Hyperbolicity and Curvature in Dynamics and Control.

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Contents

Introduction

In this thesis, we will use some techniques developed in the frame of Optimal Control Theory and some tools of Hyperbolic Dynamics to investigate problems of Hamiltonian dynamics and infinite horizon optimal control.

The intimate relation between Optimal Control Theory and Hamiltonian Dynamics became clear after the publication of Pontryagin Maximum Principle (PMP) in the 50s ([**24**]): this result in fact shows that the extremals of an optimal control problem have to be seeked among the solutions of a certain Hamiltonian system associated to the problem.

More precisely, consider the control system on a smooth manifold M

$$\dot{q}(t) = f_u(q(t)), \quad q \in M, \quad u \in U \subset \mathbb{R}^m,$$

where f is smooth with respect to q and continuous with respect to the pair (q, u).

Let $\varphi : M \times U \to \mathbb{R}$ be a function with the same regularity as f, and consider the optimal control problem with fixed terminal time T and fixed endpoints $q_0,\ q_T$

$$\text{(I.1)} \qquad \left\{ \begin{array}{ll} \min_u \int_0^T \varphi(q(t), u(t))\, dt & q(0) = q_0 \\ \dot{q} = f_u(q) & q(T) = q_T \end{array} \right. \qquad q \in M,\ u \in U.$$

For any $u \in U$, associate to the problem a Hamiltonian function $h_u^\nu : T^*M \to \mathbb{R},\ \nu \in \mathbb{R}$, defined as

$$h_u^\nu(\lambda) = \langle \lambda, f_u(q) \rangle + \nu\varphi(q, u);$$

PMP states that a trajectory $\tilde{q}(t) : [0, T] \to M$ and a control $\tilde{u} \in L^\infty([0, T], U)$ are respectively an optimal trajectory and an optimal control of the problem (I.1) only if there exists a nontrivial pair $(\nu, \lambda(t)) \neq 0$ such that $\lambda(t) \in T^*_{\tilde{q}(t)}M$, $\nu \leq 0$, and

$$\begin{array}{rcl} \dot{\lambda}(t) & = & \vec{h}^\nu_{\tilde{u}(t)}(\lambda(t)) \\ h^\nu_{\tilde{u}(t)}(\lambda(t)) & = & \max\limits_{u \in U} h_u^\nu(\lambda(t)) \quad \textit{for a.e. } t \in [0, T] \end{array}$$

(recall that $\vec{h}^\nu_{\tilde{u}(t)}$ denotes the Hamiltonian vector field associated to $h^\nu_{\tilde{u}(t)}$ by $d_\lambda h^\nu_{\tilde{u}(t)} = \sigma(\cdot, \vec{h}^\nu_{\tilde{u}(t)})$).

When we adopt this approach, we switch our description from the *base manifold* M, where the extremals of the variational problem are described by a *second order* system of ODE's (the Euler-Lagrange equations), to the *phase space* T^*M, where we describe our problem in terms of a *first order* system of ODE's (Hamilton equations), defined on a space with double dimension with respect to M. The analogy with the Hamiltonian formulation of Classical Mechanics is absolutely not occasional: PMP is actually a generalization of Least Action Principle of Classical Mechanics.

A special case of an optimal control problem is the classical *length minimization* problem on a Riemannian manifold (M, g): given two points $q_0,\ q_1$, one wants to determine the curve from q_0 to q_1 which has the minimal length. It is well known that the geodesics are *locally minimizing* curves, which means that they provide a local solution to the minimization length problem, but that any geodesic is no more minimizing after a conjugate point (see for instance [**20**]). The sectional curvature is a metric invariant of the manifold that gives a great advice in solving this optimization problem: the *Conjugate Point Comparison Theorem* states in fact that if all the sectional curvatures of the manifold are negative, than no point of the manifold has conjugate points along the geodesics.

The idea of associating a curvature-type invariant to the extremals of a generic optimal control problem is due to Agrachev and Gamkrelidze, who introduced the notion in [**6**].

The theory has been further developed by Agrachev, Zelenko and Chtcherbakova in [**1**], [**8**], [**9**], [**10**], and is completely explained in the Lecture Notes [**2**]. In its final formulation, the *generalized curvature* is actually an invariant associated to a pair *(splitting of the tangent bundle, vector field)* on a smooth manifold M, not necessarily related to a particular optimal control problem.

In this thesis we will focus on the Hamiltonian case, thus regarding the curvature as a symplectic invariant of a pair *(Lagrangian splitting of the tangent bundle, Hamiltonian vector field)* on a symplectic manifold M.

Let us now briefly recall the definition of the curvature. Let (M, σ) be a symplectic manifold, and let Λ, Π be two Lagrangian distributions that define a splitting of the tangent bundle, $TM = \Lambda \oplus \Pi$. Let $h : M \to \mathbb{R}$ be a smooth Hamiltonian function, and $\vec{h}$ is its associated Hamiltonian vector field. The pairs $(\Lambda, \vec{h})$ and $(\Pi, \vec{h})$ define, for any fixed $z \in M$, two *curves* in the Lagrange Grassmannian of T_zM, as in the following:

$$\Lambda_z(t) := e^{-t\vec{h}}{}_*\Lambda_{z_t}, \quad \Pi_z(t) := e^{-t\vec{h}}{}_*\Pi_{z_t}, \qquad z_t = e^{t\vec{h}}(z);$$

for any t, $\Lambda_z(t)$ $(\Pi_z(t))$ is a Lagrangian subspace of T_zM; then, the pair $(\Lambda, \vec{h})$ $((\Pi, \vec{h}))$ actually defines a one-parametric family of Lagrangian distributions on M.

For any $z \in M$ and any t, the curvature associated to the pair $(\Lambda \oplus \Pi, \vec{h})$ is a linear operator $R_z^{\Lambda,\Pi}(t) : \Lambda_z(t) \to \Lambda_z(t)$; it is defined as

$$R_z^{\Lambda,\Pi}(t) := -\overline{\dot{\Pi}}_z(t) \circ \overline{\dot{\Lambda}}_z(t),$$

where $\overline{\dot{\Lambda}}_z(t) : \Lambda_z(t) \to \Pi_z(t)$ is a linear operator intrinsically associated to the curve $\Lambda_z(t)$, and actually depends only on the derivative of the curve at the time t $\frac{d}{d\tau}\Lambda_z(\tau)|_{\tau=t}$. The same, with the proper substitutions, about $\overline{\dot{\Pi}}_z(t)$.

Thus defined, the curvature measures the “relative velocity” of the two curves with respect to each other.

Since the definition is intrinsic, it allows the following identity:

$$R_z^{\Lambda,\Pi}(t) = e^{-t\vec{h}}{}_* R_{z_t}^{\Lambda,\Pi}(0) e^{t\vec{h}}{}_*|_{\Lambda_z(t)}.$$

The relation above is of great importance: in fact, the knowledge of the curvature $R_{z_0}^{\Lambda,\Pi}(t)$ at a fixed initial point z_0 at any time is equivalent to the knowledge of $R_{z_t}^{\Lambda,\Pi}(0)$ along the Hamiltonian trajectory $z_t = e^{t\vec{h}}(z_0)$ for fixed time. This means that the curvature is a *local* invariant that carries information about the *global* extremal.

The definition is justified by optimal control theory; in [**6**], the authors associated to any extremal of the optimal control problem a Lagrangian curve $J_z(t) = e^{-t\vec{h}}{}_* T_z(T^*_{\pi(z)}M)$, which is called *Jacobi curve* (in analogy with Jacobi field of Riemannian geometry). We say that the parameter τ is conjugate to t if $J_z(t) \cap J_z(\tau) \neq 0$: the authors showed that an extremal cannot be a minimizer after the first conjugate parameter of its associated curve in the Lagrange Grassmannian (in complete analogy with the Riemannian case).

Moreover, given a Lagrangian distribution Λ, there is an intrinsically defined distribution Λ° such that $\Lambda \oplus \Lambda^\circ$ is a Lagrangian splitting (called *canonical splitting*) and the curvature $R_z^{\Lambda,\Lambda^\circ}$ satisfies an analogous result to Conjugate Point Comparison Theorem of Riemannian Geometry. In fact, there exists a scalar product that makes the curvature operator self-adjoint; hence its eigenvalues are real, and it is possible to talk about *positive* or *negative* curvature. In particular, the generalized Conjugate Point Comparison Theorem states that if the generalized curvature is negative, then there are no conjugate parameters along the Hamiltonian extremals, while, if the curvature is positive and bounded from below, there are estimates for the location of conjugate parameters along the extremals.

As mentioned before, it is possible to associate this curvature-type invariant to a generic pair *(Lagrangian splitting, Hamiltonian vector field)*, not necessarily arising from an Optimal Control Problem. Actually, the curvature gives a lot of information about the behaviour of the dynamical system: in [**2**] and [**5**] it is proved that the negativity of the curvature is a symptom of a hyperbolic behaviour of the Hamiltonian dynamical system.

In this thesis we will also use some classical results of Hyperbolic theory. Hyperbolic dynamical systems constitute a wide and important class of dynamical systems (or DS). They possess many important properties; the most characterizing is the existence, for any point of the state space, of a *contraction* and an *expansion* direction in the tangent space; more precisely, if $f : M \to M$ is a diffeomorphism on a smooth manifold M, it is said to be hyperbolic if there are two distributions $E^\pm = \{E_z^\pm\}_{z\in M}$ and two numbers $0 < \lambda < 1 < \mu$ such that for any z and $n > 0$

$$\begin{aligned} \|D_z f^n X\| &\leq \lambda^n \|X\| \qquad X \in E_z^- \\ \|D_z f^{-n} X\| &\leq \mu^{-n} \|X\| \qquad X \in E_z^+. \end{aligned}$$

Such systems are characterized by *sensitive dependence on initial conditions*, which means that nearly arising trajectories may diverge exponentially with time one from each other. This behaviour is the typical behaviour of chaotic systems, and in fact hyperbolic systems constitute a paradigmatic example of chaos.

To measure “chaoticity” of a dynamical system we have at our disposal many quantities; in this thesis, we will use the *Lyapunov exponents* and the *measure-theoretic (Kolmogorov-Sinai) dynamical entropy*.

We saw that a hyperbolic DS is characterized by the existence of directions of expansion and contraction under the linearization of the flow; moreover, the growth of norms of the vectors is controlled by coefficients which are uniform on the base space. Lyapunov exponents are introduced to take account of cases in which these expansion and contraction directions exist at *almost any* point of the base space (with respect to some measure), and the norms are controlled by coefficients that possibly depend on the point.

The Lyapunov exponents are in fact the coefficients that control the asymptotic evolution of the norm of the vectors, under the action of the linearization of the evolution f. Actually, where defined they determine a splitting of the tangent space into three subspaces, that are a kind of generalization of the stable, the unstable space and the central space of a hyperbolic DS (along the central space there is no expansion neither contraction, and in fact the value of the Lyapunov exponent is zero); for $z \in M$, these subspaces are denoted as E_z^-, E_z^+ and E_z^0.

The Kolmogorov-Sinai dynamical entropy measures the statistical behaviour of the orbits.

Let f be a diffeomorphism on a smooth manifold M, and μ an f-invariant probability measure on M. For any measurable set $U \subset M$, $\mu(U)$ has the meaning of the *probability* of the state of the system of belonging to U. Let $\mathscr{P} = \{P_i\}_i$ be a (finite or countable) measurable partition of M (which means that each P_i is measurable, $\mu(P_i \cap P_j) = 0$ if $i \neq j$, and $\mu\big(M \setminus \cup_i P_i\big) = 0$); the dynamical entropy associated to the partition $\mathscr{P}$ measures how precisely the evolution of the system can be predicted, when the initial condition is known only by means of the partition $\mathscr{P}$. In other words, it is the average amount of information about the system provided by the knowledge of the present state and an asymptotic future. In particular, the entropy of well-predictable systems (such as periodic ones) is zero, while a positive value of the dynamical entropy is a symptom of chaotic behaviour.

To get a partition-independent object, that shall depend only on the measure μ and the dynamics f, the Kolmogorov-Sinai dynamical entropy of the system (M, μ, f) is defined as the supremum of the KS dynamical entropies of the system associated to a particular partition.

In general, the existence of positive and negative Lyapunov exponents almost everywhere on M is a stronger requirement than the positivity of the dynamical entropy; however, Pesin's Theorem states that under some regularity conditions the two facts are equivalent, and the two quantities provide the same information. In fact, under these conditions we have that

$$h_\mu(f) = \int_M \chi \, d\mu(z),$$

where $h_\mu(f)$ is the dynamical entropy associated to (M, μ, f) and $\chi(z)$ (where defined) is the sum of the positive Lyapunov exponents, taken with multiplicity.

In this thesis we will apply the techniques exposed above in two different frames; the first result gives an estimate for the entropy of Hamiltonian flows, and constitute one more proof of the fact that the curvature operator proposed by Agrachev and Gamkrelidze is a true generalization of the sectional curvature; in the second problem, we used the generalized curvature and some facts of Hyperbolic Dynamics to state the existence of an optimal synthesis for infinite horizon variational problems. Actually, this is the first result of a wide field of research that brings a lot of interesting open questions.

The first result we will expone in this thesis is a generalization to Hamiltonian flows of the Ballmann-Wojtkowski estimate for the dynamical entropy of the geodesic flow. In [**15**], the authors proved the following relation

$$h_\mu \geq \int_{SM} \operatorname{tr}\sqrt{-K(v)} \, d\mu(v),$$

where h_μ is the KS dynamical entropy of the geodesic flow on a compact Riemannian manifold (M, g), SM is the spherical tangent bundle on M (i.e. the subbundle $SM = \{v \in TM : g(v, v) = 1\}$, $K(v)$ is defined as $K(v) = \mathcal{R}(\cdot, v)v$, where $\mathcal{R}$ denotes the Riemannian curvature, and $d\mu$ is the Liouville measure on M. The result allows under the hypothesis of nonpositivity of the curvature $\mathcal{R}$.

In the case we consider, we deal with a smooth Hamiltonian function $h : M \to \mathbb{R}$, where M is a smooth $2n$-dimensional symplectic manifold. It is well known that the Hamiltonian flow preserves the sublevels of the Hamiltonian functions: then, we have to restrict ourselves to a regular level set of the function h.

The pair $(\Lambda \oplus \Lambda^\circ, \vec{h})$ possesses another symplectic invariant, the *reduced curvature* $\widehat{R}_z^{h,\boldsymbol{h}}$, that takes account of the reduction of the system to a sublevel of the Hamiltonian; this operator was introduced by Agrachev, Zelenko and Chtcherbakova in [**10**].

Due to reduction on a sublevel of h, in this problem we deal with the reduced curvature. We obtain the following result:

THEOREM I.1. *Let N be a compact regular level set of a smooth Hamiltonian function $h : M \to \mathbb{R}$, where M is a $2n$-dimensional symplectic manifold. Let Λ be a Lagrangian distribution on $TN/\mathrm{span}\{\vec{h}\}$, and assume that the Hamiltonian vector field $\vec{h}$ is monotone with respect to Λ. Consider the Jacobi curve $\Lambda_{z_0}(t) = e^{-t\vec{h}}{}_*\Lambda_{z_t}$ and assume that the restricted curvature $\widehat{R}_z^{h,\boldsymbol{h}}$ is nonpositive.*

Then the dynamical entropy h_μ of the Hamiltonian flow on N with respect to the normalized Liouville measure μ on N satisfies the following inequality:

$$h_\mu \geq \int_N \mathrm{tr}\sqrt{-\widehat{R}_z^{h,\boldsymbol{h}}}\, d\mu. \tag{I.2}$$

The main argument we use in the proof is the Pesin Theorem, which permits us to estimate the value of the dynamical entropy computing the Lyapunov exponents. In this way, we avoid to deal with flow-invariant measures and partitions, and just have to compute the asymptotic growth of the vectors under the action of the (linearized) flow. Actually, it can be shown that the sum $\chi(z)$ of the positive Lyapunov exponents at z, if defined, is given by the formula

$$\chi(z) = \lim_{t\to\pm\infty} \frac{1}{t} \log|\det(D_z e^{t\vec{h}})|_{E_z}|, \tag{I.3}$$

where E_z is any subspace such that $E_z^+ \subset E_z \subset E_z^+ \oplus E_z^0$.

The first part of the proof is then devoted to the definition of a good candidate for such a E_z, and to check that it satisfies the requirements. We will define the space E_z in terms of the canonical splitting as the graph of a linear operator $U_z : \Lambda_z^\circ \to \Lambda_z$; this operator (to be precise, its representation in coordinates) is related to the generalized curvature by means of a Riccati equation.

In the second part, we do the computations of the term (I.3), and then, using some basic facts of ergodic theory, we arrive to the estimate for the dynamical entropy. In this part it is crucial the nonpositivity of the reduced curvature. In fact, we obtain an expression for $\chi(z)$ in terms of the representation in coordinates of U_z, which means in terms of the generalized curvature. Its negativity permits us to do some computations to get a term that depends only on $\widehat{R}_z^{h,\boldsymbol{h}}$. This leads us to the estimate (I.2).

These results have been published in [**16**].

The second topic we study deals with the existence of the optimal synthesis of infinite horizon variational problems. Infinite horizon optimal control problems have nice applications in mathematical economy, because they provide a good model for the dynamical economic system. Of great importance is the *optimal economic growth problem*:

$$\begin{cases} \max_{q(t)} \int_0^{+\infty} e^{-\alpha t}\varphi(q(t), u(t), t)\, dt \\ \dot{q}(t) = f(q(t), u(t)) \end{cases} \qquad \alpha \geq 0.$$

The functional to be maximized models the capital accumulation; there is also a formulation of the same problem involving the *minimization* of the functional, which in this case represents some cost to be minimized during a production process.

The problem is very general and can be studied in very different settings; in this thesis, we concentrated to the *smooth* case with continuous time. This means that, given a smooth manifold M and a smooth Lagrangian $\varphi : TM \to \mathbb{R}$, we focus on the functional

$$J(\gamma(\cdot)) = \int_0^\infty \varphi(\gamma(t), \dot{\gamma}(t))\, dt \tag{I.4}$$

defined on the Lipschitzian curves $\gamma : [0, +\infty) \to M$ along which the integral (I.4) converges.

More precisely, we assume the existence of an *equilibrium point* q_∞ such that $\varphi(q_\infty, 0) = 0$ and $\frac{\partial\varphi}{\partial q}(q_\infty, 0) = 0$, and, fixed some $q_0 \in M$, we want to find the minimum of the functional J in the class

$$\Gamma(q_0) = \Big\{\gamma : [0, +\infty) \to M : \gamma \text{ is Lipschitzian and } \gamma(0) = q_0, \lim_{t\to+\infty} \gamma(t) = q_\infty\Big\}.$$

Our aim is to determine the class of Lagrangians φ such that the minimization problem admits a smooth optimal synthesis, according to the following definition:

DEFINITION. *A smooth optimal synthesis is a smooth complete vector field X on M such that the point q_∞ is a stable equilibrium of the ordinary differential equation $\dot{q} = X(q)$ and for any $q_0 \in M$:*

$$J(\gamma_X) = \min_{\gamma\in\Gamma(q_0)} J(\gamma),$$

where $\dot{\gamma}_X(t) = X(\gamma_X(t))$ and $\gamma_X(0) = q_0$.

In order to do that, first we formulate the variational problem as an optimal control problem:

$$\min_u \int_0^\infty \varphi(q(t), u(t))\, dt : \qquad \dot{q} = u, \qquad \begin{cases} q(0) = q_0 \\ q(t) \to q_\infty \text{ as } t \to +\infty \end{cases} ; \tag{I.5}$$

then we associate to the problem a Hamiltonian function H on the cotangent bundle, defined as $H(\lambda) = \max_u \langle\lambda, u\rangle - \varphi(q, u)$.

The main result we obtain is the following:

THEOREM I.2. *Let M be a complete Riemannian smooth manifold, and assume that it is simply connected; let $\varphi : TM \to \mathbb{R}$ be a smooth function such that*

(H1) *φ is bounded from below and is strongly convex with respect to the second variable; moreover, we assume that φ grows superlinearly in the second variable with respect to the given Riemannian metric, i.e. $\varphi(q, u) + c > 0$ for some constant c and*

$$\frac{|u|}{\varphi(q, u) + c} \to 0 \quad \text{as } |u| \to +\infty;$$

(H2) *there is a unique point q_∞ such that*

$$\varphi(q_\infty, 0) = 0 \quad \text{and} \quad \frac{\partial\varphi}{\partial q}(q_\infty, 0) = 0;$$

(H3) *there exist constants $a,\ b > 0$ such that for any (q, u)*

$$|\partial_q\varphi(q, u)| \leq a(\varphi(q, u) + |u|) + b,$$

where ∂_q is the covariant derivative.

Let $\{\Lambda_z\}_z = \{T_z(T^*_{\pi(z)}M)\}_z$ and $\{\Pi_z\}_z$, $z \in T^*M$, be two Lagrangian distributions that provide a splitting of $T(T^*M)$; assume that the generalized curvature of $\vec{h}$ with respect to the splitting is negative definite for any z. Then the problem (I.5) with final point q_∞ admits a smooth optimal synthesis on M.

To prove the statement, we use a classical result of Optimal Control Theory, that provides a sufficient condition for the existence of a solution to an optimal control problem. This Theorem is stated in Section 3 of Chapter 1; its application to the case under consideration states that a sufficient condition for the existence of an optimal synthesis is the existence of a Lagrangian stable invariant (with respect to the Hamiltonian flow) submanifold of T^*M that projects diffeomorphically on the base space M.

The aim of the proof becomes then to find such a submanifold. The proof splits into two main parts: in the first one, we look for those Lagrangians such that their associated Hamiltonian possesses a *hyperbolic fixed point* $z_\infty \in T^*M$ (this hyperbolic fixed point in fact is a lift of the equilibrium q_∞); this, for classical results in Hyperbolic Theory, guarantees the existence of two smooth invariant submanifolds of T^*M, called the stable and the unstable manifolds. The former, denoted with $W^s(z_\infty)$, is a good candidate for the submanifold we are looking for, and it can be defined by the characterization:

$$W^s(z_\infty) = \{z \in T^*M : d(e^{t\vec{H}}(z), z_\infty) \to 0 \text{ as } t \to +\infty\}.$$

Then, in the last part of the proof we have to verify that in fact $W^s(z_\infty)$ has the required properties, i.e. it is Lagrangian and projects diffeomorphically to M.

In the whole proof the hypothesis of negativity of the generalized curvature is crucial. In particular we use this fact to establish the hyperbolicity of the equilibrium point of the Hamiltonian z_∞; the negativity of the curvature implies also that any bounded (i.e. with compact closure) Hamiltonian semitrajectory shall converge to the equilibrium with exponential rate; in other words, the system has a saddle-like behaviour.

Moreover, we formulate the same problem (I.5) on the Euclidean space, and we see that the hypotheses that φ shall satisfy are very natural: instead of the hypotheses on the curvature, to guarantee the existence of the optimal synthesis we need assumptions (H1)–(H3) and the hypotheses that φ is strictly convex (where with “strict convexity” we mean that the second derivative is always positive definite).

The next natural question is to examine what happens in the case of positive generalized curvature. The generalized Conjugate Points Comparison Theorem (Theorem 2.1) says that there cannot be optimal syntheses for the infinite horizon problem, due to presence of conjugate points. Then, we introduce a *discount* or *forgetting factor* $\alpha > 0$ and study the minimization problem associated to the functional

$$J^\alpha(\gamma(\cdot)) = \int_0^\infty e^{-\alpha t}\varphi(\gamma(t), \dot\gamma(t))\, dt. \tag{I.6}$$

The following example helps us to formulate correctly and then solve the problem: we consider the 1-dimensional case and the quadratic Lagrangian $\varphi(q, \dot q) = \frac{1}{2}(\dot q^2 - rq^2)$, where r is a constant (and is also the generalized curvature). If $r < 0$, then the problem admits an optimal synthesis for any value of α, also for $\alpha = 0$ (in fact, we are in the hypotheses of Theorem I.2).

If $r > 0$, it can be shown that the value of α is crucial for the existence of the optimal synthesis; in particular, for $\alpha < 2\sqrt{r}$, any the integral in (I.6) diverges along any Hamiltonian trajectory of the system, then there are no minimizing trajectories. Otherwise, if $\alpha > 2\sqrt{r}$, there are Hamiltonian trajectories that grow as $e^{(\frac{\alpha}{2}-\varepsilon)t}$, for some $\varepsilon \in (0, \frac{\alpha}{2})$. Along these trajectories the integral converges, and it can be shown that actually they are minimizing trajectories.

These example suggest us to look for minimizers belonging to the following class:

$$\Big\{\gamma : [0, +\infty) \to M : \gamma \text{ is locally Lipschitzian and } \gamma(0) = q_0, \lim_{t\to+\infty} e^{-\alpha t}|\dot{\gamma}(t)|^2 = 0\Big\},$$

for some fixed initial point q_0.

In this thesis it is contained the following result in this direction:

THEOREM I.3. *Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth, and strictly convex with respect to the second variable; assume that there is a compact set $K \in \mathbb{R}^{2n}$ such that the function φ is quadratic in the pair (q, u) for any $(q, u) \in \mathbb{R}^{2n} \setminus K$.*

If the curvature R^0_z of the Hamiltonian vector field $\vec{H}^0$ (i.e. associated to the system with $\alpha = 0$) with respect to the canonical splitting satisfies the inequalities $0 < R^0_z \le C$ for some constant C for any $z \in \mathbb{R}^{2n}$, then the infinite horizon variational problem without discount does not admit optimal trajectories, while the problem with discount admits an optimal synthesis of class C^1 if $\alpha > 2\sqrt{C}$.

The generalization of this Theorem to a wider class of Hamiltonians is work in progress.

This dissertation is thus organized: Chapter 1 contains an introduction to the language and the main results about Dynamical Systems Theory and Optimal Control Theory we will use.

Chapter 2 is devoted to the definition of the generalized curvature and to its properties. In particular, we will state the (generalized) *Conjugate Point Comparison Theorem*.

In Chapter 3 we will prove our results on the entropy of Hamiltonian flows.

In Chapter 4 we will introduce the infinite horizon variational problem we are studying, and prove Theorem I.2. Then, in Section 4 we will restrict the problem to the Euclidean case, and prove that the results allow also under the hypothesis of *strict convexity* of the Lagrangian φ, without any assumptions on the generalized curvature. In Section 5 we will analyze the 1-dimensional case.

Finally, Chapter 5 contains the results on the minimization problem in the presence of discount.

CHAPTER 1

Preliminaries

In this chapter, we will introduce the classical mathematical results about optimal control theory and dynamical systems used in this thesis.

In the first section there is just a brief recall of the basic tools of symplectic geometry.

In the second section, we will present some general aspects about smooth dynamical systems, with a special attention given to those which present hyperbolic behaviour.

The third section is devoted to the definition of optimal control problem and to the statement of the Pontryagin Maximum Principle, which gives a necessary condition for optimality. Then, it will stated also a sufficient condition.

1. Elements of Geometry

In this section we will just give an essential review of the basic facts on symplectic geometry we are using in this thesis; the argument is widely treated in many textbooks: we will remand to [**12**] and [**13**].

Let Σ be a $2n$-dimensional linear space endowed with a symplectic structure σ; we recall that a subspace $V \subset \Sigma$ is called *isotropic* if the symplectic form vanishes on it, i.e. $\sigma(X, Y) = 0$ for any $X, Y \in V$. An isotropic subspace is always contained in its skew-orthogonal complement, that we will denote by $V^{\angle} = \{X \in \Sigma : \sigma(X, Y) = 0 \ \forall\ Y \in V\}$; a *Lagrangian* subspace of Σ is an isotropic subspace that coincides with its skew-orthogonal complement, which implies that it has the maximal dimension, i.e. n.

Let us moreover recall that if V is a Lagrangian subspace of Σ, the symplectic structure induces an isomorphism between the dual of the space V^* and the quotient Σ/V; indeed, let, for any $\boldsymbol{v} \in \Sigma$, $[\boldsymbol{v}]$ be its equivalence class in Σ/V; then $\sigma([\boldsymbol{v}], \cdot)$ is a linear operator on V and nondegeneracy of σ guarantees that the map $[\boldsymbol{v}] \mapsto \sigma([\boldsymbol{v}], \cdot)$ is injective. Moreover, to any linear form $\varphi \in V^*$ we can find a vector $\boldsymbol{v}_\varphi \in \Sigma$ such that $\varphi(\cdot) = \sigma(\boldsymbol{v}_\varphi, \cdot)$; the proof is completely analogous to the one in the case of a symmetric scalar product. By injectivity of $[\boldsymbol{v}] \mapsto \sigma([\boldsymbol{v}], \cdot)$, there is a unique $\boldsymbol{v} \in \Sigma/V$ with this property.

Let $\Sigma = \Lambda \oplus \Pi$ be a Lagrangian splitting of Σ, namely $\Lambda \cap \Pi = 0$ and both the spaces are Lagrangian. The symplectic form identifies Π with Λ^*, and vice versa; then, the following isomorphisms allow: $\Lambda \simeq \Sigma/\Pi$ and $\Pi \simeq \Sigma/\Lambda$.

For any symplectic space Σ there exists a special basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ such that

$$\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0 \quad \text{for any } i, j \qquad \sigma(e_i, f_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker-δ; such a basis is called a *Darboux basis*. This basis actually identifies Σ with $\mathbb{R}^{n*} \times \mathbb{R}^n$: in fact, let $\boldsymbol{v}_i,\ i = 1, 2$, be two vectors in Σ, and write them as $\boldsymbol{v}_i = \sum_{j=1}^n x_i^j e_j + y_i^j f_j$; then we have that $\sigma(\boldsymbol{v}_1, \boldsymbol{v}_2) = \sum_{j=1}^n x_1^j y_2^j - x_2^j y_1^j = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle$, where $x_i,\ y_i \in \mathbb{R}^n$ for $i = 1, 2$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^n$.

Given a Lagrangian subspace $V \subset \Sigma$ and a basis $\{e_1, \dots, e_n\}$ of V, we can always use a modified version of the Gram-Schmidt algorithm to construct vectors $\{f_1, \dots, f_n\}$ in Σ in such a way that $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a Darboux basis for Σ.

The *Lagrange Grassmannian* of Σ is the set of all the Lagrangian subspaces of Σ, and has the structure of an $n(n+1)/2$-dimensional smooth manifold; we denote it with $\mathcal{L}_n(\Sigma)$.

Any space $\Delta \in \mathcal{L}_n(\Sigma)$ gives a coordinate chart on the Lagrange Grassmannian; more precisely, if $\Delta^{\pitchfork} = \{\Lambda \in G_n(\Sigma) : \Lambda \cap \Delta = 0\}$ denotes the set of n-dimensional subspaces of Σ that are transversal to Δ (here $G_n(\Sigma)$ is the Grassmannian of n-dimensional subspaces of Σ), Δ defines a coordinate chart on $\Delta^{\pitchfork} \cap \mathcal{L}_n(\Sigma)$. To show this, let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a Darboux basis of Σ such that $\Delta = \operatorname{span}\{f_1, \dots, f_n\}$; Δ is then identified with $\{(0, y) : y \in \mathbb{R}^n\}$. Then, for any $\Lambda \in \Delta^{\pitchfork}$ there is an $n \times n$ matrix S_Λ such that Λ can be parametrized as $\Lambda = \{(x, S_\Lambda x) : x \in \mathbb{R}^n\}$; Lagrangianity of the space implies that the matrix is symmetric. In this way, $\Delta^{\pitchfork} \cap \mathcal{L}_n(\Sigma)$ is identified with the spaces of $n \times n$ symmetric matrices; then the pair $(\Delta^{\pitchfork}, \Lambda \mapsto S_\Lambda)$ gives a chart of $\mathcal{L}_n(\Sigma)$.

An even dimensional smooth manifold is called symplectic if its tangent space at any point has a symplectic structure. A submanifold of a symplectic manifold is said to be Lagrangian if its tangent space at any point is a Lagrangian subspace of the tangent space to the original manifold (with respect to the same symplectic structure).

Let us now consider an n-dimensional smooth manifold M; we recall that its cotangent bundle

$$T^*M = \bigcup_{q \in M} T_q^*M \tag{1.1}$$

has a natural symplectic structure; in fact, let us denote with $\pi : T^*M \to M$ the canonical projection, and let us define, for any $\lambda \in T^*M$, the *Liouville (or tautological) form*

$$\vartheta_\lambda := \lambda \circ \pi_*, \tag{1.2}$$

which is a one-form on T^*M whose action is given by $\langle \vartheta_\lambda, \boldsymbol{x} \rangle = \langle \lambda, \pi_*(\boldsymbol{x}) \rangle$ for any $\boldsymbol{x} \in T_\lambda(T^*M)$. The symplectic structure on T^*M is given by the form

$$\sigma = d\vartheta,$$

which is in fact a closed nondegenerate skew-symmetric two-form.

If we choose a system of canonical coordinates (p, q) on T^*M, the Liouville form reads as $\vartheta_\lambda = \sum_{i=1}^n p_i dq_i$, and the canonical symplectic structure as $\sigma_\lambda = \sum_{i=1}^n dp_i \wedge dq_i$.

Let us recall moreover that to any smooth function $h : T^*M \to \mathbb{R}$ on the cotangent bundle we can associate a *Hamiltonian vector field* $\vec{h}$ on the cotangent bundle according to the relation

$$\sigma_\lambda(\vec{h}, \cdot) = -d_\lambda h.$$

Let us just say that in the following we will denote with $e^{t\vec{h}}$ the flow generated by $\vec{h}$ on M, where defined; clearly, if the Hamiltonian vector field is autonomous, the flow is defined for any $t \in \mathbb{R}$.

2. Elements of dynamical systems

In this section we will recall some fundamental results about dynamical systems, with a special emphasis on the ones that show a hyperbolic behaviour. For a complete treatment, we remand to the textbooks [**18**] and [**19**]; we also mention [**21**] for details on ergodic theory and dynamical entropy. A good reference on partial hyperbolicity is given by [**23**].

2.1. Hyperbolic Theory. Let M be a smooth manifold and $f : M \to M$ a smooth map. The pair (M, f) defines a *smooth discrete-time dynamical system*.

The class of dynamical systems we are mostly interested in is the one of *hyperbolic systems*; these system provide a paradigmatic example of the behaviour of a chaotic system.

The most topical property of hyperbolic systems is the existence of a splitting of the tangent space (to the state space) into two subspaces, called respectively the *stable* and the *unstable* space, such that on the former the vectors contract under the action of the flow, and on the latter they expand, with *exponential* rate.

Hyperbolic dynamical systems possess many other interesting properties, among which we recall *sensitive dependence on initial conditions* and *positive dynamical entropy*. The latter is studied in Subsection 2.4; the former is related to the "relative closeness" of the orbits. By the continuity with respect of the initial conditions of ODE's, we know that two solutions of the same equation arising from sufficiently near initial conditions remain near for a certain time interval; in hyperbolic dynamical system two nearly arising trajectories may diverge exponentially with time one from each other.

The class of hyperbolic dynamical system include many different examples; among them, we recall the *geodesic flow on a compact Riemannian manifold of negative curvature* (see the introduction to Chapter 3), and the celebrated *Hyperbolic Toral Automorphism*, that we will describe in the Example below.

DEFINITION 1.1. *Let q_0 be a fixed point of the map f; q_0 is a hyperbolic fixed point for f if the linear map $D_{q_0} f : T_{q_0} M \to T_{q_0} M$ is a hyperbolic map, i.e. if it has no eigenvalues with absolute value equal to 1.*

If q_0 is a hyperbolic fixed point for f, it means that there exist two constants $0 < \lambda < 1 < \mu$ such that $\mathrm{Sp}(D_{q_0} f) \cap \{z \in \mathbb{C} : \lambda < |z| < \mu\} = \emptyset$, where $\mathrm{Sp}(D_{q_0} f)$ denotes the spectrum of the map $D_{q_0} f$; this implies that $T_{q_0} M$ splits into two subspaces $E^+_{q_0}$, $E^-_{q_0}$ such that $D_{q_0} f(E^+_{q_0}) \subset E^+_{q_0}$, $D_{q_0} f(E^-_{q_0}) \subset E^-_{q_0}$, and we have that

$$\|D_{q_0} f X\| \leq \lambda \|X\| \qquad \forall\, X \in E^-_{q_0} \tag{1.3}$$

$$\|(D_{q_0} f)^{-1} X\| \leq \mu^{-1} \|X\| \qquad \forall X \in E^+_{q_0}. \tag{1.4}$$

In this case, we say that $D_{q_0} f$ admits a (λ, μ)-splitting.

Example. Let us give a 2-dimensional example; the *Hyperbolic Toral Automorphism* is the discrete-time dynamical system defined on the 2-dimensional torus $\mathbb{T}^2$ as in the following:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \mod 1,$$

where the vector $(x_n, y_n)^T \in \mathbb{T}^2$ denotes the state of the system at the time n. The matrix possesses the eigenvalues $\mu_\pm = \frac{3 \pm \sqrt{5}}{2}$, and its eigenvectors $v_\pm$ determine at *any* point $(x, y) \in \mathbb{T}^2$ a contracting and an expanding direction: this means that the system admits at *any* point a (μ_-, μ_+)-splitting. The action of the matrix is depicted in Figure 1.1: notice that there are visible the expanding and contracting directions.

This system is a paradigmatic example of *Anosov system* (i.e. a system that at any point admits a splitting such as in equations (1.7)-(1.8)), and then of a chaotic system.

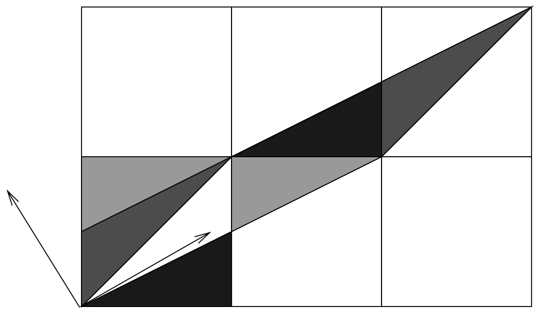

FIGURE 1.1

For maps with hyperbolic fixed points it allows the following result (see [**18**]), which gives a great advice in order to study the local behaviour of the trajectories in a neighbourhood of the fixed point:

THEOREM 1.1 (Existence of stable and unstable manifolds). *Let q_0 be a hyperbolic point of a local C^r diffeomorphism $f : U \to M$, $r \geq 1$. Then there exists two C^r embedded discs $W^s_{loc}(q_0)$, $W^u_{loc}(q_0) \subset U$ such that $T_{q_0}(W^s_{loc}(q_0)) = E^-_{q_0}$, $T_{q_0}(W^u_{loc}(q_0)) = E^+_{q_0}$, $f(W^s_{loc}(q_0)) \subset W^s_{loc}(q_0)$, $f^{-1}(W^u_{loc}(q_0)) \subset W^u_{loc}(q_0)$ and there exists $C(\delta)$ such that for any $q_s \in W^s_{loc}(q_0)$, $q_u \in W^u_{loc}(q_0)$, $n \geq 0$,*

$$d(f^n(q_s), q_0) < C(\delta)(\lambda + \delta)^n d(q_s, q_0)$$
$$d(f^{-n}(q_u), q_0) < C(\delta)(\mu{-}1 + \delta)^n d(q_u, q_0).$$

Furthermore, there exist $\delta_0 \geq 0$ such that

$$\textit{if } d(f^n(q_s), q_0) \leq \delta_0 \textit{ for } n \geq 0 \quad \textit{then} \quad q_s \in W^s_{loc}(q_0),$$
$$\textit{if } d(f^n(q_u), q_0) \leq \delta_0 \textit{ for } n \leq 0 \quad \textit{then} \quad q_u \in W^s_{loc}(q_0).$$

The manifolds $W^s_{loc}(q_0)$ and $W^u_{loc}(q_0)$ are respectively called the *(local) stable and unstable manifolds*; they are not uniquely defined. However, if there are two submanifolds $W^s_{loc}(q_0)$ and $\widetilde{W}^s_{loc}(q_0)$ that satisfy the theses of Theorem 1.1, their intersection contains a neighbourhood of q_0 in each of them. In other words, they are submanifolds of a larger submanifold we are going to define. The same allows for $W^u_{loc}(q_0)$.

DEFINITION 1.2. *The manifolds*

$$W^s(q_0) = \bigcup_{n \leq 0} f^n(W^s_{loc}(q_0)) \tag{1.5}$$

and

$$W^u(q_0) = \bigcup_{n \geq 0} f^n(W^u_{loc}(q_0)) \tag{1.6}$$

are called respectively the (global) stable and unstable manifold of f at q_0.

We remark that they are injectively immersed submanifolds of U, and that they are uniquely defined. They are also defined by the following topological characterization:

$$W^s(q_0) = \{q \in U : d(f^n(q), q_0) \to 0 \,\text{as}\, n \to +\infty\},$$

$$W^u(q_0) = \{q \in U : d(f^{-n}(q), q_0) \to 0 \,\text{as}\, n \to +\infty\}.$$

We just mention that Theorem 1.1 is a consequence of a more general result, the Hadamard-Perron Theorem:

THEOREM 1.2 (Hadamard-Perron). *Let $\lambda < \mu$, $r \geq 1$, and for each $m \in \mathbb{Z}$ let $f_m : \mathbb{R}^n \to \mathbb{R}^n$ be a surjective C^r diffeomorphism such that for each $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$*

$$f_m(x, y) = (A_m x + \alpha_m(x, y), B_m y + \beta_m(x, y))$$

for some linear maps $A_m : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and $B_m : \mathbb{R}^{n-k} \oplus \mathbb{R}^k$ with $\|A_m\| \leq \mu^{-1}$, $\|B_m\| \leq \lambda$ and $\alpha_m(0) = 0$, $\beta_m(0) = 0$.

Then there exists $\gamma_0 = \gamma_0(\lambda, \mu)$ such that for $\gamma \in (0, \gamma_0)$ there is a $\delta = \delta(\lambda, \mu, \gamma)$ with the following property: if $\|\alpha_m\|_{C^1} < \delta$ and $\|\beta_m\|_{C^1} < \delta$ for all $m \in \mathbb{Z}$ then there is

(1) *a unique family $\{W_m^+\}_{m\in\mathbb{Z}}$ of k-dimensional C^1 manifolds*

$$W_m^+ = \{(x, \varphi_m^+(x)) : x \in \mathbb{R}^k\} = \text{graph}\varphi_m^+$$

and

(2) *a unique family $\{W_m^-\}_{m\in\mathbb{Z}}$ of $(n-k)$-dimensional C^1 manifolds*

$$W_m^- = \{(\varphi_m^-(y), y) : y \in \mathbb{R}^{n-k}\} = \text{graph}\varphi_m^-,$$

where $\varphi_m^+ : \mathbb{R}^k \to \mathbb{R}^{n-k}$, $\varphi_m^- : \mathbb{R}^{n-k} \to \mathbb{R}^k$, $\sup_{m\in\mathbb{Z}} \|D\varphi_m^\pm\| < \gamma$, and the following properties hold:

- $f_m(W_m^-) = W_{m+1}^-$, $f_m(W_m^+) = W_{m+1}^+$.
- $\|f_m(z)\| < (1+\gamma)(\lambda + \delta(1+\gamma))\|z\|$ *for* $z \in W_m^-$.
  $\|f_m(z)\| < (\frac{\mu}{1+\gamma} - \delta)^{-1}\|z\|$ *for* $z \in W_m^+$.
- *Put* $\lambda' := (1+\gamma)(\lambda + \delta(1+\gamma))$ *and* $\mu' := (\frac{\mu}{1+\gamma} - \delta)$, *and let* $\lambda' < \nu < \mu'$. *If* $\|f_{m+L} \circ \ldots \circ f_m(z)\| < C\nu^L\|z\|$ *for all* $L \geq 0$ *and some* $C > 0$, *then* $z \in W_m^-$. *Similarly, if* $\|f_{m-L} \circ \ldots \circ f_{m-1}(z)\| < C\nu^{-L}\|z\|$ *for all* $L \geq 0$ *and some* $C > 0$, *then* $z \in W_m^+$.

Finally, if $\lambda < 1 < \mu$ the families $\{W_m^+\}_{m\in\mathbb{Z}}$ and $\{W_m^-\}_{m\in\mathbb{Z}}$ consist of C^r manifolds.

Remark. Let us now consider a one parametric group of diffeomorphisms $\Phi : \mathbb{R} \times M \to M$; namely, $\Phi(t, q) = \phi^t(q)$, where $\phi^t : M \to M$ is a diffeomorphism for any t and it is satisfied the group law $\phi^t \circ \phi^s = \phi^{t+s}$, $t, s \in \mathbb{R}$. The pair (M, Φ) defines a *smooth continuous-time dynamical system*. The theory of hyperbolic systems is well-developed also for this situation, as we are going to briefly mention here.

Let us introduce the definition:

DEFINITION 1.3. *Let $\Lambda \subset M$ be a compact ϕ^t-invariant set. Λ is a hyperbolic set for the flow ϕ^t if there is a Riemannian metric defined on a neighbourhood U of Λ and two constants $0 < \lambda < 1 < \mu$ such that for any $q \in \Lambda$ the tangent space T_qM is split into three subspaces $T_qM = E_q^0 \oplus E_q^- \oplus E_q^+$ such that $\dim(E_q^0) = 1$, $\frac{d}{dt}\phi^t|_{t=0} \in E_q^0 \setminus \{0\}$, $D\phi^t E_q^\pm = E_q^\pm$, and E_q^- satisfies (1.3) and E_q^+ satisfies (1.4).*

Let (M, Φ) be a continuous-time dynamical system such that $\Lambda \subset M$ is a hyperbolic set of the flow; then, using Hadamard-Perron Theorem, it can be proved a version of the *Stable and Unstable Manifold Theorem* (Theorem 1.1); this is called the *Stable and Unstable Manifold Theorem for flows*.

2.2. Partially hyperbolic dynamical systems. In this subsection we will just briefly sketch some properties of a more general class of dynamical systems, that in fact includes the class of hyperbolic systems.

They are defined as follows:

DEFINITION 1.4. *A diffeomorphism $f : M \to M$, where M is a smooth manifold, is called partially hyperbolic if there exists numbers $0 < \lambda < \mu$ and $c > 0$, and two distributions $E = \{E_q\}_{q\in M}$, $F = \{F_q\}_{q\in M}$, such that*

(1) *E_q and F_q form at each $q \in M$ an invariant splitting of the tangent space T_qM, i.e.*

$$T_qM = E_q \oplus F_q \qquad \text{and} \quad D_qf(E_q) = E_{f(q)}, \quad D_qf(F_q) = F_{f(q)};$$

(2) *for $n > 0$*

$$\|D_qf^nX\| \leq c\lambda^n\|X\| \qquad X \in E_q \tag{1.7}$$

$$\|D_qf^nX\| \geq c^{-1}\mu^n\|X\| \qquad X \in F_q \tag{1.8}$$

Remark. We have that either $\lambda < 1$ and/or $\mu > 1$. If $\lambda < 1$, the subspace E_q is stable and, analogously, if $\mu > 1$, the subspace F_q is unstable. If it happens that $\lambda < 1 < \mu$ we are in the usual hyperbolic situation.

For further purposes, we need a different definition of partial hyperbolicity, which is called in literature *partial hyperbolicity in the narrow sense*. The definition is as follows:

DEFINITION 1.5. *A diffeomorphism $f : M \to M$, where M is a smooth manifold, is called partially hyperbolic in the narrow sense if there exists a Riemannian metric $\|\cdot\|$ on M, numbers $c > 0$ and*

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3 \qquad \mu_1 < 1, \quad \lambda_3 > 1,$$

and and invariant splitting

$$T_qM = E^s_q \oplus E^c_q \oplus E^u_q$$

such that for any $n > 0$

$$c^{-1}\lambda_1^{-n}\|X\| \leq \|D_qf^nX\| \leq c\mu_1^n\|X\| \qquad X \in E^s_q \tag{1.9}$$

$$c^{-1}\lambda_2^{-n}\|X\| \leq \|D_qf^nX\| \leq c\mu_2^n\|X\| \qquad X \in E^c_q \tag{1.10}$$

$$c^{-1}\lambda_3^{-n}\|X\| \leq \|D_qf^nX\| \leq c\mu_3^n\|X\| \qquad X \in E^u_q \tag{1.11}$$

Partially hyperbolic systems satisfy the hypotheses of Hadamard-Perron Theorem, but in general the condition $\lambda < 1 < \mu$ is not satisfied; in this case, as can be seen from the statement of Hadamard-Perron Theorem, the stable and the unstable distributions (respectively E^s and E^u) are integrable, but their integral manifolds are only C^1 smooth, even if f possesses a C^r smoothness with $r > 1$. We will denote with W^s the integral foliation of the distribution E^s, and we call it *stable foliation*; analogously, W^u is called *unstable foliation* and is the integral foliation of the distribution E^u.

Otherwise, the central distribution E^c_q is in general not integrable. To have integrability we need one more hypothesis we are going to explain below.

Let $\mathcal{F}$ be a foliation of the manifold M; we say that $\mathcal{F}$ is *quasi-isometric* if there are two constants $a > 0$ and $b > 0$ such that for any x, y belonging to same leaf $\mathcal{F}_x$ we have

$$d_{\mathcal{F}}(x, y) \leq ad(x, y) + b,$$

where $d_{\mathcal{F}}$ denotes the distance along the leaf. Then we can state the result (see [**23**] for references)

THEOREM 1.3. *Let $\widetilde{M}$ denote the universal covering of M, and $\widetilde{W}^s$ and $\widetilde{W}^u$ the lifts to $\widetilde{M}$ of the manifolds W^s and W^u..*

Assume that both $\widetilde{W}^s$ and $\widetilde{W}^u$ are quasi-isometric. Then the distribution E^c is integrable and the integral foliation is unique.

2.3. Lyapunov exponents. We saw that a hyperbolic fixed point is characterized by the existence of directions of expansion and contraction under the action of the linearization of the flow; in hyperbolic sets, any point admits this contracting and expanding directions, and, moreover, the coefficients that control the norms of the vectors are uniform on the set.

There exist also dynamical systems in which at almost any point (with respect a suitable measure on M) there are these expanding and contracting directions, dominated by coefficients that possibly depend on the point. The dynamics of these systems in fact presents a behaviour dominated by sensitive (i.e. exponential) dependence on initial condition, but in general the system does not admit a (λ, μ)-splitting at any point.

In this subsection we will introduce the objects that take account of these properties of such systems: the *Lyapunov exponents*.

DEFINITION 1.6. *Let $f : M \to M$ be a diffeomorphism on a manifold M endowed with a Riemannian metric that induces the norm $\|\cdot\|$ on vectors on M. We say that $q \in M$ is a regular point of f if there exist numbers $\lambda_1(q) > \lambda_2(q) > \cdots > \lambda_m(q)$ and a decomposition $T_qM = E_1(q) \oplus \cdots \oplus E_m(q)$ such that*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|D_q f^n \boldsymbol{x}\| = \lambda_j(q) \qquad \text{for every } 0 \neq \boldsymbol{x} \in E_j(q), \quad 1 \leq j \leq m. \tag{1.12}$$

We remark that for any regular point the numbers λ_j and the decomposition in subspaces E_j are unique (see [**21**]).

DEFINITION 1.7. *The numbers $\lambda_j(q)$ are called the* Lyapunov exponents *of the map f at the regular point q.*

It is easy to check the following properties:

$$\begin{aligned} \lambda_j(f(q)) &= \lambda_j(q) \\ D_q f(E_j(q)) &= E_j(f(q)). \end{aligned}$$

Remark. If q is a regular point of f and $\boldsymbol{x} \in T_qM$, we have that

$$\lim_{k\to+\infty} \frac{1}{k} \log \|D_q f^k \boldsymbol{x}\| = \lim_{k\to-\infty} \frac{1}{k} \log \|D_q f^k \boldsymbol{x}\|,$$

i.e. the limits in the future and in the past exist and coincide.

If q is regular, we put

$$\lambda(q, \boldsymbol{x}) = \lim_{t\to\infty} \frac{1}{n} \log \|D_q f^n \boldsymbol{x}\|;$$

clearly, $\lambda(q, \boldsymbol{x}) = \lambda_j(q)$ for j such that $\boldsymbol{x} \in E_j(q)$.

Let us denote with $\mathcal{M}_f(M)$ the set of all f-invariant Borel probability measures on M (i.e. measures μ defined on the σ-algebra of Borel sets with $\mu(M) = 1$).

THEOREM 1.4 (Osedelec's Multiplicative Ergodic Theorem). *If M is compact, then the set of regular points of a diffeomorphism f is a full-measure set with respect to any measure $\mu \in \mathcal{M}_f(M)$.*

If q is a regular point, the spaces $E_j(q)$ are *asymptotically* expanding or contracting (depending on the sign of the associated Lyapunov exponent); then, in the case in which some exponents are positive and some negative, the notion of regular point in some sense generalizes the notion of hyperbolic point, taking account of the asymptotic behaviour of the linearization of the dynamics. Osedelec's Theorem says that, under some conditions, at almost any point the dynamics has this behaviour.

Example. The Hyperbolic Automorphism of the Torus indeed possesses at any point a positive and a negative Lyapunov exponent; they are $\lambda_+ = \log \mu_+ = \log\left(\frac{3+\sqrt{5}}{2}\right)$ and $\lambda_- = \log \mu_- = \log\left(\frac{3-\sqrt{5}}{2}\right)$.

The subspaces $E_\pm(x, y)$ relative to the Lyapunov exponents are in fact the subspaces determined by the eigenvectors of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Remark. The Lyapunov exponents can also be defined for a smooth flow $\Phi : \mathbb{R} \times M \to M$:

$$\lim_{t\to+\infty} \frac{1}{t} \log \|D_q \phi^t \boldsymbol{x}\| = \lambda_j(q) \qquad \textit{for every } 0 \neq \boldsymbol{x} \in E_j(q), \quad 1 \leq j \leq m; \tag{1.13}$$

all the results stated above allow also in this case.

2.4. The dynamical entropy. The Lyapunov exponents provide a measure of the instability of a dynamical system. In this subsection we will expone another approach that investigates it, focusing on the statistical behaviour of the orbits. To do that, we shall shift to a probabilistic description of the dynamics.

Let μ be a measure on M; we say that μ is an f-invariant probability measure on M if $\mu(f^{-1}(U)) = \mu(U)$ for any measurable subset $U \subset M$, and $\mu(M) = 1$. The triple (M, μ, f) defines a *(discrete-time) metric dynamical system*.

Analogously, if $\Phi : \mathbb{R} \times M \to M$, $\Phi(t, \cdot) = \phi^t(\cdot)$, is a smooth flow on M such that the measure μ on M is ϕ^t-invariant for any $t \in \mathbb{R}$, then the triple (M, μ, Φ) defines a *(continuous-time) metric dynamical system*.

Let us recall that the manifold M is the set of the states of system; the measure of a measurable subset U of M denotes the probability of the state of the system of belonging to U. Notice that the fact that f is measure-preserving takes account of the conservation of the probability; in particular, Hamiltonian systems fit this rule with respect to Liouville measure $\mu = \sigma$, where σ is the symplectic form on the manifold.

Let $\mathscr{P}$ be a finite or countable measurable partition of M, i.e. $\mathscr{P} = \{P_i\}_i$ where P_i is μ-measurable for any i, $\mu(P_i \cap P_j) = 0$ for $i \neq j$ and $\mu(M \setminus \cup_i P_i) = 0$. The elements P_i are called atoms of the partition.

The partition models the case in which the observer cannot know precisely the state of the system at a certain time, but he can only know in which atom of the partition the state is contained. Let us introduce the *information function* $I : \mathscr{P} \to \mathbb{R}$ defined as $I(P) =$

$-\log \mu(P)$, for $P \in \mathscr{P}$: it measures the *amount of information* one gains about the system when he knows that the state of the system is contained in P.

Then, let us give the following definition:

DEFINITION 1.8. *The entropy of the partition* $\mathscr{P}$ *is defined as*

$$\mathcal{H}_\mu(\mathscr{P}) = -\sum_i \mu(P_i) \log \mu(P_i),$$

where we put $0 \log 0 := 0$.

The entropy is then the *average information* that an observer can extract from the system when he knows the present state of it, or, in other words, is the *average ignorance* of an observer that can know the state of the system only by means of the partition $\mathscr{P}$.

Let $\mathscr{P}$, $\mathscr{Q}$ be two measurable partitions on M; we define the *refinement* of $\mathscr{P}$ with $\mathscr{Q}$ as the partition

$$\mathscr{P} \vee \mathscr{Q} = \{P_i \cap Q_j\}_{P_i \in \mathscr{P},\ Q_j \in \mathscr{Q}}.$$

We now define the joint partition $\mathscr{P}_n = \bigvee_{k=1}^n f^{-k}(\mathscr{P})$, that is the refinement of $\mathscr{P} = \{P_i\}$ with all the partitions $\mathscr{P}_k = f^{-k}(\mathscr{P})$ given by $\mathscr{P}_k = \{f^{-k}(P_i)\}$.

The *dynamical entropy* associated to the metric dynamical system (M, f, μ) and the partition $\mathscr{P}$ is defined as

$$h_\mu(f, \mathscr{P}) = \lim_{n \to +\infty} \frac{1}{n} \mathcal{H}_\mu(\mathscr{P}_n), \tag{1.14}$$

where $\mathcal{H}_\mu(\mathscr{P}_n) = -\sum_{i_1,\dots,i_n} \mu(P_{i_1} \cap f^{-1}(P_{i_2}) \cap \dots \cap f^{-n+1}(P_{i_n})) \log \mu(P_{i_1} \cap f^{-1}(P_{i_2}) \cap \dots \cap f^{-n+1}(P_{i_n}))$.

PROPOSITION 1.1. *The limit in (1.14) always exists (see* **[21]***).*

The entropy $h_\mu(f, \mathscr{P})$ has the meaning of the average amount of information provided by the knowledge of the present state and the asymptotic future. In other words, the value of the dynamical entropy says how precisely the evolution of the system can be predicted when the initial condition is known with the uncertainty given by the partition: if the system is well predictable (for instance it is a periodic system), the dynamical entropy is zero; otherwise, a positive value of the dynamical entropy says that the asymptotic evolution of the system cannot be predicted.

Finally, we can define the measure-theoretic dynamical entropy:

DEFINITION 1.9. *The measure-theoretic (or Kolmogorov-Sinai) dynamical entropy of the system* (M, μ, f) *is defined as*

$$h_\mu(f) = \sup_{\mathscr{P}} h_\mu(f, \mathscr{P}),$$

where the sup *is taken over all partitions of* M *that satisfies the properties above.*

We just mention here that *Kolmogorov-Sinai's Theorem* states that if a partition $\mathscr{P}$ generates the topology of M then

$$h_\mu(f, \mathscr{P}) = h_\mu(f).$$

This Theorem provides a valid tool to compute the dynamical entropy $h_\mu(f)$ of a dynamical system.

Remark. If the dynamical system is given by a smooth flow $\Phi : \mathbb{R} \times M \to M$, where M and μ are respectively a smooth manifold and a ϕ^t-invariant measure on M as above, the dynamical entropy is defined by $h_\mu(\Phi) = h_\mu(\phi^1)$.

Let us now state a result that shows the close relation between the Lyapunov exponents and the dynamical entropy of a dynamical system. First of all, define, for any regular point q of f

$$\chi(q) = \sum_{\lambda_j(q)\geq 0} \lambda_j(q) \dim E_j(q); \tag{1.15}$$

if all the Lyapunov exponents are negative at q, we put $\chi(q) = 0$. Then, we have the following result:

THEOREM 1.5 (Pesin's Formula). *Let us recall that a function f is Hölder C^1 if the function itself and its first derivative are Hölder continuous. Assume then that f is Hölder C^1 and μ is an f-invariant probability measure on M which is absolutely continuous with respect to the Lebesgue measure of M. Then*

$$h_\mu(f) = \int_M \chi \, d\mu. \tag{1.16}$$

We saw that when studying metric dynamical systems we deal with averages of functions on the state space. The following theorem is a fundamental result that, under some conditions on a function, guarantees the existence of its time average and relates its space and time averages; we remark that it allows also for continuous-time dynamical systems:

THEOREM 1.6 (Birkhoff's Ergodic Theorem). *Let (M, μ, f) be a metric dynamical system, and $g : M \to M$ a function on M. We have the following facts:*

- *if $g \in L^1(M)$, the limit*
$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(q))$$
exists μ-a.e. on M;
- *if $g \in L^p(M)$, $1 \leq p < \infty$, the function*
$$\tilde{g}(q) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(q))$$
is in $L^p(M)$; $\frac{1}{n}\sum_{k=0}^{n-1} g \circ f^k$ converges in the L^p norm to $\tilde{g}$ and we have that $\tilde{g}(q) = \tilde{g}(f(q))$ a.e. on M; $\tilde{g}$ is the time average of g;
- *for every $g \in L^p(M)$ we have that*
$$\int_M \tilde{g} \, d\mu = \int_M g \, d\mu;$$
$\int_M g \, d\mu$ is the space average of g over M.

Remark. If f is invertible, then Birkhoff Ergodic Theorem applies to f^{-1} and implies that the time-average in the past

$$\tilde{g}^{-}(q) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^{-k}(q))$$

exists μ-a.e. on M. Moreover, the time average in the future and in the past coincide:

$$\tilde{g}^{-}(q) = \tilde{g}(q) \qquad \mu\text{-a.e. on } M. \tag{1.17}$$

3. Optimal Control Theory

Let M be a smooth manifold and U a set. A *control system* on M is a family of dynamical systems

$$\dot{q} = f_u(q), \tag{1.18}$$

$q \in M$, parametrized by some parameter $u \in U$, where $f_u(q)$ is for any u a vector field on M. The parameter u is called *control parameter* (or simply *control*), the set U *set of admissible controls*.

To ensure the well-posedness of equation (1.18), we assume that $f_u(q)$ is continuous with respect to the pair (q, u) and smooth with respect to q; moreover, we assume that in any local coordinates the function $(q, u) \mapsto \frac{\partial f_u(q)}{\partial q}$ is continuous on $M \times \overline{U}$.

These assumptions assure the existence of a Carathéodory solution to (1.18); these solutions are called *trajectories* of the system.

A control problem is given by a control system as (1.18) and a set U of admissible control, where the value of u is changed in time in order to influence the evolution of the dynamical system. We will take in consideration functions $t \mapsto u(t) \in U$ which are measurable and locally bounded; such functions are called *admissible controls*. A pair $(q(\cdot), u(\cdot))$ such that $q(\cdot)$ satisfies equation (1.18) is called *admissible pair*.

In the case of the so-called *optimal control*, we are given a smooth function $\varphi : M \times U \to \mathbb{R}$ and our aim is to determine the admissible control $\tilde{u}(\cdot)$ such that the trajectory $\tilde{q}(t)$ satisfies equation (1.18) and the functional

$$J_{t_0}^{t_1}[(q(\cdot), u(\cdot))] = \int_{t_0}^{t_1} \varphi(q(t), u(t))\, dt, \tag{1.19}$$

attains at $(\tilde{q}(\cdot), \tilde{u}(\cdot))$ its minimum among all the admissible pairs $(q(\cdot), u(\cdot))$ that satisfy some prescribed boundary conditions; usually they consist in fixing the endpoints of the trajectory:

$$q(t_0) = q_0, \quad q(t_1) = q_1, \tag{1.20}$$

$q_0, q_1 \in M$ fixed.

The solution u of an optimal control problem is called *optimal control*, and the corresponding trajectory *optimal trajectory*.

The Pontryagin Maximum Principle (PMP) provides a necessary condition for a trajectory to be optimal, establishing a relation between the optimal trajectories and the solutions of a certain Hamiltonian system defined on the cotangent bundle T^*M.

In fact, to any optimal control problem we can associate a family of Hamiltonian functions

$$h_u^{\nu}(\lambda) = \langle \lambda, f_u(q) \rangle + \nu\varphi(q, u)$$

where $\lambda \in T^*M$ and ν is a real number.

The Pontryagin Maximum Principle reads as follows:

Theorem 1.7 (PMP). *Let* $(\tilde{q}(t), \tilde{u}(t))$, $t \in [t_0, t_1]$, *be an admissible pair for the problem*

$$\dot{q} = f_u(q), \quad J_{t_0}^{t_1} \to \min, \qquad q \in M,\ u \in U; \tag{1.21}$$

then $\tilde{u}(t)$ *is an optimal control, and* $\tilde{q}(t)$ *is an optimal trajectory, only if there exists a non-trivial pair*

$$(\nu, \lambda(t)) \neq 0 \qquad \lambda(t) \in T^*_{\tilde{q}(t)}M,\ \nu \in \mathbb{R},$$

such that

$$\dot{\lambda}(t) = \vec{h}^{\nu}_{\tilde{u}(t)}(\lambda(t)) \tag{1.22}$$

$$h^{\nu}_{\tilde{u}(t)}(\lambda(t)) = \max_{u \in U} h^{\nu}_u(\lambda(t)) \quad \textit{for a.e. } t \in [t_0, t_1] \tag{1.23}$$

$$\nu \leq 0. \tag{1.24}$$

The Hamiltonian $h^{\nu}_u(\lambda(t))$ is homogeneous in the pair $(\nu, \lambda(t))$, that can be then normalized; there are two distinct possibilities:

- if $\nu \neq 0$, we put $\nu = -1$; the curve $\lambda(t)$ is called in this case *normal extremal*;
- if $\nu = 0$, the curve $\lambda(t)$ is called *abnormal extremal*.

The PMP states that a necessary condition for a trajectory $q(t)$ to be optimal is to be the projection of a solution of the Hamiltonian dynamical system (1.22); actually, PMP is the generalization of the Least Action Principle of classical mechanics [**13**].

Let us now define (where possible) the maximized Hamiltonian of the optimal control problem

$$H(\lambda) = \max_{u \in U} h^{\nu}_u(\lambda); \tag{1.25}$$

the following proposition relates the solutions of the dynamical system defined by the maximized Hamiltonian with the extremal of the optimal control problem (1.21): in fact, it asserts that, under some regularity conditions on H, they coincide:

Proposition 1.2. *Assume that the maximized Hamiltonian defined as (1.25) is defined and* C^2 *on* $T^*M \setminus \{\lambda = 0\}$.

If a pair $(\tilde{u}(t), \lambda(t))$ *satisfies conditions (1.22)-(1.23), then*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)); \tag{1.26}$$

conversely, if a Lipschitzian curve $\lambda(t) \neq 0$ *is a solution of the Hamiltonian system (1.26), then there exists an admissible control* $\tilde{u}(t)$ *such that the pair* $(\tilde{u}(t), \lambda(t))$ *satisfies conditions (1.22)-(1.23).*

During this thesis, we are considering only normal extremals. For such extremals, we have also this sufficient condition for optimality:

Theorem 1.8. *Assume that the maximized Hamiltonian* (1.25) *is defined and smooth on* T^*M, *and that the Hamiltonian vector field* $\vec{H}$ *is complete.*

Let $\mathcal{L}_0$ *be a Lagrangian submanifold in* T^*M, *and let* $\mathcal{L}_t = e^{t\vec{H}}(\mathcal{L}_0)$ *be its image under the Hamiltonian flow at time* t.

Let $\pi : T^*M \to M$ *be the canonical projection, and assume that its restriction* $\pi|_{\mathcal{L}_t}$ *is a diffeomorphism for any* $t \in [t_0, t_1]$. *Then for any* $\lambda_0 \in \mathcal{L}_0$ *the normal extremal trajectory*

$$\tilde{q}(t) = \pi \circ e^{t\vec{H}}(\lambda_0), \qquad t \in [t_0, t_1],$$

realizes a strict minimum of the cost functional (1.19) among all the admissible trajectories $q(t)$, $t \in [t_0, t_1]$, *of the system (1.18) with the same boundary conditions:*

$$q(t_0) = \tilde{q}(t_0) \qquad q(t_1) = \tilde{q}(t_1).$$

CHAPTER 2

The generalized curvature

Let (M, g) be an n-dimensional Riemannian manifold, and let us consider the classical *length minimization* problem on it: given two points $q_0,\ q_1 \in M$, we want to find the curve $\gamma(\cdot) \in M$ with minimal length among all those that pass through both points q_0 and q_1.

This problem can be formulated as an optimal control problem in this way:

$$\min_{\gamma} J_0^T(\gamma(\cdot)) = \min_{\gamma} \int_0^T \Big(\sum_{i=1}^{n} u_i^2(t) \Big)^{1/2} dt \tag{2.1}$$

with

$$\dot{\gamma}(t) = \sum_{i=1}^{n} u_i(t) f_i(q), \qquad \begin{cases} \gamma(0) = q_0 \\ \gamma(T) = q_1 \end{cases},$$

where $\{f_i(q)\}_{i=1}^n$ is a smooth orthonormal frame in T_qM.

It is well known that locally any geodesic is a minimizing curve, i.e. it provides a (local) solution to problem (2.1), but that this optimality fails in presence of conjugate points: if γ is a geodesic segment from q_0 to q_1 that has an interior conjugate point $\hat{q}$, then γ is no more minimizing after $\hat{q}$. The sectional curvature is a metric invariant of the manifold (M, g) that provides lots of information about the behaviour of the geodesics on it, and, in particular, on the distribution of their conjugate points; we just recall that if all the sectional curvatures of a manifold are negative, then no point of the manifold has conjugate points along the geodesics.

In [**6**], Agrachev and Gamkrelidze introduced the notion of a curvature tensor along the extremals of an optimal control problem; this theory has been further developed by Agrachev, Zelenko and Chtcherbakova in [**8**], [**9**], [**2**] and [**10**]. This tensor is called *generalized curvature* and is a true generalization of the sectional curvature of Riemannian Geometry. It is really remarkable that this invariant provides important information about the behaviour of the extremals of an optimal control problem, without the necessity to solve any differential equation; in particular, this curvature satisfies a generalized version of Conjugate Point Comparison Theorem; for the standard version of Conjugate Point Comparison Theorem see, for instance, the textbook [**20**].

More generally, the generalized curvature is an invariant associated to a pair *(splitting of the tangent bundle, vector field)* on a smooth manifold M. In this chapter we will introduce this object within this frame, restricting ourself to the case *(Lagrangian splitting, Hamiltonian vector field)*.

First of all, in Section 1 we will introduce some basics facts about curves in the Lagrange Grassmannian of the symplectic space $T_z(T^*M),\ z \in M$.

The definition of the generalized curvature is the main topic of Section 2.

Section 3 is devoted to the description of two natural Lagrangian splittings of the cotangent bundle. We will also give some examples of explicit computations of the generalized curvature for some basics systems, in order to highlight the connection between the generalized curvature and the Riemannian curvature in a Riemannian frame, and between the curvature and the Hamiltonian of the system in a Euclidean frame.

In Section 4 we will treat the case in which the Hamiltonian systems admits a first integral of the motion. In this case, it is well known that the dynamics reduces on a sublevel of the first integral: due to this fact, a new formulation for the generalized curvature is needed, in order to take account of the reduction, to deal with an object that carries much more information on the system than the non-reduced generalized curvature.

Finally, in Section 5 we will present the notion of *canonical moving frame*, which is a family of Darboux bases of a symplectic space Σ that moves accordingly to a curve in the Lagrange Grassmannian of Σ. We will underline the relation between these frames and the generalized curvature.

1. Curves in the Lagrange Grassmannian

Let Σ be a symplectic space with symplectic form σ, and consider the curve $t \mapsto \Lambda(t)$ in the Lagrange Grassmannian $\mathcal{L}_n(\Sigma)$. We recall form Section 1 of preceding Chapter thatany $\Delta \in \mathcal{L}_n(\Sigma)$ gives a coordinate chart in the Lagrange Grassmannian. We can then write $\Lambda(t)$ in coordinates in the following way: fix $\Delta \in \mathcal{L}_n(\Sigma)$ such that $\Lambda(t)$ is transversal to Δ for any t (we will write $\Lambda(t) \in \Delta^{\pitchfork}$ for any t), which means that the curve stays for any t in the coordinate chart defined by Δ; choose a Darboux basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ in Σ such that $\Delta = \operatorname{span}\{f_1, \ldots, f_n\}$: then there is a family of symmetric $n \times n$ matrices S_t such that we can do the identification $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$ for any t.

Let $T_{\Lambda(0)}(\mathcal{L}_n(\Sigma))$ denote the tangent space at the point $\Lambda(0)$ to $\mathcal{L}_n(\Sigma)$; we can associate in a true intrinsic way to the vector $\frac{d}{dt}\Lambda(t)|_{t=0} \in T_{\Lambda(0)}(\mathcal{L}_n(\Sigma))$ a quadratic form on $\Lambda(0)$, that we will denote with $\dot{\Lambda}_0$. The definition is as follows: for any $\boldsymbol{x} \in \Lambda(0)$, we put $\dot{\Lambda}_0(\boldsymbol{x}) = \sigma(\dot{\lambda}(0), \lambda(0))$, where $\lambda(t)$ is a curve in Σ with $\lambda(t) \in \Lambda(t)$ for any t and $\lambda(0) = \boldsymbol{x}$.

The quadratic form is well defined, i.e. $\dot{\Lambda}_0(\boldsymbol{x})$ depends only on $\frac{d}{dt}\Lambda(t)|_{t=0}$ and $\boldsymbol{x}$; since the definition of $\dot{\Lambda}_0$ is intrinsic, we will show this fact in coordinates. Let then $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, and $\boldsymbol{x} = (\mathbf{x}, S_0\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$; consider a curve $\lambda(t) = (x(t), S_t x(t))$, with $x(0) = \mathbf{x}$. Then $\sigma(\dot{\lambda}(0), \lambda(0)) = \langle \dot{x}(0), S_0 x(0)\rangle_n - \langle \dot{S}_0 x(0) + S_0\dot{x}(0), x(0)\rangle_n = -\langle \dot{S}_0\mathbf{x}, \mathbf{x}\rangle_n$, by symmetricity of S_t; here $\langle\cdot,\cdot\rangle_n$ denotes the scalar product on $\mathbb{R}^n$. Then actually the result does not depend on the particular choice of the curve $\lambda(t)$.

Definition 2.1. *We say that a curve $t \mapsto \Lambda(t)$ in $\mathcal{L}_n(\Sigma)$ is regular at the point τ if the quadratic form $\dot{\Lambda}_\tau$ is nondegenerate; we say that it is monotone increasing (decreasing) if the associated quadratic form is positive (negative) definite.*

Lemma 2.1. *If the curve $t \mapsto \Lambda(t)$ is regular at the point τ, then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for any $t \neq \tau$ close to τ.*

Proof. We will prove it in coordinates. $\Lambda(t) \cap \Lambda(\tau) \neq 0$ only if $\det(S_t - S_\tau) = 0$; since $\det \dot{S}_\tau \neq 0$, there is a small neighbourhood of τ such that the condition above allows. $\square$

To any quadratic form on a vector space it is uniquely associated a self-adjoint linear operator from the vector space to its dual; this means that there is a unique self-adjoint linear operator $\overline{\dot{\Lambda}}_0 : \Lambda(0) \to \Lambda(0)^*$ such that for any $\boldsymbol{x} \in \Lambda(0)$ we have $\dot{\Lambda}_0(\boldsymbol{x}) = \langle \overline{\dot{\Lambda}}_0\boldsymbol{x}, \boldsymbol{x}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the dual action of $\Lambda(0)^*$ on $\Lambda(0)$.

In this thesis we are interested in studying the properties of Hamiltonian systems defined on the cotangent bundle T^*M of a smooth manifold M; to such systems we can associate a particular kind of curve in the Lagrange Grassmannian, that plays for Hamiltonian trajectories the role played for geodesics by the Jacobi fields; such a curve is called a *Jacobi curve*, and the definition is given below.

Let then M be a smooth manifold, let σ be the canonical symplectic structure on T^*M, and $h : T^*M \to \mathbb{R}$ a smooth Hamiltonian function. Let $\Lambda = \{\Lambda_z\}_{z\in M}$ be a Lagrangian vector distribution on M; we define the following curve:

DEFINITION 2.2. *The curve in the Lagrange Grassmannian of* $T_z(T^*M)$ *defined by*

$$\Lambda_z(t) := e^{-t\vec{h}}{}_*\Lambda_{e^{t\vec{h}}(z)} \tag{2.2}$$

is called the Jacobi curve of the curve $z \mapsto e^{t\vec{h}}(z)$ *attached at the point* $z \in T^*M$.

The Jacobi curve is defined by the push-forward (with negative time) of the vertical distribution along the integral curves of $\vec{h}$. Since the distribution is Lagrangian and the vector field is Hamiltonian, the Jacobi curve lies in $\mathfrak{L}_n(T_z(T^*M))$.

Directly from its definition it follows that

$$\Lambda_z(t) = e^{-t\vec{h}}{}_*\Lambda_{e^{t\vec{h}}(z)}(0).$$

Remark. A natural choice for the Lagrangian distribution is the so-called *vertical distribution* $\{T_z(T^*_{\pi(z)}M)\}_{z\in T^*M}$; notice that this distribution is integrable, the integral manifolds being constitute by the *vertical fibres* $T^*_{\pi(z)}M$, $z \in T^*M$. This leaves actually foliate the manifold T^*M.

Remark. We will also use this alternative definition: we say that the Hamiltonian h is *regular* (*monotone*) with respect to a distribution Λ if the Jacobi curve (2.2) is regular (monotone) for any t.

The following proposition relates the properties of regularity of the Jacobi curve (constructed by means of the vertical distribution) with the characteristics of the Hamiltonian h.

PROPOSITION 2.1. *Let* Π^h_z *be the map* $\Pi^h_z : T^*_{\pi(z)}M \to T_{\pi(z)}M$ *given by* $\Pi^h_z(z') = \pi_*(\vec{h}(z'))$*; then the Jacobi curve defined above is regular if and only if* Π^h_z *is a submersion.*

Proof. We will prove it in coordinates: let $T^*M = \{z = (p,q) : p, q \in \mathbb{R}^n\}$, with $\pi : z = (p,q) \mapsto q$; then, $T_z(T^*_{\pi(z)}M) = \operatorname{span}\{\partial_{p_1}, \ldots, \partial_{p_n}\}$. Let $\vec{h}(p,q) = \sum_{i=1}^n a_i\partial_{p_i} + b_i\partial_{q_i}$.

In coordinates, the action of Π^h_z is $\Pi^h_z(p',q') = \sum_{i=1}^n b_i(p',q')\partial_{q_i}$, hence the map is a submersion if and only if the matrix $(\frac{\partial b_i}{\partial p_j})_{i,j}$ is nondegenerate.

Then, for a vertical vector $\boldsymbol{x} = \sum_{i=1}^n \mathbf{x}_i\partial_{p_i}$, we have that $\dot{\Lambda}_z(0)(\boldsymbol{x}) = \sum_{i,j=1}^n \mathbf{x}_i\mathbf{x}_j\frac{\partial b_i}{\partial p_j}$, hence it is nondegenerate if and only if the matrix $(\frac{\partial b_i}{\partial p_j})_{i,j}$ is.

In fact, let us recall the basic formula

$$\frac{d}{dt}\left(e^{-t\vec{h}}{}_*\zeta\right) = e^{-t\vec{h}}{}_*[\vec{h},\zeta] \tag{2.3}$$

for any vector field ζ on T^*M.

Then, if $\lambda(t)$ is a curve in $J_z(t)$ such that $\lambda(0) = \boldsymbol{x}$, we have that $\dot{\lambda}(0) = [\vec{h}, \lambda(t)]|_{t=0}$ and then $\dot{\Lambda}_z(0)(\boldsymbol{x}) = \sum_{i,j=1}^n \mathbf{x}_i \mathbf{x}_j \frac{\partial b_i}{\partial p_j}$. □

Remark. All the construction above can be developed also in a non-symplectic setting (see [**6**]); in this case, we will consider an involutive distribution of rank n, namely a distribution whose spaces are tangent to the leaves of a smooth foliation of rank n of the manifold M, and a smooth vector field ζ on M. The Jacobi curve is defined as

$$J_z(t) := e^{-t\zeta}{}_* E|_{z_t}, \quad z_t = e^{t\zeta}(z),$$

where $E|_z$ denotes the subspace of the distribution contained in T_zM. We remark that all the results stated above allow also in this non-symplectic case, except of course the results concerning Lagrangianity of the Jacobi curve.

2. The generalized curvature

Let $\{\Pi_z\}_{z\in T^*M}$ be a Lagrangian distribution on T^*M transversal to $\{\Lambda_z\}_z$, namely $\Lambda_z \cap \Pi_z = 0$ for any z; then, the two distributions form a Lagrangian splitting of the tangent bundle $T(T^*M) = \Lambda \oplus \Pi$. Let h be a Hamiltonian function on T^*M.

Fix some $z_0 \in T^*M$ and put $z_t = e^{t\vec{h}}(z_0)$; call, for any $z_0 \in T^*M$, $\Lambda_{z_0}(t)$ and $\Pi_{z_0}(t)$ the two Jacobi curves defined by the distributions, $\Lambda_{z_0}(t) = e^{-t\vec{h}}{}_*\Lambda_{z_t}$ and $\Pi_{z_0}(t) = e^{-t\vec{h}}{}_*\Pi_{z_t}$. We assume that the curve $\Lambda_{z_0}(t)$ is regular.

Let us notice that the two curves are transversal at any time: $\Lambda_{z_0}(t) \cap \Pi_{z_0}(t) = 0$ for any t and for any z_0.

Due to the isomorphisms $\Lambda_{z_0}(t)^* \simeq T_{z_0}(T^*M)/\Lambda_{z_0}(t)$ and $T_{z_0}(T^*M)/\Lambda_{z_0}(t) \simeq \Pi_{z_0}(t)$, we can view the operator $\overline{\dot{\Lambda}}_{z_0}(t)$ as a linear operator from $\Lambda_{z_0}(t)$ to $\Pi_{z_0}(t)$ (and, conversely, $\overline{\dot{\Pi}}_{z_0}(t)$ as a linear operator from $\Pi_{z_0}(t)$ to $\Lambda_{z_0}(t)$). This fact permits us to endow T^*M with a Riemannian structure, as the following Proposition states:

Proposition 2.2. *Let $T_z(T^*M) = \Lambda_z \oplus \Pi_z$, $z \in T^*M$ be a Lagrangian splitting, and assume that the Jacobi curve $\Lambda_z(t)$ is monotone increasing. Then we can define a scalar product on $T_z(T^*M)$; in particular, Λ_z and Π_z are orthogonal with respect to this scalar product.*

Proof. We just define, for any $\boldsymbol{x} \in \Lambda_z$ and for any $\boldsymbol{y} \in \Pi_z$,

$$\langle \boldsymbol{x}, \boldsymbol{x}\rangle_\Lambda = |\dot{\Lambda}_z(0)(\boldsymbol{x})|, \quad \langle \boldsymbol{y}, \boldsymbol{y}\rangle_\Lambda = |\dot{\Lambda}_z(0)\Big((\overline{\dot{\Lambda}}_{z_0}(0))^{-1}\boldsymbol{y}\Big)|, \quad \langle \boldsymbol{x}, \boldsymbol{y}\rangle_\Lambda = 0.$$

□

We can now define the operator

Definition 2.3. *The operator $R^{\Lambda,\Pi}_{z_0}(t) \in gl(\Lambda_{z_0}(t))$ defined as*

$$R^{\Lambda,\Pi}_{z_0}(t) := -\overline{\dot{\Pi}}_{z_0}(t) \circ \overline{\dot{\Lambda}}_{z_0}(t) \tag{2.4}$$

is called the generalized curvature associated to the curves $\Lambda_{z_0}(t)$ and $\Pi_{z_0}(t)$ at the time t.

Since the definition is intrinsic, it allows that

$$R^{\Lambda,\Pi}_{z_0}(t) = e^{-t\vec{h}}{}_* R^{\Lambda,\Pi}_{z_t}(0) e^{t\vec{h}}{}_*|_{\Lambda_{z_0}(t)}.$$

This implies that the knowledge of $R^{\Lambda,\Pi}_z(0)$ for any z along a certain trajectory is equivalent to the knowledge of $R^{\Lambda,\Pi}_{z_0}(t)$ for any t. Then we will do the following definition:

DEFINITION 2.4. *The operator* $R_z^h \in gl(\Lambda_z(0))$ *defined as*

$$R_z^h := R_z^{\Lambda,\Pi}(0) \tag{2.5}$$

is called the generalized curvature of the Hamiltonian vector field $\vec{h}$ *associated to the splitting* $\Lambda \oplus \Pi$.

The following proposition is a valid tool to make computations:

PROPOSITION 2.3. *Let* X *be a section of* Λ. *For any* $z \in T^*M$ *we have*

$$R_z^h X(z) = -[\vec{h}, [\vec{h}, X]_\Pi]_\Lambda(z),$$

where for any vector field Y *on* T^*M *we have* $Y = Y_\Lambda + Y_\Pi$, *where* Y_Λ *is a section of* Λ *and* Y_Π *is a section of* Π.

Proof. Let us recall that by definition for any $\boldsymbol{x} \in \Lambda_{z_0}$

$$\begin{aligned} \dot{\Lambda}_{z_0}(0)(\boldsymbol{x}) &= \langle \overline{\dot{\Lambda}}_{z_0}(0)\boldsymbol{x}, \boldsymbol{x} \rangle \\ &= \sigma(\dot{\lambda}(0), \lambda(0)), \end{aligned}$$

where $\lambda(t)$ is defined as in Section 1. Then we can conclude that $\overline{\dot{\Lambda}}_{z_0}(0)(\boldsymbol{x}) = \sigma(\dot{\lambda}(0), \cdot)$, and hence $\overline{\dot{\Lambda}}_{z_0}(0)(\boldsymbol{x})$ can be identified with $[\dot{\lambda}(0)]$ (where $[\cdot]$ denotes the equivalence class of vectors of $T_{z_0}(T^*M)$ in $T_{z_0}(T^*M)/\Lambda_{z_0}$); we will chose the representative of $[\dot{\lambda}(0)]$ that lies in Π_{z_0}. Then we can put

$$\overline{\dot{\Lambda}}_{z_0}(0)(\boldsymbol{x}) = [\vec{h}, \lambda(t)]_{\Pi_{z_0}}|_{t=0}.$$

The same argument implies that

$$\overline{\dot{\Pi}}_{z_0}(0) \circ \overline{\dot{\Lambda}}_{z_0}(0) = [\vec{h}, [\vec{h}, \lambda(t)]_{\Pi_{z_0}}]_{\Lambda_{z_0}}|_{t=0}$$

and hence the thesis. □

We can conclude with the following property:

PROPOSITION 2.4. *The generalized curvature* R_z^h *is a self-adjoint operator with respect to the scalar product* $\langle \cdot, \cdot \rangle_{\dot{\Lambda}}$

3. Particular splittings

There are many different choices for the splitting $\Lambda \oplus \Pi$; in this section, we assume that a distribution $\Lambda \subset T(T^*M)$ is given, and we describe two quite natural choices for the complement Π: the first one depends on the Hamiltonian field $\vec{h}$, the second one is defined by a symmetric linear connection on the the manifold M.

3.1. The canonical splitting. Let $\Lambda_z(t)$ be the Jacobi curve associated with the distribution Λ. Let Δ be a Lagrangian subspace of $T_z(T^*M)$ transversal to $\Lambda_z(0)$, and denote with $\Lambda_z(0)^{\pitchfork}$ the set of subspaces of $T_z(T^*M)$ transversal to $\Lambda_z(0)$; call $\pi_{\Delta\Lambda_z(0)}$ the projector of $T_z(T^*M)$ onto $\Lambda_z(0)$ and parallel to Δ; then it allows that

$$\pi_{\Delta\Lambda_z(0)}|_{\Lambda_z(0)} = \text{ id} \qquad \pi_{\Delta\Lambda_z(0)}|_\Delta = 0.$$

Let us now notice that the set $\{\pi_{\Delta\Lambda_z(0)} : \Delta \in \Lambda_z(0)^{\pitchfork}\}$ has the structure of an affine subspace of $gl(T_z(T^*M))$; to prove this, notice that, for $\Delta_1,\ \Delta_2 \in \Lambda_z(0)^{\pitchfork}$, the linear combination

$$\alpha\pi_{\Delta_1\Lambda_z(0)} + (1-\alpha)\pi_{\Delta_2\Lambda_z(0)} \qquad \alpha \in \mathbb{R}$$

equals the identity on $\Lambda(0)$ and vanishes on its kernel; then, if we put $\Delta_3 := \ker(\alpha\pi_{\Delta_1\Lambda_z(0)} + (1-\alpha)\pi_{\Delta_2\Lambda_z(0)})$, we have that $\Delta_3 \in \Lambda_z(0)^{\pitchfork}$, and then $\alpha\pi_{\Delta_1\Lambda_z(0)} + (1-\alpha)\pi_{\Delta_2\Lambda_z(0)} = \pi_{\Delta_3\Lambda_z(0)}$.

Let us now recall that an affine space A over a linear space V is a set endowed with a *subtraction* operation $(u,v) \in A \times A \mapsto u - v \in V$ that satisfies the following axioms:

(1) For any $u, v, w \in A$ we have that $(u-w) + (w-v) = u - v \in V$;
(2) For any $v \in A$ and for any $x \in L$ there exists a unique $u \in A$ such that $u - v = x$.

Then, in the space $\{\pi_{\Delta\Lambda_z(0)} : \Delta \in \Lambda_z(0)^{\pitchfork}\}$ it is defined a subtraction that gives values in $gl(T_z(T^*M)$. In fact, for any $\Delta_1,\ \Delta_2 \in \Lambda_z(0)^{\pitchfork}$ and for any $\boldsymbol{x} \in \Lambda_z(0)$, we have that

$$\pi_{\Delta_1\Lambda_z(0)}(\boldsymbol{x}) - \pi_{\Delta_2\Lambda_z(0)}(\boldsymbol{x}) = 0;$$

this means that actually $\{\pi_{\Delta\Lambda_z(0)} : \Delta \in \Lambda_z(0)^{\pitchfork}\}$ is constructed over the subspace $\mathfrak{N}(\Lambda_z(0)) = \{O : T_z(T^*M) \to \Lambda_z(0) : O|_{\Lambda_z(0)} = 0\}$ of $gl(T_z(T^*M)$ of the linear operators from $T_z(T^*M))$ to $\Lambda_z(0)$ vanishing on $\Lambda_z(0)$.

Let us now fix some τ and consider the curve $\Lambda_z(\cdot)$; by regularity of the curve, $\Lambda_z(t)$ is transversal to $\Lambda_z(\tau)$ for t in a punctured neighbourhood of τ. Let us consider the operator-valued function $t \mapsto \pi_{\Lambda_z(t)\Lambda_z(\tau)}$, and let us say that it has a pole at $t = \tau$ if the function $t \mapsto \pi_{\Lambda_z(t)\Lambda_z(\tau)} - \pi_{\Delta\Lambda_z(\tau)}$ has a pole, as a function in $\mathfrak{N}(\Lambda_z(0))$, for some $\Delta \in \Lambda_z(\tau)^{\pitchfork}$.

Let us compute the Laurent expansion $\pi_{\Lambda_z(t)\Lambda_z(\tau)} = \pi_0(\tau) + \sum_{i\neq 0}\pi_i(\tau)(t-\tau)^i$; it is easy to see that actually $\pi_i(\tau) \in \mathfrak{N}(\Lambda_z(\tau))$ for any $i \neq 0$, and then $\pi_0(\tau)$ shall necessarily belong to the affine space. Then, there exists a unique $\Delta \in \Lambda_z(0)^{\pitchfork}$ such that $\pi_0(\tau) = \pi_{\Delta\Lambda_z(\tau)}$; we will denote this space by $\Lambda_z^{\circ}(\tau)$ and we will call it the *derivative element* to $\Lambda_z(\tau)$. By the axioms of the affine spaces, this elements is uniquely defined.

We can repeat this procedure for any τ, and then define the *derivative curve* $t \mapsto \Lambda_z^{\circ}(t)$. Since the definition of this curve is intrinsic, it allows:

$$\Lambda_z^{\circ}(t) = e^{-t\vec{h}}{}_*\Lambda^{\circ}_{e^{t\vec{h}}(z)}(0). \tag{2.6}$$

Easy computations (see [**2**]) show that, if we put local coordinates on $T_z(T^*M)$ in such a way that $\Lambda_z(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$ with $S_0 = 0$, then in coordinates the derivative curve reads $\Lambda_z^{\circ}(t) = \{(A_t y, y + S_t A_t y) : y \in \mathbb{R}^n\}$, where $A_t = -\frac{1}{2}\dot{S}_t^{-1}\ddot{S}_t\dot{S}_t^{-1}$. This implies that if the Jacobi curve is regular its derivative curve is smooth.

Moreover, it is easy to prove that the space $\{\pi_{\Delta\Lambda_z(0)} : \Delta \in \Lambda_z(0)^{\pitchfork} \cap \mathcal{L}_n(T_z(T^*M))\}$ is an affine subspace of $\{\pi_{\Delta\Lambda_z(0)} : \Delta \in \Lambda_z(0)^{\pitchfork}\}$ characterized by the relation

$$\Delta \in \Lambda_z(0)^{\pitchfork} \cap \mathcal{L}_n(T_z(T^*M)) \Leftrightarrow \sigma(\pi_{\Delta\Lambda_z(0)}, \cdot) + \sigma(\cdot, \pi_{\Delta\Lambda_z(0)}) = \sigma(\cdot,\cdot);$$

then, since $\pi_{\Lambda_z(t)\Lambda_z(\tau)}$ belongs to this last subspace, also does $\pi_0(\tau)$, and this implies that $\Lambda_z^{\circ}(\tau)$ is Lagrangian.

Definition 2.5. *The splitting $T_z(T^*M) = \Lambda_z(0) \oplus \Lambda_z^{\circ}(0)$ given by a curve and its derivative curve is called the canonical splitting.*

We remark here that, in these local coordinates such that $\Lambda_z(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$ and $\Lambda_z^{\circ}(t) = \{(A_t y, y + S_t A_t y) : y \in \mathbb{R}^n\}$, with A_t as above, the generalized curvature has the following coordinate expression:

$$R(t) = \frac{1}{2}\dot{S}_t^{-1}\dddot{S}_t - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2.$$

Notice that the formula above is the matrix version of the *Schwartzian derivative*, whose 1-dimensional formulation is the following:

$$\mathbb{S}(S_t) = \frac{1}{2}\dot{S}_t^{-1}\dddot{S}_t - \frac{3}{4}(\dot{S}_t^{-1}\ddot{S}_t)^2.$$

Remark. The sectional curvature in Riemannian geometry satisfies a result called *Conjugate Point Comparison Theorem* (see [**20**]), that relates the value of the curvature with the existence and the position of conjugate points of the Jacobi fields. Here we remark that the generalized curvature satisfies an analogous statement, that we report without proof (it can be found in [**6**], [**2**]).

Let us say that the numbers t_0, t_1 are conjugate parameters for the curve $\Lambda_z(\cdot)$ if $\Lambda_z(t_0) \cap \Lambda_z(t_1) \neq 0$. Then we have the following result:

THEOREM 2.1. *Let $z \in T^*M$ and let $\Lambda_z(t)$ be a curve in $\mathcal{L}_n(T_z(T^*M))$, monotone increasing for any $t > 0$; let $R_z^{\Lambda,\Lambda^\circ}(t)$ be the curvature operator associated to the canonical splitting $\Lambda \oplus \Lambda^\circ$. Then:*

- *If $R_z^{\Lambda,\Lambda^\circ}(t) \leq C\,\mathrm{id}$ for some constant $C > 0$, for any pair of conjugate parameter t_0, t_1 we have that $|t_0 - t_1| \geq \frac{\pi}{\sqrt{C}}$. In particular, if the curvature is negative definite for any t, $\Lambda_z(\cdot)$ does not possess conjugate parameters.*
- *If $\mathrm{tr}(R_z^{\Lambda,\Lambda^\circ}(t)) \geq nC$ for some $C > 0$, for any arbitrary $t_0 \leq t$ the interval $[t, t + \frac{\pi}{\sqrt{C}}]$ contains a point conjugate to t_0.*

3.2. The splitting associated to a linear connection. A linear connection on M defines a parallel transport of vectors on M along curves in M and, by duality, it defines also a parallel transport of covectors of on M. This parallel transport defines then a lift to $T(T^*M)$ of vectors on M. This lifts build a vector distribution on T^*M which is at any point transversal to the vertical distribution $T_z(T^*_{\pi(z)}M)$ (where π denotes the canonical projection of T^*M on the base manifold); the two distribution constitute a splitting of $T(T^*M)$. Let us mention moreover that this distribution is Lagrangian if and only if the connection is torsion-free.

Examples. Let us now consider the Levi-Civita connection on a Riemannian manifold (M, g) (g is the metric tensor); then, the adjoint connection to the Levi-Civita connection defines a Lagrangian splitting on $T(T^*M)$. Actually, this Lagrangian splitting coincides with the canonical splitting defined by the Hamiltonians of natural mechanical systems on the manifold (M, g), namely Hamiltonians of the form

$$h(z) = \frac{1}{2}\langle z, g^{-1}(\pi(z))z\rangle + U(\pi(z)), \tag{2.7}$$

where U is a smooth function on M, and corresponds to the potential energy of a mechanical system. In this case, the generalized curvature is given by

$$R_z^h \boldsymbol{x} = \mathcal{R}(\boldsymbol{z}, \boldsymbol{x})\boldsymbol{z} + D_{\boldsymbol{x}}(\nabla U),$$

where $\boldsymbol{z} \in TM$ is the dual to z via the metric g, $\mathcal{R}$ denotes the Riemannian curvature, and $D_{\boldsymbol{x}}$ is the covariant derivative along $\boldsymbol{x}$.

In the Euclidean case, $M = \mathbb{R}^n$ with the flat metric, we obtain:

$$h(p,q) = \frac{1}{2}|p|^2 + U(q), \qquad R^h_{(p,q)} = \frac{\partial^2 U}{\partial q^2}.$$

4. Reduction by a first integral

Let us now assume that the dynamical systems defined by the Hamiltonian $\vec{h}$ admits a first integral, i.e. that there is a smooth function $g : T^*M \to \mathbb{R}$ such that $\{h, g\} = 0$, where $\{\cdot,\cdot\}$ denotes the Poisson brackets, defined by $\{h, g\} = \sigma(\vec{h}, \vec{g})$. In this case, it is well known that $\vec{g}$ is invariant with respect to the flow $e^{t\vec{h}}{}_*$ (and vice versa), and that the flow generated by $\vec{h}$ preserves the sublevels of the function g; this implies that any trajectory$\{e^{t\vec{h}}(z) : t \in \mathbb{R}\}$ belong to a certain sublevel $g^{-1}(c)$ for any t.

Then, in presence of a first integral it is useful to restrict the analysis of the motion on its sublevels $g^{-1}(c)$; let us notice that, if $z \in g^{-1}(c)$, then $T_z(g^{-1}(c)) = \ker(d_z g)$. It is easy to see that the symplectic form restricted to the tangent space to $g^{-1}(c)$ vanishes identically when applied to $\vec{g}$; then, $\ker(d_z g)/\vec{g}(z)$ is a symplectic space with respect to the symplectic form $\sigma|_{\ker(d_z g)/\vec{g}(z)}$.

Let us remark that any Hamiltonian system admits at least one first integral, which is the Hamiltonian h.

Let now Λ be a Lagrangian distribution on M such that the curve $\Lambda_z(t)$ defined as in (2.2) is regular and that $\vec{g}(z) \notin \Lambda_z$ for any z. The curve defined as

$$J_z^{\boldsymbol{g}}(t) := e^{-t\vec{h}}{}_*(\Lambda_{z_t} \cap \ker(d_{z_t} g) + \operatorname{span}\{\vec{g}(z_t)\}) \tag{2.8}$$

is a curve in $\mathcal{L}_n(T_z(T^*M))$ and is called the *g-reduction of the curve* $J_z(t)$. Since it contains $\vec{g}(z)$ at any time, it is not regular, hence we cannot define the curvature associated to it. Then, we define the curve

$$\overline{J_z^{\boldsymbol{g}}}(t) := J_z^{\boldsymbol{g}}(t)/\operatorname{span}\{\vec{g}(z)\}, \tag{2.9}$$

which is a Lagrangian curve in the $(n-1)$-dimensional space $\ker(d_{z_t} g)/\operatorname{span}\{\vec{g}(z_t)\}$. If this Jacobi curve is regular, then the curvature operator $R_{\overline{J_z^g}}(t)$ of the curve (2.9) with respect to the canonical splitting is well defined on $\overline{J_z^{\boldsymbol{g}}}(t)$.

Let $\psi_z : T_z(T^*M) \to T_z(T^*M)/\vec{g}(z)$ denote the projection onto the factor space. Then we give the following definition:

Definition 2.6. *The operator* $\widehat{R}_{J_z^g}(t)$ *on* $J_z^{\boldsymbol{g}}(t)$ *defined as*

$$\widehat{R}_{J_z^g}(t) := (\psi|_{J_z(t)\cap\ker(d_z g)})^{-1} \circ R_{\overline{J_z^g}}(t) \circ \psi \tag{2.10}$$

is called the curvature operator of the g-reduction J_z^g *at time* t.

The operator $\widehat{R}_z^{h,\boldsymbol{g}}$ *on* $J_z^{\boldsymbol{g}}(0)$ *defined as*

$$\widehat{R}_z^{h,\boldsymbol{g}} := \widehat{R}_{J_z^g}(0)$$

is called the reduced curvature of the Hamiltonian vector field $\vec{h}$ *at the point* $z \in T^*M$.

Examples. Let (M, g) be a Riemannian manifold, and let us consider the Hamiltonian (2.7) of a mechanical system on M; the generalized curvature of the h-reduction is

$$\widehat{R}_z^{h,\boldsymbol{h}}(t)\boldsymbol{x} = R_z^h\boldsymbol{x} + \frac{3\langle\nabla_{\pi(z)}U, \boldsymbol{x}\rangle_h}{2(h(z) - U(\pi(z)))}(\nabla_{\pi(z)}U, 0)^T;$$

notice that in absence of a "potential energy" the reduction by h has no effect on the generalized curvature.

In the Euclidean case, we have

$$\widehat{R}^{h,\boldsymbol{h}}_{(p,q)} = \frac{\partial^2 U}{\partial q^2} + \frac{3}{|p|^2}(\nabla_q U, 0) \otimes (\nabla_q U, 0)^T.$$

5. The canonical moving frame

Let Σ be a symplectic space; an assignment $t \mapsto \{\varepsilon^1(t), \dots, \varepsilon^{2n}(t)\}$ for any t of a basis of Σ is called a *moving frame*; if $\Lambda(\cdot)$ is a curve in Σ, it would be useful to find a particular choice for the family $\{\varepsilon^1(t), \dots, \varepsilon^{2n}(t)\}$ such that for any t $\Lambda(t) = \operatorname{span}\{\varepsilon^1(t), \dots, \varepsilon^n(t)\}$; the possibility of this choice and some related properties are the topic of this last section.

We will restrict to the case of Jacobi curves on a smooth manifold. The first Lemma deals with a generic pair of transversal Lagrangian curves.

LEMMA 2.2. *Let $\Lambda_z(\cdot)$ and $\Pi_z(\cdot)$ be two transversal Lagrangian curves in $T_z(T^*M)$, and assume that $\Lambda_z(\cdot)$ is regular; let $\{\varepsilon^1, \dots, \varepsilon^n\}$ be a basis of $\Lambda_z(0)$. Then there exists a unique way to choose a family $\{e^1(t), \dots, e^n(t), f^1(t), \dots, f^n(t)\}$ of Darboux bases of $T_z(T^*M)$ such that $e^i(0) = \varepsilon^i$, $i = 1, \dots, n$, $\{e^1(t), \dots, e^n(t)\}$ is a basis for $\Lambda_z(t)$ for any t and $\dot{e}^i(t) \in \Pi_z(t)$, $i = 1, \dots, n$, for any t.*

Sketch of the proof. Let $\{\tilde{e}^1(t), \dots, \tilde{e}^n(t)\}$ be a basis of $\Lambda_z(t)$ such that $\tilde{e}^i(0) = \varepsilon^i$ for any $i = 1, \dots, n$; we can find for any t a complement $\{\tilde{f}^i(t)\}_{i=1}^n$ of it in such a way that $\{\tilde{e}^i(t), \tilde{f}^i(t)\}_{i=1}^n$ is a Darboux basis of $T_z(T^*M)$.

Then, we apply a symplectic transformation $\{\tilde{e}^i(t), \tilde{f}^i(t)\}_{i=1}^n \mapsto \{e^i(t), f^i(t)\}_{i=1}^n$; a proper choice of this map realizes the thesis of the lemma. □

In the case in which the complement $\Pi_z(\cdot)$ is the derivative curve of $\Lambda_z(\cdot)$, we have the following result:

PROPOSITION 2.5. *Under the hypotheses of Lemma 2.2, let $\Pi_z(\cdot) = \Lambda_z^\circ(\cdot)$. Then there exist two symmetric $n \times n$ matrices ρ and $r(t)$ such that*

$$\dot{e}^i(t) = \sum_{j=1}^n \rho_{ij} f_j(t), \qquad \dot{f}^i(t) = \sum_{j=1}^n r_{ij}(t) e_j(t), \qquad i = 1, \dots, n. \tag{2.11}$$

Notice that the matrix ρ is constant.

The proof is straightforward and consists in straight computations. We do not write it here.

Remark. Let $R(t) = \{R_{ij}(t)\}_{i,j=1}^n$ denote the representation of the curvature $R_z^{\Lambda,\Lambda^\circ}(t)$ with respect to the basis $\{e^1(t), \dots, e^n(t)\}$. Then, by straightforward application of the definition it can be proved that

$$R(t) = -\rho r(t).$$

Moreover, it can be chosen $\rho = \mathrm{id}$. With this choice, we have that $R(t) = -r(t)$ and that $\dot{e}^i(t) = f^i(t)$, $i = 1, \dots, n$ for any t, and the vectors $e^i(t)$ satisfy the following equation:

$$\ddot{e}^i(t) + \sum_{j=1}^n R_{ij}(t) e^j(t) = 0, \quad i = 1, \dots, n. \tag{2.12}$$

CHAPTER 3

Dynamical entropy of Hamiltonian flows

Riemannian manifolds of negative curvature have been widely studied for their interesting features: in particular, one of the most evident is the exponential divergence of geodesics. It is then natural to focus the attention on the properties of these curves, studying the geodesic flow and the Jacobi fields.

If (M, g) is a Riemannian manifold, the geodesic flow can be synthetically defined as the flow in the unit tangent bundle $SM = \{\boldsymbol{v} \in TM : \|\boldsymbol{v}\| = 1\}$ generated by the Lagrangian

$$L(q, \boldsymbol{v}) = \frac{1}{2} g_q(\boldsymbol{v}, \boldsymbol{v}), \qquad q \in M,\ \boldsymbol{v} \in S_q M,$$

or, more intuitively, given a vector $\boldsymbol{v} \in S_q M$, the geodesic flow is the flow that for any $t \in \mathbb{R}$ maps

$$\boldsymbol{v} \mapsto \dot{\gamma}_{\boldsymbol{v}}(t),$$

where $\gamma_{\boldsymbol{v}}(\cdot)$ is the geodesic with $\gamma_{\boldsymbol{v}}(0) = q$, $\dot{\gamma}_{\boldsymbol{v}}(0) = \boldsymbol{v}$.

It is well known (see for instance [**11**]) that the geodesic flow on closed Riemannian manifolds of negative curvature exhibits a hyperbolic behaviour; moreover, geodesic flows on compact manifolds of negative sectional curvature are *Anosov flows* (that is, the whole manifold is a hyperbolic set) (see [**13**], [**18**]).

Such dynamical systems have indeed positive definite dynamical entropies; an interesting problem is then to find an estimate for its value, knowing the curvature of the system. Ossermann and Sarnak did this in [**22**], and the result has been further generalized by Ballmann and Wojtkowski in [**15**]:

Theorem 3.1 (Ballmann-Wojtkowski). *Let M be a compact Riemannian manifold with nonpositive sectional curvature, and denote with $\mathcal{R}$ the curvature tensor on M. Then for any $q \in M$ and all unit vectors $\boldsymbol{v} \in T_q M$, $K(\boldsymbol{v}) := \mathcal{R}(\cdot, \boldsymbol{v})\boldsymbol{v}$ is a nonpositive symmetric operator on $T_q M$. Then*

$$h_\mu \geq \int_{SM} \operatorname{tr} \sqrt{-K(\boldsymbol{v})}\, d\mu(\boldsymbol{v}), \tag{3.1}$$

where h_μ is the measure-theoretic entropy of the geodesic flow on M, SM the unit tangent bundle on M and μ denotes the normalized Liouville measure on SM.

Agrachev and Chtcherbakova ([**2**], [**5**]) proved that Hamiltonian systems of negative generalized curvature exhibit hyperbolic behaviour too (for details, we remand to references); we notice moreover that the elements of the canonical moving frame satisfy an equation (2.12) analogous to the one satisfied by Jacobi fields in a Riemannian frame:

$$\ddot{Y}(t) + K(v)Y(t) = 0;$$

this equation is a crucial element used in proof of Theorem 3.1.

It is then natural to ask whether an analogous result to Theorem 3.1 allows also in a Hamiltonian context. To prove this generalization is the main topic of this chapter.

In Section 1 we will describe more properties of the canonical moving frame already introduced in Section 5 of Chapter 2; in particular, we will see that we can recover the properties of the Hamiltonian flow along a trajectory just studying the evolution of the canonical moving frame at a certain point of that trajectory.

In Section 2 we will formulate and then prove the result; in particular, the proof is split into two parts, each of them constituting the core of a subsection.

1. More properties of the canonical moving frame

Let h be a Hamiltonian defined on a smooth symplectic manifold M of dimension $2n$, and let Λ be a Lagrangian distribution on M such that the Jacobi curve $\Lambda_z(t)$ is monotone for any $z \in M$; let $\Lambda_z^\circ(t)$ denote its derivative curve. We recall from Section 5 of Chapter 2 that to any $z \in M$ we can attach a moving frame $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$ such that $\Lambda_z(t) = \operatorname{span}\{e_z^1(t), \ldots, e_z^n(t)\}$ and $\Lambda_z^\circ(t) = \operatorname{span}\{f_z^1(t), \ldots, f_z^n(t)\}$. By definition of Jacobi curve, we can choose $\{e_z^1(0), \ldots, e_z^n(0)\}$ at any z in such a way that

$$e_{z_0}^i(t) = e^{-t\vec{h}}{}_* e_{z_t}^i(0), \quad f_{z_0}^i(t) = e^{-t\vec{h}}{}_* f_{z_t}^i(0), \quad i = 1, \ldots, n, \qquad z_t = e^{t\vec{h}}(z_0). \tag{3.2}$$

Moreover, let us notice that these vectors are orthogonal with respect to the scalar product defined by Proposition 2.2.

Let us now define, for any $z \in M$, the basis $\{\varepsilon_z^i\}_{i=1}^{2n}$ as $\varepsilon_z^i = e_z^i(0)$, $\varepsilon_z^{i+n} = f_z^i(0)$ for any $i = 1, \ldots, n$; this is indeed and orthonormal basis.

Fix some z and consider a vector $\boldsymbol{x} \in T_zM$, and write its representation with respect to the latter basis and the canonical moving frame:

$$\boldsymbol{x} = \sum_{i=1}^{2n} x_i \varepsilon_z^i = \sum_{i=1}^{n} \eta_i(t) e_z^i(t) + \xi^i(t) f_z^i(t); \tag{3.3}$$

clearly, $(\eta(0), \xi(0)) = (x_1, \ldots, x_{2n})$. Recall that the vectors $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$ satisfy the equations (2.11). Then,

$$\begin{aligned} \dot{\boldsymbol{x}} &= 0 \\ &= \sum_{i=1}^{n} \Big(\dot\eta_i(t) - \sum_{j=1}^{n} (R_z(t))_{ij} \xi^j(t)\Big) e_z^i(t) + \Big(\eta^i(t) + \dot\xi^i(t)\Big) f_z^i(t), \end{aligned}$$

where we recall that $R_z(t)$ is the representation of the curvature operator $R_z^{\Lambda,\Lambda^\circ}$ with respect to the basis $\{e_z^i(t), f_z^i(t)\}_{i=1}^n$. From equation (3.4) we conclude that the coefficients $(\eta(t), \xi(t))$ satisfy the following system:

$$\left\{ \begin{array}{lcl} \dot\xi(t) &=& -\eta(t) \\ \dot\eta(t) &=& R_z(t)\xi(t) \end{array} \right. . \tag{3.4}$$

In particular, the vector $\xi(t)$ satisfy the Riccati equation

$$\ddot\xi(t) + R_z(t)\xi(t) = 0. \tag{3.5}$$

Due to the definition of the basis $\{\varepsilon_z^i\}_{i=1}^{2n}$ we have the following result:

PROPOSITION 3.1. *For any $z_0 \in T^*M$, the components of a vector $\boldsymbol{x} \in T_{z_0}(T^*M)$ with respect to the canonical moving frame $\{e^i_{z_0}(t), f^i_{z_0}(t)\}_{i=1}^n$ are equal to the components of the vector $e^{t\vec{h}}{}_*\boldsymbol{x}$ with respect to the basis $\{\varepsilon^i_{z_t}\}_{i=1}^{2n}$, for any t.*

Proof. It is just an application of equations (3.2); let $\boldsymbol{x} = \sum_{i=1}^n \eta^i(t)e^i_z(t) + \xi^i(t)\dot{e}^i_z(t)$: then

$$\begin{aligned}
e^{t\vec{h}}{}_*\boldsymbol{x} &= \sum_{i=1}^{2n} x^i \varepsilon^i_{z_t} \\
&= \sum_{i=1}^{n} \eta^i(t)(e^{t\vec{h}}{}_* e^i_{z_0}(t)) + \xi^i(t)(e^{t\vec{h}}{}_* f^i_{z_0}(t)) = \\
&= \sum_{i=1}^{n} \eta^i(t)e^i_{z_t}(0) + \xi^i(t)f^i_{z_t}(0) = \\
&= \sum_{i=1}^{n} \eta^i(t)\varepsilon^i_{z_t} + \xi^i(t)\varepsilon^{i+n}_{z_t},
\end{aligned}$$

hence the thesis. □

2. Entropy of Hamiltonian flows

In this section we will prove a generalization to Hamiltonian flows of Theorem 3.1; then, let us define the frame of the problem. Let M be a smooth $2n$-dimensional symplectic manifold, and $h : M \to \mathbb{R}$ a smooth Hamiltonian function on it.

Since the Hamiltonian flow preserves the sublevels of h, we will restrict our problem to a *compact regular* level set of the Hamiltonian, which we will call N. Let us notice that for any $z \in N$ $T_zN = \ker(d_zh)$; as seen, $\sigma|_{T_zN}(\vec{h}, \cdot)$ vanishes identically, and then the space $\ker(d_zh)/\mathrm{span}\{\vec{h}(z)\}$ is a symplectic space (with respect to the restricted symplectic form $\sigma|_{\ker(d_zh)/\mathrm{span}\{\vec{h}(z)\}}$) for any $z \in N$. Since the action of the Hamiltonian flow preserves the Hamiltonian vector field, we can restrict our analysis to the symplectic space $\Sigma_z := \ker(d_zh)/\mathrm{span}\{\vec{h}(z)\}$, $z \in N$. Let us notice that the space Σ is left invariant by the action of the Hamiltonian flow, i.e. $e^{t\vec{h}}{}_*(\Sigma_{z_0}) = \Sigma_{z_t}$, where $z_t = e^{t\vec{h}}(z_0)$.

Let us now define the (normalized) Liouville measure on N in this way:

$$d\mu := \frac{1}{\mathcal{N}} \underbrace{\sigma \wedge \cdots \wedge \sigma}_{n-1} \wedge \iota_X \sigma, \tag{3.6}$$

where X is a vector field defined on a neighbourhood of N such that $\langle dh, X\rangle = 1$, and $\mathcal{N} := \int_N \sigma \wedge \cdots \wedge \sigma \wedge \iota_X\sigma$; $\iota_X\sigma = \sigma(X, \cdot)$ is the evaluation of σ over X.

Now we state the result; we remark that in the following with σ_z we will denote the restriction $\sigma|_{\Sigma_z}$, and that $e^{t\vec{h}}$ will denote the restriction of the Hamiltonian flow to N.

THEOREM 3.2. *Let N be a compact regular level set of a smooth Hamiltonian function $h : M \to \mathbb{R}$, where M is a $2n$-dimensional symplectic manifold. Let Λ be a Lagrangian distribution on $TN/\mathrm{span}\{\vec{h}\}$, and assume that the Hamiltonian vector field $\vec{h}$ is monotone with respect to Λ. Consider the Jacobi curve $\Lambda_{z_0}(t) = e^{-t\vec{h}}{}_*\Lambda_{z_t}$ and assume that the restricted curvature $\widehat{R}^{h,\boldsymbol{h}}_z$ is nonpositive.*

Then the dynamical entropy h_μ of the Hamiltonian flow on N with respect to the normalized Liouville measure μ on N (3.6) satisfies the following inequality:

$$h_\mu \geq \int_N \operatorname{tr}\sqrt{-\widehat{R}_z^{h,\boldsymbol{h}}}\, d\mu. \tag{3.7}$$

Proof. Our main tool to estimate h_μ is represented by Pesin's formula; then, the aim will become to compute the sum of the positive Lyapunov exponents (1.15).

By Osedelec Theorem, there is a full-measure Borel set $X \subset N$ such that every $z \in X$ is a regular point for the Hamiltonian flow, that is for any $z \in X$ there is a unique splitting $T_zN = E_z^u \oplus E_z^s \oplus E_z^0$ such that

$$\begin{aligned}
\lim_{t\to\pm\infty} \frac{1}{t}\log\|D_z e^{t\vec{h}}\boldsymbol{x}\| &> 0 \qquad \boldsymbol{x} \in E_z^u\\
\lim_{t\to\pm\infty} \frac{1}{t}\log\|D_z e^{t\vec{h}}\boldsymbol{x}\| &< 0 \qquad \boldsymbol{x} \in E_z^s\\
\lim_{t\to\pm\infty} \frac{1}{t}\log\|D_z e^{t\vec{h}}\boldsymbol{x}\| &= 0 \qquad \boldsymbol{x} \in E_z^0.
\end{aligned}$$

Moreover, the limit

$$\lim_{t\to\pm\infty} \frac{1}{t}\log|\det(D_z e^{t\vec{h}})|_{E_z}| \tag{3.8}$$

exists for any $z \in X$ and is independent on E_z for any subspace E_z such that $E_z^u \subset E_z \subset E_z^u \oplus E_z^0$. Further, we have that

$$\chi(z) := \sum_{\lambda_j(q)\geq 0} \lambda_j(q)\dim E_j(q) = \lim_{t\to\pm\infty}\frac{1}{t}\log|\det(D_z e^{t\vec{h}})|_{E_z}|. \tag{3.9}$$

Since N is compact, the value of the Lyapunov exponents is independent on the choice of the norm on Σ_z; in particular, we choose the scalar product $\langle\cdot,\cdot\rangle_{\dot\Lambda}$ on Σ_z associated to the curve $\Lambda_z(\cdot)$ according to Proposition 2.2.

Then, the proof of the theorem is split into two steps: in the first one, we will look for a good candidate for the space E_z; in the second part, we will evaluate the limit (3.8) and complete the proof.

2.1. The space H_z. Let us introduce for any $z \in N$ the set $H_z \in \Sigma_z$ defined as

$$H_z := \{\boldsymbol{x} \in \Sigma_z : \frac{d}{dt}\|\pi_{\Lambda_{z_t}(0)\Lambda^\circ_{z_t}(0)}(e^{t\vec{h}}{}_*\boldsymbol{x})\| \geq 0 \quad \forall\, t\}, \tag{3.10}$$

where $z_t = e^{t\vec{h}}(z)$ and $\pi_{\Lambda_{z_t}(0)\Lambda^\circ_{z_t}(0)}$ denotes, as usual, the projector of Σ_z onto $\Lambda^\circ_{z_t}(0)$ and parallel to $\Lambda_{z_t}(0)$. Clearly, H_z is intrinsically defined and it is invariant along the trajectory $e^{t\vec{h}}(z)$.

Let $\{e_z^i(t), \dot{e}_z^i(t)\}_{i=1}^{n-1}$ be the canonical moving frame in Σ_z, with $\Lambda_z(t) = \operatorname{span}\{e_z^1(t),\ldots,e_z^{n-1}(t)\}$ and $\Lambda_z^\circ(t) = \operatorname{span}\{\dot{e}_z^1(t),\ldots,\dot{e}_z^{n-1}(t)\}$, and define the basis $\{\varepsilon_z^i\}_{i=1}^{n-1}$ as in Section 1: $\varepsilon_z^i = e_z^i(0)$ and $\varepsilon_z^{i+n-1} = \dot{e}_z^i(0)$ for $i = 1,\ldots,n-1$. Then, due to Proposition 3.1, we have for any $\boldsymbol{x} \in \Sigma_z$, $x = \sum_{i=1}^{n-1}\eta^i(t)e_z^i(t) + \xi^i(t)\dot{e}_z^i(t)$, that

$$\|\pi_{\Lambda_{z_t}(0)\Lambda^\circ_{z_t}(0)}(e^{t\vec{h}}{}_*\boldsymbol{x})\| = |\xi(t)| \qquad \|\pi_{\Lambda^\circ_{z_t}(0)\Lambda_{z_t}(0)}(e^{t\vec{h}}{}_*\boldsymbol{x})\| = |\eta^i(t)| = |\dot{\xi}^i(t)|.$$

Let us now investigate the properties of the set H_z. To simplify the notation, in the following we will put $v_t = \Lambda_{z_t}(0)$, $v_t^\circ = \Lambda_{z_t}^\circ(0)$, and $\phi^t = e^{t\vec{h}}$.

LEMMA 3.1. *H_z is a vector subspace of Σ_z.*

Proof. Notice that the function $\|\pi_{v_t v_t^\circ}({\phi^t}_* \boldsymbol{x})\|^2 = |\xi(t)|^2$ is convex; in fact, (3.5) implies that

$$\frac{d^2}{dt^2}|\xi(t)|^2 = 2[\langle\dot\xi(t),\dot\xi(t)\rangle - \langle R_z(t)\xi(t),\xi(t)\rangle] \geq 0; \tag{3.11}$$

then a vector $\boldsymbol{x} \in \Sigma_z$ belongs to H_z if and only if $\|\pi_{v_t v_t^\circ}({\phi^t}_* \boldsymbol{x})\|$ is bounded for negative times. Linear combinations of vectors satisfying this property still satisfy it. □

LEMMA 3.2. *H_z is transversal to Λ_z for any z.*

Proof. A vector $\boldsymbol{x} \in H_z$ belongs to Λ_z if $\pi_{v_0 v_0^\circ}(\boldsymbol{x}) = 0$, i.e. if $\xi(0) = 0$; for such a (nonzero) vector, inequality (3.11) is strict at the time $t = 0$, which implies that $t = 0$ is a strong minimum for $|\xi(t)|$. This is in contradiction with the definition of H_z. □

LEMMA 3.3. *H_z is a Lagrangian space.*

Proof. Put, for any $\tau \in \mathbb{R}$, $H_\tau = \{\boldsymbol{x} \in \Sigma_z : \frac{d}{dt}\|\pi_{v_t v_t^\circ}({\phi^t}_* \boldsymbol{x})\| \geq 0 \quad \forall t \geq \tau\}$; clearly, $H_{\tau_1} \subseteq H_{\tau_2}$ if $\tau_1 \leq \tau_2$ and $H_z = \cap_\tau H_\tau$.

H_τ contains at least one Lagrangian subspace for any τ. Indeed, fix τ and consider $V_\tau = \{\boldsymbol{x} \in \Sigma_z : \pi_{v_\tau v_\tau^\circ}({\phi^\tau}_* \boldsymbol{x}) = 0\}$; actually, ${\phi^\tau}_*(V_\tau) = \Lambda_{\phi^\tau(z)}(0)$, then V_τ is Lagrangian. We shall prove that it is contained in H_τ; since both sets are intrinsically defined, we can prove it in coordinates. Let then $\boldsymbol{x} = \sum_{i=1}^{n-1} -\dot\xi(t)e_z^i(t) + \xi(t)\dot e_z^i(t) \in V_\tau$; since

$$\frac{d}{dt}|\xi(t)|^2|_{t=\tau} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}|\xi(t)|^2 \geq 0 \quad \forall\, t,$$

then $\frac{d}{dt}\|\pi_{v_t v_0^\circ}({\phi^t}_* \boldsymbol{x})\| = \frac{d}{dt}|\xi(t)| \geq 0$ for any $t \geq \tau$, and $V_\tau \subset H_\tau$.

H_z contains a Lagrangian subspace too. Indeed, define for any τ $\widehat{H}_\tau = \{V \in \mathfrak{L}_{n-1}(\Sigma_z) : V \subset H_\tau\} = \mathfrak{L}_{n-1}(\Sigma_z) \cap H_\tau$, which is a compact nonempty subset of $\mathfrak{L}_{n-1}(\Sigma_z)$; moreover, since $\widehat{H}_{\tau_1} \subseteq \widehat{H}_{\tau_2}$ if $\tau_1 \leq \tau_2$, their intersection is nonempty: $\cap_\tau \widehat{H}_\tau \neq \emptyset$. Since $\widehat{H}_\tau \subset H_\tau$ for any τ, we can conclude that $\emptyset \neq \cap_\tau \widehat{H}_\tau \subset H_z$, that means that H_z contains at least a Lagrangian subspace.

From Lemma 3.2 we know that $\dim H_z \leq n-1$; then we conclude that H_z is a Lagrangian subspace. □

Since H_z is transversal to $\Lambda_z(0)$ and is Lagrangian, there shall exist a symmetric linear operator $U_z : \Lambda_z^\circ(0) \to \Lambda_z(0)$ such that H_z is the graph of this operator, i.e. any vector $\boldsymbol{x} \in H_z$ can be written as $\boldsymbol{x} = \boldsymbol{w} + U_z\boldsymbol{w}$, $\boldsymbol{w} \in \Lambda_z^\circ(0)$.

Shifting to coordinate representation, there exists a linear operator $V_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that for any vector in H_z with the coordinate representation given by Equation (3.3), we have that $\eta(0) = -V_z\xi(0)$; from (3.4) we get that $\dot\xi(0) = V_z\xi(0)$, and (3.5) implies that the linear operator V_z satisfies the equation

$$\dot V_{\phi^t(z)} + V_{\phi^t(z)}^2 + R_z(t) = 0, \tag{3.12}$$

where we recall that here $R_z(t)$ is the representation of the operator $\widehat{R}_z^{h,\boldsymbol{h}}$ with respect to the canonical moving frame. By definition of H_z, the operator V_z is nonnegative definite for any

z. In fact, for any $x \in \mathbb{R}^{n-1}$, we have that

$$\langle x, V_z x\rangle = \langle \xi(0), \dot{\xi}(0)\rangle = \frac{1}{2}\frac{d}{dt}|\xi(t)|^2|_{t=0} \geq 0,$$

where $\boldsymbol{x} = \sum_{i=1}^{n-1} -\dot{\xi}(t)e_z^i(t) + \xi(t)\dot{e}_z^i(t)$ is a vector in H_z.

LEMMA 3.4. $E_z^u \subset H_z \subset E_z^u \oplus E_z^0$.

Proof. First, we show that E_z^u and $E_z^u \oplus E_z^0$ are the skew-orthogonal (with respect to the symplectic form) complement to each other. Let then $\boldsymbol{x} \in E_z^u$ and $\boldsymbol{y} \in E_z^u \oplus E_z^0$;

$$\begin{aligned}\lim_{t\to-\infty} \frac{1}{|t|}\log|\sigma({\phi^t}_*\boldsymbol{x}, {\phi^t}_*\boldsymbol{y})| &\leq \lim_{t\to-\infty}\left[\frac{1}{|t|}\log\|\sigma\| + \frac{1}{|t|}\log\|{\phi^t}_*\boldsymbol{x}\| + \frac{1}{|t|}\log\|{\phi^t}_*\boldsymbol{y}\|\right] = \\ &= -\lambda(z,\boldsymbol{x}) - \lambda(z,\boldsymbol{y}) < 0,\end{aligned}$$

which means that $\sigma({\phi^t}_*\boldsymbol{x}, {\phi^t}_*\boldsymbol{y}) \to 0$ as $t \to -\infty$. Since the Hamiltonian flow preserves the symplectic form, we get that $\sigma(\boldsymbol{x}, \boldsymbol{y}) = 0$, i.e. E_z^u and $E_z^u \oplus E_z^0$ are skew-orthogonal. This implies that $\dim E^u(z) + \dim(E^u(z) \oplus E^0(z)) \leq 2n-2$; recall that $\dim(E^u(z)\oplus E^0(z) \oplus E^s(z)) = 2n-2$. Then, by subadditivity of the dimension of vector spaces,

$$\begin{aligned}4n-4 &= 2\dim(E^u(z)\oplus E^s(z)\oplus E^0(z)) \leq \\ &\leq \dim E^u(z) + \dim(E^s(z)\oplus E^0(z)) + \dim E^s(z) + \dim(E^0(z)\oplus E^u(z)) \leq 4n-4;\end{aligned}$$

then, in the relation above it allows the equality, and $\dim E^u(z) + \dim(E^u(z)\oplus E^0(z)) = 2n-2$, which implies that they are the skew-orthogonal complement to each other.

Let now $\boldsymbol{x} \in E_z^u$, i.e. $\lim_{t\to-\infty}\frac{1}{|t|}\log\|{\phi^t}_*\boldsymbol{x}\| < 0$; this means that $\|{\phi^t}_*\boldsymbol{x}\|$ is bounded for nonpositive times, and, consequently, also $\pi_{v_t v_t^\circ}({\phi^t}_*\boldsymbol{x})$ is: this implies, by definition, that ${\phi^t}_*\boldsymbol{x} \in H_{\phi^t(z)}$, and, by invariance of H_z along Hamiltonian trajectories, that $\boldsymbol{x} \in H_z$. Then $E_z^u \subset H_z$.

Since H_z is Lagrangian, it also allows $H_z \subset E_z^u \oplus E_z^0$. ☐

LEMMA 3.5. *For any $\boldsymbol{x} \in H_z$, $\pi_{v_0 v_0^\circ}(\boldsymbol{x}) \in \ker U_z$ if and only if $\|\pi_{v_t^\circ v_t}({\phi^t}_*\boldsymbol{x})\| = 0$ for any $t \leq 0$.*

Proof. Let us prove it in coordinates; let $\boldsymbol{x} \in H_z$ have the coordinate expression (3.3), and assume that $\xi(0) \in \ker V_z$, i.e. $\eta(0) = 0$; since $\frac{d^2}{dt^2}|\xi(t)|^2 \geq 0$ and $\frac{d}{dt}|\xi(t)|^2_{t=0} = 0$, $|\xi(t)|^2$ shall remain constant for any $t \leq 0$. This implies that $|\dot{\xi}(t)| = 0$ for any nonpositive t. Then $\|\pi_{v_t^\circ v_t}({\phi^t}_*\boldsymbol{x})\| = 0$ for any $t \leq 0$.

Conversely, if $\dot{\xi}(t) = 0$ for any $t \leq 0$, in particular we get that $\dot{\xi}(0) = -\eta(0) = 0$, and then $\xi(0) \in \ker V_z$. ☐

The space H_z is indeed a good candidate for the subspace E_z in (3.9); however, for further computation we will need the operator V_z to be strictly positive definite; this condition will be satisfied if we will restrict to a proper subspace of H_z, that is H_z^0, defined as the graph of the restriction of U_z to the orthogonal complement in $\Lambda_z^\circ(0)$ to $\ker U_z$. We have

LEMMA 3.6. *H_z^0 is invariant with respect to the Hamiltonian flow, i.e. ${\phi^t}_*(H_z^0) = H_{\phi^t(z)}^0$.*

Proof. If $\pi_{v_0 v_0^\circ}(\boldsymbol{x}) \in \ker U_z$, Lemma (3.5) implies that $\|\pi_{v_t^\circ v_t}({\phi^t}_*\boldsymbol{x})\| = 0$ for any $t \leq 0$, or, equivalently, that $\pi_{v_t v_t^\circ}({\phi^t}_*\boldsymbol{x}) \in \ker U_{\phi^t(z)}$ for any $t \leq 0$. Then, for $t \geq 0$ $\ker U_{\phi^t(z)} \subset {\phi^t}_*(\ker U_z)$, and hence ${\phi^t}_*(H_z^0) \subset (H_{\phi^t(z)}^0)$.

In particular, $\dim H^0_z$ is nondecreasing along the orbits of the Hamiltonian flow. Measurable functions which are nondecreasing along the orbits of a measurable flow that preserves a probability measure are equal, on a full-measure set, to a constant function. Then, ${\phi^t}_*(H^0_z) = (H^0_{\phi^t(z)})$. □

To restrict ${\phi^t}_*$ to H^0_z we need to prove that the space satisfies Lemma 3.4; we will call U^0_z the restriction of U_z to the orthogonal complement to $\ker U_z$ in $\Lambda^\circ_z(0)$ (in particular, notice that H^0_z is the graph of U^0_z), and respectively V^0_z and $R^0_z(t)$ the restrictions of V_z and $R_z(t)$ to the orthogonal complement of $\ker V_z$ in $\mathbb{R}^{n-1}$.

LEMMA 3.7. *$R_z(t)$ vanishes on $\ker V_{\phi^t z}$ and both $R_z(t)$ and $V_{\phi^t z}$ preserve the orthogonal complement in $\mathbb{R}^{n-1}$ to $\ker V_{\phi^t z}$.*

Proof. Call $\Delta_z(t)$ the orthogonal complement in $\mathbb{R}^{n-1}$ to $\ker V_{\phi^t(z)}$. Let $\boldsymbol{x} \in H_z$ and write it in coordinates $(-\dot\xi(t), \xi(t))$; assume that $\xi(t) \in \ker V_{\phi^t z}$; then, by previous lemma, $\dot\xi(\tau) = 0$ for $\tau \le t$, which implies the vanishing of the second derivative too, i.e. $R_z(t)\xi(t) = 0$. Let now $v \in \ker V_{\phi^t z}, v' \in \Delta_z(t)$; since $\langle v, R_z(t)v'\rangle = \langle R_z(t)v, v'\rangle = 0$, we conclude that $R_z(t)[\Delta_z(t)] \subseteq \Delta_z(t)$. In the same way we can show that $V_{\phi^t(z)}[\Delta_z(t)] \subseteq \Delta_z(t)$. □

LEMMA 3.8. *$E^u_z \subset H^0_z \subset E^u_z \oplus E^0_z$.*

Proof. Consider $\boldsymbol{x} \in H_z \setminus H^0_z$; then, by Lemma 3.5, ${\phi^t}_*\boldsymbol{x}$ is constant in norm for any $t \le 0$. Hence $\lambda(z, \boldsymbol{x}) = 0$, which implies that $\boldsymbol{x} \notin E^u_z$; then, $E^u_z \subset H^0_z$.

The fact that $H^0_z \subset E^u_z \oplus E^0_z$ is obvious, since $H^0_z \subset H_z$. □

Let us finally mention that the coefficients of vectors in H^0_z with respect to the canonical moving frame still satisfy the relation (3.5). In fact, consider $\boldsymbol{x} = \boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)} \in H_z$, with $\boldsymbol{x}^{(1)} \in H_z \setminus H^0_z$ and $\boldsymbol{x}^{(2)} \in H^0_z$; let $(-\dot\xi^{(i)}(t), \xi^{(i)}(t))$ be their components with respect to the canonical moving frame. Then, by Lemma 3.5 we get that $\dot\xi^{(1)}(t) = 0$ for any $t \le 0$, and hence $\ddot\xi^{(1)}(t) = 0$; by Lemma 3.7, $R_z(t)\xi^{(1)}(t) = 0$. Then, both the components satisfy equation (3.5).

2.2. Computation of the entropy. Since H^0_z is the graph of U^0_z, we can express the scalar product on it in terms of the canonical scalar product $\langle\cdot,\cdot\rangle_{n-1}$ on $\mathbb{R}^{n-1}$, putting

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\dot\Lambda} = \langle \xi^{\boldsymbol{x}}(0), A_z(0)\xi^{\boldsymbol{y}}(0)\rangle_{n-1}, \qquad A_z(t) = \mathrm{id} + {V^0_{\phi^t(z)}}^2.$$

Call $a_z(t) = |\det {\phi^t}_*|_{H^0_z}|$ the determinant of ${\phi^t}_*|_{H^0_z}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\dot\Lambda}$; we have that

$$a_z(t) = \sqrt{\det A_z(t)}\,|\det {\phi^t}_*|_{H^0_z}|_{n-1} = \sqrt{\det A_z(t)}\,|\det e^{\int_0^t V_{\phi^s(z)}\,ds}|_{H^0_z}|_{n-1}.$$

Define

$$r_z(t) := \frac{d}{dt}\log a_z(t) = \frac{1}{2}\mathrm{tr}\dot A_z(t)A_z^{-1}(t) + \mathrm{tr}V^0_{\phi^t(z)};$$

applying (3.12), we get by computations that $r_z(t) = \mathrm{tr}[(V^0_{\phi^t z} - R^0_z(t)V^0_{\phi^t z})(\mathrm{id} + {V^0_{\phi^t z}}^2)^{-1}]$. Since

$$\chi(z) = \lim_{t\to\infty}\frac{1}{t}\log|\det({\phi^t}_*|_{H^0_z})| = \lim_{t\to\infty}\frac{1}{t}\log a_z(t) = \lim_{t\to\infty}\frac{1}{t}\int_0^t r_z(s)\,ds,$$

by Birkhoff Ergodic Theorem (Theorem 1.6) we get that, provided that r_z is an integrable function on N,

$$h_\mu(\phi) = \int_N \chi(z)\, d\mu(z) = \int_N r_z(0)\, d\mu.$$

Now we are going to compute dynamical entropy using a different scalar product on H_z^0, after showing that we will get the same value. Call $A'_z(t) = V^0_{\phi^t(z)}$, and define the scalar product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle' = \langle \xi^{\boldsymbol{x}}(0), A'_z(0)\xi^{\boldsymbol{y}}(0)\rangle_{n-1}$; we then get that $r'_z(t) = \frac{1}{2}\mathrm{tr}[V^0_{\phi^t(z)} - R^0_z(t) {V^0_z}^{-1}]$.

The volume element on N with respect the scalar product given by A' is related to the standard volume element in this way: $d\mu' = \sqrt{\frac{\det A'}{\det A}} d\mu$. If we call $c(t) = \frac{d\mu}{d\mu'} = \sqrt{\frac{\det A(t)}{\det A'(t)}} > 1$, we find that $0 < a'(t) < a(t)c(0)$. We have that:

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t r'_z(s)\, ds = \limsup_{t\to\infty} \frac{1}{t}\log a'_z(t) \leq \lim_{t\to\infty}\frac{1}{t}\log a_z(t) = \chi(z)$$

$$\liminf_{t\to-\infty} \frac{1}{|t|}\int_t^0 r'_z(s)\, ds = -\limsup_{t\to-\infty} \frac{1}{|t|}\log a'_z(t) \geq -\lim_{t\to-\infty}\frac{1}{|t|}\log a_z(t) = \chi(z),$$

hence

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t r'_z(s)\, ds \leq \chi(z) \leq \liminf_{t\to-\infty} \frac{1}{|t|}\int_t^0 r'_z(s)\, ds.$$

r'_z is measurable on N, since continuous. Applying the following Lemma we can prove it is also integrable on N (for its proof, see [**15**]):

LEMMA 3.9. *Let ϕ^t be a measure preserving flow on a probability space (X, μ) and $f : X \to \mathbb{R}$ a measurable nonnegative function; if for almost every $x \in X$ $\limsup_{T\to+\infty} \frac{1}{T}\int_0^T f(\phi^t x)dt \leq k(x)$, where $k : X \to \mathbb{R}$ is a measurable function, then*

$$\int_X f(x)\, d\mu(x) \leq \int_X k(x)\, d\mu(x).$$

Hence, we get by Ergodic Theorem and equality of time averages in the future and in the past (equation (1.17)) that

$$\int_N r_z(0)' d\mu = \int_N \chi(z)\, d\mu(z) = h_\mu(\phi).$$

Finally, we use the following result (for the proof, see again [**15**]):

LEMMA 3.10. *Given three symmetric linear operators U, M, N on a Euclidean space such that M and N are nonnegative definite and U is strictly positive definite, we get that* $\mathrm{tr}[MU + NU^{-1}] \geq 2\mathrm{tr}\sqrt{M}\sqrt{N}$, *where equality holds iff* $\sqrt{M}U = \sqrt{N}$.

Since we have that $r'_z(t) = \frac{1}{2}\mathrm{tr}[V^0_{\phi^t(z)} - R^0_z(t){V^0_{\phi^t(z)}}^{-1}]$, where $V^0_{\phi^t(z)}$ is (strictly) positive definite and $-R^0_z(t)$ is nonnegative definite, we can apply previous lemma with $U = V^0_{\phi^t(z)}$, $M = \mathrm{id}$ and $N = -R^0_z(t)$, obtaining $\frac{1}{2}\mathrm{tr}[V^0_{\phi^t(z)} - R^0_z(t){V^0_{\phi^t(z)}}^{-1}] \geq \mathrm{tr}\sqrt{-R^0_z(t)}$, and hence

$$h_\mu(\phi) \geq \int_N \mathrm{tr}\sqrt{-R^0_z(0)}\, d\mu = \int_N \mathrm{tr}\sqrt{-R_z(0)}\, d\mu.$$

□

Remark. The estimate is sharp (i.e. we have the equality) if and only if $V^0_{\phi^t(z)} = \sqrt{-R^0_z(t)}$ for almost all $z \in N$, which implies that $V^2_{\phi^t(z)} = -R_z(t)$ almost everywhere on N, and hence, by continuity, for every $z \in N$; this means that $\dot{V}_{\phi^t(z)} = 0$ on N, i.e. all Jacobi curves are symmetric [**2**].

CHAPTER 4

Optimal synthesis for infinite horizon variational problems

Infinite horizon optimal control problem are of great interest in mathematical economy, since they provide a good model for dynamical economic systems. In particular, an important problem of this class is the *optimal economic growth problem*:

$$\begin{cases} \max_{q(t)} \int_0^{+\infty} e^{-\alpha t}\, \varphi(q(t), u(t), t)\; dt \\ \dot q(t) = f(q(t), u(t)) \end{cases} \qquad \alpha \geq 0. \tag{4.1}$$

Here the functional to be maximized is the capital accumulation, seen as the sum of utilities over a long time interval; clearly, we can also study the analogous *minimization* problem, in which we look for $\min_{q(t)} \int_0^{+\infty} e^{-\alpha t}\, \varphi(q(t), u(t), t)\; dt$: in this case the functional represents some cost to be minimized during a production process (see [**14**] for references).

The problem is very general and can be studied in many different settings, then it inspired lots of threads; we are not mentioning them here, for the largeness of the topic.

The setting we are interested in is the one of *smooth* costs with *continuous time*; this means that we will study the functional

$$J(\gamma(\cdot)) = \int_0^\infty \varphi(\gamma(t), \dot\gamma(t))\; dt, \tag{4.2}$$

where M is a smooth n-dimensional manifold, $\varphi : TM \to \mathbb{R}$ a smooth function, and the functional $J(\gamma(\cdot))$ is defined on the Lipschitzian curves $\gamma : [0, +\infty) \to M$ such that the integral in (4.2) converges.

More precisely, we assume the existence of an *equilibrium point* q_∞ for φ such that

$$\varphi(q_\infty, 0) = 0, \qquad \frac{\partial \varphi}{\partial q}(q_\infty, 0) = 0,$$

and we try to find the cost

$$c(q_0) = \min \Big\{ \int_0^\infty \varphi(\gamma(t), \dot\gamma(t))\; dt : \quad \gamma(0) = q_0,\ \lim_{t\to\infty} \gamma(t) = q_\infty \Big\}. \tag{4.3}$$

Our aim is to characterize the class of Lagrangians φ such that the minimization problem admits a smooth optimal synthesis, according to the following definition:

Definition 4.1. *A smooth optimal synthesis is a smooth complete vector field X on M such that q_∞ is a globally stable equilibrium of the ordinary differential equation $\dot q = X(q)$ and*

$$c(q_0) = \int_0^\infty \varphi(\gamma_q(t), \dot\gamma_q(t)\; dt, \qquad q \in M,$$

where $\dot\gamma_q(t) = X(\gamma_q(t))$ and $\gamma_q(0) = q$.

Any segment of a minimizing path $\gamma(t),\ t \geq 0$, is automatically a minimizer for the corresponding finite horizon functional. Finite horizon extremals satisfy the Euler-Lagrange equation associated to the functional, or the first-order Hamiltonian equation associated to the functional by PMP. To find the minimizers we are using this second approach; hence, we will formulate the variational problem as an optimal control problem. Our goal then becomes to find the conditions on the Lagrangian φ that guarantee the existence of the optimal synthesis focusing on the properties of the associated Hamiltonian system.

In Section 1, we will explain some properties of Hamiltonian system with negative curvature that will be used in the following.

In Section 2 we will state the variational problem we are studying and we will briefly explain our strategy to prove the result we are using. This strategy is suggested by some simple examples we are recalling in this section.

The result with its proof is the topic of Section 3; the proof is split into three steps, each of them is the content of a subsection. In a Euclidean frame the same results can be obtained under hypotheses of strict convexity of the Lagrangian φ with respect to the pair (q, u); this is proved treating this result as a special case. We will do it in Section 4.

Finally, Section 5 contains a generic classification for 1-dimensional problems.

Remark. In the following, with *strict convexity* of a function with respect to some variable we will mean the *positive definiteness* of the Hessian of the function with respect to the same variable.

1. Hyperbolic fixed points of Hamiltonian flows

Let M be a smooth $2n$-dimensional symplectic manifold, and $h : M \to \mathbb{R}$ be a smooth Hamiltonian function. Denote with ϕ^t the flow generated by the vector field $\vec{h}$. We give the following definition:

DEFINITION 4.2. *A point $z \in M$ is a hyperbolic fixed point of the flow ϕ^t if it is a fixed point of the flow (i.e. $\vec{h}(z) = 0$) and if there exists a ϕ^t-invariant splitting of the tangent space $T_zM = E_z^+ \oplus E_z^-$ and two positive constants γ and c such that*

$$\|D_z\phi^{\mp t}\boldsymbol{x}\| \leq ce^{-\gamma t}\|\boldsymbol{x}\| \qquad \text{for } \boldsymbol{x} \in E_z^{\pm} \text{ and } t \geq 0. \tag{4.4}$$

We now recall a result that relates the behaviour of the trajectories of a Hamiltonian flow with its generalized curvature; the result is proved in [**2**]:

THEOREM 4.1. *Let Λ_z be a Lagrangian distribution on M, and let $\Lambda_z(t)$ be the associated Jacobi curve; assume that $\Lambda_z(t)$ is regular and monotone. Let W be a compact invariant set of the flow ϕ^t. If the curvature $R_z^{\Lambda,\Lambda^\circ}$ is negative definite at any point of W, then W is a finite set and each point of W is a hyperbolic equilibrium of the field $\vec{h}$.*

Sketch of the proof. We are not going to give the proof of the Theorem here: it can be found in literature. But since in the following we are going to generalize this result, we just recall here the main steps of the proof.

The core of the proof is to find a special metric $\|\cdot\|$ on TM (which actually is the metric defined by $\dot{\Lambda}_z$, see Proposition 2.2) and a family of invariant *expanding* and *contracting* cones $C_z^{\pm} \subset T_zM$ such that

$$\|D_z\phi^{\mp t}\boldsymbol{x}\| \leq ce^{-\gamma t}\|\boldsymbol{x}\| \qquad \text{for } \boldsymbol{x} \in C_z^{\pm} \text{ and } t \geq 0, \tag{4.5}$$

for some positive constants γ and c.

Once found such cones, define for any z the sets

$$\widehat{C}_z^+ = \bigcap_{t\geq 0} {\phi^t}_*(C_{\phi^{-t}(z)}), \qquad \widehat{C}_z^- = \bigcap_{t\leq 0} {\phi^t}_*(C_{\phi^{-t}(z)}); \tag{4.6}$$

finally, prove that actually $\widehat{C}_z^+$ satisfies the conditions for E_z^+ and $\widehat{C}_z^-$ satisfies the conditions for E_z^-.

The crucial part is then the definition of the cones $C_z^{\pm}$; to do this, in [**2**] there were used the properties of the canonical moving frame. □

We say that a (semi)trajectory is *bounded* if it has compact closure; the following Corollary of Theorem 4.1 relates the boundedness of the semitrajectories of the Hamiltonian system $\dot{z} = \vec{h}(z)$ with the sign of the generalized curvature (see [**2**]):

COROLLARY 4.1. *Assume that $\Lambda_z(t)$ is regular and monotone and that $\vec{h}$ has everywhere negative curvature with respect to the canonical splitting. Then any bounded semitrajectory of the system $\dot{z} = \vec{h}(z)$ converges to an equilibrium with exponential rate, while another semitrajectory of the same trajectory must be unbounded.*

Again, this result is proved using the cones characterized by equation (4.5).

The results can be generalized using the technique that Wojtkowski proposed in [**26**]; in fact, in the cited paper it is proved the following Theorem:

THEOREM 4.2. *Le M be a smooth manifold, X a smooth vector field on it, and denote with ϕ^t the flow generated by X. Assume that there is a continuous nondegenerate quadratic form $\mathcal{Q} : TM/\mathrm{span}\{X\} \to \mathbb{R}$ such that its Lie derivative with respect to the vector field X, denoted with $\mathcal{L}_X\mathcal{Q} := \frac{d}{dt}\mathcal{Q}(De^{tX})|_{t=0}$, is positive definite. Then for any $z \in M$ there exists two cones $C_z^{\pm} \subset TM/\mathrm{span}\{X\}$ with the property exposed in equation (4.5).*

In particular, the whole M is a hyperbolic set for the flow ϕ^t, and then the flow is an Anosov flow.

Remark. For the proof, we remand to the paper [**26**]; for completeness' sake, we just recall the definition of the cones $C_z^{\pm}$:

$$C_z^+ := \{\boldsymbol{y} \in T_zM/\mathrm{span}\{X(z)\} : \mathcal{Q}(\boldsymbol{y}) \geq 0\}, \quad C_z^- := \{\boldsymbol{y} \in T_zM/\mathrm{span}\{X(z)\} : \mathcal{Q}(\boldsymbol{y}) \leq 0\};$$

the invariant cones are defined as in (4.6), and satisfy the required properties.

If we do not take any quotient of the tangent space, we still obtain the existence of the expanding and contracting cones, but we lead to a different result; this is actually the generalization of Theorem 4.1:

THEOREM 4.3. *Let M be a smooth $2n$-dimensional symplectic manifold, and $h : M \to \mathbb{R}$ be a smooth Hamiltonian function; denote with ϕ^t the flow generated by $\vec{h}$. Assume that there exists a compact invariant set W of the flow ϕ^t , and that there exists a quadratic form $\mathcal{Q} : TM \to \mathbb{R}$ such that $\mathcal{L}_{\vec{h}}\mathcal{Q} > 0$.*

Then W is a finite set and each point of W is a hyperbolic equilibrium of the field $\vec{h}$.

Of course, we can generalize Corollary 4.1 in the same way:

Corollary 4.2. *Assume that there exists a compact invariant set* W *of the flow* ϕ^t *, and that there exists a quadratic form* $\mathcal{Q} : TM \to \mathbb{R}$ *such that* $\mathcal{L}_{\vec{h}}\mathcal{Q} > 0$.

Then any bounded semitrajectory of the system $\dot{z} = \vec{h}(z)$ *converges to an equilibrium with exponential rate, while another semitrajectory of the same trajectory must be unbounded.*

Remark. As a consequence of Theorem 4.3, we get the existence of the stable and unstable manifolds $W^s(z)$, $W^u(z)$, for any $z \in W$. This is due to Hadamard-Perron Theorem.

2. Statement of the problem

Let M be a complete Riemannian n-dimensional manifold, $\varphi : TM \to \mathbb{R}$ a smooth function, and let us consider the problem (4.2); let us formulate it as an optimal control problem:

$$\min_{q(t)} \int_0^\infty \varphi(q(t), u(t))\, dt, \tag{4.7}$$

with

$$\dot{q} = u \qquad \begin{array}{rcl} q(0) & = & q_0 \\ q(t) & \to & q_\infty \quad \text{as} \quad t \to +\infty \end{array} \qquad q \in M, \quad u \in T_qM. \tag{4.8}$$

We make the following assumptions:

(H1) φ is bounded from below and is strongly convex with respect to the second variable; moreover, we assume that φ grows superlinearly in the second variable with respect to the given Riemannian metric, i.e. $\varphi(q,u) + c > 0$ for some constant c and
$$\frac{|u|}{\varphi(q,u) + c} \to 0 \quad \text{as } |u| \to +\infty;$$

(H2) there is a unique point q_∞ such that
$$\varphi(q_\infty, 0) = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial q}(q_\infty, 0) = 0;$$

(H3) there exist constants $a,\ b > 0$ such that for any (q,u)
$$|\partial_q \varphi(q,u)| \le a(\varphi(q,u) + |u|) + b,$$
where ∂_q is the covariant derivative.

Let now $H : T^*M \to \mathbb{R}$ be the maximized Hamiltonian associated to problem (4.7)-(4.8) by equation (1.25):
$$H(\lambda) = \max_{u \in T_qM} (\langle \lambda, u \rangle - \varphi(q,u));$$
assumptions (H1)-(H3) imply that the Hamiltonian H is smooth and the Hamiltonian field $\vec{H}$ is complete; the first result is a consequence of strict convexity of φ with respect to u; completeness of the vector field is a consequence of the growth assumptions.

Moreover, assumption (H2) implies that the Hamiltonian vector field $\vec{H}$ possesses a unique fixed point z_∞, with $\pi(z_\infty) = q_\infty$.

Since H is smooth, by Proposition 1.2 we know that any extremal of problem (4.7)-(4.8) shall be a solution to the Hamiltonian system

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \tag{4.9}$$

and, conversely, any solution to (4.9) satisfies PMP.

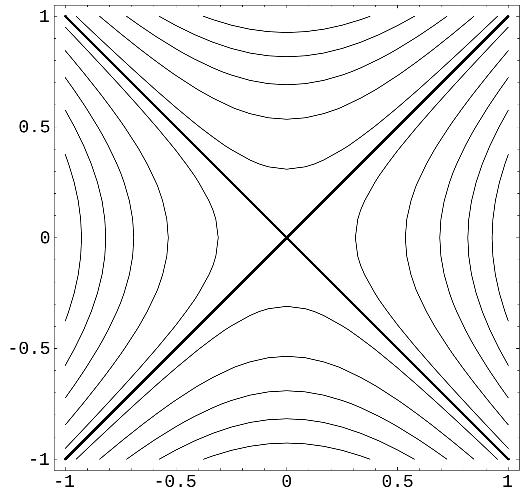

FIGURE 4.1

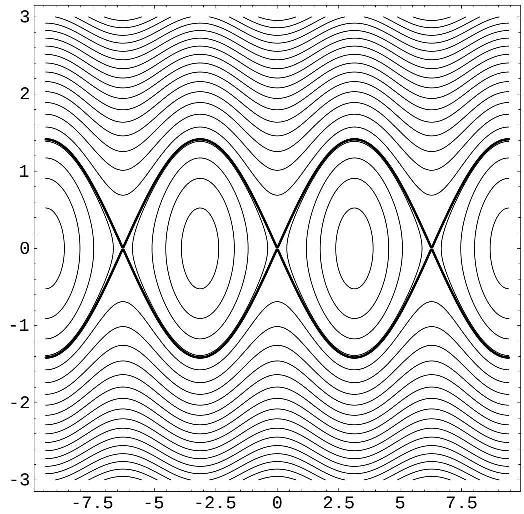

FIGURE 4.2

As anticipated in the introduction to this chapter, our strategy is to focus on the properties of the Hamiltonian system written above to establish the existence of the optimal synthesis for the problem (4.7)-(4.8); to get an idea about the direction to explore, we look at the examples explained in the following subsection.

2.1. Elementary examples. A suggestion about the direction to look forward is given by the classical quadratic Hamiltonian

$$H(p,q) = \frac{1}{2}p^2 + rq^2, \qquad p,q \in \mathbb{R},$$

originated by the Lagrangian $\varphi(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - rq^2$; for $r > 0$ the dynamical system is a *harmonic oscillator*, and the phase trajectories are ellipses centred in $(0,0)$: the corresponding optimal control problem has no optimal trajectories, since there is no trajectories reaching the origin in an infinite time.

Otherwise, for $r < 0$ the phase trajectories are hyperbolas and the semitrajectories whose initial condition lie in the bisectrix of the II and IV quadrant (except the origin) reach the origin with exponential rate; since the projection of the bisectrix is a bijection, we can guess that for any initial q_0 there is a Hamiltonian trajectory, with initial condition (p_0, q_0), that reaches the origin in an infinite time. It is easy to show that actually the integral (4.2) converges along these trajectories and then the problem admits a smooth optimal synthesis. This situation is depicted in Figure 4.1.

Actually, this is the 1-dimensional version of the case of quadratic Lagrangians such as

$$\varphi(q,\dot{q}) = \langle R\dot{q},\dot{q}\rangle + \langle Sq,q\rangle, \quad q \in \mathbb{R}^n, \quad q_\infty = 0; \tag{4.10}$$

it is known that if the matrices R and S determine positive definite quadratic forms, the problem admits a smooth optimal synthesis. Otherwise, if one of these quadratic forms is sign-indefinite, then the cost $c(q)$ is simply not defined for almost all $q \in \mathbb{R}^n$.

Let us finally consider the Hamiltonian of the *pendulum*

$$H(p,q) = \frac{1}{2}p^2 - 1 + cos(q);$$

in this case, we distinguish three different possibilities. The Hamiltonian trajectories lying on energy levels with $H > 0$ are all unstable; for $H < 0$ the trajectories are closed and do no reach the origin; but for any initial $q_0 \in (-2\pi, 2\pi)$ there is a trajectory that lies on the level $H = 0$ and reaches the origin in an infinite time, and is an optimal trajectory for the problem under investigation. If $|q_0| > 2\pi$, the trajectories on the zero level arising from q_0 do not reach the origin, but go to another fixed point of the system, and then cannot be optimal. This case is shown in Figure 4.2.

These examples suggest us that the existence of a smooth optimal synthesis shall correspond to the existence of an n-dimensional stable invariant submanifold of the Hamiltonian system. The problem is then to determine the conditions on H that guarantee the existence of such a submanifold. If we pay attention to the properties of φ, we guess that the sign of its Hessian matrix in a neighbourhood of the equilibrium point determines the existence of *local* minimizers; moreover, equation (4.10) shows that if the Lagrangian is strictly convex, the problem admits an optimal synthesis.

Since we are dealing with smooth manifolds, we need to find *intrinsic* conditions on φ that generalize the convexity condition just stated; to this purpose, we are going to use the generalized curvature of the associated Hamiltonian system.

2.2. The sufficient condition. The previous examples suggest that the minimizing trajectories of the variational problem have to be seeked among the stable trajectories of the dynamical system (4.9). Actually this intuition is supported and improved by the following Theorem, which is the infinite horizon version of Theorem 1.8:

THEOREM 4.4. *Assume that the maximized Hamiltonian* (1.25) *is defined and smooth on T^*M, and that the Hamiltonian vector field $\vec{H}$ is complete.*

Let $\mathcal{L}_0$ be a Lagrangian submanifold in T^*M, and let $\mathcal{L}_t = e^{t\vec{H}}(\mathcal{L}_0)$ be its image under the Hamiltonian flow at time t.

Let $\pi : T^*M \to M$ be the canonical projection, and assume that its restriction $\pi|_{\mathcal{L}_t}$ is a diffeomorphism for any $t \in [t_0, +\infty)$. Let $\lambda_0 \in \mathcal{L}_0$ and consider the normal extremal trajectory

$$\tilde{q}(t) = \pi \circ e^{t\vec{H}}(\lambda_0), \qquad t \in [t_0, +\infty);$$

then, for any λ_0 such that the integral $\int_0^\infty \varphi(\tilde{q}(t), \tilde{u}(t))\, dt$ converges, the trajectory $\tilde{q}(t)$ realizes a strict minimum of the cost functional (1.19) among all the admissible trajectories $q(t)$, $t \in [t_0, +\infty)$, of the system (1.18) with the same boundary conditions:

$$q(t_0) = \tilde{q}(t_0) \qquad \lim_{t\to+\infty} q(t) = \tilde{q}_\infty,$$

where $\tilde{q}_\infty = \lim_{t\to+\infty} \tilde{q}(t)$.

The proof is a straightforward adaptation of the proof of Theorem 1.8, that can be found in [**7**].

3. The results

This section is devoted to the statement of the main result of this chapter; then, we will also give a summary of the proof, which will be developed in the following subsections.

THEOREM 4.5. *Let M be a simply connected smooth manifold, and let $\varphi : TM \to \mathbb{R}$ be a smooth function that satisfies hypotheses (H1)–(H3). Let $\{\Lambda_z\}_z = \{T_z(T^*_{\pi(z)}M)\}_z$ and*

$\{\Pi_z\}_z$, $z \in T^*M$, be two Lagrangian distributions that provide a splitting of $T(T^*M)$; assume that the generalized curvature of $\vec{h}$ with respect to the splitting is negative definite for any z. Then the problem (4.7)-(4.8) with final point q_∞ admits a smooth optimal synthesis on M.

Proof. We divide the proof into three steps: in the first part, we use a result due to P. Przytycki to prove the existence of the continuous nondegenerate quadratic form needed in Wojtkowski's argument: then, by Hadamard-Perron Theorem (1.2), we get the existence of the stable manifold $W^s(z_\infty)$; notice that it is by definition invariant with respect the Hamiltonian flow. In the last two steps, we are going now to apply Theorem 4.4 to $W^s(z_\infty)$, thus proving that all the optimal trajectories for our problem are projections of stable solutions of (4.9). In particular, in the second part we prove that the stable manifold is a Lagrangian submanifold of T^*M and it is diffeomorphically projected onto its image on M; the third step is devoted to the proof of the surjectivity of the projection of the stable manifold onto M, that implies the existence of the optimal synthesis.

Once shown this, the optimal synthesis is thus constructed: we put for any $q \in M$

$$X(q) := \pi_*(\vec{H}(\lambda)), \tag{4.11}$$

with $\lambda \in W^s(z_\infty)$ and $\pi(\lambda) = q$, where $\pi : T^*M \to M$ is the canonical projection. This vector is smooth, because the Hamiltonian is smooth and $W^s(z_\infty)$ projects diffeomorphically on M; since the optimal trajectories are projections of the integral curves of $\vec{H}$, they are actually integral curves of X.

3.1. Hyperbolicity of the flow. We prove the following result:

LEMMA 4.1 (P. Przytycki). *Let $\Lambda \oplus \Pi$ be a Lagrangian splitting of $T(T^*M)$ such that the curves $\Lambda_z(t)$ and $\Pi_z(t)$ are regular, and $\Lambda_z(t)$ is monotone. Assume that the curvature R_z^H of the vector field $\vec{H}$ with respect to this splitting is negative definite for any z. Then there exists a continuous nondegenerate quadratic form $\mathcal{Q} : TM \to \mathbb{R}$ such that $\mathcal{L}_{\vec{H}}\mathcal{Q}$ is positive definite.*

Proof. First of all, note that asking the curvature R_z^H to be negative definite is equivalent to saying that the quadratic forms associated to $\dot{\Lambda}_z(t)$ and $\dot{\Pi}_z(t)$ are both regular and have opposite sign (see [**2**]); for simplicity, assume that $\Lambda_z(t)$ is monotone increasing.

Let us write any vector X on T^*M as $X = X_\Lambda + X_\Pi$, where, as before, X_Λ is a section of Λ and X_Π a section of Π; define the following quadratic form on TM :

$$\mathcal{Q}_t(X) = \sigma((D_z\phi^t X)_\Lambda, (D_z\phi^t X)_\Pi);$$

we have that

$$\begin{aligned}
\mathcal{L}_{\vec{H}}\mathcal{Q}(X) &= -\sigma([\vec{H}, X]_\Lambda, X_\Pi) - \sigma(X_\Lambda, [\vec{H}, X]_\Pi) = \\
&= \sigma([\vec{H}, X_\Lambda], X_\Lambda) + \sigma([\vec{H}, X_\Pi], X_\Lambda) + \\
&+ \sigma(X_\Pi, [\vec{H}, X_\Lambda]) + \sigma(X_\Pi, [\vec{H}, X_\Pi]) = \\
&= \sigma([\vec{H}, X_\Lambda], X_\Lambda) + \sigma(X_\Pi, [\vec{H}, X_\Pi]) = \\
&= \dot{\Lambda}_z(t)(X_\Lambda) - \dot{\Pi}_z(t)(X_\Pi) > 0,
\end{aligned}$$

□

Thanks to the Lemma above, we can apply Theorem 4.3 to the fixed point z_∞. Then we get that this point is a hyperbolic fixed point, and that there are defined the global stable and unstable manifolds. We will concentrate on the global stable manifold $W^s(z_\infty)$.

3.2. Regularity of the projection.

LEMMA 4.2. *$W^s(z_\infty)$ is a Lagrangian submanifold of T^*M*

Proof: Let $\boldsymbol{x}, \boldsymbol{y} \in T_z W^s(z_\infty)$, $z \in W^s(z_\infty)$; then, since the Hamiltonian flow preserves the symplectic form, for any t

$$\sigma(X,Y) = \sigma(D_z e^{t\vec{H}}\boldsymbol{x}, D_z e^{t\vec{H}}\boldsymbol{y}) \to 0 \quad \text{as } t \to +\infty.$$

□

LEMMA 4.3. *The restriction of the projection $\pi|_{W^s(z_\infty)}$ is a covering of its image.*

Proof: $\pi|_{W^s(z_\infty)}$ is clearly smooth map, since the stable manifold is smooth.

$\pi|_{W^s(z_\infty)}$ is also an immersion, and, in particular, a local diffeomorphism; in fact, let us assume that there is a vector $\boldsymbol{x} \in \ker \pi_* \cap T_z W^s(z_\infty)$; we get that $\mathcal{Q}_0(\boldsymbol{x}) = 0$, since the projection of $\boldsymbol{x}$ on the horizontal space vanishes. Since $\mathcal{Q}_t(\boldsymbol{x}) \to 0$ as $t \to +\infty$ and $\mathcal{L}_{\vec{H}}\mathcal{Q}$ is positive definite, we have that $\mathcal{Q}_t(\boldsymbol{x}) = 0$ for any $t \geq 0$, which implies that $D_z\phi^t\boldsymbol{x}$ is vertical (i.e. $(D_z\phi^t\boldsymbol{x})_\Pi = 0$) for nonnegative t; since $[\vec{H}(\phi^t(z)), D_z\phi^t\boldsymbol{x}] \notin T_{\phi^t(z)}(T^*_{\pi(\phi^t(z))}M)$, we get a contradiction.

Let us now prove that $\pi|_{W^s(z_\infty)}$ is a proper mapping: first of all, let $d(\cdot,\cdot)$ be the distance induced on T^*M by the scalar product given by the quadratic form $\dot{\Lambda}_z$, and let $B_r(z)$ denote the ball of radius r centered at z, for some $r > 0$, $z \in T^*M$. Let K be a compact set in $\pi(W^s(z_\infty))$, and $\{z_i\}_i$ a sequence in $\pi|^{-1}_{W^s(z_\infty)}(K)$. The sequence is bounded: in fact, let us write the z_i's in coordinates, $z_i = (p_i, q_i)$ for any i; $q_i \in K$ for any i, and hence they are bounded; the p_i shall be bounded as well, because the stable manifold lies in the level $H^{-1}(0)$, and the Hamiltonian grows to $+\infty$ when $|p| \to +\infty$. Then the sequence $\{z_i\}_i$ converges, up to a subsequence, to some $\bar{z} \in \pi^{-1}(K)$; let us assume that $\bar{z} \notin W^s(z_\infty)$. Let us consider the Hamiltonian trajectories whose initial conditions are given by these z_{i_k}'s: by continuity, for any small $\varepsilon > 0$ and any $T > 0$ we can find $\bar{k}$ such that $\phi^t(z_{i_k}) \in \phi^t(B_\varepsilon(\bar{z}))$ for $k \geq \bar{k}$ and for any $0 \leq t \leq T$.

Since we assumed that $\bar{z} \notin W^s(z_\infty)$, $\phi^t(\bar{z})$ shall go to infinity (see Corollary 4.2), which means that for any $\rho > 0$ we can always find a $t' \geq T$ such that $\phi^{t'}(\bar{z}) \notin B_{2\rho}(z_\infty)$.

Now recall that $z_{i_k} \in W^s(z_\infty)$, which means that $\phi^t(z_{i_k})$ reaches z_∞ with exponential rate, that is there are two positive constants c, γ such that $\|\vec{H}(\phi^t(z_{i_k}))\| \leq ce^{-\gamma t}\|\vec{H}(z_{i_k})\|$. So we have that

$$\int_t^{+\infty} \|\vec{H}(\phi^\tau(z_{i_k}))\| \, d\tau \leq c\,\|\vec{H}(z_{i_k})\| \int_t^{+\infty} e^{-\gamma\tau}\, d\tau = c\,\|\vec{H}(z_{i_k})\|\frac{e^{-\gamma t}}{\gamma}$$

We can chose ε so small and ρ, T so large that $c\,\|\vec{H}(z_{i_k})\|\frac{e^{-\gamma T}}{\gamma} < \rho$, which means that the trajectory $\phi^t(z_{i_k})$ cannot reach z_∞. This is a contradiction; hence $\bar{z}$ belongs to the stable manifold, and then $\pi|^{-1}_{W^s(z_\infty)}(K)$ is compact.

Since $\pi|_{W^s(z_\infty)}$ is a proper local diffeomorphism, it is also a smooth covering (of its image); if its image is simply connected, we can conclude that the map $\pi|_{W^s(z_\infty)}$ is also a global diffeomorphism (onto its image). □

3.3. Existence of the optimal synthesis. In this subsection we are proving that the restriction $\pi|_{W^s(z_\infty)}$ is a surjective map onto M, in order to apply Theorem 4.4 to get the existence of the optimal synthesis. Surjectivity of this map means that for any $q \in M$ we can find a point $z \in T_q^*M$ such that the semitrajectory $\phi^t(z)$ is stable, i.e. it reaches z_∞ with exponential rate. Since our problem satisfies the hypotheses of Corollary 4.2, it is sufficient to prove that for any $q \in M$ there is a $z \in T_q^*M$ such that the Hamiltonian trajectory emanating from it is *bounded*.

To do that, we will use the fact that any segment of a minimizing trajectory is minimizing between its endpoints (for the corresponding finite horizon problem); then, we shall prove the solutions to the finite horizon problems are equibounded with respect to the final time.

Consider the finite-horizon problem

$$J_T = \min_{q(t)} \int_0^T \varphi(q(t), \dot q(t))\, dt \qquad \begin{array}{rcl} q(0) & = & q_0, \\ q(T) & = & q_\infty \end{array} \qquad T > 0; \tag{4.12}$$

to establish the existence of this minimum in the class of Lipschitzian functions we use an approach due to Sarychev and Torres ([**25**]) and Gamkrelidze ([**17**]). Since to apply this result the function φ is needed to be strictly positive, in the following we will make the substitution $\varphi \mapsto \varphi + \alpha$, in such a way that $\varphi + \alpha > 0$; it is known that, since we are consider a finite-horizon problem, this substitution does not change the minimizers. Let us define a new time variable

$$\tau(t) = \int_0^t \varphi(q(\theta), u(\theta)) + \alpha\, d\theta, \quad t \in [0, T],$$

which is a strictly monotone (smooth) function of t, with $\tau(0) = 0$ and $\tau(T) = \tau_1$; since

$$\frac{d\tau(t)}{dt} = \varphi(q(t), u(t)) + \alpha > 0,$$

$\tau(t)$ is invertible and its inverse $t(\tau)$ is monotone on $[0, \tau_1]$.

Let us now transform the original problem (4.12) into a new one, in which τ is the new time variable, $t(\tau)$ and $z(\tau) = q(t(\tau))$ the components of the state trajectory, and $v(\tau) = u(t(\tau)) \in TM$ the new control, with

$$\left\{ \begin{array}{rcl} \dot t(\tau) & = & \frac{1}{\varphi(z(\tau), v(\tau)) + \alpha} \\ \dot z(\tau) & = & \frac{v(\tau)}{\varphi(z(\tau), v(\tau)) + \alpha} \end{array} \right. .$$

For any t and z, the set of all velocities

$$\Big\{ \Big(\frac{1}{\varphi(z(\tau), v(\tau)) + \alpha} \Big), \Big(\frac{v(\tau)}{\varphi(z(\tau), v(\tau)) + \alpha} \Big) \Big\}$$

becomes compact if we add to it the point $\{(0,0)\} \in \mathbb{R} \times T_zM$; this point corresponds to the case $v \to \infty$. Then, we compactify each fibre T_zM adding the point at infinity: we define the fibre bundle N on M with fibre given by the n-dimensional sphere S^n, we fix for any z a "north pole" $\hat w \in N_z$ (N_z fibre through z), and we define the map $\pi : N \to TM$ such that $\pi|_{N_z} : N_z \setminus \hat w \to T_zM$ is the stereographic projection.

The maps

$$\theta(z, w) = \left\{ \begin{array}{ll} \frac{1}{\varphi(z, \pi(w)) + \alpha} & \text{if} \quad w \neq \hat w \\ 0 & \text{if} \quad w = \hat w \end{array} \right.$$

and

$$\zeta(z, w) = \left\{ \begin{array}{ll} \frac{\pi(w)}{\varphi(z, \pi(w)) + \alpha} & \text{if} \quad w \neq \hat w \\ 0 & \text{if} \quad w = \hat w \end{array} \right.$$

are well-defined and continuous on N. We can then define time-optimal control problem:

$$\left\{ \begin{array}{l} \dot{t}(\tau) = \theta(z(\tau), w(\tau)) \\ \dot{z}(\tau) = \zeta(z(\tau), w(\tau)) \end{array} \right. \quad w \in N_z \quad \begin{array}{ll} t(0) = 0, & t(\tau_1) = T \\ z(0) = q_0, & z(\tau_1) = q_\infty \end{array} \quad \tau_1 \to \min . \tag{4.13}$$

This problem is equivalent to problem (4.12): in fact, to every admissible pair $(q(\cdot), \dot{q}(\cdot))$ of (4.12) there corresponds an admissible triple $(t(\cdot), z(\cdot), w(\cdot))$ of (4.13) such that $w(\cdot)$ takes values different from $\hat{w}$ almost everywhere, and the time τ_1 for this latter solution is equal to the value of the functional evaluated on $(q(\cdot), \dot{q}(\cdot))$ (see [**25**]). Sarychev and Torres showed that under the hypotheses (H1) and (H3) the time-optimal control problem (4.13) has solution with bounded optimal control, and we denote the corresponding solution to (4.12) with $q_T(t)$; denote with $\lambda_T(\cdot)$ the solution of the Hamiltonian system (4.9) such that $\pi(\lambda_T(t)) = q_T(t)$. Let $T' > T$ and put

$$\hat{q}_T(t) = \left\{ \begin{array}{ll} q_T(t) & t \in [0, T] \\ q_T(T) = q_\infty & t \in [T, T'] \end{array} \right. ;$$

we get that

$$\int_0^{T'} \varphi(\hat{q}_T(t), \dot{\hat{q}}_T(t)) + \alpha \, dt = \int_0^T \varphi(q_T(t), \dot{q}_T(t)) + \alpha \, dt + \int_T^{T'} \varphi(q_\infty, 0) + \alpha \, dt = J_T + \alpha T',$$

since $\varphi(q_\infty, 0) = 0$. Then we can argue that

$$J_{T'} = \min_{q(t)} \int_0^{T'} \varphi(q(t), \dot{q}(t)) + \alpha \, dt \leq J_T + \alpha T',$$

i.e. there exists a fixed constant $C > 0$ such that $J_T \leq CT$ for any T (at least for any T greater than some $\bar{t}$).

LEMMA 4.4. *The optimal extremals $\lambda_T(\cdot)$ are uniformly bounded with respect to T.*

Proof. The maximized Hamiltonian associated to the time-optimal problem (4.13) is

$$\widetilde{H}(p_0, \lambda, t, z) = \max_{w \in S^n} p_0 \theta(z, w) + \langle \lambda, \zeta(z, w) \rangle; \tag{4.14}$$

since the Hamiltonian is homogeneous with respect to (p_0, λ) and p_0 is constant along Hamiltonian trajectories, we can normalize the pair (p_0, λ); we have two possible cases:

- $p_0 \neq 0$: then we put $p_0 = -1$; let us notice that $\widetilde{H} = 0$ if $\lambda = 0$, and $\widetilde{H} > 0$ for nonzero values of λ;
- $p_0 = 0$: then we normalize $|\lambda| = 1$; notice that $\widetilde{H} > 0$ for any value of λ .

Let us notice that the Hamiltonian reaches the value $\widetilde{H} = 0$ only when the maximum in (4.14) is realized at $w = \hat{w}$; since the Hamiltonian is conserved along trajectories, for a trajectory lying on the zero sublevel of $\widetilde{H}$ the value of the control is constantly $w = \hat{w}$.

If the Hamiltonian assumes a small positive value, by continuity w shall live in a small neighbourhood of the point $\hat{w}$, which is equivalent to say that $\|v\|$ is very large (where $v = \pi(w)$ for $w \neq \hat{w}$). In fact, in the case in which $p_0 = -1$, if $\widetilde{H}$ is small and positive then also $|\lambda|$ is small and then, since the numerator $\langle \lambda, v_{max} \rangle - 1$ is positive, $\|v_{max}\|$ shall be large (v_{max} is the value of the control that maximixes the Hamiltonian); in the case in which $p_0 = 0$, $\widetilde{H}$ is small only if $\|v_{max}\|$ is large.

Then, we claim that for any $\delta > 0$ there exists a neighbourhood O_δ of the point $\hat{w}$ such that

$$\dot{z}(\tau) \in O_\delta \Rightarrow \frac{dt}{d\tau} = \theta(z(\tau), w(\tau)) < \delta;$$

in fact, if $\dot{z}(\tau) \in O_\delta$, then the Hamiltonian is small and positive, and hence $\dot{z}(\tau) \in O_\delta$ for any τ.

This implies that $\tau_1 \geq \frac{T}{\delta}$, and, since this allows for any δ, we can choose it in such a way that $\frac{T}{\delta} > CT$, obtaining a contradiction. Then, there shall exist a fixed $\delta > 0$ such that for any T $\dot{z}_T(\tau) \notin O_\delta$ for any τ, where $z_T(\tau) = q_T(t(\tau))$.

Hence we get that the derivatives $\dot{q}_T(t)$ of the solutions to problem (4.12) are uniformly bounded with respect to T; this implies the uniform boundedness with respect to T of $\lambda_T(t)$.

In fact, the maximum relation (1.23) implies that $\frac{\partial}{\partial \dot{q}}\big(\langle \lambda_T, \dot{q}\rangle - \varphi(q, \dot{q})\big)|_{(q,\dot{q})=(q_0,\dot{q}(0))} = 0$, and hence $\lambda_T(0)$ shall be equibounded with respect to T. Since the Hamiltonian is constant along the Hamiltonian trajectories, $H(\lambda_T(\cdot))$ shall also be equibounded.

Since $\lambda_T(\cdot)$ is optimal,

$$J_T = \int_0^T \langle \lambda_T(t), \dot{q}_T(t)\rangle - H(\lambda_T(t))\, dt = J_T = -H(\lambda_T)T + \int_0^T \langle \lambda_T(t), \dot{q}_T(t)\rangle;$$

if $\lambda_T(\cdot)$ grew with respect to T, the second term would grow more than linearly with respect to T, in contradiction with the estimate $J_T \leq CT$. □

Let now $\{t_k\}_k$ be a monotone increasing sequence of real numbers that tends to infinity, and let us consider the problem (4.12) with final time t_k; let us consider also the sequence $\{\lambda_{t_k}(0)\}$, with $\pi(\lambda_{t_k}(0)) = q_0$ for any k, of initial points of the normal extremals for the problem with finite time t_k; since the sequence is bounded, it converges (up to a subsequence) as k goes to the infinity to a point $\bar{\lambda}$, $\pi(\bar{\lambda}) = q_0$; by continuity with respect to the initial condition, also the trajectories $\lambda_{t_k}(\cdot)$ converge to the Hamiltonian trajectory $\bar{\lambda}(\cdot)$ arising from $\bar{\lambda}$.

This trajectory is bounded; in fact, let us assume the contrary, i.e. that, for any compact set K with $\bar{\lambda} \in K$, there exists some $\bar{t}$ such that $\bar{\lambda}(t) \notin K$ for any $t > \bar{t}$; then, for any k greater that some $\bar{k}$, we would also have that $\lambda_{t_k}(t) \notin K$. Choosing some $k > \bar{k}$ such that $t_k > \bar{t}$ we would get a contradiction with the fact that $\lambda_{t_k}(t)$ is the optimal extremal for the problem with finite time t_k. Hence $\bar{\lambda}(t)$ shall be bounded.

Now we apply Corollary 4.2: since the only equilibrium point is actually z_∞, this means that $\bar{\lambda}(t)$ is a stable trajectory, i.e. $\bar{\lambda} \in \pi(W^s(z_\infty))$. Hence the projection $\pi|_{W^s(z_\infty)}$ is onto. □

Remark. It is crucial to assume that the function $u \mapsto \varphi(q, u)$ has superlinear growth. In fact, assume by contradiction that there exists a direction u_j on which the function φ has linear growth for large $\|u\|$, which implies that $\partial_{u_j}\varphi$ tends to a constant as $\|u\|$ goes to infinity. Then there exists an $M > 0$ such that $\sup_u \langle \lambda, u\rangle - \varphi(q, u) = +\infty$ if the j-th component of λ exceeds M. If the maximized Hamiltonian is not defined, the flow itself is not globally defined.

4. The Euclidean case.

THEOREM 4.6. *Consider the problem*

$$\min_{q(t)} \int_0^\infty \varphi(q(t), u(t))\, dt, \tag{4.15}$$

with

$$\dot{q} = u \qquad \begin{array}{l} q(0) = q_0 \\ q(t) \to q_\infty \text{ as } t \to +\infty \end{array} \qquad q \in \mathbb{R}^n, \tag{4.16}$$

where φ is smooth and strongly convex in the pair (q,u) and the function $u \mapsto \varphi(q,u)$ has superlinear growth for any q, and there is a point q_∞ such that $\varphi(q_\infty, 0) = 0$ and $\frac{\partial \varphi}{\partial q}(q_\infty, 0) = 0$.

Then there exists a hyperbolic fixed point z_∞ with $\pi(z_\infty) = q_\infty$ of the Hamiltonian system associated to (4.15), and the problem (4.15) with final point $q_\infty = \pi(z_\infty)$ admits a smooth optimal synthesis on $\mathbb{R}^n$.

Proof: As above, PMP let us associate to problem (4.15) a Hamiltonian $h_u(p,q) = \langle p, q\rangle - \varphi(p,q)$, and its maximized $H(p,q) = \max_u h_u(p,q)$.

The hypotheses on φ imply that the maximized Hamiltonian $H(p,q)$ is smooth and that assumption (H3) is satisfied.

For any (p,q), choose $\Lambda_{(p,q)} = \operatorname{span}\{\partial_{p_1}, \ldots, \partial_{p_n}\}(p,q)$ and $\Pi_{(p,q)} = \operatorname{span}\{\partial_{q_1}, \ldots, \partial_{q_n}\}(p,q)$. We have that $\dot{\Lambda}_{(p,q)} > 0$ and $\dot{\Pi}_{(p,q)} < 0$; in fact, let $\boldsymbol{x} = \sum_{i=1}^n x_i \partial_{p_i} \in \Lambda_{(p,q)}$ and $\boldsymbol{y} = \sum_{i=1}^n y_i \partial_{q_i} \in \Pi_{(p,q)}$; then

$$\dot{\Lambda}_{(p,q)}(\boldsymbol{x}) = \sigma([\vec{H}, \boldsymbol{x}], \boldsymbol{x}) = \sum_{i=1}^n x_i x_j \frac{\partial^2 H}{\partial p_i \partial p_j},$$

$$\dot{\Pi}_{(p,q)}(\boldsymbol{y}) = \sigma([\vec{H}, \boldsymbol{y}], \boldsymbol{y}) = \sum_{i=1}^n y_i y_j \frac{\partial^2 H}{\partial q_i \partial q_j}.$$

For fixed (p,q), denote with $\bar{u}$ the value of the control that realizes the maximum of $h_u(p,q)$; by direct computation, we get that

$$\frac{\partial^2 H}{\partial p_i \partial p_j} = \sum_{l,m} \frac{\partial \bar{u}_l}{\partial p_i} \frac{\partial^2 \varphi}{\partial u_l \partial u_m}|_{u=\bar{u}} \frac{\partial \bar{u}_m}{\partial p_j},$$

which is positive definite. On the other hand,

$$\begin{aligned} \frac{\partial^2 H}{\partial q_i \partial q_j} &= \frac{\partial^2 h_u}{\partial q_i \partial q_j} - \sum_{l,m} \frac{\partial \bar{u}_l}{\partial q_i} \frac{\partial^2 h_u}{\partial u_l \partial u_m}|_{u=\bar{u}} \frac{\partial \bar{u}_m}{\partial q_j} \\ &= -\frac{\partial^2 \varphi}{\partial q_i \partial q_j} + \sum_{l,m} \frac{\partial^2 \varphi}{\partial q_j \partial u_l} \left(\frac{\partial^2 \varphi}{\partial u_l \partial u_m} \right)^{-1} \frac{\partial \varphi}{\partial q_i \partial u_m}, ; \end{aligned} \tag{4.17}$$

in fact, let us define the function $F : \mathbb{R}^{3n} \to \mathbb{R}$ by $F(p,q,u) = p - \frac{\partial \varphi}{\partial u}(q,u)$; since $\operatorname{rk}(J_u F) = n$, we can apply the Implicit Function Theorem and express locally the function $\bar{u}$ such that $\frac{\partial \varphi}{\partial u}|_{u=\bar{u}} = 0$ as a function of p and q, and moreover we have that

$$J\bar{u} = -(J_u F)^{-1}(J_{(p,q)}F) = \Big(\frac{\partial^2 \varphi}{\partial u^2}\Big)^{-1} \Big(\mathbb{I}, -\frac{\partial^2 \varphi}{\partial q \partial u}\Big).$$

Let us perform a change of variable and assume that, in the point where we are computing the derivatives, the Hessian of φ with respect to u is the unit matrix, thus reducing (4.17) to

$$\begin{aligned} \frac{\partial^2 H}{\partial q_i^2} &= -\frac{\partial^2 \varphi}{\partial q_i^2} + \sum_{l=1}^n \left(\frac{\partial u_l}{\partial q_i} \right)^2 \\ &= -\operatorname{Hess}(\varphi|_{Q=const}), \end{aligned}$$

with $Q = (q_1, \dots, \hat{q}_i, \dots, q_n)$ (i.e. we are neglecting q_i); hence $\frac{\partial^2 H}{\partial q_i^2} < 0$. Since we can perform any linear change of variable in the space of the q's obtaining the same expression (4.17), we can conclude that the second derivative of H with respect to any direction in the space of coordinates is negative, i.e. H is strictly concave with respect to q, hence $\mathrm{Hess}_q(H)$ is a negative-definite matrix.

This fact implies that the generalized curvature with respect to this splitting is negative definite; then we are under assumptions of Theorem 4.5. □

Remark. The following example shows that the convexity of $\varphi(q,u)$ is not sufficient to assure that the generalized curvature with respect to the canonical splitting is negative definite.

In fact, let

$$\varphi(q,u) = f(u) + U(q), \quad q, u \in \mathbb{R},$$

with $f''(u) > 0$ and $U''(q) > 0$.

The generalized curvature is

$$\begin{aligned} R^H(p,q) &= \frac{1}{2}\left(\frac{\partial^2 H}{\partial p^2}\right)^{-1}\left(\frac{\partial H}{\partial q}\right)^2 \frac{\partial^4 H}{\partial p^4} - \frac{1}{2}\left(\frac{\partial^2 H}{\partial p^2}\right)^{-1} \frac{\partial^2 H}{\partial q^2}\frac{\partial^3 H}{\partial p^3}\frac{\partial H}{\partial p} + \\ &+ \frac{\partial^2 H}{\partial p^2}\frac{\partial^2 H}{\partial q^2} - \frac{3}{4}\left(\frac{\partial^2 H}{\partial p^2}\right)^{-2}\left(\frac{\partial H}{\partial q}\right)^2\left(\frac{\partial^3 H}{\partial p^3}\right) = \\ &= \frac{1}{4f''(\bar{u})^4}\Big(3U'(q)^2 f^{(3)}(\bar{u})^2 - 4f''(\bar{u})^3 U''(q) - 2\bar{u} f''(\bar{u})^2 U''(q) f^{(3)}(\bar{u}) + \\ &- 2U'(q)^2 f^{(4)}(\bar{u}) f''(\bar{u})\Big); \end{aligned}$$

choosing $\varphi(q,u) = u^4 + u^2 + q^2$, we get

$$R(\bar{u}, q) = \frac{-(1+6\bar{u}^2)^2(1+12\bar{u}^2) + q^2(72\bar{u}^2 - 6)}{(1+6\bar{u}^2)^4},$$

which is not a negative function.

5. The 1-dimensional case.

Let us now consider the optimal problem (4.15)-(4.16) with $q \in \mathbb{R}$; we will investigate whether the problem admits an optimal synthesis only looking at the phase portrait of the dynamical system generated by the maximized Hamiltonian $H(p,q)$.

We assume that φ is smooth and strongly convex in the second variable, and that the maximized Hamiltonian $H(p,q) = \max_{\dot{q}} p\dot{q} - \varphi(q,\dot{q})$ is well defined for any (p,q); these hypotheses imply that $H(p,q)$ is a smooth function strongly convex in p.

At a first moment, we do the following assumption:

(A1) for any $q \in \mathbb{R}$ the function $p \mapsto H(p,q)$ has a (unique, by convexity) minimum;

this implies that there is a smooth curve γ that divides $\mathbb{R}^2$ into two disjoint regions Γ^+ and Γ^- such that $\frac{\partial H}{\partial p} > 0$ for any $(p,q) \in \Gamma^+$, $\frac{\partial H}{\partial p} < 0$ for any $(p,q) \in \Gamma^-$, and $\frac{\partial H}{\partial p} = 0$ if $(p,q) \in \gamma$; (A1) implies that γ is projected surjectively onto the horizontal axis. By strict convexity of H with respect to p, γ is never tangent to vertical lines.

We remark that hypothesis (A1) is automatically satisfied if we require φ to have superlinear growth with respect to u.

All equilibrium points of $\vec{H}$ belong to γ and, if we denote by $\vec{H}_{hor}$ the component of $\vec{H}$ along the horizontal direction, $\vec{H}_{hor}$ points in the positive directions for $(p, q) \in \Gamma^+$, in the negative direction if $(p, q) \in \Gamma^-$, and it vanishes on γ.

Let us now classify the possible phase portraits of such a Hamiltonian. We always assume that the equilibrium points are nondegenerate; if they are isolated degenerate, then we can always put things in general position provoking local differences in the phase diagram, while the global behaviour does not change much.

1 equilibrium point. Since the Hamiltonian flow preserves volumes, the equilibrium point can only be a *centre* or a *saddle*; sinks, sources, stable and unstable nodes, stable and unstable foci cannot arise.

If the equilibrium point is a centre, then there is no orbit reaching it in an infinite time, hence the problem (4.15) has never solution.

Let us now suppose that the equilibrium point is a saddle. First of all, we notice that, since the function $p \mapsto H(p, q)$ is strongly convex for any q, for any fixed q there are at most two distinct values p_1, p_2 such that (p_i, q) belongs to the same level of H for $i = 1, 2$. This implies that the semi-trajectories belonging to the stable and the unstable manifolds are unbounded: in fact, by Poincaré-Bendixon Theorem, if a semitrajectory is bounded, either it arises from another critical point, either its α-limit set is a periodic trajectory. Neither of these cases can occur; the first one because by hypothesis there is no other equilibrium points; the second one cannot occur since, by convexity of H with respect to p, each stable semi-trajectory cannot wind around the equilibrium point, otherwise we would have more than two points with the same horizontal coordinate belonging to the same level of H.

We can also prove that the stable and the unstable semi-trajectories are projected bijectively onto the horizontal axis. If the projection of a semi-trajectory that belongs to the stable or the unstable manifold is bounded, then the horizontal velocity along it vanishes while the trajectory goes to infinity, that contradicts convexity of H with respect to p; this implies surjectivity. Injectivity is a consequence of the fact that there can exist at most two points with the same horizontal coordinate that belong to the same sublevel of the Hamiltonian, and that both the stable and the unstable trajectory are projected onto the horizontal axis.

Then, the infinite-horizon problem admits a solution for any initial point q_0.

2 equilibrium points. As we saw, equilibrium points for this Hamiltonian can only assume the shape of saddles and centres, so phase portraits for two equilibria can be obtained combining these possibilities. Call the two points z_1 and z_2, with $\pi(z_1) < \pi(z_2)$.

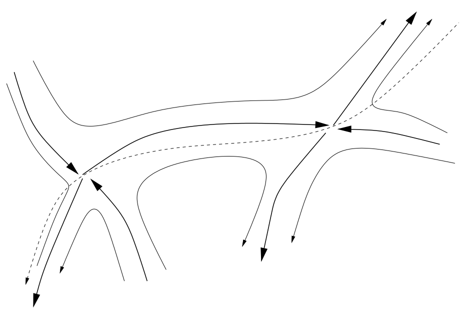

FIGURE 4.3

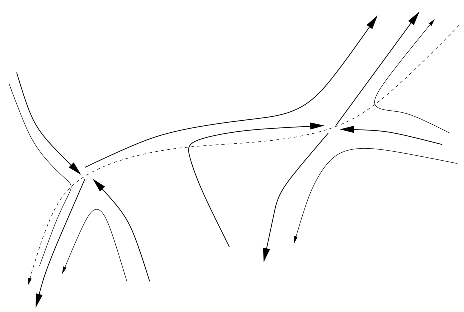

FIGURE 4.4

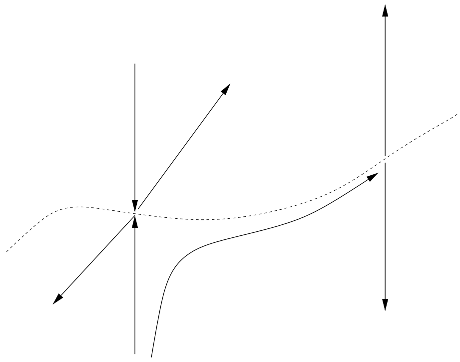

FIGURE 4.5

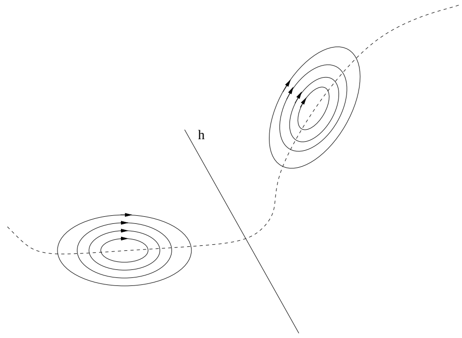

FIGURE 4.6

2 saddles. This combination is forbidden due to convexity of H w.r.t p, for this implies the non-existence of vertical trajectories and because of the fact that $\vec{H}_{hor}$ has positive verse on Γ^+ and negative verse on Γ^-. As shown in Fig. 4.3-4.4, this gives rise to trajectories that intersect each other (in particular, in Fig. 4.3 it is shown the situation in which the two equilibria belong to the same sublevel, in Fig. 4.4 the one in which they belong to different sublevels). In Fig. 4.5 it is illustrated that, if we try to avoid these intersections, we would get trajectories whose horizontal velocity has wrong verse.

2 centres. This combination is again forbidden; in fact, by the structure of the Hamiltonian, the closed trajectories around each of the equilibria are covered in the same verse (clockwise). This fact leads up to a contradiction; if, as in Fig. 4.6, the closed orbits get nearer, then there shall be a line (called h in Fig. 4.6) that separates the region of orbits around z_1 from the region of orbits around z_2 and, on this line, the component of $\vec{H}$ parallel to it shall vanish; but since this line shall cross γ and it's transversal to it, then there shall be a point on which $\vec{H}$ vanishes, which is in contradiction with the fact that H has only two critical points.

Otherwise, if there is family of closed trajectories that surround both critical points, then, getting nearer z_1 and z_2, there shall be at least another (or a continuum of) equilibrium point, as it is shown in Fig. 4.9.

1 centre and 1 saddle. In this case, there can be two kind of phase diagram; let z_1 be the saddle point and z_2 the centre. In both cases, we will consider only the infinite horizon problem with final point $q_\infty = \pi(z_1)$ (i.e. the saddle), since we know that there is no solutions if the final point is the projection of a centre.

In the first case, shown in Fig. 4.7, one unstable semitrajectory and one stable semitrajectory of z_1 act together as a separatrix between the closed orbits around z_2 and the unbounded orbits of the other part of the plane. Repeating previous arguments, we see that these trajectories are diffeomorphically projected onto the horizontal line, and then we get the existence of minima of problem (4.15) with $q_\infty = \pi(z_1)$ for any $q_0 \in \mathbb{R}^n$.

A sample Hamiltonian with this behaviour is the function $H(p,q) = \frac{1}{2}p^2 + U(q)$, where $U(q)$ behaves as $-q^2$ for $q < \pi(z_1)$ and in a neighbourhood of $\pi(z_1)$, has a local minimum in $\pi(z_2)$, and then grows monotonically and assumes values strictly less than $U(\pi(z_1))$.

In the second case, illustrated in Fig. 4.8, a stable and an unstable semitrajectory of z_1 join together in a closed separatrix σ; in the closed region surrounded by σ there are the closed orbits around z_2, in the outer there are open orbits. In this case, there is a $\hat{q}$ such that

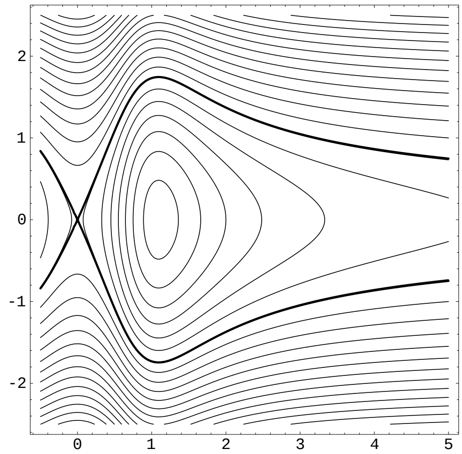

FIGURE 4.7

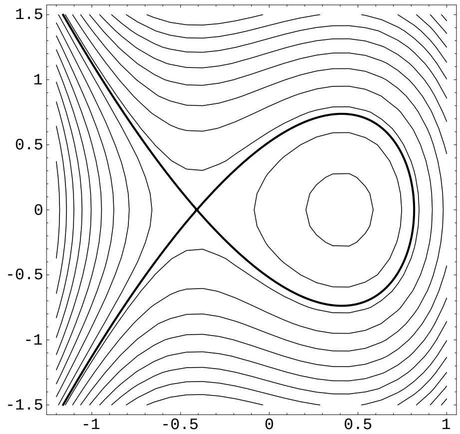

FIGURE 4.8

the infinite-horizon problem has solution only for initial point $q_0 \leq \hat{q}$ if $\pi(z_2) > \pi(z_1)$, and for $q_0 \geq \hat{q}$ if $\pi(z_2) < \pi(z_1)$.

A sample Hamiltonian in this case is the function $H(p, q) = \frac{1}{2}p^2 + q^3 - \alpha q,\ \alpha > 0$.

3 equilibrium points. As noticed all critical points lie on γ, and, moreover, we know from previous arguments that some configurations (two saddles or two centres side by side) are forbidden; hence, we can only find two situations: saddle-centre-saddle and centre-saddle-centre. Let us call z_1, z_2 and z_3 the equilibria in such a way that $\pi(z_1) < \pi(z_2) < \pi(z_3)$.

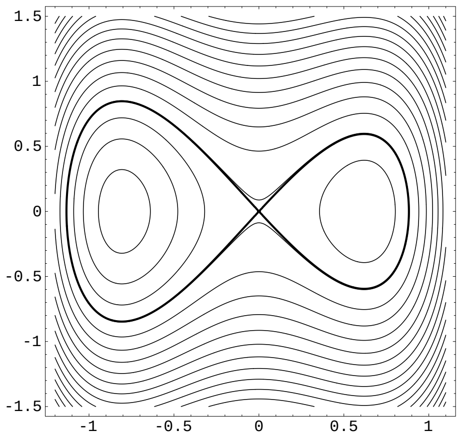

FIGURE 4.9

centre-saddle-centre. Let z_1 and z_3 be the two centres. There is a closed trajectory σ that surrounds both the centres and that passes through the saddle point that is a separatrix between the closed orbits around z_1 or z_3 and the outer orbits (that, depending on the problem, can be open or closed). In this case, the infinite-horizon problem with final point $\pi(z_2)$ has solution only for $q_0 \in \pi(\sigma)$, while there is no Hamiltonian trajectories which reach a point z such that $\pi(z) = q_1$ or $\pi(z) = q_3$ in an infinite time, thus the infinite-time problem with these conditions has no solution.

This behaviour can be generated by a Hamiltonian like $H(p,q) = \frac{1}{2}p^2 + q^4 + \alpha q^3 - \beta q^2$, $\beta > 0$.

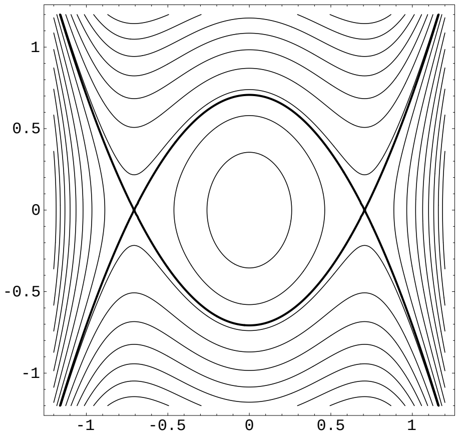

FIGURE 4.10

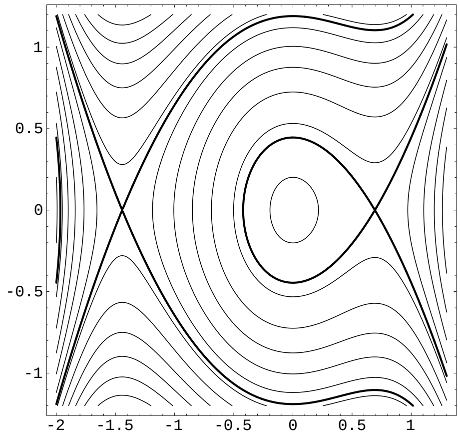

FIGURE 4.11

saddle-centre-saddle. We have to distinguish two cases: in the first one, shown in Fig. 4.10, the two saddle points (z_1 and z_3) belong to the same sublevel of H; in the other one, shown in Fig. 4.11, they belong to different sublevels.

Let us consider the former case; we notice that one stable semitrajectory arising from z_1 joins with one unstable semitrajectory of z_2, and vice-versa; these two trajectories form a separatrix between the closed orbits around the centre and the unbounded orbits. Obviously, there is no solution for the infinite-horizon problem with final point $\pi(z_2)$. Let us consider the problem with final point $\pi(z_1)$: for initial time $q_0 \leq \pi(z_1)$ the infinite-horizon solution always exists, since the unstable manifold is diffeomorphically projected onto the half-line $\{q \leq \pi(z_1)\}$; for $\pi(z_1) \leq q_0 < \pi(z_2)$, the problem has again solution, since a point on the separatrix reaches $\pi(z_1)$ in an infinite time; for $q_0 \geq \pi(z_2)$, there are no Hamiltonian trajectories reaching a point $z \in \mathbb{R}^2$ such that $\pi(z) = \pi(z_1)$ in infinite time, hence there is no solution. We can repeat the same argument to say that the infinite-horizon problem for final point $\pi(z_2)$ admits solution for $q_0 \geq \pi(z_2)$ and for $\pi(z_1) < q_0 \leq \pi(z_2)$ (on the separatrix).

A sample Hamiltonian that has this phase diagram is $H(p,q) = \frac{1}{2}p^2 - q^4 + \beta q^2$, $\beta > 0$.

Let us now focus on the second case; about the problem with final state $\pi(z_3)$, we can repeat the same arguments used in the situation depicted in Fig. 4.7 : there is a $\hat{q}$ such that the infinite-horizon problem admits solution if and only if $q \geq \hat{q}$; otherwise, problem (4.15) with final state $\pi(z_1)$ has a solution for any $q_0 \in \mathbb{R}$; as seen, there is no solution for $q_\infty = \pi(z_2)$.

This phase diagram can arise with a sample Hamiltonian as $H(p,q) = \frac{1}{2}p^2 - q^4 - \alpha q^3 + \beta q^2$, $\alpha \neq 0, \beta > 0$.

More than 3 equilibrium points. We get from previous arguments that all the equilibrium points lie on γ and they can only be aligned alternating saddle points and centres. Then all the possible configurations can be deduced by previous results.

Let us now see what happens if we remove hypothesis (A1). Since we ask that $\vec{H}$ has at least one equilibrium point, there shall exist at least one $\bar{q}$ for which the function $p \mapsto H(p, \bar{q})$ has a minimum; then, there exists a curve γ that divides $\mathbb{R}^2$ into two regions as above, and such that γ passes through the minimum of $H(\cdot, \bar{q})$. What we cannot say is that γ is projected onto the horizontal axis: it can happen that its projection is bounded. This implies that we cannot guarantee that the stable manifold is projected onto the horizontal line, hence losing the existence of the optimal synthesis (but we still have local existence of minimizers).

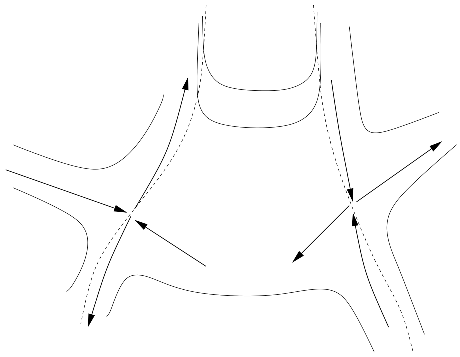

FIGURE 4.12

Moreover, we notice that there can be cases in which there exist two (or more) disjoint curves γ_i that divides the regions where $\frac{\partial H}{\partial p} > 0$ from the ones where $\frac{\partial H}{\partial p} < 0$; in such cases, it remains true that on the same curve there cannot lie two saddle points or two centres side by side, but we can have situations such as the one depicted in Fig. 4.12 in which the two critical points are saddle points.

For such cases, we just repeat that our analysis is only local and permits to prove the existence of minimizers only for some initial points q_0.

CHAPTER 5

Non-autonomous infinite horizon variational problems

Let us now consider a more general version of the optimal economic growth problem; dealing again with smooth integrands and continuous-time problems, here we study Lagrangians that depend explicitly on time: in particular, we study the functional

$$J^\alpha(\gamma) = \int_0^\infty e^{-\alpha t}\varphi(\gamma(t), \dot\gamma(t))\, dt, \tag{5.1}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function. The term $\alpha > 0$ is called *discount* or *forgetting factor*.

Our goal is to determine the trajectories that minimize the functional J^α that belong to a suitable class of admissible trajectories. In this section, we will only give some preliminary results that allow for a special class of Lagrangians φ, namely they have to be *strictly convex* with respect to the second variable and *quadratic out of a compact set*; moreover, we are interested in the case in which the Hamiltonian associated to the problem has *positive curvature*. The generalization of this results to an ampler class of Lagrangians is still work in progress.

Remark. Let us remark that, as in the preceding chapter, with *strict convexity* of a function with respect to some variable we will mean the *positive definiteness* of the Hessian of the function with respect to the given variable.

1. Introduction to the problem

We consider the functional J^α defined in (5.1), and we try to find the curve $q(\cdot)$ that minimizes it in a suitable class of trajectories to be specified later. As already done in the preceding chapter, we formulate this problem as an optimal control problem, and we associate to this problem a maximized Hamiltonian function

$$H(p, q) = \max_u \langle p, u\rangle - e^{-\alpha t}\varphi(q, u). \tag{5.2}$$

The dynamical system generated by H is of course Hamiltonian, though it is non-autonomous. As usual, we will denote the flow $e^{t\vec{H}}$ also with ϕ^t. To state the problem with more precision, we first examine the following example:

Example. Let us recall the case studied in Subsection 2.1 of preceding chapter: we had $\varphi(q, \dot q) = \frac{1}{2}(\dot q^2 - rq^2)$, $q \in \mathbb{R}$, and $H(p, q) = \frac{1}{2}(p^2 + rq^2)$, where r is a constant. We saw that the sign of r is crucial for the existence of the optimal synthesis.

Here we consider the problem with the same quadratic Lagrangian multiplied by the discount term $e^{-\alpha t}$, $\alpha > 0$; the associated non-autonomous Hamiltonian is $H(p, q) = \frac{1}{2}(e^{\alpha t}p^2 +$

$e^{-\alpha t} rq^2$); we make the following substitutions:

$$\zeta := e^{\alpha t} p \qquad H^\alpha := e^{\alpha t} H = \frac{1}{2}(\zeta^2 + rq^2); \tag{5.3}$$

the equation of motion for the variables (ζ, q) is

$$\begin{pmatrix} \dot{\zeta} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \alpha & -r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ q \end{pmatrix} \tag{5.4}$$

or, more synthetically, $(\dot{\zeta}, \dot{q}) = \vec{H}^\alpha$, where $\vec{H}^\alpha = \vec{H} + \alpha\zeta\frac{\partial}{\partial\zeta}$. Notice that this is an autonomous dynamical system, but it is not Hamiltonian.

Since the dynamical system is linear, the trajectories are completely determined by the eigenvalues of the evolution matrix, which are $\lambda_\pm = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - r}$.

If $r < 0$, then both the eigenvalues are real, with λ_- negative and λ_+ positive, and the origin is a saddle point and there are defined a stable and an unstable manifold. If we want to study a variational problem such as (4.7)-(4.8), we can repeat all the arguments previously used for the case without discount factor, and prove the existence of the optimal synthesis.

Much more interesting is the case $r > 0$; for fixed $r > 0$, the phase portrait is determined by the value of α. In fact, for $\alpha < 2\sqrt{r}$, both $\lambda_\pm$ are complex with positive real part, and then the origin is an unstable focus; in this situation, the integrand $e^{-\alpha t}\varphi(q, \dot{q})$ tends to a nonzero constant for $t \to +\infty$ along any trajectory of the dynamical system, and then the integral in (5.1) does never converge.

Otherwise, for $\alpha > 2\sqrt{r}$ both eigenvalues are real and positive, then the point $(0, 0)$ is an unstable node, but we have $\lambda_- < \alpha/2$; this implies that there is a direction (the direction determined by the eigenvector relative to the eigenvalue λ_-) along which $|(\zeta, q)|$ grows with exponential rate less than $e^{\frac{\alpha}{2}t}$. Since the Lagrangian is quadratic in $(q, \dot{q})$, the integrand converges exponentially to zero along this direction, and then the integral (5.1) converges. Along all the other trajectories the integral does not converge. Then, by PMP, the projections on $\mathbb{R}$ of these trajectories along the eigenspace relative to λ_- are the optimal trajectories of the infinite horizon problem with discount factor α. In particular, it can be shown that the problem admits an optimal synthesis.

The example is enlightening because it shows that in presence of a suitable discount factor even cases with positive r may admit an optimal solution for an infinite horizon problem (or even an optimal synthesis). We stress that in the case without discount this is forbidden, due to Theorem 2.1.

In this treatment, we will focus our attention to functions φ which are *strictly convex* with respect to the second variable and *quadratic* out of a compact in the pair of variables. For such functions we use the following definition of optimality:

DEFINITION 5.1. *We say that a locally Lipschitzian curve $\tilde{\gamma} : [0, +\infty) \to \mathbb{R}^n$ is an optimal trajectory for the infinite horizon problem with discount factor α if*

$$J^\alpha(\tilde{\gamma}) = \min\Big\{ J^\alpha(\gamma) : \gamma(0) = \tilde{\gamma}(0),\ \lim_{t\to+\infty} e^{-\alpha t}|\dot{\gamma}(t)|^2 = 0 \Big\}. \tag{5.5}$$

Notice that in this case we do not give condition on the final endpoint; in particular, in the case above shown with $r > 0$ and $\alpha > 2\sqrt{r}$, the trajectories satisfying Definition 5.1 are *unbounded*. Due to this fact, we slightly modify the definition of optimal synthesis we gave in the introduction to Chapter 4:

DEFINITION 5.2. *A smooth optimal synthesis is a smooth complete vector field X on $\mathbb{R}^n$ such that all the solutions of the equation $\dot{q} = X(q)$ are optimal trajectories (according to Definition 5.1) of the infinite variational problem under consideration.*

2. Computation of the curvature

Our aim is to determine whether the problem of minimization of the functional J^α admits a smooth optimal synthesis, according to Definition 5.2. As done in the case without discount, and as suggested by the example just given (where r was in fact the generalized curvature of the system, computed with respect to the canonical splitting associated to the vertical distribution), we need to compute the generalized curvature of the system.

We perform the non-autonomous change of variables $\zeta := e^{\alpha t}p$, $H^\alpha = e^{\alpha t}H$ (5.3), and the new dynamical system on $\mathbb{R}^{2n}$ is given by

$$\left\{ \begin{array}{rcl} \dot{\zeta} &=& \alpha\zeta - \frac{\partial H^\alpha}{\partial q} \\ \dot{q} &=& \frac{\partial H^\alpha}{\partial \zeta} \end{array} \right. ;, \tag{5.6}$$

where we put $\vec{H}^\alpha = \vec{H} + \alpha\zeta\frac{\partial}{\partial\zeta}$. We notice that this new dynamical system is autonomous, but not Hamiltonian. In fact, the flow does not preserve the symplectic form:

$$\left(e^{t\vec{H}^\alpha}\right)^* \sigma = e^{\alpha t}\sigma. \tag{5.7}$$

To prove this, let us recall that for any vector field X and any differential form ω, $\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$, where d denotes the exterior differential. We have

$$\begin{array}{rcl} \mathcal{L}_{\vec{H}^\alpha}\sigma &=& \dfrac{d}{dt}\left(e^{t\vec{H}^\alpha}\right)^*\sigma \\ &=& \iota_{\vec{H}^\alpha}(d\sigma) + d(\iota_{\vec{H}^\alpha}\sigma) = \\ &=& \displaystyle\sum_{i,k=1}^{n} -\frac{\partial^2 H}{\partial q_k \partial p_i} dp_i \wedge dq_k + \alpha dp_i \wedge dq_k + \frac{\partial^2 H}{\partial q_i \partial p_k} dp_k \wedge dq_i = \alpha\sigma, \end{array}$$

hence the thesis. However, by equation (5.7) we get that the flow preserves the Lagrangian subspaces.

Remark. In the following, we will deal with two functions – H and H^α – and their associated vector fields, respectively $\vec{H}$ and $\vec{H}^\alpha$. We notice that the former function is defined on the space $\{(p,q)\} \in \mathbb{R}^{n*} \times \mathbb{R}^n$, while the second one lives on the space $\{(\zeta,q)\} \in \mathbb{R}^{2n}$. Actually, they define *the same* dynamical system, but described in different coordinate systems; in particular, as we saw H defines a Hamiltonian non-autonomous system, H^α a non-Hamiltonian autonomous dynamical system. In the following, we will use H when working with coordinates (p,q), and H^α when using the coordinates (ζ,q).

To avoid confusion, when we will need to recall the case without discount, we will denote the Hamiltonian with H^0, and the vector field with $\vec{H}^0$.

Fix $z = (p,q) \in \mathbb{R}^{2n}$ and consider the vertical distribution $\Lambda_z = \mathbb{R}^n \times \{q\}$; we define the Jacobi curve $\Lambda_z^\alpha(t) = e^{-t\vec{H}^\alpha}{}_*\Lambda_{z_t}$, $z_t = e^{t\vec{H}^\alpha}(z)$, and we need to compute its derivative curve. Due to intrinsic definition of the derivative curve (equation (2.6)), we just need to determine the "derivative distribution" $\{\Lambda_z^{\alpha\circ}\}_{z\in\mathbb{R}^{2n}}$. To do that, we follow the construction below exposed.

Let for the moment $\alpha = 0$, and consider the canonical moving frame $\{e^i(t), f^i(t)\}_{i=1}^n$ at z associated to H^0, where $\text{span}\{e^1(0), \ldots, e^n(0)\} = \mathbb{R}^n \times \{q\}$; we already saw that these vectors satisfy the system $\dot{e}^i(t) = f^i(t)$, $\dot{f}^i(t) = -R(t)e^i(t)$, where R is the representation with respect to this basis of the curvature operator associated to $\vec{H}^0$.

For $\alpha > 0$, the frame $\{e^i(t), f^i(t)\}_{i=1}^n$ satisfies the following system:

$$\begin{cases} \dot{e}^i(t) &= f^i(t) - \alpha e^i(t) \\ \dot{f}^i(t) &= -R(t)e^i(t) \end{cases} . \tag{5.8}$$

A moving frame $\{\tilde{e}^i(t), \dot{\tilde{e}}^i(t)\}_{i=1}^n$ is the canonical moving frame associated to the canonical splitting $\Lambda^\alpha \oplus \Lambda^{\alpha\circ}$ if and only if

$$\ddot{\tilde{e}}^i(t) \in \Lambda_z^\alpha(t) \qquad \text{for any} \quad t,\ i = 1, \ldots, n. \tag{5.9}$$

To determine such vectors $\{\tilde{e}^i(t)\}$, we introduce a (invertible) linear transformation

$$X(t) : \Lambda_z^\alpha(t) \to \Lambda_z^\alpha(t), \qquad X(0) = \text{id}$$

that defines a new basis $\tilde{e}^i(t) = X(t)e(t)^i$, and we require condition (5.9) to be satisfied. Let us compute $\ddot{\tilde{e}}^i(t)$:

$$\begin{aligned} \frac{d^2}{dt^2}(X(t)e^i(t)) &= \ddot{X}(t)e^i(t) + \dot{X}(t)f^i(t) - \alpha\dot{X}(t)e^i(t) + \dot{X}(t)f^i(t) + \\ &- X(t)R(t)e^i(t) - \alpha\dot{X}(t)e^i(t) - \alpha X(t)f^i(t) + \alpha^2 X(t)e^i(t) = \\ &= (\ddot{X}(t) - 2\alpha\dot{X}(t) - X(t)R(t) + \alpha^2 X(t))e^i(t) + (2\dot{X}(t) - \alpha X(t))f^i(t). \end{aligned}$$

This implies that equation (5.9) is satisfied if and only if $\dot{X}(t) = \frac{\alpha}{2}X(t)$. The new canonical moving frame satisfies the system

$$\begin{cases} \dot{\tilde{e}}^i(t) &= \frac{d}{dt}(Xe^i)(t) &= \tilde{f}^i(t) - \frac{\alpha}{2}\tilde{e}^i(t) \\ \ddot{\tilde{e}}^i(t) &= \frac{d^2}{dt^2}(Xe^i)(t) &= -R^\alpha(t)\tilde{e}^i(t) \end{cases} , \tag{5.10}$$

where $\tilde{f}^i(t) = X(t)f^i(t)$; then we get

$$\Lambda_z^{\alpha\circ} = \text{span}\{\tilde{f}^1 - \frac{\alpha}{2}\tilde{e}^1, \ldots, \tilde{f}^n - \frac{\alpha}{2}\tilde{e}^n\}$$

and

$$R^\alpha(t) = X(t)R(t)X(t)^{-1} - \frac{\alpha^2}{4}\text{id}$$

We then conclude that $R^\alpha(t)$ is the representation with respect to the basis $\{\tilde{e}^i(t), \dot{\tilde{e}}^i(t)\}_{i=1}^n$ of the curvature operator $R_z^{\Lambda^\alpha, \Lambda^{\alpha\circ}}$ (notice that in the computations we omitted the dependence on z).

Notice that if $R(t)$ is positive and bounded from above by a constant C, for $\alpha > 2\sqrt{C}$ we get that $R^\alpha(t) < 0$.

3. Preliminary results

In the following we will denote with R_z^0 the curvature operator of H^0 (case without discount factor) with respect to the canonical splitting. The curvature operator $R_z^{H^\alpha}$ will be denoted with R_z^α.

THEOREM 5.1. *Let $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth, and strictly convex with respect to second variable, and assume that there is a compact set $K \in \mathbb{R}^{2n}$ such that the function φ is quadratic in the pair (q, u) for any $(q, u) \in \mathbb{R}^{2n} \setminus K$.*

If the curvature R^0_z of the Hamiltonian vector field $\vec{H}^0$ with respect to the canonical splitting satisfies the inequalities $0 < R^0_z \leq C$ for some constant C for any $z \in \mathbb{R}^{2n}$, then the infinite horizon variational problem without discount does not admit optimal trajectories, while the problem with discount admits an optimal synthesis of class C^1 if $\alpha > 2\sqrt{C}$.

Proof. The proof of this Theorem is analogous to the proof of Theorem 4.5: we want to find a Lagrangian invariant submanifold of $\mathbb{R}^{2n}$ such that the Hamiltonian trajectories living on it satisfy the growth assumptions in (5.5), and then we want to show that it projects diffeomorphically onto the state space $\mathbb{R}^n$. This guarantees the existence of the optimal synthesis, provided that we generalize the sufficient condition for optimality (Theorem 1.8) to this case with the discount factor.

Then the proof of the Theorem is thus organized: in Subsection 3.1 we prove the existence of a $e^{t\vec{H}^\alpha}$-invariant distribution on $\mathbb{R}^{2n}$; we use Wojtkowski's criterion to prove that the vectors on it satisfy some particular growth assumptions. In Subsection 3.2, we use Hadamard-Perron Theorem and some results on Partially Hyperbolic Theory to show that this distribution is in fact tangent to a C^1 smooth submanifold W^c; then, we prove that W^c is diffeomorphically projected onto the base space $\mathbb{R}^n$. Finally, Subsection 3.3 contains a proof of the generalization of Theorem 1.8 to the case under investigation.

3.1. The limit distribution. We construct two Lagrangian distributions through the following Lemma:

LEMMA 5.1. *If $R^\alpha_z < 0$ for any $z = (\zeta, q) \in \mathbb{R}^{2n}$, then there exist the limits $\lim_{t\to\pm\infty} \Lambda^\alpha_z(t) = \Lambda^{\alpha\pm}_z$.*

Proof. Fix z, and choose coordinates (x, y) on $T_z(\mathbb{R}^{2n})$ such that $\Lambda^\alpha_z(t) = \{(x, S(t)x) : x \in \mathbb{R}^n\}$ and $\Lambda^{\alpha\circ}_z(t) = \{(x, S^\circ(t)x) : x \in \mathbb{R}^n\}$, with $S^\circ(0) - S(0) > 0$. Since the two curves are transversal at any time, we always have $S^\circ(t) - S(t) > 0$.

By computations, we can show that the curve $\Lambda^\alpha_z(\cdot)$ is monotone increasing, and hence, since the curvature is negative, the derivative curve is monotone decreasing. This implies the existence of the limit $\lim_{t\to+\infty} \Lambda^\alpha_z(t) = \Lambda^{\alpha+}_z$. If we choose another coordinate chart such that $S^\circ(0)-S(0) < 0$, with the same argument we can prove the existence of $\lim_{t\to-\infty} \Lambda^\alpha_z(t) = \Lambda^{\alpha-}_z$.
□

Remark. Since $\Lambda^\alpha_z(\cdot)$ is Lagrangian, also its limit distributions are. Moreover, it is easy to show that by construction the distributions are invariant with respect to the flow $\phi^t = e^{t\vec{H}^\alpha}$, i.e.

$$\phi^t{}_*\Lambda^{\alpha\pm}_z = \Lambda^{\alpha\pm}_{z_t}, \quad z_t = \phi^t(z).$$

Since the curves $\Lambda^\alpha_z(\cdot)$ is monotone, we also have that the limit distributions $\Lambda^{\alpha\pm}_z$ are always transversal to $\Lambda^\alpha_z(0)$, i.e.

$$\Lambda^{\alpha+}_z \cap \Lambda^\alpha_z(0) = \Lambda^{\alpha-}_z \cap \Lambda^\alpha_z(0) = 0$$

for any z.

For these distributions the estimates written below allow:

LEMMA 5.2. *There is a norm* $\|\cdot\|$ *on* $\mathbb{R}^{2n}$ *and constants* $\rho > 0, \varepsilon \in (0, \frac{\alpha}{4})$ *such that if* $R_z^0 > 0$ *and* $R_z^\alpha < 0$ *for any* $z \in \mathbb{R}^{2n}$, *then:*

$$\frac{1}{\rho} e^{\varepsilon t} \|\boldsymbol{x}_-\| \le \|e^{t\vec{H}^\alpha}{}_* \boldsymbol{x}_-\|_{z_t} \quad \le \quad \rho e^{(\frac{\alpha}{2}-\varepsilon)t} \|\boldsymbol{x}_-\|_z \tag{5.11}$$

$$\|e^{t\vec{H}^\alpha}{}_* \boldsymbol{x}_+\|_{z_t} \quad \ge \quad \rho e^{(\frac{\alpha}{2}+\varepsilon)t} \|\boldsymbol{x}_+\|_z, \tag{5.12}$$

for any $\boldsymbol{x}_\pm \in \Lambda_z^{\alpha\pm}$ *and for* $z_t = e^{t\vec{H}^\alpha}(z)$.

Proof. Consider the dynamical system for the canonical moving frame (5.8); let us then write it as

$$\begin{aligned} \begin{pmatrix} -\alpha & 1 \\ -R(t) & 0 \end{pmatrix} &= \begin{pmatrix} -\frac{\alpha}{2} & 1 \\ -R(t) & \frac{\alpha}{2} \end{pmatrix} - \frac{\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbb{H} - \frac{\alpha}{2}\, \mathrm{id}_{2n}, \end{aligned} \tag{5.13}$$

where above with "1" we mean the $n \times n$ identity matrix.

Let us notice that the matrix $\mathbb{H}$ is Hamiltonian (that means, for a block-matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, that $A + D^T = 0$ and that B and C are symmetric matrices), while the second term is just the identity matrix on $\mathbb{R}^{2n}$ multiplied by a scalar factor.

Since this second term commutes with any other matrix, its contribution to the evolution of the vectors consists in multiplying them by the exponential factor $e^{-\frac{\alpha}{2}t}$; then we can for the moment leave it out, and focus on the action $\mathbb{H}$: we see that it defines a quadratic form $\mathcal{Q}$ that satisfies Wojtkowski's criterion and Lemma 4.1.

In fact, let $E = (e^1(0), \dots, e^n(0))^T$ and $F = (f^1(0), \dots, f^n(0))^T$, and call $\Gamma = (f^1(0) - \gamma e^1(0), \dots, f^n(0) - \gamma e^n(0))^T$; put

$$\mathcal{Q}_t(\boldsymbol{x}) = \sigma((\Phi_{\mathbb{H}}^t \boldsymbol{x})_\Gamma, (\Phi_{\mathbb{H}}^t \boldsymbol{x})_E) \tag{5.14}$$

where $\Phi_{\mathbb{H}}^t$ is the flow on $\mathbb{R}^{2n}$ generated by the equation

$$\frac{d}{dt}\left[\Phi_{\mathbb{H}}^t \begin{pmatrix} E \\ F \end{pmatrix}\right] = \mathbb{H} \begin{pmatrix} E \\ F \end{pmatrix},$$

and X_E and X_Γ denote, as usual, the sections of the vector field X such that $X_E \in \mathrm{span}\{e^1(0), \dots, e^n(0)\}$, $X_\Gamma \in \mathrm{span}\{f^1(0) - \gamma e^1(0), \dots, f^n(0) - \gamma e^n(0)\}$, and $X = X_E + X_\Gamma$. By computations, we have that

$$\frac{d}{dt} \mathcal{Q}(\Phi_{\mathbb{H}}^t \boldsymbol{x})|_{t=0} = \sigma((\boldsymbol{x}_\Gamma)^\cdot, \boldsymbol{x}_\Gamma) - \sigma((\boldsymbol{x}_E)^\cdot, \boldsymbol{x}_E),$$

where here $(\boldsymbol{x}_E)^\cdot$ denotes $\frac{d}{dt}\Phi_{\mathbb{H}}^t(\boldsymbol{x}_E)|_{t=0}$ (same for $(\boldsymbol{x}_\Gamma)^\cdot$).

By computations, for $\gamma \in \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha}{4}^2 - C}, \frac{\alpha}{2} - \sqrt{\frac{\alpha}{4}^2 + C}\right)$ we have that $\sigma((\boldsymbol{x}_\Gamma)^\cdot, \boldsymbol{x}_\Gamma)$ is positive definite; $\sigma((\boldsymbol{x}_E)^\cdot, \boldsymbol{x}_E)$ negative definite.

Then we can apply Lemma 4.1 and get the existence of the expanding and contracting cones $C_z^\pm$; this proves the existence of the splitting $\widehat{C}_z^+ \oplus \widehat{C}_z^-$ (defined as in equation (4.6)) such that

$$\|\Phi_{\mathbb{H}}^t \boldsymbol{x}\| \quad \le \quad e^{-\mu t} \|\boldsymbol{x}\| \qquad \forall\, X \in \widehat{C}_z^- \tag{5.15}$$

$$\|\Phi_{\mathbb{H}}^t \boldsymbol{x}\| \quad \ge \quad e^{\mu t} \|\boldsymbol{x}\| \qquad \forall\, X \in \widehat{C}_z^+, \tag{5.16}$$

where $\mu = \min_i\{\sqrt{\frac{\alpha}{4}^2 - r_i} : r_i \text{ eigenvalue of } R\}$.

If we consider the original flow $\phi^t = e^{t\vec{H}^\alpha}$, we shall take account of the contribution of the matrix $-\frac{\alpha}{2}\mathrm{id}_{2n}$.

Notice finally that the system (5.8) describes the evolution of the vectors $\{e^i_{z_0}(t), f^i_{z_0}(t)\} = \{\phi^{-t}{}_* e^i_{z_t}(0), \phi^{-t}{}_* f^i_{z_t}(0)\}$, i.e. the evolution for *negative* times.

Then, equations (5.15)-(5.16) reads

$$\|\phi^{-t}{}_*\boldsymbol{x}\| \leq e^{-(\mu+\frac{\alpha}{2})t}\|\boldsymbol{x}\| \qquad \forall\, \boldsymbol{x} \in \widehat{C}^-_z \tag{5.17}$$

$$\|\phi^{-t}{}_*\boldsymbol{x}\| \geq e^{(\mu-\frac{\alpha}{2})t}\|\boldsymbol{x}\| \qquad \forall\, \boldsymbol{x} \in \widehat{C}^+_z,. \tag{5.18}$$

Then we shall change the sign of time in equations (5.17)-(5.18), and also to invert the names of $\widehat{C}^+$ and $\widehat{C}^-$, putting $\widetilde{C}^+ = \widehat{C}^-$ and $\widetilde{C}^- = \widehat{C}^+$; for $\widetilde{C}^\pm$, we obtain the estimates (5.11)-(5.11).

The final step is to show that actually $\widetilde{C}^\pm_z = \Lambda^{\alpha\pm}_z$. Since $\widetilde{C}^+_z \oplus \widetilde{C}^-_z$ is a splitting, we can write any $\boldsymbol{x}$ as $\boldsymbol{x} = \boldsymbol{x}_+ + \boldsymbol{x}_-$, where obviously $\boldsymbol{x}_\pm \in \widetilde{C}^\pm_z$.

For any $\boldsymbol{x} \in T_z(\mathbb{R}^{2n})$, we have that $\|D_z\phi^t\boldsymbol{x}_-\| \to 0$ as $t \to +\infty$, and then that $D_z\phi^t\boldsymbol{x} \to D_z\phi^t\boldsymbol{x}_+$ as $t \to +\infty$; take $\boldsymbol{x} \in \Lambda^\alpha_z$: by definition of Λ^α_z, we have that $\lim_{t\to+\infty} D_z\phi^t\boldsymbol{x} \in \Lambda^{\alpha+}_z$. Since $\boldsymbol{x}_+ \neq 0$, $\lim_{t\to+\infty} D_z\phi^t\boldsymbol{x} \in \widetilde{C}^+_z$.

This, plus some dimensional considerations, implies that $\widetilde{C}^+_z = \Lambda^{\alpha+}_z$. The same for $\Lambda^{\alpha-}_z$. Then, we get that the estimates (5.15)-(5.15) allow also for $\Lambda^{\alpha+}_z$ and $\Lambda^{\alpha-}_z$. □

This Lemma has an immediate Corollary:

Corollary 5.1. *Under the hypothesis of Lemma 5.2, we have that*

$$\|\vec{H}^\alpha(z_t)\|_{z_t} \to \infty \quad \text{as} \quad t \to +\infty$$

with exponential rate for any $z \in \mathbb{R}^{2n}$. *Moreover, if* $\vec{H}^\alpha \in \Lambda^{\alpha-}_z$, *then*

$$e^{-\frac{\alpha}{2}t}\|\vec{H}^\alpha(z_t)\|_{z_t} \to 0 \quad \text{as} \quad t \to +\infty$$

with exponential rate; otherwise,

$$e^{-\frac{\alpha}{2}t}\|\vec{H}^\alpha(z_t)\|_{z_t} \to \infty \quad \text{as} \quad t \to +\infty$$

with exponential rate

3.2. Existence of the optimal synthesis. Lemma 5.2 shows that the dynamical system (5.6) is *partially hyperbolic in the narrow sense*, according to Definition 1.5. In this particular case, the distribution $\Lambda^{\alpha-}$ is the central distribution of the system, while the $\Lambda^{\alpha+}$ is the unstable distribution.

Our aim is then to establish the integrability of the central distribution; in fact, if it existed an integral manifold of $\Lambda^{\alpha-}$, any trajectory lying on it would satisfy the growth assumptions required in (5.5), while all the other Hamiltonian trajectories would grow too fast with respect to time.

Lemma 5.3. *The central distribution is integrable with integral manifold* W^c.

Proof. Thanks to Theorem 1.3, to prove the integrability of $\Lambda^{\alpha-}$ we need to prove that the lift of the integral manifold of $\Lambda^{\alpha+}$ is quasi-isometric in the universal cover to $\mathbb{R}^{2n}$.

We choose $\mathbb{R}^{2n}$ itself to be the universal covering. Call W^+ the integral manifold of the unstable distribution. Since φ is quadratic out of the compact K, there exists a compact

$\widehat{K} \subset \mathbb{R}^{2n} = \{(\zeta, q)\}$ such that H^α is quadratic on $\mathbb{R}^{2n} \setminus \widehat{K}$; on $\mathbb{R}^{2n} \setminus \widehat{K}$, the unstable manifold is a linear subspace of $\mathbb{R}^{2n}$; this implies that $d_{W^+} = d_{\mathbb{R}^{2n}}$.

Inside $\widehat{K}$, by compactness of the set, there exist two constants $a, b > 0$ such that $d_{W^+}(z_1, z_2) \leq a d_{\mathbb{R}^{2n}}(z_1, z_2) + b$.

Then W^+ is quasi-isometric, and hence we get the existence of W^c.

By Hadamard-Perron Theorem, W^c is C^1 smooth. □

Notice that W^c is Lagrangian, due to Lagrangianity of the central distribution $\Lambda^{\alpha-}$.

Since the central distribution $\Lambda^{\alpha-}$ is always transversal to the vertical distribution, we get that W^c is projected injectively on $\mathbb{R}^n$. For the same reason (transversality of $\Lambda^{\alpha-}$ to the vertical distribution), the projection $\pi|_{W^c}$ is proper, and then a diffeomorphism onto the image.

Surjectivity is a consequence of linearity of the flow out of $\widehat{K}$.

3.3. The sufficient condition. In this subsection we state and prove the generalization to our case of the sufficient condition for optimality stated in Chapter 1 (Theorem 1.8).

THEOREM 5.2. *Let W^c be the less unstable manifold defined in previous subsection; let $\pi : \mathbb{R}^{2n} \to \mathbb{R}$ be the canonical projection, and assume that its restriction $\pi|_{W^c}$ is a diffeomorphism. Then for any $\lambda_0 \in W^c$ the normal extremal trajectory*

$$\tilde{q}(t) = \pi \circ e^{t\vec{H}}(\lambda_0), \qquad t \in [0, +\infty),$$

realized a strict minimum of the cost functional (5.1) among all the locally Lipschitzian curves $q(t)$, $t \in [0, +\infty)$, such that

$$q(0) = \tilde{q}(0) \qquad \text{and} \qquad \lim_{t \to +\infty} e^{-\alpha t} |\dot{q}(t)|^2 = 0. \tag{5.19}$$

Proof. The proof is just an adaptation of the proof on Theorem 1.8, that can be found in [**7**]. Here we briefly expone the modifications to be done.

Define the set

$$\mathcal{W} := \{(\lambda, t) : \lambda \in W^c, 0 \leq t < +\infty\} \subset \mathbb{R}^{2n} \times \mathbb{R},$$

which is a smooth $(n+1)$-dimensional submanifold of $\mathbb{R}^{2n} \times \mathbb{R}$, and consider the 1-form

$$\vartheta - H\, dt,$$

where we recall that ϑ is the tautological 1-form on $\mathbb{R}^{2n}$ (see Section 1 of Chapter 1), and H is defined by (5.2).

It is straightforward to prove that $\vartheta - H\, dt$ is *exact* on $\mathcal{W}$ (see the analogous proof in [**7**], Section 17.1.1).

Then, consider a point $q_0 \in \mathbb{R}^n$ and its lift $\lambda_0 \in W^c$. Put $\tilde{q}(t) = \pi(\phi^t(\lambda_0))$, and let $q(\cdot)$ be a curve such that $q(0) = q_0$ and $\lim_{t \to +\infty} e^{-\alpha t} |\dot{q}(t)|^2 = 0$. Since W^c is diffeomorphically projected onto $\mathbb{R}^n$, there is a unique lift $\lambda(\cdot)$ of the curve $q(\cdot)$ to the less unstable manifold W^c.

Fix some $T > 0$; then

$$\begin{aligned}\int_0^T e^{-\alpha t}\varphi(q(t),\dot q(t))\,dt &= \int_0^T \langle\lambda(t),\dot q(t)\rangle - h(\lambda(t))\,dt \geq \\ &\geq \int_0^T \langle\lambda(t),\dot q(t)\rangle - H(\lambda(t))\,dt = \\ &= \int_0^T \langle\vartheta_{\lambda(t)},\dot\lambda(t)\rangle - H(\lambda(t))\,dt = \\ &= \int_\gamma \vartheta - H\,dt,\end{aligned}$$

where we recall that $h(\lambda(t)) = \langle\lambda(t),\dot q(t)\rangle - e^{-\alpha t}\varphi(q(t),\dot q(t))$, and γ is a curve in $\mathcal{W}$ defined by

$$\gamma : t \mapsto (\lambda(t), t) \in \mathcal{W}, \qquad t \in [0,T].$$

Since the form $\vartheta - H\,dt$ is exact, its integral along a curve in $\mathcal{W}$ depends only on the endpoints. Then we define the two curves

$$\tilde\gamma : t \mapsto (\phi^t(\lambda_0), t) \in \mathcal{W}, \qquad t \in [0,T]$$

and

$$\gamma_T : \tau \mapsto (\eta_T(\tau), T), \qquad \tau \in [\tau_0, \tau_f]$$

such that η_T is a curve in W^c with $\eta_T(\tau_0) = \lambda(T)$ and $\eta_T(\tau_f) = e^{T\vec H}(\lambda_0)$; the choice of η_T will be specified below. We have that

$$\int_\gamma \vartheta - H\,dt + \int_{\gamma_T} \vartheta - H\,dt = \int_{\tilde\gamma} \vartheta - H\,dt. \tag{5.20}$$

Since t is constant along γ_T, we have that

$$\int_{\gamma_T} \vartheta - H\,dt = \int_{\gamma_T}\vartheta = \int_{\eta_T} p\,dq = \int_{\tau_0}^{\tau_f} p(\tau)\dot q(\tau)\,d\tau;$$

by the growth conditions (5.19), we have that there exists some $\varepsilon > 0$ such that for $T \gg 0$ $|q(T)|,\ |\tilde q(T)| \leq |q_0|e^{(\alpha/2-\varepsilon)T}$. Then we define a curve $\hat q(\cdot) \in \mathbb{R}^n$ with $\hat q(\tau_0) = q(T),\ \hat q(\tau_f) = \tilde q(T)$, and $|\hat q(\tau)| \leq |q_0|e^{(\alpha/2-\varepsilon)T}$ for any $\tau \in [\tau_0,\tau_f]$; the curve η_T is then chosen to be the lift of $\hat q(\cdot)$ to W^c. We can choose a parametrization of the curve η_T such that $|\dot q(\tau)|$ is constant for any τ. In fact, the integral $\int_{\tau_0}^{\tau_f}|\dot q(\tau)|\,d\tau$ equals the length of the curve $\hat q(\cdot)$, and this implies that the length of the interval $(\tau_f - \tau_0)$ and the value of $|\dot q(\tau)|$ are not independent. We then fix $\dot q(\tau) = \rho$ for any τ, and $(\tau_f - \tau_0)\rho \sim Ce^{(\alpha/2-\varepsilon)T}$, for some constant C.

Since the function φ is quadratic out of K, out of this compact the dynamics is linear in the variables (ζ, q), and the space W^c is a linear space; this means that there is a linear operator A such that we can write

$$W^c \setminus (W^c \cap \widehat K) = \{(\zeta,q) \in \mathbb{R}^{2n} \setminus \widehat K : \zeta = Aq\}.$$

We can then deduce that, if T is sufficiently large, for any $(\zeta,q) \in \eta_T$ we have $|\zeta| \leq \|A\||q_0|e^{(\alpha/2-\varepsilon)T} = C'e^{(\alpha/2-\varepsilon)T}$, for some constant C'. This implies that for any $(p,q) \in \eta_T$ we have

$$|p| \leq e^{-\alpha T}C'e^{(\alpha/2-\varepsilon)T} = C'e^{(-\alpha/2-\varepsilon)T};$$

then

$$\int_{\eta_T} p\,dq \leq \int_{\tau_0}^{\tau_f} |p(\tau)||\dot{q}(\tau)|\;d\tau \leq C' e^{(-\alpha/2-\varepsilon)T} \int_{\tau_0}^{\tau_f} \rho\;d\tau = C'' e^{-2\varepsilon T}.$$

Since equation (5.20) allows for any T and $\int_{\eta_T} p\,dq \to 0$ as $T \to +\infty$, we can conclude that

$$\int_0^\infty e^{-\alpha t} \varphi(q(t), \dot{q}(t))\;dt \geq \int_0^\infty e^{-\alpha t} \varphi(\tilde{q}(t), \dot{\tilde{q}}(t))\;dt;$$

the proof that the inequality is strict is completely analogous to the one in [**7**]. □

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