have still the freedom to choose $F(q)$ so as to simplify the expression (4.2.22): for instance, if $V(q_0)$ is of the form $q_0^2 v(q_0)$ with $v(q_0)$ a polynomial, then $F=q^2$ is a suitable choice. Of course if the potential is particularly simple one may set

$$F(q) \;=\; V(q)\,. \tag{4.2.24}$$

Then

$$H_{\text{eff}} \;=\; \sqrt{2}\frac{d}{dt}(g-1)^{1/2}\int^{Q_c} dq\sqrt{V(q)}\,. \tag{4.2.25}$$

Note that H_{eff} is in general time-dependent also because of Q_c. Finally, the Lagrange multiplier is

$$u(t) = \left[-\frac{f_{,tq_0}}{V'(q_0)+f_{,q_0}f_{,q_0q_0}-H_{1,q_0}(q_0,q_\alpha,f_{,q_\alpha})}\right]_{q_0=Q_c}. \tag{4.2.26}$$

The previous results about the harmonic oscillator can be obtained from these general formulae choosing

$$f(q_0;t) = -\frac{1}{2}q_0^2\text{ctg } t\,. \tag{4.2.27}$$

Let us now apply the method discussed above to a constraint of the form

$$H = \frac{1}{2}p_0^2 - \frac{1}{2q_0^2}p_1^2 - q_0^4V_1(q_1) + V_0(q_0) = 0\,. \tag{4.2.28}$$

We will see in the next section that this case corresponds to the zero-mode of a scalar field minimally coupled to gravity. Now $Q_c(q_\alpha,p_\alpha;t)$ is defined by

$$\left[\frac{1}{2}(f_{,Q_0})^2 - Q_0^4V_1(Q_1) + V_0(Q_0) - (P_1+f_{,Q_1})^2\frac{1}{2Q_0^2}\right]_{Q_0=Q_c} = 0\,. \tag{4.2.29}$$

Choosing $f\equiv f(Q_0;t)$ and setting

$$\frac{1}{2}\big(f_{,q}(q;t)\big)^2 + V_0(q) \equiv q^4 g(t)\,, \tag{4.2.30}$$

we obtain the effective Hamiltonian

$$H_{\text{eff}} = \frac{\dot{g}}{\sqrt{2}} \int^{Q_c} dq q^2 \left(g(t) - \frac{V_0(q)}{q^4} \right)^{-1/2}, \tag{4.2.31}$$

where

$$Q_c = \left(\frac{P_1^2}{2\big(g(t) - V_1(Q_1)\big)} \right)^{1/6}. \tag{4.2.32}$$

In particular, if

$$V_0(q_0) = q_0^4\,, \tag{4.2.33}$$

eq. (4.2.31) becomes

$$H_{\text{eff}} = \pm \frac{1}{3} \frac{d}{dt} \sqrt{g-1} \frac{P_1}{\big(g(t) - V_1(Q_1)\big)^{1/2}}. \tag{4.2.34}$$

Analogously to the cases discussed before, a suitable choice for $g(t)$ simplifies the effective Hamiltonian. Clearly, (4.2.34) depends explicitly on time and is non-local. Finally the Lagrange multiplier reads

$$u(t) = \left[- \frac{q_0 f_{,tq_0}}{2V_0 + q_0 V_0' - 6q_0^4 V_1 + f_{,q_0}\big(f_{,q_0} + q_0 f_{,q_0 q_0}\big)} \right]_{q_0 = Q_c}. \tag{4.2.35}$$

Let us discuss the simple case $V_1(q_1) = 0$ and $V_0(q_0) = q_0^2$.[11] In this case it is convenient to choose

$$f(q_0;t) = \frac{1}{\sqrt{2}} q_0^2 \sinh t\,. \tag{4.2.36}$$

Using (4.2.29) we find that $Q_c(Q_\alpha, P_\alpha; t)$ is defined by

$$Q_c^2 = \pm \frac{1}{\sqrt{2}} \frac{P_1}{\cosh t}\,, \tag{4.2.37}$$

and the effective Hamiltonian (4.2.34) becomes

$$H_{\text{eff}} = \pm \frac{1}{2} P_1 \tag{4.2.38}$$

A surprising feature of (4.2.34) and (4.2.38) is that the effective Hamiltonian is linear in P_1.

[11] We will not give here a complete discussion of the gauge reduction of (4.2.28) since the coupling term $q_0^4 V_1(q_1)$ makes it difficult to obtain a simple effective Hamiltonian. However, when the system is separable, the problem of finding the proper time gauge is reduced to quadratures.

To conclude this section let us discuss briefly the quantisation *à la* Dirac. Consider for simplicity the case (4.2.1,2). In this case the Dirac method can be implemented by the canonical transformation in the sector $\{q_0, p_0\}$

$$Q_0 = \arctan \frac{q_0}{p_0}, \qquad P_0 = \frac{1}{2}(p_0^2 + q_0^2). \tag{4.2.39}$$

Using (4.2.39) the constraint becomes

$$H = P_0 - H_1(q_\alpha, p_\alpha) = 0. \tag{4.2.40}$$

The system can be quantised using the quantum measure (2.3.8) and as a consequence (2.3.9) as the operator representation.[12] The Wheeler-DeWitt equation is thus

$$\hat{H}_1(q_\alpha, \hat{p}_\alpha)\Psi(q_\alpha, Q_0) = i\frac{\partial}{\partial Q_0}\Psi(q_\alpha, Q_0). \tag{4.2.41}$$

The gauge can be fixed by the condition

$$F \equiv Q_0 - t = 0. \tag{4.2.42}$$

where t is a parameter. Using (4.2.42) the Faddeev-Popov determinant (2.3.7) is $\Delta_{FP} = 1$ and the equation (4.2.41) coincides with the Schrödinger equation (2.3.16) in the reduced space defined by the effective Hamiltonian (4.2.5). In the next section we shall apply all these results to some example of minisuperspace models.

4.3 FRW-like Universes Filled with Radiation and Matter Fields.

In this section our aim is to apply the techniques developed in the previous section to models describing FRW-like universes filled with radiation and matter fields, thus a single gravitational degree of freedom, its scale factor a (Cavaglià, de Alfaro, and Filippov, 1995a; 1995b; 1995c).

Aiming to have consistently radiation with the same symmetry as the FRW metric, we shall introduce an SU(2) Yang-Mills field and consider its zero-mode (Henneaux, 1982; Cavaglià and de Alfaro, 1994). Of course, the presence of a Yang-Mills field is very welcome, as the SU(2) Yang-Mills field is a gauge field, and gauge fields constitute a fundamental ingredient of matter. This model has the right properties of a radiation-dominated universe, i.e. a separable Wheeler-DeWitt constraint, corresponding to the classical property of radiation density scaling as a^{-4}. A further component of matter that does not spoil this simplicity is a conformal scalar field, and we shall introduce its zero-mode for completeness.[13]

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[12] Where of course $q_0 \to Q_0$, $p_0 \to P_0$.

[13] See also the recent discussion by Lemos (1996).

The model under consideration does not contain, as it stands, interaction terms producing inflation; inflation must be introduced by hand in the Lagrangian in the form of a cosmological term and we shall discuss briefly a form of gauge fixing for this case. The next step will be to consider a minimally coupled scalar field. This is of great interest as it introduces a coupling to gravitation that induces inflation in a dynamical way.

The action for Einstein gravity minimally coupled to the SU(2) Yang-Mills field A and the conformal scalar field φ is the sum of three terms:

$$S = S_{\rm GR} + S_{\rm YM} + S_{\rm CS} \,, \tag{4.3.1}$$

where $S_{\rm GR}$ represents the Einstein-Hilbert action (2.5.13) and the second and third terms represent respectively the Yang-Mills and the conformal scalar field actions

$$S_{\rm YM} = \frac{1}{2}\int \mathbf{F} \wedge {}^*\mathbf{F} \,, \tag{4.3.2a}$$

$$S_{\rm CS} = \frac{1}{2}\int d^4x\sqrt{-g}\left(\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{6}R\varphi^2\right). \tag{4.3.2b}$$

In (4.3.2a) $\mathbf{F} = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A}$ is the field strength two-form and we have set equal to one the gauge coupling constant.

We shall study a spacetime of topology $\mathbb{R}\times\Sigma$, where Σ is a homogeneous and isotropic three-surface. Following Henneaux (1982) we write

$$ds^2 = -N^2(t)dt^2 + a^2(t)\,\omega^p\otimes\omega^p \,, \tag{4.3.3}$$

where ω^p are the one-forms invariant under translations in space and $N(t)$ is the lapse function. The cosmic time corresponds to $N = 1$ and the conformal time to $N = a$. In the cases we discuss here, the ω^p's satisfy the Maurer-Cartan structure equations

$$d\omega^p = \frac{k}{2}\epsilon_{pqr}\,\omega^q\wedge\omega^r \,, \tag{4.3.4}$$

where $k = 0, 1$ and thus the metric (4.3.3) describes a flat or a closed Friedmann-Robertson-Walker universe respectively. When $k = 1$, (4.3.3) has the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ group of isometries. Using (4.3.3) and integrating over the spatial variables, apart from the (finite or infinite) space volume factor the gravitational action reads

$$S_{\rm GR} = 6\int dt\left(-\frac{a\dot{a}^2}{N} + kNa - \frac{\Lambda}{3}Na^3\right). \tag{4.3.5}$$

Introducing the conjugate momentum

$$p_a \;=\; -12\frac{a\dot{a}}{N} \,, \tag{4.3.6}$$

the action (4.3.5) can be cast in the form

$$S_{\rm GR} = \int dt \left(p_a \dot{a} \ + \ \frac{N}{a} \ H_{\rm GR} \right) , \tag{4.3.7}$$

where $H_{\rm GR}$ is

$$H_{\rm GR} \ = \ \frac{1}{12} \left(\frac{1}{2} p_a^2 + V_a(a) \right) , \tag{4.3.8}$$

and

$$V_a(a) = 72 a^2 \left(k - \frac{\Lambda}{3} a^2 \right) . \tag{4.3.9}$$

Let us now introduce a Yang-Mills field configuration with the same symmetry as the metric. We shall consider a Yang-Mills group SU(2) for simplicity and a form of the Yang-Mills field with a single degree of freedom that was proposed by Henneaux (1982) (see also Cavaglià and de Alfaro, 1994); the case of a more general group has been investigated by Bertolami et al. (1991). The form of the field, written in the dreibein cotangent space, is

$$\mathbf{A} = \frac{i}{2} \xi(t) \sigma_p \omega^p . \tag{4.3.10}$$

where σ_p are the Pauli matrices. With the definition (4.3.10) **A** is evidently left-invariant; it is also right-invariant up to a gauge transformation (Henneaux, 1982; Galt'sov and Volkov, 1991). $\xi(t)$ is the single degree of freedom of the Yang-Mills field.

The field strength F is:

$$\mathbf{F} = \frac{i}{2} \sigma_p \dot{\xi} \, dt \wedge \omega^p \ + \ \frac{i}{4} \sigma_r \epsilon_{rpq} \xi (k - \xi) \, \omega^q \wedge \omega^r . \tag{4.3.11}$$

Using (4.3.11) the action of the Yang-Mills field (4.3.2a) reads

$$S_{\rm YM} = \frac{3}{2} \int dt \left(\frac{a}{N} \dot{\xi}^2 - \frac{N}{a} \xi^2 (k - \xi)^2 \right) . \tag{4.3.12}$$

Introducing the conjugate momentum

$$p_\xi = 3 \frac{a}{N} \dot{\xi} , \tag{4.3.13}$$

the action (4.3.12) can be cast in the form

$$S_{\rm YM} = \int dt \left(p_\xi \dot{\xi} - \frac{N}{a} \ H_{\rm YM} \right) , \tag{4.3.14}$$

where

$$H_{\rm YM} = \frac{1}{3}\left(\frac{1}{2}p_\xi^2 + V_\xi(\xi)\right), \tag{4.3.15}$$

and

$$V_\xi(\xi) = \frac{9}{2}\xi^2(k-\xi)^2 . \tag{4.3.16}$$

Finally, let us discuss the conformal scalar field. Using (4.3.3) the conformal scalar field action becomes

$$S_{\rm CS} = \frac{1}{2}\int dt\left(\frac{a}{N}(a\dot\varphi + \dot a\varphi)^2 - kNa\varphi^2\right). \tag{4.3.17}$$

The action (4.3.17) can be simplified defining the rescaled scalar field $\chi = \varphi a$. In terms of χ (4.3.17) reads

$$S_{\rm CS} = \frac{1}{2}\int dt\left(\frac{a}{N}\dot\chi^2 - k\frac{N}{a}\chi^2\right). \tag{4.3.18}$$

Introducing as in the previous cases the conjugate momentum

$$p_\chi = \frac{a}{N}\dot\chi , \tag{4.3.19}$$

(4.3.18) becomes

$$S_{\rm CS} = \int dt\left(p_\chi\dot\chi - \frac{N}{a}\,H_{\rm CS}\right), \tag{4.3.20}$$

where

$$H_{\rm CS} = \frac{1}{2}(p_\chi^2 + k\chi^2) . \tag{4.3.21}$$

In the conformal gauge $N = a$, when $k = 1$, the Hamiltonian (4.3.21) describes a one-dimensional harmonic oscillator in the χ variable.

The 00 component of the Einstein equations of motion is the constraint equation

$$H_{\rm YM} + H_{\rm CS} = H_{\rm GR} . \tag{4.3.22}$$

From the equations of motion for the Yang-Mills and the conformal scalar fields it is easy to obtain

$$\left(\frac{a}{N}\dot\xi\right)^2 + \xi^2(k-\xi)^2 = K_{\rm YM}^2 , \tag{4.3.23a}$$

$$\left(\frac{a}{N}\dot\chi\right)^2 + k\chi^2 = K_{\rm CS}^2 , \tag{4.3.23b}$$

where $K_{\rm YM}$ and $K_{\rm CS}$ are constants; using (4.3.23), (4.3.22) becomes

$$\frac{a^2\dot a^2}{N^2} + \left(ka^2 - \frac{\Lambda}{3}a^4\right) = \frac{1}{12}K^2 , \tag{4.3.24}$$

where

$$K^2 = 3K^2_{\rm YM} + K^2_{\rm CS}\,. \tag{4.3.25}$$

The meaning of K^2 is made clear in the cosmic gauge. Setting $N = 1$, (4.3.24) reads

$$\frac{\dot a^2}{a^2} + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{1}{12}\frac{K^2}{a^4}\,. \tag{4.3.26}$$

We see that, as due to radiation, the energy density scales as a^{-4}. Thus

$$K^2 = 2\rho(a=1)\,, \tag{4.3.27}$$

and the meaning of K^2 is obvious: $K^2/2$ is the energy density of the radiation in a FRW universe with scale factor a of one (Planck length).

Now let us quantise the system. The complete action of our radiation filled universe is

$$S = \int dt\big(p_a\dot a + p_\xi\dot\xi + p_\chi\dot\chi - H_E\big)\,, \tag{4.3.28}$$

where the extended Hamiltonian is

$$H_E \equiv u[H_{\rm YM} + H_{\rm CS} - H_{\rm GR}]\,, \tag{4.3.29}$$

and the Lagrange multiplier is

$$u(t) = \frac{N(t)}{a}\,. \tag{4.3.30}$$

We see that we may choose the gauge-fixing in essentially different ways. A very spontaneous and physically appealing choice consists in using the gravitational degree of freedom as connected to time. Let us first discuss the case $\Lambda = 0$, i.e. the absence of the cosmological term, and a closed universe, $k = 1$. Then $H_{\rm GR}$ is a harmonic oscillator and we may use the gauge-fixing identity of the form (4.2.4),[14]

$$F \equiv p_a + 12a\,{\rm ctg}\,t_{\rm cf} = 0\,, \qquad t_{\rm cf} \equiv t. \tag{4.3.31}$$

As a consequence, the Lagrange multiplier is fixed as

$$u(t_{\rm cf}) \;=\; 1\,. \tag{4.3.32}$$

This gives the conformal time gauge condition, $N = a$.

With this choice of time, the Hamiltonian is time-independent; the Schrödinger equation on the gauge shell takes the form

$$i\frac{\partial}{\partial t_{\rm cf}}\psi(\xi,\chi;t_{\rm cf}) = \big(H_{\rm CS} + H_{\rm YM}\big)\psi(\xi,\chi;t_{\rm cf})\,. \tag{4.3.33}$$

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[14] Or (4.2.42) in the Dirac formalism.

In this gauge the classical Friedmann equation (4.3.24) for $k = 1$ and $\Lambda = 0$ reads

$$\dot{a}^2 + a^2 = \frac{K^2}{12} \equiv a_{\rm M}^2 \,, \tag{4.3.34}$$

and a solution for the classical motion is

$$a = a_{\rm M} \ \sin t_{\rm cf} \tag{4.3.35}$$

From the definition of p_a, eq. (4.3.6), we have

$$p_a = -12\dot{a} \tag{4.3.36}$$

in agreement with the gauge condition (4.3.31). Note that with the present definition of time, $t_{\rm cf} = \text{arctg}\ a/\dot{a}$, the region $0 \le t_{\rm cf} \le \pi/2$ maps the expanding phase of the closed universe. Boundary conditions are of course expressed in the form

$$\psi(\chi, \xi; t_{\rm cf} = t_0) = f(\chi, \xi) \,. \tag{4.3.37}$$

Eq. (4.3.33) expresses the evolution of the configuration in the conformal time $t_{\rm cf}$ and represents as usual in quantum mechanics the correlation amplitude for the different components of matter (χ, ξ in the present case) in the universe.

Let us point out that a gauge choice identifying a with time,[15]

$$a = \frac{|t_a|}{\sqrt{6}} \,, \qquad t_a \equiv t \,, \tag{4.3.38}$$

leads to a non-local time-dependent Hamiltonian[16]

$$H_{\rm eff} = \pm 2\sqrt{H_{\rm YM} + H_{\rm CS} - t_a^2}. \tag{4.3.39}$$

The real problem comes with a cosmological term. In that case V_a has the form (4.3.9) and we then use the form (4.2.23) with $g \to 12g$ for $H_{\rm eff}$ with the choice $F(a) = a^2$. Then

$$\begin{aligned} H_{\rm eff} &= \sqrt{6}\dot{g} \int da \ a\big(g - 6k + 2\Lambda a^2\big)^{-1/2} \\ &= \pm \frac{1}{2\sqrt{2}\lambda} \frac{\dot{g}}{g^{1/2}} \big(g(g - 6k) + 2\Lambda(H_{\rm YM} + H_{\rm CS})\big)^{1/2} \,. \end{aligned} \tag{4.3.40}$$

Let us now discuss an interesting different gauge-fixing for the present case. We could very easily choose the time so as to be connected to the conformal scalar degree of freedom, since its Hamiltonian is a simple harmonic oscillator.[17]

[15] This choice has been recently discussed by Lemos (1996).

[16] The effective Hamiltonian does not depend on time if k=0. See Lemos (1996).

[17] This leads to ambiguities, already present in the classical discussions of the Wheeler-DeWitt equation, about the self-adjointness of the quantum Hamiltonian $H_{\rm GR}$, and the differential representation of the (now) operator $\hat{p}_a$. This choice has been recently investigated by Lemos (1996).

Let us write the gauge-fixing condition as

$$F \equiv p_\chi - \chi \operatorname{ctg} t_{\mathrm{cs}} = 0 \,, \qquad t_{\mathrm{cs}} \equiv t \,. \tag{4.3.41}$$

Again this is a conformal time gauge, $u(t_{\mathrm{cs}}) = 1$. With this choice of time the Schrödinger equation takes the form

$$i \frac{\partial}{\partial t_{\mathrm{cs}}} \psi(a, \xi; t_{\mathrm{cs}}) = \big(H_{\mathrm{YM}} - H_{\mathrm{GR}}\big)\, \psi(a, \xi; t_{\mathrm{cs}}) \,. \tag{4.3.42}$$

The gauge identity (4.3.41) connects the time to a physically unclear entity as the conformal scalar field. However, this form is suited to discuss correlation between a and ξ. We will see that we may draw some consequences in agreement with the correspondence principle. Since t_{cs} is physically not well-defined, let us direct attention to stationary states. With

$$\psi(a, \xi; t_{\mathrm{cs}}) = e^{-iEt_{\mathrm{cs}}} \chi(a, \xi) \,, \tag{4.3.43}$$

we obtain the stationary Schrödinger equation

$$\big(H_{\mathrm{YM}} - H_{\mathrm{GR}}\big) \chi(a, \xi) = E \chi(a, \xi) \,. \tag{4.3.44}$$

This equation is similar to the Wheeler-DeWitt equation for pure gravity coupled to the Yang-Mills field (see Cavaglià and de Alfaro, 1994; Bertolami and Mourão, 1991), the difference being the presence of the term $E\chi$. The separability of the equation is the quantum form of the request that the density scales like a^{-4}. Let us write

$$\chi(a, \xi) = \zeta(a) \eta(\xi) \,. \tag{4.3.45}$$

The equation for the Yang-Mills wave function is

$$\frac{1}{3} \left(\frac{1}{2} p_\xi^2 + V_\xi(\xi) \right) \eta(\xi) = E_{\mathrm{YM}} \; \eta(\xi) \,. \tag{4.3.46}$$

For the gravitational degree of freedom, again in the case of a closed universe we have

$$\frac{1}{12} \left(\frac{1}{2} p_a^2 + V_a(a) \right) \zeta_n(a) = E_n^g \zeta_n(a) \,. \tag{4.3.47}$$

The gravitational degree of freedom is a harmonic oscillator. So we set the boundary condition at $a \to \infty$ by asking the square integrability of the wave function (for this suggestion see e.g. Cavaglià and de Alfaro, 1994; Conradi and Zeh, 1991; Kung, 1993). About the condition at $a = 0$, if we ask that $\zeta \to 0$, then $p_a = -id/da$ is formally Hermitian and H_{GR} is self-adjoint. Let us accept for the moment these boundary conditions. Then the spectrum is given by the odd part of a harmonic oscillator (n_g odd)

$$E_n^g = \frac{1}{2} + n_g \tag{4.3.48}$$

and the wave functions are the harmonic oscillator ones (Cavaglià and de Alfaro, 1994). The eigenvalues of the two degrees of freedom are connected by

$$E_n^g = E_n^{\mathrm{YM}} - E\,. \tag{4.3.49}$$

Let us now consider the correspondence principle with the classical gravitational motion for large oscillator quantum numbers n_g in the gauge $u(t_{\mathrm{cs}}) = 1$ (Cavaglià and de Alfaro, 1994). We interpret of course, as we are led by the Schrödinger equation, $|\psi|^2$ as the probability density for the value a for the scale factor, to be compared through the correspondence principle to the classical probability density in the conformal time gauge distribution of the physical quantity a for an ensemble of trajectories. Now this is inversely proportional to the speed of a in that time gauge. Thus (probability not normalised)

$$P_{cl}(t) = \frac{1}{da/dt}\,. \tag{4.3.50}$$

From the classical equation of motion (4.3.34) we have in the conformal time gauge

$$P_{cl}(t) = \frac{1}{\sqrt{a_{\mathrm{M}}^2 - a^2}}\,. \tag{4.3.51}$$

Now for the harmonic oscillator

$$\Sigma_{av}\ |\psi_n(a)|^2 \to \frac{1}{\sqrt{a_{\mathrm{M}}^2 - a^2}}\,. \tag{4.3.52}$$

So from the boundary conditions chosen it follows that the correspondence principle works properly in the conformal gauge $N = a$ as noted by Cavaglià and de Alfaro (1994).

The extension of the discussion of the correspondence principle to the case of a universe with a cosmological term can be carried out along the same lines.

To conclude this section, let us explore briefly the case of gravity minimally coupled to a scalar field ϕ. Its action is

$$S_{\mathrm{MS}} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \tag{4.3.53}$$

and the complete space density action in minisuperspace reduces to

$$S = \int dt\,\{p_a\dot{a} + p_\phi\dot{\phi} - H_E\}\,, \tag{4.3.54}$$

where the extended Hamiltonian is

$$H_E \equiv u(H_{\mathrm{MS}} - H_{\mathrm{GR}})\,, \qquad H_{\mathrm{MS}} = \left(\frac{1}{2}\frac{p_\phi^2}{a^2} + a^4V(\phi)\right). \tag{4.3.55}$$

Let us put, for simplicity, $V(\phi) = 0$ and consider a closed universe $k = 1$. Note that analogous results hold for the flat case $k = 0$.[18] Using the linear gauge-fixing

$$F \equiv p_a - 12a \sinh(t_{\rm ms}/\sqrt{3}) = 0\,, \qquad t_{\rm ms} \equiv t\,, \tag{4.3.56}$$

we obtain the effective Hamiltonian

$$H_{\rm eff} = \pm p_\phi\,. \tag{4.3.57}$$

We are essentially in the conformal gauge, indeed from (4.3.56) we have

$$u = \frac{1}{2\sqrt{3}\cosh(t_{\rm ms}/\sqrt{3})}\,. \tag{4.3.58}$$

Then the Schrödinger equation is

$$\left(\frac{\partial}{\partial t_{\rm ms}} \pm \frac{\partial}{\partial\phi}\right)\psi = 0\,, \tag{4.3.59}$$

which has the general solution

$$\psi = f(\phi \mp t_{\rm ms})\,. \tag{4.3.60}$$

The correspondence principle works correctly also in this case. To see this, we need the classical equations of motion in the gauge (4.3.56). The Friedmann equation is now

$$\frac{a^2\dot{a}^2}{N^2} + a^2 = \frac{1}{12a^2}K_\phi^2\,. \tag{4.3.61}$$

Substituting (4.3.58), we easily find

$$a = \left(\frac{K_\phi^2}{12}\right)^{1/4} \frac{1}{\sqrt{\cosh\,(t_{\rm ms}/\sqrt{3})}}\,. \tag{4.3.62}$$

Note that (4.3.62) agrees with the gauge condition (4.3.56). From the equation for ϕ

$$\dot{\phi} = \frac{N}{a^3}K_\phi\,, \tag{4.3.63}$$

using (4.3.58) and (4.3.62) we obtain $\dot{\phi} = \pm 1$, i.e. $\phi = \pm t_{\rm ms}$. This result agrees with the quantum solution (4.3.60). Indeed, let us consider a wave packet of the form

$$\psi = Ae^{-(\phi \mp t_{\rm ms})^2/2\sigma}\,, \tag{4.3.64}$$

[18] This case corresponds indeed to the low-energy effective string model with Λ=0 and will be discussed in detail in sect. [4.5].

where A and σ are two constants. Clearly, (4.3.64) represents a quantum localized universe whose maximum probability follows the classical trajectory $<\phi>=\pm t_{\rm ms}$.

4.4 String Cosmology.

String theory is at the moment one of the best candidates for the ultimate theory of gravity. The possibility that the short-distance modifications of general relativity due to string theory be crucial in order to understand long-standing problems of quantum gravity – as the nature of the initial singularity – has stimulated the investigation of string cosmological models during the last years. In particular critical and non-critical string cosmological solutions have been discovered and analysed in detail (see Cadoni and Cavaglià, 1995a and references therein). The typical feature of these solutions is the scale-factor duality (target-space duality) that is at the basis of the alternative picture of the birth of the universe proposed in the early 90's by Gasperini and Veneziano (1992).[19]

In the customary standard model for the thermal history of the universe, all physical quantities as curvature, energy density, temperature, increase monotonically going backward in time and become infinite at a given time in the past (initial singularity or big bang). We have already remarked that this fact is usually interpreted as a failure of general relativity. Symmetries of string theory provide an alternative picture. In this framework the birth of the universe coincides with the transition between a *pre-big bang* accelerated phase and the usual *post-big bang* decelerated phase, traced by a maximum value of the curvature (see Gasperini and Veneziano, 1992, 1993). Let us see this point in detail.

Our starting point is the low-energy effective action for string theory in D-dimensions

$$S=\frac{1}{2\lambda_s^2}\int_V d^Dx\sqrt{-g}e^{-2\phi}\left(R+4\partial_\mu\phi\partial^\mu\phi-2\Lambda\right)\,, \tag{4.4.1}$$

where $\Lambda>0$ is the cosmological constant[20] and λ_s is the fundamental string-lenght parameter.

The main argument supporting the pre-big bang picture is the property of the action (4.4.1) to be invariant under the discrete transformation

$$a_i\to a_i^{-1}\,,\qquad \phi\to\phi-\ln a_i\,, \tag{4.4.2}$$

where a_i are the scale factors of the diagonal, not necessarily isotropic, D-dimensional spatially flat metric[21]

$$ds^2=-dt^2+a_i^2(t)dx^idx^i\,. \tag{4.4.3}$$

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19 We have already met the scale-factor duality in chapt. 3, see eq. (3.3.17).

20 Λ=0 if the dimension of the spacetime is critical.

21 This transformation can be generalised in several ways. In particular it can be extended to include non-diagonal line elements. In this case the transformation mixes the components of the metric and the components of the antisymmetric tensor (see Gasperini et al., 1996).

The discrete symmetry (4.4.2) implies that for any solution with decreasing scalar curvature, there exists a solution with increasing scalar curvature. By the same transformation the Hubble parameter changes sign, so an expanding solution is mapped in a contracting one and viceversa. Further, since the solutions with decreasing monotonic behaviour of the scalar curvature are defined on a half line in the time parameter[22] one can reverse the sign of t when performing the discrete transformation (4.4.2). The final result is that we can associate to any given decelerated and monotonically decreasing scalar curvature solution an accelerated and monotonically increasing scalar curvature solution through the transformation

$$a(t) \to a^{-1}(-t)\,. \tag{4.4.4}$$

Note that the presence of the dilaton is essential. Indeed, in the case of vanishing dilaton (4.4.4) reduces to the trivial time reversal transformation

$$a(t) \to a(-t)\,. \tag{4.4.5}$$

In this case the duality symmetry is called "broken".

In this framework the initial singularity can be avoided by a transition, at $t = 0$, between the two regimes connected by the transformation (4.4.4) (graceful exit). In this case string cosmology describes the formation of a FRW-like universe with essentially the present characteristics as evolving from the string perturbative vacuum.

At the classical level the transition from the accelerated phase to the decelerated one can be obtained if one finds an exact solution joining smoothly the two regimes. However, a smooth transition requires at least the introduction of second-order terms in the low-energy action.[23] A second possibility is given by the hypotesis of a tunnelling in the quantum cosmological regime. This approach to the problem has been recently proposed by Gasperini and Veneziano (1996). The idea is very simple. In the customary quantum gravity approach to the origin of the universe (see for instance Vilenkin, 1995), use is made of the Wheeler-DeWitt equation whose solutions with appropriate boundary conditions describe the tunnelling from nothing.[24] In string cosmology the birth of the universe is described by a transition from an initial pre-big bang phase to the present one. So in the quantum picture the analogous of the string transition process is represented by a scattering and reflection of the Wheeler-DeWitt wave function in superspace (Gasperini and Veneziano, 1996; Gasperini et al., 1996). One can imagine that the transition between the pre-big bang and the post-big bang regimes be forbidden by a sort of potential barrier. However, quantum mechanically one expects a small, but not vanishing, probability of transition. The

[22] I.e., by a suitable choice of the initial conditions, for $t>0$.

[23] For the proof, see Kaloper et al. (1995).

[24] See sect. [4.1].

calculation of this probability needs of course the building up of a mathematically self-consistent quantum model, together with the determination of an univocal time parameter and the Hilbert space (Cavaglià and de Alfaro, 1996b). This programme will be the subject of the next section.

4.5 Birth of the Universe as Wave Function Reflection.

Let us start from the four-dimensional version of the action (4.4.1) and the usual FRW line element ansatz (4.3.3). Accordingly the dilaton field is assumed to depend only on time. Defining

$$\gamma = \ln a, \qquad \varphi = \phi - \frac{3}{2}\ln a\,, \tag{4.5.1}$$

(φ is usually called the *shifted dilaton field* – see Gasperini et al, 1996) and using the Lagrange multiplier

$$u = N\, e^{2\varphi}\,, \tag{4.5.2}$$

the action (4.4.1) becomes[25]

$$S = \frac{V}{2\lambda_s^2}\int dt\left[3\frac{\dot\gamma^2}{u} - 4\frac{\dot\varphi^2}{u} + ue^{-4\varphi}\left(6ke^{-2\gamma} - 2\Lambda\right)\right]\,, \tag{4.5.3}$$

where V is the spatial volume element with $a = 1$ and dots represent differentiation with respect to t. In the following we will set $V/2\lambda_s^2 = 1$. The canonical form of the action (4.5.3) is

$$S = \int dt\,\{\dot\gamma p_\gamma + \dot\varphi p_\varphi - H_E\}\,, \tag{4.5.4}$$

where

$$p_\gamma = 6\frac{\dot\gamma}{u}\,, \qquad p_\varphi = -8\frac{\dot\varphi}{u}\,, \tag{4.5.5}$$

are respectively the conjugate momenta of γ and φ, and

$$H_E \equiv uH = u\left[\frac{p_\gamma^2}{12} - \frac{p_\varphi^2}{16} + 2\Lambda e^{-4\varphi} - 6ke^{-2\gamma}\right]\,, \tag{4.5.6}$$

is the extended Hamiltonian. From now on, since the string cosmological picture requires a spatially flat manifold, we shall set $k = 0$.

4.5.a Critical Dimension Case.

Let us discuss first the case of null cosmological constant, corresponding to a string with critical dimensions (Veneziano, 1991; Gasperini and Veneziano, 1993).

[25] As usual we have integrated on surfaces at constant time and neglected inessential surface terms.

In this case the idea of wave reflection cannot be implemented because the solutions are free waves. However, it is interesting to investigate this model since all the ingredients for the discussion of the wave function reflection are already contained in it. In this case the extended Hamiltonian corresponds to the D'Alembert one

$$H_E \equiv uH = u\left[\frac{p_\gamma^2}{12} - \frac{p_\varphi^2}{16}\right]. \tag{4.5.7}$$

The finite gauge transformation (2.2.5) can be integrated explicitly (Cavaglià and de Alfaro, 1996b). The result is

$$\gamma = \gamma_0 + \frac{p_\gamma}{6}\tau\,, \tag{4.5.8a}$$

$$p_\gamma = \text{constant}\,, \tag{4.5.8b}$$

$$\varphi = -\frac{p_\varphi}{8}\tau\,, \tag{4.5.8c}$$

$$p_\varphi = \text{constant}\,, \tag{4.5.8d}$$

$$\tau = \int_{t_0}^{t} u(t')\,dt'\,, \qquad u(t') > 0\,. \tag{4.5.8e}$$

where γ_0, p_γ, and p_φ are gauge-invariant quantities.

Let us discuss briefly the solution (4.5.8) and connect the Gasperini-Veneziano (1996) parameter t_{GV} to the parameter τ. The two parameters are related by

$$dt_{\text{GV}} = e^{-2\varphi}d\tau\,, \tag{4.5.9}$$

that is

$$p_\varphi t_{\text{GV}} = 4e^{p_\varphi \tau/4}\,. \tag{4.5.10}$$

From (4.5.10) we see that the sign of t_{GV} is related to the sign of p_φ. This will be relevant to identify the pre-big bang and the post-big-bang branches. Using (4.5.10) in (4.5.8a) and recalling (4.5.1) the solution for the scale factor a in the Gasperini-Veneziano time reads

$$a = a_0(\pm t_{\text{GV}})^{2p_\gamma/3p_\varphi}\,, \tag{4.5.11}$$

where a_0 is a constant and $t_{\text{GV}} > 0$ for $p_\varphi > 0$ and $t_{\text{GV}} < 0$ for $p_\varphi < 0$. From (4.5.11) we see that the post-big bang regime ($t_{\text{GV}} > 0$ and expanding scale factor) and the pre-big bang regime ($t_{\text{GV}} < 0$ and contracting scale factor) require $p_\gamma > 0$. Thus the post-big bang phase is identified by $p_\gamma > 0$ and $p_\varphi > 0$; the pre-big bang phase by $p_\gamma > 0$ and $p_\varphi < 0$. Finally, setting the constraint to zero, we obtain the on-gauge shell solution of Gasperini et al. (1996).

We may define new variables similarly to those defined for the relativistic free massive particle of sect. [2.4] (action-angle variables)

$$\xi = 6\frac{\gamma}{p_\gamma}\,, \tag{4.5.12a}$$

$$p_\xi = \frac{1}{12}p_\gamma^2\,, \tag{4.5.12b}$$

that will be used later. The Poisson bracket of ξ and p_ξ is obviously

$$\left[\xi, p_\xi\right]_P = 1\,. \tag{4.5.13}$$

Thus $\{\varphi, p_\varphi; \xi, p_\xi\}$ form a complete set of canonically conjugate variables. Note that p_φ and p_ξ are gauge-invariant quantities and ξ tranforms by the gauge tranformation (4.5.8) as

$$\xi \to \bar{\xi} = \xi + \tau\,. \tag{4.5.14}$$

This is the reason for the interest in ξ. Eq. (4.5.14) suggests that ξ is a proper variable to fix the gauge and obtain an unitary evolution in the gauge-fixed space. We call the set $\{\varphi, p_\varphi; \xi, p_\xi\}$ *hybrid variables* because they are not the maximal gauge-invariant choice of canonical coordinates. Indeed we can identify the Shanmugadhasan variables (see sect. [2.3])

$$x = \varphi + \frac{3}{4}\frac{\gamma}{p_\gamma} p_\varphi\,, \tag{4.5.15a}$$

$$p_x \equiv p_\varphi\,, \tag{4.5.15b}$$

$$y \equiv \xi = 6\frac{\gamma}{p_\gamma}\,, \tag{4.5.15c}$$

$$p_y \equiv H = \frac{1}{12}p_\gamma^2 - \frac{1}{16}p_\varphi^2\,, \tag{4.5.15d}$$

which is a set of canonically conjugate variables.

The variables x and p_x are gauge-invariant and thus generate rigid invariance transformations. Of course the meaning of gauge invariant variables is transparent in the case of x: it is the initial value of φ. These variables and the functions $f(x, p_x)$ are the observables: "The set of the observables is isomorphic to the set of functions of the initial data" (Teitelboim, 1991).

For sake of compleneness, we write the generating function of the canonical transformation $\{\gamma, p_\gamma; \varphi, p_\varphi\} \to \{x, p_x; y, p_y\}$:

$$\mathcal{F} = -\frac{3\gamma^2}{y} + \frac{4}{y}(\varphi - x)^2\,. \tag{4.5.16}$$

Each set, $\{\varphi, p_\varphi; \xi, p_\xi\}$ or $\{x, p_x; y, p_y\}$, can be used in the quantisation programme and leads to identical results, both in the Dirac method and in the reduced method. Let us first quantise in the hybrid variables.[26]

4.5.aa Quantisation in Hybrid Variables.

Let us start by the Dirac method. Wave functions are solutions of the Wheeler-DeWitt equation. Hence, the gauge has to be fixed in the scalar product of solutions of the Wheeler-DeWitt equation.

[26] Note that the Hamiltonian (4.5.7) is essentially symmetric for $\{\gamma,p_\gamma\}\leftrightarrow\{\varphi,p_\varphi\}$. So both in the classical and quantum treatments one can exchange the $\{\varphi,p_\varphi\}$ degree of freedom with the $\{\gamma,p_\gamma\}$ one.

The first problem is the choice of variables and measure in (2.3.6). We require the measure to be gauge-invariant and invariant with respect to the rigid symmetries of the system. The choice $d[\alpha] = dp_\varphi dp_\xi$ is gauge-invariant and invariant under rigid transformations generated by p_φ and p_ξ, however it is not suitable for fixing the gauge. The suitable measure is

$$d[\alpha] = dp_\varphi d\xi \,, \tag{4.5.17}$$

which is gauge-invariant and invariant under rigid transformations generated by p_φ and p_ξ. The use of ξ in the measure allows to enforce the gauge fixing procedure.

In this representation $\{\xi, p_\varphi\}$ are differential operators. We have

$$\hat{p}_\xi \to -i\partial_\xi \,, \qquad \hat{\varphi} \to i\partial_{p_\varphi} \,, \qquad \hat{\xi} \to \xi \,, \qquad \hat{p}_\varphi \to p_\varphi \,. \tag{4.5.18}$$

Thus the Wheeler-DeWitt equation is

$$\left(-i\partial_\xi - \frac{1}{16} p_\varphi^2 \right) \Psi(\xi, p_\varphi) = 0 \,. \tag{4.5.19}$$

The solutions of (4.5.19) that are eigenstates of $\hat{p}_\varphi$ with eigenvalues $\pm k$, $k > 0$ are

$$\Psi_{\pm k}(p_\varphi, \xi) = C(k)\delta(p_\varphi \mp k) e^{ik^2\xi/16} \,. \tag{4.5.20}$$

Now we have to fix the gauge. There is a class of viable gauges for which there are no Gribov copies and the Faddeev-Popov determinant Δ_{FP} is invariant under gauge transformations (for the proof, see chapt. 5). Let us simply choose ξ as time, i.e. take

$$F(\xi, p_\varphi) \equiv \xi - t = 0 \tag{4.5.21}$$

(t is the gauge-fixed time parameter); then $\Delta_{FP} = 1$. This gauge is unique and finally the gauge-fixed scalar product is

$$(\Psi_2, \Psi_1) = \int dp_\varphi \Psi_2^*(p_\varphi, t) \Psi_1(p_\varphi, t) \,, \tag{4.5.22}$$

of course a positive-definite Hilbert space.

The gauge-fixed functions in the representation $\{\varphi, \xi = t\}$ read

$$\Psi_{\pm k}(\varphi, t) = \frac{1}{\sqrt{2\pi}} e^{\pm ik\varphi + ik^2 t/16} \,. \tag{4.5.23}$$

They are obviously orthonormal in the Fourier-transformed gauge-fixed measure $d\varphi$.

Let us discuss now the reduced method. We impose the gauge identity $F \equiv \xi - t = 0$ that gives the effective Hamiltonian

$$H_{\rm eff} = -\frac{1}{16}p_\varphi^2 \,. \qquad (4.5.24)$$

The gauge identity implies $u = 1$ since from the definition of ξ and the classical general solution of the gauge equations it follows $\xi = \tau + \text{const.}$. The Schrödinger equation is

$$i\frac{\partial}{\partial t}\psi(\xi, p_\varphi) = -\frac{1}{16}\hat{p}_\varphi^2\psi(\xi, p_\varphi) \,. \qquad (4.5.25)$$

The stationary eigenfunctions of $\hat{p}_\varphi$ coincide with (4.5.23) and are orthonormal in the reduced space measure. This proves the equivalence of the two quantisation procedures.

4.5.ab Quantisation in Shanmugadhasan Variables.

We can quantise the system also in the Shanmugadhasan representation. Performing the canonical transformation to the new variables the action becomes

$$S = \int dt\{\dot{x}p_x + \dot{y}p_y - up_y\} \,. \qquad (4.5.26)$$

Let us first quantise the system by the Dirac method. The first step is again the determination of the measure in the inner product (2.3.6). The requirement of invariance of the measure under the rigid transformations generated by p_x or x and the gauge transformation generated by p_y selects $d[\alpha] = dxdy$ (equivalently $d[\alpha] = dp_x dy$), where $-\infty < x, y, p_x < \infty$. The measure $d[\alpha] = dp_x dp_y$ cannot be chosen since the gauge fixing function must contain y. So, consider the measure $d[\alpha] = dxdy$: the conjugate variables p_x and p_y are represented as

$$\hat{p}_x \to -i\partial_x \,, \quad \hat{p}_y \to -i\partial_y \,, \quad \hat{x} \to x \,, \quad \hat{y} \to y \,, \qquad (4.5.27)$$

and the Wheeler-DeWitt equation becomes

$$-i\partial_y \Psi(x, y) = 0 \,. \qquad (4.5.28)$$

The solutions of (4.5.28) that are eigenfunctions of $\hat{p}_x$ with eigenvalues $\pm k$, $k > 0$ are

$$\Psi_{\pm k}(x) = C(k)e^{\pm ikx} \,. \qquad (4.5.29)$$

Now we introduce the gauge fixing. The convenient gauge is

$$F(x, y) \equiv y - t = 0 \,. \qquad (4.5.30)$$

Obviously this gauge is unique and $\Delta_{FP} = 1$. The wave functions (4.5.29) are of course orthonormal (choosing $C(k) = (2\pi)^{-1/2}$) in the inner product so defined.

Let us now quantise the system by the alternative method of reducing first the phase space by a canonical identity. Again the gauge fixing condition is $F \equiv y - t = 0$ which determines the Lagrange multiplier as $u = 1$. Using the constraint $H = 0$ and the gauge fixing condition, the effective Hamiltonian on the gauge shell becomes $H_{\text{eff}} = -p_y = 0$. The Schrödinger equation in the reduced space just tells that the gauge-fixed wave functions do not depend on y. Diagonalizing $\hat{p}_x$ we obtain again the wave functions (4.5.29). The two quantisation methods give identical gauge-fixed positive-norm Hilbert spaces.

We have seen that the quantisation of the system can be successfully completed both in hybrid and Shanmugadhasan variables. The two quantisation procedures are equivalent. Further, the sets of physical wave functions (4.5.23) and (4.5.29) coincide when represented in the same variables. Let us discuss this point.

In order to relate the two representations (4.5.18) and (4.5.27) we need the generating function $\mathcal{F}$ of the canonical transformation between the Shanmugadhasan and the hybrid variables:

$$\mathcal{F}(\varphi, \xi; p_x, p_y) = \varphi p_x + \xi p_y + \frac{1}{16}\xi p_x^2 \,. \tag{4.5.31}$$

The relation between the wave functions in the two representations is given by

$$\Psi(\xi, \varphi) = \int dp_x dp_y \; e^{i\mathcal{F}(\varphi,\xi;p_x,p_y)}\Psi(p_x, p_y) \,. \tag{4.5.32}$$

Substituting in (4.5.32) the Fourier transform of the wave function (4.5.29)

$$\Psi_{\pm k}(p_x, p_y) = \delta(p_x \mp k)\delta(p_y) \,, \tag{4.5.33}$$

it is straightforward to obtain (4.5.23). This proves the equivalence between hybrid and Shanmugadhasan representations.

In the Shanmugadhasan representation the reduced Hamiltonian coincides with the original H and vanishes. The reason is that after the time gauge fixing we are left with gauge-invariant variables; hence, inner products and matrix elements are purely algebraic relations because all operators are built from classical constant of the motion. The wave functions contain one less variable because there is no dependence on the gauge-fixed time.

On the contrary, the gauge-fixed wave functions for hybrid variables evolve with time, and the reduced Hamiltonian does not vanish, so these variables seem to contain more physics. However, the physical content is the same. The time dependence expresses the fact that the hybrid observables are functions of time and gauge-invariant quantities.

Let us now discuss how one can implement in this framework the idea of transition between the pre-big bang and the post-big bang phases by scattering and reflection of the wave functions. We have said before that we cannot complete the procedure in the present case because the solutions are free waves. However, all ingredients are already present in the model, so it is useful to give a closer look at it. The first step is the identification of the wave functions corresponding to the pre-big bang and post-big-bang regimes. This can be performed recalling that the classical solutions corresponding to these two regimes are identified by the sign of the canonical momentum p_φ. Hence, the wave functions corresponding to the pre-big bang and post-big bang regimes are the right and left moving waves in (4.5.23) or (4.5.29) respectively. The transition between the two regimes is obtained by a reflection of wave functions (4.5.23) or (4.5.29). Obviously there is no reflection in absence of a potential (in the Wheeler-DeWitt equation).

4.5.b Non-critical Dimension Case.

Let us consider now the case of non-vanishing cosmological constant and discuss the mechanism of wave reflection. In this case a potential is present in the Wheeler-DeWitt equation and the reflection mechanism is possible (Gasperini e Veneziano, 1996; Gasperini et al., 1996). The extended Hamiltonian is

$$H_E \equiv uH = u\left[\frac{p_\gamma^2}{12} - \frac{p_\varphi^2}{16} + 2\Lambda e^{-4\varphi}\right] . \tag{4.5.34}$$

Again the gauge equations generated by H are integrable (Cavaglià and de Alfaro, 1996b). We have

$$\gamma = \gamma_0 + \frac{p_\gamma}{6}\tau\,, \tag{4.5.35a}$$

$$p_\gamma = \text{constant}\,, \tag{4.5.35b}$$

$$e^{2\varphi} = \pm\frac{\sqrt{2\Lambda}}{\omega}\sinh(\omega\tau)\,, \tag{4.5.35c}$$

$$p_\varphi = -4\omega\coth(\omega\tau)\,, \tag{4.5.35d}$$

$$\tau = \int_{t_0}^{t} u(t')\,dt'\,, \qquad u(t') > 0\,, \tag{4.5.35e}$$

where

$$\omega = \pm\sqrt{\frac{p_\varphi^2}{16} - 2\Lambda e^{-4\varphi}}\,. \tag{4.5.36}$$

In (4.5.35c) the two signs correspond $\tau > 0$ and $\tau < 0$ respectively. γ_0, p_γ and ω are gauge-invariant quantities. On the constraint $H = 0$, $p_\gamma = \pm\sqrt{12}|\omega|$. Again the choice of positive p_γ corresponds to a pre-big bang accelerated contraction for $\tau > 0$ and a post-big bang decelerating expansion for $\tau < 0$.

Note that γ and p_γ transform very simply for the gauge transformation; formulae (4.5.8a,b) hold. Again let us connect the Gasperini-Veneziano parameter $t_{\rm GV}$ to the parameter τ. The relation between the two parameters is

$$\sinh(|\omega|\tau)\,\sinh(t_{\rm GV}2\sqrt{\Lambda}) = -1\,. \tag{4.5.37}$$

The use of τ is suggested by the simplicity of eqs. (4.5.35) with the choice (4.5.2) of the Lagrange multiplier. From these equations it is easy to obtain the on-gauge shell solution of Gasperini and Veneziano (1996)

- *pre-big bang regime,* $t_{\mathrm{GV}} < 0$*:*

$$\begin{aligned} a &= a_0 \left[\tanh \left(-\frac{t_{\mathrm{GV}}\sqrt{\Lambda}}{\sqrt{2}} \right) \right]^{-1/\sqrt{3}} , \\ 2(\varphi - \varphi_0) &= -\ln \left[\sinh \left(-t_{\mathrm{GV}}\sqrt{2\Lambda} \right) \right] ; \end{aligned} \tag{4.5.38a}$$

- *post-big bang regime,* $t_{\mathrm{GV}} > 0$*:*

$$\begin{aligned} a &= a_0 \left[\tanh \left(\frac{t_{\mathrm{GV}}\sqrt{\Lambda}}{\sqrt{2}} \right) \right]^{1/\sqrt{3}} , \\ 2(\varphi - \varphi_0) &= -\ln \left[\sinh \left(t_{\mathrm{GV}}\sqrt{2\lambda} \right) \right] . \end{aligned} \tag{4.5.38b}$$

Analogously to the D'Alembert case, we can define hybrid and Shanmugadhasan variables. The hybrid variables are those defined in (4.5.12) for the critical dimension case. The Shanmugadhasan canonical set is

$$w \equiv \omega \, , \tag{4.5.39a}$$

$$p_w = -12\omega \frac{\gamma}{p_\gamma} - 2 \,\mathrm{arcth} \left(\frac{4\omega}{p_\varphi} \right) , \tag{4.5.39b}$$

$$z \equiv \xi = 6 \frac{\gamma}{p_\gamma} , \tag{4.5.39c}$$

$$p_z \equiv H = \frac{1}{12} p_\gamma^2 - \frac{1}{16} p_\varphi^2 + 2\Lambda e^{-4\varphi} . \tag{4.5.39d}$$

All variables are gauge-invariant except z ($\delta z = \epsilon$); w and p_w generate rigid symmetry transformations. Let us now quantise the system.

4.5.ba Quantisation in Shanmugadhasan Variables.

Performing the canonical transformation to the Shanmugadhasan variables the action becomes

$$S = \int dt \{ \dot{w} p_w + \dot{z} p_z - u p_z \} . \tag{4.5.40}$$

Let us quantise first the system by the Dirac method. The requirement of invariance of the measure under both the rigid transformations generated by w or p_w and the gauge transformation generated by p_z selects the measures $d[\alpha] = dwdz$

or equivalently $d[\alpha] = dp_w dz$, where $-\infty < w, z, p_w < \infty$. Choosing for instance the first one, we have the representation

$$\hat{p}_w \to -i\partial_w\,, \quad \hat{p}_z \to -i\partial_z\,, \quad \hat{w} \to w\,, \quad \hat{z} \to z\,. \tag{4.5.41}$$

The Wheeler-DeWitt equation becomes

$$-i\partial_z \Psi(w, z) = 0\,. \tag{4.5.42}$$

The solutions of (4.5.42) that are eigenfunctions of $\hat{w}$ with eigenvalues $\pm k$, $k > 0$ are

$$\Psi_{\pm k}(w) = C(k)\delta(w \mp k)\,. \tag{4.5.43}$$

The gauge can be fixed as

$$F(w, z) \equiv z - t = 0\,. \tag{4.5.44}$$

(t is thus the gauge-fixed time). Finally, the scalar product is

$$(\Psi_2, \Psi_1) = \int dw\, \Psi_2^*(w)\Psi_1(w)\,. \tag{4.5.45}$$

Choosing $C(k) = 1$, the eigenfunctions (4.5.43) are orthonormal in the gauge-fixed measure.

Let us now quantise the system by the reduced method. Again the gauge fixing condition is (4.5.44). Analogously to the case $\Lambda = 0$, this choice determines the Lagrange multiplier as $u = 1$. Using the constraint $H = 0$ and the gauge fixing condition, the effective Hamiltonian on the gauge shell becomes $H_{\text{eff}} = -p_z = 0$ (typical of the Shanmugadhasan choice of coordinates). The wave functions do not depend on z and all matrix elements are of purely algebraic nature. Diagonalizing $\hat{w}$ we obtain again the wave functions (4.5.43) in the reduced Hilbert space. As in the D'Alembert case, this proves the equivalence of the Dirac and reduced quantisation methods in the representation used.

4.5.bb Quantisation in Hybrid Variables.

Let us start using the Dirac method. Analogously to the case of sect. [4.5aa] we have to choose the representation and establish the measure. Quite similarly, the right measure is (4.5.17). In this case it is better to work in the Fourier-transformed space, so

$$d[\alpha] = d\varphi d\xi\,. \tag{4.5.46}$$

Note that (4.5.46) is not gauge-invariant nor invariant under rigid transformations. However, it is related to (4.5.17) by a Fourier transformation.

In the representation $\{\xi, \varphi\}$ the conjugate variables to ξ and φ are differential operators. We have

$$\hat{p}_\xi \to -i\partial_\xi\,, \quad \hat{p}_\varphi \to -i\partial_\varphi\,, \quad \hat{\xi} \to \xi\,, \quad \hat{\varphi} \to \varphi\,. \tag{4.5.47}$$

The Wheeler-DeWitt equation is

$$\left(-i\partial_\xi + \frac{1}{16}\partial_\varphi^2 + 2\Lambda e^{-4\varphi}\right)\Psi(\xi,\varphi) = 0\,. \tag{4.5.48}$$

The solutions of (4.5.48) that are eigenstates of $\hat{\omega}$ with eigenvalues $\pm k$, $k > 0$ are of the form

$$\Psi_{\pm k}(\varphi,\xi) = A_\pm(k) Z_{\pm 2ik}\left(2\sqrt{2\Lambda}e^{-2\varphi}\right) e^{ik^2\xi}\,, \tag{4.5.49}$$

where Z is a generic linear combination of Bessel functions. By applying to this case similar considerations as in the D'Alembert case (see also Gasperini and Veneziano, 1996; Gasperini et al., 1996) one sees that the choices

$$\Psi_k^{(\pm)}(\varphi,\xi) = A(k) J_{\pm 2ik} e^{ik^2\xi} \tag{4.5.50}$$

represent the Wheeler-DeWitt wave functions corresponding to the post-big bang and the pre-big bang phases respectively.

Now, in order to define the inner product we must fix the gauge. Imposing the identity $F \equiv \xi - t = 0$ we have

$$(\Psi_2,\Psi_1) \;=\; \int d\varphi\, \Psi_2^*(\varphi,t)\,\Psi_1(\varphi,t) \;=\; \int \frac{dz}{z}\,\Psi_2^*(z,t)\,\Psi_1(z,t)\,. \tag{4.5.51}$$

where $z = 2\sqrt{2\Lambda}e^{2\varphi}$. Note that the choice $\gamma = t$ does not yield a positive-definite norm.

The two sets of real orthonormal functions in the gauge-fixed measure (4.5.51) are (Cavaglià, de Alfaro, and Filippov, 1996)

$$\chi_{\pm k}^{(1)}(z,t) = \sqrt{\frac{k\cosh(\pi k)}{2\sinh(\pi k)}}\left[e^{-\pi k}H_{\pm 2ik}^{(1)}(z) + e^{\pi k}H_{\pm 2ik}^{(2)}(z)\right]e^{ik^2 t}\,, \tag{4.5.52a}$$

$$\chi_{\pm k}^{(2)}(z,t) = i\sqrt{\frac{k\sinh(\pi k)}{2\cosh(\pi k)}}\left[e^{-\pi k}H_{\pm 2ik}^{(1)}(z) - e^{\pi k}H_{\pm 2ik}^{(2)}(z)\right]e^{ik^2 t}\,. \tag{4.5.52b}$$

Let us discuss now the reduced method. The gauge $F \equiv \xi - t = 0$ gives the effective Hamiltonian

$$H_{\text{eff}} = -w^2 = -\frac{p_\varphi^2}{16} + 2\Lambda e^{-2\varphi}\,, \tag{4.5.53}$$

and the Schrödinger equation coincides with (4.5.48). The stationary Schrödinger equation reads

$$\left[\frac{1}{16}\partial_\varphi^2 + \Lambda e^{-4\varphi}\right]\psi(\varphi) = E\psi(\varphi)\,, \qquad E < 0\,. \tag{4.5.54}$$

The solutions of eq. (4.5.54) are those of eq. (4.5.48) where $k = \sqrt{-E}$. They can be chosen orthonormal as in (4.5.52).

Having the Hilbert space, we can compute the probability of transition from the pre-big bang regime to the post-big bang regime. The probability amplitude coincides with that found by Gasperini and Veneziano (1996)[27]

$$R_k = \frac{|\psi_k^{(-)}(-\infty, t)|}{|\psi_k^{(+)}(-\infty, t)|} = e^{-2\pi k}\,, \tag{4.5.55}$$

where $\psi_k^{(\pm)}$ are the gauge-fixed wave functions corresponding to (4.5.50). Finally, by recalling the definition of k and (4.5.1) we can obtain the transition probability for a three-dimensional portion of space with a given initial proper volume at $t \to -\infty$ (for details and the discussion see Gasperini and Veneziano, 1996).

4.6 Beyond Minisuperspaces: Third Quantisation.

In this chapter we have seen how quantum cosmological models can be successfully used to investigate the very early universe. The mathematical framework in which this investigation has been completed is given by the theory of finite-dimensional constrained systems. Our treatment follows thus the lines of analytical mechanics.

We have considered the classical minisuperspace Lagrangian at its face value and have implemented the procedure of gauge fixing in the canonical scheme along the lines traced in chapt. 2. The reduction of the model to the physical degrees of freedom is the first step in order to obtain the Hilbert space. Thus the interpretation of wave functions is the usual one and quantum mechanical effects can be discussed. Note that this procedure amounts to a definition of time in terms of canonical coordinates.

The choice of the gauge is in general a fine art and there is no a priori rule. Within this (well-defined) approach we can solve problems about which much has been written: absence of time, absence of conserved current, choice of boundary conditions, interpretation of the Wheeler-DeWitt wave function and normalisation, and so on. Further, we have clarified the essential role of fixing the gauge (in both approaches to quantisation) in order to define a positive norm Hilbert space. We have systematically studied a wide class of gauge-fixings and shown that by a clever choice of the gauge the problem may be significantly simplified and completely solved. Finally, using maximal gauge-invariant canonical representations we have been able to prove that for integrable or partially integrable systems there exists gauge fixings leading to time-independent effective Hamiltonians.

[27] Gasperini and Veneziano do not have a Hilbert space structure nor an univocal choice of the time parameter. The procedure illustrated in this section allows to recover their result in a matematically self-consistent framework.

Of course several problems remain to be solved. In particular, our procedure is strictly related to the form of the Hamiltonian. So, the definition of time can be unstable with respect to changes of the model. Much work must be done, but this is the right direction for further investigations.

Another possibility is to investigate quantum cosmological models from the point of view of a quantum field theory (see for instance Ghoroku, 1991; Giddings and Strominger, 1989). We have already noted the similarity of minisuperspace cosmological models to the relativistic free massive particle of sect. [2.4]. It is well-known that the Klein-Gordon Lagrangian can be intepreted as a quantised field. Thus one can build by analogy a theory of "third quantised universes". In this framework processes like changes of topology and interactions between universes have been proposed to solve the cosmological constant problem (Coleman, 1988). Let us conclude this chapter discussing briefly a toy model describing interacting universes. This model may illustrate how possible third quantisation effects may play a major role on the cosmological constant problem.

The Wheeler-DeWitt equation is essentially a Klein-Gordon equation in superspace. Let us assume that a complete set of solutions do exist and label this set by an index n. To discuss interactions between universes, we assume the existence of a field theory on superspace whose free field equation is given by the Wheeler-DeWitt equation. We form linear combinations of the creation and destruction operators of universes $c^\dagger$ and c

$$\begin{aligned} \Psi(h) &= \sum_n \Psi_n(h) c_n \,, \\ \Psi^\dagger(h) &= \sum_n \Psi_n^\dagger(h) c_n^\dagger \,, \end{aligned} \tag{4.6.1}$$

where h represents the three-dimensional metric of the manifold mapping the superspace. Ψ and $\Psi^\dagger$ are field operators in the abstract occupation number Hilbert space and the operators c_n, $c_n^\dagger$ satisfy the boson commutation relations (universes are bosons). The kinetic term of the field theory action takes the form (Ghoroku, 1991; Giddings and Strominger, 1989)

$$\mathcal{S} = -\frac{1}{2} \int Dh\, \Psi^\dagger H \Psi \,. \tag{4.6.2}$$

It is not difficult to convince oneself that the interactions between universes modify the Wheeler-DeWitt equation (tree level) with a potential term (Giddings and Strominger, 1989):

$$H\Psi = -\frac{dV[\Psi]}{d\Psi}. \tag{4.6.3}$$

The idea is that an observer in a given universe could interpret the potential term as an effective potential term in the Wheeler-DeWitt equation of the universe where he lives, for instance a cosmological constant. Let us consider for simplicity the model of a FRW closed universe filled with the conformal scalar field of sect. [4.3]. Using the latter to identify time, the positive and negative frequency solutions of the Wheeler-DeWitt equation read[28]

$$\Psi_n^{(\pm)} = \chi_n(a) e^{\pm i E_n t_{\rm cs}}\,, \tag{4.6.4}$$

where χ_n is the n-th harmonic oscillator wave function that is the solution of the equation

$$H_{\rm GR}\chi_n(a) = E_n\chi_n(a)\,, \qquad E_n > 0\,. \tag{4.6.5}$$

The energy E_n is related to the energy density of the matter field. The wave functions (4.6.4) represent quantum closed spherically symmetric universes whose dimensions depend on the quantum number n. For instance, a universe with the present characteristics is described by $n \approx 10^{120}$. However, a typical solution (4.6.4) has small quantum numbers, i.e. dimensions of order of the Planck length. This is the quantum analogue to the flatness-oldness problem in classical general relativity. The conformal scalar field is a pure radiation field and its Lagrangian does not contain interaction terms producing inflation.

Suppose now the existence of a foam of baby universes (4.6.4) and a two-body interaction $v(a,a')$ between universes in the minisuperspace (Cavaglià, 1994c). In the Hartree-Fock approximation (see for instance Fetter and Walecka, 1971) eq. (4.6.5) is substituted by

$$\big(H_{\rm eff} + V_H\big)\tilde\chi_n = \tilde E_n\tilde\chi_n\,, \tag{4.6.6}$$

where V_H is the Hartree-Fock potential

$$V_H = \int da' \sum_l \tilde\chi_l^2(a') v(a,a')\,. \tag{4.6.7}$$

In particular if we consider $N-1$ universes in the foam ground state ($n=0$) and one universe in an excited state with a small quantum number, the one-body equation for the excited universe becomes ($v(a,a') = -gv(a)\delta(a-a')$, where g is a positive coupling constant $O(1)$)

$$\frac{1}{2}\left[-\frac{d^2}{da^2} + a^2 - 2g(N-1)v(a)\tilde\chi_0^2\right]\tilde\chi_n = \tilde E_n\tilde\chi_n\,. \tag{4.6.8}$$

Eq. (4.6.8) is the third-quantised modification of eq. (4.6.5). As a toy model, let us discuss a potential of the form $v(a) \approx \exp(a^2)$. In this case eq. (4.6.6) simplify

[28] They correspond respectively to having fixed the Lagrange multiplier equal to ± 1. Analogously to the Klein-Gordon case, if we want to intepret the wave functions as second-quantised fields we must consider both negative and positive values of the Lagrange multiplier, i.e. positive and negative energies. On the contrary, in the first quantisation formalism the sign of the Lagrange multiplier must be fixed.

and can be cast in the form of a Schrödinger equation for a harmonic oscillator. The energy levels are

$$\bar{E}_n \approx \tilde{E}_n + g(N-1) = (n+1/2)\,. \tag{4.6.9}$$

In absence of interactions, $\tilde{E}_n = E_n$, then $0 < \tilde{E}_n \ll g(N-1)$ and we have

$$n \approx g(N-1)\,. \tag{4.6.10}$$

Therefore, supposing a sufficiently large number of interacting universes, we see that the solutions have necessarily large quantum numbers and (4.6.5) can actually describe our (classical) universe without introducing other fields nor the cosmological constant: the third quantisation mechanism depends only on the effective Schrödinger equation. This result is of course strictly dependent on the potential $v(a)$ chosen. This problem can be overcome by assuming a condensate of baby universes in the ground state (the foam) and calculating the vertex between an excited universe and the foam in the presence of an external potential. The condensate provides the "energy reservoir" to "increase" the excited universe. Further, the presence of the condensate enhances the baby-large universe interactions with respect to the large-large universe interactions. With this procedure, a wide class of potentials produces results analogous to those of the Hartree-Fock theory discussed previously.

5

Minisuperspaces in Black Hole Physics: Quantum Black Holes

In this chapter I shall deal with minisuperspace models describing quantum black holes. In the first chapter we have seen that the investigation of the black hole physics at quantum level is very relevant for the "phenomenological" quantum gravity programme. The Hamiltonian formalism is a fundamental key to obtain a quantum description of black holes. So recently a good deal of work has been dedicated to the canonical formulation of the spherically symmetric gravity (Kuchař 1994; Kastrup and Thiemann, 1994; Lau, 1996). Kuchař has cast the dynamics of primordial Schwarzschild black holes in canonical formalism starting to discuss the quantisation procedure. A complete bibliography that covers the history of the subject is contained in his paper (Kuchař, 1994).

The outline of the chapter is the following. In the next section I shall briefly present the two main motivations for studying a quantum formalism for black holes, namely the problem of the origin of the black hole entropy and the possibility of a deep connection between black holes and wormholes. In sect. [5.2] I shall develop a canonical approach to the study of the vacuum spherically symmetric metric starting from the usual Lagrangian formulation of the Einstein equations. I shall consider a foliation in a single parameter ξ, thus the Lagrangian coordinates as functions only of ξ. This leads to a structure of minisuperspace. The theory is endowed with a gauge invariance (reparametrisation in ξ) and a constraint, as it happens for the quantum cosmological models investigated in the previous chapter. I shall investigate the properties of the operators that generate rigid symmetries of the Hamiltonian, establish the form of the invariant measure under rigid transformations, and determine the gauge fixed Hilbert space of states both in the Dirac-Wheeler-DeWitt and reduced methods. Finally, I shall prove that the two quantisation methods lead to the same Hilbert space for a suitable gauge fixing. Sect. [5.3] is devoted to the generalisation of the model to string theory. I shall show that the presence of the dilaton field changes completely the picture. In the last section I shall draw conclusions.

The content of this chapter is original work done by the author in collaboration with Prof. Vittorio de Alfaro (Università di Torino) and Prof. Alexandre T. Filippov (JINR, Dubna) (see Cavaglià, de Alfaro, and Filippov, 1995d; 1996; Cavaglià and de Alfaro, 1996a).

5.1 Quantum Black Holes: Why?.

We have mentioned that a great amount of research has been devoted to the definition of a self-consistent canonical formulation of the spherically symmetric gravity and to its quantisation. The great interest in investigating quantum black hole models is due to the hope that a self-consistent quantum treatment of the gravitational field could explain the origin of the black hole entropy (see for instance Barvinsky, Frolov, and Zelnikov, 1995). So, in the last years the canonical formalism and the quantisation procedure for spherically symmetric black holes have been widely investigated (Kuchař, 1994; Kastrup and Thiemann, 1994; Lau, 1996) also in minisuperspace approximation (Mäkëla, 1996; Louko and Mäkëla, 1996).

Classically a non-rotating, uncharged, spherically symmetric black hole is completely defined by its mass. We expect then that the Schwarzschild mass becomes a quantum observable in the quantum theory of black holes. The possibility that the mass can take only discrete values has important consequences on the spectrum properties of the Hawking radiation and can lead, at least in principle, to observational tests.[1] Let us see briefly this point.

The area of a classical Schwarzschild black hole is expressed in terms of the Schwarzschild mass by the relation[2]

$$A = \frac{16\pi G^2}{c^4} M^2 \,. \tag{5.1.1}$$

If the mass of the black hole takes only discrete values[3] the Hawking radiation must be emitted in multiples of a fundamental frequency. From (5.1.1) we see that the wavelenght of the Hawking radiation must be of the order of the radius of the black hole horizon. Thus the radiation must differ from the thermal spectrum and the difference can be in principle detected by experiments.

To complete this programme, the introduction of matter fields could be of importance in order to specify the physical degrees of freedom inaccessible to observation by an external observer, whose tracing out could explain the origin of the black hole entropy (see e.g. Barvinsky, Frolov, and Zelnikov, 1995, and references therein). Finally, the fact that in classical physics only positive masses

[1] See for instance Louko and Mäkelä (1996).

[2] We restore for a moment the Newton constant G and the velocity of light c.

[3] Or, better, "If the mass squared of the black hole...". We will see in the next section that in minisuperspace approximation the difference is not trivial.

are present may be probably related to quantum gravity effects. It is then worthwile to explore the role of the mass operator in the canonical formalism and quantisation. Perhaps some light could come.

Finally, a strictly related problem that can be addressed in a quantum formalism for black holes is the connection with wormholes. What is the final state of the black hole evolution? The conjecture that the region inside the horizon of a black hole could be connected to a second spacetime region through a wormhole was proposed by Hawking (see for instance Hawking, 1992, and references therein). Already at the classical level it is possible to speculate about the connection between black holes and wormholes (see for instance Schein and Aichelburg, 1996). However, wormholes are essentially quantum objects (see chapt. 3), so it is natural to investigate this possibility in the quantum regime.

In the next section we will see how these issues appear in the quantum minisuperspace formalism.

5.2 Quantisation of the Schwarzschild Black Hole.

The first step towards the quantum description of the Schwarzschild black hole in minisuperspace approximation is to express the Einstein equations as a canonical system in a finite-dimensional phase space. This can be done in a simple direct way by a foliation in a single parameter, analogously to what done for cosmological models.[4]

Classically the general vacuum spherically symmetric solution of the Einstein equations is locally isometric to the Schwarzschild metric. Let us consider the general static spherically symmetric line element (see for instance de Felice and Clarke, 1990)

$$ds^2 = -A(r)dt^2 + N(r)dr^2 + 2F(r)dtdr + B^2(r)d\Omega_2^2\,, \qquad (5.2.1)$$

where A, N, F, and B are real functions of r. Usually, redefining the coordinate t and fixing $B = r$, (5.2.1) is cast in the form

$$ds^2 = -A'(r)dt^2 + N'(r)dr^2 + r^2 d\Omega^2\,, \qquad (5.2.2)$$

where r is now the area coordinate since the area of the two-sphere at r and t constant is $4\pi r^2$. One is then left with two functions $A'(r)$ and $N'(r)$ that can be determined by the Einstein equations. The line element (5.2.2) is the so-called standard form of the general static isotropic metric (5.2.1) (see for instance Weinberg, 1972; de Felice and Clarke, 1990). From the above considerations we are led to choose the ansatz

$$ds^2 = -4a(\xi)d\eta^2 + 4n(\xi)d\xi^2 + 8f(\xi)d\xi d\eta + b^2(\xi)d\Omega^2\,, \qquad (5.2.3)$$

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[4] A different approach to quantisation of the Schwarzschild black hole in minisuperspace approximation has been recently discussed by Mäkelä (1996).

where a, n, b, and f only depend on the coordinate ξ. The numerical factors are chosen for later convenience in calculations.

We allow in principle for changes of signs of the components of the metric tensor but we require the signature to be Lorentzian over the whole manifold: for instance, if $f = 0$ $an > 0$. ξ can be a timelike coordinate and η spacelike over part of the manifold, so it is a matter of preference to define *a priori* ξ or η as the timelike variable. Hence, we develop a formal canonical structure in ξ in which the ξ-super Hamiltonian H is a generator of gauge canonical transformations that correspond to reparametrisations of the ξ coordinate in the Lagrangian formulation.[5] Note that the line element (5.2.3) corresponds essentially to use a Gaussian normal system of coordinates with respect to the three-surface (η, θ, ϕ), i.e. to perform the 3+1 slicing of sect. [2.5] with respect to the ξ coordinate. The variable $2\sqrt{|n|}$ plays essentially the role of the ADM lapse function (Arnowitt, Deser, and Misner, 1962) with respect to the ξ-slicing and it is just a Lagrange multiplier in the action enforcing the constraint that generates reparametrisations of ξ. Since we must allow for negative values of $n(\xi)$, we need a slight modification of the ADM formalism, similar to what has been done, for instance continuing from a Lorentzian to an Euclidean signature (see e.g. Martin, 1994).

Finally, let us stress that the line element (5.2.3) does not cover the maximal extension of spacetime since it describes only a half of the Kruskal-Szekeres plane and pure ξ-coordinate transformations do not lead to a complete covering of the Kruskal-Szekeres manifold starting from the metric (5.2.3). In spite of this, the analysis of reparametrisations from this point of view may lead to interesting consequences.

Since the metric tensor in (5.2.3) does not depend on η, no η-differentiation appears in the expression of the action; so starting from the Lagrangian we may develop a formal Hamiltonian scheme in the variable ξ and obtain the corresponding ξ-super Hamiltonian after having introduced the ξ-conjugate momenta as done in the case of cosmological models. It is easy to check that this formalism is equivalent to the Einstein equations for the static solution (Cavaglià, de Alfaro, and Filippov, 1995d).

Introducing the Ansatz (5.2.3) in the Einstein-Hilbert action (2.5.10) and neglecting surface terms one obtains (Cavaglià, de Alfaro, and Filippov, 1995d; 1996)

$$S = \int_{\eta_1}^{\eta_2} d\eta \int_{\xi_1}^{\xi_2} d\xi \, L(a, b; u) \,, \tag{5.2.4}$$

where apart from an overall constant factor 1/4 the Lagrangian L is

$$L = 2u \left(\frac{\dot{a}b\dot{b}}{u^2} + \frac{a\dot{b}^2}{u^2} + \frac{1 - \Lambda b^2}{4} \right) . \tag{5.2.5}$$

[5] Thus in the region where ξ is timelike it generates as usual the dynamics.

In eq. (5.2.5), dots denote differentiation with respect to ξ and the Lagrangian multiplier $u(\xi)$ is

$$u(\xi) = 4\sqrt{an + f^2}\,. \qquad (5.2.6)$$

Since the Lagrangian must be real, we require $an + f^2 > 0$. The signature of (5.2.3) is thus Lorentzian for any value of ξ. From (5.2.5) the Einstein equations of motion can be recovered considering formally $a(\xi)$, $n(\xi)$, $f(\xi)$ and $b(\xi)$ as Lagrangian coordinates evolving in ξ. Of course, u acts as a Lagrange multiplier (we still have the freedom of choosing $f(\xi)$ and $n(\xi)$). It is easy to see that different choices of n correspond to the Schwarzschild line element written in different coordinates (standard, Eddington-Finkelstein, etc. – see Cavaglià, de Alfaro, and Filippov, 1995d). In the following we will put $f = 0$ for simplicity. With this choice, a and n must have the same sign; we will see in a moment that on the classical solutions positive values of a represent the exterior of the black hole and negative values of a represent the region inside the horizon.[6]

Let us now set up the Hamiltonian formalism in ξ. The extended Hamiltonian H_E can be calculated by the usual Legendre transformation. Defining the canonical ξ-momenta

$$p_a = \frac{2b\dot{b}}{u}\,, \qquad (5.2.7a)$$

$$p_b = \frac{2}{u}(\dot{a}b + 2a\dot{b})\,, \qquad (5.2.7b)$$

we obtain

$$H_E \equiv uH\,, \qquad (5.2.8a)$$

$$H = \frac{1}{2b^2}[p_a(bp_b - ap_a)] - \frac{1}{2}(1 - \Lambda b^2)\,. \qquad (5.2.8b)$$

H is the generator of ξ-reparametrisations (gauge transformations). The constraint equation $H = 0$ is obviously independent of ξ, indeed there has been no gauge fixing and ξ is not determined. The identification of ξ will be obtained by connecting it to the canonical coordinates of the problem (gauge fixing) analogously to what done in the previous chapter for quantum cosmological models.[7]

Let us set from now on $\Lambda = 0$; for the case of non-zero cosmological constant the reader is referred to Cavaglià, de Alfaro, and Filippov (1996).[8]

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[6] $a=0$ defines then the horizon.

[7] Note that if we do not fix the coordinate gauge by expressing ξ in terms of the canonical coordinates, the nature of ξ is vague: for instance the trivially different fixings $b=\xi$ (area gauge) and $b=\exp\xi$ lead to obviously different values for the horizon in terms of ξ. However, this does not matter much: there is a region where ξ is timelike.

[8] The results of this chapter can be also easily extended to the Reissner-Nordström

The Hamiltonian (5.2.8b) has very interesting invariance transformations; first, the gauge transformations generated by H (see sect. [2.2], eqs. (2.2.5)) can be integrated explicitly. On the constraint shell the result is:

$$b \to \bar{b} = b + h(\xi)\,\frac{p_a}{2b}\,, \tag{5.2.9a}$$

$$p_a \to \bar{p}_a = p_a + h(\xi)\,\frac{p_a^2}{2b^2}\,, \tag{5.2.9b}$$

$$a \to \bar{a} = a + \frac{N}{b^2}\,\frac{h(\xi)/2}{1 + h(\xi)p_a/2b^2}\,, \tag{5.2.9c}$$

$$p_b \to \bar{p}_b = p_b + \frac{J}{b^2}\,\frac{h(\xi)/2}{1 + h(\xi)p_a/2b^2}\,, \tag{5.2.9d}$$

$$u(\xi) \to \bar{u}(\xi) = u(\xi) + \frac{dh}{d\xi}\,, \tag{5.2.9e}$$

where J and N are gauge-invariant quantities defined below. The simplicity of the gauge transformations of b and p_a is an essential property for the quantisation of the system and will be exploited later.

We have three gauge-invariant canonical quantities, namely

$$I \;=\; b/p_a\,, \tag{5.2.10a}$$

$$J \;=\; 2b - p_a p_b + 4bH\,, \tag{5.2.10b}$$

$$N \;\equiv\; IJ \;=\; bp_b - 2ap_a\,. \tag{5.2.10c}$$

I, J, N play a fundamental role in the theory. The algebra of H, I, J, N is simple:

$$\begin{aligned} &\big[I,H\big]_P = 0\,, \qquad &&\big[J,H\big]_P = 0\,, \qquad &&\big[J,I\big]_P = 1\,,\\ &\big[N,H\big]_P = 0\,, \qquad &&\big[N,I\big]_P = I\,, \qquad &&\big[N,J\big]_P = -J\,. \end{aligned} \tag{5.2.11}$$

So the canonical formalism allows for an interesting algebraic structure of constants of the motion.

---

case, i.e. to a static electrically charged black hole. Indeed, considering a radial electric field, and redefining suitably the Lagrange multiplier, the Hamiltonian can be cast in the form $H_{\rm RN}{=}H{+}P_A^2$, where P_A is the conjugate momentum to the electromagnetic potential one-form (Cavaglià, de Alfaro, and Filippov, 1995d). This Hamiltonian is separable and we can solve the equation of motion for the electromagnetic field. We have $P_A{=}Q$ where Q is the charge of the black hole: $H_{RN}{=}H{+}Q^2{=}0$. Thus the Reissner-Nordström case is equivalent, for a suitable choice of the Lagrange multiplier, to the Schwarzschild case with $-Q^2$ in the right-hand side of the constraint equation.

For sake of completeness, we write also the unconstrained solution of the equations of motion:[9]

$$b = \frac{\tau}{2I}\,, \tag{5.2.12a}$$

$$p_a = \frac{b}{I}\,, \tag{5.2.12b}$$

$$a = I^2\left(2H + 1 - \frac{J}{b}\right)\,, \tag{5.2.12c}$$

$$p_b = I\left(4H + 2 - \frac{J}{b}\right)\,, \tag{5.2.12d}$$

$$\tau = \int_{\xi_0}^{\xi} u(\xi')\, d\xi'\,, \qquad u(\xi') > 0\,. \tag{5.2.12e}$$

It is easy to check that eq. (5.2.12c) corresponds to the Schwarzschild solution if we set $H = 0$. Then a vanishes for $b = J$ and so $J/2 \equiv M$ is the classical canonical expression of the Schwarzschild mass M.[10] Again from (5.2.12c), remembering (5.2.3), we see that $T \equiv 2I$ is the ratio between proper and coordinate time in the asymptotic region $b \to \infty$. The Schwarzschild mass is thus expressed in the canonical formalism by a constant canonical quantity, of course gauge-invariant.[11]

There is a very important point concerning the support of the variables b and p_a. Naturally b has a definite sign, since for all classical solutions with $J \neq 0$ $b = 0$ is a singular point; we may take it positive since b^2 appears in the metric. Also the value $I = 0$ is not allowed (see eqs. (5.2.12b-d) and the next (5.2.14)). From (5.2.12a) we see that I has the sign of τ.

Now $u(\xi)$ must have a definite sign (see sect. [2.2]), or else one cannot define a unique trajectory by the boundary condition $b(\tau(\xi_1)) = b_1$, $b(\tau(\xi_2)) = b_2$. We may choose $u(\xi) > 0$ without loss of generality and accordingly we take τ positive.

Hence, from (5.2.12a) we see that $I > 0$. Furthermore, p_a never changes sign in a gauge transformation – see next (5.2.13-15) – and can be restricted to be positive. These properties will be essential in the following.

Consider the rigid symmetries generated by I, J and N. Any gauge invariant function of the canonical variables can be written as $F(H, I, J)$. The requests that it be N and I invariant, or N and J, or I and J, are equivalent as they leave us with $F(H)$, so we need only consider two of the three rigid transformations. Analogously to the model of sect. [4.5] the invariance under rigid transformations

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[9] For H=0 they have of course the same content as the gauge equations.

[10] For the Reissner-Nordström case the two roots of a=0 correspond to the two horizons of the Reissner-Nordström metric.

[11] The expression (5.2.10b) coincides with the canonical expression given by Kuchař (1994) if the t-dependence of the variables is neglected.

is an essential tool to investigate the quantum measure. Let us write down these transformations.

The finite transformations generated by I (denoted as $\mathcal{I}_f$) are:

$$
\begin{aligned}
b &\to \bar{b} = b\,, \\
p_a &\to \bar{p}_a = p_a\,, \\
a &\to \bar{a} = a - f b p_a^{-2}\,, \\
p_b &\to \bar{b}_b = p_b - f p_a^{-1}\,.
\end{aligned}
\tag{5.2.13}
$$

The finite transformations generated by J (denoted as $\mathcal{J}_q$) are:

$$
\begin{aligned}
b &\to \bar{b} = \frac{b}{1 - q/I}\,, \\
p_a &\to \bar{p}_a = \frac{p_a}{(1 - q/I)^2}\,, \\
a &\to \bar{a} = a\,(1 - q/I)^2 + \frac{N}{b} q\,(1 - q/I)^2\,, \\
p_b &\to \bar{p}_b = p_b\,(1 - q/I) + \frac{J}{b} q\,(1 - q/I)\,.
\end{aligned}
\tag{5.2.14}
$$

The finite transformations generated by N (denoted as $\mathcal{N}_g$) on the canonical variables are dilatations, due to the form of N:

$$
\begin{aligned}
b &\to \bar{b} = e^{g} b\,, \\
p_a &\to \bar{p}_a = e^{2g} p_a\,, \\
a &\to \bar{a} = e^{-2g} a\,, \\
p_b &\to \bar{p}_b = e^{-g} p_b\,.
\end{aligned}
\tag{5.2.15}
$$

Now looking at these three sets of transformations and at the gauge transformation (5.2.9) we see that the canonical variables $\{b, p_a\}$ transform separately under all transformations. So these variables will be most appropriate as coordinates in the quantum case. We also note that the J transformations may change the sign of b, in contrast with our assumption that $b > 0$. Thus we consider as fundamental symmetries $\mathcal{N}_g$ and $\mathcal{I}_f$.

Analogously to what done in sect. [4.5][12] it is useful to perform a canonical transformation to the Shanmugadhasan (1973) canonical variables $\{J, I; Y, P_Y\}$, where

$$
Y = \frac{2b^2}{p_a}\,, \qquad P_Y = H\,. \tag{5.2.16}
$$

This choice is motivated by the invariance properties of the new canonical variables: I, J, H are gauge-invariant and Y transforms linearly under the gauge

[12] See also sect. [2.3].

transformation (2.2.5). For completeness the generating function of the canonical transformation is

$$\mathcal{F} = \frac{2ab^2}{Y} - \frac{YJ}{2b} + \frac{1}{2}Y\,. \tag{5.2.17}$$

Note that we can use alternatively $N = IJ$ and $p_N = \ln I$ instead of J and I.[13] In the canonical variables $\{J, I; Y, P_Y\}$, the extended Hamiltonian reads simply

$$H_E = uP_Y\,. \tag{5.2.18}$$

The transformation properties of the Shanmugadhasan variables under gauge and rigid transformations are obvious

- $\mathcal{I}_f$:

$$\begin{aligned} I &\to \bar{I} = I\,,\\ J &\to \bar{J} = J + f\,,\\ Y &\to \bar{Y} = Y\,,\\ P_Y &\to \bar{P}_Y = P_Y\,; \end{aligned} \tag{5.2.19}$$

- $\mathcal{J}_q$:

$$\begin{aligned} I &\to \bar{I} = I - q\,,\\ J &\to \bar{J} = J\,,\\ Y &\to \bar{Y} = Y\,,\\ P_Y &\to \bar{P}_Y = P_Y\,; \end{aligned} \tag{5.2.20}$$

- $\mathcal{N}_g$:

$$\begin{aligned} I &\to \bar{I} = e^{-g}I\,,\\ J &\to \bar{J} = e^{g}J\,,\\ Y &\to \bar{Y} = Y\,,\\ P_Y &\to \bar{P}_Y = P_Y\,; \end{aligned} \tag{5.2.21}$$

- $\mathcal{H}_h$:

$$\begin{aligned} I &\to \bar{I} = I\,,\\ J &\to \bar{J} = J\,,\\ Y &\to \bar{Y} = Y + h(\xi)\,,\\ P_Y &\to \bar{P}_Y = P_Y\,. \end{aligned} \tag{5.2.22}$$

Finally, another set of canonical hybrid variables that will be used for the

[13] I is positive-defined.

quantisation is[14]

$$\begin{aligned} \alpha &= \ln |a| \,, \\ c &= 2\sqrt{|a|}b \,, \\ p_\alpha &= -\frac{N}{2} \,, \\ p_c &= \frac{p_b}{2\sqrt{|a|}} \,. \end{aligned} \tag{5.2.23}$$

Contrary to the previous set, in this case there are two different canonical transformations, for positive and negative a; in the representation (5.2.23) the constraint (5.2.8b) reads

$$H = \frac{1}{2}\left[\sigma\left(p_c^2 - 4\frac{p_\alpha^2}{c^2}\right) - 1\right] = 0 \,, \tag{5.2.24}$$

where $\sigma = a/|a|$. The gauge transformation laws of these canonical coordinates are

$$\begin{aligned} \delta p_\alpha &= 0 \,, \\ \delta\alpha &= -4\epsilon\sigma\frac{p_\alpha}{c^2} \,, \\ \delta p_c &= -4\epsilon\sigma\frac{p_\alpha^2}{c^3} \,, \\ \delta c &= \epsilon\sigma p_c \,. \end{aligned} \tag{5.2.25}$$

Thus from (5.2.25) we see that p_α is gauge-invariant.

Having introduced the classical Lagrangian and Hamiltonian, and integrated the gauge and rigid transformations, we may carry out the construction of the quantum theory. We start with the Dirac method and establish the Wheeler-DeWitt equation. For the string cosmology models of sect. [4.5] we have seen that the request of preserving at the quantum level both the gauge invariance and the classical rigid symmetries, together with the support properties of the variables used as quantum coordinates, determines completely the quantum measure and fixes the representation of the quantum operators. So we identify the solutions of the Wheeler-DeWitt equation that are eigenfunctions of the operators corresponding to the most important invariants of the classical theory. A Fourier transform will relate the results to the usual configuration space.

5.2.a Dirac Quantisation.

As usual, in order to implement the Dirac procedure, we have to choose the measure in superspace and, as a consequence, the variables to be used for the wave functions. We start with the formal commutation relations

$$\left[\hat{a},\, \hat{p}_a\right] = i \,, \tag{5.2.26a}$$

$$\left[\hat{b},\, \hat{p}_b\right] = i \,. \tag{5.2.26b}$$

[14] This set of variables is reminiscent of the hybrid variables of sect. [3.4] and sect. [4.5].

We may represent them as differential operators choosing a pair of commuting variables as coordinates and establishing the form of the (non gauge-fixed) measure $d[\alpha]$. The measure $d[\alpha]$ can be determined by the requirement of invariance under the symmetry transformations of H, namely rigid and gauge transformations.

At the end of this section we shall see that the wave functions obtained with the measure introduced above are connected by a Fourier transform to the solutions of the Wheeler-DeWitt equation that uses the covariant measure in the a, b space.

Let us come back to the algebra of $\{I, J, N, H\}$. This is a powerful inspiration for physical consequences to be found in the structure of the gauge-fixed positive-definite Hilbert space. Since the algebra of I, J, N is a dilatation algebra, it is useful to recall some important points about the self-adjointness of the dilatation operator (see for instance Messiah, 1959).

Let us consider a realisation of the dilatation algebra on differentiable functions of a single variable r. If the support of the eigenvalues of both $\hat{r}$ and $\hat{p}_r$ is $\mathbb{R}$, then $\hat{r}$ and $\hat{p}_r$ are self-adjoint while the dilatation operator $\hat{D} = (rp_r + p_r r)/2$ is not self-adjoint. If instead $r \in \mathbb{R}^+ \cup \{0\}$, then (as typical for radial variables) the dilatation generator is self-adjoint and the conjugate momentum $\hat{p}_r$ is not. So we expect that the support of the variables in the present problem will be the key to the properties of the Hilbert space.

In order to determine the quantum measure, we require that the latter be invariant under rigid and gauge transformations that preserve the sign of b. Then the measure is (we denote by x, j, y the continuous eigenvalues of $\hat{I}$, $\hat{J}$, $\hat{Y}$):

$$d[\alpha(x, y)] = \frac{dx}{x}\, dy\,. \tag{5.2.27}$$

This measure makes sense as we have seen that classically $I > 0$ since both $p_a, b > 0$. We cannot use the $\mathcal{N}_g$ and $\mathcal{J}_q$ form of the rigid symmetries, as they change the sign of b and I. The choice of implementing the rigid symmetries $\mathcal{N}_g$, $\mathcal{J}_q$ implies that b becomes negative, for which there is no basis. In that case the invariant measure would be

$$d[\alpha(j, y)] = \frac{dj}{j}\, dy\,, \tag{5.2.28}$$

that requires $j > 0$. Of course one could argue that $j > 0$ because we have to exclude negative masses, but this choice would introduce an external criterion into the discussion. We will see in a moment that the operator $\hat{J}$ is not self-adjoint in the measure (5.2.27).

The measure (5.2.27) can be obtained through different considerations, i.e. using as variables the pair $\{b, p_a\}$ whose behaviour is simple under both rigid and gauge transformations. This pair of non-conjugate variables is a basis for a representation of the gauge group and therefore b and p_a are good candidates as coordinates in the wave functions. It is straightforward to determine the form of the invariant measure in this representation. Let

$$d[\alpha(b,p_a)] = G(b,p_a)\ dbdp_a\,; \qquad d[\alpha(\bar{b},\bar{p}_a)] = G(\bar{b},\bar{p}_a)\ d\bar{b}d\bar{p}_a\,. \tag{5.2.29}$$

We have:

$$d[\alpha(\bar{b},\bar{p}_a)] \approx (1+\Delta+\mathcal{G}_{,b}\delta b+\mathcal{G}_{,p_a}\delta p_a)\,dbdp_a\,, \tag{5.2.30}$$

where

$$\mathcal{G} = \ln G\,, \qquad \frac{\partial(\bar{b},\bar{p}_a)}{\partial(b,p_a)} - 1 \approx \Delta\,, \tag{5.2.31}$$

and

$$\Delta = 3g + \frac{hp_a}{2b^2}\,. \tag{5.2.32}$$

The condition of invariance determines completely G:

$$G = \frac{b}{p_a^2}\,. \tag{5.2.33}$$

The measure invariant under the continuous transformations $\mathcal{N}_g$, $\mathcal{I}_f$ that leave H invariant is thus

$$d[\alpha(b,p_a)] = \frac{bdbdp_a}{p_a^2}\,. \tag{5.2.34}$$

It is immediate to see that it coincides with (5.2.27).

Let us consider for a moment the set of rigid transformations $\mathcal{N}_g$ and $\mathcal{J}_q$. In spite of the simple transformation properties of b, p_a under them, it is easy to see by the above method that an invariant measure of the form (5.2.29) cannot be determined. Furthermore, the measure (5.2.28) is invariant under $\mathcal{J}_q$, $\mathcal{N}_g$ and $\mathcal{H}_h$ but cannot be transformed back to the canonical variables $\{b,p_a\}$.

So let us go back to the measure (5.2.34) or (5.2.27). We can define the operators $\hat{H}$, $\hat{N}$, $\hat{J}$ both in the $\{b,p_a\}$ and $\{x,y\}$ representations. Using the first pair of coordinates we have the Hermitian operators

$$\hat{a} \;=\; i\ p_a\partial_{p_a}p_a^{-1}\,, \tag{5.2.35a}$$

$$\hat{p}_b \;=\; -i\ b^{-1/2}\partial_b\ b^{1/2}\,. \tag{5.2.35b}$$

Note that $\hat{H}$ is first-order in derivatives, as well as $\hat{N}$ and $\hat{J}$. Using the Weyl ordering we obtain

$$\hat{H} = -i\frac{p_a}{2b^2}\left(b\partial_b + p_a\partial_{p_a}\right) - \frac{1}{2}\,, \tag{5.2.36a}$$

$$\hat{N} = -i\left(b\partial_b + 2p_a\partial_{p_a}\right)\,, \tag{5.2.36b}$$

$$\hat{J} = -i\frac{p_a}{b}\left(b\partial_b + 2p_a\partial_{p_a} + \frac{1}{2}\right)\,. \tag{5.2.36c}$$

Let us first discuss the eigenfunctions of $\hat{N}$. The solution of

$$\hat{H}\Psi = 0, \qquad \hat{N}\Psi = \nu\Psi\,, \tag{5.2.37}$$

is

$$\Psi_\nu(b, p_a) \;=\; c(\nu)\; b^{-i\nu}\; p_a^{i\nu}\; e^{ib^2/p_a}\,, \tag{5.2.38a}$$

or, in terms of x, y

$$\Psi_\nu(x, y) \;=\; c(\nu)\; x^{-i\nu} e^{iy/2}\,. \tag{5.2.38b}$$

The eigenfunctions of the mass operator $\hat{J}$ are solutions of the equations

$$\hat{H}\Psi = 0, \qquad \hat{J}\Psi = j\Psi\,, \tag{5.2.39}$$

namely,

$$\Psi_j(b, p_a) \;=\; c(j)\sqrt{\frac{b}{p_a}}\; e^{ib(b-j)/p_a}\,, \tag{5.2.40a}$$

or, in the $\{x, y\}$ representation

$$\Psi_j(x, y) \;=\; c(j)\sqrt{x}\; e^{i(y/2-jx)}\,. \tag{5.2.40b}$$

For sake of completeness, let us obtain from the differential representation (5.2.27) the form of the operators $\hat{H}$, $\hat{J}$, $\hat{N}$ in the $\{x, y\}$ representation:

$$\hat{H} \;=\; \hat{P}_y \;= -i\partial_y - \frac{1}{2}\,, \tag{5.2.41a}$$

$$\hat{J} \;=\; i\sqrt{x}\partial_x\frac{1}{\sqrt{x}}\,, \tag{5.2.41b}$$

$$\hat{N} \;=\; i\frac{\partial}{\partial \ln x}\,, \tag{5.2.41c}$$

Now in order to progress we have to introduce the gauge fixing via the Faddeev-Popov method (see chapt. 2). We shall prove that there is a class of viable gauges for which there are no Gribov copies and the Faddeev-Popov determinant Δ_{FP} is invariant under gauge transformations. Indeed, let us suppose that the gauge be enforced by

$$F(x, y) = 0\,, \tag{5.2.42}$$

and let F have the form

$$F(x, y) = \psi(x, y)\,\prod_i \big(y - f_i(x)\big)\,, \tag{5.2.43}$$

where $\psi\big(x, f_i(x)\big) \neq 0$ and $f_i(x) \neq f_j(x)$ for any x. Then

$$\delta(F) = \sum_i \delta\left(y - f_i(x)\right)\; \big(\psi_i(x)\big)^{-1}\,, \tag{5.2.44}$$

where

$$\psi_i(x) = \psi\big(x, f_i(x)\big) \prod_{j\neq i} \big(f_i(x) - f_j(x)\big) . \tag{5.2.45}$$

So, finally,

$$\Delta_{FP}^{-1} \; = \; \sum_i \big(\psi_i(x)\big)^{-1} . \tag{5.2.46}$$

Note that since x is gauge-invariant, so is Δ_{FP}. The gauge-fixed invariant measure is then

$$\int \frac{dx}{x} dy \; \delta\big(F(x,y)\big) \; \Delta_{FP} . \tag{5.2.47}$$

In our case the most convenient gauge (5.2.42) is

$$F(x,y) \equiv y - \xi = \frac{2b^2}{p_a} - \xi = 0 , \tag{5.2.48}$$

where ξ is a parameter. This gauge fixing implies obviously $\Delta_{FP} = 1$ and determines uniquely the gauge. Indeed,

$$\frac{2\bar{b}^2}{\bar{p}_a} = \xi \tag{5.2.49}$$

defines uniquely $h = \xi - 2b^2/p_a$.

Now we may discuss the form of the wave functions in the gauge (5.2.48). Denote by lower-case greek letters the wave functions in the gauge-fixed representation and start from the eigenfunctions of $\hat{N}$. Choosing $c(\nu) = (2\pi)^{-1/2}$, the gauge-fixed eigenfunctions of $\hat{N}$ are

$$\psi_\nu(x) \; = \; \frac{1}{\sqrt{2\pi}} \, x^{-i\nu} e^{i\xi/2} . \tag{5.2.50}$$

They are of course orthonormal in the gauge-fixed measure:

$$(\psi_{\nu_2}, \psi_{\nu_1}) = \int_0^\infty \frac{dx}{x} \;\; \psi^*_{\nu_2}(x)\psi_{\nu_1}(x) \; = \; \delta(\nu_1 - \nu_2) . \tag{5.2.51}$$

Now consider the gauge-fixed eigenfunctions of $\hat{J}$:

$$\psi_j(x) \; = \; c'(j) \, \sqrt{x} \; e^{-ixj} . \tag{5.2.52}$$

This makes clear the important point already stressed. It is indeed immediate to verify that $\hat{J}$ is not self-adjoint in that space. We have already remarked that the situation is similar to the familiar case of the radial coordinate r in flat space: its conjugate $\hat{p}_r$ is not an essentially self-adjoint operator on the Hilbert space of the Laplace operator, although it is of course a well-defined classical quantity.

If, as it is suggested by the classical correspondence, we identify $\hat{J}$ with the mass operator, we must conclude that there is no self-adjoint mass operator in this reduced theory. In other words, with this definition the mass operator is not an observable.

Let us now investigate the operator $\hat{J}^2$. In order to be a self-adjoint operator, the eigenfunctions of $\hat{J}^2$ with eigenvalues j^2 must meet one of the two conditions:

$$\lim_{x\to 0} \frac{\psi^{(1)}_{j^2}(x)}{\sqrt{x}} = 0\,, \tag{5.2.53a}$$

or

$$\lim_{x\to 0} \left[\frac{\psi^{(2)}_{j^2}(x)}{\sqrt{x}}\right]' = 0\,. \tag{5.2.53b}$$

The two separate sets are given of course by $(j > 0)$

$$\psi^{(1)}_{j^2}(x) = \frac{1}{\sqrt{\pi j}}\,\sqrt{x}\,\sin jx\,, \tag{5.2.54a}$$

$$\psi^{(2)}_{j^2}(x) = \frac{1}{\sqrt{\pi j}}\,\sqrt{x}\,\cos jx\,. \tag{5.2.54b}$$

Either the set (5.2.54a) or the set (5.2.54b) must be chosen. The eigenfunctions of each set are orthonormal

$$\left(\psi^{(k)}_{j_2^2}, \psi^{(k)}_{j_1^2}\right) = \int_0^\infty \frac{dx}{x}\,\psi^{(k)^*}_{j_2^2}(x)\,\psi^{(k)}_{j_1^2}(x) = \delta(j_2^2 - j_1^2)\,, \qquad k = 1, 2\,. \tag{5.2.55}$$

Thus the operator $\hat{J}^2$ is self-adjoint. The effect of the non self-adjoint operator $\hat{J}$ is to transform the set (1) into the set (2) and viceversa.

5.2.b Reduced Method.

Identical results can be obtained by the reduced method using the gauge fixing condition

$$F \equiv Y - \xi = 0\,. \tag{5.2.56}$$

The gauge fixing (5.2.56) corresponds to the *area gauge* $b = \text{const}\cdot\xi$ since $Y = 2bI$. Indeed, we have $u = 1$. The effective Hamiltonian on the physical gauge shell is

$$H_{\text{eff}} = -P_Y \equiv -H = 0\,. \tag{5.2.57}$$

So the wave functions do not depend on ξ. Diagonalizing $\hat{N}$ or $\hat{J}$ using (5.2.41b,c) one obtains the gauge-fixed wave functions (5.2.50) and (5.2.52). As expected for Shanmugadhasan variables (see sect. [2.3]) this proves the equivalence of the Dirac-Wheeler-DeWitt and reduced canonical quantisation methods for the gauge fixings implemented.

5.2.c Quantisation in Hybrid Variables.

In order to make physically clear the meaning of the wave functions (5.2.38) and (5.2.40), let us follow the traditional path of determining the measure by defining the kinetic part of the Hamiltonian as a Laplace-Beltrami operator. We use the couple of variables a, b that are the "physical" variables of the system in which the metric is written. From (5.2.5) we read the covariant measure in superspace

$$d[\alpha(a,b)] = b\; da\; db\,. \tag{5.2.58}$$

The representation for $\hat{p}_a$ and $\hat{p}_b$ is

$$\hat{p}_a = -i\partial_a, \qquad \hat{p}_b = -i(\partial_b + 1/2b)\,. \tag{5.2.59}$$

In the $\{\alpha, c\}$ representation the covariant measure is $c\, d\alpha dc$ and we have

$$\hat{p}_\alpha = -i\partial_\alpha, \qquad \hat{p}_c = -i(\partial_c + 1/2c)\,. \tag{5.2.60}$$

Using the covariant Laplace-Beltrami ordering for the Hamiltonian[15] the Wheeler-DeWitt equation reads

$$[ab\partial_a\partial_b - (a\partial_a)^2 + ab^2]\Psi = 0\,, \tag{5.2.61}$$

or, in terms of α and c:

$$[-(c\partial_c)^2 + 4\partial_\alpha^2 - \sigma c^2]\Psi = 0\,, \tag{5.2.62}$$

where $\sigma = a/|a|$. The operators $\hat{J}$ and $\hat{N}$ are

$$\hat{J} = \partial_a\partial_b + 2b - \frac{1}{2b}\partial_a\,, \tag{5.2.63}$$

$$\hat{N} = -i(b\partial_b - 2a\partial_a)\,. \tag{5.2.64}$$

It is easy to check that, using a different definition of the Lagrange multiplier, the Wheeler-DeWitt differential equation (5.2.61) and the differential expressions for $\hat{J}$ and $\hat{N}$ (5.2.63,64) remain unchanged.

Now let us discuss the diagonalisation of $\hat{N}$. We have to discuss separately the cases $a > 0$ and $a < 0$. The solutions are:

$$\Psi_\nu(a,b) = c(\nu)(-a)^{-i\nu/2} K_{i\nu}(2b\sqrt{-a})\,, \tag{5.2.65a}$$

for $a < 0$, where $K_{i\nu}$ is the modified Bessel function of order $i\nu$ (Bateman, 1953).[16] For $a > 0$, we have

$$\Psi_\nu(a,b) = c'(\nu)\, a^{-i\nu/2} C_{i\nu}\big(2b\sqrt{a}\big)\,, \tag{5.2.65b}$$

[15] In this case the covariant ordering coincides with the Weyl ordering.

[16] We have chosen this solution because of its asymptotic behaviour for large argument.

where $C_{i\nu}$ is a generic linear combination of Hankel functions. Note that the general solution for the vacuum Kantowski-Sachs Euclidean wormhole (Cavaglià, 1994a)[17] corresponds to the solution of the present Wheeler-DeWitt equation obtained by diagonalizing the operator $\hat{N}$. Analogously to sect. [3.4], by suitable superpositions of the kind (3.4.9) one gets wave functions that are regular also for $b \to 0$. For the $\hat{J}$ operator, the solution with eigenvalue j is

$$\Psi_j(a,b) = \frac{K(j)}{\sqrt{|b-j|}} e^{\pm 2i\sqrt{ab(b-j)}}\,, \tag{5.2.66a}$$

in the classically allowed region ($a(b-j) > 0$, oscillating behaviour), and

$$\Psi_j(a,b) = \frac{K(j)}{\sqrt{|b-j|}} e^{-2\sqrt{ab(j-b)}}\,, \tag{5.2.66b}$$

in the classically forbidden region $a(b-j) < 0$, where we have chosen the decreasing exponential behaviour, analogously to (5.2.65a).

Now we may see that these solutions are the Fourier transforms of the solutions in the $\{b, p_a\}$ space obtained in the Shanmugadhasan representation. The Fourier transform is defined as

$$\Psi(a,b) = \int_0^\infty \frac{dp_a}{p_a^2}\ \Psi(p_a,b)\ p_a e^{iap_a}\,. \tag{5.2.67}$$

Introducing in (5.2.67) $\Psi_\nu(p_a,b)$ given in (5.2.38a) and using Bateman (1954) – Vol. I, p. 313, formula (17), one obtains (5.2.65a); (5.2.65b) is obtained by elementary analytic continuations. Analogously, introducing (5.2.40a) one obtains (5.2.66a) or (5.2.66b). This proves the equivalence of the invariant measure (5.2.34) and representation (5.2.35) with the covariant measure (5.2.58) and representation (5.2.59).

To conclude the discussion, let us investigate the gauge fixing by the reduced method. The discussion parallels that of the relativistic particle of sect. [2.4], as (5.2.62) is essentially a Klein-Gordon system in the hybrid $\{\alpha, c\}$ representation.

Using the variables α and c and the Hamiltonian (5.2.24), the equations of motion read

$$\dot{c} = u\sigma p_c\,, \quad \dot{\alpha} = -4u\sigma p_\alpha/c^2 \quad \dot{p}_\alpha = 0\,, \quad \dot{p}_c = -4u\sigma p_\alpha^2/c^3\,. \tag{5.2.68}$$

It is convenient to choose the gauge fixing canonical identity

$$F \equiv \alpha - \xi = 0\,, \tag{5.2.69}$$

[17] See chapt. 3 for the case with electromagnetic coupling.

(of course with the gauge above ξ is not the area coordinate). The effective Hamiltonian is

$$H_{\text{eff}} \;=\; -p_\alpha \;= N/2\,, \tag{5.2.70}$$

where p_α can be obtained from $H = 0$:

$$H_{\text{eff}} \;=\; \pm\frac{1}{2}\,\sqrt{c^2(p_c^2\;-\;\sigma)}\,. \tag{5.2.71}$$

Note that in the classical motion the argument in the square root never becomes negative.[18]

Let us look at the value of the Lagrange multiplier. From (5.2.69) and the equations of motion (5.2.68) we have

$$u = -\frac{c^2\sigma}{4p_\alpha}\,. \tag{5.2.72}$$

Now we must impose that $u > 0$ (see sect. [2.2]), that is $\sigma p_\alpha < 0$. This means that for $a > 0$ we must choose the positive sign in (5.2.71), while for $a < 0$ we have to choose the negative sign. Let us use (5.2.60) and the covariant ordering. First discuss $a < 0$. The eigenstate of H_{eff} with eigenvalue $E = -\nu/2,\ \nu > 0$ is obtained by solving the equation

$$[-(c\partial_c)^2 +\; c^2]\;\psi_\nu(c) \;=\; \nu^2\;\psi_\nu(c)\,. \tag{5.2.73}$$

The solution is

$$\psi_\nu(c) \;=\; \sqrt{\frac{2\nu\sinh\pi\nu}{\pi^2}}\;K_{i\nu}(c)\,. \tag{5.2.74}$$

For the case $a > 0$ we look for eigenstates of H_{eff}

$$[-(c\partial_c)^2 -\; c^2]\;\chi_\nu(c) \;=\; \nu^2\;\chi_\nu(c)\,, \tag{5.2.75}$$

with solution ($\nu > 0$)

$$\chi_\nu(c) \;= i\sqrt{\frac{\nu\sinh(\pi\nu/2)}{4\cosh(\pi\nu/2)}}\left[e^{-\pi\nu/2}H^{(1)}_{i\nu}(c) - e^{\pi\nu/2}H^{(2)}_{i\nu}(c)\right]\,. \tag{5.2.76}$$

The above solutions are orthonormal (Cavaglià, de Alfaro, and Filippov, 1996):

$$(\psi_{\nu_1},\psi_{\nu_2}) = \int_0^\infty\;\frac{dc}{c}\;\psi^*_{\nu_1}(c)\;\psi_{\nu_2}(c)\;=\;\delta(\nu_1-\nu_2)\,, \tag{5.2.77a}$$

[18] This is obvious for $a<0$. For $a>0$ it can be seen as follows: from (5.2.12c,d) we have the relation $p_b=a/I+I$ and using the definition of p_c in eqs. (5.2.23) it follows that $p_c^2=p_b^2/4a=(a/I+I)^2/4a\geq 1$.

$$(\chi_{\nu_1}, \chi_{\nu_2}) = \int_0^\infty \frac{dc}{c}\, \chi^*_{\nu_1}(c)\, \chi_{\nu_2}(c) \;=\; \delta(\nu_1 - \nu_2)\,, \tag{5.2.77b}$$

and span positive-norm Hilbert spaces.

Now let us solve the Schrödinger equation

$$i\,\frac{\partial}{\partial\alpha}\,\Psi_+(\nu; c, \alpha) \;=\; H_{\text{eff}}\;\Psi_+(\nu; c, \alpha) \tag{5.2.78}$$

for the stationary states. We have (remember $\nu > 0$)

$$\Psi_+(\nu; c, \alpha) \;=\; e^{i\alpha\nu/2}\;\psi_\nu(c) \tag{5.2.79a}$$

for $E < 0,\ a < 0$, and

$$\Psi_+(\nu; c, \alpha) \;=\; e^{-i\alpha\nu/2}\;\chi_\nu(c) \tag{5.2.79b}$$

for $E > 0,\ a > 0$. On the other hand the solutions corresponding to $u < 0$ are

$$\Psi_-(\nu; c, \alpha) \;=\; e^{-i\alpha\nu/2}\;\psi_\nu(c) \tag{5.2.80a}$$

for $E > 0,\ a < 0$, and

$$\Psi_-(\nu; c, \alpha) \;=\; e^{i\alpha\nu/2}\;\chi_\nu(c) \tag{5.2.80b}$$

for $E < 0,\ a > 0$. The solutions (5.2.79-80) are the gauge-fixed wave functions correspondent of (5.2.65). Analogously to the Klein-Gordon case the use of both positive and negative u is appropriate if one reinterprets the wave function as a quantum operator (second quantisation of black holes).[19] For instance,

$$\Psi_{\text{BH}}(\alpha, c) = \int_0^\infty d\nu\,\sqrt{\frac{2\nu\sinh\pi\nu}{\pi^2}}\;K_{i\nu}(c)\;[A^\dagger(\nu)e^{-i\nu\alpha/2} + B(\nu)e^{i\nu\alpha/2}] \tag{5.2.81}$$

is the representation of the black hole quantum field for $a < 0$.

5.3 Generalisation to String Theory.

The results of the previous section can be easily generalised to $1+1$ dimensional gravity (Filippov, 1996b) or to the string effective case. The presence of the dilaton scalar field has a very interesting consequence: the Schwarzschild or Reissner-Nordström horizons disappear and are substituted by a naked singularity.[20]

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[19] See for instance sect. [4.6].

[20] Classical solutions obtained here by the canonical formulation are known since long time (Buchdal, 1959; Janis et al., 1969; Bekenstein, 1974b); also attention has been dedicated to the coupling of scalar to gravity, even in presence of gauge fields (Lavreshlashvili and Maison, 1993; Silaev and Turyshev, 1995). While the classical solutions have been widely discussed, the quantum theory has not been fully investigated.

We have two independent geodesically complete spacetimes: one is a Kantowski-Sachs-like universe (Kompaneets and Chernov 1964; Kantowski and Sachs, 1966), the other one is a static asymptotically flat spacetime: a complete universe with a naked singularity, corresponding in the limit of vanishing dilaton to the external of a Schwarzschild black hole.

Let us start from the usual action for the four-dimensional low-energy string effective theory (see sect. [3.3] and sect. [4.4]) written in the Einstein frame. We recall it here for completeness

$$S = \int d^4x\sqrt{-g}\left[R - 2\sigma\partial_\mu\varphi\partial^\mu\varphi - 2\Lambda\right] , \qquad (5.3.1)$$

where $\sigma = \pm 1$. We shall consider both signs of sigma in order to take into account contributions from moduli fields derived from the compactified manifold (see for instance chapt. 3).

Using the ansatz (5.2.3) for the line element and correspondingly $\varphi = \varphi(\xi)$ for the scalar field, the action becomes (actually, this is the action density in η, integrated over $d\Omega$)

$$S = \int d\xi \; 2u\left[\frac{\dot{a}b\dot{b}}{u^2} + \frac{a\dot{b}^2}{u^2} - \sigma\frac{ab^2\dot{\varphi}^2}{u^2} + \frac{1}{4}\left(1 - \Lambda b^2\right)\right] , \qquad (5.3.2)$$

where u is defined as in the previous section and dots represent differentiation with respect to ξ. Again, by a Legendre transformation we obtain the extended Hamiltonian

$$H_E \equiv uH = \frac{u}{2b^2}\left[p_a\left(bp_b - ap_a\right) - \sigma p_\varphi^2/4a - b^2\left(1 - \Lambda b^2\right)\right] . \qquad (5.3.3)$$

Let us now develop the canonical formalism and obtain the general solution. We set for the moment $\Lambda = 0$. We shall see later that the absence of horizon and the completeness of the spaces hold also when Λ is different from zero. We define gauge-invariant quantities along the lines of the treatment developed in the previous section. Analogously to the Einstein case, these quantities form an interesting algebra that will be at the basis of the next discussion of the quantisation.

The gauge transformations generated by H can be integrated explicitly (Cavaglià and de Alfaro, 1996a). From the gauge equations it follows that p_φ and $N = bp_b - 2ap_a$ are gauge-invariant quantities. Then we have to discuss separately different cases, depending on the value of p_φ and N. Let us define the quantity $\tilde{H} = H + 1/2$ which reduces to $1/2$ on the gauge shell. In the following we shall also define $\tau = \int u(\xi)d\xi$ and consider without loss of generality $\tau > 0$ (see the previous section) and $\tilde{H} > 0$. We now give the general solution of the gauge equations according to the different cases.

Case 1: $\gamma^{-2} = 1 + \sigma(p_\varphi^2/N^2)$ (this is certainly true for $\sigma = 1$) and $a \geq 0$ (the solution corresponds to a static and isotropic spacetime):

$$p_a = \frac{N}{a}\left[\frac{\tilde{H}}{N}\tau - \frac{1}{2}\left(1 + \frac{1}{\gamma}\right)\right],$$

$$p_b = \frac{N}{b}\left[\frac{2\tilde{H}}{N}\tau - \frac{1}{\gamma}\right],$$

$$b = \frac{\tau}{2I}\left[1 - \frac{N}{\gamma\tilde{H}\tau}\right]^{(1-\gamma)/2}, \tag{5.3.4}$$

$$a = 2\tilde{H}I^2\left[1 - \frac{N}{\gamma\tilde{H}\tau}\right]^{\gamma},$$

$$\varphi = \Phi - \sigma\frac{p_\varphi\gamma}{2N}\ln\left[1 - \frac{N}{\gamma\tilde{H}\tau}\right],$$

where we consider for simplicity $\gamma > 0$ since the solution is invariant for $\gamma \to -\gamma$ apart from a redefinition of τ ($\tau \to \tau - N/\gamma\tilde{H}$). I and Φ are two gauge-invariant quantities defined as

$$I = \sqrt{\frac{a}{2\tilde{H}}}\left[\frac{\gamma b p_b + N}{\gamma b p_b - N}\right]^{\gamma/2},$$

$$\Phi = \varphi + \sigma\frac{p_\varphi\gamma}{2N}\ln\left[\frac{\gamma b p_b - N}{\gamma b p_b + N}\right]. \tag{5.3.5}$$

The solution (5.3.4) corresponds to the solutions given in the literature both for positive and negative σ (Buchdal, 1959; Janis et al., 1969; Bekenstein, 1974b).[21] To see it, let us fix the coordinate by going on the gauge shell, i.e. $\tilde{H} = 1/2$, and choose $u = 1$, $f = 0$. Then $\tau = \xi$ and the coordinates ξ, η can be identified respectively with a radial and a timelike variable. The solution becomes ($\xi \to r$, $\eta \to t$)

$$ds^2 = -\left[1 - \frac{2N}{\gamma r}\right]^{\gamma} dt^2 + \left[1 - \frac{2N}{\gamma r}\right]^{-\gamma} dr^2 + r^2\left[1 - \frac{2N}{\gamma r}\right]^{1-\gamma} d\Omega^2, \tag{5.3.6}$$

$$\varphi = \Phi - \sigma\frac{p_\varphi\gamma}{2N}\ln\left[1 - \frac{2N}{\gamma r}\right],$$

[21] See also Wyman (1981); Turyshev (1995); Silaev and Turyshev (1995); Beckmann and Lechtenfeld (1995); Conradi (1995); Lavreshlashvili and Maison (1993).

where we have chosen $I = 1/2$ so that t is the proper time for $r \to \infty$. Since $\gamma > 0$, by inspection of (5.3.6) we can easily see that for $r \to \infty$ the line element (5.3.6) is asymptotically flat and the spacetime is defined for $r > r_s$, where $r_s = 2N/\gamma$ for $N > 0$ and $r_s = 0$ for $N < 0$. In both cases, if $\gamma \neq 1$ $r = r_s$ is a naked curvature singularity (see for instance Virbhadra et al., 1995). When $\gamma = 1$, i.e. φ =const., for $N > 0$ (5.3.6) corresponds to the usual Schwarzschild spacetime with mass N. In this case $r = r_s$ is a coordinate singularity, so the spacetime can be continued for $r < r_s = 2N$ down to $r = 0$, where there is a curvature singularity. This picture changes completely when $\gamma \neq 1$ because in this case at $r = r_s$ the area of the two-sphere is vanishing or singular. Hence, the spacetime described by (5.3.6) when $\gamma \neq 1$ cannot be continued below r_s.

Case 2: $\gamma^{-2} = 1 + \sigma(p_\varphi^2/N^2)$ and $a \leq 0$ (this corresponds to a complete Kantowski-Sachs-like universe):

$$
\begin{aligned}
p_a &= \frac{N}{a}\left[\frac{\tilde{H}}{N}\tau - \frac{1}{2}\left(1 + \frac{1}{\gamma}\right)\right], \\
p_b &= \frac{N}{b}\left[\frac{2\tilde{H}}{N}\tau - \frac{1}{\gamma}\right], \\
b &= \frac{\tau}{2I}\left[\frac{N}{\gamma\tilde{H}\tau} - 1\right]^{(1-\gamma)/2}, \\
a &= -2\tilde{H}I^2\left[\frac{N}{\gamma\tilde{H}\tau} - 1\right]^{\gamma}, \\
\varphi &= \Phi - \sigma\frac{p_\varphi\gamma}{2N}\ln\left[\frac{N}{\gamma\tilde{H}\tau} - 1\right],
\end{aligned}
\qquad (5.3.7)
$$

where now

$$
\begin{aligned}
I &= \sqrt{\frac{-a}{2\tilde{H}}}\left[\frac{N + \gamma b p_b}{N - \gamma b p_b}\right]^{\gamma/2}, \\
\Phi &= \varphi + \sigma\frac{p_\varphi\gamma}{2N}\ln\left[\frac{N - \gamma b p_b}{N + \gamma b p_b}\right].
\end{aligned}
\qquad (5.3.8)
$$

Since $a < 0$ this is a complete Kantowski-Sachs-like spacetime, ξ is a timelike coordinate and so the metric is time-dependent. With the same choice of the Lagrange multiplier, I and $\tilde{H}$ as in the previous case, and setting $\xi \to t$ and

$\eta \to \chi$ $(0 \le \chi < 2\pi)$ in (5.2.3), the solution takes the form

$$ds^2 = -\left[\frac{2N}{\gamma t} - 1\right]^{-\gamma} dt^2 + \left[\frac{2N}{\gamma t} - 1\right]^{\gamma} d\chi^2 + t^2 \left[\frac{2N}{\gamma t} - 1\right]^{1-\gamma} d\Omega^2 ,$$

$$\varphi = \Phi - \sigma \frac{p_\varphi \gamma}{2N} \ln \left[\frac{2N}{\gamma t} - 1\right] . \tag{5.3.9}$$

Consider for simplicity $t > 0$ and $N > 0$. As in the case 1, when $\gamma = 1$ the line element coincides with the Schwarzschild metric. When $\gamma \neq 1$, we have instead a curvature singularity at $t = t_s \equiv 2N/\gamma$. Hence, for $\gamma \neq 1$ (5.3.9) represents a (complete) anisotropic Kantowski-Sachs-like universe that begins in a curvature singularity at $t = 0$ and ends at $t = t_s$ in a curvature singularity after a finite lapse of time. Conversely, when the scalar field is absent, the metric (5.3.9) reduces to the standard Schwarzschild solution and coincides (with a suitable redefinition of coordinates) with the solution (5.3.6) for $\gamma = 1$. In particular, the solution (5.3.9) reduces, for vanishing scalar field, to the internal Schwarzschild region, and solution (5.3.6) to the external Schwarzschild region. Since the singularity in $t = t_s$ $(r = r_s)$ is now a coordinate singularity, both metrics can be continued across the horizon and so coincide.

Case 3: $\gamma^{-2} = -[1 + \sigma(p_\varphi^2/N^2)]$ (this case implies $\sigma < 0$ and does not allow having a pure Schwarzschild black hole solution when the dilaton is absent) and $a \ge 0$:

$$p_a = \frac{N}{a}\left(\frac{\tilde{H}}{N}\tau - \frac{1}{2}\right) ,$$

$$p_b = \frac{2\tilde{H}\tau}{b} ,$$

$$b = \frac{|N|}{4\tilde{H}I\gamma}\left[1 + \frac{4\tilde{H}^2\gamma^2}{N^2}\tau^2\right]^{1/2} \exp\left[-\gamma \text{ arctg } \left(\frac{2\tilde{H}\gamma}{N}\tau\right)\right] , \tag{5.3.10}$$

$$a = 2\tilde{H}I^2 \exp\left[2\gamma \text{ arctg } \left(\frac{2\tilde{H}\gamma}{N}\tau\right)\right] ,$$

$$\varphi = \Phi - \sigma\frac{p_\varphi\gamma}{N}\text{arctg } \left[\frac{2\tilde{H}\gamma}{N}\tau\right] ,$$

where

$$I = \sqrt{\frac{a}{2\tilde{H}}} \exp\left[-\gamma \text{ arctg } \left(\frac{bp_b\gamma}{N}\right)\right] ,$$

$$\Phi = \varphi + \sigma\frac{p_\varphi\gamma}{N} \text{ arctg } \left(\frac{bp_b\gamma}{N}\right) . \tag{5.3.11}$$

Let us choose the Lagrange multiplier as in the previous cases. On the gauge shell the solution becomes ($\xi \to r$, $\eta \to t$)

$$\begin{aligned} ds^2 = &- \exp\left\{2\gamma\left[\text{arctg}\left(\frac{2\tilde{H}\gamma}{N}r\right) - \frac{\pi}{2}\right]\right\} dt^2 + \\ &+ \exp\left\{-2\gamma\left[\text{arctg}\left(\frac{2\tilde{H}\gamma}{N}r\right) - \frac{\pi}{2}\right]\right\} dr^2 + \\ &+ \frac{N^2}{\gamma^2}\left[1 + \frac{\gamma^2}{N^2}r^2\right] \exp\left\{-2\gamma\left[\text{arctg}\left(\frac{2\tilde{H}\gamma}{N}r\right) - \frac{\pi}{2}\right]\right\} d\Omega^2 , \end{aligned} \tag{5.3.12}$$

$$\varphi = \Phi - \sigma\frac{p_\varphi\gamma}{N}\text{arctg}\left[\frac{\gamma}{N}r\right] ,$$

where we have chosen $I = e^{-\gamma\pi/2}/2$. Analogously to the first case the spacetime described by (5.3.12) is static and asymptotically flat in the radial coordinate r. Further, it is never singular. Indeed, for $r \to 0$ the line element becomes

$$ds^2 = e^{-\pi\gamma}\left\{-dt^2 + e^{2\pi\gamma}\left[dr^2 + \left(N^2/\gamma^2\right) d\Omega^2\right]\right\} , \tag{5.2.13}$$

and the scalar field assumes its minimum value. The area of the two-sphere at $r = 0$ has a finite value different from zero. This means that the spacetime has a throat at $r = 0$. Note that the existence of the wormhole is made possible by the negative sign of σ. Indeed, in this case the scalar field has negative energy density. The spacetime (5.3.12) describes then a static wormhole (see for instance Morris and Thorne, 1988).

The gauge-invariant quantities $\{N,\, I,\, p_\varphi,\, \Phi\}$ play a fundamental role. Their Poisson algebra is

$$[N, I]_P = I , \qquad [\Phi, p_\varphi]_P = 1 . \tag{5.3.14}$$

(The remaining Poisson brackets are zero.) Thus N and $\ln I$, Φ and p_φ are canonically conjugate variables. Let us now introduce the new quantity

$$Y \equiv \frac{bp_b - ap_a}{\tilde{H}} , \tag{5.3.15}$$

which has the following Poisson brackets:

$$[Y, I]_P = [Y, N]_P = [Y, p_\varphi]_P = [Y, \Phi]_P = 0\,, \qquad [Y, H]_P = 1\,. \tag{5.3.16}$$

Thus the complete set of Shanmugadhasan variables are $\{N,\, P_N \equiv \ln I,\, \Phi,\, P_\Phi \equiv p_\varphi;\, Y,\, P_Y \equiv H\}$. Analogously to sect. [5.2] invariance under rigid transformations will be used to determine the form of the quantum measure. Performing the canonical transformation to the new variables the action becomes

$$S = \int d\xi \{\dot{N} P_N + \dot{\Phi} P_\Phi + \dot{Y} P_Y - u(P_Y - 1/2)\}\,. \tag{5.3.17}$$

The requirement of invariance of the measure under rigid transformations selects the measure $d[\alpha] = dp_N d\phi dy$. (The eigenvalues of P_N, Y, and Φ have been indicated by lower-case letters.) Given the measure, we have the following representation of the operators:

$$\begin{aligned} &\hat{P}_\Phi \to -i\partial_\phi\,, \qquad &&\hat{P}_Y \to -i\partial_y\,, \qquad &&\hat{N} \to i\partial_{p_N}\,, \\ &\hat{\Phi} \to \phi\,, &&\hat{Y} \to y\,, &&\hat{P}_N \to P_N\,. \end{aligned} \tag{5.3.18}$$

The Wheeler-DeWitt equation becomes

$$(-i\partial_y - 1/2)\,\Psi(p_N, \phi, y) = 0\,. \tag{5.3.19}$$

The solutions of eq. (5.3.19) that are eigenfunctions of $\hat{N}$ and $\hat{P}_\Phi$ with eigenvalues ν and ω are

$$\Psi_{\nu,\omega} = C(\nu, \omega)\exp[-i\nu p_N + i\omega\phi + iy/2]\,. \tag{5.3.20}$$

Now we introduce the gauge fixing via the Faddeev-Popov method. Analogously to the Einstein gravity case one proves that there is a class of viable gauges for which there are no Gribov copies and the Faddeev-Popov determinant Δ_{FP} is invariant under gauge transformations (Cavaglià and de Alfaro, 1996a). As usual when we are dealing with Shanmugadhasan variables the most convenient gauge is

$$F(p_N, y, \phi) \equiv y - \xi = (bp_b - ap_a)/H - \xi = 0\,, \tag{5.3.21}$$

where ξ is a parameter. This gauge fixing implies obviously $\Delta_{FP} = 1$ and determines uniquely the gauge. Denoting by lower-case greek letters the wave functions in the gauge-fixed representation, and choosing $C(\nu) = (2\pi)^{-1}$, the gauge-fixed eigenfunctions of $\hat{N}$ and $\hat{P}_\Phi$ are

$$\psi_{(\nu,\omega)} = \frac{1}{2\pi}\exp[-i\nu p_N + i\omega\phi + i\xi/2] \tag{5.3.22}$$

that are orthonormal. The eigenfunctions (5.3.22) coincide with those found for the Schwarzschild black hole in the previous section apart from the plane wave in ϕ.

Let us now quantise the system by the reduced method. Again the gauge fixing condition is $F \equiv Y - \xi = 0$. This determines the Lagrange multiplier as

$u = 1$ since from the definition of Y and the classical general solution of the gauge equations it follows $Y = \tau +$ constant. Using the constraint $\tilde{H} = 1/2$ and the gauge fixing condition, the effective Hamiltonian on the gauge shell becomes $H_{\text{eff}} = -P_Y = -1/2$. Thus the Schrödinger equation is

$$\left(i\frac{\partial}{\partial \xi} + \frac{1}{2} \right) \psi = 0 \,. \tag{5.3.23}$$

Diagonalizing $\hat{N}$ and $\hat{P}_\Phi$ we obtain the wave functions (5.3.22) which form a definite positive Hilbert space.

In this context a primary question concerns the effect of the cosmological constant: does it spoil the completeness of those spacetimes? We can deduce the properties of the solution without solving the equations. The method presented here can also be applied, at least in principle, to more complicate cases, for instance when a simple potential term for the dilaton is present.

Let us consider the action (5.3.2) where now $\Lambda = \sigma V(\varphi)$ is a potential term for the dilaton and redefine the Lagrange multiplier as[22]

$$u'(\xi) = u(\xi) f(b, \varphi) \,, \tag{5.3.24}$$

where

$$f(b, \varphi) = 1 - \sigma V(\varphi) b^2 > 0 \,. \tag{5.3.25}$$

Using (5.3.24) the Lagrangian becomes:

$$L = \frac{f}{u'} \left(2\dot{a}b\dot{b} + 2a\dot{b}^2 - 2\sigma a b^2 \dot{\varphi}^2 \right) + \frac{u'}{2} \,. \tag{5.3.26}$$

We easily notice that the equations of motion derived from the Lagrangian (5.3.26) can be interpreted as the geodesic equations for the minisuperspace $\tilde{M}$ with metric

$$d\tilde{s}^2 = f(b, \varphi)[b \; da db + a \; db^2 - \sigma a b^2 \; d\varphi^2] \,. \tag{5.3.27}$$

In other words, the solutions of the Einstein equations for the spacetime metric are the geodesics of the minisuperspace. Thus we can deduce the properties of the solutions of the Einstein equations from the study of the geodesics of (5.3.27). Accordingly, we are interested in establishing which points of the manifold (5.3.27) are singular. Indeed, no geodesics can cross a singular point of (5.3.27) and so this point will be an end point of all the solutions of the Einstein equations. Note that the Einstein solutions can be singular also for regular points of $\tilde{M}$, since (5.3.27) is written in a given coordinate system that can be pathological at some point, as we shall see in a next example. Hence, this method does not give in general all the curvature singularities of the spacetime manifold.

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[22] See sect. [2.2].

Let us apply these considerations and discuss some cases.

• *Vacuum case (Schwarzschild black hole, see previous section).*

In this case the manifold $\tilde{M}$ is two-dimensional, the metric reducing to the form

$$d\tilde{s}^2 = b\ dadb + a\ db^2\,. \tag{5.3.28}$$

From (5.3.28) it is straightforward to compute the curvature tensor. We find

$$R_{abab} = 0\,, \tag{5.3.29}$$

i.e. the manifold is flat. Indeed with the change of coordinates

$$x = \frac{b(a+1)}{2}\,, \qquad y = \frac{b(a-1)}{2}\,, \tag{5.3.30}$$

the line element (5.3.28) reduces to the form $d\tilde{s}^2 = dx^2 - dy^2$. The geodesics are straight lines

$$x + k_1 y = k_2\,. \tag{5.3.31}$$

However, we know that the solution of the Einstein equations is singular at $b = 0$.[23] It is easy to see that the singularity at $b = 0$ is a coordinate singularity in minisuperspace. Indeed, from (5.3.30) one sees that the transformation of coordinates is pathological at $b = 0$ since the points $(a, 0)$, $\forall a$, are mapped in the origin of the x, y plane.

• *Reissner-Nordström and cosmological constant cases.*

For the Reissner-Nordström solution the metric in the minisuperspace is (Cavaglià, de Alfaro, and Filippov, 1996d)

$$d\tilde{s} = b\ dadb + a\ db^2 + b^2 dA^2/4\,, \tag{5.3.32}$$

where A is the potential vector of the electric radial field $\mathbf{F} = dA$. Analogously to the previous case, the curvature tensor is vanishing and the minisuperspace is flat. Singularity at $b = 0$ is due to the coordinate chart.

An identical result is obtained in the presence of a negative cosmological constant. The metric of the minisuperspace is indeed

$$d\tilde{s} = (1 - \Lambda b^2)[b\ dadb + a\ db^2]\,, \tag{5.3.33}$$

and the space is flat, as it can be verified using the transformation of coordinates

$$x = \frac{b}{2}(a + 1 - \Lambda b^2/3)\,, \qquad y = \frac{b}{2}(a - 1 + \Lambda b^2/3)\,. \tag{5.3.34}$$

[23] See eq. (5.2.12c).

Again the singularity at $b = 0$ is due to the choice of coordinates.

- *Dilaton case.*

In this case, using the metric (5.3.27) with $f = 1$ it is straightforward to find the Kretschmann and Ricci scalars. We have:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4}{a^2 b^4}\,, \qquad R = -\frac{2}{ab^2}\,. \tag{5.3.35}$$

In this case we can deduce that the geometrical sections $a = 0$ and $b = 0$ of the minisuperspace metric are singular and consequently all geodesics must end there. Hence, the solutions of the Einstein equations for the spacetime metric are singular for $a = 0$ and $b = 0$.[24] Since $a = 0$ cannot be crossed, the above considerations prove that there is no horizon and the spacetimes defined by $a > 0$ or $a < 0$ are complete. Note that there could be other singular points corresponding to coordinate singularities in minisuperspace (we know from the explicit solutions that this is not the case).

- *Dilaton (modulus) plus negative cosmological constant.*

Again we have $f = 1 - \Lambda b^2$ and the Kretschmann scalar is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4\frac{e^{-2\zeta}}{a^2 b^4}\left(1 + 8\Lambda^2 b^4 e^{-2\zeta}\right)\,, \tag{5.3.36}$$

where $\zeta = \ln f$. From (5.3.36) it is easy to see that there does not exist any geodesic such that the Kretschmann scalar is finite for $a = 0$. This proves that there is no horizon.

- *General case.*

We compute the minisuperspace scalars:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{e^{-2\zeta}}{a^2 b^4}\, g(b,\varphi)\,, \tag{5.3.37a}$$

$$R = \frac{e^{-\zeta}}{ab^2}\, h(b,\varphi)\,, \tag{5.3.37b}$$

where

$$\zeta = \ln\left[1 - \sigma V(\varphi) b^2\right]\,, \tag{5.3.38a}$$

$$\begin{aligned} g = {} & 4 + \frac{\zeta_{,\varphi}^4}{4} - 4b^2\zeta_{,bb} + 4b\zeta_{,b} - 4\sigma\zeta_{,\varphi\varphi} + 4b^2\zeta_{,b}^2 - 4\sigma b\zeta_{,b}\zeta_{,\varphi\varphi} \\ & + 2\zeta_{,\varphi\varphi}^2 - 2\sigma\zeta_{,\varphi}^2 - 2\sigma b\zeta_{,b}\zeta_{,\varphi}^2 + 4\sigma b\zeta_{,b\varphi}\zeta_{,\varphi}\,, \end{aligned} \tag{5.3.38b}$$

$$h = -2 - 2b\zeta_{,b} + 2\sigma\zeta_{,\varphi\varphi} + \sigma\frac{\zeta_{,\varphi}^2}{2}\,. \tag{5.3.38c}$$

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[24] See eqs. (5.3.4) and following.

Since $V(\varphi)$ has not been restricted, it may be possible in principle to find potentials such that $g = 0$ and $h = 0$ for $a = 0$ on at least one geodesic in minisuperspace. If this happens, the traveller falling along that geodesic does not see any singularity at $a = 0$. Hence, $a = 0$ is not a priori a singular point of the minisuperspace manifold and can be crossed (one should not forget that $a = 0$ or $b = 0$ or both could be coordinate singularities in minisuperspace with the consequence of being curvature singularities in the spacetime, as it happens for $b = 0$ in the Schwarzschild solution). This implies that for suitable potentials there may be black hole solutions in spacetime (see for instance Beckmann and Lechtenfeld, 1995).

5.4 Answers or More Doubts?.

What have been the main results for the model investigated in this chapter? We have expressed the Einstein equations for the Schwarzschild black hole as a canonical system in a finite, 2×2 dimensional, phase space and integrated the gauge transformations. This has been possible thanks to a simple algebraic structure of three gauge-invariant canonical quantities that form an affine algebra. The canonical quantity corresponding to the Schwarzschild mass has been identified. Then we have quantised the system along the lines of chapt. 2. The algebra allows to determine the measure invariant for rigid and gauge transformations in the inner product and the gauge fixing with the consequent establishment of a positive-definite Hilbert space. The gauge has been fixed by the Faddeev-Popov procedure and the existence of a class of gauges has been proved. We also started by first fixing the gauge in the classical frame by a suitable canonical gauge fixing identity that contains the coordinate ξ; the results coincide with those obtained by the Dirac method.

The procedure allowed to solve for this model all conceptual problems related to the quantisation of a gravitational system; the identification of the parameter ξ (our internal time) through a gauge fixing condition that defines ξ in terms of the canonical variables leads to a unitary Hamiltonian in the reduced canonical space. Thus the quantisation of the system is non-ambiguous. Of course many problems remain unsolved.

We have discussed the eigenfunctions of the mass operator and the generator of dilatations. Using the general form of the eigenfunctions of the mass operator we have shown that they are oscillating in the classically allowed regions and exponentially decreasing in the forbidden regions. Hence, in this approach the mass plays the role of a quantum number determining wave functions; in this respect the result is in agreement with that obtained by Kuchař (1994). However, having a positive-definite Hilbert space, we are able to prove that, due to the support properties of its conjugate variable, the Hermitian operator corresponding to the Schwarzschild mass in the gauge-fixed, positive-norm, Hilbert space is not self-adjoint, while its square is a self-adjoint operator with positive eigenvalues, analogously to what happens for the radial momentum in ordinary quantum mechanics.

Of course in the classical theory the mass is perfectly defined. Again take the example of the radial momentum: although it is not a self-adjoint operator in the Hilbert space, a classical radial momentum is defined, namely $p_r = m\dot{r}$, and its square is self-adjoint. This difference between classical and quantum behaviour is due to the fact that a classical canonical quantity is a purely local entity while the definition of a self-adjoint operator conveys a general information about the Hilbert space. This possibly signals that the identification of J with the mass carried at the classical level is not the correct one in the quantum formulation. Alternatively, this may have something to do with the fact that in classical physics only positive masses are present. To look into this question in the present frame one has to construct a procedure of classical limit that yields the Schwarzschild metric and investigate the role of eigenfunctions of $\hat{J}^2$.

No quantisation of the mass appears from this theory. It is interesting to stress though that in the frame developed here quantisation of the mass squared could be achieved in a gauge-invariant way by a modification of the theory. For instance a very crude way is just to set the support condition $x < x_0$. This is a gauge-invariant cut-off that leads to quantisation of the eigenvalues of $\hat{J}^2$. Now, this cut-off is performed in the gauge $y = 1$, that is $2bx = 1$. Thus a modification of the theory for large x corresponds to a gauge-invariant modification for small b. It will be interesting to explore the consequences of less crude models leading to quantisation of the mass; this requires a reliable definition of the quantum mass operator of course.

The set of eigenfunctions of the dilatation operator coincides with that of the Kantowski-Sachs wormholes (Cavaglià, 1994a) for the region inside the horizon. This is hardly surprising. The geometry inside the horizon of a black hole coincides with the Kantowski-Sachs one, and further the foliation parameter ξ is timelike inside the horizon, so what we expose here is, for the part internal to the horizon, isomorphic to the theory of the Kantowski-Sachs spacetime. This property enforces the much discussed possibility that a black hole can be connected to a Kantowski-Sachs wormhole (see Hawking, 1992, and references therein).

Matter degrees of freedom have been introduced in the last section with the aim of possibly specify the physical degrees of freedom inaccessible for observation by an external observer, whose tracing out could explain the origin of the black hole entropy (see e.g. Barvinsky, Frolov, and Zelnikov, 1995). However, the presence of the dilaton scalar field has a very dramatic consequence: the Schwarzschild or Reissner-Nordström horizons disappear and are substituted by a naked singularity. The dilaton coupling generates two independent geodesically complete spacetimes: the first one is a Kantowski-Sachs-like universe, the second one is a static asymptotically flat spacetime: a complete universe with a naked singularity, corresponding in the limit of vanishing dilaton to the external of a Schwarzschild black hole.

In conclusion, the main result obtained from the model discussed in this chapter is the establishment of a Hilbert space for the quantum Schwarzschild black hole without any ambiguity. The Hilbert space so defined can be used as a frame in which all possible quantum effects derived from the quantum nature of the system can be investigated.

6

Conclusion and Possible Developments

In the previous chapters I have investigated several examples of minisuperspace models in canonical quantum gravity. The aim of my research project was to carry out the study of conceptual problems (prima facie questions, see chapt. 1) arising in the quantum formulation of finite-dimensional gravitational systems in which only a finite number of degrees of freedom of the gravitational and matter fields is retained.

The basic technique was introduced in chapt. 2 and is given by the theory of quantisation of constrained systems. We have seen that the quantum mechanics of time reparametrisation invariant systems involves the settlement of ambiguities and interpretation difficulties often neglected or not discussed in depth in the literature, for instance the definition of time or the equivalence between different gauges and/or different approaches to quantisation.

The final target of my project was to apply those techniques to the study of quantum cosmological models and in general of quantum gravitational systems, for instance quantum black holes and wormholes. I have already stressed that these models are crude approximations from the point of view of the ultimate theory of the gravitational field. However, the quantum formulation of black holes and the quantum formulation of the post big-bang era of the universe may provide important interesting grounds for the research in quantum gravity.

Attention has been focused essentially on three quantum gravity systems: wormholes, quantum cosmological models, and quantum black holes. Let us summarise in detail what we have learned and what could be the possible developments.

a) Wormholes.

We have seen in sect. [3.1] that wormholes are typical quantum effects of gravity. The wormhole models investigated in chapt. 3 can be used to shed light on the nature of the electric charge and on possible relations among quantum phenomena expected to arise at the Planck energy scale. Further investigations may proceed along this direction. Recently a large amount of time has been devoted to the investigation of multidimensional wormhole models, also in the framework of the Kaluza-Klein theory (Dzhunushaliev, 1996). Indeed, it has been proposed that the generation of the electric charge at Planck scales involves compactification of extra dimensions (Dzhunushaliev, 1996).

Another line of research is based on the possibility that wormholes may describe a tunnelling between oscillating and expanding universes. This is relevant to the quantum cosmology programme. Possibly there may be models in multidimensional gravity in which an instanton generates a tunnelling between a compact D-dimensional spacetime and a spacetime with three exponentially expanding spatial dimensions, as it happens in four dimensions for the instanton investigated in sect. [3.2] (Dzhunushaliev, private communication). The simplest model to be considered in order to investigate this possibility is the five-dimensional Kaluza-Klein's theory (Dzhunushaliev, work in progress).

b) Quantum Cosmological Models.

We have seen how conceptual problems related to quantum models of the early universe can be successfully solved when the techniques used for the quantisation of gauge systems are appropriately applied to cosmology. Both at the classical and quantum levels the problem of the definition of an evolution parameter for the system (the so-called problem of time) can be successfully discussed within the theory of constrained systems. In the quantum picture the definition of an evolution parameter is an indispensable step in the direction of possible detectable quantum effects of the gravitational field. Indeed, a sensible discussion about the quantum origin of structures in the early universe (see Halliwell and Hawking, 1984) can be completed only if all ambiguities (absence of time, absence of conserved current, choice of boundary conditions, interpretation of the Wheeler-DeWitt equation, normalisation of wave functions, and so on) are solved, namely if the Hilbert space of wave functions has been defined.

Along this line of research the next step is the discussion of perturbations, i.e. the investigation of models in which $g_{\mu\nu}$ depends on two or more parameters (midisuperspace models). The quantum evolution of perturbations can be successfully investigated using the quantum cosmological models of chapt. 4 plus perturbative terms. Since these models are endowed with a Hilbert space structure, the perturbative techniques of quantum mechanics can be applied and quantitative results obtained.

Similar conclusions can be drawn for models derived from string theory (sect. [4.4,5]). We have seen that string cosmology may provide a possible alternative to the standard cosmological problem and solve the initial singularity problem because of the target-space duality of the theory. The transition from the pre-big bang phase to the post-big bang phase is classically forbidden (no-go theorems) but can be allowed quantum mechanically. This property opens interesting possibilities to understand the birth of the universe. However, the formulation of a coherent quantum description of the model is prior to the search for possible observable traces of the pre-big bang era. The choice of a correct time parameter and a suitable gauge fixing, the definition of the Hilbert space, the interpretation of wave functions, etc. are unavoidable steps towards the completion of this programme. Using the techniques developed in chapt. 4, one can investigate models with a non-vanishing potential term for the dilaton (Gasperini and Veneziano, work in progress) both from theoretical and numerical points of view (Veneziano, private communication). The inclusion of a potential term may enhance the probability of tunnelling from the pre big-bang regime to the post big-bang one.

Finally, it is interesting to implement the approach to the problem of time developed in chapt. 4 in the frame of supersymmetric quantum cosmology (see Moniz, 1995; Cheng and Moniz, 1996) and for the Ashetkar formalism (Morales-Técotl, private communication). This programme has not yet fully investigated in the literature.

c) Quantum Black Holes.

I have mentioned in chapt. 5 that the formulation of a consistent quantum description of spherically symmetric black holes is an unavoidable step towards the understanding of the origin of the entropy and the Hawking effect. So my aim has been the analysis of quantum models of black holes, either in vacuum or in presence of the dilaton.

As in the case of cosmological models, a deep investigation of the gauge fixing is required to obtain a consistent formulation of the system. We have seen that a suitable choice of the gauge fixing leads naturally to the definition of the Hilbert space of states. This is the starting point for future developments. For instance, the wave functions of the black hole obtained in chapt. 5 can be used to discuss the origin and the meaning of the thermal Hawking radiation (see chapt. 1) and the connection of black holes with wormholes (see chapt. 5). In the case of the vacuum Schwarzschild black hole the mass eigenstates are linear superpositions of wormhole eigenstates. This property enforces the much discussed idea that a black hole can be connected to a wormhole and opens an intriguing possibility for the understanding of the information loss paradox. It would be worth exploring this weird and fascinating possibility. Another important issue to be investigated is the quantisation of the mass. No quantisation of the mass appears from the theory; in the vacuum case the quantisation of the mass may be achieved in a gauge-invariant way by a modification of the theory (cut-off) for small scales. It could be interesting to explore the consequences of models leading to quantisation of the mass.

The model of quantum black holes investigated in chapt. 5 can be straightforward extended to supergravity (Cavaglià and Moniz, work in progress) or to the inclusion of perturbations. Finally, a possible straightforward application of our approach to the quantum black hole is the investigation of the Hilbert space of states for the extreme and non-extreme Reissner-Nordström metrics (Cavaglià and Liberati, work in progress). It is well-known that extreme black holes may violate the laws of black hole thermodynamics since their entropy seems to be zero, while the area of the horizon does not vanish. The investigation of the differences between quantum states of extreme and non-extreme charged black holes could be of relevance in order to solve this problem.

In conclusion, the content of this thesis can be considered at two complementary levels: either as a collection of models useful to describe particular quantum gravity effects in a suitable approximation, or as a set of specific examples of the correct techniques to be used to describe reparametrisation invariant constrained systems with a finite number of degrees of freedom.

Appendix

Conventions

Metric signature in the Lorentzian regime:

$$(-,+,+,+)$$

Curvature tensor (Landau and Lifshitz, 1975):

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\mu}{}_{\nu\sigma}$$

Units (except otherwise stated):

$$c = 1,\ \hbar = 1,\ 16\pi G = 16\pi/M_{\rm Pl}^2 = 1$$

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