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NONCOMMUTATIVE GEOMETRY AND
QUANTUM GROUP SYMMETRIES

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Preface

This dissertation is based on research done at the Mathematical Physics sector of the International School for Advanced Studies of Trieste, during the period from October 2003 to June 2007. It is divided into two parts: the first part (Chapters 1-2) is an account of the general theory and a collection of some general notions and results; the second part (Chapters 3-5) contains the original work, carried out under the supervision of Prof. Ludwik Dąbrowski and Prof. Giovanni Landi. Part of the original material presented here has been published or submitted as a preprint in the following papers:

- *Local Index Formula on the Equatorial Podleś Sphere*,
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- *The Isospectral Dirac Operator on the 4-dimensional Orthogonal Quantum Sphere*,
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Notations

- \mathbb{N} natural numbers including 0.
- \mathbb{Z} integer numbers.
- $\mathcal{S}(\mathbb{Z})$ rapid decay sequences: $\{a_n\}_{n \in \mathbb{Z}} \in \mathcal{S}(\mathbb{Z})$ iff $\lim_{n \rightarrow \pm\infty} |n|^k a_n = 0 \forall k \in \mathbb{N}$.
- \mathcal{S} rapid decay matrices: $((a_{j,k}))_{j,k \in \mathbb{N}} \in \mathcal{S}$ iff $\lim_{j,k \rightarrow \infty} j^m k^n a_{j,k} = 0 \forall n, m \in \mathbb{N}$.
- $[x]$ q -analogue of $x \in \mathbb{C}$, given by $[x] := (q^x - q^{-x}) / (q - q^{-1})$.
- $[n]!$ q -factorial of $n \in \mathbb{N}$, $[n]! := [1][2][3] \cdots [n]$ if $n \geq 1$, $[0]! := 1$.
- $[n]!!$ q -double factorial of $n \in \mathbb{N} - 1$, given by $[n]!! := [1][3][5] \cdots [n]$ if $n > 0$ is odd, $[n]!! = [2][4][6] \cdots [n]$ if $n > 0$ is even, $[0]!! := 1$ and $[-1]!! := 1$.
- $\begin{bmatrix} n \\ m \end{bmatrix}$ q -binomial of $n, m \in \mathbb{N}$, $\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[m]![n-m]!}$.
- $\dot{\otimes}$ the dotted tensor product of two matrices $A = ((A_{ij}))$ and $B = ((B_{ij}))$ is the matrix $A \dot{\otimes} B$ with entries $(A \dot{\otimes} B)_{ij} = \sum_k A_{ik} \otimes B_{kj}$.
- *-algebra* = associative unital involutive algebra over \mathbb{C} .
- $\sum_{n \in \emptyset}$ an empty sum will always mean 0.
- $\prod_{n \in \emptyset}$ an empty product will always mean 1.

Landau's 'big-O' and 'little-o' notations

- $a_n = o(b_n)$ whenever $\{a_n\}$ and $\{b_n\}$ are two sequences and $\lim_{n \rightarrow \infty} a_n/b_n = 0$.
- $a_n = O(b_n)$ whenever $\{a_n\}$ and $\{b_n\}$ are two sequences and there is a constant $C > 0$ such that $|a_n| \leq C|b_n|$.
- $a_n \lesssim b_n$ another notation for $a_n = O(b_n)$.

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Chapter 1

Prologue

“... it seems that the empirical notions on which the metrical determinations of space are founded [...] cease to be valid for the infinitely small.”,
Bernhard Riemann (1826–1866) [Rie73].

It is a widespread belief that mathematics originates from the desire to understand (and eventually to formalize) some aspects of the real world. Quoting [Man07], \ll we are doing mathematics in order to understand, create, and handle things, and perhaps *this understanding* is mathematics \gg . Let me thus begin with a brief discussion of the physical ideas that motivated the development of Noncommutative Geometry and Quantum Group Theory — the areas of mathematics to which this dissertation belongs. Some physicists believe, and Einstein himself expressed this view in [Ein98a], that physics progresses in stages: there is no ‘final’ theory of Nature, but simply a sequence of theories which provide more and more accurate descriptions of the real world. This idea was made more precise [BFF⁺77, BFF⁺78a, BFF⁺78b], by saying that one passes from one stage to the next by a deformation, the meaning of which is explained for example in [Fla82] as follows: \ll Mathematically one starts with an algebraic structure which is e.g. a Lie algebra or an associative algebra and asks the question: does there exist a 1 (or n) parameter family of similar structures such that for an initial value (say zero) of the parameter we get the structure we started with? If such a field of structures exists, we call it a deformation of the original structure. \gg In this way, one can think of special relativity as a deformation of classical mechanics with deformation parameter the speed of light c : in a range of velocities much smaller than c , which corresponds *formally* to an initial value $c^{-1} = 0$ (although we recall that c is a universal constant, not a parameter) the Lorentz transformations reduce to the Galilean ones and special relativity reduces to classical mechanics. Similarly, quantum mechanics can be viewed as a deformation of classical mechanics in the framework of associative algebras, with deformation parameter the reduced Planck constant \hbar [BFF⁺77]. A typical

example is the quantization of a one-dimensional system: in the classical picture, observables are smooth real functions of the coordinate x and momentum p and the dynamic is dictated by the Poisson structure defined by the rule

$$\{x, p\} = 1$$

and by a preferred function called the Hamiltonian, depending on the system under consideration and controlling the time evolution. In the quantum picture, functions are replaced by elements of an algebra [BJ25, BHJ26] generated by two real elements \hat{x} and \hat{p} subject to the relation

$$[\hat{x}, \hat{p}] := \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar, \quad (1.1)$$

and the crucial property of this algebra is that its elements do not commute, unless ‘ $\hbar = 0$ ’ which formally corresponds to the classical limit. The phase space of quantum mechanics is the archetype of a noncommutative space: a ‘space’ which is no longer described as a set of points with some additional structure, but by a noncommutative algebra replacing (or better, deforming) the coordinate algebra of a manifold.

Of course, the deformations which give rise to special relativity and quantum mechanics do not “fall from heaven”. The development of special relativity originated with Einstein’s thought (or “gedanken”) experiment, which derived the main properties of the theory (time dilation/length contraction) from the assumption that the speed of light is finite and has the same value in every inertial frame (cf. [Ein98b] or Chapter IX and X of [Ein85]). On the other hand, one of the motivations behind the formalism of quantum mechanics (say, in the Heisenberg formulation) is Heisenberg’s gedanken experiment (see e.g. Chapter 2 of [Hei30]), which imposes limits on the localizability of a particle in phase space. This gives rise to the celebrated Heisenberg uncertainty relations, first presented in [Hei27], which are implemented mathematically by the assumption that position and momentum belong to a noncommutative algebra like (1.1). In the same way, a refined version of Heisenberg’s gedanken experiment, suitably generalized to include the effects of gravity, imposes limitations upon the localizability of a particle [SW58, Mea64, Pad85, Pad87] and suggests that at very small length scales the notion of space as a set of points is inappropriate and that space–time coordinates should be replaced by noncommuting operators [DFR94, DFR95, AC97].

While the physical ideas which led to the study of noncommutative space–times seem to be still in an embryonic stage (due to a lack of an experimental evidence which could support or falsify such ideas), the mathematics behind these ideas is now a well–established independent field of research in its own right. The geometry of noncommutative spaces has been intensively studied during recent decades by Connes and many others, and most of the tools of differential geometry have been generalized to the noncommutative framework. Until now, I have given only a vague idea of what a noncommutative space is, and one might ask for an exact definition. In fact, there is no general consensus on the meaning of these words, and often it depends on

the context. For example, the equivalence between the category of locally compact Hausdorff topological spaces and the category of commutative C^* -algebras (via the Gelfand–Naimark theorem) suggests the idea that the theory of C^* -algebras should be regarded as noncommutative generalization of topology. Similarly, from the work of Connes on von Neumann algebras, it comes the habit to consider them as the noncommutative analogue of measure spaces. Spectral triples (discussed in Chapter 2) replace Riemannian manifolds, and real spectral triples [Con95] are the algebraic counterpart of spin manifolds. Now as with any theory, one has to test the postulates of noncommutative geometry by considering a certain number of examples. This is the purpose of the research described in this dissertation, and as a source of examples, we focus on Quantum Group Theory.

What then is a ‘quantum group’? In the middle of the ‘80s, Drinfeld proposed a program of quantization of Poisson–Lie groups [Dri86]. Quantized Poisson–Lie groups, or simply *quantum groups*, are deformations of Poisson–Lie groups in the intuitive sense explained above (which we will make more precise in Chapter 2.11). They provide a large class of examples of algebras which for many reasons we would like to interpret as ‘coordinate algebras’ over noncommutative spaces (with some additional structure). One interesting feature of these examples is that it seems very difficult (if not impossible) to fulfill all the axioms of a real spectral triple, at least in their original formulation. Thus, a natural question is: are quantum groups ‘noncommutative spin manifolds’? In other words: do they satisfy the conditions needed for a real spectral triple, either in the original or in some modified form? The aim of this work is to shed some light on these issues.

The plan of the dissertation is the following. Chapter 2 is a survey of the basic notions of Noncommutative Geometry which are needed in order to discuss the examples presented in the remaining part of the dissertation. In Chapter 3 we study the family of noncommutative 2-spheres introduced by Podleś in [Pod87] as quantum homogeneous spaces for Woronowicz’ quantum $SU(2)$ group [Wor87]. In Chapter 4 we study the 4-dimensional quantum orthogonal sphere of [RTF90]. Finally in Chapter 5 we generalize some of these results to the whole family of odd-dimensional quantum spheres [VS91].

Throughout this thesis, unless otherwise stated, the term **-algebras* is used to refer to complex involutive unital algebras, and the symbol q indicates a fixed parameter in the open interval $]0, 1[$.

Chapter 2

Noncommutative Geometry and Quantum Groups

“The correspondence between geometric spaces and commutative algebras is a familiar and basic idea of algebraic geometry. The purpose of this book is to extend the correspondence to the noncommutative case in the framework of real analysis.”, Alain Connes [Con94].

As in many theoretical physics’ stories, all begins with the old dream of unifying relativity with quantum mechanics. In an attempt to find a relativistic version of Schrödinger equation, which for a free particle in 3 dimensions is

$$(i\partial_t + \sum_{i=1}^3 \partial_{x_i}^2)\psi = 0, \quad (2.1)$$

physicists noticed that the differential operator $i\partial_t + \sum_{i=1}^3 \partial_{x_i}^2$ does not commute with the natural action of the Poincaré Lie algebra on $C^\infty(\mathbb{R}^{1,3})$ via vector fields, that is: the operator is not *Poincaré invariant*. As covariance is one of the guiding principles of relativity, it became clear that a relativistic version of (2.1) should have involved a Poincaré-invariant differential operator. Klein-Gordon proposal¹ was to use the *Laplace–Beltrami operator*, which on $\mathbb{R}^{1,3}$ is

$$\Delta = -\partial_t^2 + \sum_{i=1}^3 \partial_{x_i}^2,$$

and write the relativistic version of (2.1) as $\Delta\psi = m\psi$, for a free particle of mass m . Since Δ is not of first order in the time derivative, the knowledge of a solution ψ of Klein-Gordon equation at $t = 0$ does not completely determine its time evolution. Motivated by this observation, Dirac’s

¹See [Kra44] for a historical review; however, it seems to me that Klein-Gordon equation appeared for the first time in 1923 in [Gor23] and not in 1926 as claimed in [Kra44].

idea was to search for a first order differential operator \mathcal{D} acting on vector valued functions and with square $\mathcal{D}^2 = \Delta$; the solution for $\mathbb{R}^{1,3}$ first appeared in 1928 in [Dir28a, Dir28b]. The operator \mathcal{D} , which is called *Dirac operator*, is given on $\mathbb{R}^{1,3}$ by

$$\mathcal{D} = \gamma_0 \frac{\partial}{\partial t} + \sum_{i=1}^3 \gamma_i \frac{\partial}{\partial x_i},$$

with γ_μ the well known Dirac's gamma matrices. As everybody knows Dirac equation $\mathcal{D}\psi = m\psi$ provides a description of elementary particles with spin 1/2 (and mass m), such as electrons, which is consistent with both the principles of quantum mechanics and special relativity.

We stress that the demand of symmetries played a crucial role in Dirac's construction; we will discuss the role of symmetries in noncommutative geometry in Sections 2.10–2.11. Another important point is the quest for a first order differential operator: a square root $\Delta^{1/2}$ of the Laplacian is well defined (at least in the Riemannian case), but in general is not a differential operator. We'll return on this when discussing the 'first order condition'. A difference between $\Delta^{1/2}$ and \mathcal{D} is that the former acts on scalar functions, the latter on vector valued functions (in general, on sections of the spinor bundle, or simply 'spinors') whose components are interpreted as probability amplitudes for a couple of spin 1/2 particle–antiparticle; the prediction of the existence of antiparticles, confirmed in 1933 by the experimental discovery of the positron (the antiparticle of the electron), was one of the greatest triumphs of modern theoretical physics and decreed the success of the Dirac operator in physics.

Although the importance of the Dirac operator was immediately recognized in physics, its importance in mathematics became clear much later, with Atiyah–Singer local index formula. The generalization of the Dirac operator to 'noncommutative spaces' is in part motivated by its applications to index theory, which we briefly recall in the following.

Consider an even–dimensional compact Riemannian spin manifold M with fixed spin structure, let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ the Hilbert space of L^2 -spinors and \mathcal{D} the Dirac operator on \mathcal{H} . Let γ be the natural grading operator on \mathcal{H} and $\mathcal{D}^+ := \frac{1}{4}(1-\gamma)\mathcal{D}(1+\gamma)$. If $V \rightarrow M$ is any smooth vector bundle and \mathcal{D}_V^+ a lift of \mathcal{D}^+ to V , the map sending the isomorphism class of V to the index of \mathcal{D}_V^+ gives a group homomorphism from $K^0(M) \simeq K_0(C^\infty(M))$ to the integers \mathbb{Z} . In the odd–dimensional case a similar construction gives a homomorphism from $K^1(M) \simeq K_1(C^\infty(M))$ to \mathbb{Z} . The natural generalization of this 'index map' $K_\bullet(C^\infty(M)) \rightarrow \mathbb{Z}$ to the K -groups of a generic complex involutive unital algebra \mathcal{A} ('*-algebra', for short) is possible if there exists a pre–Fredholm module over it.

Definition 2.1. *A pre–Fredholm module over a *-algebra \mathcal{A} is the data (\mathcal{H}, π, F) of a Hilbert space \mathcal{H} , a *-representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a bounded operator F on \mathcal{H} such that $F = F^*$, $F^2 = 1$ and $[F, \pi(a)]$, for all $a \in \mathcal{A}$, are compact operators. The pre–Fredholm module is even, or also 'graded', if there exists a grading operator γ on \mathcal{H} (i.e. a bounded operator satisfying $\gamma = \gamma^*$, $\gamma^2 = 1$) commuting with $\pi(a)$ for any $a \in \mathcal{A}$ and anticommuting with F . Otherwise it is odd. A Fredholm module is a pre–Fredholm module satisfying $F = F^*$ and $F^2 = 1$.*

Usually pre-Fredholm modules are called Fredholm module ‘tout court’, but since we need to distinguish between the two cases in which $F - F^*$ and $F^2 - 1$ are zero or just compact, we follow the terminology of [GBVF01] and use the prefix *pre* in the latter case.

Equivalence classes of pre-Fredholm modules form a homology theory called K -homology and dual to K -theory [Kas75]. Given an even pre-Fredholm module over \mathcal{A} , the index map $K_0(\mathcal{A}) \rightarrow \mathbb{Z}$ is constructed as follows. A class $[e] \in K_0(\mathcal{A})$ is represented by an idempotent $e = e^2 \in \text{Mat}_N(\mathcal{A})$ for some $N \in \mathbb{N}$; we can lift in the obvious way the $*$ -representation π to a $*$ -representation $\tilde{\pi}$ of $\text{Mat}_N(\mathcal{A})$ over $\mathcal{H} \otimes \mathbb{C}^N$ and form the Fredholm operator $F_e^+ : \mathcal{H}_+ \otimes \mathbb{C}^N \rightarrow \mathcal{H}_- \otimes \mathbb{C}^N$, with $\mathcal{H}_\pm := (1 \pm \gamma)\mathcal{H}$, defined by

$$F_e^+ := \frac{1}{4}(1 - \gamma)\tilde{\pi}(e)F\tilde{\pi}(e)(1 + \gamma). \quad (2.2)$$

The index map is just $[e] \mapsto \text{Index}(F_e^+)$, and gives a pairing between K -theory and K -homology.

To perform computations it is crucial that $F - F^*$ and $F^2 - 1$ are zero (not just compact) and that the Fredholm module is finite-summable: we shall assume that this is always the case. A Fredholm module is *p-summable* if $p \geq 1$ and $[F, \pi(a)]$ is in the p -th Schatten-von Neumann ideal $\mathcal{L}^p(\mathcal{H})$ for all $a \in \mathcal{A}$ (i.e. $[F, \pi(a)]^p$ is of trace class), and in this case the index of F_e^+ can be evaluated through the formula [Con85]

$$\text{Index}(F_e^+) = \frac{1}{2}(-1)^m \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^N}(\gamma F [F, \tilde{\pi}(e)]^{2m+1}) \quad (2.3)$$

for any $m \in \mathbb{N}$ such that $2m + 1 \geq p$.

The canonical example is the Fredholm module associated with a spin structure by taking $F := \mathcal{D}|\mathcal{D}|^{-1}$, the sign of the Dirac operator \mathcal{D} (which we assume for simplicity to be invertible). The fact that in this case F comes from an unbounded operator \mathcal{D} is what allows to express the index (2.3) through local formulæ, which considerably simplify the computations. This motivates, in the noncommutative case, the passage from Fredholm modules to unbounded Fredholm modules (to the best of my knowledge, this name was first used in [Con89]), later called *spectral triples* [Con95].

Definition 2.2. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over a $*$ -algebra \mathcal{A} consists of

- (i) a Hilbert space \mathcal{H} with a $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$;
- (ii) an unbounded selfadjoint operator D with dense domain in \mathcal{H} such that
 - (1) $D + i$ has a compact inverse;
 - (2) $[D, \pi(a)]$ extends to a bounded operator on \mathcal{H} for all $a \in \mathcal{A}$.

The spectral triple is even if there exists a grading operator γ on \mathcal{H} such that $D\gamma = -\gamma D$ and $\pi(a)\gamma = \gamma\pi(a)$ for all $a \in \mathcal{A}$.

The reason for the name ‘unbounded Fredholm module’ is that the data (\mathcal{H}, π, F) , with $F := D(1 + D^2)^{-1/2}$, give a pre-Fredholm module over \mathcal{A} . In fact, it was shown in [BJ83] (although they referred to them by a different name), that any pre-Fredholm module class can be constructed in this way.

The canonical example of spectral triple comes from the spin structure over a spin manifold, where $D = \mathcal{D}$ is just the Dirac operator; Definition 2.2 is the starting point for a noncommutative generalization of the theory of spin structures, which is usually called ‘noncommutative differential geometry’ [Con94]. With the additional conditions of finite metric dimension and ‘regularity’ (which will be recalled later) it is possible to develop an abstract theory of pseudo-differential operators, to give meaning to the word ‘local’ in the noncommutative setting, and to write a local formula for the index (2.3) which generalizes Atiyah-Singer index formula [CM95].

In the remaining part of this chapter we recall some general definitions of Connes’ noncommutative differential geometry, following [Con94], and collect some theorems which we shall need later. In Section 2.1 we describe how to compute the index (2.3) using an unbounded Fredholm operator. In Section 2.2 we recall some analytic properties of spectral triples which are needed in Section 2.3 to discuss local index formulæ in noncommutative geometry. In Section 2.4 we analyze in detail some ideals which will be used in the sequel. Section 2.5 is devoted to the noncommutative analogue of smooth functions. In Section 2.6 we collect some considerations about the ‘reality’ and ‘first order’ axioms. Section 2.7 is a digression on the topological dimension of quantum spaces. Section 2.8–2.9 contain some results which are used in the examples. Finally, in Sections 2.10 and 2.11 we discuss noncommutative spaces whose symmetries are described by Hopf algebras, and introduce the examples which will be studied in the subsequent chapters.

2.1 Connection theory and index computations

As discussed before, in the commutative case one constructs the index map by lifting D to vector bundles. We now explain the exact meaning of this statement by considering a more general (possibly noncommutative) case, described by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. We first need some preliminary notions about differential calculi.

To any spectral triple is associated a representation of the *universal differential calculus* (UDC) of \mathcal{A} (cf. Section III.1 of [Con94]). The UDC (Ω^\bullet, d) over \mathcal{A} is the data of the graded algebra Ω^\bullet generated by degree 0 elements $a \in \mathcal{A}$ and degree 1 elements da , $a \in \mathcal{A}$, with $d : \mathcal{A} \rightarrow \Omega^1$ linear and satisfying the relation

$$d(ab) = (da)b + a(db) \quad \forall a, b \in \mathcal{A} .$$

The involution of \mathcal{A} extends to an involution on Ω^\bullet via the rule $(da)^* = -d(a^*)$, while the

differential d on Ω^\bullet can be defined unambiguously as

$$d(a_0 da_1 \dots da_n) = da_0 da_1 \dots da_n$$

and satisfies $d^2 = 0$ and the graded Leibniz rule

$$d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^p \omega_1 (d\omega_2)$$

for $\omega_i \in \Omega$ and with p the degree of ω_1 .

Any other differential calculus can be constructed as a quotient of the universal differential calculus by a differential ideal. In particular, if $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple the representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ can be extended to an algebra morphism $\pi_D : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ as follows

$$\pi_D(a_0 da_1 \dots da_n) = \pi(a_0)[D, \pi(a_1)] \dots [D, \pi(a_n)].$$

We denote $d_D a := \pi_D(da) = [D, \pi(a)]$.

This is what is usually called *Connes' differential calculus*. Quotienting $\pi_D \Omega^\bullet$ by the differential ideal $J^\bullet := \pi_D(d \ker \pi_D)$, called ‘Junk ideal’ (cf. [Con94]), one gets a differential calculus (Ω_D^\bullet, d_D) which in the commutative case coincides with de Rham calculus. In particular there are no junk 1-forms, since the Connes' differential of any $a \in \ker \pi$ is zero; hence $\Omega_D^1 = \pi_D \Omega^1$.

The next notion we need is the analogue of Hermitian vector bundles. If M is a compact C^∞ -manifold, then by Serre-Swan theorem there is a functorial correspondence between smooth vector bundles over M and finitely generated projective $C^\infty(M)$ -modules. The latter replace vector bundles in the algebraic setting. We recall that any finitely generated (right) projective \mathcal{A} -module \mathcal{E} is of the form

$$\mathcal{E} = e\mathcal{A}^N := \left\{ \xi = (\xi_1, \dots, \xi_N) \in \mathcal{A}^N \mid e\xi := \xi \text{ i.e. } \sum_{j=1}^N e_{ij}\xi_j = \xi_i \forall i = 1, \dots, N \right\}$$

for $e = e^2 \in \text{Mat}_N(\mathcal{A})$ an idempotent and for a suitable $N \in \mathbb{N}$. The group $K_0(\mathcal{A})$ is used to attach an invariant to any finitely generated projective \mathcal{A} -module, i.e. to any idempotent $e \in \bigcup_{N \in \mathbb{N}} \text{Mat}_N(\mathcal{A})$.

An *Hermitian structure* on \mathcal{E} is a sesquilinear map $(,) : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ satisfying $(\eta a, \xi b) = a^*(\eta, \xi)b$ and $(\eta, \eta) \geq 0$, for all $\eta, \xi \in \mathcal{E}$ and $a, b \in \mathcal{A}$, and such that \mathcal{E} is selfdual with respect to $(,)$.² Hermitian structures exist on any finitely generated projective module \mathcal{E} , and when $e = e^*$ is a projection (this can be always assumed, see e.g. pages 88–89 of [GBVF01]) they are all isomorphic to the one obtained by restricting to \mathcal{E} the Hermitian structure on \mathcal{A}^N given by

$$(\eta, \xi) = \sum_{i=1}^N \eta_i^* \xi_i$$

for $\eta = (\eta_1, \dots, \eta_N)$ and $\xi = (\xi_1, \dots, \xi_N) \in \mathcal{A}^N$. A *connection* on \mathcal{E} is a linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ which satisfies the Leibniz rule

$$\nabla(\eta a) = (\nabla \eta)a + \eta \otimes d_D a$$

²This is sometimes called a *non-degenerate* Hermitian structure, see e.g. Section 4.3 of [Lan02].

and the condition

$$(\nabla\eta, \xi) - (\eta, \nabla\xi) = d_D(\eta, \xi)$$

for all $\eta, \xi \in \mathcal{E}$ and $a \in \mathcal{A}$. Here if $\eta_{1,2} \in \mathcal{E}$ and $\omega_{1,2} \in \Omega_D^1$ we define

$$(\eta_1 \otimes \omega_1, \eta_2 \otimes \omega_2) := \omega_1^*(\eta_1, \eta_2)\omega_2 \in \Omega_D^1 .$$

On any \mathcal{E} there always exists a connection ∇_e called *Grassmannian connection*, given by

$$\nabla_e \eta = ed_D \eta \quad \text{for } \eta = (\eta_1, \dots, \eta_N) \in e\mathcal{A}^N \text{ and } d_D \eta = (d_D \eta_1, \dots, d_D \eta_N) .$$

Any other connection differs from ∇_e by an element in $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1)$.

By completing $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ with respect to the inner product

$$\langle \eta_1 \otimes v_1, \eta_2 \otimes v_2 \rangle := \langle v_1, (\eta_1, \eta_2)v_2 \rangle$$

with $\eta_i \in \mathcal{E}$, $v_i \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle$ the inner product of \mathcal{H} , we get (eventually quotienting out the norm zero tensors, if the Hermitian structure is degenerate) a Hilbert space $\mathcal{H}_{\mathcal{E}}$ canonically isomorphic to $e(\mathbb{C}^N \otimes \mathcal{H})$. The lift $D_{\mathcal{E}, \nabla}$ of D to $\mathcal{H}_{\mathcal{E}}$ is defined by

$$D_{\mathcal{E}, \nabla} := 1 \otimes D + \nabla \otimes 1 ,$$

and gives a spectral triple $(\mathcal{A}, \mathcal{H}_{\mathcal{E}}, D_{\mathcal{E}, \nabla})$. The operator $D_{\mathcal{E}, \nabla}$ is an unbounded Fredholm operator and, in the even-dimensional case, a group homomorphism $\text{Ind}_D : K_0(\mathcal{A}) \rightarrow \mathbb{Z}$, called simply *index map*, is defined by

$$\text{Ind}_D([e]) = \text{Index}(D_{\mathcal{E}, \nabla}^+) , \tag{2.4}$$

with $D_{\mathcal{E}, \nabla}^+ := \frac{1}{2}(1 + \gamma)D_{\mathcal{E}, \nabla}(1 - \gamma)$ and γ the grading operator. The right hand side of (2.4) is independent of the particular connection chosen, and depends on \mathcal{E} only through its equivalence class in $K_0(\mathcal{A})$. Using the Grassmannian connection ∇_e one proves that (for a Lipschitz-regular spectral triple, cf. [Mos97])

$$\text{Ind}_D([e]) = \text{Index}(F_e^+) ,$$

where $F = D|D|^{-1}$ and F_e^+ is given by (2.2).

2.2 Analytic properties of spectral triples

The next task is to express (2.3) as a pairing between a homology theory and the dual cohomology. Before this, we need to recall some properties of spectral triples. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, with D invertible. We also assume that π is faithful (otherwise we can replace \mathcal{A} by $\mathcal{A}/\ker \pi$), we shall identify \mathcal{A} with $\pi(\mathcal{A})$ and omit the symbol π when there is no risk of confusion.

Such a spectral triple is called *Lipschitz-regular* if $[[D], a]$ is bounded for all $a \in \mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D)$ is a Lipschitz-regular spectral triple and $F := D|D|^{-1}$, by construction

$$[F, a] = ([D], a] - F[[D], a])|D|^{-1}$$

is compact for any $a \in \mathcal{A}$, being the product of a bounded operator with the compact operator $|D|^{-1}$. Hence $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module. On the other hand the condition that $[F, a]$ is compact for all $a \in \mathcal{A}$ does not guarantee that a spectral triple is Lipschitz-regular; by

$$[[D], a] = [F, a]D + F[D], a]$$

the spectral triple is Lipschitz-regular, i.e. $[[D], a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$, if and only if the operators $[F, a]|D|$ are bounded. This happens when $[F, a]$ are operators of ‘order -1 ’, a stronger condition than compactness, the meaning of which we explain below.

For each $s \in \mathbb{R}$ we let \mathcal{H}^s be the domain of $|D|^s$ (notice that $\mathcal{H}^s \subset \mathcal{H}^{s'}$ for $s \geq s'$) and $\mathcal{H}^\infty = \bigcap_{s \geq 0} \mathcal{H}^s$. The space \mathcal{H}^s is closed with respect to the norm (on the right hand side there is the norm of \mathcal{H})

$$\|v\|_s^2 = \|v\|^2 + \|D^s v\|^2.$$

In the canonical commutative example, \mathcal{H}^s are Sobolev spaces and \mathcal{H}^∞ smooth sections of the spinor bundle. For each $r \in \mathbb{R}$ a class of operators op^r is defined as the subset of (unbounded) operators with domain containing \mathcal{H}^∞ which extend to continuous operators $\mathcal{H}^s \rightarrow \mathcal{H}^{s-r}$ for all $s \in \mathbb{R}$. We have $\text{op}^r \subset \text{op}^{r'}$ for each $r \leq r'$. As the class op^r is difficult to deal with, a smaller class denoted OP^r was introduced in [CM95] as follows. Let δ be the unbounded derivation on $\mathcal{B}(\mathcal{H})$ given by

$$\delta(T) := [[D], T],$$

for all $T \in \mathcal{B}(\mathcal{H})$. Then, *order zero* operators are defined as the elements of the class

$$\text{OP}^0 := \bigcap_{j \in \mathbb{N}} \text{dom } \delta^j,$$

which is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. We remark that by definition, $T \in \text{dom } \delta$ if (i) $T \in \mathcal{B}(\mathcal{H})$, (ii) $T \cdot \text{dom } |D| \subset \text{dom } |D|$ (a necessary condition for $\delta(T) = |D|T - T|D|$ to be defined on $\text{dom } |D| = \mathcal{H}^1$), (iii) $\delta(T)$, which is *a priori* defined on $\text{dom } |D|$, extends to a bounded operator on \mathcal{H} . By iterating, $T \in \text{dom } \delta^j$ iff $T \in \text{dom } \delta^{j-1}$ and $\delta^{j-1}(T) \in \text{dom } \delta$.

Operators of *order* r , denoted by OP^r , are defined by

$$\text{OP}^r := \{T \in \mathcal{B}(\mathcal{H}) \mid |D|^{-r}T \in \text{OP}^0\}.$$

One has $\text{OP}^r \subset \text{op}^r$ for any r , and $\text{OP}^r \subset \text{OP}^{r'}$ for all $r' \geq r$. Moreover if $r \leq 0$, then OP^r is a $*$ -algebra (of bounded operators) and a two-sided $*$ -ideal in $\text{OP}^{r'}$, for all $r \leq r' \leq 0$ (cf. Appendix B of [CM95]). These classes of operators are the starting point of an abstract theory of pseudo-differential operators, which is beyond the scope of this introductory chapter.

For a spectral triple, a stronger requirement than Lipschitz-regularity is the ‘regularity’ condition [CM95]. Classically, while Lipschitz-regular elements of $C(M)$ are Lipschitz-class functions, regular elements are C^∞ functions. A spectral triple is called *regular*, or QC^∞ , if the following set inclusion holds,

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \text{OP}^0,$$

where $[D, \mathcal{A}] = \{[D, a] \mid a \in \mathcal{A}\}$. For a regular spectral triple, the algebra Ψ^0 of ‘pseudo-differential operators’ of order (less than or equal to) zero is defined as the algebra generated by $\bigcup_{k \in \mathbb{N}} \delta^k(\mathcal{A} \cup [D, \mathcal{A}])$.

Recall that, with $n \in \mathbb{R}^+$, D is called n^+ -*summable* if the operator $|D|^{-n}$ is in the Dixmier ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$, and n is called the *metric dimension* of the spectral triple if D is n^+ -summable and $\text{Tr}_\omega(|D|^{-n}) > 0$ (where Tr_ω is the Dixmier trace, which we recall later)³. If a spectral triple is regular and n -dimensional, the ‘zeta-type’ function

$$\zeta_T(s) := \text{Tr}_{\mathcal{H}}(T|D|^{-s})$$

associated to $T \in \Psi^0$ is defined (and holomorphic) for $s \in \mathbb{C}$ with $\text{Re } s > n$ and the following definition makes sense. If it happens that

- (i) for all $T \in \Psi^0$ the function $\zeta_T(s)$ has a meromorphic extension to \mathbb{C} whose only singularities are poles,
- (ii) the union of such singularities is a countable set $\Sigma \subset \mathbb{C}$,

then we call Σ the *dimension spectrum* of the spectral triple. The meaning of Σ is that its points are associated with states on the algebra. In particular if Σ is made only of simple poles, one can prove that the Wodzicki-type residue functionals

$$\tau_m(T) = \oint T|D|^{-m} := \text{Res}_{s=m} \text{Tr}(T|D|^{-s}) \quad (2.5)$$

are defined on Ψ^0 for all $m \in \Sigma$, and tracial on suitable subsets. For example, the top residue τ_n is tracial on all Ψ^0 and there coincides with the Dixmier trace (cf. Proposition 4.2.4 of [Con94]).

As in ordinary pseudodifferential calculus, computations can be simplified by neglecting *smoothing operators*, which in this abstract setting are defined as the elements of the following class

$$\text{OP}^{-\infty} := \bigcap_{r \geq 0} \text{OP}^{-r} = \{T \in \mathcal{B}(\mathcal{H}) \mid |D|^k T \in \text{OP}^0 \forall k \in \mathbb{N}\}.$$

³This definition is reminiscent of Hausdorff dimension in fractal geometry [Fal52]. In that case one has a one-parameter family of measures μ^s , $s \in \mathbb{R}^+$, and the dimension of a fractal set \mathcal{F} is the real number $\dim \mathcal{F}$ separating the open interval $\{s \in \mathbb{R}^+ \mid \mu^s(\mathcal{F}) = 0\}$ from the open interval $\{s \in \mathbb{R}^+ \mid \mu^s(\mathcal{F}) = +\infty\}$: that is $\mu^s(\mathcal{F}) = 0$ for $s < \dim \mathcal{F}$ and $\mu^s(\mathcal{F}) = +\infty$ for $s > \dim \mathcal{F}$. The metric dimension is defined analogously, with $\text{Tr}_\omega(|D|^{-s})$ replacing $\mu^s(\mathcal{F})$.

The class $\text{OP}^{-\infty}$ is a two-sided $*$ -ideal in the $*$ -algebra OP^0 (actually, in any OP^{-r} with $r \geq 0$). Moreover, if T is a smoothing operator, $\zeta_T(s)$ is holomorphic on \mathbb{C} and the ‘integral’ (2.5) vanishes. In this spirit, we say that a multilinear functional on Ψ^0 is *local* if it is insensitive to smoothing perturbations of its arguments; in particular, residues of zeta-functions are local expressions. The general philosophy is that locality makes complicated expressions computable, by ignoring irrelevant details.

2.3 Spectral triples and homology theories

Let (\mathcal{H}, F) be a p -summable Fredholm module over a $*$ -algebra \mathcal{A} , and $n = 2m \geq p - 1$ an even integer. We can get (2.3) in two steps; first we call

$$\text{ch}_n^F(a_0, \dots, a_n) = \frac{(n/2)!}{2 \cdot n!} \text{Tr}(\gamma F[F, a_0] \dots [F, a_n]), \quad a_i \in \mathcal{A}, \quad (2.6)$$

$$\text{ch}^n(e) = \sum_{k_0, k_1, \dots, k_n} (e_{k_0 k_1} - \frac{1}{2} c \delta_{k_0 k_1}) \otimes e_{k_1 k_2} \otimes \dots \otimes e_{k_n k_0}, \quad (2.7)$$

(here $c \in \mathbb{C}$ is arbitrary, as $[F, c] = 0$) and then write

$$\text{Index}(F_e^+) = \langle \text{ch}_n^F, \text{ch}^n(e) \rangle \quad (2.8)$$

where

$$\langle \tau, a_0 \otimes \dots \otimes a_n \rangle := (-1)^{n/2} \frac{n!}{(n/2)!} \tau(a_0, \dots, a_n) \quad (2.9)$$

for any multilinear functional $\tau : \mathcal{A}^{n+1} \rightarrow \mathbb{C}$ and any $a_0 \otimes \dots \otimes a_n \in \mathcal{A}^{n+1}$. The advantage of writing (2.3) in this way is that (2.6) is a cyclic cocycle, (2.7) is a cyclic cycle, and (2.9) is a pairing between the associated cohomology/homology groups. The element (2.7) is usually called Chern-Connes character.

We recall [Lod97] that Hochschild n -chains $C_n(\mathcal{A})$ are elements of \mathcal{A}^{n+1} , Hochschild n -cochains $C^n(\mathcal{A})$ are multilinear functionals $\mathcal{A}^{n+1} \rightarrow \mathbb{C}$, the Hochschild boundary $b : C_{n+1}(\mathcal{A}) \rightarrow C_n(\mathcal{A})$ is given by

$$\begin{aligned} b(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1}) &:= b'(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1}) \\ &\quad + (-1)^{n+1} a_{n+1} a_0 \otimes a_1 \otimes \dots \otimes a_n, \end{aligned}$$

with

$$b'(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1}) := \sum_{j=0}^n (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{n+1},$$

and the Hochschild coboundary is its pullback $b^* : C^n(\mathcal{A}) \rightarrow C^{n+1}(\mathcal{A})$, given by

$$\begin{aligned} (b^* \tau)(a_0, \dots, a_{n+1}) &= \sum_{j=0}^n (-1)^j \tau(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \tau(a_{n+1} a_0, a_1, \dots, a_n). \end{aligned}$$

The *Hochschild homology* $HH_\bullet(\mathcal{A})$ is defined as the homology of the complex $\{C_n(\mathcal{A}), b\}_{n \geq 0}$, while the dual *Hochschild cohomology* $HH^\bullet(\mathcal{A})$ is defined as the cohomology of the complex $\{C^n(\mathcal{A}), b^*\}_{n \geq 0}$. If $\mathcal{A} = \mathcal{A}(X)$, with X a smooth affine variety, by Hochschild-Kostant-Rosenberg theorem (generalized by Connes to smooth manifolds in [Con95]) there is an isomorphism of $\mathcal{A}(X)$ -modules $HH_n(\mathcal{A}(X)) \xrightarrow{\sim} \Omega_{\text{deRham}}^n(X)$, defined on representatives by

$$a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \dots da_n . \quad (2.10)$$

This motivates the general philosophy for which in the noncommutative framework differential forms are replaced by Hochschild chains and de Rham currents by Hochschild cochains. The image of (2.7) under (2.10), for $c = 1$, is the n th Chern character of the Grassmannian connection of e ; this explains the name ‘Chern-Connes character’ for $\text{ch}^n(e)$.

Now, while the map ch_n^F in (2.6) is a Hochschild n -cocycle, that is $b\text{ch}_n^F = 0$, the element $\text{ch}^n(e)$ in (2.7) is not always a Hochschild cycle; so, in general (2.8) is not a pairing between Hochschild homology and the dual cohomology. To understand what is the appropriate (co)homology theory, one introduces the *permutation operator* $\lambda_n : \mathcal{A}^{n+1} \rightarrow \mathcal{A}^{n+1}$,

$$\lambda_n(a_0 \otimes a_1 \otimes \dots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1} ,$$

its pull-back $\lambda_n^* : C^{n+1}(\mathcal{A}) \rightarrow C^{n+1}(\mathcal{A})$,

$$(\lambda_n^* \tau)(a_0, a_1, \dots, a_n) = (-1)^n \tau(a_n, a_0, \dots, a_{n-1}) ,$$

and notice that $\boxed{\lambda_n^* \text{ch}_n^F = \text{ch}_n^F}$. We call *cyclic n -cochains* $C_\lambda^n(\mathcal{A})$ those Hochschild n -cochains τ which satisfy $\lambda_n^* \tau = \tau$. Since

$$b(1 - \lambda_{n+1}) = (1 - \lambda_n)b' \quad (2.11)$$

we have $(1 - \lambda_{n+1}^*)b^* = b'^*(1 - \lambda_n^*)$ and b^* maps the kernel of $1 - \lambda_n^*$ into the kernel of $1 - \lambda_{n+1}^*$, that is it sends cyclic cochains into cyclic cochains. The *cyclic cohomology* $HC^\bullet(\mathcal{A})$ is the cohomology of the complex $\{C_\lambda^n(\mathcal{A}), b^*\}$. If $\tau \in C_\lambda^n(\mathcal{A})$ and $\omega \in C_n(\mathcal{A})$, the value of $\tau(\omega)$ depends only on the class of ω in $C_n(\mathcal{A})/\text{Im}(1 - \lambda_n)$. This suggests to define *cyclic homology* $HC_\bullet(\mathcal{A})$ as follows. Chains are elements of $C_n^\lambda(\mathcal{A}) := C_n(\mathcal{A})/\text{Im}(1 - \lambda_n)$. By (2.11) the map b sends the image of $1 - \lambda_{n+1}$ into the image of $1 - \lambda_n$, hence it descends to a map $b : C_{n+1}^\lambda(\mathcal{A}) \rightarrow C_n^\lambda(\mathcal{A})$. The *cyclic homology* $HH_\bullet(\mathcal{A})$ is the homology of the complex $\{C_n^\lambda(\mathcal{A}), b\}$. Remark: the one discussed here is the version of cyclic homology presented in [Con85] (cf. also [Lod97]).

If we fix $c = 0$ in (2.7), then $\text{ch}^n(e)$ is the representative of a cyclic cocycle, that is

$$b\text{ch}^n(e) = \sum_{k_1, \dots, k_n} e_{k_1 k_2} \otimes \dots \otimes e_{k_n k_1} = \frac{1}{2}(1 - \lambda_{n-1}) \sum_{k_1, \dots, k_n} e_{k_1 k_2} \otimes \dots \otimes e_{k_n k_1}$$

is in the image of $1 - \lambda_{n-1}$. Thus (2.8) is a pairing between a cyclic cycle and a cyclic cocycle, and this means that we can replace ch_n^F in (2.8) with any cyclic cycle in the same cyclic cohomology

class, if this helps to simplify computations. If we are in the fortunate case in which $\text{ch}^n(e)$ is a Hochschild cycle, i.e. $b \text{ch}^n(e) = 0$, then we can replace ch_n^F in (2.8) with any Hochschild cocycle in the same Hochschild cohomology class. Of course, the goal is to replace ch_n^F with a local functional.

If $(\mathcal{A}, \mathcal{H}, D)$ is an even k -dimensional spectral triple ($k \in 2\mathbb{N}$), a Hochschild cocycle is given by

$$\Psi_D(a_0, \dots, a_k) := \frac{(k/2)!}{k \cdot k!} \int \gamma a_0 d_D a_1 \dots d_D a_k |D|^{-k}, \quad (2.12)$$

where we recall that $d_D a := [D, a]$. For such a spectral triple the associated Fredholm module is at least $k + 1$ -summable, and the cocycles (2.12) and (2.6) have the same class in Hochschild homology (Hochschild character theorem, cf. [Con94]). So, when $\text{ch}_n(e)$ is a Hochschild cycle, we can use Ψ_D to compute the index instead of ch_n^F . In the general case, more complicated tools are needed. The idea is to look at the class of ch_n^F in a different cohomology theory, called *periodic cyclic cohomology*, HP^\bullet . Similarly to K -theory, HP^\bullet is made of two groups, HP^{ev} and HP^{odd} . The relevant group for us is HP^{ev} . An even periodic cyclic cocycle ϕ is a sequence

$$\phi = (\phi_0, \phi_2, \phi_4, \dots)$$

of multilinear functionals $\phi_{2k} : \mathcal{A}^{2k+1} \rightarrow \mathbb{C}$ which are all zero but for finitely many k (and such that ϕ is in the kernel of a suitable coboundary operator, cf. [Lod97]). An integer-valued pairing between the class $[\phi] \in HP^{\text{ev}}(\mathcal{A})$ of ϕ and the class $[e] \in K_0(\mathcal{A})$ of e is defined by

$$\langle [\phi], [e] \rangle = \phi_0(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} \phi_{2k}(e - \frac{1}{2}, e, \dots, e).$$

The trick now is to notice that any cyclic n -cocycle τ determines a periodic cyclic n -cocycle $\phi = (0, \dots, 0, \tau, 0, \dots)$, which has τ in the n th position and zero everywhere else. In particular, ch_n^F determines a periodic cyclic cocycle, denoted ch^F without subscript n , which is independent of n (for even n no less than the metric dimension). We can rewrite (2.8) as

$$\text{Index}(F_e^+) = \langle [\text{ch}^F], [e] \rangle$$

and although it is always the same expression, in this form it is clearly invariant if we replace ch^F by an element with the same class in HP^{ev} .

A general theorem (cf. Theorem II.3 in [CM95]) gives a local representative of $[\text{ch}^F]$ which can be used to compute the index of F_e^+ . We quote this theorem in the particular case we are interested in.

Theorem 2.3 (Connes-Moscovici). *Let (A, \mathcal{H}, D) be a regular, even spectral triple (with finite metric dimension n), with dimension spectrum Σ made of simple poles. Then, the following formulæ define a periodic cyclic cocycle in the same class of ch^F :*

$$\varphi_0(a_0) = \text{Res}_{s=0} s^{-1} \text{Tr}(\gamma a_0 |D|^{-2s}),$$

$$\varphi_j(a_0, \dots, a_j) = \frac{1}{2} \sum_{k \in \mathbb{N}^j} \frac{(-1)^k}{k_1! \dots k_j!} \alpha_k \int \gamma_{a_0} [D, a_1]^{(k_1)} \dots [D, a_j]^{(k_j)} |D|^{-(2|k|+j)} .$$

Here $a_i \in \mathcal{A}$, $|k| := k_1 + k_2 + \dots + k_j$, $\alpha_k^{-1} := (k_1 + 1)(k_1 + k_2 + 2) \dots (k_1 + \dots + k_j + j)$, $T^{(0)} := T$ and $T^{(i+1)} := [D^2, T^{(i)}] \forall i \in \mathbb{N}$.

The finiteness of the metric dimension guarantees that all terms in the sum with $|k| + j > n$ are zero, hence all sums are finite and only finitely many φ_j are different from zero.

All the terms φ_j with $j > 0$ are local (i.e. they are independent of smoothing contributions). The unique non-local term is φ_0 .

We close this section with a remark. In the commutative case, when \mathcal{A} is some algebra of smooth functions on a *connected* compact smooth manifold M , there is always a Hochschild cocycle associated with any projection $p = p^* = p^2$ with elements $p_{ij} \in \mathcal{A}$. Indeed, for p a projection-valued function on M , the map sending $x \in M$ to the matrix trace $\text{Tr} p(x)$ is a continuous map $M \rightarrow \mathbb{Z}$, then it is constant provided M is connected (its geometrical meaning is the rank of the bundle underlying the projective module $p\mathcal{A}^k$). Thus, $\text{Tr}(p)$ belongs to the subalgebra $\mathbb{C} \subset \mathcal{A}$, and the tensor

$$\omega_p = \text{Tr}(p - \frac{1}{2}) \dot{\otimes} p \dot{\otimes} (p - 1)$$

has vanishing Hochschild boundary

$$b\omega_p = \frac{1}{2} \text{Tr}(p \otimes 1 - 1 \otimes p) = \frac{1}{2} \{ \text{Tr}(p) \otimes 1 - 1 \otimes \text{Tr}(p) \} = 0 ,$$

the tensor product being over \mathbb{C} and $\text{Tr}(p) \in \mathbb{C}$. Here the dotted tensor product of two matrices $A = ((A_{ij}))$ and $B = ((B_{ij}))$ is the matrix $A \dot{\otimes} B$ with entries $(A \dot{\otimes} B)_{ij} = \sum_k A_{ik} \otimes B_{kj}$. Under the map (2.10) the cycle ω_p becomes the 1st Chern character (modulo a normalization constant) of the Grassmannian connection on the module $p\mathcal{A}^k$. For a generic $*$ -algebra, a sufficient condition for ω_p to be a Hochschild cycle is that $\text{Tr}(p) \in \mathbb{C}$; in the quantum group case, it is often possible to deform the matrix trace in such a way that $\text{Tr}_q(p)$ is a scalar: the resulting tensor ω_2 will be a cycle in some twisted Hochschild homology (more on this in Section 2.7).

2.4 Nested ideals in Hilbert spaces

In this section we explain how to concretely work with order r operators and other relevant ideals. In all this section, two-sided $*$ -ideals will be called simply *ideals*.

Let \mathcal{H} be a *separable* Hilbert space, and denote by $\mu_j(T)$ the j th *singular value* of the bounded operator T , which we recall is defined by

$$\mu_j(T) := \inf_{\substack{V \subset \mathcal{H}, \\ \dim V = j-1}} \sup_{v \perp V, \|v\|=1} \|Tv\| .$$

We have $\mu_j(T) \geq \mu_{j+1}(T)$, and μ_1 is the operator norm. Recall that $T \in \mathcal{B}(\mathcal{H})$ is *compact* if and only if $\mu_j(T) \rightarrow 0$ for $j \rightarrow \infty$, and let \mathcal{K} be the class of compact operators on \mathcal{H} . This is a closed ideal in $\mathcal{B}(\mathcal{H})$, and most ideals we are going to discuss are dense subspaces of \mathcal{K} . Firstly, with the inequalities (cf. [Sim05])

$$\begin{aligned} \mu_{j+k}(T_1 + T_2) &\leq \mu_j(T_1) + \mu_k(T_2) \quad \text{for all } T_1, T_2 \in \mathcal{K}, \\ \mu_j(ST) &\leq \|S\| \mu_j(T) \quad \text{and} \quad \mu_j(TS) \leq \|S\| \mu_j(T) \quad \text{for all } T \in \mathcal{K} \text{ and } S \in \mathcal{B}(\mathcal{H}), \end{aligned} \quad (2.13)$$

one proves that the Schatten-von Neumann classes $\mathcal{L}^p(\mathcal{H})$,

$$\mathcal{L}^p(\mathcal{H}) := \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \text{Tr}(|T|^p) := \sum_{n=1}^{\infty} \mu_n(T)^p < \infty \right\},$$

are ideals in $\mathcal{B}(\mathcal{H})$. Clearly $\mathcal{L}^p(\mathcal{H}) \subsetneq \mathcal{L}^{p'}(\mathcal{H})$ for all $1 \leq p < p' < \infty$. In particular, $\mathcal{L}^1(\mathcal{H})$ are the operators of *trace class*.

For the *spectral* or *quantized calculus* [Con94] the following ideals [Voi79, Voi81] are crucial ($p \geq 1$)

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) := \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \mu_n(T) = O(n^{-1/p}), \text{ as } n \rightarrow \infty \right\}.$$

We have $\mathcal{L}^p \subsetneq \mathcal{L}^{(p,\infty)} \subsetneq \mathcal{L}^{p'}$ for all $1 \leq p < p' < \infty$. In particular, the *Dixmier ideal* $\mathcal{L}^{(1,\infty)}$ is the domain of the Dixmier trace, which vanishes on \mathcal{L}^1 . For a positive $T \in \mathcal{L}^{(1,\infty)}$ the Dixmier trace Tr_ω is given by a suitable limit of the bounded sequence [Dix66]

$$\frac{1}{\log N} \sum_{n=1}^N \mu_N(T) \quad (2.14)$$

and extended to all $\mathcal{L}^{(1,\infty)}$ by linearity. Most important, if the ordinary limit of (2.14) exists, then T is called *measurable* and its Dixmier trace (if $T \geq 0$) is simply given by

$$\text{Tr}_\omega(T) = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \mu_N(T).$$

Elements of $\mathcal{L}^{(1,\infty)}$ are called for obvious reasons *infinitesimals of order 1*, being in the domain of the Dixmier trace which plays the role of a ‘noncommutative integral’. More generally, infinitesimals of order $\alpha \geq 0$ are defined by

$$\mathcal{I}_\alpha = \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \mu_n(T) = O(n^{-\alpha}) \right\},$$

and $\mathcal{I}_\alpha = \mathcal{L}^{(p,\infty)}$ if $\alpha = 1/p \leq 1$. Infinitesimals behaves as one expects [Lan02]: $\mathcal{I}_{\alpha_1} \supset \mathcal{I}_{\alpha_2}$ for all $\alpha_1 \leq \alpha_2$, $\mathcal{I}_{\alpha_1} \cdot \mathcal{I}_{\alpha_2} \subset \mathcal{I}_{\alpha_1 + \alpha_2}$ (which in particular means that \mathcal{I}_α is an ideal in $\mathcal{B}(\mathcal{H})$), and the Dixmier trace vanishes on infinitesimals of order greater than 1. We let

$$\mathcal{I}_\infty := \bigcap_{\alpha \geq 0} \mathcal{I}_\alpha = \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \lim_{n \rightarrow \infty} n^\alpha \mu_n(T) = 0 \forall \alpha \geq 0 \right\}$$

be the ideal of infinitesimals of arbitrary high order.

2.4.1 Rapid decay matrices

Up to now, we have discussed subspaces of \mathcal{K} which are ideals in $\mathcal{B}(\mathcal{H})$. We now introduce subspaces of \mathcal{K} which are ideals in algebras smaller than $\mathcal{B}(\mathcal{H})$.

Let $\{|n\rangle\}_{n \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} and

$$\mathcal{S} := \left\{ T \in \mathcal{B}(\mathcal{H}) \mid \lim_{n,m \rightarrow \infty} n^\alpha m^\beta \langle n|T|m \rangle = 0 \forall \alpha, \beta \geq 0 \right\}$$

the non-unital involutive algebra of *rapid decay matrices* relative to this basis.

Lemma 2.4. *We have $\mathcal{S} \subset \mathcal{I}_\infty$.*

Proof. We define a selfadjoint positive operator N by $N|n\rangle := (n+1)|n\rangle$, and note that $N^{-\alpha}$ is compact if $\alpha > 0$. For any $T \in \mathcal{S}$ and any $\alpha > 0$, the product TN^α is bounded. So by (2.13)

$$\mu_k(T) \leq \|TN^\alpha\| \mu_k(N^{-\alpha}) = \|TN^\alpha\| k^{-\alpha}$$

and $k^\beta \mu_k(T)$ goes to zero for any $\beta > 0$ (just chose $\alpha > \beta$ in previous inequality). This proves that $T \in \mathcal{I}_\infty$. \square

Lemma 2.5. *The algebra \mathcal{S} is not an ideal in $\mathcal{B}(\mathcal{H})$. In particular, the inclusion $\mathcal{S} \subset \mathcal{I}_\infty$ is proper.*

Proof. Let $L_q \in \mathcal{S}$, $0 < q < 1$, be the operator given by

$$L_q |n\rangle := q^n |n\rangle .$$

We now construct a $T \in \mathcal{B}(\mathcal{H})$ such that $L_q T \notin \mathcal{S}$, thus proving that \mathcal{S} is not an ideal in $\mathcal{B}(\mathcal{H})$. Since \mathcal{I}_∞ is an ideal in $\mathcal{B}(\mathcal{H})$, we conclude that \mathcal{S} does not coincide with \mathcal{I}_∞ .

Let $|v\rangle := \frac{\sqrt{6}}{\pi} \sum_n (n+1)^{-1} |n\rangle$ (it is in \mathcal{H} since by Euler's formula $\|v\|^2 = \frac{6}{\pi^2} \sum_{k \geq 1} k^{-2} = 1$), and call T the partial isometry $T := |0\rangle\langle v|$. Although T is of rank 1, it is not a rapid decay matrix. Indeed

$$\lim_{n \rightarrow \infty} n^\alpha \langle 0|T|n \rangle = \frac{\sqrt{6}}{\pi} \lim_{n \rightarrow \infty} n^{\alpha-1}$$

is zero if and only if $\alpha < 1$, while it should be zero for all α if T were in \mathcal{S} . The observation that $L_q T = T$ concludes the proof. \square

2.4.2 Properties of finite-dimensional spectral triples

We want now to discuss the relation between ‘infinitesimals’ and the classes of order r operators defined in Section 2.2, in the particular case in which $(\mathcal{A}, \mathcal{H}, D)$ is a finite-dimensional spectral triple. We assume for simplicity that D is invertible.

Lemma 2.6. *If D is d^+ -summable, then $\text{OP}^{-rd} \subset \mathcal{I}_r$ for all $r > 0$.*

Proof. For $T \in \text{OP}^{-rd}$, $|D|^{rd}T$ is bounded and $\mu_k(T) \leq \| |D|^{rd}T \| \mu_k(|D|^{-rd}) = O(k^{-r})$. Thus $T \in \mathcal{I}_r$ and this concludes the proof. \square

Since $|D|^{-1}$ is positive and compact we can find an orthonormal basis $\{|n\rangle\}_{n \in \mathbb{N}}$ of \mathcal{H} such that

$$|D|^{-1}|n\rangle = \lambda_{n+1}^{-1}|n\rangle$$

with λ_k^{-1} ($k \geq 1$) the eigenvalues of $|D|^{-1}$ in decreasing order, counting multiplicities. As for all positive compact operators, we have $\mu_k(|D|^{-1}) = \lambda_k^{-1}$ and so by d^+ -summability $\lambda_k^{-1} = O(k^{-1/d})$. Let \mathcal{S} denote rapid decay matrices relative to the eigenbasis of D .

Lemma 2.7. *For a finite-dimensional spectral triple, $\mathcal{S} \subset \text{OP}^{-\infty}$.*

Proof. Let $d \in \mathbb{R}^+$ be the metric dimension. The sequence $\lambda_n^{-1}/n^{-1/d}$ must have a non-zero $n \rightarrow \infty$ limit, otherwise λ_n^{-d} would be $o(n^{-1})$, and $\text{Tr}_\omega(|D|^{-d})$ would be zero. Therefore the inverse sequence $\lambda_n n^{-1/d}$ must be bounded by a constant C . If $T \in \mathcal{S}$, the sequence

$$\langle n | |D|^j T |D|^k |m\rangle = \lambda_n^j \lambda_m^k |\langle n | T |m\rangle| \leq C^{j+k} n^{j/d} m^{k/d} |\langle n | T |m\rangle| ,$$

is of rapid decay in n, m . Thus for all j, k , $|D|^j T |D|^k$ belongs \mathcal{S} which in particular means that it is bounded. To conclude the proof now we need just the following Lemma.

Lemma 2.8. $\{T \in \text{OP}^{-\infty}\} \iff \{|D|^j T |D|^k \in \mathcal{B}(\mathcal{H}) \forall j, k \in \mathbb{N}\} .$

Proof:

(“ \Leftarrow ”) By hypothesis, $\delta^j(|D|^r T) = \sum_{l=0}^j (-1)^l \binom{j}{l} |D|^{j+r} T |D|^{l-j} \in \mathcal{B}(\mathcal{H})$ for all $r \geq 0$, hence $|D|^r T \in \text{OP}^0$ for all $r \geq 0$ and $T \in \text{OP}^{-\infty}$.

(“ \Rightarrow ”) $|D|^j T |D|^k = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} |D|^{j+k-l} \delta^l(T)$. But $\delta^l(T) \in \text{OP}^{-\infty}$ since $\text{OP}^{-\infty}$ is δ -invariant. So all factors in the right hand side are bounded operators, and then $|D|^j T |D|^k \in \mathcal{B}(\mathcal{H})$. \square

This concludes the proof of Lemma 2.7. \square

2.4.3 The dimension spectrum

Let $(\mathcal{A}, \mathcal{H}, D)$ be a d -dimensional spectral triple, and assume for simplicity that D is invertible. If $T \in \text{OP}^{-r}$, the zeta-function $\zeta_T(s) = \text{Tr}(T|D|^{-s})$ is defined and holomorphic in the half plane $\text{Re } s > d - r$. In particular, if $T \in \text{OP}^{-\infty}$ the associated zeta-function is holomorphic on the full complex plane and gives no contribution to the dimension spectrum. As the proof is simple and instructive, we report it in the following.

Lemma 2.9. *Let $\{a_n\}$ and $\{b_n\}$ two sequences such that*

1. $0 < a_n < Cn^{-1/\ell}$, $C \in \mathbb{R}^+$;

2. $|b_n| \leq C' \in \mathbb{R}^+$ is a bounded sequence.

Then $\sum_{n \in \mathbb{N}} a_n^z b_n$ converges (pointwise) to a holomorphic function in the half plane $\operatorname{Re} z > \ell$.

Proof. The partial sums $f_N(z) = \sum_{n \leq N} a_n^z b_n$ are holomorphic on \mathbb{C} and dominated by

$$|f_N(z)| \leq C^{\operatorname{Re} z} C' \sum_{n \leq N} n^{-\operatorname{Re} z / \ell}.$$

Thus $f_N(z)$ converges pointwise in the half plane $\operatorname{Re} z > \ell$. Let E be a bounded closed domain in the half plane $\operatorname{Re} z > \ell$, and call

$$k_E = \sup_{z \in E} C^{\operatorname{Re} z} C', \quad z_0 = \inf_{z \in E} \operatorname{Re} z.$$

Notice that $z_0 > \ell$. We have the inequality

$$|f(z) - f_N(z)| \leq C^{\operatorname{Re} z} C' \sum_{n > N} n^{-\operatorname{Re} z / \ell} \leq k_E \sum_{n > N} n^{-z_0 / \ell}, \quad \forall z \in E.$$

The limit $N \rightarrow \infty$ of the right hand side is zero, since the series is convergent. So

$$\|f - f_N\|_\infty \leq k_E \sum_{n > N} n^{-z_0 / \ell} \xrightarrow{N \rightarrow \infty} 0, \quad \forall z \in E,$$

and f_N converges uniformly to f in E . By Weierstrass theorem [PF55, pag. 640] the uniform limit of a sequence of holomorphic functions is holomorphic. Thus, f is holomorphic in any bounded closed domain E of the half plane $\operatorname{Re} z > \ell$. We conclude that f is holomorphic on the entire half plane $\operatorname{Re} z > \ell$. \square

For $T \in \operatorname{OP}^{-r}$, $|D|^r T$ is bounded and so $\langle n | |D|^r T | n \rangle =: b_n$ is a bounded sequence. Called $a_n = \lambda_n^{-1}$ the n th eigenvalue of $|D|^{-1}$, applying Lemma 2.9 we prove that $\operatorname{Tr}(T |D|^{-z+r}) = \sum_{n \in \mathbb{N}} a_n^z b_n$ converges to a function $\zeta_T(z - r)$ which is holomorphic in $\operatorname{Re} z > d$. With the substitution $s = z - r$, we have proved that $\zeta_T(s)$ is holomorphic on $\operatorname{Re} s > d - r$ for any $T \in \operatorname{OP}^{-r}$ (and for all $r \in \mathbb{R}$, not necessarily positive). For $T \in \operatorname{OP}^{-\infty}$ previous claim holds for any $r > 0$, and then $\zeta_T(s)$ is holomorphic on all \mathbb{C} .

We remark that the dimension spectrum Σ contains always the metric dimension d , and any point $s \in \Sigma$ has $\operatorname{Re} s \leq d$ since $\zeta_T(s)$ is holomorphic on $\operatorname{Re} s > d$ for any T bounded.

2.4.4 Residues of zeta-type functions

Motivated by the discussion in previous section, we introduce the following classes of operators

$$\mathcal{J}_r := \{T \in \operatorname{OP}^0 \mid T |D|^{-p} \in \mathcal{L}^1(\mathcal{H}) \forall p > r\} \quad (2.15)$$

which in the finite-dimensional case contains $\operatorname{OP}^{-(n-r)}$, with n the metric dimension, and can be neglected when computing residues of zeta-functions in poles with real part bigger than r .

Lemma 2.10. *The collection \mathcal{J}_r is a two-sided ideal in OP^0 .*

Proof. Clearly \mathcal{J}_r is a vector space: if $T_1, T_2 \in \mathcal{J}_r$, that is $T_1|D|^{-p} \in \mathcal{L}^1(\mathcal{H})$, $T_2|D|^{-p} \in \mathcal{L}^1(\mathcal{H})$ for all $p > r$, then $T_1|D|^{-p} + T_2|D|^{-p} \in \mathcal{L}^1(\mathcal{H})$ for all $p > r$, which means $T_1 + T_2 \in \mathcal{J}_r$.

That \mathcal{J}_r is a left ideal is straightforward. Since $\mathcal{L}^1(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, if $T_1 \in \text{OP}^0$ and $T_2 \in \mathcal{J}_r$, for all $p > r$ we have that $T_1 \cdot T_2|D|^{-p}$ is the product of a bounded operator, T_1 , with a trace class one, $T_2|D|^{-p}$, thus it is of trace class, and $T_1T_2 \in \mathcal{J}_r$.

From Appendix B of [CM95] for any $p > 0$, we know that the bounded operator $|D|^{-p}$ maps \mathcal{H} to $\mathcal{H}^p := \text{dom}|D|^p$, that $T \in \text{OP}^0 \subset \text{op}^0$ is a bounded operator $\mathcal{H}^p \rightarrow \mathcal{H}^p$, and finally that $|D|^p$ is bounded from \mathcal{H}^p to \mathcal{H} . Thus, for $T \in \text{OP}^0$, the product $|D|^p T |D|^{-p}$ is a bounded operator on \mathcal{H} . Now, if $T_1 \in \text{OP}^0$ and $T_2 \in \mathcal{J}_r$, for all $p > r$ we can write $T_2 T_1 |D|^{-p} = T_2 |D|^{-p} \cdot |D|^p T_1 |D|^{-p}$ as the product of a bounded operator, $|D|^p T_1 |D|^{-p}$, with a trace class one, $T_2 |D|^{-p}$; thus $T_2 T_1 |D|^{-p}$ is of trace class; therefore $T_2 T_1 \in \mathcal{J}_r$ and \mathcal{J}_r is also a right ideal. \square

Of course $\text{OP}^0 = \mathcal{J}_n$. One might ask whether in the finite-dimensional case OP^{-r} and \mathcal{J}_{n-r} are the same ideal. In general they are not, and the following is a counterexample.

Let \mathcal{H} be the Hilbert space with orthonormal basis $|l, m\rangle$, with $l \in \mathbb{N}$ and $l - |m| \in \mathbb{N}$ (e.g. think of $L^2(S^2)$ in the basis of Laplace spherical harmonics), and define a positive D and a bounded \tilde{L}_q by

$$D|l, m\rangle = (l+1)|l, m\rangle, \quad \tilde{L}_q|l, m\rangle = q^{l+m}|l, m\rangle.$$

We have $\tilde{L}_q \in \mathcal{J}_1$, since

$$0 \leq \text{Tr}(\tilde{L}_q D^{-s}) = \sum_{l \in \mathbb{N}} (l+1)^{-s} \frac{1-q^{2l+1}}{1-q} \leq \frac{1}{1-q} \zeta(s), \quad \forall s > 1,$$

where $\zeta(s)$ is the Riemann zeta-function. On the other hand

$$\|D\tilde{L}_q\|^2 \geq \sup_{l,m} \langle l, m | D\tilde{L}_q | l, m \rangle \geq \sup_l \langle l, -l | D\tilde{L}_q | l, -l \rangle = \sup_l (l+1) = +\infty,$$

and then $D\tilde{L}_q$ is not bounded and $\tilde{L}_q \notin \text{OP}^{-1}$.

2.5 Smooth ‘functions’ on noncommutative spaces

As continuous functions carry all topological information about a space, C^∞ -functions carry all relevant information about its smooth structure. In the canonical commutative example of a spectral triple, we can replace $C^\infty(M)$ with any subalgebra and still have a regular spectral triple. Thus, regularity condition is not sufficient to select $C^\infty(M)$ against proper subalgebras. The characteristic property which enables to select $C^\infty(M)$ is its stability under holomorphic functional calculus.

Let \mathcal{C} be a C^* -algebra. If a function ψ is holomorphic in a neighborhood of the spectrum of $a \in \mathcal{C}$, we define an element $\psi(a) \in \mathcal{C}$ as the integral

$$\psi(a) = \oint_{\Gamma} \psi(z)(z \cdot 1 - a)^{-1} dz$$

on a rectifiable contour Γ winding once around $\mathrm{Sp}(a)$. The map $a \mapsto \psi(a)$ is called *holomorphic functional calculus*, and a dense involutive subalgebra $\mathcal{A} \subset \mathcal{C}$ is called *pre- C^* -algebra* if $\psi(a) \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and $\psi(a)$ is well-defined in \mathcal{C} .

Smooth functions on a compact manifold are characterized also by another property: they form a Fréchet space, that is a locally convex topological space closed with respect to a countable family of semi-norms. Actually, they form a Fréchet algebra, that is an algebra and a Fréchet space such that the semi-norms are submultiplicative. Given a regular spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and with the notation $\delta(a) := [[D], a]$, the algebra \mathcal{A}^∞ , completion of \mathcal{A} in the topology defined by the family of seminorms

$$\|a\|_{1,n} := \|\delta^n(a)\|, \quad \|a\|_{2,n} := \|\delta^n([D], a)\|,$$

is a Fréchet pre- C^* -algebra [RV06] and $(\mathcal{A}^\infty, \mathcal{H}, D)$ is still a regular spectral triple.

2.6 Reality and first order condition

A real structure on a spectral triple is a generalization of the charge conjugation operator acting on spinors [Con95], has application in physics to the construction of the spectral action [CC97] (see also [CCM07] and [Kas00]), and its existence is one of a list of conditions which should characterize a noncommutative spin manifold [Con96] (cf. [RV06] for an up-to-date account of this line of research). When testing these axioms on quantum groups, or quantum homogeneous spaces, things start to become complicated exactly when one arrives to the reality axiom.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. A *real structure* is an antilinear isometry J on \mathcal{H} such that for all $a, b \in \mathcal{A}$:

$$J^2 = \pm 1, \quad JD = \pm DJ, \quad [a, Jb^*J^{-1}] = 0, \quad \boxed{[[D, a], Jb^*J^{-1}] = 0}.$$

If the spectral triple is even, we require also that $J\gamma = \pm\gamma J$. The signs ‘ \pm ’ determine the KO -dimension of the spectral triple (cf. Table 9.45 of [GBVF01]), which classically corresponds to the metric dimension modulo 8. The framed condition is usually called *first order axiom*. In the canonical commutative example $(C^\infty(M), \mathcal{H}, \mathcal{D})$, with M a compact Riemannian spin manifold, \mathcal{H} the square integrable spinors and \mathcal{D} the Dirac operator, J is the charge conjugation and since $Jb^*J^{-1} = b$ the 3rd condition trivially follows from commutativity of the algebra. For \mathcal{D} a pseudo-differential operator, the condition $[\mathcal{D}, a] \in \mathcal{B}(\mathcal{H})$ implies that \mathcal{D} is of order ≤ 1 , while the further condition $[[\mathcal{D}, a], b] = 0$ is satisfied if \mathcal{D} is a first-order *differential* operator. A clarifying example is $(C^\infty(M), L^2(M), \Delta^{1/2})$, with M compact Riemannian manifold and Δ the Laplace-Beltrami operator; it is a regular spectral triple, but not real since in general $[[\Delta^{1/2}, a], b]$ is not zero. In the noncommutative case, the 3rd condition is inspired by von Neumann algebras, for which Tomita-Takesaki theorem ensures the existence of a J such that $b \mapsto Jb^*J^{-1}$ is an isomorphism from \mathcal{A} into the opposite algebra, concretely represented inside the commutant \mathcal{A}' .

The first order condition is what sometimes creates problems. Typical examples are quantum groups (and quantum homogeneous spaces), where the commutators $[[D, a], JbJ^{-1}]$ vanish only modulo some ideal of ‘infinitesimals’. Motivated by examples coming from quantum group theory, we try with the following definition.

Definition 2.11. *A weak real structure is an antilinear isometry J on \mathcal{H} such that $\forall a, b \in \mathcal{A}$,*

$$J^2 = \pm 1, \quad JD = \pm DJ, \quad [a, JbJ^{-1}] \in \mathcal{I}, \quad [[D, a], JbJ^{-1}] \in \mathcal{I},$$

where the set \mathcal{I} is a suitable two-sided ideal made of infinitesimals. If the spectral triple is even with grading γ , we impose the further relation $J\gamma = \pm\gamma J$.

A real structure corresponds to the choice $\mathcal{I} = 0$; while in examples coming from quantum groups [DLPS05, DLS⁺05a, DDLW07, DDL08] one usually takes $\mathcal{I} = \text{OP}^{-\infty}$. We stress that the term ‘weak real structure’ is not standard and is provisionally used in this dissertation to simplify the discussion.

A real spectral triple of dimension n gives an element of $KR^n(\mathcal{A} \otimes \mathcal{A}^{\text{op}}, \mathbb{C})$, and this relation with real K -homology is not clear if J is a weak real structure.

The second remark is that this weaker condition might be too weak. The ideal \mathcal{I} should be chosen with some care. For example, let M be a compact Riemannian manifold, ∇ the Levi-Civita connection, $\Delta = \nabla^* \nabla$ the Laplacian, and denote $\delta(f) = [\Delta^{1/2}, f]$ (bounded if f is Lipschitz). The commutator

$$[[\Delta, f], g] = 2(df, dg)$$

(on the right hand side we have the Hermitian structure on 1-forms) is bounded for all $f, g \in C^\infty(M)$, and then from $\Delta = \Delta^{1/2} \Delta^{1/2}$ we deduce that the operator

$$2\Delta^{1/2}[\delta(f), g] = [[\Delta, f], g] - 2\delta(f)\delta(g) + [\delta^2(f), g]$$

is bounded too. We have both $\Delta^{-1/2} \in \mathcal{L}^{(n, \infty)}(\mathcal{H}) = \mathcal{I}_{1/n}$ (by Weyl estimates where $n := \dim M$, we assume for simplicity that Δ is invertible) and $\Delta^{-1/2} \in \text{OP}^{-1}$. Thus, the commutator

$$[[\Delta^{1/2}, f], g] = [\delta(f), g] = \Delta^{-1/2} \cdot (\text{bounded operator})$$

is compact (it is even an infinitesimal of order $1/n$ and an element of OP^{-1}). This proves that the complex conjugation J is a weak real structure for $(C^\infty(M), L^2(M), \Delta^{1/2})$, with \mathcal{I} any ideal of compact operators containing either OP^{-1} or infinitesimals of order $1/n$, even if M may not be a spin manifold. This is a situation that one probably would like to avoid, by requesting for example that elements of \mathcal{I} are infinitesimals of order greater than 1.

2.7 The homological dimension of quantum spaces

One can generalize Hochschild homology by considering Hochschild homology with coefficients in a bimodule [Lod97]. A particular case is twisted Hochschild homology, which we recall briefly

here. Let \mathcal{A} be a (complex, associative) unital algebra and $\kappa \in \text{Aut}(\mathcal{A})$ a scaling automorphism (i.e. \mathcal{A} admits a linear basis in which κ is diagonal). On $\mathcal{A}^{\otimes(n+1)}$ define a linear map θ_κ by

$$\theta_\kappa(a_0 \otimes \dots \otimes a_n) = \kappa(a_0) \otimes \dots \otimes \kappa(a_n).$$

The set of twisted n -chains C_n^κ is the subset of elements of $\mathcal{A}^{\otimes(n+1)}$ in the kernel of $\theta_\kappa - 1$. Twisted Hochschild homology $HH_\bullet^\kappa(\mathcal{A})$ is defined as the homology of the complex $\{C_n^\kappa, b_\kappa\}_{n \geq 0}$, with boundary operator $b_\kappa : C_{n+1}^\kappa \rightarrow C_n^\kappa$ given by

$$\begin{aligned} b_\kappa(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1}) &= \sum_{j=0}^n (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes a_{n+1} \\ &\quad + (-1)^{n+1} \kappa(a_{n+1}) a_0 \otimes a_1 \otimes \dots \otimes a_n. \end{aligned}$$

This is not the ordinary definition, but is equivalent to it if κ is a scaling automorphism (see e.g. [Sit05]). If $\kappa = id_{\mathcal{A}}$ one gets the ‘untwisted’ groups $HH_\bullet(\mathcal{A})$.

From the dual point of view, a twisted n -cochain is a multilinear map $\phi : \mathcal{A}^{\otimes(n+1)} \rightarrow \mathbb{C}$ in the kernel of $\theta_\kappa^* - 1$, with θ_κ^* the pullback of θ_κ ,

$$\theta_\kappa^* \phi(a_0, a_1, \dots, a_n) = \phi(\kappa(a_0), \kappa(a_1), \dots, \kappa(a_n)).$$

We denote twisted n -cochains with C_n^m . Let b_κ^* be the pullback of b_κ . The twisted Hochschild cohomology $HH_\bullet^\kappa(\mathcal{A})$ is defined as the cohomology of the complex $\{C_n^m, b_\kappa^*\}_{n \geq 0}$.

The expression $\langle [\phi], [\omega] \rangle := \phi(\omega)$ is a pairing between $HH_\bullet^\kappa(\mathcal{A})$ and $HH_\bullet^\kappa(\mathcal{A})$, that is it depends only on the class of ϕ in $HH_\bullet^\kappa(\mathcal{A})$ and by the class of ω in $HH_\bullet^\kappa(\mathcal{A})$.

If \mathcal{U} is a Hopf algebra and \mathcal{A} a left \mathcal{U} -module algebra and κ commutes with the action of \mathcal{U} , the action on \mathcal{A}^{n+1} obtained by Hopf tensor product defines a structure of left \mathcal{U} -module on $C_n^\kappa(\mathcal{A})$. Similarly, the pullback of the action on \mathcal{A}^{n+1} on multilinear maps $\phi : \mathcal{A}^{n+1} \rightarrow \mathbb{C}$, given by

$$(h \triangleright \phi)(\omega) := \phi(S^{-1}(h) \triangleright \omega), \quad \forall \omega \in \mathcal{A}^{n+1}, h \in \mathcal{U},$$

turns $C_n^\kappa(\mathcal{A})$ into a left \mathcal{U} -module.

If \mathcal{A} is the algebra of smooth functions on a C^∞ manifold, the metric dimension of the space equals the Hochschild dimension $\sup\{n \mid HH_n(\mathcal{A}) \neq 0\}$. In the general case one defines the twisted Hochschild dimension as

$$\text{Hdim}^\kappa(\mathcal{A}) := \sup\{n \mid HH_n^\kappa(\mathcal{A}) \neq 0\}.$$

A common situation on q -deformations M_q of homogeneous spaces M is that the ordinary Hochschild dimension of M_q is less than the dimension of the commutative limit M , a phenomenon known as ‘dimension drop’. For example the Hochschild dimension of $SL_q(N)$ equals its rank, while it is 1 for Podleś spheres, as proved in [MNW91]. On the other hand, Hdim^κ is believed to coincide with the classical dimension if we let κ be the modular automorphism (this

was proved in the case of $SL_q(N)$ in [HK05, HK06], for Podleś spheres in [Had07] and quantum hyperplanes in [Sit05]).

Suppose now we have an even Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$, and that the automorphism κ is implemented on \mathcal{H} by the adjoint action of a (unbounded, invertible, positive, with dense domain) operator K commuting with F and γ , that is

$$\kappa(a)v = KaK^{-1}v, \quad [F, K]v = 0, \quad [\gamma, K]v = 0,$$

for any $a \in \mathcal{A}$ and any v in the domain of K . Instances of this situation are provided by subalgebras of the ‘coordinate algebra’ of a compact quantum group, with κ the modular automorphism, and $(\mathcal{A}, \mathcal{H}, F, \gamma)$ an equivariant even Fredholm module (cf. Section 2.10). Finally, suppose there exists $p \in [1, \infty)$ such that the operator

$$K^{-1}[F, a_0][F, a_1] \dots [F, a_{p-1}]$$

is of trace class on \mathcal{H} for all $a_0, \dots, a_{p-1} \in \mathcal{A}$.

Lemma 2.12. *A twisted Hochschild n -cocycle is defined by the formula*

$$\text{ch}_n^{F, \kappa}(a_0, \dots, a_n) := \text{Tr}_{\mathcal{H}}(K^{-1}\gamma F[F, a_0] \dots [F, a_n]),$$

for any even $n \geq p - 1$.

Proof. Using Leibniz rule

$$\begin{aligned} \{b_{\kappa}^* \text{ch}_n^{F, \kappa}\}(a_0, \dots, a_{n+1}) &= \text{Tr}_{\mathcal{H}}(K^{-1}\gamma F a_0 [F, a_1] \dots [F, a_{n+1}]) \\ &\quad + \text{Tr}_{\mathcal{H}}(K^{-1}\gamma F [F, a_0] \dots [F, a_n] a_{n+1}) \\ &\quad - \text{Tr}_{\mathcal{H}}(K^{-1}\gamma F [F, \kappa(a_{n+1}) a_0] \dots [F, a_n]). \end{aligned}$$

By hypothesis $K^{-1}\gamma F [F, \kappa(a_{n+1}) a_0] = \gamma F [F, a_{n+1} K^{-1} a_0]$. Then using again Leibniz rule and cyclicity of the trace we deduce that the r.h.s. vanishes, and $\text{ch}_n^{F, \kappa}$ is closed.

Similarly, from cyclicity of the trace we get

$$\begin{aligned} \text{ch}_n^{F, \kappa}(a_0, \dots, a_{n+1}) &= \text{Tr}_{\mathcal{H}}([F, a_n] K^{-1} \gamma F [F, a_0] \dots [F, a_{n-1}]) \\ &= \text{Tr}_{\mathcal{H}}(K^{-1} [F, \kappa(a_n)] \gamma F [F, a_0] \dots [F, a_{n-1}]). \end{aligned}$$

But $F^2 = 1$ implies $[F, \kappa(a_n)]F = -F[F, \kappa(a_n)]$, and

$$\text{ch}_n^{F, \kappa}(a_0, \dots, a_{n+1}) = \text{Tr}_{\mathcal{H}}(K^{-1}\gamma F [F, \kappa(a_n)] [F, a_0] \dots [F, a_{n-1}]).$$

Thus, $\text{ch}_n^{F, \kappa}$ is twisted cyclic

$$\text{ch}_n^{F, \kappa}(a_0, a_1, \dots, a_n) = \text{ch}_n^{F, \kappa}(\kappa(a_n), a_0, \dots, a_{n-1}) \tag{2.16}$$

and by iterating last identity n -times we prove that $\text{ch}_n^{F, \kappa}$ is in the kernel of $\theta_{\kappa}^* - 1$. \square

We'll show in the case of Podleś spheres S_{qs}^2 and of the quantum orthogonal 4-sphere S_q^4 how to construct a twisted Hochschild cycle which is a deformation of the volume form on S^2 and S^4 respectively. We'll also observe that on S_{qs}^2 and S_q^4 , the operators $K^{-1}[F, a_0][F, a_1] \dots [F, a_{p-1}]$ are of trace class for $p > 2$ and $p > 4$ respectively, as in the $q \rightarrow 1$ limit. This is in contrast with the fact that Fredholm modules are usually at most 1-summable on q -deformed spaces (for q real).

2.8 Some general results

Since in all our examples there is a standard strategy to prove that a collection $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is a (even, regular) spectral triple, we explain it here.

Assume \mathcal{A} is a $*$ -algebra of bounded operators on a separable graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and call $\pi_{\pm}(a) := \frac{1}{2}(1 \pm \gamma)a$ with γ the natural grading. Let $|n, k\rangle_{\pm}$ be an orthonormal basis of \mathcal{H}_{\pm} , with $n \in \mathbb{N}$ and integer k in some finite index set I_n depending on n (or eventually a multi-index, it makes no difference), and assume the cardinality of I_n grows at most polynomially in n . Let $D = F|D|$ with absolute value defined by

$$|D| |n, k\rangle_{\pm} = \lambda_n |n, k\rangle_{\pm} ,$$

where $\lambda_n > 0$ and $\lambda_n^{-1} \rightarrow 0$ for $n \rightarrow \infty$, and with sign

$$F |n, k\rangle_{\pm} = |n, k\rangle_{\mp} .$$

It is a typical situation to have D defined by its eigenvalues as above. Such a D is selfadjoint with domain the set of L^2 -vectors $\psi = \sum_{n,k} c_{n,k}^{\pm} |n, k\rangle_{\pm}$ such that $|\lambda_n c_{n,k}^{\pm}|^2$ is summable. We'll assume also that \mathcal{A} has a finite number of generators $\{x_i\}$, and that each generator is represented by a *finite* sum $\pi_{\pm}(x_i) = \sum_{j,l} x_{i,\pm}^{j,l}$ of bounded shift operators $x_{i,\pm}^{j,l}$,

$$x_{i,\pm}^{j,l} |n, k\rangle_{\epsilon} = \delta_{\epsilon,\pm} C_{i,\epsilon}^{j,l}(n, k) |n + j, k + l\rangle_{\epsilon}$$

with $\{C_{i,\pm}^{j,l}(n, k)\}_{n,k}$ a bounded sequence (for all fixed i, j, l, \pm). Finally, we assume that

$$|C_{i,+}^{j,l}(n, k) - C_{i,-}^{j,l}(n, k)| \leq K_i^{j,l} q^n \quad (2.17)$$

for some $K_i^{j,l} > 0$ and a fixed q , $0 < q < 1$.

Lemma 2.13. *The data $(\mathcal{A}, \mathcal{H}, F, \gamma)$ is a 1-summable even Fredholm module.*

Proof. By construction

$$\gamma F [F, x_{i,+}^{j,l} + x_{i,-}^{j,l}] |n, k\rangle_{\pm} = \{C_{i,+}^{j,l}(n, k) - C_{i,-}^{j,l}(n, k)\} |n + j, k + l\rangle_{\pm} \quad (2.18)$$

and then the commutators

$$[F, x_i] = (\gamma F)^{-1} \sum_{j,l} \gamma F [F, x_{i,+}^{j,l} + x_{i,-}^{j,l}]$$

because of (2.17) are of trace class for all i . Since trace-class operators form a (two-sided $*$ -)ideal in $\mathcal{B}(\mathcal{H})$, from Leibniz rule

$$[F, ab] = [F, a]b + a[F, b], \quad \forall a, b \in \mathcal{A},$$

we deduce that $[F, a]$ is of trace class for any $a \in \mathcal{A}$. \square

The next quest is for a simple sufficient condition on the data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ to have a regular even spectral triple.

Lemma 2.14. *If $\lambda_{n+1} - \lambda_n$ is a bounded sequence, the data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is a even regular spectral triple.*

Proof. Recall that $\delta := [|D|, \cdot]$ and note the identity

$$\delta^p([D, a]) = \delta^{p+1}(a)F + |D|[F, \delta^p(a)], \quad \forall p \in \mathbb{N}. \quad (2.19)$$

From

$$\delta^p(x_{i,\pm}^{j,l})|n, k\rangle_{\pm} = (\lambda_{n+j} - \lambda_n)^p x_{i,\pm}^{j,l}|n, k\rangle_{\pm} \quad (2.20)$$

we deduce that $\delta^p(x_{i,\pm}^{j,l})$ are bounded operators for any $p \in \mathbb{N}$, since $\lambda_{n+j} - \lambda_n = O(1)$. This implies that $\delta^p(x_i)$ are bounded operators for any $p \in \mathbb{N}$.

From

$$|D|[F, \delta^p(x_{i,\pm}^{j,l})]|n, k\rangle_{\pm} = (\lambda_{n+j} - \lambda_n)^p \lambda_{n+j}[F, x_{i,\pm}^{j,l}]|n, k\rangle_{\pm}, \quad (2.21)$$

from (2.18) and (2.17) we deduce that $|D|[F, \delta^p(x_i)]$ is bounded for all $p \in \mathbb{N}$. Since for $a = x_i$ both summands in (2.19) are bounded, $\delta^p([D, x_i])$ is also bounded for all $p \in \mathbb{N}$. Thus, by Leibniz rule we have the set inclusion

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \text{OP}^0$$

and $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple. That it is even is obvious. \square

As a particular case we have the following, which is a simple consequence of the observation that the series $\sum_{n \in \mathbb{N}} (n+1)^{-s}$ converges to the Riemann zeta function $\zeta(s)$ for any $s > 1$, and that $\text{Res}_{s=1} \zeta(s) = 1$.

Lemma 2.15. *If $\lambda_n = n+1$ and its multiplicity is a polynomial in n of order α , then $(\mathcal{A}, \mathcal{H}, D)$ is a regular spectral triple with metric dimension $\alpha + 1$.*

Proof. Simply

$$\text{Tr}(|D|^{-s}) \propto \sum_{n \in \mathbb{N}} \{(n+1)^{-s+\alpha} + \text{lower order monomials in } n\}.$$

Thus $|D|^{-s}$ is of trace class for $s > \alpha + 1$. The Dixmier trace of $|D|^{-\alpha-1}$ is the residue in $s = \alpha + 1$ of the meromorphic extension of $\text{Tr}(|D|^{-s})$, which is $\zeta(s - \alpha)$ plus terms which are holomorphic for $\text{Re } s > \alpha$ (coming from monomials of order $\leq \alpha - 1$ in the multiplicity of λ_n). Then $\text{Tr}_{\omega}(|D|^{-\alpha-1}) \propto \text{Res}_{s=1} \zeta(s) \neq 0$. \square

Lemma 2.16. *If $\lambda_n = O(q^{-n})$ and if, for all $j \neq 0$,*

$$|C_{i,\pm}^{j,l}(n, k)| \leq K_{i,\pm}^{j,l} q^n, \quad (2.22)$$

for some constants $K_{i,\pm}^{j,l} > 0$, the data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is a even Lipschitz-regular spectral triple. The triple is regular if λ_n diverges at most polynomially.

Proof. By (2.19) to prove the first statement it is sufficient to prove that $\delta(x_i)$ and $|D|[F, x_i]$ are bounded. As above using (2.21) with $p = 0$, (2.18) and (2.17) one proves that $|D|[F, x_{i,+}^{j,l} + x_{i,-}^{j,l}]$ are bounded operators, and then $|D|[F, x_i]$ are bounded.

Now using (2.20) we prove that $\delta^p(x_{i,\pm}^{j,l})$ is zero for $j = 0$, while for $j \neq 0$

$$\left| \pm \langle n+j, k+l | \delta^p(x_{i,\pm}^{j,l}) | n, k \rangle_{\pm} \right| \leq K_{i,\pm}^{j,l} (\lambda_{n+j} - \lambda_n)^p q^n.$$

For $p = 1$, the sequence $(\lambda_{n+j} - \lambda_n)^p q^n$ is bounded. Thus $\delta(x_{i,\pm}^{j,l})$ are bounded operators, and $\delta(x_i)$ are bounded too. This proves the first statement.

If λ_n grows at most polynomially in n , then $(\lambda_{n+j} - \lambda_n)^p q^n$ is a bounded sequence for all p , and by previous inequality $\delta^p(x_{i,\pm}^{j,l})$ are bounded operators for all p . Using (2.21), (2.18) and (2.17) one proves that also $|D|[F, \delta^p(x_{i,+}^{j,l} + x_{i,-}^{j,l})]$ are bounded operators for all p , and then $|D|[F, \delta^p(x_i)]$ are bounded. By (2.19), this proves that x_i and $[D, x_i]$ are in OP^0 , and then the spectral triple is regular. \square

2.9 Local index formula for quantum homogeneous spaces

Typically, Fredholm modules associated to isospectral Dirac operators on even-dimensional quantum homogeneous spaces satisfy $[F, a] \in \text{OP}^{-\infty}$ for all $a \in \mathcal{A}$ (cf. [DDLW07, DDL08]), and this results in a drastic simplification of the local index formula. The following proposition clarifies this last statement. We use the notations of Theorem 2.3.

Proposition 2.17. *Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be a regular even spectral triple with finite metric dimension, with dimension spectrum Σ made of simple poles only, $0 \notin \Sigma$ and $[F, a] \in \text{OP}^{-\infty}$ for all $a \in \mathcal{A}$ ($F = |D|D^{-1}$). Then, the Connes-Moscovici local cocycle φ has a unique non-vanishing component $\varphi_0 = \text{ch}_0^F$.*

Proof. Let $n \in 2\mathbb{N}$ be the metric dimension. Then $\text{Tr}(\gamma a_0 |D|^{-2s})$ is holomorphic for $\text{Re } 2s > n$, and in this half plane coincides with the function $\psi(s) := \frac{1}{2} \text{Tr}(\gamma F [F, a_0] |D|^{-2s})$ (just use $F^2 = 1$, $[F, |D|] = 0$ and the cyclicity of the trace). But $\psi(s)$ is holomorphic on \mathbb{C} , since $[F, a_0] \in \text{OP}^{-\infty}$, and then $\psi(s)$ is just the holomorphic extension on \mathbb{C} of the zeta-function $\text{Tr}(\gamma a_0 |D|^{-2s})$. This implies that

$$\varphi_0(a_0) := \text{Res}_{s=0} s^{-1} \psi(s) = \psi(0) = \text{ch}_0^F(a_0).$$

It remains to prove that all other components φ_j of the cocycle φ vanish. The generic component, for j even greater than 0, is given by:

$$\varphi_j(a_0, \dots, a_j) = \frac{1}{2} \sum_{k \in \mathbb{N}^j} \frac{(-1)^k}{k_1! \dots k_j!} \alpha_k \int \gamma a_0 [D, a_1]^{(k_1)} \dots [D, a_j]^{(k_j)} |D|^{-(2|k|+j)} .$$

Let $\chi(a) = \frac{1}{2}(a + FaF)$. Then $\chi(a)$ commutes with F and γ , and $a - \chi(a) = \frac{1}{2}F[F, a] \in \text{OP}^{-\infty}$. Smoothing operators do not contribute to φ_j , so we can replace a_i with $\chi(a_i)$ in the formula for φ_j and obtain:

$$\varphi_j(a_0, \dots, a_j) = \frac{1}{2} \sum_{k \in \mathbb{N}^j} \frac{(-1)^k}{k_1! \dots k_j!} \alpha_k \int \gamma F^j \chi(a_0) \delta \chi(a_1)^{(k_1)} \dots \delta \chi(a_j)^{(k_j)} |D|^{-(2|k|+j)} .$$

But $F^j = 1$ since j is even. Since $\chi(a)$ commutes with F and γ , it commutes also with $\sigma_{\pm} := \frac{1}{2}(1 \pm \gamma)F$. The operators σ_{\pm} commute with $|D|$ as well, so we conclude that $\xi := \chi(a_0) \delta \chi(a_1)^{(k_1)} \dots \delta \chi(a_j)^{(k_j)} |D|^{-(2|k|+j)}$ commutes with σ_{\pm} . Using the identity $\gamma = [\sigma_+, \sigma_-]$ we can write

$$\varphi_j(a_0, \dots, a_j) = \frac{1}{2} \sum_{k \in \mathbb{N}^j} \frac{(-1)^k}{k_1! \dots k_j!} \alpha_k \text{Res}_{s=0} \text{Tr}([\sigma_+, \sigma_- \xi |D|^{-s}]) .$$

Now the expression under the trace is of trace class for $\text{Re } s > n$ and in such a half plane is also traceless, being a commutator. This proves that $\varphi_j(a_0, \dots, a_n) = 0$ for all even $j > 0$. \square

For some even-dimensional quantum spaces, for example quantum complex projective spaces, there is a strong belief that the situation can drastically change by considering twisted homologies (see for example [NT05] for the case of the standard Podleś sphere $\mathbb{C}P_q^1$).

2.10 Hopf algebras and equivariance

An important notion, which enters in the construction of many noncommutative spectral triples, is the one of *equivariance*. It is inspired by the example of coset spaces. Given a (compact) Riemannian G -symmetric space M , with G a Lie group and K the isotropy group of a fixed point $p \in M$, it is natural to look for the existence of a homogeneous spin structure [Fri00, CG88, BILP03], which is a commutative diagram

$$\begin{array}{ccc} \text{Spin}(T_p M) & \xleftarrow{\widetilde{\text{Ad}}} & K \\ \downarrow \lambda & \swarrow \text{Ad} & \\ \text{SO}(T_p M) & & \end{array}$$

where the arrows are group homomorphisms: λ is the double covering of the SO group, Ad is the restriction of the adjoint action of G on its Lie algebra, and the dashed arrow is the lift one looks for. Homogeneous spin structures correspond to commutative ‘equivariant’ spectral triples. One identifies $C(M)$ with $C_K(G)$, and spinors with L^2 sections of the bundle $G \times_K V \rightarrow M$

associated to the representation V of K obtained by composing the map $\widetilde{\text{Ad}}$ with the defining representation of the spin group. The representation π of $C(M)$ on spinors is given by pointwise multiplication, while a representation U of $g \in G$ is given by the pullback of left multiplication by g^{-1} , $\{U(g)\psi\}(m) := \psi(g^{-1}m)$; the representation U is unitary since G acts by isometries. The map π intertwines the representation U of G with the natural action of G on $C(M)$:

$$U(g)\pi(f)U(g)^{-1} = \pi(L_g f), \quad (2.23)$$

where $(L_g f)(m) := f(g^{-1}m)$ for all $g \in G$, $f \in C(M)$ and $m \in M$. When a representation π satisfies (2.23) we say that it is G -covariant. A fundamental characteristic of homogeneous spin structures is that they have associated an invariant Dirac operator (i.e. commuting with $U(g)$ for all $g \in G$), and eventually an invariant grading if we are in the even-dimensional case; the charge conjugation J , instead, is not invariant but intertwines the representation U of G with the representation

$$g \mapsto U(g^{-1})^\dagger \quad (2.24)$$

(in this case $U(g^{-1})^\dagger = U(g)$ by unitarity). All these properties can be formalized to get an abstract definition of equivariant spectral triple (see e.g. [Sit03]).

Any G -covariant representation of $C(M)$ is equivalent to a representation of the ‘transformation group C^* -algebra’ $G \rtimes C(M)$ (replaced by *crossed product algebras* in the noncommutative case), and both are equivalent to Mackey’s system of imprimitivity (cf. [Lan06] and reference [60] therein), thus establishing a link with quantization of mechanical systems.

By Serre-Swan theorem, $L^2(G \times_K V)$ is the completion of a finitely generated projective $C(M)$ -module. Thus, another way to construct this equivariant spin representation π is starting from $C(M)^N$ and looking for a suitable invariant projection, i.e. a bounded linear map

$$p : C(M)^N \rightarrow C(M)^N$$

such that $p = p^2 = p^*$ and $\rho(g)p = p\rho(g)$ for all $g \in G$ and with ρ a suitable representation. The Hilbert space completion of $C(M)^N p$ will carry automatically a G -equivariant representation of $C(M)$, and one has just to guess the correct p to get spinors. This is the strategy we’ll adopt for the quantum orthogonal 4-sphere S_q^4 in Chapter 4.

We are going to discuss the noncommutative analogue of G -covariant representations (and to define equivariant spectral triples in full generality). Before this, we shall briefly discuss the objects which replace groups in the noncommutative case, i.e. Hopf algebras. Reference textbooks for this topic are [Abe80, Swe69, Maj95, Vár03, FGB05].

2.10.1 Hopf algebras and crossed products

We recall that by a $(*)$ -algebra we mean an associative algebra over \mathbb{C} with unity (and involution). Let \mathcal{A} be a finite-dimensional algebra. On the dual space $\mathcal{C} = \text{Hom}(\mathcal{A}, \mathbb{C})$ we can define

two operations as follows. Writing $f(a) =: \langle a, f \rangle$ we call $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ and $\epsilon : \mathcal{C} \rightarrow \mathbb{C}$ the linear maps uniquely defined by

$$\langle a \otimes b, \Delta f \rangle = \langle ab, f \rangle \quad (2.25a)$$

$$\epsilon(f) = \langle 1, f \rangle \quad (2.25b)$$

for all $a, b \in \mathcal{A}$ and $f \in \mathcal{C}$, and known as ‘coproduct’ and ‘counit’ respectively. The associativity and the unitality of \mathcal{A} imply, respectively, the identities

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \quad (2.26a)$$

$$(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id. \quad (2.26b)$$

A vector space with two linear maps (Δ, ϵ) satisfying (2.26) is called a ‘coalgebra’. In the infinite-dimensional case, the dual space $\text{Hom}(\mathcal{A}, \mathbb{C})$ of \mathcal{A} is not always a coalgebra with the operations defined by (2.25), since in general the map Δ sends $\text{Hom}(\mathcal{A}, \mathbb{C})$ into $\text{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathbb{C})$, which is a suitable completion of the algebraic tensor product $\text{Hom}(\mathcal{A}, \mathbb{C}) \otimes \text{Hom}(\mathcal{A}, \mathbb{C})$. This is the reason why one prefers to talk of *dual pair* $(\mathcal{A}, \mathcal{C})$ of algebra-coalgebra, meaning that \mathcal{A} is an algebra, \mathcal{C} a coalgebra, and there is a bilinear map $\langle \cdot, \cdot \rangle : \mathcal{A} \otimes \mathcal{C} \rightarrow \mathbb{C}$ such that (2.25) is satisfied. The dual pair is *non-degenerate* whenever the pairing is non-degenerate, i.e. if the unique element of \mathcal{A} (resp. \mathcal{C}) which has vanishing pairing with all the elements of \mathcal{C} (resp. \mathcal{A}) is the zero. In the finite-dimensional case there is a unique coalgebra in non-degenerate dual pairing with any \mathcal{A} , the coalgebra $\text{Hom}(\mathcal{A}, \mathbb{C})$ with operations (2.25); in the infinite-dimensional case this is no more true: there can be many non-isomorphic coalgebras in non-degenerate dual pairing with a given algebra. An example where this situation occurs is the coordinate algebra of the quantum $SU(2)$ group [KS97]. In a dual pair $(\mathcal{A}, \mathcal{C})$, \mathcal{A} is commutative if and only if the coproduct of \mathcal{C} is symmetric (we say that \mathcal{C} is ‘cocommutative’).

If \mathcal{A} is an algebra and (V, W) a non-degenerate dual pair of vector spaces, any left \mathcal{A} -module structure on W corresponds to a right \mathcal{A} -module structure on V (and vice versa) through the formula

$$\langle va, w \rangle = \langle v, aw \rangle, \quad \forall a \in \mathcal{A}, v \in V, w \in W. \quad (2.27)$$

The vector space \mathcal{A} carries a natural left/right \mathcal{A} -module structure given by left/right multiplication, sometimes called ‘left/right regular action’. If \mathcal{C} is a coalgebra in non-degenerate dual pairing with \mathcal{A} , a right/left \mathcal{A} -module structure on \mathcal{C} dual to the left/right regular action is defined using (2.27). The corresponding action is called ‘right/left canonical’, denoted \triangleleft and \triangleright respectively, and given by $(a \in \mathcal{A}, c \in \mathcal{C})$

$$c \triangleleft a = \langle a, c_{(1)} \rangle c_{(2)}, \quad a \triangleright c = c_{(1)} \langle a, c_{(2)} \rangle. \quad (2.28)$$

As usual we adopt Sweedler notation for the coproduct, $\Delta c = c_{(1)} \otimes c_{(2)}$ with summation implied.

A natural question is: if we have two representations $\rho_i : \mathcal{A} \rightarrow \text{Aut}(V_i)$ of an algebra, how to construct a representation $\psi : \mathcal{A} \rightarrow \text{Aut}(V_1 \otimes V_2)$ on the tensor product of the vector spaces

V_i starting from $\rho_{1,2}$. The tensor product $\rho_1 \otimes \rho_2$ is a representation of $\mathcal{A} \otimes \mathcal{A}$ on $V_1 \otimes V_2$, so all what we need is an algebra morphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$; the map

$$\psi := (\rho_1 \otimes \rho_2) \circ \Delta \quad (2.29)$$

gives the representation we were looking for, and is called *Hopf-tensor product* of ρ_1 and ρ_2 . With n representations ρ_i we can recursively define an n -fold tensor product, and the result does not depend on the order we multiply the ρ_i s exactly when Δ is coassociative, i.e. satisfies (2.26a). Similarly, two representations $\rho_{1,2}$ of \mathcal{A} can be constructed from a representation ψ of $\mathcal{A} \otimes \mathcal{A}$ if we have an algebra morphism $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$, simply putting $\rho_1 = \psi \circ (id \otimes \epsilon)$ and $\rho_2 = \psi \circ (\epsilon \otimes id)$; these operations are inverses of (2.29) if ϵ is a counit. Motivated by these considerations, one calls *bialgebra* an algebra \mathcal{A} which is also a coalgebra and such that coproduct and counit are algebra morphisms. Note that the dual object of a bialgebra is itself a bialgebra.

We remark that since in general Δ is not symmetric, the Hopf tensor products $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are different. The order can be crucial: the difference between a first attempt of constructing a spectral triple for $SU_q(2)$ in [Gos01] and a later successful construction in [DLS⁺05a] was exactly the order in an Hopf tensor product of representations.

A source of commutative examples of bialgebras is from group theory. If \mathcal{A} is the algebra of functions on a finite group G (or also representative functions of a compact matrix group), a coproduct and counit on \mathcal{A} can be defined by

$$(\Delta f)(g, g') := f(gg'), \quad \epsilon(f) := f(e),$$

with $f \in \mathcal{A}$, $g \in G$ and e the neutral element. A further operation $S : \mathcal{A} \rightarrow \mathcal{A}$ is obtained by dualizing the operation of taking the inverse,

$$\{S(f)\}(g) = f(g^{-1}),$$

and satisfies

$$S(f_{(1)})f_{(2)} = f_{(1)}S(f_{(2)}) = \epsilon(f), \quad \forall f \in \mathcal{A}. \quad (2.30)$$

A bialgebra \mathcal{A} with a map $S : \mathcal{A} \rightarrow \mathcal{A}$ satisfying (2.30) is called *Hopf algebra*, and S is called *antipode*. A remarkable fact is that when S exists it is unique, being the inverse of the identity map under the convolution product in $\text{End}(\mathcal{A})$, cf. equation (2.35). Given a non-degenerate dual pair of bialgebras $(\mathcal{A}, \mathcal{U})$ and an antipode S on \mathcal{A} , one defines an antipode on \mathcal{U} as

$$\langle S(a), h \rangle = \langle a, S(h) \rangle, \quad a \in \mathcal{A}, h \in \mathcal{U},$$

and talks of (non-degenerate) dual pair of Hopf-algebras.

If \mathcal{A} is a Hopf algebra, other natural actions of \mathcal{A} on itself besides the regular ones are the left/right adjoint actions, defined by

$$\text{Ad}_L(a)b := a_{(1)}bS(a_{(2)}), \quad \text{Ad}_R(a)b := S(a_{(1)})ba_{(2)},$$

for all $a, b \in \mathcal{A}$.

The last point to discuss is the involution. Suppose \mathcal{A} is a Hopf algebra, and as an algebra it possesses an involution. Let ρ_1 and ρ_2 be two $*$ -representations. Their Hopf tensor product (2.29) is a $*$ -representation (for any pair ρ_1, ρ_2) if and only if

$$\Delta(a^*) = (a_{(1)})^* \otimes (a_{(2)})^* \quad (2.31a)$$

for all $a \in \mathcal{A}$. Similarly given a $*$ -representation ψ of $\mathcal{A} \otimes \mathcal{A}$, using ϵ as explained before we get two $*$ -representations of \mathcal{A} (for any ψ) if and only if

$$\epsilon(a^*) = \overline{\epsilon(a)} \quad (2.31b)$$

for all $a \in \mathcal{A}$. If (2.31) are satisfied, we call \mathcal{A} a *Hopf $*$ -algebra*. From the axioms it automatically follows that the antipode S of a Hopf $*$ -algebra is invertible, with inverse

$$S^{-1}(a) = S(a^*)^*, \quad \forall a \in \mathcal{A}.$$

Given a non-degenerate dual pair of Hopf algebras $(\mathcal{A}, \mathcal{U})$, if the first is a Hopf $*$ -algebra, also the second becomes a Hopf $*$ -algebra with the involution

$$\langle a, h^* \rangle = \overline{\langle S(a)^*, h \rangle}, \quad \forall a \in \mathcal{A}, h \in \mathcal{U}.$$

Inspired by the (left) canonical and (left) adjoint actions of a Hopf $*$ -algebra, if $(\mathcal{U}, \Delta, \epsilon, S)$ is a Hopf $*$ -algebra and \mathcal{A} a $*$ -algebra, we say that \mathcal{A} is a (left) \mathcal{U} -*module $*$ -algebra* if there is a (left) action ‘ \triangleright ’ of \mathcal{U} on \mathcal{A} satisfying

$$h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \epsilon(h)1, \quad h \triangleright a^* = \{S(h)^* \triangleright a\}^*,$$

for all $h \in \mathcal{U}$ and $a, b \in \mathcal{A}$. The (left) *crossed product* $\mathcal{A} \rtimes \mathcal{U}$ is the $*$ -algebra generated by \mathcal{A} and \mathcal{U} with crossed commutation relations

$$ha = (h_{(1)} \triangleright a)h_{(2)}, \quad \forall h \in \mathcal{U}, a \in \mathcal{A}.$$

Right module $*$ -algebras and right crossed products are defined in a similar way.

2.10.2 Equivariant projective modules

Let \mathcal{U} be a Hopf $*$ -algebra, \mathcal{A} be an \mathcal{U} -module $*$ -algebra and $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ be an invariant faithful state (i.e. φ is linear, $\varphi(a^*a) > 0$ for all non-zero $a \in \mathcal{A}$, and $\varphi(h \triangleright a) = \epsilon(h)\varphi(a)$ for all $a \in \mathcal{A}$ and $h \in \mathcal{U}$). Suppose also that there exists $\kappa \in \text{Aut}(\mathcal{A})$ such that the ‘twisted’ cyclicity

$$\varphi(ab) = \varphi(b\kappa(a))$$

holds for all $a, b \in \mathcal{A}$. Instances of this situation are provided by subalgebras of compact quantum group algebras with φ the Haar state and κ the modular automorphism. KMS states in Thermal Quantum Field Theory provide additional examples.

For $N \in \mathbb{N}$, let $\mathcal{A}^N := \mathcal{A} \otimes \mathbb{C}^N$ be the linear space with elements $v = (v_1, \dots, v_N)$, $v_i \in \mathcal{A}$, and scalar product given by

$$\langle v, w \rangle := \sum_{i=1}^N \varphi(v_i^* w_i) . \quad (2.32)$$

Lemma 2.18. *Let $\sigma : \mathcal{U} \rightarrow \text{Mat}_N(\mathbb{C})$ be a $*$ -representation. The formulae:*

$$(a \cdot v)_i := av_i , \quad (h \cdot v)_i := \sum_{j=1}^N (h_{(1)} \triangleright v_j) \sigma_{ij}(h_{(2)}) , \quad (2.33)$$

for all $a, v_i \in \mathcal{A}$ and $h \in \mathcal{U}$ (and $i = 1, \dots, N$), define a $*$ -representation of the crossed product algebra $\mathcal{A} \rtimes \mathcal{U}$ on the linear space \mathcal{A}^N .

Proof. The inner product allows us to define the adjoint of an element of $\mathcal{A} \rtimes \mathcal{U}$ in the representation on \mathcal{A}^N . For $x \in \text{End}(\mathcal{A}^N)$ its adjoint, denoted with x^\dagger , is defined by

$$\langle x^\dagger \cdot v, w \rangle := \langle v, x \cdot w \rangle , \quad \forall v, w \in \mathcal{A}^N .$$

Recall that being a $*$ -representation means that $x^\dagger \cdot v = x^* \cdot v$ for any operator x and any $v \in \mathcal{A}^N$. The non-trivial part of the proof consists in showing that $h^\dagger \cdot v = h^* \cdot v$ for all $h \in \mathcal{U}$ and $v \in \mathcal{A}^N$. It is enough to take $N = 1$. For $N > 1$ we are considering the Hopf tensor product of the $N = 1$ representation with a matrix representation that is a $*$ -representation by hypothesis.

The \mathcal{U} -invariance of φ implies:

$$\epsilon(h) \langle v, w \rangle = \varphi(h \triangleright (v^* w)) = \varphi((h_{(1)} \triangleright v^*)(h_{(2)} \triangleright w)) .$$

But $h_{(1)} \triangleright v^* = \{S(h_{(1)})^* \triangleright v\}^*$ by definition of module $*$ -algebra. Then,

$$\epsilon(h) \langle v, w \rangle = \langle S(h_{(1)})^* \cdot v, h_{(2)} \cdot w \rangle = \langle v, S(h_{(1)})^{*\dagger} h_{(2)} \cdot w \rangle .$$

We deduce that for all $h \in \mathcal{U}$ one has that

$$S(h_{(1)})^{*\dagger} h_{(2)} = \epsilon(h) . \quad (2.34)$$

Recall that the convolution product ' \star ' for any $F, G \in \text{End}(\mathcal{U})$ is defined by

$$(F \star G)(h) := F(h_{(1)})G(h_{(2)}) \quad \forall h \in \mathcal{U} ; \quad (2.35)$$

and $(\text{End}(\mathcal{U}), \star)$ is an associative algebra with unity given by the endomorphism $h \mapsto \epsilon(h)1_{\mathcal{U}}$, with S a left and right inverse for $id_{\mathcal{U}}$ in $(\text{End}(\mathcal{U}), \star)$, that is

$$S \star id_{\mathcal{U}} = 1_{\mathcal{U}} \epsilon = id_{\mathcal{U}} \star S .$$

Let $S' \in \text{End}(\mathcal{U})$ be the composition $S' := \dagger \circ * \circ S$. Equation (2.34) implies that S' is a left inverse for $id_{\mathcal{U}}$:

$$S' \star id_{\mathcal{U}} = 1_{\mathcal{U}} \epsilon .$$

Applying $\star S$ to the right of both members of this equation and using $id_{\mathcal{U}} \star S = 1_{\mathcal{U}}\epsilon$ we get $S' = S$ as endomorphisms of \mathcal{U} , i.e. $S(h)^{\star\dagger} = S(h)$ for all $h \in \mathcal{U}$.

Now, the antipode of a Hopf \ast -algebra is invertible, with $S^{-1} = \ast \circ S \circ \ast$, thus we arrive at $h^{\star\dagger} = h$ for all $h \in \mathcal{U}$. Replacing h with h^{\ast} we prove that $h^{\dagger} = h^{\ast}$ for all $h \in \mathcal{U}$, and this concludes the proof. \square

Now, let $e = (e_{ij}) \in \text{Mat}_N(\mathcal{A})$ be an $N \times N$ matrix with entries $e_{ij} \in \mathcal{A}$. Let $\pi : \mathcal{A}^N \rightarrow \mathcal{A}^N$ be the (linear) endomorphism defined by:

$$\pi(v)_j := \sum_{i=1}^N v_i e_{ij}, \quad (2.36)$$

for all $v \in \mathcal{A}^N$ and $j = 1, \dots, N$. Since \mathcal{A} is associative, left and right multiplication commute and $\pi(av) = a\pi(v)$ for all $a \in \mathcal{A}$ and $v \in \mathcal{A}^N$. Thus we have the following Lemma.

Lemma 2.19. *The map π defined by (2.36) is an \mathcal{A} -module map.*

Recall that an endomorphism p of an inner product space V is a projection (not necessarily orthogonal) if $p \circ p = p$. A projection p is *orthogonal* if the image of p and $id_V - p$ are orthogonal with respect to the inner product of V , and this happens exactly when $p^{\dagger} = p$.

A simple computation shows that the map π in (2.36) is a projection iff $e^2 = e$, that is the matrix $e \in \text{Mat}_N(\mathcal{A})$ is an idempotent. Now we use the twisted-cyclicity of φ to deduce:

$$\langle v, \pi^{\dagger}(w) \rangle = \langle \pi(v), w \rangle = \sum_{ij} \varphi(e_{ij}^{\ast} v_i^{\ast} w_j) = \sum_{ij} \varphi(v_i^{\ast} w_j \kappa(e_{ij}^{\ast})),$$

for all $v, w \in \mathcal{A}^N$. Hence the adjoint π^{\dagger} of the endomorphism π is given by

$$\pi^{\dagger}(w)_i = \sum_{j=1}^N w_j \kappa(e_{ij}^{\ast}).$$

Let e^{\ast} be the matrix with entries $(e^{\ast})_{jk} := e_{kj}^{\ast}$. We have the following trivial Lemma.

Lemma 2.20. *The endomorphism π in (2.36) is an orthogonal projection iff $e^2 = e = \kappa(e^{\ast})$.*

Next, we determine a sufficient condition for the endomorphism π to be not only an \mathcal{A} -module map, but also an \mathcal{U} -module map.

Lemma 2.21. *With ‘ t ’ denoting transposition, if*

$$h \triangleright e = \sigma(h_{(1)})^t e \sigma(S^{-1}(h_{(2)}))^t \quad (2.37)$$

for all $h \in \mathcal{U}$, then the endomorphism π in (2.36) is an \mathcal{U} -module map.

Proof. Equation (2.37) can be rewritten as,

$$h \triangleright e_{ij} = \sum_{kl} \sigma_{ki}(h_{(1)}) e_{kl} \sigma_{jl}(S^{-1}(h_{(2)}));$$

by using it into the definition (2.33) one checks that $\pi(h \cdot v) = h \cdot \pi(v)$ for all $h \in \mathcal{U}$ and $v \in \mathcal{A}^N$.
Indeed

$$\begin{aligned} \{h \cdot \pi(v)\}_i &= \sum_{j k} (h_{(1)} \triangleright v_j e_{jk}) \sigma_{ik}(h_{(2)}) \\ &= \sum_{j k} (h_{(1)} \triangleright v_j) (h_{(2)} \triangleright e_{jk}) \sigma_{ik}(h_{(3)}) \\ &= \sum_{j k l m} (h_{(1)} \triangleright v_j) \sigma_{lj}(h_{(2)}) e_{lm} \sigma_{km}(S^{-1}(h_{(3)})) \sigma_{ik}(h_{(4)}) \\ &= \sum_{j l m} (h_{(1)} \triangleright v_j) \sigma_{lj}(h_{(2)}) e_{lm} \sigma_{im}(h_{(4)} S^{-1}(h_{(3)})) , \\ &= \sum_{l m} (h_{(1)} \cdot v)_l e_{lm} \sigma_{im}(h_{(3)} S^{-1}(h_{(2)})) , \end{aligned}$$

and since $\sigma_{im}(h_{(3)} S^{-1}(h_{(2)})) = \epsilon(h_{(3)}) \delta_{im}$, last equation is equal to

$$= \sum_l (h_{(1)} \cdot v)_l e_{li} \epsilon(h_{(2)}) = \pi(h \cdot v)_i ,$$

which concludes the proof. \square

When Lemma 2.20 and equation (2.37) are satisfied, the orthogonal projections π and $\pi^\perp = 1 - \pi$ split \mathcal{A}^N into the orthogonal sum of two $*$ -subrepresentations $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ of $\mathcal{A} \rtimes \mathcal{U}$. The next lemma gives a (quite obvious) sufficient condition for $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ to be inequivalent as representations of \mathcal{A} . Recall that an isomorphism of \mathcal{A} -modules is an invertible \mathcal{A} -linear map, so isomorphic left modules correspond to equivalent representations.

Lemma 2.22. *Let $(\mathcal{A}, \mathcal{H}, F, \gamma)$ be an even Fredholm module over \mathcal{A} , and F_e^+ the operator in (2.2). If $\text{Index}(F_e^+) \neq 0$, the left \mathcal{A} -modules $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are not equivalent.*

Proof. Suppose $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are isomorphic \mathcal{A} -modules, then $[e] = [1-e]$ and $\text{Index}(F_{1-e}^+) = \text{Index}(F_e^+)$. But from equation (2.3), $\text{Index}(F_{1-e}^+) = -\text{Index}(F_e^+)$. Hence $\text{Index}(F_e^+) = 0$, and this concludes the proof by contradiction. \square

Notice that if (2.37) is satisfied, the *right* module $e\mathcal{A}^N$ carries a left action ‘ \blacktriangleright ’ of \mathcal{U} given by

$$h \blacktriangleright v := \sigma(S^{-1}(h_{(1)}))^t h_{(2)} \triangleright v , \quad (2.38)$$

for all $h \in \mathcal{U}$ and $v = (v_1, \dots, v_N)^t \in e\mathcal{A}^N$ (with row by column multiplication implied). One can easily check that if $ev = v$, i.e. $v \in e\mathcal{A}^N$, the vector $h \blacktriangleright v$ is still in $e\mathcal{A}^N$, i.e. $e(h \blacktriangleright v) = (h \blacktriangleright v)$. Indeed, by (2.37) and (2.38)

$$\begin{aligned} h \blacktriangleright (ev) &= \sigma(S^{-1}(h_{(1)}))^t h_{(2)} \triangleright (ev) \\ &= \sigma(S^{-1}(h_{(1)}))^t (h_{(2)} \triangleright e) (h_{(3)} \triangleright v) \\ &= \sigma(S^{-1}(h_{(1)}))^t \sigma(h_{(2)})^t e \sigma(S^{-1}(h_{(3)}))^t (h_{(4)} \triangleright v) \\ &= \sigma(h_{(2)} S^{-1}(h_{(1)}))^t e \sigma(S^{-1}(h_{(3)}))^t (h_{(4)} \triangleright v) \\ &= \epsilon(h_{(1)}) e \sigma(S^{-1}(h_{(2)}))^t (h_{(3)} \triangleright v) \\ &= e \sigma(S^{-1}(h_{(1)}))^t (h_{(2)} \triangleright v) = e(h \blacktriangleright v) , \end{aligned}$$

which when $ev = v$ implies $e(h \blacktriangleright v) = (h \blacktriangleright v)$.

2.10.3 Equivariant spectral triples

Let $(\mathcal{U}, \epsilon, \Delta, S)$ be a Hopf $*$ -algebra and suppose \mathcal{A} is a (left) \mathcal{U} -module $*$ -algebra. The data $(\mathcal{A}, \mathcal{H}, D)$ is called a \mathcal{U} -equivariant spectral triple if

- (i) there is a dense subspace \mathcal{M} of \mathcal{H} carrying a $*$ -representation π of $\mathcal{A} \rtimes \mathcal{U}$,
- (ii) D is a selfadjoint operator with compact resolvent and with domain containing \mathcal{M} ,
- (iii) $\pi(a)$ and $[D, \pi(a)]$ extend to bounded operators on \mathcal{H} for all $a \in \mathcal{A}$,
- (iv) $[D, \pi(h)] = 0$ on \mathcal{M} for any $h \in \mathcal{U}$.

An equivariant spectral triple is called *even* if there exists a grading γ on \mathcal{H} (i.e. a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$) such that the Dirac operator is odd and the crossed product algebra is even:

$$\gamma D + D\gamma = 0, \quad \pi(t)\gamma = \gamma\pi(t) \quad \forall t \in \mathcal{A} \rtimes \mathcal{U}.$$

With antilinear operators the definition of equivariance is more subtle. Recall the representation (2.24): in a generic Hopf algebras the antipode plays the role of the inverse and this motivates the following definition. An antilinear operator T defined on \mathcal{M} is called *equivariant* if it intertwines the representation $h \mapsto \pi(h)$ with the representation $h \mapsto \pi(S(h)^*)$, that is

$$T\pi(h)v = \pi(S(h)^*)Tv$$

for all $h \in \mathcal{U}$ and $v \in \mathcal{M}$. Notice that if T is an equivariant antilinear operator, its square T^2 is invariant, i.e. commutes with all $h \in \mathcal{U}$. On the other hand, since

$$T^*T\pi(h) = T^*\pi(S(h)^*)T = \{\pi(S(h))T\}^*T = \{T\pi(S^2(h))^*\}^*T = \pi(S^2(h))T^*T,$$

the linear operator T^*T is not invariant, unless $S^2 = id_{\mathcal{U}}$ (which holds iff the Hopf algebra is cocommutative, i.e. has a symmetric coproduct).

A \mathcal{U} -equivariant *real* spectral triple is given by a \mathcal{U} -equivariant spectral triple $(\mathcal{A}, \mathcal{H}, D)$ together with a (weak) real structure J , with the additional condition that J is the antiunitary part of an equivariant antilinear operator.

Equivariant spectral triples have associated (left) covariant differential calculi, equivariant homologies, and so on. On the other hand, a basic object like the Dixmier trace is not invariant as one would expect. Let n be the metric dimension and consider the functional $f : \mathcal{A} \rightarrow \mathbb{C}$,

$$f(a) := \mathrm{Tr}_{\omega}(\pi(a)|D|^{-n}).$$

Using the equivariance and the cyclicity of the trace we get

$$f(h \triangleright a) = \mathrm{Tr}_{\omega}(\pi(h_{(1)})\pi(a)\pi(S(h_{(2)}))|D|^{-n}) = \mathrm{Tr}_{\omega}(\pi(S(h_{(2)})h_{(1)})\pi(a)|D|^{-n}),$$

and in general $S(h_{(2)})h_{(1)}$ is not equal to $\epsilon(h)$ due to the wrong order of the factors (unless $S^2 = id_{\mathcal{U}}$). Thus, we cannot conclude that $f(h \triangleright a)$ and $\epsilon(h)f(a)$ are equal: in general, the Dixmier trace is not invariant.

2.11 Quantum Homogeneous Spaces

A source of examples of noncommutative spaces is from quantum group theory. Quantum groups are particular examples of Hopf algebras which appear in the theory of quantum integrable systems, in the same way as their classical counterparts, Poisson-Lie groups, arise in the theory of classical integrable systems. Compact quantum homogeneous spaces are described by unital algebras \mathcal{A} which are comodule $*$ -algebras for some quantum group and such that the trivial corepresentation appears in the decomposition of \mathcal{A} with multiplicity 1 (that is, there are no other coinvariant elements in \mathcal{A} besides the constants). Reference textbooks for this topic are [CP94, Kas95, KS97].

Recall that on $C^\infty(G)$, G a semisimple Lie group, one can canonically define a Hopf algebra structure (modulo technicalities, since Δ does not map into the algebraic tensor product $C^\infty(G) \otimes C^\infty(G)$ but into some completion). If $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra \mathfrak{g} of G , a dual pairing between $\mathcal{U}(\mathfrak{g})$ and $C^\infty(G)$ is defined on generators $x \in \mathfrak{g}$ by

$$\langle x, f \rangle = \left. \frac{d}{dt} f(\exp tx) \right|_{t=0}, \quad \forall f \in C^\infty(G).$$

The algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra with operations dual to those of $C^\infty(G)$, and on an element x of the Lie algebra, the coproduct, counit and antipode are explicitly given by

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

With these, one can compute explicitly the left and the right canonical actions (2.28) of $x \in \mathfrak{g}$, and finds

$$(x \triangleright f)(g) = \left. \frac{d}{dt} f(ge^{tx}) \right|_{t=0}, \quad (f \triangleleft x)(g) = \left. \frac{d}{dt} f(e^{tx}g) \right|_{t=0}.$$

Thus, if $\{x_i\}$ is a basis of \mathfrak{g} orthonormal with respect to the Killing form, the derivations $x_i \triangleright$ and $\triangleleft x_i$ are a basis for left invariant and right invariant vector fields on G .

A Poisson-Lie group G is a Lie group with Poisson structure satisfying a suitable compatibility condition with the coproduct of $C^\infty(G)$. If G is simple, the most general Poisson bracket giving a Poisson-Lie group structure has the form [Tak94]

$$\{f_1, f_2\} = \sum r^{ij} (f_1 \triangleleft x_i \cdot f_2 \triangleleft x_j - x_i \triangleright f_1 \cdot x_j \triangleright f_2) \quad (2.39)$$

for all $f_{1,2} \in C^\infty(G)$, where $r^{ij} \in C^\infty(G)$ is a skew-symmetric tensor. Calling $c = \sum x_i^2$ the quadratic Casimir of $\mathcal{U}(\mathfrak{g})$ and

$$t = \Delta c - 1 \otimes c - c \otimes 1,$$

one has the following important characterization of simple Poisson-Lie groups, which can be taken as a definition. Equation (2.39) gives a Poisson-Lie group if and only if the element $r = \sum r^{ij} x_i \otimes x_j \in \wedge^2 \mathfrak{g}$, named *classical r -matrix*, satisfies the modified *classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = s[t_{13}, t_{23}] \quad (2.40)$$

for some constant s , and with usual notation $A_{12} := \sum A_i \otimes B_i \otimes 1$, $A_{13} := \sum A_i \otimes 1 \otimes B_i$ and $A_{23} := \sum 1 \otimes A_i \otimes B_i$ for any $A := \sum A_i \otimes B_i$.

If G is a matrix group, i.e. it is realized as a subgroup of $GL(n)$ for some n , then the algebra $\mathcal{A}(G)$ of *representative functions* on G is generated by the functions u_j^i associating to $g \in G$ its (i, j) th matrix entry (in $GL(n)$), and it is dense in $C^\infty(G)$.⁴ In this case the Poisson bracket (2.39) is explicitly given on generators by

$$\{u_j^i, u_m^l\} = [r, U \otimes U]_{j m}^{i l} \quad (2.41)$$

where $U = ((u_j^i))$ and we identify $r \in \mathfrak{g} \otimes \mathfrak{g}$ with its realization in $gl(n) \otimes gl(n)$. Now (2.41) is exactly the equation satisfied when U is a local Lax L -operator (or also the monodromy matrix) of an integrable model on a lattice [Bax82]. We can interpret such L -operators as coordinate functions on Poisson-Lie matrix groups. Similarly, the quantization of these models is associated to *non-commuting* L -operators which can be interpreted as ‘coordinates’ on some virtual ‘quantum group’. Quantum groups are ‘morally’ deformation quantizations of Poisson-Lie groups, with the additional condition that they are Hopf algebras with undeformed coproduct and counit (the unique antipode needs to be deformed since the product changes). In the next section as an illustrative example we show how to get $SU_q(2)$ by canonical quantization of the (unique) Poisson-Lie group structure on $SU(2)$. The problem with deformation quantization is compatibility of the product with the undeformed coalgebra structure, which has to be verified case by case, as well as the existence of the antipode.

A different approach is to start directly with a noncommutative algebra with a finite number of generators u_j^i . The idea is to write commutation relations among the generators in the form $\sum_{j_1 j_2} R_{j_1 j_2}^{i_1 i_2} u_{k_1}^{j_1} u_{k_2}^{j_2} = \sum_{j_1 j_2} u_{j_1}^{i_1} u_{j_2}^{i_2} R_{k_1 k_2}^{j_1 j_2}$, or in matrix form

$$R U_1 U_2 = U_1 U_2 R \quad (2.42)$$

with $U = ((u_j^i))$, $U_1 = U \otimes 1$ and $U_2 = 1 \otimes U$, and to look for a suitable 2-tensor R with matrix elements depending on a deformation parameter \hbar . The condition $R = 1 - i\hbar r + o(\hbar)$ guarantees that the semi-classical limit is (2.41), that is “ $\lim_{\hbar \rightarrow 0} [,]/i\hbar = \{ , \}$ ”. A sufficient condition (but not necessary) to get an associative algebra is that R solves the *quantum Yang-Baxter equation*

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} . \quad (2.43)$$

Although (2.43) is not linear and highly non-trivial to treat, a number of solutions were discovered in the ‘80s (see for example [RTF90]).

Thus, for any solution of the quantum Yang-Baxter equation we have an associative unital algebra $\mathcal{A}(R)$ generated by u_j^i with relations (2.42). This is always a bialgebra, with undeformed

⁴Coproduct is given on generators by $\Delta u_j^i = \sum_k u_k^i \otimes u_j^k$, and maps $\mathcal{A}(G)$ into $\mathcal{A}(G) \otimes \mathcal{A}(G)$, which then is a Hopf algebra. On the other hand $C^\infty(G)$ is not, since the image of Δ is $C^\infty(G \times G)$ and in general this is bigger than $C^\infty(G) \otimes C^\infty(G)$.

coproduct and counit

$$\Delta u_j^i = \sum_k u_k^i \otimes u_j^k, \quad \epsilon(u_j^i) = \delta_j^i.$$

So, to get a noncommutative deformation of a Poisson-Lie group, one just needs to find an antipode (and eventually an involution). The algebra $\mathcal{A}(R)$ is a noncommutative deformation of $\mathcal{A}(\text{Mat}_n)$, and to get an antipode one needs to restrict the attention at least to invertible matrices: that is, one must quotient $\mathcal{A}(R)$ by a suitable two-sided ideal. A universal way to construct quotients of $\mathcal{A}(R)$ which are quantum groups is indicated in [Man88, Chapter 7]; this approach is nevertheless quite difficult to handle. For Drinfeld-Jimbo solutions of the quantum Yang-Baxter equation, the construction of some noncommutative deformation of classical Lie groups was performed explicitly in [RTF90]. It is worth mentioning a different approach due to Woronowicz [Wor80], which studies compact quantum groups from the point of view of C^* -algebras.

The objects dual to quantum groups, which are deformations of universal enveloping algebras and are called for this reason *quantum universal enveloping algebras*, arise for example in conformal quantum field theory in the study of current algebras (see e.g. [AGGS89] and [Bab88]). A canonical deformation of $\mathcal{U}(\mathfrak{g})$ was constructed by Drinfeld and Jimbo for any finite-dimensional complex semisimple Lie algebra \mathfrak{g} (cf. [Jim85, Jim86a, Jim86b, Dri86]).

In the following we recall the definition of those quantum groups and quantum homogeneous spaces which we'll need in the original part of the work, starting with the celebrated $SU_q(2)$ and Podleś quantum spheres.

2.11.1 Podleś quantum spheres and quantum $SU(2)$

As mentioned, there is no general algorithm for finding solutions of the modified classical Yang-Baxter equation. Despite this, for any simple Lie group G a canonical r -matrix can be constructed as follows. Let $\langle \cdot, \cdot \rangle$ be the Killing form on \mathfrak{g} , and $P_{\pm} : \mathfrak{g} \rightarrow \mathfrak{n}_{\pm}$ the orthogonal projections onto its maximal nilpotent subalgebras \mathfrak{n}_{\pm} . Then, an r -matrix solving (2.40) is defined by the formula (cf. [STS94])

$$r = i \sum_j r'_j \otimes r''_j, \quad \text{with} \quad \sum_j r'_j \langle r''_j, \circ \rangle = P_+ - P_- \quad (2.44)$$

(we rescaled the r -matrix of [STS94] by a constant, which is always possible and simply means that the parameter s in (2.40) has to be rescaled accordingly). For $G = SL(2, \mathbb{C})$, with $\langle A, B \rangle = \text{Tr}_{\mathbb{C}^2}(AB)$ and Pauli matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

one has $P_{\pm} = \sigma_{\pm} \langle \sigma_{\mp}, \circ \rangle$ and then by (2.44)

$$r = i\sigma_+ \wedge \sigma_-.$$

Now let

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \text{with } \alpha\alpha^* + \beta\beta^* = 1,$$

where α and β are the generators of the coordinate $*$ -algebra of $SU(2)$; this is obtained from $\mathcal{A}(SL(2, \mathbb{C}))$ by imposing a $*$ -structure, as discussed e.g. in Chapter 4 of [KS97]. Thanks to the choice of the normalization, the Poisson structure on $SL(2, \mathbb{C})$ associated to r gives a Poisson structure on $\mathcal{A}(SU(2))$ which is compatible with the real structure, i.e.

$$\{a, b\}^* = \{a^*, b^*\}, \quad \forall a, b \in \mathcal{A}(SU(2)).$$

This is explicitly given on generators by

$$\{\alpha, \beta\} = i\alpha\beta, \quad \{\alpha, \beta^*\} = i\alpha\beta^*, \quad \{\beta, \beta^*\} = 0, \quad \{\alpha, \alpha^*\} = -2i\beta\beta^*. \quad (2.45)$$

By [CP94, Section 1.5D] this is the unique Poisson-Lie group structure on $SU(2)$ up to a multiplicative constant. Symplectic leaves are open disks D_t , defined for $t \in U(1)$ by $\beta = t|\beta|$ with $|\beta| \neq 0$, and the single points at the boundary of the disks [CP94, Section 1.5C].

The 2-sphere S^2 can be identified with the sub-manifold of matrices in $SU(2)$ with real off-diagonal elements, i.e. $\mathcal{A}(S^2)$ is the quotient of $\mathcal{A}(SU(2))$ by the ideal generated by the element $\beta - \beta^*$ (that is, the class of β is real in the quotient algebra). The class A of β and the class B of α generate $\mathcal{A}(S^2)$. The Poisson structure (2.45) induces a Poisson structure on the quotient, and S^2 splits into two non-trivial symplectic leaves: these are the two open hemispheres D_{\pm} , defined by $\pm A > 0$, and glued together along the equator S^1 , defined by $A = 0$. While on the equator the Poisson structure is trivial, on each hemisphere D_{\pm} the generators of $\mathcal{A}(S^2)$ can be rewritten in terms of canonical coordinates x, p , satisfying

$$\{x, p\} = 1,$$

via the transform ($x \geq 0$)

$$A = \pm e^{-x}, \quad B = \sqrt{1 - e^{-2x}} e^{ip}, \quad B^* = e^{-ip} \sqrt{1 - e^{-2x}}. \quad (2.46)$$

It is interesting to notice that the Lax operators for this Poisson structure are the ones associated to the Liouville model on a lattice [Tak94]. By applying the recipe of canonical quantization, that means replacing x and p with operators \hat{x} and \hat{p} satisfying the commutation relation (1.1),

$$[\hat{x}, \hat{p}] = i\hbar,$$

we get a noncommutative algebra whose underlying virtual space is known as the *equatorial Podleś sphere*, denoted S_q^2 and discovered in [Pod87]. Thus, coordinates on the equatorial Podleś sphere (actually, on its two open hemispheres) can be interpreted as matrix entries of Lax operators for the quantization of the Liouville model on a lattice. In fact, that S_q^2 is even a *strict* deformation quantization [Rie89] of S^2 was proved in [She91].⁵

⁵It is worth mentioning the appendix of [She91], where it is reviewed the work in [LW90] about the classification of $SU(2)$ -covariant Poisson structures on S^2 .

Commutation relations among generators of $\mathcal{A}(S_q^2)$ can be computed by replacing x, p with \hat{x}, \hat{p} in (2.46) (keeping the order in which the products are written), and using the Weyl-type relation $e^{-\hat{x}}e^{i\hat{p}} = e^{\hbar}e^{i\hat{p}}e^{-\hat{x}}$. They are written in the definition that follows. We set $q := e^{\hbar}$.

Definition 2.23. We call equatorial Podleś sphere the virtual space underlying the $*$ -algebra $\mathcal{A}(S_q^2)$ generated by $A = A^*$, B and B^* with relations:

$$AB = qBA, \quad BB^* + A^2 = 1, \quad B^*B + q^2A^2 = 1.$$

As usual due to symmetries of the algebra one can focus on the case $\boxed{0 < q < 1}$. In all this dissertation we'll always assume that q is in this range (unless stated otherwise).

The equatorial Podleś sphere turns out to be the actual building block for q -deformations. We'll see that working with local formulæ essentially means shifting the problem from a complicated quantum space to a number of copies of the equatorial Podleś sphere.

Similar relations yield a deformation of $\mathcal{A}(SU(2))$ which is a Hopf $*$ -algebra with undeformed coproduct Δ and counit ϵ . This is the well known quantum group $SU_q(2)$ of Woronowicz [Wor80, Wor87]. The sphere S_q^2 is 'embedded' in $SU_q(2)$ as an equator, though this is not the reason for the name 'equatorial' (it is called 'equatorial' since one can think of the space of characters of the algebra, which is a circle, as the equator of S_q^2).

Definition 2.24. The algebra $\mathcal{A}(SU_q(2))$ of polynomial functions on $SU_q(2)$ is the $*$ -algebra generated by α, β and their adjoints, with relations:

$$\beta\alpha = q\alpha\beta, \quad \beta^*\alpha = q\alpha\beta^*, \quad [\beta, \beta^*] = 0, \quad \alpha\alpha^* + \beta\beta^* = 1, \quad \alpha^*\alpha + q^2\beta^*\beta = 1.$$

It is a Hopf $*$ -algebra with coproduct, counit and antipode given by

$$\begin{aligned} \Delta \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix}, \\ \epsilon \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & -q\beta \\ \beta^* & \alpha \end{pmatrix}. \end{aligned}$$

Here the dotted tensor product is defined as $(A \dot{\otimes} B)_{ij} = \sum_k A_{ik} \otimes B_{kj}$. We use the same notation of [DLS⁺05a], but for Greek letters instead of Latin ones. In [KS97, sec. 4.1.4] they call $d = \alpha$, $c = -\beta$ and their fundamental corepresentation is related to ours by a unitary equivalence:

$$\begin{pmatrix} d^* & -qc^* \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Another quantum 2-sphere can be obtained by recalling that S^2 is diffeomorphic to the projective line $\mathbb{CP}^1 = SU(2)/U(1)$. For any q , the algebra $\mathcal{A}(SU_q(2))$ carries a covariant action of $u \in U(1)$ given by the map $\alpha \mapsto u\alpha$, $\beta \mapsto u\beta$. The algebra $\mathcal{A}(\mathbb{CP}_q^1)$ is defined as the fixed point algebra of $\mathcal{A}(SU_q(2))$ for this action of $U(1)$, and coincides with the subalgebra generated by $A = \beta\beta^*$, $B = \beta^*\alpha$ (we use again the same symbols A, B). This is called *standard Podleś*

sphere, and commutation relations among its generators can be easily computed from the ones defining $\mathcal{A}(SU_q(2))$.

Definition 2.25. We call coordinate algebra on the standard Podleś sphere $\mathbb{C}P_q^1$ the $*$ -algebra generated by $A = A^*$, B and B^* with relations

$$AB = q^2BA, \quad BB^* + A(A-1) = 0, \quad B^*B + q^2A(q^2A-1) = 0.$$

The standard and equatorial sphere are limiting cases of a family of noncommutative spheres S_{qs}^2 , depending on a parameter $s \in [0, 1]$ (q is supposed to be fixed), whose elements are all quantum homogeneous spaces for $SU_q(2)$. The spheres $\mathbb{C}P_q^1$ and $S_{q^2}^2$ correspond to the ‘extremal’ values $s = 0$ and $s = 1$, respectively.

Definition 2.26. For $0 < q < 1$ and $0 \leq s \leq 1$, the coordinate algebra on the Podleś sphere S_{qs}^2 is the $*$ -algebra generated by $A = A^*$, B and B^* with relations

$$AB = q^2BA, \quad BB^* + (A + s^2)(A-1) = 0, \quad B^*B + (q^2A + s^2)(q^2A-1) = 0.$$

For $s = 0$ we obtain the standard Podleś sphere $S_{q^0}^2 = \mathbb{C}P_q^1$, while for $s = 1$ we obtain the equatorial Podleś sphere $S_{q^1}^2 = S_{q^2}^2$.

We are using, with minor changes, the notations of [Dąb06]. The original Podleś parameter is $c = (s^{-1} - s)^{-2}$, while the original generators are $A' = (1 - s^2)^{-1}A$, $B' = (1 - s^2)^{-1}B^*$ if $s \neq 1$, and $A' = A$, $B' = B^*$ if $s = 1$ [Pod87]. The generators used in [DDLW07] are related to ours by $x_1 := \sqrt{1 + q^2}B$, $x_0 := -(1 + q^2)A + (1 - s^2)$, $x_{-1} := -q^{-1}x_1^*$.

The geometrical meaning of the parameter s is clear when one takes the $q \rightarrow 1$ limit. For $q = 1$, one gets the coordinate algebra of a 2-sphere in \mathbb{R}^3 with radius $\frac{1+s^2}{2}$ and center $(0, 0, \frac{1-s^2}{2})$, both depending on s . Notice that, for any q and s , we have an injective $*$ -algebra morphism $\mathcal{A}(S_{qs}^2) \rightarrow \mathcal{A}(SU_q(2))$ defined on generators by

$$A \mapsto s(\alpha\beta + \beta^*\alpha^*) + (1 - s^2)\beta\beta^*, \quad B \mapsto s(\alpha^2 - q(\beta^*)^2) + (1 - s^2)\beta^*\alpha, \quad (2.47)$$

and which is the noncommutative analogue of the principal bundle $SU(2) \rightarrow S^2$.

The noncommutative geometry of Podleś spheres will be studied in Chapter 3, following the papers [DD06] and [DDLW07].

2.11.2 The ‘zoo’ of quantum spaces

“We think in generalities, but we live in details”,
Alfred North Whitehead (1861 – 1947).

Generalizations of Podleś spheres and $SU_q(2)$ are possible in different directions (most of the spaces we discuss here were introduced in a more or less explicit form in [RTF90]). The equatorial Podleś sphere is the first of a family S_q^n of quantum orthogonal spheres, which are homogeneous spaces for the quantum orthogonal groups but not quotient spaces (they are not fixed

point subalgebras of $\mathcal{A}(SO_q(n+1))$ for a coaction of $SO_q(n)$, unless $q = 1$). The case $n = 4$ will be studied in Chapter 4; here we give its algebra.

Definition 2.27. We call quantum orthogonal 4-sphere the virtual space underlying the algebra $\mathcal{A}(S_q^4)$ generated by $x_0 = x_0^*$, x_1 , x_1^* , x_2 , x_2^* , with commutation relations:

$$\begin{aligned} x_i x_j &= q^2 x_j x_i, & \forall 0 \leq i < j \leq 2, \\ x_i^* x_j &= q^2 x_j x_i^*, & \forall i \neq j, \\ [x_1^*, x_1] &= (1 - q^4) x_0^2, \\ [x_2^*, x_2] &= x_1^* x_1 - q^4 x_1 x_1^*, \\ x_0^2 + x_1 x_1^* + x_2 x_2^* &= 1. \end{aligned}$$

The original notations of [RTF90, eq. (1.14)] can be obtained by defining $x'_1 := x_2^*$, $x'_2 := x_1^*$, $x'_3 := \sqrt{q(1+q^2)} x_0$, $x'_4 := x_1$, $x'_5 := x_2$ and $q' := q^2$. The notations in [HL04, eq. (2.1)] can be obtained by the replacement $x_i \mapsto x_i^*$ and $q^2 \mapsto q^{-1}$.

A family of odd-dimensional quantum spheres were introduced also in [VS91] as quantum homogeneous spaces for quantum unitary groups, and it was later recognized in [HL04, Dąb03] that these ‘unitary’ quantum spheres and the odd-dimensional orthogonal spheres were the same (up to an isomorphism).

Definition 2.28. For $\ell \geq 1$, we denote $\mathcal{A}(SU_q(\ell+1))$ the $*$ -algebra generated by the $(\ell+1)^2$ elements u_j^i ($i, j = 1, \dots, \ell+1$) with commutation relations

$$\begin{aligned} u_k^i u_k^j &= q u_k^j u_k^i & u_i^k u_j^k &= q u_j^k u_i^k & \forall i < j, \\ [u_l^i, u_k^j] &= 0 & [u_k^i, u_l^j] &= (q - q^{-1}) u_l^i u_k^j & \forall i < j, k < l, \end{aligned}$$

and determinant relation

$$\sum_{p \in S_{\ell+1}} (-q)^{|p|} u_{p(1)}^1 u_{p(2)}^2 \dots u_{p(\ell+1)}^{\ell+1} = 1$$

where the sum is over all permutations p of the $\ell+1$ elements $\{1, 2, \dots, \ell+1\}$ and $|p|$ is the (minimal) number of inversions in p . The $*$ -structure is given by

$$(u_j^i)^* = (-q)^{j-i} \sum_{p \in S_\ell} (-q)^{|p|} u_{p(m_1)}^{k_1} u_{p(m_2)}^{k_2} \dots u_{p(m_\ell)}^{k_\ell}$$

with $\{k_1, \dots, k_\ell\} = \{1, \dots, \ell+1\} \setminus \{i\}$, $\{m_1, \dots, m_\ell\} = \{1, \dots, \ell+1\} \setminus \{j\}$ (as ordered sets) and the sum is over all permutations p of the ℓ elements $\{m_1, \dots, m_\ell\}$.

This is a Hopf $*$ -algebra with standard coproduct, counit and antipode:

$$\Delta(u_j^i) = \sum_k u_k^i \otimes u_j^k, \quad \epsilon(u_j^i) = \delta_j^i, \quad S(u_j^i) = (u_i^j)^*.$$

We are using the same notations of [KS97, Section 9.2].

The coordinate algebra of the odd-dimensional quantum spheres $S_q^{2\ell+1}$ can be identified both with the subalgebra of $\mathcal{A}(SU_q(\ell+1))$ generated by the elements $u_i^{\ell+1}$, or with the one generated by the elements u_i^1 . In both cases one gets isomorphic $*$ -algebras, and both fixed point subalgebras of $\mathcal{A}(SU_q(\ell+1))$ for coactions of $SU_q(\ell)$ obtained with two different choices of the inclusion $\mathcal{A}(SU_q(\ell)) \hookrightarrow \mathcal{A}(SU_q(\ell+1))$.

Definition 2.29. We call $\mathcal{A}(S_q^{2\ell+1})$ the $*$ -algebra generated by $\{z_i, z_i^*\}_{i=1, \dots, \ell+1}$ with relations

$$\begin{aligned} z_i z_j &= q z_j z_i & \forall i < j, \\ z_i^* z_j &= q z_j z_i^* & \forall i \neq j, \\ [z_1^*, z_1] &= 0, \\ [z_{i+1}^*, z_{i+1}] &= (1 - q^2) \sum_{j=1}^i z_j z_j^* & \forall i = 1, \dots, \ell, \\ z_1 z_1^* + z_2 z_2^* + \dots + z_{\ell+1} z_{\ell+1}^* &= 1. \end{aligned}$$

The original generators of [VS91] are obtained by the replacement $z_i \rightarrow z_{\ell+1-i}$ and setting $q = e^{h/2}$. The generators of [CP07] are the same of [VS91] but numbered from 0 to ℓ instead of from 1 to $\ell+1$; there should be a q instead of q^{-1} in their second relation. The generators x_i used in [HL04] are related to ours by $x_i = z_i^*$ and by the replacement $q \rightarrow q^{-1}$.

One can check that both the maps $z_i \mapsto u_i^{\ell+1}$ and $z_{\ell+2-i}^* \mapsto q^{-i+1} u_i^1$ define $*$ -algebra morphisms. That these are also injective is proved in [VS91]. We'll spend a few words about odd-dimensional quantum spheres in Chapter 5.

Chapter 3

Noncommutative geometry of Podleś quantum spheres

“Mathematics is the part of physics where experiments are cheap.”, Vladimir I. Arnold [Arn98].

It is not yet completely clear how to generalize the axioms for commutative real spectral triples to get a theory of ‘noncommutative manifolds’. Too narrow axioms would exclude a lot of interesting examples, while too mild axioms would give a too general theory, where it is difficult to prove interesting theorems. To shed light on this problem, a fundamental step is to study the properties of a certain number of examples, which is the ‘experimental’ part in doing mathematics. The class of examples on which we focus is that of quantum homogeneous spaces.

In this chapter we study regular even spectral triples on S_{qs}^2 following [DD06, DDLW07] (recall that $0 < q < 1$ and $0 \leq s \leq 1$). We then focus on a particular spectral triple, which is a deformation of the canonical one associated to the round sphere S^2 , and whose Dirac operator has undeformed spectrum. For this, we compute dimension spectrum and the noncommutative integral, which will be simply the integral over the space of characters of the algebra. The problem of constructing a real structure is also discussed.

We recall Definition 2.23 of the coordinate algebra $\mathcal{A}(S_{qs}^2)$ of the quantum Podleś spheres, and Definition 2.24 of the $*$ -Hopf algebra $\mathcal{A}(SU_q(2))$. The former is generated by $A = A^*$, B and B^* with relations

$$AB = q^2BA, \quad BB^* + (A + s^2)(A - 1) = 0, \quad B^*B + (q^2A + s^2)(q^2A - 1) = 0.$$

The latter is generated as a $*$ -algebra by α, β, α^* and β^* , with relations

$$\beta\alpha = q\alpha\beta, \quad \beta^*\alpha = q\alpha\beta^*, \quad [\beta, \beta^*] = 0, \quad \alpha\alpha^* + \beta\beta^* = 1, \quad \alpha^*\alpha + q^2\beta^*\beta = 1.$$

3.1 The symmetry Hopf algebra $U_q(su(2))$

We call $U_q(su(2))$ the compact real form of the Drinfeld-Jimbo deformation of $sl(2)$; it is a real form of the Hopf algebra called $\check{U}_q(su_2)$ in [KS97, Section 3.1.2]. As a $*$ -algebra, $U_q(su(2))$ is generated by $K = K^*$, K^{-1} , E and $F = E^*$, with relations

$$KEK^{-1} = qE, \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

all the other relations being determined by compatibility with the involution. The Hopf algebra structure (Δ, ϵ, S) of $U_q(su(2))$ is given by

$$\begin{aligned} \Delta K &= K \otimes K, & \Delta E &= E \otimes K + K^{-1} \otimes E, \\ \epsilon(K) &= 1, & \epsilon(E) &= 0, \\ S(K) &= K^{-1}, & S(E) &= -qE. \end{aligned}$$

Using Lemma 3.2.2 of [Maj95] we write the coproduct of E^n as

$$\Delta(E^n) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} E^{n-k} K^{-k} \otimes E^k K^{n-k}, \quad (3.1)$$

where q -factorials and q -binomials are defined as:

$$[0]! := 1, \quad [n]! := [n][n-1] \dots [1] \quad \forall n \geq 1, \quad \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]}.$$

For $l \in \frac{1}{2}\mathbb{N}$, $\omega = \pm 1$, let V_l^ω be a complex vector space with orthonormal basis $|l, m\rangle$, $m = -l, -l+1, \dots, l$. Then V_l^ω carries an irreducible $*$ -representation of $U_q(su(2))$ defined by

$$\begin{aligned} K |l, m\rangle &= \omega q^m |l, m\rangle, \\ E |l, m\rangle &= \omega \sqrt{[l-m][l+m+1]} |l, m+1\rangle, \\ F |l, m\rangle &= \omega \sqrt{[l+m][l-m+1]} |l, m-1\rangle, \end{aligned} \quad (3.2)$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ is the q -analogue of $x \in \mathbb{C}$. Any finite-dimensional irreducible $*$ -representation of $U_q(su(2))$ is equivalent to one of the V_l^ω [KS97, Section 3.2.3]. In the following, we'll be interested only in $V_l := V_l^+$.

For $l \in \mathbb{N}$, let C be the *antilinear* operator on V_l defined by

$$C |l, m\rangle = (-q)^m |l, -m\rangle, \quad (3.3)$$

and which is of square equal to 1. Moreover

$$ChC = S(h)^*, \quad \forall h \in U_q(su(2)), \quad (3.4)$$

as one checks first taking $h = K, E, F$:

$$CKC |l, m\rangle = q^{-m} |l, m\rangle = K^{-1} |l, m\rangle = S(K)^* |l, m\rangle,$$

$$\begin{aligned} CEC |l, m\rangle &= -q\sqrt{[l+m][l-m+1]} |l, m-1\rangle = -qF |l, m\rangle = S(E)^* |l, m\rangle , \\ CFC |l, m\rangle &= -q^{-1}\sqrt{[l-m][l+m+1]} |l, m+1\rangle = -q^{-1}E |l, m\rangle = S(F)^* |l, m\rangle , \end{aligned}$$

and then noting that both members of the equation are algebra morphisms. Similarly, considering first $h = K, E, F$ and then using the fact that both member of the equation are algebra morphism, one proves that, for all $h \in U_q(\mathfrak{su}(2))$,

$$S^2(h) = K^2 h K^{-2} . \quad (3.5)$$

Later we need the Clebsch-Gordan coefficients for the decomposition $V_l \otimes V_{1/2} \simeq V_{l+1/2} \oplus V_{l-1/2}$ and $V_{1/2} \otimes V_l \simeq V_{l+1/2} \oplus V_{l-1/2}$. If we call $|\pm\rangle := |\frac{1}{2}, \pm\frac{1}{2}\rangle$, then [KS97]:

$$|l + \frac{1}{2}, m - \frac{1}{2}\rangle = \sqrt{\frac{q^{-(l+m)}[l-m+1]}{[2l+1]}} |l, m\rangle \otimes |-\rangle + \sqrt{\frac{q^{l-m+1}[l+m]}{[2l+1]}} |l, m-1\rangle \otimes |+\rangle , \quad (3.6a)$$

$$|l - \frac{1}{2}, m + \frac{1}{2}\rangle = \sqrt{\frac{q^{l-m}[l+m+1]}{[2l+1]}} |l, m+1\rangle \otimes |-\rangle - \sqrt{\frac{q^{-(l+m+1)}[l-m]}{[2l+1]}} |l, m\rangle \otimes |+\rangle , \quad (3.6b)$$

and

$$|l + \frac{1}{2}, m - \frac{1}{2}\rangle = \sqrt{\frac{q^{l+m}[l-m+1]}{[2l+1]}} |-\rangle \otimes |l, m\rangle + \sqrt{\frac{q^{-(l-m+1)}[l+m]}{[2l+1]}} |+\rangle \otimes |l, m-1\rangle , \quad (3.7a)$$

$$|l - \frac{1}{2}, m + \frac{1}{2}\rangle = \sqrt{\frac{q^{l+m+1}[l-m]}{[2l+1]}} |+\rangle \otimes |l, m\rangle - \sqrt{\frac{q^{-(l-m)}[l+m+1]}{[2l+1]}} |-\rangle \otimes |l, m+1\rangle . \quad (3.7b)$$

3.2 Spherical functions

A non-degenerate dual pairing $\langle , \rangle : U_q(\mathfrak{su}(2)) \times \mathcal{A}(SU_q(2)) \rightarrow \mathbb{C}$ is defined by (cf. Section 4.4.1 of [KS97], the generators there being $d = \alpha$ and $c = -\beta$)

$$\langle K, \alpha \rangle = q^{1/2} , \quad \langle K, \alpha^* \rangle = q^{-1/2} , \quad \langle E, -\beta \rangle = \langle F, q\beta^* \rangle = 1 ,$$

and $\langle h, a \rangle = 0$ for all other pairs of generators h, a . A complete proof that this pairing is non-degenerate is for example in Section VII.4 of [Kas95]. Identifying the $*$ -algebra $\mathcal{A}(S_{qs}^2)$ with its image in $\mathcal{A}(SU_q(2))$ under the $*$ -algebra morphism (2.47) one checks that $\mathcal{A}(S_{qs}^2)$ are invariant subspaces for the left canonical action (2.28), which on generators is explicitly given by

$$\begin{aligned} K \triangleright A &= A , & K \triangleright B &= qB , & K \triangleright B^* &= q^{-1}B^* , \\ E \triangleright A &= -q^{-1/2}B , & E \triangleright B^* &= q^{1/2}(1+q^2)A - q^{1/2}(1-s^2) , & E \triangleright B &= 0 , \\ F \triangleright A &= q^{-3/2}B^* , & F \triangleright B &= -q^{-1/2}(1+q^2)A + q^{-1/2}(1-s^2) , & F \triangleright B^* &= 0 . \end{aligned}$$

The algebra $\mathcal{A}(S_{qs}^2)$ is then a $U_q(\mathfrak{su}(2))$ -module $*$ -algebra for this action.

We can decompose $\mathcal{A}(S_{qs}^2)$ into irreducible representations of $U_q(\mathfrak{su}(2))$, i.e. find a basis $|l, m; \lambda\rangle^\omega$ carrying — for l, λ and ω fixed — a representation of $U_q(\mathfrak{su}(2))$ equivalent to V_l^ω . The index λ counts possible multiplicities. From (3.2) we read $K |l, m; \lambda\rangle^\omega = \omega q^m |l, m; \lambda\rangle^\omega$. Every element of $\mathcal{A}(S_{qs}^2)$ can be written as a finite sum:

$$f = \sum_{n \in \mathbb{N}} \{ B^n f_n(A) + (B^*)^{n+1} f_{-n-1}(A) \} ,$$

where f_n are polynomials in A . Then $K \triangleright f = -q^m f$ has no solution, and $K \triangleright f = q^m f$ if and only if $f = B^m f_m(A)$ if $m \in \mathbb{N}$ or $f = (B^*)^{-m} f_m(A)$ if $-m - 1 \in \mathbb{N}$. Hence, only the representations V_l^ω with $\omega = 1$ and l integer appear in the decomposition of $\mathcal{A}(S_{qs}^2)$; the index ω can be omitted.

The *lowest weight vectors* $|l, -l; \lambda\rangle = (B^*)^l f_{-l; \lambda}(A)$ are determined by the condition that

$$F |l, -l; \lambda\rangle = q^l (B^*)^l F \triangleright f_{-l; \lambda}(A) = 0 ,$$

implying that $f_{-l; \lambda}(A) = c_l$ is a normalization constant. Thus, for each $l \in \mathbb{N}$ the lowest weight vector is unique, and the representations V_l have multiplicity 1: $\mathcal{A}(S_{qs}^2) \simeq \bigoplus_{l \in \mathbb{N}} V_l$. We can omit the index λ . The equivalence $\mathcal{A}(S_{qs}^2) \rightarrow \bigoplus_{l \in \mathbb{N}} V_l$ of $U_q(su(2))$ -modules is unitary if the involution $*$ is sent to the operator C on V_l , and this in particular means

$$|l, m\rangle = (-q)^m |l, -m\rangle^* .$$

We need just to determine $|l, m\rangle$ for $m \geq 0$. For all $m = -l, -l + 1, \dots, l$ we have

$$|l, m\rangle = ([l + m][l - m + 1])^{-1/2} E |l, m - 1\rangle ,$$

which iterated $l + m$ times becomes

$$|l, m\rangle = \sqrt{\frac{[l - m]!}{[l + m]! [2l]!}} E^{l+m} |l, -l\rangle = c_l \sqrt{\frac{[l - m]!}{[l + m]! [2l]!}} E^{l+m} \triangleright (B^*)^l . \quad (3.8)$$

Last equation reduces to Rodriguez formula for Laplace spherical harmonics, in the limit $q \rightarrow 1$.

A way to determine the coefficients c_l is by imposing the normalization of $|l, m\rangle$ to 1 with respect to the inner product associated to the Haar measure on $SU_q(2)$ and restricted to S_{qs}^2 . We'll use a simpler (even if equivalent) method: we'll compute c_l at the end of this section by imposing that A is represented by a symmetric operator. In both cases, changing c_l of a phase factor does not affect the normalization neither the self-adjointness of operators (it correspond to a unitary transformation), and we can choose $c_l \in \mathbb{R}$.

The next step is to compute the action of E^{l+m} . The explicit expression for spherical functions on the generic Podleś sphere already appears in [NM90], but for completeness we prefer to compute them from scratch.

For $l \geq 1$, we call P_l^\pm the following polynomials:

$$P_l^+ = \prod_{k=1}^l (q^{2k} A + s^2) , \quad P_l^- = \prod_{k=1}^l (q^{2k} A - 1) ,$$

with the notation $\prod_{k=1}^0 := 1$ and $P_0^\pm := 1$. Then, a little induction yields

$$(B^*)^l B^l = (-1)^l P_l^- P_l^+ \quad \forall l \geq 0 . \quad (3.9)$$

The inductive step comes by observing that

$$(B^*)^l B^l = -(B^*)^{l-1} (q^2 A + s^2) (q^2 A - 1) B^{l-1} = -(B^*)^{l-1} B^{l-1} (q^{2l} A + s^2) (q^{2l} A - 1) .$$

Multiplying (3.8) from the right by B^l and using $E \triangleright B = 0$ and (3.9), we get

$$q^{l(l+m)} |l, m\rangle B^l = (-1)^l c_l \sqrt{\frac{[l-m]!}{[l+m]! [2l]!}} E^{l+m} \triangleright (P_l^- P_l^+). \quad (3.10)$$

Induction on l gives

$$E \triangleright P_l^\pm = -q^{l+1/2} [l] P_{l-1}^\pm B \quad \forall l \geq 0,$$

while induction on n yields

$$E^n \triangleright P_l^\pm = (-q^{l+1/2})^n \frac{[l]!}{[l-n]!} P_{l-n}^\pm B^n \quad \text{if } 0 \leq n \leq l, \quad (3.11a)$$

$$E^n \triangleright P_l^\pm = 0 \quad \text{if } n > l \geq 0. \quad (3.11b)$$

Using covariance of the action, together with (3.1), we obtain:

$$\begin{aligned} E^{l+m} \triangleright (P_l^- P_l^+) &= \sum_{k=0}^{l+m} \begin{bmatrix} l+m \\ k \end{bmatrix} (E^{l+m-k} \triangleright P_l^-) (E^k \triangleright P_l^+) \\ &= \sum_{\substack{0 \leq p, r \leq l, \\ p+r=l-m}} \frac{[l+m]!}{[l-p]! [l-r]!} (E^{l-p} \triangleright P_l^-) (E^{l-r} \triangleright P_l^+). \end{aligned}$$

In the second step we used the indexes $p = k - m$ and $r = l - k$, and noticed that the terms with $p, r < 0$ are zero, due to (3.11b). In turn, using (3.11a):

$$E^{l+m} \triangleright (P_l^- P_l^+) = (-q^{l+1/2})^{l+m} [l+m]! \sum_{p+r=l-m} \begin{bmatrix} l \\ p \end{bmatrix} \begin{bmatrix} l \\ r \end{bmatrix} P_p^- B^{m+r} P_r^+ B^{l-r}.$$

Now, $AB^n = q^{2n} B^n A$ implies (when $m \geq 0$, so that B^m is well-defined) that

$$P_p^- B^m = B^m \left(\prod_{h=1}^p (q^{2(m+h)} A - 1) \right), \quad B^r P_r^+ = \left(\prod_{h=1}^r (q^{2(h-r)} A + s^2) \right) B^r. \quad (3.12)$$

With this, equations (3.12) and (3.10) we get the final result, summarized in a proposition.

Proposition 3.1. *The spherical functions $|l, m\rangle$, defined by*

$$|l, m\rangle := N_{l,m} B^m p_{l,m} \quad \text{if } m \geq 0,$$

$$|l, m\rangle := (-q)^m |l, -m\rangle^* \quad \text{if } m < 0,$$

$$p_{l,m} := \sum_{p+r=l-m} \begin{bmatrix} l \\ p \end{bmatrix} \begin{bmatrix} l \\ r \end{bmatrix} \prod_{k=1}^p (q^{2(m+k)} A - 1) \prod_{h=1}^r (q^{2(h-r)} A + s^2),$$

$$N_{l,m} := c_l (-1)^{l+m} q^{-(l-m)/2} \sqrt{\frac{[l+m]! [l-m]!}{[2l]!}},$$

are an orthonormal basis of $\mathcal{A}(S_{qs}^2)$ (with respect to the inner product associated to the Haar functional). The left action of $U_q(\mathfrak{su}(2))$ on this basis is described by equations (3.2).

The normalization constants $c_l \in \mathbb{R}$ have been rescaled by $(-q)^l$ in order to have $|l, l\rangle = c_l B^l$. Their value will be determined later on. The degree $l - m$ polynomials $p_{l,m}$ are proportional to big q -Jacobi polynomials (see [KS97, Section 4.5.3] and [NM90]).

Explicit expressions of some spherical harmonics are the following:

$$\begin{aligned} (-q)^l |l, -l\rangle &= c_l (B^*)^l . \\ (-q)^l |l-1, -l+1\rangle &= -q c_{l-1} (B^*)^{l-1} . \\ (-q)^l |l, -l+1\rangle &= c_l [2l]^{-1/2} E \triangleright (B^*)^l . \\ (-q)^l |l+1, -l+1\rangle &= c_{l+1} (-q)^{-1} ([2l+1][2l+2][2])^{-1/2} E^2 \triangleright (B^*)^{l+1} , \end{aligned}$$

where $E \triangleright (B^*)^l$ and $E^2 \triangleright (B^*)^{l+1}$ are given by

$$\begin{aligned} E \triangleright (B^*)^l &= q^{1/2} \{q^l [2l] A - [l](1-s^2)\} (B^*)^{l-1} , \\ E^2 \triangleright (B^*)^{l+1} &= \left\{ q^{2l+1} [2l+1][2l+2] A^2 - q^{l+1} [l+1] ([2l] + [2l+2]) (1-s^2) A \right. \\ &\quad \left. + q[l][l+1] (1-s^2)^2 - q[2l+2] s^2 \right\} (B^*)^{l-1} , \end{aligned}$$

again proved by induction on l . Then, by direct computation one checks that the following equalities hold:

$$\begin{aligned} A |l, -l+1\rangle &= A_{l,-l+1}^+ |l+1, -l+1\rangle + A_{l,-l+1}^0 |l, -l+1\rangle + A_{l,-l+1}^- |l-1, -l+1\rangle , \\ B |l, -l\rangle &= B_{l,-l}^+ |l+1, -l+1\rangle + B_{l,-l}^0 |l, -l+1\rangle + B_{l,-l}^- |l-1, -l+1\rangle , \end{aligned}$$

with coefficients

$$\begin{aligned} A_{l,-l+1}^+ &= -\frac{c_l}{c_{l+1}} q^{-l+1/2} \sqrt{\frac{[2][2l]}{[2l+1][2l+2]}}, & B_{l,-l}^+ &= \frac{c_l}{c_{l+1}} q^{-2l} \sqrt{\frac{[2]}{[2l+1][2l+2]}}, \\ A_{l,-l+1}^0 &= (1-s^2) q^{-l} \frac{[l][l+1][2l-2]}{[2l][2l+2][l-1]}, & B_{l,-l}^0 &= -(1-s^2) \frac{1-q^2}{q^{l+1/2} \sqrt{[2l]}} \frac{[l][l+1]}{[2l+2]}, \\ A_{l,-l+1}^- &= -\frac{c_l}{c_{l-1}} q^{-l-1/2} \frac{[2l]^2 s^2 + [l]^2 (1-s^2)^2}{[2l][2l+1] \sqrt{[2l]}}, & B_{l,-l}^- &= -\frac{c_l}{c_{l-1}} \frac{[2l]^2 s^2 + [l]^2 (1-s^2)^2}{[2l][2l+1]}. \end{aligned}$$

These coefficients will be needed later. Again, by direct computation we have

$$A |l-1, -l+1\rangle = -\frac{c_{l-1}}{c_l} \frac{q^{-l+1/2}}{\sqrt{[2l]}} |l, -l+1\rangle + \text{orthogonal vectors} .$$

The representation is a $*$ -representation (i.e. $A = A^*$) only if last coefficient equals $A_{l,-l+1}^-$. From this we deduce the condition

$$\frac{c_l^2}{c_{l-1}^2} = q \frac{[2l][2l+1]}{[2l]^2 s^2 + [l]^2 (1-s^2)^2} . \quad (3.13)$$

When $q = 1$, c_0^{-2} is the area of the sphere; we fix $c_0^{-2} = 4\pi$ for all q . For $l \geq 1$, the coefficients c_l are determined by (3.13) up to a sign, that we choose equal to $+1$:

$$c_l = \sqrt{\frac{q^l [2l+1]!}{4\pi}} \left(\prod_{k=1}^l \frac{1}{[2k]^2 s^2 + [k]^2 (1-s^2)^2} \right)^{\frac{1}{2}} .$$

Summarizing, the normalization constants $N_{l,m}$ of Proposition 3.1 are given by

$$N_{l,m} = (-1)^{l+m} \sqrt{\frac{q^m [2l+1]}{4\pi}} \sqrt{[l+m]! [l-m]!} \left(\prod_{k=1}^l \frac{1}{[2k]^2 s^2 + [k]^2 (1-s^2)^2} \right)^{\frac{1}{2}}.$$

Using (3.13) we derive the final expression for the coefficients $A_{l,-l+1}^{+,0}$ and $B_{l,-l}^{\pm,0}$:

$$A_{l,-l+1}^+ = -q^{-l} \sqrt{\frac{[2l+2]^2 s^2 + [l+1]^2 (1-s^2)^2}{[2l+2]}} \sqrt{\frac{[2][2l]}{[2l+1][2l+3]}}, \quad (3.14a)$$

$$A_{l,-l+1}^0 = (1-s^2) q^{-l} \frac{[l][l+1][2l-2]}{[2l][2l+2][l-1]}, \quad (3.14b)$$

$$B_{l,-l}^+ = q^{-2l-1/2} \sqrt{\frac{[2l+2]^2 s^2 + [l+1]^2 (1-s^2)^2}{[2l+2]}} \sqrt{\frac{[2]}{[2l+1][2l+3]}}, \quad (3.14c)$$

$$B_{l,-l}^0 = -(1-s^2) \frac{1-q^2}{q^{l+1/2} \sqrt{[2l]}} \frac{[l][l+1]}{[2l+2]}, \quad (3.14d)$$

$$B_{l,-l}^- = -q^{1/2} \sqrt{\frac{[2l]^2 s^2 + [l]^2 (1-s^2)^2}{[2l][2l+1]}}. \quad (3.14e)$$

3.3 The left regular representation

Notice that, due to noncommutativity, left regular and right regular representations of $\mathcal{A}(S_{qs}^2)$ are not equivalent; indeed, they are related by the involution:

$$f |l, m\rangle = (-q)^m \{|l, -m\rangle f^*\}^*. \quad (3.15)$$

In this section we compute the left action of the generators of $\mathcal{A}(S_{qs}^2)$ on spherical functions. Since $hx = (h_{(1)} \triangleright x) h_{(2)}$, for all $h \in U_q(su(2))$ and $x \in \mathcal{A}(S_{qs}^2)$, we deduce the commutation rules:

$$KB = qBK, \quad KA = AK, \quad (3.16a)$$

$$EB = q^{-1}BE, \quad EA = AE - q^{-1/2}BK. \quad (3.16b)$$

From (3.16a) we get that $B |l, m\rangle$ is in the subspace spanned by $|l', m+1\rangle$ and $A |l, m\rangle$ is in the subspace spanned by $|l', m\rangle$. Consider

$$B |l, m\rangle = \sum_{l' \in \mathbb{N}} B_{l',m}^{l'} |l', m+1\rangle, \quad A |l, m\rangle = \sum_{l' \in \mathbb{N}} A_{l',m}^{l'} |l', m\rangle.$$

with $B_{l',m}^{l'} = 0$ if $l' < m+1$ and $A_{l',m}^{l'} = 0$ if $l' < m$. From (3.16b) we get

$$\begin{aligned} E^{l-m+1} B |l, m\rangle &= q^{-l+m-1} B E^{l-m+1} |l, m\rangle = 0, \\ \langle l', m+1 | B E^{l'+m+2} &= q^{l'+m+2} \langle l', m+1 | E^{l'+m+2} B = 0, \end{aligned}$$

and in turn,

$$\begin{aligned} B_{l,m}^{l'} \sqrt{[l' - m - 1] \dots [l' - l - 1]} \sqrt{[l' + m + 2] \dots [l' + l + 2]} &= 0, \\ \bar{B}_{l,m}^{l'} \sqrt{[l + m] \dots [l - l' - 1]} \sqrt{[l - m + 1] \dots [l + l']} &= 0. \end{aligned}$$

The square roots in the first line are all different from zero if $l' > l + 1$, while the square roots in the second line are all different from zero if $l > l' + 1$. Hence,

$$B_{l,m}^{l'} = 0 \quad \forall l' \notin \{l - 1, l, l + 1\}.$$

A similar reasoning applies to $A|l, m\rangle$. From formula (3.1) for the coproduct of E^n we deduce:

$$E^n A = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (E^k \triangleright A) E^{n-k} K^k = AE^n - [n]q^{-1/2} B E^{n-1} K.$$

This implies that $E^{l-m+2} A|l, m\rangle = 0$ and $\langle l', m| A E^{l'+m+2} = 0$. Then

$$\begin{aligned} A_{l,m}^{l'} \sqrt{[l' - m] \dots [l' - l - 1]} \sqrt{[l' + m + 1] \dots [l' + l + 2]} &= 0, \\ \bar{A}_{l,m}^{l'} \sqrt{[l + m] \dots [l - l' - 1]} \sqrt{[l - m + 1] \dots [l + l' + 2]} &= 0. \end{aligned}$$

As for B , these imply $A_{l,m}^{l'} = 0$ if $l' \notin \{l, l \pm 1\}$. We can thus state:

Proposition 3.2. *The left regular representation of $\mathcal{A}(S_{qs}^2)$ is defined by:*

$$\begin{aligned} A|l, m\rangle &= A_{l,m}^+ |l + 1, m\rangle + A_{l,m}^0 |l, m\rangle + \bar{A}_{l-1,m}^+ |l - 1, m\rangle, \\ B|l, m\rangle &= B_{l,m}^+ |l + 1, m + 1\rangle + B_{l,m}^0 |l, m + 1\rangle + B_{l,m}^- |l - 1, m + 1\rangle, \end{aligned}$$

for suitable $A_{l,m}^{0,+}, B_{l,m}^{0,\pm} \in \mathbb{C}$ and with B^* the adjoint of B .

The representation is equivariant by construction, and the equivariance condition is sufficient to determine the coefficients. In fact, (3.16b) tells us that the expressions

$$\begin{aligned} EB|l, m\rangle &= B_{l,m}^+ \sqrt{[l - m][l + m + 3]} |l + 1, m + 2\rangle + \\ &+ B_{l,m}^0 \sqrt{[l - m - 1][l + m + 2]} |l, m + 2\rangle + B_{l,m}^- \sqrt{[l - m - 2][l + m + 1]} |l - 1, m + 2\rangle, \\ EA|l, m\rangle &= A_{l,m}^+ \sqrt{[l - m + 1][l + m + 2]} |l + 1, m + 1\rangle + \\ &+ A_{l,m}^0 \sqrt{[l - m][l + m + 1]} |l, m + 1\rangle + \bar{A}_{l-1,m}^+ \sqrt{[l - m - 1][l + m]} |l - 1, m + 1\rangle, \end{aligned}$$

must be equal, respectively, to

$$\begin{aligned} q^{-1} B E|l, m\rangle &= q^{-1} \sqrt{[l - m][l + m + 1]} \left\{ B_{l,m+1}^+ |l + 1, m + 2\rangle + \right. \\ &\left. + B_{l,m+1}^0 |l, m + 2\rangle + B_{l,m+1}^- |l - 1, m + 2\rangle \right\}, \\ (A E - q^{-1/2} B K)|l, m\rangle &= \left\{ \sqrt{[l - m][l + m + 1]} A_{l,m+1}^+ - q^{m-1/2} B_{l,m}^+ \right\} |l + 1, m + 1\rangle + \\ &+ \left\{ \sqrt{[l - m][l + m + 1]} A_{l,m+1}^0 - q^{m-1/2} B_{l,m}^0 \right\} |l, m + 1\rangle + \end{aligned}$$

$$+ \left\{ \sqrt{[l-m][l+m+1]} \bar{A}_{l-1,m+1}^+ - q^{m-1/2} B_{l,m}^- \right\} |l-1, m+1\rangle .$$

From the independence of the vectors $|l, m\rangle$, we derive the conditions

$$\frac{q^{-m} B_{l,m}^+}{\sqrt{[l+m+1][l+m+2]}} = \frac{q^{-m-1} B_{l,m+1}^+}{\sqrt{[l+m+2][l+m+3]}} \quad (3.17a)$$

$$\frac{q^{-m} B_{l,m}^0}{\sqrt{[l-m][l+m+1]}} = \frac{q^{-m-1} B_{l,m+1}^0}{\sqrt{[l-m-1][l+m+2]}} \quad (3.17b)$$

$$\frac{q^{-m} B_{l,m}^-}{\sqrt{[l-m][l-m-1]}} = \frac{q^{-m-1} B_{l,m+1}^-}{\sqrt{[l-m-1][l-m-2]}} \quad (3.17c)$$

$$\sqrt{[l-m][l+m+1]} A_{l,m+1}^+ - q^{m-1/2} B_{l,m}^+ = A_{l,m}^+ \sqrt{[l-m+1][l+m+2]} \quad (3.17d)$$

$$A_{l,m+1}^0 - \frac{q^{m-1/2} B_{l,m}^0}{\sqrt{[l-m][l+m+1]}} = A_{l,m}^0 \quad (3.17e)$$

$$\sqrt{[l-m+1][l+m+2]} \bar{A}_{l,m+1}^+ - q^{m-1/2} B_{l+1,m}^- = \bar{A}_{l,m}^+ \sqrt{[l-m][l+m+1]} \quad (3.17f)$$

The first three conditions are of the form $f_{l,m} = f_{l,m+1}$, stating that they involve quantities $f_{l,m}$ which do not depend on m . These can be rewritten as ' $f_{l,m} = f_{l,-l}$ ', that is:

$$\begin{aligned} \frac{q^{-m} B_{l,m}^+}{\sqrt{[l+m+1][l+m+2]}} &= \frac{q^l B_{l,-l}^+}{\sqrt{[2]}} , \\ \frac{q^{-m} B_{l,m}^0}{\sqrt{[l-m][l+m+1]}} &= \frac{q^l B_{l,-l}^0}{\sqrt{[2l]}} , \\ \frac{q^{-m} B_{l,m}^-}{\sqrt{[l-m][l-m-1]}} &= \frac{q^l B_{l,-l}^-}{\sqrt{[2l][2l-1]}} . \end{aligned}$$

Using the boundary condition in (3.14) we arrive at the final result:

$$B_{l,m}^+ = q^{-(l-m)-1/2} \frac{\sqrt{[2l+2]^2 s^2 + [l+1]^2 (1-s^2)^2}}{[2l+2]} \sqrt{\frac{[l+m+1][l+m+2]}{[2l+1][2l+3]}} , \quad (3.18a)$$

$$B_{l,m}^0 = -(1-s^2)(1-q^2)q^{m-1/2} \frac{[l][l+1]}{[2l][2l+2]} \sqrt{[l-m][l+m+1]} , \quad (3.18b)$$

$$B_{l,m}^- = -q^{l+m+1/2} \frac{\sqrt{[2l]^2 s^2 + [l]^2 (1-s^2)^2}}{[2l]} \sqrt{\frac{[l-m][l-m-1]}{[2l-1][2l+1]}} . \quad (3.18c)$$

Equations (3.17d-3.17e) have solution

$$A_{l,m}^+ = -q^{m-1} \frac{\sqrt{[2l+2]^2 s^2 + [l+1]^2 (1-s^2)^2}}{[2l+2]} \sqrt{\frac{[l+m+1][l-m+1]}{[2l+1][2l+3]}} , \quad (3.19a)$$

$$A_{l,m}^0 = (1-s^2)q^{-1}(1+q^{2m}) \frac{[l][l+1]}{[2l][2l+2]} , \quad (3.19b)$$

while equation (3.17f) is automatically satisfied.

The next step is to lift diagonally this representation on the Hilbert space completion of $\mathcal{A}(S_{qs}^2)$, we call it \mathcal{H} , and then decompose it into two subrepresentations on \mathcal{H}_{\pm} , the Hilbert

space completion of the modules $\mathcal{A}(S_{qs}^2)^2(1 - e_s)$ and $\mathcal{A}(S_{qs}^2)^2e_s$ for a suitable invariant idempotent e_s . These will be the q -analogue of the chiral representations of $\mathcal{A}(S^2)$, and elements in \mathcal{H}_\pm will be the q -analogue of Weyl spinors.

3.4 Chiral representations

As explained in Section 2.10.2, equivariant representations correspond to invariant idempotents. Let $\sigma : U_q(su(2)) \rightarrow \text{Mat}_2(\mathbb{C})$ be the representation $V_{1/2}$, written in matrix form

$$\sigma(K) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \sigma(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.20)$$

For any s (and q) there is a 2×2 idempotent which is invariant, in the sense of (2.37), and given by

$$e_s = \frac{1}{1 + s^2} \begin{pmatrix} s^2 + A & qB \\ q^{-1}B^* & 1 - q^2A \end{pmatrix}. \quad (3.21)$$

It also satisfies $\kappa(e_s^*) = e_s$ with $\kappa(a) := K^2 \triangleright a$ the modular automorphism [KS97, Section 11.3.4] thus, by Lemma 2.20 it corresponds to an orthogonal projection via the formula (2.36). We notice that a projection (a hermitian idempotent) equivalent to e_s is given by the matrix $\sigma(K)^{-1}e_s\sigma(K)$; this was introduced in [BM98], where the authors compute its charge by pairing it with the cyclic 0-cocycle of [MNW91], and later generalized to projections of arbitrary charge first for the standard sphere in [HM99], and then for all Podleś spheres in [BM00].

Let $Y_{l,m} = |l, m\rangle$ be the spherical functions defined in Proposition 3.1 and identify $V_{1/2}$ with \mathbb{C}^2 via the map $|1/2, 1/2\rangle \mapsto (1, 0)$ and $|1/2, -1/2\rangle \mapsto (0, 1)$. Using Clebsch-Gordan coefficients in equation (3.6) we define a basis of $\mathcal{A}(S_{qs}^2)^2$,

$$|l, m\rangle_1 = \left(\sqrt{\frac{q^{l-m}[l+m]}{[2l]}} Y_{l-\frac{1}{2}, m-\frac{1}{2}}, \sqrt{\frac{q^{-(l+m)[l-m]}}{[2l]}} Y_{l-\frac{1}{2}, m+\frac{1}{2}} \right), \quad (3.22a)$$

$$|l, m\rangle_2 = \left(-\sqrt{\frac{q^{-(l+m+1)[l-m+1]}}{[2l+2]}} Y_{l+\frac{1}{2}, m-\frac{1}{2}}, \sqrt{\frac{q^{l-m+1}[l+m+1]}{[2l+2]}} Y_{l+\frac{1}{2}, m+\frac{1}{2}} \right), \quad (3.22b)$$

(with $l \in \mathbb{N} + \frac{1}{2}$ and $m = -l, -l+1, \dots, l$) which corresponds to the decomposition

$$\left(\bigoplus_{l \in \mathbb{N}} V_l \right) \otimes V_{1/2} \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} (V_l \oplus V_l).$$

Once fixed the eigenvalue of K and the spin l , the corresponding eigenvectors of $\mathcal{A}(S_{qs}^2)^2$ span a two-dimensional space. Since e_s is equivariant we can search for a basis of this two-dimensional space made of eigenvectors of e_s . In (2.36) the idempotent e_s multiplies vectors from the right, thus to find eigenvectors of the corresponding projection we must work with the right regular representation, related to the left regular one by (3.15), which means

$$|l, m\rangle A = q^{-2m} A |l, m\rangle,$$

$$\begin{aligned}
-|l, m\rangle q^{-1}B^* &= B_{l,-m}^+ |l+1, m-1\rangle + B_{l,-m}^0 |l, m-1\rangle + B_{l,-m}^- |l-1, m-1\rangle, \\
-|l, m\rangle qB &= B_{l+1,-m-1}^- |l+1, m+1\rangle + B_{l,-m-1}^0 |l, m+1\rangle + B_{l-1,-m-1}^+ |l-1, m+1\rangle.
\end{aligned}$$

A straightforward computation shows that the basis (3.22) satisfies

$$\begin{aligned}
|l, m\rangle_1 e_s &= \frac{s^2 q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}}}{1+s^2} \frac{[l+\frac{1}{2}]}{[2l+1]} |l, m\rangle_1 + c_{l+\frac{1}{2},s} \frac{1}{1+s^2} |l, m\rangle_2, \\
|l, m\rangle_2 e_s &= \frac{q^{l+\frac{1}{2}} + s^2 q^{-l-\frac{1}{2}}}{1+s^2} \frac{[l+\frac{1}{2}]}{[2l+1]} |l, m\rangle_2 + c_{l+\frac{1}{2},s} \frac{1}{1+s^2} |l, m\rangle_1,
\end{aligned}$$

where $c_{l,s}$ is defined by

$$c_{l,s} = \frac{\sqrt{[2l]^2 s^2 + [l]^2 (1-s^2)^2}}{[2l]} = \frac{[l]}{[2l]} \sqrt{(q^l s^2 + q^{-l})(q^l + q^{-l} s^2)}.$$

Then the vectors $|l, m\rangle_{\pm}$, defined by

$$\sqrt{\frac{[2l+1]}{[l+\frac{1}{2}]}} |l, m\rangle_+ = \sqrt{\frac{q^{l+\frac{1}{2}} s^2 + q^{-l-\frac{1}{2}}}{1+s^2}} |l, m\rangle_1 + \sqrt{\frac{q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}} s^2}{1+s^2}} |l, m\rangle_2, \quad (3.23a)$$

$$\sqrt{\frac{[2l+1]}{[l+\frac{1}{2}]}} |l, m\rangle_- = \sqrt{\frac{q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}} s^2}{1+s^2}} |l, m\rangle_1 - \sqrt{\frac{q^{l+\frac{1}{2}} s^2 + q^{-l-\frac{1}{2}}}{1+s^2}} |l, m\rangle_2, \quad (3.23b)$$

satisfy the equations

$$|l, m\rangle_+ (1 - e_s) = 0, \quad |l, m\rangle_- e_s = 0.$$

The ‘spinorial harmonics’ in (3.23) form an orthonormal basis of $\mathcal{H} := L^2(S_{qs}^2)^2$. We call \mathcal{H}_+ the Hilbert space with basis $|l, m\rangle_+$, and \mathcal{H}_- the Hilbert space with basis $|l, m\rangle_-$. Both are finitely generated projective modules,

$$\mathcal{H}_+ = L^2(S_{qs}^2)^2 e_s \quad \text{and} \quad \mathcal{H}_- = L^2(S_{qs}^2)^2 (1 - e_s).$$

They carry representations of $U_q(su(2))$ — each equivalent to $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$ — and ‘chiral’ representations of $\mathcal{A}(S_{qs}^2)$, which are equivariant with respect to the action of $U_q(su(2))$.

Using (3.22) and (3.23) one can write $|l, m\rangle_{\pm}$ in terms of spherical functions. Then, from Proposition 3.2 one computes the action of the generators A and B on the vectors $|l, m\rangle_{\pm}$. The result is described in the following proposition.

Proposition 3.3. *The action of generators of $\mathcal{A}(S_{qs}^2)$ on the basis $|l, m\rangle_{\pm}$ is given by:*

$$\begin{aligned}
A|l, m\rangle_+ &= -q^{m-1} \sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]} \frac{\sqrt{[l+m+1][l-m+1]}}{[2l+2]} \sqrt{(q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}} s^2)(q^{l+\frac{3}{2}} s^2 + q^{-l-\frac{3}{2}})}} |l+1, m\rangle_+ \\
&\quad - q^{m-1} \sqrt{\frac{[l-\frac{1}{2}][l+\frac{1}{2}]}{[2l-1][2l+1]} \frac{\sqrt{[l+m][l-m]}}{[2l]} \sqrt{(q^{l+\frac{1}{2}} s^2 + q^{-l-\frac{1}{2}})(q^{l-\frac{1}{2}} + q^{-l+\frac{1}{2}} s^2)}} |l-1, m\rangle_+ \\
&\quad + \left((1-s^2) \frac{(1-q^2)[l-\frac{1}{2}][l+\frac{3}{2}]+1}{1+q^2} - 1 \right) \frac{[l+m+1][l-m] - q^{-2}[l+m][l-m+1]}{[2l][2l+2]} |l, m\rangle_+
\end{aligned}$$

$$\begin{aligned}
& + \frac{1-s^2}{1+q^2} |l, m\rangle_+ , \\
A |l, m\rangle_- &= -q^{m-1} \sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]} \frac{\sqrt{[l+m+1][l-m+1]}}{[2l+2]}} \sqrt{(q^{l+\frac{1}{2}}s^2 + q^{-l-\frac{1}{2}})(q^{l+\frac{3}{2}} + q^{-l-\frac{3}{2}}s^2)} |l+1, m\rangle_- \\
& - q^{m-1} \sqrt{\frac{[l-\frac{1}{2}][l+\frac{1}{2}]}{[2l-1][2l+1]} \frac{\sqrt{[l+m][l-m]}}{[2l]}} \sqrt{(q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}}s^2)(q^{l-\frac{1}{2}}s^2 + q^{-l+\frac{1}{2}})} |l-1, m\rangle_- \\
& + \left((1-s^2) \frac{(1-q^2)[l-\frac{1}{2}][l+\frac{3}{2}]+1}{1+q^2} + s^2 \right) \frac{[l+m+1][l-m] - q^{-2}[l+m][l-m+1]}{[2l][2l+2]} |l, m\rangle_- \\
& + \frac{1-s^2}{1+q^2} |l, m\rangle_- , \\
B |l, m\rangle_+ &= q^{-(l-m)-1/2} \sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]} \frac{\sqrt{[l+m+1][l+m+2]}}{[2l+2]}} \sqrt{(q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}}s^2)(q^{l+\frac{3}{2}}s^2 + q^{-l-\frac{3}{2}})} |l+1, m+1\rangle_+ \\
& - q^{l+m+1/2} \sqrt{\frac{[l-\frac{1}{2}][l+\frac{1}{2}]}{[2l-1][2l+1]} \frac{\sqrt{[l-m][l-m-1]}}{[2l]}} \sqrt{(q^{l+\frac{1}{2}}s^2 + q^{-l-\frac{1}{2}})(q^{l-\frac{1}{2}} + q^{-l+\frac{1}{2}}s^2)} |l-1, m+1\rangle_+ \\
& + q^{m-\frac{1}{2}} \frac{\sqrt{[l+m+1][l-m]}}{[2l][2l+2]} \left\{ s^2 + q^2 - (1-s^2)(1-q^2)[l-\frac{1}{2}][l+\frac{3}{2}] \right\} |l, m+1\rangle_+ , \\
B |l, m\rangle_- &= q^{-(l-m)-1/2} \sqrt{\frac{[l+\frac{1}{2}][l+\frac{3}{2}]}{[2l+1][2l+3]} \frac{\sqrt{[l+m+1][l+m+2]}}{[2l+2]}} \sqrt{(q^{l+\frac{1}{2}}s^2 + q^{-l-\frac{1}{2}})(q^{l+\frac{3}{2}} + q^{-l-\frac{3}{2}}s^2)} |l+1, m+1\rangle_- \\
& - q^{l+m+1/2} \sqrt{\frac{[l-\frac{1}{2}][l+\frac{1}{2}]}{[2l-1][2l+1]} \frac{\sqrt{[l-m][l-m-1]}}{[2l]}} \sqrt{(q^{l+\frac{1}{2}} + q^{-l-\frac{1}{2}}s^2)(q^{l-\frac{1}{2}}s^2 + q^{-l+\frac{1}{2}})} |l-1, m+1\rangle_- \\
& - q^{m-\frac{1}{2}} \frac{\sqrt{[l+m+1][l-m]}}{[2l][2l+2]} \left\{ s^2q^2 + 1 + (1-s^2)(1-q^2)[l-\frac{1}{2}][l+\frac{3}{2}] \right\} |l, m+1\rangle_- .
\end{aligned}$$

When $s = 1$ these representations are equivalent to those considered in [DLPS05] (the generators there being $b := q^2A$ and $a := B$), and when $s = 0$ they are equivalent to those considered in [DS03] (the generators there being $A' = A$ and $B' = q^{-1}B$).

3.5 Comparison with the literature

Integrable representations of the crossed product algebra $\mathcal{A}(S_{qs}^2) \rtimes U_q(su(2))$ were classified in [SW07]. We rewrite them here. Firstly, the following set of generators for $\mathcal{A}(S_{qs}^2)$

$$x_1 = (q[2])^{\frac{1}{2}} B, \quad x_0 = -q[2]A + (1-s^2), \quad x_{-1} = -q^{-1}x_1^*, \quad (3.24)$$

satisfy the commutation rules

$$x_1x_0 - q^{-2}x_0x_1 - t(1-q^{-2})x_1 = 0, \quad (3.25a)$$

$$x_{-1}x_0 - q^2x_0x_{-1} - t(1-q^2)x_{-1} = 0, \quad (3.25b)$$

$$- [2]x_{-1}x_1 + (q^2x_0 + t)(x_0 - t) - [2]^2(1-t) = 0, \quad (3.25c)$$

$$- [2]x_1x_{-1} + (q^{-2}x_0 + t)(x_0 - t) - [2]^2(1-t) = 0. \quad (3.25d)$$

with parameter $t := 1 - s^2 \in [0, 1]$. These are exactly equations (3.1) of [DDLW07]. We rewrite the representations in Proposition 4.1 of [DDLW07] as follows.

Proposition 3.4. For fixed $N \in \frac{1}{2}\mathbb{Z}$, we call \mathcal{H}_N the Hilbert space completion of

$$\text{Span}\{ |l, m; N\rangle ; l - |N|, l - |m| \in \mathbb{N} \} ,$$

in which the vectors $|l, m; N\rangle$ are declared to be orthonormal. A $*$ -representation of $\mathcal{A}(S_{qs}^2)$ on \mathcal{H}_N is given by

$$\begin{aligned} x_1 |l, m; N\rangle &= \tilde{B}_{l,m}^+ |l+1, m+1; N\rangle + \tilde{B}_{l,m}^0 |l, m+1; N\rangle + \tilde{B}_{l,m}^- |l-1, m+1; N\rangle , \\ x_0 |l, m; N\rangle &= \tilde{A}_{l,m}^+ |l+1, m; N\rangle + \tilde{A}_{l,m}^0 |l, m; N\rangle + \tilde{A}_{l-1,m}^+ |l-1, m; N\rangle , \\ -qx_{-1} |l, m; N\rangle &= \tilde{B}_{l+1,m-1}^- |l+1, m-1; N\rangle + \tilde{B}_{l,m-1}^0 |l, m-1; N\rangle + \tilde{B}_{l-1,m-1}^+ |l-1, m-1; N\rangle , \end{aligned}$$

with coefficients

$$\begin{aligned} \tilde{A}_{l,m}^+ &= q^m \sqrt{[2][l-m+1][l+m+1]} \alpha_{N,l+1}(s) , \\ \tilde{A}_{l,m}^0 &= ([2l] - q^{l+m+1}[2][l-m]) \beta_{N,l}(s) , \\ \tilde{B}_{l,m}^+ &= q^{-l+m} \sqrt{[l+m+1][l+m+2]} \alpha_{N,l+1}(s) , \\ \tilde{B}_{l,m}^0 &= -q^{m+2} \sqrt{[2][l-m][l+m+1]} \beta_{N,l}(s) , \\ \tilde{B}_{l,m}^- &= -q^{l+m+1} \sqrt{[l-m][l-m-1]} \alpha_{N,l}(s) , \end{aligned}$$

and

$$\begin{aligned} \alpha_{N,l}(s) &= \frac{1}{[2l]} \sqrt{\frac{[2][l+N][l-N]}{[2l-1][2l+1]} \sqrt{(q^{l+N} + q^{-(l+N)}s^2)(q^{l-N}s^2 + q^{-(l-N)})}} , \\ \beta_{N,l}(s) &= \frac{([2l] - q^{l+N+1}[2][l-N])(1-s^2) + q[2][2N]s^2}{q^2[2l][2l+2]} . \end{aligned}$$

Now we want to check (i) that the representations are indeed correct, for any $N \in \frac{1}{2}\mathbb{Z}$; (ii) that these are the same equations as in Proposition 4.1 of [DDLW07]; (iii) that for $N = \pm 1/2$ we get the same representations of Proposition 3.3 by identifying $|l, m; \pm \frac{1}{2}\rangle$ with $|l, m\rangle_{\mp}$ and using the change of generators (3.24).

Concerning point (ii), the non-trivial part is to show that

$$[2l-1]^{\frac{1}{2}}[2l]^{\frac{1}{2}} \alpha_{N,l}(s) = \alpha_N(l) , \quad [2l]\beta_{N,l}(s) = \beta_N(l) , \quad (3.26)$$

where $\alpha_N(l)$ and $\beta_N(l)$ are given by equations (4.3) of [DDLW07],

$$\alpha_N(l) = \sqrt{\frac{[2][l+N][l-N][2l]}{[2l+1]} \sqrt{\frac{1-t}{[l]^2} + q^{-2N} \frac{(t-1+q^{2N})^2}{[2l]^2}}} , \quad (3.27a)$$

$$\beta_N(l) = \frac{[2N]([2] - q^{\text{sign } N} t) + t(q^{-1} - q)[l - |N|][l + |N| + 1]}{q[2l+2]} . \quad (3.27b)$$

That (3.27b) and equation (4.3a) of [DDLW07] are the same follows from the identity

$$[l][l+1] - [n][n+1] = [l-n][l+n+1] .$$

To prove (3.26) is a straightforward computation, based on the relations $[2l][l]^{-1} = q^l + q^{-l}$ and $[2](1-t) = [2] - q^{\text{sign } N}t - q^{-\text{sign } N}t$, from which one deduces the identities

$$\begin{aligned} (q^{l+N} + q^{-(l+N)}s^2)(q^{l-N}s^2 + q^{-(l-N)}) &= [2l]^2[l]^{-2}s^2 + q^{-2N}(q^{2N} - s^2)^2, \\ [2l] - q^{l+N+1}[2][l-N] &= q^{1-\text{sign } N}[2N] + (1-q^2)[l-|N|][l+|N|+1], \end{aligned}$$

and on the definition $t = 1 - s^2$. It is still not obvious that $\alpha_0^0(l, m; N)$ in (4.2b) equals $\tilde{A}_{l,m}^0$. This follows from the identity

$$[2l] - q^{l+m+1}[2][l-m] = [l+m][l-m+1] - q^2[l+m+1][l-m],$$

which concludes the check of point (ii). With previous identity, also the verification of point (iii) is a long but simple computation.

To control that the ones defined in Proposition 3.4 are representations — i.e. that equations (3.25) are satisfied — we make use of MATHEMATICA[®] 5. Now, (3.25b) is the adjoint of (3.25a). Moreover, (3.25c) and (3.25d) are identities between hermitian elements: in matrix notation, we need to compute only the upper-triangular elements. Thus, we need only to verify that the following matrix elements are zero:

$$\begin{aligned} \langle l+2, m+1 | (3.25a) | l, m \rangle &= B_{l+1,m}^+ A_{l,m}^+ - q^{-2} A_{l+1,m+1}^+ B_{l,m}^+, \\ \langle l+1, m+1 | (3.25a) | l, m \rangle &= B_{l+1,m}^0 A_{l,m}^+ + B_{l,m}^+ (A_{l,m}^0 - q^{-2} A_{l+1,m+1}^0 + tq^{-2} - t) - q^{-2} A_{l,m+1}^+ B_{l,m}^0, \\ \langle l, m+1 | (3.25a) | l, m \rangle &= B_{l-1,m}^+ A_{l-1,m}^+ - q^{-2} A_{l-1,m+1}^+ B_{l,m}^- + B_{l+1,m}^- A_{l,m}^+ - q^{-2} A_{l,m+1}^+ B_{l,m}^+ \\ &\quad + B_{l,m}^0 (A_{l,m}^0 - q^{-2} A_{l,m+1}^0 + tq^{-2} - t), \\ \langle l-1, m+1 | (3.25a) | l, m \rangle &= B_{l-1,m}^0 A_{l-1,m}^+ + B_{l,m}^- (A_{l,m}^0 - q^{-2} A_{l-1,m+1}^0 + tq^{-2} - t) - q^{-2} A_{l-1,m+1}^+ B_{l,m}^0, \\ \langle l-2, m+1 | (3.25a) | l, m \rangle &= B_{l-1,m}^- A_{l-1,m}^+ - q^{-2} A_{l-2,m+1}^+ B_{l,m}^-, \\ \langle l+2, m | (3.25c) | l, m \rangle &= q^{-1}[2] B_{l+2,m}^- B_{l,m}^+ + q^2 A_{l+1,m}^+ A_{l,m}^+, \\ \langle l+1, m | (3.25c) | l, m \rangle &= q^{-1}[2] (B_{l+1,m}^- B_{l,m}^0 + B_{l+1,m}^0 B_{l,m}^+) + q^2 A_{l,m}^+ (A_{l,m}^0 + A_{l+1,m}^0 + tq^{-2} - t), \\ \langle l, m | (3.25c) | l, m \rangle &= q^{-1}[2] \{ (B_{l,m}^+)^2 + (B_{l,m}^0)^2 + (B_{l,m}^-)^2 \} + q^2 (A_{l,m}^+)^2 + q^2 (A_{l-1,m}^+)^2 \\ &\quad + (q^2 A_{l,m}^0 + t)(A_{l,m}^0 - t) - [2]^2(1-t), \\ \langle l+2, m | (3.25d) | l, m \rangle &= q^{-1}[2] B_{l+1,m-1}^+ B_{l+1,m-1}^- + q^{-2} A_{l+1,m}^+ A_{l,m}^+, \\ \langle l+1, m | (3.25d) | l, m \rangle &= q^{-1}[2] (B_{l,m-1}^+ B_{l,m-1}^0 + B_{l+1,m-1}^0 B_{l+1,m-1}^-) + q^{-2} A_{l,m}^+ (A_{l,m}^0 + A_{l+1,m}^0 + tq^{-2} - t), \\ \langle l, m | (3.25d) | l, m \rangle &= q^{-1}[2] \{ (B_{l-1,m-1}^+)^2 + (B_{l,m-1}^0)^2 + (B_{l+1,m-1}^-)^2 \} + q^{-2} (A_{l,m}^+)^2 + q^{-2} (A_{l-1,m}^+)^2 \\ &\quad + (q^{-2} A_{l,m}^0 + t)(A_{l,m}^0 - t) - [2]^2(1-t), \end{aligned}$$

where ‘tildes’ have been removed to make the equations clearer. The inputs and outputs of MATHEMATICA[®] 5 are listed in Table 3.2: the zeroes in the outputs prove the statement. Coefficients $\{\tilde{A}_{l,m}^+, \tilde{A}_{l,m}^0, \tilde{B}_{l,m}^+, \dots\}$ are called respectively $\{\text{Apu}[l,m], \text{Azero}[l,m], \text{Bpu}[l,m], \dots\}$.

3.6 A family of equivariant spectral triples

From now on, $N \in \frac{1}{2}\mathbb{Z}$ is fixed and different from 0. Remind that the Hilbert space \mathcal{H}_N has orthonormal basis $|l, m; N\rangle$ with $l - |N|, l - |m| \in \mathbb{N}$, and carries the equivariant representation of $\mathcal{A}(S_{qs}^2)$ in Proposition 3.4. We wish to study equivariant even spectral triples on $\mathcal{H} := \mathcal{H}_{-N} \oplus \mathcal{H}_N$, with natural grading $\gamma := 1 \oplus -1$.

Let F be the unitary operator¹

$$F |l, m; \pm N\rangle := |l, m; \mp N\rangle .$$

Given any divergent sequence of *positive* numbers $\{\lambda_n\}_{n \in \mathbb{N}}$, we define a self-adjoint operator $D_\lambda = |D_\lambda|F$ on $\mathcal{H} := \mathcal{H}_N \oplus \mathcal{H}_{-N}$ by

$$|D_\lambda| |l, m; \pm N\rangle := \lambda_l |l, m; \pm N\rangle$$

with domain the linear span of basis vectors. Since $\lambda_n > 0$ and $\lambda_n^{-1} \rightarrow 0$, D_λ is invertible and with compact inverse. It commutes with the action of $U_q(su(2))$, being proportional to the identity in each irreducible representation. So, D_λ is a candidate for an equivariant Dirac operator. For a fixed N , since all D_λ have the same sign F , they correspond to different ‘metric’ structures that are ‘conformally’ equivalent (c.f. e.g. [Bär07]).

Among this family, we distinguish the operator

$$D |l, m; \pm N\rangle := (l - |N| + 1) |l, m; \mp N\rangle , \quad (3.28)$$

which for $N = \pm \frac{1}{2}$ is isospectral to the Dirac operator of the round sphere S^2 (cf. e.g. Chapter 2 of [Vár06]).

Lemma 3.5. *The coefficients $\alpha_N(l)$ and $\beta_N(l)$ in (3.27) satisfy the inequalities*

$$|\alpha_N(l) - s\sqrt{1+q^2}| \leq C_{N,s}q^l , \quad (3.29a)$$

$$|\beta_N(l) - (1-s^2)| \leq C'_{N,s}q^l , \quad (3.29b)$$

and in particular if $s = 0$

$$|\alpha_N(l)| \leq C_{N,0}q^l , \quad (3.29c)$$

with $C_{N,s}$ and $C'_{N,s}$ some positive constants.

Proof. Remind the definition of the parameter t , $t := 1 - s^2$. Equations (3.29b) and (3.29c) are an immediate consequence of the definition (3.27) and of the inequalities

$$[n] \leq (q^{-1} - q)^{-1}q^{-n} \quad \text{and} \quad [n+1]^{-1} \leq q^n , \quad \text{for all } n \geq 0 .$$

¹This operator F should not be confused with the generator of $U_q(su(2))$ denoted with the same symbol.

By (3.29c), we need to prove (3.29a) only for $s \neq 0$, i.e. $t \neq 1$. For $t \in [0, 1)$ and $l \geq 1$, set

$$u_l := \frac{[l+N][l-N][2l]}{q[2l+1][l]^2} \frac{1-t+q^{-2N}[2l]^{-2}[l]^2(t-1+q^{2N})^2}{1-t} - 1.$$

From $[l+N][l-N][2l] \leq q[2l+1][l]^2$ we obtain

$$u_l \leq \frac{q^{-2N}[2l]^{-2}[l]^2(t-1+q^{2N})^2}{1-t} = C_N(1-q^2)^2[2l]^{-2}[l]^2 \leq C_N q^{2l},$$

with $C_N := \frac{\{q^N - q^{-N}(1-t)\}^2}{(1-q^2)^2(1-t)}$. On the other hand, from the equalities $[l+N][l-N] = [l]^2 - [N]^2$ and $[2l] - q[2l+1] = -q^{2l+1}$ we get

$$u_l \geq \frac{[2l][l+N][l-N]}{q[2l+1][l]^2} - 1 = \frac{-[2l][N]^2 - q^{2l+1}[l]^2}{q[2l+1][l]^2} \geq -q^{2l-2} \frac{[N]^2}{1-q^2} - q^{4l-1}.$$

Hence there exists a constant $C'_N \in \mathbb{R}^+$ such that $|u_l| \leq C'_N q^{2l}$. Moreover from the inequality $|\sqrt{1+u} - 1| \leq |u|$, which holds for any $u \geq -1$, it follows that

$$\left| \frac{\alpha_N(l)}{\sqrt{(1+q^2)(1-t)}} - 1 \right| = |\sqrt{1+u_l} - 1| \leq |u_l| \leq C'_N q^{2l}.$$

This is exactly (3.29a). □

Lemma 3.6. *The coefficients*

$$C_{i,\pm}^{j,k}(l, m) = \langle l+j, m+k; \pm N | x_i | l, m; \pm N \rangle$$

satisfy (2.17), that is $|C_{i,+}^{j,k}(l, m) - C_{i,-}^{j,k}(l, m)| \leq K_i^{j,k} q^l$ for suitable $K_i^{j,k} > 0$. If $s = 0$, they satisfy (2.22) too, that is $|C_{i,\pm}^{j,k}(l, m)| \leq K_{i,\pm}^{j,k} q^l$ for suitable $K_{i,\pm}^{j,k} > 0$.

Proof. By Proposition 3.4 and equation (3.26) we see that $C_{i,\pm}^{j,k}(l, m)$ is zero if $k \neq i$ and $j \notin \{0, \pm 1\}$, while

$$\begin{aligned} C_{i,\pm}^{+1,i}(l, m) &= c_i^{+1}(l, m) \alpha_{\pm N}(l+1), \\ C_{i,\pm}^{0,i}(l, m) &= c_i^0(l, m) \beta_{\pm N}(l), \\ C_{i,\pm}^{-1,i}(l, m) &= c_i^{-1}(l, m) \alpha_{\pm N}(l), \end{aligned}$$

where $c_i^j(l, m)$ are independent of $\pm N$ and bounded by a constant. Thus, concerning (2.17) we just need to prove that

$$\begin{aligned} |\alpha_N(l) - \alpha_{-N}(l)| &\lesssim q^l, \\ |\beta_N(l) - \beta_{-N}(l)| &\lesssim q^l, \end{aligned}$$

which follow from (3.29a, 3.29b) and by triangle inequality:

$$\begin{aligned} |\alpha_N(l) - \alpha_{-N}(l)| &\leq |\alpha_N(l) - s\sqrt{1+q^2}| + |\alpha_{-N}(l) - s\sqrt{1+q^2}| \lesssim q^l, \\ |\beta_N(l) - \beta_{-N}(l)| &\leq |\beta_N(l) - (1-s^2)| + |\beta_{-N}(l) - (1-s^2)| \lesssim q^l. \end{aligned}$$

If $s = 0$ and for any $j \neq 0$, $C_{i,\pm}^{j,i}(l, m)$ is the product of uniformly bounded coefficients $c_i^j(l, m)$ times either $\alpha_N(l)$ or $\alpha_N(l+1)$, so by (3.29c) it follows (2.22). □

We are in the situation described in Section 2.8. By previous Lemma, for all $s \in [0, 1]$ we have Lemma 2.13 which we rewrite as a proposition,

Proposition 3.7. *The data $(\mathcal{A}(S_{qs}^2), \mathcal{H}, F, \gamma)$ is a even 1-summable Fredholm module.*

and we have also Lemma 2.14, which claims that

Proposition 3.8. *If $\lambda_{n+1} - \lambda_n$ is a bounded sequence, the data $(\mathcal{A}(S_{qs}^2), \mathcal{H}, D_\lambda, \gamma)$ is a regular even spectral triple.*

For $s = 0$, condition (2.22) is satisfied and previous result can be improved by using Lemma 2.16 as follows.

Proposition 3.9. *If $\{q^n \lambda_n\}$ is a bounded sequence, the data $(\mathcal{A}(S_{q0}^2), \mathcal{H}, D_\lambda, \gamma)$ is a even Lipschitz-regular spectral triple. The triple is regular if λ_n is at most of polynomial growth.*

Lemma 3.10. *The Fredholm module in Proposition 3.7 is non-trivial. In particular*

$$\text{Index}(F_{e_s}^+) = 2N ,$$

with e_s the idempotent in (3.21).

Proof. Recall that $N \neq 0$. We use (2.3) with $m = 0$ and find

$$\begin{aligned} \text{Index}(F_{e_s}^+) &= \frac{1}{2} \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^2}(\gamma F[F, e_s]) = \frac{1 - q^2}{2(1 + s^2)} \text{Tr}_{\mathcal{H}}(\gamma F[F, A]) \\ &= \frac{-1}{1 + s^2} \frac{1 - q^2}{1 + q^2} \sum_{l, m} \frac{[2l] - q^{l+m+1}[2][l - m]}{[2l]} \{\beta_{-N}(l) - \beta_N(l)\} \\ &= (q^{-2} - 1)[2N] \sum_{l, m} \frac{[2l] - q^{l+m+1}[2][l - m]}{[2l][2l + 2]} . \end{aligned}$$

Computing the sum in $m = -l, \dots, l$ and setting $n = l - |N| + 1$, we obtain

$$\text{Index}(F_{e_s}^+) = 2 \text{sign}(N) (1 - x^{2|N|}) \sum_{n=1}^{\infty} \left(\frac{n + |N| - 1}{1 - x^{2(n+|N|-1)}} x^{n-1} - \frac{n + |N|}{1 - x^{2(n+|N|)}} x^n \right) ,$$

where $x = q^2 \in]0, 1[$. Last sum is a telescoping series, and the sum is

$$\text{Index}(F_{e_s}^+) = 2 \text{sign}(N) (1 - x^{2|N|}) \frac{n + |N| - 1}{1 - x^{2(n+|N|-1)}} x^{n-1} \Big|_{n=1} = 2N .$$

The non-triviality of the Fredholm module comes from the non-vanishing of the index of $F_{e_s}^+$. \square

If $D_\lambda = D$ is the Dirac operator in (3.28), the multiplicity of the n th eigenvalue $n + 1$ is $4n + 4|N| + 2$ and so by Lemma 2.15,

Lemma 3.11. *The metric dimension of the linear Dirac operator D is 2.*

For this Dirac operator we now compute the dimension spectrum and the top residue. To this aim, we introduce an auxiliary representation which is obtained from simpler operators. Let $\hat{\mathcal{H}} \simeq \ell^2(\mathbb{Z} \times \mathbb{N})$ be the Hilbert space having orthonormal basis $\|n, k\rangle$, with $n \in \frac{1}{2}\mathbb{Z}$ and $n + k \in \mathbb{N}$. Consider the bounded operators $\alpha, \beta \in \mathcal{B}(\hat{\mathcal{H}})$ defined by

$$\alpha \|n, k\rangle = \sqrt{1 - q^{2(n+k+1)}} \|n + \frac{1}{2}, k + \frac{1}{2}\rangle, \quad \beta \|n, k\rangle = -q^{n+k} \|n + \frac{1}{2}, k - \frac{1}{2}\rangle.$$

These operators satisfy the defining relations of $\mathcal{A}(SU_q(2))$, as can easily be seen. The morphism $\mathcal{A}(S_{qs}^2) \hookrightarrow \mathcal{A}(SU_q(2))$ in (2.47) leads to a $*$ -representation $\pi : \mathcal{A}(S_{qs}^2) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$ given by

$$\begin{aligned} \pi(A) \|n, k\rangle &= -s q^{n+k-1} \sqrt{1 - q^{2(n+k)}} \|n - 1, k\rangle \\ &\quad + (1 - s^2) q^{2(n+k)} \|n, k\rangle \\ &\quad - s q^{n+k} \sqrt{1 - q^{2(n+k+1)}} \|n + 1, k\rangle, \end{aligned} \quad (3.30a)$$

$$\begin{aligned} \pi(B) \|n, k\rangle &= -s q^{2(n+k)+1} \|n - 1, k + 1\rangle \\ &\quad - (1 - s^2) q^{n+k+1} \sqrt{1 - q^{2(n+k+1)}} \|n, k + 1\rangle \\ &\quad + s \sqrt{1 - q^{2(n+k+1)}} \sqrt{1 - q^{2(n+k+2)}} \|n + 1, k + 1\rangle. \end{aligned} \quad (3.30b)$$

Our Hilbert space of spinors \mathcal{H} is identified with a subspace of $\hat{\mathcal{H}} \oplus \hat{\mathcal{H}}$. That is, for each N there is a partial isometry $W_N : \mathcal{H}_N \rightarrow \hat{\mathcal{H}}$, given by

$$W_N |l, m; N\rangle := \|l - |N|, m + |N|\rangle,$$

and with left inverse $W_N^* : \hat{\mathcal{H}} \rightarrow \mathcal{H}_N$, given by

$$W_N^* \|n, k\rangle = \begin{cases} |n + |N|, k - |N|; N\rangle & \text{if } n \in \mathbb{N} \text{ and } k \leq n + 2|N|, \\ 0 & \text{otherwise.} \end{cases}$$

We denote

$$Q := W_{-N} \oplus W_N, \quad P := W_{-N}^* \oplus W_N^*.$$

Since $W_N^* W_N = id_{\mathcal{H}_N}$, the Hilbert space \mathcal{H} is isomorphic to its image in $\hat{\mathcal{H}} \otimes \mathbb{C}^2 \simeq \hat{\mathcal{H}} \oplus \hat{\mathcal{H}}$ via the map Q . The Dirac operator D in (3.28) is the “restriction” of the self-adjoint operators $\hat{D} = \hat{F}|\hat{D}|$ on $\hat{\mathcal{H}} \otimes \mathbb{C}^2$ determined by

$$|\hat{D}| \|n, k\rangle = (n + 1) \|n, k\rangle, \quad \hat{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in the sense that Q intertwines D and \hat{D} (i.e. $\hat{D}Q = QD$ on \mathcal{H}). The same holds for $|\hat{D}|$ and \hat{F} . For $a \in \mathcal{A}(S_{qs}^2)$, the diagonal lift $\pi(a) \otimes id_{\mathbb{C}^2}$ of $\pi(a)$ to $\hat{\mathcal{H}} \otimes \mathbb{C}^2$ will be denoted simply by $\pi(a)$.

Lemma 3.12. *The operator $a - P\pi(a)Q$ belongs to $OP^{-\infty}(D)$ for all $a \in \mathcal{A}(S_{qs}^2)$.*

Proof. For $a \in \mathcal{A}(S_{qs}^2)$, define two bounded operators $R_a : \mathcal{H} \rightarrow \mathcal{H}$ and $T_a : \hat{\mathcal{H}} \rightarrow \mathcal{H}$ by

$$R_a := a - P\pi(a)Q, \quad T_a := P\pi(a)(1 - QP),$$

and notice that if a is a generator of $\mathcal{A}(S_{qs}^2)$, T_a has finite rank. On the other hand, if T_a and T_b have finite rank for some $a, b \in \mathcal{A}(S_{qs}^2)$, then T_{ab} has finite rank too, due to the identity

$$T_{ab} = T_a \pi(b)(1 - QP) + P\pi(a)Q T_b.$$

By the above considerations, we conclude that T_a has finite rank for all $a \in \mathcal{A}(S_{qs}^2)$. Therefore, $\text{OP}^{-\infty}$ being an ideal in OP^0 , using the identity

$$R_{ab} = aR_b + R_a P\pi(b)Q - T_a \pi(b)Q,$$

we conclude that R_{ab} is a smoothing operators whenever $R_a, R_b \in \text{OP}^{-\infty}$. It is then enough to prove this lemma for the generators of the algebra. Using (3.29) and the inequality $0 \leq 1 - \sqrt{1-u} \leq u$, valid for $u \in [0, 1]$, one proves that R_A, R_B and R_{B^*} have coefficients bounded by q^l , hence they are rapid decay matrices. This concludes the proof. \square

The next proposition is the main result of this section.

Proposition 3.13. *The dimension spectrum is $\Sigma = \{1, 2\}$.*

Proof. Let Ψ^0 be the algebra generated by $\mathcal{A}(S_{qs}^2)$, by $[D, a]$ for all $a \in \mathcal{A}(S_{qs}^2)$ and by iterated applications of the derivation δ (cf. Section 2.2). Let \mathfrak{C} be the $*$ -algebra (of bounded operators on $\hat{\mathcal{H}}$) generated by $\alpha, \beta, \alpha^*, \beta^*$ and F . By Lemma 3.12, $\mathcal{A}(S_{qt}^2) \subset P\mathfrak{C}Q + \text{OP}^{-\infty}$. Note that

$$\begin{aligned} [\hat{F}, \alpha] &= 0, & [\hat{D}, \alpha] &= \frac{1}{2} \alpha \hat{F}, & [|\hat{D}|, \alpha] &= \frac{1}{2} \alpha, \\ [\hat{F}, \beta] &= 0, & [\hat{D}, \beta] &= \frac{1}{2} \beta \hat{F}, & [|\hat{D}|, \beta] &= \frac{1}{2} \beta. \end{aligned}$$

Thus $P\mathfrak{C}Q$ is invariant under application of δ and $[D, (\cdot)]$ and hence $\Psi^0 \subset P\mathfrak{C}Q + \text{OP}^{-\infty}$.

We shall compute the singularities of zeta functions associated to the monomials

$$S := P\alpha^i \beta^j (\beta^*)^k Q F \quad \text{and} \quad T := P\alpha^i \beta^j (\beta^*)^k Q,$$

where $i \in \mathbb{Z}$ and $j, k \in \mathbb{N}$ and we use the notation $\alpha^i := (\alpha^*)^{|i|}$ for $i < 0$. From the commutation relations of α and β , it is clear that these monomials span $P\mathfrak{C}Q$.

Firstly, note that the zeta-function associated with a bounded off-diagonal operator is identically zero in the half-plane $\text{Re } z > 2$ and so is its holomorphic extension to the entire complex plane. This is the case for the monomials S due to the presence of F . The other monomials T shift the index l by $(i + j - k)/2$ and the index m by $(i - j + k)/2$ and therefore are also off-diagonal operators unless these shifts are zero, which happens when $i = 0$ and $j = k$. Hence only monomials $T = P\beta^k (\beta^*)^k Q = P(\beta\beta^*)^k Q$ contribute to the dimension spectrum.

For $k = 0$, $T = 1$ and the corresponding zeta function is

$$\zeta_1(z) = \sum_{n=1}^{\infty} (4n + 4|N| - 2)n^{-z} = 4\zeta(z-1) + (4|N| - 2)\zeta(z),$$

where $\zeta(z)$ is the Riemann zeta function, meromorphic in \mathbb{C} with a simple pole at 1 and with residue 1. Since $1 \in \Psi^0$, this shows that $2 \in \Sigma$. For $|N| \neq 1/2$, this proves also that $1 \in \Sigma$.

When $k > 0$, $T = P(\beta\beta^*)^k Q$. So $T|l, m\rangle_{\pm} = q^{2k(l+m)}|l, m\rangle_{\pm}$ and the associated ζ -function is

$$\begin{aligned} \zeta_T(z) &= \sum_{l \in \mathbb{N} + |N|}^{\infty} 2(l - |N| + 1)^{-z} \sum_{l+m=0}^{2l} (q^{2k})^{l+m} = \sum_{l \in \mathbb{N} + |N|}^{\infty} 2(l - |N| + 1)^{-z} \frac{1 - q^{2k(2l+1)}}{1 - q^{2k}} \\ &= \frac{2}{1 - q^{2k}} \zeta(z) + \text{holomorphic function}. \end{aligned}$$

Therefore Σ may contain, besides 2, at most the additional point 1. We still have to check if $1 \in \Sigma$ for $|N| = 1/2$. For this, we take $A^2 = P\pi(A)^2 Q \pmod{\text{OP}^{-\infty}(D)}$. Then, using (3.30), we get (modulo holomorphic functions)

$$\zeta_{A^2}(z) \sim 2 \sum_{l+\frac{1}{2}=1}^{\infty} (l + \frac{1}{2})^{-z} \sum_{l+m=0}^{2l} \left\{ (1 - [2]^2 s^2 + s^4) q^{4(l+m)} + [2] s^2 q^{2(l+m)-1} \right\} \sim 2 \frac{1 + s^4}{1 - q^4} \zeta(z),$$

and $\text{Res}_{z=1} \zeta_{A^2}(z) = 2 \frac{1+s^4}{1-q^4} \neq 0$ for all $s \in [0, 1]$. Thus $\Sigma = \{1, 2\}$ for any value of N . \square

Let $\mathcal{A}(S^1)$ denote the polynomial $*$ -algebra generated by the function u , given by $u(\theta) = e^{i\theta}$. A $*$ -algebra morphism $\sigma_s : \mathcal{A}(S_{qs}^2) \rightarrow \mathcal{A}(S^1)$ is defined by

$$\sigma_s(A) = 0, \quad \sigma_s(B) = su. \quad (3.31)$$

Proposition 3.14. *The top residue of $\zeta_a(z)$, $a \in \mathcal{A}(S_{qs}^2)$, is*

$$\int a|D|^{-2} = \frac{2}{\pi} \int_{S^1} \sigma_s(a)(\theta) d\theta. \quad (3.32)$$

For $s = 0$, $\sigma_0(a) \in \mathbb{C}$ is constant and equation (3.32) simplifies to $4\sigma_0(a)$.

Proof. It is sufficient to prove (3.32) for the basis elements $A^j B^k$, $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, and then extend it to $\mathcal{A}(S_{qs}^2)$ by linearity. We use the notation $B^k := (B^*)^{|k|}$ if $k < 0$.

Since $\sigma_s(A^j B^k) = \delta_{j0} (su)^k$, the right hand side of equation (3.32) is zero unless $j = k = 0$. We next show that the left hand side of (3.32) also vanishes unless $j = k = 0$. When $j = k = 0$, the relation $\zeta_1(z) = 4\zeta(z-1)$ fixes the normalization constant.

Now $A^j B^k$ is off-diagonal if $k \neq 0$ since it shifts the index m by k . It remains to prove that $\zeta_{A^j}(z) = \zeta_{P\pi(A)^j Q}(z) + \text{holomorphic function}$ has no singularity in $z = 2$. For $j \neq 0$, $P\pi(A)^j Q$ satisfies the inequality

$$|\langle l, m, \pm N | P\pi(A)^j Q | l, m, \pm N \rangle| \leq c_j q^{l+m},$$

for some positive constants c_j 's. From this inequality, we deduce that $\zeta_{A^j}(z)$ is a convergent series for all z with $\text{Re } z > 1$. In particular, it is finite for $z = 2$. \square

Last point to discuss is the existence of a real structure J . We can assume $N > 0$, as the choice $N < 0$ is equivalent to changing the sign of the grading operator. The operator T defined by $T|l, m; \pm N\rangle = (-q)^{m+N}|l, -m; \mp N\rangle$ is equivariant — one can repeat verbatim the proof of (3.4). The antiunitary part J of T is

$$J|l, m; \pm N\rangle = (-1)^{m+N}|l, -m; \mp N\rangle . \quad (3.33)$$

Proposition 3.15. *The antilinear isometry J defined by (3.33) is the antiunitary part of an antilinear operator, it satisfies $J^2 = (-1)^{2N}$, $J\gamma = -\gamma J$, $JD = DJ$ and*

$$[a, Jb^*J^{-1}] \in \text{OP}^{-\infty} , \quad [[D, a], Jb^*J^{-1}] \in \text{OP}^{-\infty} ,$$

for all $a, b \in \mathcal{A}(S_{qs}^2)$. It is then a ('weak') equivariant real structure with KO -dimension $2 \bmod 8$ (resp. $6 \bmod 8$) if $2N$ is odd (resp. even).

Proof. Since $[D, a] - \delta(a)F = |D|[F, a]$ and $(JF)b(JF)^{-1} - JbJ^{-1} = J[F, b]FJ^{-1}$ are smoothing operators for all a, b , we can replace $[D, \cdot]$ by δ and J by $J_0 := JF$ in the verification. Therefore, we want to show

$$[a, J_0b^*J_0^{-1}] \in \text{OP}^{-\infty} , \quad [\delta(a), J_0b^*J_0^{-1}] \in \text{OP}^{-\infty} .$$

Now, let \hat{J} be the antilinear operator on $\hat{\mathcal{H}}$ given by

$$\hat{J}||n, k\rangle\rangle = i^{2k}||n, -k + 2N\rangle\rangle ,$$

and note that $P\hat{J}Q = J_0$. We can replace a by $P\pi(a)Q$ since the difference is a smoothing operator, thus Lemma is proved if we prove that

$$P[\pi(a), \hat{J}\pi(b^*)\hat{J}^{-1}]Q \in \text{OP}^{-\infty} , \quad P[[\hat{D}, \pi(a)], \hat{J}\pi(b^*)\hat{J}^{-1}]Q \in \text{OP}^{-\infty} .$$

By Leibniz rule it is sufficient to prove the previous equations taking instead of $\pi(a)$ and $\pi(b^*)$ two of the generators $\{\alpha, \beta, \alpha^*, \beta^*\}$ of the enlarged algebra. These are all eigenvectors for $[[\hat{D}], \cdot]$, thus the first order condition follows from the commutant condition. From

$$[a, \hat{J}b\hat{J}^{-1}] = -[a^*, \hat{J}b^*\hat{J}^{-1}] = \hat{J}[b, \hat{J}a\hat{J}^{-1}]\hat{J}^{-1} ,$$

once the commutant condition is proved for a fixed couple of elements (a, b) , it is proved also for (a^*, b^*) and (b, a) . Therefore it is sufficient to consider the cases $(a, b) = (\alpha, \alpha), (\alpha^*, \alpha), (\beta, \alpha), (\beta^*, \alpha), (\beta, \beta)$ and (β^*, β) . By direct computation

$$[\alpha, \hat{J}\alpha\hat{J}^{-1}] = 0 , \quad [\alpha^*, \hat{J}\alpha\hat{J}^{-1}] = 0 , \quad [\beta, \hat{J}\beta\hat{J}^{-1}] = 0 ,$$

and

$$[\beta, \hat{J}\alpha\hat{J}^{-1}]||n, k\rangle\rangle = i \left(\sqrt{q^{2k} - q^{4(n+N+1)}} - \sqrt{q^{2k} - q^{4(n+N+1)-2}} \right) ||n+1, k-1\rangle\rangle ,$$

$$\begin{aligned}
[\beta^*, \hat{J}\alpha\hat{J}^{-1}] \|n, k\rangle\rangle &= i \left(\sqrt{q^{2k} - q^{4(n+N)}} - \sqrt{q^{2k} - q^{4(n+N+1)-2}} \right) \|n+1, k-1\rangle\rangle, \\
[\beta^*, \hat{J}\beta\hat{J}^{-1}] &= i(q - q^{-1})q^{2(n+N)} \|n, k+1\rangle\rangle.
\end{aligned}$$

Coefficients in the last equations are bounded by q^n , hence projected to \mathcal{H} are bounded by q^l and this concludes the proof. \square

3.7 Spectral triples for pre- C^* -algebras

Things become more complicated when one considers an algebra which is not made of polynomials in a finite number of generators. We discuss an example in this section, which is a generalization of the construction in Appendix B of [DD06].

We call $C(S_{qs}^2)$ the universal C^* -algebra associated with $\mathcal{A}(S_{qs}^2)$. The relations defining $\mathcal{A}(S_{qs}^2)$ can be rewritten as

$$BB^* + \left(A - \frac{1-s^2}{2}\right)^2 = \left(\frac{1+s^2}{2}\right)^2, \quad B^*B + \left(q^2A - \frac{1-s^2}{2}\right)^2 = \left(\frac{1+s^2}{2}\right)^2.$$

The first equation implies the following inequality for the operator norm in an arbitrary bounded $*$ -representation of the algebra:

$$\|A - \frac{1-s^2}{2}\| \leq \frac{1+s^2}{2}.$$

Hence, by triangle inequality

$$\begin{aligned}
\|q^2A - \frac{1-s^2}{2}\| &= q^2\|A - \frac{1-s^2}{2} - (q^{-2} - 1)\frac{1-s^2}{2}\| \leq q^2\|A - \frac{1-s^2}{2}\| + (1 - q^2)\frac{1-s^2}{2} \\
&\leq \frac{1+(2q^2-1)s^2}{2} < \frac{1+s^2}{2}.
\end{aligned}$$

This implies (cf. Lemma 2.2.4 of [Lan98]) that the element

$$B^*B = \left(\frac{1+s^2}{2}\right)^2 - \left(q^2A - \frac{1-s^2}{2}\right)^2$$

is invertible in the C^* -algebra. We can then construct a projection $P \in C(S_{qs}^2)$,

$$P = 1 - B(B^*B)^{-1}B^*, \tag{3.34}$$

and the corresponding representation of the algebra on the module $C(S_{qs}^2)P$. For that we work first with $\mathcal{A}(S_{qs}^2)P$, which is an $\mathcal{A}(S_{qs}^2)$ -module.

First of all, notice that if $[A, C] = 0$ and C^{-1} exists, then $[A, C^{-1}] = -C^{-1}[A, C]C^{-1} = 0$. This proves that $(B^*B)^{-1}$ commutes with A , and so in particular $AP = PA$.

From $B^*P = 0$, $(A + s^2)(1 - A) = BB^*$ and $AP = PA$ we deduce that the elements:

$$P_+ = \frac{1}{1+s^2}(A + s^2)P, \quad P_- = \frac{1}{1+s^2}(1 - A)P, \tag{3.35}$$

are projections, are mutually orthogonal ($P_+P_- = P_-P_+ = 0$) and their sum is P . Then, $\mathcal{A}(S_{qs}^2)P$ is \mathbb{Z}_2 -graded and decomposes as $\mathcal{A}(S_{qs}^2)P_+ \oplus \mathcal{A}(S_{qs}^2)P_-$.

The (left) kernel of P_+ is the left ideal generated by B^* (as $B^*P = 0$) and $1 - A$. The (left) kernel of P_- is the left ideal generated by B^* and $A + s^2$. A linear basis of both modules is then $\{B^n P_\pm\}$. For $s \neq 0$ we can normalize the vectors as follows. We take as linear basis of $\mathcal{A}(S_{qs}^2)P_\pm$ the vectors

$$|n\rangle_+ = c_n^+ B^n P_+, \quad |n\rangle_- = c_n^- B^n P_-, \quad n \in \mathbb{N}, \quad (3.36)$$

where the normalization constants are defined by $c_0^\pm := 1$ and

$$c_n^+ := \left\{ \prod_{k=1}^n (1 - q^{2k})(s^2 + q^{2k}) \right\}^{-1/2},$$

$$c_n^- := s^{-n} \left\{ \prod_{k=1}^n (1 - q^{2k})(1 + q^{2k}s^2) \right\}^{-1/2},$$

for all $n \geq 1$. The basis (3.36) is orthonormal for the inner product induced by the Haar functional, and we have a \mathbb{Z}_2 -graded $*$ -representation of $C(S_{qs}^2)$ on the Hilbert space completion, naturally isomorphic to $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, defined by

$$A |n\rangle_+ = q^{2n} |n\rangle_+ \quad B |n\rangle_+ = \sqrt{(1 - q^{2(n+1)})(s^2 + q^{2(n+1)})} |n+1\rangle_+ \quad (3.37a)$$

$$A |n\rangle_- = -s^2 q^{2n} |n\rangle_- \quad B |n\rangle_- = s \sqrt{(1 - q^{2(n+1)})(1 + q^{2(n+1)}s^2)} |n+1\rangle_- \quad (3.37b)$$

with B^* the adjoint of B . These are the well known unique irreducible infinite-dimensional representations of S_{qs}^2 . They are inequivalent and not faithful (as follows from the observation that A has positive spectrum in the former representation and negative in the latter), while their direct sum is known to be faithful [She91]. They are also equivariant with respect to the Hopf subalgebra of $U_q(su(2))$ generated by the element K ; being $U(1)$ the Lie group associated to this commutative and cocommutative Hopf algebra, these representations are usually referred to as $U(1)$ -equivariant [CP03b].

The representations (3.37) were used in [She91] to prove that the sequence

$$0 \rightarrow \mathcal{K} \oplus \mathcal{K} \rightarrow C(S_{qs}^2) \xrightarrow{\sigma_s} C(S^1) \rightarrow 0 \quad (3.38)$$

is exact (for $s \neq 0$). This is the noncommutative analogue of the sequence

$$0 \rightarrow C_0(\text{open disk}) \oplus C_0(\text{open disk}) \rightarrow C(S^2) \rightarrow C(S^1) \rightarrow 0,$$

and allows to interpret $C(S_{qs}^2)$ as the union of two noncommutative open disks \mathcal{K} glued along a commutative circle, as shown pictorially in Figure 3.1. Here $\sigma_s : C(S_{qs}^2) \rightarrow C(S^1)$ is the completion of the $*$ -algebra morphism (3.31).

When $s = 0$, the constants c_n^\pm are well defined and the ‘+’-representation is still valid, while the ‘-’-representation becomes trivial. We’ll consider only the ‘+’-representation in this case. Such a representation is faithful and irreducible, and has been used in [She91] to prove that the following sequence is exact,

$$0 \rightarrow \mathcal{K} \rightarrow C(S_{q0}^2) \xrightarrow{\sigma_0} \mathbb{C} \rightarrow 0, \quad (3.39)$$

with $\sigma_0 : C(S_{q_0}^2) \rightarrow \mathbb{C}$ the C^* -algebra morphism defined by $\sigma_0(A) = \sigma_0(B) = 0$ (and of course $\sigma_0(1) = 1$). Equation (3.39) is the noncommutative analogue of the sequence

$$0 \rightarrow C_0(\text{open disk}) \rightarrow C(S^2) \rightarrow \mathbb{C} \simeq C(\{*\}) \rightarrow 0,$$

and allows us to interpret $C(S_{q_0}^2)$ as the one-point compactification of a noncommutative disk.

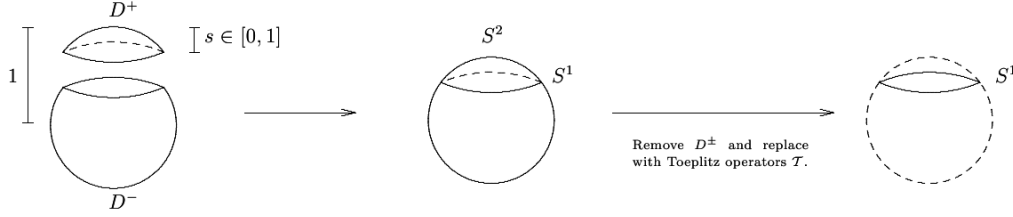


Figure 3.1: Geometric visualization of S_{qs}^2

Let us define the operators N and F as

$$N |n\rangle_{\pm} = (n+1) |n\rangle_{\pm}, \quad F |n\rangle_{\pm} = |n\rangle_{\mp}, \quad \forall n \in \mathbb{N}.$$

For $s \neq 0$, $(\mathcal{A}(S_{qs}^2), \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}), N \cdot F)$ is the $U(1) \times U(1)$ -equivariant spectral triple studied in [CP03b] (we shifted by $+1$ the spectrum of N to make it invertible), while $(\mathcal{A}(S_{q_0}^2), \ell^2(\mathbb{N}), N)$ is the analogue one for the standard Podleś sphere, and has a positive Dirac operator.

Since trace class operators do not contribute to the noncommutative integral, one immediately proves that for $s \neq 0$

$$\int a N^{-1} = \frac{1}{\pi} \int_{S^1} \sigma_s(a) d\theta \quad \forall a \in \mathcal{A}(S_{qs}^2),$$

and for $s = 0$

$$\int a N^{-1} = \sigma_0(a) \quad \forall a \in \mathcal{A}(S_{q_0}^2).$$

When considering only polynomials $\mathcal{A}(S_{qs}^2)$, one can prove that the triple is regular and the dimension spectrum is $\{1\}$. If $s \neq 0$, this result is still valid if one considers the pre- C^* -algebra $C^\infty(S_{qs}^2)$ entering in the exact sequence

$$0 \rightarrow \mathcal{S} \oplus \mathcal{S} \rightarrow C^\infty(S_{qs}^2) \xrightarrow{\sigma_s} C^\infty(S^1) \rightarrow 0. \quad (3.40)$$

where \mathcal{S} is the pre- C^* -algebra of rapid decay matrices on $\ell^2(\mathbb{N})$.

For $s = 0$ the sequence defining the algebra of ‘smooth function’ on the sphere is:

$$0 \rightarrow \mathcal{S} \rightarrow C^\infty(S_{q_0}^2) \xrightarrow{\sigma_0} \mathbb{C} \rightarrow 0. \quad (3.41)$$

In the remaining of the section we prove that the spectral triples above can be extended to regular spectral triples over $C^\infty(S_{qs}^2)$, and compute their dimension spectrum.

3.7.1 Spectral triples over the quantum disk

Let $|n\rangle$ be the canonical orthonormal basis of $\ell^2(\mathbb{N})$, and $N|n\rangle := (n+1)|n\rangle$. The unilateral shift w , defined by $w|n\rangle = |n+1\rangle$, generates the Toeplitz C^* -algebra \mathcal{T} , that can be viewed as algebra of ‘continuous functions’ on a quantum version of the closed disk [Con04].

We define $\mathcal{A}^\infty \subset \mathcal{T}$ as the linear span of the elements:

$$f = \sum_{n \in \mathbb{N}} (f_n w^n + f_{-n-1} (w^*)^{n+1}) + \sum_{j,k \in \mathbb{N}} f_{jk} w^j (1 - w w^*) (w^*)^k \quad \{f_n\} \in \mathcal{S}(\mathbb{Z}), \{f_{jk}\} \in \mathcal{S},$$

where $\mathcal{S}(\mathbb{Z})$ indicates rapid decay sequences and \mathcal{S} rapid decay matrices on $\ell^2(\mathbb{N})$.

By direct calculation one checks that \mathcal{A}^∞ is a $*$ -algebra and the map $\sigma : \mathcal{A}^\infty \rightarrow C^\infty(S^1)$, given by $f \mapsto \sigma(f)$, $\sigma(f)(\theta) := \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$, is a surjective $*$ -algebra morphism (this follows from the simple observation that, via Fourier series, $\mathcal{S}(\mathbb{Z})$ equipped with convolution product is isomorphic to $C^\infty(S^1)$). Furthermore, $\ker \sigma$ is a two-sided $*$ -ideal in \mathcal{A}^∞ isomorphic to \mathcal{S} . At the level of abstract algebras, this follows from the equality

$$w^j (1 - w w^*) (w^*)^k w^{j'} (1 - w w^*) (w^*)^{k'} = \delta_{j'k} w^j (1 - w w^*) (w^*)^{k'}$$

and becomes more evident in matrix form, as

$$\langle j|f|k\rangle = f_{j,k}, \quad \forall f = \sum_{j,k \in \mathbb{N}} f_{jk} w^j (1 - w w^*) (w^*)^k \in \ker \sigma.$$

Then, we have the short exact sequence

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{A}^\infty \xrightarrow{\sigma} C^\infty(S^1) \rightarrow 0. \quad (3.42)$$

Notice that the action of $U(1)$ on polynomials in $\{w, w^*\}$, given by $w \mapsto e^{i\theta} w$, defines a one-parameter group of automorphisms implemented on $\ell^2(\mathbb{N})$ by the unitary operators $e^{i\theta N}$ (i.e. $x \mapsto e^{i\theta N} x e^{-i\theta N}$). Being implemented by unitary operators, it extends to a (strongly continuous) action of $U(1)$ on the C^* -algebra \mathcal{T} , and then on \mathcal{A}^∞ .

We conclude with the statement,

Proposition 3.16. *\mathcal{A}^∞ is a Fréchet space and a pre- C^* -algebra.*

Proof. The direct sum of two Fréchet spaces is a Fréchet space too, and the observation that \mathcal{A}^∞ as a vector space is the direct sum of $C^\infty(S^1)$ and \mathcal{S} (the sequence (3.42) splits) proves the first claim. To prove that \mathcal{A}^∞ is a pre- C^* -algebra it is sufficient to show that it contains the inverse of each element $f \in \mathcal{A}^\infty$, whenever f is invertible in \mathcal{T} (cf. [Bos90]). To reach this conclusion, one can easily adapt the proof of Proposition 1 in [Con04] to the present case. \square

We identify $\mathcal{A}^\infty \subset \mathcal{T}$ with ‘smooth functions’ over the quantum disk.

Proposition 3.17. *The data $(\mathcal{A}^\infty, \ell^2(\mathbb{N}), N)$ is a regular spectral triple, the dimension spectrum is $\{1\}$ and the top residue is*

$$\int f N^{-1} = \frac{1}{2\pi} \int_{S^1} \sigma(f) d\theta,$$

for all $f \in \mathcal{A}^\infty$, where σ is the map in (3.42).

Proof. The operators $w^k, (w^*)^k$ are eigenvectors of the derivation $\delta = [N, \cdot]$: $\delta(w^k) = kw^k$ and $\delta((w^*)^k) = -k(w^*)^k$. If $\{f_n\} \in \mathcal{S}(\mathbb{Z}), \{f_{jk}\} \in \mathcal{S}$ then $\{nf_n\} \in \mathcal{S}(\mathbb{Z})$ and $\{(j-k)f_{jk}\} \in \mathcal{S}$ too. This proves that $\delta(f) \in \mathcal{A}^\infty$ for all $f \in \mathcal{A}^\infty$. Being \mathcal{A}^∞ bounded δ -invariant operators, $\mathcal{A}^\infty = \mathcal{A}^\infty \cup [N, \mathcal{A}^\infty] \subset \text{OP}^0$ and the triple is regular.

The algebra Ψ^0 of order ≤ 0 pseudo-differential operators coincides with \mathcal{A}^∞ . By Weierstrass theorem $\text{Tr}_{\ell^2(\mathbb{N})}(fN^{-z})$ is a holomorphic function of $z \in \mathbb{C}$ for all $f \in \mathcal{S} = \ker \sigma$, thus $\ker \sigma$ does not contribute to the dimension spectrum, and we have to consider only elements of the form

$$f = \sum_{n \in \mathbb{N}} (f_n w^n + f_{-n-1} (w^*)^{n+1}).$$

All the terms in f are off-diagonal, except the one proportional to $f_0 = \frac{1}{2\pi} \int_{S^1} \sigma(f) d\theta$. The function $\text{Tr}_{\ell^2(\mathbb{N})}(fN^{-z}) = f_0 \sum_{n \in \mathbb{N}} (n+1)^{-z}$ has meromorphic extension $f_0 \zeta(z)$, where $\zeta(z)$ is the Riemann zeta-function. This concludes the proof. \square

3.7.2 The $U(1) \times U(1)$ -equivariant spectral triple over $S_{qs}^2, s \neq 0$

In this section we consider $s \neq 0$, and introduce the symbols $\mu := \mu_+ \oplus \mu_-$ for the representation of $\mathcal{A}(S_{qs}^2)$ over $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ defined by (3.37). In terms of the operators q^N and w one has

$$\begin{aligned} \mu_+(A) &= q^{2N-2}, & \mu_+(B) &= w \sqrt{(1-q^{2N+2})(s^2+q^{2N+2})}, \\ \mu_-(A) &= -s^2 q^{2N-2}, & \mu_-(B) &= sw \sqrt{(1-q^{2N+2})(1+q^{2N+2}s^2)}. \end{aligned}$$

We have $q^N \in \mathcal{S} \subset \mathcal{A}^\infty$, with \mathcal{A}^∞ the algebra of the previous section. Since $\text{Sp}(1-q^{2N+2}) \subset \mathbb{R}^+$ and the square root is holomorphic in an angular sector containing \mathbb{R}^+ , by stability under holomorphic functional calculus we deduce that $\mu_\pm(A)$ and $\mu_\pm(B)$ belong to \mathcal{A}^∞ ; in particular, $\mu_\pm(A) \in \mathcal{S}$ and $\mu_\pm(B) = sw \pmod{\mathcal{S}}$. If we identify $\mathcal{A}(S_{qs}^2)$ with its representation μ , and call

$$C^\infty(S_{qs}^2) := \{(x_+, x_-) \in \mathcal{A}^\infty \oplus \mathcal{A}^\infty \mid \sigma(x_+) = \sigma(x_-)\},$$

then $\mathcal{A}(S_{qs}^2) \subset C^\infty(S_{qs}^2) \subset C(S_{qs}^2)$ for all $s \neq 0$. That is, ‘smooth functions’ over S_{qs}^2 contain ‘polynomial’ ones and are contained in ‘continuous’ ones.

From the direct sum of two copies of (3.42), passing to the diagonal ($\sigma(x_+) = \sigma(x_-)$) we obtain the exact sequence (3.40). The morphism $\sigma(x_+) = \sigma(x_-) \equiv \sigma_s(x)$ is the one defined by equation (3.31), for all $x = x_+ \oplus x_- \in C(S_{qs}^2)$ (the equality holds for the generators A, B and then for all the C^* -algebra, all members of the equation being C^* -algebra morphisms).

The definition of $C^\infty(S_{qs}^2)$ does not depend on s , so the quantum spheres (with $s \neq 0$) are all isomorphic already at the smooth level.

Since the smooth algebra do not depend on s , we can apply the analysis made in [DD06] for the equatorial Podleś sphere to prove that:

Proposition 3.18. *($C^\infty(S_{qs}^2), \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}), N \cdot F$) is a regular even spectral triple. The dimension spectrum is $\{1\}$. The noncommutative integral is*

$$\int a N^{-1} = \frac{1}{\pi} \int_{S^1} \sigma_s(a)(\theta) d\theta,$$

for all $a \in C^\infty(S_{qs}^2)$ and with σ_s the $*$ -algebra morphism in equation (3.31).

Proof. Concerning the first claim, the non-trivial part is to prove regularity. Let \mathfrak{B} be the algebra generated by $C^\infty(S_{qs}^2)$ and by commutators with $D = NF$, and let Ψ^0 the algebra generated by \mathfrak{B} and iterated applications of $\delta = [N, \cdot]$ to \mathfrak{B} .

Notice that for any $x = x_+ \oplus x_- \in C^\infty(S_{qs}^2)$, $x_+ - x_- \in \ker \sigma = \mathcal{S}$. Since we have

$$[N \cdot F, x] = \delta(x)F + N(x_+ - x_-)F\gamma,$$

with $\gamma = 1 \oplus -1$ the grading, and $N \cdot \mathcal{S} \subset \mathcal{S}$, then

$$\mathfrak{B} \subset \mathfrak{C} := C^\infty(S_{qs}^2) + C^\infty(S_{qs}^2) \cdot F + \mathcal{S} \cdot F\gamma.$$

Recall that $\delta(\mathcal{A}^\infty) \subset \mathcal{A}^\infty$: this means that \mathfrak{C} is δ -invariant and then belongs to the smooth domain OP^0 . This proves regularity. In particular, the algebra Ψ^0 is still contained in \mathfrak{C} .

Any $T \in \Psi^0 \subset \mathfrak{C}$ can be written as $T = x_+ \oplus x_-$ plus off-diagonal terms, with $x_\pm \in \mathcal{A}^\infty$. Therefore, for $z \in \mathbb{C}$ with sufficiently large real part, the ‘zeta-type’ function associated to T is given by

$$\zeta_T(z) = \text{Tr}_{\ell^2(\mathbb{N})}((x_+ + x_-)N^{-z}).$$

By Proposition 3.17, $\zeta_T(z)$ has meromorphic extension on $\mathbb{C} \setminus \{1\}$ and the residue in $z = 1$ is

$$\int T N^{-1} = \frac{1}{2\pi} \int_{S^1} \{\sigma(x_+)(\theta) + \sigma(x_-)(\theta)\} d\theta.$$

If $T = x_+ \oplus x_-$ belongs to the algebra $C^\infty(S_{qs}^2)$, then $\sigma(x_+) + \sigma(x_-) = 2\sigma_s(T)$, and this concludes the proof. \square

Proposition 3.19. $(C^\infty(S_{qs}^2), \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}), F)$ is a 1-summable non-trivial Fredholm module.

Proof. For all $x = x_+ \oplus x_- \in C^\infty(S_{qs}^2)$, $[F, x] = (x_+ - x_-)F\gamma$ is a rapid decay matrix, so in particular it is of trace class. Non-triviality follows from the computation

$$\text{Index}(F_{e_s}^+) = \frac{1}{2} \text{Tr}(\gamma F[F, e_s]) = \frac{1-q^2}{1+s^2} \text{Tr}(\gamma \mu(A)) = (1-q^2) \sum_{n \in \mathbb{N}} q^{2n} = +1,$$

with e_s the idempotent in (3.21). The proof of the non-triviality of this Fredholm module already appear, for $s \neq 0, 1$, in [CP03b] at the C^* -algebra level and using the projection $|0\rangle\langle 0| \oplus 0$. \square

3.7.3 The $U(1)$ -equivariant spectral triple over S_{q0}^2

In this section we consider $s = 0$ and call μ the representation of $\mathcal{A}(S_{q0}^2)$ over $\ell^2(\mathbb{N})$ defined by (3.37). In terms of the operators q^N and w one has

$$\mu(A) = q^{2N-2}, \quad \mu(B) = q^N \sqrt{1 - q^{2N}} w.$$

Thus $\mu(A), \mu(B)$ belongs to \mathcal{A}^∞ , and in particular they are in $\ker \sigma = \mathcal{S}$, so $\mathcal{A}(S_{q0}^2)$ is a subalgebra of the unitization \mathcal{S}^+ of \mathcal{S} ,

$$\mathcal{A}(S_{q0}^2) \subset C^\infty(S_{q0}^2) := \mathcal{S}^+ \subset C(S_{q0}^2).$$

By restricting Proposition 3.17 to the subalgebra \mathcal{S}^+ of \mathcal{A}^∞ we obtain the following,

Proposition 3.20. *($C^\infty(S_{q0}^2), \ell^2(\mathbb{N}), N$) is a regular spectral triple, the dimension spectrum is $\{1\}$ and the noncommutative integral is*

$$\int aN^{-1} = \sigma_0(a)$$

for all $f \in \mathcal{A}^\infty$, where σ_0 is the (restriction of the) map in (3.39).

Since the Dirac operator N is positive, in the $s = 0$ case one has no non-trivial Fredholm module associated with the triple. The spectral triples discussed in this sections appear very different for $s \neq 0$ and for $s = 0$.

3.8 The homological dimension

Let $\kappa(a) = K^2 \triangleright a$ be the modular automorphism for $\mathcal{A}(S_{qs}^2)$. We now give a simple proof that $HH_2^\kappa(\mathcal{A}(S_{qs}^2)) \neq 0$, i.e. that twisting the Hochschild homology ‘cures’ the dimension drop. The twisted Hochschild homology groups of S_{qs}^2 were computed in [Had07].

Let $\gamma_s := 2e_s - 1$ with e_s the idempotent in (3.21), and call

$$\eta := \sigma(K^2) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$

with σ the representation in (3.20). The tensor

$$\omega_2 := \frac{1}{8} \text{Tr}_{\mathbb{C}^2} \{ \eta \gamma_s \dot{\otimes} (1 + \gamma_s) \dot{\otimes} (1 - \gamma_s) \} = \frac{1}{8} \sum_{ijkl} \eta_{ij}(\gamma_s)_{jk} \otimes (1 + \gamma_s)_{kl} \otimes (1 - \gamma_s)_{li}$$

is invariant. Indeed, by (2.37),

$$\begin{aligned} h \triangleright \omega_2 &= \frac{1}{8} \text{Tr}_{\mathbb{C}^2} (\eta \sigma(h_{(1)})^t \gamma_s \dot{\otimes} (1 + \gamma_s) \dot{\otimes} (1 - \gamma_s) \sigma(S^{-1}(h_{(2)}))^t) \\ &= \frac{1}{8} \text{Tr}_{\mathbb{C}^4} (\sigma(S^{-1}(h_{(2)}))^t \eta \sigma(h_{(1)})^t \gamma_s \dot{\otimes} (1 + \gamma_s) \dot{\otimes} (1 - \gamma_s)), \end{aligned}$$

where in last step we used cyclicity of the trace. But $\eta = \sigma(K^2) = \sigma(K^2)^t$ and so

$$\sigma(S^{-1}(h_{(2)}))^t \eta \sigma(h_{(1)})^t = \sigma(h_{(1)} K^2 S^{-1}(h_{(2)})) = \sigma(h_{(1)} S(h_{(2)}) K^2) = \epsilon(h) \eta,$$

where we used (3.5). This proves the invariance of ω_2 , i.e. $h \triangleright \omega_2 = \epsilon(h) \omega_2$.

By using $\kappa(\gamma_s) \eta = \eta \gamma_s$ and $\gamma_s^2 = 1$ we compute

$$\begin{aligned} b_\kappa \omega_2 &= \frac{1}{4} \text{Tr}_{\mathbb{C}^2} (\eta \gamma_s \otimes 1 - 1 \otimes \eta \gamma_s) \\ &= \frac{1}{4} \{ \text{Tr}_{\mathbb{C}^2} (\eta \gamma_s) \otimes 1 - 1 \otimes \text{Tr}_{\mathbb{C}^2} (\eta \gamma_s) \}. \end{aligned}$$

with b_κ the boundary operator defined in Section 2.7. Since $\text{Tr}_{\mathbb{C}^2} (\eta \gamma_s) = (q^{-1} - q) \frac{1-s^2}{1+s^2}$ is a scalar, we get $b_\kappa \omega_2 = 0$ and so:

Proposition 3.21. *The tensor ω_2 is an invariant twisted Hochschild 2-cycle.*

For $q = 1$, the image of ω_2 under the isomorphism (2.10) is the volume form on S^2 . This justifies the interpretation of ω_2 as ‘twisted orientation’.

We now apply Lemma 2.12 to construct twisted cocycles. Consider the Fredholm module $(\mathcal{A}(S_{qs}^2), \mathcal{H}, F, \gamma)$ of Section 3.6. From the crossed product relation, the automorphism κ is implemented on \mathcal{H} by K^{-2} .

Lemma 3.22. *The multilinear map*

$$\text{ch}_n^{F, \kappa}(a_0, \dots, a_n) := \frac{1}{2} \text{Tr}_{\mathcal{H}}(K^{-2} \gamma F[F, a_0][F, a_1] \dots [F, a_n])$$

is a $U_q(\mathfrak{su}(2))$ -invariant twisted Hochschild n -cocycle on $\mathcal{A}(S_{qs}^2)$, for all even $n \geq 2$.

Proof. Invariance follows from (3.5), that is $S^{-1}(h)K^{-2} = K^{-2}S(h)$, and from equivariance of the representation of $\mathcal{A}(S_{qs}^2)$ on \mathcal{H} . By Lemma 2.12, to prove that $\text{ch}_n^{F, \kappa}$ is a cocycle we need just to check that it is well defined; that is, the operators

$$K^{-2}[F, a_0][F, a_1] \dots [F, a_n]$$

must be of trace class for all $n \geq 2$ and for all $a_0, \dots, a_n \in \mathcal{A}(S_{qs}^2)$. Since trace-class operators are a two-sided ideal in $\mathcal{B}(\mathcal{H})$, it is sufficient to do the computation for $n = 2$. By Lemma 3.6 the matrix coefficients of $[F, a_j]$ are bounded by q^l , thus coefficients of $K^{-2}[F, a_0][F, a_1][F, a_2]$ are bounded by $q^{3l-2m} \leq q^l$ and the operator is of trace class. \square

That $HH_2^{\kappa}(\mathcal{A}(S_{qs}^2))$ is not zero is a corollary of the following three lemmas.

Lemma 3.23. *Let $\mathcal{H}' = \mathcal{H}'_+ \oplus \mathcal{H}'_-$ be a (separable) Hilbert space and $P : \mathcal{H}'_+ \rightarrow \mathcal{H}'_-$ be a Fredholm operator with quasi-inverse P^* . Let Q be an even selfadjoint operator with dense domain in \mathcal{H}' and discrete spectrum. Let $n \in \mathbb{N}$ such that $Q(1 - PP^*)^n$ and $Q(1 - P^*P)^n$ are of trace class, and assume $[P, Q] = 0$ on \mathcal{H}'_+ . Then*

$$Q\text{-Index}(P) = \text{Tr}_{\mathcal{H}'_+} Q(1 - P^*P)^n - \text{Tr}_{\mathcal{H}'_-} Q(1 - PP^*)^n,$$

where the ‘ Q -index’ is defined by

$$Q\text{-Index}(P) := \text{Tr}_{\ker P}(Q) - \text{Tr}_{\text{coker } P}(Q).$$

Proof. Since Q commutes with P^*P (resp. with PP^*), we can find an orthonormal basis of \mathcal{H}'_+ (resp. of \mathcal{H}'_-) made of joint eigenvectors of Q and P^*P (resp. Q and PP^*). Let

$$\begin{aligned} E_0(\lambda, \mu) &= \{v \in \mathcal{H}'_+ \mid P^*Pv = \lambda v, Qv = \mu v\}, \\ E_1(\lambda, \mu) &= \{v \in \mathcal{H}'_- \mid PP^*v = \lambda v, Qv = \mu v\}. \end{aligned}$$

Since P^*P (resp. PP^*) is (the identity plus) a positive compact operator, it can be diagonalized and its eigenspaces are all finite dimensional. Since $P(P^*P) = (PP^*)P$ and Q commutes with P , the operator P maps $E_0(\lambda, \mu)$ into $E_1(\lambda, \mu)$. Similarly $P^* : E_1(\lambda, \mu) \mapsto E_0(\lambda, \mu)$. For $\lambda \neq 0$ we have the linear isomorphisms

$$\begin{aligned}\lambda = P^*P|_{E_0(\lambda, \mu)} : E_0(\lambda, \mu) &\mapsto E_1(\lambda, \mu) \mapsto E_0(\lambda, \mu), \\ \lambda = PP^*|_{E_1(\lambda, \mu)} : E_1(\lambda, \mu) &\mapsto E_0(\lambda, \mu) \mapsto E_1(\lambda, \mu),\end{aligned}$$

thus P and P^* are both bijective linear maps and $\dim E_0(\lambda, \mu) = \dim E_1(\lambda, \mu)$ for all $\lambda \neq 0$. Using this equality we compute

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}'_+} Q(1 - P^*P)^n - \mathrm{Tr}_{\mathcal{H}'_-} Q(1 - PP^*)^n &= \sum_{\lambda, \mu} \mu(1 - \lambda)^n \{ \dim E_0(\lambda, \mu) - \dim E_1(\lambda, \mu) \} \\ &= \sum_{\mu} \mu \{ \dim E_0(0, \mu) - \dim E_1(0, \mu) \} = \mathrm{Tr}_{\ker P}(Q) - \mathrm{Tr}_{\mathrm{coker} P}(Q).\end{aligned}$$

This concludes the proof. \square

Lemma 3.24. *The pairing $\mathrm{ch}_2^{F, \kappa}(\omega_2)$ is given by the ‘ q -index’ of the operator $F_{e_s}^+$ in (2.2),*

$$\mathrm{ch}_2^{F, \kappa}(\omega_2) = q\text{-Index}(F_{e_s}^+) := \mathrm{Tr}_{\ker F_{e_s}^+}(K^{-2}\eta) - \mathrm{Tr}_{\mathrm{coker} F_{e_s}^+}(K^{-2}\eta).$$

Proof. By definition

$$\mathrm{ch}_2^{F, \kappa}(\omega_2) = -\frac{1}{2} \mathrm{Tr}_{\mathcal{H} \otimes \mathbb{C}^2}(K^{-2}\eta\gamma F[F, e_s]^3).$$

Since γF and $K^{-2}\eta$ both commute with $[F, e_s]$, with some algebraic manipulation and using $[F, e_s] = e_s[F, e_s] + [F, e_s]e_s$ we get

$$\begin{aligned}\mathrm{ch}_2^{F, \kappa}(\omega_2) &= \mathrm{Tr}_{\mathcal{H} \otimes \mathbb{C}^2}(K^{-2}\eta\gamma(e_s - e_s F e_s F e_s)^2) \\ &= \mathrm{Tr}_{\mathcal{H}'_+}(K^{-2}\eta(1 - F_{e_s}^- F_{e_s}^+)^2) - \mathrm{Tr}_{\mathcal{H}'_-}(K^{-2}\eta(1 - F_{e_s}^+ F_{e_s}^-)^2).\end{aligned}$$

where $\mathcal{H}'_{\pm} = (1 \pm \gamma)e_s(\mathcal{H} \otimes \mathbb{C}^2)$, $F_{e_s}^+ : \mathcal{H}'_+ \rightarrow \mathcal{H}'_+$ is given by (2.2) and $F_{e_s}^- := (F_{e_s}^+)^*$. Now we apply Lemma 3.23 to $Q = K^{-2}\eta$, $P = F_{e_s}^+$ and $\mathcal{H}' = \mathcal{H}'_+ \oplus \mathcal{H}'_-$, and obtain the desired result (since $Q = K^{-2}\eta$ is given in diagonal form in the basis of harmonic spinors, it is immediate to extend it to a self-adjoint operator). \square

Lemma 3.25. *We have*

$$q\text{-Index}(F_{e_s}^+) = [2N].$$

In particular, the class of ω_2 in $HH_2^{\kappa}(\mathcal{A}(S_{qs}^2))$ is not zero.

Proof. The idempotent e_s satisfies (2.37) for all $h \in U_q(\mathfrak{su}(2))$, with σ the spin 1/2 representation in (3.20). As a consequence, if we define the action of $h \in U_q(\mathfrak{su}(2))$ on $w \in \mathcal{H} \otimes \mathbb{C}^2$ as in (2.38),

$$h \blacktriangleright w := \sigma(S^{-1}(h_{(1)}))^t h_{(2)} \triangleright w,$$

then

$$h \blacktriangleright (e_s w) = e_s (h \blacktriangleright w) .$$

Using the Clebsch-Gordan rules (3.7), with $|+\rangle = q^{-1/2} \binom{0}{1}$ and $|-\rangle = -q^{1/2} \binom{1}{0}$, we identify the following orthonormal basis of $\mathcal{H} \otimes \mathbb{C}^2$:

$$\begin{aligned} v_{l,m}^{\pm,\uparrow} &:= \left(\begin{array}{c} -\sqrt{\frac{q^{l+m+1}[l-m]}{[2l]}} |l - \frac{1}{2}, m + \frac{1}{2}; \mp N\rangle \\ \sqrt{\frac{q^{-(l-m+1)[l+m]}{[2l]}} |l - \frac{1}{2}, m - \frac{1}{2}; \mp N\rangle \end{array} \right) , & l = |N| + \frac{1}{2}, |N| + \frac{3}{2}, \dots , \\ v_{l,m}^{\pm,\downarrow} &:= \left(\begin{array}{c} \sqrt{\frac{q^{-(l-m)[l+m+1]}{[2l+2]}} |l + \frac{1}{2}, m + \frac{1}{2}; \mp N\rangle \\ \sqrt{\frac{q^{l+m}[l-m+1]}{[2l+2]}} |l + \frac{1}{2}, m - \frac{1}{2}; \mp N\rangle \end{array} \right) , & l = |N| - \frac{1}{2}, |N| + \frac{1}{2}, \dots , \end{aligned}$$

which transform under the ‘black’ action \blacktriangleright as in (3.2) (with $\omega = 1$). We set $v_{|N|-\frac{1}{2},m}^{\pm,\uparrow} := 0$ and start counting from $l = |N| - \frac{1}{2}$ for both v^\uparrow and v^\downarrow . Left multiplication of these vectors by e_s gives

$$e_s v_{l,m}^{\pm,\uparrow} = a_{l,s}^\pm v_{l,m}^{\pm,\uparrow} + b_{l,s}^\pm v_{l,m}^{\pm,\downarrow} , \quad (3.43a)$$

$$e_s v_{l,m}^{\pm,\downarrow} = b_{l,s}^\pm v_{l,m}^{\pm,\uparrow} + (1 - a_{l,s}^\pm) v_{l,m}^{\pm,\downarrow} , \quad (3.43b)$$

where for all $l \neq |N| - \frac{1}{2}$,

$$a_{l,s}^\pm = \frac{1 + q^2 s^2 + q^2 \beta_{\mp N}(l - \frac{1}{2})}{q[2](1 + s^2)} , \quad b_{l,s}^\pm = \frac{1}{1 + s^2} \sqrt{\frac{[2l+2]}{[2][2l+1]}} \alpha_{\mp N}(l + \frac{1}{2}) ,$$

with $\alpha_N(l)$ and $\beta_N(l)$ the coefficients in (3.27), while for $l = |N| - \frac{1}{2}$,

$$a_{|N|-\frac{1}{2},s}^\pm = \delta_{\text{sign}N, \mp 1} , \quad b_{|N|-\frac{1}{2},s}^\pm = 0 .$$

The map $N \rightarrow -N$ changes sign to the q -index (it is equivalent to changing the sign of the grading γ), thus it is sufficient to consider the case $\boxed{N > 0}$.

Since $\alpha_N(l) = 0$ if and only if $l = |N|$, we have $b_{l,s}^\pm \neq 0$ for all $l \neq |N| - \frac{1}{2}$ and the sum $(a_{l,s}^\pm)^2 + (b_{l,s}^\pm)^2$ is invertible. Therefore, the following orthonormal basis of $\mathcal{H} \otimes \mathbb{C}^2$ is well defined:

$$\begin{aligned} w_{l,m}^{\pm,\parallel} &:= \frac{1}{(a_{l,s}^\pm)^2 + (b_{l,s}^\pm)^2} (a_{l,s}^\pm v_{l,m}^{\pm,\uparrow} + b_{l,s}^\pm v_{l,m}^{\pm,\downarrow}) , & \forall l \geq |N| + \frac{1}{2} , \\ w_{l,m}^{\pm,\perp} &:= \frac{1}{(a_{l,s}^\pm)^2 + (b_{l,s}^\pm)^2} (b_{l,s}^\pm v_{l,m}^{\pm,\uparrow} - a_{l,s}^\pm v_{l,m}^{\pm,\downarrow}) , & \forall l \geq |N| + \frac{1}{2} , \\ v_m^\parallel &:= v_{|N|-\frac{1}{2},m}^{+,\downarrow} , \\ v_m^\perp &:= v_{|N|-\frac{1}{2},m}^{-,\downarrow} . \end{aligned}$$

From (3.43) it follows that this basis is made of eigenvectors of e_s :

$$e_s w_{l,m}^{\pm,\parallel} = w_{l,m}^{\pm,\parallel} , \quad e_s v_m^\parallel = v_m^\parallel , \quad e_s w_{l,m}^{\pm,\perp} = e_s v_m^\perp = 0 .$$

By definition, the kernel of $F_{e_s}^+$ is the span of the vectors $\{w_{l,m}^{+,||}, v_m^{||}\}$ which are annihilated by $e_s F$, while the cokernel is the span of the vectors $\{w_{l,m}^{-,||}\}$ which are annihilated by $e_s F$.

Since F sends v^\pm into v^\mp , we have:

$$e_s F w_{l,m}^{\pm,||} = \frac{(a_{l,s}^\mp)^2 + (b_{l,s}^\mp)^2}{(a_{l,s}^\pm)^2 + (b_{l,s}^\pm)^2} (a_{l,s}^+ a_{l,s}^- + b_{l,s}^+ b_{l,s}^-) w_{l,m}^{\mp,||}.$$

The fraction in last equation is always different from zero. If $a_{l,s}^+ a_{l,s}^- + b_{l,s}^+ b_{l,s}^- = 0$, then $w_{l,m}^{+,||}$ is in the kernel of $F_{e_s}^+$, $w_{l,m}^{-,||}$ is in the cokernel and by

$$\langle w_{l,m}^{+,||}, K^{-2} \eta w_{l,m}^{+,||} \rangle = \langle w_{l,m}^{-,||}, K^{-2} \eta w_{l,m}^{-,||} \rangle = q^{-2m}$$

their contributions to q -Index($F_{e_s}^+$) cancel each other. Since

$$e_s F v_m^{||} = e_s v_m^\perp = 0,$$

the vectors $v_m^{||} = v_{|N|-\frac{1}{2},m}^{+,\downarrow}$ are in the kernel of $F_{e_s}^+$ and we have

$$q\text{-Index}(F_{e_s}^+) = \sum_{m=-N+1/2}^{N-1/2} \langle v_{|N|-\frac{1}{2},m}^{+,\downarrow}, K^{-2} \eta v_{|N|-\frac{1}{2},m}^{+,\downarrow} \rangle = \sum_{m=-N+1/2}^{N-1/2} q^{-2m} = [2N].$$

This concludes the proof. \square

As a corollary, since $HH_2^\kappa(\mathcal{A}(S_{q_s}^2))$ has (at least) one non-zero element, it is not zero and so there is no dimension drop.

A result similar to Lemma 3.25 was obtained by [Wag07] for the standard sphere: he computed the q -index of a fixed F (coming from the Fredholm module with $N = 1/2$) twisted with the generic idempotent with charge $n \in \mathbb{Z}$; here the idempotent is fixed (is the one with charge 1) while the Fredholm module is generic and depends on the half integer N .

We conclude with some remarks. We have constructed a non-trivial element $[\omega_2] \in HH_2^\kappa$, with κ the modular automorphism, given on generators by $\kappa(A) = A$, $\kappa(B) = q^2 B$, $\kappa(B^*) = q^{-2} B^*$. From Theorem 4.6 of [Had07] we know that $HH_2^\sigma \simeq \mathbb{C}$ if σ is a positive power of the modular automorphism σ_{mod} , and $HH_2^\sigma \simeq 0$ for all other automorphisms. From equation (5) of [Had07] we deduce that his generators are related to ours by a replacement $B \leftrightarrow B^*$, and from the definition of σ_{mod} [Had07, eq. (17)] we deduce that κ and σ_{mod} are indeed the same automorphism. Thus, we can conclude that $HH_2^\kappa \simeq \mathbb{C}$ and that $[\omega_2]$ is a generator.

Dually, we proved that $\text{ch}_2^{F,\kappa}$ has non-trivial class in HH_κ^2 . We want to compare this result with [SW04]. In that paper the authors define a twisted 2-cocycle for the standard Podleś sphere, with twist σ given by the *inverse* of the modular automorphism [SW04, equation (29)]: $\sigma(a) = K^{-2} \triangleright a = \kappa^{-1}(a) = \sigma_{\text{mod}}^{-1}(a)$, for all $a \in \mathcal{A}(S_{q_0}^2)$. It is then consistent that their cocycle, as explained in Section 6 of [Had07], has a non-trivial class in cyclic cohomology but is trivial in Hochschild cohomology.

Chapter 4

The 4-dimensional quantum orthogonal sphere

In this chapter we construct an equivariant regular even spectral triple on S_q^4 following [DDL08]. This spectral triple is a deformation of the canonical one associated to the round sphere S^4 , and the Dirac operator has undeformed spectrum. We discuss analytical properties, construct a weak real structure, and prove that the noncommutative integral is the integral over the space of characters of the algebra. We give also an explicit expression of the Haar state in a basis of monomials in x_i 's, and construct a twisted Hochschild 4-cycle which may be interpreted as the 'orientation'.

We repeat here Definition 2.27 of the coordinate algebra $\mathcal{A}(S_q^4)$ of the quantum orthogonal 4-sphere. This is the $*$ -algebra generated by five elements $x_0 = x_0^*, x_i, x_i^*$ (with $i = 1, 2$), subject to the relations:

$$\begin{aligned}x_i x_j &= q^2 x_j x_i, & \forall 0 \leq i < j \leq 2, \\x_i^* x_j &= q^2 x_j x_i^*, & \forall i \neq j, \\[x_1^*, x_1] &= (1 - q^4) x_0^2, \\[x_2^*, x_2] &= x_1^* x_1 - q^4 x_1 x_1^*, \\x_0^2 + x_1 x_1^* + x_2 x_2^* &= 1.\end{aligned}$$

Recall that $0 < q < 1$.

4.1 The symmetry Hopf algebra $U_q(\mathfrak{so}(5))$

We call $U_q(\mathfrak{so}(5))$ the real form of the Drinfeld-Jimbo deformation of $\mathfrak{so}_{\mathbb{C}}(5)$, corresponding to the Euclidean signature $(+, +, +, +, +)$; it is a real form of the Hopf algebra called $\check{U}_q(\mathfrak{so}(5))$ in [KS97, Section 6.1.2]. As a $*$ -algebra, $U_q(\mathfrak{so}(5))$ is generated by the elements $K_i = K_i^*, K_i^{-1}, E_i, F_i := E_i^*, i = 1, 2$ ($i \rightarrow 3 - i$ with respect to the notations of [KS97]), with

relations:

$$[K_1, K_2] = 0, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_j^2 - K_j^{-2}}{q^j - q^{-j}},$$

$$K_i E_i K_i^{-1} = q^i E_i, \quad K_i E_j K_i^{-1} = q^{-1} E_j \text{ if } i \neq j,$$

and with the ones obtained by conjugation and Serre relations. These are explicitly given by

$$E_1 E_2^2 - (q^2 + q^{-2}) E_2 E_1 E_2 + E_2^2 E_1 = 0, \quad (4.1a)$$

$$E_1^3 E_2 - (q^2 + 1 + q^{-2})(E_1^2 E_2 E_1 - E_1 E_2 E_1^2) - E_2 E_1^3 = 0, \quad (4.1b)$$

together with their adjoints. Serre relations can be written in a more compact form by defining $[a, b]_q := q^2 ab - ba$. Then, (4.1) are equivalent to

$$[E_2, [E_1, E_2]_q]_q = 0, \quad [E_1, [E_1, [E_2, E_1]_q]_q] = 0.$$

The Hopf algebra structure (Δ, ϵ, S) of $U_q(\mathfrak{so}(5))$ is given by:

$$\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes K_i + K_i^{-1} \otimes E_i,$$

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -q^i E_i.$$

For any non-negative n_1, n_2 such that $n_2 \in \frac{1}{2}\mathbb{N}$ and $n_2 - n_1 \in \mathbb{N}$ there is an irreducible representation of $U_q(\mathfrak{so}(5))$ whose representation space we denote by $V_{(n_1, n_2)}$ [KS97]. We call it “the representation with highest weight (n_1, n_2) ”, since the highest weight vector is an eigenvector of K_1 and $K_1 K_2$ with eigenvalues q^{n_1} and q^{n_2} , respectively.

Irreducible representations with highest weight $(0, l)$ and $(\frac{1}{2}, l)$ (the ones that we need explicitly) can be found in [Cha94] and are recalled presently. We use the shorthand notation $V_l := V_{(0, l)}$ if $l \in \mathbb{N}$ and $V_l := V_{(\frac{1}{2}, l)}$ if $l \in \mathbb{N} + \frac{1}{2}$. The vector space V_l , for all $l \in \frac{1}{2}\mathbb{N}$, has orthonormal basis $|l, m_1, m_2; j\rangle$, with the labels (m_1, m_2, j) satisfying the following constraints. For $l \in \mathbb{N}$:

$$j = 0, 1, \dots, l, \quad j - |m_1| \in \mathbb{N}, \quad l - j - |m_2| \in 2\mathbb{N},$$

while for $l \in \mathbb{N} + \frac{1}{2}$:

$$j = \frac{1}{2}, \frac{3}{2}, \dots, l - 1, l, \quad j - |m_1| \in \mathbb{N}, \quad l + \frac{1}{2} - j - |m_2| \in \mathbb{N}.$$

Notice that for any admissible (l, m_1, m_2, j) there exists a unique $\epsilon \in \{0, \pm\frac{1}{2}\}$ such that $l + \epsilon - j - m_2 \in 2\mathbb{N}$ (that is, $\epsilon = 0$ if $l \in \mathbb{N}$ and $\epsilon = \frac{1}{2}(-1)^{l+\frac{1}{2}-j-m_2}$ if $l \in \mathbb{N} + \frac{1}{2}$). We shall need the coefficients,

$$a_l(j, m_2) = \frac{1}{[2]} \sqrt{\frac{[l - j - m_2 + \epsilon][l + j + m_2 + 3 + \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}}, \quad (4.2a)$$

$$b_l(j, m_2) = 2|\epsilon| \frac{\sqrt{[l - \epsilon(2j + 1) - m_2 + 1][l - \epsilon(2j + 1) + m_2 + 2]}}{[2j][2j + 2]}, \quad (4.2b)$$

$$c_l(j, m_2) = \frac{(-1)^{2\epsilon}}{[2]} \sqrt{\frac{[l - j + m_2 + 2 - \epsilon][l + j - m_2 + 1 - \epsilon]}{[2(j + |\epsilon|) - 1][2(j - |\epsilon|) + 1]}}, \quad (4.2c)$$

where, as usual, $[z] := (q^z - q^{-z})/(q - q^{-1})$ denotes the q -analogue of a number z .

The $*$ -representation $\sigma_l : U_q(\mathfrak{so}(5)) \rightarrow \text{End}(V_l)$ is defined by the assignments,

$$\begin{aligned} \sigma_l(K_1) |l, m_1, m_2; j\rangle &= q^{m_1} |l, m_1, m_2; j\rangle, \\ \sigma_l(K_2) |l, m_1, m_2; j\rangle &= q^{m_2 - m_1} |l, m_1, m_2; j\rangle, \\ \sigma_l(E_1) |l, m_1, m_2; j\rangle &= \sqrt{[j - m_1][j + m_1 + 1]} |l, m_1 + 1, m_2; j\rangle, \\ \sigma_l(E_2) |l, m_1, m_2; j\rangle &= \sqrt{[j - m_1 + 1][j - m_1 + 2]} a_l(j, m_2) |l, m_1 - 1, m_2 + 1; j + 1\rangle \\ &\quad + \sqrt{[j + m_1][j - m_1 + 1]} b_l(j, m_2) |l, m_1 - 1, m_2 + 1; j\rangle \\ &\quad + \sqrt{[j + m_1][j + m_1 - 1]} c_l(j, m_2) |l, m_1 - 1, m_2 + 1; j - 1\rangle. \end{aligned}$$

The representation symbol σ_l will be suppressed, when there is no risk of ambiguity.

For $l \in \mathbb{N}$ the representation σ_l is *real*. That is, there is an antilinear map $C : V_l \rightarrow V_l$, which satisfies $C^2 = 1$ and $C\sigma_l(h)C = \sigma_l(S(h)^*)$. It is explicitly given by

$$C |l, m_1, m_2; j\rangle := (-q)^{m_1} q^{3m_2} |l, -m_1, -m_2; j\rangle. \quad (4.3)$$

The operator

$$\mathcal{C}_1 := q^{-1}K_1^2 + qK_1^{-2} + (q - q^{-1})^2 E_1 F_1, \quad (4.4)$$

is a Casimir for the subalgebra generated by $(K_1, K_1^{-1}, E_1, F_1)$. For future reference, we note the action of \mathcal{C}_1 on a vector of V_l , with $l \in \frac{1}{2}\mathbb{N}$; it is

$$\mathcal{C}_1 |l, m_1, m_2; j\rangle = (q^{2j+1} + q^{-2j-1}) |l, m_1, m_2; j\rangle. \quad (4.5)$$

4.2 General properties of the quantum 4-sphere

In the next Propositions we summarize some well known facts [KS97].

Proposition 4.1. *The algebra $\mathcal{A}(S_q^4)$ is an $U_q(\mathfrak{so}(5))$ -module $*$ -algebra for the action given by*

$$\begin{aligned} K_1 \triangleright x_1 &= qx_1, & K_2 \triangleright x_2 &= qx_2, & K_2 \triangleright x_1 &= q^{-1}x_1, \\ E_1 \triangleright x_0 &= q^{-1/2}x_1, & E_2 \triangleright x_1 &= x_2, \\ F_1 \triangleright x_1 &= q^{1/2}[2]x_0, & F_1 \triangleright x_0 &= -q^{-3/2}x_1^* & F_2 \triangleright x_2 &= x_1, \end{aligned}$$

while the action is trivial in all other cases:

$$\begin{aligned} K_1 \triangleright x_0 &= x_0, & K_2 \triangleright x_0 &= x_0, & K_1 \triangleright x_2 &= x_2, \\ E_2 \triangleright x_0 &= E_1 \triangleright x_1 = E_1 \triangleright x_2 = E_2 \triangleright x_2 = 0, & F_2 \triangleright x_0 &= F_2 \triangleright x_1 = F_1 \triangleright x_2 = 0. \end{aligned}$$

The action on the x_i^* 's is determined by compatibility with the involution:

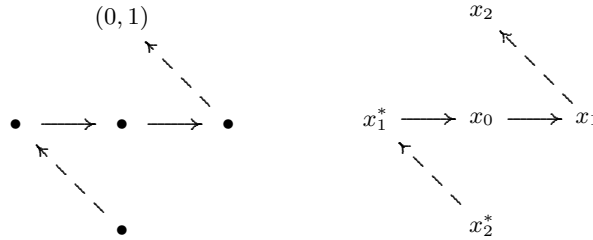
$$K_i \triangleright a^* = \{K_i^{-1} \triangleright a\}^*, \quad E_1 \triangleright a^* = \{-qF_1 \triangleright a\}^*, \quad E_2 \triangleright a^* = \{-q^2F_2 \triangleright a\}^* .$$

Proof. The algebra $\mathcal{A}(S_q^4)$ is the quotient of the $*$ -algebra $\mathbb{C}\langle x \rangle$ — the free $*$ -algebra generated by the x_i 's — by the ideal generated by suitable degree ≤ 2 polynomials. The bijective linear map from the linear span of $\{x_0, x_1, x_1^*, x_2, x_2^*\}$ to the representation space V_1 defined (modulo a global proportionality constant) by

$$x_2 \mapsto |0, 1; 0\rangle, \quad x_1 \mapsto |1, 0; 1\rangle, \quad x_0 \mapsto (q[2])^{-1/2} |0, 0; 1\rangle, \quad x_1^* \mapsto -q |-1, 0; 1\rangle, \quad x_2^* \mapsto q^3 |0, -1; 0\rangle,$$

is a unitary equivalence of $U_q(\mathfrak{so}(5))$ -modules (the real structure C on V_1 being implemented by the $*$ operation on x_i 's). This guarantee that $\mathbb{C}\langle x \rangle$ is an $U_q(\mathfrak{so}(5))$ -module $*$ -algebra.

It may help to draw the weight diagram of $V_{(0,1)}$, with each generator of the $\mathcal{A}(S_q^4)$ in the position corresponding to its weight.

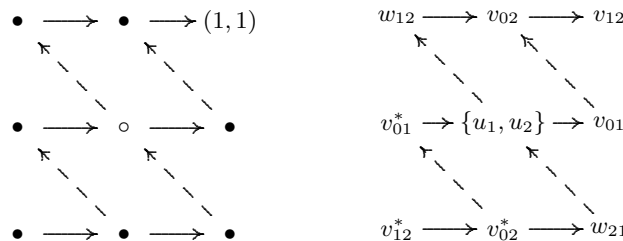


A solid arrow indicates points that can be joined by applying E_1 (the reverse arrow corresponds to F_1), a dashed arrow indicates points that can be joined by applying E_2 (the reverse arrow corresponds to F_2). In the left diagram, the highest weight vector is denoted by its weight (n_1, n_2) . A bullet indicates a weight with multiplicity 1, an empty circle a weight with multiplicity 2.

The polynomials generating the ideal which defines $\mathcal{A}(S_q^4)$, that is

$$\begin{aligned} v_{ij} &:= x_i x_j - q^2 x_j x_i, & 0 \leq i < j \leq 2, \\ w_{ij} &:= x_i^* x_j - q^2 x_j x_i^*, & i \neq j, \\ u_1 &:= [x_1^*, x_1] - (1 - q^4) x_0^2, \\ u_2 &:= [x_2^*, x_2] - x_1^* x_1 - q^4 x_1 x_1^*, \\ c &:= x_0^2 + x_1 x_1^* + x_2 x_2^* - 1, \end{aligned}$$

span the real representations V_0 and $V_{(1,1)}$ inside the tensor product $V_1 \otimes V_1$. More precisely, c is invariant and spans V_0 while the remaining polynomials are drawn in the weight diagram of the adjoint representation $V_{(1,1)}$ in the position corresponding to their weight.



As before, one moves in the direction of solid (resp. dashed) arrows by applying E_1 (resp. E_2). So, for example, one checks that $E_1 \triangleright v_{01} = 0$ and $F_1 \triangleright v_{01} = -q^{-1/2}u_1$.

Since $\mathcal{A}(S_q^4)$ is the quotient of the $U_q(\mathfrak{so}(5))$ -module $*$ -algebra $\mathbb{C}\langle x \rangle$ by an ideal which is invariant under $U_q(\mathfrak{so}(5))$, it is itself an $U_q(\mathfrak{so}(5))$ -module $*$ -algebra. \square

Proposition 4.2. *There is an isomorphism $\mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l$ of left $U_q(\mathfrak{so}(5))$ -modules.*

Proof. A linear basis for $\mathcal{A}(S_q^4)$ is made of monomials $x_0^{n_0} x_1^{n_1} (x_1^*)^{n_2} x_2^{n_3}$ with $n_0, n_1, n_2 \in \mathbb{N}$, $n_3 \in \mathbb{Z}$ and with the notation $x_2^{n_3} := (x_2^*)^{|n_3|}$ if $n_3 < 0$. Such a monomial is an eigenvector of K_1 (resp. $K_1 K_2$) with eigenvalue $n_1 - n_2$ (resp. n_3). Thus, the generic eigenvector v of K_1 and K_2 is a sum of monomials with $n_1 - n_2$ and n_3 fixed. Since E_2 sends x_1 to x_0 , being the basis monomials linearly independent, we deduce that v is annihilated by E_2 iff it contains only monomials with $n_1 = 0$:

$$v = \sum_{n_0} c_{n_0} x_0^{n_0} (x_1^*)^{n_2} x_2^{n_3} .$$

Finally, the condition $E_1 \triangleright v = 0$ implies that v must be independent of x_0 and x_1^* . We conclude that highest weight vectors are of the form x_2^l — corresponding to the weight $(0, l)$, $l \in \mathbb{N}$ — and $\mathcal{A}(S_q^4)$ decomposes as a multiplicity free direct sum of highest weight representations with weights $(0, l)$. \square

The algebra $\mathcal{A}(S_q^4)$ has two inequivalent irreducible infinite dimensional representations. The representation space is the Hilbert space $\ell^2(\mathbb{N}^2)$ and the representations are given by

$$\begin{aligned} x_0 |k_1, k_2\rangle_{\pm} &:= \pm q^{2(k_1+k_2)} |k_1, k_2\rangle_{\pm} , \\ x_1 |k_1, k_2\rangle_{\pm} &:= q^{2k_2} \sqrt{1 - q^{4(k_1+1)}} |k_1 + 1, k_2\rangle_{\pm} , \\ x_2 |k_1, k_2\rangle_{\pm} &:= \sqrt{1 - q^{4(k_2+1)}} |k_1, k_2 + 1\rangle_{\pm} . \end{aligned} \tag{4.6}$$

The direct sum of these representations, with obvious grading γ and operator F given by $F |k_1, k_2\rangle_{\pm} := |k_1, k_2\rangle_{\mp}$, constitutes a 1-summable Fredholm module over $\mathcal{A}(S_q^4)$.

Recall the Definition 2.24 of $SU_q(2)$ and the Definition 2.23 of the equatorial Podleś sphere $S_{q^2}^2 = S_{q^1}^2$ with parameter q^2 .

Proposition 4.3. *There is a $*$ -algebra morphism $\varphi : \mathcal{A}(S_q^4) \rightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_{q^2}^2)$ defined by:*

$$\begin{aligned} \varphi(x_0) &= -(\alpha\beta + \beta^* \alpha^*) \otimes A , \\ \varphi(x_1) &= (-\alpha^2 + q(\beta^*)^2) \otimes A , \\ \varphi(x_2) &= 1 \otimes B . \end{aligned} \tag{4.7}$$

Proof. One proves by direct computation that the five $\varphi(x_i), \varphi(x_i)^*$ satisfy all the defining relations of $\mathcal{A}(S_q^4)$. \square

4.3 The modules of chiral spinors

We apply the general theory of Section 2.10.2, to the case $\mathcal{A} = \mathcal{A}(S_q^4)$ and $\mathcal{U} = U_q(\mathfrak{so}(5))$. In the present case $\kappa(a) = K_1^8 K_2^6 \triangleright a$ is the modular automorphism [KS97, Section 11.3.4]. We shall use the notations of Section 4.1 for the irreducible representations (V_l, σ_l) of $U_q(\mathfrak{so}(5))$.

By Proposition 4.2 we have the equivalence $\mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l$ as left $U_q(\mathfrak{so}(5))$ -modules. Using Lemma 2.18 for $N = 1$, we deduce that on the vector space $\bigoplus_{l \in \mathbb{N}} V_l$ there exists at least one $*$ -representation of the crossed product $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$ that extends the $*$ -representation $\bigoplus_{l \in \mathbb{N}} \sigma_l$ of $U_q(\mathfrak{so}(5))$.

Let $e \in \text{Mat}_4(\mathcal{A}(S_q^4))$ be the following idempotent:

$$e := \frac{1}{2} \begin{pmatrix} 1 + x_0 & q^3 x_2 & -q x_1 & 0 \\ q^{-3} x_2^* & 1 - q^2 x_0 & 0 & q^3 x_1 \\ -q^{-1} x_1^* & 0 & 1 - q^2 x_0 & q^3 x_2 \\ 0 & q x_1^* & q^{-3} x_2^* & 1 + q^4 x_0 \end{pmatrix}. \quad (4.8)$$

By direct computation one proves that $K_1^8 K_2^6 \triangleright e^* = e = e^2$ and then, by Lemma 2.20, e defines an orthogonal projection π , by equation (2.36), on the linear space $\mathcal{A}(S_q^4)^4$ with inner product (2.32). Next, let $\sigma : U_q(\mathfrak{so}(5)) \rightarrow \text{Mat}_4(\mathbb{C})$ be the $*$ -representation defined by

$$\sigma(K_1) = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}, \quad \sigma(K_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4.9a)$$

$$\sigma(E_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma(E_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.9b)$$

Again, by direct computation one proves that:

$$K_i \triangleright e = \sigma(K_i) e \sigma(K_i)^{-1}, \quad (4.10a)$$

$$E_i \triangleright e = \sigma(F_i) e \sigma(K_i)^{-1} - q^{-i} \sigma(K_i)^{-1} e \sigma(F_i). \quad (4.10b)$$

Since $\sigma(K_i) = \sigma(K_i)^t$ and $\sigma(F_i) = \sigma(E_i)^t$, we conclude that condition (2.37) is satisfied and that π and $\pi^\perp = 1 - \pi$ project $\mathcal{A}(S_q^4)^4$ onto $*$ -subrepresentations of $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$.

We state the main Proposition of this section.

Proposition 4.4. *There exist two inequivalent representations of the crossed product algebra $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$ on $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$ that extend the representation $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} \sigma_l$ of $U_q(\mathfrak{so}(5))$.*

The proof is in two steps. We first prove (in Lemma 4.5) that $\pi(\mathcal{A}(S_q^4)^4)$ and $\pi^\perp(\mathcal{A}(S_q^4)^4)$ are not equivalent as representations of the algebra $\mathcal{A}(S_q^4)$. Then we prove (in Lemma 4.6) that as $U_q(\mathfrak{so}(5))$ representations they are both equivalent to $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$.

Lemma 4.5. *The idempotent e in (4.8) splits $\mathcal{A}(S_q^4)^4$ into two inequivalent $*$ -representations of the crossed product algebra $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$.*

Proof. To prove the statement we apply Lemma 2.22. We use the Fredholm module associated to the representation on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, and defined by equation (4.6). From (2.3) we have

$$\begin{aligned} \text{Index}(F_e^+) &= \frac{1}{2} \text{Tr}_{\ell^2(\mathbb{N}) \otimes \mathbb{C}^8}(\gamma F[F, e]) \\ &= \frac{1}{4} (1 - q^2)^2 \text{Tr}_{\ell^2(\mathbb{N}) \otimes \mathbb{C}^2}(\gamma F[F, x_0]) \\ &= (1 - q^2)^2 \sum_{k_1, k_2 \in \mathbb{N}} q^{2(k_1 + k_2)} = 1. \end{aligned}$$

By Lemma 2.22 this proves that the two representations of $\mathcal{A}(S_q^4)$ are inequivalent. The statement of the Proposition 4.4 follows from the obvious observation that if the two representations of the crossed product algebra were equivalent, their restrictions to representations of $\mathcal{A}(S_q^4)$ would be equivalent too. \square

Lemma 4.6. *$\pi(\mathcal{A}(S_q^4)^4) \simeq \pi^\perp(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$ as $U_q(\mathfrak{so}(5))$ representations.*

Proof. In this proof, ‘ \simeq ’ always means equivalence of representations of $U_q(\mathfrak{so}(5))$.

Since σ in (4.9) is just the spin representation $V_{1/2}$ in matrix form, the representation of $U_q(\mathfrak{so}(5))$ on $\mathcal{A}(S_q^4)^4$ is the Hopf tensor product of the representation over $\mathcal{A}(S_q^4)$ with the representation $V_{1/2}$. From $\mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l$ and from the decomposition $V_l \otimes V_{1/2} \simeq V_{l-\frac{1}{2}} \oplus V_{l+\frac{1}{2}}$ for all $l \in \{1, 2, 3, \dots\}$, we deduce that $\mathcal{A}(S_q^4)^4 \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} (V_l \oplus V_l)$ and then,

$$\pi(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} m_l^+ V_l, \quad \pi^\perp(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} m_l^- V_l,$$

with multiplicities m_l^\pm to be determined, such that $m_l^+ + m_l^- = 2$. For $l \in \mathbb{N} + \frac{1}{2}$, the vectors

$$v_l^\pm := x_2^{l-\frac{1}{2}} (1 \pm x_0, \pm q^3 x_2, \mp q x_1, 0),$$

are highest weight vectors, being annihilated by both E_1 and E_2 , and have weight $(\frac{1}{2}, l)$. Furthermore, $v_l^+(1 - e) = v_l^- e = 0$. Thus, $v_l^+ \in \pi(\mathcal{A}(S_q^4)^4)$ and $v_l^- \in \pi^\perp(\mathcal{A}(S_q^4)^4)$.

Then in both modules $\pi(\mathcal{A}(S_q^4)^4)$ and $\pi^\perp(\mathcal{A}(S_q^4)^4)$ each representation V_l , $l \in \mathbb{N} + \frac{1}{2}$, appears with multiplicity $m_l^\pm \geq 1$. Since $m_l^+ + m_l^- = 2$, we deduce that $m_l^\pm = 1$ for all $l \in \mathbb{N} + \frac{1}{2}$. \square

4.4 Equivariant representations

Next, we construct $U_q(\mathfrak{so}(5))$ -equivariant representations of $\mathcal{A}(S_q^4)$ which classically correspond to the left regular and chiral spinor representations. The representation spaces will be (the closure of) $\bigoplus_{l \in \mathbb{N}} V_l$ and $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$, respectively.

Equivariance of a representation means that it is a representation of the crossed product algebra $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$. The latter is defined by the crossed relations $ha = (h_{(1)} \triangleright a)h_{(2)}$ for

all $a \in \mathcal{A}(S_q^4)$ and $h \in U_q(\mathfrak{so}(5))$. Explicitly, the relations among the generators read:

$$\begin{aligned}
[K_1, x_0] &= 0, & K_1 x_1 &= q x_1 K_1, & K_1 x_2 &= x_2 K_1, \\
[K_2, x_0] &= 0, & K_2 x_1 &= q^{-1} x_1 K_2, & K_2 x_2 &= q x_2 K_2, \\
[E_1, x_0] &= q^{-1/2} x_1 K_1, & E_1 x_1 &= q^{-1} x_1 E_1, & E_1 x_2 &= x_2 E_1, \\
[F_1, x_0] &= -q^{-1/2} K_1 x_1^*, & F_1 x_1 &= q^{-1} x_1 F_1 + q^{1/2} [2] x_0 K_1, & F_1 x_2 &= x_2 F_1, \\
[E_2, x_0] &= 0, & E_2 x_1 &= q x_1 E_2 + x_2 K_2, & E_2 x_2 &= q^{-1} x_2 E_2, \\
[F_2, x_0] &= 0, & F_2 x_1 &= q x_1 F_2, & F_2 x_2 &= q^{-1} x_2 F_2 + x_1 K_2.
\end{aligned} \tag{4.11}$$

In previous Section we proved that on $\bigoplus_{l \in \mathbb{N}} V_l$ there is at least one equivariant representation, the left regular one, and that on $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$ there are at least two equivariant representations, corresponding to the projective modules $\mathcal{A}(S_q^4)^4 e$ and $\mathcal{A}(S_q^4)^4 (1 - e)$. In this section we prove that on such spaces there are no other equivariant representations (modulo unitary equivalences) besides the ones just mentioned.

Let us denote with $|l, m_1, m_2; j\rangle$ the basis of the representation space V_l of $U_q(\mathfrak{so}(5))$ as discussed in Section 4.1. From the first two lines of (4.11) we deduce that

$$x_0 |l, m_1, m_2; j\rangle = \sum_{l', j'} A_{j, j', l, l'}^{m_1, m_2} |l', m_1, m_2; j'\rangle, \tag{4.12a}$$

$$x_1 |l, m_1, m_2; j\rangle = \sum_{l', j'} B_{j, j', l, l'}^{m_1, m_2} |l', m_1 + 1, m_2; j'\rangle, \tag{4.12b}$$

$$x_2 |l, m_1, m_2; j\rangle = \sum_{l', j'} C_{j, j', l, l'}^{m_1, m_2} |l', m_1, m_2 + 1; j'\rangle, \tag{4.12c}$$

with coefficients to be determined. Notice that from the crossed relations

$$\begin{aligned}
x_1 |l, m_1, m_2; j\rangle &= (F_2 x_2 - q^{-1} x_2 F_2) K_2^{-1} |l, m_1, m_2; j\rangle, \\
x_0 |l, m_1, m_2; j\rangle &= q^{-1/2} [2]^{-1} (F_1 x_1 - q^{-1} x_1 F_1) K_1^{-1} |l, m_1, m_2; j\rangle,
\end{aligned}$$

the matrix coefficients of x_0 and x_1 can be expressed in term of the coefficients of x_2 .

Lemma 4.7. *Let $k \in \mathbb{N}$. The following formulæ hold:*

$$F_1^k |l, m_1, m_2; j\rangle = \begin{cases} = 0 & \text{if } k > j + m_1 \\ \neq 0 & \text{if } k \leq j + m_1 \end{cases} \tag{4.13a}$$

$$E_1^k |l, m_1, m_2; j\rangle = \begin{cases} = 0 & \text{if } k > j - m_1 \\ \neq 0 & \text{if } k \leq j - m_1 \end{cases}. \tag{4.13b}$$

Proof. By direct computation:

$$\begin{aligned}
F_1^k |l, m_1, m_2; j\rangle &= \sqrt{[j + m_1][j + m_1 - 1] \dots [j + m_1 - k + 1]} \times \\
&\quad \times \sqrt{[j - m_1 + 1][j - m_1 + 2] \dots [j - m_1 + k]} |l, m_1 - k, m_2; j\rangle.
\end{aligned}$$

The second square root is always different from zero since the q -analogues are in increasing order and $j - m_1 + 1 \geq 1$. In the first square root q -analogues are in decreasing order and are all different from zero if and only if $j + m_1 - k + 1 \geq 1$. This proves equation (4.13a).

In the same way one establishes (4.13b), by using the equation

$$E_1^k |l, m_1, m_2; j\rangle = \sqrt{[j - m_1][j - m_1 - 1] \dots [j - m_1 - k + 1]} \times \\ \times \sqrt{[j + m_1 + 1][j + m_1 + 2] \dots [j + m_1 + k]} |l, m_1 + k, m_2; j\rangle. \quad \square$$

Lemma 4.8. *The coefficients in (4.12) satisfy:*

$$A_{j,j',l,l'}^{m_1,m_2} = B_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } |j - j'| > 1, \quad C_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } j' \neq j.$$

Proof. From (4.11), (4.13a) and (4.13b) we derive:

$$E_1^{j-m_1+1} x_1 |l, m_1, m_2; j\rangle = q^{-j+m_1-1} x_1 E_1^{j-m_1+1} |l, m_1, m_2; j\rangle = 0, \\ F_1^{j'+m_1+2} x_1^* |l', m_1 + 1, m_2; j'\rangle = q^{j'+m_1+2} x_1^* F_1^{j'+m_1+2} |l', m_1 + 1, m_2; j'\rangle = 0.$$

We expand the left hand sides and use the independence of the vectors $E_1^{j-m_1+1} |l', m_1 + 1, m_2; j'\rangle$ and $F_1^{j'+m_1+2} |l, m_1, m_2; j\rangle$ to arrive at the conditions:

$$B_{j,j',l,l'}^{m_1,m_2} \left\{ E_1^{j-m_1+1} |l', m_1 + 1, m_2; j'\rangle \right\} = 0, \\ \bar{B}_{j,j',l,l'}^{m_1,m_2} \left\{ F_1^{j'+m_1+2} |l, m_1, m_2; j\rangle \right\} = 0.$$

By (4.13b) the curly brackets in the first line are different from zero if $j - m_1 + 1 \leq j' - m_1 - 1$, i.e. $B_{j,j',l,l'}^{m_1,m_2}$ must be zero if $j' \geq j + 2$. By (4.13a) the curly brackets in the second line is different from zero if $j' + m_1 + 2 \leq j + m_1$, i.e. $\bar{B}_{j,j',l,l'}^{m_1,m_2}$ must be zero if $j' \leq j - 2$. This proves 1/3 of the statement

$$B_{j,j',l,l'}^{m_1,m_2} = 0 \quad \forall j' \notin \{j - 1, j, j + 1\}.$$

A similar argument applies to x_0 . From the coproduct of E_1^n we deduce:

$$E_1^n x_0 = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (E_1^k \triangleright x_0) E_1^{n-k} K_1^k = x_0 E_1^n - [n] q^{-1/2} x_1 E_1^{n-1} K_1.$$

This implies that $E_1^{j-m_1+2} x_0 |l, m_1, m_2; j\rangle = 0$ and $F_1^{j'+m_1+2} x_0 |l', m_1, m_2; j\rangle = 0$. From these conditions we deduce that also x_0 shift j by $\{0, \pm 1\}$ only.

Finally, let \mathcal{C}_1 be the Casimir element in equation (4.4). Then $[\mathcal{C}_1, x_2] = 0$ and from (4.5) we deduce that x_2 is diagonal on the index j . \square

Lemma 4.9. *The coefficients in (4.12c) satisfy*

$$C_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } |l - l'| > 1 \text{ or if } |l - l'| = 0 \text{ and } l \in \mathbb{N}.$$

Proof. The elements $\{x_i, x_i^*\}$ are a basis of the irreducible representation V_1 . Covariance of the action tells that $x_i |l, m_1, m_2; j\rangle$ and $x_i^* |l, m_1, m_2; j\rangle$ are a basis of the tensor representation $V_1 \otimes V_l$. Equations (14–15) in Chapter 7 of [KS97] tells that $V_1 \otimes V_l \simeq V_{l-1} \oplus V_{l+1}$ if $l \in \mathbb{N}$ and that $V_1 \otimes V_l \simeq V_{l-1} \oplus V_l \oplus V_{l+1}$ if $l \in \mathbb{N} + \frac{1}{2}$ (with V_{l-1} omitted if $l - 1 < 0$). This Clebsh-Gordan decomposition tells that $x_2 |l, m_1, m_2; j\rangle$ is in the linear span of the basis vectors $|l', m_1, m_2 + 1; j\rangle$ with $l' - l = \pm 1$ if $l \in \mathbb{N}$ or with $l' - l = 0, \pm 1$ if $l \in \mathbb{N} + \frac{1}{2}$. This concludes the proof of the Lemma. \square

4.4.1 Computing the coefficients of x_2

From Lemma 4.9, we have to consider only the cases $j' = j$, $|l' - l| \leq 1$ if $l \in \mathbb{N} + \frac{1}{2}$ or $|l' - l| = 1$ if $l \in \mathbb{N}$. The condition $[E_1, x_2] = 0$ implies that $C_{j,j,l,l'}^{m_1, m_2} =: C_{j,l,l'}^{m_2}$ is independent of m_1 . Equations $(E_2 x_2 - q^{-1} x_2 E_2) |l, -j, m_2; j\rangle = 0$ and $(F_2 x_2^* - q x_2^* F_2) |l', j, m_2 + 1; j\rangle = 0$ imply, respectively:

$$\begin{aligned} C_{j,l,l'}^{m_2} \sqrt{[l' - j - m_2 - 1 + \epsilon'] [l' + j + m_2 + 4 + \epsilon']} &= C_{j+1,l,l'}^{m_2+1} q^{-1} \sqrt{[l - j - m_2 + \epsilon] [l + j + m_2 + 3 + \epsilon]}, \\ C_{j,l,l'}^{m_2} \sqrt{[l + j - m_2 + 3 - \epsilon] [l - j + m_2 - \epsilon]} &= C_{j+1,l,l'}^{m_2-1} q \sqrt{[l' + j - m_2 + 2 - \epsilon'] [l' - j + m_2 + 1 - \epsilon']}, \end{aligned}$$

with $\epsilon, \epsilon' \in \{0, \pm \frac{1}{2}\}$ determined by the conditions $l + \epsilon - j - m_2 \in 2\mathbb{N}$ and $l' - \epsilon' - j - m_2 \in 2\mathbb{N}$. Notice that if $l' - l \in 2\mathbb{N} + 1$ then $\epsilon' = \epsilon$, while if $l' - l \in 2\mathbb{N}$ then $\epsilon' = -\epsilon$. Looking at the cases $l' - l = \pm 1$, we deduce that

$$\frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l + j + m_2 + 3 + \epsilon]}} C_{j,l,l+1}^{m_2} \quad \text{and} \quad \frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l - j - m_2 + \epsilon]}} C_{j,l,l-1}^{m_2}$$

depend on $j + m_2$ only through their parity (i.e. they depend only on the value of ϵ). Similarly,

$$\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l - j + m_2 + 2 - \epsilon]}} C_{j,l,l+1}^{m_2} \quad \text{and} \quad \frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l + j - m_2 + 1 - \epsilon]}} C_{j,l,l-1}^{m_2}$$

depend on $j - m_2$ only through their parity. Combining these informations, we deduce that the following elements do not depend on the exact value of j, m_2 , but only on the value of ϵ ,

$$\begin{aligned} \frac{q^{-m_2}}{\sqrt{[l + j + m_2 + 3 + \epsilon] [l - j + m_2 + 2 - \epsilon]}} C_{j,l,l+1}^{m_2} &=: C_{l,l+1}(\epsilon), \\ \frac{q^{-m_2}}{\sqrt{[l - j - m_2 + \epsilon] [l + j - m_2 + 1 - \epsilon]}} C_{j,l,l-1}^{m_2} &=: C_{l,l-1}(\epsilon). \end{aligned}$$

If $l \in \mathbb{N}$ there are no other coefficients $C_{j,l,l'}^{m_2}$ to compute. If $l \notin \mathbb{N}$, we have to compute also $C_{j,l,l}^{m_2}$. In this case $\epsilon' = -\epsilon$ and we get

$$\begin{aligned} C_{j,l,l}^{m_2} \sqrt{[l - j - m_2 - 1 - \epsilon] [l + j + m_2 + 4 - \epsilon]} &= C_{j+1,l,l}^{m_2+1} q^{-1} \sqrt{[l - j - m_2 + \epsilon] [l + j + m_2 + 3 + \epsilon]}, \\ C_{j,l,l}^{m_2} \sqrt{[l + j - m_2 + 3 - \epsilon] [l - j + m_2 - \epsilon]} &= C_{j+1,l,l}^{m_2-1} q \sqrt{[l + j - m_2 + 2 + \epsilon] [l - j + m_2 + 1 + \epsilon]}. \end{aligned}$$

Again, looking at the two cases $\epsilon = \pm \frac{1}{2}$ we deduce that

$$\frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l + \frac{1}{2} - j - m_2]}} C_{j,l,l}^{m_2} \quad \text{if } \epsilon = \frac{1}{2} \quad \text{and} \quad \frac{q^{-\frac{1}{2}(j+m_2)}}{\sqrt{[l + \frac{1}{2} + j + m_2 + 2]}} C_{j,l,l}^{m_2} \quad \text{if } \epsilon = -\frac{1}{2}$$

do not depend on $j + m_2$ (this time ϵ is fixed, then the parity of $j + m_2$ is fixed). Similarly,

$$\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l + \frac{1}{2} - j + m_2 + 1]}} C_{j,l,l}^{m_2} \quad \text{if } \epsilon = \frac{1}{2} \quad \text{and} \quad \frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{[l + \frac{1}{2} + j - m_2 + 1]}} C_{j,l,l}^{m_2} \quad \text{if } \epsilon = -\frac{1}{2}$$

do not depend on $j - m_2$. Combining these informations, we deduce that the following element do not depend on the exact value of j, m_2 , but only on the value of ϵ :

$$\frac{q^{-m_2}}{\sqrt{[l - 2\epsilon j - m_2 + 1 - \epsilon][l - 2\epsilon j + m_2 + 2 - \epsilon]}} C_{j,l,l}^{m_2} =: C_{l,l}(\epsilon) .$$

The denominator of the left hand side is just $[2j][2j+2]b_l(j, m_2)$ with b_l the coefficient in equation (4.2b). The formula $C_{j,l,l}^{m_2} = q^{m_2}[2j][2j+2]b_l(j, m_2)C_{l,l}(\epsilon)$ is valid for all l , since $b_l(j, m_2)$ vanish for l integer.

Summarizing, we find that

$$C_{j,l,l+1}^{m_2} = q^{m_2} \sqrt{[l + j + m_2 + 3 + \epsilon][l - j + m_2 + 2 - \epsilon]} C_{l,l+1}(\epsilon) , \quad (4.14a)$$

$$C_{j,l,l}^{m_2} = q^{m_2}[2j][2j+2]b_l(j, m_2) C_{l,l}(\epsilon) , \quad (4.14b)$$

$$C_{j,l,l-1}^{m_2} = q^{m_2} \sqrt{[l - j - m_2 + \epsilon][l + j - m_2 + 1 - \epsilon]} C_{l,l-1}(\epsilon) , \quad (4.14c)$$

with coefficients $C_{l,l'}(\epsilon)$ to be determined.

4.4.2 Computing the coefficients of x_1

From Lemma 4.8, we have to consider only the three cases $j' = j, j \pm 1$. Using equation $E_1 x_1 = q^{-1} x_1 E_1$ we get,

$$\begin{aligned} \frac{q^{-m_1}}{\sqrt{[j + m_1 + 1][j + m_1 + 2]}} B_{j,j+1,l,l'}^{m_1,m_2} &= \frac{q^{-m_1-1}}{\sqrt{[j + m_1 + 2][j + m_1 + 3]}} B_{j,j+1,l,l'}^{m_1+1,m_2} , \\ \frac{q^{-m_1}}{\sqrt{[j - m_1][j + m_1 + 1]}} B_{j,j,l,l'}^{m_1,m_2} &= \frac{q^{-m_1-1}}{\sqrt{[j - m_1 - 1][j + m_1 + 2]}} B_{j,j,l,l'}^{m_1+1,m_2} , \\ \frac{q^{-m_1}}{\sqrt{[j - m_1][j - m_1 - 1]}} B_{j,j-1,l,l'}^{m_1,m_2} &= \frac{q^{-m_1-1}}{\sqrt{[j - m_1 - 1][j - m_1 - 2]}} B_{j,j-1,l,l'}^{m_1+1,m_2} . \end{aligned}$$

We see that the left hand sides of these three equations are independent of m_1 , and call:

$$B_{j,j+1,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{[j + m_1 + 1][j + m_1 + 2]} B_{j,j+1,l,l'}^{m_2} , \quad (4.15a)$$

$$B_{j,j,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{[j - m_1][j + m_1 + 1]} B_{j,j,l,l'}^{m_2} , \quad (4.15b)$$

$$B_{j,j-1,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{[j - m_1][j - m_1 - 1]} B_{j,j-1,l,l'}^{m_2} . \quad (4.15c)$$

Imposing the condition $x_1 K_2 = F_2 x_2 - q^{-1} x_2 F_2$ on the subspace spanned by $|l, j, m_2; j\rangle$ (so $m_1 = j$ and $B_{j,j,l,l'}^{m_1,m_2} = B_{j,j-1,l,l'}^{m_1,m_2} = 0$ on this subspace) we get:

$$q^{m_2} B_{j,j+1,l,l'}^{m_2} = c_{l'}(j + 1, m_2) C_{j,l,l'}^{m_2} - q^{-1} c_l(j + 1, m_2 - 1) C_{j+1,l,l'}^{m_2-1} .$$

From this we deduce that, that since coefficients $C_{j,l,l'}^{m_2}$ vanishes for $|l-l'| > 1$, also $B_{j,j+1,l,l'}^{m_2}$ is zero in these cases. In the remaining three cases $l' = l, l \pm 1$, using equation (4.14) we get:

$$B_{j,j+1,l,l+1}^{m_2} = (-1)^{2\epsilon} q^{l-j+m_2-\epsilon} \sqrt{\frac{[l+j+m_2+3+\epsilon][l+j-m_2+3-\epsilon]}{[2(j+|\epsilon|)+1][2(j-|\epsilon|)+3]}} C_{l,l+1}(\epsilon), \quad (4.16a)$$

$$B_{j,j+1,l,l}^{m_2} = -2\epsilon q^{2\epsilon l-j+m_2-2+3\epsilon} \sqrt{\frac{[l+\frac{1}{2}+j-2\epsilon m_2+2][l+\frac{1}{2}-j-2\epsilon m_2]}{[2j+2]}} C_{l,l}(\epsilon), \quad (4.16b)$$

$$B_{j,j+1,l,l-1}^{m_2} = (-1)^{2\epsilon+1} q^{-l-j+m_2-3+\epsilon} \sqrt{\frac{[l-j+m_2-\epsilon][l-j-m_2+\epsilon]}{[2(j+|\epsilon|)+1][2(j-|\epsilon|)+3]}} C_{l,l-1}(\epsilon). \quad (4.16c)$$

Imposing $qx_1^*K_2 = x_2^*E_2 - q^{-1}E_2x_2^*$ on the subspace spanned by $|l', -j+1, m_2; j-1\rangle$ (so $B_{j,j,l,l'}^{m_1, m_2} = B_{j,j+1,l,l'}^{m_1, m_2} = 0$ on this subspace) we get:

$$q^{m_2} B_{j,j-1,l,l'}^{m_2} = a_{l'}(j-1, m_2) C_{j,l,l'}^{m_2} - q^{-1} a_l(j-1, m_2-1) C_{j-1,l,l'}^{m_2-1}.$$

We deduce that $B_{j,j-1,l,l'}^{m_2}$ vanishes if $|l-l'| > 1$, while in the three remaining cases $l' = l, l \pm 1$ using equation (4.14) we get:

$$B_{j,j-1,l,l+1}^{m_2} = q^{l+j+m_2+1+\epsilon} \sqrt{\frac{[l-j-m_2+2+\epsilon][l-j+m_2+2-\epsilon]}{[2(j+|\epsilon|)-1][2(j-|\epsilon|)+1]}} C_{l,l+1}(\epsilon), \quad (4.17a)$$

$$B_{j,j-1,l,l}^{m_2} = -2\epsilon q^{-2\epsilon l+j+m_2-1-3\epsilon} \sqrt{\frac{[l+\frac{1}{2}+j+2\epsilon m_2+1][l+\frac{1}{2}-j+2\epsilon m_2+1]}{[2j]}} C_{l,l}(\epsilon), \quad (4.17b)$$

$$B_{j,j-1,l,l-1}^{m_2} = -q^{-l+j+m_2-2-\epsilon} \sqrt{\frac{[l+j+m_2+1+\epsilon][l+j-m_2+1-\epsilon]}{[2(j+|\epsilon|)-1][2(j-|\epsilon|)+1]}} C_{l,l-1}(\epsilon). \quad (4.17c)$$

Moreover, the condition $\langle l', j, m_2; j | x_1 K_2 + q^{-1} x_2 F_2 - F_2 x_2 | l, j-1, m_2; j \rangle = 0$ implies that

$$q^{m_2} B_{j,j,l,l'}^{m_2} = b_{l'}(j, m_2) C_{j,l,l'}^{m_2} - q^{-1} b_l(j, m_2-1) C_{j,l,l'}^{m_2-1}. \quad (4.18)$$

A further elaboration on these coefficients is postponed to after the following section.

4.4.3 Computing the coefficients of x_0

The condition $q^{1/2}[2]x_0K_1 = F_1x_1 - q^{-1}x_1F_1$ implies:

$$q^{m_1+\frac{1}{2}} [2] A_{j,j',l,l'}^{m_1, m_2} = \sqrt{[j'-m_1][j'+m_1+1]} B_{j,j',l,l'}^{m_1, m_2} - q^{-1} \sqrt{[j+m_1][j-m_1+1]} B_{j,j',l,l'}^{m_1-1, m_2}.$$

In the three non-trivial cases $j' - j = 1, 0, -1$, using (4.15), we get:

$$A_{j,j+1,l,l'}^{m_1, m_2} = q^{j+m_1-\frac{1}{2}} \sqrt{[j+m_1+1][j-m_1+1]} B_{j,j+1,l,l'}^{m_2}, \quad (4.19a)$$

$$A_{j,j,l,l'}^{m_1, m_2} = [2]^{-1} q^{-2-\frac{1}{2}} (q^{j+m_1+1} [2][j-m_1] - [2j]) B_{j,j,l,l'}^{m_2}, \quad (4.19b)$$

$$A_{j,j-1,l,l'}^{m_1, m_2} = -q^{-j+m_1-1-\frac{1}{2}} \sqrt{[j+m_1][j-m_1]} B_{j,j-1,l,l'}^{m_2}. \quad (4.19c)$$

The hermiticity condition $x_0 = x_0^*$ means that $A_{j,j+1,l,l'}^{m_1,m_2} = \bar{A}_{j+1,j,l',l}^{m_1,m_2}$ and $A_{j,j,l,l'}^{m_1,m_2} = \bar{A}_{j,j,l',l}^{m_1,m_2}$. Thus, from (4.19) follows that:

$$B_{j+1,j,l',l}^{m_2} = -q^{2j+2} \bar{B}_{j,j+1,l,l'}^{m_2}, \quad B_{j,j,l,l'}^{m_2} = \bar{B}_{j,j,l',l}^{m_2}.$$

Using (4.16), the first equation turns out to be equivalent to the following conditions:

$$C_{l+1,l}(\epsilon) = (-1)^{2\epsilon} q^{2l+4} \bar{C}_{l,l+1}(\epsilon), \quad C_{l,l}(\epsilon) = \bar{C}_{l,l}(-\epsilon). \quad (4.20a)$$

The second of equation together with (4.18) implies:

$$b_{l'}(j, m_2) C_{j,l,l'}^{m_2} - q^{-1} b_l(j, m_2 - 1) C_{j,l,l'}^{m_2-1} = b_l(j, m_2) C_{j,l',l}^{m_2} - q^{-1} b_{l'}(j, m_2 - 1) C_{j,l',l}^{m_2-1}.$$

That is, using (4.14):

$$C_{l,l+1}(\epsilon) = C_{l,l+1}(-\epsilon), \quad C_{l,l}(\epsilon) = \bar{C}_{l,l}(\epsilon). \quad (4.20b)$$

4.4.4 Again the coefficients of x_1

Now, using (4.20) together with (4.18) we are able to compute the last coefficients. Notice that from (4.2b) the coefficients b_l vanish if $\epsilon = 0$ (i.e. in the left regular representation), and then from (4.18) $B_{j,j,l,l'}^{m_2}$ vanish too if $\epsilon = 0$. Moreover, from Lemma 4.8 $B_{j,j,l,l'}^{m_2}$ vanish also if $|l - l'| > 1$. In the three cases $l' = l, l \pm 1$, using equation (4.14) we get:

$$\begin{aligned} B_{j,j,l,l+1}^{m_2} &= 2|\epsilon| [2] q^{l+m_2+1+\epsilon(2j+1)} \frac{\sqrt{[l+2\epsilon j - m_2 + 2 + \epsilon][l - 2\epsilon j + m_2 + 2 - \epsilon]}}{[2j][2j+2]} C_{l,l+1}(\epsilon), \\ B_{j,j,l,l}^{m_2} &= \frac{2|\epsilon|}{[2j][2j+2]} \left\{ [l - \epsilon(2j+1) - m_2 + 1][l - \epsilon(2j+1) + m_2 + 2] + \right. \\ &\quad \left. - q^{-2}[l + \epsilon(2j+1) - m_2 + 2][l + \epsilon(2j+1) + m_2 + 1] \right\} \\ &= -\frac{2|\epsilon|}{[2j][2j+2]} \frac{q^{-2\epsilon(2j+1)}[2l+4] - q^{2\epsilon(2j+1)}[2l+2] - [2]q^{2m_2}}{1 - q^2} C_{l,l}(\epsilon), \\ B_{j,j,l,l-1}^{m_2} &= -2|\epsilon| [2] q^{-l+m_2-2-\epsilon(2j+1)} \frac{\sqrt{[l+2\epsilon j + m_2 + 1 + \epsilon][l - 2\epsilon j - m_2 + 1 - \epsilon]}}{[2j][2j+2]} C_{l,l-1}(\epsilon). \end{aligned}$$

We have inserted the factor $2|\epsilon|$, so that the expressions remain valid also when $\epsilon = 0$.

4.4.5 The condition on the radius

Orbits for $SO(5)$ are spheres of arbitrary radius, equivariance alone not imposing constraints on the radius. Similarly, for the quantum spheres one has to impose a constraint on the radius to determine the coefficients of the representation. In fact, this will determine $C_{l,l+1}(0)$, $C_{l,l+1}(\frac{1}{2})$ and $C_{l,l}(\frac{1}{2})$ only up to a phase. Different choices of the phases correspond to unitary equivalent representations and without losing generality we choose $C_{l,l'}(\epsilon) \in \mathbb{R}$. A possible expression for

the radius is $q^8 x_0^2 + q^4 x_1^* x_1 + x_2^* x_2$, and because of the defining relations of $\mathcal{A}(S_q^4)$ this must be equal to 1. Let then,

$$r(l, m_1, m_2; j) := \langle l, m_1, m_2; j | q^8 x_0^2 + q^4 x_1^* x_1 + x_2^* x_2 | l, m_1, m_2; j \rangle .$$

All these matrix coefficients must be 1. In particular, for $l \in \mathbb{N}$ the condition $r(l, 0, l; 0) = 1$ implies (up to a phase) that

$$C_{l, l+1}(0) = \frac{q^{-l-3/2}}{\sqrt{[2l+3][2l+5]}} . \quad (4.21)$$

For $l \in \mathbb{N} + \frac{1}{2}$ we first require that $r(l, \frac{1}{2}, l; \frac{1}{2}) = r(l, -\frac{1}{2}, l; \frac{1}{2})$ obtaining two possibilities:

$$C_{l, l}(\frac{1}{2}) = \pm \frac{[2]q^{l+2}}{[2l+2]} C_{l, l+1}(\frac{1}{2}) .$$

Then imposing $r(l, \frac{1}{2}, l; \frac{1}{2}) = 1$, yields (up to a phase)

$$C_{l, l+1}(\frac{1}{2}) = \frac{q^{-l-3/2}}{[2l+4]} , \quad (4.22)$$

hence,

$$C_{l, l}(\frac{1}{2}) = \pm \frac{q^{1/2}[2]}{[2l+2][2l+4]} . \quad (4.23)$$

With these, all the coefficients are completely determined.

4.4.6 Explicit form of the representations

Let us recall what we know on the equivariant representations of the algebra $\mathcal{A}(S_q^4)$.

By the decomposition $\mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l$ into irreducible representations of $U_q(\mathfrak{so}(5))$, there exists (at least) one representation of $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$ on the vector space $\bigoplus_{l \in \mathbb{N}} V_l$ extending the representation $\bigoplus_{l \in \mathbb{N}} \sigma_l$ of $U_q(\mathfrak{so}(5))$. As we computed above, the equivariance uniquely determines (for $l \in \mathbb{N}$, up to unitary equivalence) the matrix coefficients of the representation, whose expression is characterized by (4.21). On the other hand, by Proposition 4.4 there are (at least) two inequivalent representations of $\mathcal{A}(S_q^4) \rtimes U_q(\mathfrak{so}(5))$ on the vector space $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$ extending the representation $\bigoplus_{l \in \mathbb{N} + \frac{1}{2}} \sigma_l$ of $U_q(\mathfrak{so}(5))$. These correspond, by Lemma 4.5, to the projective modules $\mathcal{A}(S_q^4)^4 e$ and $\mathcal{A}(S_q^4)^4 (1 - e)$, with e the idempotent in equation (4.8). The computation above (for $l \in \mathbb{N} + \frac{1}{2}$), which culminates in equation (4.23), tells us that there are only two possibilities for the matrix coefficients (up to unitary equivalence). Therefore, the two possible choices in (4.23) must correspond to the inequivalent representations associated with the projective modules $\mathcal{A}(S_q^4)^4 e$ and $\mathcal{A}(S_q^4)^4 (1 - e)$.

Let us summarize these results in the following two theorems, which correspond to the scalar (i.e. left regular) and chiral spinor representations, respectively.

Theorem 4.10. *The vector space $\mathcal{A}(S_q^4)$ has orthonormal basis $|l, m_1, m_2; j\rangle$ with,*

$$l \in \mathbb{N}, \quad j = 0, 1, \dots, l, \quad j - |m_1| \in \mathbb{N}, \quad l - j - |m_2| \in 2\mathbb{N}.$$

We call $L^2(S_q^4)$ the Hilbert space completion of $\mathcal{A}(S_q^4)$. Modulo a unitary equivalence, the left regular representation is given by

$$\begin{aligned} x_0 |l, m_1, m_2; j\rangle &= A_{j, m_1} C_{l, j, m_2}^+ |l + 1, m_1, m_2; j + 1\rangle \\ &\quad + A_{j, m_1} C_{l, j, m_2}^- |l - 1, m_1, m_2; j + 1\rangle \\ &\quad + A_{j-1, m_1} C_{l+1, j-1, m_2}^- |l + 1, m_1, m_2; j - 1\rangle \\ &\quad + A_{j-1, m_1} C_{l-1, j-1, m_2}^+ |l - 1, m_1, m_2; j - 1\rangle, \\ x_1 |l, m_1, m_2; j\rangle &= B_{j, m_1}^+ C_{l, j, m_2}^+ |l + 1, m_1 + 1, m_2; j + 1\rangle \\ &\quad + B_{j, m_1}^+ C_{l, j, m_2}^- |l - 1, m_1 + 1, m_2; j + 1\rangle \\ &\quad + B_{j, m_1}^- C_{l+1, j-1, m_2}^- |l + 1, m_1 + 1, m_2; j - 1\rangle \\ &\quad + B_{j, m_1}^- C_{l-1, j-1, m_2}^+ |l - 1, m_1 + 1, m_2; j - 1\rangle, \\ x_2 |l, m_1, m_2; j\rangle &= D_{l, j, m_2}^+ |l + 1, m_1, m_2 + 1; j\rangle \\ &\quad + D_{l, j, m_2}^- |l - 1, m_1, m_2 + 1; j\rangle, \end{aligned}$$

with coefficients

$$\begin{aligned} A_{j, m_1} &= q^{m_1-1} \sqrt{\frac{[j + m_1 + 1][j - m_1 + 1]}{[2j + 1][2j + 3]}}, \\ B_{j, m_1}^+ &= q^{-j+m_1-1/2} \sqrt{\frac{[j + m_1 + 1][j + m_1 + 2]}{[2j + 1][2j + 3]}}, \\ B_{j, m_1}^- &= -q^{j+m_1+1/2} \sqrt{\frac{[j - m_1][j - m_1 - 1]}{[2j - 1][2j + 1]}}. \end{aligned}$$

and

$$\begin{aligned} C_{l, j, m_2}^+ &= q^{m_2-1} \sqrt{\frac{[l + j + m_2 + 3][l + j - m_2 + 3]}{[2l + 3][2l + 5]}}, \\ C_{l, j, m_2}^- &= -q^{m_2-1} \sqrt{\frac{[l - j + m_2][l - j - m_2]}{[2l + 1][2l + 3]}}, \\ D_{l, j, m_2}^+ &= q^{-l+m_2-3/2} \sqrt{\frac{[l + j + m_2 + 3][l - j + m_2 + 2]}{[2l + 3][2l + 5]}}, \\ D_{l, j, m_2}^- &= q^{l+m_2+3/2} \sqrt{\frac{[l - j - m_2][l + j - m_2 + 1]}{[2l + 1][2l + 3]}}. \end{aligned}$$

The two chiral spinorial representations (corresponding to the sign \pm in equation (4.23)) are described in the following Theorem.

Theorem 4.11. Let \mathcal{H}_\pm be two Hilbert spaces with orthonormal basis $|l, m_1, m_2; j\rangle_\pm$, where

$$l \in \mathbb{N} + \frac{1}{2}, \quad j = \frac{1}{2}, \frac{3}{2}, \dots, l, \quad j - |m_1| \in \mathbb{N}, \quad l + \frac{1}{2} - j - |m_2| \in \mathbb{N}.$$

Let $\epsilon = \pm \frac{1}{2}$ be defined by $l + \epsilon - j - m_2 \in 2\mathbb{N}$. On each space \mathcal{H}_\pm there is an equivariant *-representation of $\mathcal{A}(S_q^4)$ defined by:

$$\begin{aligned} x_0 |l, m_1, m_2; j\rangle_\pm &= A_{j, m_1}^+ C_{l, j, m_2}^+ |l + 1, m_1, m_2; j + 1\rangle_\pm \\ &\mp A_{j, m_1}^+ C_{l, j, m_2}^0 |l, m_1, m_2; j + 1\rangle_\pm \\ &+ A_{j, m_1}^+ C_{l, j, m_2}^- |l - 1, m_1, m_2; j + 1\rangle_\pm \\ &+ A_{j, m_1}^0 H_{l, j, m_2}^+ |l + 1, m_1, m_2; j\rangle_\pm \\ &\pm A_{j, m_1}^0 H_{l, j, m_2}^0 |l, m_1, m_2; j\rangle_\pm \\ &+ A_{j, m_1}^0 H_{l-1, j, m_2}^+ |l - 1, m_1, m_2; j\rangle_\pm \\ &+ A_{j-1, m_1}^+ C_{l+1, j-1, m_2}^- |l + 1, m_1, m_2; j - 1\rangle_\pm \\ &\mp A_{j-1, m_1}^+ C_{l, j-1, m_2}^0 |l, m_1, m_2; j - 1\rangle_\pm \\ &+ A_{j-1, m_1}^+ C_{l-1, j-1, m_2}^+ |l - 1, m_1, m_2; j - 1\rangle_\pm, \\ x_1 |l, m_1, m_2; j\rangle_\pm &= B_{j, m_1}^+ C_{l, j, m_2}^+ |l + 1, m_1 + 1, m_2; j + 1\rangle_\pm \\ &\mp B_{j, m_1}^+ C_{l, j, m_2}^0 |l, m_1 + 1, m_2; j + 1\rangle_\pm \\ &+ B_{j, m_1}^+ C_{l, j, m_2}^- |l - 1, m_1 + 1, m_2; j + 1\rangle_\pm \\ &+ B_{j, m_1}^0 H_{l, j, m_2}^+ |l + 1, m_1 + 1, m_2; j\rangle_\pm \\ &\pm B_{j, m_1}^0 H_{l, j, m_2}^0 |l, m_1 + 1, m_2; j\rangle_\pm \\ &+ B_{j, m_1}^0 H_{l-1, j, m_2}^+ |l - 1, m_1 + 1, m_2; j\rangle_\pm \\ &+ B_{j, m_1}^- C_{l+1, j-1, m_2}^- |l + 1, m_1 + 1, m_2; j - 1\rangle_\pm \\ &\mp B_{j, m_1}^- C_{l, j-1, m_2}^0 |l, m_1 + 1, m_2; j - 1\rangle_\pm \\ &+ B_{j, m_1}^- C_{l-1, j-1, m_2}^+ |l - 1, m_1 + 1, m_2; j - 1\rangle_\pm, \\ x_2 |l, m_1, m_2; j\rangle_\pm &= D_{l, j, m_2}^+ |l + 1, m_1, m_2 + 1; j\rangle_\pm \\ &\pm D_{l, j, m_2}^0 |l, m_1, m_2 + 1; j\rangle_\pm \\ &+ D_{l, j, m_2}^- |l - 1, m_1, m_2 + 1; j\rangle_\pm, \end{aligned}$$

with coefficients

$$\begin{aligned} A_{j, m_1}^+ &= q^{m_1-1} \frac{\sqrt{[j + m_1 + 1][j - m_1 + 1]}}{[2j + 2]}, \\ A_{j, m_1}^0 &= q^{-2} \frac{q^{j+m_1+1}[2][j - m_1] - [2j]}{[2j][2j + 2]}, \\ B_{j, m_1}^+ &= q^{-j+m_1-1/2} \frac{\sqrt{[j + m_1 + 1][j + m_1 + 2]}}{[2j + 2]}, \\ B_{j, m_1}^0 &= (1 + q^2) q^{m_1-1/2} \frac{\sqrt{[j - m_1][j + m_1 + 1]}}{[2j][2j + 2]}, \end{aligned}$$

$$B_{j,m_1}^- = -q^{j+m_1+1/2} \frac{\sqrt{[j-m_1][j-m_1-1]}}{[2j]} .$$

and

$$\begin{aligned} C_{l,j,m_2}^+ &= -q^{m_2-1-\epsilon} \frac{\sqrt{[l+j+m_2+3+\epsilon][l+j-m_2+3-\epsilon]}}{[2l+4]} , \\ C_{l,j,m_2}^0 &= [4\epsilon] q^{2\epsilon l+m_2-1+3\epsilon} \frac{\sqrt{[l+\frac{1}{2}+j-2\epsilon m_2+2][l+\frac{1}{2}-j-2\epsilon m_2]}}{[2l+2][2l+4]} , \\ C_{l,j,m_2}^- &= -q^{m_2-1+\epsilon} \frac{\sqrt{[l-j+m_2-\epsilon][l-j-m_2+\epsilon]}}{[2l+2]} , \\ H_{l,j,m_2}^+ &= q^{m_2-1+\epsilon(2j+1)} \frac{\sqrt{[l+2\epsilon j-m_2+2+\epsilon][l-2\epsilon j+m_2+2-\epsilon]}}{[2l+4]} , \\ H_{l,j,m_2}^0 &= \frac{[l-\epsilon(2j+1)-m_2+1][l-\epsilon(2j+1)+m_2+2]-q^{-2}[l+\epsilon(2j+1)-m_2+2][l+\epsilon(2j+1)+m_2+1]}{[2l+2][2l+4]} , \\ D_{l,j,m_2}^+ &= q^{-l+m_2-3/2} \frac{\sqrt{[l+j+m_2+3+\epsilon][l-j+m_2+2-\epsilon]}}{[2l+4]} , \\ D_{l,j,m_2}^0 &= [2] q^{m_2+1/2} \frac{\sqrt{[l-2\epsilon j-m_2+1-\epsilon][l-2\epsilon j+m_2+2-\epsilon]}}{[2l+2][2l+4]} , \\ D_{l,j,m_2}^- &= -q^{l+m_2+3/2} \frac{\sqrt{[l-j-m_2+\epsilon][l+j-m_2+1-\epsilon]}}{[2l+2]} . \end{aligned}$$

These two representations are inequivalent and correspond to the projective modules $\mathcal{A}(S_q^4)^4 e$ and $\mathcal{A}(S_q^4)^4(1-e)$, with e the idempotent in equation (4.8).

4.5 The Dirac operator on the quantum orthogonal 4-sphere

We start by constructing a non-trivial Fredholm module on the quantum orthogonal sphere. With different representations a non-trivial Fredholm module was already constructed in [HL04].

Proposition 4.12. *Consider the representations of $\mathcal{A}(S_q^4)$ on \mathcal{H}_\pm given in Theorem 4.11. Then, the datum $(\mathcal{A}(S_q^4), \mathcal{H}, F, \gamma)$ is a 1-summable even Fredholm module, where $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$, γ is the natural grading and $F \in \mathcal{B}(\mathcal{H})$ is defined by*

$$F |l, m_1, m_2; j\rangle_\pm := |l, m_1, m_2; j\rangle_\mp .$$

This Fredholm module is non-trivial. In particular,

$$\text{Index}(F_e^+) = \frac{1}{2} \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^4}(\gamma F[F, e]) = 1 , \quad (4.24)$$

with e the idempotent defined by equation (4.8).

Proof. That $F = F^*$, $F^2 = 1$ and $\gamma F + F\gamma = 0$ is obvious. Then, it is enough to show that $[F, x_i] \in \mathcal{L}^1(\mathcal{H})$ for $i = 0, 1, 2$. From this and Leibniz rule it follows that $[F, a]$ is trace class, and then compact, for all $a \in \mathcal{A}(S_q^4)$.

Now, notice that

$$\begin{aligned}
[F, x_0] |l, m_1, m_2; j\rangle_{\pm} &= \mp 2A_{j, m_1}^+ C_{l, j, m_2}^0 |l, m_1, m_2; j+1\rangle_{\mp} \\
&\quad \pm 2A_{j, m_1}^0 H_{l, j, m_2}^0 |l, m_1, m_2; j\rangle_{\mp} \\
&\quad \mp 2A_{j-1, m_1}^+ C_{l, j-1, m_2}^0 |l, m_1, m_2; j-1\rangle_{\mp} , \\
[F, x_1] |l, m_1, m_2; j\rangle_{\pm} &= \mp 2B_{j, m_1}^+ C_{l, j, m_2}^0 |l, m_1+1, m_2; j+1\rangle_{\mp} \\
&\quad \pm 2B_{j, m_1}^0 H_{l, j, m_2}^0 |l, m_1+1, m_2; j\rangle_{\mp} \\
&\quad \mp 2B_{j, m_1}^- C_{l, j-1, m_2}^0 |l, m_1+1, m_2; j-1\rangle_{\mp} , \\
[F, x_2] |l, m_1, m_2; j\rangle_{\pm} &= \pm 2D_{l, j, m_2}^0 |l, m_1, m_2+1; j\rangle_{\mp} .
\end{aligned} \tag{4.25}$$

All the coefficients appearing in these equations are bounded by q^{2l} . Thus the commutators are trace class and this concludes the first part of the proof.

To prove non-triviality it is enough to prove (4.24). Substituting (4.8) into (4.24) yields

$$\text{Index}(F_e^+) = \frac{(1-q^2)^2}{4} \text{Tr}_{\mathcal{H}}(\gamma F[F, x_0]) .$$

and in turn, using equation (4.25),

$$\text{Index}(F_e^+) = (1-q^2)^2 \sum_{l, j, m_1, m_2} A_{j, m_1}^0 H_{l, j, m_2}^0 .$$

Summing over m_1 from $-j$ to j we obtain that

$$\begin{aligned}
\text{Index}(F_e^+) &= q^{-3}(1-q^2)^2 \sum_{l, j, m_2} \frac{[l + \epsilon(2j+1) - m_2 + 2][l + \epsilon(2j+1) + m_2 + 1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\quad \times \sum_{m_1} \{q^{2j+2} + q^{-2j} - [2]q^{2m_1+1}\} \\
&= \sum_{l, j} \frac{(2j+1)(q^{2j+1} + q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\quad \times \sum_{m_2} \left\{ q^{2l+2\epsilon(2j+1)+3} + q^{-2l-2\epsilon(2j+1)-3} - q^{2m_2-1} - q^{-2m_2+1} \right\} .
\end{aligned}$$

The sum over m_2 requires additional care. For ϵ fixed, $l - \epsilon - j + m_2 = 0, 2, 4, \dots, 2(l-j)$. If we call $2i := l - \epsilon - j + m_2$ and sum first over $i = 0, 1, \dots, l-j$ and then over $\epsilon = \pm 1/2$ we get:

$$\begin{aligned}
\text{Index}(F_e^+) &= \sum_{l, j} \frac{(2j+1)(q^{2j+1} + q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\quad \times \sum_{2\epsilon=\pm 1} \left\{ (l-j+1)(q^{2l+2\epsilon(2j+1)+3} + q^{-2l-2\epsilon(2j+1)-3}) - (q^{2\epsilon-1} + q^{-2\epsilon+1})[2]^{-1}[2(l-j+1)] \right\} \\
&= \sum_{l, j} \frac{(2j+1)(q^{2j+1} + q^{-2j-1}) - [2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \\
&\quad \times \left\{ (l-j+1)(q^{2l+3} + q^{-2l-3})(q^{2j+1} + q^{-2j-1}) - [2][2(l-j+1)] \right\}
\end{aligned}$$

$$=: \sum_{l,j} f_{lj}(q) =: f(q) .$$

We call $f_{lj}(q)$ the generic term of last series, explicitly written as

$$f_{lj}(q) = (1 - q^2)^4 \frac{(2j + 1)(1 + q^{4j+2}) - \frac{1+q^2}{1-q^2}(1 - q^{4j+2})}{(1 - q^{4l+4})(1 - q^{4l+8})(1 - q^{4j})(1 - q^{4j+4})} \times \\ \times q^{2l-1} \left\{ (l - j + 1)(1 + q^{4l+6})(1 + q^{4j+2}) - \frac{1+q^2}{1-q^2} q^2 (q^{4j} - q^{4l+4}) \right\} ,$$

and consider it as a function of $q \in [0, 1[$. Notice that each $f_{lj}(q)$ is a C^∞ function of q (they are rational functions whose denominators never vanish for $0 \leq q < 1$). From the inequality

$$0 \leq f_{lj}(q) \leq 4(2j + 1)q^{2l-1}$$

we deduce (using the Weierstrass M-test [Arf85, pag. 301]) that the series is absolutely (hence uniformly) convergent in each interval $[0, q_0] \subset [0, 1[$. Then, it converges to a function $f(q)$ which is continuous in $[0, 1[$. Being the index of a Fredholm operator, $f(q)$ is integer valued in $]0, 1[$; by continuity it is constant and can be computed in the limit $q \rightarrow 0$. In this limit we have $f_{lj}(q) = 2j(l - j + 1)q^{2l-1} + O(q^{2l})$. Thus, $f_{lj}(0) = \delta_{l,1/2}\delta_{j,1/2}$ and $\text{Index}(F_e^+) = f(0) = 1$. \square

The next step is to define a spectral triple whose Fredholm module is the one described in Proposition 4.12.

Proposition 4.13. *Let D be the (unbounded) operator on $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ defined by*

$$D |l, m_1, m_2; j\rangle_{\pm} := (l + \frac{3}{2}) |l, m_1, m_2; j\rangle_{\mp} .$$

Then, the datum $(\mathcal{A}(S_q^4), \mathcal{H}, D, \gamma)$ is a $U_q(\mathfrak{so}(5))$ -equivariant regular even spectral triple of metric dimension 4.

Remark: The operator D is isospectral to the classical Dirac operator on S^4 (whose spectrum has been computed in [Tra93, CH96]). When $q = 1$, this spectral triple becomes the canonical one associated to the spin structure of S^4 .

Proof. Clearly the representation of the algebra is even, D is odd, with compact resolvent and 4^+ -summable (being isospectral to the classical Dirac operator on S^4). To prove that D defines a regular spectral triple, one repeats verbatim the proof of Lemma 2.14. Finally, since D is proportional to the identity in any irreducible subrepresentation V_l of $U_q(\mathfrak{so}(5))$, it commutes with all $h \in U_q(\mathfrak{so}(5))$ and it is equivariant. \square

As a preparation for the study of the dimension spectrum in Section 4.6, let us explicitly verify the 4-summability of D . As one can easily check, the dimension of V_l is [CH96]

$$\dim V_l = \frac{2}{3}(l + \frac{5}{2})(l + \frac{3}{2})(l + \frac{1}{2}) .$$

From this we get

$$\mathrm{Tr}(|D|^{-s}) = \sum_{l \in \mathbb{N} + \frac{1}{2}} 2(l + \frac{3}{2})^{-s} \dim V_l = \frac{4}{3} \sum_{n=1}^{\infty} (n^2 - 1)n^{-s+1},$$

where $n = l + \frac{3}{2}$ (and we added the term with $n = 1$ since it is identically zero). The above series is convergent in the right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re} s > 4\}$, thus D has metric dimension 4.

Notice that $\mathrm{Tr}(|D|^{-s})$ has meromorphic extension on \mathbb{C} given by

$$\zeta_1(s) = \frac{4}{3} \{ \zeta(s-3) - \zeta(s-1) \}, \quad (4.26)$$

where $\zeta(s)$ is the Riemann zeta-function. We recall that $\zeta(s)$ has a simple pole in $s = 1$ as unique singularity and that $\mathrm{Res}_{s=1} \zeta(s) = 1$.

4.6 The dimension spectrum and residues

Let \mathcal{J}_2 be the two-sided ideal in $\mathcal{B}(\mathcal{H})$ defined in (2.15), with $r = 2$. We recall here the definition:

$$\mathcal{J}_2 := \{ T \in \mathrm{OP}^0 \mid T|D|^{-p} \in \mathcal{L}^1(\mathcal{H}) \forall p > 2 \}.$$

Clearly, if T is of trace class, so is $T|D|^{-p}$ for any positive p , and $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{J}_2$. Since $\mathrm{OP}^{-\infty} \subset \mathcal{L}^1(\mathcal{H})$, smoothing operators belongs to \mathcal{J}_2 as well. On the other hand, \mathcal{J}_2 is strictly bigger than $\mathcal{L}^1(\mathcal{H})$; indeed, the operator $L_q \in \mathcal{B}(\mathcal{H})$, given by

$$L_q |l, m_1, m_2; j\rangle_{\pm} := q^{j+\frac{1}{2}} |l, m_1, m_2; j\rangle_{\pm},$$

is not of trace class but belongs to \mathcal{J}_2 , by the following proposition.

Proposition 4.14. *For any $s \in \mathbb{C}$ with $\mathrm{Re} s > 2$ one has that*

$$\zeta_{L_q}(s) := \sum_{l, j, m_1, m_2} (l + \frac{3}{2})^{-s} q^{j+\frac{1}{2}} = \frac{4q}{(1-q)^2} \left(\zeta(s-1) - \frac{1+q}{1-q} \zeta(s) \right) + \text{holomorphic function},$$

where $\zeta(s)$ is the Riemann zeta-function. In particular, this means that L_q belongs to the ideal \mathcal{J}_2 . Furthermore, since the series $\zeta_{L_q}(0)$ is divergent, L_q is not of trace class.

Proof. Calling $n := l + \frac{3}{2}$, $k := j + \frac{1}{2}$, we have

$$\zeta_{L_q}(s) = 4 \sum_{n=2}^{\infty} n^{-s} \sum_{k=1}^{n-1} k(n-k)q^k.$$

We can sum starting from $n = 1$ and for $k = 0, \dots, n$ (we simply add zero terms) to get

$$\zeta_{L_q}(s) = 4 \sum_{n=1}^{\infty} n^{-s} \{ nq\partial_q - (q\partial_q)^2 \} \sum_{k=0}^n q^k = 4 \sum_{n=1}^{\infty} n^{-s} \{ nq\partial_q - (q\partial_q)^2 \} \frac{1 - q^{n+1}}{1 - q}.$$

Terms decaying as q^n give a holomorphic function of s , thus modulo holomorphic functions,

$$\zeta_{L_q}(s) \sim 4 \sum_{n=1}^{\infty} n^{-s} \left\{ n \frac{q}{(1-q)^2} - \frac{q(1+q)}{(1-q)^3} \right\}.$$

Last series is summable for all s with $\mathrm{Re} s > 2$, and its sum can be written in terms of the Riemann zeta-function as in the statement of the proposition. \square

4.6.1 An approximated representation

Let $\hat{\mathcal{H}}$ be a Hilbert space with orthonormal basis $\|l, m_1, m_2; j\rangle_{\pm}$ labelled by,

$$l \in \frac{1}{2}\mathbb{Z}, \quad l + j \in \mathbb{Z}, \quad j + m_1 \in \mathbb{N}, \quad l + \frac{1}{2} - j + m_2 \in \mathbb{N}.$$

Let I be the labelling set of the Hilbert space \mathcal{H}_{\pm} as in Theorem 4.11, and given by

$$I := \left\{ (l, m_1, m_2, j) \mid l \in \mathbb{N} + \frac{1}{2}, \quad j = \frac{1}{2}, \frac{3}{2}, \dots, l, \quad j - |m_1| \in \mathbb{N}, \quad l + \frac{1}{2} - j - |m_2| \in \mathbb{N} \right\}.$$

Notice that I is the subset of labels of $\hat{\mathcal{H}}$ satisfying $l \in \mathbb{N} + \frac{1}{2}$, $m_1 \leq j \leq l$ and $m_2 \leq l + \frac{1}{2} - j$. Define the inclusion $Q : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ and the adjoint projection $P : \hat{\mathcal{H}} \rightarrow \mathcal{H}$ by,

$$\begin{aligned} Q \|l, m_1, m_2; j\rangle_{\pm} &:= \|l, m_1, m_2; j\rangle_{\pm} \quad \text{for all } (l, m_1, m_2, j) \in I, \\ P \|l, m_1, m_2; j\rangle_{\pm} &:= \begin{cases} \|l, m_1, m_2; j\rangle_{\pm} & \text{if } (l, m_1, m_2, j) \in I, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, $PQ = id_{\mathcal{H}}$. The Hilbert space $\hat{\mathcal{H}}$ carries a bounded $*$ -representation of the algebra $\mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2)$ defined by,

$$\begin{aligned} \alpha \|l, m_1, m_2; j\rangle_{\pm} &= \sqrt{1 - q^{2(j+m_1+1)}} \|l + \frac{1}{2}, m_1 + \frac{1}{2}, m_2; j + \frac{1}{2}\rangle_{\pm}, \\ \beta \|l, m_1, m_2; j\rangle_{\pm} &= q^{j+m_1} \|l + \frac{1}{2}, m_1 - \frac{1}{2}, m_2; j + \frac{1}{2}\rangle_{\pm}, \\ A \|l, m_1, m_2; j\rangle_{\pm} &= q^{l-j+m_2-\epsilon} \|l, m_1, m_2; j\rangle_{\pm}, \\ B \|l, m_1, m_2; j\rangle_{\pm} &= \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}} \|l + 1, m_1, m_2 + 1; j\rangle_{\pm}, \end{aligned}$$

where, as before, $\epsilon := \frac{1}{2}(-1)^{l+\frac{1}{2}-j-m_2}$. Composition of such a representation with the algebra embedding $\mathcal{A}(S_q^4) \hookrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2)$ given in equation (4.7) results into a $*$ -representation $\pi : \mathcal{A}(S_q^4) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$. The sandwich $\tilde{\pi}(a) := P\pi(a)Q$ defines a $*$ -linear map $\tilde{\pi} : \mathcal{A}(S_q^4) \rightarrow \mathcal{B}(\mathcal{H})$.

Proposition 4.15. *With \mathcal{J}_2 the class of operators defined in equation (2.15), one has that the difference $a - \tilde{\pi}(a) \in \mathcal{J}_2$ for all $a \in \mathcal{A}(S_q^4)$.*

Proof. Define $\hat{\mathcal{J}}_2$ as the collection of bounded operators $T : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ such that $T|D|^{-p}$ is trace class for all $p > 2$. Since trace class operators are a two sided ideal in bounded operators, the space $\hat{\mathcal{J}}_2$ is stable when multiplied from the left by bounded operators: $T_1 \in \mathcal{B}(\hat{\mathcal{H}})$ and $T_2 \in \hat{\mathcal{J}}_2 \Rightarrow T_1 T_2 \in \hat{\mathcal{J}}_2$.

Next, suppose that a, b satisfy $a - \tilde{\pi}(a) \in \mathcal{J}_2$ and $b - \tilde{\pi}(b) \in \mathcal{J}_2$ and consider the following algebraic identity

$$ab - \tilde{\pi}(ab) = \{a - \tilde{\pi}(a)\}b + P\pi(a)\{Qb - \pi(b)Q\}.$$

Since \mathcal{J}_2 is a two-sided ideal in OP^0 , the first summand is in \mathcal{J}_2 . The stability of $\hat{\mathcal{J}}_2$ discussed above implies that $\pi(a)\{Qb - \pi(b)Q\} \in \hat{\mathcal{J}}_2$, but if $T \in \hat{\mathcal{J}}_2$ clearly $PT \in \mathcal{J}_2$. Hence the second summand lies in \mathcal{J}_2 too. Thus, $ab - \tilde{\pi}(ab) \in \mathcal{J}_2$ whenever this property holds for each of the

operators a, b . We conclude that it is enough to show that $a - \tilde{\pi}(a) \in \mathcal{J}_2$ when a is a generator of $\mathcal{A}(S_q^4)$. By Proposition 4.14, this amounts to prove that the matrix elements of $a - \tilde{\pi}(a)$ are bounded in modulus by $q^{j+\frac{1}{2}}$.

Let us have a close look at the coefficients of $a \in \{x_i, x_i^*\}$ in Theorem 4.11. Firstly, $A_{j,m_1}^+, B_{j,m_1}^+, B_{j,m_1}^-, q^{-2j}A_{j,m_1}^0$ and $q^{-2j}B_{j,m_1}^0$ are uniformly bounded by a constant, as one can see by writing explicitly the q -analogues in their expressions so getting:

$$\begin{aligned} A_{j,m_1}^+ &= q^{j+m_1} \frac{\sqrt{(1-q^{2(j+m_1+1)})(1-q^{2(j-m_1+1)})}}{1-q^{4j+4}}, \\ q^{-2j}A_{j,m_1}^0 &= (1-q^2) \frac{[2]q^{2(j+m_1)} - q^{4j+1} - q^{-1}}{(1-q^{4j})(1-q^{4j+4})}, \\ B_{j,m_1}^+ &= \frac{\sqrt{(1-q^{2(j+m_1+1)})(1-q^{2(j+m_1+2)})}}{1-q^{4j+4}}, \\ q^{-2j}B_{j,m_1}^0 &= (1+q^2)q^{j+m_1+1} \frac{\sqrt{(1-q^{2(j-m_1)})(1-q^{2(j+m_1+1)})}}{(1-q^{4j})(1-q^{4j+4})}, \\ B_{j,m_1}^- &= -q^{2(j+m_1)+1} \frac{\sqrt{(1-q^{2(j-m_1)})(1-q^{2(j-m_1-1)})}}{1-q^{4j}}. \end{aligned}$$

Analogously, the coefficients $q^{2j}H_{l,j,m_2}^0, C_{l,j,m_2}^0$ and D_{l,j,m_2}^0 are seen to be bounded by q^l . Thus, modulo rapid decay matrices (i.e. smoothing operators),

$$\begin{aligned} x_0 |l, m_1, m_2; j\rangle &\simeq A_{j,m_1}^+ C_{l,j,m_2}^+ |l+1, m_1, m_2; j+1\rangle \\ &\quad + A_{j,m_1}^+ C_{l,j,m_2}^- |l-1, m_1, m_2; j+1\rangle \\ &\quad + A_{j,m_1}^0 H_{l,j,m_2}^+ |l+1, m_1, m_2; j\rangle \\ &\quad + A_{j,m_1}^0 H_{l-1,j,m_2}^+ |l-1, m_1, m_2; j\rangle \\ &\quad + A_{j-1,m_1}^+ C_{l+1,j-1,m_2}^- |l+1, m_1, m_2; j-1\rangle \\ &\quad + A_{j-1,m_1}^+ C_{l-1,j-1,m_2}^+ |l-1, m_1, m_2; j-1\rangle, \end{aligned} \tag{4.27a}$$

$$\begin{aligned} x_1 |l, m_1, m_2; j\rangle &\simeq B_{j,m_1}^+ C_{l,j,m_2}^+ |l+1, m_1+1, m_2; j+1\rangle \\ &\quad + B_{j,m_1}^+ C_{l,j,m_2}^- |l-1, m_1+1, m_2; j+1\rangle \\ &\quad + B_{j,m_1}^0 H_{l,j,m_2}^+ |l+1, m_1+1, m_2; j\rangle \\ &\quad + B_{j,m_1}^0 H_{l-1,j,m_2}^+ |l-1, m_1+1, m_2; j\rangle \\ &\quad + B_{j,m_1}^- C_{l+1,j-1,m_2}^- |l+1, m_1+1, m_2; j-1\rangle \\ &\quad + B_{j,m_1}^- C_{l-1,j-1,m_2}^+ |l-1, m_1+1, m_2; j-1\rangle, \end{aligned} \tag{4.27b}$$

$$\begin{aligned} x_2 |l, m_1, m_2; j\rangle &\simeq D_{l,j,m_2}^+ |l+1, m_1, m_2+1; j\rangle \\ &\quad + D_{l,j,m_2}^- |l-1, m_1, m_2+1; j\rangle. \end{aligned} \tag{4.27c}$$

Since modulo smoothing operators the representations are the same we are omitting the label ‘ \pm ’ in the vector basis. Furthermore, using the inequalities

$$0 \leq (1-qu)^{-1} - 1 \leq q(1-q)^{-1}u, \quad 0 \leq 1 - (1-u)^{\frac{1}{2}} \leq u, \tag{4.28}$$

which are valid when $0 \leq u \leq 1$, we prove that modulo terms bounded by q^l , one has

$$C_{l,j,m_2}^+ \simeq -q^{l-j+m_2-\epsilon} \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}} , \quad (4.29a)$$

$$C_{l,j,m_2}^- \simeq -q^{l+j+m_2+1+\epsilon} \sqrt{1 - q^{2(l-j+m_2-\epsilon)}} , \quad (4.29b)$$

$$H_{l,j,m_2}^+ \simeq q^{l+m_2+1} \sqrt{q^{2\epsilon(2j+1)} - q^{2(l+m_2+2)}} , \quad (4.29c)$$

$$D_{l,j,m_2}^+ \simeq \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}} \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}} , \quad (4.29d)$$

$$D_{l,j,m_2}^- \simeq -q^{2(l+m_2)+3} . \quad (4.29e)$$

Up to now, we neglected only smoothing contributions (the above approximation will be needed when dealing with the real structure later on). We use again (4.28) to get a rougher approximation by neglecting terms bounded by q^j . This yields

$$A_{j,m_1}^+ \simeq \tilde{A}_{j,m_1}^+ := q^{j+m_1} \sqrt{1 - q^{2(j+m_1+1)}} , \quad (4.30a)$$

$$A_{j,m_1}^0 H_{l,j,m_2}^+ \simeq 0 , \quad (4.30b)$$

$$B_{j,m_1}^+ \simeq \tilde{B}_{j,m_1}^+ := \sqrt{(1 - q^{2(j+m_1+1)})(1 - q^{2(j+m_1+2)})} , \quad (4.30c)$$

$$B_{j,m_1}^0 H_{l,j,m_2}^+ \simeq 0 , \quad (4.30d)$$

$$B_{j,m_1}^- \simeq \tilde{B}_{j,m_1}^- := -q^{2(j+m_1)+1} , \quad (4.30e)$$

$$C_{l,j,m_2}^+ \simeq \tilde{C}_{l,j,m_2} := -q^{l-j+m_2-\epsilon} , \quad (4.30f)$$

$$C_{l,j,m_2}^- \simeq 0 , \quad (4.30g)$$

$$D_{l,j,m_2}^+ \simeq \tilde{D}_{l,j,m_2} := \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}} , \quad (4.30h)$$

$$D_{l,j,m_2}^- \simeq 0 . \quad (4.30i)$$

Plugging these coefficients in the equations for the x_i 's we see that, modulo operators in the ideal \mathcal{J}_2 , we get

$$\begin{aligned} x_0 |l, m_1, m_2; j\rangle &\simeq \tilde{A}_{j,m_1}^+ \tilde{C}_{l,j,m_2} |l+1, m_1, m_2; j+1\rangle \\ &\quad + \tilde{A}_{j-1,m_1}^+ \tilde{C}_{l-1,j-1,m_2} |l-1, m_1, m_2; j-1\rangle \\ &= -P(\alpha\beta A)Q |l, m_1, m_2; j\rangle - P(\beta^* \alpha^* A)Q |l, m_1, m_2; j\rangle , \\ x_1 |l, m_1, m_2; j\rangle &\simeq \tilde{B}_{j,m_1}^+ \tilde{C}_{l,j,m_2} |l+1, m_1+1, m_2; j+1\rangle \\ &\quad + \tilde{B}_{j,m_1}^- \tilde{C}_{l-1,j-1,m_2} |l-1, m_1+1, m_2; j-1\rangle \\ &= -P(\alpha^2 A)Q |l, m_1, m_2; j\rangle + P(q(\beta^*)^2 A)Q |l, m_1, m_2; j\rangle , \\ x_2 |l, m_1, m_2; j\rangle &\simeq \tilde{D}_{l,j,m_2} |l+1, m_1, m_2+1; j\rangle \\ &= PBQ |l+1, m_1, m_2+1; j\rangle . \end{aligned}$$

The observation that

$$-P(\alpha\beta + \beta^* \alpha^*)AQ = \tilde{\pi}(x_0) , \quad P(-\alpha^2 + q(\beta^*)^2)AQ = \tilde{\pi}(x_1) , \quad PBQ = \tilde{\pi}(x_2) ,$$

concludes the proof. \square

4.6.2 The dimension spectrum and the top residue

The approximation modulo \mathcal{J}_2 allows considerable simplifications when getting information on the part of the dimension spectrum contained in the half plane $\operatorname{Re} s > 2$. To study the part of the dimension spectrum in the left half plane $\operatorname{Re} s \leq 2$ would require a less drastic approximation which we are lacking at the moment.

Proposition 4.16. *In the region $\operatorname{Re} s > 2$ the dimension spectrum Σ of the spectral triple $(\mathcal{A}(S_q^4), \mathcal{H}, D, \gamma)$ given in Proposition 4.13 consists of the two points $\{3, 4\}$, which are simple poles of the zeta-functions. The top residue coincides with the integral on the subspace of classical points of S_q^4 , that is*

$$\int a|D|^{-4} = \frac{2}{3\pi} \int_0^{2\pi} \sigma(a)(\theta) d\theta, \quad (4.31)$$

with $\sigma : \mathcal{A}(S_q^4) \rightarrow \mathcal{A}(S^1)$ the $*$ -algebra morphism defined by $\sigma(x_0) = \sigma(x_1) = 0$ and $\sigma(x_2) = u$, where u , given by $u(\theta) := e^{i\theta}$, is the unitary generator of $\mathcal{A}(S^1)$.

Proof. Let Ψ^0 be the $*$ -algebra generated by $\mathcal{A}(S_q^4)$, by $[D, \mathcal{A}(S_q^4)]$ and by iterated applications of the derivation δ . Let $\mathfrak{A} \subset \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2) \otimes \operatorname{Mat}_2(\mathbb{C})$ be the $*$ -algebra generated by $\alpha, \beta, \alpha^*, \beta^*, A, B, B^*$ and F . By Proposition 4.15 there is an inclusion $\mathcal{A}(S_q^4) \subset P\mathfrak{A}Q + \mathcal{J}_2$.

A linear basis for \mathfrak{A} is given by,

$$T := \alpha^{k_1} \beta^{n_1} (\beta^*)^{n_2} A^{n_3} B^{k_2} F^h, \quad (4.32)$$

where $h \in \{0, 1\}$, $n_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$ and with the notation $\alpha^{k_1} := (\alpha^*)^{-k_1}$ if $k_1 < 0$ and $B^{k_2} := (B^*)^{-k_2}$ if $k_2 < 0$. For this operator,

$$\delta(PTQ) = \left(\frac{1}{2}(k_1 + n_1 - n_2) + k_2\right)PTQ \quad \text{and} \quad [D, PTQ] = \delta(PTQ)F.$$

Thus, $P\mathfrak{A}Q$ is invariant under application of δ and $[D, \cdot]$ and hence $\Psi^0 \subset P\mathfrak{A}Q + \mathcal{J}_2$.

For the part of the dimension spectrum in the right half plane $\operatorname{Re} s > 2$, we can neglect \mathcal{J}_2 and consider only the singularities of zeta-functions associated with elements in $P\mathfrak{A}Q$. By linearity of the zeta-functions, it is enough to consider the generic basis element in equation (4.32).

Such a T shifts l by $\frac{1}{2}(k_1 + n_1 - n_2) + k_2$, m_1 by $\frac{1}{2}(k_1 - n_1 + n_2)$, m_2 by k_2 , j by $\frac{1}{2}(k_1 + n_1 - n_2)$, and flips the chirality if $h = 1$. Thus it is off-diagonal unless $h = k_i = 0$ and $n_1 = n_2$. The zeta-function associated with a bounded off-diagonal operator is identically zero in the half-plane $\operatorname{Re} z > 4$, and so is its holomorphic extension to the entire complex plane. It remains to consider the cases $T = P(\beta\beta^*)^k A^n Q$, with $n, k \in \mathbb{N}$.

If n and k are both different from zero, one finds

$$\zeta_T(s) = 2 \sum_{l,j,m_1,m_2} \left(l + \frac{3}{2}\right)^{-s} q^{n(l-j+m_2-\epsilon)+2k(j+m_1)} = 2 \sum_{l,j,m_2} \left(l + \frac{3}{2}\right)^{-s} q^{n(l-j+m_2-\epsilon)} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}}.$$

For ϵ fixed, set $2i := l - \epsilon - j + m_2 = 0, 2, \dots, 2(l - j)$. Then,

$$\begin{aligned} \zeta_T(s) &= 2 \sum_{l,j} (l + \frac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \sum_{\epsilon=\pm 1/2} \sum_{i=0}^{l-j} q^{2ni} = 4 \sum_{l,j} (l + \frac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \frac{1 - q^{2n(l-j+1)}}{1 - q^{2n}} \\ &= 4\zeta(s-1) - 4 \frac{1 + (1 - q^{4k})^{-1} + (1 - q^{2n})^{-1}}{(1 - q^{2k})(1 - q^{2n})} \zeta(s) + \text{holomorphic function} , \end{aligned}$$

which has meromorphic extension on \mathbb{C} with simple pole in $s = \{1, 2\}$.

If $n = 0$ and $k \neq 0$,

$$\begin{aligned} \zeta_T(s) &= 4 \sum_{l,j} (l + \frac{3}{2})^{-s} (l - j + 1) \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \\ &= \frac{4}{1 - q^{2k}} \left(\frac{1}{2} \zeta(s-2) - \left(\frac{1}{2} + \frac{1}{1 - q^{4k}} \right) \zeta(s-1) + \frac{q^{4k}}{(1 - q^{4k})^2 \log q^{4k}} \zeta(s) \right) + \text{hol. function} , \end{aligned}$$

which has meromorphic extension on \mathbb{C} with simple pole in $s = \{1, 2, 3\}$.

If $n \neq 0$ and $k = 0$,

$$\begin{aligned} \zeta_T(s) &= 4 \sum_{l,j} (l + \frac{3}{2})^{-s} (2j + 1) \frac{1 - q^{2n(l-j+1)}}{1 - q^{2n}} \\ &= \frac{4}{1 - q^{2n}} \left\{ \zeta(s-2) - \left(1 + \frac{2q^{2n}}{1 - q^{2n}} \right) \zeta(s-1) + \frac{2q^{2n}}{1 - q^{2n}} \left(1 + \frac{q^{2n}}{(1 - q^{2n}) \log q^{2n}} \right) \zeta(s) \right\} + \text{hol. fun.} , \end{aligned}$$

which has meromorphic extension on \mathbb{C} with simple pole in $s = \{1, 2, 3\}$.

Finally, if both n and k are zero we get (cf. equation (4.26)), $\zeta_T(s) = \frac{4}{3} \{ \zeta(s-3) - \zeta(s-1) \}$, and this is meromorphic with simple poles in $\{2, 4\}$. Thus, the part of the dimension spectrum in the region $\text{Re } s > 2$ consists at most of the two points $\{3, 4\}$ and both are simple poles.

Since we have considered the enlarged algebra $P\mathfrak{A}Q + \mathcal{J}_2$, it suffices to prove that there exists an $a \in \Psi^0$ whose zeta-function is singular in both points $s = 3$ and $s = 4$. We take $a = x_2 x_2^*$.

From the definition

$$\tilde{\pi}(x_2 x_2^* | l, m_1, m_2; j)_{\pm} = (1 - q^{2(l-\epsilon-j+m_2)}) | l, m_1, m_2; j)_{\pm} .$$

Then, modulo functions that are holomorphic when $\text{Re } s > 2$, we have

$$\zeta_{x_2 x_2^*}(s) \sim \zeta_{\tilde{\pi}(x_2 x_2^*)}(s) = \zeta_1(s) - 2 \sum_{l,j,m_1,m_2} (l + \frac{3}{2})^{-s} q^{2(l-\epsilon-j+m_2)} \sim \frac{4}{3} \zeta(s-3) - \frac{4}{1 - q^4} \zeta(s-2) .$$

This proves the first part of the proposition, that is $\Sigma \cap \{ \text{Re } s > 2 \} = \{3, 4\}$.

The proof of equation (4.31) is based on the observation that the residue in $s = 4$ of ζ_T , for T a basis element of $P\mathfrak{A}Q$, is zero unless $T = 1$. That is, it depends only on the image of T under the map sending β, A and F to 0 while $\alpha \mapsto e^{i\phi}$ and $B \mapsto e^{i\theta}$. Composing this map with $\tilde{\pi}$ we get the morphism $\sigma : \mathcal{A}(S_q^4) \rightarrow \mathcal{A}(S^1)$ of the proposition and that

$$\int a |D|^{-4} \propto \int_0^{2\pi} \sigma(a) d\theta .$$

The equality $\int |D|^{-4} = \frac{4}{3}$ fixes the proportionality constant. □

4.7 The real structure

Classically, if $(\mathcal{A}(M), \mathcal{H}, D, \gamma)$ is the canonical spectral triple associated with a 4-dimensional spin manifold M , there exists an antilinear isometry J on \mathcal{H} , named the *real structure*, satisfying the following compatibility condition

$$J^2 = -1, \quad J\gamma = \gamma J, \quad JD = DJ. \quad (4.33)$$

There are also two additional conditions involving the coordinate algebra $\mathcal{A}(M)$:

$$[a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0, \quad \forall a, b \in \mathcal{A}(M). \quad (4.34)$$

The real structure on S^4 is equivariant and equivariance is sufficient to determine J .

In the deformed situation one has to be careful on how to implement equivariance. Let us start with the working hypothesis that equivariance for J is the requirement that it satisfies $Jh = S(h)^*J$ for all $h \in U_q(\mathfrak{so}(5))$. Then, consider the Casimir operator \mathcal{C}_1 given in equation (4.4). This operator commutes with J since $S(\mathcal{C}_1)^* = \mathcal{C}_1$ and from its expression, $\mathcal{C}_1 |l, m_1, m_2; j\rangle = (q^{2j+1} + q^{-2j-1}) |l, m_1, m_2; j\rangle$, we conclude that J leaves the index j invariant. Compatibility with γ and D in equation (4.33) and equivariance with respect to $h = K_1$ and $h' = K_2$ yields

$$J |l, m_1, m_2; j\rangle_{\pm} = c_{\pm}(l, m_1, m_2; j) |l, -m_1, -m_2; j\rangle_{\pm},$$

with some constants c_{\pm} to be determined. Equivariance with respect to $h = E_1$ implies

$$c_{\pm}(l, m_1, m_2; j) = (-1)^{m_1+1/2} q^{m_1} c_{\pm}(l, m_2; j).$$

For $h = E_2$, looking at the piece diagonal in j we deduce that the dependence on m_2 is through a factor q^{3m_2} ; and looking at the piece shifting j by ± 1 we conclude that

$$c_{\pm}(l, m_1, m_2; j) = (-1)^{j+m_1} q^{m_1+3m_2} c_{\pm}(l).$$

Such an operator J cannot be antiunitary unless $q = 1$. At $q = 1$ the antiunitarity condition requires that $c_{\pm}(l) \in U(1)$ and modulo a unitary equivalence we can choose $c_{\pm}(l) = i^{2l+1}$. In conclusion for $q = 1$ the operator

$$J |l, m_1, m_2; j\rangle_{\pm} = i^{2l+1} (-1)^{j+m_1} |l, -m_1, -m_2; j\rangle_{\pm}, \quad (4.35)$$

is *the* real structure on S^4 (modulo a unitary equivalence).

For $q \neq 1$ we keep (4.35) as the real structure and notice that conditions (4.33) are satisfied, but J no longer satisfies the requirement $Jh = S(h)^*J$ for all $h \in U_q(\mathfrak{so}(5))$. Nevertheless, J is the antiunitary part of an antilinear operator T that has this property. The antilinear operator T defined by

$$T |l, m_1, m_2; j\rangle_{\pm} = i^{2l+1} (-1)^{j+m_1} q^{m_1+3m_2} |l, -m_1, -m_2; j\rangle_{\pm},$$

has the J in (4.35) as antiunitary part and it is *equivariant*, i.e. it is such that $Th = S(h)^*T$ for all $h \in U_q(\mathfrak{so}(5))$.

Next, we turn to the conditions (4.34). In parallel with the cases of the manifold of $SU_q(2)$ in [DLS⁺05a] and of Podleś spheres in [DLPS05, DDLW07], once again we need to modify them. For instance, the commutator $[x_2, Jx_2J]$ is not zero, as one can see by computing the matrix element

$$\begin{aligned} f(l, j, m_2) &:= \pm \langle l+1, m_1, m_2; j | [x_2, Jx_2J] | l, m_1, m_2; j \rangle_{\pm} \\ &= D_{l+1, j, m_2-1}^0 D_{l, j, -m_2}^+ - D_{l, j, m_2-1}^+ D_{l, j, -m_2}^0 + D_{l+1, j, -m_2-1}^0 D_{l, j, m_2}^+ - D_{l, j, -m_2-1}^+ D_{l, j, m_2}^0, \end{aligned}$$

which for $j = \frac{1}{2}$ and $m_2 = l$ is

$$f(l, \frac{1}{2}, l) = -q^{-l-4}(1-q^2)^2 [2] \frac{(q^{l-1} + q^{-l+1}) \sqrt{[2l+3]} [l+1][l+2][l+3]}{[2l+2][2l+4]^2 [2l+6]} \neq 0.$$

It is relatively easy to prove that the two conditions are satisfied modulo the ideal \mathcal{J}_2 . It is much more cumbersome computationally to show that they are in fact satisfied modulo the smaller ideal of smoothing operators.

Proposition 4.17. *Let J be the antilinear isometry given by (4.35). Then,*

$$[a, JbJ] \in \mathcal{J}_2, \quad [[D, a], JbJ] \in \mathcal{J}_2, \quad \forall a, b \in \mathcal{A}(S_q^4).$$

Proof. We lift J and D to the Hilbert space $\hat{\mathcal{H}}$ defined in Section 4.6.1, as follows:

$$\begin{aligned} \hat{J} ||l, m_1, m_2; j\rangle_{\pm} &= i^{2l+1} (-1)^{j+m_1} ||l, -m_1, -m_2; j\rangle_{\pm}, \\ \hat{D} ||l, m_1, m_2; j\rangle_{\pm} &= (l + \frac{3}{2}) ||l, m_1, m_2; j\rangle_{\mp}. \end{aligned}$$

Notice that $\hat{J}^2 = -1$ on $\hat{\mathcal{H}}$ (thanks to the phase i^{2l+1} that is irrelevant when restricted to \mathcal{H}). Let now $\{\alpha, \beta, \alpha^*, \beta^*, A, B, B^*\}$ be the operators defined in Section 4.6.1, generators of the algebra $\mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2)$. Due to Proposition 4.15 it is enough to prove that for all pairs (a, b) of such generators, the commutators $[a, \hat{J}b\hat{J}]$ and $[[\hat{D}, a], \hat{J}b\hat{J}]$ are weighted shifts with weight which are bounded by q^{2j} . From

$$[\hat{D}, \alpha] = \frac{1}{2} \alpha \hat{F}, \quad [\hat{D}, \beta] = \frac{1}{2} \beta \hat{F}, \quad [\hat{D}, A] = 0, \quad [\hat{D}, B] = B \hat{F},$$

the condition on $[[\hat{D}, a], \hat{J}b\hat{J}]$ follows from the same condition on $[a, \hat{J}b\hat{J}]$, and we have to compute only the latter commutators.

Since $[a, \hat{J}b^*\hat{J}] = -[a^*, \hat{J}b\hat{J}]^*$ and $[b, \hat{J}a\hat{J}] = \hat{J}[a, \hat{J}b\hat{J}]\hat{J}$, we have to check the 16 combinations in the following table.

$b \backslash a$	α	α^*	β	β^*	A	B	B^*
α	•	•	×	×	•	•	•
β			•	×	•	•	•
B					•	•	•
A					•		

By direct computations one shows that bullets in the table correspond to vanishing commutators. On the other hand, the commutators corresponding to the crosses in the table are given, on the subspace with $j - |m_1| \in \mathbb{N}$, by

$$\begin{aligned} [\beta^*, \hat{J}\alpha\hat{J}] \llbracket l, m_1, m_2; j \rrbracket_{\pm} &= q^{j+m_1} \left\{ \sqrt{1 - q^{2(j-m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \right\} \llbracket l, m_1, m_2; j \rrbracket_{\pm} \\ [\beta, \hat{J}\alpha\hat{J}] \llbracket l, m_1, m_2; j \rrbracket_{\pm} &= -[\beta^*, \hat{J}\alpha\hat{J}] \llbracket l+1, m_1-1, m_2; j+1 \rrbracket_{\pm}, \\ [\beta^*, \hat{J}\beta\hat{J}] \llbracket l, m_1, m_2; j \rrbracket_{\pm} &= -[2]q^{2j} \llbracket l, m_1+1, m_2; j \rrbracket_{\pm}. \end{aligned}$$

Since $1 - u \leq \sqrt{1 - u} \leq 1$ for all $u \in [0, 1]$, we have that

$$0 \leq q^{j+m_1} \left\{ \sqrt{1 - q^{2(j-m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \right\} \leq q^{j+m_1} (1 - 1 + q^{2(j-m_1)}) \leq q^{2j}.$$

Then, all three non-zero commutators are weighted shifts with weights bounded by q^{2j} . \square

Proposition 4.18. *Let J be the antilinear isometry given by (4.35). Then,*

$$[a, JbJ] \in \text{OP}^{-\infty}, \quad [[D, a], JbJ] \in \text{OP}^{-\infty}, \quad \forall a, b \in \mathcal{A}(S_q^4).$$

Proof. By Leibniz rule, it is sufficient to prove the statement when a and b are generators of the algebra. The operators $[D, a] - \delta(a)F = |D|[F, a]$ are smoothing. Thus, it is enough to show that

$$[a, JbJ] \in \text{OP}^{-\infty}, \quad [\delta(a), JbJ] \in \text{OP}^{-\infty}, \quad (4.36)$$

for any pair (a, b) of generators. From

$$[b, JaJ] = J[a, JbJ]J, \quad [\delta(b), JaJ] = -J[\delta(a), JbJ]J + \delta([b, JaJ]),$$

it follows that if (4.36) is satisfied for a particular pair (a, b) , then it is satisfied for (b, a) too. From

$$[a^*, Jb^*J] = -[a, JbJ]^*, \quad [\delta(a^*), b^*] = [\delta(a), b]^*, \quad (4.37)$$

we see that if (4.36) is satisfied for a pair (a, b) , then it is satisfied for (a^*, b^*) too. With these symmetries we need to check only the following 9 cases out of 25:

$b \backslash a$	x_0	x_1	x_1^*	x_2	x_2^*
x_0	•	•		•	
x_1		•	•	•	•
x_2				•	•

From equations (4.27) and (4.29) we see that modulo smoothing operators

$$\begin{aligned} \frac{1}{2} \{x_0 + \delta(x_0)\} \llbracket l, m_1, m_2; j \rrbracket &\simeq \boxed{A_{j, m_1}^+ \hat{C}_{l, j, m_2}^+ \llbracket l+1, m_1, m_2; j+1 \rrbracket} \\ &+ q^{-2j} A_{j, m_1}^0 \hat{H}_{l, j, m_2}^+ \llbracket l+1, m_1, m_2; j \rrbracket \\ &+ A_{j-1, m_1}^+ \hat{C}_{l+1, j-1, m_2}^- \llbracket l+1, m_1, m_2; j-1 \rrbracket, \end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\{x_0 - \delta(x_0)\} |l, m_1, m_2; j\rangle &\simeq A_{j,m_1}^+ \hat{C}_{l,j,m_2}^- |l-1, m_1, m_2; j+1\rangle \\
&+ q^{-2j} A_{j,m_1}^0 \hat{H}_{l-1,j,m_2}^+ |l-1, m_1, m_2; j\rangle \\
&+ \boxed{A_{j-1,m_1}^+ \hat{C}_{l-1,j-1,m_2}^+ |l-1, m_1, m_2; j-1\rangle}, \\
\frac{1}{2}\{x_1 + \delta(x_1)\} |l, m_1, m_2; j\rangle &\simeq \boxed{B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ |l+1, m_1+1, m_2; j+1\rangle} \\
&+ q^{-2j} B_{j,m_1}^0 \hat{H}_{l,j,m_2}^+ |l+1, m_1+1, m_2; j\rangle \\
&+ B_{j,m_1}^- \hat{C}_{l+1,j-1,m_2}^- |l+1, m_1+1, m_2; j-1\rangle, \\
\frac{1}{2}\{x_1 - \delta(x_1)\} |l, m_1, m_2; j\rangle &\simeq B_{j,m_1}^+ \hat{C}_{l,j,m_2}^- |l-1, m_1+1, m_2; j+1\rangle \\
&+ q^{-2j} B_{j,m_1}^0 \hat{H}_{l-1,j,m_2}^+ |l-1, m_1+1, m_2; j\rangle \\
&+ \boxed{B_{j,m_1}^- \hat{C}_{l-1,j-1,m_2}^+ |l-1, m_1+1, m_2; j-1\rangle}, \\
\frac{1}{2}\{x_2 + \delta(x_2)\} |l, m_1, m_2; j\rangle &\simeq \boxed{\boxed{\hat{D}_{l,j,m_2}^+ |l+1, m_1, m_2+1; j\rangle}}, \\
\frac{1}{2}\{x_2 - \delta(x_2)\} |l, m_1, m_2; j\rangle &\simeq \hat{D}_{l,j,m_2}^- |l-1, m_1, m_2+1; j\rangle.
\end{aligned}$$

where

$$\hat{C}_{l,j,m_2}^+ = -q^{l-j+m_2-\epsilon} \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}}, \quad (4.38a)$$

$$\hat{C}_{l,j,m_2}^- = -q^{l+j+m_2+1+\epsilon} \sqrt{1 - q^{2(l-j+m_2-\epsilon)}}, \quad (4.38b)$$

$$\hat{H}_{l,j,m_2}^+ = q^{2j} q^{l+m_2+1} \sqrt{q^{2\epsilon(2j+1)} - q^{2(l+m_2+2)}}, \quad (4.38c)$$

$$\hat{D}_{l,j,m_2}^+ = \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}} \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}}, \quad (4.38d)$$

$$\hat{D}_{l,j,m_2}^- = -q^{2(l+m_2)+3}. \quad (4.38e)$$

We have divided the terms in three classes, which need to be analyzed separately.

All terms T which are not ‘boxed’ have coefficients which are uniformly bounded by q^{l+m_2} ; since the conjugation with J changes sign of the labels m_1, m_2 , for such T ’s, the coefficients of JTJ are uniformly bounded by q^{l-m_2} . They give products (and so commutators) with coefficients bounded by $q^{l+m_2} q^{l-m_2} = q^{2l}$, and so (these products) are smoothing operators.

Analogously, the coefficients of single-boxed terms are bounded by q^{l-j+m_2} , and becomes smoothing when multiplied by the J -conjugated of non-boxed terms (as $q^{l-j+m_2} q^{l-m_2} \leq q^l$), and vice versa for the product of a non-boxed term with the J -conjugated of a single-boxed one ($q^{l+m_2} q^{l-j-m_2} \leq q^l$).

Next we consider pairs of single-boxed terms. A closer look at the single-boxed terms in $x_0 \pm \delta(x_0)$ and $x_1 - \delta(x_1)$ (and then $x_1^* + \delta(x_1^*)$) shows that they have coefficients bounded by $q^{l+m_1+m_2}$, and become smoothing when multiplied by the J -conjugated of one of them ($q^{l+m_1+m_2} q^{l-m_1-m_2} = q^{2l}$). Last single-boxed term is the one in $x_1 + \delta(x_1)$ (and $x_1^* - \delta(x_1^*)$). The relevant terms for the commutators involving them are

$$\frac{1}{2}[x_1 + \delta(x_1), Jx_0J] |l, m_1, m_2; j\rangle$$

$$\begin{aligned}
& \simeq \{A_{j+1,-m_1-1}^+ \hat{C}_{l+1,j+1,-m_2}^+ B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ + \\
& \quad - A_{j,-m_1}^+ \hat{C}_{l,j,-m_2}^+ B_{j+1,m_1}^+ \hat{C}_{l+1,j+1,m_2}^+ \} |l+2, m_1+1, m_2; j+2\rangle \\
& \quad + \{A_{j,-m_1-1}^+ \hat{C}_{l,j,-m_2}^+ B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ + \\
& \quad - A_{j-1,-m_1}^+ \hat{C}_{l-1,j-1,-m_2}^+ B_{j-1,m_1}^+ \hat{C}_{l-1,j-1,m_2}^+ \} |l, m_1+1, m_2; j\rangle, \\
& \frac{1}{2}[x_1 + \delta(x_1), Jx_1J] |l, m_1, m_2; j\rangle \\
& \simeq \{B_{j+1,m_1-1}^+ \hat{C}_{l+1,j+1,m_2}^+ B_{j,-m_1}^+ \hat{C}_{l,j,-m_2}^+ + \\
& \quad - B_{j+1,-m_1-1}^+ \hat{C}_{l+1,j+1,-m_2}^+ B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ \} |l+2, m_1, m_2; j+2\rangle \\
& \quad + \{B_{j-1,m_1-1}^+ \hat{C}_{l-1,j-1,m_2}^+ B_{j,-m_1}^- \hat{C}_{l-1,j-1,-m_2}^+ + \\
& \quad - B_{j+1,-m_1-1}^- \hat{C}_{l,j,-m_2}^+ B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ \} |l, m_1, m_2; j\rangle, \\
& \frac{1}{2}[x_1 - \delta(x_1), Jx_1J] |l, m_1, m_2; j\rangle \\
& \simeq \{B_{j+1,m_1-1}^- \hat{C}_{l,j,m_2}^+ B_{j,-m_1}^+ \hat{C}_{l,j,-m_2}^+ + \\
& \quad - B_{j,m_1}^- \hat{C}_{l-1,j-1,m_2}^+ B_{j-1,-m_1-1}^+ \hat{C}_{l-1,j-1,-m_2}^+ \} |l, m_1, m_2; j\rangle, \\
& \frac{1}{2}[x_1^* + \delta(x_1^*), Jx_1J] |l, m_1, m_2; j\rangle \\
& \simeq \{B_{j+2,m_1-2}^- \hat{C}_{l+1,j+1,m_2}^+ B_{j,-m_1}^+ \hat{C}_{l,j,-m_2}^+ + \\
& \quad - B_{j+1,-m_1+1}^+ \hat{C}_{l+1,j+1,-m_2}^+ B_{j+1,m_1-1}^- \hat{C}_{l,j,m_2}^+ \} |l+2, m_1-2, m_2; j+2\rangle, \\
& \frac{1}{2}[x_1^* - \delta(x_1^*), Jx_1J] |l, m_1, m_2; j\rangle \\
& \simeq \{B_{j,m_1-2}^+ \hat{C}_{l,j,m_2}^+ B_{j,-m_1}^+ \hat{C}_{l,j,-m_2}^+ + \\
& \quad - B_{j-1,-m_1+1}^+ \hat{C}_{l-1,j-1,-m_2}^+ B_{j-1,m_1-1}^+ \hat{C}_{l-1,j-1,m_2}^+ \} |l, m_1-2, m_2; j\rangle \\
& \quad + \{B_{j-2,m_1-2}^+ \hat{C}_{l-2,j-2,m_2}^+ B_{j,-m_1}^- \hat{C}_{l-1,j-1,-m_2}^+ + \\
& \quad - B_{j-1,-m_1+1}^- \hat{C}_{l-2,j-2,-m_2}^+ B_{j-1,m_1-1}^+ \hat{C}_{l-1,j-1,m_2}^+ \} |l-2, m_1-2, m_2; j-2\rangle.
\end{aligned}$$

We need to estimate products of the form $\hat{C}_{l,j,-m_2}^+ \hat{C}_{l+i,j+i,m_2}^+$, for which, modulo smoothing operators, we find

$$\begin{aligned}
\hat{C}_{l,j,-m_2}^+ \hat{C}_{l+i,j+i,m_2}^+ & \simeq \sqrt{q^{2(l-j)} - q^{2l}q^{2(l-m_2+3-\epsilon)}} \sqrt{q^{2(l-j)} - q^{2l}q^{2(l+m_2+3+2i+\epsilon)}} \\
& \simeq \sqrt{q^{2(l-j)}} \sqrt{q^{2(l-j)}} = q^{2(l-j)}.
\end{aligned}$$

Using this we get

$$\begin{aligned}
& \frac{1}{2}[x_1 + \delta(x_1), Jx_0J] |l, m_1, m_2; j\rangle \\
& \simeq q^{2(l-j)} \{A_{j+1,-m_1-1}^+ B_{j,m_1}^+ - A_{j,-m_1}^+ B_{j+1,m_1}^+ \} |l+2, m_1+1, m_2; j+2\rangle \\
& \quad + q^{2(l-j)} \{A_{j,-m_1-1}^+ B_{j,m_1}^+ - A_{j-1,-m_1}^+ B_{j-1,m_1}^+ \} |l, m_1+1, m_2; j\rangle, \\
& \frac{1}{2}[x_1 + \delta(x_1), Jx_1J] |l, m_1, m_2; j\rangle \\
& \simeq q^{2(l-j)} \{B_{j+1,m_1-1}^+ B_{j,-m_1}^+ - B_{j+1,-m_1-1}^+ B_{j,m_1}^+ \} |l+2, m_1, m_2; j+2\rangle \\
& \quad + q^{2(l-j)} \{B_{j-1,m_1-1}^+ B_{j,-m_1}^- - B_{j+1,-m_1-1}^- B_{j,m_1}^+ \} |l, m_1, m_2; j\rangle, \\
& \frac{1}{2}[x_1 - \delta(x_1), Jx_1J] |l, m_1, m_2; j\rangle
\end{aligned}$$

$$\begin{aligned}
& \simeq q^{2(l-j)} \{ B_{j+1, m_1-1}^- B_{j, -m_1}^+ - B_{j, m_1}^- B_{j-1, -m_1-1}^+ \} |l, m_1, m_2; j\rangle, \\
\frac{1}{2} [x_1^* + \delta(x_1^*), Jx_1 J] |l, m_1, m_2; j\rangle \\
& \simeq q^{2(l-j)} \{ B_{j+2, m_1-2}^- B_{j, -m_1}^+ - B_{j+1, -m_1+1}^+ B_{j+1, m_1-1}^- \} |l+2, m_1-2, m_2; j+2\rangle, \\
\frac{1}{2} [x_1^* - \delta(x_1^*), Jx_1 J] |l, m_1, m_2; j\rangle \\
& \simeq q^{2(l-j)} \{ B_{j, m_1-2}^+ B_{j, -m_1}^+ - B_{j-1, -m_1+1}^+ B_{j-1, m_1-1}^+ \} |l, m_1-2, m_2; j\rangle \\
& + q^{2(l-j)} \{ B_{j-2, m_1-2}^+ B_{j, -m_1}^- - B_{j-1, -m_1+1}^- B_{j-1, m_1-1}^+ \} |l-2, m_1-2, m_2; j-2\rangle.
\end{aligned}$$

To prove that these commutators are smoothing we still need to check that the terms in braces are bounded by q^j (since $q^{2(l-j)}q^j \leq q^l$ is of rapid decay). This is done by using equations (4.30). For example the first two braces are identically zero, while the third one is

$$\begin{aligned}
B_{j+1, m_1-1}^+ B_{j, -m_1}^+ - B_{j+1, -m_1-1}^+ B_{j, m_1}^+ &= \tilde{B}_{j+1, m_1-1}^+ \tilde{B}_{j, -m_1}^+ - \tilde{B}_{j+1, -m_1-1}^+ \tilde{B}_{j, m_1}^+ + O(q^j) \\
&= 0 + O(q^j).
\end{aligned}$$

What remains to control are the commutators $[x_2 + \delta(x_2), JbJ]$ for $b = x_0, x_1, x_2$ and the commutators $[x_2^* - \delta(x_2^*), JbJ]$ for $b = x_1, x_2$ (which involve the ‘doubly-boxed’ term).

The operators x_2 and $\delta(x_2)$ do not shift m_1, j and have coefficients independent of m_1 . Thus, any operator acting only on the label m_1 and with coefficients depending only on m_1, j , commutes with x_2 and $\delta(x_2)$ and so can be neglected. In particular, x_0 and x_1 can be written as sums of products of operators of this kind (commuting with x_2 and $\delta(x_2)$) by operators y_i ’s,

$$\begin{aligned}
y_1 |l, m_1, m_2; j\rangle &:= \hat{C}_{l, j, m_2}^+ |l+1, m_1, m_2; j+1\rangle, \\
y_2 |l, m_1, m_2; j\rangle &:= \hat{H}_{l, j, m_2}^+ |l+1, m_1, m_2; j\rangle, \\
y_3 |l, m_1, m_2; j\rangle &:= \hat{C}_{l, j, m_2}^- |l-1, m_1, m_2; j+1\rangle,
\end{aligned}$$

and their adjoints. To prove that the commutators $[x_2 + \delta(x_2), JbJ]$, for $b = x_0, x_1, x_2$, and $[x_2^* - \delta(x_2^*), JbJ]$, for $b = x_1, x_2$, are smoothing, is sufficient to establish the same for $b = y_1, y_2, y_3$. For these operators we have

$$\begin{aligned}
\frac{1}{2} [x_2 + \delta(x_2), Jy_1 J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{C}_{l+1, j, -m_2-1}^+ \hat{D}_{l, j, m_2}^+ - \hat{D}_{l+1, j+1, m_2}^+ \hat{C}_{l, j, -m_2}^+ \} |l+2, m_1, m_2+1; j+1\rangle \\
& = \hat{C}_{l, j, -m_2}^+ \{ \hat{D}_{l, j, m_2}^+ - \hat{D}_{l+1, j+1, m_2}^+ \} |l+2, m_1, m_2+1; j+1\rangle, \\
\frac{1}{2} [x_2 + \delta(x_2), Jy_2 J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{H}_{l+1, j, -m_2-1}^+ \hat{D}_{l, j, m_2}^+ - \hat{D}_{l+1, j, m_2}^+ \hat{H}_{l, j, -m_2}^+ \} |l+2, m_1, m_2+1; j\rangle \\
& = \hat{H}_{l, j, -m_2}^+ \{ \hat{D}_{l, j, m_2}^+ - \hat{D}_{l+1, j, m_2}^+ \} |l+2, m_1, m_2+1; j\rangle, \\
\frac{1}{2} [x_2 + \delta(x_2), Jy_3 J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{C}_{l+1, j, -m_2-1}^- \hat{D}_{l, j, m_2}^+ - \hat{D}_{l-1, j+1, m_2}^+ \hat{C}_{l, j, -m_2}^- \} |l, m_1, m_2+1; j+1\rangle \\
& = \hat{C}_{l, j, -m_2}^- \{ \hat{D}_{l, j, m_2}^+ - \hat{D}_{l-1, j+1, m_2}^+ \} |l, m_1, m_2+1; j+1\rangle,
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}[x_2^* - \delta(x_2^*), Jy_1J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{C}_{l-1,j,-m_2+1}^+ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l,j+1,m_2-1}^+ \hat{C}_{l,j,-m_2}^+ \} |l, m_1, m_2 - 1; j + 1\rangle \\
& = \hat{C}_{l,j,-m_2}^+ \{ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l,j+1,m_2-1}^+ \} |l, m_1, m_2 - 1; j + 1\rangle , \\
& \frac{1}{2}[x_2^* - \delta(x_2^*), Jy_2J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{H}_{l-1,j,-m_2+1}^+ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l,j,m_2-1}^+ \hat{H}_{l,j,-m_2}^+ \} |l, m_1, m_2 - 1; j\rangle \\
& = \hat{H}_{l,j,-m_2}^+ \{ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l,j,m_2-1}^+ \} |l, m_1, m_2 - 1; j\rangle , \\
& \frac{1}{2}[x_2^* - \delta(x_2^*), Jy_3J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{C}_{l-1,j,-m_2+1}^- \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l-2,j+1,m_2-1}^+ \hat{C}_{l,j,-m_2}^- \} |l - 2, m_1, m_2 + 1; j + 1\rangle \\
& = \hat{C}_{l,j,-m_2}^- \{ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l-2,j+1,m_2-1}^+ \} |l - 2, m_1, m_2 + 1; j + 1\rangle .
\end{aligned}$$

Now $\hat{D}_{l\pm 1,j+1,m_2}^+ - \hat{D}_{l,j,m_2}^+$ is bounded by $q^{l\pm j+m_2}$, and $q^{l\pm j+m_2} \hat{C}_{l,j,-m_2}^\pm$ is bounded by q^{2l} . Next, $\hat{H}_{l,j,-m_2}^+$ is bounded by q^{l+j-m_2} , and $q^{l+j-m_2} \hat{D}_{l,j,m_2}^+ \simeq 1$. This proves that all previous commutators are smoothing. For the y_i^* 's the same statement follows from the symmetry (4.37).

We arrived at last two commutators. Modulo smoothing operators, the first one is

$$\begin{aligned}
& \frac{1}{2}[x_2 + \delta(x_2), Jx_2J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{D}_{l+1,j,-m_2-1}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2-1}^+ \hat{D}_{l,j,-m_2}^+ \} |l + 2, m_1, m_2; j\rangle \\
& \quad + \{ \hat{D}_{l+1,j,-m_2-1}^- \hat{D}_{l,j,m_2}^+ - \hat{D}_{l-1,j,m_2-1}^+ \hat{D}_{l,j,-m_2}^- \} |l, m_1, m_2; j\rangle \\
& = \hat{D}_{l,j,-m_2}^- \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l-1,j,m_2-1}^+ \} |l, m_1, m_2; j\rangle ,
\end{aligned}$$

where the second equality follows from the fact that both \hat{D}_{l,j,m_2}^+ and \hat{D}_{l,j,m_2}^- in (4.38) depend on l and m_2 only through their sum. For the same reason we have also that

$$\begin{aligned}
& \frac{1}{2}[x_2^* - \delta(x_2^*), Jx_2J] |l, m_1, m_2; j\rangle \\
& \simeq \{ \hat{D}_{l-1,j,-m_2+1}^+ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l,j,m_2-2}^+ \hat{D}_{l,j,-m_2}^+ \} |l, m_1, m_2 - 2; j\rangle \\
& \quad + \{ \hat{D}_{l-1,j,-m_2+1}^- \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l-2,j,m_2-2}^+ \hat{D}_{l,j,-m_2}^- \} |l - 2, m_1, m_2 - 2; j\rangle \\
& = \hat{D}_{l,j,-m_2}^- \{ \hat{D}_{l-1,j,m_2-1}^+ - \hat{D}_{l-2,j,m_2-2}^+ \} |l - 2, m_1, m_2 - 2; j\rangle .
\end{aligned}$$

The final observation that $\hat{D}_{l,j,-m_2}^- \hat{D}_{l-i,j,m_2-i}^+ \simeq \hat{D}_{l,j,-m_2}^-$, for $i = 0, 1, 2$, gives that these commutators vanish modulo smoothing operators. \square

4.8 An application: the Haar functional on S_q^4

The representation of $\mathcal{A}(S_q^4)$ on $L^2(S_q^4)$ is the GNS representation associated to the Haar functional φ . Then, we can recover the Haar functional from the cyclic vector 1. First, we have to identify 1 among the basis of 'harmonic functions' $|l, m_1, m_2; j\rangle$. It is the highest weight vector with weight $(0, 0)$, then $1 \propto |0, 0, 0; 0\rangle$. We'll denote it $|0\rangle$ for short. Then the Haar functional can be expressed as

$$\varphi(a) = \langle 0|a|0\rangle$$

for any $a \in \mathcal{A}(S_Q^4)$. The $\mathcal{U}_q(\mathfrak{so}(5))$ -invariance of right hand side is evident, since $h|0\rangle = \epsilon(h)|0\rangle$ is the trivial representation and by crossed product relations

$$\varphi(h \triangleright a) = \langle 0|h_{(1)}aS(h_{(2)})|0\rangle = \langle 0|a\epsilon(h_{(1)})S(h_{(2)})|0\rangle = \langle 0|ah|0\rangle = \epsilon(h)\varphi(a).$$

Let us compute explicitly φ in the basis of monomials $x_0^{n_0}x_1^{n_1}(x_1^*)^{n_2}x_2^{n_3}$, with $n_0, n_1, n_2 \in \mathbb{N}$, $n_3 \in \mathbb{Z}$ and with the notation $x_2^{n_3} := (x_2^*)^{|n_3|}$ if $n_3 < 0$. Invariance of φ implies that $\varphi(a) = 0$ unless $K_i \triangleright a = a$ for all $i = 1, 2$. Applying this condition to the generic monomial we get

$$\varphi(x_0^{n_0}x_1^{n_1}(x_1^*)^{n_2}x_2^{n_3}) = 0 \quad \text{if } n_1 \neq n_2 \text{ or } n_3 \neq 0.$$

Furthermore, for $n_1 = n_2 =: k$, $n_3 = 0$ and for odd n_0 , from $(x_1^*)^k|0\rangle \propto |k, -k, 0; k\rangle$ and using the commutation rule $x_0x_1 = q^2x_1x_0$ we get

$$\varphi(x_0^{n_0}x_1^k(x_1^*)^k) = q^{2kn_0}\langle 0|x_1^kx_0^{n_0}(x_1^*)^k|0\rangle = q^{2kn_0}\langle k, -k, 0; k|x_0^{n_0}|k, -k, 0; k\rangle$$

which is zero since $x_0^{n_0}$ is off-diagonal for odd n_0 . For $n = 2j$ even we use ‘twisted’ cyclicity. Since $\kappa^{-1}(x_1^*) = q^2x_1^*$, we have

$$\varphi(x_0^{2j-2}x_1^{k+1}(x_1^*)^{k+1}) = q^2\varphi(x_1^*x_0^{2j-2}x_1^{k+1}(x_1^*)^k) = q^{4j-2}\varphi(x_0^{2j-2}x_1^*x_1^{k+1}(x_1^*)^k). \quad (4.39)$$

By induction one proves

$$x_1^*x_1^{k+1} = x_1^{k+1}x_1^* + (q^{-4k} - q^4)x_0^2x_1^k,$$

so equation (4.39) can be rewritten as

$$\begin{aligned} \varphi(x_0^{2j}x_1^k(x_1^*)^k) &= q^{2(j+k)-3}\frac{[2j-1]}{[2k+2]}\varphi(x_0^{2j-2}x_1^{k+1}(x_1^*)^{k+1}) = \dots = \\ &= q^{2j(j+k)-3j}\frac{[2j-1]!![2k]!!}{[2(j+k)]!!}\varphi(x_1^{j+k}(x_1^*)^{j+k}). \end{aligned} \quad (4.40)$$

As usual we denote with $[n]!!$ the q -analogue of the double factorial, given by

$$[n]!! = \begin{cases} [n][n-2][n-4]\dots[2] & \text{if } n \geq 2 \text{ is even,} \\ [n][n-2][n-4]\dots[1] & \text{if } n \geq 1 \text{ is odd,} \\ 1 & \text{if } n = 0, -1. \end{cases}$$

Finally, from the equality $x_1^*|l, 0, -l; l\rangle = -q^{-1/2}\sqrt{\frac{[2l+2]}{[2l+5]}}|l+1, -l-1, 0; l+1\rangle$ we deduce that $(x_1^*)^{j+k}|0\rangle = (-1)^{j+k}q^{-(j+k)/2}\sqrt{[3]\frac{[2(j+k)]!!}{[2(j+k)+3]!!}}|j+k, -(j+k), 0; j+k\rangle$ and then

$$\varphi(x_1^{j+k}(x_1^*)^{j+k}) = q^{-(j+k)}[3]\frac{[2(j+k)]!!}{[2(j+k)+3]!!}.$$

Substituting this into equation (4.40) we get the following result.

Proposition 4.19. *The Haar functional on $\mathcal{A}(S_q^4)$ is given by:*

$$\varphi(x_0^{n_0} x_1^{n_1} (x_1^*)^{n_2} x_2^{n_3}) = \begin{cases} q^{(2j-1)(j+k)-3j} [3] \frac{[2j-1]!! [2k]!!}{[2(j+k)+3]!!} & \text{if } n_0 = 2j \in 2\mathbb{N}, n_1 = n_2 = k, n_3 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is instructive to compare the situation with the case $q = 1$. Cartesian coordinates on S^4 are expressed in spherical coordinates as

$$x_0 = \cos \theta, \quad x_1 = e^{i\alpha} \sin \theta \cos \phi, \quad x_2 = e^{i\beta} \sin \theta \sin \phi,$$

with $\alpha, \beta \in [0, 2\pi]$, $\theta \in [0, \pi]$ and $\phi \in [0, \pi/2]$. The Riemann integral is

$$\int_{S^4} d\text{Vol } S^4 = \int_{-1}^1 (\sin \theta)^2 d\cos \theta \int_0^1 \cos \phi d\cos \phi \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta,$$

and it is normalized to $8\pi^2/3$. The Haar measure on a monomial, in the non-trivial cases, yields:

$$\begin{aligned} \varphi(x_0^{2j} |x_1|^{2k}) &= \frac{3}{8\pi^2} \int_{S^4} x_0^{2j} |x_1|^{2k} d\text{Vol } S^4 \\ &= \frac{3}{2} \int_{-1}^1 (\cos \theta)^{2j} (\sin \theta)^{2k+2} d\cos \theta \int_0^1 (\cos \phi)^{2k+1} d\cos \phi \\ &= \frac{3}{2k+2} \int_0^1 \xi^{2j} (1-\xi^2)^{k+1} d\xi. \end{aligned}$$

Integrating by parts

$$\int_0^1 \xi^{2j} (1-\xi^2)^{k+1} d\xi = \int_0^1 (1-\xi^2)^{k+1} d \frac{\xi^{2j+1}}{2j+1} = \frac{2k+2}{2j+1} \int_0^1 \xi^{2j+2} (1-\xi^2)^k d\xi,$$

and by induction

$$\int_0^1 \xi^{2j} (1-\xi^2)^{k+1} d\xi = \frac{(2k+2)!! (2j-1)!!}{(2j+2k+1)!!} \int_0^1 \xi^{2j+2k+2} d\xi = \frac{(2k+2)!! (2j-1)!!}{(2j+2k+3)!!}.$$

We arrive then at the final result

$$\varphi(x_0^{2j} |x_1|^{2k}) = 3 \frac{(2k)!! (2j-1)!!}{(2j+2k+3)!!},$$

which is the $q \rightarrow 1$ limit of Proposition 4.19.

4.9 The homological dimension

With $\kappa(a) := K_1^8 K_2^6 \triangleright a$ the modular automorphism, we are able to construct a twisted Hochschild 4-cycle as follows. First of all, note that $\kappa(x_0) = x_0$, $\kappa(x_1) = q^2 x_1$ and $\kappa(x_2) = q^6 x_2$. Thus, κ is a scaling automorphism. The element $K_1^8 K_2^6$ implements the square of the antipode, that is

$$(K_1^8 K_2^6) h (K_1^8 K_2^6)^{-1} = S^2(h), \quad \forall h \in \mathcal{U}_q(\mathfrak{so}(5)), \quad (4.41)$$

as one can check on the generators E_i (a necessary and sufficient condition).

Let $\sigma : \mathcal{U}_q(\mathfrak{so}(5)) \rightarrow \text{Mat}_4(\mathbb{C})$ be the representation in (4.9) and e the idempotent in (4.8). By (2.37) we have $\kappa(e) = K_1^8 K_2^6 \triangleright e = \eta e \eta^{-1}$ where

$$\eta := \sigma(K_1^8 K_2^6) = \text{diag}(q^4, q^{-2}, q^2, q^{-4}).$$

Note that if we call $\gamma_i \in \text{Mat}_N(\mathbb{C})$ the following deformation of Dirac's gamma matrices

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2 & 0 & 0 \\ 0 & 0 & -q^2 & 0 \\ 0 & 0 & 0 & q^4 \end{pmatrix}, & \gamma_1 &= \begin{pmatrix} 0 & 0 & -q & 0 \\ 0 & 0 & 0 & q^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \gamma_{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -q^{-1} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 0 & q^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \gamma_{-2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-3} & 0 \end{pmatrix}, \end{aligned}$$

then the idempotent e can be written as

$$e = \frac{1}{2}(1 + \gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_{-1} x_1^* + \gamma_{-2} x_2^*).$$

Lemma 4.20. $\text{Tr}_{\mathbb{C}^4}(\eta \gamma_i \gamma_j \gamma_k) = 0$ for all $i, j, k \in \{0, \pm 1, \pm 2\}$.

Proof. The proof is a long and tedious computation (one has to compute 125 traces), in contrast with the $q = 1$ case where one simply uses cyclicity of the trace and the anticommutation rules of the Clifford algebra. We quote the code for MATHEMATICA[®] 5 which allows to perform the check with a personal computer.

```

gamma[0]:={1,0,0,0},{0,-q^2,0,0},{0,0,-q^2,0},{0,0,0,q^4}
gamma[1]:={0,0,-q,0},{0,0,0,q^3},{0,0,0,0},{0,0,0,0}
gamma[-1]:={0,0,0,0},{0,0,0,0},{-1/q,0,0,0},{0,q,0,0}
gamma[2]:={0,q^3,0,0},{0,0,0,0},{0,0,0,q^3},{0,0,0,0}
gamma[-2]:={0,0,0,0},{1/q^3,0,0,0},{0,0,0,0},{0,0,1/q^3,0}
eta:={q^4,0,0,0},{0,1/q^2,0,0},{0,0,q^2,0},{0,0,0,1/q^4}
f[i_,j_,k_]:=Tr[eta.gamma[i].gamma[j].gamma[k]]
For[
  counter=0;i=-2,i<3,
  For[
    j=-2,j<3,
    For[
      k=-2,k<3,
      If[f[i,j,k]==0,Null,counter++,counter++];
      k++
    ];
    j++;Clear[k]
  ];
  i++;Clear[j,k]
];Print[counter]

```

The counter 'counter' counts how many times the expression $f[i, j, k]$ ($= \text{Tr}_{\mathbb{C}^4}(\eta \gamma_i \gamma_j \gamma_k)$) fails to be zero. Running this code one gets 0, confirming that the Lemma is true. \square

Proposition 4.21. *The element*

$$\omega_4 := \text{Tr}_{\mathbb{C}^4}(\eta(e - \frac{1}{2})^{\otimes 5}) = \sum_{k_i=1}^4 \eta_{k_0, k_1}(e_{k_1, k_2} - \frac{1}{2}\delta_{k_1, k_2}) \otimes \dots \otimes (e_{k_5, k_0} - \frac{1}{2}\delta_{k_5, k_0})$$

is the representative of an $\mathcal{U}_q(\mathfrak{so}(5))$ -invariant twisted Hochschild 4-cycle.

Note that ω_4 is the sum of $4^5 = 1024$ terms, so it is very long to write it down explicitly.

Proof. Using $\kappa(e)\eta = \eta e$ we write the boundary of ω_4 as

$$\begin{aligned} b_\kappa(\omega_4) &= \sum_{j=0}^3 (-1)^j (1 + \delta_{j,0}) \text{Tr}_{\mathbb{C}^4}(\eta(e - \frac{1}{2})^{\otimes j} \otimes 1 \otimes (e - \frac{1}{2})^{\otimes (3-j)}) = \frac{1}{8} \sum_{i,j,k} \text{Tr}_{\mathbb{C}^4}(\eta\gamma_i\gamma_j\gamma_k) \times \\ &\quad \times (2 \otimes x_i \otimes x_j \otimes x_k - x_i \otimes 1 \otimes x_j \otimes x_k + x_i \otimes x_j \otimes 1 \otimes x_k - x_i \otimes x_j \otimes x_k \otimes 1), \end{aligned}$$

and this is zero by Lemma 4.20. By equation (2.37)

$$h \triangleright \omega_4 = \text{Tr}_{\mathbb{C}^4}(\eta\sigma(h_{(1)})^t(e - \frac{1}{2})^{\otimes 4} \sigma(S^{-1}(h_{(2)}))^t) = \text{Tr}_{\mathbb{C}^4}(\sigma(S^{-1}(h_{(2)}))^t \eta\sigma(h_{(1)})^t(e - \frac{1}{2})^{\otimes 4}),$$

where in last step we used cyclicity of the trace. But $\eta = \sigma(K_1^8 K_2^6) = \sigma(K_1^8 K_2^6)^t$ and by (4.41)

$$\sigma(S^{-1}(h_{(2)}))^t \eta\sigma(h_{(1)})^t = \sigma(h_{(1)} K_1^8 K_2^6 S^{-1}(h_{(2)})) = \sigma(h_{(1)} S(h_{(2)}) K_1^8 K_2^6) = \epsilon(h)\eta.$$

From this it follows the invariance of ω_4 , that is $h \triangleright \omega_4 = \epsilon(h)\omega_4$. \square

For $q = 1$ the image of ω_4 under the isomorphism (2.10) is the volume form of S^4 . For $q \neq 1$ without the factor of η Lemma 4.20 is no more valid, so the twist is essential to get a coboundary. Non triviality of ω_4 would prove that the ‘twisted’ Hochschild dimension is at least 4, i.e. there is no dimension drop. To prove non-triviality one must pair ω_4 with a 4-cocycle and prove that the pairing is not zero. A 4-cocycle is constructed in the following. Recall that $K_1^8 K_2^6 |l, m_1, m_2; j\rangle_{\pm} = q^{2m_1+6m_2} |l, m_1, m_2; j\rangle_{\pm}$ (cf. Section 4.1).

Lemma 4.22. *The operator*

$$K_1^{-8} K_2^{-6} [F, a_0] \dots [F, a_n], \quad \text{with } a_0, \dots, a_n \in \mathcal{A}(S_q^4),$$

is of trace class on \mathcal{H} for all $n \geq 3$.

Proof. Let L_q be the operator defined by $L_q |l, m_1, m_2; j\rangle_{\pm} := q^{-\frac{2}{3}m_1-2m_2} |l, m_1, m_2; j\rangle_{\pm}$. Notice that $L_q F = F L_q$ and that the map $a \mapsto L_q a L_q^{-1}$ is an automorphism of $\mathcal{A}(S_q^4)$; indeed $L_q x_0 L_q^{-1} = x_0$, $L_q x_1 L_q^{-1} = q^{-2/3} x_1$ and $L_q x_2 L_q^{-1} = q^{-2} x_2$.

Since $K_1^{-8} K_2^{-6} = L_q^3$, we have to prove that $L_q^3 [F, a_0] \dots [F, a_n]$ is of trace class for all $n \geq 3$. Since $[F, a_j]$ is of trace class, it is sufficient to prove that $L_q^3 [F, a_0] [F, a_1] [F, a_2]$ is a bounded operator. This operator can be written as:

$$L_q [F, L_q^2 a_0 L_q^{-2}] L_q [F, L_q a_1 L_q^{-1}] L_q [F, a_2] = L_q [F, a'_0] L_q [F, a'_1] L_q [F, a'_2].$$

Thus, it is sufficient to prove that $L_q[F, a]$ is bounded for all $a \in \mathcal{A}(S_q^4)$. By Leibniz rule and $[F, a^*] = -[F, a]^*$, it is sufficient to consider the cases $a = x_0, x_1, x_2$. We know that $[F, x_i]$ is a sum of independent weighted shifts with weights bounded by q^{2l} . Then, $L_q[F, x_i]$ is a sum of independent weighted shifts with weights bounded by

$$q^{2l} q^{-\frac{2}{3}m_1 - 2m_2} = q^{2(l + \frac{1}{2} - j - m_2)} q^{\frac{2}{3}(j - m_1)} q^{\frac{4}{3}j - 1} \leq q^{\frac{4}{3}j - 1} \leq q^{-1/3}$$

and this concludes the proof. \square

We are in the hypothesis of Lemma 2.12, with $K = K_1^8 K_2^6$ and $p = 4$. Thus, we can construct the ‘twisted’ analogue of Chern-Connes cocycle.

Proposition 4.23. *The multilinear map*

$$\text{ch}_n^{F, \kappa}(a_0, \dots, a_n) := \text{Tr}_{\mathcal{H}}(K_1^{-8} K_2^{-6} \gamma F[F, a_0] \dots [F, a_n])$$

is the representative on an $\mathcal{U}_q(\text{so}(5))$ -invariant twisted Hochschild n -cocycle for all even $n \geq 4$.

Proof. From Lemma 2.12, $\text{ch}_n^{F, \kappa}$ is a twisted Hochschild n -cocycle. We have only to prove invariance.

Recall that the action of $\mathcal{U}_q(\text{so}(5))$ on multilinear maps is

$$(h \triangleright \phi)(a_0, a_1, \dots, a_n) := \phi(S^{-1}(h_{(n+1)}) \triangleright a_0, S^{-1}(h_{(n)}) \triangleright a_1, \dots, S^{-1}(h_{(1)}) a_n).$$

The rules of the crossed product and the invariance of F and γ ($Fh = hF$, $F\gamma = \gamma F$) imply:

$$\gamma F[F, S^{-1}(h_{(n+1)}) \triangleright a_0] \dots [F, S^{-1}(h_{(1)}) \triangleright a_n] = S^{-1}(h_{(2)})(\gamma F[F, a_0] \dots [F, a_n])h_{(1)}.$$

Recall that $K_1^8 K_2^6$ is the element implementing the square of the antipode, $(K_1^8 K_2^6)h(K_1^8 K_2^6)^{-1} = S^2(h)$. Thus,

$$\begin{aligned} \{h \triangleright \text{ch}_n^{F, \kappa}\}(a_0, \dots, a_n) &= \text{Tr}(K_1^{-8} K_2^{-6} S^{-1}(h_{(2)}) \gamma F[F, a_0] \dots [F, a_n] h_{(1)}) \\ &= \text{Tr}(S(h_{(2)}) K_1^{-8} K_2^{-6} \gamma F[F, a_0] \dots [F, a_n] h_{(1)}). \end{aligned}$$

Cyclicity of the trace and the property $h_{(1)} S(h_{(2)}) = \epsilon(h)$ imply that $h \triangleright \text{ch}_n^{F, \kappa} = \epsilon(h) \text{ch}_n^{F, \kappa}$ for all $h \in \mathcal{U}_q(\text{so}(5))$. \square

Note that while the Fredholm module $(\mathcal{A}(S_q^4), \mathcal{H}, F, \gamma)$ is 1-summable (cf. Proposition 4.12), and one has a cocycle ch_n^F for any even $n \geq 0$, the twisted cocycle $\text{ch}_n^{F, \kappa}$ is defined only for even n , greater than 4.

Now, we prove that the Hochschild homology class of ω_4 is not zero. To this end, it is enough to prove the non-vanishing of the pairing

$$\text{ch}_4^{F, \kappa}(\omega_4) = \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^4}(\eta K_1^{-8} K_2^{-6} \gamma F[F, e]^5), \quad (4.42)$$

which depends only on the class of ω_4 and $\text{ch}_4^{F,\kappa}$ in Hochschild (co)homology.

Along the lines of Lemma 3.24, one proves that the pairing (4.42) is 2 times the q -index of the operator F_e^+ in (2.2),

$$\frac{1}{2} \text{ch}_4^{F,\kappa}(\omega_4) = \text{Tr}_{\ker F_e^+}(\eta K_1^{-8} K_2^{-6}) - \text{Tr}_{\text{coker } F_e^+}(\eta K_1^{-8} K_2^{-6}).$$

Using representation theory, it is possible to compute explicitly the index.

Lemma 4.24. *The class $[\omega_4] \in HH_4^\kappa$ is non-trivial. In particular, $\text{ch}_4^{F,\kappa}(\omega_4) = 2$.*

Proof. We lift the representation of $U_q(\mathfrak{so}(5))$ from \mathcal{H}_\pm to $v \in \mathcal{H}_\pm \otimes \mathbb{C}^4$ by taking the Hopf tensor product with $V_{1/2}$, as in (2.38):

$$h \blacktriangleright v = \sigma(S^{-1}(h_{(1)}))^t h_{(2)} \triangleright v,$$

with σ the representation (4.9). Since e satisfies (2.37), it follows that $h \blacktriangleright (ew) = e(h \blacktriangleright w)$, and the subspaces

$$\mathcal{H}_\pm^\parallel := e(\mathcal{H}_\pm \otimes \mathbb{C}^4), \quad \mathcal{H}_\pm^\perp := (1 - e)(\mathcal{H}_\pm \otimes \mathbb{C}^4),$$

carries sub-representations of $U_q(\mathfrak{so}(5))$. As $\mathcal{H}_\pm \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l$, Clebsh-Gordan decomposition

$$\begin{aligned} V_{1/2} \otimes V_{1/2} &\simeq V_0 \oplus V_1 \oplus V_{(1,1)}, \\ V_{1/2} \otimes V_l &\simeq V_{l-\frac{1}{2}} \oplus V_{l+\frac{1}{2}} \oplus V_{(1,l-\frac{1}{2})} \oplus V_{(1,l+\frac{1}{2})} \quad \text{if } l = \frac{3}{2}, \frac{5}{2}, \dots \end{aligned}$$

tells us that

$$\mathcal{H}_\pm^\parallel \simeq \bigoplus_{l \in \mathbb{N}} (m_l^{\pm,\parallel} V_l \oplus n_l^{\pm,\parallel} V_{(1,l+1)}), \quad \mathcal{H}_\pm^\perp \simeq \bigoplus_{l \in \mathbb{N}} (m_l^{\pm,\perp} V_l \oplus n_l^{\pm,\perp} V_{(1,l+1)}),$$

with multiplicities satisfying

$$m_0^{\pm,\parallel} + m_0^{\pm,\perp} = 1, \quad m_l^{\pm,\parallel} + m_l^{\pm,\perp} = 2 \forall l \geq 1, \quad n_l^{\pm,\parallel} + n_l^{\pm,\perp} = 2 \forall l \geq 0.$$

Consider the vectors ($l \geq 1$)

$$\begin{aligned} v_{(1,l)}^\pm &:= |l - \frac{1}{2}, \frac{1}{2}, l - \frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 0, 0, 1)^t, \\ v_{(0,l)}^\pm &:= |l - \frac{1}{2}, \frac{1}{2}, l - \frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 1, 0, 0)^t + |l - \frac{1}{2}, -\frac{1}{2}, l - \frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 0, 0, 1)^t, \\ v_{(0,0)}^\pm &:= |\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (1, 0, 0, 0)^t + |\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 1, 0, 0)^t \\ &\quad + |\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 0, 1, 0)^t + |\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}\rangle_\pm \otimes (0, 0, 0, 1)^t. \end{aligned}$$

These are highest weight vectors with weight $(1, l)$ and $(0, l)$ respectively. They remain highest weight vectors when multiplied by e . So,

$$v_{(n_1, n_2)}^{\pm,\parallel} := e v_{(n_1, n_2)}^\pm \quad \text{and} \quad v_{(n_1, n_2)}^{\pm,\perp} := (1 - e) v_{(n_1, n_2)}^\pm \quad (4.43)$$

are highest weight vectors with weight (n_1, n_2) , for $(n_1, n_2) = (1, l)$ and $(0, l)$. By using the explicit form of the chiral representations, one finds that $v_{(0,0)}^{-,\parallel} = ev_{(0,0)}^- = 0$ and $v_{(0,0)}^{+,\parallel} = (1 - e)v_{(0,0)}^+ = 0$, while $v_{(n_1, n_2)}^{\pm,\parallel} \neq 0$ and $v_{(n_1, n_2)}^{\pm,\perp} \neq 0$ in all other cases.

This proves that $m_l^{\pm,\parallel} = m_l^{\pm,\perp} = n_l^{\pm,\parallel} = n_l^{\pm,\perp} = 1$ for all $l \geq 1$, while $m_0^{+,\parallel} = m_0^{-,\perp} = 1$ and $m_0^{-,\parallel} = m_0^{+,\perp} = 0$. That is,

$$\mathcal{H}_s^{\parallel} \simeq (\delta_{s,+} V_0) \oplus \bigoplus_{l \geq 1} (V_l \oplus V_{(1,l)}), \quad \mathcal{H}_s^{\perp} \simeq (\delta_{s,-} V_0) \oplus \bigoplus_{l \geq 1} (V_l \oplus V_{(1,l)}),$$

with $s = \pm 1$. By equivariance, F intertwines representations with the same highest weight, while flipping the label ‘ \pm ’. The kernel and cokernel of F_e^+ are still subrepresentations of $U_q(\mathfrak{so}(5))$, thus it is enough to determine which of the highest weight vectors (4.43) are in the kernel and which in the cokernel. It turns out that $\ker F_e^+ = V_0$ is the subspace of \mathcal{H}_+ spanned by $v_{(0,0)}^+$ ($eFev_{(0,0)}^+ = eFv_{(0,0)}^+ = ev_{(0,0)}^- = 0$), and the cokernel is empty. So,

$$\frac{1}{2} \text{ch}_4^{F,\kappa}(\omega_4) = \frac{(v_{(0,0)}^+ | \eta K_1^{-8} K_2^{-6} | v_{(0,0)}^+)}{(v_{(0,0)}^+ | v_{(0,0)}^+)} = 1. \quad \square$$

4.10 Check with Mathematica[®] 5

As a further control that the left regular and chiral representations of the algebra $\mathcal{A}(S_q^4)$ are correct, we list the MATHEMATICA[®] 5 code which allows to check the commutation rules of the algebra. Before that, let me explain how the computer program works.

Left regular representation

We call $X_i(a, b, l, m, k, j)$, $i = 0, 1$, the coefficients such that

$$x_i |l, m_1, m_2; j\rangle = \sum_{a,b} X_i(a, b, l, m_1, m_2, j) |l + a, m_1 + \delta_{i1}, m_2; j + b\rangle$$

and notice that they are zero if $a, b \neq \pm 1$, while we call $X_2(a, l, m_2, j)$ the coefficients in

$$x_2 |l, m_1, m_2; j\rangle = \sum_a X_2(a, l, m_2, j) |l + a, m_1, m_2 + 1; j\rangle$$

and notice that they vanish for $a \neq \pm 1$ (they are called `Xzero[a, b, l, m, k, j]`, `Xuno[a, b, l, m, k, j]` and `Xdue[a, l, k, j]` in MATHEMATICA[®] 5). We then define the seven polynomials

$$\text{Pol}_1 = x_0 x_2 - q^2 x_2 x_0,$$

$$\text{Pol}_2 = x_1 x_2 - q^2 x_2 x_1,$$

$$\text{Pol}_3 = x_2^* x_1 - q^2 x_1 x_2^*,$$

$$\text{Pol}_4 = x_0 x_1 - q^2 x_1 x_0,$$

$$\text{Pol}_5 = [x_1, x_1^*] + (1 - q^4) x_0^2,$$

$$\begin{aligned}\text{Pol}_6 &= [x_2, x_2^*] + x_1^* x_1 - q^4 x_1 x_1^* , \\ \text{Pol}_7 &= x_0^2 + x_1 x_1^* + x_2 x_2^* - 1 .\end{aligned}$$

The vanishing of these polynomials guarantees that the commutation relations of $\mathcal{A}(S_q^4)$ are satisfied (the remaining commutation relations are obtained by conjugation). In the code listed at the end of the chapter we call

$$\text{Comm}[i, x, y] := \langle l + x, m', k', j + y | \text{Pol}_i | l, m, k, j \rangle ,$$

where m', k' are the unique values of the coefficients for which the matrix element is different from zero (m', k' depend on which polynomial one is considering). For $i = 1, 2, 3$ we have to check that $\text{Comm}[i, x, y] = 0$ in the cases $x = -2, 0, 2$ and $y = -1, +1$ (they are automatically zero in all other cases). This is done by the code

```
For[counter = 0; i = 1, i <= 3,
  For[x = -2, x <= 2,
    For[y = -1, y <= 1,
      If[Simplify[Comm[i, x, y]] == 0, Null, counter++, counter++];
      Print["i=", i, ", x=", x, ", y=", y, ", Comm=", Simplify[Comm[i, x, y]]];
      y++; y++];
    x++; x++];
  i++; Print["errors=", counter]
```

For each of the values of i, x, y we are interested in, it prints a line with the value of $\text{Comm}[i, x, y]$ (which is always zero). The counter ‘**counter**’ counts cumulatively how many times these coefficients fail to be zero, and is printed at the end of the cycle (it turns out to be zero).

For $i = 4, 5, 6, 7$ we do the same, but now we are interested in the cases $x = -2, 0, 2$ and $y = -2, 0, 2$ (in the other cases coefficients are automatically zero).

Chiral representations

Let

$$|l, j, m_1, m_2, \epsilon\rangle = \delta_{2\epsilon, (-1)^{l+\frac{1}{2}-j+m_2}} |l, m_1, m_2; j\rangle_{\pm} .$$

We first do the computation as $\epsilon \in \{\pm\frac{1}{2}\}$ were an independent variable; then we should evaluate the result on the actual value $\epsilon = \frac{1}{2}(-1)^{l+\frac{1}{2}-j+m_2}$, but this is not necessary since the result is zero even for the ‘wrong’ value of ϵ . In MATHEMATICA[®] 5 code we call $\text{Xzero}[\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{l}, \mathbf{j}, \mathbf{m}, \mathbf{k}, \mathbf{e}]$ the coefficient $X_0^{a,b,c}(l, j, m, k, \epsilon)$, etc., in the equation

$$x_i |l, j, m, k, \epsilon\rangle = \sum_{c=\pm\frac{1}{2}} \sum_{a,b=0,\pm 1} X_i^{a,b,c}(l, j, m, k, \epsilon) |l + a, j + b, m + \delta_{i1}, k + \delta_{i2}, c\rangle .$$

They are given by

$$X_i^{a,b,c}(l, j, m, k, \epsilon) = s(\epsilon, a - b + \delta_{i2}, c) Y_i^{a,b}(l, j, m, k, \epsilon) ,$$

with

$$s(\epsilon, t, c) = \begin{cases} 1 & \text{if } c = (-1)^t \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} Y_0^{a,1}(l, j, m, k, \epsilon) &= A_{j,m}^+ C_{l,j,k}^a(\epsilon), \\ Y_0^{a,0}(l, j, m, k, \epsilon) &= A_{j,m}^0 H_{l,j,k}^a(\epsilon), \\ Y_0^{a,-1}(l, j, m, k, \epsilon) &= Y_0^{-a,1}(l + a, j - 1, m, k, \epsilon), \\ Y_1^{a,1}(l, j, m, k, \epsilon) &= B_{j,m}^+ C_{l,j,k}^a(\epsilon), \\ Y_1^{a,0}(l, j, m, k, \epsilon) &= B_{j,m}^0 H_{l,j,k}^a(\epsilon), \\ Y_1^{a,-1}(l, j, m, k, \epsilon) &= B_{j,m}^- C_{l+a,p-1,k}^a(\epsilon), \\ Y_2^{a,b}(l, j, m, k, \epsilon) &= \delta_{b,0} D_{l,j,k}^a(\epsilon). \end{aligned}$$

The factor $s(\epsilon, a - b + \delta_{i2}, c)$ takes into account the fact that when $l - j + m_2$ is shifted by $a - b + \delta_{i2}$, the sign of ϵ changes or not, depending on the parity of that quantity.

With Pol_i given by the same polynomials as in the previous section, we call

$$\text{Comm}[\mathbf{i}, \mathbf{x}, \mathbf{y}, \mathbf{e}] := \langle l + x, j + y, m', k', e' | \text{Pol}_i | l, j, m, k, e \rangle,$$

where m', k', e' are the unique values of the coefficients for which the matrix elements is different from zero (they depend on which polynomial one is considering). For $i = 1, 2, 3$ we have to check that $\text{Comm}[\mathbf{i}, \mathbf{x}, \mathbf{y}, \mathbf{e}] = 0$ in the cases $x = -2, -1, 0, 1, 2$, $y = -1, 0, +1$ and $e = \pm \frac{1}{2}$ (they are automatically zero in all other cases). This is done by the code

```
For[counter = 0; i = 1, i <= 3,
  For[e = -1/2, e <= 1/2,
    For[x = -2, x <= 2,
      For[y = -1, y <= 1,
        If[Simplify[Comm[i, x, y, e]] == 0, Null, counter++, counter++];
        y++];
        x++];
        e++];
i++]; Print["errors=", counter]
```

As for the left regular representation, the value of each coefficient $\text{Comm}[\mathbf{i}, \mathbf{x}, \mathbf{y}, \mathbf{e}]$ could be printed, but we don't do it to avoid wasting paper. Instead, only the number of coefficients failing to be zero is printed at the end. For $i = 4, 5, 6, 7$ we do the same, but now we are interested in the cases $x, y = -2, -1, 0, 1, 2$ and $e = \pm \frac{1}{2}$ (in the other cases coefficients are trivially zero). As one can see by running the code, the result is always zero, confirming that the representation is correct.

MATHEMATICA[®] 5 code - left regular representation

```

$Assumptions = {0 < q < 1 && {l, m, k, j} ∈ Integers && l ≥ 0 && 0 ≤ j ≤ l && -j ≤ m ≤ j && -(l - j) ≤ k ≤ l - j}
{0 < q < 1 && {l | m | k | j} ∈ Integers && l ≥ 0 && 0 ≤ j ≤ l && -j ≤ m ≤ j && j - 1 ≤ k ≤ -j + 1}

sqa[n_] := (q^(-n) - q^n)^(1/2) (q^(-1) - q)^(-1/2)
Apiu[j_, m_] := q^(m - 1)  $\frac{sqa[j + m + 1] sqa[j - m + 1]}{sqa[2j + 1] sqa[2j + 3]}$ 
Bpiu[j_, m_] := q^(-j + m - 1/2)  $\frac{sqa[j + m + 1] sqa[j + m + 2]}{sqa[2j + 1] sqa[2j + 3]}$ 
Bmeno[j_, m_] := -q^(j + m + 1/2)  $\frac{sqa[j - m] sqa[j - m - 1]}{sqa[2j - 1] sqa[2j + 1]}$ 
Cpiu[l_, j_, k_] := q^(k - 1)  $\frac{sqa[l + j + k + 3] sqa[l + j - k + 3]}{sqa[2l + 3] sqa[2l + 5]}$ 
Cmeno[l_, j_, k_] := -q^(k - 1)  $\frac{sqa[l - j + k] sqa[l - j - k]}{sqa[2l + 1] sqa[2l + 3]}$ 
Dpiu[l_, j_, k_] := q^(-1 + k - 3/2)  $\frac{sqa[l + j + k + 3] sqa[l - j + k + 2]}{sqa[2l + 3] sqa[2l + 5]}$ 
Dmeno[l_, j_, k_] := q^(1 + k + 3/2)  $\frac{sqa[l - j - k] sqa[l + j - k + 1]}{sqa[2l + 1] sqa[2l + 3]}$ 

Xzero[1, 1, l_, m_, k_, j_] := Apiu[j, m] Cpiu[l, j, k]
Xzero[-1, 1, l_, m_, k_, j_] := Apiu[j, m] Cmeno[l, j, k]
Xzero[1, -1, l_, m_, k_, j_] := Apiu[j - 1, m] Cmeno[l + 1, j - 1, k]
Xzero[-1, -1, l_, m_, k_, j_] := Apiu[j - 1, m] Cpiu[l - 1, j - 1, k]

Xuno[1, 1, l_, m_, k_, j_] := Bpiu[j, m] Cpiu[l, j, k]
Xuno[-1, 1, l_, m_, k_, j_] := Bpiu[j, m] Cmeno[l, j, k]
Xuno[1, -1, l_, m_, k_, j_] := Bmeno[j, m] Cmeno[l + 1, j - 1, k]
Xuno[-1, -1, l_, m_, k_, j_] := Bmeno[j, m] Cpiu[l - 1, j - 1, k]

Xdue[1, l_, k_, j_] := Dpiu[l, j, k]
Xdue[-1, l_, k_, j_] := Dmeno[l, j, k]

For[j = -2, j ≤ 2,
  Xzero[i_, j, l_, m_, k_, j_] := 0;
  Xuno[i_, j, l_, m_, k_, j_] := 0;
  j++; j++]

For[s = -1, s ≤ 1,
  Xzero[3 s, i_, l_, m_, k_, j_] := 0;
  Xzero[i_, 3 s, l_, m_, k_, j_] := 0;
  Xuno[3 s, i_, l_, m_, k_, j_] := 0;
  Xuno[i_, 3 s, l_, m_, k_, j_] := 0;
  Xdue[3 s, l_, k_, j_] := 0;
  s++; s++]

Comm[1, x_, y_] := Sum[Xzero[x - a, y, l + a, m, k + 1, j] Xdue[a, l, k, j] -
  q^2 Xdue[a, l + x - a, k, j + y] Xzero[x - a, y, l, m, k, j], {a, -1, 1, 2}]

Comm[2, x_, y_] := Sum[Xuno[x - a, y, l + a, m, k + 1, j] Xdue[a, l, k, j] -
  q^2 Xdue[a, l + x - a, k, j + y] Xuno[x - a, y, l, m, k, j], {a, -1, 1, 2}]

Comm[3, x_, y_] := Sum[
  Xdue[-a, l + x, k - 1, j + y] Xuno[x - a, y, l, m, k, j] - q^2
  Xuno[x - a, y, l + a, m, k - 1, j] Xdue[-a, l + a, k - 1, j], {a, -1, 1, 2}]

```



```

Comm[4, x_, y_] := Sum[Xzero[x - a, y - b, 1 + a, m + 1, k, j + b] Xuno[a, b, 1, m, k, j] -
  q^2 Xuno[x - a, y - b, 1 + a, m, k, j + b] Xzero[a, b, 1, m, k, j], {a, -1, 1, 2}, {b, -1, 1, 2}]

Comm[5, x_, y_] := Sum[Xuno[-a, -b, 1 + x, m, k, j + y] Xuno[x - a, y - b, 1, m, k, j] -
  Xuno[x - a, y - b, 1 + a, m - 1, k, j + b] Xuno[-a, -b, 1 + a, m - 1, k, j + b] -
  (1 - q^4) Xzero[x - a, y - b, 1 + a, m, k, j + b] Xzero[a, b, 1, m, k, j], {a, -1, 1, 2}, {b, -1, 1, 2}]

Comm[6, x_, y_] := DiscreteDelta[y]
  Sum[Xdue[-a, 1 + x, k, j] Xdue[x - a, 1, k, j] - Xdue[-a, 1 + a, k - 1, j] Xdue[x - a, 1 + a, k - 1, j],
  {a, -1, 1, 2}] + Sum[-Xuno[-a, -b, 1 + x, m, k, j + y] Xuno[x - a, y - b, 1, m, k, j] + q^4
  Xuno[x - a, y - b, 1 + a, m - 1, k, j + b] Xuno[-a, -b, 1 + a, m - 1, k, j + b], {a, -1, 1, 2}, {b, -1, 1, 2}]

Comm[7, x_, y_] := Sum[Xzero[x - a, y - b, 1 + a, m, k, j + b] Xzero[a, b, 1, m, k, j] +
  Xuno[x - a, y - b, 1 + a, m - 1, k, j + b] Xuno[-a, -b, 1 + a, m - 1, k, j + b], {a, -1, 1, 2}, {b, -1, 1, 2}] ·
  DiscreteDelta[y] Sum[Xdue[-a, 1 + a, k - 1, j] Xdue[x - a, 1 + a, k - 1, j], {a, -1, 1, 2}] -
  DiscreteDelta[x] DiscreteDelta[y]

For[counter = 0; i = 1, i ≤ 3,
  For[x = -2, x ≤ 2,
    For[y = -1, y ≤ 1,
      If[Simplify[Comm[i, x, y]] == 0, Null, counter++, counter++];
      Print["i=", i, ", x=", x, ", y=", y, ", Comm=", Simplify[Comm[i, x, y]]];
      y++; y++;
      x++; x++;
      i++; Print["errors=", counter]

i=1, x=-2, y=-1, Comm=0
i=1, x=-2, y=1, Comm=0
i=1, x=0, y=-1, Comm=0
i=1, x=0, y=1, Comm=0
i=1, x=2, y=-1, Comm=0
i=1, x=2, y=1, Comm=0
i=2, x=-2, y=-1, Comm=0
i=2, x=-2, y=1, Comm=0
i=2, x=0, y=-1, Comm=0
i=2, x=0, y=1, Comm=0
i=2, x=2, y=-1, Comm=0
i=2, x=2, y=1, Comm=0
i=3, x=-2, y=-1, Comm=0
i=3, x=-2, y=1, Comm=0
i=3, x=0, y=-1, Comm=0
i=3, x=0, y=1, Comm=0
i=3, x=2, y=-1, Comm=0
i=3, x=2, y=1, Comm=0

errors=0

For[counter = 0; i = 4, i ≤ 7,
  For[x = -2, x ≤ 2,
    For[y = -2, y ≤ 2,
      If[Simplify[Comm[i, x, y]] == 0, Null, counter++, counter++];
      Print["i=", i, ", x=", x, ", y=", y, ", Comm=", Simplify[Comm[i, x, y]]];
      y++; y++;
      x++; x++;
      i++; Print["errors=", counter]

```

```
i=4, x=-2, y=-2, Comm=0
i=4, x=-2, y=0, Comm=0
i=4, x=-2, y=2, Comm=0
i=4, x=0, y=-2, Comm=0
i=4, x=0, y=0, Comm=0
i=4, x=0, y=2, Comm=0
i=4, x=2, y=-2, Comm=0
i=4, x=2, y=0, Comm=0
i=4, x=2, y=2, Comm=0
i=5, x=-2, y=-2, Comm=0
i=5, x=-2, y=0, Comm=0
i=5, x=-2, y=2, Comm=0
i=5, x=0, y=-2, Comm=0
i=5, x=0, y=0, Comm=0
i=5, x=0, y=2, Comm=0
i=5, x=2, y=-2, Comm=0
i=5, x=2, y=0, Comm=0
i=5, x=2, y=2, Comm=0
i=6, x=-2, y=-2, Comm=0
i=6, x=-2, y=0, Comm=0
i=6, x=-2, y=2, Comm=0
i=6, x=0, y=-2, Comm=0
i=6, x=0, y=0, Comm=0
i=6, x=0, y=2, Comm=0
i=6, x=2, y=-2, Comm=0
i=6, x=2, y=0, Comm=0
i=6, x=2, y=2, Comm=0
i=7, x=-2, y=-2, Comm=0
i=7, x=-2, y=0, Comm=0
i=7, x=-2, y=2, Comm=0
i=7, x=0, y=-2, Comm=0
i=7, x=0, y=0, Comm=0
i=7, x=0, y=2, Comm=0
i=7, x=2, y=-2, Comm=0
i=7, x=2, y=0, Comm=0
i=7, x=2, y=2, Comm=0
errors=0
```

MATHEMATICA® 5 code - chiral representations

```

$Assumptions = {0 < q < 1 && {1 + 1/2, m + 1/2, k + 1/2, j + 1/2} ∈ Integers &&
  1 > 0 && 0 < j ≤ 1 && -j ≤ m ≤ j && -(1 + 1/2 - j) ≤ k ≤ 1 + 1/2 - j && t^2 == 1 && (2 e)^2 == 1}

{0 < q < 1 && (1/2 + 1 | 1/2 + m | 1/2 + k | 1/2 + j) ∈ Integers &&
  1 > 0 && 0 < j ≤ 1 && -j ≤ m ≤ j && -1/2 + j - 1 ≤ k ≤ 1/2 - j + 1 && t^2 == 1 && 4 e^2 == 1}

sqa[n_] := (q^-n - q^n)^(1/2) (q^-1 - q)^(-1/2)
qa[n_] := (q^-n - q^n) (q^-1 - q)^(-1)
sn[e_, z_, c_] := DiscreteDelta[Simplify[c - e (-1)^z]]
Apiu[j_, m_] := q^(m-1)  $\frac{\text{sqa}[j+m+1] \text{sqa}[j-m+1]}{\text{qa}[2j+2]}$ 
Azero[j_, m_] := q^-2  $\frac{q^{j+m+1} \text{qa}[2] \text{qa}[j-m] - \text{qa}[2j]}{\text{qa}[2j] \text{qa}[2j+2]}$ 
Bpiu[j_, m_] := q^-j+m-1/2  $\frac{\text{sqa}[j+m+1] \text{sqa}[j+m+2]}{\text{qa}[2j+2]}$ 
Bzero[j_, m_] := (1 + q^2) q^(m-1/2)  $\frac{\text{sqa}[j-m] \text{sqa}[j+m+1]}{\text{qa}[2j] \text{qa}[2j+2]}$ 
Bmeno[j_, m_] := -q^(j+m+1/2)  $\frac{\text{sqa}[j-m] \text{sqa}[j-m-1]}{\text{qa}[2j]}$ 
CC[1, l_, j_, k_, e_] := -q^(k-1-e)  $\frac{\text{sqa}[1+j+k+3+e] \text{sqa}[1+j-k+3-e]}{\text{qa}[2l+4]}$ 
CC[0, l_, j_, k_, e_] := -2 t e qa[2] q^(2(1+3)+k-1)  $\frac{\text{sqa}[1+1/2+j+2-2ek] \text{sqa}[1+1/2-j-2ek]}{\text{qa}[2l+2] \text{qa}[2l+4]}$ 
CC[-1, l_, j_, k_, e_] := -q^(k-1+e)  $\frac{\text{sqa}[1-j+k-e] \text{sqa}[1-j-k+e]}{\text{qa}[2l+2]}$ 
HH[1, l_, j_, k_, e_] := q^(k-1+e(2j+1))  $\frac{\text{sqa}[1-k+2+e(2j+1)] \text{sqa}[1+k+2-e(2j+1)]}{\text{qa}[2l+4]}$ 
HH[0, l_, j_, k_, e_] :=
  t  $\frac{\text{qa}[1-k+1-e(2j+1)] \text{qa}[1+k+2-e(2j+1)] - q^{-2} \text{qa}[1-k+2+e(2j+1)] \text{qa}[1+k+1+e(2j+1)]}{\text{qa}[2l+2] \text{qa}[2l+4]}$ 
HH[-1, l_, j_, k_, e_] := HH[1, l-1, j, k, -e]
DD[1, l_, j_, k_, e_] := q^(1+k-3/2)  $\frac{\text{sqa}[1+j+k+3+e] \text{sqa}[1-j+k+2-e]}{\text{qa}[2l+4]}$ 
DD[0, l_, j_, k_, e_] := t qa[2] q^(k+1/2)  $\frac{\text{sqa}[1-k+1-e(2j+1)] \text{sqa}[1+k+2-e(2j+1)]}{\text{qa}[2l+2] \text{qa}[2l+4]}$ 
DD[-1, l_, j_, k_, e_] := -q^(1+k+3/2)  $\frac{\text{sqa}[1-j-k+e] \text{sqa}[1+j-k+1-e]}{\text{qa}[2l+2]}$ 

For[i = 0, i ≤ 2,
  For[s = -1, s ≤ 1,
    For[a = 2, a ≤ 3,
      Y[i, s a, b_, l_, j_, m_, k_, e_] := 0;
      a++;
      s++; s++;
    i++]
For[i = 0, i ≤ 2,
  For[s = -1, s ≤ 1,
    For[b = 2, b ≤ 3,
      Y[i, a_, s b, l_, j_, m_, k_, e_] := 0;
      b++;
      s++; s++;
    i++]

```

```

Y[0, a_, 1, l_, j_, m_, k_, e_] := Apiu[j, m] CC[a, l, j, k, e]
Y[0, a_, 0, l_, j_, m_, k_, e_] := Azero[j, m] HH[a, l, j, k, e]
Y[0, a_, -1, l_, j_, m_, k_, e_] := Y[0, -a, 1, 1+a, j-1, m, k, (-1)a+1 e]
Y[1, a_, 1, l_, j_, m_, k_, e_] := Bpiu[j, m] CC[a, l, j, k, e]
Y[1, a_, 0, l_, j_, m_, k_, e_] := Bzero[j, m] HH[a, l, j, k, e]
Y[1, a_, -1, l_, j_, m_, k_, e_] := Bmeno[j, m] CC[-a, 1+a, j-1, k, (-1)a+1 e]
Y[2, a_, b_, l_, j_, m_, k_, e_] := DiscreteDelta[b] DD[a, l, j, k, e]
Xzero[a_, b_, c_, l_, j_, m_, k_, e_] := sn[e, a-b, c] Y[0, a, b, l, j, m, k, e]
Xuno[a_, b_, c_, l_, j_, m_, k_, e_] := sn[e, a-b, c] Y[1, a, b, l, j, m, k, e]
Xdue[a_, b_, c_, l_, j_, m_, k_, e_] := sn[e, a-b+1, c] Y[2, a, b, l, j, m, k, e]

Comm[1, x_, y_, e_] :=
Sum[Xzero[x-a, y-b, (-1)x+y+1 e, 1+a, j+b, m, k+1, c] Xdue[a, b, c, l, j, m, k, e] -
q^2 Xdue[x-a, y-b, (-1)x+y+1 e, 1+a, j+b, m, k, c] Xzero[a, b, c, l, j, m, k, e],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[2, x_, y_, e_] :=
Sum[Xuno[x-a, y-b, (-1)x+y+1 e, 1+a, j+b, m, k+1, c] Xdue[a, b, c, l, j, m, k, e] -
q^2 Xdue[x-a, y-b, (-1)x+y+1 e, 1+a, j+b, m+1, k, c] Xuno[a, b, c, l, j, m, k, e],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[3, x_, y_, e_] :=
Sum[Xdue[-a, -b, c, 1+x, j+y, m+1, k-1, (-1)x+y+1 e] Xuno[x-a, y-b, c, l, j, m, k, e] -
q^2 Xuno[x-a, y-b, (-1)x+y+1 e, 1+a, j+b, m, k-1, c] Xdue[-a, -b, e, 1+a, j+b, m, k-1, c],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[4, x_, y_, e_] :=
Sum[Xzero[x-a, y-b, (-1)x+y e, 1+a, j+b, m+1, k, c] Xuno[a, b, c, l, j, m, k, e] -
q^2 Xuno[x-a, y-b, (-1)x+y e, 1+a, j+b, m, k, c] Xzero[a, b, c, l, j, m, k, e],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[5, x_, y_, e_] :=
Sum[Xuno[x-a, y-b, (-1)x+y e, 1+a, j+b, m-1, k, c] Xuno[-a, -b, e, 1+a, j+b, m-1, k, c] -
Xuno[-a, -b, c, 1+x, j+y, m, k, (-1)x+y e] Xuno[x-a, y-b, c, l, j, m, k, e] +
(1-q^4) Xzero[x-a, y-b, (-1)x+y e, 1+a, j+b, m, k, c] Xzero[a, b, c, l, j, m, k, e],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[6, x_, y_, e_] :=
Sum[Xdue[x-a, y-b, (-1)x+y e, 1+a, j+b, m, k-1, c] Xdue[-a, -b, e, 1+a, j+b, m, k-1, c] -
Xdue[-a, -b, c, 1+x, j+y, m, k, (-1)x+y e] Xdue[x-a, y-b, c, l, j, m, k, e] +
Xuno[-a, -b, c, 1+x, j+y, m, k, (-1)x+y e] Xuno[x-a, y-b, c, l, j, m, k, e] -
q^4 Xuno[x-a, y-b, (-1)x+y e, 1+a, j+b, m-1, k, c] Xuno[-a, -b, e, 1+a, j+b, m-1, k, c],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}]

Comm[7, x_, y_, e_] := Sum[Xzero[x-a, y-b, (-1)x+y e, 1+a, j+b, m, k, c] Xzero[a, b, c, l, j, m, k, e] +
Xuno[x-a, y-b, (-1)x+y e, 1+a, j+b, m-1, k, c] Xuno[-a, -b, e, 1+a, j+b, m-1, k, c] +
Xdue[x-a, y-b, (-1)x+y e, 1+a, j+b, m, k-1, c] Xdue[-a, -b, e, 1+a, j+b, m, k-1, c],
{a, -1, 1}, {b, -1, 1}, {c, -1/2, 1/2}] - DiscreteDelta[x, y]

For[counter = 0; i = 1, i ≤ 3,
For[e = -1/2, e ≤ 1/2,
For[x = -2, x ≤ 2,
For[y = -1, y ≤ 1,
If[Simplify[Comm[i, x, y, e]] == 0, Null, counter++, counter++];
y++];
x++];
e++];
i++]; Print["errors=", counter]

errors=0

```

```
For[counter = 0; i = 4, i ≤ 7,  
  For[e = -1/2, e ≤ 1/2,  
    For[x = -2, x ≤ 2,  
      For[y = -2, y ≤ 2,  
        If[Simplify[Comm[i, x, y, e]] == 0, Null, counter++, counter++];  
        y++;  
        x++;  
        e++;  
      i++]; Print["errors=", counter]  
errors=0
```


Chapter 5

Some remarks on odd-dimensional quantum spheres

“One should always generalize”,
Carl Jacobi [DH81].

The purpose of this chapter is to extend to odd-dimensional quantum spheres some of the results which hold in the case of $SU_q(2)$ [Con04, DLS⁺05b], Podleś spheres [DDLW07] and of the quantum 4-sphere [DDL08]. We recall here the definition of the odd-dimensional quantum sphere $S_q^{2\ell+1}$ of [VS91], which for $\ell > 1$ and $0 < q < 1$ is described by the $*$ -algebra $\mathcal{A}(S_q^{2\ell+1})$ generated by $\{z_i, z_i^*\}_{i=1, \dots, \ell+1}$ with relations

$$\begin{aligned} z_i z_j &= q z_j z_i & \forall i < j, \\ z_i^* z_j &= q z_j z_i^* & \forall i \neq j, \\ [z_1^*, z_1] &= 0, \\ [z_{i+1}^*, z_{i+1}] &= (1 - q^2) \sum_{j=1}^i z_j z_j^* & \forall i = 1, \dots, \ell, \\ z_1 z_1^* + z_2 z_2^* + \dots + z_{\ell+1} z_{\ell+1}^* &= 1. \end{aligned}$$

In the next section we present the left regular representation of $\mathcal{A}(S_q^{2\ell+1})$ and the ‘optimal’ Dirac operator D constructed in [CP07]. Then, in Section 5.2 we define a representation which generalizes to all odd-dimensional sphere the ‘symbol map’ of $S_q^3 = SU_q(2)$ of [Con04, DLS⁺05b] and which differs from the left regular representation only by infinitesimals, that in the present case means operators whose noncommutative integral is zero. Finally, in Section 5.3 we use the symbol map to compute the noncommutative integral associated with D . Section 5.4 contains the proof of the main result of the chapter, that is Theorem 5.3. We argue that these results may be used to construct spectral triples on quantum projective spaces, which are the ‘virtual’ spaces underlying the $*$ -algebra $\mathcal{A}(\mathbb{C}P_\ell^q) \subset \mathcal{A}(S_q^{2\ell+1})$ generated by the elements $p_{ij} := z_i^* z_j$.

5.1 The left regular representation

We use, with minor changes, the notations of [CP07]. Irreducible unitary corepresentations of $\mathcal{A}(SU_q(\ell+1))$ are labelled by Young tableaux $n = (n_1, \dots, n_\ell)$, that are ℓ -tuple of non-negative integers satisfying $n_1 \geq n_2 \geq \dots \geq n_\ell \geq 0$. We denote V_n the corresponding representation space. Basis vectors of V_n can be indexed by Gelfand-Tsetlin tableaux (GT tableaux, for short), which are arrays of integers of the form

$$\mathbf{r} = \begin{pmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,\ell-1} & r_{1,\ell} & 0 \\ r_{2,1} & r_{2,2} & \dots & r_{2,\ell-1} & r_{2,\ell} & \\ r_{3,1} & r_{3,2} & \dots & r_{3,\ell-1} & & \\ \dots & \dots & \dots & & & \\ r_{\ell,1} & r_{\ell,2} & & & & \\ r_{\ell+1,1} & & & & & \end{pmatrix} \quad (5.1)$$

with $r_{1,i} = n_i$ and $r_{i,j} \geq r_{i+1,j} \geq r_{i,j+1}$ for all i, j (from now on it is understood that $r_{ij} = 0$ if i, j are out of range). Usually GT tableaux are defined modulo a global rescaling (two arrays are equivalent if they differ by a constant); here for each equivalence class of GT tableaux, we choose the representative which has zero in the top-right corner, $r_{1,\ell+1} = 0$.

If \mathcal{H} is the Hilbert space completion of $\mathcal{A}(S_q^{2\ell+1})$ with respect to the inner product induced by the Haar state, an orthonormal basis $|n, h; \mathbf{r}\rangle$ of \mathcal{H} is indexed by $n, h \in \mathbb{N}$ and \mathbf{r} is a GT tableau with Young tableau $(r_{1,1}, \dots, r_{1,\ell}) = (n+h, h, \dots, h)$.

For $k = 1, \dots, \ell+1$ we denote

$$\Gamma^{(k)} := \{(m_1, \dots, m_k) \in \mathbb{N}^k \mid 1 \leq m_i \leq \ell+2-i \forall i = 1, \dots, k\}.$$

For $m \in \Gamma^{(k)}$ and \mathbf{r} a GT tableau, we denote $\mathbf{r}^{(m)}$ the array with elements

$$r_{ij}^{(m)} := -\delta_{m_1, \ell+1} + \begin{cases} r_{ij} + 1 & \text{if } j = m_i, 1 \leq i \leq k, \\ r_{ij} & \text{otherwise.} \end{cases}$$

The factor $-\delta_{m_1, \ell+1}$ is needed to select the representative of the GT tableau equivalence class which has $r_{1,\ell+1} = 0$.

The left regular representation $\pi : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{B}(\mathcal{H})$ can be determined by using the inclusion into $\mathcal{A}(SU_q(\ell+1))$ defined by $z_{\ell+2-i}^* \mapsto q^{-i+1}u_i^1$, with u_j^i the generators of the quantum $SU(\ell+1)$ group (cf. [CP07]). It is defined on generators by

$$\begin{aligned} \pi(z_{\ell+2-i}^*) |n, h; \mathbf{r}\rangle &= q^{-\frac{1}{2}(\ell+h)} \frac{[n+1]^{\frac{1}{2}}}{[n+h+\ell+1]^{\frac{1}{2}}} \sum_{m \in \Gamma^{(i)}: m_1=1} q^{\ell+h} q^{-i+1} C_q(i, \mathbf{r}, m) |n+1, h; \mathbf{r}^{(m)}\rangle \\ &+ q^{-\frac{1}{2}n} \frac{[h+\ell-1]^{\frac{1}{2}}}{[n+h+\ell-1]^{\frac{1}{2}}} \sum_{m \in \Gamma^{(i)}: m_1=\ell+1} q^{-i+1} C_q(i, \mathbf{r}, m) |n, h-1; \mathbf{r}^{(m)}\rangle \quad (5.2) \end{aligned}$$

where we set to zero the summands for which $\mathbf{r}^{(m)}$ is *not* a GT tableau. The coefficients $C_q(i, \mathbf{r}, \mathbf{r}^{(m)})$ are the Clebsch-Gordan coefficients in equation (5.3) of [CP07] (their sign can be found at page 220 of [KS97]). Their explicit value is

$$\begin{aligned}
C_q(i, \mathbf{r}, m) &:= q^{\frac{1}{2}(1+m_1-2m_i-r_{1,m_1}+2r_{i,m_i}-r_{i,\ell+2-i}+\sum_{k=1}^{\ell+1-i}(r_{i+1,k}-r_{i,k}))} \left\{ \prod_{k=1}^{i-1} \text{sign}(m_k - m_{k+1}) \right\} \times \\
&\times \left\{ \prod_{k=1}^{i-1} \prod_{m_k \neq j=1}^{\ell+2-k} \frac{[r_{k,j} - r_{k+1,m_{k+1}} - j + m_{k+1}]}{[r_{k,j} - r_{k,m_k} - j + m_k]} \prod_{m_{k+1} \neq j=1}^{\ell+1-k} \frac{[r_{k+1,j} - r_{k,m_k} - j + m_k - 1]}{[r_{k+1,j} - r_{k+1,m_{k+1}} - j + m_{k+1} - 1]} \right\}^{1/2} \times \\
&\times \left(\frac{\prod_{k=1}^{\ell+1-i} [r_{i+1,k} - r_{i,m_i} - k + m_i - 1]}{\prod_{m_i \neq k=1}^{\ell+2-i} [r_{i,k} - r_{i,m_i} - k + m_i]} \right)^{1/2}, \tag{5.3}
\end{aligned}$$

where we set $\text{sign}(0) := 1$, empty sums are set to 0 and empty products are set to 1. This representation together with the ‘optimal’ Dirac operator D of [CP07], given by¹

$$D |n, h; \mathbf{r}\rangle = \begin{cases} -(h+1) |n, h; \mathbf{r}\rangle & \text{if } n = 0, \\ (n+h+1) |n, h; \mathbf{r}\rangle & \text{if } n > 0, \end{cases} \tag{5.4}$$

defines a spectral triple on $S_q^{2\ell+1}$. A peculiarity of this Dirac operator is that for any other equivariant Dirac operator D' (on the same Hilbert space) there exists two positive constants a, b such that $|D'| \leq a|D| + b$, which explains the name ‘optimal’. This essentially means that D is the equivariant Dirac operator corresponding to the minimal metric dimension.

From an analysis of the case $\ell = 1$, where the bounded commutator condition fails for $q = 1$ (cf. remarks 5.1 and 5.5 in [CP03a]), we argue that these spectral triples constructed with ‘scalar functions’ instead of spinors are purely quantum objects.

Notice that the coefficients in front of the sums in (5.2) are never greater than 1. Moreover as proved in [CP07, eq. (4.10)], $C_q(i, \mathbf{r}, m)$ are bounded by a constant (which is independent of n, h, \mathbf{r}). Then, the coefficients in the first line of (5.2) are bounded by q^h . If we neglect terms bounded by q^h , the first line of (5.2) disappears. We’ll show in Section 5.3 that this ‘approximation modulo q^h ’ yields a remarkable simplification of the left regular representation.

5.2 The symbol map

Although it is a very simple example, the equatorial Podleś sphere seems to be the building block for the theory of q -deformed spaces, when doing local computations. In many examples (cf. [Con04, DLS⁺05b, DDLW07, DDL08], the last two being discussed also in Chapters 3 and 4 of this dissertation) one finds that representations of quantum spaces are described, modulo ‘infinitesimals’, by suitable tensor products of representations of the coordinate algebra $\mathcal{A}(S_q^2)$ of the equatorial Podleś sphere. This is indeed a general feature of all odd-dimensional quantum spheres, as we shall see later on.

¹We shifted the spectrum of $|D|$ in [CP07] by $+1$ to get an invertible Dirac operator.

Recall that $\mathcal{A}(S_q^2)$ is the $*$ -algebra generated by $A = A^*$, B and B^* with relations:

$$AB = qBA, \quad BB^* + A^2 = 1, \quad B^*B + q^2A^2 = 1.$$

Proposition 5.1. *Let u be the unitary generator of $\mathcal{A}(S^1)$. There is a $*$ -algebra morphism $\varphi : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{A}(S^1) \otimes \mathcal{A}(S_q^2)^{\otimes \ell}$, given by ($2 \leq i \leq \ell$)*

$$\begin{aligned} \varphi(z_1) &:= u \otimes \overbrace{A \otimes \dots \otimes A}^{\ell \text{ times}}, \\ \varphi(z_i) &:= \overbrace{1 \otimes \dots \otimes 1}^{i-1 \text{ times}} \otimes B \otimes \overbrace{A \otimes \dots \otimes A}^{\ell+1-i \text{ times}}, \\ \varphi(z_{\ell+1}) &:= \overbrace{1 \otimes \dots \otimes 1}^{\ell \text{ times}} \otimes B. \end{aligned}$$

Proof. One proves by direct computation that the elements $\{\varphi(z_j), \varphi(z_j)^*\}_{j=1, \dots, \ell+1}$ satisfy all the defining relations of $\mathcal{A}(S_q^{2\ell+1})$. \square

Let Λ be the set of $(n, h; \mathbf{a}, \mathbf{b})$ with $n, h \in \mathbb{N}$, $\mathbf{a} = (a_1, \dots, a_{\ell-1}) \in \mathbb{Z}^{\ell-1}$ and $\mathbf{b} = (b_0, \dots, b_{\ell-1}) \in \mathbb{Z}^{\ell}$ satisfying the constraints

$$0 \leq a_1 \leq a_2 \leq \dots \leq a_{\ell-1} \leq n, \quad (5.5a)$$

$$b_0 \leq b_1 \leq b_2 \leq \dots \leq b_{\ell-1} \leq h. \quad (5.5b)$$

We call $\hat{\mathcal{H}}$ the Hilbert space with orthonormal basis $\|n, h; \mathbf{a}, \mathbf{b}\rangle$ labelled by $(n, h; \mathbf{a}, \mathbf{b}) \in \Lambda$.

For $i = 1, \dots, \ell + 1$, we denote by $\underline{i} \in \{0, 1\}^{\ell}$ the array

$$\underline{i} := (\overbrace{0, 0, \dots, 0}^{i-1 \text{ times}}, \overbrace{1, 1, \dots, 1}^{\ell+1-i \text{ times}}).$$

The proof of the following proposition is a simple computation.

Proposition 5.2. *The formula*

$$\rho_1(u) \|n, h; \mathbf{a}, \mathbf{b}\rangle = \|n, h + 1; \mathbf{a}, \mathbf{b} + \underline{1}\rangle,$$

defines a $$ -representation $\rho_1 : \mathcal{A}(S^1) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$. For $2 \leq i \leq \ell + 1$, the formulæ (here $b_{\ell} := h$)*

$$\begin{aligned} \rho_i(A) \|n, h; \mathbf{a}, \mathbf{b}\rangle &= q^{b_{i-1} - b_{i-2}} \|n, h; \mathbf{a}, \mathbf{b}\rangle, \\ \rho_i(B) \|n, h; \mathbf{a}, \mathbf{b}\rangle &= \sqrt{1 - q^{2(b_{i-1} - b_{i-2} + 1)}} \|n, h + 1; \mathbf{a}, \mathbf{b} + \underline{i}\rangle, \end{aligned}$$

define a $$ -representation $\rho_i : \mathcal{A}(S_q^2) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$. Distinct representations are mutually commuting.*

Modulo multiplicities, each representation ρ_i ($i \neq 1$) is equivalent to the representation of $\mathcal{A}(S_q^2) = \mathcal{A}(S_{q^{1/2}}^2)$ given by (3.37a). By applying $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_{\ell+1}$ to $\varphi(a)$, $a \in \mathcal{A}(S_q^{2\ell+1})$, and multiplying all the factors in the tensor product we get a $*$ -representation $\rho : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$.

We want to compare this representation with the left regular representation π . As one can easily guess, they agree modulo ‘infinitesimals’.

Notice that since the spectrum of $\rho_i(A)$ is positive, the image of each ρ_i ($i \neq 1$) is a noncommutative disk $\mathcal{A}(D_q^2)$. Thus the map ρ has image in $\mathcal{A}(S^1 \times (D_q^2)^{\times \ell})$. Its extension to order zero pseudo-differential operators would be the higher-dimensional analogue of the *symbol map* of [Con04] (cf. also [DLS⁺05a]). The image of such extension would generalize to arbitrary ℓ the *cosphere bundle* of $S_q^3 = SU_q(2)$ worked out in [Con04, DLS⁺05a].

5.3 The noncommutative integral

We have a representation $\pi : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{H} \simeq L^2(S_q^{2\ell+1})$ has an orthonormal basis $|n, h; \mathbf{r}\rangle$ labelled by GT tableau \mathbf{r} with Young tableau $(r_{1,1}, \dots, r_{1,\ell}) = (n+h, h, \dots, h)$. In this basis, the operators $\pi(z_i)$ are given by (5.2). The space \mathcal{H} can be included in $\hat{\mathcal{H}}$ as follows.

A careful look at the relations defining a GT tableau shows that any GT tableau with Young tableau $(r_{1,1}, \dots, r_{1,\ell}) = (n+h, h, \dots, h)$ has $h = r_{1k} \geq r_{jk} \geq r_{1,j+k-1} = h$, for each $k \geq 2$ and $j+k \leq \ell+1$, i.e. it is of the form

$$\mathbf{r} = \begin{pmatrix} n+h & h & \dots & h & h & 0 \\ r_{2,1} & h & \dots & h & r_{2,\ell} & \\ r_{3,1} & h & \dots & r_{3,\ell-1} & & \\ \dots & \dots & \dots & & & \\ r_{\ell-1,1} & h & r_{\ell-1,3} & & & \\ r_{\ell,1} & r_{\ell,2} & & & & \\ r_{\ell+1,1} & & & & & \end{pmatrix}, \quad (5.6)$$

and we have $(n, h; \mathbf{a}(\mathbf{r}), \mathbf{b}(\mathbf{r})) \in \Lambda$, where we called

$$a_j(\mathbf{r}) := r_{\ell+1-j,1} - h, \quad b_k(\mathbf{r}) := h - r_{\ell+1-k,k+1}.$$

On the other hand, if $(n, h; \mathbf{a}, \mathbf{b}) \in \Lambda$ satisfy the additional constraints

$$b_1 \geq 0 \quad \text{and} \quad b_0 \geq -a_1, \quad (5.7)$$

then the array

$$\mathbf{r}(n, h; \mathbf{a}, \mathbf{b}) := \begin{pmatrix} n+h & h & \dots & h & h & 0 \\ a_{\ell-1}+h & h & \dots & h & h-b_{\ell-1} & \\ a_{\ell-2}+h & h & \dots & h-b_{\ell-2} & & \\ \dots & \dots & \dots & & & \\ a_2+h & h & h-b_2 & & & \\ a_1+h & h-b_1 & & & & \\ h-b_0 & & & & & \end{pmatrix} \quad (5.8)$$

is a GT tableau. We define a partial isometry $W : \mathcal{H} \rightarrow \widehat{\mathcal{H}}$ and its adjoint $W^* : \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ by

$$W |n, h; \mathbf{r}\rangle = \|\!|n, h; \mathbf{a}(\mathbf{r}), \mathbf{b}(\mathbf{r})\!\rangle\!\rangle$$

$$W^* \|\!|n, h; \mathbf{a}, \mathbf{b}\!\rangle\!\rangle = \begin{cases} |n, h; \mathbf{r}(n, h; \mathbf{a}, \mathbf{b})\rangle & \text{if } (\mathbf{a}, \mathbf{b}) \text{ satisfy (5.7),} \\ 0 & \text{otherwise .} \end{cases}$$

Clearly, $W^*W = id_{\mathcal{H}}$.

Next theorem is the main result of this chapter. In this chapter, we call *shift operator* an operator on \mathcal{H} which, in matrix form in the basis $|n, h; \mathbf{r}\rangle$, has at most one matrix element different from zero in each row, and at most one matrix element different from zero in each column. Its matrix elements will be called *weights* of the shift.

Theorem 5.3. *For all $i = 1, \dots, \ell + 1$, the operator $\pi(z_i) - W^*\rho(z_i)W$ is a finite sum of shift operators on \mathcal{H} with weights bounded by q^h .*

Since the proof is quite long, we relegate it to Section 5.4.

For $\ell > 1$, terms which we are neglecting (shifts with weights bounded by q^h) are not smoothing operators for the ‘optimal’ Dirac operator D in (5.4). Hence this approximation does not allow us to compute the dimension spectrum and the local index formula on $S_q^{2\ell+1}$.

The situation is different if we consider the subalgebra $\mathcal{A}(\mathbb{CP}_q^\ell)$, generated by the elements $p_{ij} := z_i^* z_j$. For any fixed $N \in \mathbb{N}$, let \mathcal{H}_N be the Hilbert subspace of \mathcal{H} which is the closure of the linear span

$$\text{Span}\{ |n + N, n; \mathbf{r}\rangle \mid n \geq 0 \} .$$

As one can easily check on generators, $\mathcal{H}_N \subset \mathcal{H}$ is an invariant subspace for $\mathcal{A}(\mathbb{CP}_q^\ell)$. If we call $\pi_N(a)$ the restriction of $\pi(a)$ to \mathcal{H}_N , for all $a \in \mathcal{A}(\mathbb{CP}_q^\ell)$, we get a bounded $*$ -representation $\pi_N : \mathcal{A}(\mathbb{CP}_q^\ell) \rightarrow \mathcal{B}(\mathcal{H}_N)$. A corollary of Theorem 5.3 is the following.

Corollary 5.4. *For any generator p_{ij} of $\mathcal{A}(\mathbb{CP}_q^\ell)$, $\pi_N(p_{ij}) - W^*\rho(p_{ij})W$ is a finite sum of shift operators on \mathcal{H}_N with weights bounded by q^n (which is a rapid decay matrix, and then a smoothing operator for D).*

With this, it is possible (but very involved) to construct Fredholm modules and spectral triples for quantum projective spaces.

We now use Theorem 5.3 to compute the noncommutative integral on $S_q^{2\ell+1}$. As a first step, we show that $|D|^{-2\ell-1}$ is in the Dixmier ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$ and has non-vanishing noncommutative integral; this proves that the metric dimension is $2\ell + 1$ (as argued in [CP07]).

Proposition 5.5. *The spectral triple $(\mathcal{A}(S^{2\ell+1}), \mathcal{H}, D)$ has metric dimension $2\ell + 1$, that is $|D|^{-2\ell-1} \in \mathcal{L}^{1,\infty}(\mathcal{H})$ and $\text{Tr}_\omega(|D|^{-2\ell-1}) = C_\ell \neq 0$. The normalization constant C_ℓ is given by*

$$C_\ell := \frac{1}{\ell!} \sum_{i=0}^{\ell-1} \frac{(-1)^i}{i!(\ell+i)!} . \quad (5.9)$$

Proof. From Weyl's character formula [IN66] we know that the dimension of the irreducible corepresentation of $SU_q(\ell + 1)$ with Young tableau (n_1, \dots, n_ℓ) is

$$\frac{\prod_{1 \leq j \leq k \leq \ell} \left(k - j + 1 + \sum_{i=j}^k \lambda_i \right)}{\prod_{k=1}^{\ell} k!}, \quad (5.10)$$

where $(\lambda_1, \dots, \lambda_\ell)$ is the highest weight of the representation, given by $\lambda_i = n_i - n_{i+1}$. The multiplicity μ_k of the eigenvalue $k := n + h + 1$ of $|D|$ is given by previous expression computed for $(n_1, n_2, \dots, n_\ell) = (k-1, h, \dots, h)$, i.e. $\lambda_i = (k-h-1)\delta_{i,1} + h\delta_{i,\ell}$, and summed for $h = 0, \dots, k-1$, that is

$$\begin{aligned} \mu_k &= \sum_{h=0}^{k-1} \frac{\prod_{2 \leq i \leq j \leq \ell-1} (j-i+1)}{\prod_{j=1}^{\ell} j!} (\ell+k-1) \prod_{j=1}^{\ell-1} \{j+k-h-1\} \prod_{j=2}^{\ell} \{\ell+1-j+h\} \\ &= \frac{k+\ell-1}{\ell!(\ell-1)!} \sum_{h=0}^{k-1} \prod_{j=1}^{\ell-1} (j+k-h-1)(j+h) \\ &= \frac{k+\ell-1}{\ell!(\ell-1)!} \sum_{m=1}^k \prod_{j=1}^{\ell-1} \{(j+k)(j-1) + (k+1)m - m^2\} \\ &\sim \frac{k+\ell-1}{\ell!(\ell-1)!} \sum_{m=1}^k (k+1-m)^{\ell-1} m^{\ell-1} \\ &= \frac{k+\ell-1}{\ell!(\ell-1)!} \sum_{m=1}^k \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell-1}{i} (k+1)^{\ell-1-i} m^{i+\ell-1} \\ &= \frac{k+\ell-1}{\ell!(\ell-1)!} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell-1}{i} (k+1)^{\ell-1-i} H_k^{(-r)} \Big|_{r=i+\ell-1}, \end{aligned}$$

with $H_k^{(-r)} := \sum_{m=1}^k m^r$ and ' \sim ' means modulo lower order in k . For $r \in \mathbb{N}$ by Faulhaber's formula we have

$$H_k^{(-r)} = \frac{1}{r+1} \sum_{j=0}^r \binom{r+1}{j} B_j k^{r+1-j}$$

where B_j are Bernoulli numbers; in particular $B_0 = 1$. Thus,

$$\begin{aligned} \mu_k &\sim \frac{k}{\ell!} \sum_{i=0}^{\ell-1} \sum_{j=0}^{i+\ell-1} \frac{(-1)^i}{i! j! (\ell+i-j)!} B_j k^{2\ell-j-1} \\ &\sim \frac{k^{2\ell}}{\ell!} \sum_{i=0}^{\ell-1} \frac{(-1)^i}{i! (\ell+i)!}. \end{aligned}$$

From this we deduce

$$\mathrm{Tr}(|D|^{-s}) = \sum_{k=1}^{\infty} \mu_k k^{-s} = C_\ell \zeta(s-2\ell) + \dots$$

where C_ℓ is the constant in (5.9), ζ is the Riemann zeta-function and the dots represent some function that is holomorphic on the half plane $\mathrm{Re} s > 2\ell$. This in particular means that

$|D|^{-(2\ell+1)}$ is in the Dixmier ideal. Its Dixmier trace coincides with the residue of the previous function in $s = 2\ell + 1$, which is given by

$$\mathrm{Tr}_\omega(|D|^{-(2\ell+1)}) = \mathrm{Res}_{s=2\ell+1} \mathrm{Tr}(|D|^{-s}) = C_\ell \mathrm{Res}_{s=2\ell+1} \zeta(s-2\ell) = C_\ell .$$

Let $\ell = 2n$ resp. $2n + 1$. In both cases:

$$\begin{aligned} \ell! C_\ell &\geq \sum_{i=0}^{2n-1} \frac{(-1)^i}{i! (\ell+i)!} = \sum_{i=1}^n \left(\frac{1}{(2i-2)! (\ell+2i-2)!} - \frac{1}{(2i-1)! (\ell+2i-1)!} \right) \\ &= \sum_{i=1}^n \frac{(2i-1)(\ell+2i-1) - 1}{(2i-1)! (\ell+2i-1)!} > 0 , \end{aligned}$$

being a sum of positive terms. The observation that $C_\ell > 0$ for any ℓ concludes the proof. \square

Consider now the bounded operators $L_{j,k}$ defined by

$$L_{j,k} |n, h; \mathbf{r}\rangle = q^{r_{j,k}} |n, h; \mathbf{r}\rangle , \quad (5.11)$$

and the two-sided ideal $\mathcal{J}_{2\ell} \subset \mathcal{B}(\mathcal{H})$ defined in (2.15).

Lemma 5.6. *For all $j, k = 1, \dots, \ell + 1$, we have $L_{j,k} \in \mathcal{J}_{2\ell}$.*

Proof. Since $q^{r_{j,k}} \leq q^{r_{2,\ell}}$ for all j, k , it is enough to prove the statement for $L_{2,\ell}$. We have

$$\mathrm{Tr}(L_{2,\ell} |D|^{-s}) = \sum_{k=1}^{\infty} k^{-s} \sum_{r_{2,1}=0}^{k-1} \sum_{h=0}^{r_{2,1}} \sum_{r_{2,\ell}=0}^h q^{r_{2,\ell}} c(k, r_{2,1}, h, r_{2,\ell}) ,$$

where $c(k, r_{2,1}, h, r_{2,\ell})$ is the number of \mathbf{r} of the form (5.6) with $(k, r_{2,1}, h, r_{2,\ell})$ fixed. Of course, $c(k, r_{2,1}, h, r_{2,\ell}) \leq c(k, r_{2,1}, h, 0)$ and we have $(\sum_{r_{2,\ell}=0}^h q^{r_{2,\ell}} < (1-q)^{-1})$

$$\mathrm{Tr}(L_{2,\ell} |D|^{-s}) < \frac{1}{1-q} \sum_{k=1}^{\infty} k^{-s} \sum_{i=0}^{k-1} \sum_{h=0}^i c(k, i, h, 0) .$$

Now, looking at (5.6) one realizes that $c(k, i, h, 0)$ is the dimension of the irreducible representation of $SU(\ell)$ with Young tableau (i, h, \dots, h) , so $\sum_{h=0}^i c(k, i, h, 0) = \mu_{i+1}$ where μ_{i+1} are the multiplicities appearing in the proof of Proposition 5.5, but for a replacement $\ell + 1 \rightarrow \ell$. We know that μ_{i+1} is a polynomial in $i + 1$ of order $2\ell - 2$, and so $\sum_{i=0}^{k-1} \mu_{i+1}$ is a polynomial in k of order $2\ell - 1$. The observation that

$$\sum_{k=1}^{\infty} k^{-s+2\ell-1}$$

converges for $s > 2\ell$ concludes the proof. \square

By Theorem 5.3 the difference $\pi(z_i) - W^* \rho(z_i) W$ is the product of a bounded operator by $L_{1,\ell} (q^{r_{1,\ell}} = q^h)$, hence by Lemma 5.6 it gives an element of $\mathcal{J}_{2\ell}$. Repeating verbatim the proof

of Lemma 3.12 one proves that $\pi(a) - W^* \rho(a) W \in \mathcal{J}_{2\ell}$ for all $a \in \mathcal{A}(S_q^{2\ell+1})$, which in particular means

$$\int a |D|^{-2\ell-1} := \text{Res}_{s=2\ell+1} \text{Tr}(\pi(a) |D|^{-s}) = \text{Res}_{s=2\ell+1} \text{Tr}(W^* \rho(a) W |D|^{-s}).$$

With this, it's easy to compute the noncommutative integral.

Proposition 5.7. *Let $\sigma : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{A}(S^1)$ be the $*$ -algebra morphism defined by $\sigma(z_i) := 0$ for $i \neq \ell + 1$, $\sigma(z_{\ell+1})(\theta) := e^{i\theta}$. For all $a \in \mathcal{A}(S_q^{2\ell+1})$ we have*

$$\int a |D|^{-2\ell-1} = \frac{C_\ell}{2\pi} \int_0^{2\pi} \sigma(a)(\theta) d\theta, \quad (5.12)$$

with C_ℓ given by (5.9).

Proof. It is enough to prove (5.12) for

$$a = z_1^{a_1} (z_1^*)^{a'_1} \dots z_\ell^{a_\ell} (z_\ell^*)^{a'_\ell} z_{\ell+1}^j$$

with $a_i, b_i \in \mathbb{N}$, $j \in \mathbb{Z}$ and the notation $z_{\ell+1}^j = (z_{\ell+1}^*)^{-|j|}$ if $j < 0$, since these monomials are a linear basis for the algebra. We have

$$\frac{C_\ell}{2\pi} \int_0^{2\pi} \sigma(a)(\theta) d\theta = C_\ell \delta_{a_1,0} \delta_{a'_1,0} \dots \delta_{a_\ell,0} \delta_{a'_\ell,0} \delta_{j,0}.$$

Thus if $a = 1$ the equality (5.12) follows from Proposition 5.5, while for $a \neq 1$ we have to prove that also the left hand side of (5.12) is zero. By previous Lemma we have

$$\int a |D|^{-2\ell-1} = \text{Res}_{s=2\ell+1} \text{Tr}(W^* \rho(a) W |D|^{-s}).$$

For $i \neq \ell+1$, coefficients of $\rho(z_i)$ are bounded by $q^{r_{\ell+2-i,i}}$ (cf. equation (5.17)). Thus $W^* \rho(a) W \in \mathcal{J}_{2\ell}$ and $\int a |D|^{-2\ell-1} = 0$, unless $a_1 = a'_1 = \dots = a_\ell = a'_\ell = 0$. It remains to prove that $\int z_{\ell+1}^j |D|^{-2\ell-1} = 0$ if $j \neq 0$, but this is trivial since $\rho(z_{\ell+1})$ is an off-diagonal operator. \square

5.4 On the behavior of the coefficients (5.3)

The purpose of this Section is to prove Theorem 5.3. Recall (5.2) and (5.3).

The coefficients in front of the sums in (5.2) are bounded by 1. Then, only the behavior of the coefficients inside the sums matters. Since $C_q(i, \mathbf{r}, m)$ are bounded by a constant, we immediately deduce that coefficients in the first line of (5.2) are bounded by q^h and can be neglected. We now focus on the second line, i.e. on the case $m_1 = \ell + 1$.

By equation (4.10) of [CP07] there exists a positive constant C such that

$$|C_q(i, \mathbf{r}, m)| \leq C q^{K(i, \mathbf{r}, m) + \sum_{j=m_i}^{\ell+1-i} (r_{i+1,j} - r_{i,j+1})} \leq C q^{K(i, \mathbf{r}, m)} \quad (5.13)$$

where

$$K(i, \mathbf{r}, m) := \sum_{k=1}^{i-1} \left(\sum_{j=\min(m_k, m_{k+1})}^{\max(m_k, m_{k+1})-1} (r_{k+1, j} - r_{k, j+1}) + 2 \sum_{j=m_{k+1}+1}^{m_k-1} (r_{k, j} - r_{k+1, j}) \right).$$

All terms with $K(i, \mathbf{r}, m) \geq h$ will be neglected, since in this case by (5.13) we have $|C_q(i, \mathbf{r}, m)| \leq Cq^h$. Recall that we are interested in the case $(r_{1,1}, \dots, r_{1,\ell}) = (n+h, h, \dots, h)$ and $m_1 = \ell + 1$. The first step is to prove the following Lemma.

Lemma 5.8. *Let $m_1 = \ell + 1$ and $(r_{1,1}, \dots, r_{1,\ell}) = (n+h, h, \dots, h)$. For any (admissible) $(m_2, \dots, m_i) \neq (\ell, \ell-1, \dots, \ell+2-i)$ we have $K(i, \mathbf{r}, m) \geq h$.*

Proof. We prove it by induction on i starting from $i = 2$ (for $i = 1$ it is trivial since $m = (m_1) = (\ell + 1)$ is fixed). For $i = 2$ recalling that $m_i \leq \ell + 2 - i$, we get

$$\begin{aligned} K(2, \mathbf{r}, m) &= \sum_{j=\min(m_1, m_2)}^{\max(m_1, m_2)-1} (r_{2, j} - r_{1, j+1}) + 2 \sum_{j=m_2+1}^{m_1-1} (r_{1, j} - r_{2, j}) \\ &\geq (r_{2, \ell} - r_{1, \ell+1}) + 2 \sum_{j=m_2+1}^{\ell} (r_{1, j} - r_{2, j}). \end{aligned}$$

If $m_2 = \ell$ last sum is empty, otherwise (i.e. when $m_2 < \ell$) it has at least the term with $j = \ell$ and so

$$K(2, \mathbf{r}, m) \geq (r_{2, \ell} - r_{1, \ell+1}) + 2(r_{1, \ell} - r_{2, \ell}) = r_{1, \ell} + (r_{1, \ell} - r_{2, \ell}) \geq r_{1, \ell} = h.$$

Hence $m_2 \neq \ell$ implies $K(2, \mathbf{r}, m) \geq h$.

Now we prove the inductive step. Assume the Lemma is true for i . Let $m = (m_1, \dots, m_i)$ and $m' = (m, m_{i+1})$, and notice that

$$K(i+1, \mathbf{r}, m') \geq K(i, \mathbf{r}, m).$$

If $m \neq (\ell+1, \dots, \ell+2-i)$ we have $K(i, \mathbf{r}, m) \geq h$ (inductive hypothesis) and so $K(i+1, \mathbf{r}, m') \geq h$. It remains to prove that even for $m = (\ell+1, \dots, \ell+2-i)$, if $m_{i+1} \neq \ell+1-i$ we still have $K(i+1, \mathbf{r}, m') \geq h$.

Take then m as above and $m_{i+1} < \ell+1-i = m_i - 1$. Notice that in this case

$$K(i, \mathbf{r}, m) = \sum_{k=1}^{i-1} (r_{k+1, \ell+1-k} - r_{k, \ell+2-k}) = r_{i, \ell+2-i},$$

(being a telescopic sum) and so

$$\begin{aligned} K(i+1, \mathbf{r}, m') &= K(i, \mathbf{r}, m) + \left(\sum_{j=m_{i+1}}^{m_i-1} (r_{i+1, j} - r_{i, j+1}) + 2 \sum_{j=m_{i+1}+1}^{m_i-1} (r_{i, j} - r_{i+1, j}) \right) \\ &\geq r_{i, \ell+2-i} + (r_{i+1, \ell+1-i} - r_{i, \ell+2-i}) + 2(r_{i, \ell+1-i} - r_{i+1, \ell+1-i}) \end{aligned}$$

$$= r_{i,\ell+1-i} + (r_{i,\ell+1-i} - r_{i+1,\ell+1-i}) \geq r_{i,\ell+1-i} \geq r_{1,\ell} = h .$$

In the first inequality we neglect all the terms in the sums with $j \neq m_i - 1 = \ell + 1 - i$ (both sums are non-empty since by hypothesis $m_{i+1} < \ell + 1 - i$). In the second inequality we neglect the positive term inside the parenthesis. Last inequality is due to the definition of GT tableaux. This concludes the proof. \square

By previous Lemma all terms in $\pi(z_{\ell+2-i}^*)$ have weights bounded by q^h , but for the one corresponding to $m = m^i := (\ell + 1, \ell, \dots, \ell + 2 - i)$. Thus,

$$\begin{aligned} \pi(z_{\ell+2-i}^*) |n, h; \mathbf{r}\rangle &= q^{-\frac{1}{2}n} \frac{[h + \ell - 1]^{\frac{1}{2}}}{[n + h + \ell - 1]^{\frac{1}{2}}} q^{-i+1} C_q(i, \mathbf{r}, m^i) |n, h - 1; \mathbf{r}^{(m^i)}\rangle + \dots \\ &= q^{-i+1} C_q(i, \mathbf{r}, m^i) |n, h - 1; \mathbf{r}^{(m^i)}\rangle + \dots , \end{aligned}$$

where dots denote the terms which we are neglecting. The next step is to simplify $C_q(i, \mathbf{r}, m^i)$ by neglecting terms bounded by q^h . Plugging $m_k = \ell + 2 - k$ ($k \leq i$) and $(r_{1,1}, \dots, r_{1,\ell}) = (n + h, h, \dots, h)$ into equation (5.3) we get, after some algebraic manipulations,

$$\begin{aligned} q^{-i+1} C_q(i, \mathbf{r}, m^i) &= q^{r_{i,m_i}} \left\{ \prod_{k=1}^{\ell+1-i} \frac{1 - q^{2(r_{i+1,k} - r_{i,m_i} - k + m_i - 1)}}{1 - q^{2(r_{i,k} - r_{i,m_i} - k + m_i)}} \right\}^{1/2} \\ &\times \left\{ \prod_{k=1}^{i-1} \prod_{j=1}^{\ell+1-k} \frac{1 - q^{2(r_{k,j} - r_{k+1,m_{k+1}} - j + m_{k+1})}}{1 - q^{2(r_{k,j} - r_{k,m_k} - j + m_k)}} \prod_{j=1}^{\ell-k} \frac{1 - q^{2(r_{k+1,j} - r_{k,m_k} - j + m_k - 1)}}{1 - q^{2(r_{k+1,j} - r_{k+1,m_{k+1}} - j + m_{k+1} - 1)}} \right\}^{1/2} . \end{aligned} \quad (5.14)$$

We notice that (by definition of GT tableaux) all arguments of the exponentials are non-negative (positive if in the denominator). In fact, last denominator vanishes if $j = \ell - k$ and $r_{k+1,\ell-k} = r_{k+1,\ell-k+1}$, but this ‘singularity’ is just apparent. For $r_{k+1,\ell-k} = r_{k+1,\ell-k+1}$ the move $\mathbf{r} \rightarrow \mathbf{r}^{(m^i)} =: \mathbf{r}'$ shifts $r_{k+1,\ell-k}$ by -1 and leaves $r_{k+1,\ell-k+1}$ invariant: hence the result is that $r'_{k+1,\ell-k} < r'_{k+1,\ell-k+1}$ and \mathbf{r}' is not a GT tableaux. In conclusion, the term with $r_{k+1,\ell-k} = r_{k+1,\ell-k+1}$ (for which the denominator vanishes) never appears.

We use the notation $a \lesssim b$ if $a \leq Cb$ and C is a positive constant depending only on q .

Lemma 5.9. *For any $0 < u \leq 1$ we have*

$$0 \leq 1 - (1 - u)^{\frac{1}{2}} \leq u , \quad 0 \leq (1 - qu)^{-\frac{1}{2}} - 1 \lesssim u . \quad (5.15)$$

Proof. That the differences in (5.15) are non-negative is obvious. Since $0 \leq 1 - u < 1$ we have

$$1 - u \leq \sqrt{1 - u} ,$$

and this proves the first inequality in (5.15). We have also

$$\frac{1}{\sqrt{1 - qu}} - 1 = \left(\frac{1}{\sqrt{1 - qu}} + 1 \right)^{-1} \left(\frac{1}{1 - qu} - 1 \right)$$

$$\begin{aligned}
 &= \left(\frac{1}{\sqrt{1-qu}} + 1 \right)^{-1} \frac{qu}{(1-qu)} \\
 &\leq 2^{-1} \frac{qu}{1-q} \lesssim u,
 \end{aligned}$$

and this proves the second inequality. \square

Lemma 5.10. *For any $0 < u, v \leq 1$ we have*

$$\left| 1 - \sqrt{\frac{1-u}{1-qv}} \right| \lesssim \max(u, v). \quad (5.16)$$

Proof. The algebraic identity

$$1 - \sqrt{\frac{1-u}{1-qv}} = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1-qv}} \right) (1 + \sqrt{1-u}) + \frac{1}{2} \left(1 + \frac{1}{\sqrt{1-qv}} \right) (1 - \sqrt{1-u}),$$

together with the triangle inequality and the previous Lemma implies

$$\begin{aligned}
 \left| 1 - \sqrt{\frac{1-u}{1-qv}} \right| &\lesssim \frac{1}{2} v (1 + \sqrt{1-u}) + \frac{1}{2} \left(1 + \frac{1}{\sqrt{1-qv}} \right) u \\
 &\leq v + (1-q)^{-\frac{1}{2}} u \leq (1-q)^{-\frac{1}{2}} (u+v) \leq 2(1-q)^{-\frac{1}{2}} \max(u, v).
 \end{aligned}$$

This concludes the proof. \square

Applying (5.16) to the coefficients in (5.14) we get

$$\begin{aligned}
 q^{r_i, m_i} \left| 1 - \left\{ \frac{1 - q^{2(r_{i+1, j} - r_{i, m_i} - j + m_i - 1)}}{1 - q^{2(r_{i, j} - r_{i, m_i} - j + m_i)}} \right\}^{1/2} \right| &\lesssim q^{r_{i+1, j}} \leq q^h, \quad \forall 1 \leq j \leq \ell - i, \\
 q^{r_i, m_i} \left| 1 - \left\{ \frac{1 - q^{2(r_{k, j} - r_{k+1, m_{k+1}} - j + m_{k+1})}}{1 - q^{2(r_{k, j} - r_{k, m_k} - j + m_k)}} \right\}^{1/2} \right| &\lesssim q^{r_{k, j}} \leq q^h, \quad \forall 1 \leq j \leq \ell + 1 - k, \\
 q^{r_i, m_i} \left| 1 - \left\{ \frac{1 - q^{2(r_{k+1, j} - r_{k, m_k} - j + m_k - 1)}}{1 - q^{2(r_{k+1, j} - r_{k+1, m_{k+1}} - j + m_{k+1} - 1)}} \right\}^{1/2} \right| &\lesssim q^{r_{k+1, j}} \leq q^h, \quad \forall 1 \leq j \leq \ell - k,
 \end{aligned}$$

where we used $u \leq u^{1/2}$ for $u \leq 1$, $r_{i, m_i} - r_{k, m_k} \geq 0$ and $r_{i, m_i} - r_{k+1, m_{k+1}} \geq 0$ for $k \leq i - 1$, and the fact that $r_{j, k} \geq h$ for all $j + k < \ell + 2$.

Using previous inequalities in (5.14) we deduce

$$q^{-i+1} C_q(i, \mathbf{r}, m^i) = O(q^h) + \begin{cases} q^{r_i, m_i} \left\{ \frac{1 - q^{2(r_{i+1, m_{i+1}} - r_{i, m_i})}}{1 - q^{2(r_{i, m_{i+1}} - r_{i, m_i + 1})}} \right\}^{1/2} & \text{if } 0 \leq i \leq \ell, \\ q^{r_i, m_i} & \text{if } i = \ell + 1. \end{cases}$$

Now we apply the second inequality in (5.15) to prove that

$$q^{r_i, m_i} \left| 1 - \left\{ 1 - q^{2(r_{i, m_{i+1}} - r_{i, m_i + 1})} \right\}^{-\frac{1}{2}} \right| \leq q^{r_i, m_i} q^{2(r_{i, m_{i+1}} - r_{i, m_i})} \leq q^{r_{i, m_{i+1}}} \leq q^h.$$

Recalling that $r_{1,\ell+1} = 0$, we conclude that $|q^{-i+1}C_q(i, \mathbf{r}, m^i) - \tilde{C}_q(i, \mathbf{r}, m^i)| \lesssim q^h$, where

$$\tilde{C}_q(i, \mathbf{r}, m^i) := \begin{cases} \sqrt{1 - q^{2r_{2,\ell}}} & \text{if } i = 1, \\ q^{r_{i,\ell+2-i}} \sqrt{1 - q^{2(r_{i+1,\ell+1-i} - r_{i,\ell+2-i})}} & \text{if } i = 2, \dots, \ell, \\ q^{r_{\ell+1,1}} & \text{if } i = \ell + 1. \end{cases} \quad (5.17)$$

Summarizing, we have

$$\pi(z_{\ell+2-i}^*) |n, h; \mathbf{r}\rangle = \tilde{C}_q(i, \mathbf{r}, m^i) |n, h - 1; \mathbf{r}^{(m^i)}\rangle + \dots,$$

where ‘ \dots ’ are shift operators with weights bounded by q^h .

We are now in the position to prove Theorem (5.3). Recall that we have the left regular representation $\pi : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{B}(\mathcal{H})$, given by equation (5.2), the representation $\rho : \mathcal{A}(S_q^{2\ell+1}) \rightarrow \mathcal{B}(\hat{\mathcal{H}})$ of Section 5.2, and the partial isometry $W : \mathcal{H} \rightarrow \hat{\mathcal{H}}$ of Section 5.3.

By definition of ρ and W one finds that

$$W^* \rho(z_{\ell+2-i}^*) W |n, h; \mathbf{r}\rangle = \tilde{C}_q(i, \mathbf{r}, m^i) |n, h - 1; \mathbf{r}^{(m^i)}\rangle,$$

where the coefficients $\tilde{C}_q(i, \mathbf{r}, m^i)$ are exactly the ones defined in (5.17). Then, for all $i = 1, \dots, \ell + 1$, the operator $\pi(z_i) - W^* \rho(z_i) W$ is a finite sum of shift operators on \mathcal{H} with weights bounded by q^h . This concludes the proof of the Theorem.

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