Self-propulsion in viscous fluids through shape deformation

Marco Morandotti

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To my family,
for the endless support and patience.

To my friends,
the ones that are close and the ones that are far.

To the fools
who follow, love, sustain, and tolerate me,
for sharing their life with me.

Contents

List of Figures

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Abstract

In this thesis we address the problem of modeling swimming in viscous fluids. This is a fancy way to denote a fluid dynamics problem in which a deformable object is capable to advance in a low Reynolds number flow governed by the Stokes equations. The fluid is infinitely extended around the swimming body and the propulsive viscous force and torque are those generated by the fluid-swimmer interaction. No-slip boundary conditions are imposed: the velocity of the fluid and that of the swimmer are the same at the contact surface. Moreover, a self-propulsion constraint is enforced: no external forces or torques.

The problem is treated with techniques coming from the Calculus of Variations and Continuum Mechanics, through which it is possible to define the coefficients of the ordinary differential equations that govern the position and orientation parameters of the swimmer. In a three-dimensional setting, there are six of them. Conversely, the shape of the swimmer undergoes an infinite-dimensional control. The relations between the infinite-dimensional freely adjustable shape and the six position and orientation variables is given by an explicit linear relation between viscous forces and torques, on one side, and linear and angular velocities on the other.

Suitable function spaces are defined to let the variational techniques work, both in the case of a plain viscous fluid (governed by the Stokes system) and in the case of a particulate fluid, which we model using the Brinkman equation.

Finally, a control problem for a mono-dimensional swimmer in a viscous fluid is addressed. In this part, which is still work in progress, the existence of an optimal swimming strategy is proved, and the controllability of the swimmer is achieved by showing and explicit sequence of moves to advance. At the very last, the Euler equation for characterizing the optimal chape change is set up, and some comments on its structure are made.

CHAPTER 1

Introduction

What does it mean to swim? This is the important question that E. M. Purcell addressed in his 1977 paper *Life at low Reynolds number* [35].

Motion in fluids receives particular interest from the scientific community, since a huge number of interesting phenomena takes place in a fluid environment. Scientists are tackling this kind of problems since the early Nineties, and research has proceeded covering both the theoretical and the experimental aspects.

The aim of this work is to give a contribution to the study of self-propelled micro-swimmers immersed in a viscous fluid. Before introducing the new results, we will present the "state of the art" in the study of swimming at low Reynolds numbers. After presenting the main equation that we will use to govern the fluid velocity field, we conduct a chronological development of the main important contributions and examples to the subject, to conclude pointing out the novelty of the matter of the following chapters.

1.1 The Stokes equations

When talking about fluid dynamics the first, essential element that comes into mind is the celebrated *Navier-Stokes system*. In their general form, the Navier-Stokes equations expresse the balance of linear momentum in a Newtonian incompressible fluid [17, Chapter VIII]. Let $\mathcal{F} \subset \mathbb{R}^3$ be the spatial region occupied by the fluid and let $v : \mathcal{F}\times[0,T] \to \mathbb{R}^3$ and $p : \mathcal{F}\times[0,T] \to \mathbb{R}$ be the velocity and pressure fields in the Eulerian formulation. The incompressibility constraint reads

$$\operatorname{div} v = 0 \qquad \text{in } \mathcal{F}, \tag{1.1}$$

while the above-mentioned balance of linear momentum gives the following vector equation

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = -\nabla p + \mu\Delta v + f \qquad \text{in } \mathcal{F}, \tag{1.2}$$

where $\rho > 0$ is the fluid density, $\mu > 0$ its viscosity, and f is the external force. The system of equations (1.1)-(1.2) goes under the name of incompressible Navier-Stokes equations. Using an imprecise language, we could say that these equations are good to model phenomena in “non extreme” conditions. In order to be more clear and precise about the preceding statement, we need to introduce some peculiar dimensionless quantities associated to fluid flows and to cast the equations in a non-dimensional form. To this end, let $\omega > 0$ be a frequency parameter, let $L > 0$ have the dimension of a length, and let $V > 0$ have the dimension of a velocity. We now define the dimensionless quantities

$$t^* := \omega t, \qquad x^* := \frac{x}{L}, \qquad v^* := \frac{v}{V}, \qquad p^* := \frac{L}{V\mu}p,$$

from which the following starred operators are derived

$$\nabla^* = L\nabla, \qquad \Delta^* = L^2\Delta, \qquad \frac{\partial}{\partial t^*} = \frac{1}{\omega}\frac{\partial}{\partial t}.$$

Therefore, equations (1.1) and (1.2) can be re-written in the dimensionless form (consider $f = 0$)

$$\begin{cases} \beta \mathrm{Re}\dfrac{\partial v^*}{\partial t^*} + \mathrm{Re}\, v^* \cdot \nabla^* v^* = -\nabla^* p^* + \Delta^* v^*, \\ \operatorname{div}^* v^* = 0, \end{cases} \tag{1.3}$$

where $\mathrm{Re} := \frac{LV\rho}{\mu}$ is the *Reynolds number* of the flow, and $\beta := \frac{\omega L}{V}$. Taking the formal limit as both Re and $\beta\mathrm{Re}$ tend to zero means to neglect all the inertial effects with respect to the viscous ones. In a world where viscosity dominates over inertia, the Navier-Stokes equations reduce to the *steady Stokes equations* for modeling a steady *creeping* flow

$$\begin{cases} \Delta^* v^* = \nabla^* p^*, \\ \operatorname{div}^* v^* = 0, \end{cases}$$

which are better known in their dimensional form

$$\begin{cases} \mu\Delta v = \nabla p, \\ \operatorname{div} v = 0. \end{cases} \tag{1.4}$$

This vanishing Reynolds number regime is one that could be considered as “extreme”, in the imprecise terminology used above. Viscous creeping flows are better modeled by Stokes equation, instead of the full Navier-Stokes system. It must be said that the opposite regime, in which $\mathrm{Re} \to \infty$, is suitable for modeling inviscid fluids, the set of equations governing which goes under the name of Euler equations. Since this latter case is outside of the purpose of this work, we limit ourselves to low Reynolds number flows and Stokes equations.

The mathematical theory for Stokes equations is very well established, and a number of results has been stated. Theorems proving existence and uniqueness and regularity of the solutions to the Stokes equations can be found in [14], [28], [39], [41], and in the references therein. Those that will be useful for our discussion will be cited at due time.

1.2 Swimming in viscous fluids

In 1851 G. G. Stokes derived the formula for the drag force experienced by a sphere of radius R moving linearly at a constant velocity V in an unbounded viscous fluid, its expression being

$$F = -6\pi\mu RV.$$

This formula, in its simplicity, already shows some characteristics of drag forces: they are linear with respect to the dimension and the velocity, and they depend upon the viscosity μ of the fluid.

In the mid Nineties great contributions to the field have been given by G. I. Taylor and M. J. Lighthill, who studied viscous flows and shed light on the comprehension and modeling of important phenomena. Taylor proposed a simple model for swimming in a viscous fluid, the so-called Taylor's swimming sheet [40] (see also [37] for a recent improvement), while Lighthill suggested a possible definition for swimming efficiency [30]. Another important contribution is the book by S. Childress [10], where the Taylor's swimming sheet is also discussed; for a comprehensive list of references, the reader can refer to the recent review [29]. Among the more mathematical contributions we quote [15], [25], [36], and [7].

The breakthrough in the study of self-propelled motion in viscous fluids came in the late Seventies with the paper *Life at low Reynolds number* by E. M. Purcell [35]. He proved the so-called Scallop Theorem, which states that too simple swimming strategies are not effective in a viscous fluid, and contemporarily proposed a very simple swimmer that can actually swim in those conditions. The setting is that of an infinite viscous fluid in which an object capable of deforming itself is located. The rules are that the swimmer has to perform a cyclic change of shape, without any external force acting on it. This constraint is usually referred to as *self-propulsion*. The Stokes equation together with the *no-slip boundary condition* is the model for the external fluid which is generally used

$$\begin{cases} \nu\Delta v = \nabla p, & \text{in } \Omega^{\text{ext}}, \\ \operatorname{div} v = 0, & \text{in } \Omega^{\text{ext}}, \\ v = V_{\text{swimmer}}\,, & \text{on } \partial\Omega^{\text{ext}}, \end{cases} \tag{1.5}$$

where we have written equation (1.4) using the *kinematic viscosity* $\nu = \mu/\rho$, and have called $\Omega \subset \mathbb{R}^3$ the swimmer and $\Omega^{\text{ext}} := \mathbb{R}^3 \setminus \overline{\Omega}$ the domain occupied by the fluid. It must be noticed that, up to rescaling, it is always possible to set $\nu = 1$, and this choice, as well as the notation, will be generally maintained throughout the whole thesis.

We can derive a useful mathematical expression for the self-propulsion constraint from the balance of the forces. We consider Newton's second law of dynamics and observe that the total force is the summation of external and viscous forces. Therefore,

$$ma = F = F^{\text{ext}} + F^{\text{visc}};$$

in this expression we can neglect the acceleration term, since at low Reynolds number viscosity is predominant over inertia, as well as the contribution of the external forces, because of the self-propulsion constraint. An analogous equation can be cast for the torques, so that the two of them together embody the self-propulsion constraint

$$F^{\text{visc}} = 0, \qquad M^{\text{visc}} = 0. \tag{1.6}$$

These expressions are those that will allow us to write the equations of motion for the swimmers. Generally, both quantities in (1.6) are expressed by means of boundary integrals

$$F^{\text{visc}} := \int_{\partial\Omega} \sigma(x)n(x)\,\mathrm{d}S(x), \qquad M^{\text{visc}} := \int_{\partial\Omega} x\times\sigma(x)n(x)\,\mathrm{d}S(x), \tag{1.7}$$

where $\sigma := -pI + 2\nu\mathrm{E}v$ is the stress tensor ($\mathrm{E}v$ is the symmetric part of ∇v), n is the outer unit normal to $\partial\Omega$, and $\mathrm{d}S$ is the surface measure. All these objects will be defined better when appropriate.

Before proceeding further, we give a simple proof of the Scallop Theorem and comment about the "three links swimmer" proposed by Purcell to overcome the Scallop Theorem.

Theorem 1.2.1. *A scallop cannot swim by reciprocal motion in a low Reynolds number fluid.*

Proof. Let us call c the position of the hinge of the scallop and θ the angle measuring the opening of the valves. Since the Stokes equation is linear, it follows that the viscous force depends linearly on the boundary velocity, which in turn is a combination of $\dot{c}$ and $\dot{\theta}$. Therefore, we can write

$$0 = F^{\text{visc}} = \phi_c(c,\theta)\dot{c} + \phi_\theta(c,\theta)\dot{\theta}, \tag{1.8}$$

where $\phi_c \neq 0$ and ϕ_θ are coefficients that depend only on the configuration of the scallop, that is on its shape. The zero on the left side of (1.8) comes from the self-propulsion constraint; moreover, both coefficients $\phi_c\,, \phi_\theta$ do not depend on c, by translation invariance. Thus we can solve (1.8) for $\dot{c}$ and integrate over a period of time $[0,T]$ to obtain the net displacement after a stroke. Let $V(\theta) := -\phi_\theta(\theta)/\phi_c(\theta)$, and notice that $\theta(0) = \theta(T)$, since the motion has to be T-periodic (reciprocal, using Purcell's terminology). Define

$$\Psi(\theta) := \int_0^\theta V(s)\,\mathrm{d}s. \tag{1.9}$$

Now compute the net displacement and take (1.9) into account

$$\Delta c = \int_0^T \dot{c}(t)\,\mathrm{d}t = \int_0^T V(\theta(t))\dot{\theta}(t)\,\mathrm{d}t = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\theta(t))\,\mathrm{d}t = \Psi(\theta(t)) - \Psi(\theta(0)) = 0. \quad (1.10)$$

The Scallop Theorem is proved. □

Needless to say, real scallops indeed swim. They do that by squirting water through some small holes located where the two valves are hinged together. Yet, we are not pointing out the excessive meagerness of the model. What the Scallop Theorem puts in evidence is that one variable describing the spatial position of the swimmer, c, and one variable describing its shape, θ, are not enough to obtain a profitable reciprocal motion. More degrees of freedom must be added, and a sort of symmetry breaking must occur.

The first example in this direction has been given by Purcell himself. It is the so-called "three links swimmer", consisting of three rigid rods linked together by two hinges. This system has three parameters: the center of the central link, or the barycen-

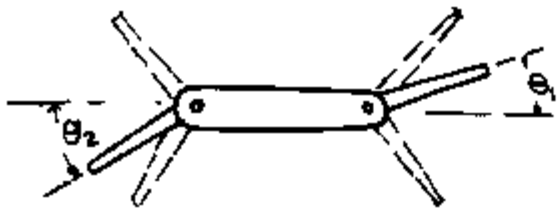

Figure 1.1: Purcell's three links swimmer.

ter of the system can be used as a position variable, while the two angles θ_1 and θ_2 are the shape variables. Again, the idea is to swim, that is to achieve a non zero net displacement, by changing shape. Purcell himself proposed a sequence of movements, originating a reciprocal motion, that allow the three links to advance. They can be better viewed in Purcell's original scheme. Performing the sequence of movements $S_1, \ldots, S_5$ as illustrated in Figure 1.2 will make the swimmer to achieve a non zero net displacement along the direction of the central rod. This easily follows by a symmetry argument, once we impose the two extremal rods to be equally long and θ_1 and θ_2 to span the same angle.

Another example of a simple swimmer has been proposed in 2004 by A. Najafi and R. Golestanian [32] and is known by the name of "Three linked spheres". Three identical spheres are lined and it is assumed that they can vary the reciprocal distance. The two

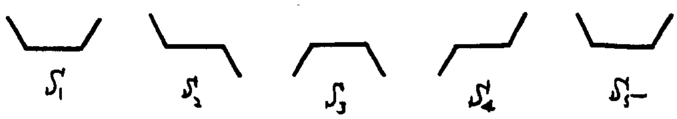

Figure 1.2: The three links swimmer's effective sequence of movements.

shape parameters are the distances between the the central sphere and the peripheral ones, while the barycenter can play the role of the position variable. The situation is very close to that of Purcell's Three links swimmer, and indeed also Najafi and Golestanian's swimmer can get to a non zero net displacement at the end of a reciprocal stroke; see Figure 1.3.

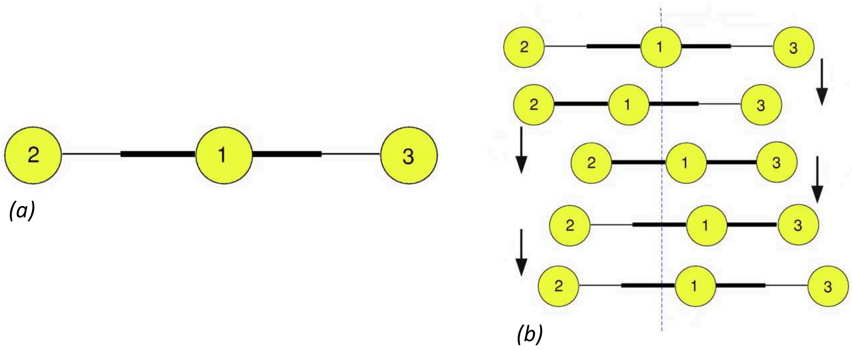

Figure 1.3: (a) The three linked spheres swimmer. (b) The sequence of movements to swim.

Some common properties between Purcell's and Najafi and Golestanian's swimmers can be pointed out. In both cases we can identify one positional variable, say the barycenter, and two shape variables, the angles in the three links swimmer or the distances between the spheres in the other case. Moreover, both swimmers, at the end of a stroke, will have advanced along a line. This is a property that is enjoyed also by a particular class of swimmers, namely the *axisymmetric swimmers*. These have been studied thoroughly in the works [2, 4]. Let us call, as before, $\Omega \subset \mathbb{R}^3$ the region occupied by the swimmer, and let Ω^{ext} be the complementary region occupied by the fluid. In order for the swimmer to be axisymmetric we have to ask Ω to have cylindrical symmetry, and let us assume, for sake of simplicity, that the axis of symmetry coincides with the x-axis of an orthonormal reference frame. Assume that the shape of the swimmer is described by N shape parameters $\xi = (\xi_1, \ldots, \xi_N)$ and let c be the position of the barycenter along the x-axis. Notice that this is enough to describe the motion of the swimmer, since its displacement is confined along the x-axis by symmetry. Recalling (1.7), and taking into account the symmetry argument, the self-propulsion constraint is expressed by

$$i \cdot \int_{\partial\Omega} \sigma n \, \mathrm{d}S = 0, \tag{1.11}$$

where i is the unit vector identifying the x-axis. Taking into account that the velocity of the fluid must be linear with respect to the boundary data given by $\dot{\xi}_1, \ldots, \dot{\xi}_N, \dot{c}$, and the translation invariance of the system, equation (1.11) can be rephrased as

$$\sum_{i=1}^{N} \phi_i(\xi)\dot{\xi}_i + \phi_{N+1}(\xi)\dot{c} = 0, \tag{1.12}$$

where the non-vanishing coefficient ϕ_{N+1} represents the drag force corresponding to a rigid translation along the symmetry axis at unit speed. Again as in the proof of the Scallop Theorem, we can solve (1.12) for $\dot{c}$ and integrate over a time period to obtain the displacement after a stroke. Therefore, the condition for swimming is

$$\Delta c = \int_0^T \sum_{i=1}^{N} V_i(\xi(t))\dot{\xi}_i(t) \, \mathrm{d}t \neq 0, \tag{1.13}$$

where $V_i(\xi) := -\phi_i(\xi)/\phi_{N+1}(\xi)$, which can be interpreted by requiring that the differential form $\gamma := \sum_{i=1}^{N} V_i \mathrm{d}\xi_i$ is not exact. Thus, only those cyclic change of shape that will produce non exact differential forms γ will be able to generate an effective motion. The non exactness of γ is the symmetry breaking condition we were mentioning beforehand.

Another example of non-trivial swimmer is the Push-Me-Pull-You swimmer. It consists of two spheres that can somehow exchange volume between themselves and get closer to one another or move away from each other [5]. Also this system is capable of producing a sequence of moves which originate an effective reciprocal stroke.

An important feature of these kinds of motion is that there is a net separation between velocities and shape parameters. The velocities enter linearly in the formulae, and this is due to the linearity of the Stokes system (1.5); on the other hand, the shape parameters enter in the coefficients of the velocities and determine the effect of those on the motion. This will become more clear in the following chapters.

As a particular case, the motion of flagella in viscous fluids is attracting great interest for the obvious applications to Biology. The study of organisms such as *Escherichia coli*, spermatozoa, the nematode *Caenorhabditis elegans* is devoted to understand how they move and their swimming strategies. For some of these organisms comparison tests have been conducted to discover whether they propel themselves more efficiently in a plain fluid or in a particulate one [22]. Among the huge amount of literature on the matter, we cite the following works [23], [27], [43], [42], and refer the reader to them and to the references therein for a more complete overview on the subject.

On the other hand, the mathematical modeling of a flagellum beating in a fluid is rather complicated: approximating the flagellum as a mono-dimensional object in a three-dimensional ambient introduces a dimensional gap when stating the boundary conditions. On the other side, it is not completely clear how to perform a limiting procedure starting from a three-dimensional, thick flagellum and letting the thickness go to zero.

For these reasons approximate theories to model flagellar motion successfully have been proposed in the past decades. We are talking about *slender body theory* [6, 24] and *resistive force theory*, also known as *local drag theory* [21]. A crash course on these approximate theories can be found in the review on the hydrodynamics of swimming micro-organisms by E. Lauga and T. Powers [29].

An interesting approach to the study of swimming, in a general setting, has been proposed by A. Shapere and F. Wilczek in [38]. They exploit a gauge field theory approach in the space of shapes. They give explicit examples in the two-dimensional case and in the case of infinitesimal deformations of a sphere. In the same spirit, axisymmetric swimmers described by finitely many shape parameters were studied in [2, 3, 4].

The novelty in the present work is that we develop a theoretical framework to study swimmers whose shape changes are completely general and genuinely infinite dimensional. This seems to clash with the old disposition to describe the shape by the lowest possible number of parameters, a tendency that must be followed keeping the Scallop Theorem clear in mind: too few parameters do not provide a useful model. With the following two definitions of swimming and self-propulsion, we propose is a framework in which an arbitrary shape is capable of deforming and moving in the fluid by exploiting the viscous fluid-structure interaction.

Definition 1.2.2. *Swimming is the ability of an organism to perform a variation of its spatial position caused by the variation of its shape, under the self-propulsion constraint.*

Definition 1.2.3. *Self-propulsion is the absence of external forces or momenta.*

We will show how it is possible to separate the contribution of the shape change from the variables that describe the spatial position and orientation of the swimmer, and how the first determines the latter. In this setting, the six parameters to locate and orient the swimmer in the three-dimensional space are determined by the infinite-dimensional shape change. We exploit a linear representation of the viscous force and torque in terms of the linear and angular velocities of the swimmer and of the velocities given by the deformation, and solve a linear system of ODE's for the former in terms of the latter. The coefficients of the systems will be determined by the shape of the swimmer and will be obtained via variational methods.

We now present the outline of the present work. In Chapter 2 we present an analytical framework to study the motion of micro-swimmers in a viscous fluid.

In Chapter 3, which contains the results of [11], we deal with the case of a swimmer immersed in a viscous fluid governed by the Stokes equation. Our main result is Theorem 3.4.4, which states that, under very mild regularity assumptions, the change of shape determines uniquely the motion of the swimmer. Thanks to the low Reynolds number regime and the self-propulsion constraint, Newton's equations of motion reduce to the vanishing of the viscous force and torque acting on the body. By exploiting an integral representation of viscous force and torque, the equations of motion can be reduced to a system of six ordinary differential equations. Variational techniques are used to prove the boundedness and measurability of its coefficients, so that classical results on ordinary differential equations can be invoked to prove existence and uniqueness of the solution. The difficulties in achieving the result are indeed in the proof of the measurability of the coefficients of the ODE's. We gave the minimal assumption for them to be measurable, instead of continuous, both to prove a more general result and to allow more general, even discontinuous in time, shape functions. The above-mentioned measurability os obtained by means of technical constructions for extending the boundary velocities to solenoidal vector fields in the interior of suitable domains.

In Chapter 4, which contains the results of [31], we turn to the case of a self-propelled micro-swimmer in a particulate viscous medium, modeled as a Brinkman fluid. Within the same analytical setting, Theorem 4.4.6 is obtained, and it extends the result obtained in Chapter 3. We use essentially the same method and we adapt the functional setting according to the Brinkman equation. Even though the equation contains an additional term, the function space needed appears to be easier to handle.

In Chapter 5, which contains the results of [12], we abandon the full generality of three-dimensional swimmer to concentrate on flagellar motion. The study is conducted in the case of a flagellum performing a planar motion in a three-dimensional fluid, and resistive force theory is used to model the drag forces and torques. Interestingly, once this approximation is assumed, the fluid becomes totally irrelevant to the computation of any physical quantities. In this case we prove three main results, namely, the existence, uniqueness, and regularity of the solution to the equations of motion; the controllability of the swimmer; the existence of an optimal beating strategy. The controllability of the system is something one could easily expect, given the huge availability of possible shapes. We were able to avoid a number of explicit computations by wisely choosing shape functions with some symmetries. Finally, we perform some computation to study the associated Euler equation, of which we highlight the general structure. The same study can be easily extended to the three-dimensional case, notations would become slightly heavier.

The main difference between the case of three-dimensional swimmers in the unbounded fluid (particulate or not) and the flagellum is that in the former the viscous force and torque are determine by the boundary integral of the normal component of the stress tensor, which is in turn obtained by solving an exterior Stokes (or Brinkman) problem in the fluid domain. On the contrary, as we have already mentioned, in the local drag theory approximation the fluid plays no active role, and the forces are defined locally in terms of the velocity on the flagellum. The same physical phenomenon, that of swimming, is therefore modeled in two different ways according to the dimension of the swimmer. Moreover, in the second case no variational machinery has to be introduced to solve the equations of motion.

1.3 Notation

We collect here the notation used throughout the work.

∇	the gradient with respect to the space variables.
$\times$	the cross product in $\mathbb{R}^3$.
$\triangle$	the symmetric difference between sets.
$^\top$	superscript: the transpose.
n	the exterior unit normal.
$\mathrm{d}S(\cdot)$	the surface measure.
$\mathbb{M}^{3\times 3}$	the Hilbert space of 3×3 real matrices.
$\sigma : \xi = \sum_{i,j=1}^{3} \sigma_{ij}\xi_{ij}$	the Euclidean norm in the space of matrices.
$a \otimes b = a_i b_j$	the dyadic product between vectors.
σ, σ_t	the stress tensor (Chapters 2, 3, and 4).
$\sigma(t)$	the position of the center of the bump (Chapter 5).
$A \subset \mathbb{R}^3$	the *reference* configuration of the swimmer.
$A_t \subset \mathbb{R}^3$	the *current* configuration of the swimmer.
$B_t \subset \mathbb{R}^3$	the *intermediate* configuration of the swimmer.
$\Omega \subset \mathbb{R}^3$	a general domain.
$\Omega^{\mathrm{ext}} = \mathbb{R}^3 \setminus \overline{\Omega}$	the exterior domain with respect to Ω.

CHAPTER 2

General setting for swimming

2.1 Shape and position

As we stated in the Introduction, swimming consists in the ability to change position by changing shape periodically and exploiting the interaction with the surrounding liquid. Shape change induces a flow in the fluid. The propulsive effect arises from the action and reaction principle: the swimmer must exert forces to set the fluid in motion and hence it receives from the fluid a propulsive force. In the absence of other actions on its body, this is the only force the swimmer can exploit (self propulsion). In what follows we will focus on the case in which the swimmer is completely immersed in the liquid.

Flows generate both inertial and viscous forces. In a Newtonian fluid, their relative importance is measured by the Reynolds number $\mathrm{Re} := \frac{VL}{\nu}$ and by the Womersley number $\alpha := (\omega \frac{L}{V}\mathrm{Re})^{1/2}$, where V is the swimming velocity, L the size of the swimmer, $\nu = \mu/\rho$ the kinematic viscosity of the fluid, and ω is the frequency of the motion. Typical swimmers move with a speed which is of the order of some body-lengths per second, and execute cyclic shape changes with frequencies not exceeding a few thousand Hertz [10, Table 1.1]. Therefore, for swimmers of sufficiently small size L, both Re and α are small, and all inertial effects are negligible.

Thus, a fish swims by accelerating the surrounding water, while bacteria and other unicellular organisms move by exploiting viscous resistance. The striking difference between these two strategies and the subtleties that follow are beautifully illustrated in [35].

In this work we deal with micro-swimmers immersed in a viscous liquid, therefore the fluid dynamics is governed by the Stokes system [10, Chapter 2].

The *motion* of a swimmer is described by a map $t \mapsto \varphi_t$, where, for every fixed t, the *state* φ_t is an orientation preserving bijective C^2 map from the *reference* configuration $A \subset \mathbb{R}^3$ into the *current* configuration $A_t \subset \mathbb{R}^3$.

Given a distinguished point $x_0 \in A$, for every fixed t, we consider the following factorization

$$\varphi_t = r_t \circ s_t \,, \tag{2.1}$$

where the *position* function r_t is a rigid deformation and the *shape* function s_t is such that

$$s_t(x_0) = x_0 \,, \tag{2.2a}$$

$$\nabla s_t(x_0) \quad \text{is symmetric.} \tag{2.2b}$$

In the applications we have in mind, one can choose the map $t \mapsto s_t$ in a suitable class of admissible shape changes and use it as a control to achieve propulsion as a consequence of the viscous reaction of the fluid. By contrast, $t \mapsto r_t$ is *a priori* unknown and it must be determined by imposing that the resulting $\varphi_t = r_t \circ s_t$ satisfies the equations of motion.

The factorization (2.1) of the motion into data (the freely adjustable shapes s_t) and unknowns (the position and orientation r_t achieved by the swimmer as a consequence of having executed some strokes) is conceptually appealing and has far reaching consequences in the analysis of biological and engineered systems. Moreover, it simplifies the problem, reducing it to a system of ordinary differential equations since $r_t(z) = y_t + R_t z$ is finite dimensional; here y_t and R_t are the translation and rotation characterizing the rigid motion r_t. Finally it is natural, because $t \mapsto s_t$ represents the motion as seen by an observer moving with the swimmer, while $t \mapsto r_t$ represents the motion of this observer with respect to a fixed frame. To establish a link with the language of [38], notice that conditions (2.2) select one special gauge for the description of the system, that s_t describes the *standard* (unlocated) shape of the swimmer, and φ_t gives its *located* shape.

The equations of motion that the map $t \mapsto \varphi_t$ must satisfy are the balance of linear and angular momentum, which, since inertia is negligible, reduce to the vanishing of total force and total torque acting on the swimmer A_t . Since we assume self propulsion, there are no external forces applied to A_t , so that the total force and torque reduce to the ones arising from the viscous resistance exerted by the fluid on the boundary ∂A_t :

$$0 = F_{A_t,\dot\varphi_t} := \int_{\partial A_t} \sigma_t(y) n(y) \, \mathrm{d}S(y), \tag{2.3a}$$

$$0 = M_{A_t,\dot\varphi_t} := \int_{\partial A_t} y \times \sigma_t(y) n(y) \, \mathrm{d}S(y). \tag{2.3b}$$

Here σ_t is the stress tensor, n is the outer unit normal to ∂A_t , $\mathrm{d}S$ indicates the integration with respect to the surface measure, and $\times$ is the cross product in $\mathbb{R}^3$. Since the Reynolds and Womersley numbers are small, stresses are computed by solving the outer Stokes problem in $A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$

$$\begin{cases} \Delta u_t(y) \;=\; \nabla p_t(y) & \text{in} \quad A_t^{\mathrm{ext}}, \\ \operatorname{div} u_t(y) \;=\; 0 & \text{in} \quad A_t^{\mathrm{ext}}, \\ u_t(y) \;=\; \dot{\varphi}_t(x)|_{x=\varphi_t^{-1}(y)} & \text{on} \quad \partial A_t\,, \\ u_t(y) \to 0 & \text{for} \quad |y| \to \infty, \end{cases}$$

where u_t is the velocity and p_t is the pressure, so that $\sigma_t n = -p_t n + (\nabla u_t + (\nabla u_t)^\top)n$ (recall that the viscosity is assumed to be 1).

Our main existence, uniqueness, and regularity results are Theorem 3.4.4 and Theorem 4.4.6 stating that for every sufficiently smooth shape change $t \mapsto s_t$, the position functions $t \mapsto r_t$ are uniquely determined by the initial conditions at $t = 0$. More precisely, there exists a unique family of rigid motions $t \mapsto r_t$ such that the state functions $t \mapsto \varphi_t := r_t \circ s_t$ satisfy the equations of motion (2.3), and φ_t (or equivalently r_t) takes a prescribed value at $t = 0$. This result provides a rigorous mathematical justification for the viewpoint pioneered in [38]: the motion of a micro-swimmer is uniquely determined by the history of its shapes.

The main ingredients in the proof are the following. By exploiting the linearity of the Stokes system, we reduce the equations of motion (2.3) to (3.22) and (4.14), namely,

$$\dot{y}_t = R_t b_t\,, \qquad \dot{R}_t = R_t \Omega_t\,,$$

a system of ordinary differential equations involving the translational and rotational velocities associated with the rigid motion $t \mapsto r_t$. The coefficients b_t and Ω_t of these equations, given in (3.21), depend only on s_t and $\dot{s}_t$. They are obtained from the shape function $t \mapsto s_t$ by solving some auxiliary outer Stokes problems on A_t^{ext}.

The main difficulty is to prove the continuity, or at least the measurability, of these coefficients. To this aim, we have to obtain the continuous dependence of the solutions to the outer Stokes and Brinkman problems on their domains and on their boundary data; the main technical issue is the fact that they both depend on time.

Once continuity of the coefficients and measurability of the data of the equations of motion are proved, our existence and uniqueness problem can be solved by using classical techniques for ordinary differential equations.

We close this section by noticing that several interesting questions related to swimming can be phrased as control problems where the function $t \mapsto \dot{s}_t$ is the input and the function $t \mapsto r_t$ is the output. For example: which net positional and orientational changes can be achieved within a given class of time-periodic shape changes? Problems of this type have been solved, e.g., in [2, 3, 4] for swimmers described by finitely many shape parameters.

In the context of control problems, it is very useful that the input variables are allowed to be discontinuous in time. This is the main reason why we have insisted in proving our result for the case of Lipschitz continuous $t \mapsto s_t$, even though a C^1 regularity in time would have simplified the proofs very much. Infinite dimensional control problems for swimmers of *fixed* shape that can control the velocity of the surrounding fluid at points in contact with the swimmer's boundary have been considered, e.g., in [15], [36]. We plan to address in future work control problems for swimmers of *variable* shape, possibly described by infinitely many shape parameters.

2.2 Kinematics

In this section we fix the notation and the assumptions for the kinematics of the swimmer. As mentioned in Section 2.1, we show that it is possible to decompose the deformation into a pure shape change followed by a time-dependent rigid motion, whose rotations and translations are Lipschitz continuous with respect to time. This holds for both a Stokes (Chapter 3) and a Brinkman (Chapter 4) fluid, as well as in the specific case of the mono-dimensional swimmer discussed in Chapter 5.

The reference configuration $A \subset \mathbb{R}^3$ is a bounded connected open set of class C^2. The time-dependent deformation of A from the point of view of an external observer is described by a function $\varphi_t : \overline{A} \to \mathbb{R}^3$. We assume that, for every t,

$$\varphi_t \in C^2(\overline{A};\mathbb{R}^3), \tag{2.4a}$$

$$\varphi_t \quad \text{is injective,} \tag{2.4b}$$

$$\det \nabla\varphi_t(x) > 0 \quad \text{for all } x \in \overline{A}. \tag{2.4c}$$

Here and henceforth ∇ denotes the gradient with respect to the space variable. Under these hypotheses the set $A_t := \varphi_t(A)$ is a bounded connected open set of class C^2 and

$$\text{the inverse} \quad \varphi_t^{-1} : \overline{A}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{A}_t;\mathbb{R}^3).$$

We assume in addition that

$$\text{the sets } \mathbb{R}^3 \setminus \overline{A}_t \text{ are connected for all } t \in [0,T]. \tag{2.5}$$

Concerning the regularity in time, we require that

$$\text{the map } t \mapsto \varphi_t \text{ belongs to } \mathrm{Lip}([0,T];C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T];C^2(\overline{A};\mathbb{R}^3)), \tag{2.6}$$

so that $\|\varphi_{t+h} - \varphi_t\|_{C^1} \leqslant L\,|h|$, for a suitable constant $L > 0$.

We now prove that for almost every t there exists $\dot\varphi_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{\varphi_{t+h} - \varphi_t}{h} \to \dot\varphi_t\,, \quad \text{uniformly on } \overline{A} \text{ as } h \to 0. \tag{2.7}$$

Indeed, condition (2.6) implies that $t \mapsto \varphi_t$ belongs to $\mathrm{Lip}([0,T];W^{1,4}(A;\mathbb{R}^3))$. Therefore, the general theory of Lipschitz functions with values in reflexive Banach spaces (see, e.g., [8, Appendix]) implies that for almost every t the difference quotient in (2.7) converges strongly in $W^{1,4}(A;\mathbb{R}^3)$ to some element $\dot\varphi_t$ of $W^{1,4}(A;\mathbb{R}^3)$. The embedding of $W^{1,4}(A;\mathbb{R}^3)$ into $C^0(A;\mathbb{R}^3)$ implies the uniform convergence considered in (2.7). Finally the bound $\|\varphi_t - \varphi_s\|_{C^1} \leqslant L\,|t-s|$ implies that $\mathrm{Lip}(\dot\varphi_t) = L$ in $\overline{A}$, where, for every function f, $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f.

It turns out that the Eulerian velocity on the boundary ∂A_t, defined by

$$U_t := \dot\varphi_t \circ \varphi_t^{-1} \tag{2.8}$$

belongs to $\mathrm{Lip}(\partial A_t;\mathbb{R}^3)$ with Lipschitz constant independent of t.

We now describe the kinematics from the point of view of the swimmer. We fix a point $x_0 \in A$ and we look for a factorization of φ_t of the form (2.1), where $s_t : A \to \mathbb{R}^3$ satisfies properties (2.2) and $r_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion of the form

$$r_t(z) = y_t + R_t z, \tag{2.9}$$

with $y_t \in \mathbb{R}^3$ and $R_t \in \mathrm{SO}(3)$, the set of orthogonal matrices with positive determinant. Conditions (2.2) allow us to interpret s_t as a pure shape change from the point of view of an observer located at x_0. Therefore, the deformation φ_t, from the point of view of an external observer, is decomposed into a shape change followed by a rigid motion.

It follows from (2.1), (2.4), and (2.9) that, for every t,

$$s_t \in C^2(\overline{A};\mathbb{R}^3), \tag{2.10a}$$

$$s_t \quad \text{is injective,} \tag{2.10b}$$

$$\det \nabla s_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{2.10c}$$

and, consequently, that

$$\text{the inverse} \quad s_t^{-1} : \overline{B}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{B}_t;\mathbb{R}^3), \tag{2.11}$$

where $B_t := s_t(A)$, see Fig. 2.1. Note that B_t is a bounded connected open set of class C^2 and that $r_t(B_t) = A_t$ and $r_t(\partial B_t) = \partial A_t$. Notice that, since A is bounded and s_t is continuous, there exists a ball Σ_ρ centered at 0 with radius ρ such that

$$A \subset\subset \Sigma_{\rho-1} \qquad \text{and} \qquad B_t \subset\subset \Sigma_{\rho-1}\,. \tag{2.12}$$

It follows from (2.5) that

$$\text{the sets } \Sigma_\rho \setminus \overline{B}_t \text{ are connected for all } t \in [0,T]. \tag{2.13}$$

Conditions (2.1), (2.2), and (2.9) imply that

$$R_t = \nabla\varphi_t(x_0)\Big[\sqrt{\nabla\varphi_t(x_0)^\top \nabla\varphi_t(x_0)}\Big]^{-1}, \tag{2.14a}$$

$$y_t = \varphi_t(x_0) - R_t x_0\,. \tag{2.14b}$$

The existence of a factorization (2.1) satisfying (2.2) and (2.9) is obtained by setting $s_t := r_t^{-1} \circ \varphi_t$, where r_t is given by (2.9) with y_t and R_t defined by (2.14). Moreover, (2.6) together with (2.14), implies that

$$t \mapsto R_t \quad \text{and} \quad t \mapsto y_t \quad \text{are Lipschitz continuous.} \tag{2.15}$$

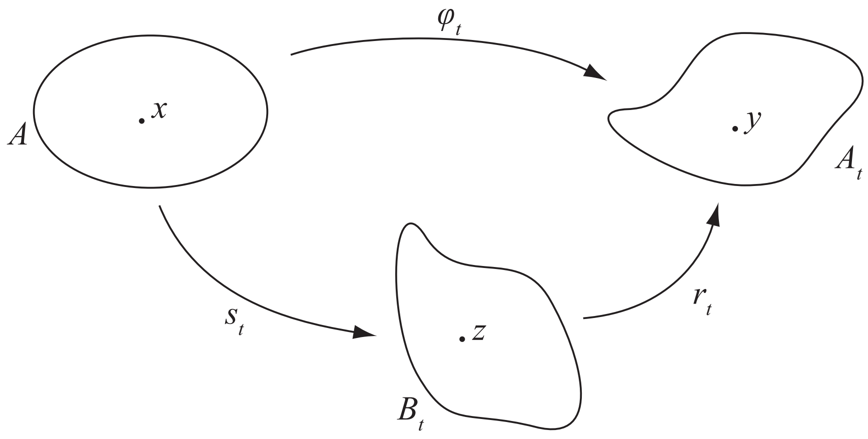

Figure 2.1: Notation for the kinematics.

Finally, since $s_t = r_t^{-1} \circ \varphi_t$,

$$\text{the map } t \mapsto s_t \text{ belongs to } \mathrm{Lip}([0,T]; C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T]; C^2(\overline{A};\mathbb{R}^3)), \tag{2.16}$$

so that $\|s_{t+h} - s_t\|_{C^1} \leqslant L\,|h|$, for a suitable constant $L > 0$. Properties (2.10c) and (2.16) imply that

$$\|s_t^{-1}\|_{C^2(\overline{B}_t;\mathbb{R}^3)} \leqslant C, \tag{2.17}$$

where $C < +\infty$ is a constant independent of t.

As for function φ_t, we can exploit condition (2.16) to prove that there exists $\dot{s}_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{s_{t+h} - s_t}{h} \to \dot{s}_t\,, \quad \text{uniformly on } \overline{A}\text{, as } h \to 0.$$

Notice that

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; W^{1,p}(\overline{A};\mathbb{R}^3)) \text{ for every } p \in [2,\infty[,$$

therefore, by the Sobolev immersions,

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; C^0(\overline{A};\mathbb{R}^3)),$$

and, by the continuous immersion of $H^1(A;\mathbb{R}^3)$ into $H^{1/2}(\partial A;\mathbb{R}^3)$,

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T]; H^{1/2}(\partial A;\mathbb{R}^3)).$$

Again as for $\dot{\varphi}_t$, we can prove that

$$\mathrm{Lip}(\dot{s}_t) \leqslant L, \qquad \text{with } L \text{ independent of } t. \tag{2.18}$$

Moreover, for any fixed $x \in \overline{A}$, the map $t \mapsto \dot{s}_t(x)$ is measurable.

Recall the definition of U_t given in (2.8) and define now $V_t(z) := R_t^\top U_t(r_t(z))$ and $W_t(z) := \dot{s}_t(s_t^{-1}(z))$, for every $z \in \partial B_t$. An elementary computation shows that for almost every $t \in [0, T]$

$$V_t(z) = R_t^\top \dot{y}_t + R_t^\top \dot{R}_t z + W_t(z) \quad \text{for every } z \in \partial B_t \,.$$

Now we would be ready for the description of the motion of the swimmer. Formally, it is the same for both the Stokes and the Brinkman cases, but the same quantities are defined in two different ways according to the underlying functional setting. This is why we will present the equations of motion in the two chapters separately.

CHAPTER 3

Swimming in an unbounded Stokes fluid

In this Chapter we develop the theory for the case of a swimmer immersed in an infinite viscous fluid governed by the Stokes equation. For this, the functional setting is presented in Section 3.1, the extension theorems are presented in Section 3.3, and the main results in Section 3.4. All these results are contained in [11].

3.1 The exterior Stokes problem

In this section we recall some known results on the exterior Stokes problem. In addition, we introduce a weak definition of the viscous force and torque, which does not require any regularity assumption on the velocity field. Finally, we prove that the solutions depend continuously on the domains for special boundary conditions.

Let Ω be an *exterior domain* with Lipschitz boundary, i.e., Ω is an unbounded, connected open set whose boundary $\partial\Omega$ is bounded and Lipschitz. The strong formulation of the *exterior* Stokes problem is

$$\begin{cases} \Delta u = \nabla p & \text{in} \quad \Omega\,, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega\,, \\ u = U & \text{on} \quad \partial\Omega\,, \\ u = 0 & \text{at} \quad \infty, \end{cases} \tag{3.1}$$

which includes a decay condition at infinity.

To write the weak formulation of this problem, we consider the *Deny-Lions space*

$$D^{1,2}(\Omega;\mathbb{R}^3) := \{u \in L^6(\Omega;\mathbb{R}^3) : \nabla u \in L^2(\Omega;\mathbb{M}^{3\times 3})\},$$

where $\mathbb{M}^{3\times 3}$ is the Hilbert space of 3×3 real matrices endowed with the Euclidean norm $\sigma : \xi := \sum_{i,j} \sigma_{ij}\xi_{ij}$. The space $D^{1,2}(\Omega;\mathbb{R}^3)$ is endowed with the norm

$$\|u\|_{D^{1,2}(\Omega;\mathbb{R}^3)} := \|\nabla u\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} . \tag{3.2}$$

It is well known that $D^{1,2}(\Omega;\mathbb{R}^3)$ is a Hilbert space and that there exists a constant $C(\Omega)$ such that

$$\|u\|_{L^6(\Omega;\mathbb{R}^3)} \leqslant C(\Omega)\,\|u\|_{D^{1,2}(\Omega;\mathbb{R}^3)} ,$$

for all $u \in D^{1,2}(\Omega;\mathbb{R}^3)$. For a thorough exposition on these spaces, see the classical work by Deny and Lions [13].

Let $\mathrm{E}u := \frac{1}{2}(\nabla u + (\nabla u)^\top)$ denote the *symmetric gradient* of u. The inequality

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leqslant C(\Omega)\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} , \tag{3.3}$$

proved in a more general setting for weighted spaces of functions defined on unbounded domains [26, Section 3, Theorem 1], shows that $\|\mathrm{E}u\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}$ is an equivalent norm on $D^{1,2}(\Omega;\mathbb{R}^3)$. Since $\partial\Omega$ is bounded, for every $u \in D^{1,2}(\Omega;\mathbb{R}^3)$ the trace of u on $\partial\Omega$, still denoted by u, belongs to $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and the trace operator is continuous between these two spaces.

The following density result plays a crucial role in the theory.

Theorem 3.1.1 (Density, [20]). *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then the space*

$$\{u \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \textit{ in } \Omega\}$$

is dense in $\{u \in D^{1,2}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \textit{ in } \Omega, u = 0 \textit{ on } \partial\Omega\}$ for the norm (3.2).

To write the weak formulation of the exterior Stokes problem, we introduce the spaces

$$\begin{aligned} \mathcal{V}(\Omega) &:= \{u \in D^{1,2}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}, \\ \mathcal{V}_0(\Omega) &:= \{u \in \mathcal{V}(\Omega) : u = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Given a function $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, which plays the role of the boundary condition, the weak formulation of (3.1) is given by

$$\begin{cases} u \in \mathcal{V}(\Omega), \quad u = U & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \,\mathrm{d}x = 0 & \text{for every } w \in \mathcal{V}_0(\Omega). \end{cases} \tag{3.4}$$

Remark: We notice that no other assumptions are to be made on the boundary velocity field. If Ω were a *bounded* domain, then the following condition would have been necessary

$$\int_{\partial\Omega} U \cdot n \,\mathrm{d}S = 0. \tag{3.5}$$

Now we state the main existence and uniqueness result for the exterior Stokes problem. Its proof is classical and can be found in the books by Galdi [14], Sohr [39], and Temam [41].

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary and let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Then problem* (3.4) *has a solution. Moreover, there exists $p \in L^2_{\mathrm{loc}}(\Omega)$, with $p \in L^2(\Omega \cap \Sigma_\rho)$ for every ball Σ_ρ centered at the origin and of radius $\rho > 0$, such that $\Delta u = \nabla p$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

If u and p are the velocity and pressure fields of problem (3.1), the stress tensor is given by

$$\sigma := -p\,\mathrm{I} + 2\mathrm{E}u, \tag{3.6}$$

where I is the identity matrix (recall, again, that the viscosity is equal to 1). Note that if u satisfies (3.4), then

$$\operatorname{div}\sigma = -\nabla p + \Delta u + \nabla(\operatorname{div} u) = 0. \tag{3.7}$$

If σn has a trace in $L^1(\partial\Omega;\mathbb{R}^3)$, then the *viscous force*, defined as the resultant of the forces acting on the boundary $\partial\Omega$, is given by

$$F := \int_{\partial\Omega} \sigma(x)n(x)\,\mathrm{d}S(x), \tag{3.8}$$

while the *viscous torque*, defined as the resultant of the corresponding momenta with respect to the origin, is given by

$$M := \int_{\partial\Omega} x\times\sigma(x)n(x)\,\mathrm{d}S(x). \tag{3.9}$$

A technical problem arises from the fact that σn has not, in general, a trace in $L^1(\partial\Omega;\mathbb{R}^3)$, even if u satisfies the outer Stokes problem as in Theorem 3.1.2, so that F and M cannot be defined via (3.8) and (3.9). Thanks to (3.7), the following definition allows us to introduce the trace of σn as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$. Through this we can define in a consistent way the *power* of the viscous force and of the torque.

Let $\mathbb{M}^{3\times3}_{\mathrm{sym}}$ be the space of 3×3 symmetric matrices. Every $\sigma \in \mathbb{M}^{3\times3}_{\mathrm{sym}}$ can be orthogonally decomposed as

$$\sigma = \tfrac{\operatorname{tr}\sigma}{3}\,\mathrm{I} + \sigma_D\,,$$

where the *deviatoric part* σ_D satisfies $\operatorname{tr}\sigma_D = 0$.

Definition 3.1.3. *Let Ω be an exterior domain with Lipschitz boundary and let $\sigma \in L^1_{\mathrm{loc}}(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ be such that $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\operatorname{div}\sigma \in L^{6/5}(\Omega;\mathbb{R}^3)$. We define the trace of σn on $\partial\Omega$, still denoted by σn, as the unique element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ satisfying*

$$\langle \sigma n, V\rangle_\Omega := \int_\Omega (\operatorname{div}\sigma)\cdot v\,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v\,\mathrm{d}x, \tag{3.10}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing between $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and v is any function in $\mathcal{V}(\Omega)$ such that $v = V$ on $\partial\Omega$.

We will drop the subscript Ω whenever the domain of integration is understood. If σ is sufficiently smooth, then an integration by parts shows that

$$\langle \sigma n, V\rangle_\Omega = \int_{\partial\Omega} \sigma n \cdot V \,\mathrm{d}S,$$

for every $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$.

Returning to the general case, it is easy to see that the right-hand side of (3.10) is well defined, since $\operatorname{div}\sigma \in L^{6/5}(\Omega;\mathbb{R}^3)$, $v \in L^6(\Omega;\mathbb{R}^3)$, $\sigma : \mathrm{E}v = \sigma_D : \mathrm{E}v$, $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, and $\mathrm{E}v \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$. Moreover, the definition of σn does not depend on the choice of v, since the right-hand side of (3.10) vanishes whenever $v \in \mathcal{V}_0(\Omega)$. This follows from the distributional definition of $\operatorname{div}\sigma$ whenever $v \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3)$ and $\operatorname{div} v = 0$, and can be obtained by approximation in the general case using the Density Theorem 3.1.1. Finally, by choosing $v \in \mathcal{V}(\Omega)$ the solution to problem (3.1) with boundary datum V on $\partial\Omega$, we conclude that (3.10) defines a continuous linear functional on $H^{1/2}(\partial\Omega;\mathbb{R}^3)$.

Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and let u be the solution to the Stokes problem (3.4) with boundary datum U and let σ be the corresponding stress tensors defined by (3.6). Since $\sigma \in L^2_{\mathrm{loc}}(\Omega;\mathbb{M}^{3\times3})$, $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$, and $\operatorname{div}\sigma = 0$ by (3.7), we can apply Definition 3.1.3 and for every $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ we obtain

$$\begin{aligned}\langle \sigma n, V\rangle &= \int_\Omega \sigma : \mathrm{E}v \,\mathrm{d}x = \int_\Omega [-pI : \mathrm{E}v + 2\mathrm{E}u : \mathrm{E}v] \,\mathrm{d}x \\ &= -\int_\Omega p \operatorname{div} v \,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v \,\mathrm{d}x = 2\int_\Omega \mathrm{E}u : \mathrm{E}v \,\mathrm{d}x,\end{aligned} \tag{3.11}$$

where v is an arbitrary element of $\mathcal{V}(\Omega)$ such that $v = V$ on $\partial\Omega$. In particular, we can take as v the solution to the Stokes problem (3.4) with boundary datum V. This leads to the *reciprocity condition*,

$$\langle \sigma n, V\rangle = \langle \tau n, U\rangle,$$

where τ is the stress tensor corresponding to v. By taking $U = V$ in (3.11), we get

$$\langle \sigma n, U\rangle = 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})}. \tag{3.12}$$

We now show that the quadratic form $\langle \sigma n, U\rangle$ is positive definite. Indeed, if $\langle \sigma n, U\rangle = 0$, by (3.12) we obtain $\mathrm{E}u = 0$ almost everywhere on Ω. This implies that that $u(x) = c + Ax$, where $c \in \mathbb{R}^3$ and A is a skew symmetric 3×3 matrix. Since $u \in L^6(\Omega;\mathbb{R}^3)$, we have $c = 0$ and $A = 0$, so that $U = 0$.

By using the duality product $\langle \sigma n, V\rangle$ for a suitable choice of V, one can define the viscous force F and the torque M in a rigorous way, extending (3.8) and (3.9) to the general case where the trace σn is not necessarily integrable on $\partial\Omega$.

Definition 3.1.4. *Let Ω be an exterior domain with Lipschitz boundary, let $u \in \mathcal{V}(\Omega)$ be the solution of the Stokes problem* (3.4) *with boundary datum $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, let σ be the corresponding stress tensor defined by* (3.6)*, and let $\sigma n \in H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ be the* *trace on $\partial\Omega$ introduced in Definition 3.1.3. The* viscous force *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $F \in \mathbb{R}^3$ such that*

$$F \cdot V = \langle \sigma n, V \rangle \quad \textit{for every } V \in \mathbb{R}^3. \tag{3.13}$$

The viscous torque *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $M \in \mathbb{R}^3$ such that*

$$M \cdot \omega = \langle \sigma n, W_\omega \rangle \quad \textit{for every } \omega \in \mathbb{R}^3, \tag{3.14}$$

where $W_\omega(x) := \omega \times x$ is the velocity field generated by the angular velocity ω.

We conclude this section by proving the continuous dependence on the domains of the solutions to the Stokes problems. To this aim, we introduce a notion of convergence for subsets of $\mathbb{R}^3$. We say that a sequence of sets $(S_k)_k$ converges to S_∞, and we write $S_k \to S_\infty$, if for every $\varepsilon > 0$ there exists m such that for every $k \geqslant m$

$$S_\infty^{-\varepsilon} \subset S_k \subset S_\infty^{+\varepsilon}, \tag{3.15}$$

where $S_\infty^{-\varepsilon} = \{y \in \mathbb{R}^3 : \operatorname{dist}(y, \mathbb{R}^3 \setminus S_\infty) \geqslant \varepsilon\}$ and $S_\infty^{+\varepsilon} = \{y \in \mathbb{R}^3 : \operatorname{dist}(y, S_\infty) \leqslant \varepsilon\}$.

Theorem 3.1.5. *For $k = 1, 2, \ldots, \infty$, let S_k be a bounded connected open set of class C^1, and let w_k be the solution to the minimum problem*

$$\min \left\{ \int_{\mathbb{R}^3} |\mathrm{E}w|^2 \, \mathrm{d}x \, : \, w \in \mathcal{V}(\mathbb{R}^3), \, w = W \textit{ on } \partial S_k \right\}, \tag{3.16}$$

where W denotes either a constant vector $a \in \mathbb{R}^3$ or the affine function $W_\omega(x) = \omega \times x$, for some $\omega \in \mathbb{R}^3$. Assume that $S_k \to S_\infty$ in the sense of (3.15)*. Then $w_k \to w_\infty$ strongly in* $\mathcal{V}(\mathbb{R}^3)$.

Notice that w_k coincides in $S_k^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{S}_k$ with the solution to the Stokes problem (3.4) in $\Omega = S_k^{\mathrm{ext}}$ with boundary condition $w_k = W$ on ∂S_k, while $w_k = W$ in S_k.

Proof. Consider a ball Σ_ρ centered at 0 and containing the closures of all the S_k's. It is possible to find a solenoidal function $\Psi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $\Psi = W$ in ∂S_k.

When W is a constant vector a, we consider a smooth closed curve Γ passing through the origin, whose tangent vector coincides with a in all points of $\Gamma \cap \Sigma_\rho$, and with curvature less than $1/(2\rho)$. In the tubular neighborhood $\Gamma + \Sigma_{2\rho}$, we consider the vector field $\Psi(x) := \psi(\operatorname{dist}(x, \Gamma))\tau(\pi_\Gamma(x))$, where π_Γ is the projection on Γ, τ returns the tangential component, and $\psi \in C_c^\infty([0, 2\rho[)$ with $\psi(r) = 1$ for $0 \leqslant r \leqslant \rho$. It is easy to see that Ψ is solenoidal, coincides with a on Σ_ρ, and vanishes near the boundary of the tubular neighborhood. Its extension by 0 provides the required function in $C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$.

In the case $W = W_\omega$, it is enough to take $\Psi(x) = \omega \times \phi(x) x$, with ϕ a radial scalar function with compact support such that $\phi(x) = 1$ for $x \in \Sigma_\rho$.

By minimality,

$$\int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}\Psi|^2 \, \mathrm{d}x, \quad \text{for } k = 1, 2, \ldots, \infty.$$

It follows that the sequence $(w_k)_k$ admits a weak limit w^* in $\mathcal{V}(\mathbb{R}^3)$.

Notice that $\Delta W = 0$ and $\operatorname{div} W = 0$ on S_k, hence $w_k = W$ on S_k for $k = 1, 2, \ldots, \infty$. Since $S_\infty^{-\varepsilon} \subset S_k$ for k large enough by the first inclusion in (3.15), we get $w^* = W$ on $S_\infty^{-\varepsilon}$. As ε is arbitrary, we conclude $w^* = W$ on S_∞, which implies that the same equality holds for the traces on ∂S_∞. Therefore, w^* is a competitor in the problem for ∂S_∞.

We now show it is also the minimum. For this, consider an admissible function v for the problem (3.16) for $k = \infty$. Then $v - \Psi \in \mathcal{V}(\mathbb{R}^3)$; it follows that $v - \Psi = 0$ on ∂S_∞. In particular, $v - \Psi \in \mathcal{V}_0(S_\infty^{\mathrm{ext}})$ and by Theorem 3.1.1 there exist functions $\varphi_\eta \in C_{\mathrm{c}}^\infty(S_\infty^{\mathrm{ext}}; \mathbb{R}^3)$ such that $\varphi_\eta \to v - \Psi$ when $\eta \to 0$. For every $\eta > 0$ the function $v_\eta := \varphi_\eta + \Psi$ coincides with W in a neighborhood of ∂S_∞. By (3.15), this implies that v_η is a competitor for problem (3.16) on ∂S_k, for k large enough. Therefore, by the minimality of w_k

$$\int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v_\eta|^2 \, \mathrm{d}x.$$

Taking the limit first as $k \to \infty$ and then as $\eta \to 0$, we get

$$\limsup_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v|^2 \, \mathrm{d}x.$$

By the lower semicontinuity of the norm in $\mathcal{V}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |\mathrm{E}w^*|^2 \, \mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \limsup_{k\to\infty} \int_{\mathbb{R}^3} |\mathrm{E}w_k|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^3} |\mathrm{E}v|^2 \, \mathrm{d}x,$$

thus proving the minimality of w^*. By uniqueness, we have $w_\infty = w^*$. The last chain of inequalities, applied with $v = w_\infty$, shows also that $\|w_k\|_{D^{1,2}} \to \|w_\infty\|_{D^{1,2}}$, hence $w_k \to w_\infty$ strongly in $\mathcal{V}(\mathbb{R}^3)$. □

3.2 The equations of motion

The motion $t \mapsto \varphi_t$ determines for almost every $t \in [0, T]$ the Eulerian velocity U_t through the formula

$$U_t(y) := \dot{\varphi}_t(\varphi_t^{-1}(y)) \quad \text{for almost every } y \in \partial A_t\,.$$

As shown in Section 2.2, A_t is of class C^2 and

$$U_t \in H^{1/2}(\partial A_t; \mathbb{R}^3) \quad \text{for almost every } t \in [0, T].$$

We can apply Theorem 3.1.2 with $\Omega = A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$ and, for almost every $t \in [0, T]$, we obtain a unique solution u_t to the problem

$$\begin{cases} u_t \in \mathcal{V}(A_t^{\mathrm{ext}}), \quad u_t = U_t & \text{on } \partial A_t\,, \\ \displaystyle\int_{A_t^{\mathrm{ext}}} \mathrm{E}u_t : \mathrm{E}w \, \mathrm{d}y = 0 & \text{for every } w \in \mathcal{V}_0(A_t^{\mathrm{ext}}). \end{cases}$$

Let F_{A_t,U_t} and M_{A_t,U_t} be the viscous force and torque determined by the velocity field U_t according to (3.13) and (3.14). Since we are neglecting inertia and imposing the self-propulsion constraint, the equations of motion reduce to the vanishing of the viscous force and torque, i.e.,

$$F_{A_t,U_t} = 0 \quad \text{and} \quad M_{A_t,U_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{3.17}$$

We assume that φ_t is written as $\varphi_t = r_t \circ s_t$, where r_t is a rigid motion as in (2.9) and $t \mapsto s_t$ is a prescribed shape function. Our aim is to find $t \mapsto r_t$ so that the equations of motion (3.17) are satisfied. More precisely, we prove Theorem 3.2.1 below, which shows that (3.17) is equivalent to a system of ordinary differential equations where the unknown functions are the translation $t \mapsto y_t$ and the rotation $t \mapsto R_t$ appearing in (2.9).

To define the coefficients of these differential equations, we consider the sets $B_t = s_t(A)$ introduced in Section 2.2 and the $3{\times}3$ matrices K_t, C_t, J_t, depending only on the geometry of B_t, whose entries are defined by

$$(K_t)_{ij} := \langle \sigma[e_j]n, e_i \rangle_{B_t^{\text{ext}}}, \tag{3.18a}$$

$$(C_t)_{ij} := \langle \sigma[e_j]n, e_i{\times}z \rangle_{B_t^{\text{ext}}}, \tag{3.18b}$$

$$(J_t)_{ij} := \langle \sigma[e_j{\times}z]n, e_i{\times}z \rangle_{B_t^{\text{ext}}}, \tag{3.18c}$$

where $B_t^{\text{ext}} := \mathbb{R}^3 \setminus \overline{B}_t$, the duality product is given in Definition 3.1.3, and $\sigma[W]$ denotes the stress tensor associated to the outer Stokes problem in B_t^{ext} with boundary datum W. The notation $\sigma[W]$ emphasizes that, by the linearity of Stokes system, the dependence of σ on W is linear. Formula (3.11) shows that K_t and J_t are symmetric. The matrix

$$\begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix}$$

is often called in the literature *grand resistance matrix*, and is invertible [19]. Let

$$\begin{bmatrix} H_t & D_t^\top \\ D_t & L_t \end{bmatrix} := \begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix}^{-1} \tag{3.19}$$

be its inverse. For almost every $t \in [0,T]$, let $W_t := \dot{s}_t \circ s_t^{-1}$, and let F_t^{sh} and M_t^{sh} be the viscous force and torque on ∂B_t determined by the boundary value W_t. According to (3.13) and (3.14), the components of F_t^{sh} and M_t^{sh} are given by

$$(F_t^{\text{sh}})_i = \langle \sigma[W_t]n, e_i \rangle_{B_t^{\text{ext}}}, \tag{3.20a}$$

$$(M_t^{\text{sh}})_i = \langle \sigma[W_t]n, e_i{\times}z \rangle_{B_t^{\text{ext}}}. \tag{3.20b}$$

Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ be the linear operator that associates to every $\omega \in \mathbb{R}^3$ the only antisymmetric matrix $\mathcal{A}(\omega)$ such that $\mathcal{A}(\omega)z = \omega{\times}z$. In other words, ω is the axial vector of $\mathcal{A}(\omega)$. Finally, we define

$$b_t := H_t F_t^{\text{sh}} + D_t^\top M_t^{\text{sh}}, \qquad \Omega_t := \mathcal{A}(D_t F_t^{\text{sh}} + L_t M_t^{\text{sh}}), \tag{3.21}$$

which depend on s_t via (3.20) and the definition of W_t.

Theorem 3.2.1. *Assume that the shape function* $t \mapsto s_t$ *satisfies* (2.10), (2.11), *and* (2.16) *and that the position function* $t \mapsto r_t$ *satisfies* (2.9) *and* (2.15). *Then the following conditions are equivalent:*

the deformation function $t \mapsto \varphi_t := r_t \circ s_t$ *satisfies the equations of motion introduced in* (3.17)*;*

the functions $t \mapsto y_t$ *and* $t \mapsto R_t$ *satisfy the system*

$$\dot{y}_t = R_t b_t\,, \qquad \dot{R}_t = R_t \Omega_t\,, \quad \textit{for almost every } t \in [0,T], \tag{3.22}$$

where b_t *and* Ω_t *are defined in* (3.21).

Proof. It is convenient to set the problem in the intermediate configuration B_t, thus assuming the point of view of the coordinate system of the shape functions.

After performing the change of variables $y = r_t(z)$, $z \in B_t^{\mathrm{ext}}$, it turns out that the velocity field $v_t(z) := R_t^\top u_t(r_t(z))$ is the solution of the Stokes problem

$$\begin{cases} v_t \in \mathcal{V}(B_t^{\mathrm{ext}}), \quad v_t = V_t & \text{on } \partial B_t\,, \\ \displaystyle\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t : \mathrm{E}w \,\mathrm{d}z = 0, & \text{for every } w \in \mathcal{V}_0(B_t^{\mathrm{ext}}), \end{cases}$$

where $V_t(z) = R_t^\top U_t(r_t(z))$, see Fig. 3.1.

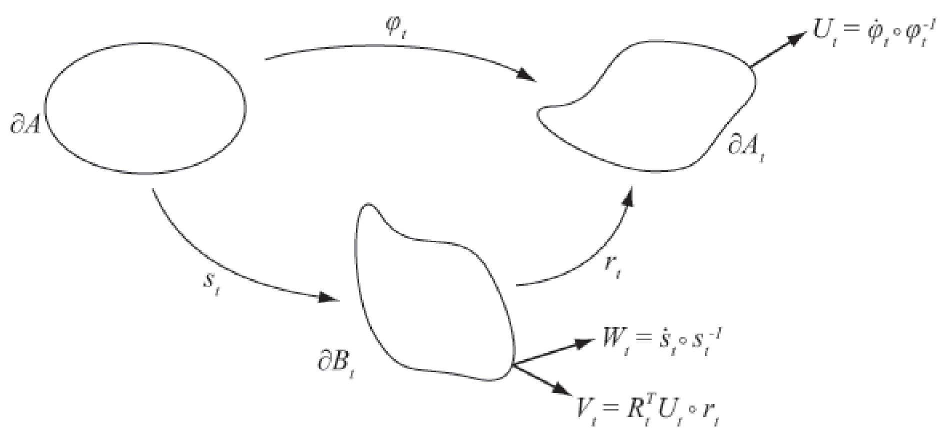

Figure 3.1: Notation for the boundary velocities.

Let F_{B_t,V_t} and M_{B_t,V_t} be the viscous force and torque on ∂B_t determined by v_t according to (3.13) and (3.14), with $\Omega = B_t^{\mathrm{ext}}$. It is easy to check that $F_{B_t,V_t} = R_t^\top F_{A_t,U_t}$ and $M_{B_t,V_t} = R_t^\top M_{A_t,U_t}$, so that the equations of motion (3.17) reduce to

$$F_{B_t,V_t} = 0 \quad \text{and} \quad M_{B_t,V_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{3.23}$$

Let ω_t be the *axial vector* of $\dot{R}_t R_t^\top$, i.e., the unique vector $\omega_t \in \mathbb{R}^3$ such that $\omega_t \times z = \dot{R}_t R_t^\top z$. It is easy to see that $R_t^\top \dot{R}_t z = (R_t^\top \omega_t) \times z$, so that

$$V_t(z) = W_t(z) + R_t^\top \dot{y}_t + (R_t^\top \omega_t) \times z \quad \text{for almost every } z \in \partial B_t,$$

where $W_t(z) = \dot{s}_t(s_t^{-1}(z))$. Let $(F_t^{\mathrm{tr}}, M_t^{\mathrm{tr}})$ and $(F_t^{\mathrm{rot}}, M_t^{\mathrm{rot}})$ be the pairs viscous force–torque on ∂B_t corresponding to the boundary values $R_t^\top \dot{y}_t$ and $(R_t^\top \omega_t)\times z$, respectively. It is well known, see, e.g., [19] that

$$\begin{aligned} F_t^{\mathrm{tr}} &= -K_t R_t^\top \dot{y}_t\,, & F_t^{\mathrm{rot}} &= -C_t^\top R_t^\top \omega_t\,,\\ M_t^{\mathrm{tr}} &= -C_t R_t^\top \dot{y}_t\,, & M_t^{\mathrm{rot}} &= -J_t R_t^\top \omega_t\,, \end{aligned}$$

where K_t, C_t, and J_t are the matrices defined in (3.18). Recalling the linearity of the equations, we get

$$\begin{bmatrix} F_{B_t,V_t} \\ M_{B_t,V_t} \end{bmatrix} = -\begin{bmatrix} K_t R_t^\top & C_t^\top R_t^\top \\ C_t R_t^\top & J_t R_t^\top \end{bmatrix} \begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} + \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix},$$

hence the equations of motion (3.23) become

$$\begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix} \begin{bmatrix} R_t^\top & 0 \\ 0 & R_t^\top \end{bmatrix} \begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix}. \tag{3.24}$$

It follows from (3.19) and (3.24) that the equations of motion (3.23) are equivalent to

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^\top \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix} \quad \text{for almost every } t \in [0,T].$$

The first equation reads

$$\dot{y}_t = R_t b_t\,, \quad \text{with } b_t = H_t F_t^{\mathrm{sh}} + D_t^\top M_t^{\mathrm{sh}}. \tag{3.25}$$

To write the second equation in the form (3.22), we use the equality $\mathcal{A}(\omega_t) = \dot{R}_t R_t^\top$. In order to rewrite the second equation

$$\omega_t = R_t(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}) \tag{3.26}$$

in a more useful way, we need a formula for $\mathcal{A}(R\omega)$ when R is an arbitrary rotation. In view of the following equalities

$$\mathcal{A}(R\omega)z = (R\omega)\times z = R\omega \times RR^\top z = R(\omega\times R^\top z) = R\mathcal{A}(\omega)R^\top z,$$

we can conclude that $\mathcal{A}(R\omega) = R\mathcal{A}(\omega)R^\top$. Therefore, by applying $\mathcal{A}$ to both members of (3.26), we get

$$\dot{R}_t R_t^\top = \mathcal{A}(\omega_t) = \mathcal{A}(R_t(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}})) = R_t \mathcal{A}(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}) R_t^\top,$$

so that, eventually, equation (3.26) reads

$$\dot{R}_t = R_t \Omega_t\,, \quad \text{with } \Omega_t = \mathcal{A}(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}). \tag{3.27}$$

This concludes the proof. □

Remark: We claim that every absolutely continuous solution to the second equation in (3.22) belongs to $\mathrm{SO}(3)$, whenever $R_0 \in \mathrm{SO}(3)$. Indeed, by differentiating $R_t R_t^\top$ with respect to time, we get

$$(R_t R_t^\top)^\cdot = \dot{R}_t R_t^\top + R_t \dot{R}_t^\top = R_t \Omega_t R_t^\top - R_t \Omega_t R_t^\top = 0,$$

where we used the fact that Ω_t is skew symmetric. This shows that the matrix $R_t R_t^\top$ is constant in time and the claim follows.

The standard theory of ordinary differential equations with possibly discontinuous coefficients [18], ensures that the Cauchy problem for (3.22) has one and only one Lipschitz solution $t \mapsto R_t$, $t \mapsto y_t$, provided that the functions $t \mapsto \Omega_t$ and $t \mapsto b_t$ are measurable and bounded. By (3.25) and (3.27), this happens when the functions

$$t \mapsto H_t\,, \quad t \mapsto D_t\,, \quad t \mapsto L_t\,, \quad t \mapsto F_t^{\mathrm{sh}}, \quad t \mapsto M_t^{\mathrm{sh}} \tag{3.28}$$

are measurable and bounded. This property for the first three functions follows from the continuity of the block elements of the grand resistance matrix

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{3.29}$$

which will be proved in the last part of this section. The proof of the measurability and boundedness of the last two functions in (3.28) requires some technical tools that will be developed in Sections 3.3 and 3.4.

To prove the continuity of the function in (3.29) we will use Theorem 3.1.5. To this aim, in the next lemma, we prove a continuity property of the set-valued function $t \mapsto B_t$.

Lemma 3.2.2. *Let s_t satisfy* (2.16). *Then if $t \to t_\infty$ the sets B_t converge to the set B_{t_∞} in the sense of* (3.15).

Proof. We recall that $B_t = s_t(A)$ for all $t \in [0,T]$. Let us prove the two inclusions separately. To see that $s_t(A) \subset (s_{t_\infty}(A))^{+\varepsilon}$, consider a point $y \in s_t(A)$: then, there exists a point $x \in A$ such that $y = s_t(x)$. We conclude if we prove that $|s_{t_\infty}(x) - s_t(x)| \leqslant \varepsilon$, for all $x \in A$ and for all t sufficiently close to t_∞.

$$\sup_{x\in A} |s_t(x) - s_{t_\infty}(x)| \leqslant \|s_t - s_{t_\infty}\|_{C^1(A;\mathbb{R}^3)} \leqslant L\,|t - t_\infty| \leqslant \varepsilon,$$

provided that $|t - t_\infty| \leqslant \varepsilon/L$. For the inclusion $(s_{t_\infty}(A))^{-\varepsilon} \subset s_t(A)$, a simple topological degree argument can be applied, so we can conclude the proof. □

We are now in a position to prove the continuity of the elements of the grand resistance matrix.

Proposition 3.2.3. *Assume that s_t satisfies* (2.10), (2.11)*, and* (2.16). *Then the functions*

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{3.30a}$$

$$t \mapsto H_t\,, \quad t \mapsto D_t\,, \quad t \mapsto L_t \tag{3.30b}$$

are continuous.

Proof. Recalling (3.18) and (3.11), we can write

$$(K_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}v_t^j : \mathrm{E}v_t^i \,\mathrm{d}z, \tag{3.31a}$$

$$(C_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}v_t^j : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{3.31b}$$

$$(J_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}\hat{v}_t^j : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{3.31c}$$

where v_t^j and $\hat{v}_t^j$ are the solutions to problem (3.16) for $S_k = B_t$, with $W = e_j$ and $W = e_j \times z$, respectively. Since the convergence of the sets B_t is guaranteed by Lemma 3.2.2, we can now apply Theorem 3.1.5 and we obtain that the functions in (3.30a) are continuous. The continuity in (3.30b) follows from (3.19). □

The proof of the measurability and boundedness of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ requires much more work, due to the fact that both the domains B_t and the boundary data $W_t = \dot{s}_t \circ s_t^{-1}$ depend on time. Moreover, the boundary value W_t might be discontinuous with respect to t, so that we cannot expect the functions $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ to be continuous.

To prove the measurability we start from an integral representation of F_t^{sh} and M_t^{sh}, similar to (3.31). As $\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S$ is not necessarily zero, we have to replace $\mathbb{R}^3$ in (3.31) by the complement of an open ball $\Sigma_\varepsilon^0 \subset\subset B_t$. Since, in general, this inclusion holds only locally in time, we first fix $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$ and select $\delta > 0$ and $\varepsilon > 0$ so that the open ball $\Sigma_\varepsilon^0 := \Sigma_\varepsilon(z^0)$ of radius ε centered at z^0 satisfies

$$\Sigma_\varepsilon^0 \subset\subset B_t\,, \quad \text{for all } t \in I_\delta(t_0) := [0,T] \cap (t_0 - \delta, t_0 + \delta). \tag{3.32}$$

This is possible thanks to the continuity properties of $t \mapsto s_t$ listed in the previous Section.

Next we consider the solution w_t to the problem

$$\min \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}w|^2 \,\mathrm{d}z,$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ such that $w = W_t$ on ∂B_t and $w = \lambda_t (z - z^0)/\varepsilon^3$ on $\partial\Sigma_\varepsilon^0$, where

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S.$$

The value of λ_t is chosen so that the flux condition (3.5) on $\partial B_t \cup \partial\Sigma_\varepsilon^0$ is satisfied.

Finally, recalling (3.20) and (3.11), we can write the following explicit integral representation of F_t^{sh} and M_t^{sh}

$$(F_t^{\mathrm{sh}})_i = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z = 2\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z, \tag{3.33a}$$

$$(M_t^{\mathrm{sh}})_i = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z = 2\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z, \tag{3.33b}$$

where v_t^i and $\hat{v}_t^i$ have been defined in the proof of Proposition 3.2.3 and where the last equalities are due to the fact that $\mathrm{E}v_t^i = \mathrm{E}\hat{v}_t^i = 0$ in B_t. We deduce from Theorem 3.1.5 and Lemma 3.2.2 that the functions $t \mapsto v_t^i$ and $t \mapsto \hat{v}_t^i$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. Therefore, the measurability and boundedness of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ will be proved if we show that the function $t \mapsto w_t$ from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ is measurable and bounded.

Even the boundedness of $\|\nabla w_t\|_{L^2}$ is an issue, since all estimates for a solenoidal extension of W_t considered so far in the literature depend on the geometry of ∂B_t. In Section 3.3 we make this dependence explicit and conclude that under our assumptions on $t \mapsto s_t$ the L^2 bound for the gradient of the solenoidal extension is uniform with respect to t. This result will be used in Section 3.4 to prove the measurability of the function $t \mapsto w_t$.

3.3 Extension operators

We give now two extension results of a function defined on ∂B_t to an open region containing ∂B_t. Lemma 3.3.2 is classical, but for our future purposes we need a solenoidal version, as stated in Proposition 3.3.3. Its proof requires a number of preliminary lemmas that are proved beforehand. The next lemma shows that, locally in time, the sets $\Sigma_\rho \setminus \overline{B}_t$ are C^2 diffeomorphic to each other.

Lemma 3.3.1. *Assume that s_t satisfies* (2.10), (2.11), *and* (2.16), *and let Σ_ρ be as in* (2.12). *Let $t_0 \in [0,T]$. Then, there exists a neighborhood $I_\delta(t_0) = [0,T] \cap (t_0-\delta, t_0+\delta)$ of t_0 with the following property: for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Phi_t^{t_0} : \Sigma_\rho \to \Sigma_\rho$, coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Phi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t. In particular, we have*

$$\Phi_t^{t_0}(B_t) = B_{t_0} \quad \textit{and} \quad \Phi_t^{t_0}(\Sigma_\rho \setminus \overline{B}_t) = \Sigma_\rho \setminus \overline{B}_{t_0}\,. \tag{3.34}$$

Moreover,

$$\left\|\Phi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} + \left\|(\Phi_t^{t_0})^{-1}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant C, \tag{3.35}$$

where C is a constant independent of t_0, t.

Proof. Recall that $B_t \subset\subset \Sigma_{\rho-1}$ by (2.12), so that $B_t \cup (\Sigma_\rho \setminus \overline{\Sigma}_{\rho-1})$ has a C^2 boundary. Therefore, it is possible to find a function $\Psi_t^{t_0} \in C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)$ such that $\Psi_t^{t_0} = s_{t_0} \circ s_t^{-1} - I$ on B_t, $\Psi_t^{t_0} = 0$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, and $\left\|\Psi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant C\left\|s_{t_0} \circ s_t^{-1} - I\right\|_{C^2(\overline{B}_t;\mathbb{R}^3)}$, where I is the identity map and C is a constant depending only on ρ and t_0 (see, e.g., [16, Theorem 6.37, page 136]). Since $s_{t_0} \circ s_t^{-1} - I \to 0$ in $C^2(\overline{B}_t;\mathbb{R}^3)$ as $t \to t_0$, there exists a neighborhood $I_\delta(t_0)$ of t_0 such that $\left\|\Psi_t^{t_0}\right\|_{C^2(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$.

For every $t \in I_\delta(t_0)$ let us define $\Phi_t^{t_0} := I + \Psi_t^{t_0}$. Then $\Phi_t^{t_0} = I$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$ and $\Phi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t, which proves the first equality in (3.34). Notice that $\left|\Phi_t^{t_0}(x) - x\right| \leqslant 1/2$

for every $x \in \Sigma_\rho$, so that this implies $\Phi_t^{t_0}(\Sigma_{\rho-1}) \subset \Sigma_\rho$. Since $\Phi_t^{t_0}(\Sigma_\rho \setminus \Sigma_{\rho-1}) = \Sigma_\rho \setminus \Sigma_{\rho-1}$, we conclude that $\Phi_t^{t_0}(\Sigma_\rho) \subset \Sigma_\rho$.

Let us prove that $\Sigma_\rho \subset \Phi_t^{t_0}(\Sigma_\rho)$. Since $\Phi_t^{t_0} = I$ on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, it is enough to show that $\Sigma_{\rho-1} \subset \Phi_t^{t_0}(\Sigma_\rho)$. To this aim we fix $y \in \Sigma_{\rho-1}$. We want to show that there exists $x \in \Sigma_\rho$ such that $x + \Psi_t^{t_0}(x) = y$. This is equivalent to solve the fixed point problem $x = y - \Psi_t^{t_0}(x)$. Since $\left\|\Psi_t^{t_0}\right\|_{C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$, the map $x \mapsto y - \Psi_t^{t_0}(x)$ is a contraction of $\overline{\Sigma}_{\rho-1/2}$ into itself. This implies the existence of a fixed point and concludes the proof of the inclusion $\Sigma_{\rho-1} \subset \Phi_t^{t_0}(\Sigma_\rho)$.

The injectivity of $\Phi_t^{t_0}$ follows easily from the inequality $\left\|\Psi_t^{t_0}\right\|_{C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)} \leqslant 1/2$. Therefore, $\Phi_t^{t_0}\colon \Sigma_\rho \to \Sigma_\rho$ is bijective. Its inverse is of class C^2 by the Local Invertibility Theorem. The second equality in (3.34) follows now from the first one.

Estimate (3.35) is a consequence of (2.16) and (2.17). □

Given two Banach spaces X and Y, the symbol $\mathcal{L}(X;Y)$ denotes the Banach space of continuous linear maps from X into Y. Given a function $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$, let us define

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} (\Phi \circ s_t^{-1}) \cdot n \,\mathrm{d}S,$$

for every $t \in [0,T]$. The constant λ_t is chosen so that if $u|_{\partial B_t} = \Phi \circ s_t^{-1}$ and $u|_{\partial\Sigma_\rho} = \lambda_t z/\left|z\right|^3$, then

$$\int_{\partial(B_t^{\mathrm{ext}}\cap\Sigma_\rho)} u \cdot n \,\mathrm{d}S = 0.$$

Lemma 3.3.2 (Extension operators). *Under the assumptions of Lemma 3.3.1, there exists a continuous function from $I_\delta(t_0)$ into $\mathcal{L}(H^{1/2}(\partial A;\mathbb{R}^3);H^1(\Sigma_\rho;\mathbb{R}^3))$, denoted $t \mapsto \mathcal{S}_t$, such that*

$$\begin{aligned}
\mathcal{S}_t(\Phi) &= \Phi \circ s_t^{-1} && \text{on } \partial B_t,\\
\mathcal{S}_t(\Phi) &= \lambda_t \frac{z}{\left|z\right|^3} && \text{on } \partial\Sigma_\rho,\\
\left\|\mathcal{S}_t(\Phi)\right\|_{H^1(\Sigma_\rho;\mathbb{R}^3)} &\leqslant C\left\|\Phi\right\|_{H^{1/2}(\partial A;\mathbb{R}^3)},
\end{aligned}$$

where the constant C is independent of t and Φ.

Proof. By known results on Sobolev spaces [33, Theorem 5.7, page 103], there exists $\mathcal{S}_{t_0} \in \mathcal{L}(H^{1/2}(\partial A;\mathbb{R}^3);H^1(\Sigma_\rho;\mathbb{R}^3))$ such that $\mathcal{S}_{t_0}(\Phi) = \Phi \circ s_{t_0}^{-1}$ on ∂B_{t_0}. Let $\Phi_t^{t_0}$ be the function given in the proof of Lemma 3.3.1. It is easy to show that $[\mathcal{S}_{t_0}(\Phi)] \circ \Phi_t^{t_0} = \Phi \circ s_t^{-1}$ on ∂B_t. It is enough to define $\mathcal{S}_t(\Phi) = [\mathcal{S}_{t_0}(\Phi)] \circ \Phi_t^{t_0}$. □

Proposition 3.3.3 (Solenoidal extension operators). *Under the assumptions of Lemma 3.3.1, let $t_0 \in [0,T]$ and let $z^0 \in B_{t_0}$. Let $\delta > 0$ and $\varepsilon > 0$ be such that* (3.32) *holds true. Then there exists a uniformly bounded family $(\mathcal{T}_t)_{t\in I_\delta(t_0)}$ of continuous linear operators*

$$\mathcal{T}_t\colon H^{1/2}(\partial A;\mathbb{R}^3) \to H^1(\Sigma_\rho \setminus \overline{\Sigma}_\varepsilon^0;\mathbb{R}^3)$$

such that

for all $t \in I_\delta(t_0)$ *and for all* $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$,

$$\begin{aligned} \mathcal{T}_t(\Phi) &= \Phi \circ s_t^{-1} \quad \text{on } \partial B_t\,, && (3.36a)\\ \mathcal{T}_t(\Phi) &= \lambda_t \frac{z}{|z|^3} \quad \text{on } \partial\Sigma_\rho\,, && (3.36b)\\ \operatorname{div}(\mathcal{T}_t(\Phi)) &= 0 \quad \text{in } \Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon\,; && (3.36c) \end{aligned}$$

for every $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$ *the map* $t \mapsto \mathcal{T}_t(\Phi)$ *is continuous from* $I_\delta(t_0)$ *into* $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. *In particular, the following estimate holds*

$$\|\mathcal{T}_t(\Phi)\|_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)} \leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \tag{3.37}$$

where the constant C *is independent of* t *and* Φ.

The proof of Proposition 3.3.3 requires the estimates contained in the following lemma, whose proof can be found in [33, page 187], [41, Proposition 1.2], [14, Exercixe III.3.3], and in [39, Lemma II.1.5.4].

Lemma 3.3.4. *For every bounded connected open set* $\Omega \subset \mathbb{R}^3$ *with Lipschitz boundary, there exists a constant* $\gamma(\Omega) > 0$ *such that*

$$\|p\|_{L^2(\Omega)} \leqslant \gamma(\Omega)\,\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,, \tag{3.38}$$

for every $p \in L^2(\Omega)$ *with* $\int_\Omega p\,\mathrm{d}x = 0$.

The constant $\gamma(\Omega)$ plays a crucial role in the following result concerning the estimate of a particular solution of the equation $\operatorname{div} u = g$ in Ω with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

Lemma 3.3.5. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded connected open set with Lipschitz boundary and let* $g \in L^2(\Omega)$ *with* $\int_\Omega g\,\mathrm{d}x = 0$. *Then there exists a unique* $u \in H^1_0(\Omega;\mathbb{R}^3)$ *such that*

$\operatorname{div} u = g$ *in* Ω,

$\int_\Omega \nabla u : \nabla v\,\mathrm{d}x = 0$ *for all* $v \in H^1_0(\Omega;\mathbb{R}^3)$ *with* $\operatorname{div} v = 0$ *in* Ω. *Moreover, the following estimate holds*

$$\|u\|_{H^1_0(\Omega;\mathbb{R}^3)} \leqslant \gamma(\Omega)\,\|g\|_{L^2(\Omega)}\,,$$

where $\gamma(\Omega)$ *is the constant in Lemma 3.3.4.*

Proof. The first part of the Lemma is classical and can be found in various texts, e.g., [41, page 22], [14, Theorem V.2.1 and Exercise V.2.1], and [39, Theorem III.1.4.1]. The estimate then follows by a straightforward computation. □

In order to prove Proposition 3.3.3 we have to show that the constants $\gamma(B_t)$ and $\gamma(\Sigma_\rho \setminus \overline{B}_t)$ are uniformly bounded with respect to t. This will be achieved through the following lemma, thanks to Lemma 3.3.1.

Lemma 3.3.6. *There exists a non decreasing function $a : [0,+\infty) \to [0,+\infty)$ such that the constant γ introduced in Lemma 3.3.4 satisfies the estimate*

$$\gamma(\Phi(\Omega)) \leqslant a\Big(\|\Phi\|_{C^2(\overline{\Omega};\mathbb{R}^3)} + \big\|\Phi^{-1}\big\|_{C^2(\Phi(\overline{\Omega});\mathbb{R}^3)} \Big)\, \gamma(\Omega), \tag{3.39}$$

for every bounded open set $\Omega \subset \mathbb{R}^3$ with C^2 boundary and for every invertible function $\Phi \in C^2(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. As shown in [33], (3.38) is a consequence of the following inequalities

$$\|p\|_{L^2(\Omega)} \leqslant \gamma_1(\Omega)\big(\|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)} + \|p\|_{H^{-1}(\Omega)} \big),$$
$$\inf_{t\in\mathbb{R}} \|p-t\|_{H^{-1}(\Omega)} \leqslant \gamma_2(\Omega)\, \|\nabla p\|_{H^{-1}(\Omega;\mathbb{R}^3)}\,,$$

valid for every $p \in L^2(\Omega)$. By a change of variables it is easy to see that $\gamma_1(\Omega)$ and $\gamma_2(\Omega)$ satisfy (3.39). The conclusion follows. □

Let Σ_ρ be as in (2.12) and t_0, z^0, δ, ε, $I_\delta(t_0)$, and Σ^0_ε be as in Proposition 3.3.3. For every $t \in I_\delta(t_0)$ let $\mathcal{U}_t : \big\{g \in L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3) : \int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z = 0\big\} \to H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ be the linear operator defined by $\mathcal{U}_t(g) = u$, where $u|_{B_t\setminus\overline{\Sigma}^0_\varepsilon}$ is the unique function in $H^1_0(B_t \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ such that

$$\operatorname{div} u = g \qquad \text{in } B_t \setminus \overline{\Sigma}^0_\varepsilon\,, \tag{3.40a}$$
$$\int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} \nabla u : \nabla v \,\mathrm{d}z = 0 \qquad \text{for all } v \in H^1_0(B_t \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3) : \operatorname{div} v = 0 \text{ in } B_t \setminus \overline{\Sigma}^0_\varepsilon\,, \tag{3.40b}$$

and $u = 0$ in $(\Sigma_\rho \setminus \overline{B}_t)$. By Lemmas 3.3.1, 3.3.5, and 3.3.6, there exists a constant M, independent of t, such that

$$\|\mathcal{U}_t\|_{\mathcal{L}_t} \leqslant M, \tag{3.41}$$

where $\mathcal{L}_t$ is the Banach space of continuous linear operators from $\big\{g \in L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3) : \int_{B_t\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z = 0\big\}$ into $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$.

Lemma 3.3.7. *Assume* (2.10), (2.11), (2.13), *and* (2.16). *Let $t_0 \in [0,T]$ and let $t_k \in I_\delta(t_0)$, $k = 1,2,\dots,\infty$, and let $g \in L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$ with $\int_{\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon} g\,\mathrm{d}z = 0$ and*

$$\operatorname{supp}(g) \subset\subset B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon \quad \textit{for every } k. \tag{3.42}$$

Assume that $t_k \to t_\infty$ as $k \to \infty$. Then $\mathcal{U}_{t_k}(g) \to \mathcal{U}_{t_\infty}(g)$ strongly in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. A similar result holds if we exchange the roles of $B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon$ and $\Sigma_\rho \setminus \overline{B}_{t_k}$ in the definition of $\mathcal{U}_t$ and in (3.42).

Proof. For $k = 1,2,\dots,\infty$, let $u_{t_k} := \mathcal{U}_{t_k}(g)$. By (3.41), the sequence $(u_{t_k})_k$ is bounded in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. Therefore a subsequence, still denoted by $(u_{t_k})_k$, converges weakly in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ to some function u^*.

We claim that $u^* \in H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. First notice that $u_{t_k} \circ (s_{t_k} \circ s^{-1}_{t_\infty}) = 0$ on ∂B_{t_∞}, hence $u_{t_k} \circ (s_{t_k} \circ s^{-1}_{t_\infty}) \in H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. Since $s_{t_k} \circ s^{-1}_{t_\infty} \to I$ in $C^1(\overline{B}_{t_\infty} \setminus \Sigma^0_\varepsilon;\mathbb{R}^3)$ as

$k \to \infty$, and $u_{t_k} \rightharpoonup u^*$ weakly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$, we obtain $u_{t_k} \circ (s_{t_k} \circ s^{-1}_{t_\infty}) \rightharpoonup u^*$ weakly in $H^1(B_{t_\infty}; \mathbb{R}^3)$. This implies that u^* belongs to $H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$ and proves the claim.

Since $\operatorname{supp}(g) \subset\subset B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon$ for every k, condition (i) in Lemma 3.3.5 gives $\operatorname{div} u_{t_k} = g$ in $\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon$ for every k, hence $\operatorname{div} u^* = g$ in Σ_ρ.

If $v \in C^\infty_c(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$ with $\operatorname{div} v = 0$, from (ii) we have

$$\int_{B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon} \nabla u_{t_k} : \nabla v \, \mathrm{d}z = 0, \quad \text{for } k \text{ large enough.}$$

Passing to the limit as $k \to \infty$, we get

$$\int_{B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon} \nabla u^* : \nabla v \, \mathrm{d}z = 0.$$

An approximation argument based on Theorem 3.1.1 gives the same equality for every $v \in H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$ with $\operatorname{div} v = 0$. By the uniqueness result proved in Lemma 3.3.5, we have $u^* = u_{t_\infty}$.

To prove the strong convergence of $(u_{t_k})_k$ in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$, we fix a connected open set B with Lipschitz boundary such that $\operatorname{supp}(g) \subset\subset B \subset\subset B_{t_k} \setminus \overline{\Sigma}^0_\varepsilon$ for every k. By Lemma 3.3.5, there exists $w \in H^1_0(B; \mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{div} w = g & \text{on } B, \\ \displaystyle\int_B \nabla w : \nabla v \, \mathrm{d}z = 0 & \text{for every } v \in H^1_0(B; \mathbb{R}^3) \text{ with } \operatorname{div} v = 0. \end{cases}$$

We extend w by setting $w = 0$ on $(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon) \setminus \overline{B}$. Since $\operatorname{supp}(g) \subset\subset B$, we have $\operatorname{div} w = g$ on $\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon$.

We take $v = u_{t_k} - w$ as test function in condition (ii) and we obtain

$$\int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_k}|^2 \, \mathrm{d}z = \int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} \nabla u_{t_k} : \nabla v \, \mathrm{d}z, \quad \text{for } k = 1, 2, \dots, \infty.$$

Since $\nabla u_{t_k} \rightharpoonup \nabla u_{t_\infty}$ in $L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{M}^{3\times 3})$, taking the limit as $k \to \infty$ we get

$$\int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_k}|^2 \, \mathrm{d}z \to \int_{\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon} |\nabla u_{t_\infty}|^2 \, \mathrm{d}z,$$

which concludes the proof of the strong convergence in $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$. □

Lemma 3.3.8. *Under the hypotheses of Lemma 3.3.7, let $t \mapsto g_t$ be a continuous function from $I_\delta(t_0)$ into $L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$, endowed with the strong topology, and let $\mathcal{U}_t$ be the operator defined in* (3.40). *Assume that*

$$\int_{B_t \setminus \overline{\Sigma}^0_\varepsilon} g_t \, \mathrm{d}z = 0 \quad \textit{for every } t \in I_\delta(t_0). \tag{3.43}$$

Then the function $t \mapsto \mathcal{U}_t(g_t)$ is continuous from $I_\delta(t_0)$ into $H^1_0(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$, endowed with the strong topology. A similar result holds if we exchange the roles of $B_t \setminus \overline{\Sigma}^0_\varepsilon$ and $\Sigma_\rho \setminus \overline{B}_t$ in the definition of $\mathcal{U}_t$ and in (3.43).

Proof. Let us fix $\tau \in I_\delta(t_0)$ and $\eta > 0$. There exists $h \in L^2(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$ with compact support in B_τ such that

$$\|h - g_\tau\|_{L^2(B_\tau \setminus \overline{\Sigma}^0_\varepsilon)} < \eta.$$

By continuity, for t sufficiently close to τ we have

$$\|h - g_t\|_{L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon)} < \eta,$$

and $\mathrm{supp}(h) \subset\subset B_t \setminus \overline{\Sigma}^0_\varepsilon$. By (3.41) we have

$$\begin{aligned}
&\|\mathcal{U}_t(g_t) - \mathcal{U}_\tau(g_\tau)\|_{H^1} \\
&\leqslant \|\mathcal{U}_t(g_t - h)\|_{H^1} + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\|_{H^1} + \|\mathcal{U}_\tau(h - g_\tau)\|_{H^1} \\
&\leqslant \|\mathcal{U}_t\|_{\mathcal{L}_t} \|g_t - h\|_{L^2(B_t \setminus \overline{\Sigma}^0_\varepsilon)} + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\|_{H^1} + \|\mathcal{U}_\tau\|_{\mathcal{L}_\tau} \|h - g_\tau\|_{L^2(B_\tau \setminus \overline{\Sigma}^0_\varepsilon)} \\
&\leqslant M\eta + \|\mathcal{U}_t(h) - \mathcal{U}_\tau(h)\| + M\eta.
\end{aligned}$$

Lemma 3.3.7 yields

$$\limsup_{t \to \tau} \|\mathcal{U}_t(g_t) - \mathcal{U}_\tau(g_\tau)\|_{H^1} \leqslant 2M\eta.$$

As η is arbitrary, we have shown that the convergence $\mathcal{U}_t(g_t) \to \mathcal{U}_\tau(g_\tau)$ is strong in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$. □

Proof of Proposition 3.3.3. For all $t \in I_\delta(t_0)$, let $\zeta_t := \mathcal{S}_t(\Phi)$ be the extension given by Lemma 3.3.2. Define g_t^{int} and g_t^{ext} as $\mathrm{div}(\zeta_t)$ restricted to $B_t \setminus \overline{\Sigma}^0_\varepsilon$ and $\Sigma_\rho \setminus \overline{B}_t$, respectively. An easy computation shows that

$$\int_{B_t \setminus \overline{\Sigma}^0_\varepsilon} g_t^{\mathrm{int}}\, \mathrm{d}z = \int_{\Sigma_\rho \setminus \overline{B}_t} g_t^{\mathrm{ext}}\, \mathrm{d}z = 0.$$

Therefore, there exist functions $u_t^{\mathrm{int}} \in H^1_0(B_t \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$ and $u_t^{\mathrm{ext}} \in H^1_0(\Sigma_\rho \setminus \overline{B}_t; \mathbb{R}^3)$ satisfying conditions (i) and (ii) of Lemma 3.3.5. One can define $u_t = \mathcal{U}_t(g_t)$ as the function defined by u_t^{int} on $B_t \setminus \overline{\Sigma}^0_\varepsilon$ and by u_t^{ext} on $\Sigma_\rho \setminus \overline{B}_t$. Notice that u_t agrees with zero on ∂B_t, on $\partial \Sigma_\rho$, and on $\partial \Sigma^0_\varepsilon$.

Consider now $\mathcal{T}_t(\Phi) := \mathcal{S}_t(\Phi) - \mathcal{U}_t(g_t) = \zeta_t - u_t$. This extension is clearly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$ and agrees with (3.36) so that (i) is satisfied. Moreover, by the continuity properties of $\mathcal{S}_t$ and $\mathcal{U}_t$, it turns out that also $\mathcal{T}_t$ is continuous from $I_\delta(t_0)$ into $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon; \mathbb{R}^3)$, so that (ii) and estimate (3.37) follow.

3.4 Dependence on the data

Using the tools developed in the preceding section, we are finally ready to prove some results concerning continuity and measurability properties of the solutions to the Stokes problems. These will lead us to the statement of Theorem 3.4.4 about the existence, uniqueness, and regularity of the rigid motion $t \mapsto r_t$ that causes the swimmer's displacement in the viscous fluid.

Proposition 3.4.1. *Assume that s_t satisfies* (2.10)*,* (2.11)*, and* (2.16)*. Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (3.32)*. Let $I_\delta(t_0)$ be given as in Lemma 3.3.1. Suppose that the map $t \mapsto \Phi_t$ belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. Define*

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} (\Phi_t \circ s_t^{-1}) \cdot n \,\mathrm{d}S.$$

Let w_t be the solution of the problem

$$\min \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w|^2 \,\mathrm{d}z, \tag{3.44}$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ such that $w = \Phi_t \circ s_t^{-1}$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma^0_\varepsilon$. Then $t \mapsto w_t$ belongs to $C^0(I_\delta(t_0); \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon))$.

Proof. Let $(t_k)_k \subset I_\delta(t_0)$ be a sequence that converges to $t_\infty \in I_\delta(t_0)$. Let ψ_{t_k} be the extension of $\Phi_{t_k} \circ s_{t_k}^{-1}$ provided by Proposition 3.3.3. It can be further extended by $\lambda_t z/|z|^3$ on $\mathbb{R}^3 \setminus \Sigma_\rho$, so that $\psi_{t_k} \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ and is a competitor in the minimum problem (3.44) corresponding to $t = t_k$; therefore,

$$\begin{aligned}\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2 \,\mathrm{d}z &\leqslant \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}\psi_{t_k}|^2 \,\mathrm{d}z \leqslant \|\psi_{t_k}\|^2_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)} \\ &\leqslant C^2(\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|)^2 \leqslant (CM)^2,\end{aligned}$$

where C is the constant in (3.37) and $M > 0$ is a uniform upper bound of $\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|$, whose existence is guaranteed by the fact that $t \mapsto \Phi_t$ belongs to $L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. Thus, the sequence $(w_{t_k})_k$ is equi-bounded in $\mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ and, up to a subsequence, it converges weakly to some $w^* \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$.

We claim that w^* is a competitor in problem (3.44) for $t = t_\infty$. First, notice that $\Phi_{t_k} \circ s_{t_\infty}^{-1} = w_{t_k} \circ (s_{t_k} \circ s_{t_\infty}^{-1})$ on ∂B_{t_∞}. Let $\Phi^{t_k}_{t_\infty}$ be the extension of $s_{t_k} \circ s_{t_\infty}^{-1}$ considered in Lemma 3.3.1. Arguing as in the proof of that lemma, we find that $\Phi^{t_k}_{t_\infty} \to I$ in $C^1(\overline{\Sigma}_\rho;\mathbb{R}^3)$ as $t_n \to t_\infty$. Since $w_{t_k} \rightharpoonup w^*$ weakly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, we obtain that $w_{t_k} \circ \Phi^{t_k}_{t_\infty} \rightharpoonup w^*$ weakly in $H^1(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$. This implies that $w_{t_k} \circ (s_{t_k} \circ s_{t_\infty}^{-1}) \rightharpoonup w^*$ weakly in $H^{1/2}(\partial B_{t_\infty};\mathbb{R}^3)$. On the other hand, $\Phi_{t_k} \circ s_{t_\infty}^{-1} \to \Phi_{t_\infty} \circ s_{t_\infty}^{-1}$ in $H^{1/2}(\partial B_{t_\infty};\mathbb{R}^3)$. As $\Phi_{t_k} \circ s_{t_\infty}^{-1} = w_{t_k} \circ (s_{t_k} \circ s_{t_\infty}^{-1})$ on ∂B_{t_∞}, we deduce that $w^* = \Phi_{t_\infty} \circ s_{t_\infty}^{-1}$ on ∂B_{t_∞}. This concludes the claim.

Let $v \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ be another competitor in problem (3.44) for $t = t_\infty$, and let $\zeta := v - \psi_{t_\infty}$, where $\psi_{t_\infty} := \mathcal{T}_{t_\infty}(\Phi_{t_\infty})$ is the extension provided by Proposition 3.3.3, extended by zero on $\mathbb{R}^3 \setminus \Sigma_\rho$. The function ζ vanishes on ∂B_{t_∞} and its restrictions to B_{t_∞} and $B^{\mathrm{ext}}_{t_\infty}$ belong to $H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$ and $\mathcal{V}_0(B^{\mathrm{ext}}_{t_\infty};\mathbb{R}^3)$, respectively. Then by the Density Theorem 3.1.1 and by a classical density result in $H^1_0(B_{t_\infty} \setminus \overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)$, for every $\eta > 0$, there exists a function $\zeta^\eta \in \mathcal{V}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$, vanishing in a neighborhood of ∂B_{t_∞}, such that $\|\zeta^\eta - \zeta\|_{D^{1,2}(\Sigma^{0,\mathrm{ext}}_\varepsilon;\mathbb{R}^3)} \leqslant \eta$. Define now $v^\eta_{t_k} := \psi_{t_k} + \zeta^\eta$, and observe that, for k large enough, it is a competitor in the minimum problem (3.44) for $t = t_k$. Therefore,

$$\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2 \,\mathrm{d}z \leqslant \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}v^\eta_{t_k}|^2 \,\mathrm{d}z = \int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}\psi_{t_k} + \mathrm{E}\zeta^\eta|^2 \,\mathrm{d}z.$$

Taking the limit first as $k \to \infty$ and then as $\eta \to 0$, we get

$$\begin{aligned}\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}w^*|^2 \,\mathrm{d}z &\leqslant \limsup_{k\to\infty} \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}w_{t_k}|^2 \,\mathrm{d}z \\ &\leqslant \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}\psi_{t_\infty} + \mathrm{E}\zeta|^2 \,\mathrm{d}z = \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}v|^2 \,\mathrm{d}z,\end{aligned}$$

where the convergence of $\mathrm{E}\psi_{t_k}$ to $\mathrm{E}\psi_{t_\infty}$ is guaranteed as a consequence of (ii) in Proposition 3.3.3. This proves that w^* is a minimum, so that $w^* = w_{t_\infty}$. By taking $v = w^*$, we get the convergence of the $D^{1,2}$ norms, therefore $w_{t_k} \to w_{t_\infty}$ strongly in $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. This concludes the proof. □

We notice that Theorem 3.1.5 turns out to be a particular case of Proposition 3.4.1, for special boundary data not depending on time. Nonetheless, we think it is useful to present both results, since the technique of the proof is much easier in Theorem 3.1.5.

As we have seen at the end of Section 3.2, Theorem 3.1.5 applied to purely linear and purely angular boundary velocities guarantees the continuity of the elements of the matrices in (3.19), while Proposition 3.4.1 will give the continuity of the known terms F_t^{sh} and M_t^{sh} in (3.24).

Theorem 3.4.2. *Assume that s_t satisfies* (2.10)*,* (2.11)*,* (2.13)*, and* (2.16)*, and let $t_0 \in [0,T]$, $z^0 \in B_{t_0}$, and let Σ_ε^0 and $I_\delta(t_0)$ be as in* (3.32)*. Assume, in addition, that $I_\delta(t_0)$ satisfies Lemma 3.3.1. Let w_t be the solution of the problem*

$$\min \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |\mathrm{E}w|^2 \,\mathrm{d}z, \tag{3.45}$$

where the minimum is taken over all functions $w \in \mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ such that $w = \dot{s}_t \circ s_t^{-1}$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma_\varepsilon^0$. Then the function $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$.

Proof. We approximate the functions $\dot{s}_t$ with the sequence Φ_t^η defined by

$$\Phi_t^\eta(x) := \int_{\mathbb{R}} \kappa_\eta(t-\tau)\dot{s}_\tau(x)\,\mathrm{d}\tau, \tag{3.46}$$

where κ_η is a regularizing kernel supported in the ball Σ_η of radius η and of unit mass. Since the function $\tau \mapsto \dot{s}_\tau$ belongs to $L^\infty(I_\delta(t_0); W^{1,p}(A;\mathbb{R}^3))$ for every $2 \leqslant p < \infty$, the integral in (3.46) can be seen as a Bochner integral in $W^{1,p}(A;\mathbb{R}^3)$. This implies that $t \mapsto \Phi_t^\eta$ belongs to $C^0(I_\delta(t_0); W^{1,p}(A;\mathbb{R}^3))$; in particular, it belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3))$. Moreover, by (2.18), we have $\mathrm{Lip}(\Phi_t^\eta) \leqslant L$. Therefore, the map $t \mapsto \Phi_t^\eta$ belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. Moreover, for almost every $t \in I_\delta(t_0)$, $\Phi_t^\eta \to \dot{s}_t$ strongly in $H^{1/2}(\partial A;\mathbb{R}^3)$.

Let w_t^η be the solutions to problems (3.45), where the minimum is now taken over all functions $w \in \mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ such that $w = \Phi_t^\eta \circ s_t^{-1}$ on ∂B_t and $w = \lambda_t(z - z^0)/\varepsilon^3$ on $\partial\Sigma_\varepsilon^0$. By the properties of the functions $t \mapsto \Phi_t^\eta$ mentioned above and by Proposition 3.4.1, the functions $t \mapsto w_t^\eta$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$.

We recall that, for almost every $t \in I_\delta(t_0)$, $\Phi_t^\eta \to \dot{s}_t$ strongly in $H^{1/2}(\partial A;\mathbb{R}^3)$. This implies that $\Phi_t^\eta \circ s_t^{-1} \to \dot{s}_t \circ s_t^{-1}$ strongly in $H^{1/2}(\partial B_t;\mathbb{R}^3)$. By the continuous dependence of the solutions on the data, we have $w_t^\eta \to w_t$ in $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ for almost every $t \in I_\delta(t_0)$. This implies the measurability of $t \mapsto w_t$. □

Theorem 3.4.3. *Under the hypotheses of Theorem 3.4.2, the vector b_t and the matrix Ω_t in* (3.21) *are bounded and measurable with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then $t \mapsto (b_t\,,\Omega_t)$ belongs to $C^0([0,T];\mathbb{R}^3{\times}\mathbb{M}^{3\times3})$.*

Proof. As noticed in Section 3.2, it is enough to prove that the functions in (3.28) are bounded and measurable, and that they are continuous under the additional assumption on $t \mapsto s_t$. Moreover, it is sufficient to prove the measurability and boundedness of these functions in a subinterval of time; the measurability and boundedness on the whole $[0,T]$ will easily follow. As for the first three functions, this property is proved in Proposition 3.2.3. The function $t \mapsto w_t$ from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ is bounded and measurable by Theorem 3.4.2. By Proposition 3.4.1 it is also continuous under the additional assumption. By formulas (3.33), this yields the boundedness and measurability of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$, and the continuity under the additional assumption on $t \mapsto s_t$, since the functions $t \mapsto v_t^i$ and $t \mapsto \hat{v}_t^i$ are continuous from $I_\delta(t_0)$ into $\mathcal{V}(\Sigma_\varepsilon^{0,\mathrm{ext}})$ by Theorem 3.1.5 and Lemma 3.2.2. □

We are now in a position to prove the main existence, uniqueness, and regularity result.

Theorem 3.4.4. *Assume that $t \mapsto s_t$ satisfies* (2.10)*,* (2.11)*,* (2.13)*, and* (2.16)*. Let $y^* \in \mathbb{R}^3$ and $R^* \in \mathrm{SO}(3)$. Then* (3.22) *has a unique absolutely continuous solution $t \mapsto (y_t\,,R_t)$ defined in $[0,T]$ with values in $\mathbb{R}^3{\times}\mathrm{SO}(3)$ such that $y_0 = y^*$ and $R_0 = R^*$. In other words, there exists a unique rigid motion $t \mapsto r_t(z) = y_t + R_t z$ such that the deformation function $t \mapsto \varphi_t = r_t \circ s_t$ satisfies the equations of motion* (3.17)*.*

Moreover this solution is Lipschitz continuous with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then the solution $t \mapsto (y_t\,,R_t)$ belongs to $C^1([0,T];\mathbb{R}^3{\times}\mathrm{SO}(3))$.

Proof. The existence and uniqueness of the solution of the Cauchy problem for (3.22) follow immediately from Theorem 3.4.3, by standard results on ordinary differential equations with bounded measurable coefficients, see, e.g., [18, Theorem I.5.1]. The assertion concerning the deformation function $t \mapsto \varphi_t$ and the equation of motion (3.17) follows from the equivalence Theorem 3.2.1. The Lipschitz continuity of the solution follows from the boundedness of the right-hand sides of the equation in (3.22).

If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then Theorem 3.4.3 ensures that the coefficients of the equations in (3.22) are continuous with respect to t, and therefore the solutions are of class C^1. □

We notice that assumptions (2.2) are not needed in Theorem 3.4.4. As a consequence, the theorem holds in a more general setting, when s_t is not a pure shape change. For instance, if s_t were a rigid motion for every t, the unique r_t given by the theorem would be $r_t = s_t^{-1}$. Consequently, φ_t would be the identity for every t and the swimmer would not move.

CHAPTER 4

Swimming in an unbounded Brinkman fluid

In this Chapter we develop the theory for the case of a swimmer immersed in an infinite viscous fluid governed by the Stokes equation. For this, the functional setting is presented in Section 4.2, the extension theorems an the main results are presented in Section 4.4. A final section about possible future directions follows. All these results are contained in [31].

4.1 Motivation

In a recent paper by S. Jung [22], the motion of *Caenorhabditis elegans* is observed in different environments: this nematode usually swims in saturated soil, and its behavior was studied in different saturation conditions as well as in a viscous fluid without solid particles. It must be noticed that the locomotion strategy of *C. elegans* is not completely understood, as it is shown by the many studies on this nematode in different conditions; nevertheless it has been taken as a model system to approach the study of many biological problems [43]. A satisfactory attempt to understand its locomotion dates back to [42], where the experiment was conducted in an environment close to the one in which *C. elegans* usually lives, yet the wet phase in which the particles are usually immersed was neglected. Other and more recent experiments have been run on agar composites [23], [27], and they could give a hint on the swimming strategies of *C. elegans*, showing that it moves more efficiently in a particulate medium rather than in a viscous fluid without particles [22].

The aim of the work presented in this chapter is to provide a theoretical framework for the motion of a body in a particulate medium. Following the approach proposed in

[22, III.C], we model the particulate medium surrounding the swimmer as a Brinkman fluid. We use the framework introduced in Chapter 2, showing that is can be adapted to the case of a Brinkman problem in an exterior domain, provided the definition of suitable function spaces. Thus, the existence, uniqueness, and regularity result contained in Theorem 4.4.6 can be considered as a generalization of Theorem 3.4.4 obtained for the Stokes system.

4.2 The exterior Brinkman problem

In this section we present some results about Brinkman equation. It was originally proposed in [9] to model a fluid flowing through a porous medium as a correction to Darcy's law by the addition of a diffusive term. A rigorous mathematical derivation from the Navier-Stokes equation via homogenization can be found in [1].

In a Lipschitz domain $\Omega \subset \mathbb{R}^3$, the Brinkman system reads

$$\begin{cases} \nu\Delta u - \alpha^2 u = \nabla p & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = U & \text{on } \partial\Omega, \\ u = 0 & \text{at infinity.} \end{cases} \tag{4.1}$$

The positive constant α takes into account the permeability properties of the porous matrix and the viscosity of the fluid, the constant ν is an effective viscosity of the fluid, while the third equation in the system is the *no-slip* boundary condition. The condition $u = 0$ at infinity is significant, and necessary, only when the domain Ω is unbounded. From now on, we will get rid of the effective viscosity, upon a redefinition of α, by setting $\nu = 1$. A brief discussion on the constant ν can be found in Brinkman's paper [9].

In order to cast equation (4.1) in the weak form, we introduce the function spaces in which we will look for the weak solution. Define

$$\mathcal{X}(\Omega) := \{u \in H^1(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}, \qquad \mathcal{X}_0(\Omega) := \{u \in H^1_0(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}.$$

Both $\mathcal{X}(\Omega)$ and $\mathcal{X}_0(\Omega)$ are equipped with the standard H^1 norm but we introduce this equivalent one

$$\|u\|^2_{\mathcal{X}(\Omega)} := \alpha^2 \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})},$$

the equivalence being a consequence of Korn's inequality.

The weak formulation of equation (4.1) is now given by

$$\begin{cases} \text{find } u \in \mathcal{X}(\Omega) \text{ such that} & u = U \text{ on } \partial\Omega, \\ 2\displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \,\mathrm{d}x + \alpha^2 \int_\Omega u \cdot w \,\mathrm{d}x = 0, & \text{for every } w \in \mathcal{X}_0(\Omega), \end{cases} \tag{4.2}$$

where the boundary velocity is a given function $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, the solution being the unique minimum in $\mathcal{X}(\Omega)$ of the strictly convex energy functional

$$\mathcal{E}(u) := 2\int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x + \alpha^2 \int_\Omega |u|^2 \,\mathrm{d}x = \|u\|^2_{\mathcal{X}(\Omega)}.$$

If we consider the term $\alpha^2 u$ as a forcing term f in system (4.1), we can invoke a classical existence and uniqueness result, see, e.g., [14], [39], or [41].

Theorem 4.2.1. *Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Then the following results hold:*

(a) Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with Lipschitz boundary. If

$$\int_{\partial\Omega} U \cdot n \, \mathrm{d}S = 0, \tag{4.3}$$

there exists a unique solution u to problem (4.2)*. Moreover, there exists $p \in L^2(\Omega)$ such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

(b) Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then problem (4.2) *has a solution. Moreover, there exists $p \in L^2_{\mathrm{loc}}(\Omega)$, with $p \in L^2(\Omega \cap \Sigma_\rho)$ for every $\rho > 0$, such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

The following density result, which is an adaptation of Theorem 3.1.1, is particularly useful when dealing with exterior domains.

Theorem 4.2.2 (Density)**.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then, the space $\{u \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}$ is dense in $\mathcal{X}(\Omega)$ for the H^1 norm.* □

The *stress tensor* associated with the velocity field u and the pressure p was defined in (3.6), and also the viscous force and torque have the same form as in (3.8) and (3.9). The same considerations that were done in Section 3.1 are still valid in the Brinkman setting.

Definition 3.1.3 of the trace of σn on $\partial\Omega$ can be rephrased adapted to the new function spaces

Definition 4.2.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary and let $\sigma \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ be such that $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$. The trace of σn, still denoted by σn, is defined as the unique element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ satisfying the equality*

$$\langle \sigma n, V\rangle_\Omega := \int_\Omega (\operatorname{div}\sigma) \cdot v \, \mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v \, \mathrm{d}x, \tag{4.4}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing between $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^3)$, and v is any function in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$.

If there is no risk of misunderstanding, the subscript Ω will be dropped whenever the domain of integration is clear. Notice that if σ is sufficiently smooth then integrating (4.4) by parts leads to the equality

$$\langle \sigma n, V\rangle_\Omega = \int_{\partial\Omega} \sigma n \cdot V \, \mathrm{d}S, \qquad \text{for every } V \in H^{1/2}(\partial\Omega;\mathbb{R}^3).$$

In the general case, the right-hand side of (4.4) is easily proved to be well defined, given the assumptions on σ. In fact, $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$ and $v \in L^2(\Omega;\mathbb{R}^3)$ make the first integral well defined, while the second one is also good since $\sigma : \mathrm{E}v = \sigma_D : \mathrm{E}v$, because of the symmetry of $\mathrm{E}v$, and both σ_D and $\mathrm{E}v$ belong to $L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}})$. Lastly, the definition is independent of the choice of $v \in \mathcal{X}(\Omega)$, since the right-hand side vanishes for every $v \in \mathcal{X}_0(\Omega)$: this follows from the very same computation for the more regular case, by the Density Theorem 4.2.2. It is easy to see that (4.4) defines a continuous linear functional on $H^{1/2}(\partial\Omega; \mathbb{R}^3)$ by choosing $v \in \mathcal{X}(\Omega)$ an extension of V.

We now proceed in showing other useful properties of the duality pairing introduced in Definition 4.2.3. Let $U \in H^{1/2}(\partial\Omega; \mathbb{R}^3)$ and let u be the solution to the Brinkman problem (4.2) with boundary datum U and let σ be the corresponding stress tensor. Since all the assumptions of Definition 4.2.3 are fulfilled, for any given $V \in H^{1/2}(\partial\Omega; \mathbb{R}^3)$ we have

$$\begin{aligned}\langle \sigma n, V\rangle &= \int_\Omega (\operatorname{div}\sigma)\cdot v\,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v\,\mathrm{d}x = \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x + \int_\Omega [-p\,\mathrm{I} : \mathrm{E}v + 2\mathrm{E}u : \mathrm{E}v]\,\mathrm{d}x \\ &= \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x - \int_\Omega p \operatorname{div} v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x \\ &= \alpha^2 \int_\Omega u\cdot v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x,\end{aligned} \tag{4.5}$$

where v is an arbitrary element in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$. If we take, in particular, v to be the solution to problem (4.2) with boundary datum V, we recover the well known *reciprocity condition* (see, e.g., [19, Section 3-5])

$$\langle \sigma n, V\rangle = \langle \tau n, U\rangle,$$

with τ being the stress tensor associated with v. Moreover, by taking $U = V$ in (4.5) we obtain

$$\langle \sigma n, U\rangle = \alpha^2 \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})} = \|u\|^2_{\mathcal{X}(\Omega)}.$$

This equality allows us to show that the quadratic form $\langle \sigma n, U\rangle$ is positive definite: if $\langle \sigma n, U\rangle = 0$, then it follows that $u = 0$, and therefore $U = 0$.

Also the weak definition of the viscous force and torque is easily adapted, keeping Definition 4.2.3 in mind.

Definition 4.2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary, let $u \in \mathcal{X}(\Omega)$ be the solution to the Brinkman problem* (4.2) *with boundary datum $U \in H^{1/2}(\partial\Omega; \mathbb{R}^3)$, let σ be the corresponding stress tensor defined by* (3.6)*, and let $\sigma n \in H^{-1/2}(\partial\Omega; \mathbb{R}^3)$ be the trace on $\partial\Omega$ defined according to* (4.4)*. The* viscous force *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $F \in \mathbb{R}^3$ such that*

$$F\cdot V = \langle \sigma n, V\rangle \quad \textit{for every } V \in \mathbb{R}^3. \tag{4.6}$$

The torque *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $M \in \mathbb{R}^3$ such that*

$$M\cdot \omega = \langle \sigma n, W_\omega\rangle \quad \textit{for every } \omega \in \mathbb{R}^3, \tag{4.7}$$

where $W_\omega(x) := \omega \times x$ is the velocity field generated by the angular velocity ω.

Notice that this definition allows us to define two different physical quantities by means of the same mathematical object, namely the duality pairing defined in (4.4).

4.3 The equations of motion

We proceed now to the description of the motion of the swimmer. The motion $t \mapsto \varphi_t$ determines for almost every $t \in [0, T]$ the Eulerian velocity U_t through the formula

$$U_t(y) := \dot{\varphi}_t(\varphi_t^{-1}(y)) \quad \text{for almost every } y \in \partial A_t\,.$$

Notice that $U_t \in H^{1/2}(\partial A_t; \mathbb{R}^3)$ for almost every $t \in [0, T]$. By applying Theorem 4.2.1 (b) with $\Omega = A_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{A}_t$ and, for almost every $t \in [0, T]$, we obtain a unique solution u_t to the problem

$$\begin{cases} \text{find } u_t \in \mathcal{X}(A_t^{\mathrm{ext}}) \text{ such that} & u_t = U_t \text{ on } \partial A_t\,, \\ 2\displaystyle\int_{A_t^{\mathrm{ext}}} \mathrm{E}u_t : \mathrm{E}w\,\mathrm{d}y + \alpha^2 \int_{A_t^{\mathrm{ext}}} u_t \cdot w\,\mathrm{d}y = 0 & \text{for every } w \in \mathcal{X}_0(A_t^{\mathrm{ext}}). \end{cases} \tag{4.8}$$

Let F_{A_t,U_t} and M_{A_t,U_t} be the viscous force and torque determined by the velocity field U_t according to (4.6) and (4.7). By neglecting inertia and imposing the self-propulsion constraint, the equations of motion reduce to the vanishing of the viscous force and torque, i.e.,

$$F_{A_t,U_t} = 0 \quad \text{and} \quad M_{A_t,U_t} = 0 \quad \text{for almost every } t \in [0, T]. \tag{4.9}$$

By assuming that φ_t is factorized as $\varphi_t = r_t \circ s_t$, where r_t is a rigid motion as in (2.9) and $t \mapsto s_t$ is a prescribed shape function, our aim is to find $t \mapsto r_t$ so that the equations of motion (4.9) are satisfied. To this extent, we present Theorem 4.3.1 below, whose result is that (4.9) is equivalent to a system of ordinary differential equations where the unknown functions are the translation $t \mapsto y_t$ and the rotation $t \mapsto R_t$ of the map $t \mapsto r_t$.

The coefficients of these differential equations are defined starting from the intermediate configuration described by the sets $B_t = s_t(A)$ introduced before and the $3{\times}3$ matrices K_t, C_t, J_t, depending only on the geometry of B_t, whose entries are defined by

$$(K_t)_{ij} := \langle \sigma[e_j]n, e_i \rangle_{B_t^{\mathrm{ext}}}\,, \tag{4.10a}$$

$$(C_t)_{ij} := \langle \sigma[e_j]n, e_i{\times}z \rangle_{B_t^{\mathrm{ext}}}\,, \tag{4.10b}$$

$$(J_t)_{ij} := \langle \sigma[e_j{\times}z]n, e_i{\times}z \rangle_{B_t^{\mathrm{ext}}}\,, \tag{4.10c}$$

where $B_t^{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{B}_t$, the duality product is given in Definition 4.2.3 by formula (4.4), and $\sigma[W]$ denotes the stress tensor associated with the outer Brinkman problem in

B_t^{ext} with boundary datum W. The notation $\sigma[W]$ is chosen to emphasize the linear dependence of σ on W. Formula (4.5) shows that K_t and J_t are symmetric. The matrix

$$\begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix}$$

is often called in the literature *grand resistance matrix*, and is symmetric and invertible. It originally arises in the case of a Stokes system [19], but the adaptation to the Brinkman system is straightforward: it only shares the structure with the original one, while the values of the entries are computed with a different formula, namely (4.5). Let

$$\begin{bmatrix} H_t & D_t^\top \\ D_t & L_t \end{bmatrix} := \begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix}^{-1} \tag{4.11}$$

be its inverse. For almost every $t \in [0,T]$, we defined $W_t = \dot{s}_t \circ s_t^{-1}$, and let F_t^{sh} and M_t^{sh} be the viscous force and torque on ∂B_t determined by the boundary velocity field W_t. The components of F_t^{sh} and M_t^{sh} are given, according to (4.6) and (4.7), by

$$(F_t^{\mathrm{sh}})_i = \langle \sigma[W_t]n, e_i \rangle_{B_t^{\mathrm{ext}}}\,, \tag{4.12a}$$

$$(M_t^{\mathrm{sh}})_i = \langle \sigma[W_t]n, e_i \times z \rangle_{B_t^{\mathrm{ext}}}\,. \tag{4.12b}$$

Consider now the linear operator $\mathcal{A} : \mathbb{R}^3 \to \mathbb{M}^{3\times 3}$ that associates to every $\omega \in \mathbb{R}^3$ the only skew-symmetric matrix $\mathcal{A}(\omega)$ such that $\mathcal{A}(\omega)z = \omega \times z$; therefore, ω is the axial vector of $\mathcal{A}(\omega)$. Finally, we define a vector b_t and a matrix Ω_t according to

$$b_t := H_t F_t^{\mathrm{sh}} + D_t^\top M_t^{\mathrm{sh}}, \qquad \Omega_t := \mathcal{A}(D_t F_t^{\mathrm{sh}} + L_t M_t^{\mathrm{sh}}), \tag{4.13}$$

which depend on s_t and, most importantly on $\dot{s}_t$, via (4.12) and the definition of W_t.

Theorem 4.3.1. *Assume that the shape function $t \mapsto s_t$ satisfies* (2.10)*,* (2.11)*, and* (2.16)*, and that the position function $t \mapsto r_t$ satisfies* (2.9) *and is Lipschitz continuous with respect to time. Then the following conditions are equivalent:*

(i) the deformation function $t \mapsto \varphi_t := r_t \circ s_t$ satisfies the equations of motion (4.9)*;*

(ii) the functions $t \mapsto y_t$ and $t \mapsto R_t$ satisfy the system

$$\dot{y}_t = R_t b_t\,, \qquad \dot{R}_t = R_t \Omega_t\,, \quad \textit{for almost every } t \in [0,T], \tag{4.14}$$

where b_t and Ω_t are defined in (4.13). □

The proof is the same as in Theorem 3.2.1 and need not be modified, so we skip it. Changing the variables according to $y = r_t(z)$, $z \in B_t^{\mathrm{ext}}$, the velocity field $v_t(z) := R_t^\top u_t(r_t(z))$ is the solution to the problem

$$\begin{cases} \text{find } v_t \in \mathcal{X}(B_t^{\mathrm{ext}}) \text{ such that} & v_t = V_t \text{ on } \partial B_t\,, \\ 2\displaystyle\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t : \mathrm{E}w \,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t \cdot w \,\mathrm{d}z = 0, & \text{for every } w \in \mathcal{X}_0(B_t^{\mathrm{ext}}), \end{cases} \tag{4.15}$$

where $V_t(z) = R_t^\top U_t(r_t(z))$, see Fig. 3.1.

Denote by F_{B_t,V_t} and M_{B_t,V_t} the viscous force and torque on ∂B_t determined by the velocity field v_t according to (4.6) and (4.7), with $\Omega = B_t^{\mathrm{ext}}$. A straightforward computation yields $F_{B_t,V_t} = R_t^\top F_{A_t,U_t}$ and $M_{B_t,V_t} = R_t^\top M_{A_t,U_t}$, so that the equations of motion (4.9) reduce to

$$F_{B_t,V_t} = 0 \quad \text{and} \quad M_{B_t,V_t} = 0 \quad \text{for almost every } t \in [0,T].$$

Again by a simple manipulation we obtain the following form of the equations of motion

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^\top \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix} \quad \text{for almost every } t \in [0,T],$$

which read, by means of (4.13), as (4.14).

Now, the standard theory of ordinary differential equations with possibly discontinuous coefficients [18] ensures that the Cauchy problem for (4.14) has one and only one Lipschitz solution $t \mapsto R_t$, $t \mapsto y_t$, provided that the functions $t \mapsto \Omega_t$ and $t \mapsto b_t$ are measurable and bounded. By (4.11) and (4.13), this happens when the functions

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \quad t \mapsto F_t^{\mathrm{sh}}, \quad t \mapsto M_t^{\mathrm{sh}} \tag{4.16}$$

are measurable and bounded. The continuity of the first three functions will be proved in the last part of this section. The proof of the measurability and boundedness of the last two functions in (4.16) requires some technical tools that will be developed in Section 4.4.

Theorem 4.3.2. *Let w_t be the solution to the exterior Brinkman problem* (4.2) *on B_t^{ext} with boundary datum W on ∂B_t, where W can be either a constant vector $a \in \mathbb{R}^3$ or the rotation $W_\omega := \omega \times z$, with $\omega \in \mathbb{R}^3$. Define $\widetilde{w}_t$ to be the extension*

$$\widetilde{w}_t := \begin{cases} W & \text{on } B_t\,, \\ w_t & \text{on } B_t^{\mathrm{ext}}, \end{cases} \tag{4.17}$$

Assume that $t \mapsto s_t$ satisfies (2.16)*. Then the map $t \mapsto \widetilde{w}_t$ is continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$.*

Proof. Let $(t_k)_k \subset [0,T]$ be a sequence that converges to $t_\infty \in [0,T]$. Lemma 3.2.2 ensures the convergence of the sets B_{t_k} to B_{t_∞} in the sense of (3.15).

Since w_{t_k} are solutions to Brinkman problems, we have the bound $2\int_{B_{t_k}^{\mathrm{ext}}} |\mathrm{E}w_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{B_{t_k}^{\mathrm{ext}}} |w_{t_k}|^2\,\mathrm{d}z \leqslant C$, which, in turn, implies that

$$2\int_{\mathbb{R}^3} |\mathrm{E}\widetilde{w}_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} |\widetilde{w}_{t_k}|^2\,\mathrm{d}z \leqslant C.$$

Therefore, $\widetilde{w}_t$ admits a subsequence that converges weakly to a function $w^* \in \mathcal{X}(\mathbb{R}^3)$. By the convergence of the B_{t_k}, it is easy to see that $w^* = W$ on B_{t_∞}. We now prove that $w^*|_{B^{\text{ext}}_{t_\infty}}$ solves the exterior Brinkman problem on B_{t_∞}. Too see it, consider a test function $\varphi \in C^\infty_{\text{c}}(B^{\text{ext}}_{t_\infty};\mathbb{R}^3)$. For k large enough, $\varphi \in C^\infty_{\text{c}}(B^{\text{ext}}_{t_k};\mathbb{R}^3)$, so that

$$2\int_{\operatorname{spt}\varphi} \mathrm{E}w_{t_k} : \mathrm{E}\varphi \,\mathrm{d}z + \alpha^2 \int_{\operatorname{spt}\varphi} w_{t_k}\cdot\varphi \,\mathrm{d}z = 0.$$

This equality passes to the limit as $k \to \infty$, showing that $w^*|_{B^{\text{ext}}_{t_\infty}}$ is a solution to the Brinkman problem at t_∞. Therefore, $w^* = \widetilde{w}_{t_\infty}$, and we have proved that $t \mapsto w_t$ is strongly continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$. □

We can now prove the following continuity result for the elements of the grand resistance matrix by means of Theorem 4.3.2.

Proposition 4.3.3. *Assume that s_t satisfies* (2.10)*,* (2.11)*, and* (2.16)*. Then the functions*

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{4.18}$$

and consequently $t \mapsto H_t$, $t \mapsto D_t$, $t \mapsto L_t$, are continuous.

Proof. Formulae (4.10) and (4.5) provide us with an explicit form for the elements of the grand resistance matrix

$$(K_t)_{ij} = 2\int_{B^{\text{ext}}_t} \mathrm{E}v^j_t : \mathrm{E}v^i_t \,\mathrm{d}z + \alpha^2 \int_{B^{\text{ext}}_t} v^j_t \cdot v^i_t \,\mathrm{d}z, \tag{4.19a}$$

$$(C_t)_{ij} = 2\int_{B^{\text{ext}}_t} \mathrm{E}v^j_t : \mathrm{E}\hat{v}^i_t \,\mathrm{d}z + \alpha^2 \int_{B^{\text{ext}}_t} v^j_t \cdot \hat{v}^i_t \,\mathrm{d}z, \tag{4.19b}$$

$$(J_t)_{ij} = 2\int_{B^{\text{ext}}_t} \mathrm{E}\hat{v}^j_t : \mathrm{E}\hat{v}^i_t \,\mathrm{d}z + \alpha^2 \int_{B^{\text{ext}}_t} \hat{v}^j_t \cdot \hat{v}^i_t \,\mathrm{d}z, \tag{4.19c}$$

where v^i_t and $\hat{v}^i_t$ are the functions defined in (4.17) with $W = e_i$ and $W = e_i \times z$, respectively. We prove the result for K_t only, since the others are similar. We write

$$(K_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}\widetilde{v}^j_t : \mathrm{E}\widetilde{v}^i_t \,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} \widetilde{v}^j_t \cdot \widetilde{v}^i_t \,\mathrm{d}z - \alpha^2 \int_{B_t} e_j \cdot e_i \,\mathrm{d}z,$$

where $\widetilde{v}^i_t$ and $\widetilde{v}^j_t$ are the extensions considered in (4.17). By Theorem 4.3.2, the first two integrals are continuous with respect to t. The continuity of the last integral is guaranteed by Lemma 3.2.2. □

The proof of the measurability and boundedness of $t \mapsto F^{\text{sh}}_t$ and $t \mapsto M^{\text{sh}}_t$ is a delicate issue. The difficulty arises from the fact that both the domains B_t and the boundary data $W_t = \dot{s}_t \circ s_t^{-1}$ depend on time. Moreover, since it is meaningful and interesting to consider boundary values W_t that might be discontinuous with respect to t, we cannot expect the functions $t \mapsto F^{\text{sh}}_t$ and $t \mapsto M^{\text{sh}}_t$ to be continuous.

To prove the measurability we start from an integral representation of F^{sh}_t and M^{sh}_t, similar to (4.19). As $\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S$ is not necessarily zero, we will not be able to compute integrals over the whole space $\mathbb{R}^3$, so we will have to work in the complement of an open ball $\Sigma^0_\varepsilon \subset\subset B_t$. Since, in general, this inclusion holds only locally in time, we first fix

$t_0 \in [0,T]$ and $z^0 \in B_{t_0}$ and select $\delta > 0$ and $\varepsilon > 0$ so that the open ball $\Sigma^0_\varepsilon := \Sigma_\varepsilon(z^0)$ of radius ε centered at z^0 satisfies

$$\Sigma^0_\varepsilon \subset\subset B_t\,, \quad \text{for all } t \in I_\delta(t_0) := [0,T] \cap (t_0-\delta, t_0+\delta). \tag{4.20}$$

This is possible thanks to the continuity properties of $t \mapsto s_t$ listed in the first part of this section.

Next we consider the solution w_t to the problem

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)} \,:\, w \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon),\ w = W_t \text{ on } \partial B_t\,,\ \text{and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma^0_\varepsilon\right\}$$

In order for the flux condition (4.3) to be fulfilled by w_t on $\partial B_t \cup \partial\Sigma^0_\varepsilon$, we choose

$$\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t} W_t\cdot n\,\mathrm{d}S.$$

Finally, putting together (4.12) and (4.5), we obtain the following explicit integral representation of F^{sh}_t and M^{sh}_t

$$(F^{\mathrm{sh}}_t)_i = 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} \mathrm{E}w_t : \mathrm{E}v^i_t\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} w_t\cdot v^i_t\,\mathrm{d}z - \alpha^2\int_{Q_{\varepsilon,t}} w_t\cdot v^i_t\,\mathrm{d}z$$

$$(M^{\mathrm{sh}}_t)_i = 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} \mathrm{E}w_t : \mathrm{E}\hat{v}^i_t\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} w_t\cdot \hat{v}^i_t\,\mathrm{d}z - \alpha^2\int_{Q_{\varepsilon,t}} w_t\cdot \hat{v}^i_t\,\mathrm{d}z$$

where v^i_t and $\hat{v}^i_t$ have been defined in the proof of Proposition 4.3.3 and $Q_{\varepsilon,t} := B_t \setminus \overline{\Sigma}^0_\varepsilon$. We deduce from Theorem 4.3.2 and Lemma 3.2.2 that the functions $t \mapsto v^i_t$ and $t \mapsto \hat{v}^i_t$ are continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$. Therefore, the measurability and boundedness of $t \mapsto F^{\mathrm{sh}}_t$ and $t \mapsto M^{\mathrm{sh}}_t$ will be proved once $t \mapsto w_t$ is proved to be measurable. We first show that $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ and eventually we will prove that the function $t \mapsto \int_{Q_{\varepsilon,t}} w_t\,\mathrm{d}z$ is continuous with respect to time. These two results are proved in the next Section.

4.4 Extensions of boundary data and main result

In order to prove the main result, some work is still to be done to prove the regularity property of the coefficients of the equations of motion (4.14). To this aim, results concerning the extension of boundary data are needed to be able to use standard variational techniques to solve the relevant minimum problem of Theorem 4.4.4. The following result is an adaptation of Proposition 3.3.3 to the Brinkman case.

Proposition 4.4.1 (Solenoidal extension operators)**.** *Assume that s_t satisfies* (2.10)*,* (2.11)*, and* (2.16)*, and let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$. Let $\delta > 0$ and $\varepsilon > 0$ be such that* (4.20) *holds true. Then there exists a uniformly bounded family $(\mathcal{T}_t)_{t\in I_\delta(t_0)}$ of continuous linear operators*

$$\mathcal{T}_t\colon H^{1/2}(\partial A;\mathbb{R}^3) \to \mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma}^0_\varepsilon)$$

such that

(i) for all $t \in I_\delta(t_0)$ and for all $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$,

$$\mathcal{T}_t(\Phi) = \Phi \circ s_t^{-1} \quad \text{on } \partial B_t\,,$$
$$\mathcal{T}_t(\Phi) = \lambda_t \frac{z}{|z|^3} \quad \text{on } \partial\Sigma_\rho\,,$$

(ii) for every $\Phi \in H^{1/2}(\partial A;\mathbb{R}^3)$ the map $t \mapsto \mathcal{T}_t(\Phi)$ is continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma^0_\varepsilon})$.

In particular, the following estimate holds

$$\|\mathcal{T}_t(\Phi)\|_{H^1(\Sigma_\rho\setminus\overline{\Sigma^0_\varepsilon};\mathbb{R}^3)} \leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \tag{4.21}$$

where the constant C is independent of t and Φ. □

Proposition 4.4.2. *Assume that s_t satisfies* (2.10), (2.11), (2.13), *and* (2.16). *Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (4.20). *Suppose, in addition, that for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi^{t_0}_t : \Sigma_\rho \to \Sigma_\rho$ coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Psi^{t_0}_t = s_{t_0} \circ s_t^{-1}$ on B_t. Let the map $t \mapsto \Phi_t$ belong to $C^0(I_\delta(t_0);H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. Let w_t be the solution to the problem*

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)} : \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon),\ w = \Phi_t \circ s_t^{-1} \text{ on } \partial B_t \text{ and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma^0_\varepsilon\right\}, \tag{4.22}$$

where $\lambda_t := -\frac{1}{4\pi}\int_{\partial B_t}(\Phi_t \circ s_t^{-1})\cdot n\,\mathrm{d}S$. Then $t \mapsto w_t$ belongs to $C^0(I_\delta(t_0);\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon))$.

Proof. The proof can be easily adapted from that of Proposition 3.4.1. The following important estimate provides a uniform bound for the norms of the w_t's in $\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ that will also be useful in the proof of Proposition 4.4.3

$$\begin{aligned} 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}w_{t_k}|^2\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |w_{t_k}|^2\,\mathrm{d}z &\leqslant 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\mathrm{E}\psi_{t_k}|^2\,\mathrm{d}z + \alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon} |\psi_{t_k}|^2\,\mathrm{d}z \\ &\leqslant \|\psi_{t_k}\|^2_{H^1(\Sigma_\rho\setminus\overline{\Sigma^0_\varepsilon};\mathbb{R}^3)} \leqslant C^2(\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|)^2 \leqslant (CM)^2, \end{aligned} \tag{4.23}$$

where $\psi_t \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is defined by

$$\psi_t := \begin{cases} \mathcal{T}_t(\Phi_t) & \text{in } \Sigma_\rho \setminus \overline{\Sigma^0_\varepsilon} \\ \lambda_t \dfrac{z}{|z|^3} & \text{in } \Sigma^{\mathrm{ext}}_\rho \end{cases}$$

and is the function provided by Proposition 4.4.1 and extended on $\Sigma^{\mathrm{ext}}_\rho$, C is the constant in (4.21), and $M > 0$ is a uniform upper bound of $\mathrm{Lip}(\Phi_{t_k}) + \max|\Phi_{t_k}|$, whose existence is guaranteed by the fact that $t \mapsto \Phi_t$ belongs to $L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. □

Proposition 4.4.3. *Under the hypotheses of Proposition 4.4.2, recalling that $Q_{\varepsilon,t} = B_t \setminus \overline{\Sigma^0_\varepsilon}$, the maps*

$$t \mapsto \int_{Q_{\varepsilon,t}} w_t\,\mathrm{d}z, \qquad t \mapsto \int_{Q_{\varepsilon,t}} z \times w_t\,\mathrm{d}z \tag{4.24}$$

where $t \mapsto w_t \in \mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is the solution to the minimum problem (4.22) *as in Proposition 4.4.2, are continuous with respect to time in $I_\delta(t_0)$.*

Proof. We check the continuity with the definition

$$\begin{aligned}\left|\int_{Q_{t+h}} w_{t+h}\,\mathrm{d}z - \int_{Q_t} w_t\,\mathrm{d}z\right| &= \left|\int_{Q_{t+h}} (w_{t+h}-w_t)\,\mathrm{d}z + \int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} w_t(\chi_{Q_{t+h}}(z)-\chi_{Q_t}(z))\,\mathrm{d}z\right| \\ &\leqslant \left(\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |w_{t+h}-w_t|^2\,\mathrm{d}z\right)^{\frac12} |Q_{t+h}|^{\frac12} + \left(\int_{\Sigma_\varepsilon^{0,\mathrm{ext}}} |w_t|^2\,\mathrm{d}z\right)^{\frac12} |Q_{t+h}\triangle Q_t|^{\frac12} \\ &\leqslant \|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})} |Q_{t+h}|^{\frac12} + \|w_t\|_{\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})} |Q_{t+h}\triangle Q_t|^{\frac12} \\ &\leqslant |\Sigma_\rho|^{\frac12} \|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})} + CM\,|Q_{t+h}\triangle Q_t|^{\frac12} \xrightarrow{h\to0} 0.\end{aligned}$$

Here, χ_Q denotes the characteristic function of the set Q, $\triangle$ is the symmetric difference operator, and CM is the uniform (with respect to t) upper bound coming from (4.23). The continuity for the second map is achieved in the same way. □

Proposition 4.4.2 and Proposition 4.4.3 combined together give the continuity of $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ with respect to time, in the case of regular boundary data $\Phi_t \circ s_t^{-1}$ on ∂B_t, where the map $t \mapsto \Phi_t$ belongs to $C^0(I_\delta(t_0); H^{1/2}(\partial A;\mathbb{R}^3)) \cap L^\infty(I_\delta(t_0); \mathrm{Lip}(\partial A;\mathbb{R}^3))$. The next results will prove that when the boundary data on ∂B_t are given by $\dot{s}_t \circ s_t^{-1}$, then the maps $t \mapsto F_t^{\mathrm{sh}}$ and $t \mapsto M_t^{\mathrm{sh}}$ are measurable and bounded.

Theorem 4.4.4. *Assume that s_t satisfies* (2.10)*,* (2.11)*,* (2.13)*, and* (2.16)*. Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ_ε^0 and $I_\delta(t_0)$ be as in* (4.20)*. Suppose, in addition, that for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi_t^{t_0} : \Sigma_\rho \to \Sigma_\rho$ coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Psi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t. Let w_t be the solution to the problem*

$$\min\left\{\|w\|^2_{\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})} : w \in \mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}}),\ w = \dot{s}_t \circ s_t^{-1} \text{ on } \partial B_t\,,\ \text{and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma_\varepsilon^0\right\}.$$

Then the function $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. Moreover, also the functions (4.24) *considered in Proposition 4.4.3 are measurable and bounded in $I_\delta(t_0)$.*

Proof. It suffices to convolve the boundary datum with a suitable regularizing kernel and to apply Propositions 4.4.2 and 4.4.3. By passing to the limit, the continuity is lost but the functions turn out to be measurable and bounded. □

Proposition 4.3.3 and Theorem 4.4.4 give the regularity result for b_t and Ω_t in (4.13), as stated in the following result.

Theorem 4.4.5. *Assume that $t \mapsto s_t$ satisfies* (2.10)*,* (2.11)*,* (2.13)*, and* (2.16)*. Then the vector b_t and the matrix Ω_t in* (4.13) *are bounded and measurable with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T]; C^1(\overline{A};\mathbb{R}^3))$, then $t \mapsto (b_t\,,\Omega_t)$ belongs to $C^0([0,T]; \mathbb{R}^3 \times \mathbb{M}^{3\times3})$.* □

We are now in a position to state the existence, uniqueness, and regularity result for the equations of motion (4.14).

Theorem 4.4.6. *Assume that* $t \mapsto s_t$ *satisfies* (2.10), (2.11), (2.13), *and* (2.16). *Let* $y^* \in \mathbb{R}^3$ *and* $R^* \in \mathrm{SO}(3)$. *Then* (4.14) *has a unique absolutely continuous solution* $t \mapsto (y_t\,, R_t)$ *defined in* $[0,T]$ *with values in* $\mathbb{R}^3{\times}\mathrm{SO}(3)$ *such that* $y_0 = y^*$ *and* $R_0 = R^*$. *In other words, there exists a unique rigid motion* $t \mapsto r_t(z) = y_t + R_t z$ *such that the deformation function* $t \mapsto \varphi_t = r_t \circ s_t$ *satisfies the equations of motion* (4.9).

Moreover this solution is Lipschitz continuous with respect to t. *If, in addition, the function* $t \mapsto s_t$ *belongs to* $C^1([0,T]; C^1(\overline{A};\mathbb{R}^3))$, *then the solution* $t \mapsto (y_t\,, R_t)$ *belongs to* $C^1([0,T];\mathbb{R}^3{\times}\mathrm{SO}(3))$. □

The proof is the same as that of Theorem 3.4.4, so we skip it. The main effort was to prove the measurability of the data.

4.5 Comments

In our model, we neglected the interactions between the solid particles and the swimmer, considering only the body-fluid phase viscous interaction. We think this is a reasonable approximation for using a simple model such as the Brinkman equation. Also, the mathematical model to describe the experiments in [22] is the same, and in that case the elastic and adhesive interactions between the nematode and the surrounding particles are neglected as well. Nevertheless, we think it can be interesting to develop more complex models to take into account also that kind of contact forces, and that could be the object of a future study.

Even though it has not been addressed in this work, we also expect our model to be able to predict, on the basis of an energy comparison, whether swimming in a particulate medium is more efficient than swimming in a plain viscous fluid; that would be an interesting theoretical check of the thesis advanced by Jung on the basis of his experimental results that *C. elegans* swims more efficiently in a particulate medium.

CHAPTER 5

Controllability of a mono-dimensional swimmer

In this chapter we draw our attention on the study of the motion of a mono-dimensional swimmer immersed in an infinite viscous three-dimensional fluid. The viscous forces and torques will be obtained from an approximate theory, and the equation of motion are obtained. Theorem 5.2.1 states that the associated initial value problem has a unique solution which depends with continuity on the initial data. Moreover, the controllability of the swimmer is proved, as well as the existence of an optimal swimming strategy, see Theorem 5.3.1. Also, the Euler equation relative to the constrained minimization of the expended power functional is derived. Yet, for the time being, the result is still partial, since the expression involved are rather complicated. We plan to address the problem of a thorough study of the Euler equation in future work, in order to derive some qualitative properties of the solutions. Nonetheless, the general structure of this equation will be presented in Section 5.5.

5.1 Introduction

As we anticipated in the Introduction, dealing with mono-dimensional bodies immersed in a three-dimensional fluid can be difficult: the dimensional gap does not allow to write boundary conditions in a proper way. Usually, singular solutions are placed along the mono-dimensional set, but these might be hard to be dealt with. Thus, two main approximation techniques have been proposed to model this case, *slender body theory* [6, 24] and *resistive force theory* [21]. We will adopt the second method to deal with the modeling of a flagellum-like swimmer immersed in a three-dimensional fluid.

Using resistive force theory means to express the force and momentum per unit length linearly with respect to the velocity. This means that the *local* tangential and normal forces per unit length acting on the flagellum are proportional to the local tangential and normal velocities of the flagellum, through the resistance coefficients $C_\parallel$ and $C_\perp$ [34]. Thus, if we let $\chi : [0,L]\times[0,T] \to \mathbb{R}^3$ denote the position of the swimmer with respect to an absolute external reference frame, the linear densities of viscous force and torque are

$$\begin{aligned} f(s,t) &= C_\parallel \dot\chi_\parallel(s,t)\chi'(s,t) + C_\perp \dot\chi_\perp(s,t) J\chi'(s,t),\\ m(s,t) &= \chi(s,t)\times(C_\parallel \dot\chi_\parallel(s,t)\chi'(s,t) + C_\perp \dot\chi_\perp(s,t) J\chi'(s,t)). \end{aligned} \tag{5.1}$$

Here, $\chi' := \partial\chi/\partial s$ denotes the partial derivative with respect to the spatial variable s, while $\dot\chi := \partial\chi/\partial t$ denotes the one with respect to time t; moreover, throughout the whole chapter, we prefer to put the time variable t in evidence, instead of writing it as a subscript as in the preceding ones. The quantities $\dot\chi_\parallel$ and $\dot\chi_\perp$ are nothing but the projection of the velocity $\dot\chi$ on the tangent and on the normal, respectively: $\dot\chi_\parallel(s,t) = \langle\dot\chi(s,t), \chi'(s,t)\rangle$ and $\dot\chi_\perp(s,t) = \langle\dot\chi(s,t), J\chi'(s,t)\rangle$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ being the rotation matrix of an angle $\pi/2$. Notice that, for the moment, we intend the partial derivatives in the distributional sense, that is, given a generic $\chi \in L^1_{\mathrm{loc}}(0,T;L^1(0,L))$, $\chi' = \partial\chi/\partial s$ and $\dot\chi = \partial\chi/\partial t$ are the functions such that, for every $\varphi \in C^\infty_{\mathrm{c}}([0,L]\times[0,T])$,

$$\int_0^L\int_0^T \chi'(s,t)\varphi(s,t)\,\mathrm{d}s\mathrm{d}t = -\int_0^L\int_0^T \chi(s,t)\frac{\partial\varphi(s,t)}{\partial s}\,\mathrm{d}s\mathrm{d}t,$$

$$\int_0^L\int_0^T \dot\chi(s,t)\varphi(s,t)\,\mathrm{d}s\mathrm{d}t = -\int_0^L\int_0^T \chi(s,t)\frac{\partial\varphi(s,t)}{\partial t}\,\mathrm{d}s\mathrm{d}t.$$

The self-propulsion constraint can be written as

$$0 = F(t) = \int_0^L f(s,t)\,\mathrm{d}s = \int_0^L K_\chi(s,t)\dot\chi(s,t)\,\mathrm{d}s, \tag{5.2a}$$

$$0 = M(t) = \int_0^L m(s,t)\,\mathrm{d}s = \int_0^L \chi(s,t)\times K_\chi(s,t)\dot\chi(s,t)\,\mathrm{d}s, \tag{5.2b}$$

where

$$K_\chi(s,t) := C_\parallel\chi'(s,t)\otimes\chi'(s,t) + C_\perp(J\chi'(s,t))\otimes(J\chi'(s,t))$$

is the matrix expressing the linear relation between viscous force and velocity.

We conclude this section by introducing the function space to which our state functions, as well as the shape functions that we will introduce later on, belong

$$\Xi := \{\chi : [0,L]\times[0,T] \to \mathbb{R}^2 : \chi \in L^\infty(0,T;H^2(0,L)), \dot\chi \in L^2(0,T;L^2(0,L))\}.$$

The space is endowed with the norm

$$\|\chi\|_\Xi := \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|\chi(\cdot,t)\|_{H^2(0,L)} + \left(\int_0^T \|\dot\chi(\cdot,t)\|^2_{L^2(0,L)}\,\mathrm{d}t\right)^{1/2},$$

with respect to which it is complete.

We shall remark that the natural inclusion $L^\infty(0,T;H^2(0,L)) \subset L^2(0,T;L^2(0,L))$ implies that $\chi \in H^1(0,T;L^2(0,L))$, and thus that there exists a continuous representative $\chi(\cdot,t)$ such that

$$\|\chi(\cdot,t)\|_{H^2(0,L)} < +\infty, \qquad \text{for every } t \in [0,T].$$

Therefore, the map $t \mapsto \chi(\cdot,t)$ is continuous from $[0,T]$ into $L^2(0,L)$ with respect to the strong topology.

Proposition 5.1.1. *Let $\chi \in \Xi$. Then the map*

$$t \mapsto \chi(\cdot,t) : [0,T] \to H^2(0,L)$$

is globally bounded and continuous with respect to the weak topology of $H^2(0,L)$ and to the strong topology of $H^1(0,L)$. Moreover, the map

$$t \mapsto \chi'(\cdot,t) : [0,T] \to L^2(0,L) \tag{5.3}$$

is strongly continuous.

Proof. To prove the first claim, we will show that if χ is the continuous representative, then there holds

$$\|\chi(\cdot,t)\|_{H^2(0,L)} \leqslant M = \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \|\chi(\cdot,t)\|_{H^2(0,L)} + \left(\int_0^T \|\dot\chi(\cdot,t)\|^2_{L^2(0,L)}\,\mathrm{d}t\right)^{1/2}.$$

Let N denote the zero measure set up to which the essential supremum is actually a supremum, and let us fix $t_0 \in [0,T]$. Consider a sequence $t_n \notin N$ which converges to t_0, such that $\|\chi(\cdot,t_n)\|_{H^2(0,L)} \leqslant M$. Then, since $\chi(\cdot,t_n) \to \chi(\cdot,t_0)$ in $L^2(0,L)$, and since the H^2 norm is lower-semicontinuous with respect to the L^2 convergence, this implies

$$\|\chi(\cdot,t_0)\|_{H^2(0,L)} \leqslant \liminf_{n\to+\infty} \|\chi(\cdot,t_n)\|_{H^2(0,L)} \leqslant M.$$

Notice that for the continuous representative the estimate holds for every $t_0 \in [0,T]$, and the essential supremum is actually a supremum.

The strong continuity in $L^2(0,L)$ implies the weak continuity in $H^2(0,L)$ which in turn implies the strong continuity of $t \mapsto \chi(\cdot,t) : [0,t] \to H^1(0,L)$, since the the embedding of $H^2(0,L)$ in $H^1(0,L)$ is compact.

This last property implies that (5.3) holds true. □

5.2 Equations of motion

The general setting introduced in the preceding section is suitable when studying fully three-dimensional motions. We restrict here to swimmer performing planar motions, but still immersed in a three-dimensional fluid. Their position with respect to an absolute reference system is given by the function $\chi : [0,L]\times[0,T] \to \mathbb{R}^2$, such that the spatial variable s is the arc-length coordinate. It follows that the tangent vector $\chi'(s,t)$ must satisfy the constraint

$$|\chi'(s,t)| \equiv 1, \quad \forall\, t \in [0,T].$$

By means of a change of reference of the type (2.1), it is possible to separate the rigid contribution of the motion from that coming from the deformation of the flagellum. Indeed, by introducing the deformation function of the flagellum with respect to its own reference system $\xi : [0,L]\times[0,T] \to \mathbb{R}^2$, we can write

$$\chi(s,t) = x(t) + R(t)\xi(s,t), \tag{5.4}$$

where $x(t)$ can be regarded as the position of the barycenter of the flagellum with respect to the absolute reference system and $R(t)$ is the rotation of angle $\theta(t)$ between the two coordinate systems. It must be noted that a necessary and sufficient condition for x to be the barycenter in the absolute reference is that

$$\int_0^L \xi(s,t)\,\mathrm{d}s = 0, \quad \forall\, t \in [0,T]. \tag{5.5}$$

Indeed, averaging (5.4) on $[0,L]$ (5.5) yields

$$x(t) = \frac{1}{L}\int_0^L \chi(s,t)\,\mathrm{d}s.$$

By the change of reference (5.4), it is possible to rephrase the self-propulsion constraint (5.2) and eventually obtain ordinary differential equations governing the time evolution of x and θ. Those will be the equations of motion of the flagellum. By differentiating (5.4) with respect to time, by noticing that $K_\chi(s,t) = R(t)K_\xi(s,t)R^\top(t)$, and by plugging all the terms in (5.2), we obtain

$$\begin{pmatrix} F(t) \\ M(t) \end{pmatrix} = \begin{bmatrix} R(t) & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} A(t) & b(t) \\ b^\top(t) & c(t) \end{bmatrix} \begin{bmatrix} R^\top(t) & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix} + \begin{pmatrix} F^{\mathrm{sh}}(t) \\ M^{\mathrm{sh}}(t) \end{pmatrix} \right\}, \tag{5.6}$$

where $\mathcal{R}(t) = \begin{bmatrix} A(t) & b(t) \\ b^\top(t) & c(t) \end{bmatrix}$ is the grand resistance matrix of [19], whose entries are given by

$$A(t) := \int_0^L K_\xi(s,t)\,\mathrm{d}s, \quad b(t) := \int_0^L K_\xi(s,t)\xi(s,t)\,\mathrm{d}s, \quad c(t) := \int_0^L \langle \xi(s,t), K_\xi(s,t)\xi(s,t)\rangle\,\mathrm{d}s,$$

and it is easy to see that they are ultimately determined by the shape of the flagellum only. The terms

$$F^{\mathrm{sh}}(t) := \int_0^L K_\xi(s,t)\dot{\xi}(s,t)\,\mathrm{d}s, \qquad M^{\mathrm{sh}}(t) := \int_0^L \langle \xi(s,t), K_\xi(s,t)\dot{\xi}(s,t)\rangle\,\mathrm{d}s, \tag{5.7}$$

are the contributions to the force and torque due to the shape deformation of the flagellum, and they also depend on the time derivative of ξ.

By setting (5.6) equal to zero and solving for $\dot{x}$ and $\dot{\theta}$, we finally obtain the equations

$$\begin{cases} \dot{x}(t) = R(t)v(t), \\ \dot{\theta}(t) = \omega(t), \end{cases} \tag{5.8}$$

where

$$v(t) := \bar{A}(t)F^{\mathrm{sh}}(t) + \bar{b}(t)M^{\mathrm{sh}}(t), \qquad \omega(t) := \bar{b}^{\top}(t)F^{\mathrm{sh}}(t) + \bar{c}(t)M^{\mathrm{sh}}(t), \tag{5.9}$$

and $\bar{A}(t)$, $\bar{b}(t)$, and $\bar{c}(t)$ are the block elements of $-\mathcal{R}^{-1}(t)$. The structure of this system of ordinary differential equations is the same as in (3.22) and (4.14). The following result holds

Theorem 5.2.1. *Let $\xi \in \Xi$. Then, given $x_0 \in \mathbb{R}^2$ and $\theta_0 \in \mathbb{R}$, the equations of motion* (5.8) *have a unique absolutely continuous solution $t \mapsto (x(t), \theta(t))$ defined in $[0,T]$ with values in $\mathbb{R}^2\times\mathbb{R}$ such that $x(0) = x_0$ and $\theta(0) = \theta_0$. In other words, there exists a unique rigid motion $t \mapsto r(t)(z) = x(t) + R(t)z$ such that the deformation function defined by* (5.4) *satisfies the equations of motion* (5.2)*.*

Proof. The result easily follows from the classical theory of ordinary differential equations, see, e.g., [18]. Indeed, the coefficients $\bar{b}^{\top}$ and $\bar{c}$ are continuous function of t, since they come from the inversion of the grand resistance matrix $\mathcal{R}$, whose entries are continuous in t. On the contrary, F^{sh} and M^{sh} are only measurable functions of time. This is enough to integrate the second equation in (5.8). By plugging the solution for θ into the first equation and by an analogous argument on the coefficients $\bar{A}$ and $\bar{b}$, also the equation for x can be integrated. $\square$

Some notes on the matrix K and on the coefficients $C_{\parallel}$ and $C_{\perp}$ are in order. First, we assume that $0 < C_{\parallel} < C_{\perp}$, secondly, we notice that the matrix K_{χ} (and therefore K_{ξ}) is symmetric and positive definite, and defines a scalar product in the space Ξ. Indeed, by introducing the power expended during the motion

$$\mathcal{P}(\chi) := \int_0^T \int_0^L \langle f(s,t), \dot{\chi}(s,t)\rangle \,\mathrm{d}s\mathrm{d}t, \tag{5.10}$$

we obtain that $\dot{\chi} = 0$ implies $\mathcal{P}(\chi) = 0$, and conversely that $\mathcal{P}(\chi) = 0$ implies $\dot{\chi} = 0$, since the resistance coefficients $C_{\parallel}$ and $C_{\perp}$ are non negative. This can be better observed if we write the power density with respect to the reference given by the tangential and normal components of the velocity. Then (5.10) reads

$$\mathcal{P}(\chi) = \int_0^T \int_0^L [C_{\parallel}\dot{\chi}_{\parallel}^2(s,t) + C_{\perp}\dot{\chi}_{\perp}^2(s,t)] \,\mathrm{d}s\mathrm{d}t,$$

and the conclusion is immediate.

Moreover, it emerges from the proof of Theorem 5.2.1 that the matrices K_{ξ} and K_{χ} belong to the space $C^0([0,T]; L^1(0,L))$.

Finally, the strict inequality assumption $C_\parallel < C_\perp$ guarantees that translational motions are "achievable". If we had $C_\parallel = C_\perp$, then $K_\chi(s,t)$ would be a multiple of the identity matrix and therefore, from (5.2a), we had

$$0 = F(t) = C_\parallel \int_0^L \dot\chi(s,t)\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L \chi(s,t)\,\mathrm{d}s\right) = \dot x(t),$$

which is expressing that the barycenter does not move as time evolves. Notice that this does not imply no motion at all; the above formula is telling us that whatever movement is performed by the swimmer, it has no net effects on its displacement. Yet, a motion which as an overall result has a rotation is achievable.

5.2.1 On the deformation function ξ

As of now, we are missing an important assumption of the shape function ξ, that will be enjoyed by χ as well. We introduce the following *external disks condition*, which prevents the flagellum from self-intersections. More precisely, what the condition states is that two different points of the flagellum cannot be too close to each other.

External disks condition – (EDC) For every $s \in [0,L]$ there exist open disks B_1, B_2 of radius $\rho > 0$ such that $B_1 \cap B_2 = \emptyset$, $\xi(s,t) \in \overline{B}_1 \cap \overline{B}_2$, and $\xi(\sigma,t) \notin B_1 \cup B_2$ for every $\sigma \in [0,L]$ and for every $t \in [0,T]$. Moreover, there exist open half disks B^-, B^+ of radius 2ρ centered at $\xi(0,t)$ and $\xi(L,t)$, respectively, whose diameters are given by the segment joining the centers of B_1 and B_2 at $s = 0, L$, and such that $\xi(\sigma,t) \notin B^- \cup B^+$ for every $\sigma \in [0,L]$ and for every $t \in [0,T]$.

An equivalent condition is the following. Consider $\rho > 0$ and define $C_{L,\rho} := [0,L] + B_\rho(0)$ the cigar-like set obtained by enlarging $[0,L]$. Call (s,y) the generic point of $C_{L,\rho}$, with $s \in (-\rho, L+\rho)$ and y being such that $(s,y) \in C_{L,\rho}$. Define now a map $\mathfrak{h}_t : C_{L,\rho} \to \mathbb{R}^2$ by

$$\mathfrak{h}_t(s,y) := \xi(s,t) + yJ\xi'(s,t).$$

Then it is easy to see that the following proposition holds.

Proposition 5.2.2. *The map $\mathfrak{h}_t$ is injective if and only if the external disks condition holds true.*

Proof. Let us assume that the external disk condition holds, and let us consider two points in $C_{L,\rho}$, $(s_1,y_1) \neq (s_2,y_2)$. From the external disks condition, there exist $B_1(s_1)$ and $B_2(s_1)$ such that $\xi(s_2,t) \notin B_1(s_1) \cup B_2(s_1)$. Notice that

$$|\mathfrak{h}_t(s_1,y_1) - \mathfrak{h}_t(s_2,y_2)| > \lim_{y\to\rho} |\mathfrak{h}_t(s_1,y) - \mathfrak{h}_t(s_2,y)| > 0,$$

which implies that $\mathfrak{h}_t$ is injective.

Let us now assume that $\mathfrak{h}_t$ is injective, and consider $(s_1,\rho) \neq (s_2,\rho)$. Let $B_1(s_1)$ and $B_2(s_1)$ be the open balls centered at $\xi(s_1,t) + \rho J\xi'(s_1,t)$ and $\xi(s_1,t) - \rho J\xi'(s_1,t)$, respectively. From the injectivity it follows that $|\xi(s_1,t) \pm \rho J\xi'(s_1,t), \xi(s_2,t)| > 0$, and therefore the external disks condition is verified.

The condition at the extremal points $s = 0, L$ can be easily verified to hold as well and the lemma is proved. □

We now prove a result stating that a bound on the angle ϑ_0 formed by the tangent with the x-axis implies the non self-intersection of the flagellum.

Lemma 5.2.3. *Let $\vartheta_0 \in C^1([0,L])$, and let $|\vartheta_0| < \pi/4$. Then, the non self-intersection condition holds.*

Proof. The C^1 regularity implies that $|\vartheta_0'| \leqslant \bar{\kappa} < +\infty$, which is a bound on the curvature of the flagellum. Therefore, it is enough to take any $\rho < \bar{\rho} := \bar{\kappa}^{-1}$ and consider the map $\mathfrak{h}_0 : C_{L,\rho} \to \mathbb{R}^2$ defined by $\mathfrak{h}_0(s,y) := \xi_0(s) + yJ\xi_0'(s)$, where ξ_0 is defined by integration

$$\xi_0(s) = \int_0^s (\cos\vartheta_0(\sigma), \sin\vartheta_0(\sigma))\,\mathrm{d}\sigma. \tag{5.11}$$

We will achieve the result by proving that $\mathcal{H}$ is injective. Let $(s_1,y_1) \neq (s_2,y_2)$. If $s_1 = s_2$, then it must be $|y_1 - y_2| > 0$, and therefore $|\mathfrak{h}_0(s_1,y_1) - \mathfrak{h}_0(s_1,y_2)| = |y_1 - y_2| > 0$. If $s_1 \neq s_2$, then again $|\mathfrak{h}_0(s_1,y_1) - \mathfrak{h}_0(s_2,y_2)| > 0$, because of the constraint on the radius of the osculating circle. Injectivity follows and the lemma is proved. □

The preceding lemma will be useful in Section 5.4 to guarantee that the deformations we construct to prove the controllability of the flagellum are good.

5.3 Optimal strokes

This section is divided into two parts. In the first one we prove Theorem 5.3.1 about the existence of the optimal beating strategy. The result is achieved by proving that a minimum problem for the power expended (5.10) has a solution. In the second part we show how it is possible to recover the optimal stroke if, for instance, it is possible to act on the curvature as a control.

5.3.1 Cost estimates

Let us recall the definition of power expended that we have already introduced in (5.10)

$$\mathcal{P}(\chi) := \int_0^L\int_0^T \langle f(s,t), \dot{\chi}(s,t)\rangle\,\mathrm{d}s\mathrm{d}t = \int_0^L\int_0^T \langle K_\chi(s,t)\dot{\chi}(s,t), \dot{\chi}(s,t)\rangle\,\mathrm{d}s\mathrm{d}t. \tag{5.12}$$

Up to a change of coordinates, it is possible to represent K_χ in diagonal form, where the entries are $C_\parallel$ and $C_\perp$. Since $C_\parallel < C_\perp$, it follows that

$$\mathcal{P}(\chi) \geqslant C_\parallel \int_0^L\int_0^T |\dot{\chi}(s,t)|^2\,\mathrm{d}s\mathrm{d}t.$$

Theorem 5.3.1. *The minimum problem*

$$\min\{\mathcal{P}(\chi) : \chi \in \Xi, \text{(5.2) and (EDC) hold}, \chi(\cdot,0) = \chi_0(\cdot), \chi(\cdot,T) = \chi_T(\cdot)\}, \tag{5.13}$$

where χ_0 *and* χ_T *are assigned states, has a solution.*

Proof. Let us consider a minimizing sequence $(\chi_k)_k \subset \Xi$ for $\mathcal{P}$. Therefore,

$$\int_0^L \int_0^T |\dot{\chi}_k(s,t)|^2 \,\mathrm{d}s\mathrm{d}t \leqslant M < +\infty, \quad \forall\, k.$$

Without loss of generality, we can assume that χ_k is parametrized by arc-length in s for every k, so that $|\chi_k'(s,t)| \equiv 1$, for all $(s,t) \in [0,L]\times[0,T]$ and for all k. Notice that the non self-intersection constraint gives a control on the second spatial derivative of χ_k. In order to prove that $(\chi_k)_k$ is uniformly bounded in Ξ, we are left with the estimation of $\int_0^L\int_0^T |\chi_k(s,t)|^2 \,\mathrm{d}s\mathrm{d}t$.

We have that $|\chi_k(s,0)| = |\chi_0(s)|$ for all $s \in [0,L]$ is uniformly bounded above by some constant C. Therefore, we can now estimate

$$|\chi_k(s,t)|^2 \leqslant |\chi_k(s,0)|^2 + \int_0^T |\dot{\chi}_k(s,t)|^2 \,\mathrm{d}t,$$

from which we easily get

$$\int_0^L |\chi_k(s,t)|^2 \,\mathrm{d}s \leqslant C^2 L + M.$$

Thus,

$$\sup_{t\in[0,T]} \|\chi_k(\cdot,t)\|^2_{L^2(0,L)} \leqslant T(C^2L + M),$$

which implies that $\chi_k \in L^\infty(0,T;L^2(0,L))$. Therefore, $(\chi_k)_k$ is uniformly bounded in Ξ, and therefore it admits a subsequence, which we do not relabel, which converges to a function χ. It is easy to see that the condition $|\chi_k'(s,t)| \equiv 1$ passes to the limit, so that s is the arc-length parameter for the limit χ. We have that

$$\chi_k \rightharpoonup \chi \qquad \text{in } H^1(0,T;L^2(0,L)), \tag{5.14}$$

since we have proved that $(\chi_k)_k$ is bounded in $H^1(0,T;L^2(0,L))$; moreover, $\dot{\chi}_k \rightharpoonup \dot{\chi}$ in $L^2(0,T;L^2(0,L))$. Once we will have proved that the limit function χ satisfies the constraints in the minimum problem, we will have proved that it is the solution we were looking for. The other following convergences hold true: $\chi_k \to \chi$ in $L^2(0,T;L^2(0,L))$. Moreover, notice that $H^1(0,T;L^2(0,L)) \subset C^0(0,T;L^2(0,L))$. Therefore, for every $g \in L^2(0,L)$, if we let $\mathrm{eval}_t(f) := \langle f(t), g\rangle_{L^2(0,L)}$, we have defined a continuous linear functional on $C^0(0,T;L^2(0,L))$. Now, $\mathrm{eval}_t(\chi_k) \to \mathrm{eval}_t(\chi)$, since $\langle \chi_k(\cdot,t), g\rangle \to \langle \chi(\cdot,t), g\rangle$ for all $g \in L^2(0,L)$ by (5.14), and this means that

$$\chi_k(\cdot,t) \rightharpoonup \chi(\cdot,t) \qquad \text{in } L^2(0,L), \text{ for all } t \in [0,T].$$

Recalling that $\|\chi_k''\|_\infty$ is uniformly bounded for every k, we get by interpolation that also $\|\chi_k'\|_\infty$ is uniformly bounded. Therefore, $(\chi_k(\cdot,t))_k$ has a weak limit in $H^2(0,L)$, which must coincide with χ. It also follows that the first spatial derivatives converge strongly in $H^1(0,L)$, so we can write

$$\chi_k(\cdot,t) \rightharpoonup \chi(\cdot,t) \qquad \text{in } H^2(0,L); \qquad\qquad \chi_k(\cdot,t) \to \chi(\cdot,t) \qquad \text{in } H^1(0,L).$$

Finally, this last convergence implies that $\chi_k(\cdot,t) \to \chi(\cdot,t)$ in $C^0(0,L)$.

Let us verify that also the other constraints pass to the limit. The self-propulsion constraint for χ_k reads, for the force,

$$0 = F_k(t) = \int_0^L K_{\chi_k}(s,t)\dot{\chi}_k(s,t)\,\mathrm{d}s.$$

The weak convergence of χ_k' to χ' in $H^1(0,L)$ is indeed strong in $L^2(0,L)$ and therefore, by dominated convergence, also $K_{\chi_k} \to K_\chi$ strongly in $L^2(0,T;L^2(0,L))$. Let now φ be a time dependent test function. We have

$$\int_0^T \varphi(t)F_k(t)\,\mathrm{d}t = \int_0^T\int_0^L \varphi(t)K_{\chi_k}(s,t)\dot{\chi}_k(s,t)\,\mathrm{d}s\mathrm{d}t \to \int_0^T 0\cdot\varphi(t)\,\mathrm{d}t$$

as $k \to \infty$ for all φ. Therefore, $F(t) = 0$ for almost every $t \in [0,T]$. The analogous result holds for the torque $M(t)$. It follows that χ is a minimizer for $\mathcal{P}$ in Ξ, as both the conditions on the initial and final times and (EDC) pass to the limit easily. $\square$

The theorem just proved states the existence of an optimal strategy to connect two different states of the flagellum, namely χ_0 and χ_T. This strategy is the one that minimizes the power expended. The next subsection contains the instruction to recover this stroke, while in Section 5.4 we will show explicitly how perform translations of a straight flagellum along its axis and rotations around its center, thus showing that the set in which we look for the minimum of the power functional $\mathcal{P}$ is not empty.

5.3.2 Recovering the stroke

We now see how it is possible to recover the optimal stroke of the flagellum once the function χ which realizes the minimum of the power expended has been selected. We fix $\xi(0,t) = 0$ and $\xi'(0,t) = e_1$, and these choices will allow us to determine the translation x and the rotation R. Indeed, observe that

$$\chi(0,t) = x(t) + R(t)\xi(0,t) = x(t)$$

and

$$\chi'(0,t) = R(t)\xi'(0,t) = R(t)e_1,$$

from which we see that the rotation matrix is such that its columns are the tangent and the normal vectors to $\chi(0,t)$:

$$R(t) = (\chi'(0,t)|J\chi'(0,t)). \tag{5.15}$$

Then, if the curvature function $\kappa : [0,L]\times[0,T] \to \mathbb{R}$ is prescribed, it is possible to reconstruct ξ via the angle it forms with the x-axis. Recall, in fact, that $\xi'(s,t) = (\cos\vartheta(s,t), \sin\vartheta(s,t))$, as in (5.11), where

$$\vartheta(s,t) = \int_0^s \kappa(\bar{s},t)\,\mathrm{d}\bar{s}. \tag{5.16}$$

Using together (5.16) to reconstruct ξ and (5.15) allows us to recover, according to (5.4), the function $x(t) = \chi(s,t) - R(t)\xi(s,t)$.

5.4 Controllability

In this Section we show that the flagellum is controllable, i.e., it is possible to prescribe a deformation that brings it from a given state χ_0 at time $t = 0$ into another given state χ_T at time $t = T$. The main effort will be to show that it is possible to produce motions to translate and rotate a straight flagellum. In addition, two homotopies will transform the flagellum from the configurations χ_0 and χ_T into a straight rod. Let $\mathcal{H}_0, \mathcal{H}_T : [0,L]\times[0,1] \to \mathbb{R}^2$ be continuous functions such that

$$\mathcal{H}_0(s,0) = \chi_0(s), \quad \mathcal{H}_0(s,1) = \Sigma_0(s), \qquad \mathcal{H}_T(s,0) = \chi_T(s), \quad \mathcal{H}_T(s,1) = \Sigma_T(s),$$

where Σ_0, Σ_T are two segments of length L in $\mathbb{R}^2$. To summarize, the whole control process is organized as follows.

$$\chi_0 \xrightarrow{\mathcal{H}_0(\cdot,t)} \Sigma_0(\cdot) \xrightarrow{\text{rotation, translation, rotation}} \Sigma_T(\cdot) \xrightarrow{\mathcal{H}_T(\cdot,1-t)} \chi_T(\cdot).$$

For sake of simplicity of notations we decide to perform the “stretching” of the flagellum from χ_0 to Σ_0 and the inverse operation from Σ_T to χ_T in a time interval outside that on which we focus for the translational and rotational motions; this is done essentially for keeping notations a little lighter.

The functions $\mathcal{H}_0$ and $\mathcal{H}_T$ exist and are unique, by virtue of Theorem 5.2.1, and are obtained by solving the equations of motion whose boundary conditions are the initial and final shapes, χ_0 and Σ_0, and χ_T and Σ_T, respectively. The main point here is that once we perform the stretching we know that the flagellum will turn into a straight rod, but the computation of its potision and orientation is not immediate; one has to let time evolve and see where are the final position and orientation of the rod. It must be pointed out that, by fixing the initial and final shapes, final position and orientation cannot be chosen freely. This is why we have to develop all the machinery for making the rod translate and rotate to pass from Σ_0 to Σ_T.

Also notice that any intermediate state $\mathcal{H}_0(s,t)$, $\mathcal{H}_T(s,t)$, for $t \in (0,1)$, is such that all the assumptions on the regularity of the shape χ, and in particular the non self-intersection property, are respected.

Once the flagellum is in a straight configuration some bumps are formed and are made slide along the flagellum itself, so that the perturbation they produce has the

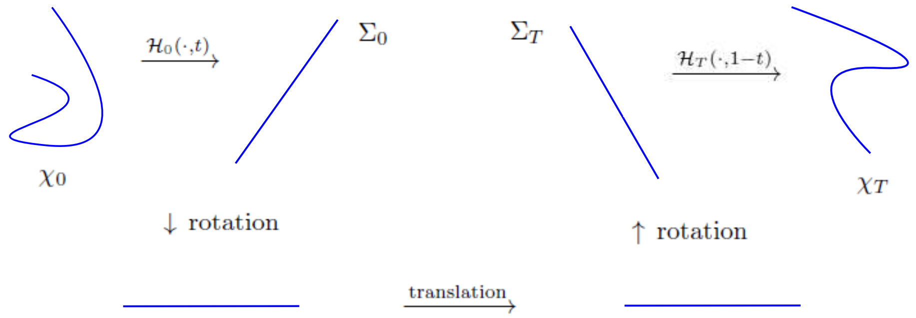

Figure 5.1: Sequence of the control process (the curved flagella might not be to scale).

effect of making it either advance along its axis or rotate around its center. We can distinguish three phases of this motion, which takes place in the time span $[0,T]$: the transient of formation of the bumps $[0,\tau]$, the translation of the bumps $[\tau,T-\tau]$, and the transient of destruction of the bumps $[T-\tau,T]$. We will show that it is possible to assign a function describing the angle that the flagellum makes with the positive x-axis, which acts as a control, through which the three phases of the motion are described. Let $\vartheta_0 : I \to \mathbb{R}$ be such function. Here, I is an interval on the real line contained in $[0,L]$, which differs between the translation and rotation cases, and which will be specified in due time. We require ϑ_0 to be smooth and with compact support, and we extend it by zero to the whole $\mathbb{R}$. Let $\gamma_0 : [0,L] \to \mathbb{R}^2$ be the reference configuration of the flagellum, so that $\gamma_0'(s) = (\cos\vartheta_0(s), \sin\vartheta_0(s))$. Roughly speaking, the bumps will be located in the region where ϑ_0 is different from zero.

Since our aim is to produce motion via the sliding of the bumps along the flagellum, we consider a function $\gamma : [0,L]\times[0,T] \to \mathbb{R}^2$ which also takes into account the time variable. To fix the ideas, let us assume that the bumps are sliding in the negative direction of the x-axis with velocity $c>0$; then we define $\gamma(s,t) = \gamma_0(s+c(t-\tau)) - c(t-\tau)e_1$, to represent the translation phase of the bumps. Here time runs in the interval $[\tau,T-\tau]$: the bumps are completely formed and are moving. It is possible to take into account the formation and destruction transient phases, by representing the function γ via integration of its tangent vector

$$\gamma(s,t) = \int_0^s (\cos\vartheta(s',t), \sin\vartheta(s',t))\, \mathrm{d}s', \qquad (s,t)\in[0,L]\times[0,T],$$

where the angle ϑ is defined, for every $s\in[0,L]$, by

$$\vartheta(s,t) = \begin{cases} \dfrac{t}{\tau}\vartheta_0\left(\dfrac{s-\sigma(\tau)}{l}\right), & t\in[0,\tau], \\ \vartheta_0\left(\dfrac{s-\sigma(t)}{l}\right), & t\in[\tau,T-\tau], \\ \dfrac{T-t}{\tau}\vartheta_0\left(\dfrac{s-\sigma(T-\tau)}{l}\right), & t\in[T-\tau,T], \end{cases} \tag{5.17}$$

where $0 < 4l < L$ is the length of the deformed part and $\sigma(t)$ describes the position of the center of the bumps at time t. The bound on l is necessary to avoid that the length of the deformed portion exceeds the total length of the flagellum; the function σ will be defined in two different ways, which we denote by σ_{transl} and σ_{rot}, according to the type of movement we are studying.

In order to make the computation easier and to have only translational movements when we are translating the flagellum and rotational movements when we are rotating it, some symmetry assumptions will be made from case to case, and of course, they will be on the function ϑ_0 which generates the bumps.

Before we proceed, we want to list and prove a number of preliminary results in Rational Mechanics that will allow us to simplify the forthcoming calculations by actually proving that some quantities vanish.

Recall that the general form of the flagellum, as seen by an observer in the lab reference, is given by the structure formula (5.4), which now reads

$$\chi(s,t) = x(t) + R(t)\gamma(s,t);$$

we prefer to change notation here and in the remainder of this section from ξ to γ to stress the fact that we are using a shape function we can manipulate. By plugging this into formulae (5.1), we have the expressions of the density of force and moment

$$f(s,t) = C_{\parallel}\langle\dot\chi(s,t),\chi'(s,t)\rangle\chi'(s,t) + C_{\perp}\langle\dot\chi(s,t),J\chi'(s,t)\rangle J\chi'(s,t), \tag{5.18a}$$

$$\begin{aligned} m(s,t) = \chi(s,t)\times f(s,t) = {} & C_{\parallel}\langle\dot\chi(s,t),\chi'(s,t)\rangle\langle J\chi(s,t),\chi'(s,t)\rangle \\ & + C_{\perp}\langle\dot\chi(s,t),J\chi'(s,t)\rangle\langle\chi(s,t),\chi'(s,t)\rangle, \end{aligned} \tag{5.18b}$$

which will be written more specifically later on according to the motion we will be studying.

The following lemma contains a well known result about the integration of even functions. The analogous result for odd functions is a triviality.

Lemma 5.4.1. *Let $f : [a,b] \to \mathbb{R}$ be a symmetric function with respect to the middle point $c := (a+b)/2$ of $[a,b]$, that is $f(x) = f(2c-x)$. Then, the integral function $F(x) := \int_c^x f(s)\,\mathrm{d}s$ is antisymmetric in $[a,b]$ with respect to c, that is $F(x) = -F(2c-x)$.*

Proof. The proof is a simple calculation

$$F(2c-x) = \int_c^{2c-x} f(s)\,\mathrm{d}s = -\int_c^x f(2c-s)\,\mathrm{d}s = -\int_c^x f(s)\,\mathrm{d}s = -F(x). \qquad \square$$

5.4.1 Translation

Now we deal with the case of translations. We will specify the choices for the interval I, the form of $\sigma = \sigma_{\mathrm{transl}}$, and the symmetry assumption on the angle function ϑ_0. Let us consider a smooth function with compact support $\vartheta_0 : I = [-2,2] \to \mathbb{R}$ such that

$$\vartheta_0 \text{ is odd: } \vartheta_0(-\sigma) = -\vartheta_0(\sigma), \text{ for all } \sigma \in [-2,2]; \tag{5.19a}$$

$$\vartheta_0(2-\sigma) = \vartheta_0(\sigma), \text{ for all } \sigma \in [0,2]; \text{ the function is even in } [0,2] \text{ and in } [-2,0]; \tag{5.19b}$$

$$\vartheta_0(1-\sigma) = -\vartheta_0(\sigma), \text{ for all } \sigma \in [0,1]; \text{ the function is odd in } [0,1]; \tag{5.19c}$$

$$\vartheta_0(2) = 0, \text{ in order to be oriented, at the extrema } \sigma = \pm 2 \text{ as in } \sigma = 0; \tag{5.19d}$$

From these assumptions, in particular from (5.19d), it also follows that

$$\int_0^1 \sin\vartheta_0(\sigma)\,\mathrm{d}\sigma = \int_0^2 \sin\vartheta_0(\sigma)\,\mathrm{d}\sigma = 0.$$

An example of a flagellum whose function ϑ_0 enjoys the properties listed above is illustrated in Figure 5.2.

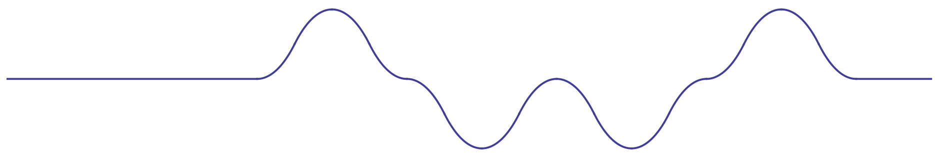

Figure 5.2: Bumps on a flagellum

When dealing with translations, we need to define $\sigma_{\mathrm{transl}}(t) := L - 2l - c(t-\tau)$. Notice that given the time interval subdivision we assumed in formula (5.17), the transient time τ and the bumps velocity c are not independent. The value of c is chosen so that $\sigma(\tau) = L - 2l$ and $\sigma(T-\tau) = 2l$. Therefore $c = (L-4l)/(T-2\tau)$.

We will show that the symmetry assumptions we made on the function ϑ_0 prevent our flagellum from rotating both when the bumps form and disappear and when they move along the flagellum. For the moment we take it for granted.

The absence of rotation allows us to fix $R(t) = I$ for every $t \in [0,T]$. Moreover, the vertical component of the total viscous force vanishes by the symmetry assumptions (5.19a)-(5.19c), so we can infer that $x_2(t) = 0$. Notice also that in principle $\dot{x}(t)$ can depend on l; to keep notations lighter, we will stress the dependance on l when needed. Plugging these information in (5.18a), we get

$$\begin{aligned} f(s,t) = C_{\parallel}\langle(\dot{x}_1(t;l),0) + \dot{\gamma}(s,t), \gamma'(s,t)\rangle\gamma'(s,t) + \\ C_{\perp}\langle(\dot{x}_1(t;l),0) + \dot{\gamma}(s,t), J\gamma'(s,t)\rangle J\gamma'(s,t), \end{aligned} \tag{5.20}$$

where $\dot{x}_1(t;l)$ is the unknown global translational velocity along the x-axis. We can expect that $\dot{x}_1(t;l)$ will have different expressions in the different phases of the translational motion. At the end of one stroke, the distance along the x-axis that the flagellum will have covered is given by

$$\Delta x_1(l) = \int_0^T \dot{x}_1(t;l)\,\mathrm{d}t = \int_0^\tau \dot{x}_{1,\mathrm{form}}(t;l)\,\mathrm{d}t + \int_\tau^{T-\tau} \dot{x}_{1,\mathrm{transl}}(t;l)\,\mathrm{d}t + \int_{T-\tau}^T \dot{x}_{1,\mathrm{destr}}(t;l)\,\mathrm{d}t.$$

Recall, from (5.17), that for $t \in [T-\tau, T]$ the prefactor in front of ϑ_0 in the expression for the angle ϑ is $(T-t)/\tau$, therefore, a change of variables $t \to T-t$ in the third integral above allows us to rewrite the formula as

$$\Delta x_1(l) = 2\int_0^\tau \dot{x}_{1,\mathrm{form}}(t'l)\,\mathrm{d}t + (L-4l)a(l),$$

where it has also been used, as it will emerge from the expression the translational velocity, that $\dot{x}_{1,\mathrm{transl}}$ does not depend on time, and the expression of c has been employed. The value of $a(l)$ will be derived in (5.21).

Figure 5.3: Configuration of the flagellum for the translation at $t \in (0,\tau)$.

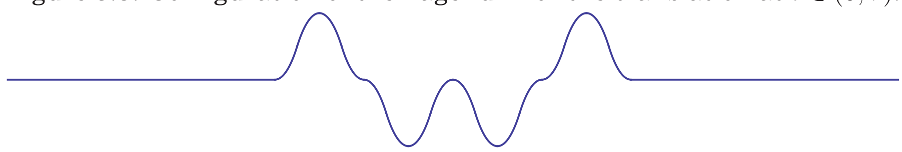

Figure 5.4: Configuration of the flagellum for the translation at $t \in (\tau, T-\tau)$.

Figure 5.5: Configuration of the flagellum for the translation at $t \in (T-\tau, T)$.

Let us now focus on the time interval $[\tau, T-\tau]$ and make the bumps translate to the left with velocity c, as illustrated in Fig. 5.4. From (5.20), we can compute the expression of the total viscous force

$$\begin{aligned}
F_{\mathrm{transl}}(t) = \int_0^L f(s,t)\,\mathrm{d}s = \int_0^{L-4l} & [C_\parallel \langle(\dot{x}_{1,\mathrm{transl}}(t),0),(1,0)\rangle(1,0) + \\
& C_\perp \langle(\dot{x}_{1,\mathrm{transl}}(t),0),(0,1)\rangle(0,1)]\,\mathrm{d}s \\
+ \int_{\sigma(t)-2l}^{\sigma(t)+2l} & [C_\parallel \langle(\dot{x}_{1,\mathrm{transl}}(t),0)+\dot{\gamma}(s,t),\gamma'(s,t)\rangle\gamma'(s,t) + \\
& C_\perp \langle(\dot{x}_{1,\mathrm{transl}}(t),0)+\dot{\gamma}(s,t),J\gamma'(s,t)\rangle J\gamma'(s,t)]\,\mathrm{d}s \\
=: & F^{(1)}_{\mathrm{transl}}(t) + F^{(2)}_{\mathrm{transl}}(t).
\end{aligned}$$

It is easy to see that $F^{(1)}_{\mathrm{transl}}(t) = ((L-4l)C_\parallel \dot{x}_{1,\mathrm{transl}}(t;l),0)$, while, upon changing variables

$\sigma = (s - \sigma(t))/l$, we get

$$
\begin{aligned}
F^{(2)}_{\mathrm{transl}}(t) =& l\dot{x}_{1,\mathrm{transl}}(t;l)\int_{-2}^{2}\begin{pmatrix} C_{\parallel}\cos^2\vartheta_0(\sigma) + C_{\perp}\sin^2\vartheta_0(\sigma) \\ 0 \end{pmatrix}\mathrm{d}\sigma \\
&+ C_{\parallel} lc\int_{-2}^{2}\left\langle \begin{pmatrix} \cos\vartheta_0(\sigma) - 1 \\ \sin\vartheta_0(\sigma) \end{pmatrix}, \begin{pmatrix} \cos\vartheta_0(\sigma) \\ \sin\vartheta_0(\sigma) \end{pmatrix}\right\rangle \begin{pmatrix} \cos\vartheta_0(\sigma) \\ \sin\vartheta_0(\sigma) \end{pmatrix}\mathrm{d}\sigma \\
&+ C_{\perp} lc\int_{-2}^{2}\left\langle \begin{pmatrix} \cos\vartheta_0(\sigma) - 1 \\ \sin\vartheta_0(\sigma) \end{pmatrix}, \begin{pmatrix} -\sin\vartheta_0(\sigma) \\ \cos\vartheta_0(\sigma) \end{pmatrix}\right\rangle \begin{pmatrix} -\sin\vartheta_0(\sigma) \\ \cos\vartheta_0(\sigma) \end{pmatrix}\mathrm{d}\sigma,
\end{aligned}
$$

from which it is also possible to see that the second component vanishes, by symmetry and by Lemma 5.4.1. Gluing all together and solving $F_{\mathrm{transl}}(t) = 0$ for $\dot{x}_{1,\mathrm{transl}}(t;l)$, we get

$$
\dot{x}_{1,\mathrm{transl}}(l) = c\,\frac{l\displaystyle\int_{-2}^{2}[C_{\parallel}(\cos^2\vartheta_0(\sigma) - \cos\vartheta_0(\sigma)) + C_{\perp}\sin^2\vartheta_0(\sigma)]\,\mathrm{d}\sigma}{C_{\parallel}(L-4l) + l\displaystyle\int_{-2}^{2}[C_{\parallel}\cos^2\vartheta_0(\sigma) + C_{\perp}\sin^2\vartheta_0(\sigma)]\,\mathrm{d}\sigma} =: ca(l). \tag{5.21}
$$

We notice, as we announced before, that the velocity $\dot{x}_{1,\mathrm{transl}}$ is constant, does not depend on time, and is linear with respect to the velocity c of the bumps. Moreover, since $C_{\perp} > C_{\parallel}$, $a(l)$ is always positive, so there is actually a positive net advancement, regardless the swimming strategy. Nonetheless, this becomes important as soon as we want to optimize the distance covered during one stroke. Also observe that the function a is continuous with respect to l and $a(0) = 0$.

In order to verify that the moment of the forces vanishes, we will exploit the symmetries of the angle function ϑ_0 and will apply Lemma 5.4.1. We will compute the moments with respect to $x_{\mathrm{transl}}(t)$, since in this way computations are a bit easier. Moreover, we will assume that $R(t) = I$ and $\omega(t) = 0$ for all t. This guess will turn out to be correct if we actually prove that $M(t) = 0$, by means of the uniqueness of the solution to the equations of motion. The moment density (5.18b) reads then

$$
\begin{aligned}
m_{\mathrm{transl}}(s,t) &= (\chi(s,t) - x_{\mathrm{transl}}(t))\times f(s,t) \\
&= C_{\parallel}\langle \dot{x}_{\mathrm{transl}}(t) + \dot{\gamma}(s,t), \gamma'(s,t)\rangle\langle J\gamma(s,t), \gamma'(s,t)\rangle + \\
&\quad C_{\perp}\langle \dot{x}_{\mathrm{transl}}(t) + \dot{\gamma}(s,t), J\gamma'(s,t)\rangle\langle \gamma(s,t), \gamma'(s,t)\rangle,
\end{aligned}
$$

and therefore, exploiting that $\dot{x}_{2,\mathrm{transl}}(t)$ vanishes, the total moment is given by

$$
\begin{aligned}
M_{\mathrm{transl}}(t) &= \int_0^L m_{\mathrm{transl}}(s,t)\,\mathrm{d}s \\
&= \int_{\sigma(t)-2l}^{\sigma(t)+2l} [C_{\parallel}\langle \dot{x}_{\mathrm{transl}}(t) + \dot{\gamma}(s,t), \gamma'(s,t)\rangle\langle J\gamma(s,t), \gamma'(s,t)\rangle + \\
&\qquad C_{\perp}\langle \dot{x}_{\mathrm{transl}}(t) + \dot{\gamma}(s,t), J\gamma'(s,t)\rangle\langle \gamma(s,t), \gamma'(s,t)\rangle]\,\mathrm{d}s
\end{aligned}
$$

which vanishes due to the symmetries of ϑ_0. The computation can be developed by writing all the terms explicitly and exploiting Lemma 5.4.1. This proves a posteriori that our choice $R(t) = I$ and $\omega(t) = 0$ for all $t \in [0, T]$, was correct.

5.4.2 Translation transients

Let us now focus on the time interval $[0, \tau]$ to study the formation transient; see Figs. 5.3 and 5.5 for the formation and the destruction transients, respectively. Starting from a straight flagellum at time $t = 0$ we will obtain a flagellum with the bumps located at the tail after the transient time $t = \tau$. According to (5.17) the tangent, the flagellum shape, and the velocity functions are described by

$$\gamma'(s,t) = \left(\cos\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right), \sin\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right)\right), \tag{5.22}$$

$$\gamma(s,t) = \int_0^s \left(\cos\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right), \sin\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right)\right) \mathrm{d}s',$$

$$\dot\gamma(s,t) = \frac{1}{\tau}\int_0^s \left(-\sin\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right), \cos\left(\frac{t}{\tau}\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right)\right)\right)\vartheta_0\left(\frac{s'-\sigma(\tau)}{l}\right) \mathrm{d}s'.$$

Here, the computations are formally the same as for the case $t \in [\tau, T-\tau]$, but in this case the velocity cannot be easily integrated. Notice that, for $s \in [0, L-4l]$ we have $\gamma'(s,t) = (1,0)$, $\gamma(s,t) = (s,0)$, and $\dot\gamma(s,t) = (0,0)$, while for $s \in [L-4l, L]$ we have γ' as in (5.22),

$$\gamma(s,t) = (L-4l, 0) + l\int_{-2}^{(s-\sigma(\tau))/l} \left(\cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right), \sin\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\right) \mathrm{d}\sigma,$$

$$\dot\gamma(s,t) = \frac{l}{\tau}\int_{-2}^{(s-\sigma(\tau))/l} \left(-\sin\left(\frac{t}{\tau}\vartheta_0(\sigma)\right), \cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\right)\vartheta_0(\sigma)\,\mathrm{d}\sigma,$$

after the change of variables $\sigma = (s' - \sigma(\tau))/l$. If one develops the computations, then the final result for the velocity will be

$$\dot x_{1,\mathrm{form}}(t;l) = \frac{-\dfrac{l^2}{\tau}\left(C_\parallel F^{(1)}_{\mathrm{form}}(t) + C_\perp F^{(2)}_{\mathrm{form}}(t)\right)}{C_\parallel(L-4l) + l\displaystyle\int_{-2}^{2}\left[C_\parallel \cos^2\left(\frac{t}{\tau}\vartheta_0(\sigma)\right) + C_\perp \sin^2\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\right]\mathrm{d}\sigma},$$

where

$$F^{(1)}_{\mathrm{form}}(t) = \int_{-2}^{2}\left\langle \int_{-2}^{\sigma}\begin{pmatrix} -\sin(t\vartheta_0(\sigma')/\tau) \\ \cos(t\vartheta_0(\sigma')/\tau)\end{pmatrix}\vartheta_0(\sigma')\,\mathrm{d}\sigma', \begin{pmatrix}\cos(t\vartheta_0(\sigma')/\tau) \\ \sin(t\vartheta_0(\sigma')/\tau)\end{pmatrix}\right\rangle \cdot$$
$$\cdot\cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\mathrm{d}\sigma,$$

$$F^{(2)}_{\mathrm{form}}(t) = \int_{-2}^{2}\left\langle \int_{-2}^{\sigma}\begin{pmatrix} \sin(t\vartheta_0(\sigma')/\tau) \\ -\cos(t\vartheta_0(\sigma')/\tau)\end{pmatrix}\vartheta_0(\sigma')\,\mathrm{d}\sigma', \begin{pmatrix}-\sin(t\vartheta_0(\sigma')/\tau) \\ \cos(t\vartheta_0(\sigma')/\tau)\end{pmatrix}\right\rangle \cdot$$
$$\cdot\sin\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\mathrm{d}\sigma,$$

Notice that in this case, by setting $\sigma(t) = \sigma(\tau) = L - 2l$, there is no influence of c on $\dot{x}_{1,\mathrm{form}}$, as one could reasonably expect since the bumps are forming and not translating. The same thing happens for the destruction transient, where $\sigma(t) = \sigma(T - \tau) = L - 2l - c(T - 2\tau) = 2l$.

It is interesting to notice the presence of the factor $1/\tau$ in the expression of $\dot{x}_{1,\mathrm{form}}$, which somehow compensates the duration of the transient. It is noteworthy that Δx_1 depends in an essential way on l, which is the size of the portion of flagellum where the deformation takes place: from the expressions of the velocities, it is evident that if no perturbation occurs, then no net displacement is achieved, and also that the displacement is a continuous function of the parameter l. Therefore, once l is fixed, the maximum displacement obtainable is given by

$$\Delta x_1(l) = 2\int_0^\tau \dot{x}_{1,\mathrm{form}}(t; l)\,\mathrm{d}t + (L - 4l)a(l).$$

Now, if a certain distance $\Delta\bar{x}$ is assigned to cover, the flagellum can perform the swimming strategy as follows. There exists an integer $k \geqslant 0$ such that $\Delta\bar{x} = k\Delta x_1(l) + \delta x_1$, where $\delta x_1 < \Delta x_1(l)$. By the continuity properties of Δx_1 as a function of l, there will exist a value l^* such that $\delta x_1 = \Delta x_1(l^*)$. Therefore, it is enough to divide the time interval $[0, T]$ in $k + 1$ subintervals, k of which of size $\Delta x_1(l)/\Delta\bar{x}$, and the last one of size $\delta x_1/\Delta\bar{x}$ and swim accordingly. This is necessary since, of course, $\Delta x_1(l)$ cannot exceed the total length L of the flagellum, and it is possible since the motion is rate independent.

5.4.3 Rotation

In the case of the rotation, we will show that it is possible to make a straight rod rotate around its center, by means of two bumps analogous to those for the translation case are used. This time, their configuration is as shown in Figure 5.7, and both bumps move towards either the center or the ends of the rod in order to achieve either a counterclockwise or a clockwise rotation, respectively. Also in this case some symmetry argument can be carried out to simplify the computations. First of all, it is more convenient to let the arclength parameter s run in $[-L/2, L/2]$. It is easy to see that the total force is zero, since the position function of such a configuration enjoys the symmetry property $\gamma(s, t) = -\gamma(-s, t)$, for $-L/2 \leqslant s \leqslant 0$. Therefore,

$$\gamma'(s, t) = \gamma'(-s, t), \qquad \dot{\gamma}(s, t) = -\dot{\gamma}(-s, t), \qquad \text{for } -L/2 \leqslant s \leqslant 0.$$

This implies that the density of force (5.18a) is an odd function in the variable s, so that $F(t) = \int_{-L/2}^{L/2} f(s, t)\,\mathrm{d}s = 0$ for all $t \in [0, T]$.

In this case, let $\vartheta_0 : [-1/2, 1/2] \to \mathbb{R}$ still describe the structure of the bump we want to form, and let $\vartheta(s, t)$ denote the angle of the tangent γ' to the curve with the positive

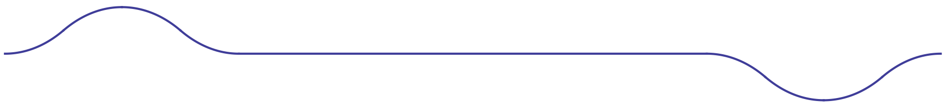

Figure 5.6: Configuration of the flagellum for the rotation at time $t \in (0, \tau)$.

Figure 5.7: Configuration of the flagellum for the rotation at time $t \in (\tau, T - \tau)$.

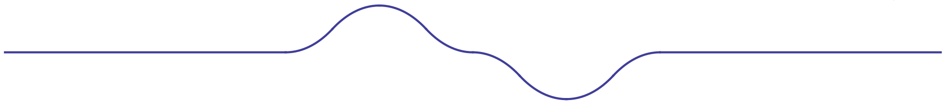

Figure 5.8: Configuration of the flagellum for the rotation at time $t \in (T - \tau, T)$.

x-axis; we extend ϑ_0 by zero outside its support and make the following assumptions

$$\begin{aligned}&\vartheta_0(-\sigma) = -\vartheta_0(\sigma), \text{ for all } \sigma \in [-1/2, 1/2],\\&\vartheta_0(\sigma) > 0 \text{ for } 0 < \sigma < l/2,\\&\vartheta_0(\sigma) \in (\pi/4, \pi/4), \text{ for all } \sigma \in [-1/2, 1/2].\end{aligned}$$

and therefore the real angle will be given again by formula (5.17), where this time $\sigma(t) = \sigma_{\mathrm{rot}}(t) = L/2 - l/2 - c(t - \tau)$ and c is chosen here so that $\sigma_{\mathrm{rot}}(\tau) = L/2 - l/2$ and $\sigma_{\mathrm{rot}}(T - \tau) = l/2$, that is $c = (L - 2l)/2(T - 2\tau)$. To be precise, we should distinguish the right bump from the left one. Two different expressions for $\sigma(t)$ would be needed, but given the symmetry of the configuration, it is enough to restrict our attention only to one bump, and we choose the one on the right.

Notice that with the symmetry assumption we have made, the term we called $b(t)$ in the grand resistance matrix $\mathcal{R}(t)$ and the term $F^{\mathrm{sh}}(t)$ defined in (5.7) vanish. Therefore, the equations of motion (5.8) read $\dot{x}(t) = 0$, $\dot{\theta}(t) = \omega(t) = \bar{c}(t)M^{\mathrm{sh}}(t)$. This implies that there is no net translation of the flagellum, therefore we can neglect $x(t)$.

Since we have proved that the total viscous force vanishes, let us concentrate on the moment. To this end, recall that $\chi(s,t) = R(t)\gamma(s,t)$, $\chi'(s,t) = R(t)\gamma'(s,t)$, and $\dot{\chi}(s,t) = \omega(t)R(t)J\gamma(t) + R(t)\dot{\gamma}(s,t)$, $\omega(t)$ being the angular velocity. The density of force

(5.18a) is given by

$$
\begin{aligned}
f(s,t) &= C_{\|}\langle \dot\chi(s,t), \chi'(s,t)\rangle \chi'(s,t) + C_{\perp}\langle \dot\chi(s,t), J\chi'(s,t)\rangle J\chi'(s,t)\\
&= C_{\|}\langle \omega(t)J\gamma(s,t) + \dot\gamma(s,t), \gamma'(s,t)\rangle R(t)\gamma'(s,t) +\\
&\quad C_{\perp}\langle \omega(t)J\gamma(s,t) + \dot\gamma(s,t), J\gamma'(s,t)\rangle R(t)J\gamma'(s,t),
\end{aligned}
$$

and therefore the moment density (5.18b) is

$$
\begin{aligned}
m(s,t) &= \gamma(s,t)\times(C_{\|}\langle \omega(t)J\gamma(s,t) + \dot\gamma(s,t), \gamma'(s,t)\rangle\gamma'(s,t)\\
&\quad + C_{\perp}\langle \omega(t)J\gamma(s,t) + \dot\gamma(s,t), J\gamma'(s,t)\rangle J\gamma'(s,t))\\
&= \gamma(s,t)\times(C_{\|}\langle \dot\gamma(s,t), \gamma'(s,t)\rangle\gamma'(s,t) + C_{\perp}\langle \dot\gamma(s,t), J\gamma'(s,t)\rangle J\gamma'(s,t))\\
&\quad + \omega(t)\gamma(s,t)\times(C_{\|}\langle J\gamma(s,t), \gamma'(s,t)\rangle\gamma'(s,t) + C_{\perp}\langle \gamma(s,t), \gamma'(s,t)\rangle J\gamma'(s,t))\\
&= C_{\|}\langle \dot\gamma(s,t), \gamma'(s,t)\rangle\langle J\gamma(s,t), \gamma'(s,t)\rangle + C_{\perp}\langle \dot\gamma(s,t), J\gamma'(s,t)\rangle\langle \gamma(s,t), \gamma'(s,t)\rangle\\
&\quad + \omega(t)[C_{\|}\langle J\gamma(s,t), \gamma'(s,t)\rangle^2 + C_{\perp}\langle \gamma(s,t), \gamma'(s,t)\rangle^2],
\end{aligned}
\tag{5.23}
$$

where we separated the contributions depending on $\omega(t)$. Now observe that the center of the bump is located in $\sigma(t) = L/2 - l/2 - c(t-\tau)$, and that the position $\gamma(s,t)$ also intervenes in the computations. The computation of the total moment of the forces can be split into three parts, according to

$$
\begin{aligned}
M(t;l) &= \int_0^{\sigma(t)-l/2} m(s,t)\,\mathrm{d}s + \int_{\sigma(t)-l/2}^{\sigma(t)+l/2} m(s,t)\,\mathrm{d}s + \int_{\sigma(t)+l/2}^{L/2} m(s,t)\,\mathrm{d}s\\
&=: \tilde M_1(t;l) + \tilde M_2(t;l) + \tilde M_3(t;l).
\end{aligned}
\tag{5.24}
$$

For $s \in [0, \sigma(t) - l/2]$, we have $\vartheta_0 = 0$, and so $\gamma'(s,t) = (1,0)$, $\gamma(s,t) = (s,0)$, and $\dot\gamma(s,t) = (0,0)$. Thus,

$$
m(s,t) = \omega_{\mathrm{rot}}(t;l)[C_{\|}\langle J\gamma(s,t), \gamma'(s,t)\rangle^2 + C_{\perp}\langle \gamma(s,t), \gamma'(s,t)\rangle^2] = \omega_{\mathrm{rot}}(t;l)C_{\perp}s^2,
$$

which implies that

$$
\tilde M_1(t;l) = \omega_{\mathrm{rot}}(t;l)M_1(t;l) = \frac{\omega_{\mathrm{rot}}(t;l)C_{\perp}}{3}\left(\frac{L}{2} - l - c(t-\tau)\right)^3.
$$

Similarly, for $s \in [\sigma(t) + l/2, L/2]$, we have

$$
\begin{aligned}
\gamma(s,t) &= \gamma(\sigma(t)+l/2,t) + \int_{\sigma(t)+l/2}^{s} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}s'\\
&= \begin{pmatrix} \sigma(t) - l/2 \\ 0 \end{pmatrix} + \int_{\sigma(t)-l/2}^{\sigma(t)+l/2} \begin{pmatrix} \cos\vartheta_0(4(s-\sigma(t))/l) \\ \sin\vartheta_0(4(s-\sigma(t))/l) \end{pmatrix} \mathrm{d}s + \begin{pmatrix} s - \sigma(t) - l/2 \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} s - l \\ 0 \end{pmatrix} + l\begin{pmatrix} \int_{-1/2}^{1/2}\cos\vartheta_0(\sigma)\,\mathrm{d}\sigma \\ 0 \end{pmatrix} =: \begin{pmatrix} s - l + lC_{\vartheta_0}(\tau) \\ 0 \end{pmatrix},
\end{aligned}
$$

where

$$
C_{\vartheta_0}(t) := 2\int_0^{1/2} \cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\mathrm{d}\sigma.
\tag{5.25}
$$

Therefore,

$$
\begin{aligned}
\tilde{M}_3(t;l) =&\omega_{\mathrm{rot}}(t;l)M_3(t;l) = \omega_{\mathrm{rot}}(t;l)C_\perp \int_{\sigma(t)+l/2}^{L/2} (s-l+lC_{\vartheta_0}(\tau))^2 \mathrm{d}s \\
=&\frac{\omega_{\mathrm{rot}}(t;l)C_\perp}{3}\left[\left(\frac{L}{2}-l+lC_{\vartheta_0}(\tau)\right)^3-\left(\frac{L}{2}-l-c(t-\tau)+lC_{\vartheta_0}(\tau)\right)^3\right].
\end{aligned}
$$

The calculations for $\tilde{M}_2(t;l)$ are slightly more cumbersome. In this case, $s \in [\sigma(t) - l/2, \sigma(t)+l/2]$, and so

$$
\gamma(s,t) = \begin{pmatrix} \sigma(t)-l/2 \\ 0 \end{pmatrix} + l\int_{-1/2}^{(s-\sigma(t))/l} \begin{pmatrix} \cos\vartheta_0(\sigma) \\ \sin\vartheta_0(\sigma) \end{pmatrix} \mathrm{d}\sigma,
$$

and the velocity reads

$$
\dot{\gamma}(s,t) = \begin{pmatrix} -c \\ 0 \end{pmatrix} + c\begin{pmatrix} \cos\vartheta_0((s-\sigma(t))/l) \\ \sin\vartheta_0((s-\sigma(t))/l) \end{pmatrix}.
$$

Consider the summands in the last line of (5.23) separately. Therefore, the contributions to the total moment $\tilde{M}_2(t;l) = cl^2 M_2^{(1)} + \omega_{\mathrm{rot}}(t;l)M_2^{(2)}(t;l)$ are

$$
\begin{aligned}
cl^2 M_2^{(1)} &= C_\parallel cl \int_{-1/2}^{1/2} [1-\cos\vartheta_0(\sigma)]\Big[\Big(\sigma(t)-\frac{l}{2}\Big)\sin\vartheta_0(\sigma) \\
&\qquad + l\Big[\sin\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\cos\vartheta_0(\sigma')\,\mathrm{d}\sigma' - \cos\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\sin\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\Big]\,\mathrm{d}\sigma \\
&\quad + C_\perp cl\int_{-1/2}^{1/2}\sin\vartheta_0(\sigma)\Big[\Big(\sigma(t)-\frac{l}{2}\Big)\cos\vartheta_0(\sigma) + l\Big[\sin\vartheta_0(\sigma)\cdot \\
&\qquad \cdot\int_{-1/2}^{\sigma}\sin\vartheta_0(\sigma')\,\mathrm{d}\sigma' + \cos\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\cos\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\Big]\,\mathrm{d}\sigma \\
&= 2C_\parallel cl^2\int_0^{1/2}[1-\cos\vartheta_0(\sigma)]\Big[\int_{-1/2}^{\sigma}\sin(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\,\mathrm{d}\sigma'\Big]\mathrm{d}\sigma \\
&\quad + 2C_\perp cl^2\int_0^{1/2}\sin\vartheta_0(\sigma)\Big[\int_{-1/2}^{\sigma}\cos(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\,\mathrm{d}\sigma'\Big]\mathrm{d}\sigma \\
&=: 2cl^2(C_\parallel\mathcal{I}_1 + C_\perp\mathcal{I}_2),
\end{aligned}
$$

where the odd terms have been dropped.

$$
\begin{aligned}
M_2^{(2)}(t;l) = C_\parallel l \int_{-1/2}^{1/2} &\Big[\Big(\sigma(t) - \frac{l}{2}\Big)\sin\vartheta_0(\sigma) + l\Big[\sin\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\cos\vartheta_0(\sigma')\,\mathrm{d}\sigma' \\
&\qquad - \cos\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\sin\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\Big]^2\mathrm{d}\sigma \\
+ C_\perp l \int_{-1/2}^{1/2} &\Big[\Big(\sigma(t) - \frac{l}{2}\Big)\cos\vartheta_0(\sigma) + l\Big[\sin\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\sin\vartheta_0(\sigma')\,\mathrm{d}\sigma' \\
&\qquad + \cos\vartheta_0(\sigma)\int_{-1/2}^{\sigma}\cos\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\Big]^2\mathrm{d}\sigma \\
= C_\parallel l \int_{-1/2}^{1/2} &\Big[\Big(\sigma(t) - \frac{l}{2}\Big)\sin\vartheta_0(\sigma) + l\int_{-1/2}^{\sigma}\sin(\vartheta_0(\sigma) - \vartheta_0(\sigma'))\,\mathrm{d}\sigma'\Big]^2\mathrm{d}\sigma \\
+ C_\perp l \int_{-1/2}^{1/2} &\Big[\Big(\sigma(t) - \frac{l}{2}\Big)\cos\vartheta_0(\sigma) + l\int_{-1/2}^{\sigma}\cos(\vartheta_0(\sigma) - \vartheta_0(\sigma'))\,\mathrm{d}\sigma'\Big]^2\mathrm{d}\sigma.
\end{aligned}
$$

Recall that solving the equation of motion for the rotation is equivalent to setting (5.24) equal to zero. This, from (5.24) , gives the value for $\omega_{\mathrm{rot}}(t;l)$

$$
\omega_{\mathrm{rot}}(t;l) = -4\frac{cl^2 M_2^{(1)}}{M_1(t;l) + M_2^{(2)}(t;l) + M_3(t;l)}, \tag{5.26}
$$

where the extra factor 2 takes into account the left bump. First, notice that the numerator has a sign. In fact, from the assumptions on ϑ_0, it is easy to see that $\mathcal{I}_2 > 0$, while to prove that also $\mathcal{I}_1 > 0$ we argue as follows. We divide the domain of integration of the inner integral in $\mathcal{I}_1$ in three parts $[-1/2,\sigma] = [-1/2,-\sigma]\cup[-\sigma,0]\cup[0,\sigma]$ and perform the change of variable in the first two integrals $\sigma' \to -\sigma'$; therefore we have

$$
\int_{-1/2}^{\sigma}\sin(\vartheta_0(\sigma) - \vartheta_0(\sigma'))\,\mathrm{d}\sigma' = \int_{\sigma}^{1/2}\sin(\vartheta_0(\sigma) + \vartheta_0(\sigma'))\,\mathrm{d}\sigma' + 2\int_0^{\sigma}\sin\vartheta_0(\sigma)\cos\vartheta_0(\sigma')\,\mathrm{d}\sigma',
$$

which is easily seen to be a positive number. Since $1 - \cos\vartheta_0(\sigma) > 0$ in $[0,1/2]$, we conclude that $\mathcal{I}_2 > 0$, which eventually yields $M_2^{(1)} > 0$.

It is interesting to point out the dependencies of ω on c and l. A closer look at the various functions that enter in the expression of ω allows us to write

$$
\omega_{\mathrm{rot}}(t;l) = -4cl^2\frac{M_2^{(1)}}{M_{\mathrm{rot}}^{\mathrm{d}}(t;l)},
$$

where $M_{\mathrm{rot}}^{\mathrm{d}}(t;l) := M_1(t;l) + M_2^{(2)}(t;l) + M_3(t;l)$ is a polynomial whose degree in l is three and in t is two, and whose constant term is $L^3/3$. From this we see that if $l = 0$, then no motion occurs, as it is reasonable to expect since there is no deformation. Moreover, $M_{\mathrm{rot}}^{\mathrm{d}}(t;l)$ is bounded away from zero as t varies. A straightforward calculation shows that $\frac{\mathrm{d}}{\mathrm{d}t}M_2^{(2)}(t;l) > 0$ in the interval $[\tau, T-\tau]$, therefore $M_2^{(2)}(t;l) \in [M_2^{(2)}(\tau;l), M_2^{(2)}(T-\tau;l)]$ and the left extremum is larger than zero. An analogous calculation shows that also $M_1 + M_3$ is increasing in $[\tau, T-\tau]$, therefore $M_1(t;l) + M_3(t;l) \in [M_1(\tau;l) + M_3(\tau;l), M_1(T-\tau;l) + M_3(T-\tau;l)]$. It follows that $M_{\mathrm{rot}}^{\mathrm{d}}(t;l) > M_{\mathrm{rot}}^{\mathrm{d}}(\tau;l) > (L/2-l)^3$. Also, $M_{\mathrm{rot}}^{\mathrm{d}}(t;l) <$

$M^{\mathrm{d}}_{\mathrm{rot}}(T-\tau;l)$. The monotonicity of $M^{\mathrm{d}}_{\mathrm{rot}}$ with respect to time implies that of ω_{rot} , which turns out to be decreasing in $[\tau, T-\tau]$. This is also reasonable, since, as t approaches $T-\tau$ the perturbation is closer to the center of the flagellum and therefore the moment arm is shorter, yielding a lesser moment.

5.4.4 Rotation formation transient

The contribution to the rotation due to the bumps formation transient (see Fig. 5.6) is computed via a very similar analysis as before, but now the following expressions hold. For $s \in [0, L/2-l]$, $\gamma'(s,t) = (1,0)$, $\gamma(s,t) = (s,0)$, and $\dot\gamma(s,t) = (0,0)$, while for $s \in [L/2-l, L/2]$

$$\gamma'(s,t) = \left(\cos\left(\frac{t}{\tau}\vartheta_0\left(\frac{s-\sigma(\tau)}{l}\right)\right), \sin\left(\frac{t}{\tau}\vartheta_0\left(\frac{s-\sigma(\tau)}{l}\right)\right)\right),$$

$$\gamma(s,t) = \left(\frac{L}{2}-l, 0\right) + l\int_{-1/2}^{(s-\sigma(\tau))/l}\left(\cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right), \sin\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\right)\mathrm{d}\sigma,$$

$$\dot\gamma(s,t) = \frac{l}{\tau}\int_{-1/2}^{(s-\sigma(\tau))/l}\left(-\sin\left(\frac{t}{\tau}\vartheta_0(\sigma)\right), \cos\left(\frac{t}{\tau}\vartheta_0(\sigma)\right)\right)\vartheta_0(\sigma)\,\mathrm{d}\sigma.$$

Recalling (5.23), we can write

$$\begin{aligned}
M_{\mathrm{form}}(t;l) &= \int_0^{L/2} m(s,t)\,\mathrm{d}s = \int_0^{L/2-l} m(s,t)\,\mathrm{d}s + \int_{L/2-l}^{L/2} m(s,t)\,\mathrm{d}s \\
&= \omega_{\mathrm{form}}(t;l)\int_0^{L/2-l}[C_\parallel\langle J\gamma(s,t),\gamma'(s,t)\rangle^2 + C_\perp\langle\gamma(s,t),\gamma'(s,t)\rangle]\,\mathrm{d}s \\
&\quad + \omega_{\mathrm{form}}(t;l)\int_{L/2-l}^{L/2}[C_\parallel\langle J\gamma(s,t),\gamma'(s,t)\rangle^2 + C_\perp\langle\gamma(s,t),\gamma'(s,t)\rangle]\,\mathrm{d}s \\
&\quad + \int_{L/2-l}^{L/2}[C_\parallel\langle J\gamma(s,t),\gamma'(s,t)\rangle\langle\dot\gamma(s,t),\gamma'(s,t)\rangle + \\
&\qquad\qquad C_\perp\langle\gamma(s,t),\gamma'(s,t)\rangle\langle\dot\gamma(s,t),J\gamma'(s,t)\rangle]\,\mathrm{d}s \\
&= \omega_{\mathrm{form}}(t;l)M^{(1)}_{\mathrm{form}}(l) + \omega_{\mathrm{form}}(t;l)M^{(2)}_{\mathrm{form}}(t;l) + M^{(3)}_{\mathrm{form}}(t;l).
\end{aligned} \tag{5.27}$$

Let us compute the three summands separately. Computations yield:

$$M^{(1)}_{\mathrm{form}}(l) = \frac{C_\perp}{3}\left(\frac{L}{2}-l\right)^3,$$

$$\begin{aligned}
M^{(2)}_{\mathrm{form}}(t;l) = {}& C_\parallel l\int_{-1/2}^{1/2}\left[\left(\frac{L}{2}-l\right)\sin\left(\frac{t\vartheta_0(\sigma)}{\tau}\right) + l\int_{-1/2}^{\sigma}\sin\left(\frac{t}{\tau}(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\right)\mathrm{d}\sigma'\right]^2\mathrm{d}\sigma \\
&+ C_\perp l\int_{-1/2}^{1/2}\left[\left(\frac{L}{2}-l\right)\cos\left(\frac{t\vartheta_0(\sigma)}{\tau}\right) + l\int_{-1/2}^{\sigma}\cos\left(\frac{t}{\tau}(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\right)\mathrm{d}\sigma'\right]^2\mathrm{d}\sigma,
\end{aligned}$$

$$
\begin{aligned}
M^{(3)}_{\mathrm{form}}(t;l) = {} & \frac{C_\parallel l^2}{\tau} \int_{-1/2}^{1/2} \Big[\int_{-1/2}^{\sigma} \sin\Big(\frac{t}{\tau}(\vartheta_0(\sigma) - \vartheta_0(\sigma')) \Big) \vartheta_0(\sigma')\, \mathrm{d}\sigma' \Big] \\
& \cdot \Big[\Big(\frac{L}{2} - l \Big) \sin\Big(\frac{t\vartheta_0(\sigma)}{\tau} \Big) + l \int_{-1/2}^{\sigma} \sin\Big(\frac{t}{\tau}(\vartheta_0(\sigma) - \vartheta_0(\sigma')) \Big)\, \mathrm{d}\sigma' \Big] \mathrm{d}\sigma \\
& + \frac{C_\perp l^2}{\tau} \int_{-1/2}^{1/2} \Big[\int_{-1/2}^{\sigma} \cos\Big(\frac{t}{\tau}(\vartheta_0(\sigma) - \vartheta_0(\sigma')) \Big) \vartheta_0(\sigma')\, \mathrm{d}\sigma' \Big] \\
& \cdot \Big[\Big(\frac{L}{2} - l \Big) \cos\Big(\frac{t\vartheta_0(\sigma)}{\tau} \Big) + l \int_{-1/2}^{\sigma} \cos\Big(\frac{t}{\tau}(\vartheta_0(\sigma) - \vartheta_0(\sigma')) \Big)\, \mathrm{d}\sigma' \Big] \mathrm{d}\sigma.
\end{aligned}
$$

Setting (5.27) equal to zero allows to find the expression for $\omega_{\mathrm{form}}(t;l)$

$$
\omega_{\mathrm{form}}(t;l) = -2 \frac{M^{(3)}_{\mathrm{form}}(t;l)}{M^{(1)}_{\mathrm{form}}(l) + M^{(2)}_{\mathrm{form}}(t;l)}. \tag{5.28}
$$

Notice that the numerator is of the form $a_1(t)l^3 + a_2(t)l^2$, whereas the denominator is a complete third degree polynomial in l, whose constant term differs from zero. This implies again that if $l \to 0$, then $\omega_{\mathrm{form}}(t;l) \to 0$ and no motion occurs. Moreover, $\omega_{\mathrm{form}}(t;l)$ goes to zero faster by a factor l with respect to $\omega_{\mathrm{rot}}(t;l)$, when $l \to 0$, whence the effects of the formation transient is negligible, for small perturbations.

5.4.5 Rotation destruction transient

For analyzing the contribution to the moment of the destruction transient (see Fig. 5.8), recall that the following expressions hold for $s \in [0, l]$

$$
\begin{aligned}
\gamma'(s,t) &= \Big(\cos\Big(\frac{T-t}{\tau} \vartheta_0 \Big(\frac{s - \sigma(T-\tau)}{l} \Big) \Big), \sin\Big(\frac{T-t}{\tau} \vartheta_0 \Big(\frac{s - \sigma(T-\tau)}{l} \Big) \Big) \Big), \\
\gamma(s,t) &= l \int_{-1/2}^{(s-\sigma(T-\tau))/l} \Big(\cos\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big), \sin\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big) \Big)\, \mathrm{d}\sigma, \\
\dot\gamma(s,t) &= \frac{l}{\tau} \int_{-1/2}^{(s-\sigma(T-\tau))/l} \Big(\sin\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big), -\cos\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big) \Big) \vartheta_0(\sigma)\, \mathrm{d}\sigma,
\end{aligned}
$$

while, for $s \in [l, L/2]$, $\gamma'(s,t) = (1,0)$ and, taking into account the symmetry property of ϑ_0, the other following expressions are valid

$$
\gamma(s,t) = l \int_{-1/2}^{1/2} \Big(\cos\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big), 0 \Big)\, \mathrm{d}\sigma + (s,0),
$$

$$
\dot\gamma(s,t) = \frac{l}{\tau} \int_{-1/2}^{1/2} \Big(\sin\Big(\frac{T-t}{\tau} \vartheta_0(\sigma) \Big), 0 \Big) \vartheta_0(\sigma)\, \mathrm{d}\sigma.
$$

Notice that in this second case $\dot\gamma(s,t) = \dot\gamma(l,t)$ does not depend on s. Recalling (5.23), the total moment will be

$$
\begin{aligned}
M_{\mathrm{destr}}(t;l) &= \int_0^{L/2} m(s,t)\, \mathrm{d}s = \int_0^{l} m(s,t)\, \mathrm{d}s + \int_l^{L/2} m(s,t)\, \mathrm{d}s \\
&=: M^{(1)}_{\mathrm{destr}}(t;l) + \omega_{\mathrm{destr}}(t;l) M^{(2)}_{\mathrm{destr}}(t;l) + \omega_{\mathrm{destr}}(t;l) M^{(3)}_{\mathrm{destr}}(l).
\end{aligned}
$$

$$
\begin{aligned}
M^{(1)}_{\mathrm{destr}}(t;l) = &-\frac{C_{\|}l^3}{\tau}\int_{1/2}^{1/2}\Big[\int_{-1/2}^{\sigma}\sin\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\right)\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\\
&\cdot\Big[\int_{-1/2}^{\sigma}\sin\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)+\vartheta_0(\sigma'))\right)\,\mathrm{d}\sigma'\Big]\,\mathrm{d}\sigma\\
&;l-\frac{C_{\perp}l^3}{\tau}\int_{1/2}^{1/2}\Big[\int_{-1/2}^{\sigma}\cos\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)+\vartheta_0(\sigma'))\right)\vartheta_0(\sigma')\,\mathrm{d}\sigma'\Big]\\
&\cdot\Big[\int_{-1/2}^{\sigma}\cos\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\right)\,\mathrm{d}\sigma'\Big]\,\mathrm{d}\sigma\\
M^{(2)}_{\mathrm{destr}}(t;l) = &\,C_{\|}l^3\int_{1/2}^{1/2}\Big[\int_{-1/2}^{\sigma}\sin\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)+\vartheta_0(\sigma'))\right)\,\mathrm{d}\sigma'\Big]^2\,\mathrm{d}\sigma\\
&+C_{\perp}l^3\int_{1/2}^{1/2}\Big[\int_{-1/2}^{\sigma}\cos\left(\frac{T-t}{\tau}(\vartheta_0(\sigma)-\vartheta_0(\sigma'))\right)\,\mathrm{d}\sigma'\Big]^2\,\mathrm{d}\sigma
\end{aligned}
$$

The last contribution is easily calculated to be

$$
M^{(3)}_{\mathrm{destr}}(l) = \frac{C_{\perp}}{3}\left[\left(\frac{L}{2}+lC_{\vartheta_0}(T-\tau)\right)^3 - l^3\left(1+C_{\vartheta_0}(T-\tau)\right)^3\right],
$$

where $C_{\vartheta_0}(T-\tau)$ is defined in (5.25). Therefore, we find that

$$
\omega_{\mathrm{destr}}(t;l) = -2\frac{M^{(1)}_{\mathrm{destr}}(t;l)}{M^{(2)}_{\mathrm{destr}}(t;l)+M^{(3)}_{\mathrm{destr}}(l)}. \tag{5.29}
$$

The same qualitative comments we have done for ω_{form} on the dependencies on l can be done for ω_{destr} as well.

Taking into account (5.28), (5.26), and (5.29), once l has been fixed, the total angle spanned by this rotation process will therefore be

$$
\Delta\Theta(l) = \int_0^{\tau}\omega_{\mathrm{form}}(t;l)\,\mathrm{d}t + \int_{\tau}^{T-\tau}\omega_{\mathrm{rot}}(t;l)\,\mathrm{d}t + \int_{T-\tau}^{T}\omega_{\mathrm{destr}}(t;l)\,\mathrm{d}t,
$$

As we argued for the translation case, we notice that $\Delta\Theta$ is a continuous function of l, which takes value zero for $l=0$. Therefore, if a certain angle $\Delta\bar{\Theta}$ is assigned to span, there exists an integer $k \geqslant 0$ such that $\Delta\bar{\Theta} = k\Delta\Theta(l)+\delta\Theta$, where $\delta\Theta < \Delta\Theta(l)$. Invoking the continuity of $\Delta\Theta$ with respect to l, there exists a value l^* such that $\delta\Theta = \Delta\Theta(l^*)$. Thus, it is enough to divide the time interval $[0,T]$ in $k+1$ subintervals, k of which of size $\Delta\Theta(l)/\Delta\bar{\Theta}$, and the last one of size $\delta\Theta/\Delta\bar{\Theta}$, and perform the swimming motion accordingly.

5.5 Euler equation

In this section we tackle the problem of studying the Euler equation associated with the constrained minimum problem (5.13). The constraints that we will give on the initial and final states are on the barycenter and the orientation of the head-tail segment. The barycenter of the flagellum at time t is given by

$$g(t) := \frac{1}{\int_0^L |\chi'(s,t)|\,\mathrm{d}s} \int_0^L \chi(s,t)\,|\chi'(s,t)|\,\mathrm{d}s, \tag{5.30}$$

where we have introduced the length element $|\chi'(s,t)| = |\xi'(s,t)|$ since it gives a non-trivial contribution to the integral whenever s is not the arc-length parameter of the curve. We have to take this into account since, in order to obtain the Euler equation, we will consider small perturbations of the shape function ξ which might not be parameterized by arc-length. We underline the fact that now $x(t)$ can be still considered as the barycenter, provided that a refined version of (5.5) holds, namely

$$\int_0^L \xi(s,t)\,|\xi'(s,t)|\,\mathrm{d}s = 0, \quad \forall\, t \in [0,T].$$

Moreover, we notice that, given an arbitrary $\xi \in \Xi$ parameterized by arc-length, it is always possible to recover a state function $\chi(s,t) = x(t) + R(t)\xi(s,t)$ for which $x(t)$ is the position of the barycenter and such that the line joining the head and the tail is the x-axis. Indeed, starting from ξ, one can construct its barycenter,

$$\hat{x}(t) := \frac{1}{L}\int_0^L \xi(s,t)\,\mathrm{d}s,$$

and a rotation $\hat{R}$ such that the line from the tail of the curve to its head is directed like the horizontal axis. The rows of the matrix $\hat{R}$ are given by $\hat{\varrho}, J\hat{\varrho}$, where

$$\hat{\varrho}(t) = \frac{\xi(L,t)-\xi(0,t)}{|\xi(L,t)-\xi(0,t)|}.$$

Now, the vector

$$\tilde{\xi}(s,t) := \hat{R}(t)(\xi(s,t)-\hat{x}(t))$$

represents the curve for which the barycenter is the origin and the tail and head lie on the horizontal axis. Notice that $|\tilde{\xi}'(s,t)| \equiv 1$ again. The procedure we have described before allows us to construct the curve

$$\begin{aligned}\chi(s,t) &:= \tilde{x}(t) + \tilde{R}(t)\tilde{\xi}(s,t)\\ &= \underbrace{\tilde{x}(t) - \tilde{R}(t)\hat{R}(t)\hat{x}(t)}_{x(t)} + \underbrace{\tilde{R}(t)\hat{R}(t)}_{R(t)}\,\xi(s,t),\end{aligned}$$

referred to the laboratory frame, such that the viscous forces and torques due to its velocity according by the resistive force theory vanish. Once again, notice that $|\chi'(s,t)| = |\xi'(s,t)| \equiv 1$. The vector x and the matrix R are uniquely determined once $\hat{x}, \hat{R}, \tilde{x}, \tilde{R}$ are known. We recall that $\tilde{x}, \tilde{R}$ are the solutions to the system of ODE's

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{R}(t)\tilde{v}(t), \\ \dot{\tilde{\theta}}(t) = \tilde{\omega}(t), \end{cases} \tag{5.31}$$

where $\tilde{\theta}$ is the angle associated with the rotation $\tilde{R}$ and the notation $\tilde{v}, \tilde{\omega}$ stresses that those elements are found starting from $\tilde{\xi}$. Theorem 5.2.1 guarantees that the initial value problem for (5.31) is well posed and has a unique solution which depends continuously from the initial data.

We denote by $\varphi(t)$ the angle that the head-tail line of χ makes with the positive horizontal axis, that is the one associated with the rotation matrix whose lines are given by ϱ and $J\varrho$, where

$$\varrho(t) = \frac{\chi(L,t) - \chi(0,t)}{|\chi(L,t) - \chi(0,t)|}.$$

Taking into account the comments on the arc-length parametrization, we rewrite the energy in equation (5.12) as

$$\mathcal{P}(\chi) = \int_0^T \int_0^L \langle K_\chi(s,t)\dot{\chi}(s,t), \dot{\chi}(s,t)\rangle \, |\chi'(s,t)| \, \mathrm{d}s\mathrm{d}t;$$

a simple computation shows that it is possible to consider $\mathcal{P}$ as a function of the shape variable ξ

$$\mathcal{P}(\xi) = \int_0^T \int_0^L \langle |\xi'(s,t)| \, K_\xi(s,t)\dot{\chi}_*(s,t), \dot{\chi}_*(s,t)\rangle \, \mathrm{d}s\mathrm{d}t,$$

where $\dot{\chi}_*(s,t) = R^\top(t)\dot{\chi}(s,t) = v(t) + \omega(t)J\xi(s,t) + \dot{\xi}(s,t)$.

Define now the map $\mathcal{G} : \Xi \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ which associates to every curve $\xi \in \Xi$ the position of its barycenter and its orientation at times $t = 0$ and $t = T$,

$$\mathcal{G}(\xi) = (g(0), g(T), \varphi(0), \varphi(T)).$$

We want to find first order conditions on ξ for it to be optimal among all the swimming strategies that minimize the energy $\mathcal{P}$ under the given constraint $(g(0), g(T), \varphi(0), \varphi(T)) = (g_0, g_T, \varphi_0, \varphi_T)$. For this, an equation involving the Lagrange multipliers will be written, namely,

$$\nabla_\xi \mathcal{P}(\xi) + \Lambda \cdot \nabla_\xi \mathcal{G}(\xi) = \mathrm{d}\mathcal{P}(\xi)[\eta] + \Lambda \cdot \mathrm{d}\mathcal{G}(\xi)[\eta] = 0, \quad \forall \eta \in \Xi, \tag{5.32}$$

in order to study which it is necessary to describe the behavior of $\mathcal{P}$ and $\mathcal{G}$ under variations of the form $\xi^\varepsilon(s,t) := \xi(s,t) + \varepsilon\eta(s,t)$, where η is the variation. For this, we need to compute the Fréchet derivatives of $\mathcal{P}$ and $\mathcal{G}$ in the direction η. It turns out, using the symmetry of K, that

$$\begin{aligned} \mathrm{d}\mathcal{P}(\xi)[\eta] &= \int_0^T \int_0^L [\langle H(s,t)\dot{\chi}_*(s,t), \dot{\chi}_*(s,t)\rangle + 2\langle \dot{X}_*(s,t), K(s,t)\dot{\chi}_*(s,t)\rangle] \, \mathrm{d}s\mathrm{d}t \\ &=: I^{(1)} + 2I^{(2)}. \end{aligned} \tag{5.33}$$

where

$$
\begin{aligned}
H(s,t) &:= \frac{\partial K^\varepsilon(s,t)\,|\xi^{\varepsilon\prime}(s,t)|}{\partial\varepsilon}\Big|_{\varepsilon=0} \\
&= -\langle\xi'(s,t),\eta'(s,t)\rangle K(s,t) + C_\parallel[\eta'(s,t)\otimes\xi'(s,t)+\xi'(s,t)\otimes\eta'(s,t)] \\
&\quad +C_\perp[(J\eta'(s,t))\otimes(J\xi'(s,t))+(J\xi'(s,t))\otimes(J\eta'(s,t))], \\
\dot\chi_*(s,t) &:= R^\top(t)\dot\chi(s,t) = v(t)+\omega(t)J\xi(s,t)+\dot\xi(s,t), \\
\dot X_*(s,t) &:= \frac{\partial\dot\chi_*^\varepsilon(s,t)}{\partial\varepsilon}\Big|_{\varepsilon=0} = z(t)+\psi(t)J\xi(s,t)+\omega(t)J\eta(s,t)+\dot\eta(s,t), \\
z(t) &:= \frac{\partial v^\varepsilon(t)}{\partial\varepsilon}\Big|_{\varepsilon=0}, \qquad \psi(t) := \frac{\partial\omega^\varepsilon(t)}{\partial\varepsilon}\Big|_{\varepsilon=0} \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}
\tag{5.34}
$$

Notice that $H(s,t)$ is a symmetric matrix. Since it will appear also in the following computations, it is useful to have a simplified expression for the terms like the first one in (5.33), namely $\langle Hv_1, v_2\rangle$. We have

$$
\begin{aligned}
\langle Hv_1,v_2\rangle =& -\langle\langle\xi',\eta'\rangle Kv_1,v_2\rangle + C_\parallel[\langle(\eta'\otimes\xi')v_1,v_2\rangle+\langle(\xi'\otimes\eta')v_1,v_2\rangle] \\
&+C_\perp[\langle(J\eta')\otimes(J\xi')v_1,v_2\rangle+\langle(J\xi')\otimes(J\eta')v_1,v_2\rangle] \\
=& -\langle\langle Kv_1,v_2\rangle\xi',\eta'\rangle + C_\parallel\langle(v_1\otimes v_2+v_2\otimes v_1)\xi',\eta'\rangle \\
&+C_\perp\langle((J^\top v_1)\otimes(J^\top v_2)+(J^\top v_2)\otimes(J^\top v_1))\xi',\eta'\rangle
\end{aligned}
\tag{5.35}
$$

Now, using (5.35) with $v_1 = v_2 = \dot\chi_*(s,t)$ yields

$$
\begin{aligned}
I^{(1)} =& \int_0^T\int_0^L \{-\langle\langle K(s,t)\dot\chi_*(s,t),\dot\chi_*(s,t)\rangle\xi'(s,t),\eta'(s,t)\rangle \\
&+2C_\parallel\langle(\dot\chi_*(s,t)\otimes\dot\chi_*(s,t))\xi'(s,t),\eta'(s,t)\rangle \\
&+2C_\perp\langle((J^\top\dot\chi_*(s,t))\otimes(J^\top\dot\chi_*(s,t))\xi'(s,t),\eta'(s,t)\rangle\}\,\mathrm{d}s\mathrm{d}t \\
=& \int_0^T\int_0^L \langle w_1^{(1)}(s,t),\eta'(s,t)\rangle\,\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{aligned}
w_1^{(1)}(s,t) =& [-\langle K(s,t)\dot\chi_*(s,t),\dot\chi_*(s,t)\rangle I + 2C_\parallel(\dot\chi_*(s,t)\otimes\dot\chi_*(s,t)) \\
&+2C_\perp((J^\top\dot\chi_*(s,t))\otimes(J^\top\dot\chi_*(s,t))]\xi'(s,t).
\end{aligned}
\tag{5.36}
$$

To compute

$$
\begin{aligned}
I^{(2)} &= \int_0^T \int_0^L \langle z(t) + \psi(t) J\xi(s,t) + \omega(t) J\eta(s,t) + \dot{\eta}(s,t), K(s,t)\dot{\chi}_*(s,t)\rangle \,\mathrm{d}s\mathrm{d}t \\
&= \int_0^T \int_0^L \langle K(s,t)\dot{\chi}_*(s,t), z(t)\rangle \,\mathrm{d}s\mathrm{d}t + \int_0^T \int_0^L \langle \xi(s,t), J^\top K(s,t)\dot{\chi}_*(s,t)\rangle \psi(t) \,\mathrm{d}s\mathrm{d}t \\
&+ \int_0^T \int_0^L \langle \omega(t) J^\top K(s,t)\dot{\chi}_*(s,t), \eta(s,t)\rangle \,\mathrm{d}s\mathrm{d}t + \int_0^T \int_0^L \langle K(s,t)\dot{\chi}_*(s,t), \dot{\eta}(s,t)\rangle \,\mathrm{d}s\mathrm{d}t \\
&= \int_0^T \langle \kappa_1(t), z(t)\rangle \,\mathrm{d}t + \int_0^T \kappa_2(t)\psi(t)\,\mathrm{d}t + \int_0^T \int_0^L \langle w_0^{(2,3)}(s,t), \eta(s,t)\rangle \,\mathrm{d}s\mathrm{d}t \\
&+ \int_0^T \int_0^L \langle w_2^{(2,4)}(s,t), \dot{\eta}(s,t)\rangle \,\mathrm{d}s\mathrm{d}t \\
&= I^{(2,1)} + I^{(2,2)} + I^{(2,3)} + I^{(2,4)},
\end{aligned}
$$

we need to evaluate $z(t)$ and $\psi(t)$, which is a cumbersome task. Here,

$$
\kappa_1(t) := \int_0^L K(s,t)\dot{\chi}_*(s,t)\,\mathrm{d}s, \qquad \kappa_2(t) := \int_0^L \langle \xi(s,t), J^\top K(s,t)\dot{\chi}_*(s,t)\rangle \,\mathrm{d}s,
$$

$$
w_0^{(2,3)}(s,t) := \omega(t) J^\top K(s,t)\dot{\chi}_*(s,t), \qquad w_2^{(2,4)}(s,t) := K(s,t)\dot{\chi}_*(s,t). \tag{5.37}
$$

To our purpose, the integrals $I^{(2,3)}$ and $I^{(2,4)}$ do not need to be modified, since they already show the explicit coefficients of η and $\dot{\eta}$. Recalling the definition of v and ω given in (5.9) and the formula for the derivative of the inverse matrix, we have

$$
\begin{aligned}
z(t) &= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{R}^{-1}(t) \left[\mathcal{S}(t)\mathcal{R}^{-1}(t) \begin{pmatrix} F^{\mathrm{sh}}(t) \\ M^{\mathrm{sh}}(t) \end{pmatrix} - \begin{pmatrix} D_F(t) \\ D_M(t) \end{pmatrix} \right] \\
&= \bar{A}(t)[\alpha(t)\Gamma(t) + \beta(t)\delta(t) - D_F(t)] + \bar{b}(t)[\beta^\top(t)\Gamma(t) + \gamma(t)\delta(t) - D_M(t)] \\
&= z_1(t) + z_2(t) + z_3(t) + z_4(t) + z_5(t) + z_6(t),
\end{aligned} \tag{5.38a}
$$

$$
\begin{aligned}
\psi(t) &= \begin{bmatrix} 0_2 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{R}^{-1}(t) \left[\mathcal{S}(t)\mathcal{R}^{-1}(t) \begin{pmatrix} F^{\mathrm{sh}}(t) \\ M^{\mathrm{sh}}(t) \end{pmatrix} - \begin{pmatrix} D_F(t) \\ D_M(t) \end{pmatrix} \right] \\
&= \bar{b}^\top(t)[\alpha(t)\Gamma(t) + \beta(t)\delta(t) - D_F(t)] + \bar{c}(t)[\beta^\top(t)\Gamma(t) + \gamma(t)\delta(t) - D_M(t)], \\
&= \psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t) + \psi_5(t) + \psi_6(t),
\end{aligned} \tag{5.38b}
$$

where

$$
\mathcal{R}^{-1}(t) = \begin{bmatrix} \bar{A}(t) & \bar{b}(t) \\ \bar{b}^\top(t) & \bar{c}(t) \end{bmatrix}, \qquad \mathcal{S}(t) := \begin{bmatrix} \alpha(t) & \beta(t) \\ \beta^\top(t) & \gamma(t) \end{bmatrix} = \frac{\partial \mathcal{R}^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0},
$$

$$
D_F(t) := \frac{\partial (F^{\mathrm{sh}})^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}, \qquad D_M(t) := \frac{\partial (M^{\mathrm{sh}})^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0},
$$

$$
\Gamma(t) := \bar{A}(t)F^{\mathrm{sh}}(t) + \bar{b}(t)M^{\mathrm{sh}}(t) \in \mathbb{R}^2, \qquad \delta(t) := \bar{b}^\top(t)F^{\mathrm{sh}}(t) + \bar{c}(t)M^{\mathrm{sh}}(t) \in \mathbb{R}.
$$

We must now compute all the Fréchet derivatives involved in these expressions. We have

$$\alpha(t) = \int_0^L H(\bar{s},t)\,\mathrm{d}\bar{s} \tag{5.39a}$$

$$\beta(t) = \int_0^L [H(\bar{s},t)J\xi(\bar{s},t) + K(\bar{s},t)J\eta(\bar{s},t)]\,\mathrm{d}\bar{s} \tag{5.39b}$$

$$\beta^\top(t) = \int_0^L [(J\xi(\bar{s},t))^\top H(\bar{s},t) + (J\eta(\bar{s},t))^\top K(\bar{s},t)]\,\mathrm{d}\bar{s} \tag{5.39c}$$

$$\gamma(t) = \int_0^L [2(J\xi(\bar{s},t))^\top K(\bar{s},t)J\eta(\bar{s},t) + (J\xi(\bar{s},t))^\top H(\bar{s},t)J\xi(\bar{s},t)]\,\mathrm{d}\bar{s} \tag{5.39d}$$

$$D_F(t) = \int_0^L [H(\bar{s},t)\dot{\xi}(\bar{s},t) + K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s} \tag{5.39e}$$

$$\begin{aligned} D_M(t) = \int_0^L &[\dot{\xi}(\bar{s},t)^\top K(\bar{s},t)(J\eta(\bar{s},t)) + (J\xi(\bar{s},t))^\top H(\bar{s},t)\dot{\xi}(\bar{s},t) \\ &+(J\xi(\bar{s},t))^\top K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s} \end{aligned} \tag{5.39f}$$

Taking (5.39) into account, the summands in (5.38a) can be written as

$$\begin{aligned}
z_1(t) &= \int_0^L \bar{A}(t)H(\bar{s},t)\Gamma(t)\,\mathrm{d}\bar{s}, \\
z_2(t) &= \int_0^L [\delta(t)\bar{A}(t)H(\bar{s},t)J\xi(\bar{s},t) + \delta(t)\bar{A}(t)K(\bar{s},t)J\eta(\bar{s},t)]\,\mathrm{d}\bar{s}, \\
z_3(t) &= -\int_0^L [\bar{A}(t)H(\bar{s},t)\dot{\xi}(\bar{s},t) + \bar{A}(\bar{s},t)K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s}, \\
z_4(t) &= \int_0^L [\bar{b}(t)(J\xi(\bar{s},t))^\top H(\bar{s},t)\Gamma(t) + \bar{b}(t)(J\eta(\bar{s},t))^\top K(\bar{s},t)\Gamma(t)]\,\mathrm{d}\bar{s}, \\
z_5(t) &= \int_0^L [2\delta(t)\bar{b}(t)(J\xi(\bar{s},t))^\top K(\bar{s},t)J\eta(\bar{s},t) \\
&\quad +\delta(t)\bar{b}(t)(J\xi(\bar{s},t))^\top H(\bar{s},t)J\xi(\bar{s},t)]\,\mathrm{d}\bar{s}, \\
z_6(t) &= -\int_0^L [\bar{b}(\bar{s},t)\dot{\xi}(\bar{s},t)^\top K(\bar{s},t)(J\eta(\bar{s},t)) + \bar{b}(\bar{s},t)(J\xi(\bar{s},t))^\top H(\bar{s},t)\dot{\xi}(\bar{s},t) \\
&\quad +\bar{b}(\bar{s},t)(J\xi(\bar{s},t))^\top K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s}.
\end{aligned}$$

Analogously, the summands in (5.38b) can be expressed in the form

$$
\begin{aligned}
\psi_1(t) &= \int_0^L \bar{b}^\top(t) H(\bar{s},t)\Gamma(t)\,\mathrm{d}\bar{s},\\
\psi_2(t) &= \int_0^L [\delta(t)\bar{b}^\top(t)H(\bar{s},t)J\xi(\bar{s},t) + \delta(t)\bar{b}^\top(t)K(\bar{s},t)J\eta(\bar{s},t)]\,\mathrm{d}\bar{s},\\
\psi_3(t) &= -\int_0^L [\bar{b}^\top(t)H(\bar{s},t)\dot{\xi}(\bar{s},t) + \bar{b}^\top(t)K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s},\\
\psi_4(t) &= \int_0^L [\bar{c}(t)(J\xi(\bar{s},t))^\top H(\bar{s},t)\Gamma(t) + \bar{c}(t)(J\eta(\bar{s},t))^\top K(\bar{s},t)\Gamma(t)]\,\mathrm{d}\bar{s},\\
\psi_5(t) &= \int_0^L [2\delta(t)\bar{c}(t)(J\xi(\bar{s},t))^\top K(\bar{s},t)J\eta(\bar{s},t)\\
&\qquad + \delta(t)\bar{c}(t)(J\xi(\bar{s},t))^\top H(\bar{s},t)J\xi(\bar{s},t)]\,\mathrm{d}\bar{s},\\
\psi_6(t) &= -\int_0^L [\bar{c}(t)\dot{\xi}^\top(\bar{s},t)K(\bar{s},t)J\eta(\bar{s},t) + \bar{c}(t)(J\xi(\bar{s},t))^\top H(\bar{s},t)\dot{\xi}(\bar{s},t)\\
&\qquad + \bar{c}(t)(J\xi(\bar{s},t))^\top K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s}.
\end{aligned}
$$

These terms can be reorganized, recalling that $v_1^\top M v_2 = \langle v_1, M v_2\rangle$, that $v_1\langle v_2, v_3\rangle = (v_1 \otimes v_2)v_3$, and invoking the symmetry of K and H, to obtain

$$
\begin{aligned}
z(t) &= z_1(t) + z_2(t) + z_3(t) + z_4(t) + z_5(t) + z_6(t)\\
&= \int_0^L [(\bar{A}(t) + \bar{b}(t)\otimes(J\xi(s,t)))H(\bar{s},t)(\Gamma(t) + \delta(t)J\xi(\bar{s},t) - \dot{\xi}(\bar{s},t))\\
&\quad + (\delta(t)\bar{A}(t) + (\bar{b}(t)\otimes(\Gamma(t) + 2\delta(t)J\xi(\bar{s},t) - \dot{\xi}(\bar{s},t))))K(\bar{s},t)J\eta(\bar{s},t)\\
&\quad - (\bar{A}(t) + \bar{b}(t)\otimes(J\xi(s,t)))K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s}\\
&= \int_0^L [Z_1(\bar{s},t)H(\bar{s},t)\zeta_1(\bar{s},t) + Z_2(\bar{s},t)K(\bar{s},t)J\eta(\bar{s},t) - Z_1(\bar{s},t)K(\bar{s},t)\dot{\eta}(\bar{s},t)]\,\mathrm{d}\bar{s},
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\psi(t) &= \psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t) + \psi_5(t) + \psi_6(t)\\
&= \int_0^L [\langle \bar{b}^\top(t) + \bar{c}(t)J\xi(\bar{s},t), H(\bar{s},t)(\Gamma(t) + \delta(t)J\xi(\bar{s},t) - \dot{\xi}(\bar{s},t))\rangle\\
&\quad + \langle \delta(t)\bar{b}^\top(t) + \bar{c}(t)\Gamma(t) + 2\delta(t)\bar{c}(t)J\xi(\bar{s},t) - \bar{c}(t)\dot{\xi}(\bar{s},t), K(\bar{s},t)J\eta(\bar{s},t)\rangle\\
&\quad - \langle \bar{b}^\top(t) + \bar{c}(t)J\xi(\bar{s},t), K(\bar{s},t)\dot{\eta}(\bar{s},t)\rangle]\,\mathrm{d}\bar{s}\\
&= \int_0^L [\langle H(\bar{s},t)\zeta_2(\bar{s},t), \zeta_1(\bar{s},t)\rangle + \langle J^\top K(\bar{s},t)\zeta_3(\bar{s},t), \eta(\bar{s},t)\rangle\\
&\quad - \langle K(\bar{s},t)\zeta_2(\bar{s},t), \dot{\eta}(\bar{s},t)\rangle]\,\mathrm{d}\bar{s},
\end{aligned}
\tag{5.41}
$$

where the following positions are made

$$Z_1(\bar{s},t) := \bar{A}(t) + \bar{b}(t)\otimes J\xi(\bar{s},t), \tag{5.42a}$$

$$Z_2(\bar{s},t) := \delta(t)\bar{A}(t) + \bar{b}(t)\otimes(\Gamma(t) + 2\delta(t)J\xi(\bar{s},t) - \dot{\xi}(\bar{s},t)), \tag{5.42b}$$

$$\zeta_1(\bar{s},t) := \Gamma(t) + \delta(t)J\xi(\bar{s},t) - \dot{\xi}(\bar{s},t), \tag{5.42c}$$

$$\zeta_2(\bar{s},t) := \bar{b}^\top(t) + \bar{c}(t)J\xi(\bar{s},t), \tag{5.42d}$$

$$\zeta_3(\bar{s},t) := \delta(t)\bar{b}^\top(t) + \bar{c}(t)\Gamma(t) + 2\delta(t)\bar{c}(t)J\xi(\bar{s},t) - \bar{c}(t)\dot{\xi}(\bar{s},t), \tag{5.42e}$$

Now, from (5.40), taking into account (5.35) we get

$$
\begin{aligned}
I^{(2,1)} &= \int_0^T \langle \kappa_1(t), z(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \int_0^L [\langle Z_1^\top(\bar{s},t)\kappa_1(t), H(\bar{s},t)\zeta_1(\bar{s},t)\rangle + \langle J^\top K(\bar{s},t) Z_2^\top(\bar{s},t)\kappa_1(t), \eta(\bar{s},t)\rangle \\
&\quad - \langle K(\bar{s},t) Z_1^\top(\bar{s},t)\kappa_1(t), \dot{\eta}(\bar{s},t)\rangle] \, \mathrm{d}\bar{s}\mathrm{d}t \\
&= \int_0^T \int_0^L [\langle w_0^{(2,1)}(\bar{s},t), \eta(\bar{s},t)\rangle + \langle w_1^{(2,1)}(\bar{s},t), \eta'(\bar{s},t)\rangle + \langle w_2^{(2,1)}(\bar{s},t), \dot{\eta}(\bar{s},t)\rangle] \, \mathrm{d}\bar{s}\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
I^{(2,2)} &= \int_0^T \kappa_2(t)\psi(t) \, \mathrm{d}t \\
&= \int_0^T \int_0^L \kappa_2(t) [\langle H(\bar{s},t)\zeta_2(\bar{s},t), \zeta_1(\bar{s},t)\rangle + \langle J^\top K(\bar{s},t)\zeta_3(\bar{s},t), \eta(\bar{s},t)\rangle \\
&\quad - \langle K(\bar{s},t)\zeta_2(\bar{s},t), \dot{\eta}(\bar{s},t)\rangle] \, \mathrm{d}\bar{s}\mathrm{d}t \\
&= \int_0^T \int_0^L \kappa_2(t) [\langle w_0^{(2,2)}(\bar{s},t), \eta(\bar{s},t)\rangle + \langle w_1^{(2,2)}(\bar{s},t), \eta'(\bar{s},t)\rangle \\
&\quad + \langle w_2^{(2,2)}(\bar{s},t), \dot{\eta}(\bar{s},t)\rangle] \, \mathrm{d}\bar{s}\mathrm{d}t,
\end{aligned}
$$

where

$$
\begin{aligned}
w_0^{(2,1)}(\bar{s},t) &:= J^\top K(\bar{s},t) Z_2^\top(\bar{s},t)\kappa_1(t), && (5.43\text{a}) \\
w_1^{(2,1)}(\bar{s},t) &:= [-\langle K(\bar{s},t)\zeta_1(\bar{s},t), Z_1^\top(\bar{s},t)\kappa_1(t)\rangle I + C_\|\zeta_1(\bar{s},t) \otimes (Z_1^\top(\bar{s},t)\kappa_1(t)) && (5.43\text{b}) \\
&\quad + C_\|(Z_1^\top(\bar{s},t)\kappa_1(t)) \otimes \zeta_1(\bar{s},t) + C_\perp(J^\top\zeta_1(\bar{s},t)) \otimes (J^\top Z_1^\top(\bar{s},t)\kappa_1(t)) \\
&\quad + C_\perp(J^\top Z_1^\top(\bar{s},t)\kappa_1(t)) \otimes (J^\top\zeta_1(\bar{s},t))]\xi'(\bar{s},t), \\
w_2^{(2,1)}(\bar{s},t) &:= -K(\bar{s},t) Z_1^\top(\bar{s},t)\kappa_1(t) && (5.43\text{c}) \\
w_0^{(2,2)}(\bar{s},t) &:= J^\top K(\bar{s},t)\zeta_3(\bar{s},t), && (5.43\text{d}) \\
w_1^{(2,2)}(\bar{s},t) &:= [-\langle K(\bar{s},t)\zeta_1(\bar{s},t), \zeta_2(\bar{s},t)\rangle I + C_\|\zeta_1(\bar{s},t) \otimes \zeta_2(\bar{s},t) && (5.43\text{e}) \\
&\quad + C_\|\zeta_2(\bar{s},t) \otimes \zeta_1(\bar{s},t) + C_\perp(J^\top\zeta_1(\bar{s},t)) \otimes (J^\top\zeta_2(\bar{s},t)) \\
&\quad + C_\perp(J^\top\zeta_2(\bar{s},t)) \otimes (J^\top\zeta_1(\bar{s},t))]\xi'(\bar{s},t) \\
w_2^{(2,2)}(\bar{s},t) &:= -K(\bar{s},t)\zeta_2(\bar{s},t) && (5.43\text{f})
\end{aligned}
$$

From these expressions, it turns out that formula (5.33) can be written as

$$
\mathrm{d}\mathcal{P}(\xi)[\eta] = \int_0^T \int_0^L [\langle w_0(s,t), \eta(s,t)\rangle + \langle w_1(s,t), \eta'(s,t)\rangle + \langle w_2(s,t), \dot{\eta}(s,t)\rangle] \, \mathrm{d}s\mathrm{d}t, \tag{5.44}
$$

where

$$
\begin{aligned}
w_0(s,t) &:= 2[w_0^{(2,1)}(s,t) + \kappa_2(t) w_0^{(2,2)}(s,t) + w_0^{(2,3)}(s,t)], \\
w_1(s,t) &:= w_1^{(1)}(s,t) + 2[w_1^{(2,1)}(s,t) + \kappa_2(t) w_1^{(2,2)}(s,t)], \\
w_2(s,t) &:= 2[w_2^{(2,1)}(s,t) + \kappa_2(t) w_2^{(2,2)}(s,t) + w_2^{(2,4)}(s,t)],
\end{aligned}
$$

and all these terms are defined in (5.36), (5.37), and (5.43).

Let us now turn our attention on the functional $\mathcal{G}$. We have to compute

$$\mathrm{d}\mathcal{G}(\xi)[\eta] = \left(\frac{\partial g^\varepsilon(0)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial g^\varepsilon(T)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial \varphi^\varepsilon(0)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \frac{\partial \varphi^\varepsilon(T)}{\partial \varepsilon}\Big|_{\varepsilon=0} \right).$$

Recalling the definitions in (5.30), the conditions on the barycenter of $\xi(\cdot,t)$ and $\eta(\cdot,t)$, and defining

$$y(t) := \frac{\partial x^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad S(t) := \frac{\partial R^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} = R(t)J\frac{\partial \theta^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} =: R(t)J\Theta(t), \tag{5.45}$$

$$\begin{cases} u_1(t) = r(t)\cos\varphi(t) \\ u_2(t) = r(t)\sin\varphi(t) \end{cases},$$

we have

$$\begin{aligned}
\frac{\partial g^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} &= -\frac{1}{L}\int_0^L g(t)\langle \xi'(s,t), \eta'(s,t)\rangle \,\mathrm{d}s + \frac{1}{L}\int_0^L \chi(s,t)\langle \xi'(s,t), \eta'(s,t)\rangle \,\mathrm{d}s \\
&\quad + \frac{1}{L}\int_0^L \Big[y(t) + S(t)\xi(s,t) + R(t)\eta(s,t) \Big] \,\mathrm{d}s \\
&= \frac{1}{L}\int_0^L [(\chi(s,t) - g(t)) \otimes \xi'(s,t)]\eta'(s,t)\,\mathrm{d}s + y(t), \\
\frac{\partial \varphi^\varepsilon(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} &= \frac{\partial \varphi^\varepsilon(t)}{\partial u_1^\varepsilon}\frac{\partial u_1^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{\partial \varphi^\varepsilon(t)}{\partial u_2^\varepsilon}\frac{\partial u_2^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} \\
&= \frac{-u_2(t)}{u_1^2(t)+u_2^2(t)}\frac{\partial u_1^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{u_1(t)}{u_1^2(t)+u_2^2(t)}\frac{\partial u_2^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} \\
&= \frac{-[R(t)(\xi(L,t)-\xi(0,t))]_2}{|\xi(L,t)-\xi(0,t)|^2}[R(t)(\Theta(t)J(\xi(L,t)-\xi(0,t)) + \eta(L,t)-\eta(0,t))]_1 \\
&\quad + \frac{[R(t)(\xi(L,t)-\xi(0,t))]_1}{|\xi(L,t)-\xi(0,t)|^2}[R(t)(\Theta(t)J(\xi(L,t)-\xi(0,t)) + \eta(L,t)-\eta(0,t))]_2 \\
&= \frac{\langle J(\xi(L,t)-\xi(0,t)), \Theta(t)J(\xi(L,t)-\xi(0,t)) + \eta(L,t)-\eta(0,t)\rangle}{|\xi(L,t)-\xi(0,t)|^2} \\
&= \frac{\langle J(\xi(L,t)-\xi(0,t)), \eta(L,t)-\eta(0,t)\rangle}{|\xi(L,t)-\xi(0,t)|^2} + \Theta(t),
\end{aligned}$$

where $u(t) = (u_1(t), u_2(t)) = \chi(L,t) - \chi(0,t) = R(t)(\xi(L,t)-\xi(0,t))$, and y and Θ satisfy the system (see (5.8) and (5.45))

$$\begin{cases} \dot{y}(t) = R(t)(\Theta(t)Jv(t) + z(t)) \\ \dot{\Theta}(t) = \psi(t) \end{cases}, \qquad y(0) = 0, \Theta(0) = 0. \tag{5.46}$$

It turns out that

$$
\begin{aligned}
\bar{g}_0 &:= \frac{\partial g^\varepsilon(0)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{1}{L}\int_0^L [(\chi(s,0)-g(0))\otimes \xi'(s,0)]\eta'(s,0)\,\mathrm{d}s,\\
\bar{g}_T &:= \frac{\partial g^\varepsilon(T)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{1}{L}\int_0^L [(\chi(s,T)-g(T))\otimes \xi'(s,T)]\eta'(s,T)\,\mathrm{d}s + y(T),\\
\bar{\varphi}_0 &:= \frac{\partial \varphi^\varepsilon(0)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\langle J(\xi(L,0)-\xi(0,0)),\eta(L,0)-\eta(0,0)\rangle}{|\xi(L,0)-\xi(0,0)|^2},\\
\bar{\varphi}_T &:= \frac{\partial \varphi^\varepsilon(T)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\langle J(\xi(L,T)-\xi(0,T)),\eta(L,T)-\eta(0,T)\rangle}{|\xi(L,T)-\xi(0,T)|^2} + \Theta(T),
\end{aligned}
$$

and the dependence on η, η', and $\dot{\eta}$ will be explicit once we find a suitable expression for $\Theta(T)$ and $y(T)$. From (5.46) and (5.41), we have

$$
\Theta(T) = \int_0^T\int_0^L [\langle w_0^{(2,2)}(s,t),\eta(s,t)\rangle + \langle w_1^{(2,2)}(s,t),\eta'(s,t)\rangle + \langle w_2^{(2,2)}(s,t),\dot{\eta}(s,t)\rangle]\,\mathrm{d}s\mathrm{d}t,
$$

where the coefficients $w_i^{(2,2)}$ are given in (5.43d), (5.43e), and (5.43f). The computation of $y(T)$ requires more effort. We have

$$
y(T) = \int_0^T \Theta(\bar{t})R(\bar{t})Jv(\bar{t})\,\mathrm{d}\bar{t} + \int_0^T R(t)z(t)\,\mathrm{d}t =: y^{(1)}(T) + y^{(2)}(T),
$$

and we compute the two contributions separately. Define

$$
V(t) := \int_0^T \mathbb{1}_{[0,\bar{t}]}(t)R(\bar{t})Jv(\bar{t})\,\mathrm{d}\bar{t}, \tag{5.47}
$$

so that we get

$$
\begin{aligned}
y^{(1)}(T) &= \int_0^T \Bigg[\int_0^T\int_0^L \mathbb{1}_{[0,\bar{t}]}(t)[\langle w_0^{(2,2)}(s,t),\eta(s,t)\rangle + \langle w_1^{(2,2)}(s,t),\eta'(s,t)\rangle\\
&\quad + \langle w_2^{(2,2)}(s,t),\dot{\eta}(s,t)\rangle]\,\mathrm{d}s\mathrm{d}t\Bigg] R(\bar{t})Jv(\bar{t})\mathrm{d}\bar{t}\\
&= \int_0^T\int_0^L [(V(t)\otimes w_0^{(2,2)}(s,t))\eta(s,t) + (V(t)\otimes w_1^{(2,2)}(s,t))\eta'(s,t)\\
&\quad + (V(t)\otimes w_2^{(2,2)}(s,t))\dot{\eta}(s,t)]\,\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
y^{(2)}(T) &= \int_0^T\int_0^L [R(t)Z_1(s,t)H(s,t)\zeta_1(s,t) + R(t)Z_2(s,t)K(s,t)J\eta(s,t)\\
&\quad - R(t)Z_1(s,t)K(s,t)\dot{\eta}(s,t)]\,\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Adding these two terms yields

$$
y(T) = \int_0^T\int_0^L [W_0^\top(s,t)\eta(s,t) + W_1^\top(s,t)\eta'(s,t) + W_2^\top(s,t)\dot{\eta}(s,t)]\,\mathrm{d}s\mathrm{d}t,
$$

where, recalling the expression for $H(s,t)$ from (5.34),

$$\begin{aligned}
W_0^\top(s,t) &= V(t)\otimes w_0^{(2,2)}(s,t) + R(t)Z_2(s,t)K(s,t)J,\\
W_1^\top(s,t) &= V(t)\otimes w_1^{(2,2)}(s,t) - R(t)Z_1(s,t)K(s,t)\zeta_1(s,t)\otimes\xi'(s,t)\\
&\quad + C_\parallel\langle\xi'(s,t),\zeta_1(s,t)\rangle R(t)Z_1(s,t) + C_\parallel R(t)Z_1(s,t)(\xi'(s,t)\otimes\zeta_1(s,t))\\
&\quad + C_\perp\langle J\xi'(s,t),\zeta_1(s,t)\rangle R(t)Z_1(s,t)J\\
&\quad + C_\perp R(t)Z_1(s,t)((J\xi'(s,t))\otimes\zeta_1(s,t))J,\\
W_2^\top(s,t) &= V(t)\otimes w_2^{(2,2)}(s,t) - R(t)Z_1(s,t)K(s,t).
\end{aligned}$$

Upon defining $\Lambda := (\lambda_0,\lambda_T,\mu_0,\mu_t)\in\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}$ and

$$G^\top(s,t) := \frac{1}{L}(\chi(s,t)-g(t))\otimes\xi'(s,t), \qquad \phi(t) := \frac{J(\xi(L,t)-\xi(0,t))}{|\xi(L,t)-\xi(0,t)|^2},$$

we can write

$$\begin{aligned}
\langle\Lambda, \mathrm{d}\mathcal{G}(\xi)[\eta]\rangle =& \langle\lambda_0,\bar g_0\rangle + \langle\lambda_T,\bar g_T\rangle + \mu_0\bar\varphi_0 + \mu_T\bar\varphi_T\\
=& \int_0^L \langle G(s,0)\lambda_0,\eta'(s,0)\rangle\,\mathrm{d}s + \int_0^L\langle G(s,T)\lambda_T,\eta'(s,T)\rangle\,\mathrm{d}s\\
&+ \mu_0\langle\phi(0),\eta(L,0)-\eta(0,0)\rangle + \mu_T\langle\phi(T),\eta(L,T)-\eta(0,T)\rangle\\
&+ \int_0^T\int_0^L [\langle W_0(s,t)\lambda_0 + \mu_T w_0^{(2,2)}(s,t),\eta(s,t)\rangle +\\
&\qquad \langle W_1(s,t)\lambda_0 + \mu_T w_1^{(2,2)}(s,t),\eta'(s,t)\rangle +\\
&\qquad \langle W_2(s,t)\lambda_0 + \mu_T w_2^{(2,2)}(s,t),\dot\eta(s,t)\rangle]\,\mathrm{d}s\mathrm{d}t
\end{aligned} \tag{5.48}$$

Finally, combining equations (5.44) and (5.48), and defining

$$h_i(s,t) := w_i(s,t) + W_i(s,t)\lambda_0 + \mu_T w_i^{(2,2)}(s,t)$$

(the convention is that h_i will be the term multiplied by the derivative of η with respect to the i-th variable), equation (5.32) becomes

$$\begin{aligned}
0 =& \int_0^T\int_0^L [\langle h_0(s,t),\eta(s,t)\rangle + \langle h_1(s,t),\eta'(s,t)\rangle + \langle h_2(s,t),\dot\eta(s,t)\rangle]\,\mathrm{d}s\mathrm{d}t\\
&+ \int_0^L\langle G(s,0)\lambda_0,\eta'(s,0)\rangle\,\mathrm{d}s + \int_0^L\langle G(s,T)\lambda_T,\eta'(s,T)\rangle\,\mathrm{d}s\\
&+ \mu_0\langle\phi(0),\eta(L,0)-\eta(0,0)\rangle + \mu_T\langle\phi(T),\eta(L,T)-\eta(0,T)\rangle,
\end{aligned} \tag{5.49}$$

for all $\eta\in\Xi$. Choosing $\eta\in\Xi$ such that $\operatorname{spt}\eta \subset\subset [0,L]\times[0,T]$, equation (5.49) gives the following condition

$$h_0 - h_1' - \dot h_2 = 0, \qquad \text{in } [0,L]\times[0,T]; \tag{5.50}$$

choosing $\eta \in \Xi$ such that $\operatorname{spt}\eta$ is compact with respect to t only and considering (5.50), we get

$$0 = \int_0^T [\langle h_1(L,t), \eta(L,t)\rangle - \langle h_1(0,t), \eta(0,t)\rangle]\,\mathrm{d}t, \quad \text{for all such } \eta\text{'s},$$

which gives

$$h_1(0,t) = h_1(L,t) = 0, \quad \text{for all } t \in [0,T]; \tag{5.51}$$

choosing $\eta \in \Xi$ such that $\operatorname{spt}\eta$ is compact with respect to s only and considering (5.50), we get

$$0 = \int_0^L [\langle h_2(s,T) - G'(s,T)\lambda_T, \eta(s,T)\rangle - \langle h_2(s,0) + G'(s,0)\lambda_0, \eta(s,0)\rangle]\,\mathrm{d}s, \quad \text{for all such } \eta,\text{'s}$$

which gives

$$h_2(s,0) + G'(s,0)\lambda_0 = h_2(s,T) - G'(s,T)\lambda_T = 0, \quad \text{for all } s \in [0,L]; \tag{5.52}$$

finally, if $\eta \in \Xi$ does not have compact support in $[0,L]\times[0,T]$, we get the contributions of the vertices of the square $[0,L]\times[0,T]$. Keeping into account (5.50),(5.51), and (5.52) we obtain

$$\begin{aligned} 0 =& \langle G(L,0)\lambda_0, \eta(L,0)\rangle - \langle G(0,0)\lambda_0, \eta(0,0)\rangle + \langle G(L,T)\lambda_T, \eta(L,T)\rangle - \langle G(0,T)\lambda_T, \eta(0,T)\rangle \\ &+ \mu_0\langle \phi(0), \eta(L,0) - \eta(0,0)\rangle + \mu_T\langle \phi(T), \eta(L,T) - \eta(0,T)\rangle. \end{aligned}$$

Thus, there must hold

$$G(0,0)\lambda_0 + \mu_0\phi(0) = G(L,0)\lambda_0 + \mu_0\phi(0) = 0, \tag{5.53a}$$

$$G(0,T)\lambda_T + \mu_T\phi(T) = G(L,T)\lambda_T + \mu_T\phi(T) = 0. \tag{5.53b}$$

These imply that

$$G(0,0) = G(L,0), \qquad G(0,T) = G(L,T). \tag{5.54}$$

Equation (5.49) is the most explicit version of (5.32) that we can give so far. The associated partial differential equation is (5.50), while (5.51), (5.52), and (5.54) are the boundary conditions.

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