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Holomorphicity and stability in string theory

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Introduction

String theory is so far the best candidate for quantization of gravity. Its very definition is however somewhat unsatisfactory, as a nonperturbative definition is still not completely clear. An important step in this direction has been to realize that the space of the states of this theory will finally include not only states coming from strings, but also from higher-dimensional extended objects, that were christened D-branes. Though in the perturbative formulation these latter objects can only be understood in terms of open strings, in the nonperturbative theory strings and branes will appear on the same footing. It is therefore compelling to understand as many properties of these objects as one can.

This is not only out of the intellectual curiosity of fully understanding the structure and formulation of an interesting branch of mathematics, as string theory has happened to become; but for the very physical relevance of the theory itself. As an example, one of the requirements for any theory to supersede the current standard model of particle physics is to explain the observed parameters of the latter. This is currently the main drawback of string theory, given the huge number of possible compactifications to four dimensions, with little or no theoretical reason of preference among them. This is the physical ground of the huge mathematical problem of describing the moduli space of string vacua. D-branes are relevant to this problem because of their above mentioned role in nonperturbative string theory, and in particular because they give additional moduli.

The best understood way to probe nonperturbative features of a supersymmetric theory is to study its BPS states; so one is lead to the study of those configurations of branes which do not break supersymmetry completely. We will study two separate applications of this method, both with the aim of understanding the nonperturbative structure of string theory.

The first is the proposal known as Matrix String Theory (MST) [62, 21]. The idea that a second-quantized theory of strings, which describes multi-strings states, can be linked with matrices, is somewhat an old one, but it was revived more recently after the emergence of Matrix Theory [3]. After the initial proposal, the essential lines of a proof were shown [12]. The main tool in this proof were BPS solutions of the theory. These turn out to be given by Hitchin equations in 2 dimensions

$$F - i[X, X^\dagger]\omega = c\omega\ , \qquad DX = 0\ , \tag{0.0.1}$$

where F is the curvature of the gauge field of the theory, D the $(1,0)$ part of its covariant derivative, and X is a matrix valued field. There is then a well-known technique to produce, out of these data, a covering of the base space, and a bundle on it. In our case the covering will be a Riemann surface with marked points; in the method of [12] these are used to reproduce the S-matrix of the Green-Schwarz string. The question arises, however, of whether this method can give the correct moduli space for string scatterings: this will be the theme of our first chapter. We will see that several interesting phenomena happen. First, there is a partial discretization of the moduli space, whose spacing goes however to zero in the limit in which MST is expected to reduce to perturbative string theory. Second, in general one has to allow these Riemann surfaces to be singular. Physically, this comes because we are embedding them in the four dimensions of spacetime, and they look singular; taking in account fluctuations in the extra dimensions, the singularities get resolved. Along the way, we will describe an interesting way of arriving to a known relationship between the Newton polygon description of a plane curve and its genus, involving toric geometry. Many of the mathematical concepts and methods introduced in this chapter will be used in the later part as well.

Another interesting way to understand nonperturbative features of the theory, and at the same time to tackle an important part of the string theory moduli problem we alluded to above, is to consider branes in Calabi-Yau manifolds. The moduli space of Calabi-Yau compactifications without branes has been already thoroughly studied (see for example [40] for a review); inclusion of branes complicates enormously the problem. In perspective, the moduli space of branes should give an infinite covering of the closed string moduli space: for each closed string background there are several allowed brane configurations. As part of such a program, we will start with an already interesting part: fixing a Calabi-Yau and considering its closed string Kähler moduli space.

In chapter 2 we will introduce the main ingredients for what follows. In particular, it will be noted a fact that will allow us to split the problem in two. This is already visible in Hitchin equations (0.0.1), which are again met in particular cases in this context: these equations consist of a real equation and of a complex holomorphic one. This splitting has a deeper meaning than one may think; it corresponds to the splitting of the equations which describe moduli spaces of vacua of supersymmetric gauge theories in D-terms and F-terms. Moreover, in the cases we will consider (varying Kähler moduli only) F-terms (holomorphic equations) are not modified [16]. It makes thus sense to consider solutions to the latter first, and to postpone analysis of the former (which, for reasons to become clear, will be referred to as a *stability* problem).

This first step will be done in chapter 3. We will see that, in two particular points of the Kähler moduli space (called *large volume limit* and *Gepner point*), branes can be described by two different kind of objects: coherent sheaves and quiver representations. Since the moduli space is connected, a continuous interpolation between them should be possible: after having noted that this is nothing but a generalized McKay correspondence, we will find this interpolation in several examples, using Bondal theorem [10], a generalization to Fano varieties of Beilinson's one [6]. The most natural formulation of this theorem is in terms of derived category, and we will argue then that this is the most natural classifying object for D-branes as long as one neglects stability matters.

These will be then the subject of chapter 4. Again, the scheme will be here to understand tractable points of the moduli space and try then to extend the analysis to the whole of it. In the large volume limit, as we have mentioned, we will find generalizations of Hitchin equations. They involve in an essential way connections on the gauge bundle E on the brane; it turns out that a property of this bundle is *equivalent* to the existence of solutions to the equations. This property is called stability and it always involves properties of subbundles E' of the given bundle E; the equivalence of stability with existence of solutions to the equations is sometimes called *Hitchin-Kobayashi correspondence*. It is interesting to note that the equations have also the meaning of physical stability of branes, giving a nice coincidence of words.

Away from the large volume limit, on a generic point of Kähler moduli space, these D-term equations will be deformed. The complete form of these deformed equations is not known, though some features were explored [58]. However, from worldsheet arguments one can argue for a deformed stability, known as Π-stability [28], which should again be equivalent to solutions of the deformed equations, whatever they are. Since this deformed stability condition reduces to usual stability in the limit, this gives in a sense a string theory rederivation of Hitchin-Kobayashi correspondence.
However, we have seen that, away from the large volume limit, the appropriate description is derived category, one would expect some kind of derived category stability to come in. The final word on this has not yet come, and this part will be more of a sketch of a program than a conclusive one. The final goal should be to understand completely what is the derived category stability coming from worldsheet arguments, and then rederive from this all the known stabilities of known equations in the tractable limits. Depending on the cases, these equations can involve, besides the connection A on the gauge bundle E on the brane, transverse scalars X (as in the Hitchin example we have seen) and/or tachyons; each of these equations has its own Hitchin-Kobayashi correspondence with a stability condition, and string theory should allow to rederive all of them as different cases of a unique derived category stability.

Contents

Chapter 1

Matrix strings

Matrix String Theory (MST) is a proposal for non-perturbative string theory, coming from the relationships of string theory with eleven dimensional theories noted in the recent years. If this proposal is to make sense, it should reproduce perturbative results in the weak string coupling limit. After having introduced the model, we will describe a sketch of a proof of this fact [12], and later concentrate on a part of it. String scattering amplitudes involve summing over moduli spaces of punctured Riemann surfaces; we will analyze the problem of recovering exactly these moduli spaces from MST.

1.1 MST and its origin

The idea is very simple. Perturbative string theory can only describe states with one string at a time. If the fields which live on the string are promoted to matrices, then they can describe several strings at a time through their eigenvalues.

In its modern form, this idea came from eleven-dimensional arguments [62, 21]. Recall indeed that non-perturbative type IIA string theory was indirectly found to be intrinsically an 11-dimensional theory, the so-called M-theory, whose low-energy limit is not surprisingly 11-d supergravity. The Kaluza-Klein modes of the 11-dimensional gravitons are D0 branes, which are indeed non-perturbative states of string theory. Note that D0 charge is quantized in energy, as it should to have a KK interpretation: bound states of N D0 branes are bound states at threshold, and thus have N times the charge of one D0.

A proposal for a description of this theory is M(atrix) theory. The idea is to consider M-theory in its infinite-momentum frame (IMF) [88]. As for field theory [3], in this limit we should not lose information, and gain however in simplicity. Indeed, if we make a boost of infinite momentum in a direction, we have that

- The only remaining degrees of freedom are the ones with very large momentum in the boosted direction; the others are effectively integrated out;
- The theory of the remaining degrees of freedom has a Galilean invariance in the directions *transverse* to the boost.

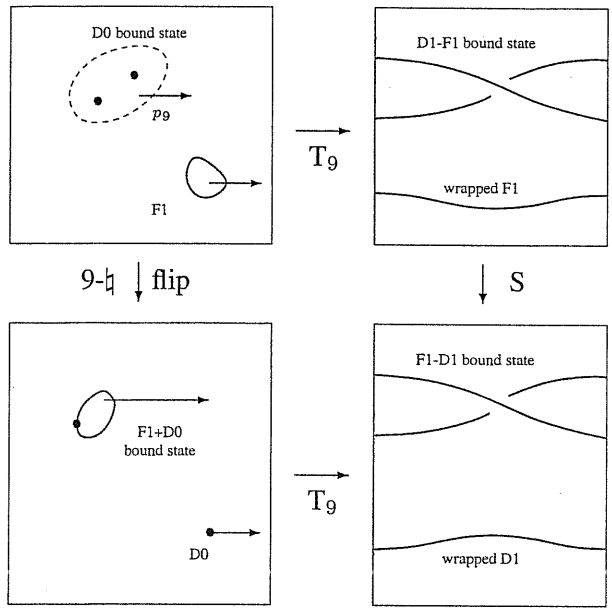
Let us apply this to a boost in the 11-th direction, which we call $x^\natural$; moreover, we compactify this direction (there is no clash between these two procedures). We have mentioned above that $p_\natural$ is proportional to D0 charge: so, because of first point above the only states remaining in our effective description are bound states of a large number of D0 branes. Moreover, because of second point, the resulting theory should be a non-relativistic theory describing 9 transverse directions. We have already such a theory: it is the nonrelativistic limit of the theory describing N D0 branes. This is a super Yang-Mills theory in 0+1 dimensions:

$$S = \frac{1}{l_s}\int dt \ \mathrm{tr}\left(\frac{1}{2g_s}(D_0X^i)^2 - i\theta^T D_0\theta + \frac{1}{4(2\pi)^2 g_s l_s^2}([X^i,X^j])^2 + \frac{1}{2\pi l_s^2}\theta^T\gamma^j[X_j,\theta]\right);$$

all the fields are hermitian $N \times N$ matrices; X^i are the 9 scalars describing fluctuations in the transverse directions, θ are their supersymmetric partners, in a spinor representation of the transverse directions (the $\underline{16}$ of $SO(9)$); γ^i are the corresponding gamma matrices; D_0 is the covariant derivative with respect to a $0+1$ dimensional gauge field A_0; l_s and g_s are the string length and coupling constant.

Since this is alleged to be a theory of non-perturbative string theory, it is not surprising that it can be massages to produce a more direct description of non-perturbative scatterings of strings. The method to achieve can be understood by compactifying the 9 direction as well and looking at what happens when one exchanges the role of directions 9 and $\natural$. Since D0 are gravitons with large $p_\natural$, after this flip we will have gravitons with large p_9; these are described in 10-dimensional terms by closed strings. It is also instructive to look at this from a chain-of-dualities perspective:

Figure 1.1: Chain of dualities and 9-$\natural$ flip



Here we have displayed for illustrative purpose the effect of dualities and of 9-$\natural$ flip on a bound state of two D0 branes and on a single fundamental string (denoted F1), both with a momentum p_9. The latter is exchanged with $D0$ charge (p_{11}), as it should. The left side of this diagram is IIA theory, the right side IIB. So, states with high D0 charge will become on the lower left corner fundamental IIA strings with high p_9 momentum.

This flip can be implemented through a careful mapping of constants, and through reinterpretation of infinite traces as integrals (as it is customary in T-duality of D-branes in the style of [77]). The result is an action defined on a cylinder $S^1 \times \mathbb{R}$:

$$S = \frac{1}{2\pi}\int \mathrm{tr}\left(D_iX^aD^iX^a + \theta^T \not{D}\theta - g_s^2F_{ij}F^{ij} - \frac{1}{g_s^2}\sum_{i,j}[X^i,X^j]^2 + \frac{1}{g_s}\theta^T\gamma_i[X^i,\theta]\right) \ ; \quad (1.1.1)$$

all the fields are matrices, X^i are 8 scalars in the vector representation of transverse $SO(8)$, θ are two-dimensional spinors as well as in the $8_s \oplus 8_c$ of $SO(8)$. This action can be recognized as a maximally extended $\mathcal{N} = (8,8)$ super Yang-Mills (not surprisingly, since the initial system of N D0 branes has 16 supercharges) with a gauge coupling constant

$$g_{YM} = \frac{1}{g_s} \ . \quad (1.1.2)$$

The important thing is that one ended up with *a matrix action*, though describing now *fundamental strings*. This gives hope that one can describe multistring states using eigenvalues of the matrices.

It can be a little surprising at first that an action as easy as (1.1.1) can describe such a complicated theory as IIA string theory, which even claims to be a theory of gravity. However, note that the perturbative regime of string theory is at small g_s, which is, by (1.1.2), strong gauge coupling for the super Yang-Mills action (1.1.1). Strong coupling regime of field theories is not under control as the weak coupling one; the idea that strong coupling is in some sense a theory of strings is in fact an old one [82, 83]. The same idea has found more recently a new avatar in the celebrated AdS/CFT correspondence; it is perhaps not often stressed that these ideas would mean a non-perturbative formulation of string theory. This is exactly the point of view taken here: super Yang-Mills action (1.1.1) should give a non-perturbative definition of type IIA string theory in flat space (once one deals carefully with the constants, in such a way that the radius of the 9 coordinate in lower-left corner in figure 1.1 goes to infinity).

Accordingly, we will now describe a sketch of a proof [12] that the strong coupling of (1.1.1) is indeed IIA theory.

1.2 Gauge strong coupling and strings

The strategy is as follows. One tries to evaluate the euclidean path integral of the theory. Usually, for that one uses the "steepest descent" method: This means that one expands the euclidean action around a classical solution,

$$S(\Phi) \cong S(\Phi) + (\delta^2 S/\delta\Phi^2)(\Phi_0)\,\phi^2\,, \qquad \phi = \Phi - \Phi_0 \tag{1.2.3}$$

and evaluates the quadratic part thanks to the fact it is Gaussian. Here, we do not want to really *evaluate* the path integral, but rather to show that it is equal to something else (to the path integral of IIA theory). So, the quadratic part in (1.2.3) will be actually left as it is, without evaluating it. This could seem a little bizarre, but recall that string theory does indeed look as a free field theory from the worldsheet point of view; it is the insertion of vertex operators that makes the story nontrivial.

The classical solution Φ_0 is also called an instanton. When one has several instantons, steepest descent methods instructs us to simply sum over all the contributions coming from all the instantons. This gives

$$Z = \sum_{\mathrm{I}} e^{-S(\Phi_{0(I)})} \int \mathcal{D}\phi\, e^{-S_{(I)}(\phi)}\,, \tag{1.2.4}$$

where I labels the instantons and $S_{(I)}$ is the quadratical piece expanded as in (1.2.3) around the instanton $\Phi_{0(I)}$. Of course this is only a symbolic expression. In particular, in general instantons will come in moduli spaces; this will have two effects on (1.2.4). First, $\sum_{\mathrm{I}}$ will then be an integral; second, this flat direction signals zero modes in the Hessian $\delta^2 S/\delta\Phi^2$; these zero modes have to be excluded when one performs the quadratical integration in (1.2.4).

All of this is not specific to the strong coupling limit. The relative weight $e^{-S(\Phi_{0(I)})}$ tells us however that only instantons for which the action is finite at strong coupling will survive in the limit. The subclass for which we are sure that this happens is the one of BPS instantons, because they are almost by definition the ones that saturate a Bogomol'nyi bound for the evaluated action. Thus we will only consider BPS instantons in what follows.

Summarizing, the strategy is

- to find BPS instantons of MST action (1.1.1)

- to apply to them the steepest descent method just reviewed.

In the first step we will discover notable properties of the BPS instantons that will allow us to complete second step, and interpreting the quadratic piece in steepest descent method as the Green-Schwarz action of the IIA string.

1.2.1 BPS instantons

For use in later chapters, we take here a more general point of view, analyzing all the BPS solutions of maximally supersymmetric super Yang-Mills actions in any dimensions. We will neglect factors of coupling constants for the time being.

Let us start with a simple example. Consider super Yang-Mills in $d = 4$ with $\mathcal{N} = 4$. In this case, one can easily verify that the condition for preserving half of supersymmetry, if the transverse scalars X are set to their vacuum values, is a slight modification of the instanton equation $F = *F$. In complex coordinates z_1, z_2, this reads

$$F_{1\bar{1}} + F_{2\bar{2}} = c\ \mathrm{Id}, \qquad F_{12} = 0 \tag{1.2.5}$$

where $F_{ij} \equiv F_{z_i,z_j}$, $F_{i\bar{j}} \equiv F_{z_i,\bar{z}_j}$, and another equation $F_{\bar{1}\bar{2}} = 0$ follows from A being antihermitian. Let us now consider the T-duals of these equations. The general procedure is the same as that of dimensional reduction, and reads $D_i \to X_i$: a connection becomes an endomorphism (a matrix). More precisely, this rule, for T-duality, means that we are but rewriting the covariant derivatives as infinite dimensional matrices; then, the expression obtained in this way will be true also in the case in which matrices are finite dimensional. The outcome of this reasoning is the same as that of dimensional reduction. If T-duality is in the $z_2, \bar{z}_2$ directions, the result is

$$F_{1\bar{1}} + [X, X^\dagger] = c\ \mathrm{Id}, \qquad D_1 X = 0 \tag{1.2.6}$$

where we have called $X_2 \equiv X$. These equations can be obtained also by considering super Yang-Mills in $d = 2$ and with $\mathcal{N} = (8, 8)$, which is indeed a dimensional reduction, or T-dual, of $d = 4$, $\mathcal{N} = 4$, and looking for solutions which preserve half supersymmetry with one complex scalar turned on. For these reasons they have already been argued to be relevant in several physical situations [8, 7, 87, 37, 12, 13]; on the mathematical side, with a proper covariantization that we will see, they have been studied by Hitchin [47].

One can look for more general BPS conditions in super Yang-Mills theories starting instead from the $d = 10$, $\mathcal{N} = 1$ case, keeping all the complex scalars on:

$$F_{1\bar{1}} + \ldots + F_{5\bar{5}} = c\mathrm{Id}, \qquad F_{ij} = 0. \tag{1.2.7}$$

Again, reducing these to lower dimensions gives equations involving complex scalars in the adjoint: for instance in 4 dimensions one gets

$$\begin{aligned} &F_{1\bar{1}} + F_{2\bar{2}} + [X_1, X_1^\dagger] + [X_2, X_2^\dagger] + [X_3, X_3^\dagger] = c\ \mathrm{Id}; \qquad F_{12} = 0; \\ &D_i X = 0, \ \ i = 1, 2; \qquad [X_a, X_b] = 0, \ \ a, b = 1, 2, 3. \end{aligned} \tag{1.2.8}$$

These are a modification of the usual self-duality conditions (1.2.5) in 4 dimensions. In chapter 4 we will consider these equations again, along with their covariantization. For instance, the case *without scalars*, in any dimension, is the well-known HYM system of equations (2.2.9).

It seems from this analysis that our BPS equations are (1.2.8). We will later show, however, that we can always reduce these equations to the case with *one* scalar only, as in the Hitchin case (1.2.6). As a consequence, the BPS solutions we are going to use are solutions to these equations. Note that the complex coordinates $z \equiv z_1$ ($\bar{z} \equiv \bar{z}_1$) are there because we are considering solutions in the euclidean; this is the reason to call them instantons.

1.2.2 Solutions to Hitchin equations

On compact Riemann surfaces Hitchin equations are very well studied (starting from [47]). On non-compact cases like this (remember we are on the cylinder), we have to be more careful, but we will see that we can still use the main ideas of the compact case.

Introducing a scheme we will use more at large in the following chapters, we will find solutions to this system starting from the holomorphic part, for which we can use directly the powerful techniques of complex geometry, and move then to the more difficult real equations, for which we will rely on the techniques we have mentioned in the introduction as stability.

Holomorphic part

The holomorphic part is here the equation $D X = 0$. This has important consequences for the matrix X. Indeed, for its characteristic polynomial $p_X(x) \equiv \det(X - x\,\mathrm{Id})$ we have

$$\begin{aligned}\partial \det(X - x\,\mathrm{Id}) &= \det(X - x\,\mathrm{Id})\,\mathrm{tr}\left((X - x\,\mathrm{Id})^{-1}\partial X\right) = \\ &\quad \det(X - x\,\mathrm{Id})\,\mathrm{tr}\left((X - x\,\mathrm{Id})^{-1}[X, A]\right) = 0\end{aligned}$$

since $(X - x\,\mathrm{Id})$ is generically invertible.

This is important in view of the interpretation of X as describing motion in a transverse direction. This is familiar with branes, and for constant X. In that case, the vacuum $X = 0$ represents N (the rank of the matrix) strings wrapped on the same locus; whereas the vacuum $X = \mathrm{diag}(x^{(1)}, \ldots, x^{(N)})$ with $x^{(1)} \neq \ldots \neq x^{(N)}$ describes N branes, but displaced from one another. We can summarize that by saying that eigenvalues of X describe positions of the branes.

Here, the difference is that X describes displacements of fundamental strings and not of branes; and that X is no longer a constant. But this is not a problem: over each point w on the base cylinder again the eigenvalues are the transverse displacement; varying w the eigenvalues will in general vary, join one another and split. If the configuration was constant, we would have N cylinders; since it is not constant, we will have in general an open Riemann surface Σ, defining a *branched* (because on some point w_0 some sheets can meet) *covering* of the cylinder $\mathcal{C}$.

This can be made more precise. The base space is a cylinder $\mathcal{C}$; the Riemann surface is described by the zeroes of the characteristic polynomial $p_X(x, w)$, where we have now explicited the dependence on w (remember that "polynomial" for the time being refers to dependence on x). This is because, on each w, the zeroes of the characteristic polynomial are the eigenvalues. We have thus a holomorphic equation $p_X(x, w) = 0$, defined in $\mathcal{C} \times \mathbb{C}$.

At $\tau = \pm\infty$, we require a mild behaviour, in the following sense. We switch to a coordinate $z = e^w$, which allows us to think of the cylinder as a $\mathbb{C}^* \subset \mathbb{P}^1$; in these terms, $p_X(x, z)$ is now required to be meromorphic on $\mathbb{P}^1$, thus not to have essential singularities on the two added points $z = 0$ and $z = \infty$. This is because we would have otherwise, in the following, too wild scattering states, as we will see.

Since we are at it, we can try to extend all the data to $\mathbb{P}^1$; this will be useful in a moment. On $\mathbb{P}^1$, the characteristic polynomial should only be meromorphic, as we just said. A general principle in algebraic geometry is the one which allows meromorphic functions to be considered as holomorphic sections of line bundles. Then, we can describe everything a little more formally in the following way. X was considered, on the cylinder, as a section of $\mathrm{End}(E)$ (with E the trivial gauge bundle). We can now consider X it as a section of $\mathrm{End}(E) \otimes L$, with L a line bundle on $\mathbb{P}^1$. As we will see again, line bundles on complex projective spaces are classified by their first Chern number; the one with $c_1 = k$ is called $\mathcal{O}(k)$. If $L = \mathcal{O}(k)$, one can see that for consistency x is a section of $\mathcal{O}(k)$, the characteristic polynomial equation is a section of $\mathcal{O}(k\,N)$, and it describes a subvariety cut by a single equation in the total space of the bundle $\mathcal{O}(k)$. Also this total space can be compactified to a complex 4-dimensional compact space, as we will see.

The real equation

Having brilliantly solved the holomorphic equation, we turn to the real one, which we rewrite here as

$$g_s^2 F_{z\bar{z}} + [X, \bar{X}] = c\,, \tag{1.2.9}$$

where we have put the appropriate factor of g_s back. We will see more extensively in chapter 4 that equations of this kind admit often a formulation in terms of a property called stability. There, we will explain the origin of this condition, proving that the latter is necessary in order to have solutions of Hitchin equations and of its generalizations we have seen in subsection (1.2.1). We will rely on the literature for sufficiency.

What we will do here is to simply formulate it, to show that we can find a solution of our equation, and sketch the method by which a solution can be really found, taking it from the general theory.

The condition of stability is on the couple (E, X), where E is the gauge bundle on $\mathbb{P}^1$ and X is a section of $\mathrm{End}(E) \otimes L$.[1] The couple is stable if, *for any subbundle E' which is X-invariant, the inequality $\mu(E') < \mu(E)$ holds*, where $\mu = c_1/rk$. Call now the stable couple (E, M); a solution is guaranteed to be found in the form

$$A = g^{-1}(\partial + A_0)g\,, \qquad X = g^{-1}Mg\,, \tag{1.2.10}$$

where A_0 is a connection on E, and g is a section of $\mathrm{End}(E)$ (better, a complex automorphism of E). This means that, starting from A_0 and M, and considering the orbit of the complexified gauge group, one will find sooner or later a solution of Hitchin equation. The moment map for the action of the non-complexified gauge group is precisely the equation itself.

We will now show that for any covering Σ we can find a solution of Hitchin equation with that assigned covering. First of all, one has to observe that under gauge transformations (1.2.10) the characteristic polynomial does not change, and so neither does Σ. Then, what we can do is to choose an appropriate starting point M in such a way that the couple (E, M) is stable. Let us try with the following simple choice:

$$M = \begin{pmatrix} -a_{N-1} & -a_{N_2} & \cdots & \cdots & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{1.2.11}$$

The a_i are by construction coefficients of the characteristic polynomial, $P_M(x) = (x^N + a_{N-1}\,x^{N-1} + \ldots + a_0$. For consistency, since x is, as we said above, a section of L, a_i is a section of L^{N-i}. This determines almost all of the bundle E. Indeed, one knows by Birkhoff theorem that all vector bundles on $\mathbb{P}^1$ are direct sums of line bundles, $L_1 \oplus \ldots \oplus L_N$. Then, the element M_{ij} of this matrix is a section of $L \otimes L_j^* \otimes L_i$. Comparing this with the bundles of which the a_i are actually sections, we obtain

$$L_k^* L_{k-1} = L$$

[1]Note that we do not need, as it was assumed in [47], that $L = K$ the canonical bundle. This requires then a little care in covariantizing, see chapter 4.

(from now on we omit $\otimes$ for line bundles, for clarity of formulas). This means that

$$E = L_1 \oplus L_1\, L^{-1} \oplus \ldots \oplus L_1\, L^{N-1} \qquad (1.2.12)$$

If the a_i are all zero, the only invariant subbundle is easily seen to be the last line bundle of the sum, $L_N = L_1\, L^{N-1}$. We can check then if the condition $\mu(L_N) < \mu(E)$ holds: we have

$$c_1(L_1) - (N-1)c_1(L) < \frac{1}{N}\left(N c_1(L_1) - \frac{N(N-1)}{2} c_1(L)\right)$$

from which one see that one should require $c_1(L) > 0$. Let us anticipate that fortunately, this coincides with the condition for a line bundle to have holomorphic sections (which then we can regard as meromorphic functions); had we obtained for instance the opposite sign, there would have been no possible a_i (but zero).

Now, the couple (E, M) is stable for $a_i = 0$; one concludes by observing [44] that stability is an open condition, and using again a particular action of the gauge group. This allows us to say that (E, M) as given by (1.2.12,1.2.11), and supplemented by the condition that L is positive, is always a stable pair.

We now know there is a g which takes this A_0 and M to a solution of Hitchin equation as in (1.2.10). Explicitly, the equation g should satisfy turns out to be (with $\Omega \equiv g\, g^\dagger$)

$$g_s^2 \partial_z(\bar{\partial}_z \Omega^{-1}\Omega) + [M, \Omega\, M^\dagger \Omega^{-1}] = 0$$

which looks like a WZNW equation; in fact it is a *reduction* of it. Stability implies that precisely this equation has a solution.

We started from the cylinder $\mathbb{C}^*$ and then extended data on $\mathbb{P}^1$; a possibly more orthodox way could be to use the more systematic theory of these extensions from noncompact to compact curves [74].

We have now solved both our equations. In principle, it would seem that the solutions have a large moduli space of solutions, similar to the Hitchin one (recall however that our equation is not exactly that one, being L different from K, see previous footnote). However, one thing we still have to do is not present in the mathematical literature: the strong (gauge) coupling limit (weak string coupling) of these solutions (after all this is the limit we are interested in checking). This has been done in [12] with subtle techniques, and it gives the result that *the only surviving moduli in the limit are the ones coming from the covering.*

1.3 Lifting the action and the moduli problem

We have thus set the stage for the steepest descent method, in the sense that we know now that the moduli space of instantons (the label (I) in (1.2.4) is the moduli space of coverings of the cylinder. As we said, we do not evaluate the remaining piece, but we transform it in a meaningful way.

The idea is to take the abelian part of the matrix action (1.1.1) expanded around a given instanton, and *lift it* to the corresponding covering Σ. Intuitively, this means the following. As we have said, the eigenvalues of X define the covering Σ. All the other fields can be divided into a part which commutes with X, and is thus diagonalizable simultaneously with it, and the rest. We call first part *Cartan modes* and second *non-Cartan.*

Now, imagine to do a trip on the cylinder around a branching point of the cover. In the branching point some eigenvalues will coincide; in the trip around that point, one can see that exactly those

eigenvalues will get permuted among themselves; this is called a *monodromy*. The Cartan modes of all the fields will get, by definition, the same monodromy.
Let us now call ϕ_A the eigenvalues of the field ϕ. On a generic point w of the cylinder we have N distinct of them, and we can imagine to assign each of the eigenvalue to a point on the covering. What will happen when we vary the point w? thanks to the above observation that the ϕ_A get the same monodromy as the eigenvalues x_A of X (which define the covering),

$$\phi_A \to U_{AB}\phi_B \ , \qquad x_A \to U_{AB}x_B \ ,$$

the assignment will be consistent all over the cylinder. This is tantamount to say that we have defined a function on the covering, or that we have *lifted* the field.

We can describe again more formally this intuitive procedure. The definition of the covering space was already formalized in previous section, through the eigenvalues; we will now give also an interpretation to the *eigenvectors*. If indeed we assign to each sheet the eigenvector v_A corresponding to that eigenvalue x_A, what we are defining on the cover is a line bundle $\mathcal{L}^2$. Take then any section ϕ of End E which commutes with X. Since ϕ and X are simultaneously diagonalizable, they have the same eigenvectors v_A; by its action

$$\phi v_A = \phi_A v_A \ ,$$

ϕ defines (since v_A is the line of the line bundle $\mathcal{L}$ over the point x_A) a morphism of line bundles from $\mathcal{L}$ to itself. This is a function, and is the lifting of the field ϕ we defined intuitively above. The main value of this formalization is that it allows things to be defined also in non-generic situations, in which the line bundle degenerates to a sheaf.

At this point, we can also give the argument we promised to reduce to the case with one scalar only, (1.2.6). The argument is this. If we consider more than one transverse scalars, again because of the equation $DX_a = 0$, all of them will have an antiholomorphic characteristic polynomial, $\partial \det(X_a - x_a) = 0$. But last equations in (1.2.8) implies, in the generic case in which the eigenvalues are not coincident, that the matrices are simultaneously diagonalizable. This in turn implies that the monodromies of the eigenvalues are the same, and thus that the characteristic polynomial are the same [13]. So there is only one covering anyway. Moreover, if the eigenspaces have generically dimension 1 (which was our assumption anyway in the above), they are all proportional to one another; there is then a redefinition that leaves again only one X in the equations.

Let us come back to the main course of the reasoning. The lifting is the main "dynamical" step in [12]. All the fields are separated into Cartan and non-Cartan modes. The part of the integral in (1.2.4) containing the latter is performed, and it is shown not to influence the rest (the Gaussian integrals of bosons and fermions exactly cancel; the deep reason of this is in supersymmetry). The rest of the path integral contains now Cartan modes, and can be lifted as we sketched above. The result is shown to consist of two parts. A first piece is what one looks for, GS action for IIA string. A second piece is a totally decoupled piece whose role is to reproduce correctly the factor $g_s^{-\chi}$ which is originally present in the string S-matrix.

The role of the "sum" over the labels (I) in (1.2.4) (the integral over the moduli space of instantons) is now to provide the integral over Riemann surfaces in the S-matrix. This is last step in the proof, and it is what we will consider now.

[2] We are supposing here to be in the generic situation, in which on a non-branching point the eigenspaces have dimension 1. We will also neglect pathological problems which may happen on the branching locus; again, generically everything works.

1.3.1 The moduli problem

The thing we will do here is to parameterize in some way the moduli space of the coverings, showing in which sense we can recover all the moduli.

Let us first describe a little more carefully the coverings. We have not emphasized so far what happens at the two punctures 0 and ∞. For instance, 0 is the infinite past $\tau \to -\infty$. Its counterimages on the covering are thus marked (or excluded) points on the Riemann surface, which describe the incoming strings. If 0 is not a branch point, the counterimages will be N; if it is a branch point, there will be less. This is important, otherwise we could only describe scatterings of N strings into themselves. We can describe what is happening thinking about a small loop around 0. The counterimages of this loop on the covering Σ represent the strings at some large and negative, but finite, time. If 0 is not a branch point, these are N loops, not surprisingly. If it is a branch point, as we said above the eigenvalue will get reshuffled among themselves by a monodromy after the loop. This means that some of these small circles will get united among themselves to form a longer incoming string. Note that this longer string will have a larger unit of momentum, recalling the proportionality between N and momentum p_9 (see figure 1.1).

Consider now the meromorphic function z on the $\mathbb{P}^1$ base. There is a projection $\pi : \Sigma \to \mathbb{P}^1$, and we can use this projection to pull back z to a meromorphic function $\pi^* z$ on Σ. This is obviously not the same as the lifting procedure which we have described above; a related comment is that $\mathcal{L}$ is *not* the pull-back of E (the ranks do not match).
This meromorphic function has – by definition – zeroes exactly in the points Z_i of Σ representing incoming strings, and poles in the points P_i representing outgoing strings (the counterimages of 0 and ∞ respectively).

This has a crucial consequence. To see it, let us recall briefly the theory of

Jacobians and the Abel map.

We had occasion to note that, on projective spaces, holomorphic line bundles are classified by their first Chern class. This is not true on higher genus Riemann surface, to an extent measured by the theory of jacobians. The jacobian is defined as a complex torus obtained from $H^1(\Sigma, \mathcal{O})^*$ (the space of linear functionals on the vector space $H^1(\Sigma, \mathcal{O})$) quotienting by the lattice Λ of functionals of the form $\int_\gamma$, with γ a closed cycle; thus

$$J(\Sigma) = \frac{H^1(\Sigma, \mathcal{O})}{H^1(\Sigma, \mathbb{Z})} .$$

This torus has dimension g, the genus of Σ, and can also be reformulated in a more concrete way as the quotient of $\mathbb{C}^g$ by the lattice generated by the periods (the integrals of the g holomorphic one forms over the periods).

Given a divisor D of total degree 0, we can associate a line bundle of zero c_1 to it. The question is now whether this gives or not the trivial line bundle; in this case the divisor is called *principal*. On projective spaces, as we said, the answer would be "yes". Here, the answer is given by so-called *Abel theorem*: $D = \sum_i Z_i - \sum_i P_i$ is principal if and only if its image under the *Jacobi map*

$$\mu(D) = (\sum_i \int_{Z_i}^{P_i} \omega_1, \ldots, \sum_i \int_{Z_i}^{P_i} \omega_g) \in J(\Sigma)$$

is zero (i.e. it belongs to the lattice Λ).

Call now P the polar divisor of $\pi^* z$, that is, $\sum_i Z_i - \sum_j P_j$. This divisor is principal, as it comes from a meromorphic function. But then, we have just seen that its image in $J(\Sigma)$ under the Abel map μ is zero. Since $J(\Sigma)$ has dimension g (the genus of Σ), this sets g constraints on the moduli space and would seem to reduce very powerfully the possible scattering problems we can describe with MST.

As an example, suppose we want to describe a one-loop scattering of two strings. The arguments tells us that, once we have fixed the incoming positions Z_1, Z_2, and one of the outgoing position P_1, last one is determined by the condition

$$\mu(Z_1 + Z_2 - P_1 - P_2) \equiv \sum_i \int_{Z_i}^{P_i} \omega = 0 \text{ in } J(\Sigma)$$

that the total polar divisor be principal (as are sometimes called divisors coming from meromorphic functions, and thus whose image in the jacobian is zero).

The way out of this is again in the branching mechanism. In our example, choose now a covering with N sheets such that the Riemann surface is again the same, and such that the three chosen positions are now branched points of order $N/2$ each. Then the polar divisor is exactly $N/2$ times what it was before, and the condition becomes

$$\frac{N}{2}\mu(Z_1 + Z_2 - P_1 - P_2) = 0 \text{ in } J(\Sigma) ;$$

being 0 in $J(\Sigma)$ means to belong to the lattice Λ, and we see that the condition is now that $\mu(Z_1 + Z_2 - P_1 - P_2)$ is a point of order $N/2$ in this torus. Points of a given order $N/2$ are themselves a lattice with narrower spacing:

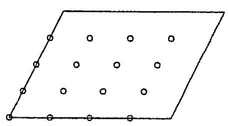


Figure 1.2: Points of given order in the Jacobian.

So, for example, the point P_2 which before was uniquely fixed, now has $(N/2)^2$ possibilities. As $N \to \infty$, this outgoing point can be put practically everywhere, with a small-at-will perturbation. The strategy is similar in the general case. For any process, we will have to use large N coverings, branched in the incoming points accordingly to the momentum p_9 we want to assign to each of them. When $N \to \infty$, the allowed points will form a narrower and narrower sublattice of the jacobian, eventually filling it and effectively making the constraint of Abel theorem ineffective.

Having solved this first problem, we feel now more confident that we can recover the whole of the moduli space. To check this, however, we have to analyze it more concretely, in particular recognizing the features of each Riemann surface Σ (the genus, the position of the punctures) from the bare characteristic polynomial (which is way Σ comes out of MST).

To do so, an effective way is to use *Newton diagrams*. In the rest of this section, we will first describe generalities about these objects and the information one can extract out of them. Then, we will use this to the discussion of the moduli problem.

1.3.2 Newton diagrams.

Let us consider a characteristic polynomial $p(x, z) = 0$. We associate to each monomial $x^\alpha y^\beta$ in it a point $p = \alpha$, $q = \beta$ in a p, q plane. We obtain a set of points called the *carrier*: its convex hull is by definition the Newton polygon associated to the curve. In figure 1.3 we have displayed the Newton polygon for the polynomial $\alpha_1 x + \alpha_2 x^3 z + \alpha_3 x^2 z + \alpha_4 x z^2 + \alpha_5 z^2$. We will see how we can deduce from this diagrams

- The structure of the singularities at 0 and ∞
- The genus of Σ.

The first task is somewhat easier. Let us begin by the singularities in the point $x = 0, z = 0$. The monomial which are important for the local structure of the singularity are the ones which lie near the origin – in the example of figure 1.3, the two dots corresponding to the points $(2, 0)$ and $(0, 1)$. Locally, this is the polynomial $\alpha_1 x + \alpha_5 z^2 = 0$. This is not singular, as there is a linear piece. Thus, we see that we have an easy graphical criterion to see whether there is a singularity at $(0, 0)$: the point is singular if neither of the dots $(1, 0)$ and $(0, 1)$ is occupied. (If the dot $(0, 0)$ is itself occupied, the point $x = 0 = z$ does not belong to Σ.) In fact, we will see in a moment that there is much more information one can extract from this.

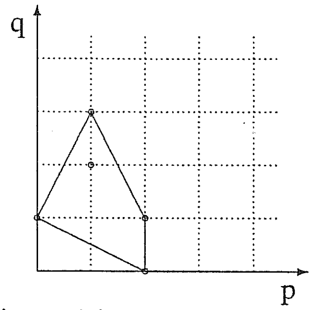


Figure 1.3: An example of Newton polygon.

The structure of singularities in other points over $z = 0$ and $z = \infty$ depends on the way we compactify the noncompact space $\mathbb{C}^* \times \mathbb{C}$ in which for the time being our open Riemann surface is defined (by the equation $p = 0$). Earlier we have already compactified $\mathbb{C}^*$ to $\mathbb{P}^1$, but now we mean to compactify the whole $\mathbb{C}^* \times \mathbb{C}$, coordinatized by z and x. There are many ways to do that. Two immediate proposals are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$; the first because plane curves are much more studied in that case, the second because it seems perhaps more natural.

Let us begin with the first. In that case, coordinates transform in the well-known way: $z' = 1/z$, $x' = x/z$. To look at the structure around for instance one of the points at infinity, one has to use this transformation rule to rewrite the polynomial in terms of the new coordinates x' and z'.
The result has fortunately a graphical interpretation that we sketch in figure 1.4. Consider homogeneous coordinates for $\mathbb{P}^2$, w_0, w_1, w_2, connected to z and x as $z = w_1/w_0$ and $x = w_2/w_0$. Then, starting from the diagram, one has to do the following thing: Draw oblique lines as in figure 1.4, starting from the dots; these oblique lines become then the vertical lines of the new diagram; moreover, perform a horizontal reflection. The result is the singularity structure around the point $(0, 1, 0)$ in homogeneous coordinates. An analogous graphical procedure holds for the singularity around the point $(0, 0, 1)$.

If one is instead to find similar things in the second case, the one in which we choose to compactify $\mathbb{C}^* \times \mathbb{C}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, it is easier to see that the graphical procedure is now to perform simple $\pi/2$ rotations, reflections and translations of the diagram.

We can summarize these graphical procedures in this way. In the first case, one draws an external triangle ◺ touching the Newton polygon and looks at the closer points not from the origin, but from the other vertices of the triangle; in the second case, the same procedure holds replacing the external triangle with an external square □.

The reader who knows toric geometry will no doubt recognize the *dual polyhedra* associated to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Toric geometry deals with manifolds which are acted on by $(\mathbb{C}^*)^d$, and which contain a dense orbit of it. Without going too much into details, we note that these polyhedra encode exactly the data of the transition functions between different open sets, which is exactly what we have used here to transform the diagrams; so this coincidence is not surprising. There are many other ways to compactify $\mathbb{C}^* \times \mathbb{C}$, and that for each of them there is a procedure like the one we have just described, with an external polygon. The occurrence of toric geometry will not stop here.

Come back indeed to the structure of singularities. Consider again the singularity around the point $(0,1,0)$. From figure 1.4 we can see it is the singularity (in the appropriate primed coordinates)

$$\alpha_2 x^3 + \alpha_4 xz + \alpha_5 z^2 = 0 \,. \tag{1.3.13}$$

This time we really have a singularity; again, we can formulate now a criterion for singularity around a corner of the polygon: this says that the point is singular only if on the two sides meeting at that vertex neither of the two nearby points is occupied. In this example, the relevant vertex is point $(p,q) = (4,0)$ in the original diagram in figure 1.3; the nearby points are at $(p,q) = (3,0)$ and $(3,1)$.

But we can go further. One can turn now to the main question: what is the genus of a curve corresponding to a given polynomial?

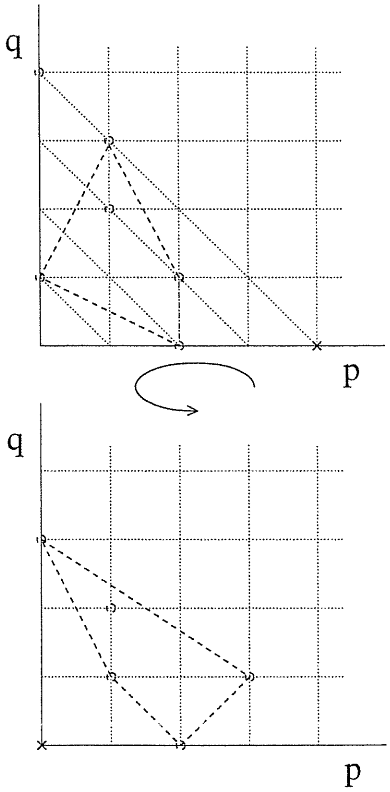


Figure 1.4: Transformation of Newton diagrams

Naively, there are lots of different answers to this. Indeed, the usual formula one learns at school for the genus of a Riemann surface in $\mathbb{P}^2$ is

$$g = \frac{(d-1)(d-2)}{2} \tag{1.3.14}$$

where d is the degree of the polynomial once one expresses it in the homogeneous coordinates. In the usual example, this degree is 3, and the genus would be 6.
However, if we compactify in $\mathbb{P}^1 \times \mathbb{P}^1$ instead, the formula can be again computed by adjunction formula, and the genus turns out to be 4. If we compactify in other ways, it may turn out to be even different numbers.

We have thus two things which depend on the compactification: the genus and the structure of singularities. It turns out that these two dependences *exactly cancel each other*, in a way which we will now explain.

The point is that the genus we have just computed is not quite what we want. These surfaces described by these polynomials have singularities, even with generic choices of the coefficients α. One may think that singular Riemann surfaces have to be excluded altogether; we will argue later that this is not possible, and even that they are a very important part of the moduli space. For the time being, let us only try to reason mathematically about possible resolutions of these singularities. A resolution, as we will also meet in later chapters (for instance in section 3.3), is essentially another manifold which is non-singular and which is the same (for algebraic geometry) as the original one outside the singular locus. There is then a definition of *geometrical genus* for singular Riemann surfaces Σ which is defined as the genus of the resolution $\tilde{\Sigma}$. This is by definition *birationally invariant*, namely invariant by resolutions and degenerations, and seems thus a little more intrinsic that the formal genus considered so far.

If we are to compute this genus, we have now to subtract from the formal genus coming from adjunction formula (for instance (1.3.14)) *a contribution for each singularity* [15]. The virtue of Newton diagrams is that we can spare ourselves the hassle of this computation, as we now describe. For each singularity, the formula instructs us to *blow it up* till it is non-singular (this can always be done). Blow up is the most popular way to resolve singularities, and works as follows.
Take a singularity embedded in $\mathbb{C}^2$. One defines a non-compact manifold $\tilde{\mathbb{C}}^2$ as the set of points in $\mathbb{C}^2 \times \mathbb{P}^1$ which satisfy

$$x\, v_1 = z\, v_0 \,, \tag{1.3.15}$$

where x, z are coordinates in $\mathbb{C}^2$ and v_0, v_1 are homogeneous coordinates in $\mathbb{P}^1$. This space is roughly a space in which the origin of $\mathbb{C}^2$ was replaces by a two sphere $\mathbb{P}^1$ (called *exceptional divisor*). It can also be described as the total space of the bundle $\mathcal{O}^1_{\mathbb{P}}(-1)$.
The idea is that, if one has a singularity embedded in $\mathbb{C}^2$, this will be "less singular" in $\tilde{\mathbb{C}}^2$. In fact, this is true, in the sense that blowing up several times always resolves the singularity after a finite number of steps. Let us see how this works in an example, again the singularity 1.3.13 displayed in the lower diagram of 1.4 around the point $x = 0 = z$. Blowing up, we have said that a $\mathbb{P}^1$ at the origin is introduced. Around a point of this $\mathbb{P}^1$, we have two coordinates: a coordinate of $\mathbb{P}^1$, and one from the transverse $\mathbb{C}^2$. For instance, let us consider the point $v_0 = 1$, $z = 0$. Around such a point one can consider the coordinates x_b and z_b, where $x_b = x$, and z_b is instead a true (non homogeneous) coordinate in the $\mathbb{P}^1$: $z_b = v_1/v_0$. By (1.3.15), we have

$$z = z_b\, x_b\ , \qquad x_b = x\ ;$$

we can now compute what is the new form of the singularity (1.3.13) in this coordinates:

$$\alpha_2\, x_b^3 + \alpha_4\, x_b^2\, z_b + \alpha_5\, x_b^2\, z_b^2 = 0\ ;$$

this, factorizing the spurious component $x_b^2 = 0$, which describes two times the exceptional divisor $\mathbb{P}^1$, becomes the innocuous

$$\alpha_2 x_b + \alpha_4 z_b + \alpha_5\, z_b^2 = 0\ ,$$

which is of course non-singular. So, in this case blowing-up was effective at first try.

We may try to give again a graphical interpretation to this. In general, the form of the singularity after a blow up can be read off from the following graphical procedure: one draws again an oblique line touching the diagram, and one then makes the oblique line of the new axes. Note that this is again rediscovering toric geometry: the effect of a blow-up on the dual polyhedron of a toric manifold is exactly a "cut" of the polyhedron by a line, as we just described.

Let us now come back to the computation of the genus. The contribution coming from each singularity, as we said, is in turn a sum of contributions coming from each blow-up. This number is $k(k-1)/2$, where k is the *multiplicity* of the singularity before the blow-up (the smallest degree of monomial appearing in it). In a moment we will interpret this formula and we will forget it. Note indeed that this has the form of the number of points inside a triangle: this triangle is the one formed by our cut of before. In our example, our cut is as follows:

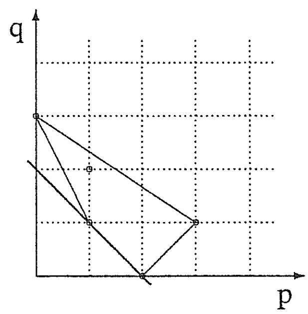


Figure 1.5: Cut of Newton polygon.

If we count also the $(1, 1)$ which is hit by the cut, we have 1, which is exactly the contribution prescribed by $k(k-1)/2$ in this case. This example may not be too convincing, but this is true in general. Even better, imagine that after a first blow-up, the singularity is still singular; we will have to resolve again, making another cut. The final result will be then that *the (negative) contribution of a singularity to the genus is given by the number of points which is contained between the axes around the singular point and the Newton diagram*. In this count, points on the axes are not valid, but points on the border of the Newton polygon are. Actually, we do not even have to transform first the diagram in such a way that the singular point is in the origin, as we did in figure 1.4; we can directly make this count around *the vertex corresponding to the singular point*; the "axes" of the above criterion are then the sides of the polygon coming out of the vertex.

What about the genus? going back to the expression (1.3.14) for the genus, we can see that also this seems to have an interpretation as number of points inside a triangle. We have a triangle in that case: the dual polyhedron to $\mathbb{P}^2$. Indeed, the number in (1.3.14) can be interpreted as the number of points inside the external triangle touching the Newton polygon we introduced before. This is also true for other compactifications! *Formal genus is the number of points which are internal to the polyhedron touching the Newton diagram externally, whatever the compact manifold chosen to compactify* $\mathbb{C}^* \times \mathbb{C}$.

We can compare now these two expressions to find the geometrical genus (the genus of the resolution, also called normalization). This is by now only a subtraction of points: the ones in the external polygon, minus the one between the external polygon and the Newton diagram. Thus the final result is that *the geometrical genus is given by the number of points internal to the Newton diagram.* Here points on the border are not valid. This remarkable results can be obtained also by more direct means [32], but it was amusing to find it in this way. This path made us also ready to appreciate the role of singularities, which we are now going to discuss, and the virtue of the geometrical genus. Note indeed that now this final result is independent of the way we compactify $\mathbb{C}^* \times \mathbb{C}$.

One word of caution is in order: this method only works if the coefficients α in this polynomial are generic. For special choices, internal singularities may appear, and one would have to compute the contribution of those singularities as well. This will have an important role in the following discussion. Our favorite example will have thus genus 2 for generic choices of the coefficients, but for special values this may sink due to other singularities.

1.3.3 The moduli problem: reprise

We have now all the information to answer the question: can we really reproduce all of the moduli space of punctured Riemann surfaces? The objects we have are polynomials $p(x, z)$, and we just learned how to compute the features of the associated open Riemann surface. In examples which follow, a way to understand the branching structure will become clear as well.

First of all, can we obtain all the possible genera with all the possible branching structures? If we used formula (1.3.14) for the genus, corresponding to $\mathbb{P}^2$, we could become worried: not all the genera are realized. Indeed, for increasing degrees, one gets genera 1,3,6,10,... Since we argued that we have to use geometrical genus, one could think we are safe; for instance, the example of previous subsection gives an illustration of genus 2. However, there are two comments about this. First, why is it right to use this genus after all? the argument we have given before is that it is the definition which is independent on compactifications; but this means one has to resolve singularities, as we said; what is the physical meaning of this? Second comment comes when, even accepting this definition, one tries to realize low genera with high N. Since N is the highest power in x of the polynomial, graphically it can be described as the height of the Newton polygon. The reader is invited to try to realize for instance genus 1 for high N; one can convince oneself that this is impossible.

First problem is solved by understanding the role of singularities. One has to remember that the instanton represents the Riemann surface *embedded in four dimensions*, whereas the physical Riemann surface will get embedded in the 10 dimensions of string theory. What achieves this are the fluctuations of the other coordinates X; we reduced ourselves to the case in which only one X was on, but this is only possible for the instanton. Fluctuations in the theory will then desingularize the Riemann surface in 10 dimensions, and this is the reason for which it is correct to consider the genus of the desingularization, which is what we have done here. The fact that singularities are unavoidable can also be realized taking the reverse point of view. Suppose we want to embed these higher (than four) dimensional processes within the instantons of the 2d field theory. The only possibility is to

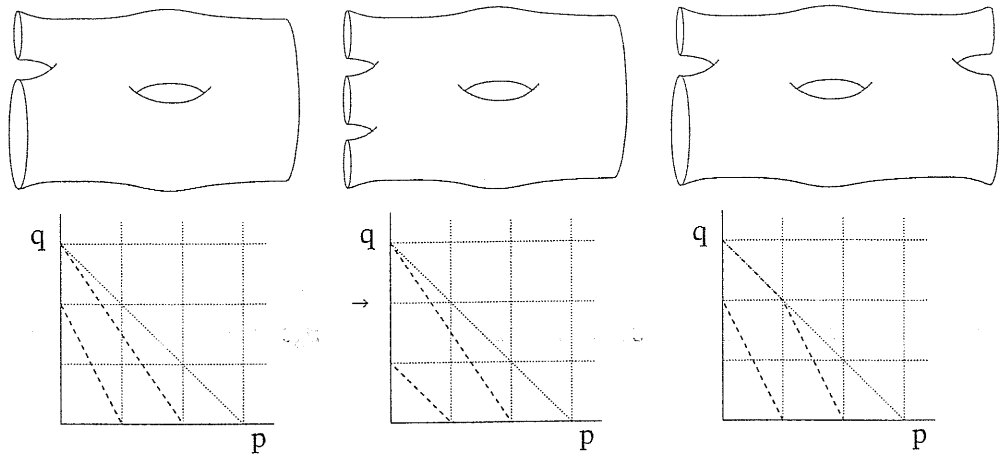


Figure 1.7: Examples of processes and their Newton diagrams.

squeeze (project) them to the appropriate four dimensions: such operation of projecting gives rise to singularities. It goes without saying that the true significance of singular plane curves is given by their representing higher (than four) dimensional processes.

Once we realize this, also the second problem can be easily overcome. Indeed, if we accepted to desingularize the singularities, we can also allow singularities at the interior, namely, singularities obtained with a careful choice of the coefficients. How this can be done will be shown shortly in some examples.

Example: smooth elliptic curve.

We start with the case $N = 3, g = 1$, for which there is already a good variety of examples. These have the advantage that one can check the results by explicitly solving the cubic algebraic equation by means of Cardano's formula. We do not write down the algebraic equations, but simply the corresponding polygons. The coefficient of the monomials within or on the border of the Newton polygon are understood to be generic, unless otherwise specified.

The simplest process one can imagine is the string self-energy. This means that we have to look for a totally branched curve over $z = 0$ and $z = \infty$. Remember that the polynomials giving the solutions over these points are given by the points of the carrier on the $q = 0$ and $p + q = N$ lines respectively. So one simple solution is given by the carrier shown in figure 1.6; the generic case will be non-singular also at finite z and so the genus will be one. The presence of the points $(1, 0)$ and $(2, 0)$ ensures the non-singularity of 0 and ∞; the local behaviour around them is given by the upper side of the inner (shaded) triangle.

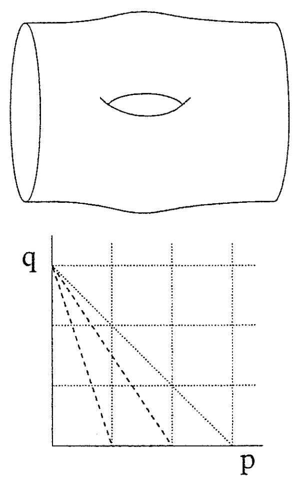


Figure 1.6: Self-energy of a string.

Next we would like to describe a joining of strings. In this case we keep $z = \infty$ totally branched, while we add a point on the q axis in order to have, at 0, a polynomial like $y^3 + y^2$ instead of y^3, so that $y = 0$ appears twice as a solution and $y = -1$ once. It is now easy to construct all combinations: figure 1.7 shows various examples and their Newton polygons — as above, with generic coefficients.

Example: singular plane curves

Of course for some choice of the parameters a singularity may appear. In this case one has simply to replace the finite hole in these figures by a hole shrunk to a point (for example see fig. 1.8); the curve becomes genus zero, i.e. a sphere with two identified points. This singularity is the simplest one, it is characterized by a non-vanishing Hessian and is called a *node*. All nodes can be viewed as two points identified: blowing up a node amounts to separating the points. For instance, consider our first case (figure 1.6): the polynomial which corresponds to the diagram can be written as

$$P = y^3 + czy + z(z - a) . \tag{1.3.16}$$

Imposing that a point be singular, one finds that a necessary (but not sufficient) condition is that its discriminant, $\delta = z^2[27(z - a)^2 - 4c^3 z]$, has a multiple root. The double root at $z = 0$ just signals that this point is another triple branch point, as we already knew; imposing that the remaining factor be a square, one finds several values, of which for instance $c = 0$ gives a triple branch at $z = a$ and no singular point, and $c = -3\,a^{1/3}$ gives instead a node.

This introduces us to our next task: to show how it is possible to describe low-genus highly branched curves. We will describe in detail the self-energy case. We take $N = 4$; since we want total branching we can choose a diagram like that in figure 1.8. The corresponding polynomial has the coefficients corresponding to the vertices of the polygon, and can also have coefficients corresponding to the points on the sides or in the interior (by the way the latter are always $(N - 1)(N - 2)/2$ in number if there is no singular point at 0 and at ∞ and count the genus of the corresponding smooth curve). Now we can look for singular cases in this family along the lines of the previous example; since already in this case computations become complicated, we restrict ourselves to the biquadratic case. In other words the polynomial we start with is

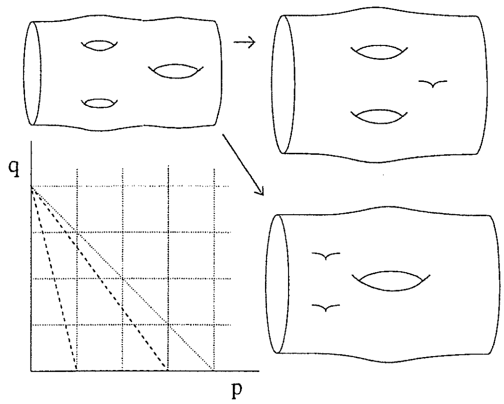


Figure 1.8: Shrinking cycles: totally branched quartics at genus two and one.

$$P = y^4 + bzy^2 + z(d + ez + fz^2) ; \tag{1.3.17}$$

its discriminant is

$$\delta = 16z^3(d + ez + fz^2)(4d + 4ez + 4fz^2 - b^2 z)^2 . \tag{1.3.18}$$

As before, the term z^3 shows that the branching at $z = 0$ is of order three, i.e. four sheets meet there. The other two terms mean the following. Solutions of a biquadratic equation are in general $\pm y_{1,2}$. Its discriminant can vanish in two cases: if $y_1 = y_2$ or $y_1 = -y_2$ — this is determined by the third term in (1.3.18) — in which case, at the corresponding value of z, there is a couple of double branch points; if $y_1 = 0$ or $y_2 = 0$, which is determined by the second term, there is a single node. If we choose the coefficients so that the third term is a fourth power, we have two nodes, and so genus one; if, instead, the coefficients are chosen so that the second term is a square, we have a single node, and so genus two. The situation is shown in figure 1.8.

If one does not wish to restrict to this particular case, one can still find examples of genus 1 and 2 curves. One notes, for instance, that imposing a node first and a total branch in $z = 0$ afterwards is computationally easier. A totally branched, non biquadratic genus one quartic is for instance given by

$$P = y^4 - 5zy^2 + 3z(z+1)y - \frac{z}{2}(z+1)^2\,, \tag{1.3.19}$$

which has two nodes at $\{z = -1,\, y = 0\}$ and $\{z = 1,\, y = 1\}$, and two regular branch points: $\delta = z^3(z+1)^2(z-1)^2(63z^2 + 62z + 63)$.

As a conclusion, all the moduli can be recovered, if one carefully considers desingularization of singularities. A careful analysis of this moduli problem beyond these examples will not be done here; we just wanted to show this is possible in principle.

Chapter 2

Branes on Calabi-Yau manifolds

One of the most interesting parts of the moduli space of string theory is the branch in which the closed string background is a Calabi-Yau manifold; not only because of expected applications to 4 dimensional physics, but also because of its mathematical richness. It is therefore interesting to analyze the open string part of this problem, again both because of physical reasons (branes might eventually be an important part of a successful string model for 4d physics, as for instance the idea of brane worlds has suggested) and because of expected mathematical richness; the latter can in turn fall back into the realm of physics by enhancing the understanding of the nonperturbative structure of string theory.

This chapter serves as a short introduction to these problems. After reviewing briefly the closed string setting, we will describe BPS branes in a limit, intuitively when the Calabi-Yau is large. In this limit we will note that BPS branes fall in two classes, and that the conditions they have to satisfy, as mentioned in the introduction, are divided into a class of holomorphic conditions and one of real conditions. As we try to tackle the much more difficult and interesting problem of extending the analysis away from the large volume limit used at the beginning, we will see that these features keep on being sensible and useful.

2.1 Closed string basics

Two points of view are in a close interplay in string theory. In the first, the *worldsheet* one, the focus is on the 2 dimensional CFT which defines the perturbative theory. In the second, called *target space*, one looks instead at the low-energy effective action (LEEA), where "low-energy" is meant with respect to l_s. This is less fundamental from the perturbative point of view, but this distinction often disappears in the nonperturbative theory. When we compactify string theory to 4 dimensions, we are in any case interested in the low-energy field theory that a 4 dimensional observer would see; "low-energy" means now with respect to the compactification scale. We will refer to this as to the effective 4 dimensional theory.

Now, a perturbative closed string theory is a 2 dimensional supersymmetric CFT. From this point of view, compactification means to consider a CFT which is a product of a simple free field theory describing propagation of strings in 4 dimensional spacetime, and of a second (call it "internal") CFT. A typical example for the latter is a nonlinear sigma model with some target manifold X. This has as fields a map x from the worldsheet Σ to the manifold X, and a "worldsheet fermion with values in the tangent bundle of X", that is a section ψ ($\tilde{\psi}$) of the bundle $K^{1/2} \otimes x^*(T_X)$ ($\bar{K}^{1/2} \otimes x^*(T_X)$); in terms of the components x^μ and ψ^μ of these two objects, one can write the action as

$$S = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z \left[\frac{1}{2}g_{\mu\nu}(x)\left(\partial_z x^\mu \partial_{\bar z} x^\nu + \psi D_{\bar z}\psi + \tilde\psi D_z \tilde\psi\right) + \frac{1}{4}R_{\mu\nu\rho\sigma}\psi^\mu\psi^\nu\tilde\psi^\rho\tilde\psi^\sigma\right] \tag{2.1.1}$$

Consistency of string theory severely constrains this manifold: absence of conformal anomaly implies $\dim(X) = 6$ and $c_1(X) = 0$[1]; if one wants, moreover, spacetime $\mathcal{N} = 1$ supersymmetry in the effective 4 dimensional theory, it is necessary [2] to take the internal nonlinear sigma model to have $\mathcal{N} = 2$ supersymmetry, and hence X to be a Kähler manifold. Putting all these conditions together, one obtains that X is a Calabi-Yau threefold; this result can be obtained also analyzing compactification of the LEEA, type II supergravity. Indeed, spinors in 6 dimensions are in the $\underline{4}$ of $Spin(6) = SU(4)$; if the holonomy is $SU(3)$, which is the case for Calabi-Yau's, the group-theoretical decomposition $\underline{3}\oplus\underline{1}$ under the holonomy tells us there is only one covariant spinor, thus breaking $3/4$ of supersymmetry. In 4 dimensional terms this is $\mathcal{N} = 2$ for the supergravity; for branes this gets broken again by half, giving $\mathcal{N} = 1$.

Actually, the most interesting case is the one in which the resulting 4 dimensional theory has exactly $\mathcal{N} = 1$ (not more), and this means that the X will have to be so-called *strictly* Calabi-Yau, meaning that the holonomy is exactly $SU(3)$ (not properly contained).

Unfortunately, Calabi-Yau threefolds are far from being unique; as we said, this is the starting point of the problem of the space of string compactifications. In the closed string part, there is freedom of choosing the topological type of X, and there is, for any such choice, a freedom of deforming it (actually the decoupling of these two issues is too naive; string theory joins often together moduli spaces of topologically different manifolds. Anyway, we will ignore this here). Deformations come from possible deformations of the sigma model action; this contains the metric and the antisymmetric B field (we neglect the dilaton here). Let us start from deformations of the metric. These should preserve its property of being Ricci flat (see, however, previous footnote); these can be divided into two classes. First one is deformations of type $\delta g_{i\bar j}$, which preserve the hermiticity of the metric; these are called *Kähler* and can be shown to be harmonic, from which it follows that they are counted by $H^{(1,1)}(X)$. The other ones are of the form δg_{ij}, with both holomorphic indices, and thus do not preserve the hermiticity of the initial metric. One can see that this can be reabsorbed into a change of the complex structure of X: thus these are called *complex structure* deformations and are counted, by Kodaira-Spencer theory, by $H^1(X, T_X) = H^{(2,1)}(X)$. More precisely, there are $(2,0)$ and $(0,2)$ deformations, so there are $h^{(2,1)}$ *complex* of them. For Kähler deformation there were instead $h^{(1,1)}$ real of them; but these couple with the same number of deformations coming from deformations of B, and make up thus together $h^{(1,1)}$ complex deformations.

Note that the number of Kähler and complex structure deformations are the only two nontrivial numbers in the so-called Hodge diamond of X:

$$\begin{array}{ccccccc} & & & h^{(0,0)} & & & \\ & & h^{(1,0)} & & h^{(0,1)} & & \\ & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\ h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\ & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\ & & h^{(3,2)} & & h^{(2,3)} & & \\ & & & h^{(3,3)} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.1.2}$$

We have considered here a local picture, but these deformations get integrated to moduli spaces (for complex structure moduli spaces, there are no obstructions; for Kähler, the issue is much more

[1] We will see shortly, actually, that this requires a more careful discussion.

subtle. See [1] and references therein). So, a first rough approximation to the moduli space of closed string theory is as a product of a Kähler moduli space and a complex structure one.
At this point we might note a couple of things. First of all, these moduli spaces have special loci in which singular behaviours may be expected; for instance, in the complex structure moduli space there are the loci in which the corresponding Calabi-Yau becomes singular (this is called *discriminant locus*). String theory can often resolve in some way these singularities, in the sense that it is well-defined anyway. Second comment requires two words about *mirror symmetry*, whose results we will be using later in chapter 3. Mirror symmetry is originally the observation that, from the CFT point of view, complex structure and Kähler deformations are very similar: They are related by a reversal of sign in a $U(1)$ charge of the e.g. right moving sector). This makes one suspect that, for any Calabi-Yau Y, there should be a mirror $\tilde{Y}$ such that *IIA theory on Y is equivalent to IIB theory on $\tilde{Y}$*. One can view this as a kind of generalized T-duality. Kähler moduli and complex structure ones are exchanged by this duality; pictorially, one can view this as a $\pi/2$ rotation in the Hodge diamond (2.1.2). So, one can individuate special loci in Kähler moduli spaces as well, by looking at the special loci of the mirror. We will meet these special loci again in the sequel; for example, this is the most rigorous interpretation of what we will call *large volume* limit later. In fact, most rigorously the Kähler moduli space in string theory should be thought of as the complex structure moduli space of the mirror [40].

2.1.1 Worldsheet techniques

Here we discuss various features of the perturbative string which will prove useful along the way. General references for the technical analysis we will sketch here are for instance [33, 40, 56].

Perturbative string theory is a 2 dimensional CFT. In the case of Calabi-Yau compactification, conformal symmetry gets enhanced to a *super*conformal algebra with $\mathcal{N} = (2,2)$. For the closed string this algebra is roughly a direct sum of two isomorphic algebras, one for right-movers and one for left-movers; each copy includes:

- the generator of conformal transformations T,
- two generators of supersymmetry G^+, G^-,
- a generator of a $U(1)$ automorphism (R-symmetry) J.

The (super-)Lie algebra structure of the modes of these currents can be as usual summarized conveniently by giving OPE's among them. We only need here to know that

$$G^+(z)G^-(w) = \frac{2\,c}{3\,(z-w)^3} + \frac{2\,J(w)}{(z-w)^2} + \frac{2\,T(w) + \partial_w J(w)}{(z-w)} + Reg.\,; \qquad (2.1.3)$$

most of the other OPEs are expressing the conformal weight h and $U(1)$ charge q of the various fields (J has $h=1$, and G have $h=1, q=3/2$).

As usual, what this algebra structure is good for is that states of the theory organize in representations. In particular, we can define in this way NS and R states in both left- and right- moving sector, generalizing their homologues for string theory in flat space.

Let us see how this works. The Gs are fermionic fields, and this calls for the usual twofold choice between periodicity and antiperiodicity. The first case is called Ramond sector and the second is called Neveu-Schwartz. Let us start from the second one.
In this case, we define a *chiral primary* state $|\psi\rangle$ to be one which is primary and annihilated by $G^+_{-1/2}$:

$$G^+_{-1/2}\,|\psi\rangle = 0\,;$$

primary states annihilated by $G^-_{-1/2}$ are instead called antiprimary. The importance of this states can be seen extracting from (2.1.3) the anticommutator

$$\{G^-_{1/2}, G^+_{1/2}\} = 2\,L_0 - J_0$$

and taking the expectation value of this expression on $|\psi\rangle$: one gets

$$h_\psi = \frac{1}{2} q_\psi \,, \qquad \text{for } \psi \text{ chiral primary.}$$

One can show that these states form a finite ring: the product of two chiral primaries is again a chiral primary. This so-called chiral ring encodes much of the properties of the given theory. Considering both the left- and right-moving sector of the theory, one can have left chiral primaries which are also right chiral primaries (these are said to form the (c, c) *ring*) or that are right antichiral primaries (called instead elements of the (c, a) ring), and so on.

Ramond sector has instead another set of relevant states, the so-called Ramond ground states, which are by definition annihilated by $G^\pm_0$. Now, the nice point is that one can moreover connect these two set of states with a worldsheet operator. For this, one employs usual bosonization techniques: the $U(1)$ current J can be written as

$$J = \sqrt{\frac{c}{3}}\,\partial\tilde{\varphi} \tag{2.1.4}$$

in terms of a worldsheet boson $\tilde{\varphi}$ (c is the central charge). Then, any state having charge q will have a factor $e^{q\sqrt{3/c}\,\tilde{\varphi}}$, which is easily seen to have the correct OPE with J as expressed in (2.1.4). Let us consider now the *spectral flow* operator

$$Q = e^{\frac{1}{2}\sqrt{\frac{c}{3}}\,\tilde{\varphi}}\,; \tag{2.1.5}$$

if we apply it to an operator of charge q, we get

$$e^{\frac{1}{2}\sqrt{\frac{c}{3}}\,\tilde{\varphi}}(z)\, e^{q\sqrt{\frac{3}{c}}\,\tilde{\varphi}}(w) = (z-w)^{q/2} e^{(q-\frac{c}{6})\sqrt{\frac{3}{c}}\,\tilde{\varphi}}\,. \tag{2.1.6}$$

This means that Q, applied to an operator of odd integer charge q, adds a square root to it. This changes its periodicity property: an R state with odd integer q becomes a NS one, and viceversa.

Since the low-energy interpretation of R and NS sectors is as fermions and bosons on the ambient manifold ($\mathbb{R}^4 \times Y$ in our case), if we only had operators with odd-integer q that would exchange bosons and fermions, and thus realizing supersymmetry. Thus, it is enough to take only operators with odd-integer q to have supersymmetry. This might seem at first a clumsy thing to do, but remember that just the same thing is happening in the more familiar superstring in flat space, where one has to introduce GSO projection in order for the theory to be supersymmetric. For this reason, this projection onto odd-integer operators is called *generalized GSO projection*. A point about which one has to be careful is that, although very often one often is dealing with the internal part of the theory, which has $c = 9$ (the CFT which describes the Calabi-Yau part), this GSO has to be applied to the whole superstring theory, which has $c = 12$. One trivial way to see that this is necessary is to look at (2.1.6) again: the $U(1)$ charge of the transformed state gets transformed as $q \to q - c/6$; with $c = 12$ everything is consistent: states with odd-integer charge get transformed into states with the same property.

Note moreover that spectral flow is a *worldsheet* operator realizing *spacetime* supersymmetry. As we anticipated, it exchanges NS chiral primaries with R ground states: this is again because, having $G^\pm$ a $U(1)$ charge $q = 1$, the spectral flow Q acts on it adding a factor of $(z-w)^{1/2}$ as in (2.1.6).

As an application of what we have just seen, one can also interpret in a more intrinsically stringy way the complex and Kähler deformations we have seen in section 2.1. The deformation problem has been indeed tackled in the context of CFT as well. Consider the addition of an operator O to the action. A necessary condition for the action to remain conformal after this addition is that its left- and right-moving conformal dimensions satisfy $h_{L,O} + h_{R,O} = 1$, a condition which is called *marginality*. There is also here something similar to obstructions, however, in the sense that this condition is not a sufficient one; operators whose addition does really preserve the property of the action being conformal are called *truly marginal*. The result we are interested in here is that [23, 22] in the case at hand a truly marginal deformation can be constructed out of any chiral or antichiral operator; more precisely, of any element of the (c, c) or (c, a) ring, considering both the left- and the right- moving sector.

In the case of primary interest so far, Calabi-Yau nonlinear sigma model, what are the (c, c) and (c, a) rings? By spectral flow, we can equivalently look for Ramond ground states; and it turns out that these have a geometrical interpretation [90] (see also [92]). In the case of strings in flat space, remember that Ramond ground states are nothing but representations of the algebra of the zero modes of the worldsheet fermions ψ. Here the theory is less trivial, and we cannot rely on such easy Fourier expansions; but the restriction to zero modes is still meaningful, as the restriction to fields which are constant as functions of the target manifold.

So, we can now take the action of the nonlinear sigma model, restrict it to constant fields and quantize it; in its Hilbert space (in the Ramond sector) we will then look by bare hands for Ramond ground states. From (2.1.1) we can read off the canonical quantization of x and ψ, $\tilde{\psi}$ which is convenient to express in terms of their holomorphic and antiholomorphic components along X, that is, splitting the index μ in indices i and $\bar{\imath}$:

$$\begin{aligned} [x^i, \partial_z x^{\bar{\jmath}}] &= g^{i\bar{\jmath}} = \{\psi^i, \psi^{\bar{\jmath}}\} , \\ [x^i, \partial_{\bar{z}} x^{\bar{\jmath}}] &= g^{i\bar{\jmath}} = \{\tilde{\psi}^i, \tilde{\psi}^{\bar{\jmath}}\}. \end{aligned} \tag{2.1.7}$$

Note that in the flat case $g = \eta$, the algebra for example of the ψ is the same as the usual algebra of creators $\psi^{\bar{\jmath}}$ and destructors ψ^i, as it should be (this is the usual flat Clifford algebra). One can therefore represent these operators on the Hilbert space of states of the form

$$\sum \alpha_{\bar{\imath}_1,\dots,\bar{\imath}_r,\bar{\jmath}_1,\dots,\bar{\jmath}_s}\, \psi^{\bar{\imath}_1}\dots\psi^{\bar{\imath}_r}\,\tilde{\psi}^{\bar{\jmath}_1}\dots\tilde{\psi}^{\bar{\jmath}_s}\,|0\rangle\ . \tag{2.1.8}$$

Having the Hilbert space, we can now look for states which are annihilated by $G_0^{\pm}$ in each sector. These can be found as usual with the Noether procedure, and turn out to be the most obvious generalization of the worldsheet supersymmetry generators for the string in flat space, namely (in the left moving sector)

$$G_0^+ = g_{\bar{\imath}j}\psi^{\bar{\imath}}\partial_z x^j\ , \qquad G_0^- = g_{i\bar{\jmath}}\psi^i\partial_z x^{\bar{\jmath}}\ .$$

The Hilbert space (2.1.8) is obviously isomorphic to the space of (r, s) forms on the manifold X. Then the creator $\psi^{\bar{\jmath}}$ is isomorphic to the operator $\wedge d\bar{z}^{\bar{\jmath}}$, and the operator ψ^i to the operator $\cdot dz^i$. Moreover, one can look at the bosonic part of (2.1.7) in the form $[f(x), g_{i\bar{\jmath}}\partial_z x^{\bar{\jmath}}] = \partial_i f(x)$, thus having that $g_{i\bar{\jmath}}\partial_z x^{\bar{\jmath}}$ one sees then that $G_0^{\pm}$ have the form $\wedge dz^{\bar{\imath}}\partial_{\bar{\imath}} \equiv \bar{\partial}$ and $\cdot dz^i\partial_i \equiv \partial^\dagger$. So, we have that the space of Ramond ground states of $U(1)$ charges (r, s) is isomorphic to the space of (r, s) harmonic forms.

The dimensions of these spaces in our case are summarized in the Hodge diamond (2.1.2). The $h^{(1,1)}$ and $h^{(2,1)}$ deformations we had seen geometrically before are thus seen to arise as the deformations associated to the elements of (c, c) and (c, a) ring, which are superpartners through spectral flow of Ramond ground states.

2.2 Open strings at large volume

When the nonperturbative importance of D-branes became clear, an analysis of D-branes on Calabi-Yau manifolds was immediately made [5]. Based on a BPS analysis of low-energy actions, the following neat results were found. BPS branes can be divided into two classes, called A and B; for each class there is a condition on the submanifold on which the brane is wrapped, and a condition on the gauge vector bundle E living on the submanifold:

A: the submanifold L is required to be *special lagrangian*; this means that the following conditions are met for the restrictions on L of the Kähler form ω and holomorphic 3-form Ω of the ambient Calabi-Yau Y:

- $\dim_{\mathbb{R}}(L) = 1/2\ \dim_{\mathbb{R}}(Y)$ and $\omega_{|_L} = 0$ (lagrangian);
- $Im\{e^{i\theta}\Omega_{|_L}\} = 0$ (special).

The vector bundle E on L has instead to be *flat*.

B: the submanifold is asked to be a holomorphic submanifold of Y; the vector bundle E is required to be a holomorphic one. In addition to this, the bundle should be such that there is a connection A on it such that its curvature F satisfies a set of equations, which are in the large volume limit *and neglecting transverse scalars* X for the time being, for simplicity, the HYM we have already mentioned:

$$\omega \cdot F = c\,, \qquad F^{(2,0)} = 0\,. \tag{2.2.9}$$

The names come from the A and B twisted topological models; these branes made an *ante litteram* appearance [89] as boundary conditions in the model with the same letter.

The conditions on A and B imply that their dimensions are respectively 3 and even. By definition IIA theory on a Calabi-Yau Y is equivalent to IIB on the mirror $\tilde{Y}$; this was originally a closed string statement, but it is natural to conjecture that the whole nonperturbative theories should be the same, and thus to extend mirror symmetry to the open string theories represented by the branes. So, A branes should be mirror of B branes and viceversa. As a rough check, note that in a strict Calabi-Yau the number of 3-cycles is $2+2b_{21}$ and that of even cycles is $2+2b_{11}$; these two numbers are correctly exchanged by mirror symmetry. We will see later that we can be more precise about the way this partnership works, establishing a more precise correspondence between the two sets of conditions.

One may think at this point the analysis is more or less complete. However, these results are only valid in a limiting situation. Indeed, many of the computations one would like to do in this context are perturbative computations in the nonlinear sigma model coupling constant l_s/R; and their validity thus is often limited by higher perturbative, or nonperturbative, corrections. As this coupling goes to zero with $R \to \infty$, we will be often talking about results which are only true in the *large volume limit*. This is the case for instance for the condition for the vanishing of the conformal anomaly we mentioned above, which is Ricci-flatness only in this approximation; the same remark applies to the BPS analysis of branes we just reviewed. This is also intuitively clear: it is hard to see how one can imagine branes to adhere to the idealized geometrical picture of "infinitely thin" objects, if the size of the ambient Calabi-Yau itself becomes comparable to typical string scales. More precisely, all this discussion about the "size" of the Calabi-Yau should be understood as choosing different closed string backgrounds, corresponding to different points in Kähler moduli space.

The description of branes as bundles on submanifolds on branes is breaking down away from the large volume limit, what will replace it? what will be the appropriate mathematical description for branes as one goes to generic points of the Kähler moduli space? This question is not completely answered yet; let us see what is known and what are the problems.

2.3 Away from the large volume limit: what remains true

The first thing we will try to do will be to see whether something of the preceding analysis can be retained at other points of the Kähler moduli space. We will see that

- the distinction between A and B remains a sensible one;
- the various conditions in their definitions keep on being split between real and holomorphic ones; only either the former or the latter are deformed.

Both of these statements are not trivial and require some explanation. For the first one we will briefly review the worldsheet approach, to be used later as well; for the second we will use instead 4 dimensional effective theories.

2.3.1 A and B: worldsheet approach

We need here to adapt the techniques outlined in 2.1.1 to open string theory. As usual this starts introducing boundary conditions. First of all, the conformal symmetry of the theory is now different, since the worldsheet will be (at tree level) the upper half plane and not the whole (punctured) plane. We can express this new conformal group using the one of the plane and requiring the boundary $z = \bar{z}$ to be fixed. As T_L and T_R generate conformal transformations in z and in $\bar{z}$ respectively, we get the boundary conditions $T_L = T_R$. More to the point, if we want to preserve a "diagonal" (between left and right) $\mathcal{N} = 2$ superconformal algebra, boundary conditions for all the currents J_I should in general have the form of

$$J_{L,I} = R_I^J J_{R,J}$$

for R_I^J an automorphism of the $\mathcal{N} = 2$ algebra, and $J_{L,I}$, $J_{R,I}$ the various currents in both sectors. These automorphisms are not too many. In particular, let us consider the current J. Since it is the only current with conformal weight one, it can only transform to a multiple of itself: $J \to R(J) = \alpha J$. The only OPE we have not yet used nor described in section 2.1.1 is the one of J with itself, which is that of a $U(1)$ Kac-Moody algebra:

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + Reg.$$

we see $\alpha^2 = 1$, namely $R(J) = \pm J$. We can likewise complete the list of all the allowed possibilities for the automorphisms, but let us concentrate on this. What we have just got is that we have the possibility of boundary conditions with

$$J_L = J_R \qquad \text{or} \qquad J_L = -J_R\,;$$

in the large volume, the first type of boundary conditions can be seen [65] to correspond to B branes, and the second to A branes. (The strange reversing of order can be explained going to the closed string channel, in which the $\pm$ present here are reversed.) A very rough account of this follows: Look at the zero-mode expression of the $U(1)$ current for the nonlinear sigma model,

$$J_L = g_{i\bar{j}}\psi^{\bar{j}}\psi^i\,; \qquad \left(J_R = g_{i\bar{j}}\tilde{\psi}^{\bar{j}}\tilde{\psi}^i\,;\right) \tag{2.3.10}$$

the boundary conditions are

$$\psi^\mu = \tilde{\psi}^\mu \ , \quad \mu \text{ Neumann} \ ; \qquad \psi^\mu = -\tilde{\psi}^\mu \ , \quad \mu \text{ Dirichlet} \ ;$$

when locally half of the coordinates are Neumann and half Dirichlet, then one of the fermions in ψ gets a plus and the other a minus sign, yielding $J_L = -J_R$; for B branes instead both fermions in each summands gets a plus or a minus sign, giving $J_L = J_R$.

Thus, we have seen that the division between A and B branes is not just a feature of large volume analysis, but finds its reason in the possible automorphisms of $\mathcal{N} = 2$ algebra, and so it will remain as we go to general Kähler moduli.

2.3.2 D and F terms, and decoupling

As we have anticipated, we will be thinking here to the 4 dimensional effective approach. Consider a brane which is totally extended in the 4 noncompact dimensions, and that may be whatever brane in the Calabi-Yau. This will be to first approximation a gauge theory; as we have said, in our situation this will turn out to have $\mathcal{N} = 1$. Now, we know the general description of $\mathcal{N} = 1, \ d = 4$ gauge theories, and we will take as much profit as we can from this description. Choosing such a theory is equivalent to choosing

- a gauge group G (with a vector multiplet in its adjoint)
- chiral multiplets Φ_i in some representations of the gauge group
- a real function $K(\Phi_i, \Phi_i^\dagger)$ of the chiral multiplets, called Kähler potential, that sets the kinetic term
- a holomorphic function $W(\Phi_i)$ of the chiral multiplets.

Given these data, one can write down an action in terms of superfields. What concerns us is the moduli space of vacua; it turns out that this can be described as the vanishing locus of a real equation

$$\sum_i \phi_i^\dagger T^a \frac{\partial K}{\partial \phi_i^\dagger} = 0 \ ,$$

(see for example [9]) with T^a the gauge generators, and of complex equations

$$\frac{\partial W}{\partial \phi_i} = 0$$

in terms of the scalars ϕ_i of the chiral multiplets. These equations are called respectively D term and F term equation, because they come from the equations of motion of the auxiliary fields of the vector and chiral multiplets, traditionally denoted with these names.

In this setting we can interpret the real and holomorphic conditions on A and B branes respectively as D and F term equations for the 4d effective gauge theory. For example, the first, real, equation in (2.2.9) is the D term; the complex one, and the requirement of complex embedding, are the F term: for example, for $F^{(2,0)} = 0$ we can even guess the superpotential, which turns out to be a holomorphic version of Chern-Simons lagrangian [89].

Now, these 4d gauge theories come from open strings. These come always with closed strings as well, and this is reflected in 4 dimensions in the fact that the effective gauge theories will couple also to the 4d effective actions of the closed strings, which are $\mathcal{N} = 2$ supergravity models, as we have seen. These couplings will be seen as a dependence of the Kähler potential and of the superpotential on the vector and hypermultiplets of the supergravity. Without repeating too explicitly the list we have just seen for the gauge theory, recall that for IIA theory vector multiplets contain the Kähler moduli and hypermultiplets contain the complex structure moduli; for IIB the converse is true. A non trivial decoupling turns out to be true [16]: F terms of the gauge theory depend only on Kähler moduli for A branes, and on complex structure moduli for B branes; and the other way around for D terms.

This statement has deep consequences. Consider B branes. Then, as we go away from the large volume limit to other points of the Kähler moduli space, the D term will change, but the superpotential will not change, because of the decoupling just discussed. This means that, as we abandon the large volume limit, the equation $F \cdot \omega = c$ will be changed, but all the holomorphicity statements – the holomorphic embedding, and the presence of a holomorphic structure on the bundle which is integrable ($F^{(2,0)} = 0$) – will not be changed.

We have thus seen that the various conditions we have seen for A and B branes in section 2.2 correspond to D and F term, and so do not get mixed but preserve their individuality; and moreover, that only some of them will change traveling in Kähler moduli space. In the case of A branes, holomorphic conditions will be deformed and real ones will not, and *vice versa* for B branes. Since holomorphicity is usually a powerful instrument, in what follows we will mostly concentrate on B branes. This is ultimately also justified by mirror symmetry, which allows A branes to be studied also as B branes on the mirror.

As a conclusion, we have now the basic lines of a program: to describe B branes at general points of Kähler moduli space. The problem can be separated into a holomorphic part, which is constant, and a real one, which depends on Kähler moduli. In what follows, we will first deal with the holomorphic part, which is already interesting in itself, and then try to tackle with more difficult real part. We will often refer to this second condition as to a stability, for reasons mentioned in the introduction and to become clearer later.

Chapter 3

Holomorphicity

As promised, we start with the easiest part of our program. Already this one will prove to be non trivial and will show interesting features of string theory, along with mathematical results.

The goal will be to understand what are the mathematical objects describing branes at all points of moduli space (disregarding stability, as we have stressed, for the time being). The naive answer would be "coherent sheaves", since these (as we will review) allow to describe very naturally bundles on submanifolds of a given manifold. We will see, however, that these objects are not enough. In a sense we have anticipated this with our remark about the thickness of branes at the end of section 2.2; but there are more solid reasons for this. We will see indeed that there is another tractable limit of the Kähler moduli space, called the *Gepner* one; and, in that limit, branes are not described by geometrical objects such as coherent sheaves but by algebraic ones, *quiver representations*[1] – in any case, different kind of objects.

A natural candidate will then arise after some thought. The idea will be to follow one brane in a path in the moduli space from one of these two special points – the large volume point and the Gepner one – to the other. The correspondence obtained in this way will be of a twofold value. First of all, the results obtained in this way are of relevance to the long-standing problem of generalizing the well-known McKay correspondence to higher dimensions. Second, the methods we will use – essentially a generalized Beilinson theorem – will strongly suggest the natural candidate one could have thought of at the beginning: the derived category. Indeed, the derived category of representations of the quiver one gets at the Gepner point is the same as the derived category of coherent sheaves of a natural Fano manifold in which the Calabi-Yau is embedded. Though we do not claim to have general theorems and complete solutions to the problems, there are many non trivial checks here, and this is part of evidence in favour of derived category.

The main part of this chapter will be devoted to explaining these results. We will try to use first a conservative approach, using derived category as little as we will be able to. At the end, we will finally explain what derived category is, and recollect the evidences in its favor as the right object for describing branes at arbitrary points of the moduli space, at least as long as we disregard stability problems.

[1]Of course algebra and geometry are strictly tight together; commutative algebra is essential in describing coherent sheaves, and quivers are related to geometry. This terminology refers only to the superficial level of these objects.

3.1 Coherent sheaves

First of all, we describe the more properly geometrical side of the story, introducing coherent sheaves and their connection to string theory.

Sheaves are mathematical objects which allow to treat in a uniform way various problems; for us their value will be that they allow to describe sort of bundles with changing rank. In general, they are not confined to algebraic geometry, though this is probably the field which has most heavily used them. Of course we will not give here definitions in their full generality (though there is no particular extra difficulty in doing so), trying instead to extract the information relevant to physics.

Let Y be a manifold. A *presheaf* of groups F is the datum of

- a group $R(U)$ for each open set U;
- for each inclusion of open sets $U \overset{\iota_{UV}}{\hookrightarrow} V$, a homomorphism (called *restriction*) $F(V) \overset{\rho_{UV}}{\to} F(U)$,

such that

$$\text{for any chain of inclusions } U \overset{\iota_{UV}}{\hookrightarrow} V \overset{\iota_{VW}}{\hookrightarrow} W \text{ we have } \rho_{UW} = \rho_{VW} \circ \rho_{UV}$$

for the corresponding restriction maps. This condition looks like the gluing condition for bundles, but note that these open sets are nested and not general intersecting ones. Note that ρ goes "the other way" with respect to ι. This is best explained thinking about the examples this definition is meant to describe, such as the sheaf of (holomorphic) functions, or of sections of a vector bundle. In these cases, what we have called here restriction is really what we know under the same name, for instance restriction of functions. An element of $F(U)$ is often called, after these examples, a *section* over U.

So far we have introduced a rather natural object (which could also be formulated, in terms of the category language whose use we hope to elucidate to the reader in section 3.5, as a contravariant functor from the category of open sets of the Y to the category of rings). The extra structure we introduce now is in a sense more significant: it can be, for instance, relaxed, leading to interesting generalization that we will only mention afterwards, again in section 3.5.

A *sheaf* of groups is now a presheaf of groups for which, for any two couple of open sets U and V, the sequence

$$0 \to F(U \cup V) \overset{(\rho_{U,U\cup V}, \rho_{V,U\cup V})}{\longrightarrow} F(U) \oplus F(V) \overset{\rho_{U\cap V,U} - \rho_{U\cap V,V}}{\longrightarrow} F(U \cap V)$$

of groups is exact. This is called gluing condition: spelling out the exactness, we see that it means that

i. if a section s on U and s' on V agree on $U \cap V$ ($s_{|U\cap V} = s'_{|U\cap V}$), there is a section s'' on $U \cup V$ whose restrictions to U and V give s and s': $s''_{|U} = s$, $s''_{|V} = s'$;

ii. if a section s on $U \cup V$ is zero when restricted to both U and V, ($s_{|U} = 0 = s_{|V}$), then it vanishes, $s = 0$.

This concept becomes powerful when applied to algebraic geometry. First of all, the definition we have given can be easily extended to sheaves of rings, just by considering rings instead of groups. A natural sheaf of rings over a complex manifold Y is its sheaf of holomorphic functions, called its *structure sheaf* and denoted by $\mathcal{O}_Y$. Second, we can now consider sheaves of modules over it. Recall that a module M over a ring R is what could also be called a representation, namely an abelian group with an action $R \times M \to M$ of R on it compatible with sum and product. A class of examples of modules over R is given by a direct sum of n copies of them:

$$\underbrace{R \oplus \ldots \oplus R}_{n \text{ times}} \equiv R^n .$$

Unlike vector spaces (which are modules over fields), however, modules do not have all this form (in which case they are said to be *free*). There can be indeed *relations*, putting some elements of R^n to zero. If there is a finite number n of generators and of relations m, the module M is said to be *of finite presentation*, and can be put in an exact sequence

$$R^m \to R^n \to M \to 0 .$$

A sheaf of modules over, in this case, the sheaf of rings $\mathcal{O}_Y$, is another slight modification of the concept of sheaves of groups. This time, to each open set U we assign a module $M(U)$ over the ring $\mathcal{O}_Y(U)$ of holomorphic functions over U, such that restrictions are now compatible with the actions of the rings over the modules.

A sheaf $\mathcal{E}$ of $\mathcal{O}_Y$-modules is called *coherent* if the following happens: each point has a neighbourhood U over which C is of finite presentation over the ring $\mathcal{O}_Y(U)$:

$$[\mathcal{O}_Y(U)]^m \to [\mathcal{O}_Y(U)]^n \to \mathcal{E}(U) \to 0 , \tag{3.1.1}$$

where again we have denoted by R^n the direct sum of n copies of R. As we have anticipated, the virtue of this definition is that it allows to describe in a uniform way bundles on Y and bundles on submanifolds of Y. For instance, suppose $\mathcal{E}$ is the sheaf of sections of a fiber bundle E.
In that case, locally $\mathcal{E}$ is $U \times \mathbb{C}^n$: sections of $\mathcal{E}$ over this open set U are simply holomorphic functions from U to $\mathbb{C}^n$. This means that $\mathcal{E}(U) = [\mathcal{O}_Y(U)]^n$. This is obviously a particular case of the definition given above, taking in (3.1.1) $m = 0$. For obvious reasons, in this case the sheaf is called *locally free.*
But we can do more. Consider a submanifold X. Then we can define a sheaf on Y out of the structure sheaf $\mathcal{O}_X$ of X in this way:

$$(i_*\mathcal{O}_X)(U) = \mathcal{O}_X(U \cap X) ,$$

where U are of course open sets in Y. This means that this sheaf $i_*(\mathcal{O}_X)$ on Y is defined to associate the empty set to the open sets of Y which do not intersect X, and the sections of $\mathcal{O}_X$ with the intersection with X in the other cases.
Now, this is a sheaf of modules over $\mathcal{O}_Y$. Indeed, there is an action of the rings $\mathcal{O}_Y(U)$ over the groups $(i_*\mathcal{O}_X)(U) = \mathcal{O}_X(U \cap X)$: it is simply given by multiplying functions on the submanifold X by functions on the ambient manifold Y. Now, it is immediate to realize that these modules are not free, but there are relations. Indeed, any two functions which differ by a function which vanishes on X will look the same when restricted to X. Suppose that locally, in the open set U we are considering, X is defined by equations $z_1 = 0, \ldots, z_m = 0$. Then the relations are exactly that $f \in \mathcal{O}_Y(U)$ is the same thing, when representing a function $i_*f \in (i_*\mathcal{O}_X)(U)$, as another function $f + z_1 g_1$, or as another function $f + z_2 g_2$, and so on. There are thus m relations; this can be written as

$$[\mathcal{O}_Y(U)]^m \to \mathcal{O}_Y(U) \to (i_*\mathcal{O}_X)(U) \to 0 ,$$

which is another example of (3.1.1), this time with $N = 1$.
Actually, sometimes we can even be more precise; for instance, if X is a divisor, we can summarize these local sequences over U with a global sequence of sheaves

$$0 \to \mathcal{O}_Y(-X) \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0 , \tag{3.1.2}$$

where $\mathcal{O}_Y(-X)$ is defined to be the sheaf of holomorphic sections on Y that vanish on X.

We have seen in this section what we wanted, namely that coherent sheaves are a good way to describe vector bundles on submanifolds. This is also the right place to note two other virtues of this formalism, namely the physical relevance of Grothendieck groups, and the correspondence between Ext groups and the fermionic spectrum.

3.1.1 Grothendieck group and projective resolutions

Grothendieck group $\mathcal{K}(Y)$ gives another definition, an algebraic-geometrical one, of K-theory. A way of defining it is as follows. Define $\mathbb{Z}[\mathcal{L}]$ to be the free abelian group generated by locally free sheaves (in other words, the group of formal sums of vector bundles). Then

$$\mathcal{K}^\bullet \equiv \mathbb{Z}[\mathcal{L}]/\mathcal{I}\,;$$

$\mathcal{I}$ is the ideal generated by elements of the form $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$ if there is an exact sequence $0 \to \mathcal{F}'' \to \mathcal{F} \to \mathcal{F}' \to 0$.

Now, any coherent sheaf on Y can be expressed as an element of $\mathcal{K}(Y)$. We have already seen this in a particular case: resolution (3.1.2) allows us to write $i_*\mathcal{O}_X$ as $\mathcal{O}_Y - \mathcal{O}_Y(-X)$.

More generally, it can be seen (using, for example, regularity and the presence of an ample invertible sheaf) that any coherent sheaf $\mathcal{E}$ can be put in a so-called (terminating) *projective resolution*, that is, in an exact sequence

$$0 \to E_{n-1} \to \ldots \to E_1 \to E_0 \to \mathcal{E} \to 0$$

where E_k are bundles. This generalizes (3.1.2). In that case the resolution terminates to the left immediately, because there is only one relation (coming from the one equation defining the submanifold X). In the general case, there are more relations; these may be in turn not independent and generate a longer sequence. Its length, however, will not exceed the dimension of Y (we are assuming here Y is smooth).

Consider now a vector bundle E on a submanifold X. This can be represented by a coherent sheaf $\mathcal{E}$ on X; as a sheaf, it can always be expressed, through extension by zero (the i_* we have seen) as a sheaf $i_*(\mathcal{E})$ on Y; since the map i_* is proper, we have that *$i_*(\mathcal{E})$ is a coherent sheaf on Y.*

The physical meaning of this is the following. This formalism has allowed us to write bundles E on submanifolds X as formal differences of bundles on Y. It transpires [85] that this formal difference can be interpreted as a brane-antibrane couple which, after tachyon condensation, gives as a product the original bundle E on the submanifold X.

3.1.2 Ext groups

Given two sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ on the manifold Y, we denote $\mathrm{Hom}_Y(\mathcal{E}_1, \mathcal{E}_2)$ the group of morphisms between them. Then one can define, with the so-called derived functor techniques, other groups $\mathrm{Ext}^i_Y(\mathcal{E}_1, \mathcal{E}_2)$ (conventions are such that $\mathrm{Ext}^0 = \mathrm{Hom}$). Their point is for us essentially that:

- They can be used to write long exact sequences involving Hom's; for instance, if we have an exact sequence of sheaves

  $$0 \to A \to B \to C \to 0$$

  we can write long exact sequences

  $$\begin{aligned} 0 &\to \mathrm{Hom}(D,A) \to \mathrm{Hom}(D,B) \to \mathrm{Hom}(D,C) \\ &\to \mathrm{Ext}^1(D,A) \to \mathrm{Ext}^1(D,B) \to \mathrm{Ext}^1(D,C) \to \mathrm{Ext}^2(D,A) \to \ldots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}(C,D) \to \mathrm{Hom}(B,D) \to \mathrm{Hom}(A,D) \\ &\to \mathrm{Ext}^1(C,D) \to \mathrm{Ext}^1(B,D) \to \mathrm{Ext}^1(A,D) \to \mathrm{Ext}^2(C,D) \to \dots\ ; \end{aligned}$$

- If F is a bundle and $\mathcal{E}$ a sheaf, one has $\mathrm{Ext}^i_Y(F,\mathcal{E}) = H^i(Y, F^* \otimes \mathcal{E})$. This means that the Ext's are nothing but a generalization of the usual bundle valued cohomology groups. This is important for the following reason. Recall the isomorphism given in section 2.1.1 between states which are Ramond ground states in both the left and the right sector, and elements of $H^{(r,s)}(X)$. For open strings stretched between two branes with bundles F and E, there is an analogous isomorphism. First modification is that the nonlinear sigma model action (2.1.1) is modified by a supersymmetric boundary term

$$\int_{\partial X} \left[A_\mu(x)dx^\mu + \frac{1}{4}F_{\mu\nu}(\psi^\mu + \tilde{\psi}^\mu)(\psi^\nu + \tilde{\psi}^\nu)\right] . \tag{3.1.3}$$

Second modification is that boundary conditions leave then only one set of worldsheet fermions. So, we have a Hilbert space with only one set of creator operators (differently from (2.1.8), in which there is ψ and $\tilde{\psi}$); moreover, the presence of (3.1.3) modifies the expression of $G_0^\pm$ by a term containing A [48]. In fact, the resulting operators have now the form of D and $D^\dagger$, when D is now $dz^i(\partial_i + A_i)$. Thus the result is that Ramond ground states belong now to $H^i(Y, F^* \otimes E)$.

In the case in which the two branes are not bundles but more general coherent sheaves, one can then show that the Ext are the right generalization; thus we get that the Ramond ground states of strings stretched between two coherent sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by the groups $\mathrm{Ext}^i(\mathcal{E}_1,\mathcal{E}_2)$.

Let us consider now the alternated sum of their dimensions $\chi(\mathcal{E}_1,\mathcal{E}_2) = \sum_i (-)^i \dim_{\mathbb{C}}(Ext^i(\mathcal{E}_1,\mathcal{E}_2)$. This number is given by an index theorem, analogously to the alternated sum of the usual cohomology groups h^i:

$$\chi(\mathcal{E}_1,\mathcal{E}_2) = \int \mathrm{ch}(\mathcal{E}_1^*)\,\mathrm{ch}(\mathcal{E}_2) Td(T) .$$

note that this is a sort of product of central charges of the two $\mathcal{E}_i$; this can be formalized [60] and is one of the arguments in favour of the K-theory classification of brane charges. Moreover, physically this χ can be seen [16] as a Witten index [91]. We will often denote $\chi(\cdot,\cdot)$ simply as $(\cdot,\cdot)$.

3.2 Quivers and branes

After having praised coherent sheaves, we will now show as anticipated that they are not enough to describe branes on Calabi-Yau's. This requires that we try to individuate another point of moduli space where things are tractable. It turns out that there is another such limit; to see this, we have to take a closer look at Kähler moduli space. An useful approach [93] is to use a model which is not conformal, but which goes to the string CFT after renormalization group flow. This model is called

3.2.1 Linear sigma model.

We prefer, rather than introducing the model in its full generality, to start with an example, which will be an important one throughout all of what follows.

Linear sigma model is an $\mathcal{N}=2$ supersymmetric gauge theory, but this time in 2 dimensions. Features of this theory are similar to the ones of $d=4$, $\mathcal{N}=1$ we have seen above, but this time this is an approximation to the worldsheet theory. As in that case, we have to choose gauge group, chiral multiplets, Kähler potentials and superpotential. In our example we take

- $G=U(1)$;
- five chiral multiplets Z_i of charge 1 and one, P, of charge -5;
- a superpotential $W=PF_5(Z_1,\ldots,z_5)$, where F_5 is a nonsingular polynomial in the Z_i of degree 5;
- we also add to the action a so-called Fayet-Iliopoulos term $r\int D$, where D is the auxiliary field of the gauge multiplet.

We are interested in determining the moduli space of vacua of this theory. This is found again by satisfying D and F term equations, which in this case read

$$|z_1|^2+\ldots+|z_5|^2-5|p|^2=r\,,\qquad \frac{\partial W}{\partial p}=F_5=0\,,\quad \frac{\partial W}{\partial z_i}=p\,\partial_i F_5=0 \tag{3.2.4}$$

and quotienting by the action of the gauge group (lower-case fields are scalars in the corresponding chiral multiplets). Let us begin with the first equation. Considering this equation in the $\mathbb{C}^6$ whose coordinates are (z_i,p), and then quotienting by the gauge group, can be recognized as a special case of what is called *symplectic quotient* or hamiltonian reduction: the equation can be seen as

$$(\text{moment map of the } U(1) \text{ action})=r\,.$$

It turns out that this can be reformulated as a quotient

$$\frac{\mathbb{C}^6-F_\Delta}{\mathbb{C}^*}\,;\qquad (z_1,\ldots,z_5,p)\sim(\lambda z_1,\ldots,\lambda z_5,\lambda^{-5}p),\quad \lambda\in\mathbb{C}^* \tag{3.2.5}$$

for an appropriate F_Δ, which depends on r. This is reasonable: instead of imposing a real equation and then quotienting by $U(1)$, we quotient by $\mathbb{C}^*=\mathbb{R}_+\times U(1)$. F_Δ is determined by the sign of r: if $r>0$, z_i cannot be all simultaneously zero, and so $F_\Delta=\{(0,\ldots,0,p)\}$. First five coordinate describe a $\mathbb{P}^4$; last one, which can also vanish, is the coordinate on the fibre of a line bundle; on projective spaces all line bundles are powers of the so-called *hyperplane line bundle* H, whose definition would have a 1 instead of a -5 in (3.2.5), or of the so-called *tautological line bundle* J, the dual of H; so our line bundle is J^5. A more uniform notation is the sheaf-theoretical one we have introduced in the previous section; with a little abuse of language we will call a vector bundle and its sheaf of sections with the same name. In this way H is also called $\mathcal{O}(H)$, or $\mathcal{O}(1)$ for short; in the same vein $J=\mathcal{O}(-1)$, and our quotient (3.2.5) is $\mathcal{O}(-5)$.
For $r<0$, the situation gets reversed: p cannot be zero. Thus we can use $\mathbb{C}^*$ action to set it to some $p_0\neq 0$; this would leave $\mathbb{C}^5$ as a quotient, but there is still a subgroup $\mathbb{Z}_5\subset U(1)$ which leaves $p=p_0$ fixed; so in this case the result is (3.2.5)$=\mathbb{C}^5/\mathbb{Z}_5$. The result so far is summarized in figure 3.1.

Let us turn now to all the other equations, the derivatives of the superpotential. Since F_5 is nonsingular, it cannot be zero together with its derivatives, unless all $z_i=0$. Thus we have

$$\{p=0,\ F_5=0\}\qquad\text{or}\qquad\{z_i=0\}\,. \tag{3.2.6}$$

Now we put all the equations together. In the $r > 0$ "phase" the D term equation describes the total space of the line bundle $\mathcal{O}_{\mathbb{P}^4}(-5)$; here the second possibility in (3.2.6) is ruled out; so we have the equation $p = 0$, which cuts the zero section of the bundle (that is nothing but the base $\mathbb{P}^4$ itself); finally, the equation F_5 cuts the locus of a quintic polynomial in this projective space. The usual adjunction formula gives that the canonical line bundle of the manifold described by this locus is trivial; so it is a Calabi-Yau threefold, the most studied one, called the *quintic* for obvious reasons.
In the $r < 0$ phase, the first possibility in (3.2.6) is ruled out instead; thus we have that the only solution is the origin in $\mathbb{C}^5/\mathbb{Z}_5$.

Summing up, we see a sort of *phase transition* in the structure of the moduli space of the theory (see again figure 3.1).

We have only written the conditions to find the zeroes of the (non negative) potential of the theory. By looking at the potential itself, in the $r > 0$ phase we can see that fields which describe fluctuations not tangent to the moduli space of vacua have masses which go to infinity as $r \to +\infty$. Thus, in this limit these extra fields go away; what we remain with is a theory which describes maps to the moduli space of vacua, which is, as we have said, a Calabi-Yau manifold, the quintic. Note that we have not broken supersymmetry. The result we get is that

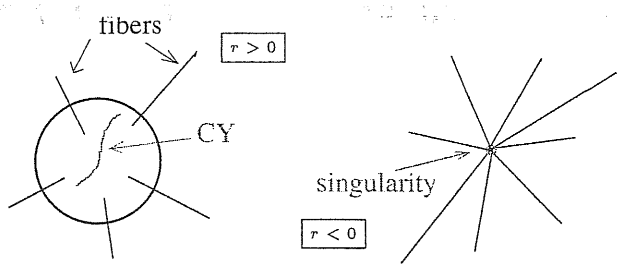


Figure 3.1: Moduli spaces of vacua of linear sigma model in different phases.

$$\lim_{r\to+\infty} (\text{linear sigma model}) = \text{nonlinear Calabi}-\text{Yau sigma model} .$$

In the $r < 0$ phase, again one can see that the mass of the field p goes to infinity with $r \to -\infty$. As for the z_i, they are massless, though the vacuum is an isolated point: this is due to the fact that the superpotential has a singular point (it "vanishes quintically") at the origin. This situation is exactly what one would usually call a supersymmetric *Landau-Ginzburg*. Recall that this is a supersymmetric theory inspired to the well-known (non supersymmetric) effective theories for critical phenomena, with chiral multiplets, some Kähler potential and a superpotential having an isolated zero somewhere. The only difference, after having integrated out the very massive field p, is that there is a $\mathbb{Z}_5$ identification on the z_i. For this reason the resulting theory is called a *Landau-Ginzburg orbifold*:

$$\lim_{r\to-\infty} (\text{linear sigma model}) = \text{Landau}-\text{Ginzburg orbifold} .$$

We have obtained in this way a one-real-parameter family of theories. As we have already seen in section 2.1, however, real moduli spaces coming from Kähler deformations get complexified by the presence of deformations of B field. The manifestation of this phenomenon here is that we can add a two-dimensional θ term here, which gets hand in hand with the FI term to give a sort of supersymmetric θ term. The result is that we have to add a phase to this line of theories, obtaining a Riemann sphere with two special points corresponding to our two limits $r \to \pm\infty$.
We have now two special points, corresponding to known theories (a nonlinear sigma model and an orbifold LG) in a family of non-conformal ones. We can, however, consider now what happens after applying a renormalization group flow to all these non conformal theories. It is reasonable that they will have a fixed point, as the special points have; thus we will get a one-complex-dimensional moduli space $\mathcal{M}_{\text{lsm}}$ of conformal theories.

Now, for this particular type of Calabi-Yau, the quintic, the Kähler moduli space $\mathcal{M}_K$ has dimension $h^{(1,1)} = 1$. Moreover, we have already seen that one of the special points in $\mathcal{M}_{\text{lsm}}$ is a nonlinear sigma model. It follows that these conformal theories we have constructed are a realization of $\mathcal{M}_K$.

To which point in $\mathcal{M}_K$ does the nonlinear sigma model special point we obtained in $\mathcal{M}_{\rm lsm}$ correspond to? to the large volume point: indeed, from first equation in (3.2.4) we can see that r is a kind of "typical radius" for the moduli space of vacua.

Summing up, we have that

- the Kähler moduli space $\mathcal{M}_K$ can be reached from a family of (generically) non conformal sigma models (linear sigma model) by a renormalization group flow;
- two points of $\mathcal{M}_K$ correspond to more usual theories; one of them is a nonlinear sigma model, the other one a Landau-Ginzburg orbifold.

This also implies that, starting from a nonlinear sigma model on a large Calabi-Yau and deforming it, at some point we will get another type of theory, a LG orbifold. There is no contradiction in this; a perturbative analysis shows that at small radii the perturbation theory of the nonlinear sigma model breaks down; LG orbifold is in a sense its "analytical continuation". We also add that, if F_5 is chosen to be $\sum_i z_i^5$, this LG orbifold can be shown to be an orbifold of a tensor product of five $\mathcal{N} = 2$ minimal models at level $k = 3$. This is called *Gepner model* and for this reason the second special point we have found is called *Gepner point.*

The theory corresponding to this point is a LG orbifold. So, the idea is that we can consider the standard procedure to treat branes supported in the singularity of an orbifold and apply it to this case. To do so, let us review how this standard procedure works.

3.2.2 Branes at the Gepner point

We will analyze, as usual, a simple example, to illustrate the main features. Consider the singularity $\mathbb{C}^2/\mathbb{Z}_3$, where the generator of $\mathbb{Z}_3$ acts on the two coordinates (z_1, z_2) as

$$(z_1, z_2) \to (\omega z_1, \omega^2 z_2)\ , \omega^3 = 1\ .$$

Now, consider a brane in the origin of this space, and extended in the other 6 directions. On the worldvolume of this brane we would have, without the orbifold action, two relevant fields, both of them matrix-valued: the gauge field A and the transverse scalars Z. Apart the spacetime action we have defined, $\mathbb{Z}_3$ can now act also on the matricial (Chan-Paton) indices of these two fields, through a representation ρ:

$$A = \rho\, A\, \rho^{-1} \qquad Z_1 = \omega\, \rho\, Z_1\, \rho^{-1} \qquad Z_2 = \omega^2\, \rho\, Z_2\, \rho^{-1}\ ; \tag{3.2.7}$$

there is no reason to choose a representation in particular; all choices for ρ are possible. In this case the classification is easy: we know that any ρ is direct sum of copies of the three one-dimensional representations: the identity, multiplication by ω, multiplication by ω^2. Concretely, the matrix in (3.2.7) can be written as

$$\rho = \mathrm{diag}\left(\underbrace{1, \ldots, 1}_{n_1 \text{ times}}, \underbrace{\omega, \ldots, \omega}_{n_2 \text{ times}}, \underbrace{\omega^2, \ldots, \omega^2}_{n_3 \text{ times}}\right)\ ;$$

all the possibilities are so far encoded in the three numbers n_i, the dimensions of the three blocks. We have to impose now invariance on the fields A and Z^1, Z^2; to do so, we decompose them into blocks A_{ij}, Z^1_{ij}, Z^2_{ij} of dimensions $n_i \times n_j$ and compute the action of ρ; (3.2.7) becomes

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} \boxed{A_{11}} & \omega^2 A_{12} & \omega A_{13} \\ \omega A_{21} & \boxed{A_{22}} & \omega^2 A_{23} \\ \omega^2 A_{31} & \omega A_{32} & \boxed{A_{33}} \end{pmatrix}\ ;$$

$$\begin{pmatrix} Z^1_{11} & Z^1_{12} & Z^1_{13} \\ Z^1_{21} & Z^1_{22} & Z^1_{23} \\ Z^1_{31} & Z^1_{32} & Z^1_{33} \end{pmatrix} = \begin{pmatrix} \omega\, Z^1_{11} & \boxed{Z^1_{12}} & \omega^2\, Z^1_{13} \\ \omega^2\, Z^1_{21} & \omega\, Z^1_{22} & \boxed{Z^1_{23}} \\ \boxed{Z^1_{31}} & \omega^2\, Z^1_{32} & \omega\, Z^1_{33} \end{pmatrix} ; \tag{3.2.8}$$

$$\begin{pmatrix} Z^2_{11} & Z^2_{12} & Z^2_{13} \\ Z^2_{21} & Z^2_{22} & Z^2_{23} \\ Z^2_{31} & Z^2_{32} & Z^2_{33} \end{pmatrix} = \begin{pmatrix} \omega^2\, Z^2_{11} & \omega\, Z^2_{12} & \boxed{Z^2_{13}} \\ \boxed{Z^2_{21}} & \omega^2\, Z^2_{22} & \omega\, Z^2_{23} \\ \omega\, Z^2_{31} & \boxed{Z^2_{32}} & \omega^2\, Z^2_{33} \end{pmatrix} .$$

In order to keep the invariant configurations, the matrices have to be chosen thus in such a way that only the $\boxed{\text{boxed}}$ blocks are non-zero. All the relevant information can now be arranged in a nice way in a diagram; the three dimensions of the blocks n_i are represented by three dots, and arrows between them represent matrices of dimensions $n_{\text{head of the arrow}} \times n_{\text{tail of the arrow}}$, as displayed in figure 3.2.

This is called a *quiver* diagram, and the assignment of the numbers to dots (and of matrices of appropriate dimensions to the arrows) is called a *quiver representation*. Summarizing, we can thus say that branes whose transverse space is an orbifold space are classified by a quiver representation. The quiver is assigned once and for all by the structure of the singularity, and in fact we will see shortly that all these branes considerations are often but a reformulation of mathematical concepts.

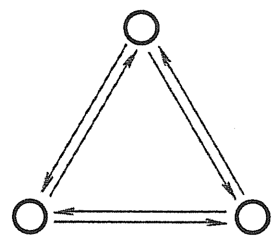


Figure 3.2: Quiver for $\mathbb{C}^2/\mathbb{Z}_3$.

All of this can be also applied to the orbifold Landau-Ginzburg we have found in the Gepner limit of the moduli space. The scalars are now not describing displacements in physical space, but in an auxiliary space; moreover, the theory is not even conformal before RG flow. But the idea is again that the holomorphic relations will be protected [19] against the flow, and thus we can pretend the theory is the conformal one and that its "branes" are the same as the actual branes of the theory after the flow.

In this sense, branes in the Gepner limit will be again quiver representations, and we can use now techniques borrowed from the part of algebra which studies these objects to ask questions about branes.

In particular, one of the important upshots of this section is that coherent sheaves are not enough, and that there is a limit in which branes are a different kind of mathematical object, a quiver representation. The question comes, what are branes away from the two limits? there should be some concept which can describe both things at will. It is instructive to arrive at the answer to such a question in the following way: a brane which is described by a given coherent sheaf in the large volume limit can be, if one neglects (as we are doing in this chapter) stability issues, followed to the Gepner point. It will then be described by some quiver representation, or bound state of them. So, it should be in some way possible to set a "dictionary" between the two kind of objects: bound states of coherent sheaves should be the same as bound states of quiver representations. This strange statement can be made precise and mathematically correct; we will do so while trying to clarify precisely the concrete form of such a dictionary. Before we do so, however, we will point out that this problem seems to be exactly a classical mathematical correspondence, known as

3.3 McKay correspondence.

First of all we will describe what is the so-called *classical* McKay correspondence. This occurs when one considers a singularity of the form $\mathbb{C}^2/\Gamma$, with Γ a finite subgroup of $SL(2,\mathbb{C})$. One can consider two sets of objects. First one is the ring of representations of Γ. Among these there is a special representation ρ_{def}, which we can call it the defining one, in the sense that is precisely the representation $\Gamma \hookrightarrow SL(2,\mathbb{C})$ which defines the singularity; and there are the irreducible representations ρ_i. The relationships between these objects can be summarized by considering all the tensor products $\rho_{\text{def}} \otimes \rho_i$; these can be in turn expanded in the irreps, which are a basis; this defines a matrix a_{ij} through the formula

$$\rho_{\text{def}} \otimes \rho_i = \oplus_j \, a_{ji} \, \rho_j \ . \tag{3.3.9}$$

We can summarize this matrix in a quiver Q with a dot for each irrep, and a_{ij} arrows between the i-th and j-th dots.

On the other side, we can consider a geometric set of objects. We can namely resolve the singularity; this means considering the singular space $\mathbb{C}^2/\Gamma$ as an algebraic variety X, and finding another, nonsingular, algebraic variety $\tilde{X}$ which is *birational* to X. Birationality is a relation typical of algebraic geometry [41]; in this case it essentially means that the two varieties are identical outside the singular locus. Intuitively, we can say that the singularity of X gets replaced by a locus E that we call the *exceptional* locus. In the cases considered here, this singular locus will always be a union of some number of complex projective lines $\mathbb{P}^1$, touching one another at some points. Again we can summarize this by a diagram, drawing a dot for each $\mathbb{P}^1$ and a number of lines between two dots equal to the intersection number of the two corresponding $\mathbb{P}^1$'s.

Now, the result is this. Take the quiver diagram we have obtained from (3.3.9); in this case this will always be unoriented, in the sense that it if there is an arrow between two dots there will always be an arrow going the other way. Take then these couples of arrows and replace them by simple lines. The resulting diagram will always be a Dynkin diagram of extended ADE type.
On the other hand, we have also obtained a diagram from the irreducible components of the resolution of the singularity. It turns out that this diagram is again a Dynkin diagram, of *exactly the same type* as that coming from representation theory, but unextended. This is what is called classical McKay correspondence, and we summarize it in this table:

Classical McKay correspondence	
Γ finite group acts on $\mathbb{C}^2$ $(\Gamma \hookrightarrow SL(2,\mathbb{C}))$	Resolution $\longrightarrow$ singularity: $\tilde{X} \longrightarrow \mathbb{C}^2/\Gamma$
ρ_i irreps of Γ	irred. comp's of $H_2(X,\mathbb{Z})$
$\rho_{\text{def}} \otimes \rho_i = \oplus_j \, a_{ji} \, \rho_j$	$\bigcap$ between components

To generalize this picture to higher dimensional singularities ($\mathbb{C}^n/\Gamma$) is a very natural thing to try, but not so immediate to do [70]. There are various approaches; essentially the questions are on the geometrical side:

- The exceptional locus of the singularity will not be now a union of components. What should generalize then the irreducible components we had in the $n = 2$ case, and the $H_2(E,\mathbb{Z})$ they generate?

- Once we have understood that, what should generalize the intersection numbers we had before?

First question has been asked in various cases, though it seems that no general theorem exists. It seems [50] that the correct group to generalize $H_2(E, \mathbb{Z})$ is the K-theory $K_c(\tilde{X})$ of sheaves supported on the exceptional locus. As to second question, we will see shortly an answer in various cases (though again no general theorem is presented) taking the physical problem of the "dictionary" between the two limits of the moduli space as a guide.

Indeed, the problem we had formulated in previous section seems to be exactly a generalization to higher dimensions of McKay correspondence. First of all, note that passing from the Gepner to the large volume phase is a resolution of the singularity of the LG orbifold. Then, branes in the Gepner phase are quiver representations. But, to irreps of the finite group Γ we can naturally associate the so-called irreps of the quiver: The irrep of the quiver associated to ρ_i is the representation which associates 1 to the i-th dot, and which sets to zero all the matrices associated to the arrows. On the geometrical side, branes are coherent sheaves supported on a Calabi-Yau contained in the exceptional locus E, which are particular cases of sheaves with support on E (though we will see that considering the Calabi-Yau is essential for computations).

Thus, we can see that McKay correspondence

representations of finite group Γ	$\longleftrightarrow$	coherent sheaves supported on the exceptional locus E

has a chance to be implemented through the physical problem

representations of quiver Q	$\longleftrightarrow$	coherent sheaves supported on the Calabi-Yau $Y \subset E$.

As a first example, here is the quiver of the singularity $\mathbb{C}^5/\mathbb{Z}_5$; recall that this should correspond to the quintic Calabi-Yau. There are 5 arrows between each couple of nearby dots, which are represented by matrices z^a_{ij}, $a = 1, \ldots, 5$. We have also displayed relations ($z^a_{ij} z^b_{jk} = z^b_{ij} z^a_{jk}$) which at this point are not justified; they will emerge later from F-term equations.

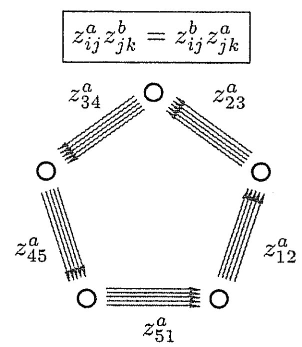


Figure 3.3: McKay quiver for the quintic.

3.4 Physical realization

Let us first of all describe the strategy to write down the dictionary we are looking for.

- First of all one has to understand the "discrete part"; namely, the relations between brane central charges in the two limits. In the Gepner limit, charge depends on the numbers n_i associated to the dots in the quiver representation; in the large volume phase, it depends on the Chern classes of the sheaf. Using mirror symmetry, we can use this to write a change of basis between the n_i and the Chern classes.

- Then we have to find the bundles S_i corresponding to the irreps ρ_i; there are several strategies to do this, all of which require previous step at least as a check.

- This is not enough, however; if we have a coherent sheaf in the geometrical phase, we want to see what is the quiver representation associated to it in the LG phase, and so far we only have the numbers n_i associated to the dots. To find the arrows as well, one has to call into use a mathematical machinery which goes under the name of theory of *helices*. This is the most interesting part.

We will describe more in detail these three steps in examples: the above mentioned quintic, and two other examples for which the Kähler moduli space has dimension two (rather than one). This is probably the place to comment on how to find Calabi-Yau manifolds. The most popular way (all the examples we consider here can be found in this way) is through toric geometry, that we already met in section 1.3.2. The idea is to take a Fano (that is, with ample anticanonical line bundle) toric manifold E of complex dimension 4, and to consider the divisor associated to the anticanonical line bundle on it. This is by adjunction formula automatically a Calabi-Yau $Y \subset E$; even better, a family of them. The only problem is that, in general, this construction will give a singular Y. If E is cleverly chosen, however, one can succeed in finding a family of nonsingular Calabi-Yau's. The condition to have this is that [4] the dual polyhedron Δ associated to the anticanonical bundle K_E^* be integral. In our case, the ambient toric manifold E will be either the easiest case of toric manifold, a weighted projective space $W\mathbb{P}$, or a toric resolution of its singularities. Although the Calabi-Yau in the $W\mathbb{P}$ will be nonsingular, we will need often to work on E, and for this reason we will prefer to resolve its singularities – a process which will not change results on the Calabi-Yau.

Thus, our other two examples will be a hypersurface of degree 8 in the weighted projective space $E = \mathbb{P}^{1,1,2,2,2}$, and one of degree 12 in $\mathbb{P}^{1,1,2,2,6}$; it is sometimes used the notation $\mathbb{P}^{1,1,2,2,2}[8]$ and $\mathbb{P}^{1,1,2,2,6}[12]$.

3.4.1 The quintic

As we have promised, we will start from the comparison of central charges. Let us explain this in more detail. Central charge Z of A and B branes can be found [60] to be:

- For A branes, which are three dimensional cycles, Z is simply the integral of the holomorphic three-form on them, $\int_A \Omega$; these are called the periods.

- For B branes, which are coherent sheaves $\mathcal{E}$, the formula is more complicated, and was historically the first way in which K-theory was found to classify brane charges. In the Calabi-Yau case it can be cast in the form

$$Z = -\int \mathrm{ch}(\mathcal{E}) e^{-t} \sqrt{Td(T_Y)}\ , \tag{3.4.10}$$

  where we have also explicitly included the dependence on the Kähler moduli $t = vol + iB$, a $(1,1)$ form, which we will need in what follows.

We have now, using mirror symmetry, all of the information we need to compare charges in the various phases. In the large volume limit, we already have the formula. In the Gepner limit, by linearity the charge of any brane will be determined by those of the irreps. What are these? There is a $\mathbb{Z}_5$ symmetry at the Gepner point, and it is not surprising that they are an orbit of this symmetry, as one can also see by comparison with boundary states techniques (that we have not reviewed here). Hence the irreps can be obtained as an orbit of the monodromy around the Gepner point starting from the "pure" D6 brane (that is, the D6 brane with no lower charge, which is, at large volume, $\mathcal{O}_Y$), which is one of them. From this we can obtain their charges as the periods Π_i of the mirror $\tilde{Y}$; summing up, the upshot is that $Z(\rho_i) = \Pi_i$ and so that

$$Z = \sum_i n_i \Pi_i \,.$$

We can now use this to find a relation between Chern classes and the n_i. First of all we work out explicitly the central charge of a B brane corresponding to a coherent sheaf $\mathcal{E}$. We need first of all $Td(T_Y)$. From the fact that $c_1(T_Y) = 0$ and the expression of the Todd polynomial in terms of the $c_i(Td(T_Y))$, one gets

$$Td(T_Y) = 1 + \frac{c_2(T_Y)}{24} \,;$$

$c_2(T_Y)$ can be computed from the adjunction formula and properties of the total Chern class $c = 1 + c_1 + c_2 + \ldots$, which give

$$c(T_Y) = \frac{(1+H)^5}{1+5\,H} \,,$$

with H the hyperplane bundle of $\mathbb{P}^4$, whence $c_2(T_Y) = 10H^2_{|Y}$. From this, and using that $H^3_{|Y} = 5$,

$$\begin{aligned} Z &= -\int_Y \left(1 - t\,H + \frac{1}{2}\,t^2\,H^2 - \frac{1}{6}\,t^3\,H^3\right)\left(rk + c_1\,H + \frac{1}{2}\,ch_2\,H^2 + \frac{1}{6}\,ch_3 H^3\right)\left(1 + \frac{5}{12}\,H^2\right) \\ &= \frac{5}{6}\,t^3 rk - \frac{5}{2}\,t^2 c_1 + \left(\frac{5}{2}\,ch_2 + \frac{25}{12}\,rk\right)t - \left(\frac{5}{6}\,ch_3 + \frac{25}{12}\,c_1\right) \end{aligned} \qquad (3.4.11)$$

where for simplicity we have written H instead of $H_{|Y}$, and $ch_i \equiv ch_i(\mathcal{E})$ for short. Periods Π_i of the mirror A branes can be instead computed using the prepotential [17]

$$F = -\frac{5}{6}t^3 - \frac{11}{4}t^2 + \frac{25}{12}t - \frac{25}{12}i\zeta(-3) + \text{nonperturbative corrections.}$$

Comparing $Z = \sum_i n_i \Pi_i$ with the above Z in 3.4.11, we get

$$n_6 = r\,, \quad n_4 = -\,c_1\,, \quad n_2 = \frac{11}{2}\,c_1 - \frac{5}{2}\,ch_2\,, \quad n_0 = \frac{50}{12}\,c_1 + \frac{5}{6}\,ch_3\,. \qquad (3.4.12)$$

Having accomplished this "zeroth" step, we have now to ask what are the sheaves which correspond to the ρ_i. Whatever method we use, the result should be such that the charge agree with respect to the change of basis just found (3.4.12).

Basically there are two methods. First one is to use beyond its proved validity a construction already found in the context of McKay correspondence by mathematicians. This is perhaps the conceptually cleaner way, but is computationally very difficult [19]. The second method is essentially to use a match of $U(1)$ charges in the two limits, and is computationally much easier.

Both methods involve the introduction of an auxiliary set of bundles, that we will call R_i. These are sometimes called *tautological* for reasons that will become clear in a moment. To find these is in both cases the hard part; from this we can find the bundles S_i corresponding to the irreps ρ_i in a relatively easy – but interesting – way.

In the present case of the quintic, the first method is not computationally difficult, whereas second one is almost trivial. We will thus concentrate here on first one, leaving second method to the other examples.

Quiver varieties and tautological bundles

It is remarkable that the construction of the quiver Q in (3.3.9) is exactly the same as the physical construction of invariant branes we sketched in the example $\mathbb{C}^2/\mathbb{Z}_3$ in section 3.2.2. Another associated mathematical construction appears very naturally in brane physics: that of *quiver variety*. Let us briefly review this again in the same example and then in general.

Almost all of the branes that we have described in 3.2.2 are confined to stay in the singularity. Consider indeed for example a brane corresponding to the choice of an irrep. If this was to move outside the singularity, it would not be an invariant configuration any longer, since it would correspond, in the covering $\mathbb{C}^2$, to a D0 in one of the three identified domains without any image in the other domains. By the way, for this reason the charge of this brane is defined to be $1/3$ of that of a D0, and the brane is called often a *fractional* one.

A true D0[2], free to move in the bulk of this singular space, outside the singular point, can be done instead putting an image on all of the three identified domains. This means, to take the preceding fractional brane along with its two images under $\mathbb{Z}_3$. This means, taking as ρ the sum of the three irreps; that is, the regular representation, $\rho = diag(1, \omega, \omega^2)$.

The moduli space of this D0 is the space itself, as usual. This can be seen again from D- and F-flatness of the brane-worldsheet theory, as in section 2.3.2:

$$[Z^1, (Z^1)^\dagger] + [Z^2, (Z^2)^\dagger] = \Theta\,, \qquad [Z^1, Z^2] = 0\,,$$

where $\Theta = diag(\theta_1, \theta_2, \theta_3)$ is a traceless diagonal matrix coming from a FI term and, in our example, Z^i are in the form given in (3.2.8):

$$Z^1 = \begin{pmatrix} & A_1 & \\ & & B_1 \\ C_1 & & \end{pmatrix}, \qquad Z^2 = \begin{pmatrix} & & C_2 \\ A_2 & & \\ & B_2 & \end{pmatrix}. \tag{3.4.13}$$

Explicitly this gives us the equations

$$\begin{array}{l} |A_1|^2 - |A_2|^2 - |C_1|^2 + |C_2|^2 = \theta_1 \\ |B_1|^2 - |B_2|^2 - |C_1|^2 + |C_2|^2 = \theta_2 \\ |A_1|^2 - |A_2|^2 - |B_1|^2 + |B_2|^2 = \theta_3 \end{array}\,, \qquad A_1\, A_2 = B_1\, B_2 = C_1\, C_2\,. \tag{3.4.14}$$

These describe the physical transverse space as their zero locus; this is what is meant when one says that space is a derived concept from the point of view of D-branes. In this case, actually, this zero locus is best analyzed in another form: the deformation due to Θ can be put in the F-term (the second set of equations in (3.4.14)); if one defines $z_i = \det(Z^i)$, $A_1\, A_2 = a$, $B_1\, B_2 = b$, $C_1\, C_2 = c$, the equation becomes eventually

$$z_1\, z_2 = a\, b\, c\,, \qquad a = b + \zeta_1 = c + \zeta_2$$

that is, $z_1\, z_2 = a\,(a - \zeta_1)\,(a - \zeta_2)$. For $\zeta_i = 0$, this is a description of the singularity $\mathbb{C}^2/\mathbb{Z}_3$ we started with; switching ζ_i on gives a resolution of it. The same can be in principle seen also in the original form of the equations (3.4.14), though in a longer way [30].

So, we have seen in this example that the locus described by D- and F-terms is exactly the transverse space singularity, and that introducing a FI term gives a resolution. This is a general feature; in fact this construction was already considered by mathematicians under the name of quiver variety.

[2]Here and in the following situations, such as Calabi-Yau compactifications, we will abuse the terminology and call for instance D0 any brane which in the transverse space under consideration appears to be a point, whatever its extension in the other directions.

To describe this, it is time to describe in more detail the higher dimensional analogue of classical McKay correspondence, the so-called generalized one. Let thus $\Gamma \subset SL_n(\mathbb{C})$. This inclusion means an action of Γ on $\mathbb{C}^n$, and this gives again a singularity $\mathbb{C}^n/\Gamma$. Consider the irreducible representations ρ_k of Γ, and the n-dimensional representation ρ_{def} given by $\Gamma \subset SL_n(\mathbb{C})$. We can again define a quiver whose dots represent ρ_k, whose arrows are given by numbers $a_{ij}^{(1)}$ given by $\rho_{\text{def}} \otimes \rho_j = \oplus\, a_{ij}^{(1)} \rho_i$. For later convenience, we supplement this quiver with relations which the X should satisfy; in the case in which Γ is abelian and cyclic, these can be written $X^i_{a,a+w_i} X^j_{a+w_i,a+w_i+w_j} = X^j_{a,a+w_j} X^i_{a+w_j,a+w_i+w_j}$ (which, in a compact notation, can be written as $[X^i, X^j] = 0$). Physically, these relations come from the F-term equations. The final quiver is called a *quiver with relations*.

The manifold described by transversal motions of branes can be found by finding the space of solutions to D and F equations for branes which correspond to the regular representation ρ, as we have just explained in the $\mathbb{C}^2/\mathbb{Z}_3$ example above., there is an image of the brane on each of the three identified sheets. The procedure we have explained above for solving these equations can then be formalized as follows. First of all, taking the invariant blocks (as in (3.4.13) in the previous example) can be formalized as considering the space $(\rho_{\text{def}} \otimes \operatorname{End} \rho)^\Gamma$; indeed, X belonging to $\rho_{\text{def}} \otimes \operatorname{End} \rho$ is nothing but a triple of X^i transforming as, for instance, in the right hand sides of (3.2.7); the notation $(\cdot)^\Gamma$ simply means taking the invariant triples – in the example, imposing the equality as in (3.2.7). The step of imposing D-term equations and of dividing by the residual gauge invariance can be formalized then by writing a so-called *Geometric Invariant Theory* quotient:

$$M \equiv \frac{\{X \in (\rho_{\text{def}} \otimes \operatorname{End} \rho)^\Gamma | [X^i, X^j] = 0\}}{\mathrm{GL}_\Gamma(\rho)}, \quad \mathrm{GL}_\Gamma(\rho) \equiv (\operatorname{End} \rho)^\Gamma . \tag{3.4.15}$$

It turns out that M is a resolution of $\mathbb{C}^n/\Gamma$, generalizing the 2-dimensional result we have seen in an example and shown in general by Kronheimer [55].

In a special case of this construction (like the resolutions obtained as Hilbert schemes of points [50]), we can go further, and define the auxiliary set of bundles R_i we are looking for. Define P as the numerator of (3.4.15), and view it as a principal fibration $P \to M$; then we can define the bundle associated to the regular representation $\mathcal{R} \equiv P \times_{\mathrm{GL}_\Gamma(\rho)} \rho$, and the ones associated to the irreps ρ_i, $\mathcal{R}_i$; both are called tautological bundles. What we will actually need in the sequel will be their restrictions $R_i = (\mathcal{R}_i)_{|E}$ to the exceptional locus.

While we are at it, we describe a way to describe the S_i as well, though we will come back on it later. Multiplication by the coordinates, which is again implemented using the defining representation ρ_{def}, defines a complex

$$\mathcal{R} \to \rho_{\text{def}} \otimes \mathcal{R} \to \Lambda^2 \rho_{\text{def}} \otimes \mathcal{R} \to \ldots \to \Lambda^n \rho_{\text{def}} \otimes \mathcal{R} \cong \mathcal{R}, \tag{3.4.16}$$

which can be decomposed as

$$\mathcal{R}_i \to \oplus\, a_{ji}^{(1)} \mathcal{R}_j \to \oplus\, a_{ji}^{(2)} \mathcal{R}_j \to \oplus\, a_{ji}^{(3)} \mathcal{R}_j \to \ldots \to \mathcal{R}_i \tag{3.4.17}$$

where $\Lambda^k \rho_{\text{def}} \otimes \rho_i = \oplus\, a_{ji}^{(k)} \rho_j$. What is important for us is that the complex (3.4.17), that we call S_i, is exact outside the exceptional locus, and thus defines an element of the K-theory supported on it; and that these S_i are dual to the $\mathcal{R}_i$ in a sense we will see below.

Thus, from this point of view we have what we wanted (the second point of the ones described at the beginning of section 3.4: a map which associates to each irrep of Γ ρ_i a K-theory class S_i on the exceptional divisor (and hence, by restriction, on the Calabi-Yau).

In the case of the quintic, the exceptional locus E of the resolution of the singularity $\mathbb{C}^5/\mathbb{Z}_5$ is $E = \mathbb{P}^4$. It is thus no surprise that the tautological bundles R_i are nothing but powers of what we have already called tautological bundle for the $\mathbb{P}^n$ case: more precisely, they are $R_i = \mathcal{O}(i)$ (there would not be much choice anyway, since on $\mathbb{P}^n$ line bundles are classified by their degree $\in \mathbb{Z}$).

We will describe now the procedure to find what we are really looking for, the S_i, the bundles (or sheaves) corresponding at large volume to the irreps ρ_i of the quiver. We have anticipated above that they are dual to the auxiliary set of bundles R_i; the precise meaning of this that $(R_i, S_j) \equiv \chi(R_i, S_j) = \delta_{ij}$ on $\mathbb{P}^4$. The question is thus now how to find the S_i that are dual to the R_i. Doing it by solving a system of equations is certainly not economical; the analogy with other inner products comes to the rescue, suggesting a Gram-Schmidt procedure.

The first one, S_1, obviously equals R_1 itself. Next, let us notice that our bundles $R_i = \mathcal{O}(i)$ have the property $(R_i, R_j) = 0 \; , \forall i > j$. Then the others S_i can be obtained as

$$\begin{aligned} S_2 &= -R_2 + (R_1, R_2)R_1, \\ S_3 &= R_3 - (R_1, R_3)R_1 - (R_2, R_3)S_2 = \\ &\quad R_3 - (R_2, R_3)R_2 + [(R_1, R_2)(R_2, R_3) - (R_1, R_3)]\, R_1, \end{aligned} \tag{3.4.18}$$

and so on. These sums are meant as sums in the Grothendieck groups of coherent sheaves reviewed in section 3.1.1. Of course near the end of the series we can use Serre to make the computations simpler.

This would be enough of course to compute the Chern classes of the S_i, but in this case we can actually do better. In fact, we have on $\mathbb{P}^4$ the Euler sequence

$$0 \to \mathcal{Q}^* \to H^0(\mathbb{P}^4, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1) \to 0 \, , \tag{3.4.19}$$

where $\mathcal{Q}^* \equiv \Omega \otimes \mathcal{O}(1)$ (Ω is the holomorphic cotangent bundle). Comparing with first equation in (3.4.19) and with the definition of Grothendieck group, we see that $S_2 = -\mathcal{Q}^*$. Similarly, repeated use of this sequence and of its exterior products yields the result

$$S_i = (-)^{i-1} \Lambda^i \mathcal{Q}^* \, .$$

We will see later that this is an example of the theory of mutations of helices; this is a first hint that that theory will be relevant.

Meanwhile, we can check that the correspondence $S_i \leftrightarrow \rho_i$ is in agreement with the change of basis we derived at the beginning of this section. To do so, on one side we compute the Chern classes of the $(S_i)_Y$ from the Euler exact sequence (3.4.19) and its exterior powers: we get

$$\begin{aligned} &\mathrm{ch}(\mathcal{Q}^*) = 5 - e^{\omega} = 4 - \omega - \tfrac{1}{2}\,\omega^2 - \tfrac{1}{6}\,\omega^3 \, , \\ &\mathrm{ch}(\Lambda^2 \mathcal{Q}^*) = 10 - 5e^{\omega} + e^{2\omega^2} = 6 - 3\omega - \tfrac{1}{2}\,\omega^2 - \tfrac{1}{2}\,\omega^3 \end{aligned} \tag{3.4.20}$$

and so on. As above, we have indicated by ω the restriction $\omega_{|Y}$, for short; for the same reason ($\omega^4_{|Y} = 0$) the expansions are truncated to order ω^3.
On the other hand, we have said that the S_i become near the orbifold point the images of the D6 under the orbifold monodromy. Using mirror symmetry computations [17] one can find the monodromy to be [16]

$$\begin{bmatrix} -4 & -1 & -8 & 5 \\ -5 & 1 & 5 & 3 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} ;$$

by repeated application of this matrix on the pure D6, one obtains states of charges (n_6, n_4, n_2, n_0) as follows:

$$\begin{aligned} v_1 &= (1,0,0,0)\ , \\ v_1 &= (-4,1,8,5)\ , \\ v_1 &= (6,3,19,-10)\ , \\ v_1 &= (-4,-3,-14,10)\ , \\ v_1 &= (1,1,3,-5)\ ; \end{aligned} \tag{3.4.21}$$

using then the change of basis (3.4.12), it is easy to check that the vectors of charges (3.4.20) and (3.4.21) indeed agree.

Now last step is remaining: we do not know yet how to reconstruct arrows from a given coherent sheaf. Again we can take profit of the fact that the exceptional divisor E in which the Calabi-Yau Y is embedded is in this case a very special space, $E = \mathbb{P}^4$; and so we can make use of

Beilinson theorem.

This construction [6], in most simple terms, allows us to decompose a bundle in terms of a "basis". The procedure works as follows: Start from a sheaf F on the projective space $\mathbb{P}^n$, and pull it back to the product $\mathbb{P}^n \times \mathbb{P}^n$. On the latter space, there is a projective resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta = \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^n$, which reads

$$0 \to \Lambda^n\left(\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{Q}^*\right) \to \ldots \to \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{Q}^* \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0\ , \tag{3.4.22}$$

where $\mathcal{Q} \equiv \mathcal{T}(-1)$ is the universal quotient bundle, where $E \boxtimes F = \pi_1^*(E) \otimes \pi_2^*(F)$ (π_i are projections on the two factors).

We can now tensor this resolution with $\pi_1^* F$; then take an injective resolution $I^{\bullet\bullet}$ of this complex (which is, by definition, a double complex), and apply to it the direct image of the second projection π_{2*}. Consider now the cohomology of the double complex $\pi_{2*} I^{\bullet\bullet}$ obtained in this way; as usual, this can be computed by spectral sequences. There are two spectral sequences, depending on which filtration one chooses; one gives the result that the cohomology of this double complex is present only in total degree zero, and its sum is the sum of the grades of a filtration of the original F; the other one has E_1 term

$$E_1^{p,q} = H^p(\mathbb{P}^n, F(q)) \otimes \Lambda^{-q} \mathcal{Q}^*. \tag{3.4.23}$$

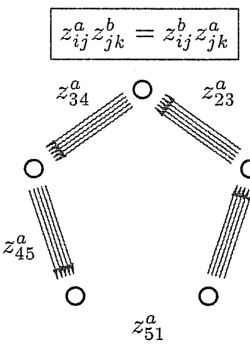


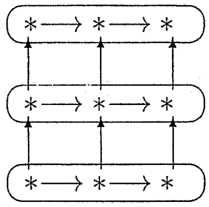
Figure 3.4: Beilinson quiver for the quintic.

To be more precise: the grades here have the range $0 \le p \le n$, $-n \le q \le 0$, and the cohomology whose sum corresponds to F is in grades $p = -q$.

As a first hint towards the physical relevance of derived category, let us anticipate here that, in that framework, this procedure has a clear interpretation as a Fourier-Mukai transform [79]. (Indeed, the whole process can be seen as $\mathbf{R}\pi_{2*}(\pi_1^* F \otimes \mathcal{O}_\Delta)$, where $\mathbf{R}$ is the derived functor within the derived category; and it is clear that it is an identity from the derived category to itself.)

What is interesting for us is that the double complex (3.4.23) has an interpretation as a complex of quiver representations. Let us make this more precise: Introduce the algebra $A \equiv \mathrm{Hom}(\oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i), \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i))$. This is the path algebra of a quiver, that we will call *Beilinson quiver* (for $\mathbb{P}^n$), and which is the quiver with $n+1$ dots, $n+1$ arrows between each pair of consecutive dots, and relations as we described for the McKay quiver $[X^i, X^j] = 0$ (see figure 3.4, where we have specialized the

general n construction to our $n = 4$ case).
Now we interpret each line of (3.4.23) as a representation of this quiver as in [29]. In detail, let each $\Lambda^i \mathcal{Q}^*$ correspond to the i-th irreducible representation of the quiver, as we have established before; then, each arrow in the line is a morphism $\in \mathrm{Hom}(\Lambda^i \mathcal{Q}^*, \Lambda^{i+1} \mathcal{Q}^*) = \mathbb{C}^{n+1}$; hence its components in a basis give matrices corresponding to the arrows between two consecutive dots.



We have pretended here that only one line is on; in general all lines of the double complex will be non zero. The interpretation of such a case is that (see figure) the whole double complex is a complex (in vertical) of representations of the Beilinson quiver. We will see that the proper place for this is the derived category.

Let us finally note that the charge can be extracted, from a complex which represents an object in the derived category, as the alternated sum of its terms. From this we can read the orbifold charges n_i, which equal $\chi(F(-i)) = \chi(\mathcal{O}(i), F) \equiv (\mathcal{O}(i), F)$.

The whole program we have described at the beginning is thus completed: we have a dictionary between branes at large volume (coherent sheaves) and branes in the Gepner limit (quiver representations). One could complain about the fact that in each line we actually gave representations of the Beilinson quiver (figure 3.4), and not of the quiver we had expected from the McKay discussion (figure 3.3). Note, however, that Beilinson one is a subquiver of the McKay quiver, in the sense that it can be obtained from it by taking away some arrows. So, if we have a representation of Beilinson quiver, we can always give a representation of the McKay one, simply by representing the "missing arrows" as zeroes. We will see that the same graphical feature remains there in more spectacular examples later.

There are two limitations of the results we have seen so far. First one is that the procedure is not general: we used heavily the fact that the quintic $Y \subset \mathbb{P}^4$. Finding a more general procedure will be the aim of next subsections. Second limitation, to which we have not found any remedy so far, is that this only works for sheaves which come from restriction of sheaves on the exceptional divisor $E = \mathbb{P}^4$.

3.4.2 $\mathbb{P}^{1,1,1,2,2}[8]$

In this example we will start seeing the salient features of the problem. The zero-th step of comparing central charges is again easily done [19].

Then, one has to find the S_i. One method is the one we have explained above involving tautological bundles R_i; this has been done for this example [19], checking that it is a correct method but also its computational difficulty. We will try instead another method, first inspired by guesswork; this involves again finding the auxiliary bundles R_i, but not describing them as tautological, and then dualizing them to find the S_i.

The idea can come noting the particular structure of this example. Resolving the weighted projective space we find a toric manifold with charge matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad (3.4.24)$$

which means that it is a $\mathbb{P}^3$ fibration over $\mathbb{P}^1$: $\mathbb{P}\left(3\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)\right)$. Let us call H the divisor corresponding to one of the first three vectors in the fan (in the same ordering of the charge matrix), and L

the divisor corresponding to the fourth (or fifth) vector. These are respectively given by an hyperplane in the fiber $\mathbb{P}^3$, and a hyperplane in the base $\mathbb{P}^1$. In these terms, the results of [19] for the $R_i \equiv \mathcal{R}_{i|_E}$ read

$$\begin{array}{ll} R_1 = \mathcal{O} & R_2 = \mathcal{O}(L) \\ R_3 = \mathcal{O}(H) & R_4 = \mathcal{O}(H+L) \\ R_5 = \mathcal{O}(2H) & R_6 = \mathcal{O}(2H+L) \\ R_1 = \mathcal{O}(3H) & R_8 = \mathcal{O}(3H+L) \end{array} \quad (3.4.25)$$

(we changed notation with respect to [19]: there the R_i are ordered differently and are the duals of (3.4.25). This is, however, taken into account by us by a change in the orthogonality condition). One immediately notices that the pattern followed is very easy: the coefficient of L (which is relative to $\mathbb{P}^1$) goes from 0 to 1, and the coefficient of H (which is relative to $\mathbb{P}^3$) goes from 0 to 3. What can one learn from this relationship between R_i and the multiple fibration structure? In next example, which has again such a structure, though more complicated, we will try to guess R_i in a form again suggested by this, but this time without the reassuring tautological bundle computation; and we will show that the guess is indeed right. After these two examples, the structure of the R_i in the multiple fibration cases will be clear; however this is only a subclass of examples, and one may wonder how is the general strategy. This will not face this here, as the most important part will be the helix structure that the guesses done for these examples will help to uncover. However, we can peep at the general strategy noting how the R_i look like when restricted to the weighted projective space. There, there is only one generator J of $H^{(1,1)}$, and our two generators go in the singular limit as $L \to J$ and $H \to 2J$. The list (3.4.25) becomes this $\mathcal{O}(k\,J)$ on $\mathbb{P}^{1,1,1,2,2}$; this regularity is precisely what helps us in the general case.

We now have to dualize the R_i, and next to find some analogue of Beilinson theorem. Remarkably, both steps use the same mathematical concepts, taken, as we have anticipated many times, from the theory of helices. Let us thus give a short account here of this piece of mathematics, giving at the same time a new interpretation to the Gram-Schmidt we would have to use to dualize the R_i.

Helices and mutations

Let us, for the time being, *assume* that further properties hold for the R_i. Namely, let us suppose that

$$\mathrm{Ext}^k(R_i, R_j) = 0 \quad \forall i > j,\ \forall k \quad (3.4.26)$$

and that

$$\mathrm{Ext}^k(R_i, R_j) = 0 \quad \forall i \leq j,\ \forall k > 0\,. \quad (3.4.27)$$

In other words, given the ordering of the R_i's, if one takes Ext of a lower sheaf with a higher one, there is only Ext^0; if one takes it in reverse order, namely of a higher sheaf with a lower one, there are no Ext at all. These conditions together make the R_i, by definition, an *exceptional series* [71]. Using the second set of them, we can interpret S_2 as being given by an exact sequence

$$0 \to S_2 \to \mathrm{Hom}(R_1, R_2) \otimes R_1 \to R_2 \to 0\,; \quad (3.4.28)$$

that is, S_2 is the kernel of the natural evaluation. This is usually called a *mutation* [71, 94, 48] of R_2 within the exceptional series $\{R_i\}$, more specifically a left mutation, and noted as LR_2. In a similar way, we can interpret S_3 as the first term in the sequence

$$0 \to S_3 \to \mathrm{Hom}(R_1, LR_3) \otimes R_1 \to \mathrm{Hom}(R_2, R_3) \otimes R_2 \to R_3 \to 0; \quad (3.4.29)$$

this sequence is obtained joining two sequences of the type (3.4.28), the first of which is

$$0 \to LR_3 \to \mathrm{Hom}(R_2, R_3) \otimes R_2 \to R_3 \to 0 \quad (3.4.30)$$

and defines LR_3. Using (3.4.29) and our assumptions we obtain

$$\begin{aligned} S_3 &= R_3 - (R_2, R_3)R_2 + (R_1, LR_3)R_1 = \\ &\quad R_3 - (R_2, R_3)R_2 + [(R_1, R_2)(R_2, R_3) - (R_1, R_3)]\, R_1 \end{aligned} \tag{3.4.31}$$

in agreement with result for S_3 in (3.4.19).

It is natural to ask oneself whether the series we constructed in our two examples are exceptional. As it turns out, even more is true: they are what is called a foundation of a *helix*. This means that the series $\{R_i\}_{i=1}^n$ can be extended infinitely in both senses, in such a way that any n consecutive elements make up an exceptional series, and that the "periodicity" condition $R_{n+1} = R_1 \otimes K^*$ holds.

Note that we have already seen an example of a helix and of a particular mutation of it; this was in our quintic example, which involved bundles $R_i = \mathcal{O}(i)$ on $\mathbb{P}^4$. Indeed, it is easy to see that

- $\mathcal{O}(i)$, $i = 0, \ldots, 4$, are a helix;
- The Euler sequence (3.4.19) is a particular case of mutation, as in (3.4.28);
- The bundles $\Lambda^i Q^*$ are another helix, obtained by successive mutations of the helix $\mathcal{O}(i)$.

We have now two things which make us curious about helices. First one is that they appear to be closely related to the Gram-Schmidt procedure we would have to use anyway. Second one is that, after dualization, our next task will be to find an analogue of Beilinson theorem; and in that theorem precisely the bundles $\mathcal{O}(i)$ and $\Lambda^i Q^*$ were the main characters, which we just saw to be helices.

These two things make one strongly suspect that the bundles R_i we consider here (and in any other example) are actually a helix. This requires a bit of technical analysis, which we will hide in an appendix at the end of next subsection. The answer is in the affirmative. This means that the R_i are a helix, but it does not automatically imply that also the S_i are: not on any manifold mutations of helices give other helices. Again, however, one can use derived categories to modify the definitions in such a way that the S_i are now a helix in this modified sense.

The check that the S_i obtained in this way are the correct correspondents to ρ_i will not be done here [19], since we will present an essentially identical computation in next example. For completeness we should note that both here and in next example these S_i are not given explicitly, but only through their K-class. It is possible to give a more detailed derived category description.

The discovery that the R_i are a helix is not computationally important for the dualization procedure, but is essential for the last point, "finding the arrows". We have already noted that two helices appear in Beilinson theorem. And indeed it turns out [10] that there is a generalization, *Bondal theorem*, to more general Fano varieties, making use *exactly of the helix property*.

An appropriate description of this again involves derived categories, and indeed can be seen very naturally as one of the main reasons for the emergence of derived category in brane physics. Anyway, we can describe here the main features of the story. The input is a helix R_i on the Fano manifold E (the exceptional divisor). Then, we can again (as for Beilinson) define the algebra $\mathcal{A}_E = \mathrm{Hom}(\oplus_i R_i, \oplus_i R_i) = \oplus_{i,j} \mathrm{Hom}(R_i, R_j)$ (product is composition of morphisms). Now, to any quiver Q one can associate an associative algebra, its *path algebra* $\mathcal{A}_Q$, in the way the name suggests: elements are paths, that is, juxtapositions of arrows with tails after heads; product of two paths is their composition when they are one a possible continuation of the other (if the tail of first path coincides with the head of first one) and zero otherwise. (This is a particular case of the construction that associates an algebra to a category; what we will do next would be better done at the level of categories.) Now, it turns out that there is a quiver Q_E whose path algebra $\mathcal{A}_{Q_E}$ is precisely $\mathcal{A}_E$.

Beilinson theorem gave as an output a double complex. We interpreted each line as a representation of a quiver, and the whole structure as a complex of representations. The same happens here: each line is replaced by a "layer" which is a quiver representation, and the whole structure is then a complex of such representations. Even without having to be too formal, we can give here the idea of how each layer is constructed.
One takes a resolution of the sheaf $\mathcal{E}$, and then, if $\mathcal{E}_k$ is the k-th object of this resolution, the k-th layer is constructed in the following way. We have all the vector spaces $\mathrm{Hom}(R_i, \mathcal{E}_k)$ representing the dots; the arrows are given by the maps

$$\mathrm{Hom}(R_i, \mathcal{E}_k) \longrightarrow \mathrm{Hom}(R_j, \mathcal{E}_k)$$

induced by composition with $\mathrm{Hom}(R_i, R_j)$. This maps give by construction exactly the number of arrows they should give, with the required relations. This generalizes Beilinson construction.

Now, let us see what is the quiver Q_E constructed in this way and and if it has to do with the expected McKay quiver. The result is given in picture 3.5: Dashed arrows are those that are set to zero in Bondal quiver. Relations are not displayed here.

In this example we described essentially all the features of the general picture. The sheaves S_i which correspond in the large volume to the irreps ρ_i should be found by successive mutations of a helix $\{R_i\}$. This helix is then needed to complete the correspondence to arrows, using Bondal theorem. All this picture is admittedly a little conjectural: does the required helix always exist? is this found by the rough method we have sketched here? we will describe here another example, leaving others to the literature [38, 59].

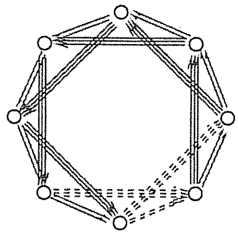


Figure 3.5: McKay and Bondal quivers for this example.

3.4.3 $\mathbb{P}^{1,1,1,2,6}[12]$

The ideology has already been described above; this example is only meant as a nontrivial check, and we will essentially present computations, with some note here and there.

As a preliminary, we need some information about the exceptional divisor E. It will be more natural here to postpone the change of basis between Chern classes and quiver numbers n_i to when we will actually need it, since the computation requires details about the manifold.

We start from the weighted projective space and we resolve it to obtain a multiple fibration of projective spaces. We do that in two steps. First we resolve the locus $z^4 = z^5 = 0$, thus adding a new homogeneous coordinate z^6; then resolve again the locus $z^2 = z^3 = z^6 = 0$. The final fan and charge matrices read

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & -6 & -3 & -1 \\ 0 & 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.4.32)$$

This means that the resulting space has the following structure: A fibration in $\mathbb{P}^1$ over a base, $F_{0,-2}$, which is itself a fibration (with a notation that generalizes the standard one for Hirzebruch surfaces) in $\mathbb{P}^2$ over $\mathbb{P}^1$. In analogy with the previous case, let us call respectively B, H and L the divisors corresponding to the hyperplanes in the fiber $\mathbb{P}^1$, in the $\mathbb{P}^2$ and in the base $\mathbb{P}^1$. In analogy with what we observed in the previous case, there should be $2 \times 3 \times 2 = 12$ R_i:

$$\begin{array}{ll} R_1 = \mathcal{O} & R_2 = \mathcal{O}(L_2) \\ R_3 = \mathcal{O}(L_1) & R_4 = \mathcal{O}(L_1 + L_2) \\ R_5 = \mathcal{O}(2L_1) & R_6 = \mathcal{O}(2L_1 + L_2) \\ R_7 = \mathcal{O}(B) & R_8 = \mathcal{O}(B + L_2) \\ R_9 = \mathcal{O}(B + L_1) & R_{10} = \mathcal{O}(B + L_1 + L_2) \\ R_{11} = \mathcal{O}(B + 2L_1) & R_{12} = \mathcal{O}(B + 2L_1 + L_2). \end{array} \tag{3.4.33}$$

This 12 is exactly the order of the singularity we started with (let us recall that the compact toric variety we are talking about here is the exceptional divisor E of the resolution of this singularity). The fact that, for instance, the coefficient of H in (3.4.33) ranges from 0 to 2 is fixed, in the framework of this procedure, by the fact that it corresponds to a hyperplane in $\mathbb{P}^2$.
It is important to note that the one we wrote down is not, obviously, the only possible resolution of the initial weighted projective space. In particular, it is not the same which was alluded to in [18]; in that case there is only one step, and the exceptional locus is a ruled surface. There are in general, indeed, other possibilities of obtaining a multiple fibration by resolving; what makes the resolution we chose more special, and having to do with the fit $12 = 12$ we obtained above, is that this one does not change the canonical class – that is, it is *crepant*. We will come back to this later.

We can describe, in any case, a rough motivation for this procedure (apart from the suggestions coming from (3.4.25)). The $\mathcal{R}_i$ of the McKay correspondence are defined as tautological bundles on M, the whole non-compact resolution of $\mathbb{C}^n/\Gamma$. The idea is that it could even be that the relevant information is already contained on the exceptional locus E; if we are able to resolve this space, in turn, in order to reveal in its interior some projective spaces, then it is reasonable to think that the tautological bundles, restricted to these projective spaces, become powers of the tautological bundle on them. We will come back again on these ideas later, after having checked the conjecture in the example we chose, and having noted a few nice fits that make the picture more plausible.

To proceed, we also need some pieces of information about the divisors in this variety and their intersections. The Picard group is generated by three divisors B, H, L; we obtain the relations

$$L^2 = 0, \quad H^2(H - 2L_2) = 0, \quad B(B - 3H) = 0, \quad B\,H^2\,L = 1. \tag{3.4.34}$$

Moreover, the anticanonical divisor is $-K = 2B$. It follows that on the Calabi-Yau submanifold Y

$$B_Y = 3H_Y, \quad H_Y^3 = 4, \quad (H^2 L)_Y = 2, \tag{3.4.35}$$

the subscript $(\ )_Y$, which we will hereafter drop when no confusion is possible, meaning restriction to Y. This matches with the results of [18], and allows us to use results in the literature which we will need. In particular, from now on we denote by h and l the generators of the curves on Y, duals to H and L in the sense that

$$\begin{array}{ll} (H \cdot h)_Y = 1, & (H \cdot l)_Y = 0 \\ (L \cdot h)_Y = 0, & (L \cdot l)_Y = 1. \end{array} \tag{3.4.36}$$

We can now find the S_i, and restrict them to Y. Since we have already described the method, let us just give the final result for the $(S_i)_Y \equiv V_i$:

$$\begin{array}{ll} \mathrm{ch}(V_1) = 1 & \mathrm{ch}(V_2) = -1 + L \\ \mathrm{ch}(V_3) = -2 + H - 2L + 2h + l + \frac{2}{3} & \mathrm{ch}(V_4) = 2 - H - l + \frac{1}{3} \\ \mathrm{ch}(V_5) = 1 - H + 2L + l - 2h + \frac{4}{3} & \mathrm{ch}(V_6) = -1 + H - L - l - \frac{1}{3} \end{array} \tag{3.4.37}$$

(of course the first, integer, numbers, are element of H^0, while the final ones, fractionary, mean elements of H^6). We have not listed the others because $V_i + V_{6+i} = 0$, a result which parallels similar ones for the other toric varieties and that matches the relation on the periods in the orbifold basis $\varpi_i + \varpi_{i+6} = 0$.
From the latter relation and the mirror map found in [53, 73], we find the monodromy to be

$$\begin{bmatrix} -1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ -1 & 1 & -1 & -1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \tag{3.4.38}$$

and as a consequence, acting repeatedly on the pure pure D6-brane state, we obtain the states

$$\begin{aligned} v_1 &= (1,0,0,0,0,0) \\ v_2 &= (-1,0,1,-2,0,0) \\ v_3 &= (-2,1,-2,-1,2,1) \\ v_4 &= (2,-1,0,4,0,-1) \\ v_5 &= (1,-1,2,-1,-2,1) \\ v_6 &= (-1,1,-1,-2,0,-1) \end{aligned} \tag{3.4.39}$$

and their negatives. The charges in (3.4.39) are listed as $(n_6, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$; to complete the check, we have to compare these charges with the ones in the Chern polynomials, the zero-th step we have described before. This is accomplished again by comparing the central charges in the two bases. We find

$$r = n_6, \quad c_1 = n_4^1 H + n_4^2 L, \quad \mathrm{ch}_2 = n_2^1 h + n_2^2 l, \quad -\mathrm{ch}_3 = n_0 + \frac{13}{2} n_4^1 + 2n_4^2, \tag{3.4.40}$$

using which the check can now be easily completed, comparing (3.4.37) and (3.4.39).

Appendix *Helices: the proof*

We will describe now the proof of the claims done in these last two subsections, that the R_i make up a foundation of a helix both in the example of this paper and in the one given in [19]. We will limit ourselves to describe the main ideas, skipping details when they become too technicals.

The latter case is easier, so let us start by that one. What we have to do is to compute cohomology groups in toric geometry; there is a standard method to do that [34], but we find it easier (and perhaps more instructive) to use a mix of this and of other techniques. In the terminology of [19], we have to check that the bundles $kH + L, kH$ for $k = 0, \dots, 3$ and $kH - L$ for $k = 1, 2, 3$ enjoy the property (which we will call acyclicity) $h^i = 0 \; \forall i > 0$.
First of all, we will use a consequence of the general method [34]: a sheaf generated by its sections on a toric variety is acyclic. The condition for this to be true, by the general theory, turns out to include all bundles $kH + L$ and kH, but not $kH - L$. To treat these, we use an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}((k-1)H + L) \;\; \to \;\; & \mathcal{O}(kH - L) \to \\ & \mathcal{O}_{|(H-2L)}(kH - L) \cong \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1) \to 0; \end{aligned} \tag{3.4.41}$$

since $\mathcal{O}_{\mathbb{P}^2}(-1)$ has no cohomology, we reduce to the case previously treated.
We have to prove, in addition, that the inverses of these line bundles have no cohomology. We use this

time two kinds of exact sequences, the first kind in which again the restriction to the divisor $H - 2L$ appears, the second one in which instead the restriction is to L. Using these, we can by a zig-zag procedure prove the result for all the bundles we need; a relevant feature is that one sees that there is a range of negative bundles with this property that is just enough bug to include those we are interested in. This is analogous to what happens with the series $\mathcal{O}(1), \ldots \mathcal{O}(n)$ in $\mathbb{P}^n$: in that case the negative bundles from $\mathcal{O}(-1)$ to $\mathcal{O}(-n)$ have no cohomology, and $\mathcal{O}(-n-1)$ starts to have it. It seems as if the conjectural method we described to find the R_i can be viewed as a means to construct helices on multiple fibrations of a certain type. We will see later why this is non trivial.

The second case, $\mathbb{P}^{6,2,2,1,1}$, is more complicated, but is conceptually similar, and we will be very sketchy. The bundles that we have to prove to be acyclic are now: a) $kH + L, kH$ for $k = 0, 1, 2$ and $kH - L$ for $k = 1, 2$; b) $B + kH + k'L$, for $k = -2, \ldots, 2$ and $k' = -1, 0, 1$. The case a) can be reduced to an analysis on a reduced fan which is nothing but a 3-dimensional analog of the $\mathbb{P}^{2,2,2,1,1}$ case that we have just seen. The case b) makes use again of a sequence very similar to (3.4.42), with the restriction to the divisor B appearing instead of that to $H - 2L$. In that case, the divisor turned out to be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and we had at our disposal known vanishing theorems for line bundles on $\mathbb{P}^n$; in this case, the divisor B is isomorphic to $\mathbb{P}^{2,2,1,1}$, and again we can use a 3-dimensional analog of the discussion above to get the vanishing theorems needed.
Finally, we have to show that the bundles inverse to those of the cases a) and b) have no cohomology. Similar techniques to the above let us get the desired results for these bundles as well.

In both cases, what we have really shown is that the R_i are an exceptional collection. To show that they are a foundation of a helix requires to check that $R^{(n-1)}R_1 = R_1(-K)$, where n is the length of the collection. About the proof of this we have nothing special to say, but that it is made easier by reformulating in terms of both left and right mutations; and that it is this last fact is very plausible from the very beginning, due to the peculiarly easy form of the series. In the $\mathbb{P}^{2,2,2,1,1}$ case, indeed, let us observe that, if one were to guess the term following R_8 in the series (3.4.25), one would naturally write $\mathcal{O}(4H)$, which is indeed K^*; the same is true for the case of $\mathbb{P}^{6,2,2,1,1}$, for which the natural guess after the series (3.4.33)would be $\mathcal{O}(2B)$, which is again K^*.

3.4.4 Some final comments

This table summarize the picture that emerged in this section.

Generalized McKay correspondence	
Γ finite group acts on $\mathbb{C}^n$ $(\Gamma \hookrightarrow SL(n, \mathbb{C}))$	Resolution $\longrightarrow$ singularity: $\tilde{X} \longrightarrow \mathbb{C}^n/\Gamma$
ρ_i irreps of Γ	$\{S_i\}$, mutation of a helix $\{R_i\}$ on $E = \pi^{-1}(0)$
McKay quiver: $\rho_{\text{def}} \otimes \rho_i = \oplus_j \, a_{ji} \, \rho_j$	Bondal quiver ($\subset$ McKay): $Q \mid \mathcal{A}_Q = \text{Hom}(\oplus_i R_i, \oplus_i R_i)$

However, there are many things still to be done; the most important conceptually is the fact that, as we have seen, a sheaf corresponds in general only to a *complex* of quiver representations. We will reconsider this later in the framework of derived categories, making the whole picture more reasonable.

We conclude this long section adding some comments about how much general the examples we have presented here are. To begin with, let us come back to our multiple fibration resolution. We already noted that the match between the number of R_i found by our method and the order of Γ is a non trivial check. We add now that another non trivial fact is that we have found a helix on the multiple fibration. The simplest examples of multiple fibrations are given by Hirzebruch surfaces F_k; in this case [71], it is known that, in the cases $F_k, k > 3$ there are no helices made up of line bundles. So our examples could seem to be special in two senses: 1) they yield the match we talked about above; 2) they allow helices on them made up of line bundles.

One special property that our resolutions share is that they are crepant resolutions of the original $E = \pi^{-1}(0)$. This is linked with another, very natural, property that toric varieties can have: that of allowing a non-singular Calabi-Yau inside them. In general, indeed, it is true that one can take in any toric variety a subvariety supported on the anticanonical divisor, and that a so chosen subvariety has formally a trivial canonical bundle; but, for most ambient toric varieties, there would be no way to find a non singular Calabi-Yau in this way. The condition to find non singular Calabi-Yau is that the polyhedron of the toric variety, with respect to its anticanonical sheaf, be integral; and this condition in turn means that the ambient toric variety has only Gorenstein singularities, which admit a partial crepant resolution [4].

So we have found that an event which seems a priori to be very unlikely, the existence on a multiple toric fibration of a helix with the right properties, seems to take place precisely when the toric fibration admits a non singular Calabi-Yau inside. This fact is somewhat surprising, from a mathematical point of view; the reason is probably that the dictionary between sheaves on the large volume Calabi-Yau and quivers is required to exist by physics. Although we concentrated in this paper on a class of examples (in general, as stated above, the crepant resolution is only partial; and the final space after the resolution could be different from a multiple fibration), the structure found here makes it probable that helices play a role in more general cases as well [59, 38].

A final remark is that we could have even guessed that our dictionary makes use of helices, because of their mirror symmetry interpretation [48, 49]. A check of consistency with that approach can also be done [84].

3.5 Derived category

We will now describe here the derived category interpretation of the results. But first we need some

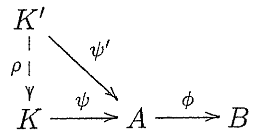
3.5.1 Mathematical background.

Let us start with a crash course on categories. Why should a physicist be bothered with such a seemingly-abstract concept? First of all, categories are not as abstract as one could think at first. The concept is only to relax the usual requirements on product structures in algebra, that require the product to be there for any couple of elements. The way to do this is to introduce first a class of useful labels, called objects. Then, one introduces for any pair of objects a, b, a set of morphisms $\mathrm{Hom}(a, b)$; let us say for the time being that these sets have the structure of a vector space. These are generalization of the elements of the algebra: there is a product $\circ$ such that a $\phi \in \mathrm{Hom}(a, b)$ and a $\phi' \in \mathrm{Hom}(a', b')$ can be multiplied only if $b = a'$: one can imagine the morphisms as arrows, and say that the product is defined only when the head of the first arrow coincides with the tail of the second one. With this important proviso, which is the whole difference between a category and an algebra, all the rest of the definition remains the same: namely, in all the $\mathrm{Hom}(a, b)$ it is required to have an unity element; there should be associativity $(\phi \circ \phi') \circ \phi'' = \phi \circ (\phi' \circ \phi'')$ whenever the products are defined; and $\circ$ should be bilinear in the two entries (distributive law).

The data we have introduced is called a linear category. Mathematicians use the same trick to define various other generalizations of algebraic concepts: if one gives up to the structure of vector space on the Hom's, for instance, one is generalizing the structure of monoid with that of a more general category (with some caveat about the fact that the category should be small, namely the class of objects should be a set); as another example, if one instead requires all the morphisms to have an inverse, one is defining a groupoid, which, as the word suggests, generalizes the concept of group, and appears both in noncommutative geometry and in the theory of gerbes.

We will now introduce other properties, which will be also needed in physics With the objects in a category we want to do more things: to add two of them, and to have a zero object. These two requirements are what is required to call a linear category an *additive* one. If one moreover wants to deal with exact sequences and homological algebra, there is instead a further set of less trivial properties, which make a category an *abelian* one. Before we can define this, we define mono- (and epi-) morphisms, and (co)-kernels, in any category.

Monomorphisms are those morphisms ϕ for which left composition is injective: $\phi \circ \alpha = \phi \circ \alpha' \Rightarrow \alpha = \alpha'$. Kernels are instead defined to be, given a map $A \xrightarrow{\phi} B$, maps $K \xrightarrow{\psi} A$ such that: i) $\psi \circ \phi = 0$, ii) for any other $K' \xrightarrow{\psi'} A$, there exists a morphism ρ such that the following diagram



commutes. (This is an example of what is called a universal property.) In the category of vector spaces, the "kernel" that we have defined here reduces to the injection of the usual kernel in the domain of the morphism ϕ. In a similar way one can define epimorphisms and cokernels.

Now, the point is that in a general category not all morphisms have kernels: a ψ satisfying the properties we have laid down might not exist. An abelian category is indeed defined as an additive category in which

- Any morphism has a kernel and a cokernel
- Any monomorphism is the kernel of its cokernel (and any epimorphism is the cokernel of its kernel); roughly speaking, this is a compatibility condition between two possible definitions of being "injective";
- Any morphism is composition of a monomorphism $\circ$ an epimorphism (this is true for instance in the category of linear spaces).

These requirements are essential to define exact sequences in our category and to start the homological algebra. We will not go too far into this here, however, and this small sentence will be enough.

In spite of the seemingly involved definition given so far, the actual examples are actually very familiar, as the category of vector spaces or of coherent sheaves on a manifold. We are now going to define, instead, out of an abelian category, a more complicated object whose behavior is less intuitive and the examples of which are less familiar.

We will try even less than before to be detailed, and try to give the idea. One first constructs the category of complexes of the given abelian category: objects are (bounded) complexes, as the name suggests; a morphism of two complexes $\phi : A_\bullet \to B_\bullet$ is a series of maps between two complexes such that all diagrams commute:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \ ; \end{array}$$

some of the entries can be zero. Remember now that such a morphism ϕ is called a quasi-isomorphism if its homology homomorphism $H^n(\phi) : H^n(A_\bullet) \to H^n(\bullet)$ is an isomorphism for all n. Now, derived category is defined to be a category in which quasi-isomorphic complexes are isomorphic, and is constructed technically constructing a new category whose morphisms are constructed out of those of the category of complexes in such a way that when the original morphisms of complexes are quasi-isomorphisms, the new ones are isomorphisms. We will not get too deeply into the technicalities of the definition of derived category, and will instead explore and underline some of its consequences.

First of all, notice that being quasi-isomorphic implies to have the same cohomology, but the converse is not true. If we were to identify all complexes with the same cohomology, we would obtain a sort of K-theory instead, with a $\mathbb{Z}$ grading instead of a $\mathbb{Z}_2$ one. For instance, we would identify a complex with its cohomology (with zero morphisms); let us see why this is not so in the derived category. If for instance a complex $A \to B$ were to be quasi-isomorphic to its cohomology, we would have a map of complexes

$$\begin{array}{ccc} A & \xrightarrow{T} & B \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \ker T & \xrightarrow{0} & \operatorname{coker} T \end{array} \quad ; \tag{3.5.42}$$

f_1 can be taken to be the natural projection of B into $B/Im(T) = \operatorname{coker} T$, and in this case $H(f_1)(= H^1(f^\bullet))$ is automatically an isomorphism. As for f_0, there is no natural map from A to $kerT$; there is one, but in the opposite direction, call it i; then the condition for having a quasi-isomorphism is $i \circ f_0 = id_{kerT}$; this is exactly the condition to split a sequence, and it does not hold in general – it may well be the case that there is no f_0 with this property. So in general a complex is not quasi-isomorphic to its cohomology. Had we tried to reverse the vertical arrows in (3.5.42), f_0 would have been automatically correct and f_1 would have been the problematic one.

We want to underline now another aspect of the derived category. To do so, let us consider an example. Take the abelian category of coherent sheaves on a manifold. For any divisor D, we have the exact sequence (3.1.2), which we will rewrite here in a handier way:

$$0 \to \mathcal{O}(-D) \overset{T}{\to} \mathcal{O} \overset{\mu}{\to} \mathcal{O}_D \to 0\,. \tag{3.5.43}$$

When we build the category of complexes, the complex $\mathcal{O}(-D) \overset{T}{\to} \mathcal{O}$ is now an object itself. Using (3.5.43) and the example with quasi-isomorphisms of the previous paragraph, we see that in the derived category we have $(\mathcal{O}(-D) \to \mathcal{O}) \cong \mathcal{O}_D[-1]$, where $\mathcal{O}_D[-1]$ is the complex having only a non-zero entry, equal to $\mathcal{O}_D$, in position 1. Let us now examine a morphism in the derived category from two points of view:

$$\begin{array}{ccccc} \mathcal{O}(-D) & \longrightarrow & \mathcal{O} & \cong & \mathcal{O}_D[-1] \ , \\ \uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle \mu}\uparrow \\ 0 & \longrightarrow & \mathcal{O} & \cong & \mathcal{O}[-1] \end{array}$$

where μ is the same map appearing in (3.5.43). On the two sides of the equivalence signs, we have displayed the same morphism using different representatives of the same equivalence classes. Now let us ask: what are the kernels and cokernels of this morphism? Looking at the left side, the morphism would clearly seem to be injective, and having a cokernel $\mathcal{O}(-D)$. On the other hand, looking at the right side this clearly seems to be impossible, and in fact the map seems instead to be surjective, and in fact having a kernel $\mathcal{O}(-D)[-1]$, from 3.5.43.

What is the resolution of this puzzle? We have warned the reader before that the existence of kernels and cokernels in general categories is not automatic, and one has to include it in the definition of an abelian category. The procedure we have done to define the derived category has destroyed this property: the derived category is not abelian. This could seem to be a disaster, because it would mean that one cannot do any homological algebra with it. Fortunately, it turns out that there is a substitute for kernel and cokernel and for exact sequences. For the latter, the substitute is called *distinguished triangle*. The general definition of these is that they are triangles (i.e. diagrams of the form $A_\bullet \to B_\bullet \to C_\bullet \to A_\bullet[1]$) satisfying some extra complicated requirement; referring again to our simple example above, the formalism of distinguished triangles allows us to use the fake kernel and cokernel above, writing in this case

$$\mathcal{O}(-D)[-1] \to \mathcal{O}[-1] \to \mathcal{O}_D[-1] \to \mathcal{O}(-D)\,.$$

You can imagine this triangle as going on forever, and one can truncate it at any four subsequent terms. Such triangles often come from exact sequences in the abelian category, as in this cases. Another way to build them is to use the so-called *cone construction*, which substitutes the concept of kernel and cokernel together. This is simple to define: out of two complexes $\ldots A_i \stackrel{d_i^A}{\to} A_{i+1} \to \ldots$, $\ldots B_i \stackrel{d_i^B}{\to} B_{i+1} \to \ldots$, and of a morphism between them consisting of maps ϕ_i, one defines the cone $\{A_\bullet \to B_\bullet\}$ as the complex

$$\ldots A_{i+1} \oplus B_i \stackrel{d_i^{\{A\to B\}}}{\longrightarrow} A_{i+2} \oplus B_{i+1} \to \ldots\,, \qquad d_i^{\{A\to B\}} \equiv \begin{pmatrix} d_{i+1}^A & 0 \\ \phi_{i+1} & d_i^B \end{pmatrix}.$$

Notice in particular for the cone over the map of two H^0 complexes (i.e. with the the only non-zero entry being the 0th one) A and B, we have $\{A \to B\} \cong (A \to B)[1]$.

3.5.2 Quivers, sheaves, derived category

We have now the background to properly reinterpret many of the results and statements of this chapter in derived category terms.

Let us start from Beilinson theorem. We have said that, in general, the double complex has to be interpreted as a complex of quiver representations with $Q = Q_{\mathbb{P}^n}$ the Beilinson quiver. We know now that this complex can be seen as an element of the derived category of quiver representations; since these representations can also be viewed as modules over its path algebra $\mathcal{A}_Q$ which we denote by $D^b(\text{mod}-\mathcal{A}_Q)$. If we do so, Beilinson theorem can be very succinctly stated as the equivalence of this quiver derived category with the derived category of coherent sheaves on $\mathbb{P}^n$:

$$D^b(\text{mod}-\mathcal{A}_{Q_{\mathbb{P}^n}}) = D^b(\mathbb{P}^n)\,. \qquad (3.5.44)$$

A similar statement holds for Bondal theorem: we can now make more precise the discussion made in subsection (3.4.2) about "layers". The original statement [10] is broken in two theorems which we will combine. If the Fano variety E is such that the anticanonical class is very ample, and we have a helix $\{R_i\}$ on it, then we can construct a quiver Q_E as we have explained above, as the quiver whose path algebra is $\mathcal{A}_E = \mathrm{Hom}(\oplus_i R_i, \oplus R_i)$; the theorem is that there is again an equivalence of derived categories

$$D^b(\mathrm{mod}-\mathcal{A}_{Q_E}) = D^b(E)\ ; \tag{3.5.45}$$

the equivalence can be described by a functor which is denoted by $\mathcal{E} \mapsto \mathbf{R}Hom(\oplus R_i, \mathcal{E})$; this is nothing but rewriting in mathematical notation we described in section 3.4.2.

Let us try to understand the meaning of this reformulation in terms of derived category for physics. We started from the statement that branes at large volume are described by coherent sheaves. Then we pointed out that this cannot be true for any point of the moduli space, since at the Gepner point branes are instead described by quiver representations. However, since the moduli space is connected, by doing a path in it (shrinking the volume of the Calabi-Yau) one should find a relationship between them: this is McKay correspondence, via Beilinson theorem. But the relationship is not directly between the two sets of objects, coherent sheaves and quiver representations, but rather between objects in the derived categories. This means that, if one supposed that, at large volume, branes were described by the derived category of coherent sheaves, there would be no contradiction with the Gepner point: this category would be precisely the same as the derived category of quiver representations, by (3.5.44) or more generally (3.5.45). It is also natural to think that actually the complete correspondence will involve the derived category of the Calabi-Yau Y itself.

The conjecture is thus that *branes are described by the derived category of coherent sheaves on* Y, $D^b(Y)$.

Some comments about this conjecture are in order. First of all, the reader may be puzzled about the word "category" itself appearing in the brane context. One could answer that the two kind of objects we started with, coherent sheaves and quiver representations, both constitute a category, and even an abelian one. But more than that, we want to point out that the occurrence of categories in brane physics is very natural in general, even without these examples. Indeed, we can think of branes in a given situation as objects in a category; $\mathrm{Hom}(a, b)$ is then the Hilbert space of strings between them. The product is given by triple correlation functions. The non-trivial thing is whether this product is associative; this in general depends also on whether this Hilbert space is taken to be the full one, or if one restricts to the subspace of physical states. In general, one may have to resort to a generalization of the concept of category which relaxes the associativity requirement, the so-called A_∞*-category* [76]. We will not enter into this here.

The category of branes is also additive: this can be seen most directly from the additivity of boundary states: note that, despite the name "states", they do not constitute a vector space, due to the nonlinear Cardy conditions; these conditions do not spoil however the simpler structure of abelian group. In other words, it makes sense add boundary states, and in particular to consider e.g. 2 times a given boundary state, though it does not make sense to consider a non-integer real number times a boundary state.

The next step would be to ask whether the category of branes is abelian; there does not seem to be any immediate reason for this; and indeed the category we are proposing, the derived category, is not abelian. Nevertheless, abelian categories of branes may play some role, as we will conjecture in section 4.1.1.

There are also other reasons to argue for the relevance of derived category. Essentially these are:

- Whereas all points on a manifold represent the same element of K-theory, derived category sees them as different. This suggests the simple idea that, while K-theory classifies the brane charge, derived category classifies the branes themselves, with moduli (such as translations) included.

- Brane monodromies turn out to be very clearly described if one supposes this hypothesis is true; this is actually strictly related to what we have seen above.
- The "anticondensation" we saw in section 3.1.1 can be seen to give an element of the derived category of Y, which is more than its image in the Grothendieck group.
- This would fit with homological mirror symmetry, among whose original motivations in fact an *ante litteram* appearance of branes appears.
- There is even a detailed analysis in the case of the torus [52]; this would actually suggest a modification of the conjecture when a B field is on, with the derived category of coherent sheaves being substituted by the derived category of modules over a certain Azumaya algebra [52]. This is very interesting because it would make noncommutativity to come into the play, but we will neglect this in the following.
- Branes in the topological string are indeed classified by derived category [26].

This last point is in a sense a proof of the conjecture. In fact, a possible reason of complain about the conjecture is that derived category would seem to be too large. After all, at large volume we know that branes are coherent sheaves, and derived category contains complexes of them, which in general are not quasi-isomorphic to a single sheaf, and so are many more objects. But remember that we have disregarded, in all of this chapter, the stability part of the problem, coming from the real (D-term) equations. This is the same as considering the twisted topological theory instead of the physical one; thus [26] tells us that, if one disregards stability, the conjecture is true.

In the physical theory, then, there should be the same objects one has in the twisted theory, but with an extra stability condition. This means that branes will be stable objects in the derived category. What this "stable" means is yet to be completely understood (though see [26]), but one already knows what one should expect in the two limits. All the extra objects in derived category which are not reducible to sheaves are expected to be unstable at large volume (and even among the single sheaves, not all of them will be stable: the concept of stability in this subclass should become that of μ-stability, as we will see). Similarly, at the Gepner point all the extra objects which are not coming from a single quiver representation will not be stable; moreover, also among the single representations not all will be stable, and stability will reduce to the so-called θ-stability [28].

Discussion of such matters will be the subject of next chapter.

Chapter 4

Stability

We have understood in last chapter several features of branes on Calabi-Yau manifolds; the most general lesson was the fact that, without considering stability matters (or, if one wants, considering the twisted topological theory instead of the physical one), branes can be considered as elements in the derived category. In the physical theory, branes should be now *stable* elements in the derived category. The point of this chapter is to understand what this stability is, and what are the equations that correspond to it (in the vein of Hitchin-Kobayashi correspondence, see introduction).

We will start by reviewing this correspondence in the case of HYM equations, and describe afterwards the modifications one should expect from string theory, both from string worldsheet and from brane worldsheet point of view.

4.1 Stability for HYM equations and its deformation

Before we come to applications in string theory, we would like to review first how stability is linked with equations. We will have to do in this chapter with various concepts of stabilities, depending on the equations one consider. All these equations will involve however a connection A on a fiber bundle E; stability will be always an inequality involving subbundles of the bundle E. Instead of being too general, however, we will present here an example.

Let us thus start from the simple equation $F_A = c\omega$ in 2 dimensions (along with $F^{(2,0)}$, as usual) where we have explicitly indicated that F_A is the curvature of a connection A on a bundle E. Taking trace and integrating, we get $c = (2\pi i)c_1(E)/(rk(E)Vol) \equiv (2\pi i)\mu(E)/Vol$. To avoid use of induction, let us consider the case in which $rk(E) = 2$. Suppose now there is a holomorphic subbundle (in this case it can only be a line bundle) $L \overset{s}{\hookrightarrow} E$, with a connection A' on it (on a line bundle we can choose it to have constant curvature, like A has). The embedding s is a section of $\mathrm{End}(L, E)$; A' and A induce on this bundle a connection B, and the condition that the subbundle be holomorphic can be explicitly expressed as $\bar{D}_B s = 0$ (here, as above, D denotes the holomorphic part of the covariant derivative). Finally, let us put a hermitian metric $(\cdot, \cdot)$ on the bundle $\mathrm{End}(L, E)$. We can now consider the equalities

$$0 = \int \partial(D_B s, s) = \int (\bar{D}_B D_B s, s) - \int (D_B s, D_B s).$$

Since, in the form notation we have been using so far, $D_B \bar{D}_B + \bar{D}_B D_B = [D_B, \bar{D}_B] = F_B^{(1,1)} = F_B$, and $\bar{D}_B s = 0$, we have

$$\int (D_B s, D_B s) = \int (F_B s, s) = (2\pi i) \frac{\int (s,s)\omega}{Vol} (\mu(E) - \mu(L));$$

but, since the (imaginary part of the) lhs is non negative, we have that if on E there is a holomorphic connection that satisfies $F = c\omega$, E satisfies the following property, called μ-semistability: *for any holomorphic subbundle $L \hookrightarrow E$, one has $\mu(E) \geq \mu(L)$.*

In this way we have understood easily enough why stability of the bundle E is *necessary* to have a solution to the proposed equation. Sufficiency is usually a little bit more difficult and involves moment maps in the infinite-dimensional space of connections, but also some analytical arguments.

This simple equation is the lowest form of life in the zoo of equations we will be considering. First of all, we can go up in the dimension. The equation becomes now the real equation in HYM,

$$\omega \cdot F = c\,,$$

and an analogue argument of the one we just presented in 2 dimensions works. More precisely, μ becomes deg/rk, where $deg = c_1 \cdot [\omega]^{n-1}$ depends now on the Kähler class $[\omega]$.

Let us now come back to physics. We have already seen in section 2.2 that HYM equations (2.2.9) are one of the conditions for a supersymmetric cycle; thus, we get that *at large volume the bundle E on a B brane should be stable.*

There are various reasons, however, for which this is not a complete statement. First one is that, as we had also mentioned in section 2.2, we have neglected transverse scalars X so far. We will take care of this later; we will argue that stability will be modified in the style of Hitchin, but that this will not affect dramatically the form of the total space of BPS states of the theory. Second problem is that, unlike for the holomorphic part (the F-term) of the equations, which was argued in section 2.3.2 not to be modified by changing Kähler moduli, the D-term (in this case $\omega \cdot F = c$) has no reason not to get deformed as we go away from the large volume limit.

4.1.1 Deformations

Indeed this has been explicitly shown in [58], where some features of the deformation have been worked out. The method was to redo the BPS analysis starting from the complete Dirac-Born-Infeld plus Chern-Simons action [58]; as a result one gets a deformation of the Hermitian-Yang-Mills equations. This obviously only works in the abelian case, as we do not have yet control over non-abelian DBI. Moreover, this will neglect possible contributions from Ricci curvature.

We can write the resulting equations in the compact form

$$[\mathrm{Im}(e^{i\theta} e^{F+\omega})]_{\mathrm{top}} = 0\,, \tag{4.1.1}$$

where ω is the Kähler form, and θ is a phase. We will call them MMMS equations. The subscript $_{\mathrm{top}}$ indicates that we only have to take the top-form part of the expansion $e^{F+\omega}$. For instance, in 4 dimensions, (4.1.1) can be written as

$$iF \wedge \omega = \mathrm{tg}(\theta) \left(\frac{1}{2} F^2 + \frac{1}{2} \omega^2 \right)\;; \tag{4.1.2}$$

here we have used the fact that F is anti-hermitian and so, in the abelian case, purely imaginary, and that ω is real. From this one can see that, at large volume ($\omega \to \infty$), one can neglect the F^2 term, and (4.1.2) goes to the $(1, 1)$ part of equations (2.2.9) in 4 dimensions, as it should.

As we have said, the equations were derived only in the abelian case. Nevertheless, (4.1.1) admit a very natural non-abelianization by treating F as a matrix, and putting an identity in the ω^n part. This will clearly be well-defined, thanks to the transformation of the curvature under gauge transformation $F \to UFU^\dagger$. In fact, we can write this non-abelianized version again in a compact form as

$$[\mathrm{Im}(e^{i\theta}e^{F+\omega\,\mathrm{Id}})]_{\mathrm{top}} = 0\ ; \qquad (4.1.3)$$

this has now to be read with a little caveat, namely that $\mathrm{Im}(\cdot)$ is now $(\,(\cdot) - (\cdot)^\dagger\,)/2i$ (this is because now F is not purely imaginary, but anti-hermitian). Practically, this means that for instance in 4 dimensions we get again the equations (4.1.2), but with a non-abelian F now. Notice that the equations come automatically with a symmetrizer, due to the fact that they are written in terms of forms.

We can now note a nice thing about (4.1.1): it bears a suggestive similarity with the condition

$$Im\{e^{i\theta}\Omega_{|_L}\} = 0 \qquad (4.1.4)$$

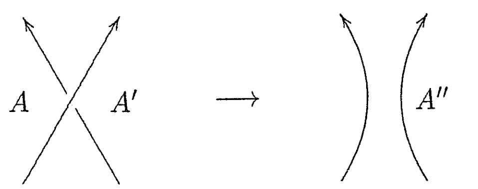
for A branes (see section 2.2). This naturally suggests that that (4.1.1) and (4.1.4) are exchanged by mirror symmetry; the natural completion of this statement would then be the exchange of the vanishing of $\omega_{|_L}$ on A branes and the vanishing of $F^{(2,0)}$ for B branes. Decisive steps towards establishing these statements have been done in [57]. A more detailed correspondence can then be drawn; see for instance [80].

This approach has problems, as we have said: nonabelianization was done by hands; possible Ricci terms have been overlooked; there will be instantonic corrections. A different method is to use worldsheet techniques. Before we consider that, however, it can be instructive to look at an example with A branes, which can be seen as a mirror counterpart of the picture we will describe later for B branes.

A branes and supersymmetry breaking.

We can already gain useful insight by considering a flat space situation. Start from two branes, which will be called B and B', wrapped on two very particular SLags (Special lagrangians): two affine linear spaces. When will the configuration of these two be stable? It turns out that this depends on the quantity $\phi \equiv 1/\pi \arg(\int \Omega)$ of the two branes. The story goes as follows:

- This configuration $A + A'$ turns out to be BPS only for $\phi(A') = \phi(A)(\mathrm{mod}\, 2)$;
- For $\phi(A) - \phi(A') < 0$ there is another SLag with the same asymptotic behavior, in which the configuration can decay:



- For $\phi(A) - \phi(A') > 0$ there is instead no such SLag. So, the configuration, though not BPS, is stable for the trivial reason that there is nothing for it to decay into.

This analysis has in this case been carried out only using geometrical arguments, but one could find the same result from worldsheet arguments. Looking at the spectrum, one will find tachyons in the second case, signaling decay in the different configuration, and massive states in the other non-BPS case, signaling that there can be no decay and the two states do not make a new one. This situation can also be described by an effective theory in which supersymmetry is spontaneously broken [51].

Worldsheet techniques

By mirror symmetry, we expect a similar picture as the one we just sketched for A branes to be there for B branes as well. This will again depend on the presence or absence of tachyons [26]. Let us consider physical states in the Hilbert space of strings stretched between two branes. On the Calabi-Yau, there are states which transform like fermions and others which transform like bosons; from the worldsheet point of view, as we have recalled in section 2.1.1, they come from Ramond and Neveu-Schwarz states respectively. Ramond ground states in a nonlinear sigma model (fermionic states), as we have already mentioned in section 3.1.2, belong to the Ext^i groups of the two branes. These are geometric quantities which only depend on complex structure, and so these states will remain massless all over the moduli space. What about bosonic states? their masses will vary with Kähler moduli.

There is an argument leading to these masses, and works as follows. In section (2.1.1) we introduced, for both left- and right-moving sectors, the $U(1)$ current $J = \sqrt{c/3}\,\partial\tilde{\phi}$, and a spectral flow $\exp(1/2\sqrt{c/3}\tilde{\varphi})$. For strings stretched between two different BPS branes B and B', the spectral flow in the right moving and left moving sector do not match exactly: they differ by a phase:

$$e^{i\sqrt{3}/2\,\tilde{\varphi}_L} = \exp[i\pi\left(\phi(B') - \phi(B)\right)]e^{-i\sqrt{3}/2\,\tilde{\varphi}_R}\,.$$

So, this allows to individuate a bosonic partner anyway, since the difference is only a phase. When the quantity $\delta\phi \equiv \phi(B') - \phi(B)$ is an integer[1], the combined brane is again a BPS one; otherwise, the mass of the bosonic state depends then on the sign of (the integer part of) this quantity. This is exactly the same situation as in the example with A branes above; in that case ϕ depended on the features of the branes as $(1/\pi)\,arg(Z)$; by mirror symmetry, it is then not surprising that this holds for B branes as well, with the only difference that now the central charge Z is given by the B brane formula (3.4.10).

This method should be used to arrive at a definition of stability in the derived category. Problems come in, however, when one tries to consider general elements in the derived category. There are two problems with this. First one is that the μ-stability requires the concept of subbundle; in the derived category there is no concept of subobject, and one should use instead distinguished triangles. Moreover, and most importantly, there are problems coming from the so-called *grading*. Although the quantity ϕ could seem to take value in $[0, 2[$, it can be seen that it actually belongs to $\mathbb{R}$. Indeed, consider whatever brane B, and the worldsheet theory of strings ending on it with both ends. It also makes sense to consider another worldsheet theory, in which all the charges of the Ramond ground states are shifted by 2. For consistency of ϕ with $U(1)$ charges, one has to consider these strings as stretched between the original brane B and a brane B' identical to this but with $\phi(B') = \phi(B) + 2$. These "shifted" branes have to be included in the spectrum because they are reachable by continuous deformations of more standard ones. Since, from section 2.1.1, $U(1)$ charges of R states are related with the index of the Ext^i, we can make contact with the derived category notation and identify B' with $B[2]$: the integer part of the $U(1)$ part gets absorbed in the Ext groups, following the derived category formula $\text{Ext}^i(A, B[j]) = \text{Ext}^{i+j}(A, B)$.

Given a brane at a point in Kähler moduli space, to decide whether it is stable or not involves thus in any case to compute its ϕ. Usual central charge computation only gives its part in $[0, 2[$. For this reason the only derived category stability known so far [26] requires to make trips in the moduli

[1]This introduces the theme of the grading, which we will see later. For the time being, let us think only about little $|\phi|$.

space. Resolution of these problems is work in progress, and should lead to a derived category stability defined *with reference to the given point of moduli space only*, without having to travel in it. This should eventually agree with all the equations we will present in this chapter, extending all the tentative definitions we will give here.

A possibility could be that, whatever derived category stability should eventually mean, it works in two steps: first identifying an abelian category relevant at the given point of moduli space (as the category of coherent sheaves at large volume), and then working in this category as an abelian category stability. Though this "two steps" stability is conjectural, it is anyway interesting to consider it. This is the early Π-stability proposal [28], which is modeled on μ-stability. Namely, given this abelian category we have just supposed to exist, a stable object in it would be an object B for which the following holds: *for any subobject B', we have the inequality $\phi(B') < \phi(B)$*.

We can check that this stability goes to μ-stability as one goes to the large volume point, noting that, for maximal branes (those which fill the Calabi-Yau),

$$arg\left(-\int e^{-t}ch(E)\sqrt{Td(T)}\right) = arg\left(\frac{1}{3!}\int t^3 - \frac{1}{2}\int t^2 c_1(E)\right) =$$
$$\operatorname{Im}\log\left(\frac{i^3}{6}\operatorname{Vol} rk(E) + deg(E)\right) = \frac{3}{2}\pi + 3\frac{deg(E)}{\operatorname{Vol} rk(E)}.$$

For lower dimensional branes, the computation is similar: one gets for instance for 4-branes X $arg[rk(E)\int_X t^2 + \int t(c_1(E) + rk(E)(\text{stuff}))]$, where (stuff) depends only on X and thus can be neglected in relative inequalities. As we have already mentioned, one can also show [28] that in the Gepner limit Π-stability goes to the natural stability condition for quivers, known as θ-stability. This is again equivalent to the existence of solutions to the D-term equation, as it should be. We will see this in more detail later.

So, Π-stability interpolates between different known stabilities in various limiting points of the moduli space; in particular, in the limit in which branes can be considered as geometric objects (holomorphic cycles with bundles on them), the inequality with ϕ reduces to the inequality with μ, and we recover the usual stability for bundles. This is clearly similar to the way in which (4.1.3) reduces to (2.2.9) in the limit of large ω (or small F). To be more precise, let us underline once again the reason for which μ is appearing in the usual stability. The reason is that μ is proportional to the constant c appearing in HYM (2.2.9). Now, we have a constant in MMMS equations (4.1.3) as well: it is θ. As in the HYM case, the way to uncover the relation of this with the topological constants is to take the trace of the equation and integrate. Here we get

$$\begin{aligned}
&\int \operatorname{Im}(e^{i\theta}e^{F+\omega})_{\text{top}} = \\
&\sin(\theta)\int(\omega^n + \omega^{n-2}\operatorname{Tr}F^2 + \ldots) + \cos(\theta)\int(\omega^{n-1}\operatorname{Tr}(-iF) + \omega^{n-3}\operatorname{Tr}(-iF^3) + \ldots) = \\
&\sin(\theta)\operatorname{Re}\int(\omega^n + \omega^{n-1}\operatorname{Tr}F + \ldots) + \cos(\theta)\operatorname{Im}\int(\omega^n + \omega^{n-1}\operatorname{Tr}F + \ldots) = \\
&\quad\operatorname{Im}\left(e^{i\theta}\int e^{F+\omega}\right)
\end{aligned}$$

The equation (4.1.3) sets this to zero; thus we find that $\theta = \arg(\int e^{F+\omega}) + \pi k i$. Since in the trivial geometries, this expression is the same as the central charge, this suggests that the corrections [27] to the equation in a nontrivial geometrical setting amount to introduction of an additional factor $e^{-F_a/2}\sqrt{\hat{A}(R)/\hat{A}(F_a)}$, where a is the connection on the normal bundle, F_a its curvature and R

the Ricci curvature on the brane. In this way we get $\theta = \arg\left(\int e^{F+\omega}e^{-F_a/2}\sqrt{\hat{A}(R)/\hat{A}(F_a)}\right) = \arg(Z) = \phi$. Unfortunately, it is now harder to reproduce the proof we saw for HYM due to the nonlinearity of equation (4.1.3). One of the central points of that proof was the appearance of F from the anticommutator of two covariant derivatives; the analogue of this is not totally clear. The lack of supersymmetric completion for CS terms involving gravitational corrections is another obstacle in this direction. One could try a different approach by attempting to generalize the usual moment map method (see e.g. [81]).

We have not given a complete solution to the problem of derived category stability. We proceed, however, to consider the inclusion in the picture of transverse scalars and of tachyons, which should come into the play at some point anyway. Even more than this, all the points of view should agree; various equations, with transverse scalars and/or tachyons [61], will have their stability conditions, and they should be recovered as various limits of derived category stability.

4.2 Inclusion of transverse scalars

Again, our arguments will be a little indirect, since we do not have control over nonabelian DBI action. The strategy we will follow will be to infer the equations from nonabelianization of the equations without scalars (which we have already considered in previous section) and its dimensional reduction. This is reasonable by comparison with manageable limits, for instance super Yang-Mills limit. We have already analyzed BPS conditions in that case in subsection 1.2.1.

These super Yang-Mills theories are known to describe the dynamics of flat branes in $\mathbb{R}^{10}$ in zero slope limit. Deformations away from this limit were discussed in the previous section. So, summarizing we have:

- the BPS conditions in the large volume limit, in which there is a super Yang-Mills description, are given by the dimensional reductions we saw in subsection 1.2.1 of the 10 dimensional equations (1.2.7), which can be rewritten as $F\omega^4 = c\omega^5$ (from now on we will omit $\wedge$ and Id if there is no danger of confusion).

- away from the large volume limit, the deformation to these equations are known in all dimensions, and in particular in 10 dimensions.

It is natural, then, to assume that dimensional reduction of the deformed equations (4.1.3) gives the correct BPS conditions in lower dimensions with transverse complex scalars turned on. This will give a deformation of the Hitchin-like equations we have shown before to arise from super Yang-Mills, for instance (1.2.6) and (1.2.8).

Before we go on and find the general deformed equations with scalars, let us write also Hitchin first equations in (1.2.6) and (1.2.8) in a covariant form. For (1.2.6) it is easier: a simple way is to write it is

$$F - i[X, X^\dagger]\omega = c\,\omega. \tag{4.2.5}$$

We could have as well made X a one form; this second choice is the version studied by Hitchin, and we will have more to say about this in section 4.2.2. As to the second, (1.2.8), there are more scalars in this case. Let us then introduce a transverse form $\omega_\perp$, with which the indices of the X_a and of the $X_{\bar{a}} \equiv X^\dagger$, which are scalars in the spacetime but vectors in the transverse directions, can contract. This is a very natural thing to do, in light of the 10 dimensional origin of the equations. Then we can write (1.2.8) in the form

$$F\omega + \frac{\omega^2}{2}(i_X + i_{X^\dagger})^2\omega_\perp = c\frac{\omega^2}{2}, \tag{4.2.6}$$

where $i_X \equiv X^a i_{e_a}, i_{X^\dagger} \equiv X^{\bar{a}} i_{e_{\bar{a}}}$ are contractions with the holomorphic and antiholomorphic parts $X, X^\dagger$ of the transverse scalars. Notice that $i_X^2, i_{X^\dagger}^2$ are zero because they reproduce the commutators $[X_a, X_b]$ and $[X_{\bar{a}}, X_{\bar{b}}]$; $i_X i_{X^\dagger} + i_{X^\dagger} i_X$ then gives the desired combination of commutators. Notice that, in order to keep $\omega_\perp$ real, we have chosen it to be of the form $i \sum_a dx^a d\bar{x}^a$.

We confine ourselves to a holomorphic setup, and thus always consider the case when both the original equations and the reductions are even-dimensional. In principle we could have as well reduced an odd number of covariant derivatives. Even though we will not do this here let us notice that such reductions could also lead to interesting equations. For instance, the odd-codimension reductions of self-duality equations yields two important equations: Nahm equations in 0+1 dimension, and the Bogomol'nyi equations for monopoles in 2+1 dimensions. Another interesting situation involving an odd-dimensional reduction is the gauge theory on the coassociative cycles in manifolds of G_2 holonomy.

4.2.1 Transverse scalars equations

Let us now turn to the reduction of the equations (4.1.3). For illustrative purposes, we will first do the reduction from 4 dimensions to 2, which can be viewed as reducing from 10 to 2 but with only one complex scalar on. Then we will tackle the case with all the complex scalars on and reduce from 10 dimensions to 4. After that we will cast the result in a dimension-independent form analogous to that of (4.1.3). The most convenient four-dimensional starting point is in the form (4.1.2). Reducing this in 2 dimensions one gets

$$\frac{1}{2}\{F(-i)[X,X^\dagger] - i[X,X^\dagger]F - i(DX^\dagger \bar{D}X - \bar{D}XDX^\dagger)\} + \omega = i\,\mathrm{tg}(\theta)(F - i[X,X^\dagger]\omega); \quad (4.2.7)$$

this is, as expected, a deformation (by the first three terms) of (4.2.5): these two equations are the same at linear order.

Coming now to the more laborious task of reducing from 10 to 4, we will not actually perform the reduction of the equations, but reduce instead the expression

$$[e^{F+\omega\,\mathrm{Id}}]_{\mathrm{top}} = \frac{1}{5!}F^5 + \frac{1}{4!}F^4\omega + \frac{1}{3!2!}F^3\omega^2 + \frac{1}{2!3!}F^2\omega^3 + \frac{1}{4!}F\omega^4 + \frac{1}{5!}\omega^5$$

and then restore the phase θ at the end. To write down the result in 4 dimensions, we introduce some little more piece of notation. Let $d_A \equiv D + \bar{D} \equiv dz^i D_i + d\bar{z}^i D_{\bar{i}}$, where $D_i, D_{\bar{i}}$ are the covariant derivatives. With the help of this we will write the result in a form halfway from an explicit and a contracted one, and then explain how to go on in either direction:

$$\mathrm{Sym}\{ \left[\left(\frac{F^2}{2!}\frac{(i_X+i_{X^\dagger})^6}{3!} + F\frac{(-)[d_A, i_X+i_{X^\dagger}]^2}{2!}\frac{(i_X+i_{X^\dagger})^4}{2!} + \right.\right.$$
$$\left.\left.\frac{(-)^2[d_A, i_X+i_{X^\dagger}]^4}{4!}(i_X+i_{X^\dagger})^2\right)\frac{\omega_\perp^3}{3!}\right] +$$
$$\left[\left(F\omega\frac{(i_X+i_{X^\dagger})^6}{3!} + \omega\frac{(-)[d_A, i_X+i_{X^\dagger}]^2}{2!}\frac{(i_X+i_{X^\dagger})^4}{2!}\right)\frac{\omega_\perp^3}{3!} + \right.$$
$$\left.\left(\frac{F^2}{2!}\frac{(i_X+i_{X^\dagger})^4}{2!} + F\frac{(-)[d_A, i_X+i_{X^\dagger}]^2}{2!}(i_X+i_{X^\dagger})^2 + \frac{(-)^2[d_A, i_X+i_{X^\dagger}]^4}{4!}\right)\frac{\omega_\perp^2}{2!}\right] +$$
$$\left[\frac{\omega^2}{2!}\frac{(i_X+i_{X^\dagger})^6}{3!}\frac{\omega_\perp^3}{3!} + \left(F\omega\frac{(i_X+i_{X^\dagger})^4}{2!} + \frac{(-)[d_A, i_X+i_{X^\dagger}]^2}{2!}(i_X+i_{X^\dagger})^2\right)\frac{\omega_\perp^2}{2!} + \right.$$

$$\left.\left(\frac{F^2}{2!}(i_X+i_{X^\dagger})^2+\frac{F(-)[d_A,i_X+i_{X^\dagger}]^2}{2!}\right)\omega_\perp\right]+ \tag{4.2.8}$$
$$\left[\frac{\omega^2}{2!}\frac{(i_X+i_{X^\dagger})^4}{2!}\frac{\omega_\perp^2}{2!}+\omega\left(F(i_X+i_{X^\dagger})^2+\frac{(-)[d_A,i_X+i_{X^\dagger}]^2}{2!}\right)\omega_\perp+\frac{F^2}{2!}\right]$$
$$\left[\frac{\omega^2}{2!}(i_X+i_{X^\dagger})^2\omega_\perp+F\omega\right]+\frac{\omega^2}{2!}\}.$$

A few explanations are in order. First of all, the symmetrizer Sym comes because reduction of terms like F^n can give rise to terms like $(i_X+i_{X^\dagger})^2F^n$ or $F(i_X+i_{X^\dagger})^2F^{n-1}$ and so on (one can see this for instance already in (4.2.7)); instead of writing all these possibilities, we have put a symmetrizer in the gauge part, which is the same (however, each term $(i_X+i_{X^\dagger})^2$ should be considered here as a unique piece; there are no odd powers of $i_X+i_{X^\dagger}$). As for expressions like $\frac{(i_X+i_{X^\dagger})^6}{3!}\frac{\omega_\perp^3}{3!}$, we have exploited here the notation introduced at the end of previous section: each power $(i_X+i_{X^\dagger})^2$ gives commutators $-i[X_a,X_{\bar d}]$, and so we have $1/(3!)\,\epsilon^{abc}\epsilon^{\bar d\bar e\bar f}[X_a,X_{\bar d}][X_b,X_{\bar e}][X_c,X_{\bar f}]=[X_1,X_1^\dagger][X_2,X_2^\dagger][X_3,X_3^\dagger]+[X_1,X_1^\dagger][X_2,X_3^\dagger][X_3,X_2^\dagger]+\ldots$ apart for an overall $(-i)^3$ factor. It can be a little harder to understand terms containing $[d_A,i_X+i_{X^\dagger}]$. For instance, we have

$$\frac{(-)^2[d_A,i_X+i_{X^\dagger}]^4}{4!}(i_X+i_{X^\dagger})^2\frac{\omega_\perp^3}{3!}=$$
$$dz^1d\bar z^1\,dz^2d\bar z^2(-)[D_1,i_{X^\dagger}][D_{\bar 1},i_X](-)[D_2,i_{X^\dagger}][D_{\bar 2},i_X](i_X+i_{X^\dagger})^2\frac{\omega_\perp^3}{3!}=$$
$$dz^1d\bar z^1dz^2d\bar z^2(-i)[D_1,X_{\bar d}][D_{\bar 1},X_a](-i)[D_2,X_{\bar e}][D_{\bar 2},X_b](-i)[X_c,X_{\bar f}]\epsilon^{abc}\epsilon^{\bar d\bar e\bar f}=$$
$$dz^1d\bar z^1dz^2d\bar z^2\left((-i)D_1X_1^\dagger D_{\bar 1}X_1\,(-i)D_2X_2^\dagger D_{\bar 2}X_2(-i)[X_3,X_3^\dagger]+\ldots\right).$$

Let us note that (4.2.8) reduces, in the case without scalars, to $[e^{F+\omega\,\mathrm{Id}}]_{\mathrm{top}}=1/2!\,F^2+F\omega+1/2!\,\omega^2$, as it should. In fact the whole formula (4.2.8) can be rewritten in a nicer and more suggestive form:

$$[e^{F+i[D+\bar D,i_X+i_{X^\dagger}]+(i_X+i_{X^\dagger})^2}\,e^{\omega+\omega_\perp}]_{\mathrm{top}}\ \overset{\text{or, alternatively,}}{=}\ [e^{F+[D,i_{X^\dagger}]+[i_X,\bar D]+(i_X+i_{X^\dagger})^2}\,e^{\omega+\omega_\perp}]_{\mathrm{top}}. \tag{4.2.9}$$

At first sight, this formula may appear a little strange since a non-homogeneous object is exponentiated. Note that formally this expression coincides with one obtained by reduction of F directly in the exponent of (4.1.3). However, the present computation, apart from clarifying the meaning of such a formal expression, constitutes a non-trivial check since in the process of reduction, some forms become scalars. For instance, from a similar formal argument (reduction of the covariant derivative) one can write the exponent of (4.2.9) in an even more compact expression, turning its two forms into respectively

$$[e^{\frac{1}{2}[D+\bar D-i(i_X+i_{X^\dagger}),D+\bar D+i(i_X+i_{X^\dagger})]}\,e^{\omega+\omega_\perp}]_{\mathrm{top}}\ =\ [e^{[D+i_X,\bar D+i_{X^\dagger}]}\,e^{\omega+\omega_\perp}]_{\mathrm{top}} \tag{4.2.10}$$

if one understands the commutator in a super-sense: i_X is treated as "fermionic" and the usual super-Lie bracket, which is an anticommutator on two fermions, is used. (Moreover, one should not forget the extra signs coming from the usual grading of forms: for instance $[A_1,A_2]=A_1A_2+A_2A_1$.) This reminds one very much of the formalism and of the expressions appearing in the computation of Chern characters with superconnections [69], as already noted in [67] in the context of the modified D-brane Chern-Simons couplings [63]. Actually, at this point we just observe a similarity; tachyons and X scalars, although related by tachyon condensation, are not quite the same object, of course. We will, nevertheless, come back to this later, arguing for the proper place of these similarities.

We can now conclude by putting back θ, and writing the equations in the form

$$\mathrm{Im}\left(e^{i\theta}\left[e^{F+[D,i_{X^\dagger}]+[i_X,\bar{D}]+(i_X+i_{X^\dagger})^2}e^{\omega+\omega_\perp}\right]_{\mathrm{top}}\right)=0. \tag{4.2.11}$$

Note that formally an expansion of (4.2.11) in n dimensions contains terms of degree $n = n_{||} + n_\perp$ in self-explanatory notation. Here we keep only the purely-longitudinal forms $n = n_{||}$ ($n_\perp = 0$). The need of defining extra rules and the presence of $\omega_\perp$ in the equations is not particularly nice, however we find this form to be the most convenient for the analysis of the next section. In section 5 we will reinterpret this equation and write it in a more conceptual form[2].

4.2.2 Geometrical considerations

All we did in this section has been done actually, so far, as if we were in flat space, though we have tried to write our formulas in the way most independent from this situation. We will try now to extrapolate the equations the geometrical cases we are interested in.

When one wraps several branes on a submanifold B of the given ambient manifold Y[3], the scalars defining transverse fluctuations of the brane – which, in the abelian case, would be sections of the normal bundle $N(B, Y)$ – become also matrices; that is, they are now sections of $N(B, Y)\otimes\mathrm{End}(E)$, where E is the gauge bundle[4]. To derive the equations for this more general case, one has to start from the analogue of (4.1.3) for a brane wrapping the whole Calabi-Yau. This can be read in eq.(3.27)(a) in [58]: its imaginary part is still identical to the covariantization of its flat space counterpart, (4.1.3), that we have already seen. Then, when the Calabi-Yau is fibred in tori over the cycle, the same logic of T-duality as in flat space applies, and we get again our equation with scalars (4.2.11). Once more we have to argue as in the flat space case: T-duality gives in principle only a rewriting of the equations, in which D's are written as infinite dimensional X's; but then the expression obtained in this way is the same for finite dimensional X. Moreover, in this case, we are supposing that the equations obtained in the case in which there is a fibration in tori will be correct even in the general case. The procedure is similar to that of [8, 7].

In principle one may imagine another method to get the equations with scalars, namely to try to directly non-abelianize relevant equations in [58]. However, this would be correct only if we had the complete κ-symmetric non-abelian action, which is not the case. So, this second method will not give the full equations; in particular, one cannot get terms with commutators $[X, X^\dagger]$.

One could wonder what the piece with $\omega_\perp$ appearing in (4.2.11) is supposed to mean in this more general case. As we mentioned at the end of the section 4.2.1, due to the appearance in the equations of the combination $\omega + \omega_\perp$, we can substitute it more generally directly with (the pull-back of) ω_Y, the Kähler form on the ambient manifold Y.

A point which deserves emphasizing is that now all the covariant derivatives we have been writing in the flat space case should not only be covariant with respect to the usual gauge part, but also contain

[2]We may also recall that the equations (4.1.1) were related in [58] to noncommutative Hermitian Yang-Mills equations via Seiberg-Witten map. The latter strictly speaking applies to abelian fields only. With a progress in finding a map applicable to more general situations it would be interesting so see if (4.2.11) can lead to a simple form of noncommutative Hitchin equations.

[3]Sometimes in the rest of the chapter our considerations will apply to cases more general than Y being a Calabi-Yau manifold. However, we will stick mostly to that case.

[4]In principle the branes can be extended or not extended in the extra flat directions; in the latter case we can have also scalars describing fluctuations in these directions, but we will ignore these issues altogether in this section.

a connection a coming from the normal bundle [43]. This is simply because, as we noted, the X are now sections of $End(E) \otimes N(B, Y)$.

Twisting

The equations we have been writing so far were all covariantized keeping X as scalars, as required by physics. However, as we noticed after (4.2.5), mathematically there would have been in principle another possibility, that of making X a form. We will see here that this corresponds in fact to the possibility of twisting the supersymmetric brane theory. The basic idea comes from [8]: consider the case in which the ambient manifold Y is a K3, and B is a divisor in it. As we have said, X are sections of the normal bundle tensor the matrix part, $N(B, Y) \otimes \mathrm{End}(E)$; but in this case, due to adjunction formula, we have $N(B, Y) = K_B$, the canonical bundle, which is in this case nothing but the cotangent Ω_B. So the X gets substituted in this case by a antiholomorphic one form[5] ϕ (that is, a $(0, 1)$ form annihilated by the holomorphic covariant derivative: $D\phi = 0$) with values in $\mathrm{End}(E)$, and the equation (1.2.6) can be written in the Hitchin form $F + [\phi, \phi^\dagger] = c\omega$.

Now, it is clear that we can do the same trick when Y_{n+1} is a higher dimensional Calabi-Yau and B_n is again a divisor in it. In that case, the only change is that the canonical bundle is now not be the cotangent: ϕ is a top antiholomorphic form on B. In this way, for instance, equation (1.2.8) with one scalar can be covariantized in the form

$$F\omega + i[\phi, \phi^\dagger] = c\frac{\omega^2}{2}.$$

It is now very natural to wonder whether this twisting can be applied also to our deformed equations (4.2.9). Let us stick to the cases that we have been considering so far, in which B is a divisor in a Calabi-Yau $n + 1$-fold Y. In this class of cases, there is only one complex scalar on. Thus, we can consider the easier equations obtained reducing from $n + 1$ dimensions to n. These equations can again be summarized by (4.2.9), but now remembering that the transverse space has dimension 2.

First of all, for terms like $[X, X^\dagger](e^{F+\omega})_{\mathrm{top}}$ it is rather easy to see how we can manage the twisting. Since X is now replaced[6] by the two form ϕ, the commutator is now $[\phi, \phi^\dagger]$, which is a top form. But there is really no great difference between a scalar and a top form, thanks to Hodge duality; so we can simply, for example, contract this top form with the rest and get a scalar equation $[\phi, \phi^\dagger] \cdot (e^{F+\omega})_{\mathrm{top}}$. It is harder to understand the twisting of terms involving $\bar{D}X$. The guiding principle in doing this is again Hodge duality, and a local (anti)holomorphic Hodge duality (see previous footnote). The result is that now, wherever $\bar{D}X$ appeared, now we have to use the $(0, n-1)$ form $\bar{D}^\dagger\phi$, where $\bar{D}^\dagger$ is the adjoint of the $\bar{D}$ operator. In this way, we have that $\bar{D}^\dagger\phi D^\dagger\phi^\dagger$ is a $(n-1, n-1)$, which can be contracted with the rest again to give a scalar equation. The whole formula becomes thus

$$\mathrm{Sym}\{(e^{F+\omega})_{2n} - i[X, X^\dagger](e^{F+\omega})_{2n} + \frac{i}{2}\left(\bar{D}XDX^\dagger - DX^\dagger\bar{D}X\right)(e^{F+\omega})_{2n-2}\}$$
$$\longrightarrow \mathrm{Sym}\{*(e^{F+\omega})_{2n} + i^{n-1}[\phi, \phi^\dagger] \cdot (e^{F+\omega})_{2n} + \frac{i^{n-1}}{2}\left(\bar{D}^\dagger\phi D^\dagger\phi^\dagger + (-)^n D^\dagger\phi^\dagger\bar{D}^\dagger\phi\right) \cdot (e^{F+\omega})_{2n-2}.$$

[5] This is due to the fact that, reducing equations like $F_{ij} = 0$, we obtain actually *anti*holomorphic scalars $D_i X_a$; see for instance (1.2.8).

[6] If the divisor were a Calabi-Yau itself, this would be not really a replacing, but rather a rearranging of the equations using a holomorphic Hodge dual [25].

4.2.3 Stability and scalars

Having obtained our equations with transverse scalars, we will now extend to them the stability considerations we have applied in section 4.1. After having obtained a Hitchin-like stability condition for linear equations, we will argue a similar condition for the full nonlinear equations with scalars (4.2.11). Using then the covering technique, which we have already seen in chapter 1, we will argue that the modification on the total space of BPS states is not so dramatic, outside some cases.

Let us thus work out first a "linearized" case. This should serve as a base for extending the modifications to the non-linear equations. To be specific, let us concentrate on the Hitchin equations in four dimensions (4.2.6) and find a stability condition for this equation. We will denote by $\langle\, ,\rangle$ the inner product $\int vol\,(\, ,)$, with vol the volume form, and make use of the Weitzenböck equalities [24]

$$\frac{1}{2}D_A^\dagger D_A = \partial_A^\dagger \partial_A - i\,\omega \cdot F_A\,.$$

As in the easier case of two dimensions and no scalars, discussed in the beginning of this subsection , we will choose for simplicity a rank 2 bundle E, and consider a sub-line-bundle L on which, without loss of generality, we can put a connection a with constant curvature. Applying a Hodge star (and changing c) we rewrite (4.2.6) in the form

$$i\,\omega \cdot F + \sum_{a=1}^{3}[X_a, X_a^\dagger] = c\,. \tag{4.2.12}$$

On the bundle $\mathrm{Hom}(L, E)$ we can now consider again the connection B induced by the connections a and A on L and E. We can apply this covariant derivative again to the holomorphic section s of $\mathrm{Hom}(L, E)$ expressing the subbundle relation (the embedding):

$$\frac{1}{2}\langle D_B^\dagger D_B\, s, s\rangle = \langle \partial_B^\dagger \partial_B\, s, s\rangle - i\,\langle \omega \cdot F_B\, s, s\rangle\,.$$

Using adjunction (that is, integrating by parts), holomorphicity of s and (4.2.12) we have

$$2\pi \frac{\langle s, s\rangle}{Vol}(\mu(E) - \mu(L)) = \langle D_B s, D_B s\rangle + \sum \langle [X_a, X_a^\dagger] s, s\rangle\,. \tag{4.2.13}$$

If we suppose that s is an eigenvector of X, $Xs = \lambda s$, we have for the last term

$$\langle [X, X^\dagger]\, s, s\rangle = \langle X^\dagger\, s, X^\dagger\, s\rangle - |\lambda|^2 \langle s, s\rangle = \langle (X^\dagger - \bar\lambda)s, (X^\dagger - \bar\lambda)s\rangle\,.$$

Since this is non-negative, from (4.2.13) we get the condition we wanted: *for any subbundle* $L \overset{s}{\hookrightarrow} E$ *which is* X*-invariant* (namely, the embedding s is eigenvector of all the X_a's) *we have the relation* $\mu(E) \geq \mu(L)$. This is the four-dimensional analogue of Hitchin stability. We have here only shown that it is necessary for solution of (4.2.6) to exist; a proof of sufficiency is more difficult, but essentially standard along the lines of other linear examples, using orbits of the complexified gauge group and analytic arguments, as for example in [86, 47].

A similar case to this one, though with a different "twisting", has been extensively studied in the mathematical literature under the name of Higgs bundles [75].

Moving now finally to our equations (4.2.9), it is very natural to propose a Hitchin-like modification of stability: for each X-invariant holomorphic subbrane B' of a given brane B, one has $\phi(B) \geq \phi(B')$. The only step left is to specify what "X-invariant" is supposed to mean, since in general (away from geometrical limits in the moduli space) branes are not bundles on cycles. For this, we notice that one can reformulate this notion in a completely abstract way by completing the embedding $B' \hookrightarrow B$ to the exact sequence

$$0 \to B' \xrightarrow{i} B \xrightarrow{\pi} B'' \to 0.$$

Using the maps i and π, one can see that being X-invariant means that $\pi \circ X \circ i = 0$ as an element of $\mathrm{Hom}(B, B'')$, where $\circ$ is composition. Though this proposal is the natural melting of Π-stability and of Hitchin's one, let us stress that again, as for the proposed connection between Π stability and (4.1.3), a complete proof is very difficult to find due to the non-linearity of the equations.

We would like to underline that these modifications à la Hitchin to the usual stabilities is substantial. Once one fixes a cycle B, a bundle E on it and a endomorphism X, it can happen in fact that even if E was unstable with respect to the usual definition, it is stable with respect to the modified one. Let us analyze for example μ-stability. Suppose that there exists only one holomorphic subbundle E' which destabilizes E, $\mu(E') > \mu(E)$. Then, if this subbundle is not X-invariant, E is still μ-Hitchin-stable. (Or more accurately, the couple (E, X) is μ-Hitchin-stable.)

Quivers and θ-stability

A logical consequence of this is also that the θ-stability that one finds going to the Gepner point [28, 54] should be modified in a similar fashion. Let us give a quick look at this here. First of all, recall that, near the Gepner point of the moduli space, branes are described to some extent by the same approach describing branes on an orbifold singularity [19]. Thus one has a supersymmetric gauge theory whose gauge content is summarized by a quiver; here, however, we will not need this explicitly, and keep all of the chiral multiplets in a set of "big" chiral fields Φ_i which are big matrices whose blocks are the matrices which represent the quiver. Then, we can write the D-term and F-term equations which describe the moduli space of the theory as

$$\sum_a [\Phi_a, \Phi_a^\dagger] = \Theta\,\mathrm{Id}; \qquad \frac{\partial W}{\partial \Phi^a} = 0 \tag{4.2.14}$$

As in the geometrical limit, the F-term equation (the $(2,0)$ part) is holomorphic and does not get modified by Kähler moduli. The equation that leads to a stability condition is again the single real equation for D-flatness, the first one in (4.2.14). The block-diagonal matrix Θ contains FI terms.

The condition for existence of solutions for these equations, again implies a stability condition. Indeed, consider a subrepresentation of the quiver. From the point of view of the total fields Φ_a (of rank, say, k), this means there is an injection i such that smaller matrices Φ'_a of rank $k' < k$, satisfying $\Phi_a \circ i = i \circ \Phi'_a$ can be found (we will omit $\circ$ from now on). It is useful to introduce as well a matrix $i^\dagger$ such that $i^\dagger i = \mathrm{Id}_{k'}$; then $ii^\dagger$ is a rank k' projector p, and we can rewrite $\Phi'_a = i^\dagger \Phi_a i$. Having introduced such a notation, let us see what happens each time we have a subrepresentation i. Then we start from a quantity manifestly non negative and expand it:

$$\begin{aligned} 0 \;\leq\; & \sum_a \mathrm{tr}\left((p\Phi_a(1-p))(p\Phi_a(1-p))^\dagger\right) = \\ & \sum_a \mathrm{tr}(p\Phi_a(1-p)\Phi_a^\dagger) = \sum_a \left(\mathrm{tr}(p\Phi_a\Phi_a^\dagger) - \mathrm{tr}(p\Phi_a p\Phi_a^\dagger)\right); \end{aligned} \tag{4.2.15}$$

but then, being

$$\begin{aligned} \mathrm{tr}(p\Phi_a p\Phi_a^\dagger) \;=\; & \mathrm{tr}(ii^\dagger\Phi_a ii^\dagger\Phi_a^\dagger) = \mathrm{tr}_{\mathrm{k}'}(i^\dagger\Phi_a i\, i^\dagger\Phi_a^\dagger i) = \mathrm{tr}_{\mathrm{k}'}(\Phi'_a\Phi'^\dagger_a) = \\ & \mathrm{tr}_{\mathrm{k}'}(\Phi'_a\Phi'^\dagger_a i^\dagger i) = \mathrm{tr}(\Phi_a p \Phi_a^\dagger), \end{aligned} \tag{4.2.16}$$

we can reexpress (4.2.15) as

$$\sum_a \mathrm{tr}(p[\Phi_a, \Phi_a^\dagger]) = \mathrm{tr}(p\Theta).$$

So, each time we have a subrepresentation of the quiver, the relation $\mathrm{tr}(p\Theta) \geq 0$ should hold. This is called θ-stability. This direct and explicit proof of necessity of stability for solving the D-term equations is exactly along the lines of the one we gave at the beginning of this long section. Now, it is not evident that the modification à la Hitchin of the stability that we have introduced will survive till this point of the moduli space; after all it is not clear to what the endomorphism which was called X in the geometrical limit will correspond in the quiver limit. In particular it could correspond to the zero endomorphism, thus giving no modification at all. But, at least mathematically, we can give an example of an equation whose solutions would imply a θ-Hitchin-stability.

Thanks to this analysis, this is now almost trivial. First, one must set the condition that the quiver representation is X-invariant – the equation $[X, \Phi_a] = 0$. Then, one finds the appropriate modification to the D-term equation in the form

$$\sum_a [\Phi_a + \alpha_a X, \Phi_a^\dagger + \alpha_a^* X^\dagger] = \Theta \,\mathrm{Id}\ . \tag{4.2.17}$$

Indeed, for a subrepresentation to be X-invariant, the injection i should satisfy the additional condition (similar to the above with Φ, Φ') that a smaller matrix X' should exist, such that $Xi = iX'$. If a subrepresentation satisfies this, there is no problem in carrying out the steps (4.2.16) in exactly the same way, with $\Phi \to \Phi + \alpha X$; otherwise this is not possible. Thus, the result is that we have the inequality $\mathrm{tr}(p\Theta) \geq 0$ only for X-invariant representations.

Thus we obtain a modification to θ-stability very similar to ones in geometric phase, which again may in particular rehabilitate representations previously discarded as unstable. One can view this as the survival of Hitchin-like modification all over the moduli space. We will see, however, that from the point of view of the moduli space of all the BPS states, this modification is not so dramatic as one could think, thanks to the fact that branes can be lifted to coverings, as we will now explain.

Coverings

In section 1.3 we have described the covering technique for the case in which the base space was either a cylinder or its compactification $\mathbb{P}^1$. Here we will apply it to less trivial geometrical situations; the base manifold will be now the submanifold B on which the brane is initially wrapped, corresponding to the vacuum $X = 0$.

Let us recall that the covering technique described, out of a section X of $\mathrm{End}(E) \otimes L$, a covering $\tilde{B}$, as the locus cut by the characteristic polynomial p_X in the total space of the line bundle L. In fact, this so-called *spectral manifold* can be defined also in the most general case, in which the base manifold B on which the non-abelian brane theory is defined is an arbitrary manifold; X is now a section of $N(B,Y) \otimes \mathrm{End}(E)$, and thus the equation $p_X(x) = 0$ becomes now an equation in the total space of the line bundle $N(B,Y)$ (or K_B, which is the same, if the ambient manifold Y is a Calabi-Yau). Now $DX = 0$ means, as we have underlined in subsection 4.2.2, $dX + [A, X] + aX = 0$, where A is the part of the connection in $\mathrm{End}(E)$ and a is a connection on the bundle $N(B,Y)$. Thus we have now $(\partial + a)p_X(x) = 0$; this can be considered as giving a holomorphic structure to the submanifold $p_X(x) = 0$ in the total space of the bundle $N(B,Y)$. With these modifications, the geometrical interpretation is intuitively the same.

Actually, what we want is not quite a submanifold defined on the total space of the normal bundle $N(B,Y)$; this is only a local (around the initial brane B) description. Then we can make $\tilde{B}$ a submanifold of the ambient manifold via the map $N(B,Y) \to Y$, as it is done in K-theory to find

Gysin map from the Thom isomorphism. Another possible version of this idea is to remain in the realm of algebraic geometry and use the so-called *normal cone deformation* (see for instance [35].

Finally, as noted many times to specify a submanifold is not enough for giving a brane, we have to provide the bundle on it as well. Also this was explained in section 1.3: the eigenvectors provide a line bundle L on the covering. This line bundle L is natural in the sense that its push-forward from $\tilde{B}$ to the base B (this operation can be defined by resorting to sheaves) is indeed the gauge bundle on the base: $\pi_* L = E$, where $\pi : \tilde{B} \to B$ is the covering map.

Coverings, stability and the BPS states

We have now a non-abelian brane theory defined on a submanifold B of a given ambient manifold Y. We have argued that stability gets modified à la Hitchin, with the condition on subbundles (or subbranes) replaced by the same condition on X-invariant subbundles.

The first case is the one in which $X = 0$. In this situation the branes are all wrapped on the base manifold B. But in this case, the modification we have proposed does not affect anything, because being X-invariant is an automatically satisfied condition.

If, on the other hand, X is a non trivial configuration, then we have seen that this describes branes on a covering $\tilde{B}$ of the initial brane B. In particular, we have seen how we can lift the gauge bundle from B to $\tilde{B}$. This is important for the following reason. If one considers BPS states on a fixed base B, there are substantial contributions coming from the presence of the scalars X, as we have seen. But, if one considers the whole moduli space of BPS states, namely the union of the former moduli space for all the base branes B, the covering technique actually shows that most of the new BPS states one obtains for one base brane B are actually copies of other BPS states pertaining to another brane $\tilde{B}$[7].

The discussion we had so far has some limitations. First, there are subtleties concerning the covering mechanism. Indeed, we have supposed so far that the eigenvalues are all distinct. If the characteristic polynomial is irreducible (which means that the covering is connect; this we can assume with no loss of generality) the only way we have to duplicate eigenvalues is to duplicate all of them, taking thus a characteristic polynomial which is a power: $p_{X'}(x) = (p_X(x))^k$. In this case, lifting yields a vector bundle $\tilde{E}$ *of rank* $\leq k$ over the covering brane $\tilde{B}$; a priori this is not guaranteed to be in the usual moduli space of BPS states pertaining to the cycle $\tilde{B}$, since $\tilde{E}$ could be not stable. But the modification of the stability à la Hitchin that we have proposed means now that the subbundles E' of the gauge bundle E on the base B should be liftable too. This way, the stability condition will get translated into a stability for the vector bundle $\tilde{E}$ over the lifted brane $\tilde{B}$.

Second, and more important, so far we have analyzed the covering mechanism in the case in which there is only one transverse scalar. When there are more, although there is an equation $[X_a, X_b] = 0$ from the undeformed $(2,0)$ part of the equations, still the commutators like $[X_a, X_b^\dagger]$ do not vanish: they appear in the deformed $(1,1)$ equations. This means that in general we are entering the realm of noncommutative – perhaps better, fuzzy – BPS states; about this, we limit ourselves to the few comments that follow.

The analysis of this case allows us to appreciate the importance of the part of the connection corresponding to the normal bundle $N(B,Y)$. Let us first of all tackle the flat space situation in

[7] We may present another somewhat indirect argument in favor of this assertion by recalling the definition of $\phi(E)$. As it was argued in [43] taking into account the non-abelian dynamics of D-branes amounts to replacing the RR coupling $C \wedge \mathcal{Y}$ by Clifford multiplication of $\mathcal{Y}$ by RR fields while leaving intact the form of the D-brane charge $\mathcal{Y}$, and thus the element in $K(Y)$. We should note however that transforming CS data into equations of motion is not straightforward in view of difference in "natural" conventions for the kinetic and CS parts. We will meet another similar clash of conventions in section 4.3.

which the brane is at least 2 dimensional but there are more than one complex scalar.

Recall the argument we had given in section 1.3 for reducing to the case with only one scalar. Let us now see what differences come in when we are in a geometrically less trivial situations; let B be a brane wrapped on a 2-cycle (or higher) on the ambient manifold Y. (In the main example we have had in mind, that of Y being a Calabi-Yau threefold, the only such case is the one in which B is a 2-cycle.) Then, as we have seen in the case with one scalar only, the equation $DX = 0$ does not imply any longer that $\partial \det(X_a - x_a) = 0$, but that $\partial_{N(B,Y)} \det(X_a - x_a) = 0$, where $\partial_{N(B,Y)}$ is a connection on $N(B,Y)$. This prevents the arguments we expounded in the paragraph above to be applied in this more general case.

Finally, we can consider instead the case in which B is a 0-cycle. Then the above considerations do not apply in flat space. The equation $DX_a = 0$ becomes now one of the already present equations $[X_a, X_b] = 0$, and so all of its consequences that we have explored in the preceding paragraph are not there any longer. In this case we are thus completely in the realm of fuzzy solutions; but we will not explore this here.

4.3 Deformed tachyon equations

In this section, we use again dimensional reduction, this time in a different way: we will obtain equations that we will argue to be relevant for the tachyon of the D$p - \overline{\mathrm{D}p}$ system.

Let us consider a pair of gauge bundles, E_1 and E_2, describing the gauge theory on the brane and on the antibrane, and a morphism of bundles T connecting them, that can be thought of as a section of $\mathrm{End}(E_1, E_2)$. In [66], a set of equations has been described that implies the equations of motion of this D$p - \overline{\mathrm{D}p}$ system, much in the same way in which the instanton equations imply the equations of motion for Yang-Mills, and more generally in the same way in which BPS conditions imply equations of motion for a supersymmetric action. These equations read

$$F_1 \cdot \omega - iTT^\dagger = \lambda_1 \, \mathrm{Id}_{N_1}, \qquad F_2 \cdot \omega + iT^\dagger T = \lambda_2 \, \mathrm{Id}_{N_2}, \qquad \partial T + A_1^{(1,0)} T - T A_2^{(1,0)} = 0, \quad (4.3.18)$$

where $N_i = rk(E_i)$; along with the usual $F^{(2,0)}$. As for the HYM equations and their deformation we have been studying so far, solution to them is equivalent to a stability condition on the triple (E_1, E_2, T).

Now, the point interesting for us is that these equations come naturally again from dimensional reduction of HYM, although in a more formal way. Namely, we reduce from complex dimension $n+1$ to n, in such a way that only one complex scalar will appear; this we will call W, as it is not one of the scalars parameterizing transverse fluctuations (for instance, the procedure we are following will give us the tachyon equations also in the maximal dimension $n = 5$, that is for the D9-$\overline{\mathrm{D}9}$ system, in which case there are no transverse fluctuations at all). Another modification to the reduction we have done before is that we have now to take a slightly more general non-abelianization of them, considering a direct sum bundle; thus, in the rhs of (2.2.9), we can be more general and write, instead of λId, $diag(\lambda_1 Id_{N_1}, \lambda_2 Id_{N_2})$ ($diag$ is here in a block sense). With this specifications, let us now do dimensional reduction of HYM (2.2.9) taking the particular choice[8]

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \qquad (4.3.19)$$

[8]This choice can be better motivated [14] if one takes the fibre to be $\mathbb{P}^1$, and chooses then the bundle on the whole fibration to be $p_1^* E_1 \oplus p_1^* E_1 \otimes p_2^* \mathcal{O}(2)$. Here we are doing a formal reduction as in section 4.1.

(remember that W has the formal role that X had in previous sections, appearing in Hitchin equations in its stead). In this way one obtains precisely the tachyon equations (4.3.18): first two come from the $(1,1)$ part, the third comes from the $(2,0)$ part.

We have obtained these tachyon equations from reduction of HYM; but we have seen that the latter get deformed by stringy corrections to (4.1.3). Thus we expect that, if we now reduce (4.1.3) in the same way, we will obtain the right deformation of the tachyon equations. This is by now an easy task: first of all, as usual, the $(2,0)$ part does not change, so we will keep on getting $DT \equiv \partial T + A_1^{(1,0)}T - TA_2^{(1,0)} = 0$. Then, let us start from our expression (4.2.9) and first of all specialize it to the case in which there is only one complex scalar. We get

$$\mathrm{Sym}\{\ (e^{F+\omega\,\mathrm{Id}})_{\mathrm{top}}(1-i[W,W^\dagger]) + \frac{i}{2}\left([D,W^\dagger][\bar D,W]-[\bar D,W][D,W^\dagger]\right)(e^{F+\omega\,\mathrm{Id}})_{\mathrm{top}-2}\ \}. \tag{4.3.20}$$

Once more, let us now extend this to the case in which the bundle is a direct sum; in this case this means to substitute $e^{i\theta}$ in (4.2.11) with $\exp\{diag(i\theta_1 Id_{N_1}, i\theta_2 Id_{N_2})\}$. Then, putting in this expression the particular choice (4.3.19), one obtains our "deformed vortex equations"

$$\begin{array}{l}\mathrm{Im}\left(e^{i\theta_1}\,\mathrm{Sym}\{(e^{F_1+\omega\,\mathrm{Id}_{N_1}})_{\mathrm{top}}(1-iTT^\dagger)-\frac{i}{2}\bar D T D T^\dagger(e^{F_1+\omega\,\mathrm{Id}_{N_1}})_{\mathrm{top}-2}\}\right)=0\,,\\ \mathrm{Im}\left(e^{i\theta_2}\,\mathrm{Sym}\{(e^{F_2+\omega\,\mathrm{Id}_{N_2}})_{\mathrm{top}}(1+iT^\dagger T)+\frac{i}{2}DT^\dagger\bar D T(e^{F_2+\omega\,\mathrm{Id}_{N_2}})_{\mathrm{top}-2}\}\right)=0\,,\end{array} \tag{4.3.21}$$

where $\bar D T \equiv \bar\partial T + A_1^{(0,1)}T - TA_2^{(0,1)}$ is the antiholomorphic covariant derivative of the tachyon. Notice that these equations are less decoupled than usual: not only does the tachyon appear in both, but also, in the deformation term with $\bar D T$, both A_1 and A_2 appear.

It is natural to speculate that (4.3.21) can be expressed in terms of superconnections. Let us try to make this expectation more precise. One would like to exploit the considerations made after (4.2.9) to write an expression which contains a superconnection in the exponent. To do so, the procedure is just like the one we have done above: i) replacing all the Id with $diag(Id_{N_1}, Id_{N_2})$, ii) taking A and T of the particular form (4.3.19). But this time, one wants to start not from the more explicit form of the equation (4.3.20), but from one of the more imaginative forms (4.2.9) or (4.2.10). Explicitly, one gets

$$\mathrm{Im}\left(\mathrm{Sym}\{\exp\begin{pmatrix} i\theta_1\,\mathrm{Id}_{N_1} & 0\\ 0 & i\theta_2\,\mathrm{Id}_{N_2}\end{pmatrix}[e^{\mathcal F}e^{\omega+J}]_{\mathrm{top}}\}\right)=0, \tag{4.3.22}$$

where

$$\mathcal F \equiv \begin{pmatrix} F_1 + i_T\, i_{T^\dagger} & i_T\,\bar D_2 - \bar D_1\, i_T\\ D_2\, i_{T^\dagger} - i_{T^\dagger}\, D_1 & F_2 + i_{T^\dagger}\, i_T\end{pmatrix} = [\mathcal D, \bar{\mathcal D}]; \qquad \mathcal D = \begin{pmatrix} D_1 & i_T\\ 0 & D_2\end{pmatrix}.$$

These equations need of course some comments. First, we remind the reader that we are using here a super-Lie bracket instead of the usual one, as explained after (4.2.10); this explains the sign of $i_{T^\dagger} i_T$ in $\mathcal F$.

Second, we introduced in (4.3.22) a symbol J which is a two-form $idwd\bar w$ in the formal transverse space spanned by W, $W^\dagger$. Once again, this is not the physical transverse space, and this J, though it has the same formal role of $\omega_\perp$ in (4.2.9) and similar equations, is a different object, introduced here only as a convenient device for bookkeeping. Given that there is only one holomorphic object T here, this choice could appear rather baroque; and one could wonder if, instead, it would not have been better to start from an expression halfway between (4.2.9) and (4.3.20); namely, an expression as compact as (4.2.9) is, which also has an exponential structure, but with i_W replaced in some way by W. In fact, such an equation does not exist. To replace i_W with W (or i_T with T in (4.3.22)) before expanding the exponential would be wrong, as one can convince oneself by noting the minus sign in the expression $[D, W^\dagger][\bar{D}, W] - [\bar{D}, W][D, W^\dagger]$ in (4.3.20). Moreover, the expression one could obtain this way will necessarily have a piece proportional to $diag(TT^\dagger, -T^\dagger T)$, which comes from a commutator $[W, W^\dagger]$ (compare with (4.3.18)). Thus, this expression would be in any case different from the superconnection arising from the CS system of the $D - \bar{D}$ system. It is hardly expected that the two superconnections are the same. Since we are using the equations of motion, there is a contribution from the DBI part. However, it is interesting to notice that the latter do not spoil completely the presence of the $\mathbb{Z}_2$-graded structures.

Finally, let us notice that it could be that (4.3.21), alias (4.3.22), is the right deformation to solve some of the problems of interpretation for stability of triples raised in [66].

Tachyons and transverse fluctuations

By expanding on the observation in section 4.2.1 on the similarity of equations involving the scalars to those appearing in the computation of Chern character with superconnections, we conclude by a comment on relation between tachyons T and transverse scalars X on D-branes. So far this is only a formal correspondence, as T and X are quite different objects. One may furthermore note that the reduction performed in this section for the tachyon equations is formal, unlike that for X which was given by T-duality. For instance, there is no reason for which the tachyon equations we propose should not be valid in the important case of D9-$\overline{\text{D9}}$ system (in which case there are no transverse scalars).

Nevertheless, there is a relation between the two objects, provided by tachyon condensation. First of all, let us begin with a couple of considerations on the general meaning of what we are going to analyze. Consider a Dp − $\overline{\text{D}p}$ system on some manifold Y. This defines, through relative K-theory, some element of the $K(Y)$. Superconnections were initially introduced [69] as a method to compute the Chern character of this element. If we consider instead a non-abelian brane theory defined on some submanifold B, by the covering mechanism this describes a brane wrapped on some other submanifold $\tilde{B}$ of Y. This is again an element of $K(Y)$. Tachyon condensation connects these two ways of obtaining a K-theory element; the result, as we will see shortly, is that T is the characteristic polynomial of X.

To be more specific, consider a D9-$\overline{\text{D}\,9}$ system on $\mathbb{C} \times Y$, where Y is now a 8 dimensional manifold. Let us first consider the case in which the bundles on the brane and the antibrane are chosen in such a way that the tachyon is $T = x$, where x is the coordinate on $\mathbb{C}$. Thus the system condenses to a D7 wrapped on Y, described by the locus $x = 0$. What we want to emphasize is that both T for the D9-$\overline{\text{D}\,9}$ and X for the D7, the vanishing locus is the same, although from a different perspective: in the first case the tachyon is a 10 dimensional field whose zeroes indicate where the resulting lower-dimensional brane will be; in the second case X it is a 8 dimensional, whose zeroes indicate in which position of the transverse direction that it parameterizes the vacuum is located. Let us go ahead with this and consider now a less trivial case: let the tachyon be $T(x, z_i)$, where z_i are coordinates on Y; let us suppose it is holomorphic, and moreover that is a polynomial in x of degree N. Now, the D7 resulting after condensation can be obtained as a classical configuration in the non-abelian brane theory with base $B = Y$. This is accomplished by the covering mechanism we have described above: it is sufficient to take a configuration X with characteristic polynomial $p_X(x, z_i) = T(x, z_i)$. Thus, more generally we can say that T is the characteristic polynomial of X.

Let us give a geometrical twist to this. We can describe this $\mathbb{C} \times Y$ as a trivial line bundle over Y; more generally we can consider a different line bundle L on Y, and take the 10 dimensional space as the total space Q of this line bundle. In this situation the locus is again described by the zeroes of the tachyon $T(x, z_i)$; but the tachyon itself now is a section of the N-th power of the bundle which is the pull-back of the bundle L to its own total space Q. This bundle has an obvious so-called tautological section, and x is to be understood now as this section.

Finally, it is now an easy generalization to consider the case in which the result of the condensation has codimension higher than 2: locally around the lower-dimensional brane, the tachyon will be in that case

$$T = \sigma_a \det(X_a - x_a), \tag{4.3.23}$$

where $\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ 0 & 0 \end{pmatrix}$ is the γ matrix relative to the holomorphic coordinate x^a. This formula makes more explicit in general the relationship between tachyon field and transverse scalars we have been using implicitly in deriving (4.3.21) and (4.3.22).

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