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Numerical methods for free-discontinuity problems based on approximations by Γ-convergence

CANDIDATE
Matteo Negri

SUPERVISORS
Prof. G. Dal Maso
Prof. M. Paolini

Thesis submitted for the degree of "Doctor Philosophiae"
Academic Year 2000/2001

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Il presente lavoro costituisce la tesi presentata dal dott. Matteo Negri, sotto la direzione del Prof. Gianni Dal Maso e del Prof. Maurizio Paolini, al fine di ottenere l'attestato di ricerca post-universitaria "Doctor Philosophiae" presso la SISSA, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni. Ai sensi del Decreto del Ministro della Pubblica Istruzione 24/4/1987, n. 419, tale diploma è equipollente al titolo di "Dottore di Ricerca in Matematica".

Trieste, a.a. 2000/2001

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Acknowledgment

First of all I would like to thank G. Dal Maso. The discussions with him have been always clarifying and very useful to develop the results of this thesis. Besides, working with him and attending his lectures gave me great opportunities to understand how to work in mathematics and to learn many interesting techniques.

I would like to thank also M. Paolini, D. Percivale, M. Morini and A. Braides because through collaborations, informal discussion and lectures I learned a lot also from them.

Finally I would like to thank P.G. Ciarlet, because his books encouraged my interest in numerical analysis and applied mathematics.

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“*Natura non facit saltus*”
Leibniz, Nuovi Saggi IV

Chapter 1

Introduction and preliminaries

1.1 Introduction

In the last years a class of problems, known as *free discontinuity problems*, has been widely studied in the contest of calculus of variation and numerical analysis. This term was introduced by De Giorgi to denote those problems where "*not only the solutions can have discontinuity points but the most difficult part consists in finding or characterizing these points*".

This framework has been used also in the formulation of some mathematical models for image processing, fracture mechanics, liquid crystals and shape optimization. In view of the numerical simulations of the models, it became interesting the development of some approximations by means of functionals defined for finite elements or finite differences. In this direction, following different approaches, many discretizations have been suggested ([3] [9] [18] [19] [20] [22] [34] [35] [38] [39] [41] [43] [44] [45]). For all these results the theory of Γ-convergence plays a crucial role, because it defines a variational convergence which (under suitable hypotheses) implies also the convergence of the minimizers.

In particular this thesis deals with some discretizations of free discontinuity problems, presenting both the theoretical results, which are based on Γ-convergence, and the numerical solutions.

The Mumford-Shah functional is nowadays one of the most studied free discontinuity problem. It was introduced, in a strong formulation, in [42] as a variational model for image segmentation. Here and in the sequel it is considered in the weak form

$$F(u) = \beta \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}^1(S_u) + \int_\Omega |u-g|^2 \, dx \,, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is an open bounded set with Lipschitz boundary, α and β are positive constants, $g \in L^\infty(\Omega)$ and $u \in SBV(\Omega)$. Inspired by [14], Chambolle and Dal Maso proposed in [20] a discretization of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega f(\varepsilon |\nabla u|^2) \, dx + \int_\Omega |u-g|^2 \, dx \,. \tag{1.2}$$

The function $f : [0,+\infty) \to [0,+\infty)$ is nondecreasing, continuous and satisfies

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 1 \qquad \lim_{t \to +\infty} f(t) = 1 \,. \tag{1.3}$$

The domain of the functional is a finite element set $V_\varepsilon^\theta(\Omega)$ which is defined in the following way: let $0 < \theta < \frac{\pi}{2}$ and, for every $\varepsilon > 0$, let $\mathcal{T}_\varepsilon^\theta$ be the family of the triangulations $\boldsymbol{T}_\varepsilon$ of $\mathbf{R}^2$ such that for each element T the length of the edges is bounded from below by ε and from above by 6ε and such that the amplitude of the internal angles is greater than or equal to θ. Then $V_\varepsilon^\theta(\Omega)$ is the union of all the finite element spaces of piecewise affine functions in Ω defined on the triangulations $\boldsymbol{T}_\varepsilon \in \mathcal{T}_\varepsilon^\theta$. Chambolle and Dal Maso proved in [20] that if θ is sufficiently small the functional (1.2) Γ-converges to the Mumford-Shah functional with $\beta = 1$ and $\alpha = \sin\theta$.

In particular this approach (see Figure 1.1) provides a triangulation $\boldsymbol{T}_\varepsilon$ in such a way that the discontinuity set S_u is contained in a tubular neighborhood. This approximation was later implemented by Bourdin and Chambolle in [11] but in practice, due to the difficulties in the generation of the adaptive triangulations, it could be effectively used only for some model problems.

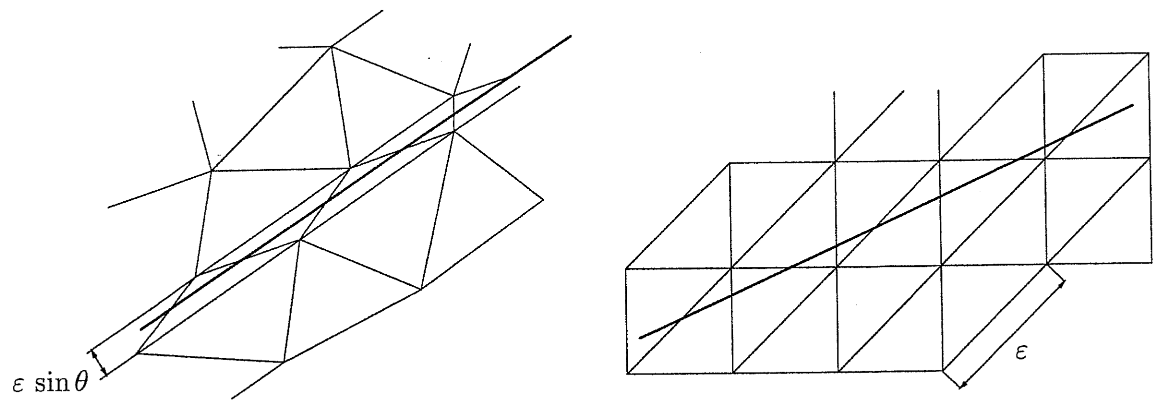


Figure 1.1: An adaptive triangulation $\boldsymbol{T}_\varepsilon$ and the structured mesh $\boldsymbol{T}_\varepsilon^1$ along a straight line.

In order to simplify this approach I studied in [43] an approximation by means of a functional which is again of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|^2)\,dx + \int_\Omega |u-g|^2\,dx \tag{1.4}$$

and which is defined on the finite element spaces $V_\varepsilon^i(\Omega)$ based on the structured triangulations $\boldsymbol{T}_\varepsilon^i$ (see Figure 1.2). In particular the elements of $\boldsymbol{T}_\varepsilon^1$ and $\boldsymbol{T}_\varepsilon^2$ are isosceles right triangles while the ones of $\boldsymbol{T}_\varepsilon^3$ are equilateral. This time the tubular neighborhood is replaced by a polyhedral set (see Figure 1.1), thus the Γ-limit will be a sort of anisotropic Mumford-Shah functional of the form

$$F(u) = \int_\Omega |\nabla u|^2\,dx + \int_{S_u} \phi_i(\nu)\,d\mathcal{H}^1 + \int_\Omega |u-g|^2\,dx\,. \tag{1.5}$$

The functions $\phi_i(\nu)$, which can be explicitly computed, take into account an anisotropy effect which is introduced by the geometry of the mesh.

Even if the functional (1.4) allows an easier numerical implementation, usually the standard algorithms fail in the attempt of finding a minimizer, in general because of the presence of many local minima. A general approach could be the simulated annealing

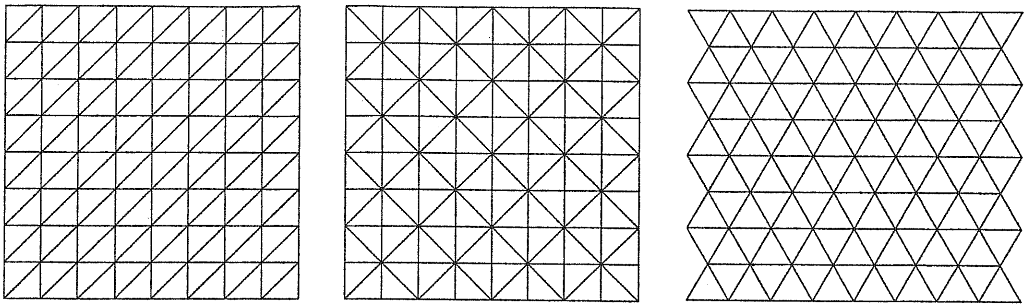


Figure 1.2: Geometries of the triangulations T_ε^i for $i = 1, 2, 3$ respectively.

algorithm. Nevertheless, in practice, being based on randomness, it is not really efficient for these problems, where the number of unknowns is large (usually 2^{16}).

These considerations suggest the use of *ad hoc* techniques which take into account the specific properties of the problems. One of these, called "*graduated non convexity*" algorithm (in short GNC), was suggested by Blake and Zissermann [15] for a similar problem. The idea, based on a sort of relaxation argument, consists basically in minimizing the discrete energy for increasing values of the parameter β, using at each time a descent algorithm. This technique can be clearly applied to our problem and gives good numerical results. Nevertheless, in collaboration with Paolini, we developed in [46] a more efficient algorithm, based on a sort of multi-scale approach. To present the motivations and the idea, let us consider, for the sake of simplicity, the one-dimensional functional

$$G_\varepsilon(u) = \frac{1}{\varepsilon}\int_I f(\varepsilon|u'|^2)\,dx + \int_I |u-g|^2\,dx\,, \tag{1.6}$$

where I is the interval $(0,1)$, $g \in L^\infty(I)$, and u belongs to the finite element space of piecewise affine functions on the subdivision $\varepsilon\mathbf{Z} \cap I$. Choosing for instance

$$f(t) = \frac{2\alpha}{\pi}\arctan\left(\frac{\beta\pi}{2\alpha}t\right),$$

the functional (1.6) Γ-converges to the one-dimensional Mumford-Shah functional

$$G(u) = \beta\int_I |u'|^2\,dx + \alpha\#(S_u) + \int_I |u-g|^2\,dx\,. \tag{1.7}$$

Since $f(t)$ behaves like $\beta t \wedge \alpha$ our functional introduces a sort of local discrete threshold, which turns out to be closely related to the presence of local minima. Indeed, considering an interval $I_i = [x_i\, x_{i+1}]$, the slope of u in I_i can be written as

$$|u'| = \frac{|u(x_i) - u(x_{i+1})|}{\varepsilon}$$

and thus

$$\frac{1}{\varepsilon}\int_{I_i} f(\varepsilon|u'|^2)\,dx \approx \begin{cases} \beta|u'|^2 & \text{if } |u(x_i)-u(x_{i+1})|^2 \le \alpha\varepsilon/\beta \\ \alpha & \text{if } |u(x_i)-u(x_{i+1})|^2 > \alpha\varepsilon/\beta. \end{cases}$$

Hence F_ε introduces a sort of local discrete threshold on the contrast $|u(x_i) - u(x_{i+1})|$. This behavior becomes dangerous when ε is small because $\alpha\varepsilon/\beta \to 0$ and then the functional will be treated numerically as if it was constant. This aspect, and the numerical solutions, suggested the use of a multiscale approach. The idea is basically to minimize the discrete functional, using a second order algorithm (like quasi-Newton), for decreasing values of the resolution parameter ε, reducing in this way the wrong thresholding effect. The implementation of this approach is faster than GNC and the quality of the numerical results is comparable.

Considering that the structure of a digital image is simply a lattice of picture elements (the so called *pixels*) it is natural to use also tecniques based on finite differences. In this contest, considering the work of Gobbino [34], Chambolle proposed in [19] a functional of the form

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbf{Z}^2} \sum_{\substack{\xi \in \mathbf{Z}^2 \\ x + \varepsilon\xi \in \Omega}} \frac{1}{\varepsilon} f\left(\frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon|\xi|^2}\right) \rho(\xi).$$

The function $f : [0, +\infty) \to [0, +\infty)$ is nondecreasing, continuous and satisfies (1.3), while the convolution term $\rho : \mathbf{Z}^2 \to [0, +\infty)$ is even and satisfies

$$\rho(0) = 0 \qquad \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) < +\infty \qquad \rho(\xi) > 0 \text{ if } |\xi| = 1 \qquad \rho(\xi) = \rho(\xi^\perp). \tag{1.8}$$

Then Chambolle proved that the Γ-limit is the functional

$$F(u) = c_\rho \int_\Omega |\nabla u|^2 \, dx + \int_{S_u} \phi(\nu) \, d\mathcal{H}^1, \tag{1.9}$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \qquad \phi(\nu) := \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) |\langle \nu, \hat{\xi} \rangle|. \tag{1.10}$$

Recently, in a joint work with Morini [41], we proved that (1.9) is the Γ-limit also of the discrete functionals

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbf{Z}^2} \sum_{\substack{\xi \in \mathbf{Z}^2 \\ x + \varepsilon\xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log\left(1 + a_\varepsilon |\xi| \frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon^2 |\xi|^2}\right) \rho(\xi), \tag{1.11}$$

where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and ρ satisfies (1.8).

A remarkable property of (1.11) is the fact it does not introduce any local discrete threshold, which caused difficulties in the minimization of (1.6) and (1.4). This aspect is confirmed by some preliminary numerical results which show that the segmentations obtained by a simple gradient descent algorithm are comparable with the ones obtained by a GNC or multi-scale technique combined with a quasi-Newton algorithm (which employs second order derivatives). Moreover this functional seems to suggest a link with a sort of anisotropic diffusion. Following Richardson and Mitter [49] we can consider, for the sake of simplicity, the functional

$$G_\varepsilon(u) = \frac{1}{a_\varepsilon} \int_\Omega \log\left(1 + a_\varepsilon |\nabla u|^2\right) dx + \int_\Omega |u - g|^2 \, dx,$$

which can heuristically be considered a continuous version of (1.11). Then its Frechet derivative is given by

$$dG_\varepsilon = \operatorname{div}\left(\frac{\nabla u}{1+a_\varepsilon|\nabla u|^2}\right) + 2(u-g)$$

and its gradient flow becomes

$$\frac{\partial u}{\partial t} = dG_\varepsilon = \operatorname{div}\left(\frac{\nabla u}{1+a_\varepsilon|\nabla u|^2}\right) + 2(u-g)\,. \tag{1.12}$$

The last equation resembles a *biased* anisotropic diffusion proposed by Nordström in [47]. As in [49] we can interpret heuristically the gradient flow as a *steepest descent* for the minimization of the continuous functional G_ε. Consequently we can expect some relationship between the minimizers of (1.11), computed by means of first order algorithms, and the behavior of the solution of (1.12), computed by a finite difference discretization and for large values of the time variable t.

In all the discrete functionals I presented before, the discontinuities were always defined implicitly by means of the behavior of ∇u and the form of the Γ-limit always depended on the geometry of the mesh, both in the adaptive and in the structured case. Recently, I studied in [45] a different type of discrete functionals, based on discontinuous finite elements, where the discontinuity set is given explicitly and the geometry of the mesh does not effect the form of the Γ-limit (allowing in this way an easier mesh refinement).

For every $\varepsilon > 0$ let $\boldsymbol{T}_\varepsilon$ be a triangulation of $\mathbf{R}^2$ and assume that the family $\{\boldsymbol{T}_\varepsilon\}$ is regular, in the sense of [21]. Let $\mathcal{B}_\varepsilon = \{\boldsymbol{B}_\varepsilon\}$ be the family of the triangulations nested in $\boldsymbol{T}_\varepsilon$ and defined in the following way (see Figure 1.3): every element $T \in \boldsymbol{T}_\varepsilon$ is divided into four sub-elements of $\boldsymbol{B}_\varepsilon$, taking on every edge $[x_\varepsilon^i\ x_\varepsilon^j]$ of T a knot x'_ε which satisfies the constraint

$$x'_\varepsilon = tx_\varepsilon^i + (1-t)x_\varepsilon^j \qquad \text{for } a_\varepsilon \le t \le 1-a_\varepsilon, \tag{1.13}$$

where a_ε is a positive infinitesimal sequence. We will say that these vertices are adaptive and we will denote by $\boldsymbol{E}_\varepsilon$ the set of edges whose extrema are both adaptive vertices.

For every $\varepsilon > 0$ and every $\boldsymbol{B}_\varepsilon \in \mathcal{B}_\varepsilon$, consider the finite element space $W_\varepsilon(\Omega, \boldsymbol{B}_\varepsilon)$ of discontinuous functions which are affine on every sub-element $T \in \boldsymbol{B}_\varepsilon$ and which can have discontinuities *only* along the edges of $\boldsymbol{E}_\varepsilon$. Then our *finite element set* $V_\varepsilon(\Omega)$ will be the union of the spaces $W_\varepsilon(\Omega, \boldsymbol{B}_\varepsilon)$ for $\boldsymbol{B}_\varepsilon \in \mathcal{B}_\varepsilon$.

Let $s : [0,+\infty) \to [0,+\infty)$ be an increasing continuous function such that

$$\lim_{t\to 0+} \frac{s(t)}{t} = 1 \qquad \lim_{t\to+\infty} s(t) = 1\,. \tag{1.14}$$

Then for $0 < q < 1$ and for a positive diverging sequence b_ε the discrete functional, defined, for $u \in W_\varepsilon(\Omega, \boldsymbol{B}_\varepsilon)$, as

$$F_\varepsilon(u) = \sum_{T\in \boldsymbol{B}_\varepsilon} \int_T |\nabla u|^2\,dx + \sum_{\zeta\in \boldsymbol{E}_\varepsilon} \int_\zeta s(b_\varepsilon|u^+ - u^-|^q)\,d\mathcal{H}^1\,, \tag{1.15}$$

Γ-converges to the Mumford-Shah functional.

Note that this approach seems to be very general and it can be applied, for instance, to functionals of the form

$$\int_\Omega f(\nabla u)\,dx + \int_{S_u} \phi(\nu_u)\,d\mathcal{H}^1\,. \tag{1.16}$$

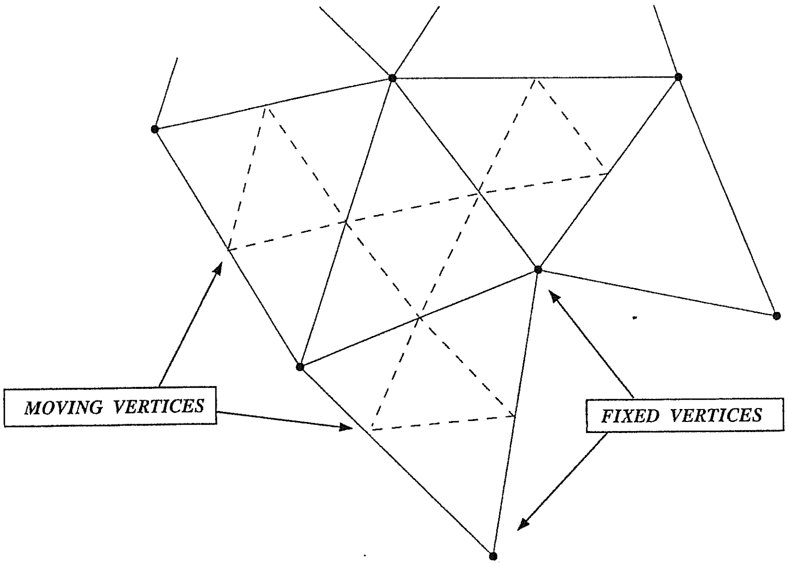


Figure 1.3: The foreground and a background triangulations T_ε and B_ε.

Finally I would like to mention another approach, based on the work [8] of Ambrosio and Tortorelli. The discretizations have been investigated by Bellettini and Coscia [9], using finite elements, and by March [38], using finite differences.

Another interesting field for the applications of free discontinuity problems is the propagation of fractures in brittle materials. An effective theory has been developed by Griffith (see [48] and [37]) using the following energy balance: let dl be the infinitesimal variation of crack length and let dE be the corresponding variation of linear elastic energy. Griffith suggested that the energy required to increase the crack should be γdl, for a parameter $\gamma > 0$ depending on material toughness. Then he postulated that the fracture evolves only when $-dE \geq \gamma dl$, which simply means that the release of elastic energy $-dE$ must be greater then the energy γdl spent to increase the fracture.

A rigorous mathematical formulation for these problems is given in the framework of the functions with bounded deformation (see [6] [29]). Indeed, for these functions the symmetric part of the derivative (in the sense of distributions) is the measure

$$Eu = Du^T + Du = \mathcal{E}u\,\mathcal{L}^n + (u^+ - u^-) \otimes \nu\,\mathcal{H}^{n-1} \llcorner J_u\,. \tag{1.17}$$

In this setting, for a two dimensional isotropic material, the linearized elastic energy is given by the Hooke low

$$\int_\Omega W(\mathcal{E}u)\,dx = \int_\Omega \mu|\mathcal{E}u|^2 + \frac{\lambda}{2}|tr(\mathcal{E}u)|^2\,dx\,,$$

where $\mu > 0$ and $\lambda > 0$ are the Lamé coefficients; while the fracture energy is given by

$$\gamma\mathcal{H}^1(J_u)\,,$$

where $\gamma > 0$ is called the material toughness. Then the Griffith energy becomes

$$\int_\Omega W(\mathcal{E}u)\,dx + \gamma\mathcal{H}^1(J_u)\,. \tag{1.18}$$

In particular for the applications it is interesting to consider the evolution of the fracture when the displacement u_t (at time t) satisfies a Dirichlet boundary condition $u_t = g(t)$ on a set $\partial\Omega_D \subset \partial\Omega$ and when $\partial\Omega \setminus \partial\Omega_D$ is traction free (for the rigorous mathematical treatise of this problem see [26], [31] and recently [17]). In this situation, knowing the displacement u_t and the corresponding fracture J_{u_t} at time t, the behavior at time $t + dt$ is determined as

$$u_{t+dt} \in \operatorname{argmin}\Big\{ \int_\Omega W(\mathcal{E}v)\,dx + \gamma\mathcal{H}^1(J_v \setminus J_{u_t})\Big\} \tag{1.19}$$

under the boundary condition $v(x) = g(t + dt, x)$ in $\partial\Omega_D$ and the irreversibility condition $J_v \supset J_{u_t}$. Indeed, being u_{t+dt} a minimum point, for every function $w(x)$ such that $w(x) = g(t + dt, x)$ in $\partial\Omega_D$ and $J_w = J_{u_t}$ we have

$$\int_\Omega W(\mathcal{E}u_{t+dt}) + \gamma\mathcal{H}^1(J_{u_{t+dt}} \setminus J_{u_t}) \leq \int_\Omega W(\mathcal{E}w)\,dx$$

and then

$$\gamma\mathcal{H}^1(J_{u_{t+dt}} \setminus J_{u_t}) \leq -\Big(\int_\Omega W(\mathcal{E}u_{t+dt}) - W(\mathcal{E}w)\,dx\Big)$$

which implies Griffith's criterion $-dE \geq \gamma dl$.

Clearly, in view of the numerical implementation of the model we need a discretization of the energy (1.18). Following [20] and [43], I studied in [44] an approximation of (1.18) by means of functionals defined on finite elements. Let $V^i_\varepsilon(\Omega, \mathbf{R}^2)$ and $V^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ be the sets of piecewise affine functions in Ω taking values in $\mathbf{R}^2$ and defined on the triangulations $\boldsymbol{T}^i_\varepsilon$ and $\boldsymbol{T}_\varepsilon$ defined before. The discrete functionals are of the form

$$G_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f(\varepsilon, Du)\,dx\,.$$

The function $f : \Omega \times \mathrm{M}^{2\times 2} \to [0, +\infty)$ this time behaves as

$$f(\varepsilon, Du) \approx \begin{cases} \varepsilon W(Eu) & \text{if } \varepsilon|Du|^2 < \eta \\ f_\infty & \text{if } \varepsilon|Du|^2 \geq \eta\,, \end{cases}$$

for a given constant $\eta > 0$. In analogy with [20] and [43] the Γ-limits will be of the form

$$\int_\Omega W(\mathcal{E}u)\,dx + \gamma\int_{J_u} \phi(\nu)\,d\mathcal{H}^1\,, \tag{1.20}$$

and

$$\int_\Omega W(\mathcal{E}u)\,dx + \gamma\mathcal{H}^1(J_u)\,, \tag{1.21}$$

depending on the choice of the space. Note that, due to the lack of a density property in $SBD^2(\Omega, \mathbf{R}^2)$, this convergence result holds only in $SBV(\Omega, \mathbf{R}^2)$. Nevertheless this approach seems to be sufficiently general to predict a realistic physical behavior, as shown by the numerical solutions.

Finally, Chapter 7 deals with a recent result, obtained in collaboration with Dal Maso and Percivale [25], which is not directly related to the main subject of this thesis. Indeed, by means of a Γ-convergence approach it justifies, from a variational point of view, the linearization of elastic energies, under the constitutive assumption which seem more natural in this contest. We consider the case of hyperelastic materials whose reference configuration is an open, bounded region with Lipschitz boundary. For these kind of materials the elastic energy can be written in terms of the deformation gradient ∇v as

$$\int_\Omega W(x, \nabla v)\, dx\,, \tag{1.22}$$

where the energy density $W(x, F)$ is defined for $x \in \Omega$ and F in the space $\mathbb{M}^{n\times n}$ of $n\times n$ matrices. The stress tensor corresponding to the deformation gradient ∇v is then given by $T(x, \nabla v) = \partial_F W(x, \nabla v)$.

By frame indifference we can express $W(x, \nabla v)$ in terms of the right Cauchy-Green strain tensor $C(v) := \nabla v^T \nabla v$ or, equivalently, in terms of the Green-St.Venant tensor $\frac{1}{2}(C(v) - I)$, where I is the identity matrix. Thus we can write $W(x, \nabla v) = V(x, \frac{1}{2}(C(v) - I))$ for a suitable function $V(x, E)$ defined for $x \in \Omega$ and E in the space $\mathbb{M}^{n\times n}_{sym}$ of symmetric $n\times n$ matrices.

We prefer to express these quantities in terms of the displacement u, defined by $u(x) := v(x) - x$. As $\nabla v = I + \nabla u$ the Green-St.Venant tensor $\frac{1}{2}(C(v) - I)$ can be written as $E(u) := e(u) + \frac{1}{2}C(u)$, where $e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$ is the symmetric part of the displacement gradient.

We assume that the reference configuration is stress free, i.e., $T(x, I) = 0$, and thus $\partial_F W(x, I) = \partial_E V(x, 0) = 0$. As $W(x, \cdot)$ and $V(x, \cdot)$ are defined up to an additive constant, it is not restrictive to assume also that $W(x, 0) = V(x, 0) = 0$.

Since the displacement $u = 0$ is an equilibrium configuration when no external loads are applied, it is natural to expect small displacements for small external loads. It is then convenient to rescale the variables and to write the load as $\varepsilon\ell$ and the displacement as εu for a suitable (adimensional) small parameter $\varepsilon > 0$. Thus we have $v(x) = x + \varepsilon u(x)$, and the equilibrium configurations are stationary points of the total energy

$$\int_\Omega W(x, I + \varepsilon\nabla u)\, dx - \varepsilon^2 \int_\Omega \ell u\, dx\,. \tag{1.23}$$

As

$$W(x, I + \varepsilon\nabla u) = V(x, \varepsilon e(u) + \tfrac{1}{2}\varepsilon^2 C(u))\,, \tag{1.24}$$

if ∇u is bounded we have, by Taylor expansion,

$$W(x, I + \varepsilon\nabla u) = \varepsilon^2 \tfrac{1}{2}\partial_E^2 V(x, 0)[e(u), e(u)] + o(\varepsilon^2)\,, \tag{1.25}$$

where $\partial_E^2 V(x, \cdot)$ denotes the second derivative of $V(x, \cdot)$ on $\mathbb{M}^{n\times n}_{sym}$, and $o(\varepsilon^2)$ is uniform with respect to x. The tensor $\mathsf{A}(x) := \partial_E^2 V(x, 0)$ is called the elasticity tensor, and the linearized elastic energy is then defined as

$$\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\, dx\,.$$

The previous discussion shows that, if we rescale the total energy given by (1.23), we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big(\int_\Omega W(x, I + \varepsilon\nabla u)\, dx - \varepsilon^2 \int_\Omega \ell u\, dx\Big) = \frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u), e(u)]\, dx - \int_\Omega \ell u\, dx \tag{1.26}$$

for every Lipschitz function u. This equality is usually considered as the main justification of linearized elasticity.

Note that this argument does not prove that the minimizers u_ε of (1.23), satisfying suitable boundary conditions, actually converge to the minimizer of the corresponding problem

$$\frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u),e(u)]\,dx - \int_\Omega \ell u\,dx\,.$$

Indeed we shall see that this is not always true.

Given a load $\ell \in L^2(\Omega,\mathbf{R}^n)$, a boundary value $g \in W^{1,\infty}(\Omega,\mathbf{R}^n)$, and a closed subset $\partial\Omega_D$ of $\partial\Omega$ with $\mathcal{H}^{n-1}(\partial\Omega_D) > 0$, we consider the minimum problems

$$\min_{u\in H^1_{g,\partial\Omega_D}} \Big\{ \int_\Omega W(x, I+\varepsilon\nabla u)\,dx - \varepsilon^2\int_\Omega \ell u\,dx \Big\}\,, \tag{1.27}$$

where $H^1_{g,\partial\Omega_D}$ denotes the closure in $H^1(\Omega,\mathbf{R}^n)$ of the space of functions $u \in W^{1,\infty}(\Omega,\mathbf{R}^n)$ such that $u = g$ on $\partial\Omega_D$. Suppose that, for every $\varepsilon > 0$, there exists a solution u_ε of (1.27) which satisfies the orientation preserving condition $\det(I+\varepsilon\nabla u_\varepsilon) > 0$. Under some natural hypotheses on V, we prove that u_ε converges weakly in $H^1(\Omega,\mathbf{R}^n)$ to the (unique) minimizer u_0 of the problem

$$\min_{u\in H^1_{g,\partial\Omega_D}} \Big\{ \frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u),e(u)]\,dx - \int_\Omega \ell u\,dx \Big\}\,.$$

Moreover we prove the convergence of the rescaled energies, i.e.,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\Big\{ \int_\Omega W(x, I+\varepsilon\nabla u_\varepsilon)\,dx - \varepsilon^2\int_\Omega \ell u_\varepsilon\,dx \Big\} = \frac{1}{2}\int_\Omega \mathsf{A}(x)[e(u_0),e(u_0)]\,dx - \int_\Omega \ell u_0\,dx\,. \tag{1.28}$$

More generally, the same results hold if $\det(I+\varepsilon\nabla u_\varepsilon) > 0$ and

$$\int_\Omega W(x, I+\varepsilon\nabla u_\varepsilon)\,dx - \varepsilon^2\int_\Omega \ell u_\varepsilon\,dx = \mathcal{J}_\varepsilon + o(\varepsilon^2)\,,$$

where $\mathcal{J}_\varepsilon$ is the (possibly not attained) infimum of problem (1.27). This provides a full variational justification of linearized elasticity.

These results are proved under the following additional hypotheses on V:

(a) $\inf\limits_{|E|\geq\rho}\ \inf\limits_{x\in\Omega} V(x,E) > 0$ for every $\rho > 0$;

(b) there exist $\alpha > 0$ and $\rho > 0$ such that $\inf\limits_{x\in\Omega} V(x,E) \geq \alpha|E|^2$ for every $|E| \leq \rho$;

(c) $\liminf\limits_{|E|\to+\infty} \dfrac{1}{|E|}\inf\limits_{x\in\Omega} V(x,E) > 0$.

These conditions say that 0 is the unique minimizer of $V(x,\cdot)$ (with a uniform estimate with respect to x) and that $V(x,\cdot)$ grows more than quadratically near the origin and more than linearly at infinity.

If (c) is replaced by the slightly stronger condition

(c') $\liminf\limits_{|E|\to+\infty} \dfrac{1}{|E|^p}\inf\limits_{x\in\Omega} V(x,E) > 0$

for some exponent $p > 1$, then we prove also that u_ε converges to u_0 strongly in $W^{1,q}(\Omega, \mathbf{R}^n)$ for every $q < 2$.

The proof is obtained in two steps. First we show that the sequence u_ε is compact in the weak topology of $H^1(\Omega, \mathbf{R}^n)$, using a recent lemma proved by Friesecke, James, and Müller [32]. Then we prove that the functionals

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_\Omega V(x, \varepsilon e(u) + \tfrac{1}{2}\varepsilon^2 C(u))\, dx$$

Γ-converge to the functional

$$\mathcal{F}(u) = \frac{1}{2} \int_\Omega \mathsf{A}(x)[e(u), e(u)]\, dx\,.$$

These two facts lead to the weak convergence of the solutions in $H^1(\Omega, \mathbf{R}^n)$ and to the convergence of the rescaled energies expressed by (1.28). The strong convergence in $W^{1,q}(\Omega, \mathbf{R}^n)$ for $q < 2$ is obtained from (1.28).

1.2 Preliminaries

1.2.1 Γ-convergence

We summarize in this section the definition of Γ-convergence and its main properties in the case of functionals defined in metric spaces (in the sequel denoted by X). For a comprehensive treatise on Γ-convergence we refer to Dal Maso [24].

Definition 1.2.1 *Let $F_j : X \to [-\infty, +\infty]$ be a sequence of functionals. For every $u \in X$ we define*

$$F'(u) := \Gamma\text{-}\liminf_{j\to\infty} F_j(u) = \inf\{\liminf_{j\to\infty} F_j(u_j) \ : \ \textit{for every } u_j \to u\}\,,$$

$$F''(u) := \Gamma\text{-}\limsup_{j\to\infty} F_j(u) = \inf\{\limsup_{j\to\infty} F_j(u_j) \ : \ \textit{for every } u_j \to u\}\,.$$

Note that F' and F'' are lower semicontinuous.

Definition 1.2.2 *A sequence of functionals $F_j : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if and only if for every $u \in X$ we have*

$$F''(u) \le F(u) \le F'(u)\,.$$

In this case we write $F = \Gamma\text{-}\lim_{j\to\infty} F_j$.

It follows easily by Definition 1.2.1 that F_j Γ-converges to F if for every sequence $u_j \to u$

$$F(u) \le \liminf_{j\to\infty} F_j(u_j)\,,$$

and if there exists a (recovery) sequence $u_j \to u$ such that

$$\limsup_{j\to\infty} F_j(u_j) \le F(u)\,.$$

One of the most important results in the theory of Γ-convergence is the following.

Theorem 1.2.3 *Assume that F_j Γ-converges to F and let u_j be a sequence such that $\lim_j F_j(u_j) = \lim_j \inf_X F_j$. Then, if u_j converges, its limit is a minimum point for F.*

1.2.2 Supremum of a family of measures

Using the regularity of (positive) Borel measures it is not difficult to prove the following result [13].

Proposition 1.2.4 *We denote by $\mathcal{A}(\Omega)$ the topology of Ω. Let $\mu : \mathcal{A}(\Omega) \to [0,+\infty)$ be super-additive on open sets with disjoint compact closures and let λ be a positive Borel measure in Ω. Let ψ_i be a family of positive Borel functions such that*

$$\int_A \psi_i \, d\lambda \leq \mu(A) \quad \text{for all } A \in \mathcal{A}(\Omega)$$

Then

$$\int_A \sup_i \psi_i(x) \, d\lambda \leq \mu(A) \quad \text{for all } A \in \mathcal{A}(\Omega)\,.$$

1.2.3 Spaces of functions with bounded variation

This section presents the main definitions and properties of the spaces of functions with bounded variation which appear in the free discontinuity problems treated in the sequel. For the comprehensive theories we refer to [7], [6], [29]. We will always assume that Ω is an open bounded set with Lipschitz boundary.

Definition 1.2.5 *We say that $u \in L^1(\Omega)$ is a special function of bounded variation if its distributional derivative is a measure with finite total variation which can be written as*

$$Du = \nabla u \, \mathcal{L}^n + (u^+ - u^-)\nu_u \, \mathcal{H}^{n-1} \llcorner S_u\,,$$

where ∇u is the approximate gradient, $\mathcal{L}^n$ is the Lebesgue measure, S_u is the set of discontinuity points of u, $\mathcal{H}^{n-1}$ is the Hausdorff measure, while u^+ and u^- are the traces (in a measure theoretic sense) along S_u. The space of special functions of bounded variation on Ω is denoted by $SBV(\Omega)$.

The following compactness and lower semicontinuity theorem for SBV functions is due to Ambrosio [4].

Theorem 1.2.6 *Let $\psi : [0,+\infty) \to [0,+\infty]$ be a convex non-decreasing function such that*

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty$$

and let $\theta : [0,+\infty) \to [0,+\infty]$ be a concave non-decreasing function such that

$$\lim_{t \to 0^+} \frac{\theta(t)}{t} = +\infty\,.$$

Let u_k be a sequence in $SBV(\Omega)$ such that

$$\sup_k \left\{ \int_\Omega \psi(|\nabla u_k|) \, dx + \int_{S_{u_k}} \theta(|u_k^+ - u_k^-|) \, d\mathcal{H}^{n-1} + \|u_k\|_\infty \right\} < +\infty\,. \tag{1.29}$$

Then we can extract a subsequence u_j which converges in $L^1(\Omega)$ to a function $u \in SBV(\Omega)$. Moreover for every function $\phi : S^{n-1} \to [0, +\infty)$, which is convex pair and positively 1-homogeneous, we have

$$\int_{S_u} \theta(|u^+ - u^-|)\phi(\nu)\, d\mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{S_{u_j}} \theta(|u_j^+ - u_j^-|)\phi(\nu)\, d\mathcal{H}^{n-1}. \tag{1.30}$$

Finally ∇u_j converges weakly to ∇u in $L^1(\Omega, \mathbf{R}^n)$, hence, as ψ is convex, we have

$$\int_\Omega \psi(|\nabla u|)\, dx \leq \liminf_{j\to+\infty} \int_\Omega \psi(|\nabla u_j|)\, dx. \tag{1.31}$$

We recall the slicing properties for functions in $SBV(\Omega)$. Let $\xi \in S^{n-1}$ and let $\Pi_\xi = \{y \in \mathbf{R}^n : \langle y, \xi\rangle = 0\}$, then we define

$$\Omega_\xi^y := \{t \in \mathbf{R} : y + t\xi \in \Omega\}.$$

Moreover, for $u : \Omega \to \mathbf{R}$ we denote by $u_\xi^y : \Omega_\xi^y \to \mathbf{R}$ the function

$$u_\xi^y(t) := u(y + t\xi).$$

Theorem 1.2.7 *Let $u \in SBV(\Omega)$. For every $\xi \in S^{n-1}$ the function u_ξ^y belongs to $SBV(\Omega_\xi^y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$. Moreover for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$ we have*

$$\langle \nabla u(y + t\xi), \xi\rangle = (u_\xi^y)'(t) \quad \textit{for a.e } t \in \Omega_\xi^y,$$

$$S_{u_\xi^y} = (S_u)_\xi^y = \{t \in \mathbf{R} : y + t\xi \in S_u\},$$

$$(u_\xi^y)^\pm(t) = u^\pm(y + t\xi) \quad \forall t \in S_{u_\xi^y},$$

$$\int_{\Pi_\xi} \left(\#(S_u)_\xi^y\right) d\mathcal{H}^{n-1}(y) = \int_{S_u} |\langle \nu_u, \xi\rangle| d\mathcal{H}^{n-1}.$$

Conversely, if $u \in L^1(\Omega)$ and if for every $\xi \in \{\hat{e}_1, \dots, \hat{e}_n\}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$ we have $u_\xi^y \in SBV(\Omega_\xi^y)$ and

$$\int_{\Pi_\xi} |Du_\xi^y|(\Omega_\xi^y)\, d\mathcal{H}^{n-1}(y) < +\infty,$$

then $u \in SBV(\Omega)$.

Given a family $\mathbf{F}$ of functions, for every $\xi \in S^{n-1}$ and $y \in \Pi_\xi$ we set $\mathbf{F}_\xi^y := \{u_\xi^y : u \in \mathbf{F}\}$; moreover we say that a family $\mathbf{F}'$ is δ-close to $\mathbf{F}$ if $\mathbf{F}'$ is contained in a δ-neighborhood of $\mathbf{F}$. The following is due to Alberti, Bouchitté and Seppecher [2].

Lemma 1.2.8 *Let $\mathbf{F}$ be a family of equiintegrable functions belonging to $L^1(A)$ and assume that there exists a basis of unit vectors $\{\xi_1, \dots, \xi_n\}$ with the property that for every $i = 1, ..., n$, for every $\delta > 0$, there exists a family $\mathbf{F}_\delta$ δ-close to $\mathbf{F}$ such that $(\mathbf{F}_\delta)_{\xi_i}^y$ is precompact in $L^1(A_{\xi_i}^y)$ for $\mathcal{H}^{n-1}$-a.e $y \in A_{\xi_i}$. Then $\mathbf{F}$ is precompact in $L^1(A)$.*

Definition 1.2.9 *A function $u \in L^1(\Omega)$ is called a generalized special function of bounded variation if for each $T > 0$ the truncated function $u_T = (-T) \vee (T \wedge u)$ belongs to $SBV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$. The space of these functions will be denoted by $GSBV(\Omega)$.*

In the following we recall a compactness and lower semicontinuity result in $GSBV(\Omega)$.

Theorem 1.2.10 *Let ψ, θ, and ϕ as in Theorem 1.2.6. Let u_k be a sequence in $GSBV(\Omega)$ such that*

$$\sup_k \left\{ \int_\Omega \psi(|\nabla u_k|)\, dx + \int_{S_{u_k}} \theta(|u_k^+ - u_k^-|)\, d\mathcal{H}^{n-1} + \int_\Omega |u_k|\, dx \right\} < +\infty.$$

Then there is subsequence u_j of u_k such that u_j converges in measure to a function $u \in GSBV(\Omega)$ and the lower semicontinuity inequalities (1.30) and (1.31) hold.

Definition 1.2.11 *For $1 \le p < +\infty$ we define the spaces*

$$GSBV^p(\Omega) := \{u \in GSBV(\Omega) \,:\, \nabla u \in L^p(\Omega, \mathbf{R}^n),\ \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Definition 1.2.12 *We denote by $SBV(\Omega, \mathbf{R}^m)$ the space of functions $u \in L^1(\Omega, \mathbf{R}^m)$ whose distributional derivative is a measure with finite total variation which can be written as*

$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S_u\,,$$

Definition 1.2.13 *For $1 \le p < +\infty$ we define the spaces*

$$SBV^p(\Omega, \mathbf{R}^m) := \{u \in SBV(\Omega, \mathbf{R}^m) \,:\, \nabla u \in L^p(\Omega, \mathbf{M}^{m\times n}),\ \mathcal{H}^{n-1}(S_u) < +\infty\}\,.$$

Definition 1.2.14 *Let $\mathcal{W}(\Omega, \mathbf{R}^m)$ be the set of $u \in SBV(\Omega, \mathbf{R}^m)$ such that*

1. *S_u is essentially closed, i.e. $\mathcal{H}^1(\overline{S_u} \setminus S_u) = 0$*
2. *$\overline{S_u}$ is the union of a finite number of $(n-1)$-simplexes*
3. *$u \in W^{k,\infty}(\Omega \setminus \overline{S_u}, \mathbf{R}^m)$ $\forall k \in \mathbf{N}$.*

The following lemma, proved in [23], shows the density property of the space $\mathcal{W}(\Omega, \mathbf{R}^m)$ for free discontinuity problems.

Lemma 1.2.15 *Let $g : \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{S}^{n-1} \to [0, +\infty)$ be a continuous function satisfying $g(a, b, \nu) = g(b, a, -\nu)$ for every $a, b \in \mathbf{R}^m$ and $\nu \in S^{n-1}$. Let $u \in SBV^p(\Omega, \mathbf{R}^n) \cap L^\infty(\Omega, \mathbf{R}^m)$ for $p > 1$, then there is a sequence $w_k \in \mathcal{W}(\Omega, \mathbf{R}^m)$ such that*

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega, \mathbf{R}^m)\,, \tag{1.32}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbf{M}^{m\times 2})\,, \tag{1.33}$$

$$\limsup_{k\to+\infty} \|w_k\|_\infty \le \|u\|_\infty\,, \tag{1.34}$$

$$\limsup_{k\to+\infty} \int_{S_{w_k}} g(w_k^+, w_k^-, \nu) d\mathcal{H}^1 \le \int_{S_u} g(u^+, u^-, \nu) d\mathcal{H}^1\,. \tag{1.35}$$

Remark 1.2.16 *Under the additional assumption that* $1 < p \leq 2$ *and using a* C_p*-capacity argument we can find a sequence* w_k *such that for every* k *the jump set* S_{w_k} *is given by the union of a finite number of disjoint simplexes.*

Using a truncation argument it is easy to prove also the following density result in $GSBV^p(\Omega)$.

Lemma 1.2.17 *Let* $g : \mathbf{R} \times \mathbf{S}^{n-1} \to [0, +\infty)$ *be a continuous function increasing with respect to the first variable. Let* $u \in GSBV^p(\Omega)$ *for* $p > 1$*, then there is a sequence* $w_k \in \mathcal{W}(\Omega)$ *such that*

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega) \tag{1.36}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbf{R}^n) \tag{1.37}$$

$$\limsup_{k \to +\infty} \int_{S_{w_k}} g(|w_k^+ - w_k^-|, \nu) d\mathcal{H}^1 \leq \int_{S_u} g(|u^+ - u^-|, \nu) d\mathcal{H}^1 . \tag{1.38}$$

1.2.4 Spaces of functions with bounded deformation

Definition 1.2.18 *We say that* $u \in L^1(\Omega, \mathbf{R}^n)$ *is a special function with bounded deformation in* Ω*, if the symmetric part of its distributional derivative is a measure with finite total variation, which can be written as*

$$Eu := \frac{1}{2}(Du + Du^T) = \mathcal{E}u\mathcal{L}^n + (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u ,$$

where $\mathcal{E}u$ *is the approximate symmetric derivative,* J_u *is the set of jump points and* $\odot$ *denotes the symmetrized tensor product. The space of these functions will be denoted by* $SBD(\Omega, \mathbf{R}^n)$.

The following compactness and lower semicontinuity result [45] slightly generalizes Theorem 1.1 and Corollary 1.2 in [10] to the case of anisotropic energies in $SBD^2(\Omega, \mathbf{R}^2)$.

Proposition 1.2.19 *Let* Ω *be an open bounded set in* $\mathbf{R}^2$*, let* $\phi : \mathbf{R}^2 \to [0, +\infty)$ *be convex, positively 1-homogeneous and pair, let* $\psi : \mathbf{M}^{2\times 2}_{sym} \to \mathbf{R}$ *be convex and lower semicontinuous. Let* u_j *be a sequence in* $SBD^2(\Omega, \mathbf{R}^2)$ *such that*

$$\int_\Omega |\mathcal{E}u_j|^2 dx + \mathcal{H}^1(J_{u_j}) + \|u_j\|_\infty \leq c < +\infty , \tag{1.39}$$

then there exists a function $u \in SBD(\Omega, \mathbf{R}^2)$ *and subsequence (not relabelled) such that*

$$u_j \to u \quad \text{in } L^1_{loc}(\Omega, \mathbf{R}^2) , \tag{1.40}$$

$$\int_\Omega \psi(\mathcal{E}u) dx \leq \liminf_{j \to +\infty} \int_\Omega \psi(\mathcal{E}u_j) dx , \tag{1.41}$$

$$\mathcal{H}^1(J_u) \leq \liminf_{j \to +\infty} \mathcal{H}^1(J_{u_j}) , \tag{1.42}$$

$$\int_{J_u} \phi(\nu) d\mathcal{H}^1 \leq \liminf_{j \to +\infty} \int_{J_{u_j}} \phi(\nu) d\mathcal{H}^1 . \tag{1.43}$$

Proof. From the compactness and lower semicontinuity result contained in [10] follows the existence of a subsequence, denoted as u_j such that $u_j \to u$ in $L^1_{loc}(\Omega, \mathbf{R}^2)$ and such that inequalities (1.41) and (1.42) holds.

To prove (1.43) let us first consider a function ϕ defined as $\phi(v) = |\langle v, \zeta \rangle|$ for $\zeta \in S^1$. For $\xi \in S^1$ let $\Pi_\xi = \{x \in \mathbf{R}^2 : \langle x, \xi \rangle = 0\}$, let Ω_ξ be the projection of Ω on the hyperplane Π_ξ and $\Omega^y_\xi = \{t \in \mathbf{R} : \text{for} y \in \Pi_\xi \quad y + t\xi \in \Omega\}$. Moreover for $y \in \Pi_\xi$ and $\xi \in S^1$, given $v \in SBD(\Omega, \mathbf{R}^2)$ let $J_{v,\xi} = \{x \in J_v : \langle v^+(x) - v^-(x), \xi \rangle \neq 0\}$, $v^y_\xi : \Omega^y_\xi \to \mathbf{R}$ be defined as $v^y_\xi = \langle v(y + t\xi), \xi \rangle$ and finally let $A^y_\xi(v)$ and $B^y_\xi(v)$ be respectively the total variation of the absolutely continuous part and the counting measure of singular part of $(v^y_\xi)'$ in Ω^y_ξ, namely

$$A^y_\xi(v) = |(v^y_\xi)'|(\Omega^y_\xi) \qquad B^y_\xi(v) = \mathcal{H}^0(J_{v^y_\xi}) = \#(J_{v^y_\xi}) .$$

Being $\{u_j\} \subset SBD(\Omega, \mathbf{R}^2)$ then by Proposition 2.1 in [10] for every $\xi \in S^1$ and for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$ we have $(u_j)^y_\xi \in SBV(\Omega^y_\xi)$ for every j and

$$\int_{\Omega_\xi} A^y_\xi(u_j) d\mathcal{H}^1(y) < +\infty . \tag{1.44}$$

Moreover for $\mathcal{H}^1$- a.e. $\xi \in S^1$ we have $J_{u_j,\xi} = J_{u_j}$. Now take $\xi \in S^1$ such that $J_{u_j,\xi} = J_{u_j}$ for every j, then

$$\int_{J_{u_j}} |\langle \nu, \xi \rangle| d\mathcal{H}^1(y) = \int_{\Omega_\xi} B^y_\xi(u_j) d\mathcal{H}^1(y) , \tag{1.45}$$

moreover, being $\mathcal{H}^1(J_{u_j}) < +\infty$, there exist a subsequence, denoted $\{u_k\}$, such that

$$\lim_{k \to +\infty} \int_{\Omega_\xi} B^y_\xi(u_k) d\mathcal{H}^1(y) = \liminf_{j \to +\infty} \int_{\Omega_\xi} B^y_\xi(u_j) d\mathcal{H}^1(y) < +\infty .$$

Take $\varepsilon \in (0,1)$ and take a subsequence $\{u_l\}$ of $\{u_k\}$, such that

$$\lim_{l \to +\infty} \int_{\Omega_\xi} \varepsilon A^y_\xi(u_j) + B^y_\xi(u_j) d\mathcal{H}^1(y) =$$

$$\liminf_{j \to +\infty} \int_{\Omega_\xi} \varepsilon A^y_\xi(u_j) + B^y_\xi(u_j) d\mathcal{H}^1(y) < +\infty$$

and such that for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$

$$u^y_\xi \in SBV(\Omega^y_\xi) \qquad (u_l)^y_\xi \to u^y_\xi \text{ strongly in } L^1_{loc}(\Omega^y_\xi) .$$

By Fatou's Lemma we get that

$$\liminf_{l \to +\infty} \left(\varepsilon A^y_\xi(u_l) + B^y_\xi(u_l) \right) < +\infty$$

for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$. Let $y \in \Omega_\xi$ such that the previous inequality is satisfied, then there exists a further subsequence $\{u_n\}$ such that

$$\lim_{n \to +\infty} \left(\varepsilon A^y_\xi(u_n) + B^y_\xi(u_n) \right) = \liminf_{l \to +\infty} \left(\varepsilon A^y_\xi(u_l) + B^y_\xi(u_l) \right) < +\infty ,$$

then by Ambrosio's compactness and lower semicontinuity Theorem in $SBV(\Omega_\xi^y)$ [7] there exists a subsequence $\{u_h\}$ such that

$$\begin{aligned}
B_\xi^y(u) &\leq \liminf_{h\to+\infty} B_\xi^y(u_h) \\
&\leq \liminf_{h\to+\infty} \left(\varepsilon A_\xi^y(u_h) + B_\xi^y(u_h)\right) \\
&\leq \lim_{n\to+\infty} \left(\varepsilon A_\xi^y(u_n) + B_\xi^y(u_n)\right) \\
&\leq \liminf_{l\to+\infty} \left(\varepsilon A_\xi^y(u_l) + B_\xi^y(u_l)\right) < +\infty\,.
\end{aligned}$$

The previous inequality holds for $\mathcal{H}^1$-a.e. $y \in \Omega_\xi$ so by Fatou's Lemma

$$\begin{aligned}
\int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1(y) &= \int_{\Omega_\xi} B_\xi^y(u) d\mathcal{H}^1(y) \\
&\leq \liminf_{l\to+\infty} \int_{\Omega_\xi} \varepsilon A_\xi^y(u_l) + B_\xi^y(u_l) d\mathcal{H}^1(y) \\
&\leq \varepsilon \limsup_{l\to+\infty} \int_{\Omega_\xi} A_\xi^y(u_l) d\mathcal{H}^1(y) + \liminf_{l\to+\infty} \int_{\Omega_\xi} B_\xi^y(u_l) d\mathcal{H}^1(y) \\
&\leq c\varepsilon + \lim_{k\to+\infty} \int_{\Omega_\xi} B_\xi^y(u_k) d\mathcal{H}^1(y) \\
&\leq c\varepsilon + \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1\,. \qquad (1.46)
\end{aligned}$$

It remains to prove (1.46) for every $\xi \in S^1$. Let $\zeta \in S^1$ and let $\delta > 0$ then there exist ξ_δ such that $|\zeta - \xi_\delta| < \delta$ and

$$\int_{J_u} |\langle \nu, \xi\rangle| d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1\,. \qquad (1.47)$$

It follows that

$$\begin{aligned}
\int_{J_u} |\langle \nu, \zeta\rangle| d\mathcal{H}^1 &\leq \int_{J_u} |\langle \nu, \xi_\delta\rangle| d\mathcal{H}^1 + c_1\delta \\
&\leq \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi_\delta\rangle| d\mathcal{H}^1 + c_1\delta \\
&\leq \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \xi_\delta - \zeta\rangle| + |\langle \nu, \zeta\rangle| d\mathcal{H}^1 + c_1\delta \\
&\leq \liminf_{j\to+\infty} \int_{J_{u_j}} |\langle \nu, \zeta\rangle| d\mathcal{H}^1 + c_2\delta
\end{aligned}$$

where c_1 and c_2 does no depend on δ.

At this point the lower semicontinuity inequality is proved for every function $\phi(v) = |\langle v, \zeta\rangle|$ for $\zeta \in S^1$. If ϕ is convex, 1-homogeneous and pair it can be written as

$$\phi(v) = \sup\{\psi(v) \,:\, \psi(u) = \langle u, \eta\rangle - c \text{ and } \psi(u) \leq \phi(u)\} \qquad (1.48)$$

In (1.48) it is not restrictive to take ψ linear, indeed for every $\xi \in S^1$, being ϕ 1-homogeneous, if $\langle u, \eta\rangle - c \leq \phi(u)$ for every u then $\langle u, \eta\rangle \leq \phi(u)$. Thus we can define a set Θ such that

$$\phi(v) = \sup_{\xi \in \Theta} |\langle v, \xi\rangle| .$$

By the lower semicontinuity we have for every $\xi \in \Theta$

$$\begin{aligned} \int_{J_u} |\langle \nu, \xi\rangle| d\mathcal{H}^1 &\leq \liminf_{j \to +\infty} \int_{J_{u_j}} |\langle \nu, \xi\rangle| d\mathcal{H}^1 \\ &\leq \liminf_{j \to +\infty} \int_{J_{u_j}} \phi(\nu) d\mathcal{H}^1 \end{aligned}$$

and by a supremum of measures argument it is easy to deduce that

$$\int_{J_u} \phi(\nu) d\mathcal{H}^1 \leq \liminf_{j \to +\infty} \int_{J_{u_j}} \phi(\nu) d\mathcal{H}^1$$

which gives inequality (1.43). ∎

Chapter 2

A finite element approximation of the Mumford-Shah functional

2.1 Statement of the main result

Let $\Omega \subset \mathbf{R}^2$ be an open bounded set with Lipschitz boundary, let $\boldsymbol{T}_\varepsilon^i$ for $i = 1, 2, 3$ be the triangulations of $\mathbf{R}^2$ defined in Figure 1.2 and let ε denote the diameter of the elements. Let $V_\varepsilon^i(\Omega)$ be the usual finite element space of piecewise affine functions in Ω defined on $\boldsymbol{T}_\varepsilon^i$. Let $f : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function such that

$$\lim_{t \to 0+} \frac{f(t)}{t} = 1 \qquad \lim_{t \to +\infty} f(t) = f_\infty < +\infty . \tag{2.1}$$

Finally let $g \in L^\infty(\Omega)$. The main result is summarized in the following theorem.

Theorem 2.1.1 *For $i = 1, \dots, 3$, there exists a convex, 1-homogeneous function ϕ_i : $\mathbf{R}^2 \to [0, +\infty)$ such that, for every positive sequence $\varepsilon_j \to 0$ the functionals*

$$G_{\varepsilon_j}(u) = \begin{cases} \dfrac{1}{\varepsilon_j} \displaystyle\int_\Omega f(\varepsilon_j |\nabla u|^2) dx + \int_\Omega |u - g|^2 dx & \text{if } u \in V_{\varepsilon_j}^i(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus V_{\varepsilon_j}^i(\Omega) , \end{cases}$$

Γ-converge, respect to the strong L^2-topology, to the anisotropic Mumford-Shah functional, defined by

$$G(u) = \int_\Omega |\nabla u|^2 dx + f_\infty \int_{S_u} \phi_i(\nu_u) d\mathcal{H}^1 + \int_\Omega |u - g|^2 dx \tag{2.2}$$

if $u \in L^2(\Omega) \cap SBV^2(\Omega)$ and $G(u) = +\infty$ if $u \in L^2(\Omega) \setminus SBV^2(\Omega)$.

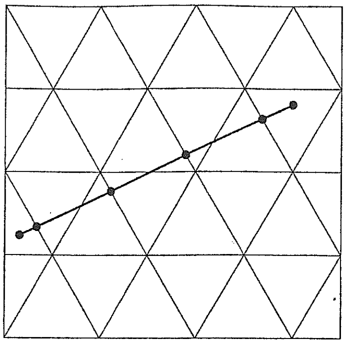
2.2 Mesh anisotropies

In this section we deal with the anisotropy functions $\phi_i(\nu)$ appearing in (2.2). Let $S \subset \Omega$ be a segment with unit normal ν. Consider the set

$$Q = \{x \in \Omega : x = s + t\nu \text{ for } s \in S \text{ and } t \geq 0\} \tag{2.3}$$

and define a neighborhood of S in $\boldsymbol{T}_{\varepsilon_j}^i$ as

$$S_{\varepsilon_j}^i(S, \nu) = \{T \in \boldsymbol{T}_{\varepsilon_j}^i : T \cap S \neq \emptyset \text{ and } |T \cap Q| \neq 0\} . \tag{2.4}$$



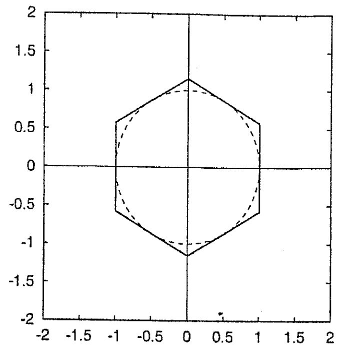


Figure 2.1: A segment in $\boldsymbol{T}^3_\varepsilon$ and the level curve $\{\phi_3(v) = 1\}$

In the following, for $i = 1, \dots, 3$ we define a convex, 1-homogeneous, pair function $\phi_i(\nu)$ in such a way that for every segment S with normal ν we have

$$\phi_i(\nu)\mathcal{H}^1(S) = \lim_{\varepsilon_j \to 0} \frac{1}{\varepsilon_j} \sum_{T \in \boldsymbol{S}^i_{\varepsilon_j}} |T| \,. \tag{2.5}$$

Lemma 2.2.1 *Let $i = 3$ and let $\nu = (-\sin\alpha, \cos\alpha)$. Let $\xi^3_1 = (1, 0)$, $\xi^3_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\xi^3_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and let the function $\phi_3(\nu) : S^1 \to [0, +\infty)$ be defined as*

$$\phi_3(\nu) = \begin{cases} |\langle \nu, \xi^3_1 \rangle| & \text{if } \frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3} \text{ and } -\frac{2\pi}{3} \leq \alpha \leq -\frac{\pi}{3} \\ |\langle \nu, \xi^3_2 \rangle| & \text{if } \frac{2\pi}{3} \leq \alpha \leq \pi \text{ and } -\frac{\pi}{3} \leq \alpha \leq 0 \\ |\langle \nu, \xi^3_3 \rangle| & \text{if } 0 \leq \alpha \leq \frac{\pi}{3} \text{ and } -\pi \leq \alpha \leq -\frac{2\pi}{3} \end{cases}$$

then (2.5) holds for every segment S with unit normal ν.

Proof. By symmetry it is sufficient to consider the case $0 \leq \alpha \leq \frac{\pi}{3}$. Moreover it will be clear from the computation that it is not restrictive to suppose $\mathcal{H}^1(S) = 1$. For every $T \in \boldsymbol{T}^3_{\varepsilon_j}$ we have $|T| = \varepsilon_j^2\sqrt{3}/4$, then

$$\frac{\sum_{T \in \boldsymbol{S}^3_{\varepsilon_j}} |T|}{\varepsilon_j} = \#(\boldsymbol{S}^3_{\varepsilon_j}) \frac{\sqrt{3}}{4} \varepsilon_j \,,$$

where $\#(\cdot)$ denotes the cardinality. We will prove that

$$\phi_3(\nu) = \lim_{\varepsilon_j \to 0} \#(\boldsymbol{S}^3_{\varepsilon_j}) \frac{\sqrt{3}}{4} \varepsilon_j \,. \tag{2.6}$$

Fix α and ε_j, take an orientation on S and let x_0 be the first point, x_{m+1} the last and $x_1, \dots, x_m$ be the intersection points between S and the edges with slope $\tan(\frac{2\pi}{3})$ (see Figure 2.1). It's clear that for $n = 1, \dots, m-1$ each segment $[x_n\, x_{n+1}]$ intersects exactly two triangles, while $[x_0\, x_1]$ and $[x_m\, x_{m+1}]$ can intersect either one or two. So we have the following estimate

$$2(m-1) + 2 \leq \#(\boldsymbol{S}^3_{\varepsilon_j}) \leq 2(m-1) + 4 \,. \tag{2.7}$$

To estimate m, let $h(\varepsilon_j) = \mathcal{H}^1([x_1\, x_2])$ and note that for $n = 1, \dots, m-1$ each segment $[x_n\, x_{n+1}]$ has the same length, then

$$\left[\frac{1}{h(\varepsilon_j)}\right] - 1 \le m \le \left[\frac{1}{h(\varepsilon_j)}\right] + 1\,, \tag{2.8}$$

where $[\cdot]$ denotes the integer part. Joining (2.7) and (2.8) we get

$$2\left[\frac{1}{h(\varepsilon_j)}\right] - 2 \le \#(S^3_{\varepsilon_j}) \le 2\left[\frac{1}{h(\varepsilon_j)}\right] + 4\,. \tag{2.9}$$

At this point it should be clear that (2.6) does not depend on the particular choice of S, indeed the estimate on $\#(S^3_{\varepsilon_j})$ holds for every segment with the same normal, even when it contains a vertex of the mesh, due to the careful definition of the set $S^3_{\varepsilon_j}$. Then we have

$$\lim_{\varepsilon_j \to 0} \#(S^3_{\varepsilon_j})\frac{\sqrt{3}}{4}\varepsilon_j = \lim_{\varepsilon_j \to 0}\left[\frac{1}{h(\varepsilon_j)}\right]\frac{\sqrt{3}}{2}\varepsilon_j = \lim_{\varepsilon_j \to 0}\frac{\sqrt{3}}{2}\frac{\varepsilon_j}{h(\varepsilon_j)}\,. \tag{2.10}$$

By a simple trigonometric argument we get $h(\varepsilon_j)\langle \xi^3_3, \nu\rangle = \varepsilon_j\sqrt{3}/2$, then

$$\lim_{\varepsilon_j \to 0} \#(S^3_{\varepsilon_j})\frac{\sqrt{3}}{4}\varepsilon_j = \langle \xi^3_3, \nu\rangle \tag{2.11}$$

and the proof is concluded. ∎

Remark 2.2.2 *The function $\phi_3(\nu)$ can clearly be extended to any vector $v \in \mathbf{R}^2$ by 1-homogeneity. Moreover it's easy to check that $\phi_3(v)$ will be convex and its level curve $\{\phi_3(v) = 1\}$ is an hexagon, represented in Figure 2.1. For the applications it is useful to measure the effect of the anisotropy, thus following [17] we compute the ratio $a = \frac{M}{m}$ where $M = \max_{|\nu|=1}\phi(\nu)$ and $m = \min_{|\nu|=1}\phi(\nu)$. Obviously, the greatest a is the more the triangulation is anisotropic, in this case we have $a_3 = \frac{2\sqrt{3}}{3} \simeq 1.154$.*

Lemma 2.2.3 *Let $i = 1$ and let $\nu = (-\sin\alpha, \cos\alpha)$. Let $\xi^1_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\xi^1_2 = (0,1)$, $\xi^1_3 = (-1,0)$, and let the function $\phi_1(\nu) : S^1 \to [0,+\infty)$ be defined as*

$$\phi_1(\nu) = \begin{cases} |\langle \nu, \xi^1_1\rangle| & \text{if } \frac{\pi}{2} \le \alpha \le \pi \text{ and } -\frac{\pi}{2} \le \alpha \le 0 \\[2ex] \frac{\sqrt{2}}{2}|\langle \nu, \xi^1_2\rangle| & \text{if } 0 \le \alpha \le \frac{\pi}{4} \text{ and } -\pi \le \alpha \le -\frac{3\pi}{4} \\[2ex] \frac{\sqrt{2}}{2}|\langle \nu, \xi^1_3\rangle| & \text{if } \frac{\pi}{4} \le \alpha \le \frac{\pi}{2} \text{ and } -\frac{3\pi}{4} \le \alpha \le -\frac{\pi}{2} \end{cases}$$

then (2.5) holds for every segment S with unit normal ν.

Proof. The proof follows the same argument used in the previous Lemma. Assume that $\mathcal{H}^1(S) = 1$ and consider for α fixed a set of points $x_0, \cdots, x_{m+1}$ such that for $n = 1, \dots, m-1$ each segment $[x_n\, x_{n+1}]$ intersect exactly two triangles, then, using an estimate like (2.6) and (2.9) we prove that

$$\phi_1(\nu) = \lim_{\varepsilon_j \to 0} \#(S^1_{\varepsilon_j})\,\frac{\varepsilon_j}{4} = \lim_{\varepsilon_j \to 0}\frac{1}{2}\frac{\varepsilon_j}{h(\varepsilon_j)}\,. \tag{2.12}$$

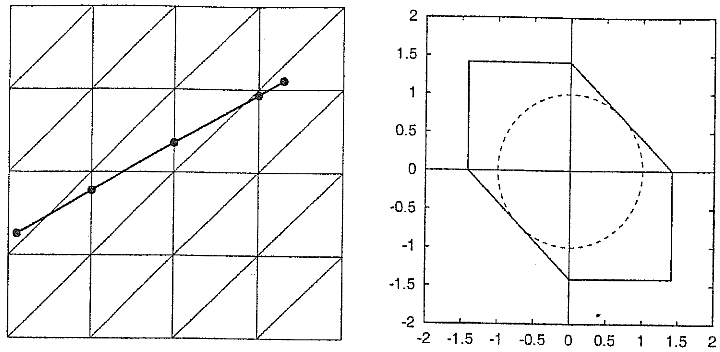


Figure 2.2: A segment in T^1_ε and the level curve $\{\phi_1(v) = 1\}$.

Take first $0 \le \alpha \le \frac{\pi}{4}$, the set of points $x_1, \cdots, x_m$ is defined as the intersection between S and vertical edges (see Figure 2.2). In this way, from $h(\varepsilon_j)|\langle \xi^1_2, \nu\rangle| = \varepsilon_j\sqrt{2}/2$ follows

$$\lim_{\varepsilon_j \to 0} \#(S^1_{\varepsilon_j}) \frac{\varepsilon_j}{4} = \frac{\sqrt{2}}{2} |\langle \xi^1_2, \nu\rangle| = \phi_1(\nu)\,. \tag{2.13}$$

For $\frac{\pi}{4} \le \alpha \le \frac{\pi}{2}$ consider the set of intersection points between S and horizontal edges, this time $h(\varepsilon_j)|\langle \xi^1_3, \nu\rangle| = \varepsilon\sqrt{2}/2$ then

$$\lim_{\varepsilon_j \to 0} \#(S^1_{\varepsilon_j}) \frac{\varepsilon_j}{4} = \frac{\sqrt{2}}{2} |\langle \xi^1_3, \nu\rangle| = \phi_1(\nu)\,. \tag{2.14}$$

Finally, for $\frac{\pi}{2} \le \alpha \le \pi$ define $x_1, \ldots, x_m$ as the intersection between S and the edges with slope $\tan(\frac{\pi}{4})$. In this case we have $h(\varepsilon_j)|\langle \xi^1_1, \nu\rangle| = \varepsilon_j/2$, then

$$\lim_{\varepsilon_j \to 0} \#(S^1_{\varepsilon_j}) \frac{\varepsilon_j}{4} = |\langle \xi^1_1, \nu\rangle| = \phi_1(\nu) \tag{2.15}$$

and this concludes the proof. ■

Remark 2.2.4 *As before $\phi_1(\nu)$ can be defined for $v \in \mathbf{R}^2$ in a 1-homogeneous convex function, whose level curve $\{\phi_1(v) = 1\}$ is represented in Figure 2.2. This time the anisotropy factor $a = \frac{M}{m}$ is much greater, due to the particular orientation of the triangles, indeed we have $M = 1$ and $m = \frac{\sqrt{2}}{2}$ so $a_1 = \sqrt{2} \simeq 1.414$.*

Lemma 2.2.5 *Let $i = 2$ and let $\nu = (-\sin\alpha, \cos\alpha)$. Let the function $\phi_2(\nu) : S^1 \to [0, +\infty)$ be defined as*

$$\phi_2(\nu) = \frac{1}{2}\Big(\phi_1(\nu) + \phi_1(\nu')\Big)\,,$$

where ν' denotes the symmetric vector with respect to the y-axis. Then (2.5) holds for every segment S with unit normal ν.

Proof. The idea is to reduce the computation to the previous case. For this reason we will consider two variants of the triangulations T^1_ε and T^2_ε depending on the orientation of the elements. They are represented in Figure 2.3 and 2.4. Consider the squares Q obtained

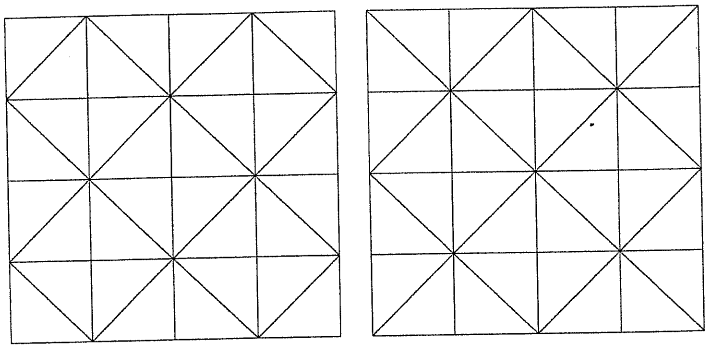


Figure 2.3: The triangulations $\boldsymbol{T}_{\varepsilon}^{2a}$ and $\boldsymbol{T}_{\varepsilon}^{2b}$.

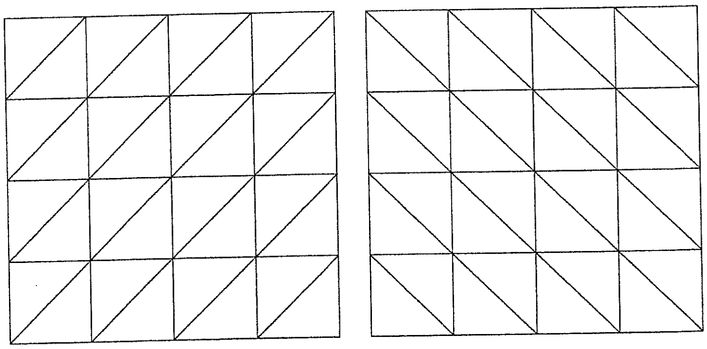


Figure 2.4: The triangulations $\boldsymbol{T}_{\varepsilon}^{1a}$ and $\boldsymbol{T}_{\varepsilon}^{1b}$.

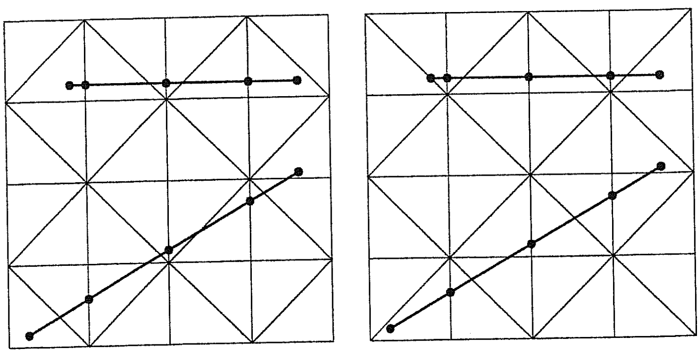


Figure 2.5: Two segments in $\boldsymbol{T}_{\varepsilon}^{2a}$ and $\boldsymbol{T}_{\varepsilon}^{2b}$.

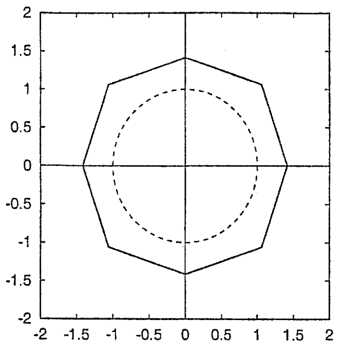


Figure 2.6: The level curve $\{\phi_2(v) = 1\}$

from the vertices of the mesh and note that the orientation induces a correspondence between the triangles of T_ε^{2a} contained in Q and the ones of T_ε^{1a} or T_ε^{1b}. It should be clear that for every segment S

$$\#(S_{\varepsilon_j}^{1a}(S,\nu)) + \#(S_{\varepsilon_j}^{1b}(S,\nu)) = \#(S_{\varepsilon_j}^{2a}(S,\nu)) + \#(S_{\varepsilon_j}^{2b}(S,\nu)). \tag{2.16}$$

Obviously by the symmetry it is sufficient to consider $-\frac{\pi}{2} \leq \alpha \leq -\frac{\pi}{4}$, in this case we show that

$$\#(S_{\varepsilon_j}^{2a}(S,\nu)) - 6 \leq \#(S_{\varepsilon_j}^{2b}(S,\nu)) \leq \#(S_{\varepsilon_j}^{2a}(S,\nu)) + 6. \tag{2.17}$$

Being $-90 \leq \alpha \leq -45°$, taking the set of points $x_1, \ldots, x_m$ obtained by intersection with vertical edges, it is not difficult to see that for $n = 1, \cdots, m-1$ each segment $[x_n\, x_{n+1}]$ intersects the same number of triangles in T_ε^{2a} and in T_ε^{2b} (see Figure 2.5). This is enough to prove (2.17) because in $[x_0\, x_1]$ and in $[x_m\, x_{m+1}]$ there are at most three elements. In conclusion

$$\lim_{\varepsilon_j \to 0} \#(S_{\varepsilon_j}^{2a}(S,\nu))\frac{1}{4}\varepsilon_j = \lim_{\varepsilon_j \to 0} \#(S_{\varepsilon_j}^{2b}(S,\nu))\frac{1}{4}\varepsilon_j$$

then, by (2.16),

$$\begin{aligned}\lim_{\varepsilon_j \to 0} \#(S_{\varepsilon_j}^{2a}(S,\nu))\,\frac{1}{4}\varepsilon_j &= \frac{1}{2}\lim_{\varepsilon_j \to 0}\Big(\#(S_{\varepsilon_j}^{1a}(S,\nu))\,\frac{1}{4}\varepsilon_j + \#(S_{\varepsilon_j}^{1b}(S,\nu))\,\frac{1}{4}\varepsilon_j\Big)\\ &= \frac{1}{2}\Big(\phi_1(\nu) + \phi_1(\nu')\Big)\end{aligned} \tag{2.18}$$

where ν' is the reflection of ν respect to the y-axis. ■

Remark 2.2.6 *Again, the function ϕ_2 can be extended in a 1-homogeneous convex way to $\mathbf{R}^2$. This time the anisotropy factor a_2 is much less then a_1, infact $M = \frac{\sqrt{5}}{3}$ and $m = \frac{2}{3}$ so $a_2 = \frac{\sqrt{5}}{3} \simeq 1.118$.*

Remark 2.2.7 *In the sequel, to prove the Γ-liminf inequality, it will be very useful to have a representation of the functions ϕ_i in terms of the scalar products $\langle \nu, \xi_k^i \rangle$. For $i = 3$, let $c_k^3 = 1\, \forall k$, then*

$$\phi_3(\nu) = \max_k\, c_k^3 |\langle \nu, \xi_k^3 \rangle|. \tag{2.19}$$

For $i=1$, let $c_1^1=1$ and $c_2^1=c_3^1=\sqrt{2}/2$, then

$$\phi_1(\nu) = \max_k \; c_k^1 |\langle \nu, \xi_k^1 \rangle| \,. \tag{2.20}$$

For $i=2$, let $c_k^2 = c_k^1 \;\; \forall k$, then we have

$$\begin{aligned} \phi_2(\nu) &= \frac{1}{2}\Big(\phi_1(\nu) + \phi_1(\nu')\Big) \\ &= \frac{1}{2}\max_{k_1} \; c_{k_1}^2 |\langle \nu, \zeta_{k_1}\rangle| + \frac{1}{2}\max_{k_2} \; c_{k_2}^2 |\langle \nu; \zeta'_{k_2}\rangle| \\ &= \frac{1}{2}\max_{k_1,k_2} \; \Big(c_{k_1}^2 |\langle \nu, \zeta_{k_1}\rangle| + c_{k_2}^2 |\langle \nu, \zeta'_{k_2}\rangle|\Big) \,, \end{aligned} \tag{2.21}$$

where ζ'_k denotes the reflection of ζ_k with respect to the y-axis. Note that the max *in (2.21) can be reduced over the pairs (k_1, k_2) such that $\langle \zeta_{k_1}, \zeta'_{k_2}\rangle = \sqrt{2}/2$.*

2.3 Γ-limsup inequality

In the sequel we will use the following notations, given $u \in L^2(\Omega)$ and a sequence $\varepsilon_j \searrow 0$, let

$$F_{\varepsilon_j}(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\displaystyle\int_\Omega f(\varepsilon_j |\nabla u|^2)\, dx & \text{if } u \in V^i_{\varepsilon_j}(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus V^i_{\varepsilon_j}(\Omega) \,, \end{cases} \tag{2.22}$$

$$F'(u) = \Gamma\text{-}\liminf_{j\to+\infty} \; F_{\varepsilon_j}(u) \,, \tag{2.23}$$

$$F''(u) = \Gamma\text{-}\limsup_{j\to+\infty} \; F_{\varepsilon_j}(u) \,. \tag{2.24}$$

In this section we prove that for any $u \in L^2(\Omega)$ the Γ-lim sup inequality holds respect to the strong topology of $L^2(\Omega)$. The proof is similar to the one presented in [20]. In order to prove Proposition 2.3.2 we need the following result (for the proof see Remark 3.5 in [20]).

Lemma 2.3.1 *Let T be a triangle with angles θ_i and edges ζ_k. Let $v : T \to \mathbf{R}$ be an affine function, then for every pair ζ_i, ζ_j we have the following estimate on the gradient*

$$|\nabla v| \le C \; \max\{|\nabla_{\zeta_i} v|, |\nabla_{\zeta_j} v|\} \,, \tag{2.25}$$

where

$$C = \max_{i=1,2,3} \frac{\sqrt{\sin^2\theta_i + 4}}{|\sin\theta_i|} \,.$$

In particular for C_i defined by

$$C_i = \begin{cases} 3 & \textit{for } i = 3 \\ \sqrt{5} & \textit{for } i = 1,2 \end{cases}$$

inequality (2.25) holds for every $\varepsilon_j > 0$ and for every $T \in \boldsymbol{T}^i_{\varepsilon_j}$.

Proposition 2.3.2 *Let $w \in \mathcal{W}(\Omega)$, then for $i = 1, 2, 3$ there exists a sequence $v_j \in V^i_{\varepsilon_j}(\Omega)$ such that*

$$\|v_j - w\|_{L^2(\Omega)} \le c\,\varepsilon_j^2\,,$$

$$\limsup_{j\to+\infty} F_{\varepsilon_j}(v_j) \le \int_\Omega |\nabla w|^2 dx + f_\infty \int_{S_w} \phi_i(\nu_w)\, d\mathcal{H}^1\,,$$

where the constant c does not depend on ε_j.

Proof. Fix i and let $S_u = \cup_{m=1}^p S_m$ where S_m are disjoint segments. Fix for every m a normal ν_m to S_m and let $S_j = \cup_{m=1}^p S^i_{\varepsilon_j}(S_m, \nu_m)$ be the union of the coverings as defined before. By the regularity of w we have $w \in C^\infty(\overline{\Omega \setminus S_j})$ for every j. Then the sequence v_j, defined as the Lagrange interpolation of w in $V^i_{\varepsilon_j}(\Omega)$, satisfies the required properties. First of all by the regularity of w and by standard results on finite elements, see for instance [21] Theorem 3.1.6, there exists a constant c such that the inequalities

$$\|v_j - w\|_{L^2(T)} \le c\,\varepsilon_j^2 |w|_{H^2(T)}$$

$$\|v_j - w\|_{H^1(T)} \le c\,\varepsilon_j |w|_{H^2(T)}$$

hold for every w, for every ε_j and for every T such that $T \not\subset S_j$. Now let $\Omega_j = \Omega \setminus S_j$ then we can write

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_\Omega f(\varepsilon_j|\nabla v_j|^2)\,dx &\le \sum_{T\in\Omega_j} |T|\frac{1}{\varepsilon_j} f(\varepsilon_j|\nabla v_j|^2) + f_\infty \sum_{T\in S_j} |T|\frac{1}{\varepsilon_j}\\ &\le \sum_{T\in\Omega_j} |T|\frac{1}{\varepsilon_j} f(\varepsilon_j|\nabla v_j|^2) + f_\infty \sum_{m=1}^{p}\Big(\sum_{T\in S^i_{\varepsilon_j}(S_m,\nu_m)} |T|\frac{1}{\varepsilon_j}\Big).\end{aligned}$$

The proof follows from the inequalities

$$\limsup_{j\to+\infty} \sum_{T\in\Omega_j} |T|\frac{1}{\varepsilon_j} f(\varepsilon_j|\nabla v_j|^2) \le \int_\Omega |\nabla u|^2\,dx\,, \tag{2.26}$$

$$\limsup_{j\to+\infty} \sum_{T\in S^i_{\varepsilon_j}} |T|\frac{1}{\varepsilon_j} = \int_{S_m} \phi_i(\nu_m)\,d\mathcal{H}^1\,. \tag{2.27}$$

The first can be proved as in [20] while the second follows easily from the properties of the functions $\phi_i(\nu)$. ■

The previous result proves the Γ-limsup inequality for $u \in \mathcal{W}(\Omega)$, it remains to extend it to $u \in L^2(\Omega)$. Clearly it is not restrictive to assume that $u \in GSBV^2(\Omega)$. Then by Lemma 1.2.17 there exists a sequence $w_k \in \mathcal{W}(\Omega)$ such that

$$w_k \longrightarrow u \text{ strongly in } L^2(\Omega)\,,$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^2(\Omega, \mathbf{R}^2)\,,$$

$$\limsup_{k\to+\infty} \int_{S_{w_k}} \phi_i(\nu_{w_k})\,d\mathcal{H}^1 \le \int_{S_u} \phi_i(\nu_u)\,d\mathcal{H}^1\,.$$

Then by the lower semicontinuity of F'' respect to the topology of $L^2(\Omega)$ it follows that

$$F''(u) \le \liminf_{k\to+\infty} F''(w_k) \le \limsup_{k\to+\infty} F(w_k) \le F(u)$$

and thus Γ-limsup inequality is proved completely.

2.4 Γ-liminf inequality

In this section we will prove that the Γ-liminf inequality $F(u) \leq F'(u)$ holds for every $u \in L^2(\Omega)$.

Proposition 2.4.1 *Let $\varepsilon_j \searrow 0$ and let $u \in L^2(\Omega)$. If $F'(u) < +\infty$ then $u \in SBV^2(\Omega)$ and*

$$\int_\Omega |\nabla u|^2 dx + f_\infty \int_{S_u} \phi_i(\nu_u) d\mathcal{H}^1 \leq \liminf_{j\to+\infty} \frac{1}{\varepsilon_j} \int_\Omega f(\varepsilon_j |\nabla u_j|^2) dx \ ,, \tag{2.28}$$

for every sequence $u_j \in V^i_{\varepsilon_j}(\Omega)$ converging strongly to u in $L^2(\Omega)$.

The proof is based on the one presented in [20]. First of all, for every open set $A \subset \Omega$, we define the localized functionals

$$F_{\varepsilon_j}(u, A) = \begin{cases} \dfrac{1}{\varepsilon_j} \displaystyle\int_A f(\varepsilon_j |\nabla u|^2)\, dx & \text{if } u \in V^i_{\varepsilon_j}(\Omega) \\ +\infty, & \text{if } u \in L^2(\Omega) \backslash V^i_{\varepsilon_j}(\Omega) \end{cases}$$

$$F'(u, A) = \Gamma\text{-}\liminf_{j\to+\infty} F_{\varepsilon_j}(u, A)\,,$$

then we prove the following Proposition.

Proposition 2.4.2 *Let $i = 1, \ldots, 3$, $u \in L^2(\Omega)$ and A an open set in Ω. If $F'(u, A) < +\infty$ then $u \in SBV^2(A)$ and*

$$\int_A |\nabla u|^2 dx \leq F'(u, A)\,. \tag{2.29}$$

Moreover for $i = 1, 3$

$$f_\infty \int_{S_u \cap A} c^i_k |\langle \nu_u, \xi^i_k \rangle| d\mathcal{H}^1 \leq F'(u, A) \quad \text{for } k = 1, \ldots, 3\,, \tag{2.30}$$

while for $i = 2$

$$f_\infty \int_{S_u \cap A} \frac{1}{2}\left(c^3_{k_1} |\langle \nu_u, \zeta_{k_1} \rangle| + c^3_{k_2} |\langle \nu_u, \zeta'_{k_2} \rangle| \right) d\mathcal{H}^1 \leq F'(u, A)\,, \tag{2.31}$$

for every pair $\zeta_{k_1}, \zeta'_{k_2}$ such that $\langle \zeta_{k_1}, \zeta'_{k_2} \rangle = \sqrt{2}/2$.

In order to prove Proposition 2.4.2 we need some of technical lemma.

Lemma 2.4.3 *Let A be an open set in Ω and let $i = 1, \ldots, 3$. For every $\delta \in (0,1)$ there exist $\alpha_\delta > 0$ and $\beta_\delta > 0$ such that for every $u \in V^i_{\varepsilon_j}(\Omega)$ it is possible to find a function $v \in SBV^2(\Omega)$ satisfying*

$$(1-\delta) \int_{A_{\varepsilon_j}} |\nabla v|^2 dx + \alpha_\delta \mathcal{H}^1(S_v \cap A_{\varepsilon_j}) \leq F_{\varepsilon_j}(u, A)\,,$$

$$|\{x \in \Omega \ : \ u(x) \neq v(x)\}| \leq \beta_\delta\, \varepsilon_j\, F_{\varepsilon_j}(u, A)\,,$$

where $A_{\varepsilon_j} = \{x \in A \ : \ d(x, \partial A) > \varepsilon_j\}$.

Proof. See [20], Proposition 3.3.

Lemma 2.4.4 *Let A be an open set in Ω and $i = 1, 3$. For every $\delta \in (0,1)$ there exist $\alpha_\delta > 0$, $\beta_\delta > 0$, and $\gamma_\delta > 0$ such that for every index k and for every $u \in V^i_{\varepsilon_j}(\Omega)$ it is possible to find a function $v \in SBV^2(\Omega)$ satisfying*

$$\alpha_\delta \int_{A_{\varepsilon_j}} |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap A_{\varepsilon_j}) \leq F_{\varepsilon_j}(u, A)\,, \tag{2.32}$$

$$|\{x \in \Omega \ : \ u(x) \neq v(x)\}| \leq \gamma_\delta \, \varepsilon_j \, F_{\varepsilon_j}(u, A)\,, \tag{2.33}$$

$$(1-\delta) f_\infty \int_{S_v \cap A_{\varepsilon_j}} c^i_k |\langle \nu_v, \xi^i_k \rangle| d\mathcal{H}^1 \leq F_{\varepsilon_j}(u, A)\,. \tag{2.34}$$

Proof. From the properties of the function f we deduce the existence of a positive constant c such that

$$\min(ct, (1-\delta)f_\infty) \leq f(t) \quad \forall\, t \geq 0\,.$$

Let $\mathcal{T}_A = \{T \in \boldsymbol{T}^i_{\varepsilon_j} : T \subset A\}$ and let $\sigma = \sqrt{(1-\delta)f_\infty / c}$. To build the function v we need a careful classification of the triangles in $\mathcal{T}_A$ according to the behavior of ∇u_T: let $\mathcal{T}_{A,m}$ for $m = 0, \ldots, 3$ be the set of triangles $T \in \mathcal{T}_A$ such that the slope of u is greater than $\sigma / \sqrt{\varepsilon_j}$ exactly on m edges. So we can write

$$\begin{aligned} F_{\varepsilon_j}(u, A) &\geq \sum_{T \in \mathcal{T}_A} \frac{|T|}{\varepsilon_j} \min(c\varepsilon_j |\nabla u_T|^2, (1-\delta) f_\infty) = \\ &= c \sum_{T \in \mathcal{T}_{A,0}} |T| \min(|\nabla u_T|^2, \frac{\sigma^2}{\varepsilon_j}) + (1-\delta) f_\infty \sum_{T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}} \frac{|T|}{\varepsilon_j}\,. \end{aligned}$$

If $T \in \mathcal{T}_{A,0}$ then $|\nabla u_T| \leq C_i \frac{\sigma}{\sqrt{\varepsilon_j}}$, where C_i is the constant defined in Lemma 2.3.1, thus we have a lower bound on σ^2/ε_j in terms of $|\nabla u_T|^2$, in particular for $c' = 1/C_i^2$ we get

$$F_{\varepsilon_j}(u, A) \geq c\,c' \sum_{T \in \mathcal{T}_{A,0}} |T||\nabla u_T|^2 + (1-\delta) f_\infty \sum_{T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}} \frac{|T|}{\varepsilon_j}\,. \tag{2.35}$$

Now, if $T \in \mathcal{T}_{A,0}$ we take $v = u$, in this way inequality (2.33) holds with $\gamma_\delta = 1/(1-\delta)f_\infty$, while if $T \in \mathcal{T}_A \setminus \mathcal{T}_{A,0}$ the function v and the constants α_δ and β_δ are defined in order to satisfy the inequalities

$$\frac{|T|}{\varepsilon_j} \geq c^i_k \int_{S_v \cap T} |\langle \nu_v, \xi^i_k \rangle| \mathrm{d}\mathcal{H}^1 \quad \forall k\,, \tag{2.36}$$

$$(1-\delta) f_\infty \frac{|T|}{\varepsilon_j} \geq \alpha_\delta \int_T |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap T)\,, \tag{2.37}$$

which gives respectively (2.32) and (2.34). The function v can be defined as in [20], and repeating the proof of Proposition 3.4 in [20] it is not difficult to see that the same choice of v, α_δ and β_δ satisfies all the previous inequalities. ■

Lemma 2.4.5 *Let A be an open set in Ω and let $i = 2$. For every $\delta \in (0,1)$ there exist $\alpha_\delta > 0$, $\beta_\delta > 0$, and $\gamma_\delta > 0$ such that for every pair k_1, k_2 with $\langle \zeta_{k_1}, \zeta'_{k_2} \rangle = \sqrt{2}/2$ and for every $u \in V^i_{\varepsilon_j}(\Omega)$ it is possible to find a function $v \in SBV^2(\Omega)$ satisfying (2.33) and*

$$\alpha_\delta \int_{A_{2\varepsilon_j}} |\nabla v|^2 dx + \beta_\delta \mathcal{H}^1(S_v \cap A_{2\varepsilon_j}) \leq F_{\varepsilon_j}(u, A)\,, \tag{2.38}$$

$$(1-\delta)f_\infty \int_{S_v \cap A_{2\varepsilon_j}} \frac{1}{2}\Big(c^3_{k_1}|\langle \nu_v, \zeta_{k_1}\rangle| + c^3_{k_2}|\langle \nu_v, \zeta'_{k_2}\rangle|\Big)\, d\mathcal{H}^1 \le F_{\varepsilon_j}(u, A). \qquad (2.39)$$

Proof. In this case it is not possible to define a function v on a single element in such a way that (2.39) is satisfied. It will be necessary to consider a group of triangles as the one represented in Figure 2.7a, which is the smallest periodic structure of the mesh. Let us call Q this set and let $\mathcal{Q}_A$ be the set of squares Q in $meshepsj^2$ such that $Q \subset A$. It is clear that

$$A_{2\varepsilon_j} \subset \Big(\cup_{Q\in\mathcal{Q}_A} Q\Big) \subset A.$$

First v is defined on ∂Q, thus for every edge $L \subset \partial Q$, let x_1, x_2 be its vertices, x_{12} its middle point and let u'_L be the slope of u on L, then

- $v \equiv u(x_1)$ on the segment $[x_1\, x_{12}]$ and $v \equiv u(x_2)$ on the segment $[x_{12}\, x_2]$ if $|u'_L| > \sigma/\sqrt{\varepsilon_j}$
- $v = u$ if $|u'_L| \le \sigma/\sqrt{\varepsilon_j}$.

Note that in this way v is no longer continuous on ∂Q but its slope now is piecewise uniformly bounded by $\sigma/\sqrt{\varepsilon_j}$.

Now, let T be a triangle in Q, if $|\nabla u_T| \le \sigma/\sqrt{\varepsilon_j}$ we take $v = u$ on T, hence by an estimate like (2.35) inequality (2.33) is still satisfied. Moreover this definition agrees with the one given on ∂Q, indeed on the edges of T we have $|u'_L| \le |\nabla u_T| \le \sigma/\sqrt{\varepsilon_j}$.

At this point we have defined v on the triangles $T \subset Q$ where $|\nabla u_T| \le \sigma/\sqrt{\varepsilon_j}$ and on the boundary ∂Q. Figure 2.7b presents a set Q where on the lower left triangle $v = u$ (the colored triangle and the solid lines denotes where v is already given).

Call E the set where v is still to be defined, note that on ∂E the function is assigned and its slope is piecewise controlled by $\sigma/\sqrt{\varepsilon_j}$. In the following the idea is to extend the values on ∂E inside E introducing a jump set S_v, which takes into account the discontinuities on ∂Q and such that $\mathcal{H}^1(S_v \cap \partial Q) = \emptyset$, and keeping a control on the gradient. Unfortunately it is not possible to find a unique procedure for all the cases, thus we need to divide $\mathcal{Q}_A$ into five classes $\mathcal{Q}_{A,m}$ for $m = 0, \dots, 4$ according to the number of elements $T \subset Q$ on which $|\nabla u_T| > \sigma/\sqrt{\varepsilon_j}$. To avoid a repetitious proof we omit the easy part of verifying (2.39) and (2.38) and every time we describe only the construction of v. In particular the existence of the constants α_δ and β_δ will follow easily from the uniform bound on $|\nabla v|$ and $\mathcal{H}^1(S_v)$. Moreover it is sufficient to prove inequality (2.39) for the pair $\zeta_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\zeta'_3 = (1,0)$.

$Q \in \mathcal{Q}_{A,0}$. Obviously we take $v = u$ on Q, then (2.39) is satisfied being $S_v = \emptyset$, and the gradient is controlled by definition being $|\nabla v_T| = |\nabla u_T| \le \sigma/\sqrt{\varepsilon_j}$.

$Q \in \mathcal{Q}_{A,1}$. In this case we have four possibilities but, without loss of generality, we can consider only the one represented in Figure 2.7c, all the other cases are equivalent by a symmetry argument. Suppose that in the lower right triangle T we have $|\nabla u_T| > \sigma/\sqrt{\varepsilon_j}$, in the others (colored) we have $v = u$. Moreover note that on $\partial T \cap \mathring{Q}$ the slope of v, being equal to the slope of u, is bounded by $\sigma/\sqrt{\varepsilon_j}$. Now consider the triangles T_1 and T_2. In both the triangles v is already given on two edges, where its slope is controlled; thus by linear extension v is defined on T_1 and T_2 and the gradient will be controlled by $C\sigma/\sqrt{\varepsilon_j}$. It remain to consider T_3, also in this case we can apply the previous reasoning because v now is defined on two edges. This time we have $|\nabla v_{T_3}| \le C^2\sigma/\sqrt{\varepsilon_j}$. By the construction

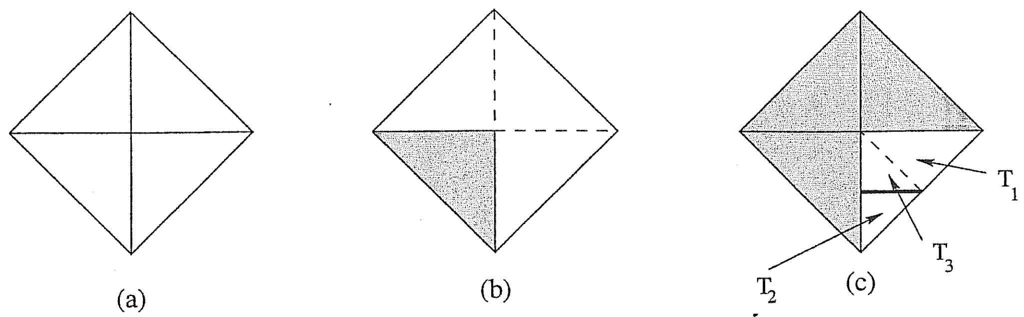


Figure 2.7: The set Q, an example and $Q \in \mathcal{Q}_{A,1}$.

it is clear that v can be discontinuous only on the bold segment and it is not difficult to see that (2.39) is satisfied.

$Q \in \mathcal{Q}_{A,2}$. Taking into account the symmetries of Q and v, we can consider only the three possibilities, represented in Figure 2.8. For the first it is not necessary to give details, indeed it is sufficient to adapt the definition of v given for $Q \in \mathcal{Q}_{A,1}$.

In the second case v is defined first on T_1, T_2 and T_3, then, using the values of T_2 and T_3 on the dashed lines we complete the procedure on T_4. The discontinuity, represented by the bold segment, satisfies (2.39) and the gradient controlled in the worst case by $C^2\sigma/\sqrt{\varepsilon_j}$.

The last case in Figure 2.8c can be reduced to the previous by means of a rotation, note that the scalar products in (2.39) changes but the inequality is still satisfied.

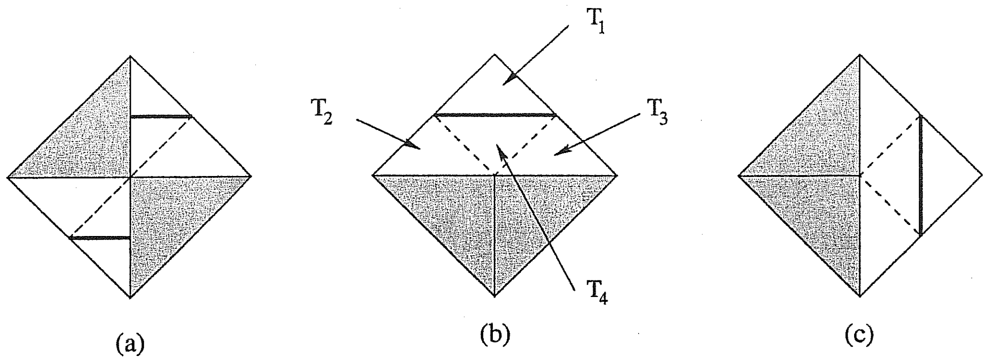


Figure 2.8: The three cases for $Q \in \mathcal{Q}_{A,2}$.

$Q \in \mathcal{Q}_{A,3}$. In this case we should consider two possibilities, see Figure 2.9, nevertheless it is easy to verify by symmetry that the same construction hold for both, thus we will give details only for the first. As before v is built by linear extension first on the triangles T_i for $i = 1, \dots, 4$, then on T_5 and T_6 where it is known on one edge. By construction and by Lemma 2.3.1 we see that $|\nabla v_{T_i}| \leq C\sigma/\sqrt{\varepsilon_j}$ if $i = 1, \dots, 4$ and $|\nabla v_{T_j}| \leq C^2\sigma/\sqrt{\varepsilon_j}$ for $j = 5, 6$.

$Q \in \mathcal{Q}_{A,4}$. This time it is very easy to find a good function v with the required properties, indeed it is enough to extend the values on ∂Q in the sets E_i with an affine function such that $|\nabla v_{E_i}| \leq C\sigma/\sqrt{\varepsilon_j}$ for every i. ∎

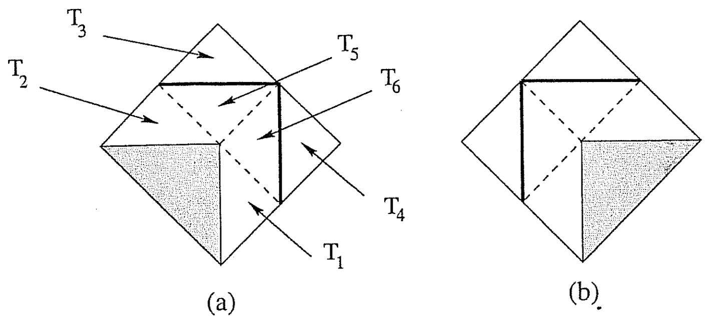


Figure 2.9: The two cases for $Q \in \mathcal{Q}_{A,3}$.

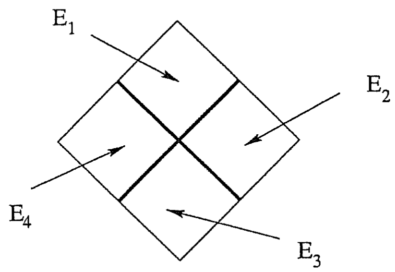


Figure 2.10: $Q \in \mathcal{Q}_{A,4}$.

Proof of Proposition 2.4.2. Consider a sequence u_j converging to u in $L^2(\Omega)$ such that $u_j \in V^i_{\varepsilon_j}(\Omega)$ and $\liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A) < +\infty$. Up to extracting a subsequence we can assume that $\sup_j F_{\varepsilon_j}(u_j, A) \le c < +\infty$. Now, take $\delta \in (0,1)$, for every j let $v_j \in SBV^2(\Omega)$ be as in Lemma 2.4.3. Let $\lambda \in \mathbf{N}$ be a truncation level, we define

$$v_j^\lambda = (-\lambda \vee v_j) \wedge \lambda \quad \text{and} \quad u_j^\lambda = (-\lambda \vee u_j) \wedge \lambda\,.$$

It's easy to see that u_j^λ converges to u^λ in $L^2(\Omega)$, then

$$\int_\Omega |v_j^\lambda - u_j^\lambda| dx \le 2\lambda\beta_\delta\varepsilon_j F_{\varepsilon_j}(u_j, A) \le (2\lambda\beta_\delta c)\varepsilon_j$$

and we conclude that also v_j^λ converges to u^λ for every λ. Now, let $\eta > 0$, if ε_j is small enough $A_\eta \subset A_{\varepsilon_j}$ and

$$(1-\delta)\int_{A_\eta} |\nabla v_j^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{v_j^\lambda} \cap A_\eta) \le F_{\varepsilon_j}(u_j, A) \le c\,.$$

By Ambrosio's theorem we have that $u^\lambda \in SBV(A_\eta)$ for every λ and

$$(1-\delta)\int_{A_\eta} |\nabla u^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{u^\lambda} \cap A_\eta) \le \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j, A)\,.$$

Since the previous inequality holds for every sequence u_j and for every η we have $u^\lambda \in SBV^2(\Omega)$ and

$$(1-\delta)\int_A |\nabla u^\lambda|^2 dx + \alpha_\delta \mathcal{H}^1(S_{u^\lambda} \cap A) \le F'(u, A)\,.$$

For $\lambda \to +\infty$ and $\delta \to 0$ we get (2.29). It remain to consider inequality (2.30), it's proved exactly in the same way using this time the function v_j given by Lemma 2.4.4. ■

Proof of proposition 2.4.1. From Proposition 2.4.4 follows (2.28) by Proposition 1.2.4 and by the representations of $\phi_i(\nu)$ given in Remark 2.2.7. ■

2.5 Numerical examples

This section presents some numerical results for artificial images obtained minimizing the discrete functional

$$G_{\varepsilon_j}(u) = \frac{1}{\varepsilon_j} \int_\Omega \alpha \frac{2}{\pi} \arctan\left(\varepsilon_j \frac{\beta}{\alpha} \frac{\pi}{2} |\nabla u|^2\right) dx + \int_\Omega |u-g|^2\, dx\,, \tag{2.40}$$

where $\Omega = (0,1) \times (0,1)$ and the function g takes values in the interval $[0,1]$. Details about the minimizing procedure are given in the next chapter. By Theorem 2.1.1 we know that G_{ε_j} Γ-converge to the functional

$$G(u) = \beta \int_\Omega |\nabla u|^2 dx + \alpha \int_{S_u} \phi_i(\nu_u) d\mathcal{H}^1 + \int_\Omega |u-g|^2 dx\,, \tag{2.41}$$

thus the solutions of (2.40) should present for ε_j small enough the properties of the minima of (2.41). In particular, according to [15], the amplitude of jump along discontinuities should be greater than a minimum contrast threshold which can be explicitly computed in terms of the coefficients α and β. For our functional, due to the presence of anisotropy, in general it is no longer possible to have a unique value of this threshold, nevertheless for rectilinear discontinuities the estimate becomes

$$c = \sqrt{\frac{2\alpha\phi(\nu)}{\sqrt{\beta}}}. \tag{2.42}$$

Let us consider now Figure 2.11. Here both the original image and the parameters were chosen in order to stress the effect of anisotropy. Indeed on the white part we have $g \equiv 1$ and on the black one $g \equiv 0$, hence the original contrast is 1. Taking $\beta = 0.07$ and $\alpha = 0.15$, for $\boldsymbol{T}^1_\varepsilon$ the anisotropy function takes the value 1 if $\nu = (\sqrt{2}/2, \sqrt{2}/2)$ and 0.5 if $\nu = (-\sqrt{2}/2, \sqrt{2}/2)$ and so the constant c is respectively 1.06 and 0.75. For $\boldsymbol{T}^2_\varepsilon$ there is a unique value, which is 0.9. According to these estimates, in the first case the solution presents a discontinuity on one side and a smooth region on the other, while in the second the original symmetry is preserved.

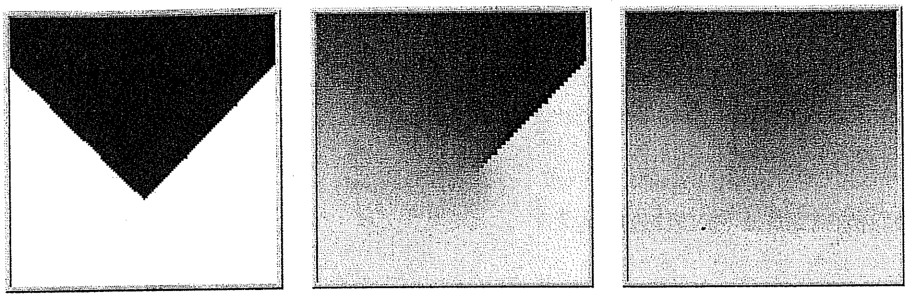


Figure 2.11: The original image and the numerical solutions using $\boldsymbol{T}_\varepsilon^1$ and $\boldsymbol{T}_\varepsilon^2$.

Chapter 3

Numerical solution of the Mumford-Shah functional

3.1 Discrete functional and Γ-limit

Let $\Omega = (0,1) \times (0,1)$ and let T^i_ε for $i = 1,2$ be the triangulations represented in Figure 1.2 (ε is the diameter of the triangles). Moreover let $V^i_\varepsilon(\Omega)$ be the finite element space of piecewise affine functions in Ω defined on the mesh T^i_ε.

In order to fit the domain preserving the geometry of the mesh, let ε_j be a sequence such that $\varepsilon_j \searrow 0$ and $(\sqrt{2}/\varepsilon_j) \in \mathbf{N}\ \forall j$, then, by the previous chapter, for $i = 1,2$ the discrete functionals

$$G_{\varepsilon_j}(u) = \begin{cases} \dfrac{1}{\varepsilon_j}\displaystyle\int_\Omega \alpha\frac{2}{\pi}\arctan\left(\varepsilon_j\frac{\beta}{\alpha}\frac{\pi}{2}|\nabla u|^2\right)dx & \text{if } u \in V^i_{\varepsilon_j}(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus V^i_{\varepsilon_j}(\Omega) \end{cases}$$

Γ-converge respect to strong topology of $L^2(\Omega)$ to the functional

$$G(u) = \begin{cases} \beta\displaystyle\int_\Omega |\nabla u|^2 dx + \alpha\int_{S_u}\phi_i(\nu_u)d\mathcal{H}^1 & \text{if } u \in L^2(\Omega) \cap SBV^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus SBV^2(\Omega). \end{cases}$$

From this result it follows easily by continuity that the functionals

$$F_{\varepsilon_j}(u) = G_{\varepsilon_j}(u) + \int_\Omega |u-g|^2 dx$$

Γ-converge in the same topology to the *anisotropic Mumford-Shah functional*

$$F(u) = G(u) + \int_\Omega |u-g|^2 dx\,.$$

3.2 The quasi-Newton minimizing algorithm

Let us consider in general the problem of finding a minimizer for a smooth functional $F : \mathbf{R}^n \to [0,+\infty)$. In abstract the idea is to produce a sequence $\{u_k\} \subset \mathbf{R}^n$ that converges to a minimum point of F. In practice, given an initial point u_0, the algorithm generates a finite sequence, defined by induction as $u_{k+1} = u_k + t_k d_k$, where t_k is a scalar in $[0,1]$ and $d_k \in \mathbf{R}^n$ is a descent direction. In the following we briefly recall the classical quasi-Newton algorithm (for a detailed treatise see [27] or [33]).

Step 1. Compute the gradient $\nabla F(u_k)$ and the Hessian $HF(u_k)$. Then check if the Hessian is positive definite and (if it is not) introduce a small perturbation of its diagonal in order to have a matrix $\widetilde{HF}(u_k) = HF(u_k) + D_k$ which is positive definite. In general this can be done in different ways, for instance through an incomplete and modified Cholesky factorization, but for our functional, due to the huge number of unknowns, a better choice is the following. Let h_{ij} be the elements of $HF(u_k)$, by Gershgorin's theorem the eigenvalues are contained in the union of the intervals

$$\left(h_{ii} - \sum_{j \neq i} |h_{ij}|, h_{ii} + \sum_{j \neq i} |h_{ij}|\right).$$

Thus to ensure the positive definiteness it is sufficient to replace h_{ii} with $\tilde{h}_{ii} = h_{ii} + \delta_i$ in such a way that

$$(h_{ii} + \delta) - \sum_{j \neq i} |h_{ij}| > 0.$$

Moreover to prevent ill-conditioning it is better to require

$$(h_{ii} + \delta) - \sum_{j \neq i} |h_{ij}| > \eta,$$

where η is a suitable positive parameter. Finally check the gradient: if $\nabla F(u_k) \neq 0$ we jump to the following step, if $\nabla F(u_k) = 0$ and $HF(u_k)$ is positive definite then the algorithm has found a minimizer, otherwise introduce a small perturbation in u_k and restart.

Step 2. Compute a preconditioner for $\widetilde{HF}(u_k)$, for our functional an efficient choice is the incomplete Cholesky factorization, which preserves the sparsity of the original matrix.

Step 3. The descent direction is defined by the linear system

$$\widetilde{HF}(u_k) d_k = -\nabla F(u_k)$$

because

$$\langle \nabla F(u_k), d_k \rangle = -\langle \widetilde{HF}(u_k) d_k, d_k \rangle < 0.$$

The linear system can be solved using any preconditioned method for symmetric positive definite matrices. Nevertheless for a fast implementation it is sufficient to solve the system $\widetilde{DHF}(u_k) d_k = -\nabla F(u_k)$ where $\widetilde{DHF}$ denotes the diagonal of $\widetilde{HF}$.

Step 4. Determine the step length t_k. If $HF(u_k)$ is positive definite then we take $t_k = 1$, as in the classical Newton algorithm, otherwise t_k is computed minimizing the one dimensional restriction of F to the interval $\{u_k + t d_k \text{ for } t \in [0,1]\}$.

Step 5. The algorithm is stopped if

$$|u_{k+1} - u_k| = |t_k d_k| < \mu,$$

where μ is a suitable positive value. Otherwise it goes back to the first step.

Note that the algorithm in general converges to a relative minimum of F depending on the choice of the starting point u_0 and that, by Step 4, near the solution it should converge quadratically as the classical Newton method does.

3.3 The model problem in dimension one

In this section we will consider the one dimensional version of the Mumford-Shah functional and we will prove some existence result about its global and local minima. Let $I = (0,1)$ and let $g \in L^\infty(I)$, taking values in $[0,1]$, then the one dimensional Mumford-Shah functional is

$$F(u) = \beta \int_I |u'|^2 dx + \alpha \#(S_u) + \int_I |u-g|^2 dx \tag{3.1}$$

and the related segmentation problem will be $\min\{F(u) : u \in SBV^2(I)\}$.

In particular let us first consider a function g having one discontinuity: let $I_1 = (0, 0.5)$, $I_2 = (0.5, 1)$, $h \in (0,1]$ and

$$g = \begin{cases} h & \text{in } I_1 \\ 0 & \text{in } I_2\,. \end{cases} \tag{3.2}$$

In this case we can compute explicitly the solutions of the problems $\min\{F(u) : u \in W^{1,2}(I)\}$ and $\min\{F(u) : u \in SBV^2(I) \setminus W^{1,2}(I)\}$. Let $\lambda = \sqrt{\beta}$, the function $u_c \in C^1(I)$ defined as

$$u_c(x) = \begin{cases} h - \dfrac{h\cosh(x/\lambda)}{2\cosh(1/2\lambda)} & \text{in } I_1 \\[2ex] \dfrac{h\cosh((x-1)/\lambda))}{2\cosh(1/2\lambda)} & \text{in } I_2 \end{cases} \tag{3.3}$$

is the solution of the problem

$$\begin{cases} -\lambda^2 u'' + u = g & \text{in } I \\ u'(0) = u'(1) = 0\,, \end{cases}$$

which is the Euler equation of $F(u)$. Thus u_c is the minimum in $W^{1,2}(I)$. Let $u \in SBV^2(I) \setminus W^{1,2}(I)$, being $S_u \neq \emptyset$ it's easy to verify that the minimum is the function $u_d \equiv g$. The values of the energy relative to the previous functions are

$$\begin{aligned} E_d &= F(u_d) = \alpha \\ E_c &= F(u_c) = \frac{h^2\lambda\sinh(1/\lambda)}{4\cosh^2(1/2\lambda)}. \end{aligned}$$

At this point it is very easy to find the minimum, indeed it sufficient to compare the values of E_d and E_c, so

$$u_{min} = \begin{cases} u_d & \text{if } E_d < E_c \\ u_c & \text{if } E_d > E_c\,. \end{cases} \tag{3.4}$$

In particular when $\lambda \ll 1$, as it is in the applications, we have

$$E_c = \frac{h^2\lambda\sinh(1/\lambda)}{4\cosh^2(1/2\lambda)} \simeq \frac{h^2\lambda}{2},$$

then

$$E_d = E_c \Leftrightarrow \alpha = \frac{h^2\lambda}{2} \Leftrightarrow h = \sqrt{\frac{2\alpha}{\lambda}}.$$

So we recover the estimate for the contrast threshold

$$h_0 = \sqrt{\frac{2\alpha}{\sqrt{\beta}}} \tag{3.5}$$

as it is reported in [15]. Similarly for $\lambda \ll 1$ (3.4) becomes

$$u_{min} = \begin{cases} u_d & \text{if } h < h_0 \\ u_c & \text{if } h > h_0 \end{cases} . \tag{3.6}$$

The functions u_d and u_c are also local minima of (3.1) as it is proved in the following propositions.

Proposition 3.3.1 *Let $h \in (0,1]$, g defined as in (3.2) and $\delta \in (0, \frac{h}{2})$. Then for every $\alpha > 0$ and $\beta > 0$ we have*

$$F(u_d) = \min\{F(u) \, : \, u \in B_{\delta,\infty}(u_d)\}$$

where $B_{\delta,\infty}(u_d) = \{u \in SBV^2(I) \, : \, 0 \leq u \leq 1 \ \ \|u - u_d\|_{L^\infty(\Omega)} < \delta\}$.

Proof. Being $\delta < \frac{h}{2}$ for every $u \in B_{\delta,\infty}$ we have $u(\frac{1}{2}^-) > u(\frac{1}{2}^+)$, then $\#(S_u) \geq 1$ and $F(u) \geq \alpha = F(u_d)$. ∎

Proposition 3.3.2 *Let $h \in (0,1]$, g defined as in (3.2). Then for every $\alpha > 0$ and $\beta > 0$ there exists $\delta > 0$ (sufficiently small) such that*

$$F(u_c) = \min\{F(u) \, : \, u \in B_{\delta,\infty}(u_c)\}$$

where $B_{\delta,\infty}(u_c) = \{u \in SBV^2(I) \, : \, 0 \leq u \leq 1 \ \ \|u - u_c\|_{L^\infty(\Omega)} < \delta\}$.

Proof. Let $\delta > 0$ and let $B_{\delta,\infty} = B^0_{\delta,\infty} \cup B^1_{\delta,\infty}$ where

$$B^0_{\delta,\infty} = \{u \in B_{\delta,\infty} \, : \, S_u = \emptyset\} \qquad B^1_{\delta,\infty} = B_{\delta,\infty} \setminus B^0_{\delta,\infty} .$$

It's easy to see that for every δ,

$$F(u_c) = \min\{F(u) \, : \, u \in B^0_{\delta,\infty}\} ,$$

thus it is sufficient to prove that for δ sufficiently small

$$F(u_c) = \min\{F(u) \, : \, u \in B^1_{\delta,\infty}\} .$$

Let $k \in \mathbf{N}$ such that

$$k\alpha \leq F(u_c) \qquad (k+1)\alpha > F(u_c) .$$

Consider $u \in B^1_{\delta,\infty}$, since we are interested in the minimum points, we can suppose that $S_u = \{x_1, x_2, \ldots, x_{n-1}\}$ with $2 \leq n \leq (k+1)$ and we set $x_0 = 0$ and $x_n = 1$. First of all we will prove that there exists $\delta > 0$

$$\beta \int_I |u'|^2 dx + (n-1)\alpha > \beta \int_I |u_c'|^2 dx . \tag{3.7}$$

It is easy to see that there exists $a > 0$ such that for every subinterval $(y_1, y_2) \subset I$ with $|y_2 - y_1| \leq a$ we have

$$\left(\frac{n-1}{n}\right)\alpha > \beta \int_{y_1}^{y_2} |u_c'|^2 dx . \tag{3.8}$$

Moreover, given a, for δ sufficiently small we have $u_c(y_1) - \delta > u_c(y_2) + \delta$ for every interval (y_1, y_2) such that $|y_2 - y_1| \geq a$ (because u_c is decreasing). Let $u \in B^1_{\delta,\infty}$ and let $S_u = \{x_1, x_2, \ldots, x_{n-1}\}$. If for every index $1 \leq i \leq n$ we have $|x_{i-1} - x_i| \leq a$ then by (3.8) inequality (3.7) holds. If for some index i we have $|x_{i-1} - x_i| > a$ then we define on (x_{i-1}, x_i) a function v_i, which is the solution of the minimum problem

$$\int_{x_{i-1}}^{x_i} |v_i'|^2 dx = \min\Big\{ \int_{x_{i-1}}^{x_i} |u'|^2 dx \ : \ u_c(x) - \delta \leq u(x) \leq u_c(x) + \delta \Big\}.$$

Being u_c decreasing, it is natural to suppose that $v_i(x_{i-1}) = u_c(x_{i-1}) - \delta$ and $v_i(x_i) = u_c(x_i) + \delta$. Moreover let x_a and x_b such that $x_{i-1} \leq x_a < x_b \leq x_i$. If the line

$$r_i(x) = \Big(\frac{v_i(x_b) - v_i(x_a)}{x_b - x_a}\Big)(x - x_a) + u(x_a)$$

satisfies $u_c(x) - \delta \leq r_i(x) \leq u_c(x) + \delta$ for $x_a \leq x \leq x_b$ then the new function defined as

$$w_i(x) = \begin{cases} v_i(x) & \text{if } x_{i-1} \leq x < x_a \\ r_i(x) & \text{if } x_a \leq x \leq x_b \\ v_i(x) & \text{if } x_b < x \leq x_i \end{cases}$$

satisfies

$$\int_{x_{i-1}}^{x_i} |w_i'|^2 dx \leq \int_{x_{i-1}}^{x_i} |v_i'|^2 dx .$$

Then there exist y_{i-1} and y_i such that $x_{i-1} \leq y_{i-1} < y_i \leq x_i$ and

$$v_i(x) = \begin{cases} u_c(x) - \delta & \text{if } x_{i-1} \leq y_{i-1} \\ l(x) & \text{if } y_{i-1} < x < y_i \\ u_c(x) + \delta & \text{if } y_i \leq x_i , \end{cases} \tag{3.9}$$

where $l(x)$ is the line which interpolates the points $(y_{i-1}, u_c(y_{i-1}) - \delta)$ and $(y_i, u_c(y_i) + \delta)$. Note that the points y_{i-1} and y_i depends on δ, in particular due to the minimality of v_i we have that y_{i-1} is non-decreasing and y_i is non-increasing respect to δ. Then there exist the limits

$$\lim_{\delta \to 0} y_{i-1} = y_a \qquad \lim_{\delta \to 0} y_i = y_b$$

and $y_a = y_b$. Indeed if $y_a \neq y_b$ then, due to uniform convergence, $u_c(x)$ should be affine on the interval (y_a, y_b) and it is impossible by its definition.

Now we will prove that there exists a value $\delta > 0$ such that

$$\beta \int_{x_{i-1}}^{x_i} |v_i'|^2 dx + \Big(\frac{n-1}{n}\Big)\alpha > \beta \int_{x_{i-1}}^{x_i} |u_c'|^2 dx . \tag{3.10}$$

Considering (3.9) we deduce that

$$\int_{x_{i-1}}^{x_i} |v_i'|^2 dx = \int_{x_{i-1}}^{y_{i-1}} |u_c'|^2 dx + \int_{y_{i-1}}^{y_i} |v_i'|^2 dx + \int_{y_i}^{x_i} |u_c'|^2 dx ,$$

then (3.10) becomes

$$\beta \int_{y_{i-1}}^{y_i} |u_c'|^2 dx - \beta \int_{y_{i-1}}^{y_i} |v_i'|^2 dx < \Big(\frac{n-1}{n}\Big)\alpha .$$

This is for sure satisfied for some positive δ because

$$\lim_{\delta\to 0}\int_{y_{i-1}}^{y_i}|u_c'|^2dx=\lim_{\delta\to 0}\int_{y_{i-1}}^{y_i}|v_i'|^2dx=0\,.$$

Moreover δ can be chosen in order to satisfy inequality (3.10) on every subinterval (x_α, x_β) such that $|x_\alpha - x_\beta| > a$. Then for every $u \in B^1_{\delta,\infty}$ and for δ sufficiently small

$$\min_{u\in B^1_{\delta,\infty}}\left\{\beta\int_I|u'|^2dx+\alpha\#S_u\right\}>F(u_c),$$

which concludes the proof. ∎

3.4 Numerical solutions

First of all we present the numerical results obtained minimizing the discrete one dimensional functional

$$F_\varepsilon(u)=\frac{1}{\varepsilon}\int_\Omega\alpha\frac{\pi}{2}\arctan\left(\varepsilon\frac{\beta}{\alpha}\frac{2}{\pi}|u'|^2\right)dx+\int_\Omega|u-g|^2dx\,, \tag{3.11}$$

using the standard quasi-Newton algorithm described in Section 3.2. Let $x_0 = 0$, $x_n = 1$ and $x_0 < x_1 < \ldots < x_n$ a uniform subdivision with size ε. We consider F_ε defined over the space $V_\varepsilon(I)$ of affine finite elements on the subdivision. The minimization of F_ε presents in general some difficulties due to the presence of local minima. For instance, let g be the characteristic function of the interval $(0, 0.5)$, and let $\beta = 0.01$ and $\alpha = 0.1$. With this choice the contrast threshold is $h_0 = \sqrt{2}$ and by (3.6) the minimum is the function u_c defined in (3.3). The solutions obtained numerically by the quasi-Newton algorithm, taking g as initial point, (see Figure 3.1), are the function u_c if $\varepsilon = 0.02$ and a solution u_s which behaves likes u_d if $\varepsilon = 0.004$.

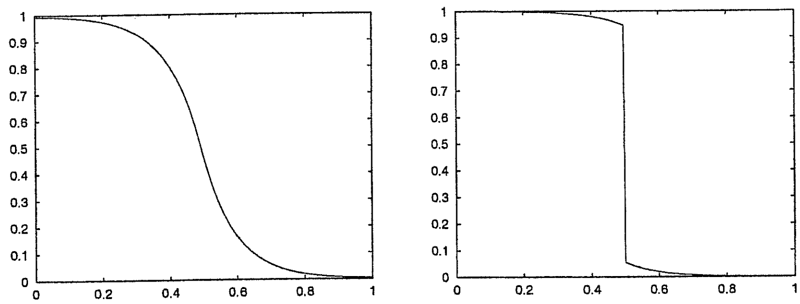


Figure 3.1: Solutions with $\varepsilon = 0.02$ and $\varepsilon = 0.004$ respectively.

The same phenomenon occurs also changing the coefficients (see Figure 3.2). For $\varepsilon = 0.008$

- if $\alpha = 4\cdot 10^{-2}$ and $\beta = 16\cdot 10^{-4}$ then $h_0 = \sqrt{2}$ and the numerical solution is correct
- if $\alpha = 4\cdot 10^{-1}$ and $\beta = 16\cdot 10^{-2}$ then $h_0 = \sqrt{2}$ but the solution u_s is is again similar to u_d.

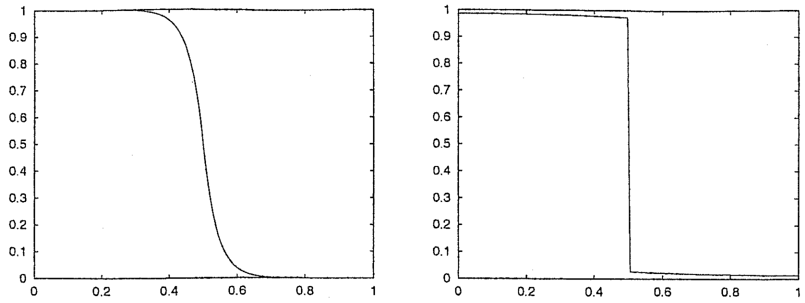


Figure 3.2: Solutions with different coefficients and same contrast threshold.

Note that in both the cases the eigenvalues of the Hessian computed in u_s are all positive, so that the numerical solutions turn out to be relative minima for the discrete functional (corresponding to the relative minimum u_d of the limit functional).

Following a heuristic reasoning, we want to show for which values of ε, α, β the discrete functional presents these local minima. Let $f(x) = \min\{\beta x, \alpha\}$ and consider that the functional

$$\frac{1}{\varepsilon}\int_I f(\varepsilon|u'|^2)dx + \int_I |u-g|^2 dx\,,$$

which behaves like (3.11) and Γ-converges to the one dimensional Mumford-Shah functional as well. Let $I_i = (x_i, x_{i+1})$ be an interval of the subdivision and let $[u_i] = |u(x_i) - u(x_{i+1})|$ be the variation of u in I_i, then

$$\frac{1}{\varepsilon}\int_{I_i} f(\varepsilon|u'|^2)dx = \begin{cases} \beta|u'|^2 & \text{if } [u_i]^2 \leq \alpha\varepsilon/\beta \\ \alpha & \text{if } [u_i]^2 > \alpha\varepsilon/\beta\,. \end{cases}$$

As a consequence, for the discrete functional the variation $[u_i]$ is considered a jump when $[u_i]^2 > \alpha\varepsilon/\beta$, thus F_ε introduces a local discrete threshold, given by

$$h_\varepsilon = \sqrt{\frac{\alpha\varepsilon}{\beta}}\,,$$

Note that h_ε depends on ε and in general it is different from h_0. In particular when $h_\varepsilon \ll h_0 \Leftrightarrow \varepsilon \ll 2\sqrt{\beta}$ then F_ε has a local minimum corresponding to the discontinuous function u_d. Indeed let I_k be the interval containing the jump of g and let $u = g$ then, being $h_\varepsilon \ll [u_k]$, we have

$$\frac{1}{\varepsilon}\int_{I_k} f(\varepsilon|u'|^2)dx + \int_{I_k} |u-g|^2 dx = \frac{1}{\varepsilon}\int_{I_k} f(\varepsilon|u'|^2)dx = \alpha\,.$$

Moreover for every interval $I_i \neq I_k$ we have

$$\frac{1}{\varepsilon}\int_{I_i} f(\varepsilon|u'|^2)dx + \int_{I_i} |u-g|^2 dx = 0\,.$$

Thus for every function v such that $h_\varepsilon < [v_k]$ we have

$$\frac{1}{\varepsilon}\int_{I_k} f(\varepsilon|v'|^2)dx + \int_{I_k} |v-g|^2 dx = \alpha + \int_{I_k} |v-g|^2 dx > \alpha\,.$$

This proves that g is a relative minimum, at least respect to small perturbations in the variation.

3.4.1 Minimizing algorithms

The analysis developed in the previous section shows that for certain choices of the coefficients, that often occur in the applications, the functional F_ε has a relative minimum near the initial point and clearly the algorithm cannot find the global minimizer. An easy solution should be the choice of another initial point but this idea does not work well in general, in particular for real life images. In order to overcome the difficulty we developed two minimizing algorithms that should be able to avoid this kind of relative minima. Roughly speaking the idea is to force the condition $\varepsilon = 2\sqrt{\beta}$ which gives the equality $h_\varepsilon = h_0$ between the local discrete threshold and the contrast threshold. This can be done either changing the coefficient β, as in the first algorithm, or changing the size parameter ε, as in second.

Algorithm 1. Let the coefficients $\beta_1 < \beta_2 < \ldots < \beta_n$ be defined as

$$\beta_1 = \varepsilon^2/4 \qquad \beta_j = 2\beta_{j-1} \text{ for } j < n \qquad \beta_n = \beta$$

and the corresponding α_j as $\alpha_j = \alpha\sqrt{\beta_j}/\sqrt{\beta}$. Then we consider the functionals

$$F^j(u) = \beta_j \int_I |u'|^2 dx + \alpha_j \#(S_u) + \int_I |u-g|^2 dx$$

and their discretizations

$$F^j_\varepsilon(u) = \frac{1}{\varepsilon}\int_I \alpha_j \frac{2}{\pi} \arctan\left(\varepsilon \frac{\beta_j}{\alpha_j}\frac{\pi}{2}|u'|^2\right) dx + \int_I |u-g|^2 dx \ . \tag{3.12}$$

The algorithm gives a sequence of functions u_j for $j = 1, \ldots, n$ defined by induction in the following way:

- u_1 is the minimum of F^1_ε computed by the quasi-Newton algorithm with guess point g
- u_{j+1} is the minimum of F^{j+1}_ε computed with initial point u_j.

In this way, at the first minimization we have $\varepsilon = 2\sqrt{\beta_1}$, thus the quasi-Newton algorithm should find the global minimum of F^1_ε. In particular the discontinuities are determined according to the original contrast threshold, because

$$h_j = \sqrt{\frac{2\alpha_j}{\sqrt{\beta_j}}} = \sqrt{\frac{2\alpha}{\sqrt{\beta}}}\,.$$

For the same reason in the successive iterations the behavior of the discontinuities is preserved and in the region of continuity the solutions get closer to the minimum of F_ε as α_j and β_j tend to α and β.

Remark 3.4.1 *This procedure turns out to be similar to the one proposed in [15] and known as Graduated Non-Convexity algorithm (in short GNC). In [15] this minimizing strategy was applied to a similar functional but on a sort of relaxation argument. Reformulated in our case the GNC algorithm consists in minimizing a sequence of functionals F_j with the same choice of β_j but taking $\alpha_j \equiv \alpha$. In the sequel we will call GNCL this "graduated non convexity like" algorithm.*

Algorithm 2. Let $\varepsilon_1 > \ldots > \varepsilon_n$ be defined in such a way that

$$\varepsilon_1 = 2\sqrt{\beta} \qquad \varepsilon_j = \varepsilon_{j-1}/2 \text{ for } j < n \qquad \varepsilon_n = \varepsilon\,.$$

In this case the coefficients are not changed and the corresponding discrete functionals are

$$F_{\varepsilon_j}(u) = \frac{1}{\varepsilon_j}\int_I \alpha\frac{2}{\pi}\arctan\left(\varepsilon_j\frac{\beta}{\alpha}\frac{\pi}{2}|u'|^2\right)dx + \int_\Omega |u-g|^2 dx\,. \tag{3.13}$$

The procedure is similar to the previous one and gives a sequence of functions u_j defined in the following way:

- u_1 is the minimum of F_{ε_1} computed by the quasi-Newton algorithm with guess point g
- u_j is the minimum of F_{ε_j} computed with initial point u_{j-1}.

As before, for $j = 1$ we expect the solution u_1 to be the global minimum because $\varepsilon_1 = 2\sqrt{\beta}$, then for ε_j decreasing the solution is improved with details "living" at smaller scales. Note that, being $\varepsilon_j < \varepsilon$, the dimension of the finite element spaces $V_{\varepsilon_j}(I)$ is lower than the dimension of $V_\varepsilon(I)$. Thus the implementation is very fast because the number of unknown is considerably reduced. Note that for $\varepsilon_j \neq \varepsilon$ we must define a new function g_{ε_j}. The easiest idea is to re-interpolate g_ε but for noisy data and in particular for real life images a better choice is an average or a median filtering of g_ε. We will call this algorithm *multi-scale algorithm* (in short MS).

In Figure 3.3 are reported the numerical solution obtained by the previous algorithms for the model problem in the case $\varepsilon = 0.004$, $\alpha = 0.1$ and $\beta = 0.001$. They both represent the correct solution u_c.

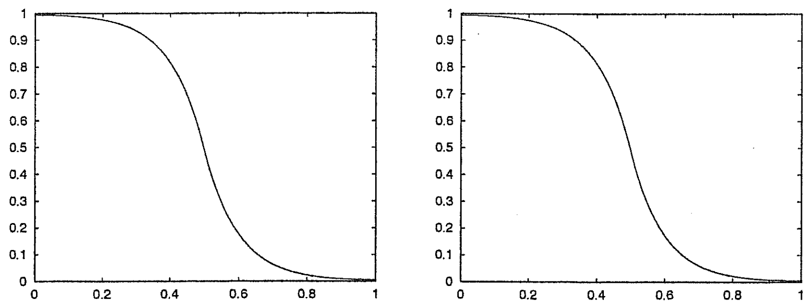


Figure 3.3: Numerical solutions with GNCL and MS.

Remark 3.4.2 *As explained in [15] the solution of the Mumford-Shah functional has a characteristic scale $\lambda = \sqrt{\beta}$, which heuristically gives an upper bound to the distance of interacting points.*

We remark that in the GNCL algorithm we have at the first step $\lambda_1 = \sqrt{\beta_1} = \varepsilon/2$ which means that short distance interactions are taken into accounts. Then, for j increasing, also the range of the interactions increases until the original one is reached.

On the contrary, the MS algorithm begins with the largest interaction on a scale $\varepsilon_1 > \varepsilon$ and successively improves the solution at smaller scales.

In the sequel this observation will be confirmed by the numerical solutions.

3.4.2 Detection of single discontinuities

In this section we want to test our algorithms as regards precision in the detection of discontinuities. Many numerical experiments have been done in this direction, here it is reported the one that better represents the general behavior. Let $\varepsilon = 0.004$, $\alpha = 0.015$ and $\beta = 0.01$. Let $h \in (0,1)$ and

$$g(x) = \begin{cases} h & \text{if } x \leq 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

For this choice of α and β the contrast threshold is $h_0 = 0.547$, hence the exact solution of the Mumford-Shah functional should have a jump when $h > 0.547$. We want to see for which values of h the numerical solutions present a discontinuity: GNCL finds a discontinuity if $h \geq 0.659$ and MS if $h \geq 0.734$.

The difference from the expected threshold is quite big but the reason is partially due to the discretization, indeed the absolute minimum of F_ε for $h \leq 0.680$ is given by the continuous solution. Moreover for ε smaller the precision of the first method is improved, on the contrary this is not possible for the second because the solution depends strongly on the ones obtained for bigger values of ε.

3.4.3 Model problem for interacting discontinuities

The minimizing procedures were built to improve the solutions in the model case of a single discontinuity, nevertheless the minima of the Mumford-Shah functional presents an important feature related to the effect of interacting discontinuities. Indeed if the datum g presents more than one jump in a region of width smaller than $\sqrt{\beta}$ then the minimum should behave as if the datum had only a jump, given by the sum of two.

Let us consider a function g such as

$$g(x) = \begin{cases} 0.4 & \text{if } x \leq 0.55 \\ 0 & \text{if } 0.55 < x < 0.6 \\ 0.7 & \text{if } x \geq 0.6 \end{cases}$$

and let $\alpha = 0.015$ and $\beta = 0.01$.

The solutions in Figure 3.4 show that only the multi-scale method can reproduce the correct behavior while GNCL seems to consider the discontinuities as if they were isolated. This difference is explained by Remark 3.4.2: when an interaction occurs the solution depends essentially on large scales which are sufficiently taken into account only in the second algorithm when ε large.

3.4.4 A model problem with noise

An important property of the Mumford-Shah functional, which is very useful in image segmentation, is its cleaning effect on noisy data. Consider a datum g with noise r given by

$$g(x) = \begin{cases} 0.7 + r(x) & \text{if } x \leq 0.5 \\ r(x) & \text{if } x > 0.5 \end{cases}.$$

and take $\alpha = 0.01$, $\beta = 0.01$ and $\varepsilon = 0.004$. The numerical solutions are reported in Figure 3.5 and 3.6.

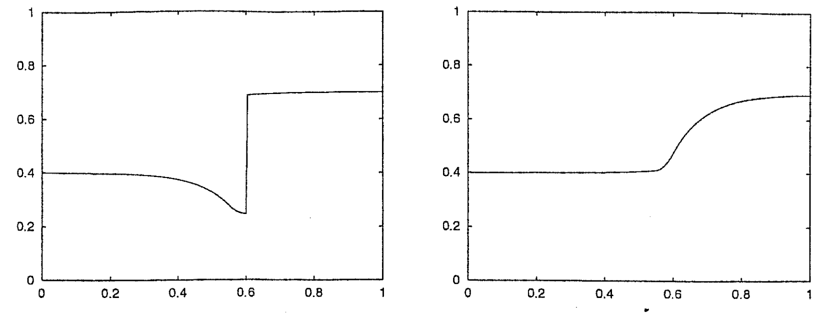


Figure 3.4: Solutions for interacting discontinuities with $\varepsilon = 0.004$.

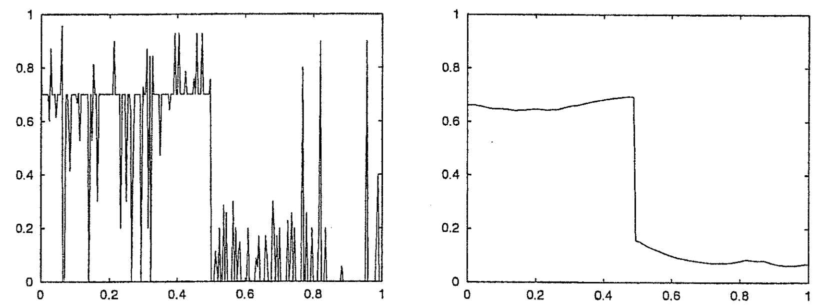


Figure 3.5: The original datum and the solution using MS.

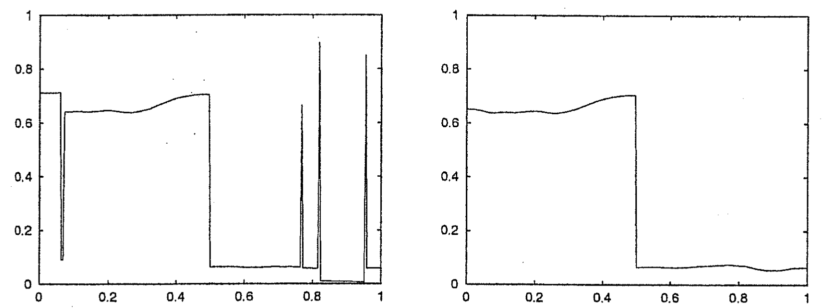


Figure 3.6: Numerical solutions with GNCL and its regularization.

It is clear that the first method cannot eliminate all the noise where it is concentrated in a peak. This phenomenon seems to be strange, nevertheless it can be explained by the same heuristic reasoning used to show the existence of local minima. For $h_\varepsilon \ll h_0$ let I_k and I_{k+1} be the subintervals containing the support of a peak in the datum. For every function v such that $h_\varepsilon < [v_k]$ and $h_\varepsilon < [v_{k+1}]$ we have

$$\frac{1}{\varepsilon}\int_{I_k \cup I_{k+1}} f(\varepsilon|v'|^2)dx + \int_{I_k \cup I_{k+1}} |v-g|^2 dx = 2\alpha + \int_{I_k \cup I_{k+1}} |v-g|^2 dx$$

which is a quadratic functional with minimum in g. To improve the solution, eliminating all the noise, it is possible to perform a minimization of F_ε restricted to $I_k \cup I_{k+1}$ replacing the value of u in the peak with the value in a neighboring vertex.

3.5 Numerical results for real images

In the sequel we will not repeat the detailed analysis developed in the previous Section. All the considerations are still valid, with obvious changes due to higher dimension. We suppose that the domain is the square $(0,1) \times (0,1)$ and that the vertices of the triangulations coincides with the pixels of the image (so we have $\varepsilon = 1/256 \simeq 0.004$). Moreover the function g, which represents the gray level, is normalized to take values in the interval $[0,1]$. The left image is segmented with coefficients $\alpha = 2 \cdot 10^{-4}$ and $\beta = 2.4 \cdot 10^{-4}$ and the right one with and $\alpha = 1.4 \cdot 10^{-4}$ and $\beta = 2.4 \cdot 10^{-4}$.

In both the images the segmentation obtained with the GNCL algorithm is more detailed and finds more contours. As shown in the one dimensional framework, some of them are not correct because, the interactions between discontinuities are not properly taken into account. Nevertheless MS is affected by a sort of *step like* effect on contours which is caused by the passage from lower to higher resolutions.

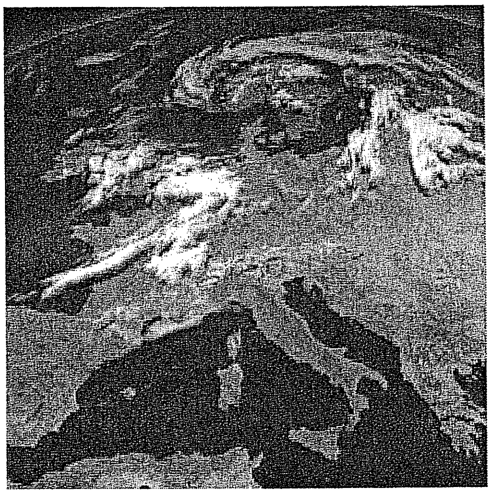


Figure 3.7: Original images.

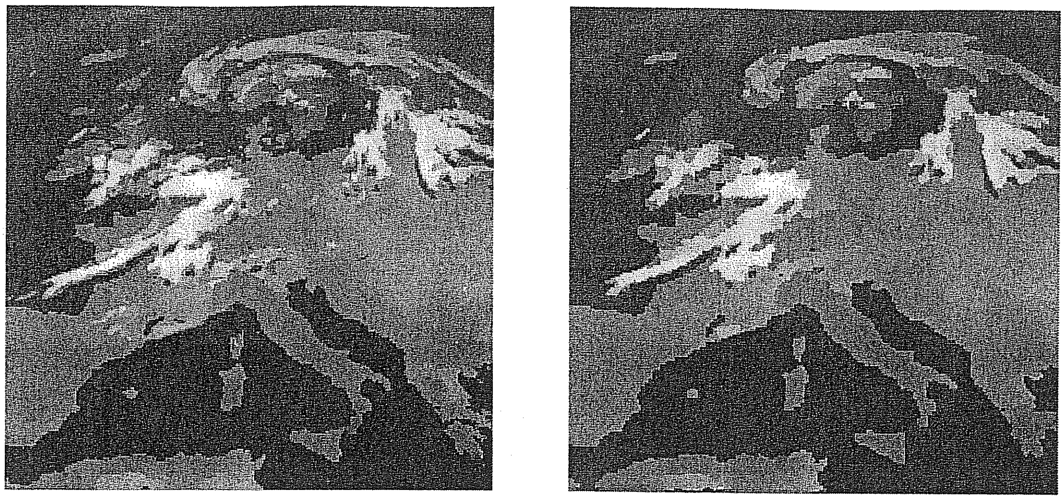


Figure 3.8: Segmentations with GNCL and MS respectively.

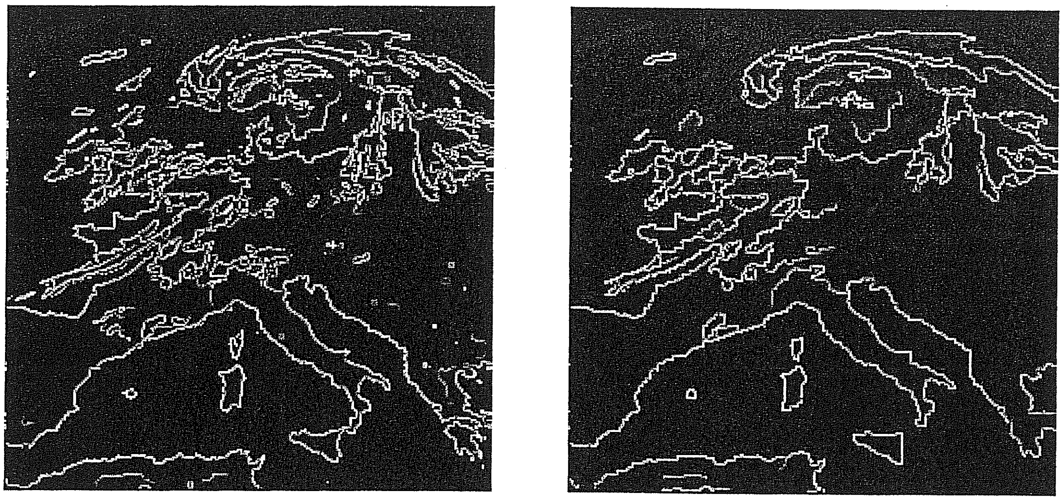


Figure 3.9: Detected contours with GNCL and MS respectively.

Figure 3.10: Segmentations with GNCL and MS respectively.

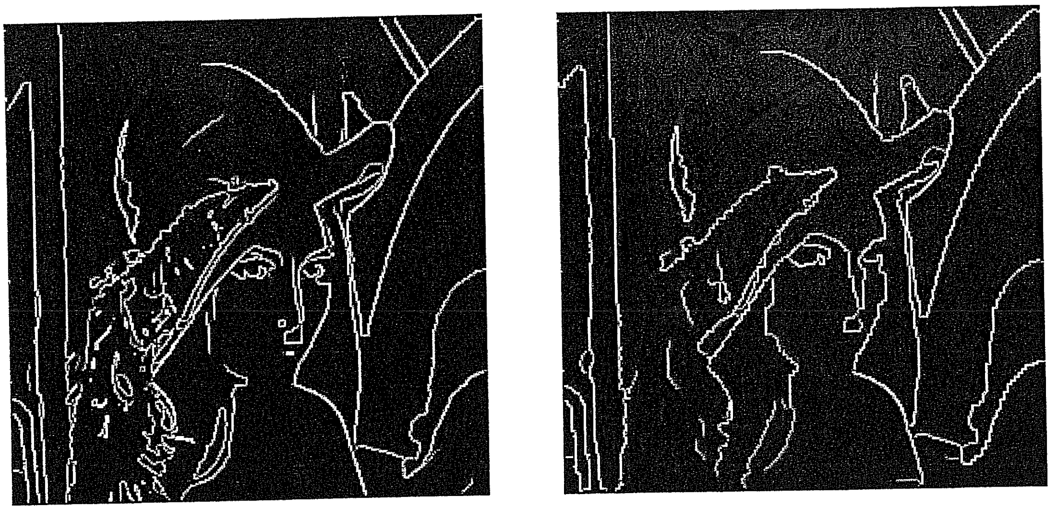


Figure 3.11: Detected contours with GNCL and MS respectively.

Chapter 4

On the relationship between Mumford-Shah functional and Perona-Malik anisotropic diffusion

4.1 Notations and statement of the main result

Given a vector $\tau \in \mathbf{R}^2$ let

$$\mathbf{Z}^2_\tau = \{x \in \mathbf{R}^2 : x = m\tau + n\tau^\perp \text{ for } (m,n) \in \mathbf{Z}^2\}, \tag{4.1}$$

$$C_\tau = \{x \in \mathbf{Z}^2 : x = s\tau + r\tau^\perp \text{ for } (s,r) \in [0,1) \times [0,1)\}. \tag{4.2}$$

For every open subset $A \subseteq \mathbf{R}^2$ we denote

$$l^1(\mathbf{Z}^2_\tau \cap A) := \left\{ v : \mathbf{Z}^2_\tau \cap A \to \mathbf{R} \text{ such that } \sum_{x \in \mathbf{Z}^2_\tau \cap A} |v(x)| < +\infty \right\};$$

in the following every function $v \in l^1(\mathbf{Z}^2_\tau \cap A)$ will be identified with the function $\tilde{v} \in L^1(A)$ which takes the constant value $v(x)$ in the square $x + C_\tau$ if $x \in \mathbf{Z}^2_\tau \cap A$, and zero otherwise. So, having in mind this identification, given a sequence $v_\varepsilon \in l^1(\mathbf{Z}^2_{\tau_\varepsilon} \cap A)$ and a function $v \in L^1(A)$, we will often write, with a slight abuse of notation, $v_\varepsilon \to v$ instead of $\tilde{v}_\varepsilon \to v$ in $L^1(A)$. Given a vector τ we will denote $\hat{\tau} := \frac{\tau}{|\tau|}$.

Let $\Omega \subset \mathbf{R}^2$ be a bounded open domain with Lipschitz boundary and for every $\varepsilon > 0$ consider the following functional

$$F_\varepsilon(u) := \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon\mathbf{Z}^2} \sum_{\substack{\xi \in \mathbf{Z}^2 \\ x + \varepsilon\xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log\left(1 + a_\varepsilon |\xi| \frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon^2 |\xi|^2}\right) \rho(\xi) \tag{4.3}$$

if $u \in l^1(\varepsilon\mathbf{Z}^2 \cap \Omega)$, and $F_\varepsilon(u) := +\infty$ otherwise in $L^1(\Omega)$, where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and $\rho : \mathbf{Z}^2 \to [0, +\infty)$ satisfies

$$\sum_{\xi \in \mathbf{Z}^2} \rho(\xi) < +\infty \qquad \text{and} \qquad \rho(\xi) = \rho(\xi^\perp) \ \forall \xi \in \mathbf{Z}^2. \tag{4.4}$$

In this chapter we will prove the following theorem.

Theorem 4.1.1 *The functionals F_ε Γ-converge (as $\varepsilon \to 0$) with respect to the L^1-norm to the anisotropic Mumford-Shah functional F given by*

$$F(u) := \begin{cases} c_\rho \int_\Omega |\nabla u|^2\, dx + \int_{S_u} \phi(\nu_u)\, d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \qquad \text{and} \qquad \phi(\nu) := \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|. \tag{4.5}$$

Moreover, every sequence (u_ε) satisfying $\sup_\varepsilon (F_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty) < +\infty$ is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.

The proof of the theorem will be split in the next sections.

4.2 Estimate from below of the Γ-limit for $N = 1$

In this section we study the one-dimensional version of the functionals defined above. Given a bounded open subset $I \subset \mathbf{R}$ we define

$$I_\varepsilon := \{x \in I \cap \varepsilon\mathbf{Z} : \ x + \varepsilon \in I\},$$

and, for every $u : I \cap \varepsilon\mathbf{Z} \to \mathbf{R}$, we define

$$\mathcal{F}_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon} \log\left(1 + a_\varepsilon \frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2}\right),$$

where, as above, $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$. As usual we will identify every function $u : I \cap \varepsilon\mathbf{Z} \to \mathbf{R}$ (briefly $u \in l^1(I \cap \varepsilon\mathbf{Z})$) with the piecewise constant function u of $L^1(I)$ given by

$$u(x) := \begin{cases} u\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right) & \text{if } \varepsilon \left[\frac{x}{\varepsilon}\right] \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Our aim is to prove the following proposition.

Proposition 4.2.1 *Let $u_\varepsilon \in l^1(I \cap \varepsilon\mathbf{Z})$ such that $u_\varepsilon \to u$ in $L^1(I)$ as $\varepsilon \to 0^+$ and $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Then $u \in SBV(I)$ and*

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2\, dx + \mathcal{H}^0(S_u).$$

We postpone the proof of the proposition after proving some useful lemmas.

Lemma 4.2.2 *Let $p(\varepsilon) > 0$ be such that $\lim_{\varepsilon \to 0^+} p(\varepsilon) = 0$ and*

$$\lim_{\varepsilon \to 0^+} \left(p(\varepsilon) \log \frac{1}{\varepsilon} - \log\log \frac{1}{\varepsilon}\right) = +\infty,$$

and set $c_\varepsilon := \varepsilon^{p(\varepsilon)}$. Then the following properties hold true:

a) $\lim_{\varepsilon \to 0^+} c_\varepsilon = 0;$

b) $\lim_{\varepsilon\to 0^+} \dfrac{\log c_\varepsilon}{\log\frac{1}{\varepsilon}} = 0;$

c) $\lim_{\varepsilon\to 0^+} \dfrac{c_\varepsilon^2}{\varepsilon} \log\dfrac{1}{\varepsilon} = +\infty;$

d) $\lim_{\varepsilon\to 0^+} c_\varepsilon \log\dfrac{1}{\varepsilon} = 0;$

e) $\lim_{\varepsilon\to 0^+} \dfrac{\log\left(1 + a_\varepsilon\frac{c_\varepsilon^2}{\varepsilon^2}\right)}{\log\frac{1}{\varepsilon}} = 1.$

Proof. . Properties *a), b), c)*, and *d)* follow immediately. Let us check only *e)*. Recalling the definition of a_ε, we have

$$\lim_{\varepsilon\to 0^+} \frac{\log\left(1 + a_\varepsilon\frac{c_\varepsilon^2}{\varepsilon^2}\right)}{\log\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0^+} \frac{\log\left(1 + \frac{\log\frac{1}{\varepsilon}c_\varepsilon^2}{\varepsilon}\right)}{\log\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0^+} \frac{\log\left(\frac{\log\frac{1}{\varepsilon}c_\varepsilon^2}{\varepsilon}\right)}{\log\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0^+} \left(1 + \frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}} + \frac{2\log c_\varepsilon}{\log\frac{1}{\varepsilon}}\right) = 1,$$

where the second equality follows from *c)* while the last from *b)*. ■

Lemma 4.2.3 *Let $u_\varepsilon \in l^1(I \cap \varepsilon\mathbf{Z})$ be such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < K < +\infty$ and let c_ε be as in the previous lemma. Set $b_\varepsilon^2 := \sqrt{c_\varepsilon \log\frac{1}{\varepsilon}}$ and consider the following set*

$$D_\varepsilon := \left\{x \in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} > \frac{b_\varepsilon}{\sqrt{a_\varepsilon}}\right\}.$$

Then

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^0(D_\varepsilon)c_\varepsilon = 0.$$

Proof. By our assumptions and recalling the definition of a_ε, we have

$$K > \frac{\varepsilon}{a_\varepsilon}\sum_{x\in D_\varepsilon} \log\left(1 + a_\varepsilon\frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2}\right) \geq \frac{\log(1+b_\varepsilon^2)}{\log\frac{1}{\varepsilon}}\mathcal{H}^0(D_\varepsilon)$$

so that, substituting the expression of b_ε^2, if ε is small enough, from *d)* of Lemma 4.2.2 we get

$$\mathcal{H}^0(D_\varepsilon)c_\varepsilon \leq K\frac{c_\varepsilon\log\frac{1}{\varepsilon}}{\log\left(1+\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}\right)} \leq K'\frac{c_\varepsilon\log\frac{1}{\varepsilon}}{\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}} = K'\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}.$$

Again *d)* implies now the thesis. ■

Lemma 4.2.4 *Let $v_\varepsilon \in SBV(I)$ such that $\lim_{\varepsilon\to 0^+}\|v_\varepsilon'\|_\infty\sqrt{a_\varepsilon} = 0$. Then, for every $\delta > 0$ there exists $\overline{\varepsilon} > 0$ such that*

$$\frac{1}{a_\varepsilon}\int_I \log(1 + a_\varepsilon|v_\varepsilon'|^2)\,dx \geq (1-\delta)\int_I |v_\varepsilon'|^2\,dx,$$

for every $\varepsilon \leq \overline{\varepsilon}$.

Proof. Fix $\delta > 0$ and note that there exists $T_\delta > 0$ such that

$$\log(1 + a_\varepsilon t^2) \geq (1-\delta) a_\varepsilon t^2 \qquad \forall t \in \left[0, \frac{T_\delta}{\sqrt{a_\varepsilon}}\right];$$

by the assumptions, if ε is small enough we have $\|v'_\varepsilon\|_\infty \leq T_\delta/\sqrt{a_\varepsilon}$ and therefore

$$\frac{1}{a_\varepsilon}\int_I \log(1 + a_\varepsilon |v'_\varepsilon|^2)\, dx \geq (1-\delta)\int_I |v'_\varepsilon|^2\, dx.$$

■

We are now in a position to prove Proposition 4.2.1.

Proof of Proposition 4.2.1. Let b_ε and c_ε be as in Lemma 4.2.3 and set

$$B_\varepsilon(u_\varepsilon) := \left\{x \in I_\varepsilon : \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} < \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} < \frac{c_\varepsilon}{\varepsilon}\right\} = \{x^1_\varepsilon, x^2_\varepsilon, \ldots, x^{m_\varepsilon}_\varepsilon\},$$

where $x^1_\varepsilon < x^2_\varepsilon < \ldots < x^m_\varepsilon$ and $m_\varepsilon := \mathcal{H}^0(B_\varepsilon)$. Now we want to replace the sequence u_ε with a new one $\tilde{u}_\varepsilon$, still converging to u, such that $B_\varepsilon(\tilde{u}_\varepsilon)$ is empty and $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. Setting $v^0_\varepsilon = u_\varepsilon$ and for $k = 1, ..., m_\varepsilon - 1$ we define by induction the functions

$$v^{k+1}_\varepsilon(t) := \begin{cases} v^k_\varepsilon(t) & \text{for } t < x^{k+1}_\varepsilon, \\ v^k_\varepsilon(t) - [v^k_\varepsilon(x^{k+1}_\varepsilon) - v^k_\varepsilon(x^k_\varepsilon)] & \text{for } t \geq x^{k+1}_\varepsilon, \end{cases}$$

and finally we set $\tilde{u}_\varepsilon := v^{m_\varepsilon}_\varepsilon$ (see Figure 4.1).

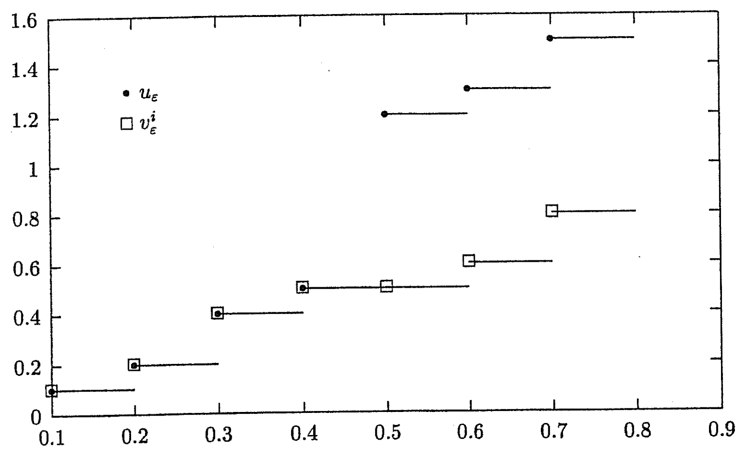


Figure 4.1: The construction of $\tilde{u}_\varepsilon$.

First of all, using the fact that for every $\varepsilon > 0$ and for every $i = 1, \ldots, m_\varepsilon$ we get

$$\int_I |v^i_\varepsilon - v^{i-1}_\varepsilon|\, dx \leq |v^{i-1}_\varepsilon(x^i_\varepsilon) - v^{i-1}_\varepsilon(x^{i-1}_\varepsilon)|\,|I| = |u_\varepsilon(x^i_\varepsilon) - u_\varepsilon(x^{i-1}_\varepsilon)|\,|I| \leq c_\varepsilon\,|I|,$$

then we can estimate

$$\int_I |\tilde{u}_\varepsilon - u_\varepsilon|\, dx \leq \sum_{i=1}^{m_\varepsilon}\int_I |v^i_\varepsilon - v^{i-1}_\varepsilon|\, dx \leq \sum_{i=1}^{m_\varepsilon} c_\varepsilon|I| \leq \mathcal{H}^0(B_\varepsilon(u_\varepsilon))c_\varepsilon|I| \leq \mathcal{H}^0(D_\varepsilon)c_\varepsilon|I|.$$

Therefore, by Lemma 4.2.3, we get $\tilde{u}_\varepsilon \to u$ in $L^1(I)$. Moreover, by construction, we clearly have that $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. We set

$$I_\varepsilon^\flat := \left\{ x \in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} \leq \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} \right\}$$

$$I_\varepsilon^\sharp := \left\{ x \in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} \geq \frac{c_\varepsilon}{\varepsilon} \right\}$$

and we call w_ε the function belonging to $SBV(I)$ defined by

$$w_\varepsilon(x) := \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right) + \frac{\tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon\right) - \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right)}{\varepsilon}\left(x - \varepsilon\left[\frac{x}{\varepsilon}\right]\right) \qquad \text{if } \varepsilon\left[\tfrac{x}{\varepsilon}\right] \in I_\varepsilon^\flat,$$

$$w_\varepsilon(x) := \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right) \qquad \text{if } \varepsilon\left[\tfrac{x}{\varepsilon}\right] \in I_\varepsilon^\sharp \text{ or } \varepsilon\left[\tfrac{x}{\varepsilon}\right] + \varepsilon \notin (\varepsilon\mathbf{Z} \cap I),$$

and $w_\varepsilon(x) := 0$ otherwise. Roughly speaking w_ε coincides with the affine interpolation of $\tilde{u}_\varepsilon$ in the intervals $(y, y+\varepsilon)$ with $y \in I_\varepsilon^\flat$ while takes the constant value $\tilde{u}_\varepsilon(y)$ in the intervals $(y, y+\varepsilon)$ with $y \in I_\varepsilon^\sharp$; it is clear that

$$w_\varepsilon \to u \text{ in } L^1 \qquad \sqrt{a_\varepsilon}\|w_\varepsilon'\|_\infty \leq b_\varepsilon \to 0 \qquad \text{and} \qquad S_{w_\varepsilon} = I_\varepsilon^\sharp + \varepsilon. \tag{4.6}$$

Now we can estimate

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^\flat} \log\left(1 + a_\varepsilon \frac{|\tilde{u}_\varepsilon(x+\varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2}\right) \tag{4.7}$$

$$+ \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^\sharp} \log\left(1 + a_\varepsilon \frac{|\tilde{u}_\varepsilon(x+\varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2}\right)$$

$$\geq \frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |w_\varepsilon'|^2)\, dx + \mathcal{H}^0(I_\varepsilon^\sharp) \frac{\varepsilon}{a_\varepsilon} \log\left(1 + a_\varepsilon \frac{c_\varepsilon^2}{\varepsilon^2}\right). \tag{4.8}$$

Fix $\delta \in (0,1)$; recalling (4.6) and the definition of a_ε, by Lemma 4.2.4 and by *e)* of Lemma 4.2.2, from (4.8) we deduce the existence of $\bar{\varepsilon}$ such that

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq (1-\delta)\left(\int_I |w_\varepsilon'|^2\, dx + \mathcal{H}^0(S_{w_\varepsilon})\right);$$

by the Ambrosio Semicontinuity Theorem we therefore obtain that $u \in SBV(I)$ and

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq (1-\delta)\left(\int_I |u'|^2\, dx + \mathcal{H}^0(S_u)\right),$$

which concludes the proof of the proposition since δ is arbitrary. ■

We conclude this section with a remark that will be useful in the sequel.

Remark 4.2.5 *Fix $t \in \mathbf{R}$ and for $u \in l^1(\varepsilon\mathbf{Z} \cap I)$ define*

$$\mathcal{F}_\varepsilon^t(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^t} \log\left(1 + a_\varepsilon \frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2}\right),$$

where

$$I_\varepsilon^t := \{x \in I \cap \varepsilon(t + \mathbf{Z}) : x + \varepsilon \in I\};$$

then we have that Proposition 4.2.1 is still valid with $\mathcal{F}_\varepsilon^t$ instead of $\mathcal{F}_\varepsilon$ (without changes in the proof).

4.3 Estimate from below of the Γ-limit for $N = 2$

Lemma 4.3.1 *Let $u_\varepsilon \in l^1(\varepsilon\mathbf{Z}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\mathbf{R}^2)$. For y, $\xi \in \mathbf{Z}^2$, let $v^y_{\varepsilon,\xi} \in l^1(\varepsilon(y + \mathbf{Z}^2_\xi))$ be defined as $v^y_{\varepsilon,\xi}(x) := u_\varepsilon(x)$ for every $x \in \varepsilon(y + \mathbf{Z}^2_\xi)$. Then $v^y_{\varepsilon,\xi} \to u$ in $L^1(\mathbf{R}^2)$.*

Proof. We call Q_ξ the unit cell of the lattice $\mathbf{Z}^2_\xi$, i.e.

$$Q_\xi := C_\xi \cap \mathbf{Z}^2 = \{\tau^1, \dots, \tau^k\}, \tag{4.9}$$

where C_ξ is the set defined in (4.2). For $j = 1, ..., k$ we set $u^j_\varepsilon(x) = u_\varepsilon(x - \varepsilon\tau^j)$. Since

$$\begin{aligned}\int_{\mathbf{R}^2} |u^j_\varepsilon(x) - u(x)|\, dx &\le \int_{\mathbf{R}^2} |u_\varepsilon(x - \varepsilon\tau^j) - u(x - \varepsilon\tau^j)|\, dx + \int_{\mathbf{R}^2} |u(x - \varepsilon\tau^j) - u(x)|\, dx \\ &= \int_{\mathbf{R}^2} |u_\varepsilon(x) - u(x)|\, dx + \int_{\mathbf{R}^2} |u(x - \varepsilon\tau^j) - u(x)|\, dx\,,\end{aligned}$$

we have that $u^j_\varepsilon \to u$ in $L^1(I)$ as $\varepsilon \to 0^+$, for every $j \in \{1, ..., k\}$; therefore, up to passing to a subsequence, we can suppose that

- there exists $N \subset \mathbf{R}^2$ with $\mathcal{L}^2(N) = 0$ such that $u^j_\varepsilon \to u$ pointwise in $\mathbf{R}^2 \setminus N$ for $j = 1, ..., k$;
- $|u^j_\varepsilon| \le g^j$ almost everywhere where g^j is a L^1 function, for $j = 1, ..., k$.

Since for every $x \in \mathbf{R}^2 \setminus N$ there exists $j \in \{1, ..., k\}$ such that $v^y_{\varepsilon,\xi}(x) = u^j_\varepsilon(x)$, we get $v^y_{\varepsilon,\xi} \to u$ pointwise in $\mathbf{R}^2 \setminus N$; moreover $|v^y_{\varepsilon,\xi}| \le g_1 + ... + g_k$ and therefore, by the Dominated Convergence Theorem, $v^y_{\varepsilon,\xi} \to u$ in L^1. As the same argument can be repeated for every subsequence, the lemma is proved. ∎

We will need also the following lemma, whose proof is elementary (see Figure 4.2).

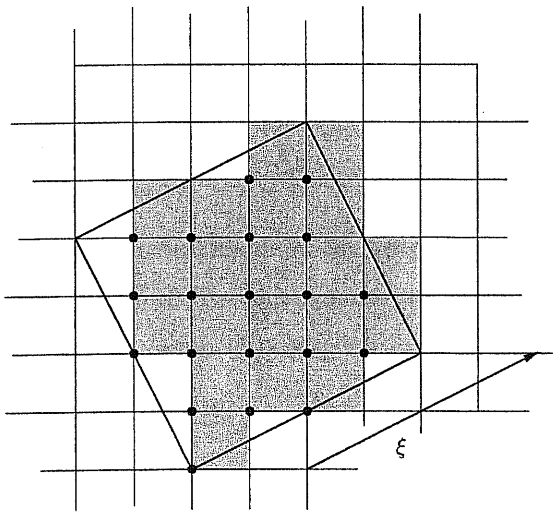


Figure 4.2: The set Q_ξ and the shaded region.

Lemma 4.3.2 *Let Q_ξ the unit cell of the lattice $\mathbf{Z}^2_\xi$ as defined in (4.9). Then $\mathcal{H}^0(Q_\xi) = |\xi|^2$.*

Proof. We refer to Figure 4.2. We associate every point $x \in Q_\xi$ with the square $x + [0,1] \times [0,1]$. The area of the shaded region, which is the union of such squares, coincides with the cardinality of Q_ξ and it is clear from the Figure that it is equal to the area of the set C_ξ (defined in (4.2)). ∎

Before starting the proof of the Γ-liminf inequality it is convenient to rewrite the functional F_ε in a suitable way. After observing that

$$\bigcup_{y \in Q_\xi} (\mathbf{Z}^2_\xi + y) = \mathbf{Z}^2,$$

we can write, for every $u \in l^1(\varepsilon\mathbf{Z}^2 \cap \Omega)$,

$$\begin{aligned} F_\varepsilon(u) &= \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbf{Z}^2} \sum_{\substack{\xi \in \mathbf{Z}^2 \\ x + \varepsilon\xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log\left(1 + a_\varepsilon |\xi| \frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon^2 |\xi|^2}\right) \rho(\xi) \\ &= \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} G^y_{\varepsilon,\xi}(u), \end{aligned} \tag{4.10}$$

where

$$G^y_{\varepsilon,\xi}(u) := \varepsilon^2 \sum_{\substack{x \in \varepsilon(y + \mathbf{Z}^2_\xi) \cap \Omega \\ x + \varepsilon\xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log\left(1 + a_\varepsilon |\xi| \frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon^2 |\xi|^2}\right).$$

Let $u_\varepsilon \to u$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Taking u_ε and u equal to zero outside $(\varepsilon\mathbf{Z}^2 \cap \Omega)$, we can suppose that $u_\varepsilon \in l^1(\varepsilon\mathbf{Z}^2)$, $u \in L^1(\mathbf{R}^2)$, and $u_\varepsilon \to u$ in $L^1(\mathbf{R}^2)$. If we are able to prove that $u \in GSBV(\Omega)$ and

$$\liminf_{\varepsilon \to 0+} G^y_{\varepsilon,\xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, d\mathcal{H}^1\right), \tag{4.11}$$

for every $\xi \in \mathbf{Z}^2$ and every $y \in Q_\xi$, then, by (4.10), Lemma 4.3.2, (4.4) and (4.5), we have

$$\begin{aligned} \liminf_{\varepsilon \to 0+} F_\varepsilon(u_\varepsilon) &\geq \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \liminf_{\varepsilon \to 0+} G^y_{\varepsilon,\xi}(u_\varepsilon) \\ &\geq \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, d\mathcal{H}^1\right) \\ &= \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) \left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, d\mathcal{H}^1\right) \\ &= \int_\Omega \frac{1}{2} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi)(|\nabla u \cdot \hat{\xi}|^2 + |\nabla u \cdot \hat{\xi^\perp}|^2) \, dx + \int_{S_u} \sum_{\xi \in \mathbf{Z}^2} \rho(\xi) |\nu_u \cdot \hat{\xi}| \, d\mathcal{H}^1 \\ &= c_\rho \int_\Omega |\nabla u|^2 \, dx + \int_{S_u} \phi(\nu_u) \, d\mathcal{H}^1. \end{aligned} \tag{4.12}$$

Following the notation introduced in the Preliminaries, we denote the hyperplane orthogonal to ξ by Π_ξ and Ω_ξ the projection of Ω on Π_ξ. For every $w \in \Pi_\xi$ we set $\Omega_\xi^w := \{t \in \mathbf{R} : w + t\hat{\xi} \in \Omega\}$ and, given a function, we define $f_\xi^w(t) := f(w + t\hat{\xi})$. We also define $O_{\varepsilon,\xi} := \Omega_\xi \cap \varepsilon\mathbf{Z}^2$ (see Figure 4.3) and for every $x \in \mathbf{R}^2$

$$O_{\varepsilon,\xi}^x := \{y \in x + \varepsilon\xi\mathbf{Z} : y + \varepsilon\xi \in \Omega\}.$$

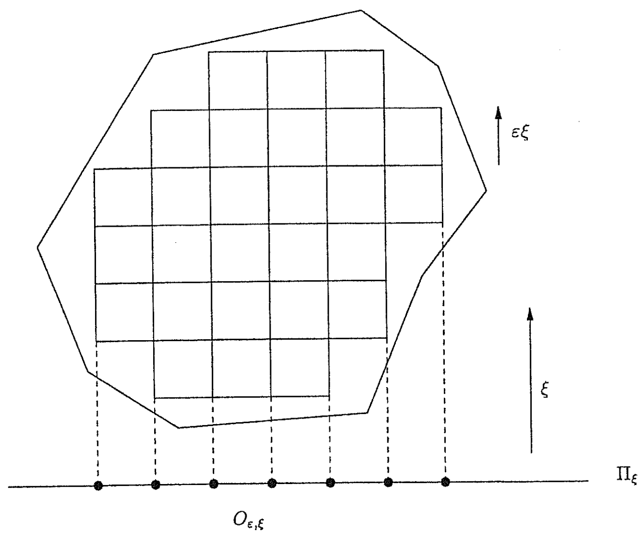


Figure 4.3: The sets Π_ξ and $O_{\varepsilon,\xi}$.

Note that we can write

$$\begin{aligned} G_{\varepsilon,\xi}^y(u_\varepsilon) &= \varepsilon^2 \sum_{w\in O_{\varepsilon,\xi}} \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \frac{1}{a_\varepsilon|\xi|} \log\left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right) \\ &= \frac{1}{|\xi|^2} \sum_{w\in O_{\varepsilon,\xi}} \varepsilon|\xi| \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \log\left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right) \\ &= \frac{1}{|\xi|^2} \int_{\Omega_\xi} \left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \log\left(1 + a_\varepsilon|\xi| \frac{|v_{\varepsilon,\xi}^y(x+\varepsilon\xi) - v_{\varepsilon,\xi}^y(x)|^2}{\varepsilon^2|\xi|^2}\right)\right] d\mathcal{H}^1(w), \quad (4.13) \end{aligned}$$

where $v_{\varepsilon,\xi}^y$ is the sequence defined in Lemma 4.3.1. Set $\eta := \varepsilon|\xi|$, $w_{\eta,\xi}^y := v_{\varepsilon,\xi}^y$, $z := y/|\xi|$ and observe that

$$\lim_{\varepsilon\to0^+} \frac{a_\eta}{a_\varepsilon|\xi|} = 1. \quad (4.14)$$

Fix $\delta \in (0,1)$; by (4.13), by Fatou's Lemma, and by (4.14), we obtain

$$\liminf_{\varepsilon\to0^+} G_{\varepsilon,\xi}^y(u_\varepsilon) \geq \frac{1}{|\xi|^2} \int_{\Omega_\xi} \liminf_{\varepsilon\to0^+} \left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \log\left(1 + a_\varepsilon|\xi| \frac{|v_{\varepsilon,\xi}^y(x+\varepsilon\xi) - v_{\varepsilon,\xi}^y(x)|^2}{\varepsilon^2|\xi|^2}\right)\right] d\mathcal{H}^1(w)$$

$$\geq \frac{1}{|\xi|^2}\int_{\Omega_\xi} \liminf_{\eta\to 0^+}\left[\frac{\eta}{a_\eta}\sum_{O^{w+\eta z}_{\eta,\hat\xi}} \log\left(1+\delta a_\eta \frac{|w^y_{\eta,\xi}(x+\eta\hat\xi)-w^y_{\eta,\xi}(x)|^2}{\eta^2}\right)\right] d\mathcal{H}^1(w)$$

$$= \frac{1}{|\xi|^2}\int_{\Omega_\xi} \liminf_{\eta\to 0^+} \mathcal{F}^t_\eta\Big((\sqrt{\delta} w^y_{\eta,\xi})^w_\xi, \Omega^w_\xi\Big)\, d\mathcal{H}^1(w)\,,$$

where $t := z\cdot\hat\xi$ and $\mathcal{F}^t_\eta$ is the functional defined in Remark 4.2.5. Since $(w^y_{\eta,\xi})^w_\xi \to u^w_\xi$ for $\mathcal{H}^1$-a.e. $w\in\Pi_\xi$, as $\eta\to 0$ (thanks to Lemma 4.3.1), by Proposition 4.2.1 and Remark 4.2.5 we deduce

$$\liminf_{\varepsilon\to 0^+} G^y_{\varepsilon,\xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2}\int_{\Omega_\xi}\left(\int_{\Omega^w_\xi} \delta|(u^w_\xi)'|^2\,dt + \mathcal{H}^0(S_{u^w_\xi})\right) d\mathcal{H}^1(w),$$

from which (4.11) follows by letting $\delta \uparrow 1$ and by applying Theorem 1.2.7.

4.4 Estimate from above of the Γ-limit

Thanks to a standard approximation argument based on the use of Theorem 1.2.15, it will be enough to prove the Γ-limsup inequality for a function $u\in\mathcal{W}(\Omega)$ whose discontinuity set consists of the union of a finite family $\{S_1,\dots,S_k\}$ of disjoint segments compactly contained in Ω. Let $\varepsilon_n\to 0$ and set, for every $u\in L^1(\Omega)$, $F''(u) := \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u)$; we aim to prove that

$$F''(u) \leq F(u). \tag{4.15}$$

We begin by assuming that

$$S_i\cap\varepsilon_n\mathbf{Z}^2=\emptyset \qquad \forall n\in\mathbf{N},\ \forall i\in\{1,..,k\}. \tag{4.16}$$

As for the proof of the Γ-liminf inequality, the thesis is achieved once we have shown that for a suitable sequence (u_n) converging to u, we have

$$\limsup_{n\to\infty} G^y_{\varepsilon_n,\xi}(u_n) \leq \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u\cdot\hat\xi|^2\,dx + \int_{S_u}|\nu_u\cdot\hat\xi|\,d\mathcal{H}^1\right) \quad \forall\xi\in\mathbf{Z}^2,\ y\in Q_\xi. \tag{4.17}$$

To simplify the notation we will prove (4.17) only for $y=0$. In the sequel, given x_1 and x_2 in $\mathbf{R}^2$, we denote by $[x_1,x_2]$ the segment joining the two points. Let us define the following sets:

$$A_n := \left\{x\in\varepsilon_n\xi\mathbf{Z}^2\cap\Omega:\ x+\varepsilon_n\xi\in\Omega,\ [x,x+\varepsilon_n\xi]\cap S_j=\emptyset \text{ for } j=1,\dots,k\right\},$$

and

$$B^j_n := \left\{x\in\varepsilon_n\mathbf{Z}^2_\xi:\ [x,x+\varepsilon_n\xi]\cap S_j\neq\emptyset\right\} \qquad j=1,\dots,k.$$

Clearly for n large enough, $B^j_n\cap B^i_n=\emptyset$ if $i\neq j$. Note now that we can write

$$G^y_{\varepsilon_n,\xi}(u_n) = \varepsilon_n^2\sum_{A_n}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)$$
$$+\sum_{j=1}^k\varepsilon_n^2\sum_{B^j_n}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)$$

$$= \underbrace{\frac{1}{|\xi|^2}\int_\Omega \frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|v^0_{n,\xi}(x+\varepsilon_n\xi)-v^0_{n,\xi}(x)|^2}{\varepsilon_n^2|\xi|^2}\right)\chi_{(A_n+\varepsilon_n C_\xi)}}_{\overset{||}{I_{n,1}}}$$

$$+\underbrace{\frac{1}{|\xi|^2}\sum_{j=1}^k \varepsilon_n^2|\xi|^2\sum_{B_n^j}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)}_{\overset{||}{I_{n,2}}},$$

where $v^0_{n,\xi}$ is the sequence defined in Lemma 4.3.1, while C_ξ is the set defined in (4.2). It is immediate to see that

$$\chi_{(A_n+\varepsilon_n C_\xi)} \to \chi_{\Omega\setminus S_u}. \tag{4.18}$$

Take $x \in \Omega\setminus S_u$ and let $y_n \in \varepsilon_n \mathbf{Z}^2_\xi$ be such that $x \in y_n+\varepsilon_n C_\xi$; by Lagrange's Theorem it turns out that

$$\begin{aligned}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|v^0_{n,\xi}(x+\varepsilon_n\xi)-v^0_{n,\xi}(x)|^2}{\varepsilon_n^2|\xi|^2}\right) &= \log\left(1+a_{\varepsilon_n}|\xi|\frac{|u(y_n+\varepsilon_n\xi)-u(y_n)|^2}{\varepsilon_n^2|\xi|^2}\right)\\ &= \log\left(1+a_{\varepsilon_n}|\xi||\nabla u(\xi_n)\cdot\hat\xi|^2\right)\\ &\le a_{\varepsilon_n}|\xi||\nabla u(\xi_n)\cdot\hat\xi|^2,\end{aligned}$$

where $\xi_n \in [y_n, y_n+\varepsilon_n\xi]$ and therefore $\xi_n \to x$. Taking into account the continuity of ∇u and recalling (4.18), we deduce that

$$\limsup_{n\to\infty} I_{n,1} \le \frac{1}{|\xi|^2}\int_\Omega |\nabla u(x)\cdot\hat\xi|^2\,dx. \tag{4.19}$$

Moreover, for every $x \in B_n^j$, we have

$$\frac{\varepsilon_n|\xi|}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right) \le \frac{\varepsilon_n}{a_{\varepsilon_n}}\log\left(1+a_{\varepsilon_n}\frac{4\|u\|^2_\infty}{\varepsilon_n^2}\right) \to 1, \tag{4.20}$$

where the last limit follows from the definition of a_{ε_n}. Denote by $l_\xi(S_j)$ the length of the projection of S_j on Π_ξ; using the fact that $l_\xi(S_j) = \int_{S_j}\nu_u\cdot\hat\xi\,d\mathcal{H}^1$, we easily obtain (see Figure 4.4 below)

$$\mathcal{H}^0(B_n^j) \le \frac{l_\xi(S_j)}{\varepsilon_n|\xi|}+1 \le \frac{1}{\varepsilon_n|\xi|}\int_{S_j}\nu_u\cdot\hat\xi\,d\mathcal{H}^1+1; \tag{4.21}$$

therefore from (4.20) and (4.21) we get

$$\limsup_{n\to\infty} I_{2,n} \le \frac{1}{|\xi|^2}\limsup_{n\to\infty}\sum_{j=1}^k \varepsilon_n|\xi|\mathcal{H}^0(B_n^j) \le \frac{1}{|\xi|^2}\sum_{j=1}^k\int_{S_j}\nu_u\cdot\hat\xi\,d\mathcal{H}^1 = \frac{1}{|\xi|^2}\int_{S_u}\nu_u\cdot\hat\xi\,d\mathcal{H}^1,$$

which, combined with (4.19), gives (4.17) and therefore (4.15).

If (4.16) is not true we can argue in the following way. We first observe that it is possible to find a sequence $(\tau_k)\subset\mathbf{R}^2$ such that $\tau_k\to 0$ and $S_u+\tau_k$ satisfy (4.16) for every

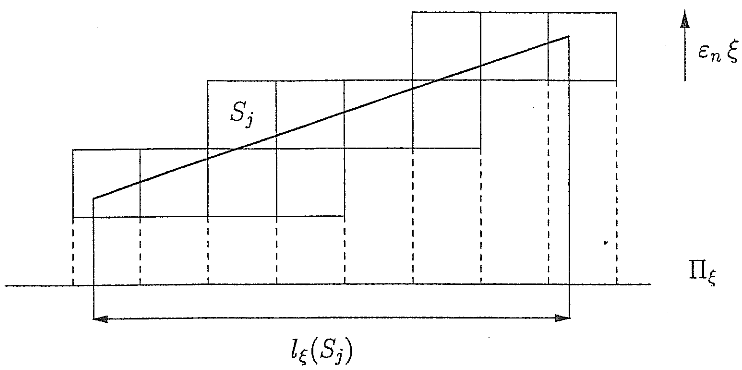


Figure 4.4: The projection of S_j on Π_ξ.

k. Let $u_k(x) := u(x - \tau_k)$, then $u_k \to u$, $S_{u_k} = S_u + \tau_k$ satisfies 4.16, and $F(u_k) \to F(u)$; using he previous step and the semicontinuity of F'', we have

$$F''(u) \leq \liminf_{k\to\infty} F''(u_k) \leq \lim_{k\to\infty} F(u_k) = F(u),$$

which concludes the proof. ■

4.5 Compactness

In this section we prove the equicoerciveness of the approximating functionals F_ε. We will use the L^1-precompactness criterion by slicing introduced by Alberti, Bouchitté & Seppecher (see Lemma 1.2.8).

Proposition 4.5.1 *Let (u_ε) be a sequence of equibounded functions such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < M < +\infty$; then (u_ε) is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.*

Proof. Clearly it is enough to prove the precompactness in L^1. Let $\{e_1, e_2\}$ be the canonical basis in $\mathbf{R}^2$. Since for $\xi = e_i$ (for $i = 1, 2$) the function $v^y_{\varepsilon,\xi}$ defined in Lemma 4.3.1 coincides with u_ε, from (4.13) we have

$$\begin{aligned} M &> \sup_\varepsilon F_\varepsilon(u_\varepsilon) \geq \sup_\varepsilon G^0_{\varepsilon,e_i} \\ &\geq \sup_\varepsilon \int_{\Omega_{e_i}} \mathcal{F}_\varepsilon\Big((u_\varepsilon)^w_{e_i}, \Omega^w_{e_i}\Big)\, d\mathcal{H}^1(w) = \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w)\, d\mathcal{H}^1(w), \end{aligned} \tag{4.22}$$

where

$$f_\varepsilon(w) := \mathcal{F}_\varepsilon\Big((u_\varepsilon)^w_{e_i}, \Omega^w_{e_i}\Big).$$

Fix $\delta \in (0,1)$ and choose $k > 0$ so large that

$$M \frac{\sup_\varepsilon \|u_\varepsilon\|_\infty}{k} \operatorname{diam}(\Omega) < \delta; \tag{4.23}$$

setting $A^k_{\varepsilon,i} := \{w \in \Omega_{e_i} : f_\varepsilon(w) > k\}$, by Chebychev Inequality and 4.22, we can estimate

$$|A^k_{\varepsilon,i}| \leq \frac{1}{k} \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w)\, d\mathcal{H}^1(w) \leq \frac{M}{k}. \tag{4.24}$$

Let $z_{\varepsilon,\delta}$ be such that $z_{\varepsilon,\delta}(x) = 0$ if the projection of x on Ω_{e_i} belongs to $A^k_{\varepsilon,i}$ and $z_{\varepsilon,\delta}(x) = u_\varepsilon(x)$ otherwise. We clearly have

$$\|z_{\varepsilon,\delta} - u_\varepsilon\|_{L^1} \leq \sup_\varepsilon \|u_\varepsilon\|_\infty |A^k_{\varepsilon,i}| \operatorname{diam}(\Omega) \leq \delta,$$

where the last inequality follows from 4.24 and 4.23. Moreover $\mathcal{F}_\varepsilon\Big((z_{\varepsilon,\delta})^w_{e_i}, \Omega^w_{e_i}\Big) \leq f_\varepsilon(w)(1 - \chi_{A^k_{\varepsilon,i}}) \leq k$ for every $w \in \Omega_{e_i}$ and therefore $((z_{\varepsilon,\delta})^w_{e_i})$, by the one dimensional result, is precompact in $L^1(\Omega^w_{e_i})$ for every $w \in \Omega_{e_i}$. Thus we have constructed a sequence which is δ-closed to (u_ε) and such that the one-dimensional sections in the e_i-direction are precompact, for $i = 1, 2$. The thesis follows from Lemma 1.2.8. ■

Chapter 5

A discontinuous finite element approach for the approximation of free discontinuity problems

5.1 Convergence result

Let $\Omega \subset \mathbf{R}^2$ be an open bounded polyhedral set. Let $f : \mathbf{R}^2 \to [0, +\infty)$ be a convex function such that for $c_1, c_2 > 0$ and for $p > 1$

$$c_1 |z|^p \leq f(z) \leq c_2 |z|^p \quad \forall z \in \mathbf{R}^2 . \tag{5.1}$$

Let $\phi : \mathbf{R}^2 \to [0, +\infty)$ be convex, positively 1-homogeneous and such that

$$\phi(\nu) = \phi(-\nu) \quad \text{and} \quad 0 < c \leq \phi(\nu) \quad \text{for } \nu \in S^1. \tag{5.2}$$

For $u \in GSBV^p(\Omega)$ we consider the functional

$$F(u) := \int_\Omega f(\nabla u)\, dx + \int_{S_u} \phi(\nu_u)\, d\mathcal{H}^1 . \tag{5.3}$$

For every $h > 0$ let $\boldsymbol{T}_h$ be a triangulation of $\mathbf{R}^2$. Let h_T and r_T be respectively the diameter and the internal radius of the triangle T. Assume that the family $\{\boldsymbol{T}_h\}$ is regular, i.e. for every $h > 0$ and for every triangle $T \in \boldsymbol{T}_h$ we have

$$h \leq h_T \leq C_1 h \qquad r_T \geq C_2 h \tag{5.4}$$

for some constants $C_1 > 0$ and $C_2 > 0$ independent of h and T. Moreover, for the sake of simplicity, we assume that Ω can be represented by the union of the triangles $T \in \boldsymbol{T}_h$ contained in Ω.

Remark 5.1.1 *For every triangle T let ζ^i and θ^i (for $i = 1, \dots, 3$) be respectively the edges and the internal angles. It is well known that the regularity condition (5.4) implies a lower and an upper bound on the amplitude of the internal angles (the Zlamal's condition). Thus there exist $0 < \theta_0 < \theta_1$, independent of h, such that $\theta_0 \leq \theta^i \leq \theta_1$. Moreover it is not difficult to see that there exists a constant $C_3 > 0$ such that for every $T \in \boldsymbol{T}_h$ and for every edge ζ^i we have $C_3 h \leq \mathcal{H}^1(\zeta^i)$.*

Let $a_h \leq \frac{1}{2}$ be a positive infinitesimal sequence. For every $h > 0$ let $\mathcal{B}_h = \{\boldsymbol{B}_h\}$ the family of the triangulations nested in $\boldsymbol{T}_h$ and defined in the following way (see Figure 5.1): every element $T \in \boldsymbol{T}_h$ is divided into four sub-elements $T_1, \ldots, T_4$, taking on every edge $[x_h^i\, x_h^j]$ of T a knot x_h' of $\boldsymbol{B}_h$ which satisfies the constraint

$$x_h' = t x_h^i + (1-t) x_h^j \qquad \text{for } a_h \leq t \leq 1 - a_h. \tag{5.5}$$

We will say that these vertices are adaptive. Finally, for every mesh $\boldsymbol{B}_h$, we denote by $\boldsymbol{E}_h$ the set of edges whose extrema are both adaptive vertices.

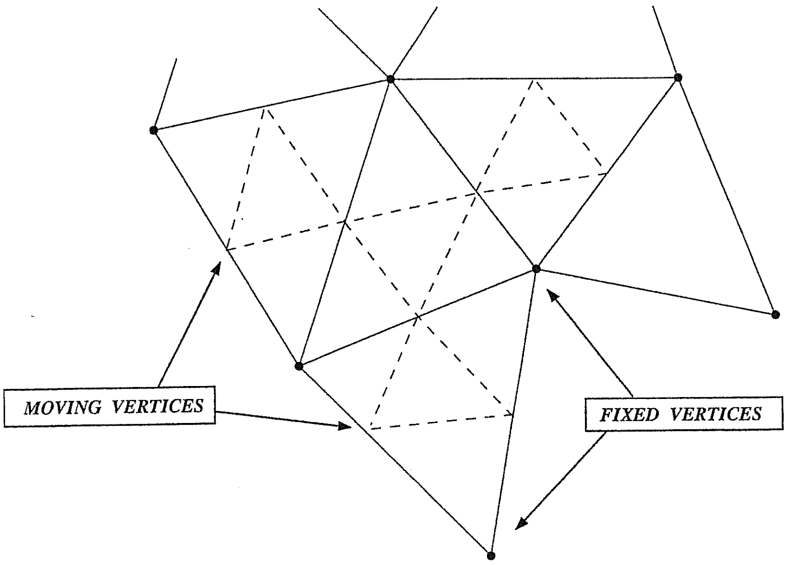


Figure 5.1: The foreground and a background triangulations $\boldsymbol{T}_h$ and $\boldsymbol{B}_h$.

For every $h > 0$ and every $\boldsymbol{B}_h \in \mathcal{B}_h$, let $W_h(\Omega, \boldsymbol{B}_h)$ be the finite element space of discontinuous functions which are affine on every sub-element $T \in \boldsymbol{B}_h$ and which can have discontinuities only along the edges belonging to $\boldsymbol{E}_h$. Then our *finite element set* $V_h(\Omega)$ will be the union of the spaces $W_h(\Omega, \boldsymbol{B}_h)$ for $\boldsymbol{B}_h \in \mathcal{B}_h$.

Theorem 5.1.2 *Let b_h be a positive diverging sequence and let $s : [0, +\infty) \to [0, +\infty)$ be an increasing continuous function such that*

$$\lim_{t \to 0^+} \frac{s(t)}{t} = 1 \qquad \lim_{t \to +\infty} s(t) = 1\,. \tag{5.6}$$

Let $0 < q < 1$, for $u \in L^1(\Omega)$ let

$$F_h(u) = \sum_{T \in \boldsymbol{B}_h} \int_T f(\nabla u)\, dx + \sum_{\zeta \in \boldsymbol{E}_h} \int_\zeta s(b_h |u^+ - u^-|^q) \phi(\nu_\zeta)\, d\mathcal{H}^1\,, \tag{5.7}$$

if $u \in W_h(\Omega, \boldsymbol{B}_h)$ and let $F_h(u) = +\infty$ otherwise. Then F_h Γ-converges, in the strong topology of $L^1(\Omega)$, to the functional given by

$$F(u) = \int_\Omega f(\nabla u)\, dx + \int_{S_u} \phi(\nu_u)\, d\mathcal{H}^1\,, \tag{5.8}$$

if $u \in GSBV^p(\Omega)$ and $F(u) = +\infty$ otherwise.

5.2 Γ-limsup inequality

Lemma 5.2.1 *Let $\{\boldsymbol{T}_h\}$ be a regular family of triangulations of $\mathbf{R}^2$. Let S be a segment with $\mathcal{H}^1(S) < +\infty$ and let $\mathbf{S}_h = \{T \in \boldsymbol{T}_h : T \cap S \neq \emptyset\}$, then there is a constant C such that, for h sufficiently small, $\#(\mathbf{S}_h) \leq C\mathcal{H}^1(S)/h$.*

Proof. By the regularity of the mesh $\boldsymbol{T}_h$, for every element T we have $|T| \geq \pi r_T^2 \geq \pi C h^2$. Let C_1 be the constant appearing in (5.4) and let $U(S, 2C_1 h) = \{x \in \Omega : d(x, S) < 2C_1 h\}$, then $U(S, 2C_1 h)$ contains all the elements $T \in \mathbf{S}_h$. Being S a segment

$$\frac{|U(S, 2C_1 h)|}{h} \leq 4C_1 \mathcal{H}^1(S) + o(h) .$$

By the lower bound on $|T|$, it follows that

$$\#(\mathbf{S}_h) \leq \frac{|U(S, 2C_1 h)|}{Ch^2} \leq C\frac{\mathcal{H}^1(S)}{h} + o(1) ,$$

which gives the required estimate for h sufficiently small. ■

Let $S \subset \Omega$ be a segment with unit normal ν and let $\mathbf{S}_h$ be as in the previous lemma. For every positive h we define a discretization of S in $\mathcal{B}_h$ by means of a piecewise linear curve S_h which can be represented by the edges $\zeta \in \boldsymbol{E}_h$. For our purpose, as it will be clear in the sequel, it is not restrictive to assume that the line $\tilde{S}$, which contains S, does not intersects any knot of the mesh $\boldsymbol{T}_h$. For every $T \in \mathbf{S}_h$ let ζ be an edge of ∂T such that $(\tilde{S} \cap \zeta) = \{p\}$. Then $p = x_h^i + t(x_h^j - x_h^i)$, where x_h^i and x_h^j are the extrema of ζ and $t \in (0,1)$. Let p_h be defined by

$$p_h = \begin{cases} x_h^i + a_h(x_h^j - x_h^i) & \text{if } 0 < t \leq a_h \\ p & \text{if } a_h \leq t \leq 1 - a_h \\ x_h^i + (1 - a_h)(x_h^j - x_h^i) & \text{if } 1 - a_h \leq t < 1. \end{cases} \tag{5.9}$$

Roughly speaking p_h is the projection of p on the segment $[x_h^i + a_h(x_h^j - x_h^i), x_h^i + (1 - a_h)(x_h^j - x_h^i]$. In this way, as $\#(\tilde{S} \cap \partial T) = 2$, for every $T \in \mathbf{S}_h$ we defined two points p_h and q_h (contained in ∂T) and then we define $S_h \cap T$ as the segment $[p_h\, q_h]$. The set S_h will be clearly given by the union of the segments $S_h \cap T$ for $T \in \mathbf{S}_h$ (it's not difficult to see that with this definition S_h is a piecewise linear curve).

Lemma 5.2.2 *Let S be a segment, S_h (for every $h > 0$) its discretization in $\mathcal{B}_h$, $T \in \mathbf{S}_h$ such that $\#(\partial T \cap S) = 2$. Then*

$$|\mathcal{H}^1(S \cap T) - \mathcal{H}^1(S_h \cap T)| = O(a_h h) , \tag{5.10}$$

$$\left| \int_{S \cap T} \phi(\nu)\, d\mathcal{H}^1 - \int_{S_h \cap T} \phi(\nu_h)\, d\mathcal{H}^1 \right| = O(a_h h) . \tag{5.11}$$

Proof. Let p and q be the extrema of $S \cap T$, while p_h and q_h are the ones of $S_h \cap T$. Note that $d(p, p_h) \leq C a_h h$, then from

$$\mathcal{H}^1(S_h \cap T) \leq d(p_h, p) + \mathcal{H}^1(S \cap T) + d(q, q_h) ,$$

$$\mathcal{H}^1(S \cap T) \leq d(p, p_h) + \mathcal{H}^1(S_h \cap T) + d(q_h, q) ,$$

it follows easily that

$$|\mathcal{H}^1(S_h \cap T) - \mathcal{H}^1(S \cap T)| \le C a_h h = O(a_h h) .$$

Let ν and ν_h be the unit normal to $S \cap T$ and $S_h \cap T$ respectively. Considering that ν_h and ν are constant in T we get

$$\Big| \int_{S_h \cap T} \phi(\nu_h)\, d\mathcal{H}^1 - \int_{S \cap T} \phi(\nu)\, d\mathcal{H}^1 \Big| \le |\phi(\nu_h)\mathcal{H}^1(S_h \cap T) - \phi(\nu)\mathcal{H}^1(S \cap T)|$$

$$\le |\phi(\nu_h)\mathcal{H}^1(S_h \cap T) - \phi(\nu_h)\mathcal{H}^1(S \cap T)| + |\phi(\nu_h)\mathcal{H}^1(S \cap T) - \phi(\nu)\mathcal{H}^1(S \cap T)|$$

$$\le |\phi(\nu_h)| C a_h h + |\phi(\nu) - \phi(\nu_h)|\mathcal{H}^1(S \cap T) . \tag{5.12}$$

Denote by τ and τ_h the tangent vectors to S and S_h. Being $\phi(\nu) = \phi(-\nu)$ it is not restrictive to suppose that in (5.11) $\arccos(\langle \nu, \nu_h \rangle) = \arccos(\langle \tau, \tau_h \rangle) \le \frac{\pi}{2}$. By the uniform continuity of ϕ it follows that

$$|\phi(\nu_h) - \phi(\nu)|\mathcal{H}^1(S \cap T) \le C|\nu - \nu_h|\mathcal{H}^1(S \cap T) \le C|\tau - \tau_h|\mathcal{H}^1(S \cap T)$$

$$\le C|\tau\mathcal{H}^1(S \cap T) - \tau_h\mathcal{H}^1(S_h \cap T)| + C|\tau_h\mathcal{H}^1(S_h \cap T) - \tau_h\mathcal{H}^1(S \cap T)|$$

$$\le C|\overline{pq} - \overline{p_h q_h}| + C|\mathcal{H}^1(S_h \cap T) - \mathcal{H}^1(S \cap T)|$$

$$\le C|\overline{pq} - \overline{p_h p} - \overline{pq} - \overline{q_h q}| + O(a_h h)$$

$$\le C|\overline{p p_h}| + C|\overline{q q_h}| + O(a_h h) = O(a_h h) . \tag{5.13}$$

Joining inequalities (5.12) and (5.13) we conclude the proof. ■

From the estimate on the cardinality $\#(\mathbf{S}_h)$ and the previous Lemma the following global error estimates come easily.

Lemma 5.2.3 *Let S be a segment and let S_h (for every $h > 0$) be its discretization in $\mathcal{B}_h$, then*

$$|\mathcal{H}^1(S) - \mathcal{H}^1(S_h)| = O(a_h + h) , \tag{5.14}$$

$$\Big| \int_S \phi(\nu)\, d\mathcal{H}^1 - \int_{S_h} \phi(\nu_h)\, d\mathcal{H}^1 \Big| = O(a_h + h) . \tag{5.15}$$

Lemma 5.2.4 *Let $L : \mathbf{R}^2 \to \mathbf{R}^2$ an affine application and let ζ_i and ζ_j be unit vectors such that $\arccos(\langle \zeta_i, \zeta_j \rangle) \ge \theta_0 > 0$. Then*

$$|DL| \le C\Big(|D_{\zeta_i}L| + |D_{\zeta_j}L|\Big) , \tag{5.16}$$

where C depends only on θ_0.

Proof. Arguing by components the proof follows easily from Remark 3.7 in [20]. ■

Lemma 5.2.5 *Let $S \subset \Omega$ be a segment. Let $\tilde{S}$ be the line which contain S and assume that it does not intersects any vertex of $\mathcal{T}_h$. Let $T \in \mathcal{T}_h$ such that $\#(\partial T \cap S) = 2$. Let S_h be the approximation of S in $\mathcal{B}_h$. Consider the triangles T_k and T'_k as in Figure 5.2. Denote by L_k the affine maps of T_k onto T'_k which keeps the natural correspondence between the vertices. Then $|DL_k| \leq C$, where C depends only on the constants C_1 and C_2 appearing in (5.4).*

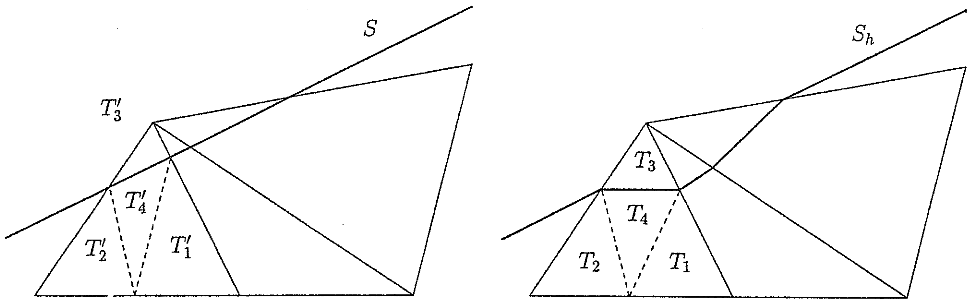


Figure 5.2: The triangles T'_k and T_k of Lemma 5.2.5.

Proof. Consider first the behavior of the triangles T_k for $k = 1, \ldots, 3$. Let $x_{k,h}$ be the vertex of T_k belonging to $\mathcal{T}_h$ and take a system of coordinates with origin in $x_{k,h}$. We denote by ζ_k^i, for $i = 1, 2$, the unit vectors corresponding to the edges of T_k containing $x_{k,h}$. Then L_k is a linear map and ζ_k^i are its eigenvectors. Hence, for $i = 1, 2$ we have

$$|D_{\zeta_k^i} L_k| = |L_k \, \zeta_k^i| = |\lambda_k^i| \, ,$$

where λ_k^i are the eigenvalues of L_k. Considering the three cases in (5.9) we get either $|\lambda_k^i| \leq 1$ or

$$|\lambda_k^i| \leq \frac{C_1 h}{C_3 h (1 - a_h)} \leq C \, .$$

Then by Lemma 5.2.4 we have $|DL_k| \leq C$, where C depends only on θ_0 and thus, by Remark 5.1.1, on the constants appearing in (5.4).

Finally, since $L_4 = L_k$ on $\partial T_4 \cap \partial T_k$ (for $k = 1, 2, 3$), it is sufficient to choose a couple of edges of ∂T_4 which form an angle greater than or equal to θ_0. Then by Lemma 5.2.4 we have again $|DL_4| \leq C$. ■

Proposition 5.2.6 *Let F and F_h as in Theorem 5.1.2. Let $u \in \mathcal{W}(\Omega) \cap L^\infty(\Omega)$, then for every $h > 0$ there exists $u_h \in V_h(\Omega)$ such that*

$$u_h \longrightarrow u \text{ in } L^1(\Omega) \, , \tag{5.17}$$

$$\limsup_{h \to 0} F_h(u_h) \leq \int_\Omega f(\nabla u) \, dx + \int_{S_u} \phi(\nu_u) \, d\mathcal{H}^1 = F(u) \, . \tag{5.18}$$

Proof. Step 1. Let $Q \subset \mathbf{R}^2$ be an open square such that $\Omega \subset\subset Q$. Since $\partial\Omega$ is piecewise linear and compact and $u \in W^{k,q}(\Omega \setminus \overline{S_u})$ for every integer k and for $1 \le q \le \infty$ by the standard extension and embedding results for Sobolev spaces, see e.g. [1], there exists $v \in SBV(Q)$ such that $v = u$ in Ω, $\overline{S_v}$ is the union of segments S_i for $i = 1, \dots, m$ and $v \in W^{2,\infty}(Q \setminus \overline{S_v})$.

Let τ_i denote the unit vectors parallel to S_i. Let $\xi \in \mathbf{S}^1$ such that $\xi \neq \pm\tau_i$ for every $i = 1, \dots, m$. Denote by K_h the set of knots of the mesh $\boldsymbol{T}_h$ and define the set $Z_h = \{x_h \in K_h : x \notin \overline{S_v}\}$. Clearly $d(Z_h, \overline{S_v}) > 0$. Then for some $0 < t_h < C_1 h$ we have $(\overline{S_v} + t\xi) \cap K_h = \emptyset$. We will denote the function $v(x - t_h\xi)$ by $v^t(x)$, the set $\overline{S_{v^t}} = (\overline{S_v} + t_h\xi)$ by S^t and the sets $(S_i + t_h\xi)$ by S_i^t .

First of all, for every $h > 0$, we define a mesh $\boldsymbol{B}_h \in \mathcal{B}_h$ which fits to the discontinuity set S^t. Let $I = \{x \in \Omega : x = S_i^t \cap S_j^t\}$. For the sake of simplicity it will be useful to suppose (without loss of generality) that the segments intersect in their extrema. Moreover we assume that h is sufficiently small in such a way that we can localize the constructions.

Consider $p_k \in I$ and assume, for simplicity of notations, that it is the intersection point of the sets S_i^t for $1 \le i \le j$. It is easy to see that it is possible to choose $r_k > 0$ in such a way that for every $h > 0$ the sets $U(S_i^t, C_1 h) \setminus B(p_k, r_k h)$ are pairwise disjoint, for $1 \le i \le j$. We will denote by $B_{k,h}$ the ball $B(p_k, r_k h)$. Let $C_{k,h}$ be the union of elements $T \in \boldsymbol{T}_h$ such that $T \cap B_{k,h} \neq \emptyset$. Consider the segments $S_i^t \setminus C_{k,h}$ and let $S_{i,h}^t$ be their approximations in $\mathcal{B}_h$. Then it is possible to find a polygonal region $P_{k,h} \subset C_{k,h}$ (see Figure 5.3) which contains p_k and such that $\partial P_{k,h}$ is representable in $\mathcal{B}_h$ and it connects the sets $S_{1,h}^t, \dots, S_{j,h}^t$.

The polygonals $\partial P_{k,h}$ and the piecewise linear curves $S_{j,h}^t$ define some vertices of $\boldsymbol{B}_h$, all the others will be given by $x_h' = \frac{1}{2}x_h^i + \frac{1}{2}x_h^j$ (if x_h' belongs to the edge $[x_h^i\, x_h^j]$).

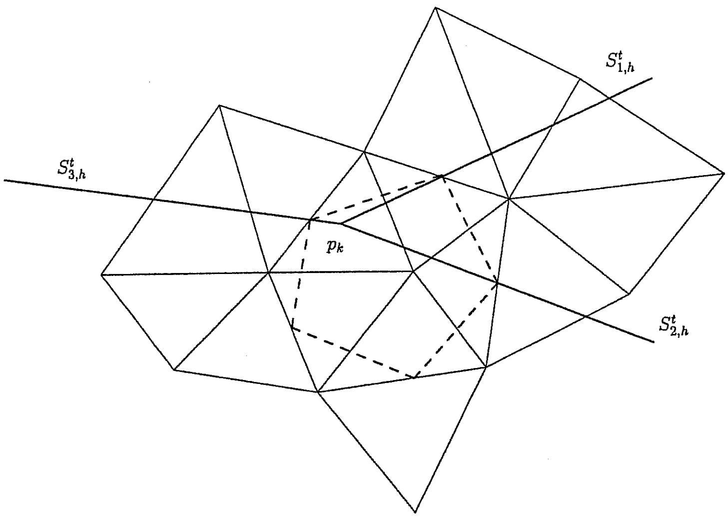


Figure 5.3: The construction around the point p_k.

Step 2. Now we can define the function u_h. Let Ω_h^s be the union of the elements $T \in \boldsymbol{T}_h$ such that $T \subset C_{k,h}$ or $T \cap S_{k,h}^t \neq \emptyset$ for some k. We denote the set $\Omega \setminus \Omega_h^s$ by Ω_h^c. Note that $|\Omega_h^s| \le Ch\mathcal{H}^1(S_v) + \#(I)Ch^2$ and thus $|\Omega_h^s| \to 0$.

We consider first the set Ω_h^c. Since v^t is continuous in Ω_h^c we can define u_h as the Lagrange interpolation of v^t in $\boldsymbol{B}_h$. Note that, by its definition, the restriction of $\boldsymbol{B}_h$ to the set Ω_h^c is regular in the sense of (5.4). Hence, by a classical result in the theory of finite elements (see for instance [21] Theorem 3.1.6), we know that there exists a constant C, independent of h, such that, for every element $T \in \boldsymbol{B}_h$ with $T \subset \Omega_h^c$, and for every $p, q \in [1, \infty]$ we have

$$\|v^t - v_h\|_{m,q,T} \leq C|T|^{\frac{1}{q}-\frac{1}{p}} h^{2-m} |v^t|_{2,p,T} .$$

In particular for $p = \infty$, $q = 1$ and $m = 0$ we have

$$\|v^t - v_h\|_{1,T} \leq Ch^4 |v^t|_{2,\infty,T} , \tag{5.19}$$

while for $p = \infty$, $q = \infty$ and $m = 1$ we have

$$\|v^t - v_h\|_{1,\infty,T} \leq Ch|v^t|_{2,\infty,T} , \tag{5.20}$$

for a suitable constant C independent of h and T.

Consider now the set Ω_h^s. Let $T \in \boldsymbol{T}_h$ such that $T \cap S_{j,h}^t \neq \emptyset$. Consider the notations of Lemma 5.2.5 and Figure 5.2. For $k = 1, \ldots, 4$, as $T_k' \subset (\Omega \setminus \overline{S^t})$ we have $v^t \in W^{2,\infty}(T_k')$. Thus we can define an auxiliary affine function $v_h : T_k' \to \mathbf{R}$ as the Lagrange interpolation of v in T_k'. It's easy to see that

$$\|\nabla v_h\|_{\infty,T_k'} \leq \|\nabla v\|_{\infty,T_k'} \leq \|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})} .$$

Then we define $u_h : T_k \to \mathbf{R}$ as $u_h(x) = v_h(L_k x)$. By Lemma 5.2.5 it follows that

$$\|\nabla u_h\|_{\infty,T_k} \leq |DL_k| \, \|\nabla v_h\|_{\infty,T_k'} \leq C\|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})} .$$

Finally let $T \subset C_{k,h}$ and consider the sub-elements $T_k \in \boldsymbol{B}_h$. If $T_k \subset P_{k,h}$ we define $u_h \equiv 0$, while if $T_k \not\subset P_{k,h}$ we define u_h as the Lagrange interpolation of v^t in T_k. As before we have

$$\|\nabla u_h\|_{\infty,T_k'} \leq \|\nabla v^t\|_{\infty,(\Omega\setminus\overline{S^t})} .$$

Step 3. Let us see that the sequence u_h verifies (5.17) and (5.18). It's clear that $v^t \to u$ strongly in $L^1(\Omega)$, moreover by (5.19) it follows that

$$\begin{aligned}
\int_\Omega |u_h - v^t| \, dx &\leq \int_{\Omega_h^s} |u_h - v^t| \, dx + \int_{\Omega_h^c} |u_h - v^t| \, dx \\
&\leq 2\,|\Omega_h^s| \, \|v^t\|_{\infty,\Omega} + \sum_{T \subset \Omega_h^c} \int_T |u_h - v^t| \, dx \\
&\leq O(h) + \sum_{T \subset \Omega_h^c} Ch^4 |v^t|_{2,\infty,T} \\
&\leq O(h) + Ch^2 |v|_{2,\infty,(\Omega\setminus\overline{S_v})} ,
\end{aligned}$$

which implies (5.17). The proof of the Γ-limsup inequality follows from

$$\limsup_{h\to 0} \int_\Omega f(\nabla u_h) \, dx = \int_\Omega f(\nabla u) \, dx , \tag{5.21}$$

$$\limsup_{h\to 0} \sum_{\zeta\in E_h} \int_\zeta s(b_h|u_h^+ - u_h^-|^q)\phi(\nu_\zeta)\, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu_u)\, d\mathcal{H}^1 . \tag{5.22}$$

Let $1_{\Omega_h^c}$ be the indicator function of Ω_h^c. By the previous step, for every $T \in \boldsymbol{B}_h$ we have $\|\nabla u_h\|_{\infty,T} \le \|\nabla v^t\|_{\infty,T} < +\infty$ and thus by (5.1) the function $f(\nabla u_h)$ is bounded from above. Since $|\Omega_h^s| \to 0$ by (5.20) it follows that $f(\nabla u_h)$ converges pointwise to $f(\nabla u)$. Thus by dominated convergence we have

$$\lim_{h\to 0} \sum_{T\in B_h} \int_T f(\nabla u_h)\, dx = \int_\Omega \lim_{h\to 0} f(\nabla u_h)\, dx = \int_\Omega f(\nabla u)\, dx . \tag{5.23}$$

Moreover, as $s(b_h|u_h^+ - u_h^-|^q) \le 1$ we have

$$\begin{aligned} \sum_{\zeta\in E_h} \int_\zeta s(b_h|u_h^+ - u_h^-|^q)\phi(\nu)\, d\mathcal{H}^1 &\le \sum_{j=1}^m \int_{S_{j,h}^t} \phi(\nu)\, d\mathcal{H}^1 + C \sum_{p_k\in I} \mathcal{H}^1(\partial P_{k,h}) \\ &\le \sum_{j=1}^m \int_{S_{j,h}} \phi(\nu_h)\, d\mathcal{H}^1 + O(h) . \end{aligned} \tag{5.24}$$

Then by Lemma 5.2.3

$$\begin{aligned} \limsup_{h\to 0} \sum_{\zeta\in E_h} \int_\zeta s(b_h|u_h^+ - u_h^-|^q)\phi(\nu_\zeta)\, d\mathcal{H}^1 &\le \sum_{j=1}^m \limsup_{h\to 0} \int_{S_{j,h}} \phi(\nu_h)\, d\mathcal{H}^1 \\ &\le \int_{S_u} \phi(\nu)\, d\mathcal{H}^1 . \end{aligned}$$

and the proof is concluded. ■

Proposition 5.2.7 *For every $u \in GSBV^p(\Omega)$ we have $F''(u) \le F(u)$.*

Proof. By Lemma 1.2.17 we have a sequence $w_k \in \mathcal{W}(\Omega)$ such that

$$w_k \longrightarrow u \text{ strongly in } L^1(\Omega) \tag{5.25}$$

$$\nabla w_k \longrightarrow \nabla u \text{ strongly in } L^p(\Omega, \mathbf{R}^2) \tag{5.26}$$

$$\limsup_{k\to 0} \int_{S_{w_k}} \phi(\nu)\, d\mathcal{H}^1 \le \int_{S_u} \phi(\nu)\, d\mathcal{H}^1 . \tag{5.27}$$

Then by (5.26) and by a standard result on Nemitskii operators we have

$$\int_\Omega f(\nabla w_k)\, dx \longrightarrow \int_\Omega f(\nabla u)\, dx. \tag{5.28}$$

By Proposition 5.2.6, the lower-semicontinuity of $F''(u)$, (5.27) and (5.28) it follows that

$$F''(u) \le \liminf_{k\to+\infty} F''(w_k) \le \limsup_{k\to+\infty} F(w_k) \le F(u) .$$

which concludes the proof. ■

5.3 Γ-liminf inequality and compactness

Proposition 5.3.1 *Let h_n be a positive sequence such that* $\lim_{n\to\infty} h_n = 0$. *Let* $u \in L^1(\Omega)$ *and* $u_n \in V_{h_n}(\Omega)$ *such that* $u_n \to u$ *in* $L^1(\Omega)$. *Then*

$$F(u) \le \liminf_{n\to\infty} F_{h_n}(u_n)\,. \tag{5.29}$$

Proof. We denote the functional F_{h_n} by F_n. Clearly it is not restrictive to assume that $\liminf_{n\to\infty} F_n(u_n) < +\infty$. Then there exists a subsequence h_j of h_n, such that $\sup_j F_j(u_j) < +\infty$ and

$$\liminf_{n\to\infty} F_n(u_n) = \lim_{j\to\infty} F_j(u_j)\,. \tag{5.30}$$

Moreover, up to a further subsequence (not relabelled), the limits

$$\lim_{j\to\infty}\int_\Omega f(\nabla u_j)\,dx \qquad \lim_{j\to\infty}\int_{S_j} s(b_j|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1$$

exist and clearly (5.30) is still valid.

Let $m \in \mathbf{N}$ and j sufficiently large in such a way that $b_j > m$. Since the function $s(t)$ is increasing, we can write

$$\int_{S_j} s(m|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1 \le \int_{S_j} s(b_j|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1 \le C\,. \tag{5.31}$$

Then by Theorem 1.2.10 we have a subsequence u_k of u_j and a function $v \in GSBV(\Omega)$ such that u_k converges to v in measure and such that

$$\int_\Omega f(\nabla v)\,dx \le \liminf_{k\to\infty}\int_\Omega f(\nabla u_k)\,dx\,,$$

$$\int_{S_v} s(m|v^+ - v^-|^q)\phi(\nu_v)\,d\mathcal{H}^1 \le \liminf_{k\to\infty}\int_{S_k} s(m|u_k^+ - u_k^-|^q)\phi(\nu)\,d\mathcal{H}^1\,.$$

As $u_k \to u$ in $L^1(\Omega)$, the functions v and u must coincide, hence

$$\int_\Omega f(\nabla u)\,dx \le \liminf_{k\to\infty}\int_\Omega f(\nabla u_k)\,dx = \lim_{j\to\infty}\int_\Omega f(\nabla u_j)\,dx\,. \tag{5.32}$$

$$\begin{aligned}\int_{S_u} s(m|u^+ - u^-|^q)\phi(\nu)\,d\mathcal{H}^1 &\le \liminf_{k\to\infty}\int_{S_k} s(m|u_k^+ - u_k^-|^q)\phi(\nu)\,d\mathcal{H}^1\\ &\le \lim_{j\to\infty}\int_{S_j} s(b_j|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1\,.\end{aligned}$$

Being

$$\lim_{m\to+\infty} s(m|a-b|^q) = \begin{cases} 1 \text{ if } a \ne b\\ 0 \text{ if } a = b,\end{cases}$$

by monotone convergence we have

$$\begin{aligned}\int_{S_u}\phi(\nu)\,d\mathcal{H}^1 &\le \sup_m \int_{S_u} s(m|u^+ - u^-|^q)\phi(\nu)\,d\mathcal{H}^1\\ &\le \lim_{j\to\infty}\int_{S_j} s(b_j|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1\end{aligned} \tag{5.33}$$

Joining inequalities (5.32) and (5.33) with (5.30) we have

$$\begin{aligned} G(u) &= \int_\Omega f(\nabla u)\,dx + \int_{S_u} \phi(\nu)\,d\mathcal{H}^1 \\ &\leq \lim_{j\to\infty} \int_\Omega f(\nabla u_j)\,dx + \lim_{j\to\infty} \int_{S_j} s(b_j|u_j^+ - u_j^-|^q)\phi(\nu)\,d\mathcal{H}^1 \\ &= \liminf_{n\to\infty} F_n(u_n) \end{aligned}$$

and we conclude the proof. ∎

Proposition 5.3.2 *Let h_n be a positive sequence such that $\lim_{n\to\infty} h_n = 0$. Let $u_n \in V_{h_n}(\Omega)$ such that*

$$F_n(u_n) + \int_\Omega |u_n|\,dx$$

is equibounded. Then u_n is strongly precompact in $L^1(\Omega)$.

Proof. The proof follows easily by (5.31) and Theorem 1.2.10. ∎

Chapter 6

A finite element approximation of the Griffith's model in fracture mechanics

6.1 Statement of the convergence result

Let Ω be an open bounded Lipschitz set in $\mathbf{R}^2$. For a positive constant k, denote by $K(\Omega)$ the set of functions $u \in L^1(\Omega, \mathbf{R}^2)$ such that $|u(x)| \leq k$ for a.e. $x \in \Omega$. The set Ω represents the reference configuration and K the constraint. Moreover let $\mathbf{M}^{2\times 2}$ be the space of 2×2 matrices with the norm $|M|^2 = \sum_{i,j} |m_{ij}|^2$ and let $\mathbf{M}^{2\times 2}_{sym}$ be the subspace of symmetric matrices.

For $i = 1, 2, 3$, let $\boldsymbol{T}^i_\varepsilon$ be the triangulations of $\mathbf{R}^2$ having the geometries represented in Figure 1.2. The corresponding finite element spaces, denoted by $V^i_\varepsilon(\Omega, \mathbf{R}^2)$, are the classical spaces of piecewise affine functions on $\boldsymbol{T}^i_\varepsilon$ restricted to Ω. Moreover given $\theta \in (0, \frac{\pi}{2})$ and an infinitesimal sequence $d_\varepsilon \geq 6\varepsilon$, let $\mathcal{T}^\theta_\varepsilon$ be the family of triangulations $\boldsymbol{T}_\varepsilon$ such that for every element the amplitude of the internal angles θ_i and the length of the edges ζ_i satisfy

$$\theta \leq \theta_i \qquad \varepsilon \leq \mathcal{H}^1(\zeta_j) \leq d_\varepsilon\,. \tag{6.1}$$

The corresponding finite element *set*, given by the union of the spaces $V^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ defined on $\boldsymbol{T}_\varepsilon$, will be denoted by $\mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$.

Let $\psi : [0, +\infty) \to [0, 1]$ be a non decreasing function such that

$$\psi(t) = o(t) \quad \text{for } t \to 0\,, \tag{6.2}$$

$$(1 - \psi(t)) = o\Big(\frac{1}{t}\Big) \quad \text{for } t \to +\infty\,, \tag{6.3}$$

and such that for t large the function $(1 - \psi(t))t$ is non-increasing. Given $M \in \mathbf{M}^{2\times 2}$ let the strain energy density be defined as

$$W(M^{sym}) = \mu|M^{sym}|^2 + \frac{\lambda}{2}|tr(M^{sym})|^2\,, \tag{6.4}$$

for $\mu > 0$ and $\lambda > 0$, and let

$$f(\varepsilon, M) = \varepsilon W(M^{sym})(1 - \psi(\varepsilon|M|^2)) + \gamma\psi(\varepsilon|M|^2)\,. \tag{6.5}$$

Using the structured triangulations $\boldsymbol{T}_\varepsilon^i$ (for $i = 1, 2, 3$), the discrete functionals $F_\varepsilon^i(u)$ are defined as

$$F_\varepsilon^i(u) = \sum_{T \in \boldsymbol{T}_\varepsilon^i} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx \tag{6.6}$$

if $v_\varepsilon \in V_\varepsilon^i(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F_\varepsilon^i(u) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. The convergence result is the following.

Theorem 6.1.1 *For every mesh $\boldsymbol{T}_\varepsilon^i$ let $\phi_i : \mathbf{R}^2 \to [0, +\infty)$ be the anisotropy function (depending only on the geometry of the triangulation) defined in Section 2.2. Let the limit functional be given by*

$$F^i(u) = \int_\Omega W(Eu) dx + \gamma \int_{J_u} \phi_i(\nu_u) d\mathcal{H}^1 \tag{6.7}$$

if $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $F^i(u) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. Then for every $u \in L^1(\Omega, \mathbf{R}^2)$ and for every sequence $v_{\varepsilon_j} \in V_{\varepsilon_j}^i(\Omega, \mathbf{R}^2)$, converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$, we have

$$F^i(u) \leq \liminf_{\varepsilon_j \to 0} F_{\varepsilon_j}^i(v_{\varepsilon_j}) .$$

Moreover for every $u \in SBV^2(\Omega, \mathbf{R}^2)$ there exists a sequence $v_{\varepsilon_j} \in V_{\varepsilon_j}(\Omega, \mathbf{R}^2)$, converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$, such that

$$F^i(u) \geq \limsup_{\varepsilon_j \to 0} F_{\varepsilon_j}^i(v_{\varepsilon_j}) .$$

For the isotropic case, we consider the functional

$$\mathcal{F}_{\varepsilon_j}^\theta(v_{\varepsilon_j}) = \sum_{T \in \boldsymbol{T}_\varepsilon^\theta} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx \tag{6.8}$$

if $v_\varepsilon \in \mathcal{V}_\varepsilon^\theta(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $\mathcal{F}_\varepsilon^\theta(v_\varepsilon) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. Then the convergence result is the following.

Theorem 6.1.2 *Let the limit functional be*

$$\mathcal{F}^\theta(u) = \int_\Omega W(Eu) dx + \gamma \sin\theta\, \mathcal{H}^1(J_u) \tag{6.9}$$

if $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and $\mathcal{F}^\theta(u) = +\infty$ otherwise in $L^1(\Omega, \mathbf{R}^2)$. Then for every $u \in L^1(\Omega, \mathbf{R}^2)$ and for every sequence $v_{\varepsilon_j} \in \mathcal{V}_{\varepsilon_j}^\theta(\Omega, \mathbf{R}^2)$, converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$, we have

$$\mathcal{F}^\theta(u) \leq \liminf_{\varepsilon_j \to 0} \mathcal{F}_{\varepsilon_j}^\theta(v_{\varepsilon_j}) .$$

Moreover for every $u \in SBV^2(\Omega, \mathbf{R}^2)$ there exists a sequence $v_{\varepsilon_j} \in \mathcal{V}_{\varepsilon_j}^\theta(\Omega, \mathbf{R}^2)$, converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$, such that

$$\mathcal{F}^\theta(u) \geq \limsup_{\varepsilon_j \to 0} \mathcal{F}_{\varepsilon_j}^\theta(v_{\varepsilon_j}) .$$

Remark 6.1.3 *The easiest choice for the function* ψ *is given by*

$$\psi(t) = \begin{cases} 0 \text{ if } t < \delta \\ 1 \text{ otherwise,} \end{cases}$$

nevertheless conditions (6.2) and (6.3) allow the use of smooth functions (such as $\frac{2}{\pi}\arctan(t^n)$ *for* $n \geq 2$*) which are much better for the numerical implementation.*

Note that the Γ-limsup inequality is not complete because the proof is based on a density argument which is not yet known for the case $u \in SBD^2(\Omega, \mathbf{R}^2) \setminus SBV^2(\Omega, \mathbf{R}^2)$. Complete convergence results can be stated as follows (but these formulations cannot ensure compactness for sequences of minima).

Remark 6.1.4 *For* $i = 1, 2, 3$ *let the discrete functionals* $F^i_\varepsilon(u)$ *be defined as*

$$F^i_\varepsilon(u) = \sum_{T \in T^i_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx$$

if $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $F^i_\varepsilon(u) = +\infty$ *otherwise in* $SBV^2(\Omega, \mathbf{R}^2)$*. The functionals* F^i_ε *Γ-converge (as* $\varepsilon \to 0$*), respect to the strong topology of* $L^1(\Omega, \mathbf{R}^2)$*, to the functional*

$$F^i(u) = \int_\Omega W(Eu) dx + \gamma \int_{J_u} \phi_i(\nu_u) d\mathcal{H}^1$$

if $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $F^i(u) = +\infty$ *otherwise in* $SBV^2(\Omega, \mathbf{R}^2)$*.*

Remark 6.1.5 *Let the isotropic limit functional be given by*

$$\mathcal{F}^\theta_\varepsilon(v_\varepsilon) = \sum_{T \in T^\theta_\varepsilon} \frac{1}{\varepsilon} \int_{T \cap \Omega} f(\varepsilon, Dv_\varepsilon) dx$$

if $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $\mathcal{F}^\theta_\varepsilon(v_\varepsilon) = +\infty$*otherwise in* $SBV^2(\Omega, \mathbf{R}^2)$*. The functionals* $\mathcal{F}^\theta_\varepsilon$ *Γ-converge (as* $\varepsilon \to 0$*), respect to the strong topology of* $L^1(\Omega, \mathbf{R}^2)$*, to the functional*

$$\mathcal{F}(u) = \int_\Omega W(Eu) dx + \gamma \sin\theta \, \mathcal{H}^1(J_u)$$

if $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ *and* $\mathcal{F}(u) = +\infty$ *otherwise in* $SBV^2(\Omega, \mathbf{R}^2)$*.*

6.2 The anisotropy functions

The anisotropy functions appearing in (6.7) have been explicitly computed and studied in Chapter 2 for the Mumford-Shah functional, here we report only the main properties useful in the sequel.

These functions are convex, positively 1-homogeneous and pair. They have an easy representation in terms of scalar products. Indeed let

$$\xi_{1,1} = (1, 0) \quad \xi_{1,2} = (\sqrt{2}/2, \sqrt{2}/2) \quad \xi_{1,3} = (0, 1),$$

$$c_{1,1} = c_{1,3} = \sqrt{2}/2 \quad c_{1,2} = 1,$$

then we can write

$$\phi_1(\nu) = \max_{1\le k\le 3} c_{1,k}|\langle \nu, \xi_{1,k}\rangle| . \tag{6.10}$$

Moreover let

$$\xi_{2,k} = \xi_{1,k} \text{ for } k=1,2,3 \quad \xi_{2,4} = (-\sqrt{2}/2, \sqrt{2}/2) \quad \xi_{2,5} = (-1,0),$$

$$c_{2,k} = c_{1,k}/2 \text{ for } k=1,2,3 \quad c_{2,4} = c_{2,2} \quad c_{2,5} = c_{2,1},$$

then

$$\phi_2(\nu) = \max_{1\le k\le 4} \{c_{2,k}|\langle \nu, \xi_{2,k}\rangle| + c_{2,k+1}|\langle \nu, \xi_{2,k+1}\rangle|\} . \tag{6.11}$$

Finally let

$$\xi_{3,1} = (-1,0) \quad \xi_{3,2} = (1/2, \sqrt{3}/2) \quad \xi_{3,3} = (-1/2, \sqrt{3}/2),$$

$$c_{3,k} = 1 \text{ for } k=1,2,3,$$

then we have

$$\phi_3(\nu) = \max_{1\le k\le 3} c_{3,k}|\langle \nu, \xi_{3,k}\rangle| . \tag{6.12}$$

On the contrary the idea for the isotropic approximation is to orient the elements along discontinuities (see Figure 1.1) in order to have a tubular neighborhood and consequently an isotropic approximation of the Hausdorff measure. This property is explained in the following lemma (for the proof see Appendix A in [20]).

Lemma 6.2.1 *Let S be the union of a finite number of disjoint segments S_m, then there exists a family of triangulations $\boldsymbol{T}_\varepsilon \in \mathcal{T}_\varepsilon^\theta$ such that*

$$\limsup_{\varepsilon\to 0} \frac{|S_\varepsilon^\theta|}{\varepsilon} = \sin\theta\, \mathcal{H}^1(S), \tag{6.13}$$

where S_ε^θ is the covering of S in $\boldsymbol{T}_\varepsilon$.

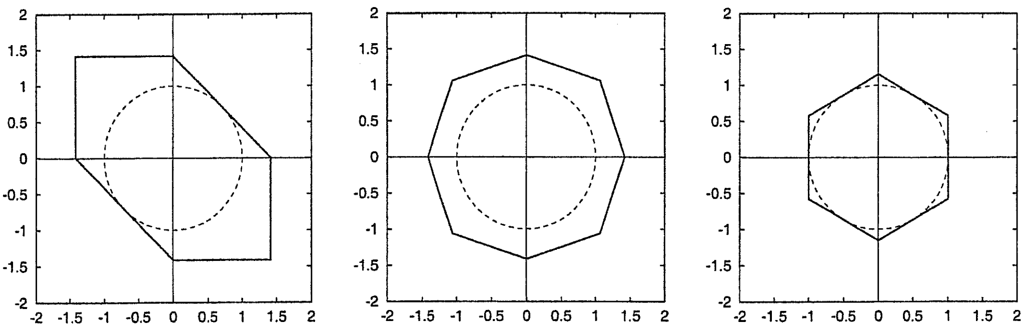


Figure 6.1: The level curves compared with the unit circle.

6.3 Γ-limsup inequality

Proposition 6.3.1 *Let $i = 1, 2, 3$ then for every $u \in SBV^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ there exists a "sequence" $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ such that*

$$v_\varepsilon \longrightarrow u \text{ strongly in } L^1(\Omega, \mathbf{R}^2) \ , \tag{6.14}$$

$$\limsup_{\varepsilon \to 0} F^i_\varepsilon(v_\varepsilon) \leq F^i(u) \,. \tag{6.15}$$

Proof. Step 1. Consider first $u \in K(\Omega) \cap \mathcal{W}(\Omega, \mathbf{R}^2)$ with compact support, then, by Definition 1.2.14, $\overline{S_u}$ is the union of the disjoint segments S_m, for $1 \leq m \leq k$. Let $S^i_{m,\varepsilon}$ be the coverings of S_m and consider ε sufficiently small in such a way that they are pairwise disjoint. Let $S^i_{u,\varepsilon}$ be their union and $\Omega^i_\varepsilon = \Omega \setminus S^i_{u,\varepsilon}$. Being $\overline{S_u} \subset S^i_{u,\varepsilon}$ by regularity we have $u \in C^\infty(\overline{\Omega^i_\varepsilon}, \mathbf{R}^2)$, thus v_ε can be defined in $\overline{\Omega^i_\varepsilon}$ as the Lagrange interpolation of u. Moreover $\overline{\Omega^i_\varepsilon}$ contains all the knots of the mesh $\boldsymbol{T}^i_\varepsilon$ because the sets $S^i_{m,\varepsilon}$ are disjoint and their interior do not contain any vertex by definition. Thus the function v_ε is defined in the whole set Ω, it clearly belongs to $V^i_\varepsilon(\Omega, \mathbf{R}^2)$ and it satisfies also the constraint $v_\varepsilon \in K(\Omega)$. By a standard result on finite elements (see [21]) there exists a constant C, which does not depend on ε and u, such that

$$\|u - v_\varepsilon\|_{m,q,T} \leq C\, |T|^{\frac{1}{q} - \frac{1}{p}} \varepsilon^{2-m}\, |u|_{2,p,T} \,. \tag{6.16}$$

Then for every triangle $T \not\subset S^i_{u,\varepsilon}$, for $m = 0$, $q = 1$ and $p = \infty$ we have

$$\int_T |v_\varepsilon - u| \, dx \leq C\, |T|\, \varepsilon^2\, |u|_{2,\infty,(\Omega \setminus \overline{S_u})} \ .$$

Considering that $\|v_\varepsilon\|_\infty \leq \|u\|_\infty$ and that $|S^i_{u,\varepsilon}| \to 0$ it follows easily that v_ε converges strongly to u in $L^1(\Omega, \mathbf{R}^2)$. Moreover for $m = 1$, $q = 2$ and $p = \infty$ for every triangle $T \not\subset S^i_{u,\varepsilon}$ we have

$$\int_T |Dv_\varepsilon - Du|^2 \, dx \leq C\, |T|\, \varepsilon^2\, |u|^2_{2,\infty,(\Omega \setminus \overline{S_u})} \ .$$

Denote by D_ε the function $Dv_\varepsilon 1_{\Omega^i_\varepsilon}$ (where $1_{\Omega^i_\varepsilon}$ is the characteristic function of Ω^i_ε). Then D_ε converges strongly to Du in $L^2(\Omega, \mathbf{M}^{2\times 2})$, indeed from $|S^i_{u,\varepsilon}| = O(\varepsilon)$, the regularity of u and the previous inequality it follows that

$$\begin{aligned} \int_\Omega |D_\varepsilon - Du|^2 \, dx &\leq \int_{\Omega^i_\varepsilon} |Dv_\varepsilon - Du|^2 \, dx + \int_{S^i_{u,\varepsilon}} |Du|^2 \, dx \\ &\leq c\varepsilon^2\, |u|^2_{2,\infty,(\Omega \setminus \overline{S_u})} + |S^i_{u,\varepsilon}|\, |u|^2_{1,\infty,(\Omega \setminus \overline{S_u})} \,. \end{aligned} \tag{6.17}$$

To prove (6.15) we must consider separately the behavior in Ω^i_ε and $S^i_{u,\varepsilon}$. We start with Ω^i_ε. By the previous inequality it follows that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \int_{\Omega^i_\varepsilon} W(Dv^{sym}_\varepsilon) \, dx &= \limsup_{\varepsilon \to 0} \int_\Omega W(D^{sym}_\varepsilon) \, dx \\ &= \int_\Omega W(Eu) \, dx \,. \end{aligned} \tag{6.18}$$

Moreover, being $(1-\psi(t)) \leq 1$, we have

$$\limsup_{\varepsilon\to 0}\int_{\Omega^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx \leq \limsup_{\varepsilon\to 0}\int_{\Omega^i_\varepsilon} W(Dv^{sym}_\varepsilon)\,dx$$
$$\leq \int_\Omega W(Eu)\,dx\,. \tag{6.19}$$

As $u \in W^{1,\infty}(\Omega\setminus\overline{S_u},\mathbf{R}^2)$ then $|Dv_\varepsilon| \leq c$ uniformly in Ω^i_ε and thus by (6.2)

$$\limsup_{\varepsilon\to 0}\int_{\Omega^i_\varepsilon}\frac{\gamma}{\varepsilon}\psi(\varepsilon|Dv_\varepsilon|^2)\,dx \leq \limsup_{\varepsilon\to 0}\int_{\Omega^i_\varepsilon}\frac{\gamma}{\varepsilon}\psi(\varepsilon c^2)\,dx = 0\,. \tag{6.20}$$

Let us consider now the behavior in $S^i_{u,\varepsilon}$. If $S\subset\Omega$ is a segment and S^i_ε is its covering, then, being $(1-\psi(t))t$ bounded in $[0,+\infty)$, it follows that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{S^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx\\
&\leq \limsup_{\varepsilon\to 0}\int_{S^i_\varepsilon}\frac{c}{\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)\varepsilon|Dv_\varepsilon|^2\,dx\\
&\leq c\limsup_{\varepsilon\to 0}\frac{|S^i_\varepsilon|}{\varepsilon} \leq c\mathcal{H}^1(S)\,.
\end{aligned}\tag{6.21}$$

Let now $\delta > 0$, let $S^\delta_u = \{x\in S_u\,:\,|u^+-u^-|\geq\delta\}$ and $(S^\delta_u)^i_\varepsilon$ be its covering. Being $u\in W^{1,\infty}(\overline{\Omega\setminus S_u},\mathbf{R}^2)$, for ε sufficiently small we have $|Dv_\varepsilon| \geq \frac{\delta}{4\varepsilon}$ for $T\subset(S^\delta_u)^i_\varepsilon$. Then, considering that $\varepsilon|Dv_\varepsilon|^2$ diverges in $(S^\delta_u)^i_\varepsilon$ and that $(1-\psi(t))t$ is decreasing for t large we deduce that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{(S^\delta_u)^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx\\
&\leq \limsup_{\varepsilon\to 0}\int_{(S^\delta_u)^i_\varepsilon}\frac{1}{\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)c\varepsilon|Dv_\varepsilon|^2\,dx\\
&\leq \limsup_{\varepsilon\to 0}\frac{|(S_u)^i_\varepsilon|}{\varepsilon}\Big(1-\psi\Big(\frac{\delta^2}{16\varepsilon}\Big)\Big)\frac{\delta^2}{16\varepsilon}c\\
&\leq c\mathcal{H}^1(S^\delta_u)\limsup_{\varepsilon\to 0}\Big(1-\psi\Big(\frac{\delta^2}{16\varepsilon}\Big)\Big)\frac{\delta^2}{16\varepsilon} = 0\,.
\end{aligned}\tag{6.22}$$

Then, for every $\delta > 0$, by inequalities (6.21) and (6.22), we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{(S_u)^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx\\
&\leq \limsup_{\varepsilon\to 0}\int_{(S^\delta_u)^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx\,+\\
&\quad+\limsup_{\varepsilon\to 0}\int_{(S_u)^i_\varepsilon\setminus(S^\delta_u)^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx\\
&\leq C\mathcal{H}^1(S_u\setminus S^\delta_u)\,,
\end{aligned}$$

which proves (for $\delta\to 0$) that

$$\limsup_{\varepsilon\to 0}\int_{(S_u)^i_\varepsilon}\Big(1-\psi(\varepsilon|Dv_\varepsilon|^2)\Big)W(Dv^{sym}_\varepsilon)\,dx = 0\,. \tag{6.23}$$

Finally, for every segment S_m, from (2.5) follows

$$\limsup_{\varepsilon\to 0}\int_{S^i_{m,\varepsilon}}\gamma\frac{\psi(\varepsilon|Dv_\varepsilon|^2)}{\varepsilon}\,dx\le\gamma\limsup_{\varepsilon\to 0}\frac{|(S_m)^i_\varepsilon|}{\varepsilon}=\gamma\int_{S_m}\phi_i(\nu)d\mathcal{H}^1\,. \tag{6.24}$$

Then by inequalities (6.19)-(6.20) and (6.23)-(6.24) it follows that

$$\limsup_{\varepsilon\to 0}F^i_\varepsilon(v_\varepsilon)\le\int_\Omega W(Eu)dx+\gamma\int_{J_u}\phi_i(\nu_u)d\mathcal{H}^1=F^i(u)\,. \tag{6.25}$$

So the Γ-limsup inequality is proved for $u\in K(\Omega)\cap\mathcal{W}(\Omega,\mathbf{R}^2)$ with compact support. Here the compact support is not strictly necessary, it just prevents technical problems near the boundary $\partial\Omega$.

Step 2. Denote by $\overline{F^i}(u)$ the Γ-limsup. By the previous step, if $u\in K(\Omega)\cap\mathcal{W}(\Omega,\mathbf{R}^2)$ with compact support then $\overline{F^i}(u)\le F^i(u)$. Consider now $u\in K(\Omega)\cap SBV^2(\Omega,\mathbf{R}^2)$ with compact support. By Proposition 1.2.15 there exists a sequence of functions $w_k\in K(\Omega)\cap\mathcal{W}(\Omega,\mathbf{R}^2)$ with compact support such that (1.32)-(1.35) hold. Then by the lower semicontinuity of $\overline{F^i}$ it follows that

$$\overline{F^i}(u)\le\liminf_{k\to+\infty}\overline{F^i}(w_k)\le\limsup_{k\to+\infty}\overline{F^i}(w_k)\le\limsup_{k\to+\infty}F^i(w_k)\le F^i(u)\,.$$

It remains to remove the hypothesis on the compact support. Let $u\in K(\Omega)\cap SBV^2(\Omega,\mathbf{R}^2)$, from Lemma 4.2 in [20] follows the existence of a function $u'\in SBV^2(\mathbf{R}^2,\mathbf{R}^2)$ with compact support such that $u'=u$ in Ω, $\|u'\|_\infty=\|u\|_\infty$ and $\mathcal{H}^1(S_{u'}\cap\partial\Omega)=\emptyset$. Let Ω' be a rectangle containing the support of u', then there exists a sequence $v_\varepsilon\in V^i_\varepsilon(\Omega,\mathbf{R}^2)$ such that $\limsup_{\varepsilon\to 0}F^i(v_\varepsilon,\Omega')\le F^i(u,\Omega')$. Considering the Γ-liminf inequality, we have

$$\begin{aligned}F^i\varepsilon(u,\Omega')\ge\limsup_{\varepsilon\to 0}F^i_\varepsilon(v_\varepsilon,\Omega')&\ge\limsup_{\varepsilon\to 0}F^i_\varepsilon(v_\varepsilon,\Omega)+\liminf_{\varepsilon\to 0}F^i_\varepsilon(v_\varepsilon,\Omega'\setminus\overline{\Omega})\\&\ge\overline{F^i}(u,\Omega)+F^i(u,\Omega'\setminus\overline{\Omega})\,.\end{aligned}$$

Then

$$\overline{F^i}(u)=\overline{F^i}(u,\Omega)\le F^i(u,\Omega')-F^i(u,\Omega'\setminus\overline{\Omega})=F^i(u,\Omega)=F^i(u)\,,$$

which completes the proof. ∎

Consider now the isotropic approximation.

Proposition 6.3.2 *For every $u\in SBV^2(\Omega,\mathbf{R}^2)\cap K(\Omega)$ there exists a "sequence" $v_\varepsilon\in\mathcal{V}^\theta_\varepsilon(\Omega)$ such that*

$$v_\varepsilon\longrightarrow u \text{ strongly in } L^1(\Omega,\mathbf{R}^2)\,, \tag{6.26}$$

$$\limsup_{\varepsilon\to 0}\mathcal{F}^\theta_\varepsilon(v_\varepsilon)\le\mathcal{F}^\theta(u)\,. \tag{6.27}$$

Proof. Step 1. Let $u\in K(\Omega)\cap\mathcal{W}(\Omega,\mathbf{R}^2)$ with compact support, and let $\overline{S_u}$ be the union of the disjoint segments S_m. By Lemma 6.2.1 there exists a mesh $T_\varepsilon\in\mathcal{T}^\theta_\varepsilon$ such that (6.13) holds. Using this mesh we can repeat the proof of Proposition 6.3.1 and we get (6.19),(6.20) and (6.23). Finally, by (6.13) we have

$$\limsup_{\varepsilon\to 0}\int_{S^i_{m,\varepsilon}}\gamma\frac{\psi(\varepsilon|Dv_\varepsilon|^2)}{\varepsilon}\,dx\le\gamma\limsup_{\varepsilon\to 0}\frac{|(S_m)^i_\varepsilon|}{\varepsilon}=\gamma\,\sin\theta\,\mathcal{H}^1(S_u)\,,$$

and then the Γ-liminf inequality is proved for $u\in K(\Omega)\cap\mathcal{W}(\Omega,\mathbf{R}^2)$ with compact support.
Step 2. See the proof of Proposition 6.3.1. ∎

6.4 Γ-liminf inequality

The proof of the Γ-liminf inequality is based on the measure theoretic argument, presented in Proposition 1.2.4, which requires the localization of the functionals.

Definition 6.4.1 *Let $A \subset \Omega$ be an open set, the localized functionals are defined as*

$$F^i_\varepsilon(v_\varepsilon, A) = \sum_{T\in T^i_\varepsilon} \frac{1}{\varepsilon}\int_{T\cap A} f(\varepsilon, Dv_\varepsilon)\,dx$$

if $v_\varepsilon \in V^i_\varepsilon(\Omega,\mathbf{R}^2)\cap K(\Omega)$ and $F^i_\varepsilon(v_\varepsilon, A) = +\infty$ otherwise in $L^1(\Omega,\mathbf{R}^2)$.

We consider first the case of the structured triangulations.

Proposition 6.4.2 *For $i = 1,2,3$, denote by $\underline{F}^i(u)$ the $\Gamma\text{-}\liminf_{\varepsilon\to 0} F^i_\varepsilon(u)$ and take $u \in L^1(\Omega,\mathbf{R}^2)$ such that $\underline{F}^i(u) < +\infty$, then $u \in K(\Omega)\cap SBD^2(\Omega,\mathbf{R}^2)$ and*

$$\int_\Omega W(Eu)\,dx + \gamma\int_{J_u}\phi_i(\nu_u)\,d\mathcal{H}^1 \le \liminf_{\varepsilon\to 0} F^i_\varepsilon(v_\varepsilon) \tag{6.28}$$

for every sequence $v_\varepsilon \in V^i_\varepsilon(\Omega,\mathbf{R}^2)$ converging strongly to u in $L^1(\Omega,\mathbf{R}^2)$.

The proof of Proposition 6.4.2 requires some preliminary lemmas on the localized functionals.

Lemma 6.4.3 *For some positive constants α,β, for every open set $A\subset\Omega$ and for every function $v_\varepsilon \in V^i_\varepsilon(\Omega,\mathbf{R}^2)$ there exists $v \in SBD^2(\Omega,\mathbf{R}^2)$ satisfying*

$$\int_{A_\varepsilon} W(Ev)\,dx + \alpha\,\mathcal{H}^1\left(J_v\cap A_\varepsilon\right) \le F^i_\varepsilon(v_\varepsilon, A)\,, \tag{6.29}$$

$$|\{x\in\Omega\ :\ v_\varepsilon(x)\neq v(x)\}| \le \beta\varepsilon F^i_\varepsilon(v_\varepsilon, A)\,, \tag{6.30}$$

where $A_\varepsilon = \{x\in A\ :\ d(x,\partial A) > \varepsilon\}$.

Proof. Let $\tau > 0$ such that

$$\sup\{\varepsilon W(M^{sym}) \text{ for } \varepsilon|M| < \tau\} \le \gamma$$

and define

$$\tilde\psi(t) = \begin{cases} 0 & \text{for } t < \tau \\ \psi(t) & \text{otherwise.}\end{cases} \tag{6.31}$$

Then the function

$$\tilde f(\varepsilon, M) = \varepsilon W(M^{sym}) = \varepsilon W(M^{sym})\Big(1-\tilde\psi(\varepsilon|M|^2)\Big) + \gamma\tilde\psi(\varepsilon|M|^2) \tag{6.32}$$

satisfies $f(\varepsilon,M)\ge\tilde f(\varepsilon,M)$. Indeed if $\varepsilon|M|^2\ge\tau$ then $\tilde\psi(\varepsilon|M|^2) = \psi(\varepsilon|M|^2)$, while for $\varepsilon|M|^2<\tau$, being $\varepsilon W(M^{sym})\le\gamma$ we have

$$\begin{aligned}\tilde f(\varepsilon, M) &= \varepsilon W(M^{sym})\Big(1-\psi(\varepsilon|M|^2)\Big) + \varepsilon W(M^{sym})\psi(\varepsilon|M|^2)\\ &\le \varepsilon W(M^{sym})\Big(1-\psi(\varepsilon|M|^2)\Big) + \gamma\psi(\varepsilon|M|^2) = f(\varepsilon, Du)\,.\end{aligned}$$

Moreover, being $\psi(t)$ non decreasing, $\tilde{\psi}(t) \geq \psi(\tau)$ for $t \geq \tau$ so that for $\varepsilon|M|^2 \geq \tau$

$$\tilde{f}(\varepsilon, M) = \varepsilon W(M^{sym})\Big(1 - \psi(\varepsilon|M|^2)\Big) + \gamma\psi(\varepsilon|M|^2) \geq \gamma\psi(\tau)\,.$$

Finally, given an open set $A \subset \Omega$ and given $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ let $\mathbf{A}^i_\varepsilon = \{T \in \boldsymbol{T}^i_\varepsilon \,:\, T \subset A\}$ and

$$\mathbf{A}^{i,\flat}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, \varepsilon|Dv_\varepsilon|^2 \leq \tau\} \qquad \mathbf{A}^{i,\sharp}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \,:\, T \notin \mathbf{A}^{i,\flat}_\varepsilon\}\,.$$

Define $A^{i,\flat}_\varepsilon \subset \Omega$ and $A^{i,\sharp}_\varepsilon \subset \Omega$ as the union of the elements belonging to $\mathbf{A}^{i,\flat}_\varepsilon$ and $\mathbf{A}^{i,\sharp}_\varepsilon$ respectively. Then it follows by the previous inequalities that

$$\begin{aligned} F^i_\varepsilon(v_\varepsilon, A) &\geq \sum_{T \in \mathbf{A}^i_\varepsilon} \frac{1}{\varepsilon} f(\varepsilon, Dv_\varepsilon)|T| \\ &\geq \sum_{T \in \mathbf{A}^{i,\flat}_\varepsilon} \frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon)|T| + \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon)|T| \\ &\geq \sum_{T \in \mathbf{A}^{i,\flat}_\varepsilon} W(Ev_\varepsilon)|T| + \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon}\gamma\psi(\tau)|T|\,. \end{aligned} \tag{6.33}$$

Let $v \in SBD^2(\Omega, \mathbf{R}^2)$ be defined as

$$v = \begin{cases} v_\varepsilon & \text{in } A^{i,\flat}_\varepsilon \\ 0 & \text{in } \Omega \setminus A^{i,\flat}_\varepsilon\,, \end{cases}$$

then from (6.33) follows

$$|\{x \in \Omega \,:\, v_\varepsilon(x) \neq v(x)\}| = |A^{i,\sharp}_\varepsilon| = \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} |T| \leq \frac{\varepsilon}{\gamma\psi(\tau)} F^i_\varepsilon(v_\varepsilon, A)\,,$$

which proves (6.30) for $\beta = 1/(\gamma\psi(\tau))$. Moreover, being $\mathcal{H}^1(\partial T) \leq c_i\varepsilon$, it easy to check that for a positive value of α we have

$$\frac{1}{\varepsilon}|T|\gamma\psi(\tau) \geq \alpha\mathcal{H}^1(\partial T)\,.$$

Thus from (6.33) follows

$$\begin{aligned} F^i_\varepsilon(v_\varepsilon, A) &\geq \sum_{T \in \mathbf{A}^{i,\flat}_\varepsilon} W(Ev)|T| + \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} \alpha\mathcal{H}^1(\partial T) \\ &\geq \int_{A_\varepsilon} W(Ev)\,dx + \alpha\mathcal{H}^1(J_v \cap A_\varepsilon)\,, \end{aligned}$$

which proves inequality (6.29). ∎

Lemma 6.4.4 *Let $i = 1, 2, 3$, for every $\delta \in (0,1)$ there are some positive constants α, β, η (depending only on δ) such that for every $v_\varepsilon \in V^i_\varepsilon(\Omega, \mathbf{R}^2)$ and for every vector $\xi_{i,k}$ (appearing in (6.10) (6.10) (6.10)) there exists $v \in SBD^2(\Omega, \mathbf{R}^2)$ satisfying*

$$\alpha\int_{A_\varepsilon} W(Ev)\,dx + \beta\mathcal{H}^1(J_v \cap A_\varepsilon) \leq F^i_\varepsilon(v_\varepsilon, A)\,, \tag{6.34}$$

$$|\{x \in \Omega \ : \ v_\varepsilon(x) \neq v(x)\}| \leq \eta\, \varepsilon\, F^i_\varepsilon(v_\varepsilon, A)\,, \tag{6.35}$$

and for $i = 1$ and $i = 3$

$$(1-\delta)\gamma \int_{J_v \cap A_\varepsilon} c_{i,k} |\langle \nu_v, \xi_{i,k}\rangle|\, d\mathcal{H}^1 \leq F^i_\varepsilon(v_\varepsilon, A)\,, \tag{6.36}$$

while for $i = 2$

$$(1-\delta)\gamma \int_{J_v \cap A} \Big(c_{i,k} |\langle \nu_v, \xi_{i,k}\rangle| + c_{i,k+1} |\langle \nu_v, \xi_{i,k+1}\rangle| \Big)\, d\mathcal{H}^1 \leq F^i_\varepsilon(v_\varepsilon, A)\,. \tag{6.37}$$

Proof. Step 1. For a given $\delta \in (0,1)$ let τ_δ such that $(1-\delta) < \psi(t)$ for $t \geq \tau_\delta$. Let $\alpha_1 < 1$ such that for $\varepsilon |M|^2 < \tau_\delta$ we have $\alpha_1 \varepsilon W(M^{sym}) < \gamma$. Moreover define

$$\tilde{\psi}(t) = \begin{cases} 0 & \text{for } t \leq \tau_\delta \\ \psi(t) & \text{otherwise,} \end{cases} \tag{6.38}$$

and

$$\tilde{f}(\varepsilon, M) = \alpha_1 \varepsilon W(M^{sym}) \Big(1 - \tilde{\psi}(\varepsilon |M|^2)\Big) + \gamma \tilde{\psi}(\varepsilon |M|^2)\,.$$

Being $f(\varepsilon, M)$ a convex combination of $\varepsilon W(M^{sym})$ and γ then $f(\varepsilon, M) \geq \min\{\varepsilon W(M^{sym}), \gamma\}$. By the choice of τ_δ and α_1 it follows that for $\varepsilon |M|^2 < \tau_\delta$ we have

$$\tilde{f}(\varepsilon, M) = \alpha_1 \varepsilon W(M^{sym}) \leq \min\{\varepsilon W(M^{sym}), \gamma\} \leq f(\varepsilon, M)\,.$$

Clearly $\tilde{f}(\varepsilon, M) \leq f(\varepsilon, M)$ also for $\varepsilon |M|^2 \geq \tau_\delta$, being $\tilde{\psi}(\varepsilon |M|^2) = \psi(\varepsilon |M|^2)$ and $\alpha_1 < 1$. As in the previous proof let $\mathbf{A}^i_\varepsilon = \{T \in \boldsymbol{T}^i_\varepsilon \ : \ T \subset A\}$. Let

$$\mathbf{A}^{i,\flat}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \ : \ \varepsilon |Dv_\varepsilon|^2 \leq \tau_\delta\}\,,$$

$$\mathbf{A}^{i,\sharp}_\varepsilon = \{T \in \mathbf{A}^i_\varepsilon \ : \ T \notin \mathbf{A}^{i,\flat}_\varepsilon\}\,,$$

and define $A^{i,\flat}_\varepsilon$ and $A^{i,\sharp}_\varepsilon$ as the union of their elements. For $T \in \mathbf{A}^{i,\flat}_\varepsilon$ we have $\tilde{\psi}(\varepsilon |Dv_\varepsilon|^2) = 0$, then

$$\frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon) \geq \alpha_1 W(Ev_\varepsilon)\,.$$

While for $T \in \mathbf{A}^{i,\sharp}_\varepsilon$ we have $\varepsilon |Dv_\varepsilon|^2 > \tau_\delta$ and then

$$\frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon) \geq \frac{1}{\varepsilon} \gamma \psi(\varepsilon |Dv_\varepsilon|^2) \geq \frac{1}{\varepsilon} \gamma (1-\delta)\,.$$

Thus, arguing as in the previous Lemma we can write

$$\begin{aligned} F^i_\varepsilon(v_\varepsilon, A) &\geq \sum_{T \in \mathbf{A}^i_\varepsilon} \frac{1}{\varepsilon} \tilde{f}(\varepsilon, Dv_\varepsilon)|T| \\ &\geq \sum_{T \in \mathbf{A}^{i,\flat}_\varepsilon} \alpha_1 W(Ev_\varepsilon)|T| + \gamma \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon} \psi(\varepsilon |Dv_\varepsilon|^2)|T| \\ &\geq \sum_{T \in \mathbf{A}^{i,\flat}_\varepsilon} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{T \in \mathbf{A}^{i,\sharp}_\varepsilon} \frac{1}{\varepsilon} |T|\,. \end{aligned} \tag{6.39}$$

Step 2. Consider the case $i = 1$ and $i = 3$. The function v is defined as $v = v_\varepsilon$ on $A_\varepsilon^{i,\flat}$ thus by (6.39) follow inequality (6.35) for $\eta = 1/\gamma(1-\delta)$ and inequality

$$\alpha_1 \int_{A_\varepsilon^{i,\flat}} W(Ev)\,dx \le F_\varepsilon^i(v, A_\varepsilon^{i,\flat})\,. \tag{6.40}$$

On $A_\varepsilon^{i,\sharp}$ we proceed element by element and component by component, defining v first on the boundary of the element and then in its interior, in such a way that, for a suitable choice of β_1 and α_2 the following inequalities hold

$$\alpha_2 \int_T W(Ev)\,dx \le \frac{|T|}{\varepsilon}\,, \tag{6.41}$$

$$S_v \cap \partial T = \emptyset\,, \tag{6.42}$$

$$\beta_1 \mathcal{H}^1(J_v \cap T) \le \frac{|T|}{\varepsilon}\,, \tag{6.43}$$

$$\int_{J_v \cap T} c_{i,k} |\langle \nu_v, \xi_{i,k}\rangle|\,d\mathcal{H}^1 \le \frac{|T|}{\varepsilon}\,. \tag{6.44}$$

Then from (6.39), (6.42) and (6.44) follows (6.36) while from (6.41)-(6.43) follows

$$\alpha_2\gamma(1-\delta) \int_{A_\varepsilon^{i,\sharp}} W(Ev)\,dx + \beta_1 \mathcal{H}^1(J_v \cap A_\varepsilon^{i,\sharp}) \le F_\varepsilon^i(u, A_\varepsilon^{i,\sharp})$$

and thus by (6.40) follows the existence of α and β such that (6.34) holds.

Let ζ_j denote the edges of ∂T and let a_j and b_j be the endpoints of ζ_j. We proceed by components. Let v_ε^n be the n$^{\text{th}}$ component of v_ε and let $\partial_j v_\varepsilon^n$ be the slope of v_ε^n along ζ_j. If $T \in \mathbf{A}_\varepsilon^{i,\flat}$ it's clear that

$$\varepsilon|\partial_j v_\varepsilon^n|^2 \le 2\tau_\delta\,. \tag{6.45}$$

Now, consider a triangle $T \in \mathbf{A}_\varepsilon^{i,\sharp}$ and an edge ζ_j then, proceeding by components, we set $v^n = v_\varepsilon^n$ on ζ_j if $\varepsilon|\partial_j v_\varepsilon^n|^2 \le 2\tau_\delta$, otherwise we set

$$v^n(ta_j + (1-t)b_j) = \begin{cases} v_\varepsilon(b_j) & \text{if } t < 1/2 \\ v_\varepsilon(a_j) & \text{if } t \ge 1/2. \end{cases} \tag{6.46}$$

In this way v^n is no longer continuous on ∂T but now its slope is uniformly controlled on $\partial T \setminus \{m_1, m_2, m_3\}$, where m_j denotes the middle point of the edges ζ_j.

Given $\xi_{i,k}$ let $J_{i,k}$ be the bold set represented in Figure 6.2 for $\boldsymbol{T}_\varepsilon^1$ and in Figure 6.3 for $\boldsymbol{T}_\varepsilon^3$. It's easy to see by a simple trigonometric argument that for every k we have

$$\int_{J_{i,k}} c_{i,k}|\langle \nu, \xi_{i,k}\rangle|\,d\mathcal{H}^1 \le \frac{|T|}{\varepsilon}\,, \tag{6.47}$$

and that for a sufficiently small parameter $\beta_1 > 0$ we get

$$\beta_1 \mathcal{H}^1(J_{i,k} \cap T) \le \frac{|T|}{\varepsilon}\,. \tag{6.48}$$

Given $\xi_{i,k}$, both the components of v are defined is such a way that the discontinuity set $J_{v^n} \subset J_{i,k}$, so that $J_v \subset J_{i,k}$ and consequently inequalities (6.43) and (6.44) hold as a consequence of (6.47) and (6.48). The construction is the same for all the choices of $\xi_{i,k}$.

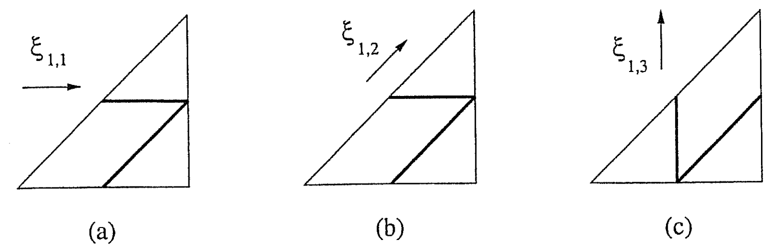


Figure 6.2: The sets of discontinuity for $\mathcal{T}_\varepsilon^1$.

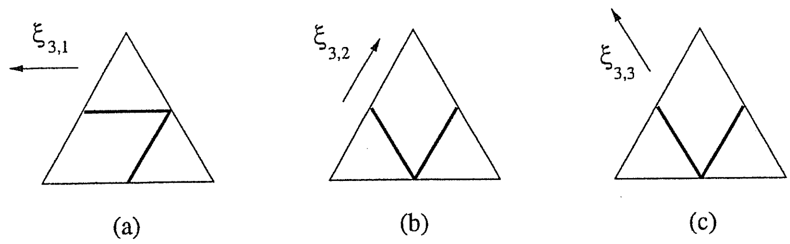


Figure 6.3: The sets of discontinuity for $\mathcal{T}_\varepsilon^3$.

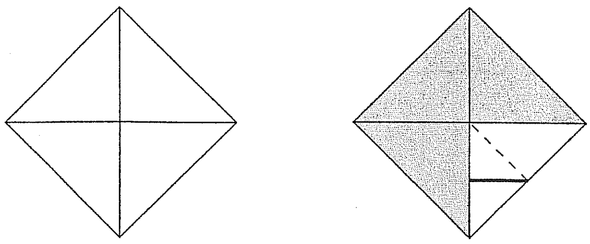
Let R_m for $m = 1, 2, 3$ be the regions of $T \setminus J_{i,k}$. On $\partial R_m \setminus J_{i,k} = \partial R \cap \partial T$ the component v^n is already assigned, by construction it is continuous because the middle points are all separated, and its slope is uniformly bounded by $\sqrt{2\tau_\delta/\varepsilon}$. As a consequence, its value on ∂T defines in each region an affine function v^n such that $|\nabla v^n|^2 \leq c/\varepsilon$ (see [43] for details), where c depends only on δ and on the mesh. Then, for a suitable constant α_2, inequality (6.41) holds and the proof is concluded.

Step 3. Consider now the case $i = 2$. The function v is defined again as $v = v_\varepsilon$ on $A_\varepsilon^{i,\flat}$ thus by (6.39) follows inequality (6.35) for $\eta = 1/\gamma(1-\delta)$ and (6.40) holds. As before we can proceed component by component but this time it is not possible to define v^n element by element because the anisotropy is the result of the orientation of all the triangles contained in the squares Q (see Figure 6.4 (a)) which represent the smallest periodic structure of the mesh. Thus let $\mathbf{Q}_\varepsilon$ be the set of squares $Q \subset A$. We partition $\mathbf{Q}_\varepsilon$ into the subsets $\mathbf{Q}_{\varepsilon,m}$ for $m = 0, \ldots, 4$, according to the number of triangles $T \subset Q$ belonging to $\mathbf{A}_\varepsilon^{i,\sharp}$. In particular (6.39) becomes

$$
\begin{aligned}
F_\varepsilon^i(v_\varepsilon, A) &\geq \sum_{T \in \mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{T \in \mathbf{A}_\varepsilon^{i,\sharp}} \frac{1}{\varepsilon}|T| \\
&\geq \sum_{T \in \mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{Q \in \mathbf{Q}_\varepsilon} \Big(\sum_{T \subset Q : T \in \mathbf{A}_\varepsilon^{i,\sharp}} \frac{|T|}{\varepsilon} \Big) \\
&\geq \sum_{T \in \mathbf{A}_\varepsilon^{i,\flat}} \alpha_1 W(Ev_\varepsilon)|T| + \gamma(1-\delta) \sum_{m=1}^{4} \Big(\sum_{Q \in \mathbf{Q}_{\varepsilon,m}} \frac{m|T|}{\varepsilon} \Big) .
\end{aligned}
\tag{6.49}
$$

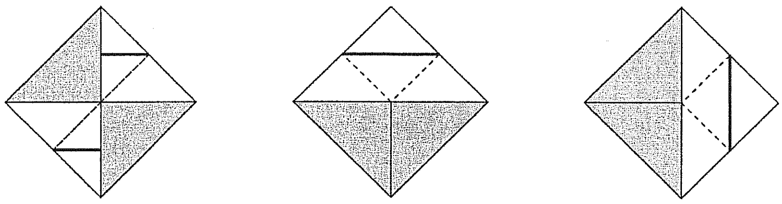
Given $k = 1, \ldots, 4$ the function v^n will be defined in such a way that for $Q \in \mathbf{Q}_{\varepsilon,m}$

$$
J_v \cap \partial Q = \emptyset\,, \tag{6.50}
$$



(a) (b)

Figure 6.4: The periodic structure of T_ε^2. Discontinuity set J_1 for the case $Q \in \mathbf{Q}_{\varepsilon,1}$.



(a) (b) (c)

Figure 6.5: Discontinuity set J_2 for the case $Q \in \mathbf{Q}_{\varepsilon,2}$

$$\alpha_2 \int_Q W(Ev)dx \leq \frac{m|T|}{\varepsilon}, \tag{6.51}$$

$$\beta_1 \mathcal{H}^1(S_v \cap Q) \leq \frac{m|T|}{\varepsilon}, \tag{6.52}$$

$$\int_{S_v \cap Q} \Big(c_{2,k} |\langle \nu_v, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu_v, \xi_{2,k+1} \rangle| \Big) d\mathcal{H}^1 \leq m \frac{|T|}{\varepsilon}. \tag{6.53}$$

If all the previous inequalities are satisfied then the proof is concluded, indeed, considering (6.49), from (6.50) and (6.53) follows (6.37) while from (6.50)-(6.52) follows the existence of α and β such that (6.34) holds.

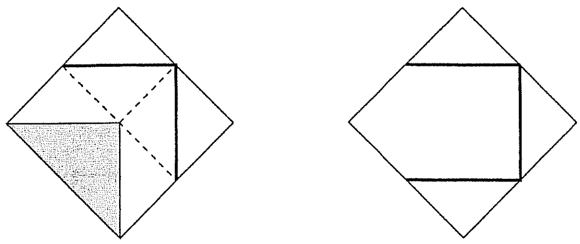
First of all note that by symmetry it is sufficient to consider the case $k = 1$ ($\xi_{2,1} = (1,0)$, $\xi_{2,2} = (\sqrt{2}/2, \sqrt{2}/2)$). As before, we proceed by components, defining v^n first in ∂Q and then the interior. Note that v is already defined in $Q \cap A_\varepsilon^{i,\flat}$. Let $T \subset Q$ such that $T \in \mathbf{A}_\varepsilon^{i,\sharp}$ and let ζ_j be the edge of $\partial T \cap \partial Q$. If $\varepsilon |\partial_j v_\varepsilon^n|^2 > 2\tau_\delta$ then v_ε^n is defined in ζ_j as in (6.46) otherwise we take $v^n = v_\varepsilon^n$. Let J_m be the sets represented in Figure 6.4-6.6. By a simple trigonometric argument it easy to check that

$$\int_{J_m \cap Q} \Big(c_{2,k} |\langle \nu, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu, \xi_{2,k+1} \rangle| \Big) d\mathcal{H}^1 \leq m \frac{|T|}{\varepsilon}, \tag{6.54}$$

and clearly there exists β_1 such that

$$\beta_1 \mathcal{H}^1(J_m \cap Q) \leq \frac{m|T|}{\varepsilon}. \tag{6.55}$$

Note that the sets J_m are defined in such a way that for every connected component C of $Q \setminus A_\varepsilon^{i,\flat}$ the slope of v^n is uniformly bounded by $\sqrt{2\tau_\delta/\varepsilon}$ on $\partial C \cap \partial Q$. Thus we can extend



(a) (b)

Figure 6.6: Discontinuity sets J_3 and J_4 for the cases $Q \in \mathbf{Q}_{\varepsilon,3}$ and $Q \in \mathbf{Q}_{\varepsilon,4}$ respectively.

the values of v^n inside C in such a way that $J_{v^n} \subset J_m$ and

$$\int_Q |\nabla v^n|^2 \, dx \leq c \frac{m|T|}{\varepsilon} . \tag{6.56}$$

Once the components are defined in this way, property (6.50) is clearly satisfied, (6.51) and (6.52) follows easily and finally inequality (6.53) is proved from (6.54). ■

Proposition 6.4.5 *Let $u \in L^1(\Omega, \mathbf{R}^2)$ and let A be an open set in Ω, if $\underline{F}^i(u, A) < +\infty$ then $u \in SBD^2(\Omega, \mathbf{R}^2) \cap K(\Omega)$ and*

$$\mathcal{H}^1(J_u \cap A) < +\infty , \tag{6.57}$$

$$\int_A W(Eu)\, dx \leq \underline{F}^i(u, A) . \tag{6.58}$$

Moreover for $i = 1$ and $i = 3$ and for every $k = 1, \dots, 3$ we have

$$\gamma \int_{J_u \cap A} c_{i,k} |\langle \nu_u, \xi_{i,k} \rangle| \, d\mathcal{H}^1 \leq \underline{F}^i(u, A) . \tag{6.59}$$

Finally for $i = 2$ and for every $k = 1, \dots, 4$ we have

$$\gamma \int_{J_u \cap A} \Big(c_{2,k} |\langle \nu_u, \xi_{2,k} \rangle| + c_{2,k+1} |\langle \nu_u, \xi_{2,k+1} \rangle| \Big) \, d\mathcal{H}^1 \leq \underline{F}^i(u, A) . \tag{6.60}$$

Proof. Let $\varepsilon_j \searrow 0$ and $v_{\varepsilon_j} \in V^i_{\varepsilon_j}(\Omega, \mathbf{R}^2)$ such that $v_{\varepsilon_j} \to u$ in $L^1(\Omega, \mathbf{R}^2)$ and $\liminf_{\varepsilon_j \to 0} F^i_{\varepsilon_j}(u_{\varepsilon_j}, A) < +\infty$. Up to taking a subsequence (denoted again as v_{ε_j}) it is not restrictive to suppose that $F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \leq c < +\infty$.

Let us first prove inequalities (6.57) and (6.58). For every v_{ε_j} let $v_j \in SBD^2(\Omega, \mathbf{R}^2)$ be the function given by Lemma 6.4.3. From the convergence in $L^1(\Omega, \mathbf{R}^2)$ of v_{ε_j} and from (6.30) it follows that v_j converges to u and by (6.29) that

$$\int_{A_{\varepsilon_j}} W(Ev_j)\, dx + \alpha \mathcal{H}^1(J_{v_j} \cap A_{\varepsilon_j}) \leq F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \leq c .$$

Let $\eta > 0$, if ε_j is small enough then $A_\eta \subset A_{\varepsilon_j}$ and then

$$\int_{A_\eta} W(Ev_j)\, dx + \alpha \mathcal{H}^1(J_{v_j} \cap A_\eta) \leq F^i_{\varepsilon_j}(v_{\varepsilon_j}, A) \leq c .$$

Then by the compactness and lower semicontinuity result of Proposition 1.2.19 we have that $u \in SBD^2(A_\eta)$ and

$$\int_{A_\eta} W(Eu)\,dx + \alpha\mathcal{H}^1(J_u \cap A_\eta) \le \liminf_{j\to+\infty} F^i_{\varepsilon_j}(v_{\varepsilon_j}, A)\,.$$

Since the previous inequality holds for every η we have

$$\int_A W(Eu)\,dx \le \liminf_{j\to+\infty} F^i_{\varepsilon_j}(v_{\varepsilon_j}, A)\,,$$

$$\mathcal{H}^1(J_u \cap A) < +\infty\,. \tag{6.61}$$

Applying the same reasoning for every sequence $\varepsilon_j \searrow 0$ and for every sequence v_{ε_j} it follows that

$$\int_A W(Eu)\,dx \le \underline{F}^i(u, A)\,. \tag{6.62}$$

Finally, to show inequalities (6.59) and (6.60), let this time v_j be the function given by Lemma 6.4.4 then, as for the previous inequalities, (6.59) and (6.60) will follow from the lower semicontinuity inequality (1.43). ■

Proof of Proposition 6.4.2. The Γ-liminf inequality follows applying the usual supremum of measures argument and considering the representations (6.10)-(6.12) of the anisotropy functions. The constrain $u \in K(\Omega)$ follows by pointwise convergence. ■

Consider now the isotropic case.

Proposition 6.4.6 *Denote by $\underline{\mathcal{F}}^\theta(u)$ the Γ-$\liminf_{\varepsilon\to 0} \mathcal{F}^\theta_\varepsilon(u)$ and take $u \in L^1(\Omega, \mathbf{R}^2)$ such that $\underline{\mathcal{F}}^\theta(u) < +\infty$. Then $u \in K(\Omega) \cap SBD^2(\Omega, \mathbf{R}^2)$ and*

$$\int_\Omega W(Eu)\,dx + \gamma\mathcal{H}^1(S_u) \le \liminf_{\varepsilon\to 0} \mathcal{F}^\theta_\varepsilon(v_\varepsilon)\,, \tag{6.63}$$

for every sequence $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ converging strongly to u in $L^1(\Omega, \mathbf{R}^2)$.

Proof. Following the proof of Lemma 6.4.3 we can easily obtain (6.29) and (6.30). Arguing as in the proof of Proposition 6.4.2, we get (6.61) and (6.62). It remains to consider the $\mathcal{H}^1$-term. We prove first a result on the localized functional similar to Lemma 6.4.4. For every $\delta \in (0,1)$ there are some positive constants α, β, η such that for every $v_\varepsilon \in \mathcal{V}^\theta_\varepsilon(\Omega, \mathbf{R}^2)$ and for every $\xi \in S^1$ there exists $v \in SBD^2(\Omega, \mathbf{R}^2)$ satisfying

$$\alpha \int_{A_\varepsilon} W(Ev)\,dx + \beta\mathcal{H}^1(J_v \cap A_\varepsilon) \le \mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,, \tag{6.64}$$

$$|\{x \in \Omega \ : \ v_\varepsilon(x) \neq v(x)\}| \le \eta\,\varepsilon\,\mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,, \tag{6.65}$$

$$\sin\theta\,(1-\delta)\gamma \int_{J_v \cap A_\varepsilon} |\langle \nu_v, \xi\rangle|\,d\mathcal{H}^1 \le \mathcal{F}^\theta_\varepsilon(v_\varepsilon, A)\,. \tag{6.66}$$

Following exactly the proof of Proposition 6.4.4 we get again inequality (6.39). Define $v_\varepsilon = v$ on $A^{i,b}_\varepsilon$, so that from (6.39) we have (6.65) and

$$\alpha_1 \int_{A^{4,b}_\varepsilon} W(Ev)\,dx \le \mathcal{F}^\theta_\varepsilon(v, A^{\theta,b}_\varepsilon)\,.$$

We want to define v on $A_\varepsilon^{i,\sharp}$ in such a way that (6.41)-(6.43) and

$$\sin\theta \int_{J_v \cap T} |\langle \nu_v, \xi\rangle| \, d\mathcal{H}^1 \le \frac{|T|}{\varepsilon} \tag{6.67}$$

are satisfied. We can proceed element by element and component by component. Note that for a constant $c > 0$ if $T \in \mathbf{A}_\varepsilon^{\theta,\flat}$ then $\varepsilon|\partial_j v_\varepsilon^n|^2 \le c\tau_\delta$ for every edge $\zeta_j \subset \partial T$. Let now $T \in \mathbf{A}_\varepsilon^{\theta,\sharp}$ and let $\zeta_j \subset \partial T$. If $\varepsilon|\partial_j v_\varepsilon^n|^2 > c\tau_\delta$ then we define v^n on ζ_j as in (6.46) otherwise we set $v^n = v_\varepsilon^n$.

Given $\xi \in S^1$, following the idea of [20], let the edges of ∂T be ordered according to

$$\langle m_1, \xi\rangle \le \langle m_2, \xi\rangle \le \langle m_3, \xi\rangle ,$$

where m_j denotes the middle point of ζ_j. We define the discontinuity set J as the union of the segments $[m_1, m_2]$ and $[m_2, m_3]$. Then by [20] we have

$$\mathcal{H}^1(J) \le \frac{2|T|}{\varepsilon \sin\theta}, \tag{6.68}$$

$$\int_J |\langle \nu_v, \xi\rangle| \, d\mathcal{H}^1 \le \frac{|T|}{\varepsilon \sin\theta}. \tag{6.69}$$

Note that J contains all the middle points m_j and that the components v^n are continuous on $\partial T \setminus \{m_1, m_2, m_3\}$ and that their slope is uniformly bounded by $\sqrt{c\tau_\delta/\varepsilon}$. Consequently for every connected component of $T \setminus J$ the value of v^n on ∂T defines an affine function whose gradient is controlled by c/ε (see [20] Remark 3.5) and for a suitable choice of α_1 we have (6.64). Moreover in this way $J_{v^n} \subset J$ and then (6.67) follows from (6.69). Then the Γ-liminf inequality is obtained following the proof of Proposition 6.4.2. ∎

6.5 Numerical results for a quasi-static evolution of a pre-existing fracture

Let $\Omega \subset \mathbf{R}^2$ be an open, bounded, and connected set with polyhedral boundary and let $\partial\Omega_D \subset \partial\Omega$ with $\mathcal{H}^1(\partial\Omega_D) > 0$. Let the boundary condition be given in $\partial\Omega_D$ by a monotonically increasing function $g(t,x) = t\hat{g}(x)$ for $\hat{g}(x) \in C^0(\partial\Omega_D, \mathbf{R}^2)$. Let $S \subset \Omega$ be a segment representing the initial fracture and let $0 = t_0 < t_1 < \ldots < t_n = T$ be a uniform subdivision of the time interval $[0, T]$.

For the sake of simplicity we will consider only the triangulations $\boldsymbol{T}_\varepsilon^3$, which have been used in the numerical experiments, and we assume that Ω and $\partial\Omega_D$ can be represented exactly in $\boldsymbol{T}_\varepsilon^3$. Let $\hat{g}_\varepsilon$ be the Lagrange interpolation of $\hat{g}(x)$. For a constant $k > \|g\|_\infty$ and for $t = t_1, \ldots, t_n$ let the discrete constraint be defined as

$$K_{\varepsilon,t}(\Omega) = \{u \in L^1(\Omega, \mathbf{R}^2) \,:\, \|u\|_\infty \le tk \text{ and } u = t\,\hat{g}_\varepsilon \text{ in } \partial\Omega_D\}.$$

For $t = t_1$ let $\boldsymbol{S}_{\varepsilon,t_1} = \{T \in \boldsymbol{T}_\varepsilon^3 \,:\, T \cap S \neq \emptyset\}$ then the discrete functional is given by

$$G_{\varepsilon,t_1}(v_\varepsilon) = \frac{1}{\varepsilon} \sum_{T \in \boldsymbol{T}_\varepsilon^3 \setminus \boldsymbol{S}_{\varepsilon,t_1}} \int_{T\cap\Omega} f(\varepsilon, Dv_\varepsilon)\, dx. \tag{6.70}$$

Let $w_{\varepsilon,t_1} \in \mathrm{argmin}\{G_{\varepsilon,t_1}(v_\varepsilon)$ for $v_\varepsilon \in V_\varepsilon^3(\Omega, \mathbf{R}^2) \cap K_{\varepsilon,t_1}(\Omega)\ \}$. The discontinuity set of w_{ε_j} is defined implicitly by an energy balance; since $f(\varepsilon, Dw_{\varepsilon_j})$ is a convex combination of

$\varepsilon W(Ew_{\varepsilon,0})$ and γ, the fracture will be represented by the set $J_{\varepsilon,1}$ of the elements $T \in T_\varepsilon^3$ where the "*local Griffith's criterion*"

$$\varepsilon W(Ew_{\varepsilon,t_1}) > \gamma$$

is satisfied. Moreover, in order to ensure the irreversibility of the fracture, we define $S_{\varepsilon,t_2} = S_{\varepsilon,t_1} \cup J_{\varepsilon,t_1}$. Proceeding by induction, the quasi-static evolution will be given by the sequence of functions w_{ε,t_i} for $i = 1, \dots, n$.

By a rescaling argument we can choose $\gamma = 1$. For $c = (\mu + \frac{\lambda}{2})$ and $s > 1$ a good choice for the function $\psi(z)$ is given by

$$\psi(z) = \tfrac{2}{\pi} \arctan((cz)^s) \tag{6.71}$$

which gives

$$\psi(\varepsilon|Dv_\varepsilon|^2) = \tfrac{2}{\pi} \arctan\left(\varepsilon^s \left(\mu + \frac{\lambda}{2}\right)^s |Dv_\varepsilon|^{2s}\right).$$

Indeed, considering $(\mu + \frac{\lambda}{2})|Dv_\varepsilon|^2$ as an approximation of the energy $W(Ev_\varepsilon)$, the "*local Griffith's criterion*" becomes

$$\varepsilon\left(\mu + \frac{\lambda}{2}\right)|Dv_\varepsilon|^2 > 1 = \gamma,$$

suggesting that the function $\psi(z)$ should change its behavior for $z = 1$.

From the numerical point of view the difficulties come from the non-convexity of the function G_ε. Indeed in order to reproduce accurately the Griffith's criterion we should use a function ψ with a fast transition from 0 to 1, which is obtained taking s large. Indeed in this way the bulk and surface energies are computed carefully, because $f(\varepsilon, Dv_\varepsilon)$ is close to the function

$$\tilde{f}(\varepsilon, Dv_\varepsilon) = \begin{cases} \varepsilon W(Ev_\varepsilon) & \text{for } \varepsilon c|Dv_\varepsilon|^2 < 1 \\ \gamma & \text{otherwise.} \end{cases}$$

Unfortunately the numerical minimization for s large is very difficult due to the sharp layer of ψ at $z = 1$, indeed the algorithm seems to be unable to overcome the layer and the solution does not exhibit any motion of the crack. For this reason we adopted a sort of graduated non-convexity strategy. For every time t_j let $1.5 = s_1 < \dots < s_8 = 8.5$ with $s_{n+1} - s_n = 1$ and let $G^{s_k}_{\varepsilon,t_j}$ be the discrete functional obtained with the exponent s_k in (6.71). For every time t_j, starting from s_1 we compute a solution of $G^{s_k}_{\varepsilon,t_j}$ taking as initial guess the solution of $G^{s_{k-1}}_{\varepsilon,t_j}$. Clearly for $G^{s_1}_{\varepsilon,t_j}$ the initial guess will be the solution at time t_{j-1}.

For every time t_j and every value s_k the minimization is performed by a quasi-Newton algorithm for non-convex functions using a quadratic back-tracking as line search strategy (we refer to [46] and to the references therein for the details).

Our model problem is defined in the set $\Omega = (0,2) \times (0,1)$, the initial fracture is the segment with extrema $(0, 0.5)$ and $(0.55, 0.5)$. The boundary condition $\hat{g}(x)$ is assigned on the sets $\partial\Omega_D^{up} = \{(x_1, 1) \text{ for } x_1 \in (0,2)\}$ and $\partial\Omega_D^{down} = \{(x_1, 0) \text{ for } x_1 \in (0,2)\}$ and it is defined as

$$\hat{g}(x) = \begin{cases} (0, 0.5) & \text{for } x \in \partial\Omega_D^{up}, \\ \\ (0, -0.5) & \text{for } x \in \partial\Omega_D^{down}. \end{cases}$$

The Lamé constants are $\mu = 9$ and $\lambda = 12$ and the toughness is $\gamma = 1$. Let us try to give a rough estimate of the critical time t_c when the motion of the fracture should start.

Suppose that the crack tip is located at the point $(L, 0.5)$, we expect the fracture to evolve horizontally from left to right. We will restrict our analysis to the set $(L, 2) \times (0, 1)$ because the value of the energy in $(0, L) \times (0, 1)$ remains basically constant until loss of cohesion occurs. Considering the geometrical symmetries of the problem we can approximate the value of the elastic energy by

$$\Big(\mu + \frac{\lambda}{2}\Big)\Big|\frac{\partial \hat{u}}{\partial x_2}\Big|^2 t^2(2 - L),$$

where $\hat{u}(x)$ is the affine function having boundary condition $\hat{g}(x)$ on $\partial\Omega_D$. Let $c_e = (\mu + \frac{\lambda}{2})|\frac{\partial \hat{u}}{\partial x_2}|^2$ and let dl, the increase in fracture length, be the unknown. Then the Griffith's energy is reduced to

$$G_t(dl) = c_e t^2(2 - L - dl) + \gamma dl = dl(\gamma - c_e t^2) + c_e t^2(2 - L)$$

which is a linear function in dl. The minimum problem becomes

$$\min_{0 \leq dl \leq (2-L)} dl(\gamma - c_e t^2) + c_e t^2(2 - L).$$

Thus for $(\gamma - c_e t^2) > 0$ the minimum is attained in $dl = 0$ (the crack does not move) while for $(\gamma - c_e t^2) < 0$ it is attained at $dl = (2 - L)$ (loss of cohesion). Even if the real behavior is not so simple, because of the influence of lower order terms, we can take the value t_c such that $\gamma = c_e t_c^2$ as an approximation of the critical time. In our case $t_c \simeq 0.258$. The numerical results seem to obey to this estimate, being the numerical critical time $t_c^n = 0.252$.

Figure 6.8 and 6.9 show the evolution of the fracture and the behavior of the energies.

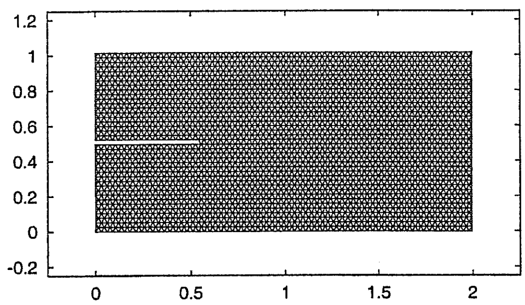


Figure 6.7: Initial configuration.

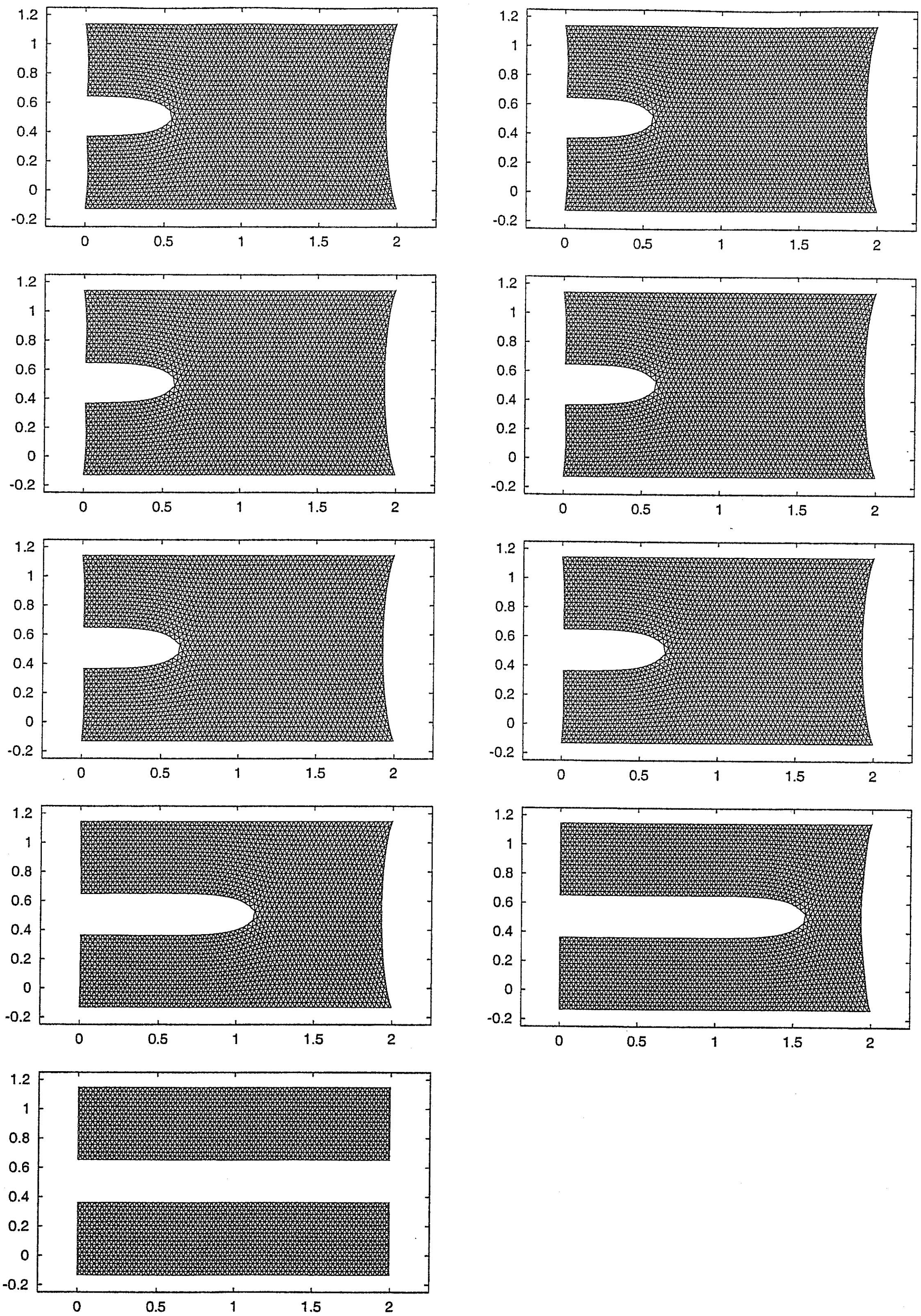


Figure 6.8: Configurations from $t = 0.250$ to $t = 0.266$ with $dt = 0.002$.

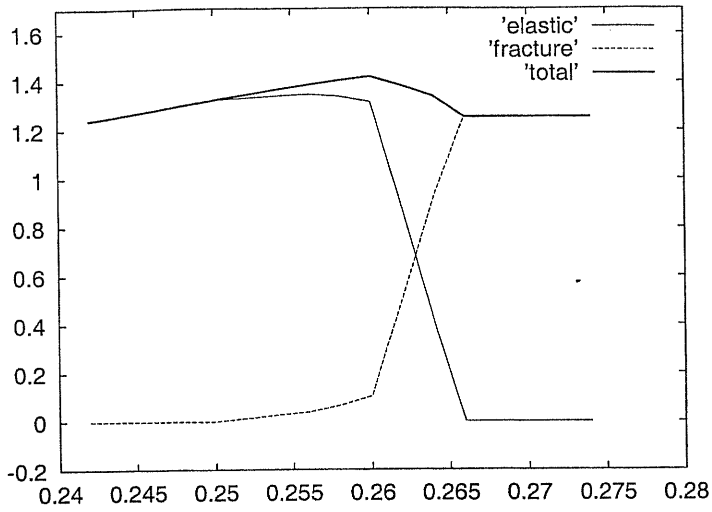


Figure 6.9: Comparison of elastic, fracture and total energies.

Chapter 7

Linearized elasticity as Γ-limit of finite elasticity

7.1 The main results

Let the reference configuration be an open, bounded, connected domain $\Omega \subset \mathbf{R}^n$, for $n \geq 2$, having Lipschitz boundary. Let $|F|^2 = \sum_{i,j} |F_{ij}|^2$ be the norm in the space $\mathbf{M}^{n\times n}$ and let $SO(n)$ be the subset of rotations (orthogonal matrices with positive determinant).

We will assume that the material is hyperelastic, i.e., there exists a stored energy density $W : \Omega \times \mathbf{M}^{n\times n} \to [0, +\infty]$ such that for a.e. $x \in \Omega$ we have

$$W(x, F) = +\infty \quad \text{if } \det F \leq 0 \tag{7.1}$$

(orientation preserving condition), and such that for a.e. $x \in \Omega$

$$W(x, F) < +\infty \tag{7.2}$$

for F in a neighborhood U of the identity I independent of x (so that small deformation of the reference configuration have finite energy). By frame indifference the stored energy density can be written as

$$W(x, F) = V(x, \tfrac{1}{2}(F^T F - I)) \,. \tag{7.3}$$

where F^T denotes the transpose of the matrix F. We suppose that $V : \Omega \times \mathbf{M}^{n\times n}_{sym} \to \mathbf{R}$ is $\mathcal{L}^n \times \mathcal{B}^n$-measurable (where $\mathcal{L}^n$ and $\mathcal{B}^n$ are the σ-algebras of Lebesgue measurable and Borel measurable subsets of $\mathbf{R}^n$) and that, for some $\delta > 0$, the function $B \to V(x, B)$ is of class C^2 for $|B| < \delta$ and for a.e. $x \in \Omega$. Moreover we will assume that the reference configuration has zero energy and is stress free, which means that for a.e. $x \in \Omega$

$$V(x, 0) = 0 \qquad \partial_E V(x, 0) = 0 \,. \tag{7.4}$$

Finally we require the coercivity assumptions (a), (b), (c) and for a.e. $x \in \Omega$ the upper bound

$$|\partial^2_E V(x, E)[T, T]| \leq 2\gamma |T|^2 \quad \text{for } |E| < \delta \text{ and } T \in \mathbf{M}^{n\times n}_{sym} \,, \tag{7.5}$$

for some constant $\gamma > 0$ independent of x.

From (7.4) it is easy to deduce by Taylor expansion that for a.e. $x \in \Omega$

$$V(x, E) = \frac{1}{2} \partial^2_E V(x, tE)[E, E] \tag{7.6}$$

for some $t \in (0,1)$ depending on x, hence

$$|V(x,E)| \leq \gamma |E|^2 \quad \forall E \in \mathbf{M}^{n\times n}_{sym} \text{ with } |E| < \delta\,. \tag{7.7}$$

Let $\mathsf{A}(x) := \partial^2_E V(x,0)$. From (7.6) and (b) it follows that for a.e. $x \in \Omega$

$$\mathsf{A}(x)[E,E] = \partial^2_E V(x,0)\,[E,E] \geq 2\alpha |E|^2 \quad \forall E \in \mathbf{M}^{n\times n}_{sym}\,. \tag{7.8}$$

Finally for every $x \in \Omega$ and $F \in \mathbf{M}^{n\times n}$ let $F_{sym} = (F + F^T)/2$ and

$$W_\varepsilon(x,F) := \frac{1}{\varepsilon^2}\, W(x, I + \varepsilon F) = \frac{1}{\varepsilon^2}\, V(x, \varepsilon\, F_{sym} + \tfrac{1}{2}\varepsilon^2 F^T F)\,. \tag{7.9}$$

It is easy to see that for a.e. $x \in \Omega$

$$\begin{aligned} \lim_{\varepsilon\to 0} W_\varepsilon(x,F) &= \frac{1}{2}\partial^2_F W(x,I)[F,F] \\ &= \frac{1}{2}\partial^2_E V(x,0)[F_{sym},F_{sym}] = \frac{1}{2}\mathsf{A}(x)[F_{sym},F_{sym}]\,. \end{aligned} \tag{7.10}$$

We consider the functional $\mathcal{F}_\varepsilon : H^1(\Omega,\mathbf{R}^n) \to [0,+\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \int_\Omega W_\varepsilon(x,\nabla u)\, dx\,, \tag{7.11}$$

and the functional $\mathcal{F} : H^1(\Omega,\mathbf{R}^n) \to [0,+\infty)$ given by

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega \mathsf{A}(x)\,[e(u),e(u)]\, dx\,. \tag{7.12}$$

Let $\partial\Omega_D$ a closed subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\partial\Omega_D) > 0$ and let $g \in W^{1,\infty}(\Omega,\mathbf{R}^n)$. Let $H^1_{g,\partial\Omega_D}$ be the closure in $H^1(\Omega,\mathbf{R}^n)$ of the space of functions $u \in W^{1,\infty}(\Omega,\mathbf{R}^n)$ such that $u = g$ on $\partial\Omega_D$. By strong (resp. weak) topology in $H^1_{g,\partial\Omega_D}$ we mean the restriction of the strong (resp. weak) topology of $H^1(\Omega,\mathbf{R}^n)$. Let $\mathcal{L} : H^1(\Omega,\mathbf{R}^n) \to \mathbf{R}$ be a continuous linear operator, representing the work of the (rescaled) loads. We define the functionals $\mathcal{G}_\varepsilon, \mathcal{G} : H^1_{g,\Omega_D} \to [0,+\infty]$ as $\mathcal{G}_\varepsilon(u) = \mathcal{F}_\varepsilon(u) - \mathcal{L}(u)$ and $\mathcal{G}(u) = \mathcal{F}(u) - \mathcal{L}(u)$.

The main convergence results, proved in Section 7.4, are the following.

Theorem 7.1.1 *Assume that $V : \Omega \times \mathbf{M}^{n\times n}_{sym} \to [0,+\infty]$ satisfies conditions (a), (b), (c), (7.1), (7.4), and (7.5). If u_ε satisfies*

$$\mathcal{G}(u_\varepsilon) = \inf_{u\in H^1_{g,\partial\Omega_D}} \mathcal{G}_\varepsilon(u) + o(1) \tag{7.13}$$

then u_ε converges weakly to the (unique) solution u_0 of

$$\min_{u\in H^1_{g,\partial\Omega_D}} \mathcal{G}(u)\,.$$

Theorem 7.1.2 *Under the hypotheses of the previous theorem, if condition (c') is satisfied then u_ε converges to u_0 strongly in $W^{1,q}(\Omega,\mathbf{R}^n)$ for $1 \leq q < 2$.*

The proof follows basically from the following results, contained in Section 7.2 and 7.3 respectively.

Proposition 7.1.3 *If $\varepsilon_j \to 0$ and $u_{\varepsilon_j} \in H^1_{g,\partial\Omega_D}$ is a sequence such that*

$$\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \leq C < +\infty\,,$$

then u_{ε_j} is equibounded in $H^1(\Omega, \mathbf{R}^n)$.

Proposition 7.1.4 *Let $\varepsilon_j \to 0$. The functionals $\mathcal{G}_{\varepsilon_j}$ Γ-converge to $\mathcal{G}$ in the weak topology of $H^1_{g,\partial\Omega_D}$.*

7.2 Compactness

From conditions (a), (b) and (c) it follows easily that there exists a non-decreasing, continuous function $\phi(t)$, of the form

$$\phi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq c \\ \alpha c^2 & \text{for } c \leq t \leq d \\ (\alpha c^2 d^{-1})t & \text{for } d \leq t\,, \end{cases}$$

such that $\phi(|E|) \leq V(x, E)$ for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$. For a positive β let $\psi(t)$ be the function defined as

$$\psi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq \beta\,, \\ (2\alpha\beta)t - (\alpha\beta^2) & \text{for } t \geq \beta\,. \end{cases} \tag{7.14}$$

It is easy to check that $\psi(t)$ is increasing, C^1, and convex. Moreover, since

$$\lim_{\beta\to 0} 2\alpha\beta = 0\,,$$

for β sufficiently small we have $\psi(t) \leq \phi(t)$ and then

$$V(x, E) \geq \psi(|E|) \tag{7.15}$$

for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$.

Lemma 7.2.1 *Let $\varepsilon > 0$ and $u_\varepsilon \in H^1(\Omega, \mathbf{R}^n)$. Denote the rescaled deformation $x+\varepsilon u_\varepsilon(x)$ by $v_\varepsilon(x)$. Then there exists a function $R_\varepsilon : \Omega \to SO(n)$ such that*

$$\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|^2\, dx \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,, \tag{7.16}$$

where C depends only on the function ψ (in particular it does not depend on ε or v_ε).

Proof. We may assume $\mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$, so that $\det \nabla v_\varepsilon > 0$ a.e. in Ω by (7.1). Considering that

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \int_\Omega W_\varepsilon(x, \nabla u_\varepsilon)\, dx = \frac{1}{\varepsilon^2}\int_\Omega V(x, \tfrac{1}{2}(\nabla v_\varepsilon^T \nabla v_\varepsilon - I))\, dx \tag{7.17}$$

and using (7.15) we get

$$\int_\Omega \psi(\tfrac{1}{2}|\nabla v_\varepsilon^T \nabla v_\varepsilon - I|)\, dx \leq \int_\Omega V(x, \tfrac{1}{2}(\nabla v_\varepsilon^T \nabla v_\varepsilon - I)) \leq \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,. \tag{7.18}$$

As $\det \nabla v_\varepsilon > 0$ a.e. in Ω by polar decomposition (see for instance [21]) for a.e. $x \in \Omega$ there exists a rotation R_ε and a symmetric positive definite matrix U_ε such that $\nabla v_\varepsilon = R_\varepsilon U_\varepsilon$. In particular $\nabla v_\varepsilon^T \nabla v_\varepsilon = U_\varepsilon^2$, hence

$$|\nabla v_\varepsilon^T \nabla v_\varepsilon - I| = |U_\varepsilon^2 - I|. \tag{7.19}$$

Since U_ε is symmetric and positive definite, using an orthonormal basis in which U_ε is diagonal, we can prove that

$$|U_\varepsilon - I| \leq |U_\varepsilon^2 - I|.$$

Thus, by the definition of ψ, it follows that for $\frac{1}{2}|U_\varepsilon^2 - I| \leq \beta$

$$\tfrac{\alpha}{4}|U_\varepsilon - I|^2 = \psi(\tfrac{1}{2}|U_\varepsilon^2 - I|).$$

Moreover for a suitable constant c_1, depending on β,

$$c_1|U_\varepsilon - I|^2 \leq |U_\varepsilon^2 - I| \qquad \text{for } \tfrac{1}{2}|U_\varepsilon^2 - I| \geq \beta.$$

Indeed, using again the diagonal form, we can write

$$\begin{aligned}\sum_{i=1}^n (\lambda_i - 1)^2 &\leq \sum_{i=1}^n \lambda_i^2 + n = \\ &= \sum_{i=1}^n (\lambda_i^2 - 1) + \frac{2n}{\beta}\beta \leq \left(1 + \frac{n}{\beta}\right) \sum_{i=1}^n |\lambda_i^2 - 1|.\end{aligned}$$

Moreover there is a constant c_2 such that $2c_2 t \leq \psi(t)$ for $t \geq \beta$, hence for $\frac{1}{2}|U_\varepsilon^2 - I| \geq \beta$

$$c_1 c_2 |U_\varepsilon - I|^2 \leq c_2|U_\varepsilon^2 - I| \leq \psi(\tfrac{1}{2}|U_\varepsilon^2 - I|).$$

By this inequality and by (7.18) and (7.19) there exists a constant c_3, depending only on ψ, such that

$$\int_\Omega |U_\varepsilon - I|^2\, dx \leq c_3 \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon).$$

Finally considering that for a.e. $x \in \Omega$ we have $\nabla v_\varepsilon = R_\varepsilon U_\varepsilon$ we can write

$$\int_\Omega |\nabla v_\varepsilon - R_\varepsilon|^2\, dx = \int_\Omega |U_\varepsilon - I|^2\, dx \leq c_3 \varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon).$$

which is the required estimate. ■

The following Lemma (for which we refer to [32]) will be crucial in our proof.

Lemma 7.2.2 *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with Lipschitz boundary. There exists a constant C such that for every $v \in H^1(\Omega, \mathbf{R}^n)$ there exists a constant rotation $R \in SO(n)$ such that*

$$\int_\Omega |\nabla v(x) - R|^2\, dx \leq C \int_\Omega \operatorname{dist}(\nabla v(x), SO(n))^2\, dx, \tag{7.20}$$

where $\operatorname{dist}(F, SO(n))$ *denotes the distance from the matrix F to the set $SO(n)$.*

Moreover we will need the following result.

Lemma 7.2.3 *Let $S \subset \mathbf{R}^n$ be a bounded $\mathcal{H}^n$-measurable set with $0 < \mathcal{H}^m(S) < +\infty$, for some $m > 0$. Then*

$$|F|_S := \Big(\min_{\zeta \in \mathbf{R}^n} \int_S |Fx - \zeta|^2 d\mathcal{H}^m(x) \Big)^{\frac{1}{2}}$$

is a seminorm on $\mathbf{M}^{n\times n}$.

Let S_0 be the set of points $x \in S$ such that $\mathcal{H}^m(S \cap B_\rho(x)) > 0$, and let $\mathrm{aff}(S_0)$ be the smallest affine space containing S_0. Let $\mathbf{K} \subset \mathbf{M}^{n\times n}$ be a closed cone such that for every $F \in \mathbf{K}$ with $F \neq 0$

$$\dim(\ker(F)) < \dim(\mathrm{aff}(S_0))\,. \tag{7.21}$$

Then there exists a constant $C > 0$ such that

$$C|F| \leq |F|_S \tag{7.22}$$

for every $F \in \mathbf{K}$.

Proof. It is not difficult to check that $|F|_S$ is a seminorm and the minimum is attained for $\zeta = ⨍_S Fx\, d\mathcal{H}^m$. We will prove (7.22) by contradiction. Suppose that for every integer k it is possible to find a matrix $F_k \in \mathbf{K}$ with $|F_k| = 1$ such that

$$\frac{1}{k} = \frac{1}{k}|F_k|^2 > \int_S |F_k x - \zeta_k|^2 d\mathcal{H}^m \geq 0\,, \tag{7.23}$$

with $\zeta_k := ⨍_S F_k x\, d\mathcal{H}^m$. It is not restrictive to assume that F_k converges to $F \in \mathbf{K}$, with $|F| = 1$. Then by (7.23) and by continuity it follows that

$$\int_S |Fx - \zeta|^2 d\mathcal{H}^m = 0\,.$$

for $\zeta = ⨍_S Fx\, d\mathcal{H}^m$. Then $Fx = \zeta$ for $\mathcal{H}^m$-a.e. $x \in S$ and hence for every $x \in S_0$. By continuity and linearity $Fx = \zeta$ for every $x \in \mathrm{aff}(S_0)$. Then $\dim(\ker(F)) \geq \dim(\mathrm{aff}(S_0))$ and thus, by (7.21), $F = 0$. This is clearly impossible because $|F| = 1$. ∎

Now we are ready to prove the following compactness result.

Proposition 7.2.4 *Let u_ε be a sequence in $H^1_{g,\partial\Omega_D}$. Then*

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \leq C\mathcal{F}_\varepsilon(u_\varepsilon) + C\int_{\partial\Omega_D} |g|^2\, d\mathcal{H}^{n-1} \tag{7.24}$$

where C depends only on ψ, Ω, and $\partial\Omega_D$.

Proof. By Lemma 7.2.1 we have

$$\int_\Omega \mathrm{dist}(\nabla v_\varepsilon(x), SO(n))^2\, dx \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)$$

and by Lemma 7.2.2 there exists a constant rotation R_ε such that

$$\int_\Omega |\nabla v_\varepsilon(x) - R_\varepsilon|^2\, dx \leq C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,. \tag{7.25}$$

If $\zeta_\varepsilon = \fint_\Omega (v_\varepsilon(x) - R_\varepsilon x)\, dx$, then by the Poincaré inequality

$$\|v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C \int_\Omega |\nabla v_\varepsilon(x) - R_\varepsilon|^2\, dx \le C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,.$$

Moreover by the continuity of the traces

$$\int_{\partial\Omega_D} |v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1} \le C\|v_\varepsilon(x) - R_\varepsilon x - \zeta_\varepsilon\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon)\,.$$

Considering that on $\partial\Omega_D$ we have $v_\varepsilon(x) = x + \varepsilon g(x)$ we can write

$$\int_{\partial\Omega_D} |x - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1} \le C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2 \int_{\partial\Omega_D} |g|^2\, d\mathcal{H}^{n-1}\,. \tag{7.26}$$

Let $\mathbf{K}$ be the closed cone generated by $SO(n) - I$, which is the union of the cone generated by $SO(n) - I$ and of the space of antisymmetric matrices. Therefore, $\dim(\ker(F)) < n-1$ if $F \in \mathbf{K}$ and $F \neq 0$. Let $S := \partial\Omega_D$. As S is contained in the Lipschitz manifold $\partial\Omega$ and $\mathcal{H}^{n-1}(S) > 0$, we have $\mathcal{H}^{n-1}(S_0) > 0$. This implies that $\dim(\mathrm{aff}(S_0)) \ge n-1$ and thus condition (7.21) is satisfied. Using Lemma 7.2.3 and the previous inequality we obtain

$$|I - R_\varepsilon|^2 \le C|I - R_\varepsilon|^2_S = C \int_{\partial\Omega_D} |x - R_\varepsilon x - \zeta_\varepsilon|^2 d\mathcal{H}^{n-1}$$

and thus by (7.26)

$$\int_\Omega |I - R_\varepsilon|^2\, dx \le C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2 \int_{\partial\Omega_D} |g|^2\, d\mathcal{H}^{n-1}\,. \tag{7.27}$$

By (7.25) and (7.27) we have easily

$$\int_\Omega |\nabla v_\varepsilon - I|^2\, dx \le C\varepsilon^2 \mathcal{F}_\varepsilon(u_\varepsilon) + C\varepsilon^2 \int_{\partial\Omega_D} |g|^2\, d\mathcal{H}^{n-1}\,.$$

Substituting $\nabla v_\varepsilon = I + \varepsilon\nabla u_\varepsilon$ in the previous inequality we get (7.24). ∎

Proof of Proposition 7.1.3. Using Proposition 7.2.4 we have

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\, dx \le C\mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) + C \int_{\partial\Omega_D} |g|^2\, d\mathcal{H}^{n-1}\,.$$

Hence we can write

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2\, dx \le C(\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) + \mathcal{L}(u_{\varepsilon_j}) + 1)\,,$$

and by the Poincaré and the Holder inequality it follows that

$$\|u_{\varepsilon_j}\|^2_{H^1(\Omega,\mathbf{R}^n)} \le C + C\|u_{\varepsilon_j}\|_{H^1(\Omega,\mathbf{R}^n)}\,,$$

which gives the boundedness of u_{ε_j} in $H^1(\Omega,\mathbf{R}^n)$. ∎

Finally we remark that for $n = 2$ and for a sequence $u_{\varepsilon_j} \in H^1_0(\Omega,\mathbf{R}^2)$ we can prove the compactness result in a more elementary way without using Lemma 7.2.2. Indeed for every ε_j let $R_{\varepsilon_j}: \Omega \to SO(2)$ be given by Lemma 7.2.1. Define $M_{\varepsilon_j} = (R_{\varepsilon_j} - I)/\varepsilon_j$. Then, substituting $\nabla v_{\varepsilon_j} = I + \varepsilon_j \nabla u_{\varepsilon_j}$ in (7.16), we get

$$\int_\Omega |\nabla u_{\varepsilon_j} - M_{\varepsilon_j}|^2 \, dx \leq C_1 \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) .$$

Note that M_{ε_j} has the form

$$M_{\varepsilon_j} = \begin{pmatrix} a_{\varepsilon_j} & -b_{\varepsilon_j} \\ b_{\varepsilon_j} & a_{\varepsilon_j} \end{pmatrix}$$

for some real functions a_{ε_j} and b_{ε_j}. Denote the components of u by u^i. By a linear combination we obtain

$$\int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2 \, dx = \int_\Omega |(\nabla_1 u^1_{\varepsilon_j} - a_{\varepsilon_j}) - (\nabla_2 u^2_{\varepsilon_j} - a_{\varepsilon_j})|^2 \, dx \leq C_2 \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) ,$$

$$\int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2 \, dx \int_\Omega |(\nabla_2 u^1_{\varepsilon_j} + b_{\varepsilon_j}) + (\nabla_1 u^2_{\varepsilon_j} - b_{\varepsilon_j})|^2 \, dx \leq C_3 \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) .$$

Moreover, being $n = 2$, we can write

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2 \, dx = \int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2 \, dx + \int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2 \, dx + 2 \int_\Omega \det \nabla u_{\varepsilon_j} \, dx .$$

As $u_{\varepsilon_j} \in H^1_0(\Omega, \mathbf{R}^2)$ we have (see e.g. [21])

$$\int_\Omega \det \nabla u_{\varepsilon_j} \, dx = 0 .$$

Then by the previous inequalities we get

$$\int_\Omega |\nabla u_{\varepsilon_j}|^2 \, dx = \int_\Omega |\nabla_1 u^1_{\varepsilon_j} - \nabla_2 u^2_{\varepsilon_j}|^2 \, dx + \int_\Omega |\nabla_2 u^1_{\varepsilon_j} + \nabla_1 u^2_{\varepsilon_j}|^2 \, dx \leq C \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j})$$

and thus u_{ε_j} is bounded in $H^1_0(\Omega, \mathbf{R}^2)$.

The following example shows that, if other potential wells are present, with the same value of the energy, we might lose compactness of solutions.

Example 7.2.5 *Let $\Omega = (-1,1) \times (-1,1)$, $\ell = 1$ and $w \in H^1_0(\Omega, \mathbf{R}^2)$ defined as $w^1(x_1, x_2) = -\max\{|x_1|, |x_2|\} + 1$ and $w^2(x_1, x_2) = 0$. Let $\varepsilon_j \to 0$, $w_{\varepsilon_j}(x) = w(x)/\varepsilon_j$ and $v_{\varepsilon_j}(x) = x + \varepsilon_j w_{\varepsilon_j}(x)$. Then $\nabla v_{\varepsilon_j} = I + \varepsilon_j \nabla w_{\varepsilon_j} = I + \nabla w$ does not depend on ε_j and takes only four values, denoted by $F_1, \ldots, F_4$. Let $E_i = \frac{1}{2}(F_i^T F_i - I)$, for $i = 1, \ldots, 4$. Let V be the function satisfying conditions (b) and (c) and such that $V(x, 0) = V(x, E_i) = 0$ for $i = 1, \ldots, 4$. Then*

$$\begin{aligned} \inf\{\mathcal{G}_{\varepsilon_j}(u) \,:\, u \in H^1_0(\Omega)\} &\leq \frac{1}{\varepsilon_j^2} \int_\Omega V(x, \tfrac{1}{2}(\nabla v_{\varepsilon_j}^T \nabla v_{\varepsilon_j} - I)) \, dx - \int_\Omega w_{\varepsilon_j} \, dx \\ &= -\frac{1}{\varepsilon_j} \|w\|_{L^1(\Omega, \mathbf{R}^n)} . \end{aligned}$$

If u_{ε_j} is a sequence satisfying (7.13) then

$$-\frac{1}{\varepsilon_j} \|w\|_{L^1(\Omega, \mathbf{R}^n)} + o(1) \geq \mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \geq -\|u_{\varepsilon_j}\|_{L^1(\Omega, \mathbf{R}^n)} ,$$

hence $\|u_{\varepsilon_j}\|_{L^1(\Omega, \mathbf{R}^2)}$ diverges.

7.3 Γ-convergence

For $x \in \Omega$ and $E \in \mathbf{M}^{n\times n}_{sym}$ let $|E|_{\mathsf{A}(x)}$ be the norm defined by

$$|E|_{\mathsf{A}(x)} = \Big\{\frac{1}{2}\mathsf{A}(x)[E,E]\Big\}^{\frac{1}{2}} = \Big\{\frac{1}{2}\partial^2_E V(x,0)[E,E]\Big\}^{\frac{1}{2}}. \tag{7.28}$$

Note that by (7.5) and (7.8) we have

$$\alpha|E|^2 \leq |E|^2_{\mathsf{A}(x)} \leq \gamma|E|^2. \tag{7.29}$$

If $\Phi : \Omega \to \mathbf{M}^{n\times n}_{sym}$ is a measurable map, the function $x \mapsto |\Phi(x)|_{\mathsf{A}(x)}$ is denoted by $|\Phi|_{\mathsf{A}}$.

Let us fix a sequence $\varepsilon_j \to 0$. By Proposition 7.1.3 the functionals $\mathcal{G}_{\varepsilon_j}$ are equicoercive in $H^1_{g,\partial\Omega_D}$ and by Proposition 8.10 in [24] we can characterize the Γ-limit in the weak topology of $H^1_{g,\partial\Omega_D}$ in terms of weakly converging sequences. In particular we have

$$\mathcal{G}'(u) := \Gamma\text{-}\liminf_{\varepsilon_j\to 0} \mathcal{G}_\varepsilon(u) = \inf\{\liminf_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(u_j) \,:\, \text{for } u_j \rightharpoonup u \text{ in } H^1_{g,\partial\Omega_D}\},$$

$$\mathcal{G}''(u) := \Gamma\text{-}\limsup_{\varepsilon_j\to 0} \mathcal{G}_\varepsilon(u) = \inf\{\limsup_{j\to+\infty} \mathcal{G}_{\varepsilon_j}(u_j) \,:\, \text{for } u_j \rightharpoonup u \text{ in } H^1_{g,\partial\Omega_D}\}.$$

We will prove that for every function $u \in H^1_{g,\partial\Omega_D}$ we have $\mathcal{G}''(u) \leq \mathcal{G}(u) \leq \mathcal{G}'(u)$, from which Proposition 7.1.4 follows.

Proposition 7.3.1 *For every $u \in H^1_{g,\partial\Omega_D}$ we have $\mathcal{G}''(u) \leq \mathcal{G}(u)$.*

Proof. Consider first the case $u \in W^{1,\infty}(\Omega,\mathbf{R}^n)$. By (7.10) it follows that for a.e. $x \in \Omega$

$$\lim_{\varepsilon_j\to 0} W_{\varepsilon_j}(x,\nabla u) = \frac{1}{2}\mathsf{A}(x)\,[e(u),e(u)].$$

Using the upper bound (7.7) we deduce that $V_{\varepsilon_j}(x,\nabla u)$ is equi-bounded in $L^\infty(\Omega)$. Then taking the sequence $u_{\varepsilon_j} = u$, by dominated convergence it follows that

$$\limsup_{\varepsilon_j\to 0} \mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) = \lim_{\varepsilon_j\to 0}\int_\Omega V_{\varepsilon_j}(x,\nabla u)\,dx - \mathcal{L}(u) = \frac{1}{2}\int_\Omega \mathsf{A}(x)\,[e(u),e(u)]\,dx - \mathcal{L}(u). \tag{7.30}$$

If $u \notin W^{1,\infty}(\Omega,\mathbf{R}^n)$ by the definition of $H^1_{g,\partial\Omega_D}$ there exists a sequence u_k in $W^{1,\infty}(\Omega,\mathbf{R}^n)$, which satisfy the boundary condition $u_k = g$ on $\partial\Omega_D$ and converge to u strongly in $H^1(\Omega,\mathbf{R}^n)$. Since, by (7.30), $\mathcal{G}''(u_k) \leq \mathcal{G}(u_k)$, the lower semicontinuity of the Γ-limsup and the continuity of $\mathcal{G}$ respect to strong convergence imply that

$$\mathcal{G}''(u) \leq \liminf_{k\to\infty} \mathcal{G}''(u_k) \leq \liminf_{k\to\infty} \mathcal{G}(u_k) = \mathcal{G}(u)$$

and the proof is concluded. ∎

Lemma 7.3.2 *Let $\varepsilon_j \to 0$ be a decreasing sequence. For every $k \in \mathbf{N}$ there exist an increasing sequence of Caratheodory functions $V^k_j : \Omega\times\mathbf{M}^{n\times n}_{sym} \to [0,+\infty)$ and a measurable function $\mu^k : \Omega \to (0,+\infty)$ such that $V^k_j(x,\cdot)$ is convex for a.e. $x \in \Omega$ and satisfies*

$$V^k_j(x,E) \leq V(x,\varepsilon_j E)/\varepsilon_j^2 \qquad \forall E \in \mathbf{M}^{n\times n}_{sym}, \tag{7.31}$$

$$V^k_j(x,E) = \Big(1-\frac{1}{k}\Big)\,|E|^2_{\mathsf{A}(x)} \qquad \text{for } |E|_{\mathsf{A}(x)} \leq \mu^k(x)/\varepsilon_j. \tag{7.32}$$

Proof. By Taylor's formula, from (7.4) and (7.29) it follows that for a.e. $x \in \Omega$ and every $k \in \mathbf{N}$ there exists $r^k(x) > 0$ such that

$$\Big(1 - \frac{1}{k}\Big)|E|^2_{\mathsf{A}(x)} \leq V(x, E) \quad \text{for } |E|_{\mathsf{A}(x)} \leq r^k(x) \,. \tag{7.33}$$

Let us consider the function $h^k : \Omega \times \mathbf{M}^{n\times n}_{sym} \longrightarrow \mathbf{R}$ defined by

$$h^k(x, E) = \begin{cases} (1 - \frac{1}{k})|E|^2_{\mathsf{A}(x)} & \text{for } |E|_{\mathsf{A}(x)} \leq r^k(x) \,, \\ \psi(\gamma^{-\frac{1}{2}}|E|_{\mathsf{A}(x)}) & \text{for } |E|_{\mathsf{A}(x)} > r^k(x) \,, \end{cases}$$

which is less than or equal to $V(x, E)$ by (7.33), (7.29), and (7.15).

For a suitable choice of $\mu^k(x) > 0$ the function

$$\phi^k(x, t) = \begin{cases} (1 - \frac{1}{k})t^2 & \text{for } 0 \leq t \leq \mu^k(x) \,, \\ 2(1 - \frac{1}{k})\mu^k(x)t - (1 - \frac{1}{k})(\mu^k(x))^2 & \text{for } t \geq \mu^k(x) \,, \end{cases}$$

is convex in t and satisfies $\phi^k(x, |E|_{\mathsf{A}(x)}) \leq h^k(x, E) \leq V(x, E)$. To conclude the proof it is enough to define $V^k_j(x, E) := \phi^k(x, \varepsilon_j|E|_{\mathsf{A}(x)})/\varepsilon_j^2$. From the special form of $\phi^k(x, \cdot)$ it is easy to see that $V^k_j(x, \cdot)$ is increasing with respect to j and that (7.32) holds, while (7.31) follows from the inequality $\phi^k(x, |E|_{\mathsf{A}(x)}) \leq V(x, E)$. ■

Lemma 7.3.3 *Let $g_j : \Omega \times \mathbf{R}^m \to [0, +\infty)$ be Caratheodory functions such that $g_j(x, \cdot)$ is convex. Let $g_j(x, \xi)$ be increasing in j and pointwise converging to a function $g(x, \xi)$. If w_j converges weakly to w in $L^1(\Omega, \mathbf{R}^m)$, then*

$$\int_\Omega g(x, w)\, dx \leq \liminf_{j\to+\infty} \int_\Omega g_j(x, w_j)\, dx \,. \tag{7.34}$$

Proof. As $g_i(x, w_j) \leq g_j(x, w_j)$ for $j \geq i$, by the lower semicontinuity of the functional $\int_\Omega g_i(x, v)\, dx$ we have

$$\int_\Omega g_i(x, w)\, dx \leq \liminf_{j\to+\infty} \int_\Omega g_i(x, w_j)\, dx \leq \liminf_{j\to+\infty} \int_\Omega g_j(x, w_j)\, dx \,,$$

which proves (7.34) for $i \to \infty$. ■

Proposition 7.3.4 *For every $u \in H^1_{g,\partial\Omega_D}$ and every sequence $u_j \in H^1_{g,\partial\Omega_D}$ weakly converging to u, we have the Γ-liminf inequality*

$$\frac{1}{2}\int_\Omega \mathsf{A}(x)\,[e(u), e(u)]\, dx \leq \liminf_{\varepsilon_j\to 0} \mathcal{G}_{\varepsilon_j}(u_j) \,, \tag{7.35}$$

from which it follows that $\mathcal{G}(u) \leq \mathcal{G}'(u)$.

Proof. For every $k \in \mathbf{N}$ let $V^k_j(x, E)$ be the sequence given by Lemma 7.3.2. Note that by (7.32) for every $E \in \mathbf{M}^{n\times n}_{sym}$ we have

$$\lim_{j\to+\infty} V^k_j(x, E) = \Big(1 - \frac{1}{k}\Big)\, |E|^2_{\mathsf{A}(x)} \,. \tag{7.36}$$

Then inequality (7.31) gives

$$\begin{aligned} W_{\varepsilon_j}(x, \nabla u_{\varepsilon_j}) &= \frac{1}{\varepsilon_j^2} V(x, \varepsilon_j e(u_{\varepsilon_j}) + \tfrac{1}{2}{\varepsilon_j}^2 \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j}) \\ &\geq V_j^k(x, e(u_{\varepsilon_j}) + \tfrac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j}) \,. \end{aligned}$$

Since $\nabla u_{\varepsilon_j} \rightharpoonup \nabla u$ in $L^2(\Omega, \mathrm{M}^{n\times n})$ we have that $\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j} \longrightarrow 0$ strongly in $L^1(\Omega, \mathrm{M}^{n\times n})$, hence $e(u_{\varepsilon_j}) + \frac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j} \rightharpoonup e(u)$ weakly in $L^1(\Omega, \mathrm{M}^{n\times n})$. Then by Lemma 7.3.3 and (7.36) for every $k \in \mathbf{N}$ we have

$$\begin{aligned} \liminf_{\varepsilon_j \to 0} \int_\Omega W_{\varepsilon_j}(x, \nabla u_{\varepsilon_j})\, dx &\geq \liminf_{j\to+\infty} \int_\Omega V_j^k(x, e(u_{\varepsilon_j}) + \tfrac{1}{2}\varepsilon_j \nabla u_{\varepsilon_j}^T \nabla u_{\varepsilon_j})\, dx \\ &\geq \frac{1}{2}\int_\Omega \Big(1 - \frac{1}{k}\Big) \mathsf{A}(x)\,[e(u), e(u)]\, dx \,. \end{aligned} \tag{7.37}$$

Taking the supremum as $k \to \infty$ and considering the weak continuity of $\mathcal{L}$ we deduce inequality (7.35). ■

7.4 Convergence of minimizers

We are now in a position to prove Theorem 7.1.1.

Proof of Theorem 7.1.1. It is enough to prove the statement for evey sequence $\varepsilon_j \to 0$. Since $\mathcal{G}_{\varepsilon_j}(g) \leq C < +\infty$, we have $\mathcal{G}_{\varepsilon_j}(u_{\varepsilon_j}) \leq C < +\infty$, then by Proposition 7.1.3 u_{ε_j} is equibounded in $H^1(\Omega, \mathbf{R}^n)$. Thus there exists a subsequence u_{ε_k} converging weakly to some limit $w \in H^1_{g,\partial\Omega_D}$. By Γ-convergence we know that w must be the minimizer u_0 of the limit functional $\mathcal{G}$ (see, e.g., [24], Corollary 7.17)

Finally, as the limit w depends neither on the subsequence w_{ε_k} nor on the sequence ε_j, the whole sequence w_ε converges weakly to u in $H^1_{g,\partial\Omega_D}$. ■

In the sequel we will assume that $V(x,E)$ satisfies conditions (a), (b), and (c'). It is not restrictive to assume that $1 < p < 2$. Let α be the constant appearing in (b). From (a), (b), and (c') it follows that there exists a nondecreasing, continuous function $\phi(t)$ of the form

$$\phi(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq c\,, \\ \alpha c^2 & \text{for } c \leq t \leq d\,, \\ (\alpha c^2 d^{-p}) t^p & \text{for } d \leq t\,, \end{cases} \qquad \text{for } 0 < c < d\,,$$

such that $\phi(|E|) \leq V(x,E)$ for a.e. $x \in \Omega$. Consider the function $\psi_p(t)$ defined as

$$\psi_p(t) = \begin{cases} \alpha t^2 & \text{for } 0 \leq t \leq \mu\,, \\ a(t-b)^p & \text{for } \mu \leq t\,, \end{cases} \tag{7.38}$$

for $a = \alpha\, p^{-p}\, 2^p\, \mu^{2-p}$ and $b = (1 - \frac{p}{2})\mu$. It is not difficult to check that $\psi_p(t)$ is increasing, C^1, and convex. As $1 < p < 2$, we have

$$\lim_{\mu\to 0} \alpha\, p^{-p}\, 2^p\, \mu^{2-p} = 0\,, \tag{7.39}$$

thus for μ sufficiently small $\psi_p(t) \leq \phi(t)$ for every $t \geq 0$ and then $\psi_p(|E|) \leq V(x,E)$ for a.e. $x \in \Omega$ and every $E \in \mathrm{M}^{n\times n}_{sym}$.

Lemma 7.4.1 *Let $\varepsilon_j \to 0$. For every $k \in \mathbf{N}$ there exists an increasing sequence of Caratheodory functions $V_j^k : \Omega \times \mathbf{M}^{n\times n}_{sym} \to [0,+\infty)$ and a measurable function $\mu^k : \Omega \to (0,+\infty)$ such that for a.e. $x \in \Omega$ the function $V_j^k(x,\cdot)^{\frac{1}{p}}$ is convex and (7.31) and (7.32) hold.*

Proof. We follow the proof of Lemma 7.3.2, with ψ replaced by ψ_p, and we consider the functions

$$\phi_p^k(x,t) = \begin{cases} (1-\frac{1}{k})t^2 & \text{for } 0 \le t \le \mu^k(x)\,, \\ a(x)(t-b(x))^p & \text{for } t \ge \mu^k(x)\,. \end{cases}$$

Note that $\phi_p^k(x,t)^{\frac{1}{p}}$ is convex for $a(x) = (1-\frac{1}{k})2^p p^{-p}(\mu^k(x))^{2-p}$ and $b(x) = (1-\frac{p}{2})\mu^k(x)$. By (7.39) for $\mu^k(x)$ sufficiently small we have that

$$\phi_p^k(x, |E|_{\mathsf{A}(x)}) \le V(x,E)$$

for a.e. $x \in \Omega$ and every $E \in \mathbf{M}^{n\times n}_{sym}$. Then the sequence defined by $V_j^k(x,E) := \phi_p^k(x, \varepsilon_j|E|_{\mathsf{A}(x)})/\varepsilon_j^2$ satisfies (7.31) and (7.32), is increasing with respect to j, and $V_j^k(x,\cdot)^{\frac{1}{p}}$ is convex for a.e. $x \in \Omega$. ∎

Lemma 7.4.2 *Let $\Phi_n \rightharpoonup \Phi$ weakly in $L^1(\Omega, \mathbf{M}^{n\times n})$ such that $|\Phi_n|_{\mathsf{A}}$ converges to $|\Phi|_{\mathsf{A}}$ in measure. Then Φ_n converges to Φ in measure.*

Proof. By passing to a subsequence and to a suitable measurable subdomain, it is not restrictive to suppose that $\Phi(x) \neq 0$ for every $x \in \Omega$ and that $|\Phi_n|_{\mathsf{A}}$ converges to $|\Phi|_{\mathsf{A}}$ pointwise.

By (7.29) and by weak convergence we have

$$\int_\Omega \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n - \Phi\rangle_{\mathsf{A}}\, dx \longrightarrow 0\,, \tag{7.40}$$

where $\langle\cdot,\cdot\rangle_{\mathsf{A}}$ is the scalar product associated with the norm $|\cdot|_{\mathsf{A}}$, i.e.,

$$\langle \Psi_1, \Psi_2\rangle_{\mathsf{A}} = \frac{1}{2}\mathsf{A}(x)[\Psi_1(x), \Psi_2(x)]\,.$$

Moreover by the Schwarz inequality

$$\int_\Omega \left(\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n - \Phi\rangle_{\mathsf{A}}\right)^+ dx \le \int_\Omega (|\Phi_n|_{\mathsf{A}} - |\Phi|_{\mathsf{A}})^+\, dx\,.$$

As $|\Phi_n|_{\mathsf{A}}$ is equiintegrable and converges to $|\Phi|_{\mathsf{A}}$ in measure, it converges also in $L^1(\Omega)$. Thus

$$\int_\Omega \left(\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n - \Phi\rangle_{\mathsf{A}}\right)^+ dx \longrightarrow 0\,.$$

Then by (7.40) $\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n - \Phi\rangle_{\mathsf{A}} \longrightarrow 0$ in $L^1(\Omega)$ and, up to a subsequence, it converges for a.e. $x \in \Omega$, hence $\langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n\rangle_{\mathsf{A}} \longrightarrow |\Phi|_{\mathsf{A}}$ pointwise a.e. in Ω.

Considering the identity

$$|\Phi_n - \Phi|^2_{\mathsf{A}} = \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n - \Phi\rangle^2_{\mathsf{A}} + |\Phi_n|^2_{\mathsf{A}} - \langle \frac{\Phi}{|\Phi|_{\mathsf{A}}}, \Phi_n\rangle^2_{\mathsf{A}}\,,$$

we deduce that $|\Phi_n - \Phi|_{\mathsf{A}}^2 \longrightarrow 0$ pointwise for a.e. $x \in \Omega$.

Since for every subsequence of Φ_n we can find a further subsequence converging pointwise to Φ, it follows that Φ_n converges to Φ in measure. ∎

Proposition 7.4.3 *Let $\varepsilon_j \to 0^+$ and let $u_j \rightharpoonup u$ weakly in $H^1(\Omega, \mathbf{R}^n)$ such that*

$$\frac{1}{{\varepsilon_j}^2} \int_\Omega V(x, \varepsilon_j e(u_j) + \tfrac{1}{2}{\varepsilon_j}^2 C(u_j))\, dx \longrightarrow \int_\Omega |e(u)|_{\mathsf{A}}^2\, dx\,. \tag{7.41}$$

Then $u_j \longrightarrow u$ strongly in $W^{1,q}(\Omega, \mathbf{R}^n)$ for $1 \le q < 2$.

Proof. For every k let $V_j^k(x, E)$ be the sequence given by Lemma 7.4.1. Denote $e(u_j) + \frac{1}{2}\varepsilon_j C(u_j)$ by Φ_j. By (7.31) for every k and every j we have

$$V_j^k(x, \Phi_j) \le \frac{1}{{\varepsilon_j}^2} V(x, \varepsilon_j e(u_j) + \tfrac{1}{2}{\varepsilon_j}^2 C(u_j))\,. \tag{7.42}$$

By (7.41) it follows that, for every k, $V_j^k(x, \Phi_j)^{\frac{1}{p}}$ is bounded in $L^p(\Omega)$ uniformly with respect to j and k. Being $p > 1$, by a diagonal argument there exists a sequence $j_m \to \infty$ such that for every k

$$V_{j_m}^k(x, \Phi_{j_m})^{\frac{1}{p}} \rightharpoonup w^k \qquad \text{weakly in } L^p(\Omega)\,, \tag{7.43}$$

for a suitable function $w^k \in L^p(\Omega)$. Moreover by the weak convergence of u_j it follows that Φ_j converges weakly to $e(u)$ in $L^1(\Omega, \mathrm{M}^{n\times n})$. Since the functions $V_j^k(x, \xi)^{\frac{1}{p}}$ are convex in ξ, then by Lemma 7.3.3 and by (7.32), for every Borel set $B \subset \Omega$ we have

$$\left(1 - \frac{1}{k}\right)^{\frac{1}{p}} \int_B |e(u)|_{\mathsf{A}}^{\frac{2}{p}}\, dx \le \liminf_{m\to\infty} \int_B V_{j_m}^k(x, \Phi_{j_m})^{\frac{1}{p}}\, dx = \int_B w^k\, dx\,.$$

Thus

$$w^k \ge \left(1 - \frac{1}{k}\right)^{\frac{1}{p}} |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{a.e. in } \Omega\,. \tag{7.44}$$

Moreover, by the weak lower semicontinuity of the norm, from (7.41), (7.42), and (7.43) it follows that

$$\int_\Omega (w^k)^p\, dx \le \int_\Omega |e(u)|_{\mathsf{A}}^2\, dx\,. \tag{7.45}$$

Being $p > 1$, there exists $w \in L^p(\Omega)$ and a subsequence of w^k which converges weakly to w in $L^p(\Omega)$. Then passing to the limit in (7.44) we get

$$(w)^p \ge |e(u)|_{\mathsf{A}}^2$$

for a.e. $x \in \Omega$, and by (7.45) we have

$$\int_\Omega w^p\, dx \le \int_\Omega |e(u)|_{\mathsf{A}}^2\, dx\,.$$

These inequalities imply that $w = |e(u)|_{\mathsf{A}}^{\frac{2}{p}}$. Being the limit independent of the subsequence, we have proved for whole sequence w^k that

$$w^k \rightharpoonup |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{weakly in } L^p(\Omega). \tag{7.46}$$

Let $\mu^k(x)$ be the functions defined in Lemma 7.4.1. As $\mu^k > 0$ a.e. in Ω, there exists a decreasing sequence of constants η_k such that

$$\text{meas}(\{x \in \Omega : \mu^k(x) < \eta^k\}) \leq \frac{1}{k}. \tag{7.47}$$

Considering that $V^k_{j_m}(x, \Phi_{j_m})^{\frac{1}{p}}$ is bounded in $L^p(\Omega)$ uniformly with respect to m and k, and hence we can use a metric equivalent to the weak topology, by (7.43) and (7.46) we can extract a subsequence i_k of j_m such that, writing for simplicity ε_k instead of ε_{i_k}, we have

$$\frac{\eta^k}{\varepsilon_k} > k \qquad \text{and} \qquad V^k_{i_k}(x, \Phi_{i_k})^{\frac{1}{p}} \rightharpoonup |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \quad \text{weakly in } L^p(\Omega). \tag{7.48}$$

Then by (7.41) and (7.42) we have

$$\limsup_{k\to\infty} \int_\Omega V^k_{i_k}(x, \Phi_{i_k}) \leq \int_\Omega |e(u)|^2_{\mathsf{A}}\, dx.$$

By the uniform convexity of the $L^p(\Omega)$ space this implies that

$$V^k_{i_k}(x, \Phi_{i_k})^{\frac{1}{p}} \longrightarrow |e(u)|_{\mathsf{A}}^{\frac{2}{p}} \qquad \text{strongly in } L^p(\Omega).$$

Then we have

$$V^k_{i_k}(x, \Phi_{i_k}) \longrightarrow |e(u)|^2_{\mathsf{A}}$$

strongly in $L^1(\Omega)$ and a.e. in Ω.

Now we can prove that $|e(u_{i_k})|_{\mathsf{A}}$ converges in measure to $|e(u)|_{\mathsf{A}}$. Indeed, for every $\delta > 0$ the set $\{|\,|e(u_{i_k})|_{\mathsf{A}} - |e(u)|_{\mathsf{A}}\,| > \delta\}$ is contained in

$$\Big\{ \big||e(u_{i_k})|_{\mathsf{A}} - |e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}}\big| > \tfrac{\delta}{2} \Big\} \cup \Big\{ \big||e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} - |e(u)|_{\mathsf{A}}\big| > \tfrac{\delta}{2} \Big\}. \tag{7.49}$$

The first set is contained in $\{|\frac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} > \frac{\delta}{2}\}$, whose measure tends to zero since $\varepsilon_k C(u_{i_k}) \longrightarrow 0$ in $L^1(\Omega, \mathrm{M}^{n\times n})$. Note that for $x \in \{\mu^k(x) > \eta^k\}$ if

$$|e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} < k$$

then by (7.32) and (7.48) we have

$$V^k_{i_k}(x, \Phi_{i_k}) = \frac{k-1}{k} |e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|^2_{\mathsf{A}}.$$

Then the second set in (7.49) is contained in

$$\{\mu^k(x) < \eta^k\} \cup \{|e(u_{i_k}) + \tfrac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}} > k\} \cup \{|(\tfrac{k}{k-1} V^k_{i_k}(x, \Phi_{i_k}))^{\frac{1}{2}} - |e(u)|_{\mathsf{A}}| > \tfrac{\delta}{2}\}.$$

The measure of all these sets tends to zero as $k \to +\infty$. The first one by (7.47), the second one since $|e(u_{i_k}) + \frac{1}{2}\varepsilon_k C(u_{i_k})|_{\mathsf{A}}$ is equibounded in $L^1(\Omega)$, and the third one because $(\frac{k}{k-1} V^k_{i_k}(x, \Phi_{u_k}))^{\frac{1}{2}} \longrightarrow |e(u)|_{\mathsf{A}}$ pointwise. This concludes the proof of the convergence in measure of $|e(u_{i_k})|_{\mathsf{A}}$ to $|e(u)|_{\mathsf{A}}$.

Then by Lemma 7.4.2 it follows that $e(u_{i_k})$ converges in measure to $e(u)$. As $e(u_{i_k})$ is bounded in $L^2(\Omega, \mathrm{M}^{n\times n})$, we deduce that $e(u_{i_k})$ converges strongly to $e(u)$ in $L^q(\Omega, \mathrm{M}^{n\times n})$

for $1 \leq q < 2$. Since the limit does not depend on the subsequence we have that $e(u_j)$ converges strongly to $e(u)$ in $L^q(\Omega, \mathbf{M}^{n\times n})$.

By the Korn inequality (see e.g. [50]) there exists a constant C_q such that

$$\int_\Omega |\nabla(u - u_j)|^q \, dx \leq C_q \int_\Omega |e(u - u_j)|^q \, dx + C_q \int_\Omega |u - u_j|^q \, dx \,.$$

As $e(u_j)$ converges strongly to $e(u)$ in $L^q(\Omega, \mathbf{M}^{n\times n})$ and u_j converges strongly to u in $L^q(\Omega, \mathbf{R}^n)$ by the Rellich theorem, we deduce that u_j converges to u in the strong topology of $W^{1,q}(\Omega, \mathbf{R}^n)$. ■

Proof of Theorem 7.1.2. Let $\varepsilon_j \to 0$. By Proposition 7.1.3 u_{ε_j} converges weakly to u in $H^1(\Omega, \mathbf{R}^n)$ and by Γ-convergence we have $\mathcal{G}(u_{\varepsilon_j}) \longrightarrow \mathcal{G}(u)$ (see e.g. [24] Corollary 7.17). By weak continuity we have $\mathcal{L}(u_{\varepsilon_j}) \longrightarrow \mathcal{L}(u)$, so that (7.41) holds. The conclusion follows from Proposition 7.4.3. ■

Bibliography

[1] R. A. Adams: *Sobolev spaces*, Academic Press, New York (1975)

[2] G. Alberti, G. Bouchitté, P. Seppecher: Phase transition with the line-tension effect *Arch. Rational Mech. Anal.*, **144** (1998) 1–46

[3] R. Alicandro, M. Focardi, M. S. Gelli: Finite-difference approximation of energies in fracture mechanics, *Preprint SISSA 91/99/M* `http://cvgmt.sns.it/papers/alifocgel99/`

[4] L. Ambrosio: A compactness theorem for a new class of functions of bounded variation, *Boll. Unione Mat. Ital.* **3-B** (1989) 857–881

[5] L. Ambrosio, A. Braides: Energies in SBV and variational problems in fracture mechanics, *Proceedings of the International Conference "Homogenization and applications to material science"* Nice, June (1995)

[6] L. Ambrosio, A. Coscia, G. Dal Maso: Fine properties of Functions with Bounded Deformation, *Arch. Rational Mech. Anal.*, **139** (1997) 201–238

[7] L. Ambrosio, N. Fusco, D. Pallara: *Special Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford (2000)

[8] L. Ambrosio, V. M. Tortorelli: Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990) 999–1036

[9] G. Bellettini, A. Coscia: Discrete approximation of a free discontinuity problem, *Num. Funct. An. Opt.* **15** (1994) 201–224

[10] G. Bellettini, A. Coscia, G. Dal Maso: Compactness and lower semicontinuity properties in $SBD(\Omega)$, *Math. Z.* **228** (1998) 337–351

[11] B. Bourdin, A. Chambolle: Implementation of an adaptive finite element approximation of the Mumford-Shah functional, *Numer. Math.* **85** (2000) 609–646

[12] B. Bourdin, G. Francfort, J.J. Marigo: Numerical experiments in revisited brittle fracture *J. Mech. Phys. Solids* **48** (2000) 797–826

[13] A. Braides: *Approximation of free discontinuity problems*, Springer, Berlin (1998)

[14] A. Braides, G. Dal Maso: Non-Local Approximation of the Mumford-Shah Functional, *Calc. Var. Partial Differ. Equ.* **5** (1997) 293–322

[15] A. Blake, A. Zisserman: *Visual Reconstruction*, MIT Press, Cambridge (1987)

[16] M. Buliga: Energy minimizing brittle crack propagation, *J. Elasticity* **52** (1999) 201–238

[17] A. Chambolle: Personal communication, *Giornate di Lavoro sul Calcolo delle Variazioni*, Levico Terme (2001)

[18] A. Chambolle: Image segmentation by variational methods: Mumford and Shah functional and the discrete approximations, *Siam J. Appl. Math.* **55** 827–863

[19] A. Chambolle: Finite differences discretization of the Mumford-Shah functional, *Math. Model. Numer. Anal.* **33** (1999) 261–288

[20] A. Chambolle, G. Dal Maso: Discrete approximation of the Mumford-Shah functional in dimension two, *Math. Model. Numer. Anal.* **33** (1999) 651-672

[21] P.G. Ciarlet: *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam (1978)

[22] G. Cortesani, R. Toader: Finite element approximation of non-isotropic free-discontinuity problems, *Numer. Funct. Anal. Optimization* **18** (1997) 921–940

[23] G. Cortesani, R. Toader: A density result in SBV with respect to non-isotropic energies, *Nonlinear Anal.* **38** (1999) 585–604

[24] G. Dal Maso: *An Introduction to* Γ*-convergence*, Birkhäuser, Boston (1993)

[25] G. Dal Maso, M. Negri, D. Percivale: (to appear)

[26] G. Dal Maso, R. Toader: A model for the quasi-static growth of a brittle fracture: existence and approximation results, *Arch. Rational Mech. Anal.* (to appear) `http://www.sissa.it/fa/publications/pprint00.html`

[27] J. E. Dennis, R. B. Schnabel: *Numerical methods for unconstrained optimization and nonlinear equations*, Prentice Hall (1983)

[28] F. Dibos, E. Séré : An approximation result for the minimizers of the Mumford-Shah functional, *Boll. Unione Mat. Ital.* **11-A** (1997), 149–162

[29] F. Ebobisse : *Fine properties of functions of bounded deformation and applications in variational problems*, Ph. D. Thesis, University of Pisa (1999), `http://cvgmt.sns.it/papers/ebo99b/`

[30] F. Ferro, R. Goglione, M. Paolini: Finite element minimization of curvature functionals with anisotropy, *Calcolo* **31** (1994) 191–210

[31] G. Francfort, J.J. Marigo: Revisiting brittle fracture as an energy minimization problem, *J. Mech. Phys. Solids* **46** (1998) 1319–1342

[32] G. Friesecke, R. D. James, S. Müller: Rigorous derivation of nonlinear plate theory and geometric rigidity, *C. R. Acad. Sci. I* (to appear)

[33] P. E. Gill, W. Murray: *Practical Optimization*, Academic Press (1989)

[34] M. Gobbino: Finite difference approximation of the Mumford-Shah functional, *Comm. Pure Appl. Math.* **51** (1998) 197–228

[35] M. Gobbino, M.G. Mora: Finite difference approximation of free discontinuity problems, *Proc. Royal Soc. Edinburgh* **131** (2001) 567–595

[36] M. E. Gurtin: *An introduction to continuum mechanics* Academic Press, New York (1981)

[37] H. Liebowitz, G.C. Sih: Mathematical theories in brittle fracture, *Fracture: An advanced treatise, vol. II: Mathematical Fundamentals, ed. Liebowitz* 67–190, Academic Press, New York, 1968

[38] R. March: Visual reconstruction with discontinuities using variational methods, *Image Vision Comput.* **10** (1992) 705–712

[39] R. March, M. Dozio: A variational method for the recovery of smooth boundaries, *Image Vision Comput.*, **15** (1997) 705–712

[40] J.M. Morel, S. Solimini: *Variational Models in Image Segmentation*, Birkhäuser, Boston (1995)

[41] M. Morini, M. Negri: On the relationship between Mumford-Shah functional and Perona-Malik anisotropic diffusion, (in preparation)

[42] D. Mumford, J. Shah: Optimal approximation by piecewise smooth functions and associated problems, *Comm. Pure Appl. Math.* **17** (1989) 577–685

[43] M. Negri: The anisotropy introduced by the mesh in the finite element approximation of the Mumford-Shah functional, *Numer. Funct. Anal. Optimization*, **20** (1999) 957–982

[44] M. Negri: A finite element approximation of the Griffith's model in fracture mechanics *Numer. Math.* (submitted)

[45] M. Negri: A discontinuous finite element approach for the approximation of free discontinuity problems, (in preparation)

[46] M. Negri, M. Paolini: Numerical minimization of the Mumford-Shah functional *Calcolo*, **38** (2001) 957–982

[47] N. Nordström: Biased anisotropic diffusion - a unified regularization and diffusion approach to edge detection, *Image Vision Comput.* **8** (1990) 318–327

[48] J. Rice: Mathematical analysis in the mechanics of fracture, *Fracture: An advanced treatise, vol. II: Mathematical Fundamentals, ed. Liebowitz* 191–311, Academic Press, New York (1968)

[49] T. Richardson, S. Mitter: Approximation, computation, and distortion in the variational formulation, *Geometry-driven diffusion in computer vision* B. M. ter Haar Romeny ed., Kluwer Academic Publisher, Dordrecht (1994)

[50] R. Temam: *Problèmes mathématiques en plasticité*, Gauthier-Villars, Paris (1983)

Ringraziamenti personali

Per quest'ultima pagina mi piace l'idea di ricordare e ringraziare alcune delle persone con cui ho trascorso il periodo del Ph.D. Qualcuno non c'entra proprio niente con il contenuto di questa tesi, qualcuno invece ha avuto anche un "ruolo scientifico" e altri posso considerarli veramente come amici.

Senza troppi preamboli, partendo da Trieste ci sono i compagni della SISSA, il Marelli, Max & Mary, i colleghi d'ufficio, Roberto, il Bosciani e Guy, le "ragazze della porta accanto" Chiara & Maria Stella e Nadia, e poi i compagni di appartamento, Christian, Antonio, il Latitanzi, lo Zoccolan, Oronzo, Caruso Pavlosky e infine la "donna della mattina", la squadra di basket e tutti gli "artisti" del Carrozzone.

Poi non posso proprio dimenticarmi di Pavia, degli amici, Andrea, Luca, Rupert, del Pampa, e della mia famiglia, della nonna, del Ricky e della mia ex Donatella. Per non tralasciare la parentesi del servizio civile vorrei ringraziare anche Alfredo della Croce Verde di Quinto.

Per concludere, anche se è quasi scontato, un ringraziamento speciale lo riservo a Frau Doctor perché, nonostante tutto il tempo speso a scrivere, con lei ho passato veramente una bellissima estate.

Trieste, 13/9/2001