Geometric PDEs on compact Riemannian manifolds

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Contents

Introduction

In this thesis, some nonlinear problems coming from conformal geometry and physics, namely the prescription of Q-curvature, T-curvature ones and the generalized 2×2 Toda system are studied. We study also the existence of extremal functions of two Moser-Trudinger type inequalities (which is a common feature of those problems) due to Fontana[40] and Chang-Yang[23].

0.1 Introduction of the problems

The study of the relationships between conformally covariant operators on compact closed Riemmannian manifolds, their associated conformal invariants and the related partial differential equations has received much attention in the last decades.

A model example is the Laplace-Beltrami operator on compact surfaces (Σ, g), which governs the transformation laws of the Gauss curvature. In fact under the conformal change of metric $g_u = e^{2u}g$, we have

$$\Delta_{g_u} = e^{-2u}\Delta_g; \qquad -\Delta_g u + K_g = K_{g_u}e^{2u}, \tag{1}$$

where Δ_g and K_g (resp. Δ_{g_u} and K_{g_u}) are the Laplace-Beltrami operator and the Gauss curvature of (Σ, g) (resp. of (Σ, g_u)).
Moreover, we have the Gauss-Bonnet formula which relates $\int_\Sigma K_g dV_g$ and the topology of Σ :

$$\int_\Sigma K_g dV_g = 2\pi\chi(\Sigma);$$

where $\chi(\Sigma)$ is the Euler-Poincaré characteristic of Σ. From this, we have that $\int_\Sigma K_g dV_g$ is a topological invariant (hence also a conformal one).

There exists also another example of a conformally covariant differential operator on four dimensional compact closed Riemannian manifolds called the Paneitz operator, and to which is associated a natural concept of curvature. This operator, discovered by Paneitz in 1983 (see [74]) and the corresponding Q-curvature introduced by Branson (see [11]) are defined in terms of Ricci tensor Ric_g and scalar curvature R_g of the Riemannian manifold (M, g) as follows

$$P_g\varphi = \Delta_g^2\varphi + div_g\left((\frac{2}{3}R_g g - 2Ric_g)d\varphi\right); \tag{2}$$

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|Ric_g|^2),$$

where φ is any smooth function on M.

As the Laplace-Beltrami operator governs the transformation laws of the Gauss curvature, we also have that the Paneitz operator does the same for the Q-curvature. Indeed under a conformal change of metric $g_u = e^{2u}g$ we have

$$P_{g_u} = e^{-4u}P_g; \qquad P_g u + 2Q_g = 2Q_{g_u}e^{4u}.$$

Apart from this analogy, we also have an extension of the Gauss-Bonnet formula which is the Gauss-Bonnet-Chern formula

$$\int_M (Q_g + \frac{|W_g|^2}{8}) dV_g = 4\pi^2 \chi(M),$$

where W_g denotes the Weyl tensor of (M, g), see [33]. Hence, from the pointwise conformal invariance of $|W_g|^2 dV_g$, it follows that the integral of Q_g over M is also a conformal invariant.

On the other hand, there are high-order analogues to the Laplace-Beltrami operator and to the Paneitz operator for high dimensional compact closed Riemannian manifolds and also to the associated curvatures. More precisely, given a compact closed n-dimensional Riemannian manifold (M, g), in [47] it was introduced a family of conformally covariant differential operators P^n_{2m} (for every positive integer m if n is odd and for every positive integer m such that $2m \leq n$ if n is even) whose leading term is $(-\Delta_g)^m$. These operators are usually referred to as the GJMS operators. Moreover after passing to stereographic projection P^n_{2m} coincides with $(-\Delta_g)^m$ if M is the sphere and g its standard metric. In [9], some curvature invariants Q^n_{2m} were defined, naturally associated to P^n_{2m}.
Now for n even let us set

$$P^n_g = P^n_n; \qquad Q^n_g = Q^n_n.$$

Then in low dimensions we have the following relations

$$P^2_g = \Delta_g; \quad Q^2_g = K_g,$$

and

$$P^4_g = P_g; \quad Q^4_g = 2Q_g.$$

It turns out that P^n_g is self-adjoint and annihilates constants. Furthermore as for the Laplace-Beltrami operator on compact closed Riemannian surfaces and the Paneitz operator on compact closed four dimensional Riemannian manifolds, for every compact n-dimensional Riemannian manifold (M, g) with n even, we have that after a conformal change of metric $g_u = e^{2u} g$

$$P^n_{g_u} = e^{-nu} P^n_g; \qquad P^n_g u + Q^n_g = Q^n_{g_u} e^{nu}. \tag{3}$$

We remark, that due to equation (3) and to the fact that P^n_g is self-adjoint and annihilates constants, $\int_M Q^n_g dV_g$ is conformally invariant and will be denoted by κ_{P^n}.

In the paper of Fefferman and Graham, see [38], it was developed a tool which is referred to as FG ambient metric construction and allows them to show existence of scalar conformal invariants. Later the same tool was used to derive the GJMS operators. On the other hand, Branson [11] defined the Q-curvature in the even dimensional case via a continuation argument in the dimension, while in the paper of Graham an Zworsky, see [48], Q^n_g was derived by an analytic continuation in a spectral parameter. Furthermore, inspired by this work, Fefferman and Graham derived the Q-curvature by solving some Lapace problem associated to the formal Poincaré metric in the ambient space, and considering formal asymptotics of the solutions. Moreover this new approach of Fefferman and Graham to derive the Q-curvature allows them to define analogues of P^n_g and Q^n_g also when n is odd. In this case, P^n_g and Q^n_g enjo y several properties similar to their counterparts in even dimension. More precisely P^n_g is self-adjoint and also annihilates constants. Moreover P^n_g governs the transformation laws of Q^n_g. On the other hand there is a difference because in the odd case P^n_g turns out to be a pseudodifferential operator. In the context of conformal geometry, the role of P^n_g and Q^n_g is not clear yet since the definition of P^n_g and Q^n_g does not only depend on the conformal class of the boundary of the ambient space but also on the extension of the formal Poincaré metric to a metric in the interior.

As for the case of compact closed Riemannian manifolds, many works have been done also in the study of conformally covariant differential (pseudodifferential) operators on compact smooth

Riemannian manifolds with smooth boundary, their associated curvature invariants, the corresponding boundary operators and curvatures in order also to understand the relationship between analytic and geometric properties of such objects.

A model example is the Laplace-Beltrami operator on compact surfaces with boundary (Σ, g), and the Neumann operator on the boundary. Under a conformal change of metric the couple constituted by the Laplace-Beltrami operator and the Neumann operator govern the transformation laws of the Gauss curvature and the geodesic curvature. In fact, under the conformal change of metric $g_u = e^{2u}g$, we have

$$\begin{cases} \Delta_{g_u} = e^{-2u}\Delta_g; \\ \dfrac{\partial}{\partial n_{g_u}} = e^{-u}\dfrac{\partial}{\partial n_g}; \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_g u + K_g = K_{g_u}e^{2u} \text{ in } \Sigma; \\ \dfrac{\partial u}{\partial n_g} + k_g = k_{g_u}e^u \text{ on } \partial\Sigma. \end{cases}$$

where Δ_g (resp. Δ_{g_u}) is the Laplace-Beltrami operator of (Σ, g) (resp. (Σ,g_u)) and K_g (resp. K_{g_u}) is the Gauss curvature of (Σ, g) (resp. of (Σ,g_u)), $\frac{\partial}{\partial n_g}$ (resp $\frac{\partial}{\partial n_{g_u}}$) is the Neumann operator of (Σ, g) (resp. of (Σ,g_u)) and k_g (resp. k_{g_u}) is the geodesic curvature of $(\partial\Sigma, g)$ (resp of $(\partial\Sigma,g_u)$). Moreover we have the Gauss-Bonnet formula which relates $\int_\Sigma K_g dV_g + \int_{\partial\Sigma} k_g dS_g$ and the topology of Σ

$$\int_\Sigma K_g dV_g + \int_{\partial\Sigma} k_g dS_g = 2\pi\chi(\Sigma), \tag{4}$$

where $\chi(\Sigma)$ is the Euler-Poincaré characteristic of Σ, dV_g is the element area of Σ and dS_g is the line element of $\partial\Sigma$. Thus $\int_\Sigma K_g dV_g + \int_{\partial\Sigma} k_g dS_g$ is a topological invariant, hence a conformal one.

The Paneitz operator and the Q-curvature discussed above exist also on four dimensional Riemannian ma Onifolds with boundary and enjoy the same conformal invariance properties that we recall below.

On the other hand, Chang and Qing, see [18], have discovered a boundary operator P^3_g defined on the boundary of compact four dimensional smooth Riemannian manifolds and a natural third-order curvature T_g associated to P^3_g as follows

$$P^3_g\varphi = \frac{1}{2}\frac{\partial\Delta_g\varphi}{\partial n_g} + \Delta_{\hat g}\frac{\partial\varphi}{\partial n_g} - 2H_g\Delta_{\hat g}\varphi + (L_g)_{ab}(\nabla_{\hat g})_a(\nabla_{\hat g})_b + \nabla_{\hat g}H_g.\nabla_{\hat g}\varphi + (F - \frac{R_g}{3})\frac{\partial\varphi}{\partial n_g}.$$

$$T_g = -\frac{1}{12}\frac{\partial R_g}{\partial n_g} + \frac{1}{2}R_gH_g - < G_g, L_g > +3H_g^3 - \frac{1}{3}Tr(L^3) + \Delta_{\hat g}H_g,$$

where φ is any smooth function on M, $\hat g$ is the metric induced by g on ∂M, $L_g = (L_g)_{ab} = -\frac{1}{2}\frac{\partial g_{ab}}{\partial n_g}$ is the second fundamental form of ∂M, $H_g = \frac{1}{3}tr(L_g) = \frac{1}{3}g^{ab}L_{ab}$ (g^{ab} are the entries of the inverse g^{-1} of the metric g) is the mean curvature of ∂M, R^k_{bcd} is the ambient (extrinsic) Riemann curvature tensor $F = R^a_{nan}$, $R_{abcd} = g_{ak}R^k_{bcd}$ (g_{ak} are the entries of the metric g) and $< G_g, L_g >= R_{anbn}(L_g)_{ab}$.

As for closed Riemannian manifolds, we have that, as the Laplace-Beltrami operator and the Neumann operator govern the transformation laws of the Gauss curvature and the geodesic curvature on compact surfaces with boundary under conformal change of metric, the couple (P^4_g, P^3_g) does the same for (Q_g, T_g) on compact four dimensional smooth Riemannian manifolds with boundary. In fact, after a conformal change of metric $g_u = e^{2u}g$ we have that

$$\begin{cases} P^4_{g_u} = e^{-4u}P^4_g; \\ P^3_{g_u} = e^{-3u}P^3_g; \end{cases} \quad \text{and} \quad \begin{cases} P^4_g + 2Q_g = 2Q_{g_u}e^{4u} \text{ in } M \\ P^3_g + T_g = T_{g_u}e^{3u} \text{ on } \partial M. \end{cases} \tag{5}$$

Apart from this analogy, as in the case of closed compact four dimensional Riemannian manifolds, there holds also an extension of the Gauss-Bonnet formula (4) which is known as the Gauss-Bonnet-Chern formula

$$\int_M (Q_g + \frac{|W_g|^2}{8})dV_g + \int_{\partial M}(T + Z)dS_g = 4\pi^2\chi(M) \tag{6}$$

where W_g denotes the Weyl tensor of (M,g) and ZdS_g (for the definition of Z see [18]) are pointwise conformally invariant. Moreover, it turns out that Z vanishes when the boundary is totally geodesic (by totally geodesic we mean that the boundary ∂M is umbilic and minimal). Setting

$$\kappa_{P_g^4} = \int_M Q_g dV_g, \qquad \kappa_{P_g^3} = \int_{\partial M} T_g dS_g;$$

we have that thanks to (6), and to the fact that $W_g dV_g$ and ZdS_g are pointwise conformally invariant, also $\kappa_{P_g^4} + \kappa_{P_g^3}$ is conformally invariant, and will be denoted by

$$\kappa_{(P^4,P^3)} = \kappa_{P_g^4} + \kappa_{P_g^3}. \tag{7}$$

We have three *Uniformization* type problems related to equations (3), and (5) that we describe in more details in the next Subsections.

The application of the method of nonlinear partial differential equations in the study of conformal structures on manifolds can be trace back to Poincaré. Indeed, using the later method Poincaré solved the Classical *Uniformization* problem for closed Riemannian surfaces of genus greater than 1. The analogous question for surfaces of positive curvature was first succesfully studied by Moser, in which he obtained with precise constant a sharp version of a limiting case of Sobolev inequality that is commonly referred to as the Moser-Trudinger inequality. This inequality was the crucial analytical tool in Moser's argument. The role played by the Moser-Trudinger inequality in Moser's variational approach is due to the exponential nonlinearity and not to the fact that the problem is the one of prescribing Gaussian curvature. Thus such ideas can be applied to deal with variational problems with exponential nonlinearities. The later type of problems are very well-know to be models for many physical phenomenon. A celebrated example is the following system called *Toda system* defined on a domain $\Omega \subseteq \mathbb{R}^2$,

$$-\Delta u_i = \sum_{j=1}^{N} a_{ij} e^{u_j}, \qquad i = 1, \dots, N, \tag{8}$$

where $A = (a_{ij})_{ij}$ is the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}, \tag{9}$$

and arising in the study of non-abelian Chern-Simons theory, see for example [35] or [84].

The system

$$-\Delta u_i = \sum_{j=1}^{N} \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} dV_g} - 1 \right), \qquad i = 1, \dots, N, \tag{10}$$

where h_i are smooth and positive functions on the surface Σ (of volume 1) is a generalized version of (8). When $N = 2$, the system becomes

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right); \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right), \end{cases} \quad \text{on } \Sigma, \tag{11}$$

and is reffered as the generalized 2×2 Toda system.

The solvability of the system (11) is a very important question in physics, and is investigated in this thesis. We will discuss it in more details in the next Subsection.

A common feature of the above problems is a phenomena called *bubbling*. As said above, the crucial analytical tools which allow to deal with it are Moser-Trudinger type inequalities. There are two important objects in the study of Moser-Trudinger type inequalities: one is to find the best constant and the other is to determine whether there exists extremal functions. In this thesis, we study the problem of the existence of extremal functions for two Moser-Trudinger type inequalities due to Fontana[40] and Chang-Yang[23] that we discuss in more details below.

0.1.1 The prescribed Q-curvature problem in arbitrary dimensions

The prescribed Q-curvature problem for compact closed Riemannian manifold (M, g) of arbitrary dimension n, consist of finding metrics conformal to g such that the corresponding Q-curvature is a constant. Due to equation (3), the problem is equivalent to finding a solution of the equation

$$P^n_g u + Q^n_g = \bar{Q}e^{nu} \qquad in \ \ M; \tag{12}$$

where $\bar{Q}$ is a real constant.
The problem has a variational structure. Hence in view of standard elliptic regularity theory, solutions can be found as critical points of the following functional

$$II_A(u) = n\left\langle P^n_g u, u\right\rangle + 2n\int_M Q^n_g u dV_g - 2\kappa_{P^n}\log\int_M e^{nu}dV_g; \quad u \in H^{\frac{n}{2}}(M). \tag{13}$$

Since this Euler-Lagrange functional is in general unbounded from above and below, it is necessary to find extremas which are possibly saddle points. We will use a min-max scheme following the method of Djadli and Malchiodi in [33]. By classical arguments, a min-max scheme yields a *Palais-Smale sequence*, namely a sequence $(u_l)_l \in H^{\frac{n}{2}}(M)$ satisfying the following properties

$$II_A(u_l) \to c \in \mathbb{R}; \quad II^{'}_A(u_l) \to 0 \ \text{ as } l \to +\infty. \tag{14}$$

Then, to recover existence, one should prove for example that $(u_l)_l$ is bounded, or a similar compactness criterion. But since we do not know if the Palais-Smale condition holds or even if Palais-Smale sequences are bounded, we will employ a monotonicity argument due to Struwe , see [80]. This consists in studying compactness of solutions to perturbations of (12), like

$$P^n_g u_l + Q_l = \bar{Q}_l e^{nu_l} \qquad in \ \ M; \tag{15}$$

where

$$\bar{Q}_l \longrightarrow \bar{Q}_0 \quad in \ \ C^1(M); \tag{16}$$

$$Q_l \longrightarrow Q_0 \quad in \ \ C^1(M); \tag{17}$$

$$\bar{Q}_0 > 0. \tag{18}$$

Adopting the standard terminology in geometric analysis, we say that a sequence (u_l) of solutions to (15) *blows up* if the following holds:

$$\text{there exist } \ x_l \in M \ \text{ such that } \ u_l(x_l) \to +\infty \ \text{ as } \ l \to +\infty, \tag{19}$$

To give some geometric applications, we discuss three results proven by Gursky, [49] , and by Chang, Gursky and Yang, [17] , [20] for the four dimensional case. If a manifold which

has a conformal metric of positive constant scalar curvature satisfies $\int_M Q_g dV_g > 0$, then its first Betti number vanishes. Moreover up to a conformal metric it has positive Ricci tensor, and hence M has a finite fundamental group. Furthermore, if the quantitative assumption $\int_M Q_g dV_g > \frac{1}{8}\int_M |W_g|^2 dV_g$ holds, then M must be diffeomorphic to the four-sphere or to the projective space. In particular the last result is an improvement of a theorem by Margerin, [68] with a conformally invariant assumption, while the one of Margerin assumes pointwise pinching conditions on the Ricci tensor in terms of W_g.
Finally, we also point out that the Paneitz operator, the Q-curvature and their high-dimensional analogues, see [9] , [10], appear in the study of Moser-Trudinger type inequalities, log-determinant formulas and the compactification of locally conformally flat manifolds, see [12], [20], [21], [22].

For the four dimensional case, problem (12) has been solved in [23] under the assumption that P_g is a non-negative operator and $\int_M Q_g dV_g < 8\pi^2$ ($8\pi^2$ is the integral of the Q-curvature on the standard sphere). Under these assumptions by the Adams inequality (see [22])

$$\log \int_M e^{4(u-\bar{u})} dV_g \leq \frac{1}{8\pi^2}\langle P_g u, u\rangle + C, \quad u \in H^2(M),$$

where $\bar{u}$ is the average of u and where C depends only on M, the functional II_A is bounded from below, coercive and lower semicontinuous, hence solutions can be found as global minima using the Direct Methods of the Calculus of Variations. A first sufficient condition to ensure these hypotheses was given by Gursky in [49]. He proved that if the Yamabe invariant of (M, g) is non-negative, and if $\int_M Q_g dV_g \geq 0$, then P_g is non-negative with trivial kernel, and moreover $\int_M Q_g dV_g \leq 8\pi^2$, with the equality holding if and only if M is conformally equivalent to S^4. More recently Djadli and Malchiodi (see [33]) proved existence of solutions for (12) still in the four-dimensional case under generic assumptions. More precisely they proved existence of solutions when P_g has no kernel and $\int_M Q_g dV_g \notin 8\pi^2\mathbb{N}$. These conditions include manifolds with negative curvature or negative Yamabe class, for which $\int_M Q_g dV_g$ can be bigger than $8\pi^2$.

For the n-dimensional case with n even, problem (12) has been solved under the condition that P_g^n is a non-negative operator with trivial kernel and $\kappa_{P^n} < (n-1)!\omega_n$ ($(n-1)!\omega_n$ is the value of κ_{P^n} on the standard sphere) using a geometric flow (see [13]). On the other hand, since under these assumptions by a Moser-Trudinger type inequality (see Chapter 1), the functional is bounded from below, coercive and lower semicontinuous, then solutions can be found also by Minimization via Weierstrass theorem in the Calculus of Variations, as for the case of [23].

0.1.2 The prescribed Q-curvature and T-curvature problem on four manifolds with boundary

When considering the problem of prescribing the Q-curvature and the boundary T-curvature of a compact four dimensional Riemannian manifold with boundary (M, g), of particular relevance are two cases. The first one consist of finding metrics conformal to g such that the corresponding Q-curvature is constant and the T-curvature zero, and the second one to search for metrics in the conformal class of the background metric g for which the T-curvature is constant and the Q-curvature vanishes.
In this thesis, due to PDEs reasons, we will focus on two particular cases:
a): To search for conformal metrics in $[g]$ with constant Q-curvature, zero T-curvature and zero mean curvature,
b): To find metrics conformally related to g with constant T-curvature, zero Q-curvature and vanishing mean curvature.
From the fact that the Neumann operator governs the transformation law (under conformal changes) of the mean curvature and (5), we have that problem **a)** is equivalent to solving the

following BVP:

$$\begin{cases} P_g^4 u + 2Q_g = 2\bar{Q}e^{4u} & \text{in } M; \\ P_g^3 u + T_g = 0 & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} - H_g u = 0 & \text{on } \partial M, \end{cases}$$

where $\bar{Q}$ is a fixed real number and $\frac{\partial}{\partial n_g}$ is the inward normal derivative with respect to g.
Problem **b)** reduces to solving

$$\begin{cases} P_g^4 u + 2Q_g = 0 & \text{in } M; \\ P_g^3 u + T_g = \bar{T}e^{3u} & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} - H_g u = 0 & \text{on } \partial M, \end{cases}$$

where $\bar{T}$ is a fixed real number and $\frac{\partial}{\partial n_g}$ still denoting the inward normal derivative with respect to g.
Due to a result by Escobar, [36], and to the fact that we are interested in solving the problem under conformally invariant assumptions, it is not restrictive to assume $H_g = 0$, since this can always be obtained through a conformal transformation of the background metric. Thus, to solve problem **a)**, we are led to solve the following BVP with Neumann homogeneous boundary condition:

$$\begin{cases} P_g^4 u + 2Q_g = 2\bar{Q}e^{4u} & \text{in } M; \\ P_g^3 u + T_g = 0 & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} = 0 & \text{on } \partial M, \end{cases} \tag{20}$$

and problem **b)** to solve

$$\begin{cases} P_g^4 u + 2Q_g = 0 & \text{in } M; \\ P_g^3 u + T_g = \bar{T}e^{3u} & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{21}$$

Defining $H_{\frac{\partial}{\partial n}}$ as

$$H_{\frac{\partial}{\partial n}} = \left\{ u \in H^2(M) : \quad \frac{\partial u}{\partial n_g} = 0 \right\};$$

and $P_g^{4,3}$ as follows, for every $u, v \in H_{\frac{\partial}{\partial n}}$

$$\langle P_g^{4,3} u, v \rangle_{L^2(M)} = \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \nabla_g v \right) dV_g - 2 \int_M Ric_g(\nabla_g u, \nabla_g v) dV_g$$
$$-2 \int_{\partial M} L_g(\nabla_{\hat{g}} u, \nabla_{\hat{g}} v) dS_g,$$

we have that, by the regularity result in Proposition 0.3.5 below, critical points of the functional

$$II_Q(u) = \langle P^{4,3} u, u \rangle_{L^2(M)} + 4 \int_M Q_g u dV_g + 4 \int_{\partial M} T_g u dS_g - \kappa_{(P^4, P^3)} \log \int_M e^{4u} dV_g; \quad u \in H_{\frac{\partial}{\partial n}},$$

which are weak solutions of (20) are also smooth and hence strong solutions. Furthermore by the regularity result in Proposition 0.3.8 below, critical points of the functional

$$II_T(u) = \langle P^{4,3} u, u \rangle_{L^2(M)} + 4 \int_M Q_g u dV_g + 4 \int_{\partial M} T_g u dS_g - \frac{4}{3} \kappa_{(P^4, P^3)} \log \int_{\partial M} e^{3u} dS_g; \quad u \in H_{\frac{\partial}{\partial n}},$$

which are weak solutions of (21) are also smooth and hence strong solutions.

For the same reasons as in the problem of finding constant Q-curvature conformal metrics on compact closed Riemannian manifolds, to solve these two problems, we use mix-max arguments and Struwe's monotonicity method. Therefore, to find solutions for BVP (20), we have to study compactness of solutions to perturbations of (20) of the form,

$$\begin{cases} P^4_g u_l + 2Q_l = 2\bar{Q}_l e^{4u} & \text{in } M; \\ P^3_g u + T_l = 0 & \text{on } \partial M; \\ \dfrac{\partial u_l}{\partial n_g} = 0 & \text{on } \partial M, \end{cases} \tag{22}$$

where

$$\bar{Q}_l \longrightarrow \bar{Q}_0 > 0 \quad in \;\; C^2(M); \;\; Q_l \longrightarrow Q_0 \quad in \;\; C^2(M); \;\; T_l \longrightarrow T_0 \quad in \;\; C^2(\partial M); \tag{23}$$

and for BVP (21) to study compactness of solutions to perturbations of (21) like

$$\begin{cases} P^4_g u_l + 2Q_l = 0 & \text{in } M; \\ P^3_g u_l + T_l = \bar{T}_l e^{3u_l} & \text{on } \partial M; \\ \dfrac{\partial u_l}{\partial n_g} = 0 & \text{on } \partial M, \end{cases} \tag{24}$$

where

$$\bar{T}_l \longrightarrow \bar{T}_0 > 0 \quad in \;\; C^2(\partial M) \quad T_l \longrightarrow T_0 \quad in \;\; C^2(\partial M) \quad Q_l \longrightarrow Q_0 \quad in \;\; C^2(M); \tag{25}$$

As in the case of the prescribed Q-curvature problem in arbitrary dimensions, here we also adopt the standard terminology in geometric analysis, and we say that a sequence (u_l) of solutions to (22) *blows up* if the following holds:

$$\text{there exist } \; x_l \in M \; \text{ such that } \; u_l(x_l) \to +\infty \; \text{ as } \; l \to +\infty. \tag{26}$$

On the other hand, from the Green representation formula given in Lemma 0.3.3 below, we have that if u_l is a sequence of solutions to (24), then u_l satisfies

$$u_l(x) = -2\int_M G(x,y)Q_l(y)dV_g - 2\int_{\partial M} G(x,y)T_l(y)dS_g(y) + 2\int_{\partial M} G(x,y)\bar{T}_l(y)e^{3u_l(y)}dS_g(y).$$

Therefore, under the assumption (23), if $\sup_{\partial M} u_l \leq C$, then we have u_l is bounded in $C^{4+\alpha}$ for every $\alpha \in (0,1)$.
Hence in this context, we say that a sequence (u_l) of solutions to (24) *blows up* if the following holds:

$$\text{there exist } \; x_l \in \partial M \; \text{ such that } \; u_l(x_l) \to +\infty \; \text{ as } \; l \to +\infty. \tag{27}$$

To mention some geometric applications, we discuss two results which can be found in the survey [24]. The first one is a rigidity type result saying that if (M,g) has a constant positive scalar curvature and ∂M has zero mean curvature, then $\kappa_{(P^4,P^3)} \leq 4\pi^2$; and the equality holds if $(M,\partial M)$ is conformally equivalent to the upper hemisphere (S^4_+, S^3). The second one is a classification of the pairs $(M,\partial M)$ with $Q=0$ and $T=0$. Indeed it says that, if $(M,\partial M)$ is locally conformally flat with umbilic boundary ∂M, $Q=0, T=0$, $Y(g)>0$ (where $Y(g) = \inf < L_c u, u >$ where the infimum is taken over all metrics conformal to g with the same volume as g

and zero mean curvature and $L_c = -6\Delta_g + R$ is the conformal Laplacian) and $\chi(M) = 0$ then either $(M, \partial M) = (S^1 \times S^3_+, S^1 \times S^2)$, or $(M, \partial M) = (I \times S^3, \partial I \times S^3)$ whe re I is an interval.

To the best of our knowledge, the first existence results for problem **a)** have been obtained by Chang and Qing, see [19] under the assumptions that $P^{4,3}_g$ is non-negative, $KerP^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} < 4\pi^2$, and no existence results is known for the problem **b)**.

In the case of closed four dimensional Riemannian manifold M, it is well-known that the Q-curvature equation is intimately related to a fully nonlinear PDE called the σ_2-equation ($\sigma_2(A_g) = 2Q_g + \frac{1}{6}\Delta_g R_g$ is the second symmetric function of the Shouten tensor A_g), see [17],[20]. A study of the latter PDE has given important geometric applications of the Q-curvature. In [17],[20], it is proven that if the underlying Riemannian manifold has a conformal metric of positive constant scalar curvature and $\int_M Q_g dV_g > 0$, then its first Betti number vanishes. Moreover up to a conformal metric it has positive Ricci tensor, and hence M has a finite fundamental group. Furthermore, as said in the previous Subsection, if the quantitative assumption $\int_M Q_g dV_g > \frac{1}{8}\int_M |W_g|^2 dV_g$ holds then M must be diffeomorphic to the four-sphere or to the projective space.
In the case when M has a boundary, Chen [25] has studied an analogue of the σ_2-equation which turns out to be a fully nonlinear BVP. Among other results, she obtained that if the Yamabe invariant $Y(M, \partial M, [g])$ (for the definition, see [25]) and $\kappa_{(P^4,P^3)}$ are both positive and M umbilic then there exists a metric g_u in the conformal class of g such that $\sigma_2(A_{g_u})$ is a positive constant (hence Q_{g_u} constant), $T_{g_u} = H_{g_u} = 0$, hence giving another existence result for the problem **a)**. Furthermore g_u can be taken so that the Ricci curvature Ric_{g_u} is positive, hence M has a finite fundamental group.

Remark 0.1.1. *We point out that due to the rigidity type result above, the assumptions under which Chen obtained existence results for problem* **a)**, *we have that implicitely* $\kappa_{(P^4,P^3)} \leq 4\pi^2$ *(even if the the boundary is not umbilic).*

0.1.3 The generalized 2×2 Toda system

The generalized 2×2 Toda system is the following system:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right); \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right); \end{cases} \quad \text{on } \Sigma,$$

where h_i are smooth and positive functions on the Riemannian surface Σ which we assume to have unit volume. Problem (11) is variational, and solutions can be found as critical points of the functional $II_\rho : H^1(\Sigma) \times H^1(\Sigma)$, $\rho = (\rho_1, \rho_2)$ defined as

$$II_\rho(u_1, u_2) = \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g \right] + \sum_{i=1}^{2} \rho_i \int_\Sigma u_i dV_g - \sum_{i=1}^{2} \rho_i \log \int_\Sigma h_i e^{u_i} dV_g.$$

Here a^{ij} are the entries of the inverse matrix A^{-1} (where A is as in (9)).

The structure of the functional II_ρ strongly depends on the values of ρ_1 and ρ_2. For example, the condition $\rho_i \leq 4\pi$ for both $i = 1, 2$ has been proven in [44] to be necessary and sufficient for II_ρ to be bounded from below, see Theorem 1.3.6 (we refer also to [77] and [78]). In particular, for ρ_1 and ρ_2 strictly less than 4π, II_ρ becomes coercive (once we factor out the constants, since II_ρ is invariant under the transformation $u_i \mapsto u_i + c_i$, $c_i \in \mathbb{R}$) and solutions of (11) can be found as global minima.

The case in which one of the ρ_i's becomes equal to 4π (or both of them) is more subtle since the functional is still bounded from below but not coercive anymore. In [43] and [58] some conditions for existence are given in this case, and the proofs involve a delicate analysis of the limit behavior of the solutions when the ρ_i's converge to 4π from below.

On the other hand, when some of the ρ_i's are bigger than 4π, II_ρ is unbounded from below and solutions should be found as saddle points. In [63], [72] and [73] some existence results are given and it is proved that if $h_i \equiv 1$, and if some additional assumptions are satisfied, then $(0,0)$ is a local minimizer for II_ρ, so the functional has a mountain pass structure and some corresponding critical points. Furthermore in [43] a very refined blow-up behavior of solutions is given (below we report Theorem 2.4.1 as a consequence of this analysis) and existence is proved if Σ has positive genus and if ρ_1, ρ_2 satisfy either **(i)** $\rho_1 < 4\pi$, $\rho_2 \in (4\pi, 8\pi)$ (and viceversa), or **(ii)** $\rho_1, \rho_2 \in (4\pi, 8\pi)$.

In this thesis we investigate the solvability of the generalized 2×2 Toda system in the case where one of the ρ_i can be very large and the other one less the 4π.

0.1.4 Extremals for Fontana and Chang-Yang inequalities

In his study of the extension of the results of Adams[1] to compact closed Riemannian manifolds, L. Fontana[40] has proved among other things the following optimal inequality

$$\int_M e^{32\pi^2 u^2} dV_g \leq C \;\; \forall u \in H^2(M) \;\text{ such that }\; \int_M |\Delta u|^2 dV_g \leq 1 \;\text{ and }\; \int_M u dV_g = 0. \qquad (28)$$

Likewise, in their study of the extremals of the log-determinant functional on four dimensional closed Riemannian manifolds, Chang and Yang[23] have proved an optimal inequality involving the Paneitz operator. Precisely, they showed that if the Paneitz operator P^4_g is non-negative with trivial kernel, then there holds

$$\int_M e^{32\pi^2 u^2} dV_g \leq C \;\; \forall u \in H^2(M) \;\text{ such that }\; < P^4_g u, u > \leq 1 \;\text{ and }\; \int_M u dV_g = 0. \qquad (29)$$

The problem of extremals for Fontana's inequality (resp Chang-Yang's inequality), is the one of finding whether there exists an extremal for the maximization problem

$$\sup_{u \in \mathcal{H}_1} \int_M e^{32\pi^2 u^2} dV_g, \quad \text{where} \;\; \mathcal{H}_1 = \{u \in H^2(M) : \overline{u} = 0, \;\; \int_M |\Delta_g u|^2 dV_g = 1\}.$$

respectively

$$\sup_{u \in \mathcal{H}_2} \int_M e^{32\pi^2 u^2} dV_g, \quad \text{where} \;\; \mathcal{H}_2 = \{u \in H^2(M) : \overline{u} = 0, \;\; < P^4_g u, u >= 1\}$$

We recall that for the Sobolev inequality, the related extremal problem has no solution if the domain is a ball of the Euclidean space. However, Carleson and Chang[16] proved a suprising result by showing that indeed for the associated Moser-Trudinger inequality on the unit ball in Euclidean space, there is a solution. This result was later extended to every connected domain in two dimensional Euclidean space by Flucher[42]. In 2001 Li[59] proved the existence of extremal functions for Moser-Trudinger inequality on every compact closed Riemannian surface.
We remark that in all these problems, the Euler-Lagrange equations associated are second order in contrast to the probems of finding extremals for Fontana and Chang-Yang inequalities.

0.2 Content of the thesis

In this thesis, we study the four problems described above. The ones of prescribing Q-curvature of a compact closed manifold of arbitrary dimension, of prescribing the Q-curvature and boundary T-curvature of a compact four dimensional manifold with boundary and the generalized 2×2 Toda system are *non compact* variational problems. By *non compact*, we mean that the standard compactness conditions like Palais-Smale one fail to hold. We tackled them using min-max method and refined blow-up analysis combined with a monotonicity method introduced by Struwe. The problem of finding extremals for Fontana's inequality and Chang-Yang's one is solved through blow-up techniques combined with Pohozaev type identity and capacity estimates to overcome the lack of a good maximum principle for fourth order PDEs and the fact that truncations are not allowed in H^2. We remark that the crucial analytical tools for the study of the problems of prescription of Q-curvature in arbitrary dimension, of prescribing the Q-curvature and boundary T-curvature of a compact four manifold with boundary and the generalized 2×2 Toda system are Moser-Trudinger type inequalities. We divide the thesis into three main Chapters. In the first one, we recall some classical Moser-Trudinger type inequalities, give some new ones and their improvement used to tackle the problem of prescribing Q-curvature in arbitrary dimension, Q-curvature and T-curvature of a four manifold with boundary, and the generalized 2×2 Toda system, and the proof of the existence of extremals for Fontana's inequality and Chang-Yang's one. The second Chapter is concerned with the blow-up analysis of perturbations of the PDEs (BVPs) involved in the problems of prescription of Q-curvature, T-curvature and the generalized 2×2 Toda system, and will be used to overcome the lack of compactness in their study. The last Chapter deals with the min-max scheme to get existence results for the problems of prescribing constant Q-curvature in arbitrary dimensions, constant Q-curvature, constant T-curvature on four dimensional manifolds with boundary, and the generalized 2×2 Toda system on compact closed surfaces.

0.2.1 Existence of extremals for Fontana and Chang-Yang inequalities

In Chapter 1, the main results we obtain are the existence of extremal functions for Fontana and Chang-Yang inequalities. Precisely, we prove the following two theorems:

Theorem 0.2.1. *Let (M,g) be a compact closed smooth 4-dimensional Riemannian manifold. Then setting*

$$\mathcal{H}_1=\{u\in H^2(M)\ \text{ such that }\ \overline{u}=0\ \text{ and }\ \int_M|\Delta_g u|^2dV_g=1\}$$

we have that

$$\sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2u^2}dV_g$$

is attained.

Theorem 0.2.2. *Let (M,g) be a compact closed smooth 4-dimensional Riemannian manifold. Assuming that P_g^4 is non-negative and $KerP_g^4\simeq\mathbb{R}$, then setting*

$$\mathcal{H}_2=\{u\in H^2(M)\ \text{ such that }\ \overline{u}=0\ \text{ and }\ <P_g^4u,u>=1\}$$

we have that

$$\sup_{u\in\mathcal{H}_2}\int_M e^{32\pi^2u^2}dV_g$$

is attained.

These results are obtained in collaboration with Yuxiang Li and are contained in the paper[61].

Remark 0.2.3. *Since the leading term of P_g^4 is Δ_g^2, then the proof of the two Theorems are quite similar. We point out that the same proof is valid for both, except for some trivial adaptations, hence we will give a full proof of Theorem 0.2.1 and only a sketch of the proof of Theorem 0.2.2.*

Remark 0.2.4. *As already said in the discussion of the prescribed Q-curvature problem in arbitrary dimensions, we recall that due to a result by Gursky, see [49] if both the Yamabe invariant of (M,g) and $\int_M Q_gdV_g$ are non-negative, then we have that P_g^4 is non-negative and $KerP_g^4\simeq\mathbb{R}$, hence we have the assumptions of Theorem 0.2.2.*

We are going to describe our approach to prove Theorem 0.2.1. We use Blow-up analysis. First of all we take a sequence $(\alpha_k)_k$ such that $\alpha_k \nearrow 32\pi^2$, and by using Direct Methods of the Calculus of variations we can find $u_k \in \mathcal{H}_1$ such that

$$\int_M e^{\alpha_k u_k^2} dV_g = \sup_{v \in \mathcal{H}_1} \int_M e^{\alpha_k v^2} dV_g.$$

see Lemma 1.5.1. Moreover using the Lagrange multiplier rule we have that $(u_k)_k$ satisfies the equation:

$$\Delta_g^2 u_k = \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} - \gamma_k \tag{30}$$

for some constants λ_k and γ_k.

Now, it is easy to see that if there exists $\alpha > 32\pi^2$ such that $\int_M e^{\alpha u_k^2} dV_g$ is bounded, then by using Lagrange formula, Young's inequality and Rellich compactness Theorem, we obtain that the weak limit of u_k becomes an extremizer. On the other hand if

$$c_k = \max_M |u_k| = |u_k|(x_k);$$

is bounded, then from standard elliptic regularity theory, u_k is compact, and thus converges uniformly to an extremizer. Hence assuming that Theorem 0.2.1 does not hold, we get

1)

$$\forall \alpha > 32\pi^2 \quad \lim_{k \to +\infty} \int_M e^{\alpha u_k^2} dV_g \to +\infty;$$

2)

$$c_k \to +\infty.$$

We will follow the same method as in [59] up to some extents.

In [59] where the author deals with a second order problem, the function sequence studied is the following:

$$-\Delta_g u_k = \frac{u_k}{\lambda_k} e^{\alpha'_k u_k^2} - \gamma_k,$$

where $\alpha'_k \nearrow 4\pi$, and u_k attains $\sup\limits_{\int_M |\nabla_g u|^2 dV_g = 1, \bar{u} = 0} \int_M e^{\alpha'_k u^2} dV_g$. The author also assumed $c_k \to +\infty$. Then he showed that

$$2\alpha_k c_k (u_k(x_k + r_k x) - c_k) \to -2\log(1 + \pi |x|^2) \tag{31}$$

for suitable choices of r_k, x_k with $r_k \to 0$. Next he proved the following

$$\lim_{k \to +\infty} \int_{\{u_k \le \frac{c_k}{A}\}} |\nabla_g u_k|^2 dV_g = \frac{1}{A} \;\; \forall A > 1, \tag{32}$$

which implies that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g = Vol_g(M) + \lim_{k \to +\infty} \frac{\lambda_k}{c_k^2},$$

and that $c_k u_k$ converges to some Green function weakly. In the end, using capacity arguments which consist in evaluating the energy of u_k on a annulus around the blow-up point, he got an upper bound of $\frac{\lambda_k}{c_k^2}$.

Remark 0.2.5. *(31) was first noticed by Struwe in [81], and (32) also appeared in [2].*

However there are two main differences between the present case and the one in [59]. One is that there is no direct maximum principle for equation (30) and the other one is that truncations are not allowed in the space $H^2(M)$. Hence to get a counterpart of (31) and (32) is not easy.

To solve the first difficulty, we replace $c_k(u_k(x_k+r_kx)-c_k)$ with $\beta_k(u_k(exp_{x_k}(r_kx))-c_k)$, where

$$1/\beta_k=\int_M\frac{|u_k|}{\lambda_k}e^{\alpha_k u_k^2}dV_g.$$

By using the strength of the Green representation formula, we get that the profile of u_k is either a constant function or a standard bubble. The second difficulty will be solved by applying capacity and Pohozaev type identity. In more detail we will prove that $\beta_k u_k\rightharpoonup G$ (see Lemma 1.5.6) which satisfies

$$\begin{cases}\Delta_g^2G=\tau(\delta_{x_0}-Vol_g(M))\\ \int_M G=0.\end{cases}$$

for some $\tau\in(0,1]$. Then we can derive from a Pohozaev type identity (see Lemma 1.5.7) that

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g=Vol_g(M)+\lim_{k\to+\infty}\tau^2\frac{\lambda_k}{\beta_k^2}.$$

In order to apply the capacity, we will follow some ideas in [57]. Concretely, we will show that up to a small term, the energy of u_k on some annulus is bounded below by the Euclidean one (see Lemma 1.5.10). Moreover, one can prove the existence of U_k (see Lemma 1.5.11) such that the energy of U_k is comparable to the Euclidean energy of u_k , and the Dirichlet datum and Neumann datum of U_k at the boundary of the annulus are also comparable to those of u_k. In this sense, we simplify the calculation of capacity in [60]. Now using capacity techniques we get $\frac{c_k}{\beta_k}\to d$ and $d\tau=1$, see Proposition 1.5.12. Furthermore we have that

$$\lim_{k\to+\infty}\tau^2\frac{\lambda_k}{\beta_k^2}\le\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2S_0}.$$

Hence we arrive to

$$\sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2u^2}dV_g\le Vol_g(M)+\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2S_0}. \tag{33}$$

In the end, we will find test functions in order to contradict (33). We will simplify the arguments in [59]. Indeed we use carefully the regular part of G to avoid cut-off functions and hence making the calculations simpler.

0.2.2 Some compactness results for Q-curvature, $Q-T$-curvature equations and generalized 2×2 Toda system

In Chapter 2, we study the compactness issue of some perturbations of the Q-curvature equations on compact closed Riemannian manifolds of arbitrary dimension, of the Q-curvature and T-curvature equations on compact four dimensional Riemannian manifolds with boundary. Furthermore, using a result of Jost-Lin-Wang[43] and Yanyan Li[52], we prove a compactness result for the generalized 2×2 Toda system.
The main results obtained in Chapter 2 are the following:

Theorem 0.2.6. *Let (M,g) be a compact closed smooth n-dimensional Riemannian manifold ($n\ge 3$). Suppose $KerP_g^n\simeq\mathbb{R}$ and that (u_l) is a sequence of solutions of* (15) *with $\bar{Q}_l$ satisfying* (16) *Q_l satisfying* (17)*, and $\bar{Q}_0$ satisfying* (18)*. Assuming that $(u_l)_l$ blows up (in the sens* (19)*), there exists $N\in\mathbb{N}^*$ such that*

$$\int_M Q_0dV_g=N(n-1)!\omega_n. \tag{34}$$

From this and standard elliptic regularity theory, we derive the following corollary:

Corollary 0.2.7. *Let (M, g) be a compact closed smooth n-dimensional Riemannian manifold($n \geq 3$) and suppose $Ker P_g^n \simeq \mathbb{R}$.*
a) Let (u_l) be a sequence of solutions of (15) with $\bar{Q}_l$ satisfying (16), Q_l satisfying (17) and $\bar{Q}_0$ satisfying (18). Assume also that

$$k_0 = \int_M Q_0 dV_g \neq k(n-1)!\omega_n \quad k = 1, 2, 3, \dots. \tag{35}$$

Then $(u_l)_l$ is bounded in $C^{\alpha}(M)$ for any $\alpha \in (0, 1)$.

b) Let (u_l) be a sequence of solutions to (12) for a fixed value of the constant $\bar{Q}$. Assume that $\kappa_{P^n} \neq k(n-1)!\omega_n$ $k = 1, 2, 3, \dots$, then $(u_l)_l$ is bounded in $C^m(M)$ for every positive integer m.

c) Let (u_{ρ_k}) $\{\rho_k \to 1\}$ be a family of solutions to (12) with Q_g^n replaced by $\rho_k Q_g^n$,and $\bar{Q}$ by $\rho_k \bar{Q}$ for a fixed value of the constant $\bar{Q}$. Assume also that $\kappa_{P^n} \neq (n-1)!k\omega_n$, then $(u_{\rho_k})_k$ is bounded in $C^m(M)$ for every positive integer m .

d) If $\kappa_{P^n} \neq k(n-1)!\omega_n$ $k = 1, 2, 3, \dots$, then the set of metrics conformal to g with constant Q-curvature and of unit volume is compact in $C^m(M)$ for positive integer m.

Theorem 0.2.13 and corollary 0.2.7 are contained in the paper[69].

We are going to describe our strategy to prove Theorem 0.2.6. Our method follows up to some extent [34] and [64]. However some new ideas are needed since some of the arguments in [34] and [64] rely on the fact of being in low dimensions (more precise comments are given below). We study equation (15) as an integral one. This is possible since one can show that P_g^n admits a Green's function $G(x, y)$ which is symmetric and for which $G(x, y) \sim \frac{1}{c_n} \log \frac{1}{d_g(x,y)}$ for $x \sim y$. Hence from the existence of the Green's function, we have that equation (15) can be written as

$$u_l(x) - \bar{u}_l = \int_M G(x, y)(\bar{Q}_l e^{nu_l(y)} - Q_l(y))dV_g(y) \quad x \in M. \tag{36}$$

As a first issue in the proof of Theorem 0.2.6 we determine the profile of solutions near blow-up points. To do this in [34] and [64], a scaling argument and a classification result by C.S Lin [62] is used. Unfortunately this classification result for entire solutions of $(-\Delta)^{\frac{n}{2}} u = e^{nu}$ (without growth condition at infinity) is available only in dimension 2 and 4.
In higher dimensions, it is convenient instead to use the full strength of (36) and still after a scaling argument to arrive to the following integral equation on $\mathbb{R}^n$

$$u(x) = \int_{\mathbb{R}^n} \sigma_n \log(\frac{|y|}{|x-y|})e^{nu}dy - \frac{1}{n}\log(k_n). \tag{37}$$

Assuming only that $\int_{\mathbb{R}^n} e^{nu}dx < \infty$, solutions of (37) have been classified by X. Xu in [89] as standard bubbles and this allows us to deduce the profile of blow ups of (12). Moreover using a generalized Pohozaev equality proven by X. Xu in [89] we derive a volume quantization near the blow ups points.

At this stage the analysis is only local, and the next issue is to obtain a global volume quantization as in the statement of Theorem 0.2.6. After proving a Harnack type inequality, one is reduced to study the behavior of the radial average $\bar{u}_l(r) = \frac{1}{Vol_g(\partial B_{x_l}(r))} \int_{\partial B_{x_l}(r)} u_l d\sigma_g$. For doing this in [64] this function was studied by an ODE analysis while in [34] it was mainly done using a classification results of some singular solutions to a PDE in $\mathbb{R}^4$.
On the other hand, one can still exploit the properties of (36) entirely. Here indeed we can also *radialize* (36) and study the radial function $\bar{u}_l(r)$ as a solution of a suitable integral inequality in one variable. This approach seems rather natural.

The next compactness result obtained deals with the prescribed Q-curvature equation on compact four dimensional Riemannian manifolds with boundary.

Theorem 0.2.8. *Let* (M,g) *be a compact smooth four dimensional Riemannian manifold with smooth boundary. Suppose* $KerP_g^{4,3} \simeq \mathbb{R}$ *and that* (u_l) *is a sequence of solutions to* (22) *with* $\bar{Q}_l$, Q_l *and* T_l *satisfying* (23). *Assuming also that* $(u_l)_l$ *blows up (in the sense of* (26)*) and*

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g + o_l(1) = \int_M \bar{Q}_l e^{4u_l} dV_g; \tag{38}$$

then there exists $N \in \mathbb{N} \setminus \{0\}$ *such that*

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g = 4N\pi^2.$$

From this, and from the regularity result in Proposition 0.3.5 below, we derive the following corollary .

Corollary 0.2.9. *Let* (M,g) *be a compact smooth four dimensional Riemannian manifold with smooth boundary, and suppose* $KerP_g^{4,3} \simeq \mathbb{R}$.
a) Let (u_l) *be a sequence of solutions to* (22) *with* $\bar{Q}_l$, Q_l *and* T_l *satisfying* (23). *Assume also that*

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g + o_l(1) = \int_M \bar{Q}_l e^{4u_l} dV_g;$$

and

$$k_0 = \int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g \neq 4k\pi^2 \quad k = 1,2,3,\ldots.$$

then $(u_l)_l$ *is bounded in* $C^{4+\alpha}(M)$ *for any* $\alpha \in (0,1)$.

b) Let (u_l) *be a sequence of solutions to* (20) *for a fixed value of the constant* $\bar{Q}$. *Assume also that* $\kappa_{(P^4,P^3)} \neq 4k\pi^2$, *then* $(u_l)_l$ *is bounded in* $C^m(M)$ *for every positive integer* m.

c) Let (u_{ρ_k}) $\{\rho_k \to 1\}$ *be a family of solutions to* (20) *with* T_g *replaced by* $\rho_k T_g$, Q_g *by* $\rho_k Q_g$ *and* $\bar{Q}$ *by* $\rho_k \bar{Q}$ *for a fixed value of the constant* $\bar{Q}$. *Assume also that* $\kappa_{(P^4,P^3)} \neq 4k\pi^2$, *then* $(u_{\rho_k})_k$ *is bounded in* $C^m(M)$ *for every positive integer* m.

d) If $\kappa_{(P^4,P^3)} \neq 4k\pi^2 \quad k = 1,2,3,\ldots$, *then the set of metrics conformal to* g *with constant* Q*-curvature, zero* T*-curvature, zero mean curvature and of unit interior volume is compact in* $C^m(M)$ *for positive integer* m.

Theorem 0.2.16 and Corollary 0.2.9 are contained in the paper[70].

Now we describe our approach to prove Theorem 0.2.8. We use a strategy related to that in [34], but in our case, we have to consider possible blow-ups at the boundary. We recall that a variant of this method was used to prove Theorem 0.2.6, and it relies strongly on the Green representation formula, transforming the PDE into an integral equation. For this case, we will employ a similar method since for the BVP one can prove the existence of a Green representation formula as well (using the method of the parametrix) with the difference that we have a boundary term, see Lemma 0.3.3. We consider the same scaling as in [34] and in the proof of Theorem 0.2.6. When we deal with the situation of interior blow-up points, we use the same argument as in the proof of Theorem 0.2.6 to get that the limit function V_0 which describes the profile near the blow-up point satisfies the following conformally invariant integral equation

$$\tilde{V}_0(x) = \int_{\mathbb{R}^4} \frac{3}{4\pi^2} \log\left(\frac{|z|}{|x-z|}\right) e^{4\tilde{V}_0(z)} dz - \frac{1}{4}\log(3). \tag{39}$$

Hence using the same argument as in the proof of Theorem 0.2.6, based on a classification result of X. Xu [89], we deduce that V_0 is a *standard bubble* and the local volume is $8\pi^2$. On the other hand when the blow-up happens at the boundary, we obtain that the limiting function satisfies the integral equation on the upper half space $\mathbb{R}^4_+$

$$V_0(x) = \int_{\mathbb{R}^4_+} \frac{3}{4\pi^2} \left(\log \frac{|z|}{|x-z|} + \log \frac{|z|}{|x-\bar{z}|} \right) e^{4V_0(z)} dz - \frac{1}{4} \log(3).$$

So from this we are able to deduce that the normal derivative of V_0 vanishes. Thus using Alexandrov reflection principle, we infer that the even reflection across $\partial\mathbb{R}^4_+$ $\bar{V}_0$ of V_0 solves the conformally invariant integral equation on the entire space $\mathbb{R}^4$ as in (39).
In this way, we can use the classification result of X. Xu (mentioned above) to deduce that $\bar{V}_0$ is a standard bubble and that the local volume associated is $8\pi^2$. Hence we find that the profile near such blow-up points (boundary) are half of a standard bubble and that the local volume associated is $4\pi^2$. At this stage to conclude we argue, as in the proof of Theorem 0.2.6, to show that the residual volume tends to zero, and obtain quantization. We point out that, by the above discussion, the volume of an interior blow-up is double with respect to the one at the boundary.

Next, we give a compactness result which deals with the prescribed T-curvature equation on compact four dimensional Riemannian manifolds with boundary.

Theorem 0.2.10. *Let (M, g) be a compact four dimensional Riemannian manifold with smooth boundary. Suppose $KerP_g^{4,3} \simeq \mathbb{R}$ and that (u_l) is a sequence of solutions to* (24) *with $\bar{T}_l$, T_l and Q_l satisfying* (25)*. Assuming that $(u_l)_l$ blows up (in the sense of* (26)*) and*

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g + o_l(1) = \int_{\partial M} \bar{T}_l e^{3u_l} dS_g; \tag{40}$$

then there exists $N \in \mathbb{N} \setminus \{0\}$ such that

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g = 4N\pi^2.$$

From this and the regularity result in Proposition 0.3.8 below, we derive the following corollary.

Corollary 0.2.11. *Let (M, g) be a compact smooth four dimensional Riemannian manifold with smooth boundary, and suppose that $KerP_g^{4,3} \simeq \mathbb{R}$.*
a) Let (u_l) be a sequence of solutions to (24) *with $\bar{T}_l$, T_l and Q_l satisfying* (25)*. Assume also that*

$$\int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g + o_l(1) = \int_{\partial M} \bar{T}_l e^{3u_l} dV_g;$$

and

$$k_0 = \int_M Q_0 dV_g + \int_{\partial M} T_0 dS_g \neq 4k\pi^2 \quad k = 1, 2, 3, \dots.$$

then $(u_l)_l$ is bounded in $C^{4+\alpha}(M)$ for any $\alpha \in (0, 1)$.

b) Let (u_l) be a sequence of solutions to (21) *for a fixed value of the constant $\bar{T}$. Assume also that $\kappa_{(P^4,P^3)} \neq 4k\pi^2$, then $(u_l)_l$ is bounded in $C^m(M)$ for every positive integer m.*

c) Let (u_{ρ_k}) $\{\rho_k \to 1\}$ be a family of solutions to (21) *with T_g replaced by $\rho_k T_g$, Q_g by $\rho_k Q_g$ and $\bar{T}$ by $\rho_k \bar{T}$ for a fixed value of the constant $\bar{T}$. Assume also that $\kappa_{(P^4,P^3)} \neq 4k\pi^2$, then $(u_{\rho_k})_k$ is bounded in $C^m(M)$ for every positive integer m.*

d) If $\kappa_{(P^4,P^3)} \neq 4k\pi^2$ $k = 1, 2, 3, \dots$, then the set of metrics conformal to g with constant T-curvature, zero Q-curvature, zero mean curvature and of unit boundary volume is compact in $C^m(M)$ for positive integer m.

Theorem 0.2.10 and Corollary 0.2.11 are contained in the paper[71].

To prove Theorem 0.2.10 we use the same argument as in the proof of Theorem 0.2.6 and Theorem 0.2.8, and the fact that due to the Green representation formula blow-up is equivalent to blow-up at the boundary .

The last compactness result deals with the generalized 2×2 Toda system on a compact closed Riemannian surface (Σ, g) of unit volume.

Theorem 0.2.12. *Suppose h_1, h_2 are smooth positive functions on Σ, and consider the sequence of solutions of the system*

$$\begin{cases} -\Delta u_{1,k} = 2\rho_{1,k}\left(\frac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} dV_g} - 1\right) - \rho_{2,k}\left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1\right); \\ -\Delta u_{2,k} = 2\rho_{2,k}\left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1\right) - \rho_{1,k}\left(\frac{h_1 e^{u_{1,k}}}{\int_\Sigma h_1 e^{u_{1,k}} dV_g} - 1\right), \end{cases} \quad on\ \Sigma. \tag{41}$$

Suppose $(\rho_{1,k})_k$ lie in a compact set K_1 of $\cup_{i=1}^{\infty}(4i\pi, 4(i+1)\pi)$, and that $(\rho_{2,k})_k$ lie in a compact set K_2 of $(-\infty, 4\pi)$. Then, if $\int_\Sigma u_{i,k} dV_g = 0$ for $i = 1, 2$ and for $k \in \mathbb{N}$, the functions $(u_{1,k}, u_{2,k})$ of (41) *stay uniformly bounded in $L^\infty(\Sigma) \times L^\infty(\Sigma)$.*

To prove Theorem 0.2.12, we exploit the blow-up analysis in [43] when ρ_2 stays positive and away from zero. On the other hand, for $\rho_2 \in (-\infty, \delta]$ with δ positive and small, we use an argument inspired by Brezis and Merle, [15], combined with a compactness result in [52].

0.2.3 Existence of constant Q-curvature conformal metrics in arbitrary dimensions

In Chapter 3, we prove a high-dimensional analogue of the classical *uniformization* Theorem for compact closed Riemannian surfaces. Precisely, we prove that, given a compact closed Riemannian manifold (M, g) of dimension n, there exists a metric conformally related to g of constant Q-curvature under generic and conformally invariant assumptions. Indeed we obtain the following theorem:

Theorem 0.2.13. *Let (M, g) be a compact closed smooth n-dimensional Riemannian manifold with $n \geq 3$. Suppose $KerP^n_g \simeq \mathbb{R}$, and assume that $\kappa_{P^n} \neq k(n-1)!\omega_n$ for $k = 1, 2, \ldots..$ Then M admits a conformal metric with constant Q-curvature.*

Remark 0.2.14. *(a) Our assumptions are conformally invariant and generic, so the result applies to a large class of compact closed smooth n-dimensional Riemannian manifolds.*

(b) Under these assumptions, by Corollary 0.2.7 *above, we have that blow ups of sequences of solutions to* (12) *is not possible. Indeed, these turn out to be bounded in $C^m(M)$ for every integer m.*

Our assumptions include those made in [13] (and its counterpart in the *odd* dimensional case) and (one) of the following two possibilities (or both)

$$\kappa_{P^n} \in (k(n-1)!\omega_n \ , \ (k+1)(n-1)!\omega_n), \quad \text{for some} \quad k \in \mathbb{N} \tag{42}$$

$$P^n_g \text{ possesses } \bar{k} \text{ negative eigenvalues (counted with multiplicity).} \tag{43}$$

Theorem 0.2.13 is the main result in the paper[69].

Remark 0.2.15. *a) For the sake of simplicity of the exposition, we will give the proof of Theorem* 0.2.13 *in the case where P^n_g is non-negative and* (42) *holds. In Chapter 3 after the proof of the main Theorems, we will make discussions to settle the general case.*

We are going to give the main ideas for the proof of Theorem 0.2.13 assuming that $\bar{k} = 0$ and (42) holds. Using an improvement of an appropriate Moser-Trudinger type inequality (see (1.4.1)), we show that if the conformal volume e^{nu} is *spread* into $(k+1)$ distinct sets where k is as in (42), then the functional II_A stays bounded from below. As a consequence, we deduce that if k is given as in (42) and if $II_A(u_l) \to -\infty$ along a sequence, then e^{nu_l} has to concentrate near at most k points of M. Hence, if we assume the normalization $\int_M e^{nu_l} dV_g = 1$, then $e^{nu_l} \simeq \sum_{i=1}^k t_i \delta_{x_i}$, where $t_i \geq 0$, $x_i \in M$, $\sum_{i=1}^k t_i = 1$ for $II_A(u_l) \to -\infty$. Therefore, as in [33] we can map e^{nu_l} onto M_k for l large, where M_k is the *formal set o f barycenters* of M of order k. Precisely for $L \gg 1$ we can define a continuous projection $\Psi : \{II_A \leq -L\} \to M_k$ which is homotopically non-trivial. The non-triviality of this map comes from the fact that M_k is non-contractible and from the existence of another map $\Phi_{\bar{\lambda}}$ such that $\Phi_{\bar{\lambda}} \circ \Psi$ is homotopic to the identity on M_k. Furthermore, the map $\Phi_{\bar{\lambda}}$ is such that $II_A(\Phi_{\bar{\lambda}}(M_k))$ can become arbitrary large negative, so that Ψ is well-defined on its image. Hence from this discussion we derive that for L large enough $\{II_A < -L\}$ has the same homology as M_k. Using the non contractibility of M_k, we define a min-max scheme for a perturbed functional $II_{A,\rho}$, ρ close to 1, finding a P-S sequence at some levels c_ρ. Applying the monotonicity procedure of Struwe, we can show existence of critical points of $II_{A,\rho}$ for a.e ρ, and we reduce ourselves to the assumptions of Theorem 0.2.7

Some comments in the construction of the map $\Phi_{\bar{\lambda}}$ are in order. We basically use the same function as in [33]. However, we point out that in [33] the estimates of $II_A(\varphi_{\bar{\lambda}})$ were done by explicit calculations which was possible since the dimension was fixed and low. Here instead, since we want to let n be arbitrary, we need a more systematic approach, which both simplifies and extends that in [33], see Lemma 3.2.26 and its proof.

0.2.4 Existence of constant Q-curvature conformal metrics on four manifolds with boundary

In Chapter 3, we prove a fourth order uniformization result for compact four dimensional Riemannian manifolds with boundary. We prove that on any compact four dimensional smooth Riemannian manifold with boundary, there exists a metric of constant Q-curvature, zero T-curvature and zero mean curvature within a given conformal class under generic and conformally invariant assumptions. Precisely we prove the following theorem:

Theorem 0.2.16. *Let (M,g) be a compact smooth four dimensional Riemannian manifold with smooth boundary and suppose $Ker P^{4,3}_g \simeq \mathbb{R}$. Then assuming $\kappa_{(P^4,P^3)} \neq k4\pi^2$ for $k = 1, 2, \cdots$, we have that (M,g) admits a conformal metric with constant Q-curvature, zero T-curvature and zero mean curvature.*

Remark 0.2.17. *a) As in Theorem* 0.2.13*, also here our assumptions are conformally invariant and generic, so the result applies to a large class of compact* 4*-dimensional Riemannian manifolds with boundary.*
b) From the Gauss-Bonnet-Chern formula, see (6) *we have that Theorem* 0.2.16 *does NOT cover the case of locally conformally flat manifolds with totally geodesic boundary and positive integer Euler-Poincaré characteristic.*
c) For the boundary Yamabe problem in low dimension (less than 5*) existence of solutions was obtained only under the assumption of local conformal flatness of the manifold and umbilicity of the boundary. However in our Theorem, we point out that no umbilicity condition for the boundary ∂M and no flatness condition for M are assumed.*

Our assumptions include the two following situations:

$$\kappa_{(P^4,P^3)} < 4\pi^2 \text{ and (or) } P^{4,3}_g \text{ possesses } \bar{k} \text{ negative eigenvalues (counted with multiplicity)} \quad (44)$$

$$\kappa_{(P^4,P^3)} \in \left(4k\pi^2\ ,\ 4(k+1)\pi^2\right), \quad \text{for some } k \in \mathbb{N}^* \text{ and (or) } P^{4,3}_g \text{ possesses } \bar{k} \text{ negative eigenvalues (counted with multiplicity)} \quad (45)$$

Theorem 0.2.16 is contained in the paper[70].

Remark 0.2.18. *As in the case of Theorem* 0.2.13*, in order to simplify the exposition, we will also give the proof of Theorem* 0.2.16 *in the case where we are in situation* (45) *and* $\bar{k} = 0$ *(namely* $P_g^{4,3}$ *is non-negative). At the end of Chapter 3, a discussion to settle the general case* (45) *and also case* (44) *is made.*

We are going to describe the main ideas in the proof of Theorem 0.2.16. We use the same strategy as in the proof of Theorem 0.2.13 above. However in the present case, there are some differences. These consists in the fact that M_k might be contractible and also boundary concentration can appear, hence new ideas are needed. Using a more refined improvement of an appropriate Moser-Trudinger inequality (see (1.4.2)), we first study how big can be the number of possible boundary and interior blow-up points for the conformal volume e^{4u}, $u \in \{v \in H_{\frac{\partial}{\partial n}} \int_M e^{4v} dV_g = 1; \quad II_Q(v) \leq -L\}$ with L large enough. From this study, we derive that if k is as in (45) and if $II_Q(u_l) \to -\infty$ along a sequence u_l with $\int_M e^{4u_l} dV_g = 1$, then e^{4u_l} has to concentrate near at most h interior points and l boundary points with $2h + l \leq k$ and $e^{4u_l} \simeq \sigma = \sum_{i=1}^h t_i \delta_{x_i} + \sum_{i=1}^l s_i \delta_{y_i}, \quad t_i \geq 0, \quad \sum_{i=1}^h t_i + \sum_{i=1}^l s_i = 1; \; x_i \in int(M), \quad y_i \in \partial M$. Therefore, instead of M_k, it is natural to consider the barycentric set $(M_\partial)_k$ (for the definition see Section Notation) which is a good candidate for describing the homology of large negative sublevels of II_Q. In order to do this, one needs to map (nontrivially) the large negative sublevels into $(M_\partial)_k$, and to do the opposite, namely to map $(M_\partial)_k$ (nontrivially) onto low sublevels of II_Q. If the composition of these two maps is homotopic to the identity, we derive information in the topology of the low sublevels of II_Q, in terms of the number of concentration points of the conformal volume e^{4u}. To find the projection onto $(M_\partial)_k$, we can use some of the arguments in [33], but with evident differences , because of the presence of the boundary. Taking advantage of the fact that the functions we are dealing with have zero normal derivatives, we use a doubling argument, which consists of constructing a new C^1 manifold DM, and using the Alexandrov reflection principle . We then use some suitable test functions to find the desired homotopy equivalence.
Using the Mayers-Vietoris Theorem, one can prove that $(M_\partial)_k$ is non-contractible. At this stage, we define a min-max scheme as in the proof of Theorem 0.2.13, and we reduce ourselves to the assumptions of Theorem 0.2.9.

0.2.5 Existence of constant T-curvature conformal metrics on four manifolds with boundary

In Chapter 3, we also prove that, given any four dimensional Riemannian manifold with boundary (M, g), there exists a metric in the conformal class of the background metric $[g]$ with constant T-curvature, zero Q-curvature and zero mean curvature, still under generic and conformally invariant assumptions. We obtain the following theorem:

Theorem 0.2.19. *Let* (M, g) *be a compact smooth four dimensional Riemannian manifold with smooth boundary, and suppose* $KerP_g^{4,3} \simeq \mathbb{R}$*. Then assuming* $\kappa_{(P^4,P^3)} \neq k4\pi^2$ *for* $k = 1, 2, \dots,$ *we have that* (M, g) *admits a conformal metric with constant* T*-curvature, zero* Q*-curvature and zero mean curvature.*

Remark 0.2.20. *a) As in Theorem* 0.2.13*, and Theorem* 0.2.16*, also here our assumptions are conformally invariant and generic, so that the result applies to a large class of compact four dimensional Riemannian manifolds with boundary.*
b) From the Gauss-Bonnet-Chern formula, see (6) *we have that Theorem* 0.2.19 *does NOT cover the case of locally conformally flat manifolds with totally geodesic boundary and positive integer Euler-Poincaré characteristic.*

Our assumptions include the following two situations:

$$\kappa_{(P^4,P^3)} < 4\pi^2 \text{ and (or) } P_g^{4,3} \text{ possesses } \bar{k} \text{ negative eigenvalues (counted with multiplicity)} \tag{46}$$

$$\kappa_{(P^4,P^3)} \in \left(4k\pi^2 \;,\; 4(k+1)\pi^2\right), \quad \text{for some } k \in \mathbb{N}^* \text{ and (or) } P_g^{4,3} \text{ possesses } \bar{k} \text{ negative eigenvalues (counted with multiplicity)} \tag{47}$$

Theorem 0.2.19 is contained in the paper[71].

Remark 0.2.21. *Here also, to simplify the exposition, we will give the proof of Theorem* 0.2.19 *in the case where we are in situation* (47) *and* $\bar{k} = 0$ *(namely* $P_g^{4,3}$ *is non-negative). At the end of Chapter 3 a discussion to settle the general case* (47) *and also case* (46) *will be done.*

To prove Theorem 0.2.19 we use the same idea as the one used in Theorem 0.2.13, namely in the case without boundary. The only difference is that, here instead of working with M_k , we use ∂M_k.

0.2.6 Existence results for the generalized 2×2 Toda system on compact closed surfaces

The last result in this thesis is contained in Chapter 3. It deals with the existence of solutions for the generalized 2×2 Toda system in the case that one of the parameter is allowed to be large and the other one is subcritical (i.e less than 4π). Indeed we prove

Theorem 0.2.22. *Let* (Σ, g) *be a compact closed Riemannian surface with unit volume. Suppose* m *is a positive integer, and let* $h_1, h_2 : \Sigma \to \mathbb{R}$ *be smooth positive functions. Then for* $\rho_1 \in (4\pi m, 4\pi(m+1))$ *and for* $\rho_2 < 4\pi$ *problem* (11) *is solvable.*

Theorem 0.2.22 is obtained in a joint work With Andrea Malchiodi[66].

We are going to describe the main ideas to prove Theorem 0.2.22. We use the same methods as in the proof of Theorem 0.2.13. For the sake of clarity, we will repeat the arguments and point out the adaptations to the system. Again, a main ingredient in our proof is an improved version of the Moser-Trudinger inequality for systems, which was given in [44], see Theorem 1.3.6. From the improved inequality, we derive the following consequence: if $\rho_1 \in (4\pi m, 4\pi(m+1))$, if $\rho_2 < 4\pi$ and if $II_\rho(u_{1,l}, u_{2,l}) \to -\infty$ along a sequence $(u_{1,l}, u_{2,l})$, then $e^{u_{1,l}}$ has to concentrate near at most m points of Σ. Therefore, as for the prescribed Q-curvature problem in arbitrary dimensions, we can map $e^{u_{1,l}}$ onto Σ_m for l large. Precisely, for $L \gg 1$ we can define a continuous projection $\Psi : \{II_\rho \leq -L\} \to \Sigma_m$ which is homotopically non-trivial. Indeed, recalling that Σ_m is non-contractible, there exists a map Φ such that $\Psi \circ \Phi$ is homotopic to the identity and such that $II_\rho(\Phi(\Sigma_m))$ can become arbitrarily large negative, so that Ψ is well-defined on its image. Hence we obtain characterization of low energy sublevels of II_ρ as in the scalar case.

Some comments on the construction of the map Φ are in order. If we want to obtain low values of II_ρ on a couple (u_1, u_2), since e^{u_1} has necessarily to concentrate near at most m points of Σ, a natural choice of the test functions (u_1, u_2) is $\left(\varphi_{\lambda,\sigma}, -\frac{1}{2}\varphi_{\lambda,\sigma}\right)$, where σ is any element of Σ_m, and where $\varphi_{\lambda,\sigma}$ is given in (3.96). In fact, as λ tends to infinity, $e^{\varphi_{\lambda,\sigma}}$ converges to σ in the weak sense of distributions, while the choice of u_2 is done in such a way to obtain the best possible cancellation in the quadratic part of the functional, see Remark 3.2.38. .

At this point, using the non-contractibility of Σ_m , we run a min-max scheme as in the proof of Theorem 0.2.13, and reduces ourselfs to the conditions of Theorem 0.2.12.

0.3 Notation and Preliminaries

0.3.1 Notation

- $\mathbb{R}^n$, is the standard n-dimensional Euclidean space, and $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in R^n : x_n > 0\}$.
- $\mathbb{N}$ denotes the set of non-negative integers, and $\mathbb{N}^*$ for the set of positive integers.
- $B_p(r)$, the open geodesic ball of radius r and center p, in the Riemannian manifold (M, g).
- $B^0(r)$, the open ball of center 0 and radius r in $\mathbb{R}^n$.
- $B_p^+(r) = B_p(r) \cap M$, and $B_x^{0,+}(r) = B_x^0(r) \cap \mathbb{R}^n_+$.
- Given (M, g) a compact four dimensional Riemannian manifold with boundary ∂M, we denote by $B_p^{\partial M}(r)$ the ball of center $p \in \partial M$ and of radius r with respect to the intrinsic Riemannian structure of ∂M.
- $d_g(\cdot, \cdot)$ stands for the geodesic distance in (M, g).
- $H^s(M)$, for $s \in \mathbb{R}$ denotes the usual Sobolev space of functions on M which are of class H^s in each coordinate system.
- M^2 stands for the cartesian product $M \times M$, while $Diag(M)$ is the diagonal of M^2.
- $inj_g(M)$, is the injectivity radius of (M, g).
- ω_n stands for the volume of the unit sphere in $\mathbb{R}^{n+1}$.
- $A_l = o_l(1)$ means that $A_l \longrightarrow 0$ as the integer $l \longrightarrow +\infty$.
- $A_\epsilon = o_\epsilon(1)$ means that $A_\epsilon \longrightarrow 0$ as the real number $\epsilon \longrightarrow 0$.
- $A_\delta = o_\delta(1)$ means that $A_\delta \longrightarrow 0$ as the real number $\delta \longrightarrow 0$.
- $A_l = O(B_l)$ means that $A_l \leq C B_l$ for some fixed constant C..
- dV_g denotes the Riemannian measure associated to the metric g on the Riemannian manifold (M, g).
- $d\sigma_g$ stands for the induced volume form on geodesic spheres associated to g.
- For (M, g) a compact four dimensional Riemannian manifold with boundary ∂M , we denote by dS_g the volume form of ∂M given by the induced metric $\hat{g}$.
- Given a compact closed Riemannian manifold (M, g) and a function $u \in L^1(M)$, we denote by $\overline{u}$ the mean value of u, namely $\overline{u} = Vol_g(M)^{-1} \int_M u dV_g$ where $Vol_g(M) = \int_M dV_g$.
- Given an operator P acting on functions $u(x, y)$ defined on M^2, P_y means the action of P with respect to the variable $y \in M$.
- For (M, g) a compact four dimensional Riemannian manifold with boundary, given $u \in L^1(M)$ (resp. $L^1(\partial M)$), we denote by $\overline{u}$ (resp. $\overline{u}_{\partial M}$) by the following quantities $\overline{u} = Vol_g(M)^{-1} \int_M u dV_g$, and $\overline{u}_{\partial M} = Vol_g(\partial M) \int_{\partial M} u dS_g$ where $Vol_g(\partial M) = \int_{\partial M} dS_g$.

Given (M,g) a compact closed n-dimensional Riemannian manifold and k a positive integer, we set

$$M_k = \{\sum_{i=1}^{k} t_i\delta_{x_i},\ \ t_i \geq 0,\ \ \sum_{i=1}^{k} t_i = 1; x_i \in M\}. \tag{48}$$

M_k is known in the literature as the set of *formal barycenters* relative to M of order k (for more details see [33] and the references therein) . We recall that M_k is a stratified set namely a union of sets of different dimension with maximum one equal to $nk-1$.

M_k will be endowed with the weak topology of distributions. To carry out some computations, we will use on M_k the metric given by $C^1(M)^*$, which induces the same topology, and which will be denoted by $d(\cdot,\cdot)$.

Next, given $\sigma \in M_k$, $\sigma = \sum_{i=1}^{k} t_i\delta_{x_i}$ with $x_i \in M$, and $\varphi \in C^1(M)$, we denote the action of σ on φ as

$$< \sigma, \varphi >= \sum_{i=1}^{k} t_i\varphi(x_i)$$

Given f a nonnegative L^1 function on M with $\int_M f dV_g = 1$ and $S \subset M_k$ we define the distance of f from S as follows

$$d(f,S) = \inf_{\sigma\in S} d(f,\sigma);$$

Now we consider a four dimensional compact Riemannian manifold with smooth boundary (M,g).

For $\epsilon > 0$ we set

$$(\partial M)^{\epsilon} = \{x \in M \quad d_g(x,\partial M) \leq \epsilon\}.$$

We set also

$$\tilde{k} = \left[\frac{k}{2}\right]$$

where $\left[\frac{k}{2}\right]$ stands for the integer part of $\frac{k}{2}$.

Given $\delta > 0$ a small positive constant we set

$$M_\delta = M \setminus \partial M \times [0,\delta].$$

Let $h \in \mathbb{N}, l \in \mathbb{N}$ such that $h \leq \tilde{k}, l \leq k$ and $2h + l \leq k$ we define $M_{h,l}$ as follows

$$M_{h,l} = \{\sum_{i=1}^{h} t_i\delta_{x_i} + \sum_{i=1}^{l} s_i\delta_{y_i},\ \ t_i \geq 0,\ \ \sum_{i=1}^{h} t_i + \sum_{i=1}^{l} s_i = 1;\ x_i \in int(M),\ \ y_i \in \partial M\}; \tag{49}$$

We set also

$$(M_\partial)_k = \cup_{h,l} M_{h,l}.$$

As for the case of compact closed Riemannian manifolds, $(M_\partial)_k$ will be endowed with the weak topology of distributions. To carry out some computations, we will use on $(M_\partial)_k$ the metric given by $C^1(M)^*$, which induces the same topology, and which will be denoted by $d_M(\cdot,\cdot)$.

Now let us introduce some further definitions.

Given $\sigma \in (M_\partial)_k$, $\sigma = \sum_{i=1}^{h} t_i\delta_{x_i} + \sum_{i=1}^{l} s_i\delta_{y_i}$ with $x_i \in int(M), y_i \in \partial M$ and $2h + l \leq k$ we set

$$\sigma_{int} = \sum_{i=1}^{h} t_i\delta_{x_i};$$

and

$$\sigma_{bdry} = \sum_{i=1}^{l} s_i\delta_{y_i}.$$

Next for $\varphi \in C^1(M)$ and $\sigma = \sigma_{int} + \sigma_{bdry} \in (M_\partial)_k$, similar to the case without boundary, we denote the action of σ on φ as

$$< \sigma, \varphi >= \sum_{i=1}^{h} t_i \varphi(x_i) + \sum_{i=1}^{l} s_i \varphi(y_i)$$

where $\sigma_{int} = \sum_{i=1}^{h} t_i \delta_{x_i}$ and $\sigma_{bdry} = \sum_{i=1}^{l} s_i \delta_{y_i}$.
Next if f is a nonnegative L^1 function on M with $\int_M f dV_g = 1$ and $S \subset (M_\partial)_k$, again similar to the case without boundary, we define the distance of f from S as follows

$$d_M(f, S) = \inf_{\sigma \in S} d_M(f, \sigma);$$

0.3.2 Geometric background

Given a positive integer n, a n-dimensional Riemannian manifold (M, g), and a system of co-ordinates (U, φ), $U \subset M$, $\varphi : U \to \mathbb{R}^n$, we denote by g_{ij} the components of the metric g in these co-ordinates.
The Riemannian measure or volume form of M with respect to g is defined as follows

$$dV_g = \sqrt{|g|}dx,$$

where $|g|$ stands for the determinant of the matrix (g_{ij}) and dx the standard n-dimensional Lebesgue measure.
We denote by g^{ij} the component of the inverse g^{-1} of g, and by Γ_{ij}^l the Cristoffel symbols which are given by the following formula

$$\Gamma_{ij}^l = \frac{1}{2} \left(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij} \right) g^{kl}.$$

By means of the Cristoffel symbols, one obtain the components of the Riemann curvature tensor Riem_{kij}^l as follows

$$\mathrm{Riem}_{kij}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m.$$

The Ricci curvature tensor denoted by Ric_{ij}, is obtained by contracting the full curvature tensor Riem, namely

$$\mathrm{Ric}_{ij} = \mathrm{Riem}_{ilj}^l.$$

The scalar curvature R, is a scalar function arising from the contraction of the Ricci tensor

$$R = \mathrm{Ric}_{ij} g^{ij}.$$

The Weyl tensor (W_{ijkl}) is defined by

$$W_{ijkl} = \mathrm{Riem}_{ijkl} - \frac{1}{n-2} \left(\mathrm{Ric}_{ik} g_{jl} - \mathrm{Ric}_{il} g_{jk} + \mathrm{Ric}_{jl} g_{ik} + \frac{R}{n-1} (g_{jl} g_{ik} - g_{jk} g_{il}) \right).$$

Given a smooth function $u : M \to \mathbb{R}$, we denote by $\nabla_g u$ the vector with components $\nabla_g u^i$ defined by

$$\nabla_g u^i = g^{ij} \partial_j u.$$

The Laplace-Beltrami operator Δ_g is the second order linear differential operator acting on smooth functions $u : M \to \mathbb{R}$, as follows

$$\Delta_g u = \frac{1}{\sqrt{|g|}} \partial_i \left(g^{ij} \partial_j u \sqrt{|g|} \right).$$

Given k a positive integer Δ_g^k, is the $2k$-th order linear differential operator defined by the iterative formula

$$\Delta_g^k u = \Delta_g (\Delta_g^{k-1} u),$$

for all smooth functions $u : M \to \mathbb{R}$.
Given a real number s, Δ_g^s stands for the s-th power of the Laplace-Beltrami operator: it is a Pseudodifferential operator with symbol $|\zeta|^{2s}$ (for more details see [41] and the references therein).

0.3.3 Preliminary results

In this Subsection we give some preliminary results like the Green function for Δ_g^2, P_g^n , for the couple (P_g^4, P_g^3), with homogeneous Neumann boundary condition, and some regularity results.

We start by stating a lemma giving the existence of the Green function of Δ_g^2 and its asymptotics near its singularities.

Lemma 0.3.1. *Let (M, g) be a compact closed smooth four dimensional Riemannian manifold. We have that the Green function $F(x, y)$ of Δ_g^2 exists in the following sense :*
a) For all functions $u \in C^2(M)$, we have

$$u(x) - \bar{u} = \int_M F(x, y)\Delta_g^2 u(y) dV_g(y) \qquad x \neq y \in M$$

b)

$$F(x, y) = H(x, y) + K(x, y)$$

is smooth on $M^2 \setminus Diag(M^2)$, K extends to a $C^{1+\alpha}$ function on M^2 and

$$H(x, y) = \frac{1}{8\pi^2} f(r) \log \frac{1}{r}$$

where, $r = d_g(x, y)$ is the geodesic distance from x to y; $f(r)$ is a C^∞ positive decreasing function, $f(r) = 1$ in a neighborhood of $r = 0$ and $f(r) = 0$ for $r \geq inj_g(M)$. Moreover we have that the following estimates holds

$$|\nabla_g F(x, y)| \leq C \frac{1}{d_g(x, y)} \qquad |\nabla_g^2 F(x, y)| \leq C \frac{1}{d_g(x, y)^2}.$$

Proof. For the proof see [23] and the proof of Lemma 2.3 in [64]. □

Next we state a Proposition giving the existence of the Green function of P_g^n and its asymptotics near its singularities.

Proposition 0.3.2. *Suppose (M, g) is a compact smooth closed n-dimensional Riemannian manifold with $n \geq 3$, and suppose $KerP_g^n \simeq \mathbb{R}$. Then the Green function $G(x, y)$ of P_g^n exists in the following sense :*
a) For all functions $u \in C^n(M)$, we have

$$u(x) - \bar{u} = \int_M G(x, y) P_g^n u(y) dV_g(y) \qquad x \neq y \in M \tag{50}$$

b)

$$G(x, y) = H(x, y) + K(x, y) \tag{51}$$

is smooth on $M^2 \setminus Diag(M^2)$, K extends to a $C^{2+\alpha}$ function on M^2 and

$$H(x, y) = \frac{1}{c_n} \log \left(\frac{1}{r}\right) f(r) \tag{52}$$

where $2c_n = (n-1)!\omega_n$, $r = d_g(x, y)$ is the geodesic distance from x to y; $f(r)$ is a C^∞ positive decreasing function , $f(r) = 1$ in a neighborhood of $r = 0$ and $f(r) = 0$ for $r \geq inj_g(M)$.

Proof. Let $x \in M$ be fixed and m be a positive integer large enough. By [51] Theorem 5.1, there exists a metric $\tilde{g}$ conformal to g and a coordinate system around x called conformal normal coordinate such that in the latter coordinate system we have that

$$|\tilde{g}(x, y)| = 1 + O(r^m) \qquad \text{for} \quad y \text{ close to } x. \tag{53}$$

Now in coordinates $\Delta_{\tilde{g},y}$ has the following expression,

$$\Delta_{\tilde{g},y}v = \frac{1}{\sqrt{|\tilde{g}|}}\partial_i\left(\tilde{g}^{i,j}\sqrt{|\tilde{g}|}\partial_j v\right).$$

On the other hand in conformal normal coordinates we have that

$$\tilde{g}^{i,j} = \delta_{i,j} + O(r^2),$$

and

$$\partial_i\tilde{g}^{i,j} = O(r).$$

Now to continue the proof, we divide the remainder into two cases.

Case n even

In the above coordinate system, we have by easy calculations that the following holds:

$$|P_g^n H(x,y)| \leq C|P_{\tilde{g}}^n H(x,y)| \leq Cr^{2-n} \quad for \;\; r \leq C^{-1}inj_g(M). \tag{54}$$

On the other hand by considering the expression,

$$\int_{M\setminus B_x(\epsilon)} H(x,y)P_{\tilde{g}}^n u(y)dV_{\tilde{g}}(y) - \int_{M\setminus B_x(\epsilon)} u(y)P_{\tilde{g}}^n H(x,y)u(y)dV_{\tilde{g}}(y);$$

we have by integration by parts that,

$$\begin{aligned}\int_{M\setminus B_x(\epsilon)} H(x,y)P_{\tilde{g}}^n u(y)dV_{\tilde{g}}(y) - \int_{M\setminus B_x(\epsilon)} u(y)P_{\tilde{g}}^n H(x,y)dV_{\tilde{g}}(y) = \\ \int_{\partial B_x(\epsilon)} \partial_\nu(-\Delta)^{\frac{n}{2}-1}H(x,y)u(y)dV_{\tilde{g}}(y) + o_\epsilon(1).\end{aligned} \tag{55}$$

Now by using the fact that close to x in conformal normal coordinate Δ_g is *close to* $\Delta_{\mathbb{R}^n}$, we obtain by letting ϵ go to 0

$$u(x) = \int_M H(x,y)P_{\tilde{g}}^n u(y)dV_{\tilde{g}}(y) - \int_M P_{\tilde{g}}^n H(x,y)u(y)dV_{\tilde{g}}(y). \tag{56}$$

Hence using the conformal invariance property of P_g^n, for every $x \in M$ we obtain

$$u(x) = \int_M H(x,y)P_g^n u(y)dV_g(y) - \int_M P_g^n H(x,y)u(y)dV_g(y). \tag{57}$$

Next we can apply the same method as in [8] (Theorem 4.13) to construct a parametrix for the Green's function. We set

$$G(x,y) = H(x,y) + \sum_{i=1}^{q} Z_i(x,y) + F(x,y);$$

where $q > \frac{n}{2}$,

$$Z_i(x,y) = \int_M \Gamma_i(x,\zeta)H(\zeta,y)dV_g(\zeta); \tag{58}$$

and Γ_i are defined inductively as follows,

$$\Gamma_{i+1}(x,y) = \int_M \Gamma_i(x,\zeta)\Gamma(\zeta,y)dV_g(\zeta);$$

with

$$\Gamma_1(x,y) = \Gamma(x,y) = -P_{g,y}^n H(x,y);$$

and F being the solution of the equation

$$P^n_{g,y}F(x,y) = \Gamma_{k+1}(x,y) - \frac{1}{Vol_g(M)}. \tag{59}$$

From (72) we have that Z_i satisfies

$$P^n_{g,y}Z_i(x,y) = \Gamma_i(x,y) - \Gamma_{i+1}(x,y). \tag{60}$$

We observe that the following estimate holds for $\Gamma(x,y)$,

$$|\Gamma(x,y)| \leq Cr^{2-n}; \tag{61}$$

hence by using the results in [8] (Proposition 4.12), we obtain the following estimate for $\Gamma_i(x,y)$,

$$|\Gamma_i(x,y)| \leq Cr^{2i-n}. \tag{62}$$

So arriving at this stage by still the same result in [8] (Proposition 4.12), we have that $\Gamma_q(x,y)$ and $\Gamma_{q+1}(x,y)$ are continuous hence using elliptic regularity we get $Z_q(x,y)$ and $F(x,y)$ are in $C^{n-1+\alpha}(M^2)$. The regularity in both the variables x and y can be deduced by the symmetry of G, which follows from the self adjointness of P^n_g and reasoning as in [8] (Proposition 4.13). Further from (73) we deduce that $\Gamma_i \in L^p$ with $\frac{n-2}{n} < p < \frac{n}{n-2}$ for all $i = 1, \ldots, q-1$. Hence by using standard elliptic regularity we infer that $Z_i(x,y) \in H^{n,p}$. So from the Sobolev embedding theorem and the fact that $\frac{n-2}{n} < p < \frac{n}{n-2}$ we get $Z_i(x,y) \in C^{2+\alpha}(M^2)$ for all $i = 1, \ldots, q-1$ for some α. Hence setting $K(x,y) = \sum_{i=1}^q Z_i(x,y) + F(x,y)$, the Lemma is proved for n even.

Case n odd

We remark that if the analogues of (54) and (57) are valid, namely if the following properties

$$|P^n_g H(x,y)| \leq Cr^{2-n} \quad for \;\; r \leq C^{-1}inj_g(M); \tag{63}$$

$$u(x) = \int_M H(x,y)P^n_g u(y)dV_g(y) - \int_M P^n_g H(x,y)u(y)dV_g(y), \tag{64}$$

hold, then the proof for the even case can be easily adapted. Hence to finish the proof of the Proposition, we need only to prove (63)-(64).

We first start by the second one which is less technical. Using the self adjointness of P^n_g we have

$$\int_{M\setminus B_x(\epsilon)} P^n_{\tilde g}H(x,y)u(y)dV_{\tilde g}(y) = \int_M H(x,y)P^n_{\tilde g}u(y)dV_{\tilde g}(y) - \int_{B_x(\epsilon)} P^n_{\tilde g}H(x,y)u(y)dV_{\tilde g}(y). \tag{65}$$

Letting $\epsilon \to 0$ we are done. Now let us prove the first one. Writting $n = 2k+1$ and recalling we are working in conformal normal coordinates around x, up to errors terms we can suppose we are on flat space and that we have to compute $(-\Delta)^{\frac{1}{2}}(-\Delta)^k H$. First, reasoning as in the even case we have the following estimate for $(-\Delta)^k H(r)$

$$(-\Delta)^k H(r) = O(r^{2-2k}). \tag{66}$$

Now we recall a well known formula for Fourier transform of radial functions, see [79] (Theorem 3.3) that we will use to continue our analysis. Given $f \in L^1(\mathbb{R}^n)$ radial, it is well known that its Fourier transform that we denote by $\hat f$ is still radial and verifies the following formula

$$\hat f(r) = 2\pi r^{-\frac{n-2}{2}} \int_0^\infty f(s)J_{\frac{n-2}{2}}(2\pi rs)s^{\frac{n}{2}}ds, \tag{67}$$

where $J_{\frac{n-2}{2}}$ is the Bessel function of first kind and of order $\frac{n-2}{2}$. On the other hand $J_{\frac{n-2}{2}}$ has the following asymptotics at 0

$$J_{\frac{n-2}{2}}(t) = t^{\frac{n-2}{2}}(b_n + o_t(1)), \tag{68}$$

where $o_t(1) \to 0$ as $t \to 0$ and b_n is a dimensional constant. Furthermore it has also the following asymptotics at infinity

$$J_{\frac{n-2}{2}}(t) = O(t^{-\frac{1}{2}}). \tag{69}$$

For reference about the asymptotics at zero and infinity of $J_{\frac{n-2}{2}}$, see [79] (Lemma 3.11).
Now using (66)-(69), by easily calculations we obtain

$$\widehat{(-\Delta)^k H}(r) = O(r^{2k-2-n}); \tag{70}$$

where $\widehat{(-\Delta)^k H}(r)$ stands for the Fourier transform of $(-\Delta)^k H(r)$.
On the other hand using the definition of $(-\Delta)^{\frac{1}{2}}$, again (67)-(69) and (70) we have that

$$(-\Delta)^{\frac{1}{2}}(-\Delta)^k H = O(r^{1-2k}). \tag{71}$$

Hence from the trivial identity $2k-1 = n-2$, we are done. So this conclude also the proof of the Proposition. ■

Now we state a Proposition which asserts the existence of the Green function of (P_g^4, P_g^3) with homogeneous Neumann condition. Moreover we give its asymptotics near its singularities.

Proposition 0.3.3. *Suppose (M, g) is a compact four dimensional Riemannian manifold with boundary and $KerP_g^{4,3} \simeq \mathbb{R}$. Then the Green function $G(x, y)$ of (P_g^4, P_g^3) exists in the following sense :*
a) For all functions $u \in C^2(M)$, $\frac{\partial u}{\partial n_g} = 0$, we have

$$u(x) - \bar{u} = \int_M G(x,y) P_g^4 u(y) dV_g(y) + 2\int_{\partial M} G(x,y') P_g^3 u(t) dS_g(y') \qquad x \in M$$

b)

$$G(x, y) = H(x, y) + K(x, y)$$

is smooth on $M^2 \setminus Diag(M^2)$, K extends to a $C^{2+\alpha}$ function on M^2 and

$$H(x,y) = \begin{cases} \frac{1}{8\pi^2} f(r) \log \frac{1}{r} & \text{if} \quad B_x(\delta) \cap \partial M = \emptyset; \\ \\ \frac{1}{8\pi^2} f(r) (\log \frac{1}{r} + \log \frac{1}{\bar{r}}) & \text{otherwise.} \end{cases}$$

where $f(\cdot) = 1$ in $[-\frac{\delta}{2}, \frac{\delta}{2}]$ and $f(\cdot) \in C_0^\infty(-\delta, \delta)$, $\delta \leq \frac{1}{2}\min\{\delta_1, \delta_2\}$, δ_1 is the injectivity radius of M in $\tilde{M}$, and $\delta_2 = \frac{\delta_0}{2}$, $r = d_g(x, y)$ and $\bar{r} = d_g(x, \bar{y})$.

To give the proof of the Proposition we need a Lemma which can be found in [19] (Proposition A.1)

Lemma 0.3.4. *There exists an extension of (M, g) into $(\tilde{M}, \tilde{g})$ which is a closed smooth four dimensional Riemannian manifold such that*
1) M is an open submanifold of $\tilde{M}$,
2) $\tilde{g}|M = g$,
3) In $\tilde{M}$, ∂M has a smooth tubular neighborhood T of width δ_0, such that, for any $x \in T \cap M$ there exists an unique $\bar{x} \in T \setminus M$ with $d_{\bar{g}}(x, \partial M) = d_{\bar{g}}(\bar{x}, \partial M)$, and for $x \in \partial M$, $x = \bar{x}$, where $d_{\bar{g}}$ denotes the Riemannian distance associated to $\bar{g}$.

PROOF of Proposition 0.3.3
We use the same strategy as in the proof of the Proposition 0.3.2. For the convenience of the reader we add more details.
Let $x \in M$ be fixed, it is well known that in normal coordinate around x the following holds

$$|g(y)| = 1 + O(r^2) \qquad \text{for} \quad y \text{ close to } x.$$

Now working in this normal coordinate system around x we have that

$$|P_g^4 H(x,y)| \leq Cr^{-2} \quad for \;\; r \leq C^{-1} inj_g(M).$$

and

$$|P_g^3 H(x,y)| \leq Cr^{-1} \quad for \;\; r \leq C^{-1} inj_g(M).$$

On the other hand, by considering the expression

$$\int_{M\setminus B_x(\epsilon)} H(x,y)P_g^4 u(y)dV_{\tilde{g}}(y) - \int_{M\setminus B_x(\epsilon)} u(y)P_g^4 H(x,y)u(y)dV_{\tilde{g}}(y);$$

we have by integration by parts that,

$$\int_{M\setminus B_x(\epsilon)} H(x,y)P_g^4 u(y)dV_{\tilde{g}}(y) - \int_{M\setminus B_x(\epsilon)} u(y)P_g^4 H(x,y)u(y)dV_g(y) =$$
$$-\int_{\partial B_x(\epsilon)} \frac{\partial \Delta_{\bar{g},y}}{\partial n_g} H(x,y)u(y)dV_g(y) + 2\int_{\partial M} H(x,y')P_g^3 dS_{\bar{g}}(y') + o_\epsilon(1)$$

Now by using the fact that close to x in conformal normal coordinate Δ_g is *close to* $\Delta_{\mathbb{R}^4}$, we obtain by letting ϵ go to 0

$$u(x) = \int_M H(x,y)P_g^4 u(y)dV_g(y) - \int_M P_g^4 H(x,y)u(y)dV_{\tilde{g}}(y) + 2\int_{\partial M} H(x,y')P_g^3 u(y')dS_g(y').$$

Hence, for every $x \in M$ we obtain

$$u(x) = \int_M H(x,y)P_g^4 u(y)dV_g(y) - \int_M P_g^4 H(x,y)u(y)dV_g(y) + 2\int_{\partial M} H(x,y')P_g^3 u(y')dS_g(y'). \quad (72)$$

Now we can apply the same method as in [8] (Theorem 4.13) to construct a parametrix for the Green's function. We set

$$G(x,y) = H(x,y) + \sum_{i=1}^{q} Z_i(x,y) + F(x,y);$$

where $q > 2$,

$$Z_i(x,y) = \int_M \Gamma_i(x,\zeta)H(\zeta,y)dV_g(\zeta);$$

and Γ_i are defined inductively as follows,

$$\Gamma_{i+1}(x,y) = \int_M \Gamma_i(x,\zeta)\Gamma(\zeta,y)dV_g(\zeta);$$

with

$$\Gamma_1(x,y) = \Gamma(x,y) = -P_{g,y}^4 H(x,y);$$

and F being the solution of the equation

$$\begin{cases} P_{g,y}^4 F(x,y) = \Gamma_{k+1}(x,y) - \frac{1}{Vol_g(M)}. & \text{in } M; \\ P_{g,y}^3 F(x,y) = -P_{g,y}^3 H(x,y) & \text{on } \partial M; \\ \frac{\partial F(x,y)}{\partial n_{g,y}} = 0 & \text{on } \partial M. \end{cases}$$

Now from (72) we have that Z_i satisfies

$$\begin{cases} P_{g,y}^4 Z_i(x,y) = \Gamma_i(x,y) - \Gamma_{i+1}(x,y) & \text{in } M; \\ P_{g,y}^3 Z_i(x,y) = 0 & \text{on } \partial M; \\ \frac{\partial Z_i(x,y)}{\partial n_{g,y}} = 0 & \text{on } \partial M. \end{cases}$$

We observe that the following estimate holds for $\Gamma(x,y)$,

$$|\Gamma(x,y)| \leq Cr^{-2};$$

hence by using the results in [8] (Proposition 4.12), we obtain the following estimate for $\Gamma_i(x,y)$,

$$|\Gamma_i(x,y)| \leq Cr^{2i-4} \tag{73}$$

So arriving at this stage by still the same result in [8] (Proposition 4.12), we have that $\Gamma_q(x,y)$ and $\Gamma_{q+1}(x,y)$ are continuous hence using elliptic regularity we get $Z_q(x,y)$ and $F(x,y)$ are in $C^{3+\alpha}(M^2)$. The regularity in both the variables x and y can be deduced by the symmetry of G, which follows from the self adjointness of $P_g^{4,3}$ and reasoning as in [8] (Proposition 4.13). Further from (73) we deduce that $\Gamma_i \in L^p$ with $\frac{1}{2} < p < 2$ for all $i = 1, \dots, q-1$. Hence by using standard elliptic regularity we infer that $Z_i(x,y) \in H^{4,p}$. So from the Sobolev embedding theorem and the fact that $\frac{1}{2} < p < 2$ we get $Z_i(x,y) \in C^{2+\alpha}(M^2)$ for all $i = 1, \dots, q-1$ for some α. Hence setting $K(x,y) = \sum_{i=1}^{q} Z_i(x,y) + F(x,y)$, the Lemma is proved. ■

Next we give a regularity result corresponding to boundary value problems of the type of BVP (20) and high order *a priori* estimates for sequences of solutions to BVP like (22) when they are bounded from above.

Proposition 0.3.5. *Let $u \in H_{\frac{\partial}{\partial n}}$ be a weak solution to*

$$\begin{cases} P_g^4 u + f = \bar{f} e^{4u} & \text{in } M; \\ P_g^3 u = h & \text{on } \partial M. \end{cases}$$

with $f \in C^\infty(M)$, $h \in C^\infty(\partial M)$ and $\bar{f}$ a real constant. Then we have that $u \in C^\infty(M)$.
Let $u_l \in H_{\frac{\partial}{\partial n}}$ be a sequence of weak solutions to

$$\begin{cases} P_g^4 u_l + f_l = \bar{f}_l e^{4u_l} & \text{in } M; \\ P_g^3 u_l = h_l & \text{on } \partial M. \end{cases}$$

with $f_l \to f_0$ in $C^k(M)$, $\bar{f}_l \to \bar{f}_0$ in $C^k(M)$ and $h \to h_0$ in $C^k(\partial M)$ for some fixed $k \in \mathbb{N}^*$. Assuming $\sup_M u_l \leq C$ we have that

$$||u_l||_{C^{k-1+\alpha}(M)} \leq C$$

for any $\alpha \in (0,1)$.

Before making the proof of Proposition 0.3.5 we give some Lemmas that will be needed. We first state a Lemma which is a direct consequence of Lemma 0.3.4. Next we recall a Lemma giving the existence of a Green function for Paneitz operator on compact closed four dimensional smooth Riemannian manifold.

Lemma 0.3.6. *Adopting the same notations as in Lemma (0.3.4), we have that there exists a closed compact smooth four dimensional submanifold N of $(\widetilde{M}, \tilde{g})$ such that $M \subset N$. Moreover the following holds:*
$\forall x \in N \setminus M$ there exists a unique $\bar{x} \in M \cap T$ such that

$$d_{\tilde{g}}(x, \partial M) = d_{\tilde{g}}(\bar{x}, \partial M).$$

As said above, we state a Lemma giving the existence of the Green function for $P_{\tilde{g}}^4$. It is a particular case of Proposition 0.3.2.

Lemma 0.3.7. *Suppose* $KerP^4_{\tilde{g}} \simeq \mathbb{R}$. *Then the Green function* $\tilde{G}(x,y)$ *of* $P^4_{\tilde{g}}$ *exists in the following sense :*
a) For all functions $u \in C^2(N)$, *we have*

$$u(x) - \bar{u} = \int_M \tilde{G}(x,y) P^4_{\tilde{g}} u(y) dV_{\tilde{g}}(y) \qquad \forall x \in N;$$

b)

$$\tilde{G}(x,y) = H_0(x,y) + K_0(x,y) \quad \forall x \neq y;$$

is smooth on $N^2 \setminus Diag(N^2)$, K *extends to a* $C^{2+\alpha}$ *function on* N^2 *and*

$$H(x,y) = \frac{1}{8\pi^2} f(r) \log \frac{1}{r}$$

where, $r = d_{\tilde{g}}(x,y)$ *is the geodesic distance from* x *to* y; $f(r)$ *is a* C^∞ *positive decreasing function,* $f(r) = 1$ *in a neighborhood of* $r = 0$ *and* $f(r) = 0$ *for* $r \geq inj_{\tilde{g}}(N)$.

Now we are ready to make the proof of Proposition 0.3.5.
Proof of Proposition 0.3.5
We have that by assumption $u \in H_{\partial n}$ is a weak solution to

$$\begin{cases} P^4_g u + f = \bar{f} e^{4u} & \text{in } M; \\ P^3_g u = h & \text{on } \partial M. \end{cases}$$

Then using Lemma 0.3.3 we obtain that

$$u(x) - \bar{u} = \int_M G(x,y)(\bar{f}e^{4u} - f) dV_g(y) + 2\int_{\partial M} G(x,y') h(y') dS_g(y').$$

Now let us define the following auxiliary functions

$$w(x) = \int_M G(x,y) \bar{f} e^{4u(y)} dV_g(y) \;\; x \in M;$$

and

$$v(x) = -\int_M G(x,y) f dV_g(y) + 2\int_{\partial M} G(x,y') h(y') dS_g(y'). \tag{74}$$

Then it is trivially seen that

$$w(x) = u(x) - v(x) \;\; x \in M. \tag{75}$$

On the other hand since $f \in C^\infty(M)$ and $h \in C^\infty(\partial M)$, then one can check easily that

$$v \in C^\infty(M). \tag{76}$$

Now using the relation (75) we obtain w satisfies the following integral equation

$$w(x) = \int_M G(x,y) e^{-4v(y)} \bar{f} e^{4w(y)} dV_g(y) \;\; x \in M; \tag{77}$$

and

$$\frac{\partial w}{\partial n_g} = 0 \;\; \text{on } \partial M.$$

Now let us define the *even* reflection of w through ∂M

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in M; \\ w(\bar{x}) & \text{if } x \in N \setminus M; \end{cases} \tag{78}$$

where N is the closed 4-manifold given by Lemma 0.3.6.
Thanks to the fact that $\frac{\partial w}{\partial n_g} = 0$, we have that $\tilde{w} \in H^2(N)$. Moreover using the integral equation solved by w (see (77)), one can check easily that $\tilde{w}$ satisfies

$$\tilde{w}(x) = \int_N \tilde{G}(x,y) e^{-4\tilde{v}(y)} \bar{f} e^{4\tilde{w}(y)} dV_{\tilde{g}}(y) \;\; x \in N.$$

where $\tilde{G}$ is the Green function of $P^4_{\tilde{g}}$ (see Lemma 0.3.7) and $\tilde{v}$ is the *even* reflection of v through ∂M, namely

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in M; \\ v(\bar{x}) & \text{if } x \in N \setminus M. \end{cases}$$

Furthermore from (74) and the fact that f and h are smooth, we derive that $\tilde{v} \in C^1(N)$.
On the other from the assumption $KerP^{4,3}_g \simeq \mathbb{R}$, it is easily seen that $KerP^4_{\tilde{g}} \simeq \mathbb{R}$. Hence using Lemma 0.3.7 we have that $\tilde{w}$ is a weak solution to

$$P^4_{\tilde{g}} = \bar{f} e^{-4\tilde{v}} e^{4\tilde{w}} \;\; \text{on } \; N.$$

Thus from a regularity result due to Uhlenbeck and Viaclovsky, see [88], we infer that $\tilde{w} \in C^\infty(N)$. Now restricting back to M we obtain $w \in C^\infty(M)$. So using (75), (76) and the fact that w is smooth on M, we have that $u \in C^\infty(M)$. The last part of the proposition follows from the same argument.
Hence the proof of the proposition is complete. ■

Now we give a regularity result corresponding to boundary value problems of the type of BVP (21) and high order *a priori* estimates for sequences of solutions to BVP like (24) when they are bounded from above. Its proof is the same as the one of Proposition 0.3.5, hence will be omitted.

Proposition 0.3.8. *Let* $u \in H_{\frac{\partial}{\partial n}}$ *be a weak solution to*

$$\begin{cases} P^4_g u = h & \text{in } M; \\ P^3_g u + f = \bar{f} e^{3u} & \text{on } \partial M. \end{cases}$$

with $f \in C^\infty(\partial M)$, $h \in C^\infty(M)$ *and* $\bar{f}$ *a real constant. Then we have that* $u \in C^\infty(M)$.
Let $u_l \in H_{\frac{\partial}{\partial n}}$ *be a sequence of weak solutions to*

$$\begin{cases} P^4_g u_l = h_l & \text{in } M; \\ P^3_g u_l + f_l = \bar{f}_l e^{4u_l} & \text{on } \partial M. \end{cases}$$

with $f_l \to f_0$ *in* $C^k(\partial M)$, $\bar{f}_l \to \bar{f}_0$ *in* $C^k(\partial M)$ *and* $h_l \to h_0$ *in* $C^k(M)$ *for some fixed* $k \in \mathbb{N}^*$. *Assuming* $\sup_{\partial M} u_l \leq C$ *we have that*

$$||u_l||_{C^{k-1+\alpha}(M)} \leq C$$

for any $\alpha \in (0,1)$.

Chapter 1

Moser-Trudinger type inequalities

In this Chapter we recall some classical Moser-Trudinger type inequalities, present some new ones involving the Paneitz-GJMS-Fefferman-Graham operators and the Chang-Qing one. Moreover we give some improvements of new inequalities.

1.1 Some Classical Moser-Trudinger type inequalities

In this Section we recall some classical Moser-Trudinger type inequalities. We start with the one due to Trudinger[86].
In 1967 Trudinger proved the following result:

Theorem 1.1.1. *Given $n \geq 2$ and k two positive integers with $k < n$, Ω an open bounded subset of $\mathbb{R}^n$, there exists a constant $\beta > 0$ and $C = C(n,k) > 0$, such that*

$$\int_\Omega e^{\beta |u|^{\frac{n}{n-k}}} dx \leq C|\Omega|$$

for all $u \in W_0^{k,\frac{n}{k}}(\Omega)$ such that $||\nabla^k u||_{L^{\frac{n}{k}}} \leq 1$.

Later in 1971 Moser[65] show the existence of the best constant β for the case $k = 1$, and give an explicit expression for it. Precisely he proved

Theorem 1.1.2. *Given $n \geq 2$, Ω an open bounded subset of $\mathbb{R}^n$, there exists a constant $C = C(n) > 0$, such that*

$$\int_\Omega e^{n w_{n-1}^{\frac{1}{n-1}} |u|^{\frac{n}{n-1}}} dx \leq C|\Omega|$$

for all $u \in W_0^{1,n}(\Omega)$ such that $||\nabla u||_{L^n} \leq 1$. Moreover the constant $n w_{n-1}^{\frac{1}{n-1}}$ is optimal in the sens that if we replace it by an other one bigger, we can not find such a C independent of u.

In 1983 D.R Adams[1] extends Moser's results to every $k < n$.

Theorem 1.1.3. *If $n \geq 2$ and k are two positive integers with $k < n$, Ω an open bounded subset of $\mathbb{R}^n$, then there exists a constant $\beta_0 = \beta_0(n,k)$ and $C = C(n,k) > 0$, such that*

$$\int_\Omega e^{\beta |u|^{\frac{n}{n-k}}} dx \leq C|\Omega|$$

for all $u \in W_0^{k,\frac{n}{k}}(\Omega)$ such that $||\nabla^k u||_{L^{\frac{n}{k}}} \leq 1$ and for all $\beta \leq \beta_0$, where

$$\beta_0 = \begin{cases} \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k+1}{2})}{\Gamma(\frac{n-k+1}{2}} \right)^{\frac{n}{n-k}}, & \text{if } k \text{ is odd;} \\[2ex] \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k}{2})}{\Gamma(\frac{n-k}{2}} \right)^{\frac{n}{n-k}}, & \text{if } k \text{ is even.} \end{cases}$$

Furthermore, if $\beta > \beta_0$*, then there exists a smooth function supported in* Ω *with* $||\nabla^k u||_{L^{\frac{n}{k}}} \leq 1$ *for which the integral can be made as large as desired.*

1.2 Fontana, Chang-Yang and Chang-Qing inequalities

In 1993, L. Fontana[40] extends the results of D.R Adams to curved spaces with the particularity that the best constant is the same as in the Euclidean setting. Precisely he proved

Theorem 1.2.1. *If* $n \geq 2, k$ *are two positive integers* $k < n$*, and* (M, g) *a compact closed smooth* n*-dimensional Riemannian manifold, then there exists a positive constant* $\beta_0 = \beta_0(n, k)$ *and* $C = C(n, k, M, g) > 0$*, such that*

$$\int_M e^{\beta|u|^{\frac{n}{n-k}}} dV_g \leq C$$

for all $u \in W^{k,\frac{n}{k}}(M)$ *such that* $||\nabla^k_g ku||_{L^{\frac{n}{k}}} \leq 1$ *and* $\int_M udV_g = 0$*, and for all* $\beta \leq \beta_0$*, where*

$$\beta_0 = \begin{cases} \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k+1}{2})}{\Gamma(\frac{n-k+1}{2})} \right)^{\frac{n}{n-k}}, & \text{if } k \text{ is odd;} \\ \dfrac{n}{\omega_{n-1}} \left(\dfrac{\pi^{\frac{n}{2}} 2^k \Gamma(\frac{k}{2})}{\Gamma(\frac{n-k}{2})} \right)^{\frac{n}{n-k}}, & \text{if } k \text{ is even.} \end{cases}$$

Furthermore, if $\beta > \beta_0$*, then there exists a smooth function with* $||\nabla^k_g u||_{L^{\frac{n}{k}}} \leq 1$ *and* $\int_M udV_g = 0$ *for which the integral can be made as large as desired.*

In their study of extremals of log-determinant functional on compact closed four dimensional Riemannian manifolds, Chang and Yang have derived a Moser-Trudinger type inequality involving the Paneitz operator. Precisely they proved

Theorem 1.2.2. *If* (M, g) *is a smooth compact closed four dimensional Riemannian manifold and the Paneitz operator* P^4_g *is non-negative with trivial kernel, then there exists a positive constant* $C = C(M, g)$*, such that for all* $u \in H^2(M)$ *with* $\langle P^4_g u, u \rangle \leq 1$ *and* $\int_M udV_g = 0$ *there holds*

$$\int_M e^{32\pi^2 u^2} dV_g \leq C.$$

1.3 Some new Moser-Trudinger type inequalities

This Section deals with some new Moser-Trudinger type inequalities. We start with an extension of Chang-Yang inequality to every dimensions. The same inequality was derived also by Brendle, see Section 3 in [13]. For the seek of completeness we provide a proof which is also similar to the one of Brendle.

Proposition 1.3.1. *Let* (M, g) *be a compact closed* n*-dimensional smooth Riemannian manifold with* $n \geq 3$*. Assume* P^n_g *is a non-negative operator with* $KerP^n_g \simeq \mathbb{R}$*. Then there exists a positive constant* $C = C(M, g)$ *so that*

$$\int_M e^{\frac{nc_n(u-\bar{u})^2}{\langle P^n_g u, u \rangle}} dV_g \leq C, \tag{1.1}$$

for all $u \in H^{\frac{n}{2}}(M)$*, and hence*

$$\log \int_M e^{n(u-\bar{u})} \leq C + \frac{n}{4c_n} \langle P^n_g u, u \rangle. \tag{1.2}$$

PROOF. Since P_g^n is a nonnegative operator with $KerP_g^n \simeq \mathbb{R}$ then $\sqrt{P_g^n}$ is well defined see [23] (in that case the authors are concerned with the four dimensional case but the same construction remains true for all n). Moreover from the point **a)** of the Lemma 0.3.2 and the self adjointness of P_g^n we obtain,

$$u(x) - \bar{u} = \int_M \sqrt{P_g^n} G(x,y) \sqrt{P_g^n} u(y) dV_g(y); \quad \forall \ u \ C^n(M). \tag{1.3}$$

Hence $\tilde{G}(x,y) = \sqrt{P_g^n} G(x,y)$ is the Green function of $\sqrt{P_g^n} G(x,y)$(see [13] (Section 3 in the proof of the boundedness of ω in $H^{\frac{n}{2}}$). Moreover it is a well known fact in the theory of pseudodifferential operator that $\sqrt{P_g^n}$ is a pseudodifferential operator of order $\frac{n}{2}$ and whose leading order symbol is as the one of $(-\Delta)^{\frac{n}{4}}$(see [41]). Hence, the leading term in the asymptotic expansion of its kernel $\tilde{G}(x,y)$ coincide with that of the Green's function for the operator $(-\Delta)^{\frac{n}{4}}$ in $\mathbb{R}^n$. So by a well know formula for Fourier transform of radial functions (see [79], Theorem 3.3) we infer that the leading term is $a_n r^{-\frac{n}{4}}$ where a_n is a dimensional constant . Hence arriving at this step we can follow the same proof as in [40] (Propostion 2.2) to conclude the first inequality. Moreover from the basic inequality

$$nab \le a^2 c_n + \frac{nb^2}{4c_n} \quad \forall a, \ b \in \mathbb{R}; \tag{1.4}$$

setting $a = u - \bar{u}$ and $b = \langle P_g^n u, u \rangle$, taking the exponential and integrating we obtain the last one. ■

In their study of extremals for the log-determinant functional on compact four dimensional Riemannian manifolds with boundary, Chang and Qing have proved a Moser-Trudinger type inequality. Precisely they showed the following theorem whose proof can be found in[19]:

Theorem 1.3.2. *If (M,g) is a smooth compact four dimensional Riemannian manifold with smooth boundary, then for all $\alpha < 16\pi^2$, there exists a constant $C = C(M,g,\alpha)$, such that for all $u \in H^2(M)$ with $\int_M |\Delta_g u|^2 \le 1$ and $\int_M u dV_g = 0$ there holds*

$$\int_M e^{\alpha u^2} dV_g \le C.$$

For the case of four manifolds with boundary, we prove a Moser-Trudinger type inequality similar to the one of Chang and Yang involving the Paneitz operator and the Chang and Qing one.

Proposition 1.3.3. *Let (M,g) be a compact four dimensional Riemannian manifold with boundary, and assume $P_g^{4,3}$ is a non-negative operator with $KerP_g^{4,3} \simeq \mathbb{R}$. Then we have that for all $\alpha < 16\pi^2$ there exists a constant $C = C(M,g,\alpha)$ such that*

$$\int_M e^{\frac{\alpha (u-\bar{u})^2}{\langle P_g^{4,3} u, u \rangle_{L^2(M)}}} dV_g \le C,$$

for all $u \in H_{\frac{\partial}{\partial n}}$, and hence

$$\log \int_M e^{4(u-\bar{u})} \le C + \frac{4}{\alpha} \langle P_g^{4,3} u, u \rangle_{L^2(M)} \quad \forall u \in H_{\frac{\partial}{\partial n}}.$$

In order to make the proof of Proposition 1.3.3 we will need a technical Lemma. It says that under the assumptions $KerP_g^{4,3} \simeq \mathbb{R}$ and $P_g^{4,3}$ non-negative, the map

$$u \in H_{\frac{\partial}{\partial n}} \longrightarrow ||u||_{P_g^{4,3}} = \langle P_g^{4,3} u, u \rangle_{L^2(M)}^{\frac{1}{2}}$$

induces an equivalent norm to the standard norm of $H^2(M)$ on $\{u \in H_{\frac{\partial}{\partial n}} \quad \bar{u} = 0\}$. More precisely we have the following

Lemma 1.3.4. *Suppose $KerP_g^{4,3} \simeq \mathbb{R}$ and $P_g^{4,3}$ non-negative then we have that $||\cdot||_{P_g^{4,3}}$ is an equivalent norm to $||\cdot||_{H^2}$ on $\{u \in H_{\frac{\partial}{\partial n}} \;\; \bar{u} = 0\}$*

Proof. First of all we have that $u \to (\int_M |\Delta_g u|^2 dV_g)^{\frac{1}{2}}$ is an equivalent norm to the standard norm of $H^2(M)$ on $\{u \in H_{\frac{\partial}{\partial n}} \;\; \bar{u} = 0\}$.
Now with this, to prove the Lemma it is sufficient to show that $||u||_{P_g^{4,3}}$ and $(\int_M |\Delta_g u|^2 dV_g)^{\frac{1}{2}}$ are equivalent norms on $\{u \in H_{\frac{\partial}{\partial n}} \;\; \bar{u} = 0\}$.
To do so we will use a compactness argument. First of all using the definition of $P_g^{4,3}$ one can check easily that the following holds

$$||u||_{P_g^{4,3}} \leq C(\int_M |\Delta_g u|^2 dV_g)^{\frac{1}{2}}. \tag{1.5}$$

Now let us show that

$$(\int_M |\Delta_g u|^2 dV_g)^{\frac{1}{2}} \leq C||u||_{P_g^{4,3}} \qquad \forall \;\; u \in \{u \in H_{\frac{\partial}{\partial n}} \;\; \bar{u} = 0\}. \tag{1.6}$$

We argue by contradiction, suppose (1.6) does not hold, then there exists $u_l \in \{u \in H_{\frac{\partial}{\partial n}} \;\; \bar{u} = 0\}$ such that

$$\int_M (|\Delta_g u_l|^2 dV_g)^{\frac{1}{2}} = 1 \;\text{ and }\; ||u_l||_{P_g^{4,3}} \to 0. \tag{1.7}$$

Now using the fact that $\int_M (|\Delta_g u_l|^2 dV_g)^{\frac{1}{2}} = 1$, we get that (up to a subsequence) $u_l \rightharpoonup u^*$. Moreover using the fact that $KerP_g^{4,3} \simeq \mathbb{R}$, $P_g^{4,3}$ is a non-negative, $||u_l||_{P_g^{4,3}} \to 0$ and Rellich compactness theorem we infer that

$$u^* = 0. \tag{1.8}$$

Next using again the fact that $||u_l||_{P_g^{4,3}} \to 0$ and the definition of $P_g^{4,3}$ we infer that

$$\int_M |\Delta_g u_l|^2 dV_g + \frac{2}{3} R_g |\nabla_g u_l| dV_g - 2\int_M Ric_g(\nabla_g u_l, \nabla_g u_l) dV_g - 2\int_{\partial M} L_g(\nabla_{\hat{g}} u_l, \nabla_{\hat{g}} u_l) dS_g = o_l(1). \tag{1.9}$$

Furthermore still by using Rellich compactness theorem we obtain

$$\frac{2}{3}\int_M R_g |\nabla_g u_l| dV_g - 2\int_M Ric_g(\nabla_g u_l, \nabla_g u_l) dV_g = o_l(1). \tag{1.10}$$

Now let $\epsilon > 0$ and small then by Lemma 2.3 in [19] and also Rellich compactness theorem we have that

$$-2\int_{\partial M} L_g(\nabla_{\hat{g}} u_l, \nabla_{\hat{g}} u_l) dS_g \geq -\epsilon \int_M |\Delta_g u_l| dV_g - o_l(1). \tag{1.11}$$

So using (1.7), (1.9), (1.10) and (1.11) we get

$$o_l(1) \geq 1 - \epsilon + o_l(1).$$

Thus since ϵ is small we arrive to a contradiction. So (1.6) is true. Hence (1.5) and (1.6) imply that the Lemma is proved. ∎

Now we are ready to make the proof of Proposition 1.3.3.

Proof of Proposition 1.3.3
First of all let us set

$$\mathcal{H} = \{u \in H_{\frac{\partial}{\partial n}}, \;\; \bar{u} = 0, \;\; \langle P_g^{4,3} u, u \rangle_{L^2(M)} = 1\}$$

and for $\alpha > 0$

$$J_\alpha(u) = \int_M e^{\alpha u^2} dV_g \;, \;\; u \in \mathcal{H}.$$

We have that from Theorem 1.3.2 and Lemma 1.3.4 there exists $\alpha > 0$ such that

$$\sup_{u\in\mathcal{H}} J_\alpha(u) < +\infty.$$

Hence

$$\alpha_0 = \sup\{\alpha > 0 : \sup_{u\in\mathcal{H}} J_\alpha(u) < +\infty\}$$

is well defined and $0 < \alpha_0 \leq \infty$.
To prove the proposition it is sufficient to show that

$$\alpha_0 \geq 16\pi^2$$

Suppose by contradiction that $\alpha_0 < 16\pi^2$ and let us argue for a contradiction.
We have that by definition of α_0 there exists a family u_ϵ, $\epsilon > 0$ such that

$$J_{\alpha_0+\epsilon}(u_\epsilon) \to +\infty.$$

On the other hand, using a covering argument there exists a point $p \in M$ such that for all $r > 0$

$$\int_{B_p(r)} e^{(\alpha_0+\epsilon)u_\epsilon^2} dV_g \to +\infty \quad \text{as } \epsilon \to 0.. \tag{1.12}$$

Moreover from the fact that $u_\epsilon \in \mathcal{H}$ and Lemma 1.3.4, we can assume without lost of generality that $u_\epsilon \rightharpoonup u_0$. Now we claim that $u_0 = 0$. Suppose not, then by using the property of the inner product we get

$$||u_\epsilon - u_0||_{P_g^{4,3}} < \beta$$

for some $\beta < 1$ and for ϵ small. Hence using Thoerem 1.3.2 and Lemma 1.3.4 we infer that

$$J_{\alpha_1}(u_\epsilon - u_0) \leq C$$

for some $\alpha_1 > \alpha_0$. Next using Cauchy inequality it is easily seen that

$$J_{\alpha_2}(u_\epsilon) \leq C$$

for some $\alpha_2 > \alpha_0$. Thus a contradiction to (1.12). Hence $u_0 = 0$.
Now suppose $p \in \partial M$
Let us take a cut-off function $\eta \in C_0^\infty(B_p(\delta))$, $\eta = 1$ on $B_p(\frac{\delta}{2})$ where $\delta > 0$ is a fixed positive and small number. Using Leibniz rule we obtain

$$\int_{B_p(\frac{\delta}{2})_+} P_g^{4,3}(\eta u_\epsilon)(\eta u_\epsilon) dV_g \leq ||\eta u_\epsilon||_{P_g^{4,3}} \leq 1 + \epsilon^{'}, \tag{1.13}$$

for some $\epsilon^{'} > 0$ such that $\frac{16\pi^2}{1+\epsilon^{'}} > \alpha_0$. Now let us set

$$\tilde{u}_\epsilon(s,t) = \begin{cases} (\eta u_\epsilon) \circ exp_p(s,t), & t \geq 0; \\ (\eta u_\epsilon) \circ exp_p(s,-t), & t \leq 0. \end{cases}$$

Then from 1.13 we derive that

$$\int_{B^0(\delta)} |\Delta_0 \tilde{u}_\epsilon|^2 dx \leq 2 + \epsilon^{"};$$

for some $\epsilon^{"}$ small where Δ_0 denotes the Euclidean Laplacian.
Hence by Adams inequality, see Theorem 1.1.3, we get

$$\int_{B^0(\delta)} e^{\alpha_3 \tilde{u}_\epsilon^2} dx \leq C$$

for some $\alpha_3 > 16\pi^2$. Thus we arrive to

$$\int_{B_p(\frac{\delta}{2})} e^{\alpha_3 u_\epsilon^2} dV_g \leq C \int_{B^0(\delta)} e^{\alpha_3 \tilde{u}_\epsilon^2} dx \leq C.$$

Hence reaching a contradiction to (1.12).
Now suppose $p \in int(M)$.
In this case, following the same method as above (and in a simpler way since we do not need to use $\tilde{u}_\epsilon$, but u_ϵ its self) one gets the same contradiction. Hence the proof of the Proposition is complete. ■

Moreover we also prove a trace analogue of the previous Moser-Trudinger type inequality.

Proposition 1.3.5. *Let (M, g) be a compact smooth four dimensional Riemannian manifold with smooth boundary, and assume $P_g^{4,3}$ is a non-negative operator with $KerP_g^{4,3} \simeq \mathbb{R}$. Then we have that for all $\alpha < 12\pi^2$ there exists a constant $C = C(M, g, \alpha)$ such that*

$$\int_{\partial M} e^{\frac{\alpha(u-\bar{u}_{\partial M})^2}{\langle P_g^{4,3}u,u\rangle_{L^2(M)}}} dS_g \leq C, \tag{1.14}$$

for all $u \in H_{\frac{\partial}{\partial n}}$, and hence

$$\log \int_{\partial M} e^{3(u-\bar{u})} dS_g \leq C + \frac{9}{4\alpha} \langle P_g^{4,3}u, u\rangle_{L^2(M,g)} \quad \forall u \in H_{\frac{\partial}{\partial n}}. \tag{1.15}$$

Proof. First of all, without loss of generality we can assume $\bar{u}_{\partial M} = 0$. Following the same argument as in Lemma 2.2 in [19], we get $\forall \beta < 16\pi^2$ there exists a positive constant $C = C(\beta, M, g)$

$$\int_M e^{\frac{\beta v^2}{\int_M |\Delta_g v|^2 dV_g}} dV_g \leq C, \quad \forall v \in H_{\frac{\partial}{\partial n}} \text{ with } \bar{v}_{\partial M} = 0.$$

From this, using the same reasoning as in Proposition 1.3.3 , we derive

$$\int_M e^{\frac{\beta v^2}{\langle P_g^{4,3}v,v\rangle_{L^2(M)}}} dV_g \leq C, \quad \forall v \in H_{\frac{\partial}{\partial n}} \text{ with } \bar{v}_{\partial M} = 0. \tag{1.16}$$

Now let X be a vector field extending the the outward normal at the boundary ∂M. Using the divergence theorem we obtain

$$\int_{\partial M} e^{\alpha u^2} dS_g = \int_M div_g \left(X e^{\alpha u^2}\right) dV_g.$$

Using the formula for the divergence of the product of a vector field and a function we get

$$\int_{\partial M} e^{\alpha u^2} dS_g = \int_M (div_g X + 2u\alpha \nabla_g u \nabla_g X) e^{\alpha u^2} dV_g. \tag{1.17}$$

Now we suppose $< P_g^{4,3}u, u >_{L^2(M)} \leq 1$, then since the vector field X is smooth we have

$$\left| \int_M div_g X e^{\alpha u^2} dV_g \right| \leq C; \tag{1.18}$$

thanks to (1.16). Next let us show that

$$\left| \int_M 2\alpha u \nabla_g u \nabla_g X e^{\alpha u^2} dV_g. \right| \leq C$$

Let $\epsilon > 0$ small and let us set

$$p_1 = \frac{4}{3-\epsilon}, \;\; p_2 = 4, \;\; p_3 = \frac{4}{\epsilon}.$$

It is easy to check that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.$$

Using Young's inequality we obtain

$$\left|\int_M 2\alpha u \nabla_g u \nabla_g X e^{\alpha u^2} dV_g\right| \leq C ||u||_{L^{\frac{4}{\epsilon}}} ||\nabla_g u||_{L^4} \left(\int_M e^{\alpha \frac{4}{3-\epsilon} u^2} dV_g\right)^{\frac{3-\epsilon}{4}}.$$

On the other hand, Lemma 1.3.4 and Sobolev embedding theorem imply

$$||u||_{L^{\frac{4}{\epsilon}}} \leq C;$$

and

$$||\nabla_g u||_{L^4} \leq C.$$

Furthermore from the fact that $\alpha < 12\pi^2$, by taking ϵ sufficiently small and using (1.16), we obtain

$$\left(\int_M e^{\alpha \frac{4}{3-\epsilon} u^2} dV_g\right)^{\frac{3-\epsilon}{4}}.$$

Thus we arrive to

$$\left|\int_M 2\alpha u \nabla_g u \nabla_g X e^{\alpha u^2} dV_g\right| \leq C. \tag{1.19}$$

Hence (1.17), (1.18) and (3.2.3) imply

$$\int_{\partial M} e^{\alpha u^2} dS_g \leq C,$$

as desired. So the first point of the Lemma is proved.
Now using the algebraic inequality

$$3ab \leq 3\gamma^2 a^2 + \frac{3b^2}{4\gamma^2},$$

we have that the second point follows directly from the first one. Hence the Lemma is proved. ■

Next we recall a Moser-Trudinger type inequality for system due to Jost and Wang[44]

Theorem 1.3.6. *([44]) For $\rho = (\rho_1, \rho_2)$ the functional $II_\rho : H^1(\Sigma) \times H^1(\Sigma)$ is bounded from below if and only if both ρ_1 and ρ_2 satisfy the inequality $\rho_i \leq 4\pi$.*

1.4 Improvement of Moser-Trudinger type inequalities

In this Section, we present some improvement of Proposition 1.3.1, Proposition 1.3.3, Proposition 1.3.5, and Theorem 1.3.6.

We start by stating a result which gives an improvement of Proposition 1.3.1. Its proof is the same as the one Lemma 2.2 in [33] when n is *even* and in the *odd* case only one step is modified. Hence will not repeat the proof but just sketch the arguments and show the modification in the *odd* case.

Proposition 1.4.1. *Let (M,g) be a compact smooth closed n-dimensional Riemannian manifold with $n \geq 3$, and suppose P_g^n non-negative with $KerP_g^n \simeq \mathbb{R}$. Let $l \in \mathbb{N}$, and $S_1 \cdots S_{l+1}$ be subsets of M satisfying $dist(S_i,S_j) \geq \delta_0$ for $i \neq j$. Moreover assume $\gamma_0 \in (0, \frac{1}{l+1})$, then, for any $\bar{\epsilon} > 0$, there exists a constant $C = C(\bar{\epsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_M e^{n(u-\bar{u})} \leq C + \frac{n}{4c_n(l+1) - \bar{\epsilon}} \left\langle P_g^n u, u \right\rangle \tag{1.20}$$

for all the functions $u \in H^{\frac{n}{2}}(M)$ satisfying

$$\frac{\int_{S_i} e^{nu} dVg}{\int_M e^{nu} dVg} \geq \gamma_0, \quad i \in \{1,..,l+1\}. \tag{1.21}$$

Proof. As already said the proof follows that of Lemma 2.2 in [33]. We recall the arguments which apply to the *even* case, and after show the modification to get the *odd* case. The argument is based on constructing some cutoff functions g_i which are identically 1 on S_i; and which have disjoint support. Then $\forall i$ by (1.21) we have that

$$\int_M e^{nu} dV_g \leq \frac{1}{\gamma_0} \int_{S_i} e^{nu} dV_g \leq \frac{C_M}{\gamma_0} \int_M e^{n g_i u} dV_g \tag{1.22}$$

On the other hand using the Leibniz rule and interpolation inequalities we obtain

$$\left\langle P_g^n g_i v, g_i v \right\rangle \leq \int_M g_i^2 (P_g^n v, v) dV_g + \epsilon \left\langle P_g^n v, v \right\rangle + C_{\epsilon, \delta_0} \int_M v^2 dV_g. \tag{1.23}$$

Applying Moser-Trudinger inequality (see (1.3.1)) to ug_j, choosing i such that $\int_M g_i^2(P_g^n v, v) dV_g = \min_j \int_M g_j^2 (P_g^n v, v) dV_g$, and by using interpolation inequalities we obtain the required statement.

We point out that in the *odd* case P_g^n being a pseudodifferential operator does not verify Leibniz rule, hence to get counterpart of (1.23), we need a different argument. We will use the pseudodifferential calculus. Indeed for every $v \in H^{\frac{n}{2}}(M)$ we have that

$$\left\langle P_g^n g_i v, g_i v \right\rangle = \int_M g_i^2 (P_g^n v, v) dV_g + \left\langle P_g^n g_i v - g_i P_g^n v, g_i v \right\rangle. \tag{1.24}$$

On the other hand by using the property of the duality pairing, we obtain

$$\left\langle P_g^n g_i v - g_i P_g^n v, g_i v \right\rangle \leq ||P_g^n g_i v - g_i P_g^n v||_{H^{-\frac{n}{2}}} ||g_i v||_{H^{\frac{n}{2}}}. \tag{1.25}$$

Now using the property of commutators, (see [85] Corollary 4.2) we have that

$$||P_g^n g_i v - g_i P_g^n v||_{H^{-\frac{n}{2}}} \leq C ||v||_{H^{\frac{n}{2}-1}}; \tag{1.26}$$

so using interpolations as in the *even* case we obtain

$$\left\langle P_g^n g_i v, g_i v \right\rangle \leq \int_M g_i^2 (P_g^n v, v) dV_g + \epsilon \left\langle P_g^n v, v \right\rangle + C_{\epsilon, \delta_0} \int_M v^2 dV_g. \tag{1.27}$$

As soon as we get a counterpart of (1.23), all the other steps apply as in the *even* case. ■

Next we give an improvement of Proposition 1.3.3.

Proposition 1.4.2. *Let (M,g) be a compact smooth four dimensional Riemannian manifold with smooth boundary, and assume $P_g^{4,3}$ non-negative with $KerP_g^{4,3} \simeq \mathbb{R}$. For a fixed l_1, $l_2 \in \mathbb{N}$, $l_1 + l_2 \neq 0$ and $\delta > 0$, let $S_1 \cdots S_{l_1}$, $\Omega_1 \cdots \Omega_{l_2}$ be subsets of M satisfying $\overline{S}_i \subset\subset M_\delta$, $dist(S_i,S_j) \geq \delta$ for $i \neq j$, $dist(\Omega_i, \Omega_j) \geq \delta$,, $\Omega_i \cap \partial M \neq \emptyset$, $\bar{\Omega}_i \subset\subset \partial M \times [0,\delta]$ and let $\gamma_0 \in (0, \frac{1}{l_1+l_2})$.*

Then, for any $\bar{\epsilon} > 0$*, there exists a constant* $C = C(\bar{\epsilon}, \gamma_0, l_1, l_2, M, \delta)$ *such that the following holds 1)*

$$\log \int_M e^{4(u-\bar{u})} \leq C + \frac{1}{4\pi^2}(\frac{1}{2l_1 + l_2 - \bar{\epsilon}}) \left\langle P_g^{4,3} u, u \right\rangle_{L^2(M)};$$

for all the functions $u \in H_{\frac{\partial}{\partial n}}$ *satisfying*

$$\frac{\int_{S_i} e^{4u} dVg}{\int_M e^{4u} dVg} \geq \gamma_0, \quad i \in \{1, .., l_1\}. \tag{1.28}$$

and

$$\frac{\int_{\Omega_i} e^{4u} dVg}{\int_M e^{4u} dVg} \geq \gamma_0, \quad i \in \{1, .., l_2\}. \tag{1.29}$$

Proof. We modify the argument in [29] and [33]. First of all we can assume without loss of generality that $\bar{u} = 0$. On the other hand by the properties verified by the sets S_i and Ω_i we have that there exists

$$N_\delta \subset M \text{ closed submanifold of dimension four, } \overline{\cup S_i} \subset\subset N_\delta \subset int(M), \ \overline{\cup \Omega_i} \subset\subset M \setminus N_\delta.$$

Fwe can find $l_1 + l_2$ functions $g_1, \cdots, g_{l_1}$ and $h_1, \cdots, h_{l_2}$ such that

$$\begin{cases} g_i(x) \in [0,1] & \text{for every } x \in M, \ i = 1, \cdots, l_1; \\ g_i(x) = 1 & \text{for } x \in S_i, \ i = 1, \cdots, l_1; \\ g_i(x) = 0 & \text{if } dist(x, S_i) \geq \frac{\delta}{4}; \ i = 1, \cdots, l_1; \\ supp(g_i) \subset N_\delta \\ ||g_i||_{C^4(M)}, \leq C_\delta \text{ for } i = 1, \cdots, l_1; \\ \sum_{i=1}^{l_1} g_i = 1 \text{ on } N_\delta \end{cases} \tag{1.30}$$

and

$$\begin{cases} h_i(x) \in [0,1] & \text{for every } x \in M, \ i = 1, \cdots, l_2; \\ h_i(x) = 1 & \text{for } x \in \Omega_i, \ i = 1, \cdots, l_2; \\ h_i(x) = 0 & \text{if } dist(x, \Omega_i) \geq \frac{\delta}{4}; \ i = 1, \cdots, l_2 \\ ||h_i||_{C^4(M)}, \leq C_\delta \text{ for } i = 1, \cdots, l_2. \\ \sum_{i=1}^{l_2} h_i = 1 \text{ on } M \setminus N_\delta \end{cases} \tag{1.31}$$

where C_δ is a positive constant depending only on δ. Moreover we can choose the functions g_i and h_i such that they have (mutually) disjoint supports.
We remark that the submanifold N_δ depends only on δ. But since in our analysis, only its volume is involved when we apply Moser-Trudinger inequality to $g_i u_2$ see (1.36), then we can omit the dependence to δ.
Using Leibniz rule, Schwartz inequality and interpolation, we obtain that for every $\epsilon > 0$ there exists $C_{\epsilon,\delta}$ (depending only on ϵ and δ) such that $\forall v \in H^2(M)$, for any $i = 1, \cdots, l_1$ and $j = 1, \cdots, l_2$ there holds

$$\left\langle P_g^{4,3} g_i v, g_i v \right\rangle \leq \int_M g_i^2 (P_g^{4,3} v, v) dV_g + \epsilon \left\langle P_g^{4,3} v, v \right\rangle_{L^2(M)} + C_{\epsilon,\delta} \int_M v^2 dV_g. \tag{1.32}$$

and

$$\left\langle P_g^{4,3} h_j v, h_j v \right\rangle \leq \int_M h_j^2 (P_g^4 v, v) dV_g + \epsilon \left\langle P_g^{4,3} v, v \right\rangle_{L^2(M)} + C_{\epsilon,\delta} \int_M v^2 dV_g. \tag{1.33}$$

Next we decompose u in Fourier mode, namely we decompose u into low and high modes by setting $u = u_1 + u_2$ with $u_1 \in L^\infty(M)$ (u_1 represents the low mode and u_2 the high one). Hence from our assumptions, see (1.28) and (1.29) we derive that

$$\int_{S_i} e^{4u_2} dV_g \geq e^{-4||u_1||_{L^\infty}} \gamma_0 \int_M e^{4u} dV_g, \quad i = 1, \cdots, l_1; \tag{1.34}$$

and

$$\int_{\Omega_i} e^{4u_2} dV_g \geq e^{-4||u_1||_{L^\infty}} \gamma_0 \int_M e^{4u} dV_g, \quad i = 1, \cdots, l_2; \tag{1.35}$$

Now using (1.34), (1.35) and the trivial identity

$$\log \int_M e^{4u} dV_g = \frac{2l_1}{2l_1 + l_2} \log \int_M e^{4u} dV_g + \frac{l_1}{2l_1 + l2} \log \int_M e^{4u} dV_g$$

we obtain

$$\log \int_M e^{4u} dV_g \leq \log \frac{1}{\gamma_0} + 4||u_1||_{L^\infty} + \frac{2l_1}{2l_1 + l_2} \log \int_N e^{g_i 4u_2} + \frac{l_2}{2l_1 + l_2} \log \int_M e^{4h_j u_2} dV_g + C.$$

where C depends only on M. On the other hand by Chang-Yang inequality (see Theorem 1.2.2), we get

$$\log \int_N e^{g_i 4u_2} \leq C_M + \frac{1}{8\pi^2} \left\langle P^4_{g,N}(g_i u_2), g_i u_2 \right\rangle + 4\overline{g_i u_2}; \tag{1.36}$$

where $P^4_{g,N}$ denotes the Paneitz operator associated to the close 4-manifold N endowed with the induced metric from g, and C_M depends only on $Vol_g(M)$.
Now let $\alpha < 16\pi^2$ (to be fixed latter), from Propostion 1.3.3 we infer

$$\log \int_M e^{4u} dV_g \leq \log \frac{1}{\gamma_0} + 4||u_1||_{L^\infty} + \frac{4}{2\alpha}\left(\frac{2l_1}{2l_1 + l_2}\right) \left\langle P^4_{g,N}(g_i u_2), g_i u_2 \right\rangle + \frac{4}{\alpha}\left(\frac{l_2}{2l_1 + l_2}\right) \left\langle P^{4,3}_g(h_j u_2), h_j u_2 \right\rangle$$
$$+ 4\left(\frac{2l_1}{2l_1 + l_2}\right)\overline{g_i u_2} + 4\left(\frac{l_2}{2l_1 + l_2}\right)\overline{h_j u_2} + C_{\alpha,M,l_1,l_2}. \tag{1.37}$$

Where C_{α,M,l_1,l_2} depends only on α, l_1, l_2 and M. We now choose i and j such that

$$\int_N g_i^2 (P^4_{g,N} u_2, u_2) dV_g \leq \int_N g_p^2 (P^4_{g,N} u_2, u_2) dV_g \text{ for every } p = 1, \cdots, l_1;$$

and

$$\int_M h_j^2 (P^4, 3g u_2, u_2) dV_g \leq \int_M h_q^2 (P^{4,3}_g u_2, u_2) dV_g; \text{for every } q = 1, \cdots, l_2.$$

Hence since the functions g_p, h_q have disjoint supports and verify (1.30) and (1.31)), then by (1.32), (1.33) and (1.37) we get

$$\log \int_M e^{4u} dV_g \leq \log \frac{1}{\gamma_0} + 4||u_1||_{L^\infty} + \frac{4}{\alpha}\left(\frac{1}{2l_1 + l_2} + \epsilon\right) \left\langle P^{4,3}_g u_2, u_2 \right\rangle + C_{\epsilon,\delta_0} \int_M u_2^2 dV_g$$
$$+ 4\left(\frac{2l_1}{2l_1 + l_2}\right)\overline{g_i u_2} + 4\left(\frac{l_2}{2l_1 + l_2}\right)\overline{h_j u_2} + C_{\alpha,M,l_1,l_2}. \tag{1.38}$$

Now we choose $\lambda_{\epsilon,\delta}$ to be an eigenvalue of $P^{4,3}_g$ such that $\frac{C_{\epsilon,\delta}}{\lambda_{\epsilon,\delta}} < \epsilon$ and we set

$$u_1 = P_{V_{\epsilon,\delta}} u; \qquad u_2 = P_{V^\perp_{\epsilon,\delta}} u; \tag{1.39}$$

where $V_{\epsilon,\delta}$ is the direct sum of the eigenspaces of $P^{4,3}_g$ with eigenvalues less or equal to $\lambda_{\epsilon,\delta}$, and $P_{V_{\epsilon,\delta}}$, $P_{V^\perp_{\epsilon,\delta}}$ denote the projections onto $V_{\epsilon,\delta}$ and $V^\perp_{\epsilon,\delta}$ respectively. Since $\bar{u} = 0$, then the

L^2-norm and the L^∞-norm on $V_{\epsilon,\delta_0,\delta}$ are equivalent (with a proportionality factor which depends on ϵ and δ). Hence by the choice of u_1 and u_2, see (1.39) we have that

$$||u_1||_{L^\infty} \leq \tilde{C}_{\epsilon,\delta} \left\langle P_g^{4,3} u_1, u_1 \right\rangle^{\frac{1}{2}}$$

and

$$C_{\epsilon,\delta} \int_M u_2^2 dV_g < \epsilon \left\langle P_g^{4,3} u_2, u_2 \right\rangle ;$$

where $\tilde{C}_{\epsilon,\delta}$ depends on ϵ and δ. Furthermore by Hölder inequality and Lemma 1.3.4 we have that

$$\overline{g_i u_2} \leq C \left\langle P_g^{4,3} u, u \right\rangle^{\frac{1}{2}} ;$$

and

$$\overline{h_j u_2} \leq C \left\langle P_g^{4,3} u, u \right\rangle^{\frac{1}{2}} .$$

So (1.38) becomes

$$\log \int_M e^{4u} dV_g \leq 2 \log \frac{1}{\gamma_0} + \hat{C}_{\epsilon,\delta} \left\langle P_g^{4,3} u_1, u_1 \right\rangle^{\frac{1}{2}} + \frac{4}{\alpha} \left(\frac{1}{2l_1 + l_2} + \epsilon \right) \left\langle P_g^{4,3} u_2, u_2 \right\rangle + \epsilon \left\langle P_g^{4,3} u_2, u_2 \right\rangle$$
$$+ C_{l_1,l_2} \left\langle P_g^{4,3} u, u \right\rangle^{\frac{1}{2}} + C_{\alpha,M,l_1,l_2} .$$

where $\hat{C}_{\epsilon,\delta} = 4\tilde{C}_{\epsilon,\delta}$. Thus by using Cauchy inequality we get

$$\log \int_M e^{4u} dV_g \leq C_{\epsilon,\delta,\gamma_0,\alpha,l_1,l_2,M} + \frac{4}{\alpha} \left(\frac{1}{2l_1 + l_2} + 3\epsilon \right) \left\langle P_g^{4,3} u_2, u_2 \right\rangle .$$

Now setting $\alpha = 16\pi^2 - 4\epsilon$ we obtain

$$\log \int_M e^{4u} dV_g \leq C_{\epsilon,\delta,\gamma_0,l_1,l_2,M} + \frac{1}{4\pi^2 - \epsilon} \left(\frac{1}{2l_1 + l_2} + 3\epsilon \right) \left\langle P_g^{4,3} u_2, u_2 \right\rangle .$$

So choosing ϵ such that $\frac{1}{4\pi^2-\epsilon}(\frac{1}{2l_1+l_2} + 3\epsilon) \leq \frac{1}{4\pi^2}(\frac{1}{2l_1+l_2-\bar{\epsilon}})$ we get

$$\log \int_M e^{4u} dV_g \leq C_{\epsilon,\delta,\gamma_0,l_1,l_2,M} + \frac{1}{4\pi^2} \left(\frac{1}{2l_1 + l_2 - \bar{\epsilon}} \right) \left\langle P_g^{4,3} u_2, u_2 \right\rangle .$$

Hence the Lemma is proved. ■

Next we give an improvement of Proposition 1.3.5. Its proof is the same as the one of Lemma 2.2 in [33], hence will be omitted.

Proposition 1.4.3. *Let* (M, g) *be a compact smooth four dimensional Riemannian manifold with smooth boundary, and assume* $P_g^{4,3}$ *is non-negative with* $Ker P_g^{4,3} \simeq \mathbb{R}$*. For a fixed* $l \in \mathbb{N}$*, let* $S_1 \cdots S_{l+1}$*, be subsets of* ∂M *satisfying,* $dist(S_i, S_j) \geq \delta_0$ *for* $i \neq j$*, let* $\gamma_0 \in (0, \frac{1}{l+l})$*.*
Then, for any $\bar{\epsilon} > 0$*, there exists a constant* $C = C(\bar{\epsilon}, \delta_0, \gamma_0, l, M,)$ *such that the following hods*
1)

$$\log \int_{\partial M} e^{3(u - \bar{u}_{\partial M})} \leq C + \frac{3}{16\pi^2} \left(\frac{1}{l + 1 - \bar{\epsilon}} \right) \left\langle P_g^{4,3} u, u \right\rangle_{L^2(M)} ;$$

for all the functions $u \in H_{\frac{\partial}{\partial n}}$ *satisfying*

$$\frac{\int_{S_i} e^{3u} dSg}{\int_{\partial M} e^{3u} dSg} \geq \gamma_0, \quad i \in \{1, .., l+1\}. \tag{1.40}$$

Now we give the last Proposition of this Section. It gives an improvement of Theorem 1.3.6.

Proposition 1.4.4. *Let (Σ, g) be a compact closed Riemannian surface, let $\delta_0 > 0$, $\ell \in \mathbb{N}$, and let $S_1, \dots, S_\ell$ be subsets of Σ satisfying $dist(S_i, S_j) \geq \delta_0$ for $i \neq j$. Let $\gamma \in \left(0, \frac{1}{\ell}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma, \ell, \Sigma)$ such that*

$$\ell \log \int_\Sigma e^{(u_1 - \overline{u}_1)} dV_g + \log \int_\Sigma e^{(u_2 - \overline{u}_2)} dV_g \leq C + \frac{1}{4\pi - \tilde{\varepsilon}} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla u_i \cdot \nabla u_j dV_g \right]$$

provided the function u_1 satisfies the relations

$$\frac{\int_{S_i} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \gamma, \qquad i \in \{1, \dots, \ell\}. \tag{1.41}$$

Before proving the Proposition, we state a particular case of Fontana's inequality (see Theorem 1.2.1 with $n = 2$), an improvement of it and a preliminary lemma which will be proved later. As already said we start by recalling the following particular case of Fontana's inequality.

Lemma 1.4.5. *Let (Σ, g) be a compact closed Riemannian surface. We have that there exists a constant $C = C(\Sigma, g) > 0$ such that*

$$\log \int_\Sigma e^{(u - \overline{u})} dV_g \leq C + \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 dV_g, \quad \textit{for all } u \in H^1(\Sigma). \tag{1.42}$$

Next we give an improvement of the latter inequality.

Proposition 1.4.6. *Let (Σ, g) be a compact closed Riemannian surface, $S_1, \dots, S_\ell$ be subsets of Σ satisfying $dist(S_i, S_j) \geq \delta_0$ for $i \neq j$, and let $\gamma \in \left(0, \frac{1}{\ell}\right)$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma)$ such that*

$$\log \int_\Sigma e^{(u - \overline{u})} dV_g \leq C + \frac{1}{16\ell\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla u|^2 dV_g$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{S_i} e^{u} dV_g}{\int_\Sigma e^{u} dV_g} \geq \gamma; \qquad i \in \{1, \dots, \ell\}.$$

The proof is the same as the one of Proposition 1.4.1. We also refer the reader to [33].
Now we give an auxilliary Lemma whose proof is postponed at the end.

Lemma 1.4.7. *Let (Σ, g) be a compact closed Riemannian surface. Under the assumptions of Proposition 1.4.4, there exists numbers $\tilde{\gamma}_0, \tilde{\delta}_0 > 0$, depending only on γ, δ_0, Σ, and ℓ sets $\tilde{S}_1, \dots, \tilde{S}_\ell$ such that $d(\tilde{S}_i, \tilde{S}_j) \geq \tilde{\delta}_0$ for $i \neq j$ and such that*

$$\frac{\int_{\tilde{S}_1} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{S}_1} e^{u_2} dV_g}{\int_\Sigma e^{u_2} dV_g} \geq \tilde{\gamma}_0; \qquad \frac{\int_{\tilde{S}_i} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \tilde{\gamma}_0, \quad i \in \{2, \dots, \ell\}.$$

Proof of Proposition 1.4.4. We modify the argument in [29] and [33]. Let $\tilde{S}_1, \dots, \tilde{S}_\ell$ be given by Lemma 1.4.7. Assuming without loss of generality that $\overline{u}_1 = \overline{u}_2 = 0$, we can find ℓ functions $g_1, \dots, g_\ell$ satisfying the properties

$$\begin{cases} g_i(x) \in [0,1] & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \tilde{S}_i, i = 1, \dots, \ell; \\ supp(g_i) \cap supp(g_j) = \emptyset, & \text{for } i \neq j; \\ \|g_i\|_{C^2(\Sigma)} \leq C_{\tilde{\delta}_0}, \end{cases} \tag{1.43}$$

where $C_{\tilde{\delta}_0}$ is a positive constant depending only on $\tilde{\delta}_0$. We decompose the functions u_1 and u_2 in the following way

$$u_1 = \hat{u}_1 + \tilde{u}_1; \qquad u_2 = \hat{u}_2 + \tilde{u}_2, \qquad \hat{u}_1, \hat{u}_2 \in L^\infty(\Sigma). \tag{1.44}$$

The explicit decomposition (via some truncation in the Fourier modes) will be chosen later on. Using Lemma 1.4.7, for any $b \in 2, \dots, \ell$ we can write that

$$\begin{aligned}
\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &= \log \left[\left(\int_\Sigma e^{u_1} dV_g \int_\Sigma e^{u_2} dV_g \right) \left(\int_\Sigma e^{u_1} dV_g \right)^{\ell-1} \right] \\
&\leq \left[\left(\int_{\tilde{S}_1} e^{u_1} dV_g \int_{\tilde{S}_1} e^{u_2} dV_g \right) \left(\int_{\tilde{S}_b} e^{u_1} dV_g \right)^{\ell-1} \right] - \ell \log \tilde{\gamma}_0 \\
&\leq \log \left[\left(\int_\Sigma e^{g_1 u_1} dV_g \int_\Sigma e^{g_1 u_2} dV_g \right) \left(\int_\Sigma e^{g_b u_1} dV_g \right)^{\ell-1} \right] \\
&- \ell \log \tilde{\gamma}_0,
\end{aligned}$$

where C is independent of u_1 and u_2.

Now, using the fact that $\hat{u}_1$ and $\hat{u}_2$ belong to $L^\infty(\Sigma)$, we also write

$$\begin{aligned}
\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \log \left[\left(\int_\Sigma e^{g_1 \tilde{u}_1} dV_g \int_\Sigma e^{g_1 \tilde{u}_2} dV_g \right) \left(\int_\Sigma e^{g_b \tilde{u}_1} dV_g \right)^{\ell-1} \right] \\
&- \ell \log \tilde{\gamma}_0 + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g &\leq \log \int_\Sigma e^{g_1 \tilde{u}_1} dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} dV_g + (\ell - 1) \int_\Sigma e^{g_b \tilde{u}_2} dV_g \\
&- \ell \log \tilde{\gamma}_0 + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).
\end{aligned} \tag{1.45}$$

At this point we can use Theorem 1.3.6 with parameters $(4\pi, 4\pi)$, applied to the couple $(g_1 \tilde{u}_1, g_1 \tilde{u}_2)$, and the standard Moser-Trudinger inequality (1.42) to $g_b \tilde{u}_1$ to get the following estimates

$$\begin{aligned}
\log \int_\Sigma e^{g_1 \tilde{u}_1} dV_g + \log \int_\Sigma e^{g_1 \tilde{u}_2} dV_g &\leq \frac{1}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla (g_1 \tilde{u}_i) \cdot \nabla (g_1 \tilde{u}_j) dV_g \right] \\
&+ \left(\overline{g_1 \tilde{u}_1} + \overline{g_1 \tilde{u}_2} \right) + C;
\end{aligned} \tag{1.46}$$

$$(\ell - 1) \int_\Sigma e^{g_b \tilde{u}_1} dV_g \leq \frac{(\ell - 1)}{16\pi} \int_\Sigma |\nabla (g_b \tilde{u}_1)|^2 dV_g + (\ell - 1) \overline{g_b \tilde{u}_1} + (\ell - 1) C.$$

Now we notice that for $N = 2$ one has

$$a^{ij} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Therefore, using elementary inequalities (completion of squares) one can check that for every point $x \in \Sigma$ there holds

$$\frac{1}{2} \sum_{i,j} a^{ij} g(\xi_i, \xi_j) \geq \frac{1}{4} g(\xi_1, \xi_1) \qquad \text{for every couple } (\xi_1, \xi_2) \in T_x\Sigma \times T_x\Sigma. \tag{1.47}$$

This can be checked for example using orthonormal coordinates at x, so that the metric g just becomes the identity at this point. Applying this inequality to the couple $(\nabla(g_b \tilde{u}_1), \nabla(g_b \tilde{u}_2))$ and integrating one finds

$$\frac{(\ell - 1)}{16\pi} \int_\Sigma |\nabla (g_b \tilde{u}_1)|^2 dV_g \leq \frac{(\ell - 1)}{4\pi} \left[\frac{1}{2} \sum_{i,j=1}^{2} \int_\Sigma a^{ij} \nabla (g_b \tilde{u}_i) \cdot \nabla (g_b \tilde{u}_j) dV_g \right]. \tag{1.48}$$

Putting together (1.45)-(1.48) we then obtain

$$\begin{aligned}\ell\log\int_\Sigma e^{u_1}dV_g+\log\int_\Sigma e^{u_2}dV_g &\leq \frac{1}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_1\tilde{u}_i)\cdot\nabla(g_1\tilde{u}_j)dV_g\right]\\ &+ \frac{(\ell-1)}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_b\tilde{u}_i)\cdot\nabla(g_b\tilde{u}_j)dV_g\right] \qquad (1.49)\\ &+ \left(\overline{g_1\tilde{u}_1}+\overline{g_1\tilde{u}_2}\right)+(\ell-1)\overline{g_b\tilde{u}_1}+\ell C-\ell\log\tilde{\gamma}_0\\ &+ \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)}+\|\hat{u}_2\|_{L^\infty(\Sigma)}).\end{aligned}$$

Now we notice that, by interpolation, for any $\varepsilon>0$ there exists $C_{\varepsilon,\tilde{\delta}_0}$ (depending only on ε and $\tilde{\delta}_0$) such that

$$\begin{aligned}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla(g_1\tilde{u}_i)\cdot\nabla(g_1\tilde{u}_j)dV_g\right] &\leq \left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_1^2a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_jdV_g\right]\\ &+ \varepsilon\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_jdV_g\right]+C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g.\end{aligned}$$

Inserting this inequality into (1.49) we get

$$\begin{aligned}\ell\log\int_\Sigma e^{u_1}dV_g+\log\int_\Sigma e^{u_2}dV_g &\leq \frac{1}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_1^2a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_jdV_g\right]\\ &+ \frac{(\ell-1)}{4\pi}\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_b^2a^{ij}\nabla u_i\cdot\nabla\tilde{u}_jdV_g\right]\\ &+ \frac{\ell}{4\pi}\varepsilon\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_jdV_g\right]+\ell C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g\\ &+ \left(\overline{g_1\tilde{u}_1}+\overline{g_1\tilde{u}_2}\right)+(\ell-1)\overline{g_b\tilde{u}_1}+\ell C-\ell\log\tilde{\gamma}_0\\ &+ \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)}+\|\hat{u}_2\|_{L^\infty(\Sigma)}),\end{aligned}$$

for $b=2,\dots,\ell$.

We now choose $b\in\{2,\dots,\ell\}$ such that

$$\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma g_b^2a^{ij}\nabla u_i\cdot\nabla\tilde{u}_jdV_g\leq\frac{1}{\ell-1}\frac{1}{2}\sum_{i,j=1}^{2}\int_{\cup_{s=1+1}^{\ell}supp(g_s)}a^{ij}\nabla u_i\cdot\nabla\tilde{u}_jdV_g.$$

Since the $g_i's$ have disjoint supports, see (1.43), the last formula yields

$$\begin{aligned}\ell\log\int_\Sigma e^{u_1}dV_g+\log\int_\Sigma e^{u_2}dV_g &\leq \frac{1}{4\pi}(1+\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i\cdot\nabla\tilde{u}_jdV_g\right]\\ &+ \ell C_{\varepsilon,\tilde{\delta}_0}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g+\left(\overline{g_1\tilde{u}_1}+\overline{g_1\tilde{u}_2}\right)+(\ell-1)\overline{g_b\tilde{u}_1}\\ &+ \ell C-\ell\log\tilde{\gamma}_0+\ell(\|\hat{u}_1\|_{L^\infty(\Sigma)}+\|\hat{u}_2\|_{L^\infty(\Sigma)}).\end{aligned}$$

Now, by elementary estimates we find

$$\begin{aligned}\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g \quad &\leq \quad \frac{1}{4\pi}(1+\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i \cdot \nabla\tilde{u}_j dV_g\right] \\ &+ \quad C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g \\ &+ \quad C_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0} + \ell(\|\hat{u}_1\|_{L^\infty(\Sigma)} + \|\hat{u}_2\|_{L^\infty(\Sigma)}).\end{aligned}$$

Now comes the choice of $\hat{u}_1$ and $\hat{u}_2$, see (1.44). We choose $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$ to be so large that the following property holds

$$C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma(v_1^2+v_2^2)dV_g < \frac{\varepsilon}{2}\int_\Sigma a^{ij}\nabla v_i\cdot\nabla v_j dV_g, \qquad \forall v_1, v_2 \in V_{\varepsilon,\tilde{\delta}_0,\ell},$$

where $V_{\varepsilon,\tilde{\delta}_0,\ell}$ denotes the span of the eigenfunctions of the Laplacian on Σ corresponding to eigenvalues bigger than $\tilde{C}_{\varepsilon,\tilde{\delta}_0,\ell}$.

Then we set

$$\hat{u}_i = P_{V_{\varepsilon,\tilde{\delta}_0,\ell}} u_i; \qquad\qquad \tilde{u}_i = P_{V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}} u_i,$$

where $P_{V_{\varepsilon,\tilde{\delta}_0,\ell}}$ (resp. $P_{V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}}$) stands for the orthogonal projection onto $V_{\varepsilon,\tilde{\delta}_0,\ell}$ (resp. $V^\perp_{\varepsilon,\tilde{\delta}_0,\ell}$). Since $\overline{u}_i = 0$, the H^1-norm and the L^∞-norm on $V_{\varepsilon,\tilde{\delta}_0,\ell}$ are equivalent (with a proportionality factor which depends on $\varepsilon, \tilde{\delta}_0$ and ℓ), hence by our choice of u_1 and u_2 there holds

$$\|\hat{u}_i\|^2_{L^\infty(\Sigma)} \leq \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell}\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla u_i\cdot\nabla u_j dV_g; \qquad C_{\varepsilon,\tilde{\delta}_0,\ell}\int_\Sigma(\tilde{u}_1^2+\tilde{u}_2^2)dV_g < \frac{\varepsilon}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla v\cdot\nabla v_j dV_g.$$

Hence the last formulas imply

$$\begin{aligned}\ell \log \int_\Sigma e^{u_1} dV_g + \log \int_\Sigma e^{u_2} dV_g \quad &\leq \quad \frac{1}{4\pi}(1+3\ell\varepsilon)\left[\frac{1}{2}\sum_{i,j=1}^{2}\int_\Sigma a^{ij}\nabla\tilde{u}_i \cdot \nabla\tilde{u}_j dV_g\right] \\ &+ \quad \hat{C}_{\varepsilon,\tilde{\delta}_0,\ell,\tilde{\gamma}_0}.\end{aligned}$$

This concludes the proof. ■

Proof of Lemma 1.4.7. First of all we fix a number $r_0 < \frac{\delta_0}{80}$. Then we cover Σ with a finite union of metric balls $(B_{r_0}(x_l))_l$. The number of these balls can be bounded by an integer N_{r_0} which depends only on r_0 (and Σ).

Next we cover the closure $\overline{S}_i$ of every set S_i by a finite number of these balls, and we choose a point $y_i \in \cup_l\{x_l\}$ such that

$$\int_{B_{r_0}(y_i)} e^{u_1} dV_g = \max\left\{\int_{B_{r_0}(x_l)} e^{u_1} dV_g \ : \ B_{r_0}(x_l)\cap\overline{S}_i \neq \emptyset\right\}.$$

We also choose $y \in \cup_l\{x_l\}$ such that

$$\int_{B_{r_0}(y)} e^{u_2} dV_g = \max_l \int_{B_{r_0}(x_l)} e^{u_2} dV_g.$$

Since the total number of balls is bounded by N_{r_0} and since by our assumption the integral of e^{u_1} over S_i is greater or equal than γ, it follows that

$$\frac{\int_{B_{r_0}(y_i)} e^{u_1} dV_g}{\int_\Sigma e^{u_1} dV_g} \geq \frac{\gamma}{N_{r_0}}; \qquad\qquad \frac{\int_{B_{r_0}(y)} e^{u_2} dV_g}{\int_\Sigma e^{u_2} dV_g} \geq \frac{1}{N_{r_0}}. \tag{1.50}$$

By the properties of the sets S_i, we have that

$$B_{20r_0}(y_i) \cap B_{r_0}(y_j) \text{ for } i \neq j; \qquad card\{y_s \ : \ B_{r_0}(y_s) \cap B_{20r_0}(y) \neq \emptyset\} \leq 1.$$

In other words, if we fix y_i, the ball $B_{20r_0}(y_i)$ intersects no other of the balls $B_{r_0}(y_j)$ except $B_{r_0}(y_i)$, and given y, $B_{20r_0}(y)$ intersects at most one of the balls $B_{r_0}(y_i)$.

Now, by a relabeling of the points, we can assume that one of the following two possibilities occur

(a) $B_{20r_0}(y) \cap B_{r_0}(y_1) \neq \emptyset$ (and hence that $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for $i > 1$)

(b) $B_{20r_0}(y) \cap B_{r_0}(y_i) = \emptyset$ for every $i = 1, \dots, \ell$.

In case **(a)** we define the sets $\tilde{S}_i$ as

$$\tilde{S}_i = B_{30r_0}(y_i), \qquad \text{for } i = 1, \dots, \ell.$$

while in case **(b)** we define

$$\tilde{S}_i = \begin{cases} B_{10r_0}(y_1) \cup B_{10r_0}(y) & \text{for } i = 1; \\ B_{10r_0}(y_i), & \text{for } i = 2 \dots \ell, \end{cases}$$

We also set $\tilde{\gamma}_0 = \frac{\gamma}{N_{r_0}}$ and $\tilde{\delta}_0 = 5r_0$. We notice that $\tilde{\gamma}_0$ and $\tilde{\delta}_0$ depend only on γ, δ_0 and Σ, as claimed, and that the sets $\tilde{S}_i$ satisfy the required conditions. This concludes the proof of the lemma. ■

1.5 Existence of extremals for Fontana and Chang-Yang inequalities

In this Section, we give the proof of the existence of extremals for Fontana's inequality in the particular case $n = 4$ and also for Chang-Yang's one. As said in Remark 0.2.3, we will give only a full proof of Theorem 0.2.1 and a sketch of the proof of Theorem 0.2.2.
We start with the following Lemma:

Lemma 1.5.1. *Let α_k be an increasing sequence converging to $32\pi^2$. Then for every k there exists $u_k \in \mathcal{H}_1$ such that*

$$\int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in \mathcal{H}_1} \int_M e^{\alpha_k u^2} dV_g.$$

Moreover u_k satisfies the following equation

$$\Delta_g^2 u_k = \frac{1}{\lambda_k} u_k e^{\alpha_k u_k^2} - \gamma_k \tag{1.51}$$

where

$$\lambda_k = \int_M u_k^2 e^{\alpha_k u_k^2} dV_g$$

and

$$\gamma_k = \frac{1}{\lambda_k Vol_g(M)} \int_M u_k e^{\alpha_k u_k^2} dV_g.$$

Moreover we have $u_k \in C^\infty(M)$.

PROOF. First of all using the inequality in Theorem 1.2.1, one can check easily that the functional

$$I_k(u) = \int_M e^{\alpha_k u^2} dV_g;$$

is weakly ccontinuous Hence using Direct Methods of the Calculus of Variations we get the existence of maximizer say u_k. On the other hand using the Lagrange multiplier rule one get the equation (1.51). Moreover integrating the equation (1.51) and after mmultiplyingit by u_k and integrating again, we get the value of γ_k and λ_k respectively. Moreover using standard elliptic regularity we get that $u_k \in C^\infty(M)$. Hence the Lemma is proved. ■

Now we are ready to give the proof of Theorem 0.2.1. From now on we suppose by contradiction that Theorem 0.2.1 does not hold. Hence from the same considerations as in the Introduction we have that :
1)

$$\forall \alpha > 32\pi^2 \quad \lim_{k \to +\infty} \int_M e^{\alpha u_k^2} dV_g \to +\infty \tag{1.52}$$

2)

$$c_k = \max_M |u_k| = |u_k|(x_k) \to +\infty$$

We will divide the reminder of the proof into six Subsections.

1.5.1 Concentration behavior and profile of u_k

This Subsection is concerned about two main ingredients. The first one is the study of the concentration phenomenon of the energy corresponding to u_k. The second one is the description of the profile of $\beta_k u_k$ as $k \to +\infty$, where β_k is given by the relation

$$1/\beta_k = \int_M \frac{|u_k|}{\lambda_k} e^{\alpha_k u_k^2} dV_g.$$

We start by giving an energy concentration lemma which is inspired from P.L.Lions'work.

Lemma 1.5.2. u_k *verifies :*

$$u_k \rightharpoonup 0 \ \ in \ \ H^2(M);$$

and

$$|\Delta_g u_k|^2 \rightharpoonup \delta_{x_0}$$

for some $x_0 \in M$.

PROOF. First of all from the fact that $u_k \in \mathcal{H}_1$ we can assume without loss of generality that

$$u_k \rightharpoonup u_0 \ \text{ in } \ H^2(M). \tag{1.53}$$

Now let us show that $u_0 = 0$.
We have the trivial identity

$$\int_M |\Delta_g(u_k - u_0)|^2 dV_g = \int_M |\Delta_g u_k|^2 dV_g + \int_M |\Delta_g u_0|^2 dV_g - 2\int_M \Delta_g u_k \Delta_g u_0 dV_g.$$

Hence using the fact that $\int_M |\Delta_g u_k|^2 dV_g = 1$ we derive

$$\int_M |\Delta_g(u_k - u_0)|^2 dV_g = 1 + \int_M |\Delta_g u_0|^2 dV_g - 2\int_M \Delta_g u_k \Delta_g u_0 dV_g$$

So using (1.53) we get

$$\lim_{k\to 0}\int_M |\Delta_g(u_k-u_0)|^2 dV_g = 1-\int_M \Delta_g u_0\Delta_g u_0 dV_g$$

Now suppose that $u_0\neq 0$ and let us argue for a contradiction. Then there exists some $\beta<1$ such that for k large enough the following holds

$$\int_M |\Delta_g(u_k-u_0)|^2 dV_g < \beta.$$

Hence using Fontana's result see Theorem 1.2.1 we obtain that

$$\int_M e^{\alpha_1(u_k-u_0)^2} dV_g \leq C \text{ for some } \alpha_1 > 32\pi^2.$$

Now using Cauchy inequality one can check easily that

$$\int_M e^{\alpha_2 u_k^2} dV_g \leq C \text{ for some } \alpha_2 > 32\pi^2.$$

Hence reaching a contradiction to (1.52).
On the other hand without lost of generality we can assume that

$$|\Delta_g u_k| dV_g \rightharpoonup \mu.$$

Now suppose $\mu\neq\delta_p$ for every $p\in M$ and let us argue for a contradiction to (1.52) again. First of all let us take a cut-off function $\eta\in C_0^\infty(B_x(\delta))$, $\eta=1$ on $B_x(\frac{\delta}{2})$ where x is a fixed point in M and δ a fixed positive and small number.
We have that

$$\limsup_{k\to+\infty}\int_{B_x(\delta)} |\Delta_g u_k|^2 dV_g < 1.$$

Now working in a normal coordinate system around x and using standard elliptic regularity theory we get

$$\int_{B^{\tilde{x}}(\delta)} |\Delta_0 \widetilde{\eta u_k}|^2 dV_g \leq (1+o_\delta(1))\int_{B_x(\delta)} |\Delta_g u_k|^2 dV_g;$$

where $\tilde{x}$ is the point corresponding to x in $\mathbb{R}^4$ and $\widetilde{\eta u_k}$ the expression of ηu_k on the normal coordinate system. Hence for δ small we get

$$\int_{B^{\tilde{x}}(\delta)} |\Delta_0 \widetilde{\eta u_k}|^2 dV_g < 1$$

Thus using the Adams result see [1] we have that

$$\int_{B^{\tilde{x}}(\delta)} e^{\tilde{\alpha}\widetilde{(\eta u_k)^2}} dx \leq C \text{ for some } \tilde{\alpha} > 32\pi^2.$$

Hence using a covering argument we infer that

$$\int_M e^{\bar{\alpha} u_k^2} dV_g \leq C \text{for some} \bar{\alpha} > 32\pi^2,$$

so reaching a contradiction. Hence the Lemma is proved. ■

Lemma 1.5.3. *We have the following hold:*

$$\lim_{k\to+\infty}\lambda_k = +\infty, \qquad \lim_{k\to+\infty}\gamma_k = 0.$$

PROOF. Let $N > 0$ be large eenough By using the definition of λ_k we have that

$$\lambda_k = \int_M u_k^2 e^{\alpha_k u_k^2} dV_g \geq N^2 \int_{\{u_k \geq N\}} e^{\alpha_k u_k^2} dV_g = N^2 (\int_M e^{\alpha_k u_k^2} dV_g - \int_{\{u_k \leq N\}} e^{\alpha_k u_k^2} dV_g).$$

On the other hand

$$\lim_{k \to +\infty} \left(\int_M e^{\alpha_k u_k^2} dV_g - \int_{\{u_k \leq N\}} e^{\alpha_k u_k^2} dV_g \right) = \lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g - Vol_g(M).$$

Hence using the fact that

$$\lim_{k \to +\infty} \int_M e^{\alpha_k u_k^2} dV_g = \sup_{u \in \mathcal{H}_1} \int_M e^{32\pi^2 u^2} dV_g > Vol_g(M)$$

we have that 1) holds. Now we prove 2). using the definition of γ_k , we get

$$|\gamma_k| \leq \frac{N}{\lambda_k} N e^{32\pi^2 N^2} + \frac{1}{Vol_g(M)} \frac{1}{N}.$$

Hence by using point 1 and letting $k \to +\infty$ and after $N \to +\infty$ we get point 2. So the Lemma is proved. ■

Next let us set

$$\tau_k = \int_M \frac{\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2}.$$

One can check easily the following

Lemma 1.5.4. *With the definition above we have that* $0 \leq \beta_k \leq c_k$, $|\tau_k| \leq 1$ *and* $\beta_k \gamma_k$ *is bounded. Moreover up to a subsequence and up to changing* u_k *to* $-u_k$

$$\tau_k \to \tau \geq 0.$$

The next Lemma gives some Lebesgue estimates on Ball in terms of the radius with constant independent of the ball. As a corollary we get the profile of $\beta_k u_k$ as $k \to +\infty$.

Lemma 1.5.5. *There are constants* $C_1(p)$,*and* $C_2(p)$ *depending only on* p *and* M *such that, for* r *sufficiently small and for any* $x \in M$ *there holds*

$$\int_{B_x(r)} |\nabla_g^2 \beta_k u_k|^p dV_g \leq C_2(p) r^{4-2p};$$

and

$$\int_{B_x(r)} |\nabla_g \beta_k u_k|^p dV_g \leq C_1(p) r^{4-p}$$

where, respectively, $p < 2$, *and* $p < 4$.

PROOF. First of all using the Green representation formula we have

$$u_k(x) = \int_M F(x,y) \Delta_g^2 u_k dV_g(y) \quad \forall x \in M.$$

Hence using the equation we get

$$u_k(x) = \int_M F(x,y) \left(\frac{1}{\lambda_k} u_k e^{\alpha_k u_k^2} \right) dV_g(y) - \int_M F(x,y) \gamma_k dV_g(y).$$

Now by differentiating with respect to x for every $m = 1,2$ we have that

$$|\nabla_g^m u_k(x)| \leq \int_M |\nabla_g^m F(x,y)| \left(\frac{1}{\lambda_k} \right) |u_k| e^{\alpha_k u_k^2} dV_g(y) + \int_M |\nabla_g^m F(x,y)| \, |\gamma_k| \, .$$

Hence we get

$$|\nabla_g^m(\beta_k u_k(x))| \le \int_M |\nabla_g^m F(x,y)|\beta_k \left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g(y) + \int_M |\nabla_g^m F(x,y)|\beta_k\,|\gamma_k|\,.$$

Taking the p-th power in both side of the inequality and using the basic inequality

$$(a+b)^p \le 2^{p-1}(a^p+b^p) \quad \text{for} \quad a \ge 0 \quad \text{and} \quad b \ge 0$$

we obtain

$$|\nabla_g^m(\beta_k u_k(x))|^p \le 2^{p-1}\left[\int_M |\nabla_g^m F(x,y)|\beta_m \left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g(y)\right]^p + 2^{p-1}\left[\int_M |\nabla_g^m F(x,y)|\beta_k\,|\gamma_k|\right]^p$$

Now integrating both sides of the inequality we obtain

$$\int_{B_x(r)} |\nabla_g^m(\beta_k u_k(z))| dV_g(z) \le 2^{p-1}\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|\beta_k \left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g(y)\right]^p dV_g(z) + 2^{p-1}\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|\beta_k\,|\gamma_k|\right]^p dV_g(z).$$

First let us estimate the second term in the right hand side of the inequality

$$\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|\beta_k\,|\gamma_k|\right]^p dV_g(z) \le C\int_{B_x(r)} \sup_{y\in M}\frac{1}{d_g(z,y)^{pm}}dV_g(z) \le C(M)r^{4-mp}$$

Thanks to the fact that $\beta_k\gamma_k$ is bounded, to the asymptotics of the Green function and to Jensen's inequality. Now let us estimates the second term. First of all we define the following auxiliary measure

$$m_k = \beta_k\left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g$$

We have that m_k is a probability measure. On the other hand we can write

$$\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|\beta_k \left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g(y)\right]^p dV_g(z) = \int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|dm_k(y)\right]^p dV_g(z). \tag{1.54}$$

Now by using Jensen's inequality we have that

$$\left[\int_M |\nabla_g^m F(z,y)|dm_k(y)\right]^p \le \left[\int_M |\nabla_g^m F(z,y)|^p dm_k(y)\right]$$

Thus with the (1.54) we have that

$$\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|\beta_k \left(\frac{1}{\lambda_k}\right)|u_k|e^{\alpha_k u_k^2}dV_g(y)\right]^p dV_g(z) \le \int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|^p dm_k(y)\right]dV_g(z).$$

Now by using again the same argument as in the first term we obtain

$$\int_{B_x(r)}\left[\int_M |\nabla_g^m F(z,y)|^p dm_k(y)\right]dV_g(z) \le C(M)r^{4-mp}.$$

Hence the Lemma is proved. ■

Next we give a corollary of this Lemma.

Corollary 1.5.6. *We have* $\beta_k u_k \rightharpoonup G \;\; W^{2,p}(M)$ *for* $p \in (1,2)$, $\beta_k u_k \rightharpoonup G$ *smoothly in* $M \setminus B_{x_0}(\delta)$ *where* δ *is small and* G *satisfies*

$$\begin{cases} \Delta_g^2 G = \tau(\delta_{x_0} - \frac{1}{Vol_g(M)}) \;\; in \;\; M; \\ \overline{G} = 0 \end{cases}$$

Moreover

$$G(x) = \frac{\tau}{8\pi^2}\log\frac{1}{r} + \tau S(x)$$

with $r = d_g(x, x_0)$. $S = S_0 + S_1(x)$, $S_0 = S(x_0)$ *and* $S \in W^{2,q}(M)$ *for every* $q \geq 1$.

Proof. By Lemma 1.5.5 we have that

$$\beta_k u_k \rightharpoonup G \;\; W^{2,p}(M) \;\; p \in (1,2)$$

On the other hand using Lemma 1.5.2 we get $e^{\alpha_k u_k^2}$ is bounded in $L^p(M \setminus B_{x_0}(\delta))$. Hence the standard elliptic regularity implies that

$$\beta_k u_k \to G \;\text{ smoothly in }\; M \setminus B_{x_0}(\delta). \tag{1.55}$$

So to end the proof of the proposition we need only to show that

$$\frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} \rightharpoonup \tau\delta_{x_0}. \tag{1.56}$$

To do this let us take $\varphi \in C^\infty(M)$ then we have

$$\int_M \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = \int_{M\setminus B_{x_0}(\delta)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g + \int_{B_{x_0}(\delta)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g$$

Using (1.55) we have that

$$\int_{M\setminus B_{x_0}(\delta)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = O(\frac{1}{\lambda_k}).$$

On the other hand, we can write inside the ball $B_{x_0}(\delta)$

$$\begin{aligned} \textstyle\int_{B_{x_0}(\delta)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g &= (\varphi(x_0) + o_\delta(1)) \int_{B_{x_0}(\delta)} \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g \\ &= (\varphi(x_0) + o_\delta(1)) \left(\tau - \int_{M\setminus B_{x_0}(\delta)} \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g \right) \end{aligned}$$

Now using again (1.55) we derive

$$\int_{M\setminus B_{x_0}(\delta)} \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} = O(\frac{1}{\lambda_k}).$$

Hence we arrive to

$$\int_{B_{x_0}(\delta)} \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = \tau\varphi(x_0) + o_{k,\delta}(1).$$

Thus we get

$$\int_M \varphi \frac{\beta_k}{\lambda_k} u_k e^{\alpha_k u_k^2} dV_g = O(\frac{1}{\lambda_k}) + \tau\varphi(x_0) + o_{k,\delta}(1).$$

Hence from Lemma 1.5.3 we conclude the proof of claim (1.56))and of the Corollary too. ■

1.5.2 Pohozaev type identity and application

As it is already said in the introduction this Subsection deals with the derivation of a Pohozaev type identity. And as corollary we give the limit of $\int_M e^{\alpha_k u_k^2} dV_g$ in terms of $Vol_g(M)$, λ_k, β_k and τ

Lemma 1.5.7. *Setting* $U_k = \Delta_g u_k$ *we have the following holds*

$$-\frac{2}{\alpha_k\lambda_k}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} dV_g = -\frac{\delta}{2}\int_{\partial B_{x_k}(\delta)} U_k^2 dS_g - \delta\int_{\partial B_{x_k}(\delta)} \nabla_g u_k \nabla_g U_k dV_g + 2\int_{\partial B_{x_k}(\delta)} U_k \frac{\partial u_k}{\partial r}$$
$$+2\delta\int_{\partial B_{x_k}(\delta)} \frac{\partial U_k}{\partial r}\frac{\partial u_k}{\partial r} dS_g + \int_{B_{x_k}(\delta)} O(r^2)\nabla_g u_k \nabla_g U_k dV_g$$
$$+\int_{B_{x_k}(\delta)} O(r^2) U_k^2 dV_g + \int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} O(r^2) dV_g - \frac{\delta}{2\lambda_k\alpha_k}\int_{\partial B_{x_k}(\delta)} e^{\alpha_k u_k^2} dV_g + O(\frac{\delta}{\beta_k^2}).$$

where δ *is small and fixed real number.*

PROOF. The proof relies on the divergence formula and the asymptotics of the metric g in normal coordinates around x_k.
By the definition of U_k we have that

$$\begin{cases} \Delta_g u_k = U_k \\ \Delta_g U_k = \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} - \gamma_k. \end{cases}$$

The first issue is to compute $\int_{B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\Delta_g u_k$ in two different ways, where $r(x) = d_g(x, x_k)$. On one side we obtain

$$\int_{B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\Delta_g u_k dV_g = -\int_{B_{x_k}(\delta)} (\nabla_g U_k \nabla_g u_k + r\frac{\partial \nabla_g U_k}{\partial r}\nabla_g u_k) dV_g + \int_{\partial B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\frac{\partial u_k}{\partial r} dS_g.$$

On the other side we get

$$\begin{aligned}\int_{B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\Delta_g u_k dV_g &= \int_{B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r} U_k dV_g \\ &= \int_0^\delta 2\pi^2 \int_{\partial B_r(x_k)} \frac{\partial U_k}{\partial r} U_k \sqrt{|g|} r^4 dS dr \\ &= \tfrac{\delta}{2}\int_{\partial B_{x_k}(\delta)} U_k^2 dS_g - 2\int_{B_{x_k}(\delta)} U_k^2(1+O(r^2)) dV_g.\end{aligned}$$

Thus we have

$$\begin{aligned}\tfrac{\delta}{2}\int_{\partial B_{x_k}(\delta)} U_k^2 dS_g - 2\int_{B_{x_k}(\delta)} U_k^2 dV_g &= -\int_{B_{x_k}(\delta)} (\nabla_g U_k \nabla_g u_k + r\frac{\partial \nabla_g U_k}{\partial r}\nabla_g u_k) dV_g \\ &\quad + \int_{\partial B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\frac{\partial v_k}{\partial r} dS_g + \int_{B_{x_k}(\delta)} O(r^2) U_k^2 dV_g\end{aligned}$$

In the same way we obtain

$$\tfrac{\delta}{2\lambda_k\alpha_k}\int_{\partial B_{x_k}(\delta)} e^{\alpha_k u_k^2} dS_g - \frac{2}{\lambda_k\alpha_k}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2}(1+O(r^2)) dV_g$$
$$\sigma\sigma\sigma = -\int_{B_{x_k}(\delta)} (\nabla_g U_k \nabla_g u_k + r\frac{\partial \nabla_g u_k}{\partial r}\nabla_g U_k) dV_g + \int_{\partial B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\frac{\partial u_k}{\partial r} dS_g + O(\frac{\delta}{\beta_k^2}).$$

Hence by summing this two last lines we arrive to

$$\begin{aligned}
&\tfrac{\delta}{2\lambda_k\alpha_k}\int_{\partial B_{x_k}(\delta)} e^{\alpha_k u_k^2} dS_g - \frac{2}{\lambda_k\alpha_k}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} dV_g + \frac{\delta}{2}\int_{\partial B_{x_k}(\delta)} U_k^2 dS_g - 2\int_{B_{x_k}(\delta)} U_k^2 dV_g \\
&\sigma\sigma = -\int_{B_{x_k}(\delta)} (2\nabla_g U_k \nabla_g u_k + r\frac{\partial}{\partial r}\nabla_g u_k \nabla_g U_k) dV_g + 2\int_{\partial B_{x_k}(\delta)} r\frac{\partial U_k}{\partial r}\frac{\partial u_k}{\partial r} dS_g \\
&\sigma\sigma\sigma\sigma + \int_{B_{x_k}(\delta)} O(r^2) U_k^2 dV_g + \int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} O(r^2) dV_g + O(\frac{\delta}{\beta_k^2}).
\end{aligned} \tag{1.57}$$

On the other hand using the same method one can check easily that

$$\begin{aligned}
\int_{B_{x_k}(\delta)} r\frac{\partial}{\partial r}\nabla_g u_k \nabla_g U_k dV_g &= \delta\int_{\partial B_{x_k}(\delta)} \nabla_g u_k \nabla_g U_k dV_g - 4\int_{B_{x_k}(\delta)} \nabla_g u_k \nabla_g U_k dV_g \\
&+\int_{B_{x_k}(\delta)} O(r^2)\nabla_g u_k \nabla_g U_k dV_g
\end{aligned} \tag{1.58}$$

and

$$\begin{aligned}
\int_{B_{x_k}(\delta)} \nabla_g U_k \nabla_g u_k dV_g &= -\int_{B_{x_k}(\delta)} U_k \Delta_g u_k dV_g + \int_{\partial B_{x_k}(\delta)} U_k \frac{\partial u_k}{\partial r} dS_g \\
&= -\int_{B_{x_k}(\delta)} U_k^2 dV_g + \int_{\partial B_{x_k}(\delta)} U_k \frac{\partial u_k}{\partial r} dS_g,
\end{aligned} \tag{1.59}$$

So using (1.57),(1.58) and (1.59) we arrive to

$$\begin{aligned}
-\tfrac{2}{\alpha_k\lambda_k}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} dV_g &= -\tfrac{\delta}{2}\int_{\partial B_{x_k}(\delta)} U_k^2 dS_g - \delta\int_{\partial B_{x_k}(\delta)} \nabla_g u_k \nabla_g U_k dV_g + 2\int_{\partial B_{x_k}(\delta)} U_k \frac{\partial u_k}{\partial r} \\
&+2\delta\int_{\partial B_{x_k}(\delta)} \frac{\partial U_k}{\partial r}\frac{\partial u_k}{\partial r} dS_g + \int_{B_{x_k}(\delta)} O(r^2)\nabla_g u_k \nabla_g U_k dV_g \\
&+\int_{B_{x_k}(\delta)} O(r^2) U_k^2 dV_g + \int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2} O(r^2) dV_g \\
&-\tfrac{\delta}{2\lambda_k\alpha_k}\int_{\partial B_\delta(x_k)} e^{\alpha_k u_k^2} dV_g + O(\frac{\delta}{\beta_k^2}).
\end{aligned}$$

Thus the Lemma is proved ■

Corollary 1.5.8. *We have that*

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2} = Vol_g(M) + \tau^2 \lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}.$$

Moreover we have that

$$\tau \in (0,1].$$

Proof. First of all we have that the sequence $(\frac{\lambda_k}{\beta_k^2})_k$ is bounded. Indeed using the definition of β_k we have that

$$\frac{\lambda_k}{\beta_k^2} = \frac{1}{\lambda_k}(\int_M |u_k| e^{\alpha_k u_k^2} dV_g)^2.$$

Hence using Jensen's inequality we obtain

$$\frac{\lambda_k}{\beta_k^2} \le \frac{1}{\lambda_k}\int_M e^{\alpha_k u_k^2} dV_g \int_M u_k^2 e^{\alpha_k u_k^2} dV_g.$$

Thus using the definition of λ_k we have that

$$\frac{\lambda_k}{\beta_k^2} \le \int_M e^{\alpha_k u_k^2} dV_g.$$

On the other hand one can check easily that

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = \sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2 u^2}dV_g < \infty.$$

Hence we derive that $(\frac{\lambda_k}{\beta_k^2})_k$ is bounded. So we can suppose without lost of generality that $(\frac{\lambda_k}{\beta_k^2})_k$ converges.
Now from Lemma 1.5.7 we have that

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g &= 16\pi^2\lim_{k\to+\infty}\tfrac{\lambda_k}{\beta_k^2}(\tfrac{\delta}{2}\int_{\partial B_{x_k}(\delta)}(\beta_k U_k)^2 dS_g\\ &+\delta\int_{\partial B_{x_k}(\delta)}\nabla_g(\beta_k u_k)\nabla_g(\beta_k U_k)dS_g - 2\int_{\partial B_{x_k}(\delta)}(\beta_k U_k)\frac{\partial(\beta_k u_k)}{\partial r}\\ &-2\delta\int_{\partial B_{x_k}(\delta)}\frac{\partial(\beta_k U_k)}{\partial r}\frac{\partial(\beta_k u_k)}{\partial r}dS_g + O(\delta)).\end{aligned}$$

So using Lemma 1.5.6 we obtain

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g &= 16\pi^2\lim_{k\to+\infty}\tfrac{\lambda_k}{\beta_k^2}(\tfrac{\delta}{2}\int_{\partial B_{x_0}(\delta)}|\Delta_g G|^2 dS_g\\ &+\delta\int_{\partial B_\delta(x0)}\nabla_g G\nabla_g(\Delta_g G)dS_g - 2\int_{\partial B_{x_0}(\delta)}\Delta_g G\frac{\partial G}{\partial r}\\ &-2\delta\int_{\partial B_{x_0}(\delta)}\frac{\partial\Delta_g G}{\partial r}\frac{\partial G}{\partial r}dS_g + O(\delta)).\end{aligned}$$

Moreover by trivial calculations we get

$$\int_{\partial B_{x_0}(\delta)}|\Delta_g G|^2 dS_g = \frac{\tau^2}{8\pi^2\delta} + O(1);$$

$$\int_{\partial B_{x_0}(\delta)}\nabla_g G\nabla_g(\Delta_g G)dS_g = -\frac{\tau^2}{8\pi^2\delta} + O(1);$$

$$\int_{\partial B_{x_0}(\delta)}\Delta_g G\frac{\partial G}{\partial r} = \frac{\tau^2}{16\pi^2} + O(\delta);$$

and

$$\int_{\partial B_{x_0}(\delta)}\frac{\partial\Delta_g G}{\partial r}\frac{\partial G}{\partial r}dS_g = -\frac{\tau^2}{8\pi^2\delta} + O(1)$$

Hence with this we obtain

$$\lim_{k\to+\infty}\int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g = \tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2} + O(\delta).$$

On the other hand we have that

$$\int_M e^{\alpha_k u_k^2}dV_g = \int_{B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g + \int_{M\setminus B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g$$

Moreover by Lemma 1.5.2 we have that

$$\int_{M\setminus B_{x_k}(\delta)} e^{\alpha_k u_k^2}dV_g = Vol_g(M) + o_{k,\delta}(1).$$

Thus we derive that

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = Vol_g(M) + \tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2} + o_\delta(1).$$

Hence letting $\delta \to 0$ we obtain

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = Vol_g(M) + \tau^2 \lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}.$$

Now suppose $\tau = 0$ then we get

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = Vol_g(M).$$

On the other hand we have that

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = \sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2 u^2}dV_g > Vol_g(M);$$

hence a contradiction. Thus $\tau \neq 0$ and the Corollary is proved. ■

1.5.3 Blow-up analysis

In this Subsection we perform the Blow-up analysis and show that the asymptotic profile of u_k is either the zero function or a standard Bubble.
First of all let us introduce some notations.
We set

$$r_k^4 = \frac{\lambda_k}{\beta_k c_k}e^{-\alpha_k c_k^2}.$$

Now for $x \in B^{r_k^{-1}\delta}(0)$ with $\delta > 0$ small we set

$$w_k(x) = 2\alpha_k\beta_k\left(u_k(exp_{x_k}(r_k x)) - c_k\right);$$

$$v_k(x) = \frac{1}{c_k}u_k(exp_{x_k}(r_k x));$$

$$g_k(x) = (exp_{x_k}^* g)(r_k x).$$

Next we define

$$d_k = \frac{c_k}{\beta_k} \quad d = \lim_{k\to+\infty} d_k.$$

Proposition 1.5.9. *The following hold:*
We have

$$\text{if } \; d < +\infty \;\text{ then }\; w_k \to w(x) := \frac{4}{d}\log\left(\frac{1}{1+\sqrt{\frac{d}{6}}|x|^2}\right) \;\text{ in }\; C^2_{loc}(\mathbb{R}^4);$$

and

$$\text{if } \; d = \infty \;\text{ then }\; w_k \to w = 0 \;\text{ in }\; C^2_{loc}(\mathbb{R}^4).$$

PROOF. *First of all we recall that*

$$g_k \to dx^2 \;\text{ in }\; C^2_{loc}(\mathbb{R}^4).$$

Since $(\frac{\lambda_k}{\beta_k^2})$, $(\frac{\beta_k}{c_k})$ *are bounded and* $c_k \to +\infty$, *then we infer that*

$$r_k \to 0 \;\text{ as }\; k \to 0.$$

Now using the Green representation formula for Δ_g^2 *(see Lemma* 0.3.1*) we have that*

$$u_k(x) = \int_M F(x,y)\Delta_g^2 u_k dV_g(y) \quad \forall x \in M.$$

Now using equation and differentiating with respect to x *we obtain that for* $m = 1, 2$

$$|\nabla_g^m u_k(x)| \leq \int_M \left|\nabla_g^m F(x,y)\right| \left|\frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} - \gamma_k\right| dV_g(y).$$

Hence from the fact that $\beta_k \gamma_k$ *is bounded see Lemma* 1.5.4 *we get*

$$|\nabla_g^m u_k(x)| \leq \int_M \left|\nabla_g^m F(x,y)\right| \left|\frac{u_k}{\lambda_k} e^{\alpha_k u_k^2}\right| dV_g(y) + O(\beta_k^{-1}).$$

Now for $y_k \in B_{x_k}(Lr_k)$, $L > 0$ *fixed we write that*

$$\begin{aligned}\int_M \left|\nabla_g^m F(y_k,y)\right| \frac{|u_k|}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) &= O\left(r_k^{-m} \int_{M \setminus B_{Lr_k}(y_k)} \frac{|u_k|}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y)\right) \\ &+ O\left(\frac{c_k}{\lambda_k} e^{\alpha_k c_k^2} \int_{B_{Lr_k}(y_k)} d_g(y_k,y)^{-m} dV_g(y)\right) \\ &= O(r_k^{-m}\beta_k^{-1}).\end{aligned}$$

thanks to the fact that $|u_k| \leq c_k$ *to the definition of* r_k.
Now it is not worth remarking that $c_k = u_k(x_k)$ *since we have taken* $\tau \geq 0$ *(see Lemma* 1.5.4*).*
Hence we have that

$$w_k(x) \leq w_k(0) = 0 \quad \forall\, x \in \mathbb{R}^4.$$

So we get from the estimate above that w_k *is uniformly bounded in* $C^2(K)$ *for every compact subset* K *of* $\mathbb{R}^4$. *Thus by Arzéla-Ascoli Theorem we infer that*

$$w_k \longrightarrow w \in C^1_{loc}(\mathbb{R}^4).$$

Clearly w *is a Lipschitz function since the constant which bounds the gradient of* w_k *is independent of the compact set* K.
On the other hand from the Green representation formula we have for $x \in \mathbb{R}^4$ *fixed and for* L *big enough such that* $x \in B^0(L)$

$$u_k(exp_{x_k}(r_k x)) = \int_M F(exp_{x_k}(r_k x), y) \Delta_g^2 u_k(y) dV_g(y).$$

Now remarking that

$$u_k(x_k) = u_k(exp_{x_k}(r_k 0));$$

we have that

$$u_k(exp_{x_k}(r_k x)) - u_k(x_k) = \int_M \left(F(exp_{x_\epsilon}(r_k x), y) - F(exp_{x_k}(0), y)\right) \Delta_g^2 u_k(y) dV_g(y).$$

Hence using (1.51) *we obtain*

$$\begin{aligned}u_k(exp_{x_k}(r_k x)) - u_k(x_k) &= \int_M \left(F(exp_{x_k}(r_k x), y) - F(exp_{x_k}(0), y)\right) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) \\ &- \int_M \left(F(exp_{x_k}(r_k x), y) - F(exp_{x_k}(0), y)\right) (\gamma_k) dV_g(y).\end{aligned}$$

Now setting

$$I_k(x) = \int_{B_{x_k}(Lr_k)} \left(F(exp_{x_k}(r_k x), y) - F(exp_{x_k}(0), y)\right) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y);$$

$$II_k(x) = \int_{M \setminus B_{x_k}(Lr_k)} \left(F(exp_{x_k}(r_k x), y) - F(exp_{x_k}(0), y)\right) \frac{u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y)$$

and

$$III_k(x) = \int_M \left(F(exp_{x_k}(r_k x), y) - F(exp_{x_k}(0), y)\right)(\gamma_k) dV_g(y);$$

we find

$$u_k(exp_{x_k}(r_k x)) - u_k(x_k) = I_k(x) + II_k(x) + III_k(x).$$

So using the definition of w_k *we arrive to*

$$w_k = 2\alpha_k\beta_k \left(I_k(x) + II_k(x) + III_k(x)\right).$$

Now to continue the proof we consider two cases:
Case 1: $d < +\infty$
First of all let us study each of the terms $2\alpha_k\beta_k I_k(x)$, $2\alpha_k\beta_k II_k(x)$, $2\alpha_k\beta_k III_k(x)$ *separately. Using the change of variables* $y = exp_{x_k}(r_k z)$ *we have*

$$2\alpha_k\beta_k I_k(x) = \int_{B^L(0)} \left(F(exp_{x_k}(r_k x), exp_{x_k}(r_k z)) - F(exp_{x_k}(0), exp_{x_k}(r_k z))\right) \frac{2\alpha_k\beta_k u_k(exp_{x_k}(r_k z))}{\lambda_k} e^{\alpha_k u_\epsilon^2(exp_{x_k}(r_k z))} r_k^4 dV_{g_k}(z).$$

Hence using the definition of r_k *and* v_k *one can check easily that the following holds*

$$2\alpha_k\beta_k I_k(x) = 2\alpha_k \int_{B^0(L)} \left(G(exp_{x_\epsilon}(r_k x), exp_{x_\epsilon}(r_k z)) - G(exp_{x_\epsilon}(0), exp_{x_\epsilon}(r_k z))\right) v_k(z) e^{\frac{d_k}{2}(w_k(z)(1+v_k))} dV_{g_k}(z).$$

Moreover from the asymptotics of the Green function see Lemma 0.3.1 *we have that*

$$2\alpha_k\beta_k I_k(x) = 2\alpha_k \int_{B^0(L)} \left(\frac{1}{8\pi^2}\log\frac{|z|}{|x-z|} + K_k(x,z)\right) v_k(z) e^{\frac{d_k}{2}(w_k(z)(1+v_k(z)))} dV_{g_k}(z).$$

where

$$K_k(x,z) = \left[K(exp_{x_k}(r_k x), exp_{x_k}(r_k z)) - (K(exp_{x_k}(0), exp_{x_k}(r_k z)\right].$$

Hence since K *is of class* C^1 *on* M^2 *and* $g_k \to dx^2$ *in* $C^2_{loc}(\mathbb{R}^4)$ *and* $v_k \to 1$ *then letting* $k \to +\infty$ *we derive*

$$\lim_{k\to+\infty} 2\alpha_k\beta_k I_k(x) = 8\int_{B^0(L)} \log\frac{|z|}{|x-z|} e^{dw(z)} dz.$$

Now to estimate $\alpha_k\beta_k II_k(x)$ *we write for* k *large enough*

$$\alpha_k\beta_k II_k(x) = \int_{M\setminus B_{x_k}(Lr_k)} \frac{1}{8\pi^2}\log\left(\frac{d_g(exp_{x_k}(0), y)}{d_g(exp_{x_k}(r_k x), y)}\right)\frac{2\alpha_k\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y) + \int_{M\setminus B_{x_k}(Lr_k)} \bar{K}_k(x,y)\frac{2\alpha_k\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y),$$

where

$$\bar{K}_k(x,y) = \left(K(exp_{x_k}(r_k x), y) - K(exp_{x_k}(0), y)\right).$$

Taking the absolute value in both sides of the equality and using the change of variable $y = exp_{x_k}(r_k z)$ *and the fact that* $K \in C^1$ *we obtain,*

$$|2\alpha_k\beta_k II_k(x)| \le \int_{\mathbb{R}^4\setminus B^L(0)} 8\left|\log\left(\frac{|z|}{|x-z|}\right)\right| |v_k|(z) e^{\frac{d_k}{2}(w_k(z)(1+v_k(z)))} dV_{g_k}(z) + Lr_k \int_{M\setminus B_{x_k}(Lr_k)} \frac{2\alpha_k\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g(y).$$

Hence letting $k \to +\infty$ *we deduce that*

$$\limsup_{k\to+\infty} |2\alpha_k\beta_k II_k(x)| = o_L(1).$$

Now using the same method one proves that

$$2\alpha_k\beta_k III_k(x) \to 0 \ \ as \ \ k \to +\infty.$$

So we have that

$$w(x) = \int_{B^L(R)} 8\log\left(\frac{|z|}{|x-z|}\right) e^{dw(z)}dz + \lim_{k\to+\infty} 2\alpha_k\beta_k II_k(x).$$

Hence letting $L \to +\infty$ *we obtain that* w *is a solution of the following integral equation*

$$w(x) = \int_{\mathbb{R}^4} 8\log\left(\frac{|z|}{|x-z|}\right) e^{dw(z)}dz. \tag{1.60}$$

Now since w *is Lipschitz then the theory of singular integral operator gives that* $w \in C^1(\mathbb{R}^4)$. *Since*

$$\lim_{k\to+\infty} \int_{B_{x_k}(Lr_k)} \frac{2\alpha_k\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g = 64\pi^2 \int_{B^0(L)} e^{dw(x)}dx.$$

and

$$\int_{B_{x_k}(Lr_k)} \frac{2\alpha_k\beta_k u_k}{\lambda_k} e^{\alpha_k u_k^2} dV_g \le 64\pi^2,$$

then we get

$$\lim_{L\to+\infty} \int_{B^0(L)} e^{dw(x)}dx = \int_{\mathbb{R}^4} e^{dw(x)}dx \le 1.$$

Now setting

$$\tilde{w}(x) = \frac{d}{4}w(x) + \frac{1}{4}\log(\frac{8\pi^2 d}{3});$$

we have that $\tilde{w}$ *satisfies the following conformally invariant integral equation*

$$\tilde{w}(x) = \int_{\mathbb{R}^4} \frac{6}{8\pi^2}\log\left(\frac{|z|}{|x-z|}\right) e^{\tilde{w}(z)}dz + \frac{1}{4}\log(\frac{8\pi^2 d}{3}), \tag{1.61}$$

and

$$\int_{\mathbb{R}^4} e^{4\tilde{w}(x)}dx < +\infty.$$

Hence from the classification result by X.Xu see Theorem 1.2 in [89] we derive that

$$\tilde{w}(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x-x_0|^2}\right)$$

for some $\lambda > 0$ *and* $x_0 \in \mathbb{R}^4$.
¿From the fact that

$$w(x) \le w(0) = 0 \quad \forall x \in \mathbb{R}^4;$$

we obtain

$$\tilde{w}(x) \le \tilde{w}(0) = \frac{1}{4}\log(\frac{8\pi^2 d}{3}) \quad \forall x \in \mathbb{R}^4.$$

Then we derive

$$x_0 = 0, \ \ \lambda = 2(\frac{8\pi^2 d}{3})^{-\frac{1}{4}}$$

Hence by trivial calculations we get

$$w(x) = \frac{4}{d}\log\left(\frac{1}{1+\sqrt{\frac{d}{6}}|x|^2}\right).$$

Case 2: $d = +\infty$.
In this case using the same argument we get

$$\limsup_{k\to+\infty} |\alpha_k\beta_k II_k(x)| = o_L(1);$$

and

$$\alpha_k\beta_k III_k(x) = o_k(1),$$

Now let us show that

$$\alpha_k\beta_k I_k(x) = o_k(1)$$

By using the same arguments as in Case 1 we get

$$\alpha_k\beta_k I_k(x) = \int_{B^0(L)}\left(\frac{1}{8\pi^2}\log\frac{|z|}{|x-z|} + K_k(x,z)\right)v_k(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z)$$

Now since K *is* C^1 *we need only to show that*

$$\int_{B^0(L)}\frac{1}{8\pi^2}\log\frac{|z|}{|x-z|}v_k(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z) = o_k(1).$$

By using the trivial inequality

$$\int_{B_{x_k}(Lr_k)}\frac{u_k^2}{\lambda_k}e^{\alpha_k u_k^2}dV_g \le 1;$$

and the change of variables as above, we obtain

$$\int_{B^0(L)} v_k^2(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z) = O(\frac{1}{d_k}) = o_k(1).$$

On the other hand using the property of v_k *one can check easily that*

$$\int_{B^0(L)} v_k(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z) = \int_{B^0(L)} v_k^2(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z) + o_k(1).$$

Thus we arrive to

$$\int_{B^0(L)}\frac{1}{8\pi^2}\log\frac{|z|}{|x-z|}v_k(z)e^{d_k(w_k(z)(1+v_k(z)))}dV_{g_k}(z) = o_k(1)$$

So we get

$$\alpha_k\beta_k I_k(x) = o_k(1)$$

Thus letting $k \to +\infty$, *we obtain*

$$w(x) = 0 \;\; \forall x \in \mathbb{R}^4.$$

Hence the Proposition is proved. ■

1.5.4 Capacity estimates

This Subsection deals with some capacity-type estimates which allow us to get an upper bound of $\tau^2 \lim_{k\to+\infty} \frac{\lambda_k}{\beta_k^2}$. We start by giving a first Lemma to show that we can basically work on Euclidean space in order to get the capacity estimates as already said in the Introduction.

Lemma 1.5.10. *There is a constant B which is independent of k, L and δ s.t.*

$$\int_{B^\delta(0)\setminus B^0(Lr_k)} |(1-B|x|^2)\Delta_0 \tilde{u}_k|^2 dx \le \int_{B_{x_k}(\delta)\setminus B_{x_k}(Lr_k)} |\Delta_g u_k|^2 dV_g + \frac{J_1(k,L,\delta)}{\beta_k^2},$$

where

$$\tilde{u}(x) = u_k(exp_{x_k}(x)).$$

Moreover we have that

$$\lim_{\delta\to 0}\lim_{k\to+\infty} J_1(k,L,\delta) = 0.$$

Proof. First of all by using the definition of Δ_g ie

$$\Delta_g = \frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}g^{rs}\partial_s);$$

we get

$$\begin{aligned} |\Delta_g \beta_k u_k|^2 &= |g^{rs}\beta_k \tfrac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s} + O(|\nabla \beta_k \tilde{u}_k|)|^2 \\ &= |g^{rs}\beta_k \tfrac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2 + O(|\nabla^2 \beta_k \tilde{u}_k||\nabla \beta_k \tilde{u}_k|)) + O((|\nabla \beta_k \tilde{u}_k|)^2) \end{aligned}$$

On the other hand using the fact that (see Corollary 1.5.6))

$$\beta_k \tilde{u}_k \rightharpoonup \tilde{G} \quad \text{in } W^{2,p}(M);$$

where $p \in (1,2)$; and $\tilde{G}(x) = G(exp_{x_0}(x))$; we obtain

$$\begin{aligned} &\int_{B^0(\delta)\setminus B^0(Lr_k)} O(|\nabla^2 \beta_k \tilde{u}_k||\nabla \beta_k \tilde{u}_k|) + O((|\nabla \beta_k \tilde{u}_k|)^2) \\ &\le C||\tilde{G}||_{W^{1,2}(B^0(\delta)\setminus B^0(Lr_k))} \\ &= J_2(k,L,\delta), \end{aligned}$$

and it is clear that

$$\lim_{\delta\to 0}\lim_{k\to+\infty} J_2(k,L,\delta) = 0$$

Now let us estimate $\int_{B^0(\delta)\setminus B^0(Lr_k)} |g^{rs}\beta_k \frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2$. To do this, we first write the inverse of the metric in the following form

$$g^{rs} = \delta^{rs} + A^{rs}$$

with

$$|A^{rs}| \le C|x|^2.$$

We can write

$$|g^{rs}\frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}|^2 |\Delta_0 \tilde{u}_k|^2 + 2\sum_{p,q} A^{pq}\Delta_0 \tilde{u}_k \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q} + \sum_{r,s,p,q} A^{rs}A^{pq}\frac{\partial^2 \tilde{u}_k}{\partial x^r \partial x^s}\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}$$

Furthermore we derive

$$\sum_{p,q} 2\int_{B^0(\delta)\setminus B^0(Lr_k)} |A^{pq}\Delta_0 \tilde{u}_k \frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}| dV_g \le C\int_{B^0(\delta)\setminus B^0(Lr_k)} (|x|^2|\Delta_0 \tilde{u}_k|^2 + \sum_{p,q}|x|^2|\frac{\partial^2 \tilde{u}_k}{\partial x^p \partial x^q}|^2) dx$$

On the other hand we have that

$$\sum_{p,q}\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^2|\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}|^2dx\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^2\frac{\partial^2\tilde{u}_k}{\partial x^s\partial x^s}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^p}dx$$
$$+\int_{B^0(\delta)\setminus B^0(Lr_k)}O(|\nabla\tilde{u}_k||\nabla^2\tilde{u}_k|)dx+\int_{\partial(B^0(\delta)\setminus B^{Lr_k}(0))}|x|^2\frac{\partial\tilde{u}_k}{\partial x^q}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}\left\langle\frac{\partial}{\partial x^p},\frac{\partial}{\partial r}\right\rangle dS$$
$$+\int_{\partial(B^0(\delta)\setminus B^0(Lr_k))}|x|^2\frac{\partial\tilde{u}_k}{\partial x^q}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^p}\left\langle\frac{\partial}{\partial x^q},\frac{\partial}{\partial r}\right\rangle dS.$$

So setting

$$\frac{J_3(k,L,\delta)}{\beta_k^2}=\int_{B^0(\delta)\setminus B^0(Lr_k)}O(|\nabla\tilde{u}_k||\nabla^2\tilde{u}_k|)dx+\int_{\partial(B^0(\delta)\setminus B^0(Lr_k))}|x|^2\frac{\partial\tilde{u}_k}{\partial x^q}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}\left\langle\frac{\partial}{\partial x^p},\frac{\partial}{\partial r}\right\rangle dS$$
$$+\int_{\partial(B^0(\delta)\setminus B^0(Lr_k))}|x|^2\frac{\partial\tilde{u}_k}{\partial x^q}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^p})\left\langle\frac{\partial}{\partial x^q},\frac{\partial}{\partial r}\right\rangle dS$$

We obtain

$$\sum_{p,q}\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^2|\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}|^2=\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^2\frac{\partial^2\tilde{u}_k}{\partial x^q\partial x^q}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^p}dx+\frac{J_3(k,L,\delta)}{\beta_k^2}.$$

Moreover we have that

$$\lim_{\delta\to 0}\lim_{k\to+\infty}J_3(k,L,\delta)=0.$$

Hence we get

$$2\sum_{p,q}\int_{B^0(\delta)\setminus B^0(Lr_k)}|A^{pq}\frac{\partial^2\tilde{u}_k}{\partial x^s\partial x^s}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}|\le C\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^2|\Delta_0\tilde{u}_k|^2dx+\frac{J_4(k,L,\delta)}{\beta_k^2}$$

with

$$\lim_{\delta\to 0}\lim_{k\to+\infty}J_4(k,L,\delta)=0.$$

On the other hand using similar arguments we get

$$\int_{B^0(\delta)\setminus B^0(Lr_k)}\sum_{r,s,p,q}A^{rs}A^{pq}\frac{\partial^2\tilde{u}_k}{\partial x^r\partial x^s}\frac{\partial^2\tilde{u}_k}{\partial x^p\partial x^q}\le C\int_{B^0(\delta)\setminus B^0(Lr_k)}|x|^4|\Delta_0\tilde{u}_k|^2dx+\frac{J_5(k,L,\delta)}{\beta_k^2}.$$

with

$$\lim_{\delta\to 0}\lim_{k\to+\infty}J_5(k,L,\delta)=0.$$

So we arrive to

$$\int_{B_{x_k}(\delta)\setminus B_{x_k}(Lr_k)}|\Delta_g u_k|^2dV_g\le\int_{B^0(\delta)\setminus B^0(Lr_k)}(1+C|x|^2+C|x|^4)|\Delta_0\tilde{u}_k|^2dx+\frac{J_6(k,L,\delta)}{\beta_k^2};$$

with

$$\lim_{\delta\to 0}\lim_{k\to+\infty}J_6(k,L,\delta)=0$$

Hence we can find a constant B_1 independent of k, L and δ s.t

$$\int_{B_{x_k}(\delta)\setminus B_{Lr_k}}|\Delta_g u_k|^2dV_g\ge\int_{B^0(\delta)\setminus B^0(Lr_k)}(1-B_1|x|^2)|\Delta_0\tilde{u}_k|^2dx+\frac{J_7(k,L,\delta)}{\beta_k^2}.$$

So setting

$$J_1(k,L,\delta)=-J_7(k,L,\delta)\quad\text{and}\quad B=B_1$$

we have the proved the Lemma. ■

Next we give a technical Lemma

Lemma 1.5.11. *There exists a sequence of functions* $U_k \in W^{2,2}(B^0(\delta)\setminus B^0(Lr_k))$ *s.t*

$$U_k|_{\partial B^0(\delta)} = \tau\frac{-\frac{1}{16\pi^2}\log\delta + S_0}{\beta_k}, \sigma U_k|_{\partial B^0(Lr_k)} = \frac{w(L)}{2\alpha_k\beta_k} + c_k;$$

and

$$\frac{\partial U_k}{\partial r}|_{\partial B_\delta(0)} = -\frac{\tau}{8\pi^2\delta\beta_k}, \sigma\frac{\partial U_k}{\partial r}|_{\partial B^0(Lr_k)} = \frac{w'(L)}{2\alpha_k\beta_k r_k}.$$

Moreover there holds

$$\lim_{\delta\to 0}\lim_{k\to+\infty}\beta_k^2\left(\int_{B^0(\delta)\setminus B^0(Lr_k)}|\Delta_0(1-B|x|^2)U_k|^2dx - \int_{B^\delta\setminus B^0(Lr_k)}|(1-B|x|^2)\Delta_0\tilde{u}_k|^2dx\right) = 0.$$

Proof. First of all let us set

$$h_k(x) = u_k(exp_{x_k}(r_kx)).$$

and u_k' to be the solution of

$$\begin{cases} \Delta_0^2 u_k' = \Delta_0^2 h_k \\ \frac{\partial u_k'}{\partial n}|_{\partial B^0(2L)}\frac{\partial h_k}{\partial n}|_{\partial B^0(2L)}, \sigma u_k'|_{\partial B^0(2L)} = h_k|_{\partial B^0(2L)} \\ \frac{\partial u_k'}{\partial n}|_{\partial B^0(L)}\frac{1}{2\alpha_k\beta_k}\frac{\partial w}{\partial n}|_{\partial B^0(L)}, \sigma u_k'|_{\partial B^0(L)} = \frac{w}{2\alpha_k\beta_k}|_{\partial B^L(0)}. \end{cases}$$

Next let us define

$$U_k' = \begin{cases} u_k'(\frac{x}{r_k}) & Lr_k \le |x| \le 2Lr_k \\ \tilde{u}_k(x) & 2Lr_k \le |x|. \end{cases}$$

Clearly we have that

$$\lim_{k\to+\infty}\int_{B^0(2Lr_k)\setminus B^0(Lr_k)}(1-B|x|^2)(|\Delta_0 U_k'|^2 - |\Delta_0\tilde{u}_k|^2)dx = 0,$$

and

$$\lim_{k\to+\infty}|U_k' - \tilde{u}_k'|_{C^0(B^0(2Lr_k)\setminus B^0(Lr_k))} = 0.$$

Now let η be a smooth function which satisfies

$$\eta(t) = \begin{cases} 1 & t \le 1/2 \\ 0 & t > 2/3 \end{cases}$$

and set

$$G_k = \eta(\frac{|x|}{\delta})(\tilde{u}_k - \tau S_0 + \frac{\tau}{8\pi^2}\log|x|) - \frac{\tau}{8\pi^2}\log|x| + \tau S_0.$$

Then we have that

$$G_k \to -\frac{\tau}{8\pi^2}\log|x| + \tau S_0 + \tau\eta(\frac{|x|}{\delta})\tilde{S}_1(x);$$

where $\tilde{S}_1(x) = S_1(exp_{x_0}(x))$.
Furthermore we obtain

$$\beta_k\tilde{u}_k - G_k \to \tau\left(1 - \eta(\frac{|x|}{\delta})\right)S_1(x),$$

then

$$\lim_{\epsilon\to 0}\left|\int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0\beta_k\tilde{u}_k|^2dx - \int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0 G_k|^2dx\right| \le \Sigma.$$

where

$$\begin{aligned}\Sigma &= \sqrt{\int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0(1-\eta(\frac{|x|}{\delta}))\tilde{S}_1(x)|^2dx\int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0(\tilde{G} - \frac{1}{8\pi^2}\log|x| + \eta(\frac{|x|}{\delta})\tilde{S}_1(x))|^2dx} \\ &\le C\delta\sqrt{|\log\delta|}.\end{aligned}$$

So we get

$$\lim_{\epsilon\to 0}\Big|\int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0\beta_k\tilde{u}_k|^2dx-\int_{B^0(\delta)\setminus B^0(\frac{\delta}{2})}|\Delta_0 G_k|^2dx\Big|\le C\delta\sqrt{|\log\delta|}.$$

Hence setting

$$U_k=\begin{cases} U_k'(x) & |x|\le\frac{\delta}{2}\\ G_k(x) & \delta/2\le|x|\le\delta\end{cases}$$

we have proved the Lemma. ■

Proposition 1.5.12. *We have the following holds*

$$\tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}\le\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2S_0};$$

and

$$d\tau=1.$$

PROOF. First using Lemma 1.5.10 and Lemma 1.5.11 we get

$$\int_{B^0(\delta)\setminus B^0(Lr_k)}|\Delta_0(1-B|x|^2)U_k|^2dx\le 1-\frac{\int_{B_L(x_0)}|\Delta w|^2+\int_{M\setminus B_{x_0}(\delta)}|\Delta G|^2+J_0(k,L,\delta)}{\beta_k^2}.\quad(1.62)$$

with

$$\lim_{\delta\to 0}\lim_{k\to+\infty}J_0(k,L,\delta)=0.$$

Next we will apply capacity to give a lower boundary of $\int_{B^0(\delta)\setminus B^0(Lr_k)}|\Delta_0(1-B|x|^2)U_k|^2dx$. Hence we need to calculate

$$\inf_{\Phi|_{\partial B^0(r)}=P_1,\Phi|_{\partial B^0(R)}=P_2,\frac{\partial\Phi}{\partial r}|_{\partial B^0(r)}=Q_1,\frac{\partial\Phi}{\partial r}|_{\partial B^0(R)}=Q_2}\int_{B^0(R)\setminus B^0(r)}|\Delta_0\Phi|^2dx,$$

where P_1, P_2, Q_1, Q_2 are constants.
It is obvious that the infimum is attained by the function Φ which satisfies

$$\begin{cases}\Delta_0^2\Phi=0\\ \Phi|_{\partial B^0(r)}=P_1\ ,\Phi|_{\partial B^0(R)}=P_2\ ,\frac{\partial\Phi}{\partial r}|_{\partial B^0(r)}=Q_1\ ,\frac{\partial\Phi}{\partial r}|_{\partial B^0(R)}=Q_2.\end{cases}$$

Moreover we can require the function Φ to be of the form

$$\Phi=A\log r+Br^2+\frac{C}{r^2}+D,$$

where A, B, C, D are all constants which satisfies the following linear system of equations

$$\begin{cases}A\log r+Br^2+\frac{C}{r^2}+D=P_1\\ A\log R+BR^2+\frac{C}{R^2}+D=P_2\\ \frac{A}{r}+2Br-2\frac{C}{r^3}=Q_1\\ \frac{A}{R}+2BR-2\frac{C}{R^3}=Q_2\end{cases}$$

Now by straightforward calculations we obtain the explicit expression of A and B

$$\begin{cases}A=\frac{P_1-P_2+\frac{\varrho}{2}rQ_1+\frac{\varrho}{2}RQ_2}{\log r/R+\varrho}\\ B=\frac{-2P_1+2P_2-rQ_1(1+\frac{2r^2}{R^2-r^2}\log r/R)+RQ_2(1+\frac{2R^2}{R^2-r^2}\log r/R)}{4(R^2+r^2)(\log r/R+\varrho)}\end{cases}$$

Where $\varrho = \frac{R^2-r^2}{R^2+r^2}$. Furthermore we have

$$\int_{B^0(R)\setminus B^0(r)} |\Delta_0\Phi|^2 dx = -8\pi^2 A^2 \log r/R + 32\pi^2 AB(R^2-r^2) + 32\pi^2 B^2(R^4-r^4) \tag{1.63}$$

In our case in which we have that

$$R = \delta \quad r = Lr_k,$$

$$P_1 = c_k + \frac{w(L)}{2\alpha_k\beta_k} + O(r_k c_k) \quad P_2 \frac{-\frac{\tau}{8\pi^2}\log\delta + \tau S_0 + O(\delta\log\delta)}{\beta_k}$$

$$Q_1 = \frac{w'(L) + O(r_k c_k)}{2\alpha_k\beta_k r_k} \quad Q_2 = -\frac{\tau + O(\delta\log\delta)}{8\pi^2\beta_k\delta}.$$

Then by the formula giving A we obtain by trivial calculations

$$A = \frac{c_k + \frac{N_k + \frac{\tau}{8\pi^2}\log\delta}{\beta_k}}{-\log\delta + \log L + \frac{\log\frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} + 1 + O(r_k^2)}$$

where

$$N_k = \frac{w(L)}{2\alpha_k} - \tau S_0 + \frac{w'(L)L}{4\alpha_k} - \frac{\tau}{16\pi^2} + O(\delta\log\delta) + O(r_k c_k^2).$$

Moreover using the the fact that the sequence $(\frac{\lambda_k}{\beta_k^2})_k$ is bounded it is easily seen that

$$A = O(\frac{1}{c_k}).$$

Furthermore using the formula of B we get still by trivial calculations

$$B = \frac{-2c_k + \frac{\alpha_k c_k^2}{8\pi^2\beta_k}\frac{\tau}{2} + O(\frac{1}{\beta_k})}{\delta^2(-\alpha_k c_k^2 + \log\frac{\lambda_k}{\beta_k c_k})}.$$

and then

$$B = O(\frac{1}{\beta_k})\frac{1}{\delta^2}.$$

Now let compute $8\pi^2 A^2 \log r/R$. By using the expression of A, r and R, we have that

$$-8\pi^2 A^2 \log(\frac{r}{R}) = -8\pi^2 (\frac{c_k + \frac{N_k + \frac{\tau}{8\pi^2}\log\delta}{\beta_k}}{-\log\delta + \log L + \frac{\log\frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} + 1 + O(r_k^2)})^2 (\frac{\log\frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} - \log\delta + \log L)$$

Now using the relation

$$(\frac{\alpha_k c_k^2}{4})^2 \left(1 - \frac{1}{\alpha_k c_k^2}(-4\log\delta + 4\log L + \log\frac{\lambda_k}{\beta_k c_k} + 4 + O(r_k^2))\right)^2 = \left(-\log\delta + \log L + \frac{\log\frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} + 1 + O(r_k^2)\right)^2$$

we derive

$$-8\pi^2 A^2 \log(\frac{r}{R}) = -8\pi^2 (\frac{c_k + \frac{N_k + \frac{\tau}{8\pi^2}\log\delta}{\beta_k}}{\frac{\alpha_k c_k^2}{4}})^2 \left(1 - \frac{1}{\alpha_k c_k^2}(-4\log\delta + 4\log L + \log\frac{\lambda_k}{\beta_k c_k} + 4 + O(r_k^2))\right)^{-2}$$
$$\times (\frac{\log\frac{\lambda_k}{\beta_k c_k} - \alpha_k c_k^2}{4} - \log\delta + \log L).$$

On the other hand using Taylor expansion we have the following identity

$$\left(1-\frac{1}{\alpha_k c_k^2}(-4\log\delta+4\log L+\log\frac{\lambda_k}{\beta_k c_k}+4+O(r_k^2))\right)^{-2}=1+2\frac{\log\frac{\lambda_k}{\beta_k c_k}+4-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4});$$

hence we get

$$-8\pi^2A^2\log(\frac{r}{R})=-8\pi^2(\frac{c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k}}{\frac{\alpha_k c_k^2}{4}})^2(\frac{\log\frac{\lambda_k}{\beta_k c_k}-\alpha_k c_k^2}{4}-\log\delta+\log L)$$
$$\times(1+2\frac{\log\frac{\lambda_k}{\beta_k c_k}+4-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))$$

On the other hand using the relation

$$-8\pi^2(\frac{c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k}}{\frac{\alpha_k c_k^2}{4}})^2(\frac{\log\frac{\lambda_k}{\beta_k c_k}-\alpha_k c_k^2}{4}-\log\delta+\log L)=$$
$$\frac{32\pi^2}{\alpha_k}\frac{1}{c_k^2}(c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k})^2(1-\frac{\log\frac{\lambda_k}{\beta_k c_k}-4\log\delta+4\log L}{\alpha_k c_k^2})$$

we obtain

$$-8\pi^2A^2\log(\frac{r}{R})=\frac{32\pi^2}{\alpha_k}\frac{1}{c_k^2}(c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k})^2(1+2\frac{\log\frac{\lambda_k}{\beta_k c_k}+4-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))$$
$$\times(1-\frac{\log\frac{\lambda_k}{\beta_k c_k}-4\log\delta+4\log L}{\alpha_k c_k^2})$$

Moreover using again the trivial relation

$$(1+2\frac{\log\frac{\lambda_k}{\beta_k c_k}+4-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))(1-\frac{\log\frac{\lambda_k}{\beta_k c_k}-4\log\delta+4\log L}{\alpha_k c_k^2})=$$
$$(1+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))$$

we arrive to

$$-8\pi^2A^2\log(\frac{r}{R})=\frac{32\pi^2}{\alpha_k}\frac{1}{c_k^2}(c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k})^2(1+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))$$

On the other hand one can check easily that the following holds

$$(c_k+\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k})^2(1+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k c_k^2}+O(\frac{\log^2 c_k}{c_k^4}))=$$
$$\left(c_k^2+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k}+2c_k\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k}+O(\frac{\log c_k}{c_k^2})+O(\frac{1}{\beta_k^2})\right);$$

thus we obtain

$$-8\pi^2A^2\log(\frac{r}{R})=\frac{32\pi^2}{\alpha_k}\frac{1}{c_k^2}\left(c_k^2+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k}+2c_k\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k}\right)$$
$$+\frac{32\pi^2}{\alpha_k}\frac{1}{c_k^2}\left(O(\frac{\log c_k}{c_k^2})+O(\frac{1}{\beta_k^2})\right)$$

Furthermore using the relation

$$\left(c_k^2+\frac{\log\frac{\lambda_k}{\beta_k c_k}+8-4\log\delta+4\log L}{\alpha_k}+2c_k\frac{N_k+\frac{\tau}{8\pi^2}\log\delta}{\beta_k}+O(\frac{\log c_k}{c_k^2})+O(\frac{1}{\beta_k^2})\right)=$$
$$\left(c_k^2+\frac{1}{\alpha_k}\log\frac{\lambda_k}{\beta_k c_k}-\frac{4}{\alpha_k}\log\delta+\frac{1}{4\pi^2}d_k\tau\log\delta+2d_kN_k+\frac{4\log L}{\alpha_k}+\frac{8}{\alpha_k}+o_k(1)\right)$$

we get

$$-8\pi^2A^2\log(\frac{r}{R})=\frac{32\pi^2}{\alpha_k^2}\frac{1}{c_k^2}\left(c_k^2+\frac{1}{\alpha_k}\log\frac{\lambda_k}{\beta_k c_k}-\frac{4}{\alpha_k}\log\delta+\frac{1}{4\pi^2}d_k\tau\log\delta+2d_kN_k+\frac{4\log L}{\alpha_k}+\frac{8}{\alpha_k}\right)$$
$$+\frac{32\pi^2}{\alpha_k^2}\frac{1}{c_k^2}o_k(1) \tag{1.64}$$

Next we will evaluate $\int_{M\setminus B_{x_0}(\delta)}\Delta_gG\Delta_gGdV_g$. We have that by Green formula

$$\int_{M\setminus B_{x_0}(\delta)}\Delta_gG\Delta_gGdV_g=\int_{M\setminus B_{x_0}(\delta)}G\Delta_g^2GdV_g-\int_{\partial B_{x_0}(\delta)}\frac{\partial G}{\partial r}\Delta_gG+\int_{\partial B_{x_0}(\delta)}G\frac{\partial\Delta_gG}{\partial r}.$$

Thus using the equation solved by G we get

$$\int_{M\setminus B_{x_0}(\delta)}\Delta_gG\Delta_gGdV_g=-\frac{\tau}{\mu(M)}\int_{M\setminus B_\delta(p)}GdV_g-\frac{\tau^2}{64\pi^4}\int_{\partial B_{x_0}(\delta)}\frac{\partial(-\log r)}{\partial r}\Delta_0(-\log r)$$
$$+\int_{\partial B_{x_0}(\delta)}(-\frac{\tau}{8\pi^2}\log r+S_0)\frac{\partial\Delta_0(-\frac{\tau}{8\pi^2}\log r)}{\partial r}+O(\delta\log\delta)$$

Hence we obtain

$$\int_{M\setminus B_{x_0}(\delta)}\Delta_gG\Delta_gGdV_g=-\frac{\tau^2}{16\pi^2}-\frac{\tau^2}{8\pi^2}\log\delta+\tau^2S_0+O(\delta\log\delta),$$

Now let us set

$$P(L)=\int_{B^0(L)}|\Delta_0w|^2dx/(2\times 32\pi^2)^2.$$

Hence using (1.62), (1.63), (1.64), we derive that

$$\frac{32\pi^2}{\alpha_k}\left(c_k^2+\frac{1}{\alpha_k}\log\frac{\lambda_k}{\beta_k c_k}-\frac{4}{\alpha_k}\log\delta+\frac{1}{4\pi^2}d_k\tau\log\delta+2d_kN_k+\frac{4\log L}{\alpha_k}+\frac{8}{\alpha_k}\right)$$
$$\leq c_k^2(1-\frac{P(L)-\frac{\tau^2}{16\pi^2}-\frac{\tau^2}{8\pi^2}\log\delta+\tau S_0+O(\delta\log\delta)+o_{k,\delta}(1)}{\beta_k^2})+\delta^2O(c_k^2AB)+\delta^4O(c_k^2B^2).$$

Moreover by isolating the term $\frac{32\pi^2}{\alpha_k^2}\log\frac{\lambda_k}{\beta_k c_k}$ in the left and transposing all the other in the right we get

$$\frac{32\pi^2}{\alpha_k^2}\log\frac{\lambda_k}{\beta_k c_k}\leq\frac{1}{8\pi^2}(d_k^2\tau^2-\frac{64}{\alpha_k}d_k\tau+(\frac{32\pi}{\alpha_k})^2)\log\delta-\frac{32\pi^2}{\alpha_k}(2d_kN_k+\frac{4\log L}{\alpha_k}+\frac{8}{\alpha_k})$$
$$-d_k^2(P(L)+\tau S_0-\frac{\tau^2}{16\pi^2}+O(\delta\log\delta)+o_k(1))+\delta^2O(c_k^2AB)+\delta^4O(c_k^2B^2). \tag{1.65}$$

Hence using the trivial identity

$$\log\frac{\lambda_k}{\beta_k^2}=\log\frac{\lambda_k}{\beta_k c_k}+\log d_k$$

we get

$$\frac{32\pi^2}{\alpha_k^2}\log\frac{\lambda_k}{\beta_k^2} \leq \frac{1}{8\pi^2}(d_k^2\tau^2 - \frac{64}{\alpha_k}d_k\tau + (\frac{32\pi}{\alpha_k})^2)\log\delta - \frac{32\pi^2}{\alpha_k}(2d_kN_k + \frac{2+4\log L}{\alpha_k} + \frac{2}{\alpha_k})$$
$$-d_k^2(P(L) + \tau S_0 - \frac{\tau^2}{16\pi^2} + O(\delta\log\delta) + o_k(1)) + \frac{32\pi^2}{\alpha_k^2}\log d_k + O(d_k^2).$$

Now suppose $d = +\infty$, letting $\delta \to 0$, then we have that

$$\lim_{k\to+\infty}\log\frac{\lambda_k}{\beta_k^2} = -\infty,$$

thus we derive

$$\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2} = 0$$

Hence using Corollary 1.5.8 we obtain a contradiction. So d must be finite.
On the other hand one can check easily that the following holds

$$\frac{32\pi^2}{\alpha_k^2}\log\frac{\lambda_k}{\beta_k^2} \leq \frac{1}{8\pi^2}(d_k\tau - \frac{32\pi^2}{\alpha_k})^2\log\delta + O(1)(d_k^2 + d_k + \log d_k) + O(1).$$

Hence we derive

$$d_k\tau \to 1;$$

otherwise we reach the same contradiction. So we have that

$$d\tau = 1.$$

Hence by using this we can rewrite B as follows

$$B = \frac{-2c_k + \delta(-\frac{1}{8\pi^2c_k\delta}2^{\frac{-\alpha_kc_k^2}{4}}) + O(1/c_k)}{\delta^2(-\alpha_kc_k^2) + O(1)} = \frac{o_k(1)}{c_k}.$$

Thus we obtain

$$32\pi^2AB(R^2 - r^2) + 32\pi^2B^2(R^4 - r^4) = \frac{o_k(1)}{c_k^2}.$$

On the other hand since $d < +\infty$, we have that by Lemma 1.5.9

$$w = -\frac{4\log(1 + \sqrt{\frac{d}{6}}\pi|x|^2)}{d}.$$

Moreover by trivial calculations we get

$$P(L) = \frac{1}{96d^2\pi^2} + \frac{\log(1 + \sqrt{\frac{d}{6}}\pi L^2)}{16d^2\pi^2}.$$

Furthermore by taking the limit as $k \to +\infty$ in (1.65) we obtain

$$\lim_{k\to+\infty}\log\frac{\lambda_k}{\beta_kc_k} \leq -\frac{25}{3} + 4d\tau + 2d^2\tau^2 + 32\pi^2S_0 + \frac{4\sqrt{\frac{d}{6}}\pi L^2}{1 + \sqrt{\frac{d}{6}}\pi L^2} + 2\log(1 + \sqrt{\frac{d}{6}}\pi L^2) - 4\log L$$

Now letting $L \to +\infty$, we get

$$\lim_{k\to+\infty}\log\frac{\lambda_k}{\beta_kc_k} \leq \frac{5}{3} - \log 6 + \log\pi^2 + \log d.$$

Hence by remarking the trivial identity

$$\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k c_k}\frac{1}{d}\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}$$

we get

$$\tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}\leq\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2S_0}.$$

So the proof of the proposition is done. ■

1.5.5 The test function

This Subsection deals with the construction of some test functions in order to reach a contradiction. Now let $\epsilon>0$, $c>0$, $L>0$ and set

$$f_\epsilon(x)=\begin{cases} c+\frac{\Lambda+Bd_g(x,x_0)^2-4\log\left(1+\lambda(\frac{d_g(x,x_0)}{\epsilon})^2\right)}{64\pi^2c}+\frac{S(x)}{c} & d_g(x,x_0)\leq L\epsilon\\ \frac{G(x)}{c} & d_g(x,x_0)>L\epsilon\end{cases}$$

where

$$\lambda=\frac{\pi}{\sqrt{6}},\sigma B=-\frac{4}{L^2\epsilon^2(1+\lambda L^2)}$$

and

$$\Lambda=-64\pi^2c^2-BL^2\epsilon^2-8\log(L\epsilon)+4\log(1+\lambda L^2). \tag{1.66}$$

Proposition 1.5.13. *We have that for ϵ small, there exist suitable c and L such that*

$$\int_M|\Delta_g f_\epsilon|^2dV_g=1;$$

and

$$\limsup_{\epsilon\to0}\int_M e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2}dV_g>Vol(M)+\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2S_0}.$$

PROOF. First of all using the expansion of g in normal coordinates we get

$$\int_{B_{L\epsilon}(x_0)}|\Delta_g f_\epsilon|^2dV_g\int_{B^{L\epsilon}(0)}|\Delta_0\tilde{f}_\epsilon|^2(1+O(L\epsilon)^2)dx+\int_{B^{L\epsilon}(0)}O(r^2|\nabla_0\tilde{f}_\epsilon|^2)dx$$

where

$$\tilde{f}_\epsilon(x)=f_\epsilon(exp_{x_0}(x)).$$

On the other hand by direct calculations owe obtain

$$\int_{B^{L\epsilon}(0)}|\Delta_0\tilde{f}_\epsilon|^2dx=\frac{12+\lambda L^2(30+\lambda L^2(21+\lambda L^2))+6(1+\lambda L^3)^3\log(1+\lambda L^2)}{96c^2(1+\lambda L^2)^3\pi^2}$$

Hence we arrive to

$$\begin{aligned}\int_{B_{L\epsilon}(x_0)}|\Delta_g f_\epsilon|^2dV_g &= (1+O(L\epsilon)^2)\frac{12+\lambda L^2(30+\lambda L^2(21+\lambda L^2))+6(1+\lambda L^3)^3\log(1+\lambda L^2)}{96c^2(1+\lambda L^2)^3\pi^2}\\ &= \frac{\frac{1}{3}+4\log(1+\lambda L^2)+O(\frac{1}{L^2})+O((L\epsilon)^2\log L\epsilon)}{32c^2\pi^2}\end{aligned}$$

Furthermore, by direct computation, we have

$$\int_{B^{L\epsilon}(0)}r^2|\nabla_0\tilde{f}_\epsilon|^2dx=O(\frac{L^4\epsilon^4}{c^2}).$$

Moreover using Green formula we get

$$\begin{aligned}\int_{M\setminus B_{L\epsilon}(x_0)}|\Delta_g G|^2 dV_g &= \int_{M\setminus B_{L\epsilon}(x_0)} G dV_g - \int_{\partial B_{L\epsilon}(x_0)} \frac{\partial G}{\partial r}\Delta_g G dS_g + \int_{\partial B_{L\epsilon}} G\frac{\partial \Delta_g G}{\partial r} dS_g \\ &= -\tfrac{1}{16\pi^2} + S_0 - \tfrac{\log L\epsilon}{8\pi^2} + O(L\epsilon \log L\epsilon)\end{aligned}$$

Now let us find a condition to have $\int_M |\Delta_g f_\epsilon|^2 dV_g = 1$. By trivial calculations we can see that it is equivalent to

$$\frac{1}{32\pi^2 c^2}\left(-\frac{5}{3} + 2\log(1+\lambda L^2) + 32\pi^2 S_0 - 4\log L\epsilon + O(\frac{1}{L^2}) + O(L\epsilon\log L\epsilon)\right) = 1.$$

i.e.

$$32\pi^2 c^2 = -\frac{5}{3} + 2\log(1+\lambda L^2) + 32\pi^2 S_0 - 4\log L\epsilon + O(\frac{1}{L^2}) + O(L\epsilon\log L\epsilon).$$

Hence by (1.66) Λ take the following form

$$\Lambda = \frac{10}{3} - 64\pi^2 S_0 + O(\frac{1}{L^2}) + O(L\epsilon\log L\epsilon).$$

On the other hand it is easily seen that

$$\int_{B_{L\epsilon}(x_0)} f_\epsilon dV_g = O(c(L\epsilon)^4);$$

and

$$\int_{M\setminus B_{L\epsilon}(x_0)} f_\epsilon dV_g = -\int_{B_{L\epsilon}} \frac{G}{c} = O(\frac{(L\epsilon)^4\log L\epsilon}{c}).$$

hence

$$\bar{f}_\epsilon = O(c(L\epsilon)^4).$$

Furthermore by trivial calculations one gets that in $B_{L\epsilon}(x_0)$

$$\begin{aligned}(f_\epsilon - \bar{f}_\epsilon)^2 &\geq c^2 + \tfrac{2}{64\pi^2}\left(\Lambda + Br^2 - 4\log(1+\lambda(\tfrac{r}{\epsilon})^2) + 64\pi^2 S_0 + O(L\epsilon) + O(c^2(L\epsilon)^4)\right) \\ &= c^2 + \tfrac{5}{48\pi^2} - \tfrac{\log(1+\lambda(r/\epsilon)^2)}{8\pi^2} + O(\tfrac{1}{L^2}) + O(L\epsilon\log L\epsilon) + O(c^2(L\epsilon)^4);\end{aligned}$$

hence

$$\begin{aligned}\int_{B_{L\epsilon}(x_0)} e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2} dV_g &\geq (1+O(L\epsilon)^2)\int_{B_{L\epsilon}(x_0)} e^{32\pi^2\left(c^2+\frac{5}{48\pi^2}-\frac{\log(1+\lambda(r/\epsilon)^2}{8\pi^2}\right)+O(\frac{1}{L^2})+O(L\epsilon\log L\epsilon)+O(c^2(L\epsilon)^4)} dx \\ &= \epsilon^4 e^{\frac{10}{3}+32\pi^2c^2+O(\frac{1}{L^2})+O(L\epsilon\log L\epsilon)+O(c^2(L\epsilon)^4)}\left(\pi^2\tfrac{L^6}{1+\lambda L^6} + O(L\epsilon)^2\right) \\ &= \epsilon^4 e^{\frac{10}{3}+32\pi^2c^2}\pi^2(1+O(\tfrac{1}{L^2}) + O(L\epsilon\log L\epsilon) + O(L\epsilon)^2) \\ &= \tfrac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0}(1+O(L\epsilon\log L\epsilon) + O(\tfrac{1}{L^2}) + O(c^2(L\epsilon)^4)).\end{aligned}$$

on the other hand

$$\begin{aligned}\int_{M\setminus B_{L\epsilon}(x_0)} e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2} dV_g &\geq \int_{M\setminus B_{L\epsilon}(x_0)} (1+32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2) dV_g \\ &\geq Vol(M\setminus B_{L\epsilon}(x_0)) + \frac{\int_{M\setminus B_{L\epsilon}(x_0)} 32\pi^2 G^2 dV_g + O(c(L\epsilon)^4)}{c^2} \\ &= Vol(M) + \frac{\int_M 32\pi^2 G^2 dV_g}{c^2} + O(L\epsilon)^4\log L\epsilon\end{aligned}$$

Thus we arrive to

$$\int_M e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2} dV_g \geq Vol(M) + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0} + \frac{\int_{M\setminus B_{L\epsilon}(x_0)} 32\pi^2 G^2 dV_g}{c^2};$$
$$+O(L\epsilon\log(L\epsilon)) + O(\frac{1}{L^2}) + O(c^2(L\epsilon)^4)$$

and factorizing by $\frac{1}{c^2}$ we get

$$\int_M e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2}dV_g \geq Vol(M)+\tfrac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0} + \tfrac{1}{c^2}\left(\int_M 32\pi^2 G^2 dV_g + O(c^2 L\epsilon\log(L\epsilon)) + O(\frac{c^2}{L^2}) + O(c^4(L\epsilon)^4)\right).$$

On the other hand setting

$$L = \log\frac{1}{\epsilon}$$

we get

$$O(c^2L\epsilon\log(L\epsilon)) + O(\frac{c^2}{L^2}) + O(c^4(L\epsilon)^4) \to 0 \text{ as } \epsilon\to 0.$$

Hence the Proposition is proved.

1.5.6 Proof of Theorem 0.2.1

This small Subsection is concerned about the proof of Theorem 0.2.1.
First of all by corollary we have that

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2} = Vol_g(M) + \tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2}$$

with $\tau\neq 0$.
On the other hand from Proposition 1.5.12 we get

$$\tau^2\lim_{k\to+\infty}\frac{\lambda_k}{\beta_k^2} \leq \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0}.$$

Hence we obtain

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2} \leq Vol_g(M) + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0}.$$

Thus using the relation

$$\lim_{k\to+\infty}\int_M e^{\alpha_k u_k^2}dV_g = \sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2u^2}dV_g.$$

we derive

$$\sup_{u\in\mathcal{H}_1}\int_M e^{32\pi^2u^2}dV_g \leq Vol_g(M) + \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 S_0}.$$

On the other hand from Proposition 1.5.13 we have the existence of a family of function f_ϵ such that

$$\int_M |\Delta_g f_\epsilon|^2 dV_g = 1;$$

and

$$\limsup_{\epsilon\to 0}\int_M e^{32\pi^2(f_\epsilon-\bar{f}_\epsilon)^2}dV_g > Vol(M) + \frac{1}{6}e^{\frac{5}{3}+32\pi^2 S_0}\pi^2.$$

Hence we reach a contradiction. So the proof of Theorem 0.2.1 is completed. ■

1.5.7 Proof of Theorem 0.2.2

As already said in the Introduction, in this brief Subsection we will explain how the proof of Theorem 0.2.1 remains valid for Theorem 0.2.2.
First of all we remark that all the analysis above have been possible due to the following facts
1)
$\int_M |\Delta_g u|^2 dV_g$ is an equivalent norm to the standard norm of $H^2(M)$ on $\mathcal{H}_1$.
2)
The existence of the Green function for Δ_g^2.
3)
The result of Fontana.
On the other hand we have a counterpart of 2) and 3). Moreover it is easy to see that $\left\langle P_g^4 u, u \right\rangle$ is also an equivalent norm to the standard norm of $H^2(M)$ on $\mathcal{H}_2$. Notice that for a blowing-up sequence u_k we have that

$$\left\langle P_g^4 u_k, u_k \right\rangle = \int_M |\Delta_g u_k|^2 dV_g + o_k(1); \tag{1.67}$$

then it is easy to see that the same proof is valid up to the Subsection of test functions. Notice that (1.67) holds for the test functions f_ϵ , then it is easy to see that continuing the same proof we get Theorem 0.2.2.

Chapter 2

Blow-up analysis

In this Chapter, we perform the Blow-up analysis of some perturbations of the prescribed Q-curvature equation in arbitrary dimensions, the prescribed Q-curvature and T-curvature equations on four dimensional compact Riemannian manifolds with boundary. Precisely we give the proof of Theorem 0.2.6, Theorem 0.2.8, and Theorem 0.2.10 announced in the Introduction. Moreover we give also the proof of Theorem 0.2.12.

2.1 Proof of Theorem 0.2.6

In this Section we give the proof of Theorem 0.2.10.

First integrating (15) we get

$$\int_M Q_0 dV_g + o_l(1) = \int_M \bar{Q}_l e^{nu_l} dV_g \tag{2.1}$$

We recall now the following result of X. Xu (Theorem 1.2 in [89]).

Theorem 2.1.1. *([89]) There exists a dimensional constant $\sigma_n > 0$ such that, if $u \in C^1(\mathbb{R}^n)$ is solution of the integral equation*

$$u(x) = \int_{\mathbb{R}^n} \sigma_n \log\left(\frac{|y|}{|x-y|}\right) e^{nu(y)} dy + c_0,$$

where c_0 is a real number, then $e^u \in L^n(\mathbb{R}^n)$ implies, there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x-x_0|^2}\right).$$

Now, if c_n is given in Proposition 0.3.2 and σ_n in Theorem 2.3.1 we set $k_n = \sigma_n c_n$ and $\gamma_n = 2(k_n)^n$

The proof is divided into five steps.

Step 1

There exists $N \in \mathbb{N}^*$, N converging points $(x_{i,l})$ $i = 1, ..., N$, N sequences $(\mu_{i,l})$ $i = 1; ...; N$; of positive real numbers converging to 0 such that the following hold:

a)

$$\frac{d_g(x_{i,l},x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i\neq j \;\; i,j=1,..,N \quad \textit{and} \quad \bar{Q}_l(x_{i,l})\mu_{i.l}^n e^{nu_l(x_{i,l})}=1;$$

b)

$$v_{i,l}(x)=u_l(exp_{x_{i,l}}(\mu_{i,l}x))-u_l(x_{i,l})-\frac{1}{n}\log(k_n)\longrightarrow V_0(x):=\log(\frac{4\gamma_n}{4\gamma_n^2+|x|^2}) \quad in \quad C^1_{loc}(\mathbb{R}^n);$$

c)

$$\forall i=1,..,N \;\; we\; have, \; \lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B_{x_{i,l}}(R\mu_{i,l})}\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)=(n-1)!\omega_n;$$

d)

$$There\; exists\;\; C>0\;\; such\; that \inf_{i=1,\dots,N} d_g(x_{i,l},x)^n e^{nu_l(x)}\leq C \quad \forall x\in M,\;\; \forall l\in\mathbb{N}.$$

Proof of Step 1

Let $x_l\in M$ be such that $u_l(x_l)=\max_{x\in M}u_l(x)$, then we have that $u_l(x_l)\longrightarrow+\infty$.
Let $\mu_l>0$ be such that $\bar{Q}_l(x_l)\mu_l^n e^{nu_l(x_l)}=1$. Since $\bar{Q}_l\longrightarrow\bar{Q}_0$ $C^1(M)$, $\bar{Q}_0>0$ and $u_l(x_l)\longrightarrow+\infty$, we have that $\mu_l\longrightarrow 0$.
Now let $B^0(\delta\mu_l^{-1})$ be the euclidean ball of center 0 and radius $\delta\mu_l^{-1}$, with $\delta>0$ small fixed . For $x\in B^0(\delta\mu_l^{-1})$, we set

$$v_l(x)=u_l(exp_{x_l}(\mu_l x))-u_l(x_l)-\frac{1}{n}\log(k_n); \tag{2.2}$$

$$\tilde{Q}_l(x)=Q_l(exp_{x_l}(\mu_l x)); \tag{2.3}$$

$$\tilde{\bar{Q}}_l(x)=\bar{Q}_l(exp_{x_l}(\mu_l x)); \tag{2.4}$$

$$g_l(x)=\left(exp_{x_l}^*g\right)(\mu_l x). \tag{2.5}$$

We have that $g_l\longrightarrow dx^2$ $C^2_{loc}(\mathbb{R}^n)$ as $l\longrightarrow+\infty$.
Now from the Green representation formula we have,

$$u_l(x)-\bar{u}_l=\int_M G(x,y)P_g^n u_l(y)dV_g(y) \quad \forall x\in M, \tag{2.6}$$

where G is the Green function of P_g^n (see Proposition 0.3.2).
Now using equation (15) and differentiating (2.6) with respect to x we obtain that for $k=1,2$

$$\begin{aligned}|\nabla^k u_l|_g(x)&\leq\int_M|\nabla^k G(x,y)|_g|\bar{Q}_l(y)e^{nu_l(y)}-Q_l(y)|dV_g\\&\leq\int_M|\nabla^k G(x,y)|_g\bar{Q}_l(y)e^{nu_l(y)}dV_g+O(1),\end{aligned} \tag{2.7}$$

since $Q_l\longrightarrow Q_0$ in $C^1(M)$.
Now for $y_l\in B_{x_l}(R\mu_l)$, $R>0$ fixed we write that,

$$\begin{aligned}\int_M|\nabla^k G(y_l,y)|_g e^{nu_l(y)}dV_g(y)=&O\left(\mu_l^{-k}\int_{M\setminus B_{y_l}(\mu_l)}e^{nu_l}dV_g\right)\\&+O\left(e^{nu_l(x_l)}\int_{B_{y_l}(\mu_l)}d_g(y_l,y)^{-k}dv_g(y)\right)=O(\mu_l^{-k}).\end{aligned} \tag{2.8}$$

thanks to the fact that $u_l \leq u_l(x_l)$, to the relation $\bar{Q}_l(x_l)\mu_l^n e^{nu_l(x_l)} = 1$ to (2.1) and Proposition (0.3.2).
Together with the definition of v_l (see (2.2)) and the fact the fact that $v_l(x) \leq v_l(0) = -\frac{1}{n}\log(k_n)$ $\forall x \in \mathbb{R}^n$, we obtain $(v_l)_l$ is uniformly bounded in $C^2(K)$ for all compact subsets K of $\mathbb{R}^n$. Hence by Arzelà-Ascoli theorem we infer that

$$v_l \longrightarrow V_0 \quad in \;\; C^1_{loc}(\mathbb{R}^n), \tag{2.9}$$

hence we have that $V_0(x) \leq V_0(0) = -\frac{1}{n}\log(k_n)$ $\forall x \in \mathbb{R}^n$.
Clearly V_0 is a Lipschitz function since the constant which bounds the gradient of v_l is independent of the compact set K.
On the other hand from the Green's representation formula we have for $x \in \mathbb{R}^n$ fixed and for R big enough such that $x \in B^0(R)$

$$u_l(exp_{x_l}(\mu_l x)) - \bar{u}_l = \int_M G(exp_{x_l}(\mu_l x), y) P_g^n u_l(y) dV_g(y). \tag{2.10}$$

Now remarking that

$$u_l(exp_{x_l}(\mu_l x)) - u_l(x_l) = u_l(exp_{x_l}(\mu_l x)) - u_l(exp_{x_l}(0)),$$

we have the following relation

$$u_l(exp_{x_l}(\mu_l x)) - u_l(x_l) = ((u_l(exp_{x_l}(\mu_l x)) - \bar{u}_l) - (u_l(exp_{x_l}(0)) - \bar{u}_l).$$

Hence (2.10) gives

$$u_l(exp_{x_l}(\mu_l x)) - u_l(x_l) = \int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) P_g^n u_l(y) dV_g(y).$$

Moreover using (15) we obtain

$$u_l(exp_{x_l}(\mu_l x)) - u_l(x_l) = \int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) \tag{2.11}$$

$$- \int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) Q_l(y) dV_g(y). \tag{2.12}$$

Now setting

$$I_l(x) = \int_{B_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{nu_l(y)} dV_g(y); \tag{2.13}$$

$$\mathrm{II}_l(x) = \int_{M \setminus B_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{nu_l(y)} dV_g(y); \tag{2.14}$$

$$\mathrm{III}_l(x) = \int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) Q_l(y) dV_g(y); \tag{2.15}$$

we find

$$u_l(exp_{x_l}(\mu_l x)) - u_l(x_l) = I_l(x) + \mathrm{II}_l(x) + \mathrm{III}_l(x). \tag{2.16}$$

So using the definition of the v_l's we arrive to

$$v_l(x) = I_l(x) + \mathrm{II}\;_l(x) + \mathrm{III}_l(x) - \frac{1}{n}\log(k_n). \tag{2.17}$$

Now let study each of the terms $I_l(x), II_l(x), III_l(x)$ separately.
Using the change of variables $y = exp_{x_l}(\mu_l z)$ and setting

$$\overline{G}_l(x, z) = \left(G(exp_{x_l}(\mu_l x), exp_{x_l}(\mu_l z)) - G(exp_{x_l}(0), exp_{x_l}(\mu_l z))\right),$$

we have

$$I_l(x) = \int_{B_0(R)} \overline{G}_l(x,z)\bar{Q}_l(\exp_{x_l}(\mu_l z))e^{nu_l(exp_{x_l}(\mu_l z))}\mu_l^n dV_{g_l}(z). \tag{2.18}$$

Now using the relation $\bar{Q}(x_l)\mu_l^n e^{nu_l(x_l)} = 1$ and (2.2)-(2.5), we obtain

$$I_l(x) = \int_{B_0(R)} k_n \left(G(exp_{x_l}(\mu_l x), exp_{x_l}(\mu_l z)) - G(exp_{x_l}(0), exp_{x_l}(\mu_l z)\right) \frac{\tilde{\bar{Q}}_l(z)}{\bar{Q}(x_l)} e^{nv_l(z)} dV_{g_l}(z). \tag{2.19}$$

Now from the asymptotics of the Green's function (see Proposition (0.3.2)) we have,

$$I_l(x) = \int_{B_0(R)} k_n \left(\frac{1}{c_n}\log\left(\frac{|z|}{|x-z|}\right) + K_l(x,y)\right) \frac{\tilde{\bar{Q}}_l(z)}{\bar{Q}(x_l)} e^{nv_l(z)} dV_{g_l}(z) \quad \text{for } l \text{ large enough}, \tag{2.20}$$

with

$$K_l(x,z) = \left(K(exp_{x_l}(\mu_l x), exp_{x_l}(\mu_l z)) - K(exp_{x_l}(0), exp_{x_l}(\mu_l z))\right). \tag{2.21}$$

Hence since K is of class C^1 on M^2 and $g_l \longrightarrow dx^2$ in $C^2_{loc}(\mathbb{R}^n)$, then letting $l \longrightarrow +\infty$ we derive the following equality

$$\lim_l I_l(x) = \int_{B^0(R)} \sigma_n \log\left(\frac{|z|}{|x-z|}\right) e^{nV_0(z)} dz. \tag{2.22}$$

Now to estimate $\mathrm{II}_l(x)$ we write for l large

$$\mathrm{II}_l(x) = \int_{M\setminus B_{x_l}(R\mu_l)} \frac{1}{c_n} \log\left(\frac{d_g(exp_{x_l}(0), y)}{d_g(exp_{x_l}(\mu_l x), y)}\right) \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) \tag{2.23}$$
$$+ \int_{M\setminus B_{x_l}(R\mu_l)} \bar{K}_l(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y),$$

where

$$\bar{K}_l(x,y) = \left(K(exp_{x_l}(\mu_l x), y) - K(exp_{x_l}(0), y)\right). \tag{2.24}$$

Taking the absolute value in both sides of the equality (2.23) and using the change of variable $y = exp_{x_l}(\mu_l z)$ and the fact that $K \in C^1$ we obtain,

$$|\mathrm{II}_l(x)| \le \int_{\mathbb{R}^n\setminus B_0(R)} \frac{1}{c_n}\left|\log\left(\frac{|z|}{|x-z|}\right)\right| \frac{\tilde{\bar{Q}}_l(z)}{\bar{Q}(x_l)} e^{nv_l(z)} dV_{g_l}(z) + R\mu_l \int_{M\setminus B_{x_l}(R\mu_l)} \bar{Q}_l(y)e^{nu_l(y)}dV_g(z). \tag{2.25}$$

Hence letting $l \longrightarrow +\infty$ we deduce by (2.1) that

$$\limsup_l \mathrm{II}_l(x) = o_R(1). \tag{2.26}$$

Now using the same method one proves that

$$\mathrm{III}_l(x) \longrightarrow 0 \ \ as \ \ l \longrightarrow +\infty. \tag{2.27}$$

So we have that

$$V_0(x) = \int_{B^0(R)} \sigma_n \log\left(\frac{|z|}{|x-z|}\right) e^{nV_0(z)} dz - \frac{1}{n}\log(k_n) + \lim_l \mathrm{II}_l(x). \tag{2.28}$$

Hence letting $R \longrightarrow +\infty$ we obtain that V_0 solve the following conformally invariant integral equation

$$V_0(x) = \int_{\mathbb{R}^n} \sigma_n \log\left(\frac{|z|}{|x-z|}\right) e^{nV_0(z)} dz - \frac{1}{n}\log(k_n). \tag{2.29}$$

Now since V_0 is Lipschitz then the theory of singular integral operator gives that $V_0 \in C^1(\mathbb{R}^n)$. Moreover by using a change of variables and the fact that g_l converges to the Euclidean metric in $C^2_{loc}(\mathbb{R}^n)$ we obtain,

$$\lim_{l \longrightarrow +\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l e^{nu_l} dV_g = k_n \int_{B_0(R)} e^{nV_0} dx; \tag{2.30}$$

hence (2.1) implies that $e^{V_0} \in L^n(\mathbb{R}^n)$.
So by a classification result by X.Xu for the solutions of (2.29) (see Theorem 2.3.1) we get that,

$$V_0(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\right) \tag{2.31}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.
On the other hand from $V_0(x) \leq V_0(0) = -\frac{1}{n}\log(k_n)$ $\forall x \in \mathbb{R}^n$, we have that $\lambda = 2k_n$ and $x_0 = 0$ namely,

$$V_0(x) = \log\left(\frac{4\gamma_n}{4\gamma_n^2 + |x|^2}\right). \tag{2.32}$$

It is then easily checked that,

$$\lim_{R\to+\infty} \lim_{l\to+\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = k_n \int_{\mathbb{R}^n} e^{nV_0} dx. \tag{2.33}$$

Furthermore from a generalized Pohozaev identity by X.Xu (see Theorem 1.1) in [89] for the conformally invariant integral equation (2.29) we obtain that

$$\sigma_n \int_{\mathbb{R}^n} e^{nV_0(y)} dy = 2,$$

hence we derives that

$$\lim_{R\to+\infty} \lim_{l\to+\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = 2c_n = (n-1)!\omega_n. \tag{2.34}$$

Now for $k \geq 1$ we say that (H_k) holds if there exists k converging points $(x_{i,l})_l$ $i = 1, ..., k$, k sequences $(\mu_{i,l})$ $i = 1, ..., k$ of positive real numbers converging to 0 such that the following hold
(A_k^1)

$$\frac{d_g(x_{i,l}, x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i \neq j \quad i, j = 1, .., k \; and \quad \bar{Q}_l(x_{i,l}) \mu_{i.l}^n e^{nu_l(x_{i,l})} = 1;$$

(A_k^2)

$$v_{i,l}(x) = u_l(exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \frac{1}{n}\log(k_n) \longrightarrow V_0(x) = \log\left(\frac{4\gamma_n}{4\gamma_n^2 + |x|^2}\right) \; in \quad C^1_{loc}(\mathbb{R}^n) \quad \forall i$$

(A_k^3)

$$\forall \; i = 1, .., k, \text{ one has } \lim_{R\to+\infty} \lim_{l\to+\infty} \int_{B_{x_{i,l}}(R\mu_{i,l})} \bar{Q}_l(y) e^{nu_l(y)} = (n-1)!\omega_n.$$

Clearly, by the above arguments (H_1) holds. We let now $k \geq 1$ and assume that (H_k) holds. We also assume that

$$\sup_M R_{k,l}(x)^n e^{nu_l(x)} \longrightarrow +\infty \quad as \quad l \longrightarrow +\infty, \tag{2.35}$$

where

$$R_{k,l}(x) = \min_{i=1;..;k} d_g(x_{i,l}, x).$$

We prove in the following that in this situation (H_{k+1}) holds. For this purpose we let $x_{k+1,l} \in M$ be such that

$$R_{k,l}(x_{k+1,l})^n e^{nu_l(x_{k+1,l})} = \sup_M R_{k,l}(x)^n e^{nu_l(x)}, \tag{2.36}$$

and we set

$$\mu_{k+1,l} = \left(\frac{1}{\bar{Q}(x_{k+1,l})e^{nu_l(x_{k+1,l})}}\right)^{\frac{1}{n}}. \tag{2.37}$$

Since M is compact then (2.195), (2.36) and (2.37) imply that

$$\mu_{k+1,l} \longrightarrow +\infty \;\; as \;\; l \longrightarrow +\infty; \tag{2.38}$$

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{k+1,l}} \longrightarrow +\infty \;\; as \;\; l \longrightarrow +\infty \;\; \forall i = 1, ..., k. \tag{2.39}$$

Indeed from (2.195) we have that

$$R_{k,l}(x_{k+1,l})^n e^{nu_l(x_{k+1,l})} \longrightarrow +\infty,$$

and since $R_{k,l}(x_{k+1,l})$ is bounded because M compact then we obtain that,

$$e^{nu_l(x_{k+1,l})} \longrightarrow +\infty.$$

Now from (2.37), $\bar{Q}_l \longrightarrow \bar{Q}_0$ in $C^0(M)$ and $\bar{Q}_0 > 0$ we infer that

$$\mu_{k+1,l} \longrightarrow 0.$$

On the other hand we have that

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{k+1,l}} \geq \frac{R_{k,l}(x_{k+1,l})}{\mu_{k+1,l}} = (R_{k,l}(x_{k+1,l})^n \bar{Q}(x_{k+1,l})e^{nu_l(x_{k+1,l})})^{\frac{1}{n}},$$

hence (2.195) and (2.36) give that

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{k+1,l}} \longrightarrow +\infty.$$

Now thanks to (A_k^2), we can prove that

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{i,l}} \longrightarrow +\infty \;\; as \;\; l \longrightarrow +\infty \;\; \forall i = 1, ..., k. \tag{2.40}$$

Indeed if $d_g(x_{i,l}, x_{k+1,l})$ stays away from 0 then since $\mu_{i,l} \longrightarrow 0$, we are done. So suppose that $d_g(x_{i,l}, x_{k+1,l}) \leq \epsilon$, ϵ small enough and set,

$$\tilde{x}_{k+1,l} = \frac{exp^{-1}_{x_{i,l}}(x_{k+1,l})}{\mu_{i,l}}.$$

We have that,

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{i,l}} = \frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{k+1,l}} \frac{\mu_{k+1,l}}{\mu_{i,l}}.$$

On the other hand we have also that

$$(\frac{\mu_{k+1,l}}{\mu_{i,l}})^n = \frac{\bar{Q}(x_{k,l})}{\bar{Q}(x_{k+1,l})e^{nu_k(x_{k+1,l})-u_k(x_{k,l})}} = \frac{\bar{Q}(x_{k,l})}{\bar{Q}(x_{k+1,l})e^{nv_{i,l}(\tilde{x}_{k+1,l})}}.$$

Hence if $(\tilde{x}_{k+1,l})_l$ is bounded in $\mathbb{R}^n$ we have thanks to (A_k^2) that $\frac{\mu_{k+1,l}}{\mu_{i,l}}$ converges to a positive number hence we are done. If $(\tilde{x}_{k+1,l})_l$ were not bounded, then the relation

$$d_g(x_{i,l}, x_{k+1,l}) = \mu_{i,l}\|\tilde{x}_{k+1,l}\|$$

shows that

$$\frac{d_g(x_{i,l}, x_{k+1,l})}{\mu_{i,l}} \longrightarrow +\infty \;\; as \;\; l \longrightarrow +\infty;$$

hence $\left(A_{k+1}^1\right)$ holds.
Moreover it follows from (2.36) and (A_{k+1}^1) that

$$\lim_{l\longrightarrow+\infty} \sup_{z\in B_{x_{k+1,l}}(R\mu_{k+1,l})} (u_l(z) - u_l(x_{k+1,l})) = 0. \tag{2.41}$$

Indeed from (2.36) we have that ,

$$R_{k,l}(x_{k+1,l})^n e^{nu_l(x_{k+1,l})} \geq R_{k,l}(x)^n e^{nu_l(x)} \;\; \forall x \in M;$$

hence the following holds

$$R_{k,l}(x_{k+1,l})^n e^{nu_l(x_{k+1,l})} \geq R_{k,l}(z)^n e^{nu_l(z)} \;\; \forall z \in B_{x_{k+1,l}}(R\mu_{k+1,l}).$$

So taking the n-th root in both sides of the inequality we obtain that

$$R_{k,l}(x_{k+1,l}) e^{u_l(x_{k+1,l})} \geq R_{k,l}(z) e^{u_l(z)} \;\; \forall z \in B'_{x_{k+1,l}}(R\mu_{k+1,l});$$

hence dividing by $e^{u_l(x_{k+1,l})}R_{k,l}(z)$ in both sides we get

$$e^{u_l(z)-u_l(x_{k+1,l})} \leq \frac{R_{k,l}(x_{k+1,l})}{R_{k,l}(z)}.$$

Now let $z_{k+1,l} \in B'_{x_{k+1,l}}(R\mu_{k+1,l})$ be such that

$$u_l(z_{k+1,l}) - u_l(x_{k+1,l}) = \sup_{z\in B_{x_{k+1,l}}(R\mu_{k+1,l})} (u_l(z) - u_l(x_{k+1,l}));$$

so we have

$$e^{u_l(z_{k+1,l})-u_l(x_{k+1,l})} \leq \frac{R_{k,l}(x_{k+1,l})}{R_{k,l}(z_{k+1,l})},$$

and let $i_{k+1,l} \in \{1, ..., k\}$ be such that,

$$R_{k,l}(z_{k+1,l}) = d_g(x_{i_{k+1,l},l}, z_{k+1,l});$$

so we have that

$$e^{u_l(z_{k+1,l})-u_l(x_{k+1,l})} \leq \frac{R_{k,l}(x_{k+1,l})}{d_g(x_{i_{k+1,l},l}, z_{k+1,l})} \leq \frac{d_g(x_{i_{k+1,l},l}, x_{k+1,l})}{d_g(x_{i_{k+1,l},l}, z_{k+1,l})} \tag{2.42}$$

$$e^{u_l(z_{k+1,l})-u_l(x_{k+1,l})} \leq 1 + \frac{d_g(z, x_{k+1,l})}{d_g(x_{i_{k+1,l},l}, z_{k+1,l})} \leq 1 + \frac{R\mu_{k+1,l}}{d_g(x_{i_{k+1,l},l}, z_{k+1,l})}. \tag{2.43}$$

On the other hand the following chain of inequality holds

$$\frac{d_g(x_{i_{k+1,l},l}, z_{k+1,l})}{\mu_{k+1,l}} \geq \frac{d_g(x_{i_{k+1,l},l}, x_{k+1,l})}{\mu_{k+1,l}} - \frac{d_g(x_{k+1,l}, z_{k+1,l})}{\mu_{k+1,l}} \geq \frac{d_g(x_{i_{k+1,l},l}, x_{k+1,l})}{\mu_{k+1,l}} - R;$$

but from (A^1_{k+1}) we deduce that,

$$\frac{d_g(x_{i_{k+1,l},l}, x_{k+1,l})}{\mu_{k+1,l}} \longrightarrow +\infty;$$

hence

$$\frac{d_g(x_{i_{k+1,l},l}, z_{k+1,l})}{R\mu_{k+1,l}} \longrightarrow +\infty;$$

which imply with (2.43) that

$$\limsup_l \left(u_l(z_{k+1,l}) - u_k(x_{k+1,l})\right) \leq 0;$$

and since

$$\left(u_l(z_{k+1,l}) - u_k(x_{k+1,l})\right) \geq 0;$$

then we have proved that,

$$\lim_{l\longrightarrow+\infty} \sup_{z\in B_{x_{k+1,l}}(R\mu_{k+1,l})} \left(u_l(z) - u_l(x_{k+1,l})\right) = 0.$$

Now mimicking what we did above thanks to the Green's representation formula (see in particular formula (2.8)) and using (2.41) then one proves that up to a subsequence,

$$v_{k+1,l}(x) = u_l(exp_{x_{k+1,l}}(\mu_{k+1,l}x)) - u_l(x_{k+1,l}) - \frac{1}{n}\log(k_n) \longrightarrow V_0(x) = \log\left(\frac{4\gamma_n}{4\gamma_n^2 + |x|^2}\right) \ \ in \ \ C^1_{loc}(\mathbb{R}^n),$$

and

$$\lim_{R\rightarrow+\infty}\lim_{l\rightarrow+\infty}\int_{B_{x_{k+1,l}}(R\mu_{k+1,l})} \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) = (n-1)!\omega_n.$$

Hence recollecting the informations above, one gets that (H_{k+1}) holds. Moreover since $\left(A^1_k\right)$ and $\left(A^3_k\right)$ of H_k imply that

$$\int_M \bar{Q}(y)e^{nu_l(y)}dV_g(y) \geq (n-1)!\omega_n k + o_l(1),$$

then we easily get thanks to (2.1) that there exists a maximal k , $1 \leq k \leq \frac{1}{(n-1)!\omega_n}\int_M Q_0(y)dV_g(y)$, such that (H_k) holds. Arriving to this maximal k, we get that (2.195) cannot hold. Hence setting $N = k$ the proof of Step 1 is done.

Step 2

There exists a constant $C > 0$ such that

$$R_l(x)|\nabla u_l|_g(x) \leq C \qquad \forall x \in M \ \ and \ \ \forall l \in N; \tag{2.44}$$

where

$$R_l(x) = \min_{i=1,..,N} d_g(x_{i,l}, x);$$

and the $x_{i,l}$'s are as in Step 1.

Proof of Step 2

We use again the Green's representation formula for u_l, that we differentiate. We let $x_l \in M$ be such that $x_l \neq x_{i,l}$ for all $i =, .., N$. Note that, for $x_l = x_{i,l}$, the estimates of the proposition are obvious. We write thanks to the asymptotics of the Green function of P_g^n see (Proposition 0.3.2) that

$$|\nabla u_l|_g(x_l) = O\left(\int_M \frac{1}{d_g(x_l, y)} e^{nu_l(y)} dV_g(y)\right) + O(1). \tag{2.45}$$

Now for $i = 1, ...N$, we set

$$\Omega_{i,l} = \{y \in M, R_l(y) = d_g(x_{i,l}, y)\}; \tag{2.46}$$

and we write that

$$\int_{\Omega_{i,l}} \frac{1}{(d_g(x_l, y))} e^{nu_l(y)} dV_g(y) = I_{i,l} + \mathrm{II}_{i,l} + \mathrm{III}_{i,l}; \tag{2.47}$$

with

$$I_{i,l} = \int_{\Omega_{i,l} \cap B_{x_{i,l}}(\frac{d_g(x_l, x_{i,l})}{2})} \frac{1}{(d_g(x_l, y))} e^{nu_l(y)} dV_g(y); \tag{2.48}$$

$$\mathrm{II}_{i,l} = \int_{\Omega_{i,l} \setminus B_{x_l}(5d_g(x_l, x_{i,l}))} \frac{1}{(d_g(x_l, y))} e^{nu_l(y)} dV_g(y); \tag{2.49}$$

and

$$\mathrm{III}_{i,l} = \int_{\Omega_{i,l} \cap B_{x_l}(5d_g(x_l, x_{i,l})) \setminus B_{x_{i,l}}(\frac{d_g(x_l, x_{i,l})}{2})} \frac{1}{(d_g(x_l, y))} e^{nu_l(y)} dV_g(y). \tag{2.50}$$

To estimate $I_{i,l}$ we use the fact that $y \in B_{x_{i,l}}(\frac{d_g(x_l, x_{i,l})}{2})$, the triangle inequality and equation (3) to find that

$$I_{i,l} = O\left(\frac{1}{(d_g(x_l, x_{i,l}))}\right). \tag{2.51}$$

On the other hand using the fact that $y \notin B_{x_l}(5d_g(x_{i,l}, x_l))$, and equation (0.3.2) we have that

$$\mathrm{II}_{i,l} = O\left(\frac{1}{(d_g(x_l, x_{i,l}))}\right). \tag{2.52}$$

Moreover using the fact that we are in $\Omega_{i,l}$, assumption *d)* of Step 1 implies that

$$\mathrm{III}_{i,l} = O\left(\int_{B_{x_l}(5d_g(x_l, x_{i,l})) \setminus B_{x_{i,l}} \frac{d_g(x_l, x_{i,l})}{2})} \frac{1}{(d_g(x_l, y) d_g(x_{i,l}, y)^n)}\right); \tag{2.53}$$

hence using the fact that $y \notin B_{x_{i,l}}(\frac{d_g(x_l, x_{i,l})}{2})$, we obtain

$$\mathrm{III}_{i,l} = O\left(\frac{1}{(d_g(x_l, x_{i,l}))^n} \int_{B_{x_l}(5d_g(x_l, x_{i,l}))} \frac{1}{(d_g(x_l, y)}\right). \tag{2.54}$$

Now working in geodesic polar coordinatesat x_l we have that

$$\int_{B_{x_l}(5d_g(x_l, x_{i,l}))} \frac{1}{d_g(x_l, y)} = O\left((d_g(x_{i,l}, x_l))^{n-1}\right); \tag{2.55}$$

hence we derive

$$\mathrm{III}_{i,l} = O\left(\frac{1}{d_g(x_l, x_{i,l})}\right). \tag{2.56}$$

So we have
$$\int_{\Omega_{i,l}} \frac{1}{(d_g(x_l,y))} e^{nu_l(y)} dV_g(y) = O\left(\frac{1}{(d_g(x_l,x_{i,l}))}\right); \quad (2.57)$$
hence Step 2 clearly follows.

Step 3

Set
$$R_{i,l} = \min_{i \neq j} d_g(x_{i,l}, x_{j,l}); \quad (2.58)$$
we have that

1) There exists a constant $C > 0$ such that $\forall\, r \in (0, R_{i,l}]$ $\forall\, s \in (\frac{r}{4}, r]$
$$|u_l\left(exp_{x_{i,l}}(rx)\right) - u_l\left(exp_{x_{i,l}}(sy)\right)| \le C \quad for\ \ all\ \ x, y \in \mathbb{R}^n\ \ such\ \ that\ \ |x|,\ |y| \le \frac{3}{2}. \quad (2.59)$$

2) If $d_{i,l}$ is such that $0 < d_{i,l} \le \frac{R_{i,l}}{2}$ and $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$ then we have that,if
$$\int_{B_{x_{i,l}}(d_{i,l})} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = (n-1)!\omega_n + o_l(1); \quad (2.60)$$
then
$$\int_{B_{x_{i,l}}(2d_{i,l})} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = (n-1)!\omega_n + o_l(1).$$

3) Let R be large and fixed. If $d_{i,l} > 0$ is such that $d_{i,l} \longrightarrow 0$, $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, $d_{i,l} < \frac{R_{i,l}}{4R}$ and
$$\int_{B_{x_{i,l}}(\frac{d_{i,l}}{2R})} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = (n-1)!\omega_n + o_l(1);$$
then by setting
$$\tilde{u}_l(x) = u_l(exp_{x_{i,l}}(d_{i,l}x)); \qquad x \in A_{2R};$$
where $A_{2R} = B^0(2R) \setminus B^0(\frac{1}{2R})$, we have that,
$$||d_{i,l}^n e^{n\tilde{u}_l}||_{C^\alpha(A_R)} \longrightarrow 0 \ \ as\ \ l \longrightarrow +\infty;$$
for some $\alpha \in (0,1)$ where $A_R = B^0(R) \setminus B^0(\frac{1}{R})$.

Proof of Step 3

Property 1) follows immediately from Step 2 and the definition of $R_{i,l}$.
In fact we can join rx to sy by a curve whose length is bounded by a constant proportional to r.
On the other hand from $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, point c) of Step1 and (2.60) we have that
$$\int_{B_{x_{i,l}}(d_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{nu_l(y)} dV_g(y) = o_l(1). \quad (2.61)$$
Now from (2.59),by taking $s = \frac{r}{2}$ and $r = 2d_{i,l}$ we obtain that
$$\int_{B_{x_{i,l}}(2d_{i,l}) \setminus B_{x_{i,l}}(d_{i,l})} e^{nu_l(y)} dV_g(y) \le C \int_{B_{x_{i,l}}(d_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{nu_l(y)} dV_g(y);$$

hence

$$\int_{B_{x_{i,l}}(2d_{i,l})\setminus B_{x_{i,l}}(d_{i,l})} e^{nu_l(y)}dV_g(y) = o_l(1).$$

So also point 2) of the step is proved.
Now let us prove point 3 . First of all applying point 2) of the step a finite number of times we obtain

$$\int_{B_{x_{i,l}}(2Rd_{i,l})\setminus B_{x_{i,l}}(\frac{d_{i,l}}{2R})} e^{nu_l(y)}dV_g(y) = o_l(1); \tag{2.62}$$

hence since $\bar{Q}_l \longrightarrow \bar{Q}_0$ $C^1(M)$ then we obtain from (2.62) that,

$$\int_{B_{x_{i,l}}(2Rd_{i,l})\setminus B_{x_{i,l}}(\frac{d_{i,l}}{2R})} \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) = o_l(1). \tag{2.63}$$

On the other hand using the change of variable $y = exp_{x_{i,l}}(d_{i,l}x)$ and letting $J_{d_{i,l}}(x)$ denote the Jacobian of the exponential map at the point $x_{i,l}$ applied to the vector $d_{i,l}x$ we have that

$$\int_{B_{x_{i,l}}(2Rd_{i,l})\setminus B_{x_{i,l}}(\frac{d_{i,l}}{2R})} \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) = \int_{A_{2R}} \bar{Q}_{d_{i,l}}(x)e^{n\tilde{u}_l(x)}d_{i,l}^n J_{d_{i,l}}(x)dV_{g_{d_{i,l}}}(x) \tag{2.64}$$

where

$$g_{d_{i,l}}(x) = (exp^*_{x_{i,l}}g)(d_{i,l}x); \tag{2.65}$$

$$\bar{Q}_{d_{i,l}}(x) = \bar{Q}_l(d_{i,l}x); \tag{2.66}$$

Hence (2.63) implies that

$$\int_{A_{2R}} \bar{Q}_{d_{i,l}}(x)e^{n\tilde{u}_l(x)}d_{i,l}^n J_{d_{i,l}}(x)dV_{g_{d_{i,l}}}(x) = o_l(1). \tag{2.67}$$

Now let fix p so big that $H^{1,p}(A_R)$ is continuously embedded into $C^\alpha(A_R)$ where α is given by the Sobolev embedding theorem, that is $\alpha = \frac{p-n}{p}$.
Remarking that since $d_{i,l} \longrightarrow 0$ then $g_{d_{i,l}} \longrightarrow dx^2$ in every $C^k(A_R)$, then the embedding constant can be chosen independent of l.
On the other hand, using an argument of Brezis and Merle see [15](Theorem 1) we have that

$$||d_{i,l}^n e^{n\tilde{u}_l}||_{L^p(A_R)} = o_l(1).$$

Indeed from the Green representation formula for u_l we have that

$$u_l(x) = \bar{u}_l + \tilde{O}(1) + \int_M G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y) \qquad x \in B_{x_{i,l}}(Rd_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{R}).$$

Here $\tilde{O}(1)$ stands for a quantity bounded from above and from below uniformly in l.
Now defining $B_{i,l} = B_{x_{i,l}}(2Rd_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{2R})$, we obtain

$$u_l(x) = \bar{u}_l + \tilde{O}(1) + \int_{B_{i,l}} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y) + \int_{M\setminus B_{i,l}} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y). \tag{2.68}$$

Hence setting

$$\hat{u}_l(x) = \bar{u}_l(x) + \int_{M\setminus B_{i,l}} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y) \qquad x \in B_{i,l};$$

we have that (2.68) becomes,

$$u_l(x) = \hat{u}_l(x) + \tilde{O}(1) + \int_{B_{i,l}} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y). \tag{2.69}$$

Now let us estimate $\int_{B_{i,l}} e^{n\hat{u}_l(x)}dV_g(x)$.
From (2.69) we obtain,

$$e^{nu_l(x)} \geq Ce^{n\hat{u}_l(x)}e^{\int_{B_{i,l}} nG(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)}; \tag{2.70}$$

hence using the asymptotics of the Green's function (see Proposition (0.3.2)), we find that

$$e^{nu_l(x)} \geq C\frac{e^{n\hat{u}_l(x)}}{d_{i,l}^{\frac{n}{c_n}\int_{B_{i,l}}\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)}}; \tag{2.71}$$

so integrating we obtain,

$$\int_{B_{i,l}} e^{nu_l(x)}dV_g(x) \geq C\frac{\int_{B_{i,l}} e^{n\hat{u}_l(x)}dV_g(x)}{d_{i,l}^{\frac{n}{c_n}\int_{B_{i,l}}\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)}}; \tag{2.72}$$

hence from (2.62) we arrive to the following estimate

$$\int_{B_{i,l}} e^{n\hat{u}_l(x)}dV_g(x) = o_l\left(d_{i,l}^{\frac{n}{c_n}\int_{B_{i,l}}\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)}\right). \tag{2.73}$$

Now let us estimate $||e^{n\tilde{u}_l}||_{L^p(A_R)}$. From equation (2.69) we have that,

$$npu_l(x) = np\hat{u}_l(x) + \tilde{O}(1) + \int_{B_{i,l}} npG(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y) \tag{2.74}$$

hence

$$npu_l(exp_{x_{i,l}}(d_{i,l}x)) = np\hat{u}_l(exp_{x_{i,l}}(d_{i,l}x) + \tilde{O}(1) + \int_{B_{i,l}} npG(exp_{x_{i,l}}(d_{i,l}x),y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y);$$

so using the change of variable $y = exp_{x_{i,l}}(d_{i,l}z)$ and setting $\hat{u}_{d_{i,l}}(x) = \hat{u}_l(exp_{x_{i,l}}(d_{i,l}x))$, we obtain that,

$$np\tilde{u}_l(x) = np\hat{u}_{d_{i,l}}(x)+\tilde{O}(1)+\int_{B_{i,l}} npd_{i,l}^n J_{d_{i,l}}(z)G(expx_{i,l}(d_{i,l}x), exp_{x_{i,l}}(d_{i,l}z))\bar{Q}_{d_{i,l}}(z)e^{n\tilde{u}_l(z)}dV_g(z). \tag{2.75}$$

Now by using the Harnack-type inequality for u_l, see (2.59) and the asymptotics of the Green function in Proposition 0.3.2 we have an Harnack-type inequality for for $\hat{u}_l$. Namely there exist a positive constant C such that

$$|\hat{u}_l(x_1) - \hat{u}_l(x_2)| \leq C \quad \forall\ x_1\ x_2 \in\ B_{i,l};$$

hence the following holds,

$$e^{n\hat{u}_{d_{i,l}}(x)} \leq C\frac{\int_{A_R} d_{i,l}^n J_{d_{i,l}}(y)e^{n\hat{u}_{d_{i,l}}(y)}dV_{g_{d_{i,l}}}(y)}{d_{i,l}^n}. \tag{2.76}$$

On the other hand by taking the exponential and integrating on both sides of equation (2.75), using Jensen's inequality, the asymptotics of the Green's function (see Proposition 0.3.2), and Fubini theorem, we arrive to

$$\int_{A_R} e^{np\tilde{u}_l(x)}dV_{g_{d_{i,l}}} \leq C\frac{(\int_{A_R} d_{i,l}^n J_{d_{i,l}}(z)e^{n\hat{u}_{d_{i,l}}(z)}dV_{g_{d_{i,l}}}(z))^p}{d_{i,l}^{np}d_{i,l}^{\frac{np}{c_n}\int_{A_{2R}} d_{i,l}^n J_{d_{i,l}}(z)\bar{Q}_{d_{i,l}}(z)e^{n\tilde{u}_l(z)}dV_{g_{d_{i,l}}}(z)}}I_{d_{i,l}} \tag{2.77}$$

where

$$I_{d_{i,l}} = \sup_{y \in A_{2R}} \int_{A_R} \frac{1}{|x-y|^{\frac{np}{cn} \int_{A_{2R}} d_{i,l}^n J_{d_{i,l}}(z) \bar{Q}_{d_{i,l}}(y) e^{n\tilde{u}_l(z)} dV_{g_{d_{i,l}}}(z)}} dV_{g_{d_{i,l}}}(x).$$

Hence taking the pth-root in both sides we find

$$||e^{n\tilde{u}_l}||_{L^p(A_R)} \leq C \frac{\int_{A_R} d_{i,l}^n J_{d_{i,l}}(z) e^{n\hat{u}_{d_{i,l}}(z)} dV_{g_{d_{i,l}}}(z)}{d_{i,l}^n d_{i,l}^{\frac{n}{cn} \int_{A_{2R}} d_{i,l}^n J_{d_{i,l}}(z) \bar{Q}_{d_{i,l}}(z) e^{n\tilde{u}_l(z)} dV_{g_{d_{i,l}}}(z)}} I_{d_{i,l}}^{\frac{1}{p}}. \tag{2.78}$$

From (2.63) and (2.64) we derive that

$$\int_{A_{2R}} d_{i,l}^n J_{d_{i,l}}(z) \bar{Q}_{d_{i,l}}(z) e^{n\tilde{u}_l(z)} dV_{g_{d_{i,l}}}(z) = o_l(1), \tag{2.79}$$

and hence

$$|I_{d_{i,l}}^{\frac{1}{p}}| \leq C. \tag{2.80}$$

Furthermore by a change of variables we have easily that

$$\int_{A_R} d_{i,l}^n J_{d_{i,l}}(y) e^{n\hat{u}_{d_{i,l}}(y)} dV_{g_{d_{i,l}}}(y) = \int_{B_{i,l}} e^{\hat{u}_l(x)} dV_g(x). \tag{2.81}$$

From (2.73) we obtain

$$||e^{n\tilde{u}_l}||_{L^p(A_R)} = o_l(\frac{1}{d_{i,l}^n}); \tag{2.82}$$

hence

$$||d_{i,l}^n e^{n\tilde{u}_l}||_{L^p(A_R)} = o_l(1). \tag{2.83}$$

On the other hand remarking that from Step 2 we have that $||\nabla \tilde{u}_l||_{L^\infty} = O(1)$, then we deduce that

$$||\nabla(d_{i,l}^n e^{n\tilde{u}_l})||_{L^p(A_R)} \leq C||d_{i,l} e^{n\tilde{u}_l}||_{L^p(A_R)}; \tag{2.84}$$

hence (2.83) implies

$$||\nabla(d_{i,l}^n e^{n\tilde{u}_l})||_{L^p(A_R)} = o_l(1); \tag{2.85}$$

so from (2.83) and (2.85) we obtain,

$$||d_{i,l}^n e^{n\tilde{u}_l}||_{H^{1,p}(A_R)} = o_l(1). \tag{2.86}$$

Hence from the Sobolev embedding we arrive to

$$||d_{i,l}^n e^{n\tilde{u}_l}||_{C^\alpha(A_R)} = o_l(1); \tag{2.87}$$

so end of point 3 and Step also.

Step 4

There exists a positive constant C independent of l and i such that $r_{i,l} \geq \frac{R_{i,l}}{C}$ and

$$\int_{B_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l(y) e^{nu_l(y)} dV_g(y) = (n-1)!\omega_n + o_l(1). \tag{2.88}$$

Proof of Step 4

First of all fix $\frac{1}{n} < \nu < \frac{2}{n}$ and set for $i = 1, ..., N$,

$$\bar{u}_{i,l}(r) = Vol_g(\partial B_{x_{i,l}}(r))^{-1} \int_{\partial B_{x_{i,l}}(r)} u_l(x) d\sigma_g(x) \quad \forall 0 \leq r < inj_g(M); \tag{2.89}$$

$$\varphi_{i,l}(r) = r^{n\nu} exp\,(\bar{u}_{i,l}(r)) \quad \forall 0 \leq r < inj_g(M). \tag{2.90}$$

By assumption *b)* of Step 1 we have that there exists R_ν such that,

$$\forall R \geq R_\nu \quad \varphi^{'}_{i,l}(R\mu_{i,l}) < 0 \quad \forall l \;\; sufficiently \;\;\; large \;\; (depending \;\; on \;\; R). \tag{2.91}$$

Now we define $r_{i,l}$ by

$$r_{i,l} = \sup\{R_\nu \mu_{i,l} \leq r \leq \frac{R_{i,l}}{2} \;\; s.t \;\; \varphi^{'}_{i,l}(\cdot) < 0 \;\; in \;\;\; [R_\nu, r)\}. \tag{2.92}$$

Hence (2.91) implies that

$$\frac{r_{i,l}}{\mu_{i,l}} \longrightarrow +\infty \quad as \;\; l \longrightarrow +\infty. \tag{2.93}$$

Now to prove the step it suffices to show that $\frac{R_{i,l}}{r_{i,l}} \not\longrightarrow +\infty$ as $l \longrightarrow +\infty$.
Indeed if $\frac{R_{i,l}}{r_{i,l}} \not\longrightarrow +\infty$, we have that there exist a positive constant C such that $\frac{R_{i,l}}{C} \leq r_{i,l}$.
On the other hand from the Harnack type inequality (2.59), point b) of Step 1, and (2.92) we have that for any $\eta > 0$, there exists $R_\eta > 0$ such that for any $R > R_\eta$, we have that

$$d_g(x, x_{i,l})^{n\nu} e^{nu_l} \leq \eta \mu_{i,l}^{n(\nu-1)} \quad \forall x \in B_{x_{i,l}}(r_{i,l}) \setminus B_{x_{i,l}}(R\mu_{i,l}). \tag{2.94}$$

Since $\frac{r_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$ see (2.93) and $\frac{R_{i,l}}{2} \geq r_{i,l}$ see (2.92), we have $\frac{R_{i,l}}{C\mu_{i,l}} \longrightarrow +\infty$, hence point c) of Step 1 implies that

$$\int_{B_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l e^{nu_l} = (n-1)!\omega_n + o_l(1).$$

On the other hand, by continuity and by the definition of $r_{i,l}$ it follows that

$$\varphi^{'}_{i,l}(r_{i,l}) = 0. \tag{2.95}$$

Let us assume by contradiction that $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$. We will show next that $\varphi^{'}_{i,l}(r_{i,l}) < 0$ for l large contradicting the above equality (2.95). To do so we will study $\bar{u}_{i,l}(\cdot)$.
First let us remark that since M is compact then $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$ implies that $r_{i,l} \longrightarrow 0$.
From the Green's representation formula for u_l we have the following equation,

$$u_l(x) = \int_M G(x,y) P^n_g u_l(y) dV_g(y) + \bar{u}_l = \int_M G(x,y)\bar{Q}_l(y) e^{nu_l(y)} dV_g(y) + \bar{u}_l - \int_M G(x,y) Q_l(y) dV_g(y).$$

Hence

$$\bar{u}_{i,l}(r) = (Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_M G(x,y)\bar{Q}_l(y) e^{nu_l(y)} dV_g(y) d\sigma_g(x) + \bar{u}_l \tag{2.96}$$

$$-(Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_M G(x,y) Q_l(y) dV_g(y) d\sigma_g(x). \tag{2.97}$$

Setting

$$F_{i,l}(r) = (Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_M G(x,y) Q_l(y) dV_g(y) d\sigma_g(x);$$

we obtain

$$\bar{u}_{i,l} = (Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_M G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)d\sigma_g(x) + \bar{u}_l - F_{i,l}(r).$$

Since $Q_l \longrightarrow Q_0$ in $C^1(M)$ we have that $F_{i,l}$ is of class C^1 for all i,l and moreover,

$$|F^{'}_{i,l}(r)| \leq C; \quad \forall\ r \in (0, \frac{inj_g(M)}{4}). \tag{2.98}$$

Now let $\frac{inj_g(M)}{4} < A < \frac{inj_g(M)}{2}$ be fixed: we have that

$$\int_M G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y) = \int_{B_{x_{i,l}}(A)} G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y) + \int_{M\setminus B_{x_{i,l}}(A)} G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y). \tag{2.99}$$

So

$$\bar{u}_{i,l}(r) = Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_{B_{x_{i,l}}(A)} \left(G(x,y) - K(x,y)\right) \bar{Q}_l(y)e^{nu_l(y)}dV_g(y)d\sigma_g(x) + \bar{u}_l - F_{i,l}(r) + H_{i,l}(r); \tag{2.100}$$

with

$$H_{i,l}(r) = Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_{M\setminus B_{x_{i,l}}(A)} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)d\sigma_g(x) \tag{2.101}$$
$$+Vol_g(\partial B_{x_{i,l}}(r)))^{-1} \int_{\partial B_{x_{i,l}}(r)} \int_{B_{x_{i,l}}(A)} K(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y)d\sigma_g(x).$$

Since G is smooth out of $Diag(M)$, then for all i,l; $H_{i,l} \in C^1\left(0, \frac{inj_g(M)}{4}\right)$ and moreover,

$$|H^{'}_{i,l}(r)| \leq C \quad \forall\ r \in (0, \frac{inj_g(M)}{4}). \tag{2.102}$$

Now using the change of variable $x = r\theta$ and $y = s\tilde{\theta}$ we obtain

$$\bar{u}_{i,l} = (Vol(S^{n-1}))^{-1} \int_{S^{n-1}} \int_{S^{n-1}} \int_0^A f(r,\theta)\left(G(r\theta, s\tilde{\theta}) - K(r\theta, s\tilde{\theta})\right)\bar{Q}(s\tilde{\theta})e^{nu_l(s\tilde{\theta})}s^{n-1}f(s,\tilde{\theta})dsd\tilde{\theta}d\theta + \bar{u}_l - F_{i,l}(r) + H_{i,l}(r).$$

So differentiating with respect to r and setting

$$\Gamma(r,\theta,\tilde{\theta},s) = \frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\right)$$

we have that

$$\bar{u}^{'}_{i,l}(r) = (Vol(S^{n-1}))^{-1} \int_{S^{n-1}} \int_{S^{n-1}} \int_0^A \Gamma(r,\theta,\tilde{\theta},s)\bar{Q}(s\tilde{\theta})e^{nu_l(s\tilde{\theta})}s^{n-1}f(s,\tilde{\theta})dsd\tilde{\theta}d\theta - F^{'}_{i,l}(r) + H^{'}_{i,l}(r).$$

From the asymptotics of $G(\cdot,\cdot)$ (see Proposition (0.3.2)) and the fact that f is bounded in C^2, it follows that

$$(Vol(S^{n-1}))^{-1} \int_{S^{n-1}} \int_{S^{n-1}} \left(G(r\theta, s\tilde{\theta}) - K(r\theta, s\tilde{\theta})\right) d\tilde{\theta}d\theta = \hat{f}(r,s)\log(\frac{1}{|r-s|}) + H(r,s); \tag{2.103}$$

with $H(\cdot,\cdot)$ of class C^α and $\hat{f}(\cdot,\cdot)$ of class C^2.
Hence setting

$$\tilde{G}(r,s)=(Vol(S^{n-1}))^{-1}\int_{S^{n-1}}\int_{S^{n-1}}\frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta})-K(r\theta,s\tilde{\theta}))\right)\bar{Q}(s\tilde{\theta})f(s,\tilde{\theta})d\tilde{\theta}d\theta. \tag{2.104}$$

we obtain

$$\tilde{G}(r,s)=\hat{f}(r,s)\frac{1}{r-s}+\tilde{H}(r,s); \tag{2.105}$$

where $\tilde{H}(r,\cdot)$ is integrable for every r fixed.
On the other hand using the Harnack type inequality (see (2.59)) we have that,

$$u_l(s\tilde{\theta})\leq \bar{u}_{i,l}(s)+C\quad uniformly\ in\ \tilde{\theta},$$

hence we obtain

$$\bar{u}^{'}_{i,l}(r)\leq C\int_0^A s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds-F^{'}_{i,l}(r)+H^{'}_{i,l}(r).$$

Now le study $\int_0^A s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds$. To do so let R so large such that $r_{i,l}\leq\frac{R_{i,l}}{4R}$(this is possible because of the assumption of contradiction). Now let us split the integral in the following way,

$$\begin{aligned}\int_0^A s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds&=\int_0^{\frac{r_{i,l}}{R}} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds+\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds\\&+\int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds+\int_{\frac{R_{i,l}}{C}}^{A} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds.\end{aligned}$$

Using the fact that we are at the scale $\frac{r_{i,l}}{R}$ then b) of Step 1 implies that we have the following estimates for the first term of the equality above with $r=r_{i,l}$,

$$\int_0^{\frac{r_{i,l}}{R}} s^{n-1}\tilde{G}(r_{i,l},s)e^{n\bar{u}_{i,l}(s)}ds=-\frac{2}{r_{i,l}}+o_l(1)\frac{1}{r_{i,l}}$$

On the other hand using assumption *d)* of Step 1 we obtain the following estimates for the third term of the equality above with $r=r_{i,l}$

$$\int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^{n-1}\tilde{G}(r_{i,l},s)e^{n\bar{u}_{i,l}(s)}ds=o_l(1)\frac{1}{r_{i,l}}.$$

We have also using assumption *d)* of Step 1 and the fact that $\frac{R_{i,l}}{r_{i,l}}\longrightarrow+\infty$ the following estimate for the fourth still with $r=r_{i,l}$,

$$\int_{\frac{R_{i,l}}{C}}^{A} s^{n-1}\tilde{G}(r_{i,l},s)e^{n\bar{u}_{i,l}(s)}ds=o_l(1)\frac{1}{r_{i,l}}.$$

Now let us estimate the second term. For this we will use the point 3) of Step 3. First we recall that $r_{i,l}$ and R verify the assumption of the latter. Hence the following holds

$$||r_{i,l}^n e^{n\tilde{u}_l}||_{C^\alpha(A_R)}=o_l(1) \tag{2.106}$$

for the definition of A_R and $\tilde{u}_l$ see statement of the point 3) of Step 3 where $d_{i,l}$ is replaced by $r_{i,l}$. On the other hand performing a change of variable say $r_{i,l}y=s$ we obtain the following equality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds=\int_{\frac{1}{R}}^{R} y^{n-1}\hat{G}_{i,l}(y)r_{i,l}^n e^{n\hat{u}_{i,l}(y)}dy, \tag{2.107}$$

where

$$\hat{u}_{i,l}(y) = \bar{u}_{i,l}(r_{i,l}y)$$
$$\hat{G}_{i,l}(y) = \tilde{G}(r_{i,l}, r_{i,l}y)$$

From the asymptotics of $\tilde{G}(\cdot,\cdot)$ (see (2.105)) we deduce the following one for $\hat{G}_{i,l}(\cdot,\cdot)$,

$$\hat{G}_{i,l}(y) = \hat{f}_{i,l}(y)\frac{1}{r_{i,l}(1-y)} + \hat{H}_{i,l}(y); \tag{2.108}$$

where $\hat{H}_{i,l}(\cdot)$ is integrable and $\hat{f}_{i,l}(\cdot)$ of class C^2.
Hence by using (2.107) and (2.108) we obtain the following inequality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^{n-1}\tilde{G}(r_{i,l}, s)e^{n\bar{u}_{i,l}(s)}ds = \frac{1}{r_{i,l}}\int_{\frac{1}{R}}^{R} y^{n-1}\left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right) r_{i,l}^n e^{n\hat{u}_{i,l}(y)}dy. \tag{2.109}$$

Moreover using Harnack-type inequality for u_l (see) and (2.106) we have that,

$$||r_{i,l}^n e^{n\hat{u}_{i,l}}||_{C^\alpha(]\frac{1}{R},R[)} = o_l(1); \tag{2.110}$$

so using techniques of the theory of singular integral operators as in Lemma 4.4 ([46]) to have Holder estimates, we obtain

$$\int_{\frac{1}{R}}^{R} y^{n-1}\left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right) r_{i,l}^n e^{n\hat{u}_{i,l}(y)}dy = o_l(1); \tag{2.111}$$

hence with (2.107) we deduce that

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^{n-1}\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds = o_l(\frac{1}{r_{i,l}}). \tag{2.112}$$

So we obtain

$$\bar{u}_{i,l}^{'}(r_{i,l}) \le -2C\frac{1}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}} - F_{i,l}^{'}(r_{i,l}) + H_{i,l}^{'}(r). \tag{2.113}$$

Now let compute $\varphi_{i,l}^{'}(r_{i,l})$. From straightforward computations we have,

$$\varphi_{i,l}^{'}(r_{i,l}) = (r_{i,l})^{n\nu-1}exp(\bar{u}_{i,l}(r_{i,l}))\left(n\nu + r_{i,l}\bar{u}_{i,l}^{'}(r_{i,l})\right).$$

Hence using (2.113) we arrive to the following inequality,

$$\varphi_{i,l}^{'}(r_{i,l}) \le (r_{i,l})^{n\nu-1}exp(\bar{u}_{i,l}(r_{i,l})\left(n\nu - 2C + o_l(1) - r_{i,l}F_{i,l}^{'}(r_{i,l}) + r_{i,l}H_{i,l}^{'}(r_{i,l})\right);$$

so $\nu < \frac{2}{n}$ implies $n\nu - 2C + o_l(1) < 0$ for l sufficiently large.
Hence since $F_{i,l}^{'}$ and $H_{i,l}^{'}$ are bounded in $(0, \frac{inj_g(M)}{4})$ uniformly in l and $r_{i,l} \longrightarrow 0$ we have that for l big enough,

$$\varphi_{i,l}^{'}(r_{i,l}) < 0;$$

hence we reach the desired contradiction and we conclude the proof of the step.

Step 5 :Proof of Theorem 0.2.6

We show first the following estimate

$$\int_{M\setminus\cup_{i=1}^{i=N}B_{x_{i,l}}(\frac{R_{i,l}}{C})} e^{nu_l(y)}dV_g(y) = o_l(1). \tag{2.114}$$

For this we first start by proving

$$\bar{u}_l \longrightarrow -\infty \quad as \quad l \longrightarrow +\infty. \tag{2.115}$$

In fact, using the Green's representation formula for u_l we have that for every $x \in M$,

$$u_l(x) = \bar{u}_l + \int_M G(x,y)\left(\bar{Q}(y)e^{nu_l(y)} - Q_l(y)\right)dV_g(y) \geq \bar{u}_l - C + \int_M G(x,y)\bar{Q}(y)e^{nu_l(y)}dV_g(y).$$

By assumption c) of Step1 we have given any $\epsilon > 0$, there exists R_ϵ such that for l sufficiently large

$$\int_{B_{x_{1,l}}(R_\epsilon \mu_{1,l})} \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) \geq (n-1)!\omega_n - \frac{n\epsilon}{16}(n-1)!\omega_n.$$

Hence the last two formulas and the asymptotics of the Green's function implie that

$$e^{nu_l}(x) \geq C^{-1}e^{n\bar{u}_l}\frac{1}{|x-x_{1,l}|^{2n-\epsilon}} \quad for \quad |x-x_{1,l}| \geq 2R_\epsilon\mu_{1,l} \qquad for \quad l \quad large;$$

From this it follows that

$$\begin{aligned}\int_M e^{nu_l(y)}dV_g(y) &\geq \int_{B_{x_{1,l}}(inj_g(M))\setminus B_{x_{1,l}}(2R_\epsilon\mu_{1,l})} e^{nu_l(y)}dV_g(y) \\ &\geq C^{-1}e^{n\bar{u}_l}\int_{2R_\epsilon\mu_{1,l}}^{inj_g(M)} s^{\epsilon-(n+1)}ds \geq C^{-1}e^{n\bar{u}_l}\left(2R_\epsilon\mu_{1,l}\right)^{\epsilon-n}.\end{aligned} \tag{2.116}$$

So if ϵ is small enough we have from (2.1) that

$$\bar{u}_l \longrightarrow -\infty, \tag{2.117}$$

hence we are done .
Now by assumption d) of Step 1 we can cover $M \setminus \cup_{i=1}^{i=N} B_{x_{i,l}}(\frac{R_{i,l}}{C})$ with a finite number of balls $B_{y_k}(r_k)$ such that for any k there holds ,

$$\int_{B_{y_k}(2r_k)} \bar{Q}_l e^{nu_l(y)}dV_g(y) \leq \frac{c_n}{2}.$$

Now set $B_k = B_{y_k}(2r_k)$ and $\tilde{B}_k = B_{y_k}(r_k)$ so using again the Green representation formula for u_l we have $\forall x \in \tilde{B}_k$

$$u_l(x) = \bar{u}_l + \int_M G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y) - \int_M G(x,y)Q_l(y)dV_g(y).$$

hence

$$\begin{aligned}u_l(x) \leq \bar{u}_l + C + \int_M G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y) &= \bar{u}_l + C + \int_{B_k} G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y) \\ &+ \int_{M\setminus B_k} G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y).\end{aligned}$$

So since G is smooth out of the diagonal we have that

$$u_l(x) \leq \bar{u}_l + C + \int_{B_k} G(x,y)\bar{Q}_l(y)e^{nu_l(y)}dV_g(y).$$

Now using Jensen's inequality we obtain ,

$$exp\left(\int_{B_k} G(x,y)\bar{Q}_l e^{nu_l(y)}dV_g(y)\right) \leq \int_M exp\left(||\bar{Q}e^{nu_l}\chi_{B_k}||_{L^1(M)}|G(x,y)|\right)\frac{\bar{Q}_l(y)e^{nu_l(y)}\chi_{B_k}(y)}{||\bar{Q}e^{nu_l}\chi_{B_k}||_{L^1(M)}}dV_g(y).$$

Hence using Fubini theorem we have

$$\int_{\tilde{B}_k} e^{nu_l}(y)dv_g(x) \leq Ce^{n\bar{u}_l} \sup_{y\in M,k} \int_M \left(\frac{1}{d_g(x,y)}\right)^{\frac{n}{c_n}||\bar{Q}e^{nu_l}\chi_{B_k}||_{L^1(M)}} dV_g(x).$$

So from $\int_{B_k} \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) \leq \frac{c_n}{2}$ and (2.115) we have that,

$$\int_{\tilde{B}_k} e^{nu_l(y)}dV_g(y) = o_l(1) \quad \forall \ k.$$

Hence

$$\int_{M\setminus\cup_{i=1}^{i=N} B_{x_{i,l}}(\frac{R_{i,l}}{C})} e^{nu_l(y)}dV_g(y) = o_l(1).$$

So since $B_{x_{i,l}}(\frac{R_{i,l}}{C})$ are disjoint then the Step 4 implies that,

$$\int_M \bar{Q}_l(y)e^{nu_l(y)}dV_g(y) = N(n-1)!\omega_n + o_l(1),$$

hence (2.1) implies that

$$\int_M Q_0(y)dV_g(y) = N(n-1)!\omega_n.$$

ending the proof of Theorem 0.2.6.

2.2 Proof of Theorem 0.2.8

In this Section, we give the proof of Theorem 0.2.8. For convenience we divide the proof into five steps as in the previous Section.

Step 1

There exists $N \in \mathbb{N}^*$, N converging points $(x_{i,l})$ $i = 1,...,N$, N sequences $(\mu_{i,l})$ $i = 1;...;N$; of positive real numbers converging to 0 such that the following hold:

a)

$$\frac{d_g(x_{i,l},x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i \neq j \ \ i,j = 1,..,N \quad and \quad \bar{Q}_l(x_{i,l})\mu_{i.l}^4 e^{4u_l(x_{i,l})} = 1;$$

b)

There exists $C > 0$ *such that* $\inf_{i=1,...,N} d_g(x_{i,l},x)^4 e^{4u_l(x)} \leq C \quad \forall x \in M, \ \forall l \in \mathbb{N}.$

c)
For every $i = 1, \cdots, N$
either
$c_1^i)$

$$x_{i,l} \to \bar{x}_i \in int(M);$$

$$v_{i,l}(x) = u_l(exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \frac{1}{4}\log(3) \longrightarrow V_0(x) := \log(\frac{324}{162^2 + |x|^2}) \quad in \quad C^1_{loc}(\mathbb{R}^4);$$

and

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B_{x_{i,l}}(R\mu_{i,l})}\bar{Q}_l(y)e^{4u_l(y)}dV_g(y)=8\pi^2;$$

or

c_2^i)

$$x_{i,l}\to\bar{x}_i\in\partial M;$$

$$v_{i,l}(x)=u_l(exp_{x_{i,l}}(\mu_{i,l}x))-u_l(x_{i,l})-\frac{1}{4}\log(3)\longrightarrow V_0(x):=\log(\frac{324}{162^2+|x|^2})\quad in\quad C^1_{loc}(\mathbb{R}^4_+);$$

and

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B^+_{x_{i,l}}(R\mu_{i,l})}\bar{Q}_l(y)e^{4u_l(y)}dV_g(y)=4\pi^2;$$

Proof of Step 1

First of all let $x_l\in M$ be such that $u_l(x_l)=\max_{x\in M}u_l(x)$, then using the fact that u_l blows up we infer $u_l(x_l)\longrightarrow+\infty$.
Now let $\mu_l>0$ be such that $\bar{Q}_l(x_l)\mu_l^4e^{4u_l(x_l)}=1$. Since $\bar{Q}_l\longrightarrow\bar{Q}_0$ $C^1(M)$, $\bar{Q}_0>0$ and $u_l(x_l)\longrightarrow+\infty$, we have that $\mu_l\longrightarrow0$.
Now suppose $x_l\to\bar{x}\in int(M)$ and let $B^0(\delta\mu_l^{-1})$ be the Euclidean ball of center 0 and radius $\delta\mu_l^{-1}$, with $\delta>0$ small fixed . For $x\in B^0(\delta\mu_l^{-1})$, we set

$$v_l(x)=u_l(exp_{x_l}(\mu_lx))-u_l(x_l)-\frac{1}{4}\log(3);\tag{2.118}$$

$$\tilde{Q}_l(x)=Q_l(exp_{x_l}(\mu_lx));\tag{2.119}$$

$$\tilde{\bar{Q}}_l(x)=\bar{Q}_l(exp_{x_l}(\mu_lx));\tag{2.120}$$

$$g_l(x)=\left(exp^*_{x_l}g\right)(\mu_lx).\tag{2.121}$$

Now from the Green representation formula we have,

$$u_l(x)-\bar{u}_l=\int_MG(x,y)P^4_gu_l(y)dV_g(y)+2\int_{\partial M}G(x,y')P^3_gu_l(y')dS_g(y');\quad\forall x\in M,\tag{2.122}$$

where G is the Green function of (P^4_g,P^3_g) (see Proposition 0.3.3).
Now using equation (22) and differentiating (2.122) with respect to x we obtain that for $k=1,2$

$$|\nabla^ku_l|_g(x)\leq\int_M|\nabla^kG(x,y)|_g\bar{Q}_l(y)e^{4u_l(y)}dV_g+O(1),$$

since $Q_l\longrightarrow Q_0$ in $C^1(M)$ and $T_l\to T_0$.
Now let $y_l\in B_{x_l}(R\mu_l)$, $R>0$ fixed, by using the same argument as in the proof of Theorem 0.2.6(formula (2.8)), we obtain

$$\int_M|\nabla^kG(y_l,y)|_ge^{4u_l(y)}dV_g(y)=O(\mu_l^{-k})\tag{2.123}$$

Hence we get

$$|\nabla^kv_l|_g(x)\leq C.\tag{2.124}$$

Furthermore from the definition of v_l (see (2.118)), we get

$$v_l(x) \leq v_l(0) = -\frac{1}{4}\log(3) \;\; \forall x \in \mathbb{R}^4 \tag{2.125}$$

Thus we infer that $(v_l)_l$ is uniformly bounded in $C^2(K)$ for all compact subsets K of $\mathbb{R}^4$. Hence by Arzelà-Ascoli theorem we derive that

$$v_l \longrightarrow V_0 \quad in \;\; C^1_{loc}(\mathbb{R}^4), \tag{2.126}$$

On the other hand (2.125) and (2.126) imply that

$$V_0(x) \leq V_0(0) = -\frac{1}{4}\log(3) \;\; \forall x \in \mathbb{R}^4. \tag{2.127}$$

Moreover from (2.124) and (2.126) we have that V_0 is Lipschitz.
On the other hand using the Green's representation formula for (P^4_g, P^3_g) we obtain that for $x \in \mathbb{R}^4$ fixed and for R big enough such that $x \in B^0(R)$

$$u_l(exp_{x_l}(\mu_l x)) - \bar{u}_l = \int_M G(exp_{x_l}(\mu_l x), y) P^4_g u_l(y) dV_g(y) + 2\int_{\partial M} G(exp_{x_l}(\mu_l x), y') P^3_g u_l(y') dS_g(y'). \tag{2.128}$$

Now let us set

$$I_l(x) = 2\int_{B_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y);$$

$$\mathrm{II}_l(x) = 2\int_{M \setminus B_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y);$$

$$\mathrm{III}_l(x) = 2\int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) Q_l(y) dV_g(y);$$

and

$$\mathrm{IIII}_l(x) = \int_{\partial M} \left(G(exp_{x_l}(\mu_l x), y') - G(exp_{x_l}(0), y')\right) T_l(y') dS_g(y').$$

Using again the same argument as in the proof of Theorem 0.2.6 (see formula (2.10)- formula (2.16)) we get

$$v_l(x) = I_l(x) + \mathrm{II}_l(x) - \mathrm{III}_l(x) - \mathrm{IIII}_l(x) - \frac{1}{4}\log(3). \tag{2.129}$$

Moreover following the same methods as in the proof of Theorem 0.2.6(see formula (2.18)- formula (2.28)) we obtain

$$\lim_l I_l(x) = \int_{B^0(R)} \frac{3}{4\pi^2}\log\left(\frac{|z|}{|x-z|}\right) e^{4V_0(z)} dz. \tag{2.130}$$

$$\limsup_l \mathrm{II}_l(x) = o_R(1). \tag{2.131}$$

$$\mathrm{III}_l(x) = o_l(1) \tag{2.132}$$

and

$$\mathrm{IIII}_l(x) = o_l(1). \tag{2.133}$$

Hence from (2.126), (2.129)-(2.133) by letting l tends to infinity and after R tends to infinity, we obtain V_0 satisfies the following conformally invariant integral equation

$$V_0(x) = \int_{\mathbb{R}^4} \frac{3}{4\pi^2}\log\left(\frac{|z|}{|x-z|}\right) e^{4V_0(z)} dz - \frac{1}{4}\log(3). \tag{2.134}$$

Now since V_0 is Lipschitz then the theory of singular integral operator gives that $V_0 \in C^1(\mathbb{R}^4)$. On the other hand by using the change of variable $y = exp_{x_l}(\mu_l x)$, one can check that the following holds

$$\lim_{l \longrightarrow +\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l e^{4u_l} dV_g = 3 \int_{B_0(R)} e^{4V_0} dx; \tag{2.135}$$

Hence (38) implies that $e^{V_0} \in L^4(\mathbb{R}^4)$.
Furthermore by a classification result by X. Xu, see [89](Theorem 1.2) for the solutions of (2.134) we derive that

$$V_0(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\right) \tag{2.136}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^4$.
Moreover from $V_0(x) \le V_0(0) = -\frac{1}{4}\log(3)$ $\forall x \in \mathbb{R}^4$, we have that $\lambda = 162$ and $x_0 = 0$ namely,

$$V_0(x) = \log\left(\frac{324}{162^2 + |x|^2}\right).$$

On the other hand by letting R tends to infinity in (2.135) we obtain

$$\lim_{R \to +\infty} \lim_{l \to +\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 3 \int_{\mathbb{R}^4} e^{4V_0} dx. \tag{2.137}$$

Moreover from a generalized Pohozaev type identity by X.Xu [89] (see Theorem 1.1) we get

$$\frac{3}{4\pi^2} \int_{\mathbb{R}^4} e^{4V_0(y)} dy = 2,$$

hence using (2.137) we derive that

$$\lim_{R \to +\infty} \lim_{l \to +\infty} \int_{B_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 8\pi^2$$

Next suppose $x_l \to \bar{x} \in \partial M$ and let let $B^0_+(\delta\mu_l^{-1})$ be the upper half euclidean ball of center 0 and radius $\delta\mu_l^{-1}$, with $\delta > 0$ small fixed . For $x \in B^0_+(\delta\mu_l^{-1})$, we consider $v_l(x), \tilde{Q}_l(x), \tilde{\tilde{Q}}_l(x)$ and $g_l(x)$ as in (2.118)- (2.121).
Repeating the same argument as above we get v_l is uniformly bounded in $C^2(K)$ for every compact set K of $\mathbb{R}^4_+$. Moreover we obtain

$$v_l \longrightarrow V_0 \quad in \;\; C^1_{loc}(\mathbb{R}^4_+), \tag{2.138}$$

$$V_0(x) \le V_0(0) = -\frac{1}{3}\log(3) \;\; \forall x \in \mathbb{R}^4_+;$$

and V_0 is Lipschitz.
Now let us define

$$I_l(x) = 2 \int_{B^+_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y);$$

$$II_l(x) = 2 \int_{M \setminus B^+_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y);$$

$$III_l(x) = 2 \int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right) Q_l(y) dV_g(y);$$

and

$$\mathrm{IIII}_l(x) = \int_{\partial M} \left(G(exp_{x_l}(\mu_l x), y') - G(exp_{x_l}(0), y')\right) T_l(y') dS_g(y').$$

By still the same argument as above we obtain

$$v_l(x) = I_l(x) + \mathrm{II}_l(x) - \mathrm{III}_l(x) - \mathrm{IIII}_l(x) - \frac{1}{4}\log(3).$$

Moreover we have that

$$\lim_l I_l(x) = \int_{B^0_+(R)} \frac{3}{4\pi^2}\left(\log\frac{|z|}{|x-z|} + \log\frac{|z|}{|x-\bar{z}|}\right) e^{4V_0(z)} dz.$$

$$\limsup_l \mathrm{II}_l(x) = o_R(1).$$

$$\mathrm{III}_l(x) = o_l(1)$$

and

$$\mathrm{IIII}_l(x) = o_l(1).$$

Hence letting l tends to infinity and after R tending to infinity, we derive that V_0 satisfies the following integral equation

$$V_0(x) = \int_{\mathbb{R}^4_+} \frac{3}{4\pi^2}\left(\log\frac{|z|}{|x-z|} + \log\frac{|z|}{|x-\bar{z}|}\right) e^{4V_0(z)} dz - \frac{1}{4}\log(3). \tag{2.139}$$

On the other hand from (2.139), it is easily seen that

$$\frac{\partial V_0}{\partial t} = 0 \ \text{ on } \ \partial\mathbb{R}^4_+.$$

Now using Alexandrov reflection principle and denoting $\tilde{V}_0$ the even reflection of V_0 through the plane $\partial\mathbb{R}^4_+$, we obtain $\tilde{V}_0$ solves the following conformally invariant integral equation

$$\tilde{V}_0(x) = \int_{\mathbb{R}^4} \frac{3}{4\pi^2}\log\left(\frac{|z|}{|x-z|}\right) e^{4\tilde{V}_0(z)} dz - \frac{1}{4}\log(3). \tag{2.140}$$

On the other hand since V_0 was Lipschitz then $\tilde{V}_0$ is also. Thus using the theory of singular integral operator we infer that $\tilde{V}_0$ is of class C^1. Moreover using again the change of variable $y = exp_{x_l}(\mu_l x)$ we get

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B^+_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 3\int_{\mathbb{R}^4_+} e^{4V_0(x)} dx \tag{2.141}$$

So from (38) we infer that $\int_{\mathbb{R}^4_+} e^{4V_0(x)} dx < +\infty$. Thus $e^{4\tilde{V}_0} \in L^1(\mathbb{R}^4)$. Now arguing as above we obtain

$$\tilde{V}_0(x) = \log\left(\frac{324}{162^2 + |x|^2}\right).$$

and

$$\frac{3}{4\pi^2}\int_{\mathbb{R}^4} e^{4\tilde{V}_0(y)} dy = 2. \tag{2.142}$$

Hence from the fact the $\tilde{V}_0$ is the even reflection of V_0 through $\partial\mathbb{R}^4_+$, (2.141) and (2.142) we get

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B^+_{x_l}(R\mu_l)} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 4\pi^2.$$

Now for $k \geq 1$ we say that (H_k) holds if there exists k converging points $(x_{i,l})_l$ $i = 1, ..., k$, k sequences $(\mu_{i,l})$ $i = 1, ..., k$ of positive real numbers converging to 0 such that the following hold

(A_k^1)

$$\frac{d_g(x_{i,l}, x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i \neq j \quad i,j = 1,..,k \; and \quad \bar{Q}_l(x_{i,l})\mu_{i.l}^4 e^{4u_l(x_{i,l})} = 1;$$

(A_k^2)

For every $i = 1, \cdot, k$

either

$\left(A_{k,1}^{2,i}\right)$

$$x_{i,l} \to \bar{x}_i \in int(M);$$

$$v_{i,l}(x) = u_l(exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \frac{1}{4}\log(3) \longrightarrow V_0(x) := \log(\frac{324}{162^2 + |x|^2}) \quad in \quad C^1_{loc}(\mathbb{R}^4)$$

and

$$\lim_{R \to +\infty} \lim_{l \to +\infty} \int_{B_{x_{i,l}}(R\mu_{i,l})} \bar{Q}_l(y) e^{4u_l(y)} = 8\pi^2$$

or

$\left(A_{k,2}^{2,i}\right)$

$$x_{i,l} \to \bar{x}_i \in \partial M;$$

$$v_{i,l}(x) = u_l(exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \frac{1}{4}\log(3) \longrightarrow V_0(x) := \log(\frac{324}{162^2 + |x|^2}) \quad in \quad C^1_{loc}(\mathbb{R}^4_+)$$

and

$$\lim_{R \to +\infty} \lim_{l \to +\infty} \int_{B^+_{x_{i,l}}(R\mu_{i,l})} \bar{Q}_l(y) e^{4u_l(y)} = 4\pi^2$$

Clearly, by the above arguments (H_1) holds. We let now $k \geq 1$ and assume that (H_k) holds. We also assume that

$$\sup_M R_{k,l}(x)^4 e^{4u_l(x)} \longrightarrow +\infty \quad as \quad l \longrightarrow +\infty, \tag{2.143}$$

Now using the same argument as in the proof of Theorem 0.2.6, one can see easily that (H_{k+1}). Hence since (A_k^1) and (A_k^2) of H_k imply that

$$\int_M \bar{Q}(y) e^{4u_l(y)} dV_g(y) \geq (2k_1 + k_2)4\pi^2 + o_l(1),$$

with $k_1,\ k_2 \in \mathbb{N}$ and $2k_1 + k_2 = k$. Thus we easily get thanks to (38) that there exists a maximal k , $1 \leq k \leq \frac{1}{4\pi^2}\left(\int_M Q_0(y)dV_g(y) + \int_{\partial M} T_0(y')dS_g(y')\right)$, such that (H_k) holds. Arriving to this maximal k, we get that (2.143) cannot hold. Hence setting $N = k$ the proof of Step 1 is done.

Step 2

There exists a constant $C > 0$ such that

$$R_l(x)|\nabla u_l|_g(x) \leq C \qquad \forall x \in M \;\; and \;\; \forall l \in N; \tag{2.144}$$

where

$$R_l(x) = \min_{i=1,..,N} d_g(x_{i,l}, x);$$

and the $x_{i,l}$'s are as in Step 1.

Proof of Step 2
First of all using the Green representation formula for (P_g^4, P_g^3) see Proposition 0.3.3 we obtain

$$u_l(x) - \bar{u}_l = \int_M G(x,y) P_g^4 u_l(y) dV_g(y) + 2 \int_{\partial M} G(x,y') P_g^3 u_l(y') dS_g(y').$$

Now using the BVP (20) we get

$$u_l(x) - \bar{u}_l = 2 \int_M G(x,y)(\bar{Q}_l(y) e^{4u_l(y)} - Q_l) dV_g(y) - 2 \int_{\partial M} G(x,y') T_l(y') u_l(y') dS_g(y'). \quad (2.145)$$

Thus differentiating with respect to x (2.145) and using the fact that $Q_l \to Q_0$, $\bar{Q}_l \to \bar{Q}_0$ and $T_l \to T_0$ in C^1, we have that for $x_l \in M$

$$|\nabla u_l(x_l)|_g = O\left(\int_M \frac{1}{d_g(x_l,y)} e^{4u_l(y)} dV_g(y)\right) + O(1).$$

Hence at this stage following the same argument as in the proof of Theorem 0.2.6, Step 2, we obtain

$$\int_M \frac{1}{(d_g(x_l,y))} e^{4u_l(y)} dV_g(y) = O\left(\frac{1}{R_l(x_l)}\right);$$

hence since x_l is arbitrary, then the proof of Step 2 is complete.

Step 3
Set

$$R_{i,l} = \min_{i \neq j} d_g(x_{i,l}, x_{j,l});$$

we have that
1) There exists a constant $C > 0$ such that $\forall\, r \in (0, R_{i,l}]$ $\forall\, s \in (\frac{r}{4}, r]$
if $\bar{x}_i \in int(M)$ then

$$|u_l\left(exp_{x_{i,l}}(rx)\right) - u_l\left(exp_{x_{i,l}}(sy)\right)| \leq C \quad for\ all\ x, y \in \mathbb{R}^4\ such\ that\ |x|,\ |y| \leq \frac{3}{2}. \quad (2.146)$$

and if $\bar{x}_i \in \partial M$ then

$$|u_l\left(exp_{x_{i,l}}(rx)\right) - u_l\left(exp_{x_{i,l}}(sy)\right)| \leq C \quad for\ all\ x, y \in \mathbb{R}^4_+\ such\ that\ |x|,\ |y| \leq \frac{3}{2}. \quad (2.147)$$

2) If $d_{i,l}$ is such that $0 < d_{i,l} \leq \frac{R_{i,l}}{2}$ and $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$ then we have that
if $\bar{x}_i \in int(M)$ and

$$\int_{B_{x_{i,l}}(d_{i,l})} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 8\pi^2 + o_l(1); \quad (2.148)$$

then

$$\int_{B_{x_{i,l}}(2d_{i,l})} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 8\pi^2 + o_l(1).$$

if $\bar{x}_i \in \partial M$ and

$$\int_{B^+_{x_{i,l}}(d_{i,l})} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) = 4\pi^2 + o_l(1); \quad (2.149)$$

then

$$\int_{B^+_{x_{i,l}}(2d_{i,l})} \bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = 4\pi^2 + o_l(1).$$

3) Let R be large and fixed. If $d_{i,l} > 0$ is such that $d_{i,l} \longrightarrow 0$, $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, $d_{i,l} < \frac{R_{i,l}}{4R}$ then if $\bar{x}_i \in int(M)$ and

$$\int_{B_{x_{i,l}}(\frac{d_{i,l}}{2R})} \bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = 8\pi^2 + o_l(1);$$

then by setting

$$\tilde{u}_l(x) = u_l(exp_{x_{i,l}}(d_{i,l}x)); \qquad x \in A_{2R};$$

where $A_{2R} = B^0(2R) \setminus B^0(\frac{1}{2R})$, we have that,

$$||d^4_{i,l}e^{4\tilde{u}_l}||_{C^\alpha(A_R)} \longrightarrow 0 \ \ as \ \ l \longrightarrow +\infty;$$

for some $\alpha \in (0,1)$ where $A_R = B^0(R) \setminus B^0(\frac{1}{R})$.
and
if $\bar{x}_i \in \partial M$ and

$$\int_{B^+_{x_{i,l}}(\frac{d_{i,l}}{2R})} \bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = 4\pi^2 + o_l(1);$$

then by setting

$$\tilde{u}_l(x) = u_l(exp_{x_{i,l}}(d_{i,l}x)); \qquad x \in A^+_{2R};$$

where $A^+_{2R} = B^0_+(2R) \setminus B^0_+(\frac{1}{2R})$, we have that,

$$||d^4_{i,l}e^{4\tilde{u}_l}||_{C^\alpha(A^+_R)} \longrightarrow 0 \ \ as \ \ l \longrightarrow +\infty;$$

for some $\alpha \in (0,1)$ where $A^+_R = B^0_+(R) \setminus B^0_+(\frac{1}{R})$.

Proof of Step 3
We have that property 1 follows immediately from Step 2 and the definition of $R_{i,l}$. In fact we can join rx to sy by a curve whose length is bounded by a constant proportional to r.
Now let us show point 2. First suppose $\bar{x}_i \in int(M)$. From $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, point c) of Step 1 and (2.148) we have that

$$\int_{B_{x_{i,l}}(d_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{4u_l(y)}dV_g(y) = o_l(1). \tag{2.150}$$

Hence from (2.146),by taking $s = \frac{r}{2}$ and $r = 2d_{i,l}$ we obtain that

$$\int_{B_{x_{i,l}}(2d_{i,l}) \setminus B_{x_{i,l}}(d_{i,l})} e^{4u_l(y)}dV_g(y) \leq C \int_{B_{x_{i,l}}(d_{i,l}) \setminus B_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{4u_l(y)}dV_g(y);$$

Thus we get

$$\int_{B_{x_{i,l}}(2d_{i,l}) \setminus B_{x_{i,l}}(d_{i,l})} e^{4u_l(y)}dV_g(y) = o_l(1).$$

Next assume $\bar{x}_i \in \partial M$.Thanks to $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, point c) of Step 1 and (2.149) we have that

$$\int_{B^+_{x_{i,l}}(d_{i,l}) \setminus B^+_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{4u_l(y)}dV_g(y) = o_l(1). \tag{2.151}$$

Thus using (2.147), with $s = \frac{r}{2}$ and $r = 2d_{i,l}$ we get

$$\int_{B^+_{x_{i,l}}(2d_{i,l}) \setminus B^+_{x_{i,l}}(d_{i,l})} e^{4u_l(y)}dV_g(y) \leq C \int_{B^+_{x_{i,l}}(d_{i,l}) \setminus B^+_{x_{i,l}}(\frac{d_{i,l}}{2})} e^{4u_l(y)}dV_g(y);$$

Hence we arrive
$$\int_{B^+_{x_{i,l}}(2d_{i,l})\setminus B^+_{x_{i,l}}(d_{i,l})} e^{4u_l(y)}dV_g(y) = o_l(1).$$
So the proof of point 2 is done. On the other hand by following in a straightforward way the proof of point 3 in Step 3 of Theorem 0.2.6 one gets easily point 3. Hence the proof of Step 3 is complete.

Step 4
There exists a positive constant C independent of l and i such that
if $\bar{x}_i \in int(M)$ then
$$\int_{B_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = 8\pi^2 + o_l(1).$$
and
if $\bar{x}_i \in \partial M$ then
$$\int_{B^+_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = 4\pi^2 + o_l(1).$$

Proof of Step 4
The proof is an adaptation of the arguments in Step 4 of the one of Theorem 0.2.6, but for the readers convenience we will make it.

First of all fix $\frac{1}{4} < \nu < \frac{1}{2}$ and for $i = 1, ..., N$
if $\bar{x}_i \in int(M)$ then set
$$\bar{u}_{i,l}(r) = Vol_g(\partial B_{x_{i,l}}(r)))^{-1}\int_{\partial B_{x_{i,l}}(r)} u_l(x)d\sigma_g(x) \quad \forall 0 \leq r < inj_g(M);$$
$$\varphi_{i,l}(r) = r^{4\nu}exp\left(\bar{u}_{i,l}(r)\right) \quad \forall 0 \leq r < inj_g(M).$$
if $\bar{x}_i \in \partial M$ then set
$$\bar{u}_{i,l}(r) = Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1}\int_{\partial B^+_{x_{i,l}}(r)} u_l(x)d\sigma_g(x) \quad \forall 0 \leq r < inj_g(M);$$
$$\varphi_{i,l}(r) = r^{4\nu}exp\left(\bar{u}_{i,l}(r)\right) \quad \forall 0 \leq r < inj_g(M).$$
By assumption *c)* or *d)* of Step 1 we have that there exists R_ν such that,
$$\forall R \geq R_\nu \quad \varphi^{'}_{i,l}(R\mu_{i,l}) < 0 \quad \forall l \ sufficiently \quad large \ (depending \ on \ R). \tag{2.152}$$
Now we define $r_{i,l}$ by
$$r_{i,l} = \sup\{R_\nu\mu_{i,l} \leq r \leq \frac{R_{i,l}}{2} \ \ s.t \ \ \varphi^{'}_{i,l}(\cdot) < 0 \ \ in \ \ [R_\nu, r)\}. \tag{2.153}$$
Hence (2.152) implies that
$$\frac{r_{i,l}}{\mu_{i,l}} \longrightarrow +\infty \quad as \ \ l \longrightarrow +\infty. \tag{2.154}$$
Now to prove the step it suffices to show that $\frac{R_{i,l}}{r_{i,l}} \not\longrightarrow +\infty$ as $l \longrightarrow +\infty$.
Indeed if $\frac{R_{i,l}}{r_{i,l}} \not\longrightarrow +\infty$, we have that there exists a positive constant C such that
$$\frac{R_{i,l}}{C} \leq r_{i,l}. \tag{2.155}$$

On the other hand from the Harnack type inequality (2.146) or (2.147), point c) or d) of Step 1, and (2.153) we have that for any $\eta > 0$, there exists $R_\eta > 0$ such that for any $R > R_\eta$, we have that

$$d_g(x, x_{i,l})^{4\nu} e^{4u_l} \leq \eta \mu_{i,l}^{4(\nu-1)} \quad \forall x \in (B^+_{x_{i,l}}(r_{i,l}) \setminus B^+_{x_{i,l}}(R\mu_{i,l})). \tag{2.156}$$

Since $\frac{r_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$ see (2.154) and $\frac{R_{i,l}}{2} \geq r_{i,l}$ see (2.153), we have $\frac{R_{i,l}}{C\mu_{i,l}} \longrightarrow +\infty$, hence point c) or d) of Step 1 (2.156) and (2.155) imply that if $\bar{x}_i \in int(M)$ then

$$\int_{B_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l e^{4u_l} = 8\pi^2 + o_l(1);$$

and
if $\bar{x}_i \in \partial M$ then

$$\int_{B^+_{x_{i,l}}(\frac{R_{i,l}}{C})} \bar{Q}_l e^{4u_l} = 4\pi^2 + o_l(1).$$

On the other hand, by continuity and by the definition of $r_{i,l}$ it follows that

$$\varphi'_{i,l}(r_{i,l}) = 0. \tag{2.157}$$

equation Let us assume by contradiction that $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$. We will show next that $\varphi'_{i,l}(r_{i,l}) < 0$ for l large contradicting the above equality (2.157). To do so we will study $\bar{u}_{i,l}(\cdot)$.
First let us remark that since M is compact then $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$ implies that $r_{i,l} \longrightarrow 0$.
From the Green's representation formula for u_l we have the following equation,

$$u_l(x) = \int_M G(x,y) P^4_g u_l(y) dV_g(y) + \bar{u}_l + 2\int_M G(x,y,) P^3_g u_l(y') dS_g(y') = \int_M G(x,y) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y)$$
$$+\bar{u}_l - \int_M G(x,y) Q_l(y) dV_g(y) - 2\int_{\partial M} G(x,y') P^3_g u_l(y') dS_g(y').$$

Hence

$$\bar{u}_{i,l}(r) = 2(Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_M G(x,y) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) d\sigma_g(x) + \bar{u}_l$$
$$-2(Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_M G(x,y) Q_l(y) dV_g(y) d\sigma_g(x)$$
$$-(Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_{\partial M} G(x,y) T_l(y) dS_g(y) d\sigma_g(x).$$

Setting

$$F_{i,l}(r) = 2(Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_M G(x,y) Q_l(y) dV_g(y) d\sigma_g(x)$$
$$+(Vol_g(\partial B^+_{x_{i,l}}(r))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_{\partial M} G(x,y) T_l(y) dS_g(y) d\sigma_g(x);$$

we obtain

$$\bar{u}_{i,l} = 2(Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1} \int_{\partial B^+_{x_{i,l}}(r)} \int_M G(x,y) \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) d\sigma_g(x) + \bar{u}_l - F_{i,l}(r).$$

Since $Q_l \longrightarrow Q_0$ in $C^1(M)$ and $T_l \longrightarrow T_0$ in $C^1(\partial M)$ then we have that $F_{i,l}$ is of class C^1 for all i, l and moreover,

$$|F'_{i,l}(r)| \leq C; \quad \forall \ r \in (0, \frac{inj_g(M)}{4}). \tag{2.158}$$

Now let $\frac{inj_g(M)}{4} < A < \frac{inj_g(M)}{2}$ be fixed: we have that

$$\int_M G(x,y)\bar{Q}_l(y)e^{4u_l(y)}dV_g(y) = \int_{B^+_{x_{i,l}}(A)} G(x,y)\bar{Q}_l e^{4u_l(y)}dV_g(y) + \int_{M\setminus B_{x_{i,l}}(A)} G(x,y)\bar{Q}_l e^{4u_l(y)}dV_g(y).$$

So

$$\bar{u}_{i,l}(r) = 2Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1}\int_{\partial B^+_{x_{i,l}}(r)}\int_{B^+_{x_{i,l}}(A)} (G(x,y)-K(x,y))\,\bar{Q}_l(y)e^{4u_l(y)}dV_g(y)d\sigma_g(x) + \bar{u}_l$$
$$-F_{i,l}(r) + H_{i,l}(r);$$

with

$$H_{i,l}(r) = 2Vol_g(\partial B^+_{x_{i,l}}(r)))^{-1}\int_{\partial B^+_{x_{i,l}}(r)}\int_{M\setminus B_{x_{i,l}}(A)} G(x,y)\bar{Q}_l(y)e^{4u_l(y)}dV_g(y)d\sigma_g(x)$$
$$+2Vol_g(\partial B^+_{x_{i,l}}(r)\cap M))^{-1}\int_{\partial B^+_{x_{i,l}}(r)}\int_{B^+_{x_{i,l}}(A)} K(x,y)\bar{Q}_l(y)e^{4u_l(y)}dV_g(y)d\sigma_g(x).$$

Since G is smooth out of $Diag(M)$, then for all i,l; $H_{i,l}\in C^1\left(0,\frac{inj_g(M)}{4}\right)$ and moreover,

$$|H^{'}_{i,l}(r)| \le C \quad \forall\ r\in(0,\frac{inj_g(M)}{4}). \tag{2.159}$$

To continue the proof of the Step we divide it into two cases
Case 1 $\bar{x}_i \in int(M)$
First of all using the change of variable $x = r\theta$ and $y = s\tilde{\theta}$ we obtain

$$\bar{u}_{i,l} = (Vol(S^3))^{-1}\int_{S^3}\int_{S^3}\int_0^A f(r,\theta)\left(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta})\right)\bar{Q}(s\tilde{\theta})e^{4u_l(s\tilde{\theta})}s^3 f(s,\tilde{\theta})dsd\tilde{\theta}d\theta$$
$$+\bar{u}_l - F_{i,l}(r) + H_{i,l}(r).$$

So differentiating with respect to r we have that

$$\bar{u}^{'}_{i,l}(r) = (Vol(S^3))^{-1}\int_{S^3}\int_{S^3}\int_0^A \frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\right)\bar{Q}(s\tilde{\theta})e^{4u_l(s\tilde{\theta})}s^3 f(s,\tilde{\theta})dsd\tilde{\theta}d\theta$$
$$-F^{'}_{i,l}(r) + H^{'}_{i,l}(r).$$

From the asymptotics of $G(\cdot,\cdot)$ (see Proposition (0.3.3)) and the fact that f is bounded in C^2, it follows that

$$(Vol(S^3))^{-1}\int_{S^3}\int_{S^3}\left(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta})\right)d\tilde{\theta}d\theta = \hat{f}(r,s)\log(\frac{1}{|r-s|}) + H(r,s);$$

with $H(\cdot,\cdot)$ of class C^α and $\hat{f}(\cdot,\cdot)$ of class C^2.
Hence setting

$$\tilde{G}(r,s) = (Vol(S^3))^{-1}\int_{S^3}\int_{S^3}\frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\right)\bar{Q}(s\tilde{\theta})f(s,\tilde{\theta})d\tilde{\theta}d\theta.$$

we obtain

$$\tilde{G}(r,s) = \hat{f}(r,s)\frac{1}{r-s} + \tilde{H}(r,s); \tag{2.160}$$

where $\tilde{H}(r,\cdot)$ is integrable for every r fixed.
On the other hand using the Harnack type inequality (see (2.146)) we have that,

$$u_l(s\tilde{\theta}) \le \bar{u}_{i,l}(s) + C \quad uniformly\ \ in\ \ \tilde{\theta},$$

hence we obtain

$$\bar{u}_{i,l}(r) \leq C\int_0^A s^3\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds - F_{i,l}^{'}(r) + H_{i,l}^{'}(r).$$

Now le study $\int_0^A s^3\tilde{G}(r,s)e^{4\bar{u}_{i,l}(s)}ds$. To do so let R so large such that $r_{i,l} \leq \frac{R_{i,l}}{4R}$ (this is possible because of the assumption of contradiction). Now let us split the integral in the following way,

$$\int_0^A s^3\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds = \int_0^{\frac{r_{i,l}}{R}} s^3\tilde{G}(r,s)e^{4\bar{u}_{i,l}(s)}ds + \int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3\tilde{G}(r,s)e^{4\bar{u}_{i,l}(s)}ds$$
$$+\int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^3\tilde{G}(r,s)e^{4\bar{u}_{i,l}(s)}ds + \int_{\frac{R_{i,l}}{C}}^{A} s^3\tilde{G}(r,s)e^{n\bar{u}_{i,l}(s)}ds.$$

Using the fact that we are at the scale $\frac{r_{i,l}}{R}$ then *c)* of Step 1 implies that we have the following estimates for the first term of the equality above with $r = r_{i,l}$,

$$\int_0^{\frac{r_{i,l}}{R}} s^3\tilde{G}(r_{i,l},s)e^{4\bar{u}_{i,l}(s)}ds = -\frac{2}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}}$$

On the other hand using assumption *b)* of Step 1 we obtain the following estimate for the third term of the equality above with $r = r_{i,l}$

$$\int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^3\tilde{G}(r_{i,l},s)e^{4\bar{u}_{i,l}(s)}ds = o_l(1)\frac{1}{r_{i,l}}.$$

We have also using assumption *b)* of Step 1 and the fact that $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$ the following estimate for the fourth still with $r = r_{i,l}$,

$$\int_{\frac{R_{i,l}}{C}}^{A} s^3\tilde{G}(r_{i,l},s)e^{4\bar{u}_{i,l}(s)}ds = o_l(1)\frac{1}{r_{i,l}}.$$

Now let us estimate the second term. For this we will use the point 3) of Step 3. First we recall that $r_{i,l}$ and R verify the assumption of the latter. Hence the following holds

$$||r_{i,l}^4 e^{4\tilde{u}_l}||_{C^\alpha(A_R)} = o_l(1) \tag{2.161}$$

for the definition of A_R and $\tilde{u}_l$ see statement of the point 3) of Step 3 where $d_{i,l}$ is replaced by $r_{i,l}$. On the other hand performing a change of variable say $r_{i,l}y = s$ we obtain the following equality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3\tilde{G}(r,s)e^{4\bar{u}_{i,l}(s)}ds = \int_{\frac{1}{R}}^{R} y^3\hat{G}_{i,l}(y)r_{i,l}^4 e^{4\hat{u}_{i,l}(y)}dy, \tag{2.162}$$

where

$$\hat{u}_{i,l}(y) = \bar{u}_{i,l}(r_{i,l}y)$$
$$\hat{G}_{i,l}(y) = \tilde{G}(r_{i,l}, r_{i,l}y)$$

From the asymptotics of $\tilde{G}(\cdot,\cdot)$ (see (2.160)) we deduce the following one for $\hat{G}_{i,l}(\cdot,\cdot)$,

$$\hat{G}_{i,l}(y) = \hat{f}_{i,l}(y)\frac{1}{r_{i,l}(1-y)} + \hat{H}_{i,l}(y); \tag{2.163}$$

where $\hat{H}_{i,l}(\cdot)$ is integrable and $\hat{f}_{i,l}(\cdot)$ of class C^2.
Hence by using (2.162) and (2.170) we obtain the following inequality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3\tilde{G}(r_{i,l},s)e^{4\bar{u}_{i,l}(s)}ds = \frac{1}{r_{i,l}}\int_{\frac{1}{R}}^{R} y^3\left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right)r_{i,l}^4 e^{4\hat{u}_{i,l}(y)}dy. \tag{2.164}$$

Moreover using Harnack-type inequality for u_l (see (2.146)) and (2.161) we have that,

$$||r_{i,l}^4 e^{4\hat{u}_{i,l}}||_{C^\alpha(]\frac{1}{R},R[)} = o_l(1). \tag{2.165}$$

So using techniques of the theory of singular integral operators as in Lemma 4.4 ([46]) to have Holder estimates, we obtain

$$\int_{\frac{1}{R}}^{R} y^3 \left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right) r_{i,l}^4 e^{4\hat{u}_{i,l}(y)} dy = o_l(1);$$

hence with (2.162) we deduce that

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds = o_l(\frac{1}{r_{i,l}}).$$

So we obtain

$$\bar{u}_{i,l}^{'}(r_{i,l}) \leq -2C\frac{1}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}} - F_{i,l}^{'}(r_{i,l}) + H_{i,l}^{'}(r). \tag{2.166}$$

Case 2 $\bar{x}_i \in \partial M$

We will follow the same strategy up to some trivial adaptations. First using the change of variable $x = r\theta$ and $y = s\tilde{\theta}$ we obtain

$$\bar{u}_{i,l} = (Vol(S_+^3))^{-1} \int_{S_+^3}\int_{S_+^3}\int_0^A f(r,\theta)\left(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta})\right)\bar{Q}(s\tilde{\theta})e^{4u_l(s\tilde{\theta})}s^3 f(s,\tilde{\theta})dsd\tilde{\theta}d\theta$$
$$+\bar{u}_l - F_{i,l}(r) + H_{i,l}(r).$$

So differentiating with respect to r we have that

$$\bar{u}_{i,l}^{'}(r) = (Vol(S_+^3))^{-1} \int_{S_+^3}\int_{S_+^3}\int_0^A \frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\right)\bar{Q}(s\tilde{\theta})e^{4u_l(s\tilde{\theta})}s^3 f(s,\tilde{\theta})dsd\tilde{\theta}d\theta$$
$$-F_{i,l}^{'}(r) + H_{i,l}^{'}(r).$$

From the asymptotics of $G(\cdot,\cdot)$ (see Proposition (0.3.3)) and the fact that f is bounded in C^2, it follows that

$$(Vol(S^3))^{-1}\int_{S_+^3}\int_{S_+^3}\left(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta})\right)d\tilde{\theta}d\theta = \hat{f}(r,s)\log(\frac{1}{|r-s|}) + H(r,s);$$

with $H(\cdot,\cdot)$ of class C^α and $\hat{f}(\cdot,\cdot)$ of class C^2.

Hence setting

$$\tilde{G}(r,s) = (Vol(S_+^3))^{-1}\int_{S_+^3}\int_{S_+^3}\frac{\partial}{\partial r}\left(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\right)\bar{Q}(s\tilde{\theta})f(s,\tilde{\theta})d\tilde{\theta}d\theta.$$

we obtain

$$\tilde{G}(r,s) = \hat{f}(r,s)\frac{1}{r-s} + \tilde{H}(r,s); \tag{2.167}$$

where $\tilde{H}(r,\cdot)$ is integrable for every r fixed.

On the other hand using the Harnack type inequality (see (2.147)) we have that,

$$u_l(s\tilde{\theta}) \leq \bar{u}_{i,l}(s) + C \quad uniformly \ in \ \tilde{\theta},$$

hence we obtain

$$\bar{u}_{i,l}(r) \le C \int_0^A s^3 \tilde{G}(r,s) e^{n\bar{u}_{i,l}(s)} ds - F'_{i,l}(r) + H'_{i,l}(r).$$

Now le study $\int_0^A s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds$. To do so let R so large such that $r_{i,l} \le \frac{R_{i,l}}{4R}$ (this is possible because of the assumption of contradiction). Now let us split the integral in the following way,

$$\int_0^A s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds = \int_0^{\frac{r_{i,l}}{R}} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds + \int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds$$
$$+ \int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds + \int_{\frac{R_{i,l}}{C}}^{A} s^3 \tilde{G}(r,s) e^{n\bar{u}_{i,l}(s)} ds.$$

Using the fact that we are at the scale $\frac{r_{i,l}}{R}$ then *d)* of Step 1 implies that we have the following estimates for the first term of the equality above with $r = r_{i,l}$,

$$\int_0^{\frac{r_{i,l}}{R}} s^3 \tilde{G}(r_{i,l},s) e^{4\bar{u}_{i,l}(s)} ds = -\frac{2}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}}$$

On the other hand using assumption *b)* of Step 1 we obtain the following estimates for the third term of the equality above with $r = r_{i,l}$

$$\int_{r_{i,l}R}^{\frac{R_{i,l}}{C}} s^3 \tilde{G}(r_{i,l},s) e^{4\bar{u}_{i,l}(s)} ds = o_l(1)\frac{1}{r_{i,l}}.$$

We have also using assumption *d)* of Step 1 and the fact that $\frac{R_{i,l}}{r_{i,l}} \longrightarrow +\infty$ the following estimate for the fourth still with $r = r_{i,l}$,

$$\int_{\frac{R_{i,l}}{C}}^{A} s^3 \tilde{G}(r_{i,l},s) e^{4\bar{u}_{i,l}(s)} ds = o_l(1)\frac{1}{r_{i,l}}.$$

Now let us estimate the second term. For this we will use the point 3) of Step 3. First we recall that $r_{i,l}$ and R verify the assumption of the latter. Hence the following holds

$$||r_{i,l}^4 e^{4\tilde{u}_l}||_{C^\alpha(A_R)} = o_l(1) \tag{2.168}$$

for the definition of A_R and $\tilde{u}_l$ see statement of the point 3) of Step 3 where $d_{i,l}$ is replaced by $r_{i,l}$. On the other hand performing a change of variable say $r_{i,l}y = s$ we obtain the following equality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds = \int_{\frac{1}{R}}^{R} y^3 \hat{G}_{i,l}(y) r_{i,l}^4 e^{4\hat{u}_{i,l}(y)} dy, \tag{2.169}$$

where

$$\hat{u}_{i,l}(y) = \bar{u}_{i,l}(r_{i,l}y)$$
$$\hat{G}_{i,l}(y) = \tilde{G}(r_{i,l}, r_{i,l}y)$$

From the asymptotics of $\tilde{G}(\cdot,\cdot)$ (see (2.167)) we deduce the following one for $\hat{G}_{i,l}(\cdot,\cdot)$,

$$\hat{G}_{i,l}(y) = \hat{f}_{i,l}(y)\frac{1}{r_{i,l}(1-y)} + \hat{H}_{i,l}(y); \tag{2.170}$$

where $\hat{H}_{i,l}(\cdot)$ is integrable and $\hat{f}_{i,l}(\cdot)$ of class C^2.
Hence by using (2.169) and (2.170) we obtain the following inequality

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3 \tilde{G}(r_{i,l},s) e^{4\bar{u}_{i,l}(s)} ds = \frac{1}{r_{i,l}} \int_{\frac{1}{R}}^{R} y^3 \left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y) \right) r_{i,l}^4 e^{4\hat{u}_{i,l}(y)} dy. \tag{2.171}$$

Moreover using Harnack-type inequality for u_l (see (2.147)) and (2.168) we have that,

$$||r_{i,l}^4 e^{4\hat{u}_{i,l}}||_{C^\alpha(]\frac{1}{R},R[)} = o_l(1); \tag{2.172}$$

So using techniques of the theory of singular integral operators as in Lemma 4.4 ([46]) to have Holder estimates, we obtain

$$\int_{\frac{1}{R}}^{R} y^3 \left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right) r_{i,l}^4 e^{4\hat{u}_{i,l}(y)} dy = o_l(1);$$

hence with (2.171) we deduce that

$$\int_{\frac{r_{i,l}}{R}}^{r_{i,l}R} s^3 \tilde{G}(r,s) e^{4\bar{u}_{i,l}(s)} ds = o_l(\frac{1}{r_{i,l}}).$$

So we obtain

$$\bar{u}_{i,l}^{'}(r_{i,l}) \leq -2C\frac{1}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}} - F_{i,l}^{'}(r_{i,l}) + H_{i,l}^{'}(r). \tag{2.173}$$

Hence in both case we get

$$\bar{u}_{i,l}^{'}(r_{i,l}) \leq -2C\frac{1}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}} - F_{i,l}^{'}(r_{i,l}) + H_{i,l}^{'}(r). \tag{2.174}$$

Now let compute $\varphi_{i,l}^{'}(r_{i,l})$. From straightforward computations we have,

$$\varphi_{i,l}^{'}(r_{i,l}) = (r_{i,l})^{4\nu-1} exp(\bar{u}_{i,l}(r_{i,l})) \left(4\nu + r_{i,l}\bar{u}_{i,l}^{'}(r_{i,l})\right).$$

So using (2.173) we arrive to the following inequality,

$$\varphi_{i,l}^{'}(r_{i,l}) \leq (r_{i,l})^{4\nu-1} exp(\bar{u}_{i,l}(r_{i,l}) \left(4\nu - 2C + o_l(1) - r_{i,l}F_{i,l}^{'}(r_{i,l}) + r_{i,l}H_{i,l}^{'}(r_{i,l})\right);$$

so $\nu < \frac{1}{2}$ implies $4\nu - 2C + o_l(1) < 0$ for l sufficiently large.
Thus since $F_{i,l}^{'}$ and $H_{i,l}^{'}$ are bounded in $(0, \frac{inj_g(M)}{4})$ uniformly in l and $r_{i,l} \longrightarrow 0$ we have that for l big enough,

$$\varphi_{i,l}^{'}(r_{i,l}) < 0;$$

hence we reach the desired contradiction and we conclude the proof of the step.

Step 5 :Proof of Theorem 0.2.8

We show first the following estimate

$$\int_{M\setminus\cup_{i=1}^{i=N} B_{x_{i,l}}(\frac{R_{i,l}}{C})} e^{4u_l(y)} dV_g(y) = o_l(1).$$

For this we first start by proving

$$\bar{u}_l \longrightarrow -\infty \quad as \quad l \longrightarrow +\infty. \tag{2.175}$$

In fact, using the Green's representation formula for u_l (see Proposition 0.3.3) we have that for every $x \in M$,

$$u_l(x) = \bar{u}_l + 2\int_M G(x,y)\left(\bar{Q}(y)e^{4u_l(y)} - Q_l(y)\right) dV_g(y) + -\int_M G(x,y')T_l(y')dS_g(y')$$

$$\geq \bar{u}_l - C + 2\int_M G(x,y)\bar{Q}(y)e^{4u_l(y)} dV_g(y).$$

By assumption c) or d) of Step1 we have given any $\epsilon > 0$, there exists R_ϵ such that for l sufficiently large
if $\bar{x}_i \in int(M)$ then

$$\int_{B_{x_{1,l}}(R_\epsilon \mu_{1,l})} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) \geq 8\pi^2 - \frac{\epsilon}{32\pi^2}$$

and
if $\bar{x}_i \in \partial M$ then

$$\int_{B^+_{x_{1,l}}(R_\epsilon \mu_{1,l})} \bar{Q}_l(y) e^{4u_l(y)} dV_g(y) \geq 4\pi^2 - \frac{\epsilon}{16\pi^2}$$

Hence the last three formulas and the asymptotics of the Green's function of (P^4_g, P^3_g) imply that if $\bar{x}_1 \in int(M)$ then

$$e^{4u_l}(x) \geq C^{-1} e^{4\bar{u}_l} \frac{1}{|x - x_{1,l}|^{8-\epsilon}} \quad for \ \ |x - x_{1,l}| \geq 2R_\epsilon \mu_{1,l} \qquad for \ \ l \ \ large;$$

and if $\bar{x}_1 \in \partial M$ then

$$e^{4u_l}(x) \geq C^{-1} e^{4\bar{u}_l} \frac{1}{|x - x_{1,l}|^{8-\epsilon}} \quad for \ \ |x - x_{1,l}| \geq 2R_\epsilon \mu_{1,l} \qquad for \ \ l \ \ large;$$

From this it follows that

$$\begin{aligned}\int_M e^{4u_l(y)} dV_g(y) &\geq \int_{(B^+_{x_{1,l}}(inj_g(M)) \setminus B^+_{x_{1,l}}(2R_\epsilon \mu_{1,l}))} e^{4u_l(y)} dV_g(y) \\ &\geq C^{-1} e^{4\bar{u}_l} \int_{2R_\epsilon \mu_{1,l}}^{inj_g(M)} s^{\epsilon-(5)} ds \geq C^{-1} e^{4\bar{u}_l} \left(2R_\epsilon \mu_{1,l}\right)^{\epsilon-4}.\end{aligned}$$

So if ϵ is small enough we have from (38) that

$$\bar{u}_l \longrightarrow -\infty,$$

hence we are done .
Now by assumption b) of Step 1 we can cover $M \setminus \cup_{i=1}^{i=N} B_{x_{i,l}}(\frac{R_{i,l}}{C})$ with a finite number of balls $B_{y_k}(r_k)$ such that for any k there holds ,

$$\int_{B^+_{y_k}(2r_k)} \bar{Q}_l e^{4u_l(y)} dV_g(y) \leq 4\pi^2.$$

Now set $B_k = B_{y_k}(2r_k)$ and $\tilde{B}_k = B_{y_k}(r_k)$ so using again the Green representation formula for u_l we have $\forall x \in \tilde{B}_k$

$$u_l(x) = \bar{u}_l + 2\int_M G(x,y) \bar{Q}_l e^{4u_l(y)} dV_g(y) - \int_M G(x,y) Q_l(y) dV_g(y) - \int_{\partial M} G(x,y') T_l(y') dS_g(y').$$

hence

$$\begin{aligned}u_l(x) \leq \bar{u}_l + C + 2\int_M G(x,y) \bar{Q}_l e^{4u_l(y)} dV_g(y) &= \bar{u}_l + C + 2\int_{B_k} G(x,y) \bar{Q}_l e^{4u_l(y)} dV_g(y) \\ &+ 2\int_{M \setminus B_k} G(x,y) \bar{Q}_l e^{4u_l(y)} dV_g(y).\end{aligned}$$

So since G is smooth out of the diagonal we have that

$$u_l(x) \leq \bar{u}_l + C + 2\int_{B_k} G(x,y) \bar{Q}_l(y) e^{nu_l(y)} dV_g(y).$$

Now using Jensen's inequality we obtain ,

$$exp\left(\int_{B_k} G(x,y)\bar{Q}_l e^{4u_l(y)} dV_g(y)\right) \leq \int_M exp\left(||\bar{Q}e^{4u_l}\chi_{B_k}||_{L^1(M)}|G(x,y)|\right) \frac{\bar{Q}_l(y)e^{4u_l(y)}\chi_{B_k}(y)}{||\bar{Q}e^{4u_l}\chi_{B_k}||_{L^1(M)}} dV_g(y).$$

Hence using Fubini theorem we have

$$\int_{\tilde{B}_k} e^{4u_l}(y) dv_g(x) \leq Ce^{4\bar{u}_l} \sup_{y\in M,k} \int_M \left(\frac{1}{d_g(x,y)}\right)^{\frac{1}{2\pi^2}||\bar{Q}e^{4u_l}\chi_{B_k}||_{L^1(M)}} dV_g(x).$$

So from $\int_{B_k} \bar{Q}_l(y)e^{4u_l(y)} dV_g(y) \leq 4\pi^2$ and (2.175) we have that,

$$\int_{\tilde{B}_k} e^{4u_l(y)} dV_g(y) = o_l(1) \quad \forall \; k.$$

Hence

$$\int_{M\setminus\cup_{i=1}^{i=N} B_{x_{i,l}}(\frac{R_{i,l}}{C})} e^{4u_l(y)} dV_g(y) = o_l(1).$$

So since $B_{x_{i,l}}(\frac{R_{i,l}}{C})$ are disjoint then the Step 4 implies that,

$$\int_M \bar{Q}_l(y)e^{4u_l(y)} dV_g(y) = 4N\pi^2 + o_l(1),$$

hence (38) implies that

$$\int_M Q_0(y) dV_g(y) + \int_{\partial M} T_0(y') dS_g(y') = 4N\pi^2.$$

ending the proof of Theorem 0.2.8.

2.3 Proof of Theorem 0.2.10

In this Section, we give the proof of Theorem 0.2.10. We will use the same strategy as in the proof of Theorem 0.2.6 and Theorem 0.2.8, hence in many arguments we will be sketchy.
First of all, we recall the following particular case of the result of X. Xu (Theorem 1.2 in [89]).

Theorem 2.3.1. *([89]) There exists a dimensional constant $\sigma_3 > 0$ such that, if $u \in C^1(\mathbb{R}^3)$ is solution of the integral equation*

$$u(x) = \int_{\mathbb{R}^3} \sigma_3 \log\left(\frac{|y|}{|x-y|}\right) e^{3u(y)} dy + c_0,$$

where c_0 is a real number, then $e^u \in L^3(\mathbb{R}^3)$ implies, there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^3$ such that

$$u(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x-x_0|^2}\right).$$

Now, if σ_3 is as in Theorem 2.3.1, then we set $k_3 = 2\pi^2\sigma_3$ and $\gamma_3 = 2(k_3)^3$
We divide the proof in 5-steps as in [69].

Step 1

There exists $N \in \mathbb{N}^*$, N converging points $(x_{i,l}) \subset \partial M$ $i = 1, ..., N$, with limit points $x_i \in \partial M$, N sequences $(\mu_{i,l})$ $i = 1; ...; N$; of positive real numbers converging to 0 such that the following hold:

$a)$

$$\frac{d_g(x_{i,l},x_{j,l})}{\mu_{i,l}}\longrightarrow+\infty\quad i\neq j\;\; i,j=1,..,N\quad and\quad \bar{T}_l(x_{i,l})\mu_{i.l}^3e^{3u_l(x_{i,l})}=1;$$

$b)$For every i

$$v_{i,l}(x)=u_l(exp_{x_{i,l}}(\mu_{i,l}x))-u_l(x_{i,l})-\frac{1}{3}\log(k_3)\longrightarrow V_0(x)\qquad in\quad C^1_{loc}(\mathbb{R}^4_+),\; V_{0|\partial\mathbb{R}^4_+}(x):=\log(\frac{4\gamma_3}{4\gamma_3^2+|x|^2});$$

and

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B^+_{x_{i,l}}(R\mu_{i,l})\cap\partial M}\bar{T}_l(y)e^{3u_l(y)}ds_g(y)=4\pi^2;$$

$c)$

$$There\ exists\ \ C>0\ \ such\ that\ \inf_{i=1,\dots,N}d_g(x_{i,l},x)^3e^{3u_l(x)}\leq C\quad\forall x\in\partial M,\;\;\forall l\in\mathbb{N}.$$

Proof of Step 1

First of all let $x_l\in\partial M$ be such that $u_l(x_l)=\max_{x\in\partial M}u_l(x)$, then using the fact that u_l blows up we infer $u_l(x_l)\longrightarrow+\infty$.
Now since ∂M is compact, without loss of generality we can assume that $x_l\to\bar{x}\in\partial M$.
Next let $\mu_l>0$ be such that $\bar{T}_l(x_l)\mu_l^3e^{3u_l(x_l)}=1$. Since $\bar{T}_l\longrightarrow\bar{T}_0$ $C^1(\partial M)$, $\bar{T}_0>0$ and $u_l(x_l)\longrightarrow+\infty$, we have that $\mu_l\longrightarrow0$.
Let $B^0_+(\delta\mu_l^{-1})$ be the half Euclidean ball of center 0 and radius $\delta\mu_l^{-1}$, with $\delta>0$ small fixed .
For $x\in B^0_+(\delta\mu_l^{-1})$, we set

$$v_l(x)=u_l(exp_{x_l}(\mu_lx))-u_l(x_l)-\frac{1}{3}\log(k_3);\tag{2.175}$$

$$\tilde{Q}_l(x)=Q_l(exp_{x_l}(\mu_lx));\tag{2.176}$$

$$\tilde{\bar{Q}}_l(x)=\bar{Q}_l(exp_{x_l}(\mu_lx));\tag{2.177}$$

$$g_l(x)=\left(exp^*_{x_l}g\right)(\mu_lx).\tag{2.178}$$

Now from the Green representation formula we have,

$$u_l(x)-\bar{u}_l=\int_MG(x,y)P^4_gu_l(y)dV_g(y)+2\int_{\partial M}G(x,y')P^3_gu_l(y')dS_g(y');\quad\forall x\in M,\tag{2.179}$$

where G is the Green function of (P^4_g,P^3_g) (see Lemma 0.3.3).
Now using equation (15) and differentiating (2.179) with respect to x we obtain that for $k=1,2$

$$|\nabla^ku_l|_g(x)\leq\int_{\partial M}|\nabla^kG(x,y)|_g\bar{T}_l(y)e^{3u_l(y)}dV_g+O(1),$$

since $T_l\longrightarrow T_0$ in $C^1(\partial M)$ and $Q_l\to Q_0$ in $C^1(M)$.
Now let $y_l\in B^+_{x_l}(R\mu_l)$, $R>0$ fixed, by using the same argument as in [69] (formula 43 page 11) we obtain

$$\int_{\partial M}|\nabla^kG(y_l,y)|_ge^{3u_l(y)}dV_g(y)=O(\mu_l^{-k})\tag{2.180}$$

Hence we get

$$|\nabla^k v_l|_g(x) \le C. \tag{2.181}$$

Furthermore from the definition of v_l (see (2.175)), we get

$$v_l(x) \le v_l(0) = -\frac{1}{3}\log(k_3) \ \ \forall x \in \mathbb{R}^4_+ \tag{2.182}$$

Thus we infer that $(v_l)_l$ is uniformly bounded in $C^2(K)$ for all compact subsets K of $\mathbb{R}^4_+$. Hence by Arzelà-Ascoli theorem we derive that

$$v_l \longrightarrow V_0 \quad in \ C^1_{loc}(\mathbb{R}^4_+), \tag{2.183}$$

On the other hand (2.182) and (2.183) imply that

$$V_0(x) \le V_0(0) = -\frac{1}{3}\log(k_3) \ \ \forall x \in \mathbb{R}^4_+. \tag{2.184}$$

Moreover from (2.181) and (2.183) we have that V_0 is Lipschitz.
On the other hand using the Green's representation formula for (P^4_g, P^3_g) we obtain that for $x \in \mathbb{R}^4_+$ fixed and for R big enough such that $x \in B^0_+(R)$

$$u_l(exp_{x_l}(\mu_l x)) - \bar{u}_l = \int_M G(exp_{x_l}(\mu_l x), y)P^4_g u_l(y)dV_g(y) + 2\int_{\partial M} G(exp_{x_l}(\mu_l x), y')P^3_g u_l(y')dS_g(y'). \tag{2.185}$$

Now let us set

$$I_l(x) = 2\int_{B^+_{x_l}(R\mu_l)\cap\partial M} \left(G(exp_{x_l}(\mu_l x), y') - G(exp_{x_l}(0), y')\right)\bar{T}_l(y)e^{3u_l(y)}dS_g(y');$$

$$\mathrm{II}_l(x) = 2\int_{\partial M\setminus(B^+_{x_l}(R\mu_l)} \left(G(exp_{x_l}(\mu_l x), y') - G(exp_{x_l}(0), y')\right)\bar{T}_l(y')e^{3u_l(y)}dS_g(y');$$

$$\mathrm{III}_l(x) = 2\int_{\partial M} \left(G(exp_{x_l}(\mu_l x), y') - G(exp_{x_l}(0), y')\right)T_l(y)dS_g(y');$$

and

$$\mathrm{IIII}_l(x) = 2\int_M \left(G(exp_{x_l}(\mu_l x), y) - G(exp_{x_l}(0), y)\right)Q_l(y)dV_g(y).$$

Using again the same argument as in [69] (see formula (45)- formula (51)) we get

$$v_l(x) = I_l(x) + \mathrm{II}_l(x) - \mathrm{III}_l(x) - \mathrm{IIII}_l(x) - \frac{1}{4}\log(3). \tag{2.186}$$

Moreover following the same methods as in [69](see formula (53)-formula (62)) we obtain

$$\lim_l I_l(x) = \int_{B^0_+(R)\cap\partial R^4_+} \sigma_3 \log\left(\frac{|z|}{|x-z|}\right)e^{3V_0(z)}dz. \tag{2.187}$$

$$\limsup_l \mathrm{II}_l(x) = o_R(1). \tag{2.188}$$

$$\mathrm{III}_l(x) = o_l(1) \tag{2.189}$$

and

$$\mathrm{IIII}_l(x) = o_l(1). \tag{2.190}$$

Hence from (2.183), (2.186)-(2.190) by letting l tends to infinity and after R tends to infinity, we obtain $V_{0|\mathbb{R}^3}$(that for simplicity we will always write by V_0) satisfies the following conformally invariant integral equation on $\mathbb{R}^3$

$$V_0(x) = \int_{\mathbb{R}^3} \sigma_3 \log\left(\frac{|z|}{|x-z|}\right)e^{3V_0(z)}dz - \frac{1}{3}\log(k_3). \tag{2.191}$$

Now since V_0 is Lipschitz then the theory of singular integral operator gives that $V_0 \in C^1(\mathbb{R}^3)$.
On the other hand by using the change of variable $y = exp_{x_l}(\mu_l x)$, one can check that the following holds

$$\lim_{l \longrightarrow +\infty} \int_{B^+_{x_l}(R\mu_l)\cap\partial M} \bar{T}_l e^{3u_l} dV_g = k_3 \int_{B^+_0(R)\cap\partial R^4_+} e^{3V_0} dx; \tag{2.192}$$

Hence (40) implies that $e^{V_0} \in L^3(\mathbb{R}^3)$.
Furthermore by a classification result by X. Xu, see Theorem 2.3.1 for the solutions of (2.191) we derive that

$$V_0(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x - x_0|^2}\right) \tag{2.193}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^3$.
Moreover from $V_0(x) \leq V_0(0) = -\frac{1}{3}\log(k_3)$ $\forall x \in \mathbb{R}^3$, we have that $\lambda = 2k_3$ and $x_0 = 0$ namely,

$$V_0(x) = \log(\frac{4\gamma_3}{4\gamma_3^2 + |x|^2}).$$

On the other hand by letting R tends to infinity in (2.192) we obtain

$$\lim_{R\to+\infty}\lim_{l\to+\infty} \int_{B^+_{x_l}(R\mu_l)\cap\partial R^4_+} \bar{T}_l(y) e^{3u_l(y)} dS_g(y) = k_3 \int_{\mathbb{R}^3} e^{3V_0} dx. \tag{2.194}$$

Moreover from a generalized Pohozaev type identity by X.Xu [89] (see Theorem 1.1) we get

$$\sigma_3 \int_{\mathbb{R}^3} e^{3V_0(y)} dy = 2,$$

hence using (2.194) we derive that

$$\lim_{R\to+\infty}\lim_{l\to+\infty} \int_{B^+_{x_l}(R\mu_l)\cap\partial M} \bar{T}_l(y) e^{3u_l(y)} dS_g(y) = 4\pi^2$$

Now for $k \geq 1$ we say that (H_k) holds if there exists k converging points $(x_{i,l})_l \subset \partial M$ $i = 1, ..., k,$ k
sequences $(\mu_{i,l})$ $i = 1, ..., k$ of positive real numbers converging to 0 such that the following hold
(A^1_k)

$$\frac{d_g(x_{i,l}, x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i \neq j \quad i, j = 1, .., k \; and \quad \bar{T}_l(x_{i,l})\mu^3_{i.l} e^{3u_l(x_{i,l})} = 1;$$

(A^2_k)
For every $i = 1, \cdot, k$

$$x_{i,l} \to \bar{x}_i \in \partial M;$$

$$v_{i,l}(x) = u_l(exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \frac{1}{3}\log(k_3) \longrightarrow V_0(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}^4_+), \quad V_{0|\partial\mathbb{R}^4_+} := \log(\frac{4\gamma_3}{4\gamma_3^2 + |x|^2})$$

and

$$\lim_{R\to+\infty}\lim_{l\to+\infty} \int_{B^+_{x_{i,l}}(R\mu_{i,l})\cap\partial M} \bar{T}_l(y) e^{3u_l(y)} = 4\pi^2$$

Clearly, by the above arguments (H_1) holds. We let now $k \geq 1$ and assume that (H_k) holds. We also assume that

$$\sup_{\partial M} R_{k,l}(x)^3 e^{3u_l(x)} \longrightarrow +\infty \quad as \quad l \longrightarrow +\infty, \tag{2.195}$$

where

$$R_{k,l}(x) = \min_{i=1;..;k} d_g(x_{i,l}, x).$$

Now using the same argument as in [34],[69] and the arguments which have rule out the possibility of interior blow up above that also apply for local maxima, one can see easily that (H_{k+1}). Hence since $\left(A_k^1\right)$ and $\left(A_k^2\right)$ of H_k imply that

$$\int_{\partial M} \bar{T}_l(y) e^{3u_l(y)} dS_g(y) \geq k4\pi^2 + o_l(1).$$

Thus (40) imply that there exists a maximal k , $1 \leq k \leq \frac{1}{4\pi^2}\left(\int_M Q_0(y)dV_g(y) + \int_{\partial M} T_0(y')dS_g(y')\right)$, such that (H_k) holds. Arriving to this maximal k, we get that (2.195) cannot hold. Hence setting $N = k$ the proof of Step 1 is done.

Step 2

There exists a constant $C > 0$ such that

$$R_l(x)|\nabla_g u_l|_g(x) \leq C \qquad \forall x \in M \;\; and \;\; \forall l \in N; \quad \forall\, x \in \partial M \tag{2.196}$$

where

$$R_l(x) = \min_{i=1,..,N} d_g(x_{i,l}, x);$$

and the $x_{i,l}$'s are as in Step 1.

Proof of Step 2

First of all using the Green representation formula for (P_g^4, P_g^3) see Lemma 0.3.3 we obtain

$$u_l(x) - \bar{u}_l = \int_M G(x,y) P_g^4 u_l(y) dV_g(y) + 2\int_{\partial M} G(x,y') P_g^3 u_l(y') dS_g(y').$$

Now using the BVP (**??**) we get

$$\begin{aligned} u_l(x) - \bar{u}_l = -2\int_M G(x,y) Q_l dV_g(y) - 2\int_{\partial M} G(x,y') T_l(y') u_l(y') dS_g(y') \\ +2\int_{\partial M} G(x,y) \bar{T}_l(y') e^{3u_l(y')} dS_g(y'). \end{aligned} \tag{2.197}$$

Thus differentiating with respect to x (2.197) and using the fact that $Q_l \to Q_0$, $\bar{Q}_l \to \bar{Q}_0$ and $T_l \to T_0$ in C^1, we have that for $x_l \in \partial M$

$$|\nabla_g u_l(x_l)|_g = O\left(\int_{\partial M} \frac{1}{d_g(x_l, y)} e^{3u_l(y)} dS_g(y)\right) + O(1).$$

Hence at this stage following the same argument as in the proof of Theorem 1.3, Step 2 in [69], we obtain

$$\int_{\partial M} \frac{1}{(d_g(x_l, y))} e^{3u_l(y)} dV_g(y) = O\left(\frac{1}{R_l(x_l)}\right);$$

hence since x_l is arbitrary, then the proof of Step 2 is complete.

Step 3
Set

$$R_{i,l} = \min_{i\neq j} d_g(x_{i,l}, x_{j,l});$$

we have that
1) There exists a constant $C > 0$ such that $\forall\, r \in (0, R_{i,l}]$ $\forall\, s \in (\frac{r}{4}, r]$

$$|u_l\left(exp_{x_{i,l}}(rx)\right) - u_l\left(exp_{x_{i,l}}(sy)\right)| \leq C \quad for\ \ all\ \ x, y \in \partial\mathbb{R}^4_+\ \ such\ \ that\ \ |x|,\ |y| \leq \frac{3}{2}. \tag{2.198}$$

2) If $d_{i,l}$ is such that $0 < d_{i,l} \leq \frac{R_{i,l}}{2}$ and $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$ then we have that
if

$$\int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M} \bar{T}_l(y) e^{3u_l(y)} dS_g(y) = 4\pi^2 + o_l(1); \tag{2.199}$$

then

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M} \bar{T}_l(y) e^{3u_l(y)} ds_g(y) = 4\pi^2 + o_l(1).$$

3) Let R be large and fixed. If $d_{i,l} > 0$ is such that $d_{i,l} \longrightarrow 0$, $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, and $d_{i,l} < \frac{R_{i,l}}{4R}$
then if

$$\int_{B^+_{x_{i,l}}(\frac{d_{i,l}}{2R})\cap\partial M} \bar{Q}_l(y) e^{3u_l(y)} dS_g(y) = 4\pi^2 + o_l(1);$$

then by setting

$$\tilde{u}_l(x) = u_l(exp_{x_{i,l}}(d_{i,l}x)); \qquad x \in A^+_{2R};$$

where $A^+_{2R} = (B^0_+(2R) \setminus B^0_+(\frac{1}{2R})) \cap \partial\mathbb{R}^4_+$, we have that,

$$||d^4_{i,l} e^{3\tilde{u}_l}||_{C^\alpha(A^+_R)} \to 0 \ \ as\ \ l \to +\infty;$$

for some $\alpha \in (0,1)$ where $A^+_R = (B^0_+(R) \setminus B^0_+(\frac{1}{R})) \cap \partial\mathbb{R}^4_+$.

Proof of Step 3
We have that property 1 follows immediately from Step 2 and the definition of $R_{i,l}$. In fact we can join rx to sy by a curve whose length is bounded by a constant proportional to r.
Now let us show point 2. Thanks to $\frac{d_{i,l}}{\mu_{i,l}} \longrightarrow +\infty$, point c) of Step 1 and (2.199) we have that

$$\int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(\frac{d_{i,l}}{2})\cap\partial M} e^{3u_l(y)} dS_g(y) = o_l(1). \tag{2.200}$$

Thus using (2.198), with $s = \frac{r}{2}$ and $r = 2d_{i,l}$ we get

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(d_{i,l})\cap\partial M} e^{3u_l(y)} ds_g(y) \leq C \int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(\frac{d_{i,l}}{2})\cap\partial M} e^{3u_l(y)} dS_g(y);$$

Hence we arrive

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(d_{i,l})\cap\partial M} e^{3u_l(y)} dS_g(y) = o_l(1).$$

So the proof of point 2 is done. On the other hand by following in a straightforward way the proof of point 3 in Step 3 of Theorem 1.3 in [69] one gets easily point 3. Hence the proof of Step 3 is complete.

Step 4
There exists a positive constant C independent of l and i such that

$$\int_{B^+_{x_{i,l}}(\frac{R_{i,l}}{C})\cap\partial M} \bar{T}_l(y)e^{3u_l(y)}dS_g(y) = 4\pi^2 + o_l(1).$$

Proof of Step 4
The proof is an adaptation of the arguments in Step 4 ([69])

Step 5 :Proof of Theorem 0.2.10

Following the same argument as in Step 5([69]) we have

$$\int_{\partial M\setminus(\cup_{i=1}^{i=N} B^+_{x_{i,l}}(\frac{R_{i,l}}{C})\cap\partial M)} e^{3u_l(y)}dS_g(y) = o_l(1).$$

So since $B^+_{x_{i,l}}(\frac{R_{i,l}}{C})\cap\partial M$ are disjoint then the Step 4 implies that,

$$\int_{\partial M} \bar{T}_l(y)e^{3u_l(y)}dS_g(y) = 4N\pi^2 + o_l(1),$$

hence (40) implies that

$$\int_M Q_0(y)dV_g(y) + \int_{\partial M} T_0(y')dS_g(y') = 4N\pi^2.$$

ending the proof of Theorem 0.2.10.

2.4 Proof of Theorem 0.2.12

In this Section, we give the proof of Theorem 0.2.12. As already said in the previous Chapter, in order to prove the latter theorem, we exploit a result of Jost-Lin-Wang[43] and an other one of Li[52] that we recall

Theorem 2.4.1. *([43]) Let m_1, m_2 be two non-negative integers, and suppose Λ_1, Λ_2 are two compact sets of the intervals $(4\pi m_1, 4\pi(m_1+1))$ and $(4\pi m_2, 4\pi(m_2+1))$ respectively. Then if $\rho_1 \in \Lambda_1$ and $\rho_2 \in \Lambda_2$ and if we impose $\int_\Sigma u_i dV_g = 0$, $i = 1, 2$, the solutions of (11) stay uniformly bounded in $L^\infty(\Sigma)$ (actually in every $C^l(\Sigma)$ with $l \in \mathbb{N}$).*

This theorem, as stated in [43], requires m_1 and m_2 to be positive. However it is clear from the blow-up analysis there that one can allow also zero values of m_1 or of m_2.

Theorem 2.4.2. *([52]) Let $(u_k)_k$ be a sequence of solutions of the equations*

$$-\Delta u_k = \lambda_k\left(\frac{V_k e^{u_k}}{\int_\Sigma V_k e^{u_k} dV_g} - W_k\right),$$

where $(V_k)_k$ and $(W_k)_k$ satisfy

$$\int_\Sigma W_k dV_g = 1; \qquad \|W_k\|_{C^1(\Sigma)} \le C; \qquad |\log V_k| \le C; \qquad \|\nabla V_k\|_{L^\infty(\Sigma)} \le C,$$

and where $\lambda_k \to \lambda_0 > 0$, $\lambda_0 \neq 8k\pi$ for $k = 1, 2, \ldots$. Then, under the additional constraint $\int_\Sigma u_k dV_g = 1$, $(u_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$.

PROOF of Theorem 0.2.12
First of all we claim that the following property holds true: for any $p > 1$ there exists $\overline{\rho} > 0$ (depending on p, K_1, K_2, h_1 and h_2) such that for $\rho_{2,k} \leq \overline{\rho}$ the solutions of $(e^{u_{2,k}})_k$ stay uniformly bounded in $L^p(\Sigma)$.

The proof of this claim follows an argument in [15]: using the Green's representation formula and the fact that $\rho_1 > 0$ we find (recall that $\int_\Sigma u_{2,k} dV_g = 0$)

$$u_{2,k}(x) \leq C + \int_\Sigma G(x,y) \left(2\rho_{2,k} \frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} \right) dV_g(y),$$

where $G(x,y)$ is the Green's function of $-\Delta$ on Σ. Using the Jensen's inequality we then find

$$e^{pu_{2,k}(x)} \leq C \int_\Sigma \exp(2p\rho_{2,k} G(x,y)) \frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} dV_g(y).$$

Recalling that $G(x,y) \simeq \frac{1}{2\pi} \log \left(\frac{1}{d(x,y)} \right)$ and using also the Fubini theorem we get

$$\int_\Sigma e^{pu_{2,k}} dV_g \leq C \sup_{x \in \Sigma} \int_\Sigma \frac{1}{d(x,y)^{\frac{p\rho_{2,k}}{\pi}}} dV_g(y).$$

Now it is sufficient to take $\overline{\rho} = \frac{\pi}{2p}$ in order to obtain the claim.

For proving the proposition, in the case $\rho_{2,k} \geq \overline{\rho}$ we simply use Theorem 2.4.1, while for $\rho_{2,k} \leq \overline{\rho}$ we employ the above claim. In fact, from uniform L^p bounds on $e^{u_{2,k}}$ and from elliptic regularity theory, we obtain uniform $W^{2,p}$ bounds on the sequence $(v_k)_k$, where v_k is defined as the unique (we can assume that every v_k has zero average) solution of

$$-\Delta v_k = -\rho_{2,k} \left(\frac{h_2 e^{u_{2,k}}}{\int_\Sigma h_2 e^{u_{2,k}} dV_g} - 1 \right).$$

Taking p sufficiently large, by the Sobolev embedding, we also obtain uniform $C^{1,\alpha}$ bounds on $(v_k)_k$ (and hence on $(e^{v_k})_k$). Now we write $u_{1,k} = w_{1,k} + v_k$, so that $w_{1,k}$ satisfies

$$-\Delta w_{1,k} = 2\rho_{1,k} \left(\frac{h_1 e^{v_k} e^{w_{1,k}}}{\int_\Sigma h_1 e^{v_k} e^{w_{1,k}} dV_g} - 1 \right).$$

Moreover, since we are assuming $\int_\Sigma u_{1,k} dV_g = 0$ and since $\int_\Sigma v_k dV_g = 0$ as well, we have that also $\int_\Sigma w_{1,k} dV_g = 0$. Hence, applying Theorem 2.4.2 with $u_k = w_{1,k}$, $\lambda_k = 2\rho_{1,k}$, $V_k = h_1 e^{v_k}$ and $W_k \equiv 1$, we obtain uniform bounds on $w_{1,k}$ in $L^\infty(\Sigma)$. Since $(v_k)_k$ stays uniformly bounded in $L^\infty(\Sigma)$, we also get uniform bounds on $u_{1,k}$ in $L^\infty(\Sigma)$. Then, from the second equation in (41) we also achieve uniform bounds on $u_{2,k}$ in $W^{2,p}(\Sigma)$ (and hence in $L^\infty(\Sigma)$ taking p large enough). This concludes the proof. ■

Chapter 3

Existence results

3.1 A general min-max scheme and Struwe's monotonicity argument

Great part of this thesis deals with variational problems with *lack of compactness* and unbounded Euler-Lagrange functional . In order to get existence results, we use min-max method combined with Struwe's monotonicity argument. Since they turn out to be one of the main ingredients in this Chapter, then we decide to recall their abstract formulation.

We first give a general min-max scheme.

Theorem 3.1.1. *Let X be a Hilbert space and $J \in C^1(X, \mathbb{R})$ a functional on X. Let A_0 be a topological subspace of X and $\mathcal{A} \subset \mathcal{P}(X)$ be a collection of topological subspaces of X such that $\partial A \simeq A_0$ for all $A \in \mathcal{A}$. Suppose that there exists a positive constant β such that for all $A \in \mathcal{A}$*

$$I(u) > \beta + \sup_{v \in A_0} J(v) \;\; \forall u \in A \in \mathcal{A}, \tag{3.1}$$

then setting

$$c_J = \inf_{A \in \mathcal{A}} \sup_{u \in A} J(u)$$

we have that if $(PS)_{c_J}$ holds then c_J is a critical level of J

Remark 3.1.1. *We remark that the condition* (3.1) *produces Palais-Smale sequence at level c_J.*

In his study of surfaces of constant mean curvature with free boundary, M. Struwe has introduced a monotonicity argument to overcome the failure of (PS) condition. Later Ding-Jost-Li-Wang[30] have used the same strategy to study the mean field equation on compact closed Riemannian surfaces. We recall the general strategy here, since such a argument will be always used .

Theorem 3.1.2. *Let X be a Hilbert space and J_μ, $\mu \in \mathbb{R}$ be a family of C^1 functional on X. Let A_0 be a topological subspace of X and $\mathcal{A} \subset \mathcal{P}(X)$ be a collection of topological subspaces of X such that $\partial A \simeq A_0$ for all $A \in \mathcal{A}$. Suppose that J_μ have the following form*

$$J_\mu(u) = \frac{1}{2}||u|| - \mu F(u),$$

where F is such that ∇F is a compact operator.
Setting

$$c_{J_\mu} = \inf_{A \in \mathcal{A}} \sup_{u \in A} J_\mu(u)$$

we have that if the map $\mu \to \frac{C_{J_\mu}}{\mu}$ *is monotone in a neighborhood of* $]\mu_0-\epsilon,\mu_0+\epsilon[$ *of* μ_0*, then if* $\mu \in]\mu_0-\epsilon,\mu_0+\epsilon[$ *is a point of differentiability of the latter map, then any Palais-Smale sequence of* J_μ *at level* c_{J_μ} *is bounded.*

Remark 3.1.3. *We point out that in some cases to apply Theorem* 3.1.2 *we will do it with some modifications (see last Subsection).*

3.2 Topology of large negative sublevels of II_A, II_Q, II_T, II_ρ

In this Section, we discuss the topological structure of some large negatives sublevel of the Euler-Lagrange functionals II_A, II_Q, II_T, and II_ρ. The topological characterization of those sublevels will be used to get existence of solutions for the corresponding problems via the application of the abstract min-max Theorem 3.1.1 above and the monotonicity procedure given by Theorem 3.1.2.

3.2.1 Applications of the improved Moser-Trudinger type inequalities

In this Subsection, we give some applications of the improved Moser-Trudinger type inequalities of Chapter 1.
We start by giving a Lemma which show a criterion which implies the situation described in the first condition in (1.21). The result is proven in [33] Lemma 2.3.

Lemma 3.2.1. *Let* (M,g) *be an* n*-dimensional compact closed Riemannian manifold,* l *be a positive integer, and suppose that* ϵ *and* r *are positive numbers. Suppose that for a non-negative function* $f \in L^1(M)$ *with* $\|f\|_{L^1(M)} = 1$ *there holds*

$$\int_{\cup_{i=1}^{\ell} B_r(p_i)} f dV_g < 1-\varepsilon \qquad \text{for every } \ell\text{-tuples } p_1,\dots,p_\ell \in M$$

Then there exist $\overline{\varepsilon} > 0$ *and* $\overline{r} > 0$*, depending only on* ε, r, ℓ *and* M *(but not on* f*), and* $\ell+1$ *points* $\overline{p}_1,\dots,\overline{p}_{\ell+1} \in M$ *(which depend on* f*) satisfying*

$$\int_{B_{\overline{r}}(\overline{p}_1)} f dV_g > \overline{\varepsilon}, \ \dots, \ \int_{B_{\overline{r}}(\overline{p}_{\ell+1})} f dV_g > \overline{\varepsilon}; \qquad B_{2\overline{r}}(\overline{p}_i) \cap B_{2\overline{r}}(\overline{p}_j) = \emptyset \text{ for } i \neq j.$$

In the next Lemma we show a criterion which implies the situation described in the conditions in (1.28) and (1.29). The proof is a trivial adaptation of the arguments of Lemma 2.3 in [33].

Lemma 3.2.2. *Let* (M,g) *be a four dimensional compact closed Riemannian manifold with boundary,* h *and* l *be positive integer, and suppose that* ϵ*,* r *and* δ *are positive numbers. Assume* $f \in L^1(M)$ *is a non-negative function such that* $\|f\|_{L^1(M)} = 1$*, then we have the following*
1) If $\int_{M\setminus M_{4\delta}} f dV_g < \epsilon$ *then there holds*
If

$$\int_{M_{4\delta}\cap(\cup_{i=1}^{h} B_{p_i}(r))} f dV_g < \int_{M_{4\delta}} f dV_g - \varepsilon \qquad \text{for every } h\text{-tuples } p_1,\dots,p_h \in M_{4\delta} \text{ such that } B_{p_i}(2r) \subset M_{2\delta}$$

then there exist $\overline{\varepsilon} > 0$ *and* $\overline{r} > 0$*, depending only on* $\varepsilon, r, \tilde{h}$*,* δ *and* M *(but not on* f*), and points* $\overline{p}_1,\dots,\overline{p}_{h+1} \in M_{4\delta}$*, satisfying*

$$\int_{B_{\overline{p}_1}(\overline{r})} f dV_g > \overline{\varepsilon}, \ \dots, \ \int_{B_{\overline{p}_{\tilde{h}}}(\overline{r})} f dV_g > \overline{\varepsilon}; \qquad B_{\overline{p}_i}(2\overline{r}) \cap B_{\overline{p}_j}(2\overline{r}) = \emptyset \quad \text{for } i \neq j, \ B_{\overline{p}_j}(2\overline{r}) \subset M_{2\delta}. \tag{3.2}$$

2) If $\int_{M_{\frac{\delta}{4}}} f dV_g < \epsilon$ then there holds:
If

$$\int_{\partial M \times [0,\frac{\delta}{4}[\cap(\cup_{i=1}^{h} B_{q_i}^{+}(r))} f dV_g < \int_{\partial M\times[0,\frac{\delta}{4}]} f dV_g - \varepsilon \qquad \text{for every } l\text{-tuples } q_1,\dots,q_l \in \partial M,$$

$$B_{q_j}^{+}(2r) \subset \partial M \times [0,\frac{\delta}{2}]$$

then there exist $\overline{\varepsilon} > 0$,and $\overline{r} > 0$, depending only on $\varepsilon, r, \tilde{l}$ and M (but not on f), and points $\overline{q}_1,\dots,\overline{q}_{l+1} \in \partial M$, $B_{\overline{q}_j}^{+}(2r) \subset \partial M \times [0,\frac{\delta}{2}]$ satisfying

$$\int_{B_{\overline{q}_1}^{+}(\overline{r})} f dV_g > \overline{\varepsilon}, \;\dots,\; \int_{B_{\overline{q}_{\tilde{l}}}^{+}(\overline{r})} f dV_g > \overline{\varepsilon}; \qquad B_{\overline{q}_i}^{+}(2\overline{r}) \cap B_{\overline{q}_j}^{+}(2\overline{r}) = \emptyset \text{ for } i \neq j. \tag{3.3}$$

3) If $\int_{M\setminus M_{4\delta}} f dV_g \geq \epsilon$ and $\int_{M_{\frac{\delta}{4}}} f dV_g \geq \epsilon$ then there holds

$$\int_{M_{4\delta}\cap(\cup_{i=1}^{h} B_{p_i}(r))} f dV_g < \int_{M_{4\delta}} f dV_g - \qquad \text{for every } h\text{-tuples } p_1,\dots,p_h \in M_{4\delta} \text{ such that}$$

$$B_{p_i}(2r) \subset M_{2\delta}$$

and

$$\int_{\partial M \times [0,\frac{\delta}{4}[\cap(\cup_{i=1}^{h} B_{q_i}^{+}(r))} f dV_g < \int_{\partial M\times[0,\frac{\delta}{4}]} f dV_g - \varepsilon \qquad \text{for every } l\text{-tuples } q_1,\dots,q_l \in \partial M,$$

$$B_{q_j}^{+}(2r) \subset \partial M \times [0,\frac{\delta}{2}]$$

then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on $\varepsilon, r, \tilde{h}$, l, δ and M (but not on f), points $\overline{p}_1,\dots,\overline{p}_{h+1} \in M_{4\delta}$,and points $\overline{q}_1,\dots,\overline{q}_{l+1} \in \partial M$,
$B_{\overline{q}_j}^{+}(2r) \subset \partial M \times [0,\frac{\delta}{2}]$ satisfying

$$\int_{B_{\overline{p}_1}(\overline{r})} f dV_g > \overline{\varepsilon}, \;\dots,\; \int_{B_{\overline{p}_{\tilde{h}}}(\overline{r})} f dV_g > \overline{\varepsilon}; \qquad B_{\overline{p}_i}(2\overline{r}) \cap B_{\overline{p}_j}(2\overline{r}) = \emptyset \quad \text{for } i \neq j,\; B_{\overline{p}_j}(2\overline{r}) \subset M_{2\delta}. \tag{3.4}$$

and

$$\int_{B_{\overline{q}_1}^{+}(\overline{r})} f dV_g > \overline{\varepsilon}, \;\dots,\; \int_{B_{\overline{q}_{\tilde{l}}}^{+}(\overline{r})} f dV_g > \overline{\varepsilon}; \qquad B_{\overline{q}_i}^{+}(2\overline{r}) \cap B_{\overline{q}_j}^{+}(2\overline{r}) = \emptyset \text{ for } i \neq j. \tag{3.5}$$

In the next Lemma we show a criterion which implies the situation described in the first condition in (1.40). The result is proven in [33] Lemma 2.3.

Lemma 3.2.3. *Let (M,g) be a compact four dimensional Riemannian manifold with boundary, l be a given positive integer, and suppose that ϵ and r are positive numbers. Suppose that for a non-negative function $f \in L^1(\partial M)$ with $\|f\|_{L^1(\partial M)} = 1$ there holds*

$$\int_{\cup_{i=1}^{\ell} B_r^{\partial M}(p_i)} f dS_g < 1 - \varepsilon \qquad \text{for every } \ell\text{-tuples } p_1,\dots,p_\ell \in \partial M$$

Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on ε, r, ℓ and ∂M (but not on f), and $\ell+1$ points $\overline{p}_1,\dots,\overline{p}_{\ell+1} \in \partial M$ (which depend on f) satisfying

$$\int_{B_{\overline{r}}^{\partial M}(\overline{p}_1)} f dS_g > \overline{\varepsilon}, \;\dots,\; \int_{B_{\overline{r}}^{\partial M}(\overline{p}_{\ell+1})} f dS_g > \overline{\varepsilon}; \qquad B_{2\overline{r}}^{\partial M}(\overline{p}_i) \cap B_{2\overline{r}}^{\partial M}(\overline{p}_j) = \emptyset \text{ for } i \neq j.$$

Next we use the improved versions of The Moser-Trudinger type inequalities in Chapter 1 combined with the above Lemmas to characterize large negative sublevels of the functional $II_A, II_Q, II_T,$ and II_ρ.
We start with II_A.

Lemma 3.2.4. *Let (M, g) be a compact closed Riemannian manifold of arbitrary dimension n. Under the assumptions of Theorem 0.2.13, and for $k \geq 1$ given by (42), and for $\bar{k} = 0$, the following property holds. For any $\epsilon > 0$ and any $r > 0$ there exists large positive $L = L(\epsilon, r)$ such that for any $u \in H^{\frac{n}{2}}(M)$ with $II_A(u) \leq -L$, $\int_M e^{nu} dV_g = 1$ there exists k points $p_{1,u}, \ldots, p_{k,u} \in M$ such that*

$$\int_{M \setminus \cup_{i=1}^k B_{p_{i,u}}(r)} e^{nu} dV_g < \epsilon \tag{3.6}$$

Proof. Suppose by contradiction that the statement is not true. Then we can apply Lemma 3.2.1 with $l = k$, $f = e^{nu}$, and in turn Lemma 1.4.1 with $\delta_0 = 2\bar{r}$, $S_1 = B_{\bar{p}_1}(\bar{r}), \ldots, S_{k+1} = B_{\bar{p}_{k+1}}(\bar{r})$. This implies

$$II_A(u) \geq \frac{n}{2} \left\langle P_g^n u, u \right\rangle + n \int_M Q_n u dV_g - C\kappa_{P^n} - \frac{\kappa_{P^n} n}{4c_n(k+1) - \bar{\epsilon}} \left\langle P_g^n u, u \right\rangle - n\kappa_{P^n} \bar{u}.$$

Since $\kappa_{P^n} < 2c_n(k+1)$, we can choose $\bar{\epsilon} > 0$ so small that $\frac{n}{2} - \frac{\kappa_{P^n} n}{4c_n(k+1) - \bar{\epsilon}} > \delta > 0$. Hence using also the Poincaré inequality we deduce

$$\begin{aligned} II_A(u) &\geq \delta \left\langle P_g^n u, u \right\rangle + n \int_M Q_n (u - \bar{u}) dV_g - C\kappa_{P^n} \\ &\geq \delta \left\langle P_g^n u, u \right\rangle - nC \left\langle P_g^n u, u \right\rangle^{\frac{1}{2}} - C\kappa_{P^n} \geq -C. \end{aligned} \tag{3.7}$$

This concludes the proof. ■

Next we consider the functional II_Q.

Lemma 3.2.5. *Let (M, g) be a compact four dimensional Riemannian manifold with boundary. Under the assumptions of Theorem 0.2.16, and for $k \geq 1$ given by (45), and for $\bar{k} = 0$, the following property holds. For any $\epsilon > 0$, and $r > 0$ (all small) there exists large positive $L = L(\epsilon, r)$ such that for any $u \in H_{\frac{\partial}{\partial n}}$ with $II_Q(u) \leq -L$, $\int_M e^{4u} dV_g = 1$ the following holds, $\forall \delta > 0$ (small)*
1) If $\int_{M \setminus M_{4\delta}} e^{4u} dV_g < \epsilon$ then we have there exists $\tilde{k}$ points $p_{1,u}, \ldots, p_{\tilde{k},u} \in M_{4\delta}$ $B_{p_{i,u}}(2r) \subset M_{2\delta}$ such that

$$\int_{M_{4\delta} \setminus \cup_{i=1}^h B_{p_{i,u}}(r)} e^{4u} dV_g < \epsilon; \tag{3.8}$$

2) If $\int_{M_{\frac{\delta}{4}}} e^{4u} dVg < \epsilon$ then there exists k points $q_{1,u}, \ldots, q_{k,u} \in \partial M$, $B^+_{q_{i,u}}(2r) \subset \partial M \times [0, \frac{\delta}{2}]$ such that

$$\int_{\partial M \times [0, \frac{\delta}{4}] \setminus \cup_{i=1}^l B^+_{q_{i,u}}(r)} e^{4u} dV_g < \epsilon.$$

If If $\int_{M \setminus M_{4\delta}} e^{4u} dV_g \geq \epsilon$ and $\int_{M_{\frac{\delta}{4}}} e^{4u} dVg \geq \epsilon$ then there exists $(h, l) \in \mathbb{N}^* \times \mathbb{N}^*, 2h + l \leq k$, h points
$p_{1,u}, \ldots, p_{h,u} \in M_{4\delta}$ $B_{p_{i,u}}(2r) \subset M_{2\delta}$ and l points $q_{1,u}, \ldots, q_{l,u} \in \partial M$, $B^+_{q_{i,u}}(2r) \subset \partial M \times [0, \frac{\delta}{2}]$ such that

$$\int_{M_{4\delta} \setminus \cup_{i=1}^h B_{p_{i,u}}(r)} e^{4u} dV_g < \epsilon; \tag{3.9}$$

and

$$\int_{\partial M\times[0,\frac{\delta}{4}]\setminus\cup_{i=1}^{l}B^{+}_{q_{i,u}}(r)} e^{4u}dV_g < \epsilon.$$

PROOF. Suppose that by contradiction the statement is not true. Then there exists $\epsilon > 0, r > 0, \delta > 0$ and a sequence $(u_n) \in H_{\partial n}$ such that $\int_M e^{4u_n}dV_g = 1$, $II_Q(u_n) \to -\infty$ as $n \to +\infty$ and such that
Either 1)
$\int_{M\setminus M_{4\delta}} e^{4u_n}dV_g < \epsilon$ and $\tilde{k}$ tuples of points $p_1, \ldots, p_k \in M_{4\delta}$ and $B_{p_i}(2r) \subset M_{2\delta}$,we have

$$\int_{M_{4\delta}\cap(\cup_{i=1}^{h}B_{p_{i,u}}(r))} e^{4u}dV_g < \int_{M_{4\delta}} fdV_g - \epsilon; \tag{3.10}$$

Or
2)
$\int_{M_{\frac{\delta}{4}}} e^{4u_n}dV_g < \epsilon$ and $\forall k$ tuples of points $q_1, \ldots, q_k \in \partial M$ we have

$$\int_{\partial M\times[0,\frac{\delta}{4}\cap(\cup_{i=1}^{l}B^{+}_{q_{i,u}}(r))} e^{4u}dV_g < \int_{\partial M\times[0,\frac{\delta}{4}]} fdV_g - \epsilon.$$

Or
3)
$\int_{M\setminus M_{4\delta}} e^{4u_n}dV_g \geq \epsilon$, $\int_{M_{\frac{\delta}{4}}} e^{4u_n}dV_g \geq \epsilon$ and $\forall (h,l) \in \mathbb{N}^* \times \mathbb{N}^*$, $2h + l \leq k$, for every h tuples of points $p_1, \ldots, p_h \in M_{4\delta}$ and $B_{p_i}(2r) \subset M_{2\delta}$ and for every l tuples of points $q_1, \ldots, q_l \in \partial M$ we have

$$\int_{M_{4\delta}\cap(\cup_{i=1}^{h}B_{p_{i,u}}(r))} e^{4u}dV_g < \int_{M_{4\delta}} fdV_g - \epsilon; \tag{3.11}$$

and

$$\int_{\partial M\times[0,\frac{\delta}{4}\cap(\cup_{i=1}^{l}B^{+}_{q_{i,u}}(r))} e^{4u}dV_g < \int_{\partial M\times[0,\frac{\delta}{4}]} fdV_g - \epsilon.$$

Now since the arguments we will carried out work for all the three cases, then we will focus only on the case 3. We assume that this is the case and we apply Lemma 3.2.2 with $f = e^{4u_n}$, and in turn Lemma 1.4.2 with $\delta_0 = 2\bar{r}$, $S_i = B_{\bar{p}_i}(\bar{r})$, $\Omega_j = B^{+}_{\bar{q}_j}(\bar{r})$ and $\gamma_0 = \bar{\epsilon}$ where $\bar{\epsilon}$, $\bar{r}$, $\bar{p}_i$ and $\bar{q}_i$ are given as in Lemma 3.2.2. Thus we have for every $\tilde{\epsilon} > 0$ there exists C depending on ϵ, r, δ and $\tilde{\epsilon}$ such that

$$II_Q(u_n) \geq \left\langle P_g^{4,3}u_n, u_n\right\rangle + 4\int_M Q_g u_n dV_g + 4\int_{\partial M} T_g u_n dS_g - \frac{\kappa_{P^4,P^3}}{4\pi^2(2\tilde{h} + \tilde{l} - \tilde{\epsilon})}\left\langle P_g^{4,3}u_n, u_n\right\rangle$$
$$-C\kappa_{P^4,P^3} - 4\kappa_{P^4,P^3}\overline{u_n}$$

where $\tilde{h}$ and $\tilde{l}$ are given as in Lemma 3.2.2 and C is independent of n. On the other hand, using the fact that $2\tilde{h} + \tilde{l} \geq k + 1$ we have that

$$II_Q(u_n) \geq \left\langle P_g^{4,3}u_n, u_n\right\rangle + 4\int_M Q_g u_n dV_g + 4\int_{\partial M} T_g u_n dS_g - \frac{\kappa_{P^4,P^3}}{4\pi^2(k + 1 - \tilde{\epsilon})}\left\langle P_g^{4,3}u_n, u_n\right\rangle$$
$$-C\kappa_{P^4,P^3} - 4\kappa_{P^4,P^3}\overline{u_n}.$$

So, since $\kappa_{P^4,P^3} < (k+1)4\pi^2$, by choosing $\tilde{\epsilon}$ small we get

$$II_Q(u_n) \geq \beta\left\langle P_g^{4,3}u_n, u_n\right\rangle - 4C\left\langle P_g^{4,3}u_n, u_n\right\rangle^{\frac{1}{2}} - C\kappa_{P^4,P^3};$$

thanks to Hölder inequality, Sobolev embedding, trace Sobolev embedding and to the fact that $KerP_{g_0}^{4,3} \simeq \mathbb{R}$ (where $\beta = 1 - \frac{\kappa_{P^4,P^3}}{4\pi^2(k+1-\tilde{\epsilon})} > 0$). Thus we arrive to

$$II_Q(u_n) \geq -C.$$

So we reach a contradiction. Hence the Lemma is proved. ■

Now we consider the functional II_T.

Lemma 3.2.6. *Let (M, g) be a compact four dimensional Riemannian manifold with boundary. Under the assumptions of Theorem 0.2.16, and for $k \geq 1$ given by (47), and for $\bar{k} = 0$, the following property holds. For any $\epsilon > 0$ and any $r > 0$ there exists large positive $L = L(\epsilon, r)$ such that for any $u \in H_{\frac{\partial}{\partial n}}$ with $II_T(u) \leq -L$, $\int_{\partial M} e^{3u} dS_g = 1$ there exists k points $p_{1,u}, \dots, p_{k,u} \in \partial M$ such that*

$$\int_{\partial M \setminus \cup_{i=1}^{k} B^{\partial M}_{p_{i,u}}(r)} e^{3u} dS_g < \epsilon \tag{3.12}$$

Proof. The proof is same as the one of Lemma in [33]. For the reader convenience we repeat it. Suppose that by contradiction the statement is not true. Then there exists $\epsilon > 0, r > 0$, and a sequence $(u_n) \in H_{\partial n}$ such that $\int_{\partial M} e^{3u_n} dS_g = 1$, $II_T(u_n) \to -\infty$ as $n \to +\infty$ and such that for any k tuples of points $p_1, \dots, p_k \in \partial M$,we have

$$\int_{(\cup_{i=1}^{k} B^{\partial M}_{p_{i,u}}(r))} e^{3u} dS_g < 1 - \epsilon; \tag{3.13}$$

Now applying Lemma 3.2.3 with $f = e^{3u_n}$, and after Lemma 1.4.3 with $\delta_0 = 2\bar{r}$, $S_i = B^{\partial M}_{\bar{p}_i}(\bar{r})$, and $\gamma_0 = \bar{\epsilon}$ where $\bar{\epsilon}$, $\bar{r}, \bar{p}_i$ are given as in Lemma 3.2.3, we have for every $\tilde{\epsilon} > 0$ there exists C depending on ϵ, r, and $\tilde{\epsilon}$ such that

$$II_T(u_n) \geq \left\langle P^{4,3}_g u_n, u_n \right\rangle + 4\int_M Q_g u_n dV_g + 4\int_{\partial M} T_g u_n dS_g - \frac{4}{3}\kappa_{(P^4,P^3)} \frac{3}{16\pi^2(k+1-\tilde{\epsilon})} \left\langle P^{4,3}_g u_n, u_n \right\rangle$$
$$-C\kappa_{(P^4,P^3)} - 4\kappa_{(P^4,P^3)} \overline{u_n}_{\partial M}$$

where C is independent of n. Using elementary simplifications, the above inequality becomes

$$II_T(u_n) \geq \left\langle P^{4,3}_g u_n, u_n \right\rangle + 4\int_M Q_g u_n dV_g + 4\int_{\partial M} T_g u_n dS_g - \frac{\kappa_{P^4,P^3}}{4\pi^2(k+1-\tilde{\epsilon})} \left\langle P^{4,3}_g u_n, u_n \right\rangle$$
$$-C\kappa_{P^4,P^3} - 4\kappa_{P^4,P^3} \overline{u_n}_{\partial M}.$$

So, since $\kappa_{P^4,P^3} < (k+1)4\pi^2$, by choosing $\tilde{\epsilon}$ small we get

$$II_T(u_n) \geq \beta \left\langle P^{4,3}_g u_n, u_n \right\rangle - 4C \left\langle P^{4,3}_g u_n, u_n \right\rangle^{\frac{1}{2}} - C\kappa_{P^4,P^3};$$

thanks to Hölder inequality, Sobolev embedding, trace Sobolev embedding and to the fact that $Ker P^{4,3}_{g_0} \simeq \mathbb{R}$ (where $\beta = 1 - \frac{\kappa_{P^4,P^3}}{4\pi^2(k+1-\tilde{\epsilon})} > 0$). Thus we arrive to

$$II_T(u_n) \geq -C.$$

So we reach a contradiction. Hence the Lemma is proved. ■

Finally we consider the functional II_ρ.

Lemma 3.2.7. *let (Σ, g) be a compact closed Riemannian surface and suppose $\rho_2 < 4\pi$ and that $\rho_1 \in (4\pi m, 4\pi(m+1))$. Then for any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r)$ such that for every $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with $II_\rho(u) \leq -L$ and with $\int_\Sigma e^{u_i} dV_g = 1$, $i = 1, 2$, there exists m points $p_{1,u_1}, \dots, p_{m,u_1} \in \Sigma$ such that*

$$\int_{\Sigma \setminus \cup_{i=1}^{m} B_r(p_{i,u_1})} e^{u_1} dV_g < \varepsilon. \tag{3.14}$$

PROOF. Suppose by contradiction that the statement is not true. Then we can apply Lemma 3.2.1 with $\ell = m+1$ and $f = e^{u_1}$ to obtain $\hat{\delta}_0$, $\hat{\gamma}_0$ and sets $\hat{S}_1, \dots \hat{S}_{m+1}$ such that

$$d(\hat{S}_i, \hat{S}_j) \geq \hat{\delta}_0, \qquad i \neq j;$$

$$\int_{\hat{S}_i} e^{u_1} dV_g > \hat{\gamma}_0 \int_\Sigma e^{u_1} dV_g, \qquad i = 1, \dots, m+1.$$

Now we notice that, by the Jensen's inequality, there holds $\int_\Sigma u_i dV_g \leq 0$ for $i = 1, 2$, and that two cases may occur

(a) $\rho_2 \leq 0$;

(b) $\rho_2 > 0$.

In case **(a)** we have that $\rho_2 \int_\Sigma u_2 dV_g \geq 0$. Using also inequality (1.47) to find

$$II_\rho(u_1, u_2) \geq \frac{1}{4} \int_\Sigma |\nabla u_1|^2 dV_g + \rho_1 \int_\Sigma u_1 dV_g - C.$$

Now it is sufficient to use Proposition 1.4.6 with $\ell = m+1$, $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $S_j = \hat{S}_j$, $j = 1, \dots, m+1$ and $\tilde{\varepsilon} \in (0, 16\pi(m+1) - 4\rho_1)$, to get

$$\begin{aligned} II_\rho(u_1, u_2) &\geq \frac{1}{4} \int_\Sigma |\nabla u_1|^2 dV_g - \frac{\rho_1}{16\pi(m+1) - \tilde{\varepsilon}} \int_\Sigma |\nabla u_1|^2 dV_g - C \\ &\geq \frac{16\pi(m+1) - 4\rho_1 - \tilde{\varepsilon}}{4\,[16\pi(m+1) - \tilde{\varepsilon}]} \int_\Sigma |\nabla u_1|^2 dV_g - \tilde{C}, \end{aligned}$$

where $\tilde{C}$ is independent of (u_1, u_2).

In case **(b)** we use Proposition 1.4.4 with $\delta_0 = \hat{\delta}_0$, $\gamma_0 = \hat{\gamma}_0$, $\ell = m+1$, $S_j = \hat{S}_j$ and $\tilde{\varepsilon}$ such that $(4\pi - \tilde{\varepsilon})(m+1) > \rho_1$ and such that $4\pi - \tilde{\varepsilon} > \rho_2$ (recall that ρ_1 is strictly less than $4\pi(m+1)$ and that $\mu_2 < 4\pi$), to deduce that

$$\begin{aligned} II_\rho(u_1, u_2) &\geq (4\pi - \tilde{\varepsilon})\,[-(m+1)\overline{u}_1 - \overline{u}_2] + \rho_1 \overline{u}_1 + \rho_2 \overline{u}_2 \\ &= (\rho_1 - (m+1)(4\pi - \tilde{\varepsilon}))\,\overline{u}_1 + (\rho_2 - 4\pi + \tilde{\varepsilon})\,\overline{u}_2 - C \geq -C, \end{aligned}$$

by the Jensen inequality, where, again, $\tilde{C}$ is independent of (u_1, u_2). This concludes the proof. ∎

Next we give some corollaries which are direct consequences of Lemma 3.2.4-Lemma 3.2.7 . Loosely speaking it gives the distance of some (suitably) normalized functions belonging to large negative sublevels of II_A, II_Q, II_T,and II_ρ to some barycentric sets.

We start with II_A. We state a result which gives the distance of the functions e^{nu} from M_k for u belonging to low energy levels of II_A such that $\int_M e^{nu} dV_g = 1$ and $II_A(u) < -L$ with L large. Its proof is similar to the one of the next corollary, hence we ommit it.

Corollary 3.2.8. *Let (M, g) be a compact n-dimensional closed Riemannian manifold with P^n_g non-negative and $KerP^n_g \simeq \mathbb{R}$. Let $\overline{\varepsilon}$ be a (small) arbitrary positive number and k be given as in (42). Then there exists $L > 0$ such that, if $II_A(u) \leq -L$ and $\int_M e^{nu} dV_g = 1$, then we have that $d(e^{nu}, M_k) \leq \overline{\varepsilon}$.*

Next we consider the functional II_Q. We give a corollary which provides the distance of the functions e^{4u} (suitably normalized) from $(M_\partial)_k$.

Corollary 3.2.9. *Let (M, g) be a compact four dimensional Riemannian manifold with boundary such that $P^{4,3}_g$ non-negative and $KerP^{4,3}_g \simeq \mathbb{R}$. Let $\overline{\varepsilon}$ be a (small) arbitrary positive number and k be given as in (45). Then there exists $L > 0$ such that, if $II_Q(u) \leq -L$ and $\int_M e^{4u} dV_g = 1$, then we have that $d_M(e^{4u}, (M_\partial)_k) \leq \overline{\varepsilon}$.*

PROOF. Let $\epsilon > 0$, $r > 0$ (to be fixed later) and let L be the corresponding constant given by Lemma 3.2.5. Now let $\delta > 0$, then by Lemma 3.2.5 we have that the following 3 situations:
a)
The conclusion 1 in Lemma 3.2.5 holds
Or
b)
The conclusion 2 in Lemma3.2.5 holds
Or
c)
The conclusion in Lemma 3.2.5 hold.

Suppose that **a)** holds. Since the same arguments can be carried out for the other cases, then we will only consider this case . We have that by Lemma (3.2.5), there exists $\tilde{k}$ points $p_1, \cdots, p_{\tilde{k}}$ verifying (3.8). Next we define $\sigma \in (M_\partial)_k$ as follows

$$\sigma = \sum_{i=1}^{\tilde{k}} t_i \delta_{p_i} \text{ where } t_i = \int_{A_{r,i}} e^{4u} dV_j, \;\; A_{r,i} := B_{p_i}(r) \setminus \cup_{s=1}^{i-1} B_{p_s}(r), \; i = 1, \cdots, \tilde{k}-1, \; t_{\tilde{k}} = 1 - \sum_{i=1}^{\tilde{k}-1} t_i.$$

By construction we have $A_{r,i}$ are disjoint and $\cup_{i=1}^{\tilde{k}-1} A_{r,i} = \cup_{i=1}^{\tilde{k}-1} B_{p_i}(r)$. Now let $\varphi \in C^1(M)$ be such that $||\varphi||_{C^1(M)} \leq 1$. By triangle inequality, the mean value theorem and the integral estimate in Lemma 3.2.5 we have that the following estimate holds

$$\left| \int_M e^{4u} \varphi - < \sigma, \varphi > \right| \leq C_M \left(r + \epsilon + \int_{M_{2\delta} \setminus M_{4\delta}} e^{4u} dV_g \right)$$

where C_M is a constant depending only on M. So, letting δ tend to zero and choosing ϵ and r so small that $C_M(r + \epsilon) < \frac{\bar{\epsilon}}{2}$, we obtain

$$d_M(e^{4u}, (M_\partial)_k) < \bar{\epsilon};$$

as desired. ■

Next we turn to the functional II_T. We give a corollary which provides the distance of the functions e^{3u} from ∂M_k for u belonging to low energy levels of II_T such that $\int_{\partial M} e^{3u} dS_g = 1$. Its proof is the same as the one above.

Corollary 3.2.10. *Let (M, g) be a compact four dimensional Riemannian manifold with boundary such that $Ker P_g^{4,3} \simeq \mathbb{R}$ and $P_g^{4,3}$ non-negative. Let $\bar{\varepsilon}$ be a (small) arbitrary positive number and k be given as in (47). Then there exists $L > 0$ such that, if $II_T(u) \leq -L$ and $\int_{\partial M} e^{3u} dS_g = 1$, then we have that $d_{\partial M}(e^{3u}, \partial M_k) \leq \bar{\varepsilon}$.*

Now we consider the remaining functional, namely II_ρ. Using the same argument as in the above corollaries, we get following result regarding the distance of the functions e^{u_1} (suitably normalized) from Σ_m.

Corollary 3.2.11. *Let $\bar{\varepsilon}$ be a (small) arbitrary positive number, and let $\rho_1 \in (4\pi m, 4\pi(m+1))$, $\rho_2 < 4\pi$. Then there exists $L > 0$ such that, if $II_\rho(u_1, u_2) \leq -L$ and if $\int_\Sigma e^{u_1} dV_g = 1$, we have $d(e^{u_1}, \Sigma_m) < \bar{\varepsilon}$.*

3.2.2 Projections of large negative sublevels of II_A, II_Q, II_T, and II_ρ onto barycentric sets

In this Subsection we show how to map nontrivialy large negative sublevels of the functionals II_A, II_Q, II_T and II_ρ onto appropriate barycentric sets.
We first discuss the topology of some sets which will be used to do that. We start by a Proposition whose proof can be found in [33].

Proposition 3.2.12. *For every closed compact n-dimensional Riemannian manifold M and for every positive integer k, the* set of formal barycenters *M_k is non-contractible. Indeed we have $H_{(n+1)k-1}(M_k;\mathbb{Z}_2) \neq 0$.*

Next we give a proposition which asserts the non-contractibility of the barycentric set $(M_\partial)_k$.

Proposition 3.2.13. *For every compact four dimensional Riemannian manifold (M,g) with smooth boundary, and for every positive integer k, the barycentric set $(M_\partial)_k$ is non-contractible.*

To prove the Proposition we need an auxiliary Lemma. It is a trivial consequence of normal geodesics at the boundary.

Lemma 3.2.14. *Let (M,g) be a compact four dimensional Riemannian manifold with boundary. Then there exists a small $\epsilon_0 > 0$ such that a continuous projection*

$$P_{\partial M} : (\partial M)^{\epsilon_0} \longrightarrow \partial M$$

exists.

Proof of Proposition 3.2.13
Suppose that the following claim is true, $(\partial M)_k$ is a deformation retract of some of its open neighborhood U in $(M_\partial)_k$ such that setting $V = (M_\partial)_k \backslash (\partial M)_k$, we have that $X = U \cup int(V) \simeq (M_\partial)_k$. Now assuming that the claim holds we have that

$$H_{4k-1}(X;\mathbb{Z}_2) \simeq H_{4k-1}((M_\partial)_k;\mathbb{Z}_2); \tag{3.15}$$

and

$$H_{4k-1}(U;\mathbb{Z}_2) \simeq H_{4k-1}((\partial M)_k;\mathbb{Z}_2). \tag{3.16}$$

Next let us denote

$$i : U \cap V \to U, \;\; j : U \cap V \to V, \;\; m : U \to X, \;\; t : V \to X$$

the canonical injections and by i_*, j_*, m_*, t_* the corresponding homomorphism on homology groups.
We have that by Mayers-Vietoris Theorem there exists an homomorphism $\Delta : H_p((M_\partial)_k)) \to H_{p-1}((M_\partial)_k)$
(where p is a generic positive integer number) such that the following sequence is exact

$$\begin{aligned} \cdots \overset{\Delta}{\to} H_{4k-1}(U \cap V;\mathbb{Z}_2) \overset{(i_*,j_*)}{\to} H_{4k-1}(U;\mathbb{Z}_2) \oplus H_{4k-1}(V;\mathbb{Z}_2) \overset{m_*-t_*}{\to} H_{4k-1}(X;\mathbb{Z}_2) \\ \overset{\Delta}{\to} H_{4k-2}(U \cap V;\mathbb{Z}_2) \overset{(i_*,j_*)}{\to} \cdots \end{aligned} \tag{3.17}$$

Now for $h \in \mathbb{N}, l \in \mathbb{N}$ such that $h \leq \tilde{k}, l \leq k$ and $2h + l \leq k$ we recall that $M_{h,l}$ (for the definition see section 2) is a stratified set, namely a union of sets of different dimension. The maximal dimension is $5h + 4l - 1$, when all the points are distinct and the coefficients belongs to $(0,1)$. Hence the following holds

$$dim(M_{h,l} \cap V) \leq 5h + 4l - 1;$$

and if $h = 0$ then

$$M_{0,l} \cap V = \emptyset.$$

Hence from the trivial identity $5h + 4l - 1 < 4k - 2$ for such a (h,l) with $h \neq 0$, we infer that

$$H_{4k-1}(U \cap V;\mathbb{Z}_2) = H_{4k-1}(V;\mathbb{Z}_2) = H_{4k-2}(U \cap V;\mathbb{Z}_2) = 0$$

Thus from (3.17) we deduce that

$$H_{4k-1}(U;\mathbb{Z}_2) \simeq H_{4k-1}(X;\mathbb{Z}_2)$$

So using Proposition 3.2.12, and the formulas (3.15) and (3.16) we get

$$H_{4k-1}((M_\partial)_k;\mathbb{Z}_2)\neq 0$$

Hence to complete the proof of the Lemma it is sufficient to prove the claim. Now let us make its proof.
First of all it is easy to see that there exists $\epsilon>0$ ($4\epsilon<\epsilon_0$) small enough and a continuous map

$$X_\partial:[0,1]\times(\partial M)^{2\epsilon}\longrightarrow(\partial M)^{2\epsilon}$$

such that

$$X_\partial(0,\cdot)=Id_{(\partial M)^{2\epsilon}}(\cdot);\qquad X_\partial(1,\cdot)=P_{\partial M}(\cdot).$$

where P_∂ is given by Lemma 3.2.14.
Next, we define a homotopy $F:[0,1]\times B_{2\epsilon,k}\longrightarrow B_{2\epsilon,k}$ (for the definition of $B_{2\epsilon,k}$ see section 2) whose construction is based on the following idea. Given $\sigma=\sigma_{int}+\sigma_{bdry}\in B_{2\epsilon,k}$ $\sigma_{int}=\sum_{i=1}^h t_i\delta_{x_i}$, $\sigma_{bdry}=\sum_{i=1}^l s_i\delta_{y_i}$ $h\leq\tilde{k}$, $l\leq k$, $2h+l\leq k$, we fixed the boundary part, namely $\sigma_{bdry}\in B_{2\epsilon,k}$. And for the interior part σ_{int}, we argue as follows if x_i is closed to the boundary at distance less than ϵ we send $t_i\delta_{x_i}$, to $t_i\delta_{P_{\partial M}(x_i)}$, if it is far from the boundary, say at distance bigger than 2ϵ we squeeze and in the intermediate regime we use an homotopy argument reflecting the possibility between squeezing and projection to boundary via $P_{\partial M}$ since the distance is less or equal than 2ϵ. More precisely w e define the homotopy $F:[0,1]\times B_{2\epsilon,k}\longrightarrow B_{2\epsilon,k}$ as follows
For every $\sigma=\sigma_{int}+\sigma_{bdry}\in B_{2\epsilon,k}$ with $\sigma_{int}=\sum_{i=1}^h t_i\delta_{x_i}$, $\sigma_{bdry}=\sum_{i=1}^l s_i\delta_{y_i}$ and $s\in[0,1]$ we set

$$F(\sigma,s)=\sigma(s)+\sigma_{bdry}$$

where $\sigma(s)$ is defined as

$$\sigma(s)=\sum_{i=1}t_i(s)\delta_{x_i(s)}$$

and

$$t_i(s)\delta_{x_i(s)}=\begin{cases}(1-\frac{s}{2})\gamma(s)t_i\delta_{X_\partial(s,x_i)} & \text{if } dist(x_i,\partial M)\leq\epsilon;\\ (1-s\frac{dist(x_i,\partial M)}{2\epsilon})\gamma(s)t_i\delta_{X_\partial(2-\frac{dist(x_i,\partial M)}{\epsilon},x_i)} & \text{if } \epsilon\leq dist(x_i,\partial M)\leq 2\epsilon;\\ (1-s)\gamma(s)t_i\delta_{x_i} & \text{if } dist(x_i,\partial M)\geq 2\epsilon;\end{cases}$$

where $\gamma(s)$ is such that we have the normalization $\sum_{i=1}^h t_i(s)+\sum_{i=1}^l s_i=1$. Thus by trivial calculations we obtain

$$\gamma(s)=\frac{\sum_{i-=1}^l t_i}{\sum_{d(x_i,\partial M)<\epsilon}((1-\frac{s}{2})t_i)+\sum_{\epsilon\leq d(x_i,\partial M)<2\epsilon}((1-s\frac{dist(x_i,\partial M)}{2\epsilon})t_i))+\sum_{d(x_i,\partial M)\geq 2\epsilon}((1-s)t_i)}.$$

So by setting $U=B_{2\epsilon,k}$ we have that the claim is proved. Hence the proof of the proposition is complete. ■

Using the barycentric sets, we give a first step in describing the topology of large negative sublevels of the functionals II_A, II_Q, II_T and II_ρ. We start with the functional II_A.

Proposition 3.2.15. *Let (M,g) be a compact closed n-dimensional Riemannian manifold with P_g^n nonegative and $KerP_g^n\simeq\mathbb{R}$. For $k\geq 1$ given as in (42), there exists a large $L>0$ and a continuous map Ψ from the sublevel $\{u\in H^{\frac{n}{2}}(M):\ II_A(u)<-L,$ and $\int_M e^{nu}dV_g=1\}$ into M_k which is topologically non-trivial.*

Remark 3.2.16. *a) By topologically non-trivial, we means that the map carry some homology.*
b) The non triviality of the map will come from the non-contractibility of M_k and the Proposition 3.2.25 below.

To prove the Proposition, we need the following Lemma, whose proof comes from the arguments of Proposition 3.1 in [33].

Lemma 3.2.17. *Let (M, g) be a compact closed n-dimensional smooth Riemannian manifold and l be a positive integer. Then there exists ϵ_l such that for all $\epsilon \leq \epsilon_l$, there exists a continuous nontrivial map $\Pi_{\epsilon,l} : D_{\epsilon,l} \to M_l$.*

Proof of Proposition 3.2.15
We fix ε_k so small that Proposition 3.2.17 applies with $l = k$. Then we apply Corollary 3.2.10 with $\overline{\varepsilon} = \varepsilon_k$. We let L be the corresponding large number, so that if $u \in \{v \in H^{\frac{n}{2}}(M) : II_A(v) < -L$, and $\int_M e^{nv} dV_g = 1\}$, then $d(e^{nu}, M_k) < \varepsilon_k$. Hence for these ranges of u, since the map $u \mapsto e^{nu}$ is continuous from $H^{\frac{n}{2}}(M)$ into $L^1(M)$, the projections $\Pi_{\epsilon_k,k}$ from $H^{\frac{n}{2}}(M)$ onto M_k is well defined and continuous. ■

Now we consider the functional II_Q.

Proposition 3.2.18. *Let (M, g) be a compact four dimensional smooth Riemannian manifold with smooth boundary such that $P_g^{4,3}$ non-negative and $KerP_g^{4,3} \simeq \mathbb{R}$. For $k \geq 1$ given as in* (45)*, there exists a large $L > 0$ and a continuous map Ψ from the sublevel $\{II_Q(u) < -L,\ \int_M e^{4u} dV_g = 1\}$ onto $(M_\partial)_k$ which is topologically non-trivial.*

Remark 3.2.19. *As in Proposition* 3.2.15*, here also topologically non-trivial means the same thing. In this case the non triviality of the map will come from the non-contractibility of $(M_\partial)_k$ and Proposition* 3.2.18 *below.*

To prove Proposition 3.2.18, we need two auxiliary Lemmas. We start with the one which states (roughly) that M can be embedded smoothly in Euclidean space (with large dimension) such that its interior lies in the interior of the positive half space and its boundary at the one (boundary) of that half space. Since the proof works for all dimensions, we will give the Lemma for a general finite-dimensional compact smooth Riemannian manifold with smooth boundary. Precisely, we have

Lemma 3.2.20. *Suppose N is a smooth n-dimensional compact manifold with smooth boundary. Then there exists $m \in \mathbb{N}^*$ (large enough) and $T : N \to \mathbb{R}^{m+1}$ an embedding such that, $T(\partial N) \subset \partial\mathbb{R}^{m+1}_+$, $T(int(N)) \subset int(\mathbb{R}^{m+1}_+)$ and $T : int(N) \to int(\mathbb{R}^{m+1}_+)$ is smooth. Furthermore, there holds for all $x \in \partial N$, the vector ν_x with origin $T(x)$ and parallel to the x_{m+1}-axis is the normal vector of $T(\partial N)$ at $T(x)$.*

Proof. First of all, by Whitney's embedding theorem we have that there exists $m \in \mathbb{N}^*$ such that N is smoothly embedded in $\mathbb{R}^m$, namely there exists $\hat{T} : N \to \mathbb{R}^m$ a smooth embedding. Now, we extend N by adding a nice tubular neighborhood such that the resulting object is a compact smooth manifold that we denote by $\hat{N}$. Using the compactness of $\hat{N}$, we can find a finite open covering $\{\Theta_i\}_{1=1}^k$ of N and a finite number of smooth functions $\varphi_i : \Theta_i \to \mathbb{R}^n$ such that $\{(\Theta_i, \varphi_i)\}_1^k$ are local coordinates for N and $\Theta_i \subset \hat{N}$. Moreover, we can take Θ_i such that if $\Theta_i \cap \partial N \neq \emptyset$ then the associated φ_i verifies the following properties:

$$\varphi_i : \Theta_i \to [-1, 1]^n;$$

$$\varphi_i : \Theta_i \cap N \to [-1, 1]^n \cap \{x_n > 0\};$$

$$\varphi_i : \Theta_i \cap \partial N \to [-1, 1]^n \cap \{x_n = 0\};$$

and furthermore φ_i maps the outward normal vectors on ∂N to the outward normal vectors at $\partial\mathbb{R}^n_+$ of $[-1, 1]^n \cap \{x_n = 0\}$.
Now to the covering $\{\Theta_i\}_{i=1}^k$, we associate a finite number of functions $\{h_i\}_{1=1}^k$ $h_i : \Theta_i \to \mathbb{R}$ as follows

$$h_i(x) = \begin{cases} 1 & x \in \Theta_i \subset \text{int}(N); \\ h \circ \varphi_i(x) & x \in \Theta_i \cap \partial N. \end{cases}$$

where $h : [-1,1]^n \to \mathbb{R}$ is defined as follows

$$h(x) = \begin{cases} 0 & \text{if } x \in [-1,1]^n \cap \{x_n < 0\}; \\ \sqrt{1-(x_n-1)^2} & \text{if } x \in [-1,1]^n \cap \{x_n \geq 0\}. \end{cases}$$

Next we choose a partition of unity $\{g_i\}_1^k$ subordinated to the covering $\{\Theta_i\}_1^k$. Therefore the g_i's satisfy

$$\begin{cases} g_i \in C_c^\infty(\Theta_i) \quad 1 \leq i \leq k; \\ 0 \leq g_i \leq 1 \;\text{ on }\; N \;\; \forall i; \\ \sum_{i=1}^k g_i = 1 \quad \text{on} \;\; N. \end{cases}$$

With this partition of unity and the functions h_i, we set

$$u(x) = \sum_{i=1}^k g_i(x) h_i(x).$$

Using the definition of h_i one can check easily that u verifies the following properties

$$u(x) > 0 \;\; \forall x \in \text{int}(N) \quad \text{and} \quad u \in C^\infty(\text{int}(N));$$

$$u(x) = 0 \;\; \forall x \in \partial N \quad \text{and} \quad \frac{\partial u}{\partial n} = +\infty \;\text{ on }\; \partial N;$$

where $\frac{\partial}{\partial n}$ stands for the inward normal derivative at ∂N.
Now for $x \in N$ we define $T : N \to \mathbb{R}^{m+1}$ as follows

$$T(x) = (\hat{T}(x), u(x)).$$

where $\hat{T}$ is given by the Whitney embedding theorem.
It is obvious that T is an embedding, smooth in $\text{int}(N)$ and satisfies the properties of the Lemma. Hence the proof is completed. ■

Next we will used the previous Lemma to define a special doubling of M such that it is C^1. First of all applying Lemma 3.2.20 to M we get the existence of an embedding $T : M \to \mathbb{R}^{m+1}$ (given by Lemma 3.2.20).
Now we define the reflection $\tilde{T}$ of T as follows

$$\tilde{T}(x) = (T^1(x), \cdots, T^m(x), -T^{m+1}(x));$$

where $T(x) = (T^1(x), \cdots, T^m(x), T^{m+1}(x))$. From the properties of T, it is easily seen that $\tilde{T}$ is also an embedding of M.
With the embeddings T and $\tilde{T}$, we can define the desired doubling of M. To do so, we start by making some notations. We set

$$DM^+ = T(M) \quad \text{and} \quad DM^- = \tilde{T}(M).$$

By the properties of T and $\tilde{T}$ (see Lemma 3.2.20) we have that DM^+ and DM^- have a commun boundary which is ∂M. Moreover they have the same normal vectors at their commun boundary. Now we are ready to define the doubling of M and denote it by DM as follows

$$DM = \widetilde{DM^+ \cup DM^-}.$$

where the notation $\tilde{}$ means we identify $T(x)$ and $\tilde{T}(x)$ for $x \in \partial M$.
Using the fact that DM^+ and DM^- have the same normal at ∂M and by considering the reflection $\bar{g}$ of g through ∂M, we derive that $(DM, \bar{g})$ is a C^1 closed 4-dimensional Riemannian

manifold with lipschitz metric.
Next we introduce some further definitions.
Given a point $x = (x_1, \cdots, x_{m+1}) \in DM$, we define the even reflection of x accross ∂M and denote it by $\hat{x}$ as follows

$$\hat{x} = (x_1, \cdots, x_m, -x_{m+1}). \tag{3.18}$$

For a function $u \in H^2(M)$ and identifying DM^+ to M, we define the *even* reflection of u accross ∂M as follows

$$u_{DM}(x) = \begin{cases} u(x) & \text{if } x \in DM^+; \\ u(\hat{x}) & \text{if } x \in DM^-; \end{cases}$$

We say that a function $u \in L^1(DM)$ is *even* with respect to the boundary ∂M if

$$u(x) = u(\hat{x}) \quad \text{for a.e } x \in DM. \tag{3.19}$$

We denote by DM_k the k barycenters relative to DM of order k, namely

$$DM_k = \{\sum_{i=1}^k t_i \delta_{x_i},\ \ x_i \in DM,\ \ \sum_{i=1}^k t_i = 1\}.$$

We have that DM_k is a stratified set, namely a union of sets of different dimension with maximal dimension being $5k - 1$ (for more details see [33]). It will be endowed with the weak topology of distributions. To prove the Proposition 3.2.15 we will need at one stage to (roughly speaking) evaluate the distance of some suitable functions to DM_k (see formula (3.21) below). To do this, we will adopt the metric distance given by $C^1(DM)^*$ and inducing the same topology as the weak topology of distributions and will be denoted by $d_{DM}(\cdot, \cdot)$.
For $\epsilon > 0$, we set

$$D_{\epsilon,k,DM} = \{f \in L^1(DM),\ \ f \geq 0,\ \ \int_{DM} f dV_{\bar{g}} = 1 \text{ and } d_{DM}(f, DM_k) \leq \epsilon\}.$$

The next discussion concern the way of defining convex combination of points of DM belonging to a small metric ball. To do so we use the embedding of DM in $\mathbb{R}^{m+1}$ discussed above in the following way. Given points $x_i \in DM$, $i = 1, \cdots, l$, which belongs to a small metric ball and $\alpha_i \geq 0$, $i = 1, \cdots, l$, $\sum_{i=1}^l \alpha_i = 1$, we define their convex combination denoted by $\sum_{i=1}^l \alpha_i x_i$ by considering the convex combination of their image under the embedding and after project the result on the image of DM (which is also identified to DM). Hence in this way we have that for such a type of points, the convex combination is well defined and if $d(x_i, x_j) \leq \beta$ then we obtain $d(x_i, \sum_{j=1}^l \alpha_j x_j) \leq 2\beta$.

We recall that the arguments which has lead to Proposition 3.1 in [33] are based on the construction of some partial projections on some suitable subsets $M_j(\epsilon_j)$ (obtained by removing singularities) of M_k and gluing method based on the construction of a suitable homotopy. The construction of the latter homotopy which is not trivial is based on some weighted convex combinations and the fact that the underlying manifold does not have corners.
Using the notion of convex combinations discussed above and the fact that DM is a C^1 closed Riemannian manifold with lipschitz metric which rule out the presence of corners, and an adaptation of the arguments of Proposition 3.1 in [33], we have the following Lemma:

Lemma 3.2.21. *Let $k \geq 1$ be as in* (45) *and DM be as above. Then there exists $\epsilon_{k,DM}$ such that for every $\epsilon \leq \epsilon_{k,DM}$, we have the existence of a non-trivial continuous projection*

$$P_{\epsilon,k,DM} : D_{\epsilon,k,DM} \to DM_k;$$

with the following property:
For every $u \in D_{\epsilon,k,DM}$ even *(in the sens of* (3.19) *) if $P_{\epsilon,k,DM}(u) = \sum_{i=1}^k t_i \delta_{x_i}$ then*

$$\forall x_i \notin \partial M \text{ there exists } j \neq i \text{ such that } x_j = \hat{x_i} \text{ and } t_j = t_i.$$

Now we are ready to make the proof of the Proposition 3.2.18.

PROOF of Proposition 3.2.18
To begin, we let ϵ_k be so small that Lemma 3.2.21 holds with $\epsilon = \epsilon_k$. Next applying corollary 3.2.10 with $\bar{\epsilon} = \frac{\epsilon_k}{4}$, we obtain the existence of L (large enough) such that

$$\forall u \in H_{\frac{\partial}{\partial n}}, \quad \int_M e^{4u} dV_g = 1, \text{ and } II_Q(u) \leq -L$$

there holds

$$4d(e^{4u}, (M_\partial)_k) \leq \epsilon_k.$$

Now since for $u \in H_{\frac{\partial}{\partial \eta}}$, we have by definition of $H_{\frac{\partial}{\partial \eta}}$ that

$$\frac{\partial u}{\partial n_g} = 0;$$

then we infer that the *even* reflection u_{DM} of u belongs to $H^2(DM)$. Moreover we have also that the map

$$u \in H^2(M) \to u_{DM} \in H^2(DM) \text{ is continuous.} \tag{3.20}$$

On the other hand, one can easily check (using the eveness of $\bar{g}$) that the following distance estimate holds

$$d_{DM}\left(\frac{e^{4u_{DM}}}{\int_{DM} e^{4u_{DM}} dV_{\bar{g}}}, DM_k\right) < \epsilon_k. \tag{3.21}$$

Therefore, by Lemma 3.2.21, we have that $P_{\epsilon_k,k,DM}(\frac{e^{4u_{DM}}}{\int_{DM} e^{4u_{DM}} dV_{\bar{g}}})$ is well defined and belongs to DM_k. Moreover, still from Lemma 3.2.21 we have that if

$$P_{\epsilon_k,k,DM}\left(\frac{e^{4u_{DM}}}{\int_{DM} e^{4u_{DM}} dV_{\bar{g}}}\right) = \sum_{i=1}^{k} t_i \delta_{x_i};$$

then the following holds

$$\forall x_i \notin \partial M \text{ there exists } j \neq i \text{ such that } \quad x_j = \hat{x_i} \text{ and } t_j = t_i.$$

Thus setting

$$\Psi(u) = \frac{1}{\sum_{x_a \in \text{ int}(DM^+)} t_a + \sum_{x_b \in \partial M} t_b} \left(\sum_{x_i \in \text{ int}(DM^+)} t_i \delta_{x_i} + \sum_{x_j \in \partial M} t_j \delta_{x_j} \right).$$

we get $\Psi(u) \in (M_\partial)_k$. On the other hand, since the map $v \in H^2(DM) \to e^{4v} \in L^1(DM)$ is continuous, then from (3.20) we derive that the map $u \in H_{\frac{\partial}{\partial \eta}} \to e^{4u_{DM}} \in L^1(DM)$ is continuous, too. Thus from the continuity of $P_{\epsilon_k,k,DM}$ we infer that, Ψ is also continuous. Hence the proof of the proposition is complete. ■

Next we consider the functional II_T, and we have the following Proposition whose proof is the same as the proof of Proposition 3.2.15.

Proposition 3.2.22. *Let (M, g) be a compact four dimensional smooth Riemannian manifold with smooth boundary such that $P_g^{4,3}$ is non-negative and $KerP_g^{4,3} \simeq \mathbb{R}$. For $k \geq 1$ given as in (47), there exists a large $L > 0$ and a continuous map Ψ from the sublevel $\{u : II_T(u) < -L, \int_{\partial M} e^{3u} dS_g = 1\}$ into ∂M_k which is topologically non-trivial.*

Remark 3.2.23. *Here also by topologically non-trivial we mean the same thing as in the previous cases. The non-triviality of the map will come from the non-contractibility of ∂M_k and Proposition 3.2.33.*

Finally we arrive to the functional II_ρ, and we have the following Proposition.

Proposition 3.2.24. *Suppose m is a positive integer, and suppose that $\rho_1 \in (4\pi m, 4\pi(m+1))$, and that $\rho_2 < 4\pi$. Then there exists a large $L > 0$ and a continuous projection Ψ from $\{II_\rho \leq -L\} \cap \left\{\int_\Sigma e^{u_1} dV_g = 1\right\}$ (with the natural topology of $H^1(\Sigma) \times H^1(\Sigma)$) onto Σ_m which is topologically non-trivial.*

Remark 3.2.1. *As for the other functionals, here aslo topologically non-trivial means the same thing, and will come from non-contractibility of Σ_m and Proposition 3.2.37 below.*

Its proof is similar to the one of Proposition 3.2.15. For the seek of completeness we give the details.
Proof of Proposition 3.2.24
We fix ε_m so small that Proposition 3.2.17 applies with $l = m$. Then we apply Corollary 3.2.11 with $\bar{\varepsilon} = \varepsilon_m$. We let L be the corresponding large number, so that if $II_\rho(u) \leq -L$, then $d(e^{u_1}, \Sigma_m) < \varepsilon_m$. Hence for these ranges of u_1 and u_2, since the map $u \mapsto e^u$ is continuous from $H^1(\Sigma)$ into $L^1(\Sigma)$, the projections $\Pi_{\epsilon_m,m}$ from $H^1(\Sigma)$ onto Σ_m is well defined and continuous. ■

3.2.3 Projections of barycentric sets onto large negative sublevels of the functionals II_A, II_Q, II_T and II_ρ

In this Subsection, we prove that some suitable barycentric sets can be map in a nontrivial way to some large negative sublevels of the Euler-Lagrange functionals II_A, II_Q, II_T, and II_ρ. From this results and the one of the previous Subsection, we have as a corollary that those negative sublevels have the same homology as the corresponding barycentric sets. Hence the knowledge of the homology of the barycentric set implies the one of the sublevels. For the purpose of clarity of the exposition, we divide this Subsection into four Subsubsections devoted each to the treatment of an Euler-Lagrange functional.

The case of II_A

Proposition 3.2.25. *Let (M, g) be a compact closed n-dimensional smooth Riemannian manifold with P^n_g non-negative and $KerP^n_g \simeq \mathbb{R}$. Let Ψ be the map defined in Proposition 3.2.15 . Then assuming $k \geq 1$ (given as in (42)), for every $L > 0$ sufficiently large (such that Proposition 3.2.15 applies), there exists a map*

$$\Phi_{\bar{\lambda}} : M_k \longrightarrow H^{\frac{n}{2}}(M) \tag{3.22}$$

with the following properties
a)

$$II_A(\Phi_{\bar{\lambda}}(z)) \leq -L \ \ for\ any\ \ z \in M_k; \tag{3.23}$$

b)
$\Psi \circ \Phi_{\bar{\lambda}}$ is homotopic to the identity on M_k.

We are going to make the proof of Proposition 3.2.25. For doing this, we start with some technical estimates.

Technical estimates for Mapping M_k into large negative sublevels of II_A
In this Subparagraph we will define some test functions depending on a parameter λ and give estimate of the quadratic part of the functional II_A on those functions as λ tends to infinity. And as a corollary we define a continuous map from M_k into large negative sublevels of II_A.

For $\delta > 0$ small, consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following properties (see [33]):

$$\begin{cases} \chi_\delta(t) = t, & for\ \ t \in [0, \delta]; \\ \chi_\delta(t) = 2\delta, & for\ \ t \geq 2\delta; \\ \chi_\delta(t) \in [\delta, 2\delta], & for\ \ t \in [\delta, 2\delta]. \end{cases} \tag{3.24}$$

Then, given $\sigma \in M_k$, $\sigma = \sum_{i=1}^k t_i \delta_{x_i}$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ by

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{n} \log \sum_{i=1}^k t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2(d_i(y))} \right)^n \tag{3.25}$$

where we have set

$$d_i(y) = d_g(y, x_i), \qquad x_i, y \in M,$$

with $d_g(\cdot, \cdot)$ denoting the distance function on M. We define also

$$d_{min}(y) = \min_i d_i(y). \tag{3.26}$$

When $n = 4m$ we set

$$T_n \varphi_{\lambda,\sigma} = (-\Delta)^m \varphi_{\lambda,\sigma}, \tag{3.27}$$

when $n = 4m + 2$ we set

$$T_n \varphi_{\lambda,\sigma} = \nabla((-\Delta)^m \varphi_{\lambda,\sigma}) \tag{3.28}$$

when $m = 4m + 1$ we set

$$T_n \varphi_{\lambda,\sigma} = (-\Delta)^{\frac{1}{4}} (-\Delta)^m \varphi_{\lambda,\sigma}, \tag{3.29}$$

and when $n = 4m + 3$ we set

$$T_n \varphi_{\lambda,\sigma} = (-\Delta)^{\frac{3}{4}} (-\Delta)^m \varphi_{\lambda,\sigma}, \tag{3.30}$$

Now we state a Lemma giving an estimate (uniform in $\sigma \in M_k$) of $\left\langle P_g^n \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \right\rangle$ as $\lambda \to +\infty$.

Lemma 3.2.26. *Under the assumptions of Proposition* 3.2.25*, and for* $\varphi_{\lambda,\sigma}$ *as in* (3.25)*, let* $\epsilon > 0$ *small enough. Then as* $\lambda \to +\infty$ *one has*

$$\left\langle P_g^n \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \right\rangle \leq (4kc_n + \epsilon + o_\delta(1)) \log \lambda + C_{\epsilon,\delta} \tag{3.31}$$

Proof. We divide the proof into two cases.

Case n even

We first give an estimate of $\int_M (T_n \varphi_{\lambda,\sigma})^2 \, dV_g$ and after use interpolation inequalities to conclude. Let Θ be large and fixed, then by induction in the degree of differentiation we have that the following pointwise estimates holds in $\cup_{i=1}^k B_{x_i}(\frac{\Theta}{\lambda})$:

$$|T_n \varphi_{\lambda,\sigma}| \leq C \lambda^{\frac{n}{2}}, \tag{3.32}$$

hence we obtain

$$\int_{\cup_{i=1}^k B_{x_i}(\frac{\Theta}{\lambda})} (T_n \varphi_{\lambda,\sigma})^2 \, dV_g \leq C \Theta^n \tag{3.33}$$

Now to have a further simplification of the expression of $\varphi_{\sigma,\lambda}$, it is convenient to get rid of the cutoff functions χ_δ. In order to do this, we divide the set of points $\{x_1, \ldots, x_k\}$ in a suitable way. Since the number k is fixed, there exists $\hat{\delta}$ and sets $B_1, \ldots, B_j, j \leq k$ with the following properties

$$\begin{cases} C_k^{-1} \delta \leq \hat{\delta} \leq \frac{\delta}{16}; \\ B_1 \cup \cdots \cup B_j = \{x_1, \ldots, x_k\}; \\ dist(x_i, x_s) \leq \hat{\delta} \qquad \text{if } x_i, x_s \in B_a; \\ dist(x_i, x_s) \geq 4\hat{\delta} \qquad \text{if } x_i \in B_a, x_s \in B_b, a \neq b, \end{cases} \tag{3.34}$$

where C_k is a positive constant depending only on k. Now we define

$$\hat{B}_a = \{y \in M : dist(y, B_a) \leq 2\hat{\delta}\}. \tag{3.35}$$

By definition of $\hat{\delta}$ it follows that

$$\chi_\delta(d_i(y)) = d_i(y), \quad \text{for } x_i \in B_a, \ y \in \hat{B}_a, \tag{3.36}$$

and

$$\chi_\delta(d_i(y)) \geq 2\hat{\delta}, \quad \text{for } x_i \in B_a, \ y \notin \hat{B}_a. \tag{3.37}$$

Furthermore one has

$$\hat{B}_a \cap \hat{B}_b = \emptyset \text{ for } a \neq b \tag{3.38}$$

On the other hand it is also easy to see that the following holds,

$$|T_n \varphi_{\lambda,\sigma}| \leq C_{\hat{\delta}} \quad \text{in } M \setminus \cup_{a=1}^{j} \hat{B}_a. \tag{3.39}$$

Now set $M_{\Theta,\sigma,\lambda.\hat{\delta}} = \left(M \setminus \cup_{i=1}^{k} B_{x_i}(\frac{\Theta}{\lambda})\right) \cup \left(\cup_{a=1}^{j} B_a\right)$. Since we are taking Θ large, then in the set $M_{\Theta,\sigma,\lambda\hat{\delta}}$ the following estimates hold:

$$(1+\lambda^2 d_i^2) = (1+o_{\delta,\Theta}(1))\lambda^2 d_i^2, \quad \partial^\beta(1+\lambda^2 d_i^2) = (1+o_{\delta,\Theta}(1))\lambda^2 \partial^\beta d_i^2; \quad \text{for all muti-indices } \beta \ : |\beta| \leq \frac{n}{2} \tag{3.40}$$

First let suppose $k=1$ and after we treat the case $k>1$. In the case $k=1$ we have $\varphi_{\sigma,\lambda}$ takes the simple form

$$\varphi_{\sigma,\lambda}(x) = \log \frac{2\lambda}{1+\lambda^2 d_1^2(x)} \quad \text{in } M_{\Theta,\sigma,\lambda\hat{\delta}}. \tag{3.41}$$

Hence from (3.40) we obtain

$$\partial^\beta \varphi_{\sigma,\lambda} = 2\partial^\beta \log \frac{1}{d_1} + o_{\delta,\Theta}(1)\frac{1}{d_1^{|\beta|}} \quad \text{for all multi-indices } \beta \ : \ |\beta| \leq \frac{n}{2} \tag{3.42}$$

So we have that

$$(T_n \varphi_{\sigma,\lambda})^2 = 4(T_n \log \frac{1}{d_1})^2 + o_{\delta,\Theta}(1)(\frac{1}{d_1^n}). \tag{3.43}$$

On the other hand we have that in geodesic coordinates around x_1

$$(T_n \log \frac{1}{d_1})^2 \leq \frac{c_n}{\omega_{n-1} r^n}(1+o_r(1)); \tag{3.44}$$

Hence working now in geodesic polar coordinates we obtain

$$\int_{M_{\Theta,\sigma,\lambda.\hat{\delta}}} (T_n \varphi_{\sigma,\lambda})^2 \leq 4c_n \log \lambda(1+o_{\delta,\Theta}(1)) + C_{\delta,\Theta}. \tag{3.45}$$

So with what is said above we have that by fixing Θ large we arrive to

$$\int_M (T_n \varphi_{\sigma,\lambda})^2 \leq 4c_n \log \lambda(1+o_\delta(1)) + C_\delta. \tag{3.46}$$

Now let treat the case $k>1$. For this let $\overline{C}$ large and let $a_{j,\sigma,\Theta,\lambda}(x) = t_j \left(\frac{2\lambda}{1+\lambda^2 \chi_\delta^2(d_i(x))}\right)^n$. Next for $i \in \{1,\dots,k\}$ define the set $A_{\lambda,\sigma,i,\overline{C}}$ by the following formula.

$$A_{\lambda,\sigma,i,\Theta,\overline{C}} = \{x \in M_{\Theta,\sigma,\lambda,hat\delta} \ / \ a_{i,\sigma,\lambda}(x) > \overline{C} a_{j,\sigma,\lambda}(x) \text{ for all } j \neq i\}. \tag{3.47}$$

By definition of $\varphi_{\sigma,\lambda}(x)$ and $a_{j,\sigma,\lambda}(x)$ we have that,

$$\varphi_{\lambda,\sigma}(x) = \frac{1}{n}\log(\sum_{j=1}^{k} a_{j,\sigma,\lambda}(x)) = \frac{1}{n}\log(a_{i,\sigma,\lambda}(x)) + \frac{1}{n}\log\left(1+\sum_{j\neq i}\frac{a_{j,\sigma,\lambda}(x)}{a_{i,\sigma,\lambda}(x)}\right) \quad \text{in } A_{\lambda,\sigma,i,\Theta,\overline{C}}. \tag{3.48}$$

Moreover the following holds :

$$\sum_{j\neq i}\frac{a_{j,\sigma,\lambda}(x)}{a_{i,\sigma,\lambda}(x)}=\sum_{j\neq i}\frac{t_j}{t_i}(\frac{1+\lambda^2\chi_\delta(d_i)^2}{1+\lambda^2\chi_\delta(d_j)^2})^n. \tag{3.49}$$

So By the above arguments we have that the following holds in $A_{\lambda,\sigma,i,\Theta,\overline{C}}$

$$\sum_{j\neq i}\frac{a_{j,\sigma,\lambda}(x)}{a_{i,\sigma,\lambda}(x)}=\sum_{j\neq i}\frac{t_j}{t_i}(\frac{1+\lambda^2d_i^2}{1+\lambda^2d_j^2})^n=o_{\Theta,\overline{C}}(1), \tag{3.50}$$

hence from (3.40) we deduce that

$$\sum_{j\neq i}\frac{t_j}{t_i}\frac{d_j^{2n}}{d_i^{2n}}=o_{\delta,\Theta,\overline{C}}(1). \tag{3.51}$$

By differentiation and reasoning as in (3.40) we obtain

$$\partial^\beta\varphi_{\lambda,\sigma}(x)=2\partial^\beta\log(\frac{1}{d_i})+\partial^\beta\sum_{j\neq i=k}\frac{t_j}{t_i}(\frac{1+\lambda^2d_i^2}{1+\lambda^2d_j^2})^n+o_{\Theta,\overline{C}}(1)(\frac{1}{d_i^{|\beta|}})\ \text{ for all multi-indices }\ \beta:\ |\beta|\leq\frac{n}{2};$$

where $o_{\Theta,\overline{C}}(1)\to 0$ as $\Theta,\overline{C}\to+\infty$. Hence using again (3.40) we obtain

$$\partial^\beta\varphi_{\lambda,\sigma}(x)=2\partial^\beta\log(\frac{1}{d_i})+(1+o_{\delta,\Theta}(1))\sum_{j\neq i}\frac{t_j}{t_i}\partial^\beta(\frac{d_i^2}{d_j^2})^n+o_{\Theta,\overline{C}}(1)(\frac{1}{d_i^{|\beta|}})\ \text{ for all multi-indices }\ \beta:\ |\beta|\leq\frac{n}{2} \tag{3.52}$$

Moreover by easy calculations we have that the folowing holds,

$$\partial^\beta(\frac{d_i^2}{d_j^2})^n=O((\frac{d_i^2}{d_j^2})^n\frac{1}{d_{min}^{|\beta|}})\ \text{ for all multi-index }\ \beta\ \text{ such that }\ |\beta|\leq\frac{n}{2}. \tag{3.53}$$

Hence we infer that

$$\sum_{j\neq i}\frac{t_j}{t_i}\partial^\beta(\frac{d_i^2}{d_j^2})^n=\sum_{j\neq i}\frac{t_j}{t_i}\frac{d_j^{2n}}{d_i^{2n}}O(\frac{1}{d_{min}^{|\beta|}}), \tag{3.54}$$

so from (3.2.3) we obtain

$$\sum_{j\neq i}\frac{t_j}{t_i}\partial^\beta(\frac{d_i^2}{d_j^2})^n=o_{\delta,\Theta,\overline{C}}(1)(\frac{1}{d_{min}^{|\beta|}}). \tag{3.55}$$

Hence we have that,

$$\partial^\beta\varphi_{\lambda,\sigma}(x)=2\partial^\beta\log(\frac{1}{d_i})+o_{\delta,\Theta,\overline{C}}(1)(\frac{1}{d_{min}^{|\beta|}})\ \text{ for all multi-index }\ \beta\ \text{ such that }\ |\beta|\leq\frac{n}{2}. \tag{3.56}$$

Now define the set $\widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}=\cup_{i=1}^kA_{i,\sigma,\lambda,\Theta,\overline{C}}$. Since $(A_{i,\sigma,\lambda,\Theta,\overline{C}})_{i=1,\dots,k}$ are disjoint, then we have that

$$\int_{\widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}}(T_n\varphi_{\lambda,\sigma})^2\,dV_g=\sum_{i=1}^k\int_{A_{\lambda,\sigma,i,\Theta,\overline{C}}}(T_n\varphi_{\lambda,\sigma})^2\,dV_g=\sum_{i=1}^k\int_{A_{\lambda,\sigma,i,\Theta,\overline{C}}\cap(\{d_i\geq\frac{\Theta}{\lambda}\})}(T_n\varphi_{\lambda,\sigma})^2\,dV_g. \tag{3.57}$$

From (3.56) we have that,

$$\int_{\widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}} (T_n\varphi_{\lambda,\sigma})^2 \, dV_g = \sum_{i=1}^{k} \int_{A_{\lambda,\sigma,i,\Theta,\overline{C}} \cap (\{d_i \geq \frac{\Theta}{\lambda}\})} \left(4(T_n \log(\frac{1}{d_i}))^2 + o_{\delta,\Theta,\overline{C}}(1)(\frac{1}{d_{min}^n}) \right) dV_g. \tag{3.58}$$

On the other hand working in polar coordinates we have that

$$(T_n \log(\frac{1}{d_i}))^2 \leq \frac{c_n}{\omega_{n-1} r^n}(1 + o_r(1)); \tag{3.59}$$

hence we obtain

$$\int_{A_{\lambda,\sigma,i,\theta,C} \cap (\{d_i \geq \frac{\Theta}{\lambda}\})} \left[4(T_n \log(\frac{1}{d_i}))^2 + o_{\delta,\Theta,\overline{C}}(1)(\frac{1}{d_{min}^n}) \right] dV_g \leq (4c_n + o_{\delta,\Theta,\overline{C}}(1)) \log \lambda + C_{\delta,\hat{\delta},\Theta,\overline{C}}, \tag{3.60}$$

hence we have that

$$\int_{\widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}} (T_n\varphi_{\lambda,\sigma})^2 \, dV_g \leq (4kc_n + o_{\delta,\Theta,\overline{C}}(1)) \log \lambda + C_{\delta,\Theta,\overline{C}}. \tag{3.61}$$

Now let us estimate $\int_{M_{\Theta,\sigma,\lambda,\hat{\delta}} \setminus \widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}} (T_n\varphi_{\lambda,\sigma})^2 \, dV_g$
First of all we give a characterization of the set $M_{\Theta,\sigma,\lambda} \setminus \widetilde{M}_{\Theta,\sigma,\lambda}$. We have that the following holds

$$M_{\Theta,\sigma,\lambda,\hat{\delta}} \setminus \widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}} = \{x \in M_{\Theta,\sigma,\lambda} : \ \forall i \text{ there exists an index } j \neq i \text{ such that } \ a_{i,\sigma,\lambda}(x) \leq \overline{C} a_{j,\sigma,\lambda}(x)\}.$$

Hence we have that $x \in M_{\Theta,\sigma,\lambda,\hat{\delta}} \setminus \widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}$ is equivalent also to the fact that

$$\forall i \text{ there exists an index } j \neq i \text{ such that } d_i^2(x) \geq \frac{t_i^{\frac{1}{n}}}{\overline{C}^{\frac{1}{n}} t_j^{\frac{1}{n}}}(1 + o_{\delta,\Theta}(1)) d_j^2(x).$$

So from this fact an using an iterative argument we have that if $x \in M_{\Theta,\sigma,\lambda,\hat{\delta}} \setminus \widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}}$ then

$$\exists j \neq i \text{ such that } \ C^{-1} \frac{t_j^{\frac{1}{n}}}{t_i^{\frac{1}{n}}} d_i^2(x)(1 + o_{\Theta,\overline{C}}(1)) \leq d_j^2(x) \leq C \frac{t_j^{\frac{1}{n}}}{t_i^{\frac{1}{n}}} d_i^2(x)(1 + o_{\Theta,\overline{C}}(1)). \tag{3.62}$$

Hence the following holds :

$$\text{there exists } \ l = l(k) \in \mathbb{N} \ \text{ such that } \ M_{\Theta,\sigma,\lambda,\hat{\delta}} \setminus \widetilde{M}_{\Theta,\sigma,\lambda,\hat{\delta}} \subset \cup_{i=1}^{k} A_i,$$

where A_i is the annulus

$$A_i = B_{y_i}(b_i) \setminus B_{y_i}(a_i),$$

with $y_i \in \{x_1, \dots, x_k\}$ and $\frac{b_i}{a_i} \leq C_{\Theta,\overline{C},k}$.
On the other hand reasoning as in (3.53) we have that

$$|T_n\varphi_{\lambda,\sigma}| = O(\frac{1}{d_{min}^{\frac{n}{2}}}) \quad \text{in} \quad M_{\Theta,\sigma,\lambda} \setminus \widetilde{M}_{\Theta,\sigma,\lambda}. \tag{3.63}$$

Hence working again on polar coordinates as for (3.64) we find that

$$\int_{M_{\Theta,\sigma,\lambda} \setminus \widetilde{M}_{\Theta,\sigma,\lambda}} (T_n\varphi_{\lambda,\sigma})^2 \, dV_g \leq C_{\overline{C},\Theta}. \tag{3.64}$$

So from (3.33), (3.61) and (3.64), by fixing $\overline{C}$ and Θ large enough we obtain we obtain

$$\int_M (T_n\varphi_{\lambda,\sigma})^2 \, dV_g \leq (4kc_n + o_\delta(1)) \log \lambda + C_\delta. \tag{3.65}$$

Hence we obtain for every $k \geq 1$

$$\int_M (T_n \varphi_{\lambda,\sigma})^2 \, dV_g \leq (4kc_n + o_\delta(1)) \log \lambda + C_\delta. \tag{3.66}$$

Now let us estimate $\langle P_g^n \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle$. We have from the self-adjointness of P_g^n and the fact that it annihilates constants that the following holds,

$$\langle P_g^n \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle = \langle P_g^n (\varphi_{\lambda,\sigma} - \overline{\varphi_{\lambda,\sigma}}), \varphi_{\lambda,\sigma} - \overline{\varphi_{\lambda,\sigma}} \rangle .$$

Hence using interpolation inequalities (see [55]) we have that

$$\langle P_g^n \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle \leq (1+\epsilon) \int_M (T_n \varphi_{\lambda,\sigma})^2 \, dV_g + C_\epsilon \int_M |\varphi_{\lambda,\sigma} - \overline{\varphi_{\lambda,\sigma}}|^2 dV_g. \tag{3.67}$$

We notice first that the following fact holds true as one can check easily,

$$\varphi_{\lambda,\sigma}(x) = \log \frac{2\lambda}{1+4\lambda^2\delta^2}, \quad \text{for } y \in M \setminus \cup_{i=1}^k B_{x_i}(2\delta); \tag{3.68}$$

$$\log \frac{2\lambda}{1+4\lambda^2\delta^2} \leq \varphi_{\lambda,\sigma}(x) \leq \log 2\lambda \quad \text{in } \cup_{i=1}^k B_{x_i}(2\delta); \tag{3.69}$$

and

$$\log \frac{2\lambda}{1+4\lambda^2\delta^2} \leq \varphi_{\lambda,\sigma}(x) \leq \log \frac{2\lambda}{1+\chi_\delta^2(d_{min}(x))}. \tag{3.70}$$

Next let us estimate $\int_M |\varphi_{\lambda,\sigma} - \overline{\varphi_{\lambda,\sigma}}|^2 dV_g$. By remarking the trivial identity

$$\overline{\varphi_{\sigma,\lambda}} - \log \frac{2\lambda}{1+4\lambda^2\delta^2} = \frac{1}{Vol_g(M)} \int_M (\overline{\varphi_{\sigma,\lambda}} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g \tag{3.71}$$

we have, by the bilinearity of the inner product that the following holds

$$\begin{aligned}\int_M (\varphi_{\sigma,\lambda} - \overline{\varphi_{\sigma,\lambda}})^2 dV_g = &\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2})^2 dV_g - \frac{2}{Vol_g(M)} \left(\int_M \varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2} dV_g \right)^2 \\ &+ Vol_g(M) |\log \frac{2\lambda}{1+4\lambda^2\delta^2} - \overline{\varphi_{\sigma,\lambda}}|^2,\end{aligned}$$

hence we find

$$\int_M (\varphi_{\sigma,\lambda} - \overline{\varphi_{\sigma,\lambda}})^2 dV_g \leq \int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2})^2 dV_g + Vol_g(M) |\log \frac{2\lambda}{1+4\lambda^2\delta^2} - \overline{\varphi_{\sigma,\lambda}}|^2. \tag{3.72}$$

So in order to estimate $\int_M (\varphi_{\sigma,\lambda} - \overline{\varphi_{\sigma,\lambda}})^2 \, dV_g$ it suffices to do it for $\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2})^2 dV_g$ and for $\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g$.
Let us first estimate $\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g$. From (3.68) the following holds

$$\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g = \int_{\cup_{i=1}^k B_{x_i}(2\delta)} (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g. \tag{3.73}$$

Using (3.70) we have that the following holds

$$\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g \leq \sum_{i=1}^k \int_{B_{x_i}(2\delta)} (\log \frac{2\lambda}{1+4\lambda^2\chi_\delta^2(d_i)} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g. \tag{3.74}$$

Now working in geodesic normal coordinates around the points x_i we find

$$\begin{aligned}\int_{B_{x_i}(2\delta)} (\log \frac{2\lambda}{1+4\lambda^2\chi_\delta^2(d_i)} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g \leq &C \int_0^\delta s^{n-1} \left(\log \frac{1+4\lambda^2\delta^2}{1+\lambda^2 s^2} \right) ds \\ &+ C \int_\delta^{2\delta} \left(\log \frac{1+4\lambda^2\delta^2}{1+\lambda^2\chi_\delta^2(s)} \right) ds.\end{aligned} \tag{3.75}$$

Now recalling that χ_δ is non-decreasing we have that

$$\int_{B_{x_i}(2\delta)} (\log \frac{2\lambda}{1+4\lambda^2\chi_\delta^2(d_i)} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g \leq C \int_0^\delta s^{n-1} \left(\log \frac{1+4\lambda^2\delta^2}{1+\lambda^2 s^2} \right) ds + O(\delta^n). \quad (3.76)$$

On the other hand by performing the change of variables $\lambda s = z$ we obtain

$$\int_0^\delta s^{n-1} \left(\log \frac{1+4\lambda^2\delta^2}{1+\lambda^2 s^2} ds \right) \leq \frac{1}{\lambda^n} \int_0^{\delta\sqrt{\lambda}} z^{n-1} \log \frac{1+4\lambda^2\delta^2}{1+z^2} dz + \frac{1}{\lambda^n} \int_{\delta\sqrt{\lambda}}^{\delta\lambda} z^{n-1} \log \frac{1+4\lambda^2\delta^2}{1+z^2} dz \quad (3.77)$$

It is easy to see that the following holds

$$\frac{1}{\lambda^n} \int_0^{\delta\sqrt{\lambda}} z^{n-1} \log \frac{1+4\lambda^2\delta^2}{1+z^2} dz = O(\delta^n \lambda^{-\frac{n}{2}} \log \lambda), \quad (3.78)$$

and

$$\frac{1}{\lambda^n} \int_{\delta\sqrt{\lambda}}^{\delta\lambda} z^{n-1} \log \frac{1+4\lambda^2\delta^2}{1+z^2} dz = O(\delta^n). \quad (3.79)$$

Therefore we obtain

$$\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2}) dV_g \leq O(\delta^n \lambda^{-\frac{n}{2}} \log \lambda) + O(\delta^n). \quad (3.80)$$

Furthermore using the same procedure one finds

$$\int_M (\varphi_{\sigma,\lambda} - \log \frac{2\lambda}{1+4\lambda^2\delta^2})^2 dV_g \leq O(\delta^n \lambda^{-\frac{n}{2}} (\log \lambda)^2) + O(\delta^n). \quad (3.81)$$

Hence using (3.80), (3.81) and (3.71) we obtain

$$\int_M (\varphi_{\sigma,\lambda} - \overline{\varphi_{\sigma,\lambda}})^2 dV_g \leq o_\delta(1) \log \lambda + C_\delta. \quad (3.82)$$

From (3.82), (3.67) and (3.65), the Lemma is proved.

Case n odd

We first remark that as soon we have the formula (3.66) in the *even* case the same proof holds. Now following the proof of the *even* case, we have that everything remain true up formula (3.42), that is

$$\partial^\beta \varphi_{\sigma,\lambda} = 2\partial^\beta \log \frac{1}{d_1} + o_{\delta,\Theta}(1) \frac{1}{d_1^{|\beta|}} \quad \text{for all multi-indices } \beta \; : \; |\beta| \leq \frac{n}{2}. \quad (3.83)$$

Hence we obtain

$$(T_n \varphi_{\sigma,\lambda})^2 = 4(T_n \log \frac{1}{d_1})^2 + o_{\delta,\Theta}(1)(T_n \frac{1}{d_1})^2. \quad (3.84)$$

On the other hand working in geodesic polar coordinates and reasoning as in the proof of the asymptotics of the Green function P_g^n in the *odd* case, we obtain

$$(T_n \log \frac{1}{d_1})^2 \leq \frac{c_n}{\omega_{n-1} r^n} (1 + o_r(1)). \quad (3.85)$$

Now by using the definition of $(-\Delta)^{\frac{1}{4}}$ or $(-\Delta)^{\frac{3}{4}}$ and still by reasoning as in the *odd* case for the asymptotics for the Green function for P_g^n, we find by easy calculations

$$T_n \frac{1}{d_1} = o(\frac{1}{r^{\frac{n}{2}}}). \quad (3.86)$$

Hence at this step we can continue the proof of the *even* case to get the estimate for the case $k = 1$. Now let show the adaptations to do to get the case $k > 1$. Focusing on two steps, we follow the proof in the *even* case up to formula 3.56 that we recall

$$\partial^\beta \varphi_{\lambda,\sigma}(x) = 2\partial^\beta \log(\frac{1}{d_i}) + o_{\delta,\Theta,\overline{C}}(1)(\frac{1}{d_{min}^{|\beta|}}) \text{ for all multi-index } \beta \text{ such that } |\beta| \leq \frac{n}{2}. \quad (3.87)$$

Hence from this we obtain

$$(T_n\varphi_{\lambda,\sigma})^2 = 4(T_n \log(\frac{1}{d_i}))^2 + o_{\delta,\Theta,\overline{C}}(1)((T_n\frac{1}{d_{min}})^2). \quad (3.88)$$

So reasoning as in the case $k = 1$ we can continue the proof in the *even* case up to arriving to the formula (3.61). Moreover to continue the proof of the *even* case we need only one more adaptation to obtain our result which is the formula (3.63). To do this we still argue as in the case $k = 1$. Hence continuing to adapt the proof for the *even* case we get our Lemma. ■

Next we state a lemma giving estimates of the remainder part of the functional II_A along $\varphi_{\sigma,\lambda}$. The proof is the same as formula 40 and formula 41 in the proof of Lemma 4.3 in [33].

Lemma 3.2.27. *Suppose* $\varphi_{\sigma,\lambda}$ *as in* (3.25)*. Then as* $\lambda \to +\infty$ *one has*

$$\int_M Q_g^n \varphi_{\sigma,\lambda} = -\kappa_{P^n} \log\lambda + O(\delta^n \log\lambda) + O(\log\delta) + O(1); \quad (3.89)$$

$$\log\int_M e^{n\varphi_{\sigma,\lambda}} = O(1). \quad (3.90)$$

Now for $\lambda > 0$ we define the map $\Phi_\lambda : M_k \to H^{\frac{n}{2}}(M)$ by the following formula

$$\forall\ \sigma \in M_k \quad \Phi_\lambda(\sigma) = \varphi_{\sigma,\lambda}. \quad (3.91)$$

We have the following Lemma which is a trivial application of Lemmas 3.2.26 and 3.2.27.

Lemma 3.2.28. *Under the assumptions of Proposition* 3.2.25*, and for* $k \geq 1$ *(given as in* (42) *), given any* $L > 0$ *, there exists a small* δ *and a large* $\bar{\lambda}$ *such that* $II(\Phi_{\bar{\lambda}}(\sigma)) \leq -L$ *for every* $\sigma \in M_k$*.*

Now we are ready to make the proof of Proposition
PROOF. The statement *(a)* follows from Lemma 3.2.28. To prove *(b)* it is sufficient to consider the family of maps $T_\lambda : M_k \to M_k$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi(\sigma)), \quad \sigma \in M_k$$

We recall that when λ is sufficiently large this composition is well defined. Therefore , since $\frac{e^{n\varphi_{\sigma,\lambda}}}{\int_M e^{n\varphi_{\sigma,\lambda}}dV_g} \rightharpoonup \sigma$ in the weak sens of distributions, letting $\lambda \to +\infty$ we obtain an homotopy between $\Psi \circ \Phi$ and Id_{M_k}. This conclude the proof. ■

The case of II_Q

Proposition 3.2.29. *Let* (M, g) *be a compact four dimensional smooth Riemannian manifold with smooth boundary such that* $P_g^{4,3}$ *is non-negative and* $KerP_g^{4,3} \simeq \mathbb{R}$*. Let* Ψ *be the map defined in Proposition* 3.2.18 *. Then assuming* $k \geq 1$ *(given as in* (45)*), for every* $L > 0$ *sufficiently large (such that Proposition* 3.2.18 *applies), there exists a map*

$$\Phi_{\bar{\lambda}} : (M_\partial)_k \longrightarrow H_{\frac{\partial}{\partial n}}$$

with the following properties
a)

$$II(\Phi_{\bar{\lambda}}(z)) \leq -L \text{ for any } z \in (M_\partial)_k;$$

b)
$\Psi \circ \Phi_{\bar{\lambda}}$ *is homotopic to the identity on* $(M_\partial)_k$*.*

To prove Proposition 3.2.29, we start as in the previous Proposition by giving some technical estimates.

Some technical estimates
As in the case of II_A, we are going to define some test functions depending on a real parameter λ and give estimate of the quadratic part of the functional II_Q on those functions as λ tends to infinity. And as a corollary we define a continuous map from $(M_\partial)_k$ into large negative sublevels of II_Q.
For $\delta > 0$ small, consider a smooth non-decreasing cut-off function $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ defined as above see (case II_A). Then, given $\sigma = \sigma_{int} + \sigma_{bdry} \in (M_\partial)_k$, $\sigma_{int} = \sum_{i=1}^h t_i\delta_{x_i}$, $\sigma_{bdry} = \sum_{i=1}^l s_i\delta_{q_i}$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma,int} : M \to \mathbb{R}$, $\varphi_{\lambda,\sigma,bdry} : M \to \mathbb{R}$ and $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ as follows

$$\varphi_{\lambda,\sigma,int}(y) = \frac{1}{4}\log\left[\sum_{i=1}^h t_i\left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(d_{1,i}(y))}\right)^4\right];$$

$$\varphi_{\lambda,\sigma,bdry}(y) = \frac{1}{4}\log\left[\sum_{i=1}^l s_i\left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(d_{2,i}(y))}\right)^4\right]$$

and

$$\varphi_{\lambda,\sigma} = \varphi_{\lambda,\sigma,int} + \varphi_{\lambda,\sigma,bdry} \tag{3.92}$$

where we have set

$$d_{1,i}(y) = d_g(y, x_i), \qquad x_i \in int(M), y \in M, ;$$

$$d_{2,i}(y) = d_g(y, q_i), \qquad q_i \in \partial M, y \in M, ;$$

with $d_g(\cdot,\cdot)$ denoting the Riemannian distance on M.
Now we state a Lemma giving an estimate (uniform in $\sigma \in (M_\partial)_k$) of the quadratic part $\left\langle P_g^{4,3}\varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma}\right\rangle$ of the Euler functional II as $\lambda \to +\infty$. Its proof is a straightforward adaptation of the arguments in the case of II_A with the dimension beeing 4.

Lemma 3.2.30. *Under the assumptions of Proposition* 3.2.29 *and for* $\varphi_{\lambda,\sigma}$ *as in* (3.92)*, let* $\epsilon > 0$ *small enough. Then as* $\lambda \to +\infty$ *one has*

$$\left\langle P_g^{4,3}\varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma}\right\rangle \le (16\pi^2 k + \epsilon + o_\delta(1))\log\lambda + C_{\epsilon,\delta} \tag{3.93}$$

Next we state a lemma giving estimates of the remainder part of the functional II_Q along $\varphi_{\sigma,\lambda}$. The proof is the same as the one of formulas (40) and (41) in the proof of Lemma 4.3 in [33].

Lemma 3.2.31. *Suppose* $\varphi_{\sigma,\lambda}$ *as in* (3.92)*. Then as* $\lambda \to +\infty$ *one has*

$$\int_M Q_g\varphi_{\sigma,\lambda}dV_g = -\kappa_{P_g^4}\log\lambda + O(\delta^4\log\lambda) + O(\log\delta) + O(1);$$

$$\int_{\partial M} T_g\varphi_{\sigma,\lambda}dV_g = -\kappa_{P_g^3}\log\lambda + O(\delta^3\log\lambda) + O(\log\delta) + O(1);$$

and

$$\log\int_M e^{4\varphi_{\sigma,\lambda}} = O(1).$$

Now for $\lambda > 0$ we define the map $\Phi_\lambda : (M_\partial)_k \to H_{\frac{\partial}{\partial n}}$ by the following formula

$$\forall\ \sigma \in M_k \quad \Phi_\lambda(\sigma) = \varphi_{\sigma,\lambda}.$$

We have the following Lemma which is a trivial application of Lemmas 3.2.30 and 3.2.31.

Lemma 3.2.32. *Under the assumptions of Proposition* 3.2.29 *and for* $k \geq 1$ *(given as in* (45) *), given any* $L > 0$ *large enought, there exists a small* δ *and a large* $\bar{\lambda}$ *such that* $II(\Phi_{\bar{\lambda}}(\sigma)) \leq -L$ *for every* $\sigma \in (M_\partial)_k$.

Now we are ready to give the proof of Proposition.
PROOF. The statement *(a)* follows from Lemma 3.2.32. To prove *(b)* it is sufficient to consider the family of maps $T_\lambda : (M_\partial)_k \to (M_\partial)_k$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi_\lambda(\sigma)), \quad \sigma \in M_k$$

We recall that when λ is sufficiently large, then this composition is well defined. Therefore , since $\frac{e^{4\varphi_{\sigma,\lambda}}}{\int_M e^{4\varphi_{\sigma,\lambda}} dV_g} \rightharpoonup \sigma$ in the weak sens of distributions, letting $\lambda \to +\infty$ we obtain an homotopy between $\Psi \circ \Phi$ and $\mathrm{Id}_{(M_\partial)_k}$. This concludes the proof. ■

The case of II_T

Proposition 3.2.33. *Let* (M, g) *be a compact four dimensional smooth Riemannian manifold with smooth boundary such that* $P_g^{4,3}$ *is non-negative and* $KerP_g^{4,3} \simeq \mathbb{R}$. *Let* Ψ *be the map defined in Proposition* 3.2.15 . *Then assuming* $k \geq 1$ *(given as in* (47)*), for every* $L > 0$ *sufficiently large (such that Proposition* 3.2.15 *applies), there exists a map*

$$\Phi_{\bar{\lambda}} : \partial M_k \longrightarrow H_{\frac{\partial}{\partial n}}$$

with the following properties
a)

$$II(\Phi_{\bar{\lambda}}(z)) \leq -L \ \text{for any} \ z \in \partial M_k;$$

b)
$\Psi \circ \Phi_{\bar{\lambda}}$ *is homotopic to the identity on* ∂M_k.

Some technical estimates
As above, we are going to define some test functions depending on a real parameter λ and give estimate of the quadratic part of the functional II_T on those functions as λ tends to infinity. And as a corollary we define a continuous map from ∂M_k into large negative sublevels of II_T.
For $\delta > 0$ small, let $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ be as in the case of II_A.
Then, given $\sigma = \in \partial M_k$, $\sigma = \sum_{i=1}^k t_i \delta_{x_i}$ and $\lambda > 0$, we define the function ;$\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ as follows

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{3} \log \left[\sum_{i=1}^k t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2(d_i(y))} \right)^3 \right]; \tag{3.94}$$

where we have set

$$d_i(y) = d_g(y, x_i), \qquad x_i \in \partial M, y \in M, ;$$

with $d_g(\cdot, \cdot)$ denoting the Riemannian distance on M.
Now we state a Lemma giving an estimate (uniform in $\sigma \in \partial M_k$) of the quadratic part $\left\langle P_g^{4,3} \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \right\rangle$ of the Euler functional II as $\lambda \to +\infty$. Its proof is a straightforward adaptation of the arguments in Lemma 4.5 in [69].

Lemma 3.2.34. *Under the assumptions of Proposition* 3.2.33 *and for* $\varphi_{\lambda,\sigma}$ *as in* (3.94), *let* $\epsilon > 0$ *small enough. Then as* $\lambda \to +\infty$ *one has*

$$\left\langle P_g^{4,3} \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \right\rangle \leq (16\pi^2 k + \epsilon + o_\delta(1)) \log \lambda + C_{\epsilon,\delta} \tag{3.95}$$

Next we state a lemma giving estimates of the remainder part of the functional II_T along $\varphi_{\sigma,\lambda}$. The proof is the same as the one of formulas (40) and (41) in the proof of Lemma 4.3 in [33].

Lemma 3.2.35. *Suppose* $\varphi_{\sigma,\lambda}$ *as in* (3.94). *Then as* $\lambda \to +\infty$ *one has*

$$\int_M Q_g\varphi_{\sigma,\lambda}dV_g = -\kappa_{P_g^4}\log\lambda + O(\delta^4\log\lambda) + O(\log\delta) + O(1);$$

$$\int_{\partial M} T_g\varphi_{\sigma,\lambda}dV_g = -\kappa_{P_g^3}\log\lambda + O(\delta^3\log\lambda) + O(\log\delta) + O(1);$$

and

$$\log\int_{\partial M} e^{3\varphi_{\sigma,\lambda}} = O(1).$$

Now for $\lambda > 0$ we define the map $\Phi_\lambda : \partial M_k \to H_{\frac{\partial}{\partial n}}$ by the following formula

$$\forall\ \sigma\in\partial M_k \quad \Phi_\lambda(\sigma) = \varphi_{\sigma,\lambda}.$$

We have the following Lemma which is a trivial application of Lemmas 3.2.34 and 3.2.35.

Lemma 3.2.36. *Under the assumptions of Proposition* 3.2.33 *and for* $k \geq 1$ *(given as in* (47) *), given any* $L > 0$ *large enough there exists a small* δ *and a large* $\bar\lambda$ *such that* $II(\Phi_{\bar\lambda}(\sigma)) \leq -L$ *for every* $\sigma\in\partial M_k$.

Now we are ready to make the proof fo Proposition.

Proof. The statement *(a)* follows from Lemma 3.2.36. To prove *(b)* it is sufficient to consider the family of maps $T_\lambda : \partial M_k \to \partial M_k$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi_\lambda(\sigma)), \quad \sigma\in\partial M_k$$

We recall that when λ is sufficiently large, then this composition is well defined. Therefore , since $\frac{e^{3\varphi_{\sigma,\lambda}}}{\int_{\partial M} e^{3\varphi_{\sigma,\lambda}}dS_g} \rightharpoonup \sigma$ in the weak sens of distributions, letting $\lambda \to +\infty$ we obtain an homotopy between $\Psi\circ\Phi$ and $\mathrm{Id}_{\partial M_k}$. This concludes the proof. ■

The case of II_ρ

As in the other cases, here also our goal is to map non trivially Σ_m into arbitrarily negative sublevels of II_ρ. In order to do this, we need some preliminary notation. Given $\sigma\in\Sigma_m$, $\sigma = \sum_{i=1}^m t_i\delta_{x_i}$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : \Sigma \to \mathbb{R}$ by

$$\varphi_{\lambda,\sigma}(y) = \log\sum_{i=1}^m t_i\left(\frac{\lambda}{1+\lambda^2 d_i^2(y)}\right)^2, \tag{3.96}$$

where we have set

$$d_i(y) = d_g(y, x_i), \qquad x_i, y\in\Sigma.$$

We point out that, since the distance from a fixed point of Σ is a Lipschitz function, $\varphi_{\lambda,\sigma}(y)$ is also Lipschitz in y, and hence it belongs to $H^1(\Sigma)$.

Proposition 3.2.37. *Suppose* m *is a positive integer, and suppose that* $\rho_1 \in (4\pi m, 4\pi(m+1))$, *and that* $\rho_2 < 4\pi$. *For* $\lambda > 0$ *and for* $\sigma\in\Sigma_m$, *we define* $\Phi : \Sigma_m \to H^1(\Sigma)\times H^1(\Sigma)$ *as*

$$(\Phi(\sigma))(\cdot) = (\Phi(\sigma)_1(\cdot), \Phi(\sigma)_2(\cdot)) := \left(\varphi_{\lambda,\sigma}(\cdot), -\frac{1}{2}\varphi_{\lambda,\sigma}(\cdot)\right), \tag{3.97}$$

where $\varphi_{\lambda,\sigma}$ *is given in* (3.96). *Then for* L *sufficiently large there exists* $\lambda > 0$ *such that*

(i) $II_\rho(\Phi(\sigma)) \leq -L$ *uniformly in* $\sigma\in\Sigma_m$;

(ii) *$\Psi \circ \Phi$ is homotopic to the identity on Σ_m,*

where Ψ is defined in Proposition 3.2.24, and where we assume L to be so large that Ψ is well defined on $\{II_\rho \leq -L\}$.

Proof. The main ideas follow the strategy in the case II_A, and in [33], but for the reader's convenience we present here a simplified argument (for the H^2 setting in [33] and $H^{\frac{n}{2}}$ as above, it was necessary to introduce a cutoff function on the distances d_i which made the computations more involved).

The proof of **(i)** relies on showing the following two pointwise estimates on the gradient of $\varphi_{\lambda,\sigma}$

$$|\nabla \varphi_{\lambda,\sigma}(y)| \leq C\lambda; \qquad \text{for every } y \in \Sigma, \tag{3.98}$$

where C is a constant independent of σ and λ, and

$$|\nabla \varphi_{\lambda,\sigma}(y)| \leq \frac{4}{d_{min}(y)} \qquad \text{where} \qquad d_{min}(y) = \min_{i=1,\dots,m} d(y, x_i). \tag{3.99}$$

For proving (3.98) we notice that the following inequality holds

$$\frac{\lambda^2 d(y,x_i)}{1+\lambda^2 d^2(y,x_i)} \leq C\lambda, \qquad i = 1,\dots,m, \tag{3.100}$$

where C is a fixed constant (independent of λ and x_i). Moreover we have

$$\nabla \varphi_{\lambda,\sigma}(y) = -2\lambda^2 \frac{\sum_i t_i (1+\lambda^2 d_i^2(y))^{-3} \nabla_y (d_i^2(y))}{\sum_j t_j (1+\lambda^2 d_j^2(y))^{-2}}. \tag{3.101}$$

Using the fact that $|\nabla_y(d_i^2(y))| \leq 2 d_i(y)$ and inserting (3.100) into (3.101) we obtain immediately (3.98). Similarly we find

$$\begin{aligned} |\nabla \varphi_{\lambda,\sigma}(y)| &\leq 4\lambda^2 \frac{\sum_i t_i (1+\lambda^2 d_i^2(y))^{-3} d_i(y)}{\sum_j t_j (1+\lambda^2 d_j^2(y))^{-2}} \leq 4\lambda^2 \frac{\sum_i t_i (1+\lambda^2 d_i^2(y))^{-2} \frac{d_i(y)}{\lambda^2 d_i^2(y)}}{\sum_j t_j (1+\lambda^2 d_j^2(y))^{-2}} \\ &\leq 4 \frac{\sum_i t_i (1+\lambda^2 d_i^2(y))^{-2} \frac{1}{d_{min}(y)}}{\sum_j t_j (1+\lambda^2 d_j^2(y))^{-2}} \leq \frac{4}{d_{min}(y)}, \end{aligned}$$

which is (3.99).

Now, using (3.98), (3.99) and the fact that $\nabla\Phi(\sigma)_2 = -\frac{1}{2}\nabla\Phi(\sigma)_1$, one easily finds that

$$\frac{1}{2}\sum_{i,j=1}^{2} \int_\Sigma a^{ij} (\nabla\Phi(\sigma)_i) \cdot (\nabla\Phi(\sigma)_j) dV_g \leq C + 4 \int_{\Sigma \setminus \cup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{min}^2(y)} dV_g(y).$$

Reasoning as in [33] one can show that

$$\int_{\Sigma \setminus \cup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{min}^2(y)} dV_g(y) \leq 8\pi m (1 + o_\lambda(1)) \log \lambda, \qquad (o_\lambda(1) \to 0 \text{ as } \lambda \to +\infty),$$

and that

$$\int_\Sigma \varphi_{\lambda,\sigma} dV_g = -2(1+o_\lambda(1))\log\lambda; \quad \log \int_\Sigma e^{\varphi_{\lambda,\sigma}} dV_g = O(1); \quad \log \int_\Sigma e^{-\frac{1}{2}\varphi_{\lambda,\sigma}} dV_g = (1+o_\lambda(1))\log\lambda.$$

Using the last four inequalities one then obtains

$$II_\rho(\Phi(\sigma)) \leq (8m\pi - 2\rho_1 + o_\lambda(1)) \log \lambda + C,$$

where C is independent of λ and σ. Since we are assuming that ρ_1 is bigger than $4m\pi$, we achieve **(i)**.

To prove **(ii)** it is sufficient to consider the family of maps $T_\lambda : \Sigma_m \to \Sigma_m$ defined by

$$T_\lambda(\sigma) = \Psi(\Phi_\lambda(\sigma)), \qquad \sigma \in \Sigma_m.$$

We recall that when λ is sufficiently large this composition is well defined. Therefore, since $\frac{e^{\varphi_{\lambda,\sigma}}}{\int_\Sigma e^{\varphi_{\lambda,\sigma}} dV_g} \rightharpoonup \sigma$ in the weak sense of distributions, letting $\lambda \to \infty$ we obtain an homotopy between $\Psi \circ \Phi$ and Id_{Σ_m}. This concludes the proof. ■

Remark 3.2.38. *We point out that, fixing $\xi_1 \in \mathbb{R}^2$, the choice of ξ_2 which minimizes the quadratic form $\sum_{i,j} a^{ij}\xi_1 \cdot \xi_j$ is $\xi_2 = -\frac{1}{2}\xi_1$. This motivates the coefficient $-\frac{1}{2}$ in the second component of Φ.*

3.3 Min-max schemes for existence of solutions

In this Section, we perform the min-max schemes in order to get the existence results corresponding to the problems of prescribing Q-curvature in arbitrary dimensions, Q-curvature and boundary T-curvature of four manifolds with boundary and the generalized 2×2 Toda system. For the purpose of clarity, we will divide it into four Subsections. The first one is concerned about the prescribed Q-curvature problem in arbitrary dimensions. The second one deals with the probem of finding constant Q-curvature conformal metrics on four dimensional manifolds with boundary. In the third one, we threat the problem of existence of constant T-curvature conformal metrics on four dimensional Riemannian manifolds with boundary. And finally in the last one, we deal with the generalized 2×2 Toda system.

As said above, we start with the prescribed Q-curvature problem in arbitrary dimensions.

3.3.1 Min-max for the existence of constant Q-curvature metrics in arbitrary dimensions

In this Subsection we provide the proof of Theorem 0.2.13. As said in the Introduction we will suppose that P^n_g is non-negative and (42) holds.

First of all, we introduce the min-max scheme which provides existence of solutions . Let $\widehat{M_k}$ denote the (contractible) cone over M_k, which can be represented as $\widehat{M_k} = (M_k \times [0,1])$ with $M_k \times 0$ collapsed to a single point. Next let L be so large that Proposition 3.2.15 applies with $\frac{L}{4}$, and then let $\bar\lambda$ be so large (that Proposition 3.2.25 applies for this value of L). Fixing $\bar\lambda$, we define the following class.

$$II_{A,\bar\lambda} = \{\pi : \widehat{M_k} \to H^{\frac{n}{2}}(M) : \pi \text{ is continuous and } \pi(\cdot \times 1) = \Phi_{\bar\lambda}(\cdot)\}. \tag{3.102}$$

Then we have the following properties.

Lemma 3.3.1. *The set $II_{A,\bar\lambda}$ is non-empty and moreover, letting*

$$\overline{II}_{A,\bar\lambda} = \inf_{\pi \in II_{A,\bar\lambda}} \sup_{m \in \widehat{M_k},} II_A(\pi(m)), \qquad \textit{there holds } \overline{II}_{A,\bar\lambda} > -\frac{L}{2}.$$

Proof. The proof is the same as the one of Lemma 5.1 in [33]. But we will repeat it for the reader's convenience.
To prove that $\overline{II}_{A,\bar\lambda}$ is non-empty, we just notice that the following map

$$\bar\pi(\cdot, t) = t\Phi_{\bar\lambda}(\cdot) \tag{3.102}$$

belongs to $II_{A,\bar\lambda}$. Now to prove that $\overline{II}_{A,\bar\lambda} > -\frac{L}{2}$, let us argue by contradiction. Suppose that $\overline{II}_{A,\bar\lambda} \leq -\frac{L}{2}$: then there exists a map $\pi \in II_{A,\bar\lambda}$ such that $\sup_{m\in\widehat{M_k}} II(\pi(m)) \leq -\frac{3}{8}L$. Hence since Proposition 3.2.15 applies with $\frac{L}{4}$, writing $m = (z,t)$ with $z \in M_k$ we have that the map

$$t \to \Psi \circ \pi(\cdot, t)$$

is an homotopy in M_k between $\Psi \circ \Phi_{\bar\lambda}$ and a constant map. But this is impossible since M_k is non-contractible and $\Psi \circ \Phi_{\bar\lambda}$ is homotopic to the identity by Proposition 3.2.25.
■

Next we introduce a variant of the above minimax scheme, following [33] and [80]. For μ in a small neighborhood of 1, $[1-\mu_0, 1+\mu_0]$, we define the modified functional $II_{A,\mu} : H^{\frac{n}{2}}(M) \to \mathbb{R}$

$$II_{A,\mu}(u) = n\left\langle P_g^n u, u\right\rangle + 2n\mu \int_M Q_g^n u dV_g - 2\mu\kappa_{P^n} \log \int_M e^{nu} dV_g; \quad u \in H^{\frac{n}{2}}(M). \tag{3.103}$$

Following the estimates of the previous section, one easily checks that the above minimax scheme applies uniformly for $\mu \in [1-\mu_0, 1+\mu_0]$ and for $\bar\lambda$ sufficiently large. More precisely, given any large number $L > 0$, there exist $\bar\lambda$ sufficiently large and μ_0 sufficiently small such that

$$\sup_{\pi\in II_{A,\bar\lambda}} \sup_{m\in\partial\widehat{M_k}} II_{A,\mu}(\pi(m)) < -2L; \quad \overline{II}_{A,\mu,\bar\lambda} = \inf_{\pi\in II_{A,\bar\lambda}} \sup_{m\in\widehat{M_k}} II_{A,\mu}(\pi(m)) > -\frac{L}{2}; \qquad \mu \in [1-\mu_0, 1+\mu_0], \tag{3.104}$$

where $II_{A,\bar\lambda}$ is defined as in (3.102). Moreover, using for example the test map, one shows that for μ_0 sufficiently small there exists a large constant $\bar L$ such that

$$\overline{II}_{A,\mu,\bar\lambda} \leq \bar L, \qquad \text{for every } \mu \in [1-\mu_0, 1+\mu_0]. \tag{3.105}$$

We have the following result regarding the dependence in μ of the minimax value $\overline{II}_{A,\mu,\bar\lambda}$.

Lemma 3.3.2. *Let $\bar\lambda$ and μ_0 such that (3.104) holds. Then the function*

$$\mu \to \frac{\overline{II}_{A,\mu,\bar\lambda}}{\mu} \qquad \textit{is non-increasing in } [1-\mu_0, 1+1-\mu_0]$$

PROOF. For $\mu \geq \mu^{'}$, there holds

$$\frac{II_{A,\mu,\bar\lambda}(u)}{\mu} - \frac{II_{A,\mu^{'},\bar\lambda}(u)}{\mu^{'}} = \frac{n}{2}\left(\frac{1}{\mu} - \frac{1}{\mu^{'}}\right)\left\langle P_g^n u, u\right\rangle \tag{3.106}$$

Therefore it follows easily that also

$$\frac{\overline{II}_{A,\mu,\bar\lambda}}{\mu} - \frac{\overline{II}_{A,\mu^{'},\bar\lambda}}{\mu^{'}} \leq 0, \tag{3.107}$$

hence the Lemma is proved. ■

From this Lemma it follows that the function $\mu \to \frac{\overline{II}_{A,\mu,\bar\lambda}}{\mu}$ is a.e. differentiable in $[1-\mu_0, 1+\mu_0]$, and we obtain the following corollary.

Corollary 3.3.3. *Let $\bar\lambda$ and μ_0 be as in Lemma 3.3.2, and let $\Lambda \subset [1-\mu_0, 1+\mu_0]$ be the (dense) set of μ for which the function $\frac{\overline{II}_{A,\mu,\bar\lambda}}{\mu}$ is differentiable. Then for $\mu \in \Lambda$ the functional $II_{A,\mu}$ possesses a bounded Palais-Smale sequence $(u_l)_l$ at level $\overline{II}_{A,\mu,\bar\lambda}$.*

PROOF. The existence of Palais-Smale sequence $(u_l)_l$ at level $\overline{II}_{A,\mu,\bar{\lambda}}$ follows from the estimates (3.104) and the Remark 3.1.1. Now applying Theorem 3.1.2, we get the boundedness. ■

Next we state a Proposition saying that bounded Palais-Smale sequence of $II_{A,\mu}$ converges weakly (up to a subsequence) to a solution of the perturbed problem. The proof is the same as the one of Proposition 5.5 in [33].

Proposition 3.3.4. *Suppose $(u_l)_l \subset H^{\frac{n}{2}}(M)$ is a sequence for which*

$$II_{A,\mu}(u_l) \to c \in \mathbb{R}; \quad II_{A,\mu}^{'}[u_l] \to 0; \quad \int_M e^{nu_l} dV_g = 1 \quad \|u_l\|_{H^{\frac{n}{2}}(M)} \leq C.$$

Then (u_l) has a weak limit u_0 (up to a subsequence) which satisfies the following equation:

$$P_g^n u + \mu Q_g^n = \mu \kappa_{P^n} e^{nu} \qquad \text{in } M.$$

Now we are ready to make the proof of Theorem 0.2.13.

PROOFOF THEOREM 0.2.13

By Lemma 3.3.2, Corollary 3.3.3 and Proposition 3.3.4, we have that there exists a sequence $\mu_l \to 1$ and u_l such that the following holds :

$$P_g^n u + \mu_l Q_g^n = \mu_l \kappa_{P^n} e^{nu_l} \qquad \text{in } M.$$

Now since $\kappa_{P^n} = \int_M Q_g^n dV_g$ then applying corollary 0.2.7 with $Q_l = \mu_l Q_g^n$ and $\bar{Q}_l = \mu_l \kappa_{P^n}$ we have that u_l is bounded in C^α for every $\alpha \in (0,1)$. Hence up to a subsequence it converges uniformly to a solution of (12). Hence Theorem 0.2.13 is proved. ■

Next, we discuss the min-max scheme for the prescribed Q-curvature problem on four manifolds with boundary.

3.3.2 Min-max for the existence of constant Q-curvature metrics on four manifolds with boundary

In this Subsection we give the proof of Theorem 0.2.16. As already said in the Introduction, we suppose that $P_g^{4,3}$ is non-negative and (45) holds.

We start by definning the min-max scheme. To do so, we let $\widehat{(M\partial)_k}$ denote the (contractible) cone over $(M\partial)_k$, which can be represented as $\widehat{(M\partial)_k} = ((M\partial)_k \times [0,1])$ with $(M\partial)_k \times 0$ collapsed to a single point. Next, we choose L be so large that Proposition 3.2.18 applies with $\frac{L}{4}$, and then let $\bar{\lambda}$ be so large that Proposition 3.2.29 applies for this value of L). Fixing $\bar{\lambda}$, we define the following class.

$$II_{Q,\bar{\lambda}} = \{\pi : \widehat{(M\partial)_k} \to H_{\frac{\partial}{\partial n}} : \pi \text{ is continuous and } \pi(\cdot \times 1) = \Phi_{\bar{\lambda}}(\cdot)\}. \tag{3.108}$$

We then have the following properties.

Lemma 3.3.5. *The set $II_{Q,\bar{\lambda}}$ is non-empty and moreover, letting*

$$\overline{II}_{Q,\bar{\lambda}} = \inf_{\pi \in II_{Q,\bar{\lambda}}} \sup_{m \in \widehat{(M\partial)_k},} II_Q(\pi(m)), \qquad \textit{there holds } \overline{II}_{Q,\bar{\lambda}} > -\frac{L}{2}.$$

PROOF. The proof is the same as the one of Lemma 5.1 in [33]. But we will repeat it for the reader's convenience.

To prove that $\overline{II}_{Q,\bar{\lambda}}$ is non-empty, we just notice that the following map

$$\bar{\pi}(\cdot, t) = t\Phi_{\bar{\lambda}}(\cdot)$$

belongs to $II_{Q,\bar\lambda}$. Now to prove that $\overline{II}_{Q,\bar\lambda} > -\frac{L}{2}$, let us argue by contradiction. Suppose that $\overline{II}_{Q,\bar\lambda} \leq -\frac{L}{2}$: then there exists a map $\pi \in II_{Q,\bar\lambda}$ such that $\sup_{m\in\widehat{(M_\partial)_k}} II_Q(\pi(m)) \leq -\frac{3}{8}L$. Hence since Proposition 3.2.15 applies with $\frac{L}{4}$, writing $m=(z,t)$ with $z\in(M_\partial)_k$ we have that the map

$$t \rightarrow \Psi\circ\pi(\cdot,t)$$

is an homotopy in $(M_\partial)_k$ between $\Psi\circ\Phi_{\bar\lambda}$ and a constant map. But this is impossible since $(M_\partial)_k$ is non-contractible and $\Psi\circ\Phi_{\bar\lambda}$ is homotopic to the identity by Proposition 3.2.29.
■

Next we introduce a variant of the above minimax scheme as in the previous subsection. For μ in a small neighborhood of 1, $[1-\mu_0, 1+\mu_0]$, we define the modified functional $II_{Q,\mu} : H_{\frac{\partial}{\partial n}} \rightarrow \mathbb{R}$

$$II_{Q,\mu}(u) = \langle P_g^{4,3}u,u\rangle + 4\mu\int_M Q_g u dV_g + 4\mu\int_{\partial M} T_g u dS_g - 4\mu\kappa_{(P^4,P^3)}\log\int_M e^{4u}dV_g; \quad u\in H_{\frac{\partial}{\partial n}}. \tag{3.108}$$

Following the estimates of the previous section, one easily checks that the above minimax scheme applies uniformly for $\mu\in[1-\mu_0,1+\mu_0]$ and for $\bar\lambda$ sufficiently large. More precisely, given any large number $L>0$, there exist $\bar\lambda$ sufficiently large and μ_0 sufficiently small such that

$$\sup_{\pi\in II_{Q,\bar\lambda}}\ \sup_{m\in\partial\widehat{(M_\partial)_k}} II_{Q,\mu}(\pi(m)) < -2L; \quad \overline{II}_{Q,\mu,\bar\lambda} = \inf_{\pi\in II_{Q,\bar\lambda}}\ \sup_{m\in\widehat{(M_\partial)_k}} II_{Q,\mu}(\pi(m)) > -\frac{L}{2}; \quad \mu\in[1-\mu_0,1+\mu_0], \tag{3.109}$$

where $II_{Q,\bar\lambda}$ is defined as in (3.108). Moreover, using for example the test map, one shows that for μ_0 sufficiently small there exists a large constant $\bar L$ such that

$$\overline{II}_{Q,\mu,\bar\lambda} \leq \bar L, \qquad \text{for every } \mu\in[1-\mu_0,1+\mu_0]. \tag{3.110}$$

We have the following result regarding the dependence in μ of the minimax value $\overline{II}_{Q,\mu,\bar\lambda}$.

Lemma 3.3.6. *Let $\bar\lambda$ and μ_0 such that (3.109) holds. Then the function*

$$\mu \rightarrow \frac{\overline{II}_{Q,\mu,\bar\lambda}}{\mu} \qquad \textit{is non-increasing in } [1-\mu_0, 1+1-\mu_0]$$

Proof. For $\mu\geq\mu'$, there holds

$$\frac{II_{Q,\mu,\bar\lambda}(u)}{\mu} - \frac{II_{Q,\mu',\bar\lambda}(u)}{\mu'} = \left(\frac{1}{\mu}-\frac{1}{\mu'}\right)\langle P_g^{4,3}u,u\rangle$$

Therefore it follows easily that also

$$\frac{\overline{II}_{Q,\mu,\bar\lambda}}{\mu} - \frac{\overline{II}_{Q,\mu',\bar\lambda}}{\mu'} \leq 0,$$

hence the Lemma is proved. ■

From this Lemma it follows that the function $\mu\rightarrow\frac{\overline{II}_{Q,\mu,\bar\lambda}}{\mu}$ is a.e. differentiable in $[1-\mu_0,1+\mu_0]$, and we obtain the following corollary.

Corollary 3.3.7. *Let $\bar\lambda$ and μ_0 be as in Lemma 3.3.6, and let $\Lambda\subset[1-\mu_0,1+\mu_0]$ be the (dense) set of μ for which the function $\frac{\overline{II}_{Q,\mu,\bar\lambda}}{\mu}$ is differentiable. Then for $\mu\in\Lambda$ the functional $II_{Q,\mu}$ possesses a bounded Palais-Smale sequence $(u_l)_l$ at level $\overline{II}_{Q,\mu,\bar\lambda}$.*

Proof. As for the case of II_A, we have aso here the existence of Palais-Smale sequence $(u_l)_l$ at level $\overline{II}_{A,\mu,\bar{\lambda}}$ follows from the estimates (3.109) and the Remark 3.1.1. Now applying Theorem 3.1.2, we get the boundedness. ∎

Next we state a Proposition saying that bounded Palais-Smale sequence of $II_{Q,\mu}$ converges weakly (up to a subsequence) to a solution of the perturbed problem. The proof is the same as the one of Proposition 5.5 in [33].

Proposition 3.3.8. *Suppose* $(u_l)_l \subset H_{\frac{\partial}{\partial n}}$ *is a sequence for which*

$$II_{Q,\mu}(u_l) \to c \in \mathbb{R}; \quad II^{'}_{Q,\mu}[u_l] \to 0; \quad \int_M e^{4u_l} dV_g = 1 \quad \|u_l\|_{H^2(M)} \leq C.$$

Then (u_l) *has a weak limit* u *(up to a subsequence) which satisfies the following equation:*

$$\begin{cases} P^4_g u + 2\mu Q_g = 2\mu\kappa_{(P^4,P^3)} e^{4u} & \text{in } M; \\ P^3_g u + \mu T_g = 0 & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} = 0 & \text{on } \partial M. \end{cases}$$

Now we are ready to make the proof of Theorem 0.2.16.

Proofof Theorem 0.2.16

By Lemma 3.3.6, Corollary 3.3.7 and Proposition 3.3.8, we have that there exists a sequence $\mu_l \to 1$ and u_l such that the following holds :

$$\begin{cases} P^4_g u_l + 2\mu_l Q_g = 2\mu\kappa_{(P^4,P^3)} e^{4u_l} & \text{in } M; \\ P^3_g u_l + \mu_l T_g = 0 & \text{on } \partial M; \\ \dfrac{\partial u_l}{\partial n_g} = 0 & \text{on } \partial M. \end{cases}$$

Now since $\kappa_{(P^4,P^3)} = \int_M Q_g dV_g + \int_{\partial M} dS_g$ then applying corollary 0.2.9 with $Q_l = \mu_l Q_g$, $T_l = \mu_l T_g$ and $\bar{Q}_l = \mu_l \kappa_{(P^4,P^3)}$ we have that u_l is bounded in $C^{1+\alpha}$ for every $\alpha \in (0,1)$. Hence up to a subsequence it converges in $C^1(M)$ to a solution of (20). Hence Theorem 0.2.16 is proved ∎

Next we discuss the problem of finding conformal metrics with constant T-curvature on four manifolds with boundary.

3.3.3 Min-max for the existence of constant T-curvature metrics on four manifolds with boundary

In this Subsection we give the proof of Theorem 0.2.19. As already said in the Introduction, here also we assume $P^{4,3}_g$ is non-negative and (47) holds.
As done in the other Subsections, we start by defining the min-max scheme. For doing this, we denot by $\widehat{\partial M_k}$ the (contractible) cone over ∂M_k, which can be represented as $\widehat{\partial M_k} = (\partial M_k \times [0,1])$ with $\partial M_k \times 0$ collapsed to a single point. Next let L be so large that Proposition 3.2.22 applies with $\frac{L}{4}$, and then let $\bar{\lambda}$ be so large that Proposition 3.2.33 applies for this value of L. Fixing $\bar{\lambda}$, we define the following class.

$$II_{T,\bar{\lambda}} = \{\pi : \widehat{\partial M_k} \to H_{\frac{\partial}{\partial n}} : \pi \text{ is continuous and } \pi(\cdot \times 1) = \Phi_{\bar{\lambda}}(\cdot)\}. \tag{3.111}$$

We then have the following properties.

Lemma 3.3.9. *The set $II_{T,\bar\lambda}$ is non-empty and moreover, letting*

$$\overline{II}_{T,\bar\lambda} = \inf_{\pi \in II_{\bar\lambda}} \sup_{m \in \widehat{\partial M_{k,}}} II_T(\pi(m)), \qquad \text{there holds } \overline{II}_{T,\bar\lambda} > -\frac{L}{2}.$$

Proof. The proof is the same as the one of Lemma 5.1 in [33]. But we will repeat it for the reader's convenience.
To prove that $\overline{II}_{T,\bar\lambda}$ is non-empty, we just notice that the following map

$$\bar\pi(\cdot, t) = t\Phi_{\bar\lambda}(\cdot)$$

belongs to $II_{T,\bar\lambda}$. Now to prove that $\overline{II}_{T,\bar\lambda} > -\frac{L}{2}$, let us argue by contradiction. Suppose that $\overline{II}_{T,\bar\lambda} \leq -\frac{L}{2}$: then there exists a map $\pi \in II_{T,\bar\lambda}$ such that $\sup_{m\in\widehat{\partial M_k}} II(\pi(m)) \leq -\frac{3}{8}L$. Hence since Proposition 3.2.15 applies with $\frac{L}{4}$, writing $m = (z,t)$ with $z \in \partial M_k$ we have that the map

$$t \to \Psi \circ \pi(\cdot, t)$$

is an homotopy in ∂M_k between $\Psi \circ \Phi_{\bar\lambda}$ and a constant map. But this is impossible since ∂M_k is non-contractible and $\Psi \circ \Phi_{\bar\lambda}$ is homotopic to the identity by Proposition 3.2.33.
■

Next we introduce a variant of the above minimax scheme, following [33] [80] and[69]. For μ in a small neighborhood of 1, $[1-\mu_0, 1+\mu_0]$, we define the modified functional $II_{T,\mu} : H_{\frac{\partial}{\partial n}} \to \mathbb{R}$

$$II_{T,\mu}(u) = \left\langle P_g^{4,3}u, u\right\rangle + 4\mu \int_M Q_g u dV_g + 4\mu \int_{\partial M} T_g u dS_g - \frac{4}{3}\mu\kappa_{(P^4,P^3)} \log \int_{\partial M} e^{3u} dS_g; \quad u \in H_{\frac{\partial}{\partial n}}. \tag{3.111}$$

Following the estimates of the previous section, one easily checks that the above minimax scheme applies uniformly for $\mu \in [1-\mu_0, 1+\mu_0]$ and for $\bar\lambda$ sufficiently large. More precisely, given any large number $L > 0$, there exist $\bar\lambda$ sufficiently large and μ_0 sufficiently small such that

$$\sup_{\pi \in II_{T,\bar\lambda}} \sup_{m \in \partial\widehat{\partial M_k}} II_{T,\mu}(\pi(m)) < -2L; \quad \overline{II}_{T,\mu,\bar\lambda} = \inf_{\pi \in II_{T,\bar\lambda}} \sup_{m \in \widehat{\partial M_k}} II_{T,\mu}(\pi(m)) > -\frac{L}{2}; \qquad \mu \in [1-\mu_0, 1+\mu_0], \tag{3.112}$$

where $II_{\bar\lambda}$ is defined as in (3.111). Moreover, using for example the test map, one shows that for μ_0 sufficiently small there exists a large constant $\bar L$ such that

$$\overline{II}_{T,\mu,\bar\lambda} \leq \bar L, \qquad \text{for every } \mu \in [1-\mu_0, 1+\mu_0]. \tag{3.113}$$

We have the following result regarding the dependence in μ of the minimax value $\overline{II}_{T,\mu,\bar\lambda}$.

Lemma 3.3.10. *Let $\bar\lambda$ and μ_0 such that* (3.112) *holds. Then the function*

$$\mu \to \frac{\overline{II}_{T,\mu,\bar\lambda}}{\mu} \qquad \text{is non-increasing in } [1-\mu_0, 1+1-\mu_0]$$

Proof. For $\mu \geq \mu'$, there holds

$$\frac{II_{T,\mu,\bar\lambda}(u)}{\mu} - \frac{II_{T,\mu',\bar\lambda}(u)}{\mu'} = \left(\frac{1}{\mu} - \frac{1}{\mu'}\right)\left\langle P_g^{4,3}u, u\right\rangle$$

Therefore it follows easily that also

$$\frac{\overline{II}_{T,\mu,\bar\lambda}}{\mu} - \frac{\overline{II}_{T,\mu',\bar\lambda}}{\mu'} \leq 0,$$

hence the Lemma is proved. ■

From this Lemma it follows that the function $\mu \to \frac{\overline{II}_{T,\mu,\bar\lambda}}{\mu}$ is a.e. differentiable in $[1-\rho_0, 1+\mu_0]$, and we obtain the following corollary.

Corollary 3.3.11. *Let* $\bar\lambda$ *and* μ_0 *be as in Lemma* 3.3.10, *and let* $\Lambda \subset [1-\mu_0, 1+\mu_0]$ *be the (dense) set of* μ *for which the function* $\frac{\overline{II}_{T,\mu,\bar\lambda}}{\mu}$ *is differentiable. Then for* $\mu \in \Lambda$ *the functional* $II_{T,\mu}$ *possesses a bounded Palais-Smale sequence* $(u_l)_l$ *at level* $\overline{II}_{T,\mu,\bar\lambda}$.

PROOF. As for the case of II_A and II_Q, we have also here the existence of Palais-Smale sequence $(u_l)_l$ at level $\overline{II}_{A,\mu,\bar\lambda}$ follows from the estimates (3.112) and the Remark 3.1.1. Now applying Theorem 3.1.2, we get the boundedness. ■

Next we state a Proposition saying that bounded Palais-Smale sequence of $II_{T,\mu}$ converges weakly (up to a subsequence) to a solution of the perturbed problem. The proof is the same as the one of Proposition 5.5 in [33].

Proposition 3.3.12. *Suppose* $(u_l)_l \subset H_{\frac{\partial}{\partial n}}$ *is a sequence for which*

$$II_{T,\mu}(u_l) \to c \in \mathbb{R}; \quad II^{'}_{T,\mu}[u_l] \to 0; \quad \int_{\partial M} e^{3u_l} dS_g = 1 \quad \|u_l\|_{H^2(M)} \le C.$$

Then (u_l) *has a weak limit* u *(up to a subsequence) which satisfies the following equation:*

$$\begin{cases} P^4_g u + 2\mu Q_g = 0 & \text{in } M; \\ P^3_g u + \mu T_g = \mu\kappa_{(P4,P^3)} e^{3u} & \text{on } \partial M; \\ \dfrac{\partial u}{\partial n_g} = 0 & \text{on } \partial M. \end{cases}$$

Now we are ready to make the proof of Theorem 0.2.16.

PROOFOF THEOREM 0.2.16

By Lemma 3.3.10, Corollary 3.3.11 and Proposition 3.3.12, we have that there exists a sequence $\mu_l \to 1$ and u_l such that the following holds :

$$\begin{cases} P^4_g u_l + 2\mu_l Q_g = 0 & \text{in } M; \\ P^3_g u_l + \mu_l T_g = \mu\kappa_{(P4,P^3)} e^{3u_l}; & \text{on } \partial M; \\ \dfrac{\partial u_l}{\partial n_g} = 0 & \text{on } \partial M. \end{cases}$$

Now since $\kappa_{(P^4,P^3)} = \int_M Q_g dV_g + \int_{\partial M} T_g dS_g$ then applying corollary 0.2.11 with $Q_l = \mu_l Q_g$, $T_l = \mu_l T_g$ and $\bar{T}_l = \mu_l \kappa_{(P^4,P^3)}$ we have that u_l is bounded in $C^{4+\alpha}$ for every $\alpha \in (0,1)$. Hence up to a subsequence it converges in $C^1(M)$ to a solution of (21). Hence Theorem 0.2.16 is proved. ■

The next and last discussion concerns some existence results for the 2×2 Toda system.

3.3.4 Min-max for the existence results for the generalized 2×2 Toda system on compact closed surfaces

In this Subsection, we give the proof of Theorem 0.2.22. As done above, we start by definding the scheme. To do this, we denote by K_m the topological cone over Σ_m defined as in the other

Subsections. Next let L be so large that Proposition 3.2.24 applies with $\frac{L}{4}$, and choose then Φ such that Proposition 3.2.37 applies for L. Fixing L and Φ, we define the class of maps

$$\Pi_\Phi = \left\{\pi : K_m \to H^1(\Sigma) \times H^1(\Sigma) \ : \ \pi \text{ is continuous and } \pi|_{\Sigma_m(=\partial K_m)} = \Phi\right\}. \tag{3.114}$$

Then we have the following properties.

Lemma 3.3.13. *The set* Π_Φ *is non-empty and moreover, letting*

$$\alpha_\rho = \inf_{\pi\in\Pi_\Phi} \sup_{m\in K_m} II_{\rho_1,\rho_2}(\pi(m)), \qquad \textit{there holds} \qquad \alpha_\rho > -\frac{L}{2}.$$

Proof. To prove that $\Pi_\Phi \neq \emptyset$, we just notice that the following map

$$\overline{\pi}(\sigma, t) = t\Phi(\sigma); \qquad \sigma \in \Sigma_m, t \in [0,1] \quad ((\sigma, t) \in K_m) \tag{3.115}$$

belongs to Π_Φ. Assuming by contradiction that $\alpha_\rho \leq -\frac{L}{2}$, there would exist a map $\pi \in \Pi_\Phi$ with $\sup_{\tilde\sigma\in K_m} II_\rho(\pi(\tilde\sigma)) \leq -\frac{3}{8}L$. Then, since Proposition 3.2.24 applies with $\frac{L}{4}$, writing $\tilde\sigma = (\sigma, t)$, with $\sigma \in \Sigma_m$, the map

$$t \mapsto \Psi \circ \pi(\cdot, t)$$

would be an homotopy in Σ_m between $\Psi\circ\Phi$ and a constant map. But this is impossible since Σ_m is non-contractible and since $\Psi\circ\Phi$ is homotopic to the identity, by Proposition 3.2.37. Therefore we deduce $\overline{\Pi}_\Phi > -\frac{L}{2}$. ■

As in the case of II_A, II_Q, and II_T, we introduce a variant of the above minimax scheme. For t close to 1, we consider the functional

$$\begin{aligned} J_{t\rho_1,t\rho_2}(u) &= \frac{1}{2}\sum_{i,j}\int_\Sigma a^{ij}\nabla u_i \cdot \nabla u_j dV_g + t\rho_1 \int_\Sigma u_1 dV_g + t\rho_2 \int_\Sigma u_2 dV_g \\ &- t\rho_1 \log \int_\Sigma h_1 e^{u_1} dV_g - t\rho_2 \log\int_\Sigma h_2 e^{u_2} dV_g. \end{aligned}$$

Repeating the estimates of the previous sections, one easily checks that the above minimax scheme applies uniformly for $t \in [1-t_0, 1+t_0]$ with t_0 sufficiently small. More precisely, given $L > 0$ as before, for t_0 sufficiently small we have

$$\sup_{\pi\in\Pi_\Phi}\sup_{m\in\partial K_m} J_{t\rho_1,t\rho_2}(\pi(m)) < -2L; \quad \alpha_{t\rho} := \inf_{\pi\in\Pi_\Phi}\sup_{m\in K_m} J_{t\rho_1,t\rho_2}(\pi(m)) > -\frac{L}{2}; \quad \text{for every } t\in[1-t_0, 1+t_0], \tag{3.116}$$

where Π_Φ is defined in (3.114).

Next we notice that for $t' \geq t$ there holds

$$\frac{J_{t\rho_1,t\rho_2}(u)}{t} - \frac{J_{t'\rho_1,t'\rho_2}(u)}{t'} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t'}\right)\int_\Sigma a^{ij}\nabla u_i\cdot\nabla u_j dV_g \geq 0, \qquad u \in H^1(\Sigma)\times H^1(\Sigma).$$

Therefore it follows easily that also

$$\frac{\alpha_{t\rho}}{t} - \frac{\alpha_{t'\rho}}{t'} \geq 0,$$

namely the function $t \mapsto \frac{\alpha_{t\mu}}{t}$ is non-increasing, and hence is almost everywhere differentiable. Using (3.116), Remark 3.1.1 and Theorem 3.1.2, one can see that at the points where $\frac{\alpha_{t\rho}}{t}$ is differentiable $J_{t\rho_1,t\rho_2}$ admits a bounded Palais-Smale sequence at level $\alpha_{t\rho}$, which converges to a critical point of $J_{t\rho_1,t\rho_2}$. Therefore, since the points with differentiability fill densely the interval $[1-t_0, 1+t_0]$, there exists $t_k \to 1$ such that the following system has a solution $(u_{1,k}, u_{2,k})$

$$-\Delta u_{i,k} = \sum_{j=1}^N t_k\rho_j a_{ij}\left(\frac{h_j e^{u_{j,k}}}{\int_\Sigma h_j e^{u_{j,k}} dV_g} - 1\right), \qquad i = 1, 2. \tag{3.117}$$

Now it is sufficient to apply Proposition 0.2.12 to obtain a limit (u_1, u_2) which is a solution of (11). This concludes the proof.

3.3.5 Adaptations for the generic cases

As said in the Introduction, the condition P_g^n (resp $P_g^{4,3}$) non-negative in the proofs of Theorem 0.2.13, Theorem 0.2.16 and Theorem 0.2.19 is only required to make the exposition clear. In this small Subsection we show how to deal with the general case. Since the same considerations hold for all the three Theorems, then we will make the discussion only for Theorem 0.2.13. Assuming we are dealing with the later Theorem, we divide the discussion into the three differents remaining cases.

Case $\bar{k} = 0$ and $\kappa_{P^n} < (n-1)!\omega_n$

This case was proven by Brendle[13](in the *even* dimensional case) using geometric flows. However using Direct Methods in the Calculus of variations it can be obtained (both in the *even* and *odd* dimensional cases) thanks to the Moser-Trudinger type inequality (see Proposition 1.3.1).

Case $\bar{k} \neq 0$ and $\kappa_{P^n} < (n-1)!\omega_n$

In this case, we have that P_g^n has some negative eigenvalues. We change the arguments as follows. To obtain Moser-Trudinger type inequality we impose the additional condition $\|\hat{u}\| \leq C$ where $\hat{u}$ is the component of u in the direct sum of the negative eigenspaces. Thus we have that the only way that the functional go to negative infinity is that $||\hat{u}||$ tends to infinity. Hence to run the min-max scheme we substitute M_k with $S^{\bar{k}-1}$, the boundary of the unit ball in the $\bar{k}$-dimensional Euclidean space. Moreover an other modification for the min-max scheme is the monotonicity formula which becomes

$$\rho \to \frac{\overline{II_{A\mu}}}{\mu} - C\mu \quad \text{is non-increasing in} \quad [1-\mu_0, 1+\mu_0];$$

for a fixed constant $C > 0$

Case $\bar{k} \neq 0$ and $\kappa_{P^n} \in ((n-1)!k\omega_n, (n-1)!(k+1)\omega_n)$, $k \geq 1$

In this case we mix the ideas of the case $\bar{k} = 0$ and $\kappa_{P^n} \in ((n-1)!k\omega_n, (n-1)!(k+1)\omega_n)$, and the Case $\bar{k} \neq 0$ and $\kappa_{P^n} < (n-1)!\omega_n$. Precisely to obtain the Moser-Trudinger inequality and its improvement, we impose the additional condition $\|\hat{u}\| \leq C$ where $\hat{u}$ is the component of u in the direct sum of the negative eigenspaces. Furthermore another aspect has to be considered that is not only e^{nu} can concentrate but also $\|\hat{u}\|$ can also tend to infinity. And to deal with this we have to substitute the set M_k with an other one, $A_{k,\bar{k}}$ which is defined in terms of the integer k (given in (42)) and the number $\bar{k}$ of negative eigenvalues of P_g^n, as done in [33]. This also requires suitable adaptation of the min-max scheme and of the monotonicity formula , which in general becomes

$$\rho \to \frac{\overline{II_{A\mu}}}{\mu} - C\mu \quad \text{is non-increasing in} \quad [1-\mu_0, 1+\mu_0];$$

for a fixed constant $C > 0$

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