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ISAS
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES
Trieste – ITALY

Thesis submitted for the degree of Doctor Philosophiæ

New perspectives for the first second: preheating of fermions and extra–dimensions

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ACADEMIC YEAR 1999-2000

Contents

Introduction

In this thesis I discuss two separate issues related to the very early Universe. The first of them is *preheating of fermions.* Creation of matter from the inflaton field is one of the most active areas of research of the inflationary scenario. In the last ten years, it has been realized that the first stages of this process are typically governed by relevant nonperturbative production of particles. Creation of bosons is very efficient, since it is characterized by stimulated particle emission into energy bands with large occupation number. It was very recently realized that preheating of fermions can be in some cases even more relevant, despite large occupation numbers are forbidden by Pauli principle. In the first part of the thesis I review the analyses (first numerical and then analytical) which have lead to this conclusion. I then discuss the possible implications for leptogenesis.

The second part of the thesis is instead devoted to *extra dimensions.* The last two and a half years have witnessed a formidable interest in models (I will simply refer to them as to "brane models") where the extra–dimensions manifest themselves not too far from the electroweak scale. The initial motivation for these studies was to provide a solution to the hierarchy problem. I discuss this issue in the first chapter of this part, where brane models are introduced. While these scenarios are compatible with the Newton law at the distances we presently probe, they can lead to a very interesting new phenomenology both in accelerator physics and in cosmology. Concerning the latter, I discuss possible modifications of the standard baryogenesis scenarios and of the cosmological evolution of the Universe.

Part I

Preheating of fermions

It is commonly believed that the Universe underwent an early era of cosmological inflation (for a recent review, see [1]). The flatness and the horizon problems of the standard big bang cosmology are indeed elegantly solved if, during the evolution of the early Universe, the energy density happens to be dominated by the vacuum energy of a scalar field ϕ – the inflaton – and physical scales grow quasi-exponentially.

At the end of inflation the Universe was in a cold, low-entropy state with few degrees of freedom, very much unlike the present hot, high-entropy Universe. The process which leads from the former to the latter is known as *reheating* and it constitutes one of the most active areas of research of inflationary cosmology.

In Guth's original model for inflation [2] (now usually referred to as "old inflation") the period of exponential expansion occurs while the scalar field ϕ is trapped in a metastable (positive energy) vacuum. Inflation ends when the field ϕ tunnels towards the absolute (zero energy) minimum of its potential and bubbles of true vacuum nucleate and expand. The energy of the false vacuum is thus converted into kinetic energy of the walls of the bubbles which expansion speeds quickly approaches the one of light. The universe is then thought to be reheated by the particle production that occurs during the subsequent phase of collision of the different bubbles which have formed (some analyses of the process can be found in refs. [3, 4, 5, 6]).

As it is well known, this old inflationary scenario suffers from the so called "graceful exit problem" [7, 8] and some modifications have been considered (for a review, see [9]). An alternative possibility is offered by the framework proposed by both Linde [10], and Albrecht and Steinhardt [11], often referred to as "new inflation", or, more appropriately, "slow–rollover inflation". In this scenario, inflation occurs while the VEV of the inflaton field ϕ slowly evolves towards the minimum of its potential. As $\langle\phi\rangle$ approaches and then overshoots the minimum it begins to oscillate about it. The vacuum energy of the field then exists in form of coherent oscillations of the ϕ field, corresponding to a condensate of zero–momentum ϕ particles. The decay to these particles to other, lighter fields to which ϕ couples damps the coherent oscillations. Finally, the produced particles thermalize and the Universe begins its radiation dominated stage.

While the first studies of this process considered only perturbative decays of the quanta of ϕ, it was later pointed out by Traschen and Brandenberger [12], and by Kofman, Linde, and Starobinsky [13], that nonperturbative processes can play a very prominent role. The condensate $\langle\phi\rangle$ cannot indeed be considered just as the sum of independent quanta, but collective effects due to the coherent oscillations give rise to particle creation through parametric resonance [12]. This phenomenon was named *preheating* in [13], since (with the exception of some very recent versions [14]) it is usually followed by a stage of (ordinary) perturbative reheating.

The first studies on preheating focused on production of bosons [13, 15, 16]. They show that in the first stage of reheating nonlinear quantum effects may lead to extremely effective dissipative dynamics and explosive particle production, even when single particle decay is kinematically forbidden. This production mainly occurs as stimulated particle emissions into energy bands with large occupancy numbers.

Less attention was initially devoted to nonperturbative production of fermions, since Pauli blocking prevents the possibility of stimulated emission. As a consequence, preheating of fermions was thought unable to be as significant as the bosonic one. However, it was first pointed out in ref. [17] that this is not the case. In that work it is shown that fermion production in an expanding Universe can be extremely efficient, in a mass range even broader than the one for heavy bosons: fermions may be generated up to masses of order of 10^{18} GeV (two or three orders of magnitude more than in the bosonic case [18, 19, 20]). What distinguishes the production of very massive fermions and bosons in an oscillating background is the expression for the total mass. For bosons, the total mass can never vanish (at least for a minimal Yukawa coupling $\phi^2\chi^2$ of the boson χ to the inflaton) and the production reaches the maximum when the amplitude of the inflaton goes through zero. For fermions, the total mass can vanish for particular values of the inflaton field, rendering particle creation much easier.

Parametric creation of spin 1/2 fermions has been the subject of several works in the past. Pure gravitational production has been examined in refs. [21, 22], while creation by an oscillating background field is instead considered in the works [23, 24, 25, 14, 17, 26, 27, 28]. References [23, 24] report results for creation in a Minkowski space. Reference [25] studies the production of massless fermions after a $\lambda\phi^4$ inflation, exploiting the fact that this case can also be reconducted to a static one. In this work, production in a static Universe after chaotic inflation is also considered, and some conjectures on the effects of the expansion are made. The full calculation of preheating of massive fermions after chaotic inflation in an expanding Universe has been performed numerically in ref. [17], and finally analytically in refs. [27, 28].

These last works have been followed by several recent studies. For example, their results turned also useful to the study of gravitinos production at preheating [29, 30, 31, 32]. This issue is particularly important, since gravitinos can easily overclose the Universe (if they are stable) or (if they decay) spoil the successful predictions of primordial nucleosynthesis through photodissociation of the light elements. Gravitinos can be thermally produced during the stage of reheating. To avoid this overproduction, the reheating temperature $T_{\rm RH}$ after inflation cannot be larger than $\sim (10^8 - 10^9)$ GeV [33, 34, 35, 36]. However, it has been realized [29, 30, 31, 32] that the non-thermal production of helicity $\pm 1/2$ gravitinos (whose equation of motion can be reconducted to the one of an ordinary spin 1/2 Dirac particle) can be easily more efficient than the thermal one, and this in general leads to more stringent upper bounds on $T_{\rm RH}$. Several papers related to the works [29, 30, 31, 32] have recently appeared [37, 38, 39, 40, 41, 42, 43].

Another important implication of preheating of fermions is constituted by leptogenesis, as the work [17] and the related papers [44, 45, 46, 47, 48, 49] show. In this scheme [50], a leptonic asymmetry is first created from the decay of right-handed neutrinos, and then partially converted to baryon asymmetry through sphaleronic interactions. Since leptogenesis is very sensitive to both the mechanism of creation of the heavy neutrinos and to the neutrino mass matrices, it could constitute an interesting link between preheating and the experiments on neutrino oscillations. In particular, perturbative production of the right-handed neutrinos is viable only if their mass is smaller than the inflaton one, that is 10^{13} GeV in chaotic inflation [51]. Creation of these particles at preheating may thus be a forced choice if their mass overcome this limit.

Other phenomenological implications of these studies appear in refs. [52, 53], with preheating as a possible mechanism for creating superheavy relic particles responsible for the ultrahigh energy cosmic rays, and in ref. [54], where the possible impact of fermions produced *during* inflation on the microwave background anisotropies and on the large-structure surveys is considered. Fermionic production can also play an interesting role in hybrid inflationary models [43, 55]. Finally, fermion preheating in presence of several scalar fields has been considered in ref. [56].

In the following we review some of these results, mainly focusing on the works [17, 28]. In the first chapter we report the numerical analysis of preheating of massive fermions in an expanding Universe. The second chapter is instead devoted to the analytical study. As we show, the two approaches give results which are in excellent agreement with each other. In the third chapter we present a detailed discussion of leptogenesis in general and leptogenesis after preheating, which constitutes, as we said, one of the most interesting applications of these above studies.

Chapter 1

Numerical approach

In this chapter we describe the basic physics underlying the mechanism of heavy fermions generation during the preheating stage and we perform the relevant numerical calculations. As in the next chapters, we focus on the model of chaotic inflation [51], with a massive inflaton ϕ having quadratic potential $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$. Here $m_\phi \sim 10^{13}$ GeV is fixed by the COBE normalization of the cosmic microwave background anisotropy.

We suppose that the inflaton field is coupled to a very massive Dirac fermion X with bare mass m_X via the Yukawa coupling

$$\mathcal{L}_Y = g\phi\bar{X}X. \tag{1.1}$$

The total mass of the fermion X is then given by

$$m(t) = m_X + g\phi(t). \tag{1.2}$$

From this coupling the fermions acquire a time varying mass. Nonperturbative production occurs when this mass (and the related frequency) varies in a non adiabatic way. In case of very massive fermions, the non-adiabaticity condition can be satisfied only when their total mass vanishes, and the production occurs at discrete intervals, until the inflaton oscillations become too small for the total fermionic mass to vanish.

In the first section we review a quite standard formalism to properly define the occupation number of created fermions. As we show, this can be related to the Bogolyubov coefficients of the transformation which diagonalizes the hamiltonian of the system at any given time. In the second section we present the results of a numerical evaluation of the occupation number. This study is performed neglecting the backreaction of the produced particles on the evolution of the inflaton field and on the scale factor. Despite the difficulty of a more complete treatment, backreaction effects can be understood at least in the Hartree approximation. We do this in the third section, where we evaluate at which coupling of the fermionic field to the inflaton the backreaction starts to be important.

1.1 The basic formalism

We start (see *e.g.* discussion in ref. [21]) by canonically quantizing the action of a massive fermionic field X in curved space with Friedmann-Robertson-Walker metric. In the system of coordinates in which the line element is given by $ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2)$, where a is the scale factor of the expanding Universe and η is the conformal time defined as $d\eta = dt/a$, the Dirac equation becomes

$$\left(\frac{i}{a}\gamma^\mu\partial_\mu + i\frac{3}{2}H\gamma^0 - m\right) X = 0. \tag{1.3}$$

Here $H = (a'/a^2)$ is the Hubble rate, the prime denotes derivative with respect to conformal time, and the γ-matrices are defined in *flat* space-time. By defining $\chi = a^{-3/2}X$, eq. (1.3) can be reduced to the more familiar form

$$(i\gamma^\mu\partial_\mu - a\,m)\,\chi = 0. \tag{1.4}$$

Since a is a function of η, but not of $\vec{x}$, spatial translations are symmetries of space-time, and we can separate the variables using the decomposition

$$\chi\,(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}} \sum_r \left[u_r(k,\eta)a_r(k) + v_r(k,\eta)b_r^\dagger(-k)\right], \tag{1.5}$$

where the summation is over spin, and $v_r(k) = C\bar{u}_r^T(-k)$. We impose the canonical anticommutation relations on the creation and annihilation operators

$$\left\{a_r(k), a_s^\dagger(k')\right\} = \left\{b_r(k), b_s^\dagger(k')\right\} = \delta_{rs}\delta(\vec{k} - \vec{k}') \tag{1.6}$$

which, together with the quantization conditions, determine the normalization of the spinors u,

$$u_r^\dagger(k,\eta)u_s(k,\eta) = v_r^\dagger(k,\eta)v_s(k,\eta) = \delta_{rs}, \qquad u_r^\dagger(k,\eta)v_s(k,\eta) = 0. \tag{1.7}$$

Equations (1.7) are valid at any conformal time, since they are preserved by the evolution.

In the representation in which $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and with the definition $u \equiv \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$, the equation of motion (1.4) can be written as

$$\frac{d}{d\eta}u_\pm = i\,k\,u_\mp \mp i\,a\,m\,u_\pm\;, \tag{1.8}$$

or, equivalently, as a set of uncoupled second-order differential equations,

$$\left[\frac{d^2}{d\eta^2} + \omega^2 \pm i(a'm + am')\right] u_\pm(k) = 0 \tag{1.9}$$

$$\omega^2 = k^2 + m^2 a^2. \tag{1.10}$$

We can now write the Hamiltonian as

$$\begin{aligned} H(\eta) = \int d^3x \chi^\dagger (-i\partial_0) \chi &= \int d^3k \sum_r \{E_k(\eta) \left[a_r^\dagger(k) a_r(k) - b_r(k) b_r^\dagger(k)\right] \\ &+ F_k(\eta)\, b_r(-k) a_r(k) + F_k^*(\eta)\, a_r^\dagger(k) b_r^\dagger(-k)\} \end{aligned} \tag{1.11}$$

By using the equations of motion, we find

$$\begin{aligned} E_k &= 2k \, \mathrm{Re}(u_+^* u_-) + am\left(1 - 2u_+^* u_+\right), \\ F_k &= k\left(u_+^2 - u_-^2\right) + 2am\, u_+ u_-, \\ E_k^2 &+ |F_k|^2 = \omega^2. \end{aligned} \tag{1.12}$$

Here we have chosen the momentum k along the third axis, and selected the gamma-matrix representation in which $\gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is always possible to choose an initial configuration with the hamiltonian in the standard (diagonal) form. If we take, for example,

$$u_\pm(\eta_0) = \left(1 \mp \frac{am}{\omega}\right)^{1/2} e^{i\phi}, \quad \phi \text{ arbitrary phase}, \tag{1.13}$$

we have $E_k(\eta_0) = \omega$, $F_k(\eta_0) = 0$.

However, the evolution equations (1.8) drive F_k different from zero. In order to give a "quasi-particle" interpretation, we diagonalize the Hamiltonian in eq. (1.11) with a time-dependent Bogolyubov canonical transformation, and define the new creation and annihilation operators

$$\begin{aligned} \hat{a}(k,\eta) &= \alpha(k,\eta) a(k) + \beta(k,\eta) b^\dagger(-k) \\ \hat{b}^\dagger(k,\eta) &= -\beta^*(k,\eta) a(k) + \alpha^*(k,\eta) b^\dagger(-k). \end{aligned} \tag{1.14}$$

Imposing canonical anticommutation relations on the operators $\hat{a}$ and $\hat{b}$, we find $|\alpha|^2 + |\beta|^2 = 1$. For

$$\frac{\alpha}{\beta} = \frac{E_k + \omega}{F_k^*}, \qquad |\beta|^2 = \frac{|F_k|^2}{2\omega(\omega + E_k)} = \frac{\omega - E_k}{2\omega}, \tag{1.15}$$

the normal-ordered Hamiltonian in terms of the "quasi-particle" operators is diagonal,

$$H(\eta) = \int d^3k \sum_r \omega(\eta) \left[\hat{a}_r^\dagger(k)\hat{a}_r(k) + \hat{b}_r^\dagger(k)\hat{b}_r(k)\right]. \tag{1.16}$$

Next, we define a "quasi-particle" vacuum, such that

$$\hat{a}|0_\eta\rangle = \hat{b}|0_\eta\rangle = 0 . \tag{1.17}$$

The total number of produced particles up to time η (equal to the number of produced antiparticles) is given by the vacuum expectation value of the particle number operator N,

$$n(\eta) = \langle 0_\eta|N|0_\eta\rangle = \frac{1}{\pi^2 a^3(\eta)} \int_0^\infty dk\; k^2\; |\beta|^2. \tag{1.18}$$

The density of produced particles is then computed by integrating [1] the equations of motion (1.9) with initial conditions at time $\eta = 0$ given by eqs. (1.13) and (1.8). This conditions correspond to $E_k = \omega$, $F_k = 0$ at $\eta = 0$ or, in other words, to an initial vanishing particle density.

1.2 Numerical Results

The equations of motion (1.9) describe oscillators with time-varying complex frequency. If m is constant, the time dependence enters only through the scale factor a. The corresponding gravitational creation of heavy fermions was studied numerically in ref. [22]. However, of particular interest is the case in which $m = m(t)$ is a (quasi) periodic function of time. This is realized when the scalar field ϕ, coupled to X as in eq. (1.1), is homogeneous and oscillates in time with frequency $V''(\phi)$. It is useful to write the equations of motion in terms of dimensionless variables. We introduce a dimensionless time $\tau \equiv m_\phi \eta$, as well as a dimensionless field $\varphi \equiv \phi/\phi(0)$, so that the scalar field is normalized by the condition $\varphi(0) = 1$. We start the numerical evaluation at $\phi(0) = 0.28 M_p$, short after inflation, $\phi'(0) = -0.15 M_p m_\phi$ (as follows from a numerical evaluation of the inflaton alone during inflation), and $a(0) = 1$. With the above redefinitions, the equation of motion for the background field φ does not contain any parameter, while the fermion mass is measured in units of m_ϕ and the strength of the fermion coupling to the external background is determined by

[1] In practice, one has to solve only the evolution equation for u_+, since u_- is determined by the equation $u_- = (amu_+ - iu'_+)/k$.

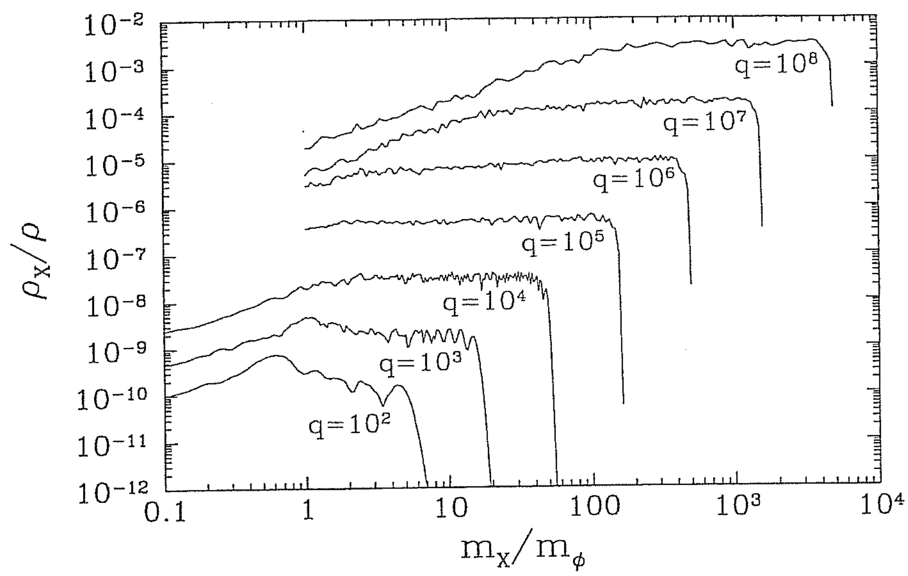

Figure 1.1: The fraction of the energy density of produced fermions with respect to the total energy density, as a function of the fermion mass m_X in units of the inflaton mass, for various values of q.

the dimensionless combination $g\phi(0)/m_\phi$. For the sake of correspondence with the bosonic case, we introduce the parameter

$$q \equiv g^2\phi^2(0)/4m_\phi^2 \,. \tag{1.19}$$

By analogy with the bosonic case [16, 57, 58], it is expected that an efficient production of very massive fermions will require a very large value of q.

The result of the numerical integration is best summarized in fig. 1.1. For different values of the fermion mass m_X and of the q parameter, fig. 1.1 shows ρ_X/ρ, the fraction of the inflaton energy density ρ which ends up in the fermionic energy density ρ_X. In this simulation, the back reaction of the produced fermions on the evolution of the inflaton field is neglected.

Fermion production is extremely efficient up to a time at which the ratio ρ_X/ρ freezes out. For fixed q, the larger m_X is, the earlier this freeze-out occurs. In particular, near the cut-off of ρ_X/ρ at large m_X (we denote with $(m_X)_{\rm th}$ the value of m_X at which this cut-off occurs), the production ceases just after a few oscillations of the inflaton field. As we will show in the next section, particle production occurs only in very short intervals about the points in which the total fermionic mass vanishes. It is then reasonable to assume that the saturation of the production occurs when the total mass (1.2) stops vanishing and that the sharp cut-off in the particle production at large m_X that we observe in fig. 1.1 corresponds to a situation in which the condition $m(t) = 0$ cannot be satisfied even during the first oscillation of the inflaton field.

To verify this assumption, let us write $m(t)$, with the help of eq. (1.19), as

$$m = m_X + 2\sqrt{q}m_\phi \frac{\phi}{\phi(0)} \simeq m_X + \frac{\sqrt{q}m_\phi}{\pi N} \cos(2\pi N)\,. \tag{1.20}$$

Here we have taken into account that $\phi \propto t^{-1}$, when the energy density of the Universe is dominated by the oscillating inflaton field, and we have denoted with N the number of oscillations of the inflaton field $N = m_\phi t/(2\,\pi)$.

At large m_X, the minimum of m is reached around $N = 1/2$. Therefore, if

$$m_X > m_\phi \frac{2}{\pi} \sqrt{q}, \tag{1.21}$$

the total mass m never vanishes [2]. The value of $(m_X)_{\rm th}$ given in eq. (1.21) is already in good agreement with our numerical results. However, eq. (1.20) cannot be trusted at small N. Actually, our numerical integration shows that the minimum of the inflaton amplitude ϕ_0 during the first oscillation is given by

$$\frac{\phi_0}{\phi(0)} \simeq -0.25. \tag{1.22}$$

This means that the cut-off value of the mass is at

$$(m_X)_{\rm th} \simeq m_\phi \frac{\sqrt{q}}{2}, \tag{1.23}$$

in perfect agreement with the results presented in fig. 1.1.

While for m close to the cut-off value the production saturates very quickly, for $m \ll (m_X)_{\rm th}$ the total mass vanishes many times and the process is very prolonged. Due to limited numerical resource, the curves which appear in fig. 1.1 correspond however to an evolution up to 20 inflaton oscillations. It is thus possible that the curves underestimate the production at m_X much smaller than $(m_X)_{\rm th}$. [3]

To qualitatively appreciate this feature we fix $m_x = 100\, m_\phi$ and we show how the production evolves for two different values of q. In fig. 1.2 the case $q = 10^6$ and the productions after 1 and 3 oscillations are presented. For $q = 10^6$ the value $m_x = 100\, m_\phi$ is quite close to $(m_X)_{\rm th}$ (cfr. fig. 1.1) and the fermionic production is expected to quickly freeze-out. Indeed the distribution after 3 oscillations presented

[2] We are considering here the case in which $g\phi(0)$ and m_X have the same sign. If these two terms have a relative minus sign, particle production can be extended to even larger values of m_X.

[3] In the next chapter, a detailed analytical study will allow us to better understand this point and to give a limit in m_X above which the results presented in fig. 1.1 can be trusted.

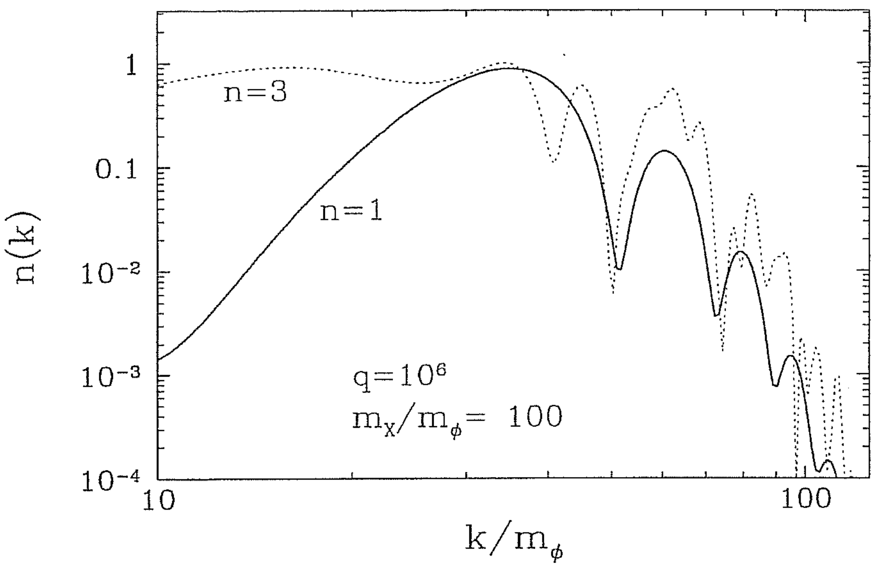

Figure 1.2: The phase-space density of produced particles after $n = 1$ and $n = 3$ inflaton oscillations for $q = 10^6$ and for X-fermions 100 times heavier than the inflaton. The distribution at $n = 3$ coincides with the final distribution.

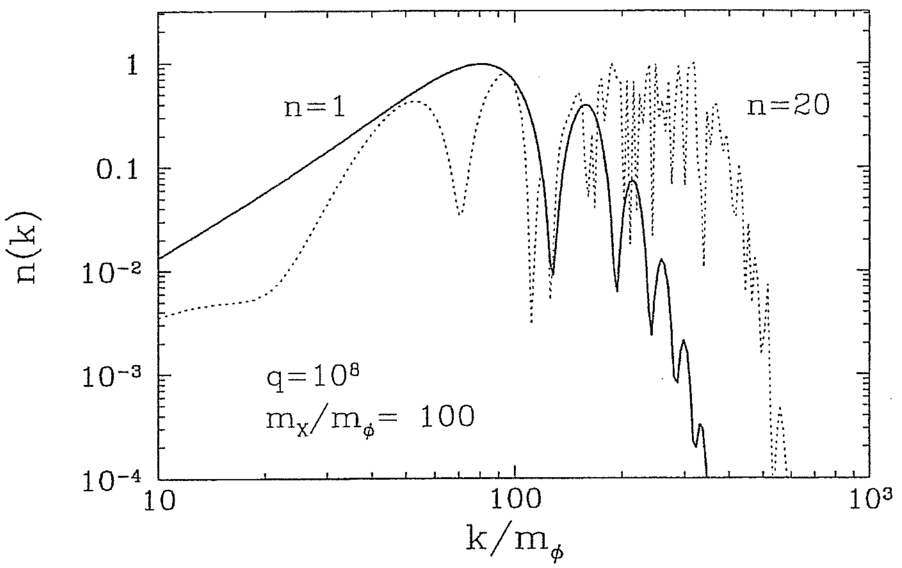

Figure 1.3: The phase-space density of produced particles after $n = 1$ and $n = 20$ inflaton oscillations for $q = 10^8$ and for X-fermions 100 times heavier than the inflaton. The distribution at $n = 20$ coincides with the final distribution.

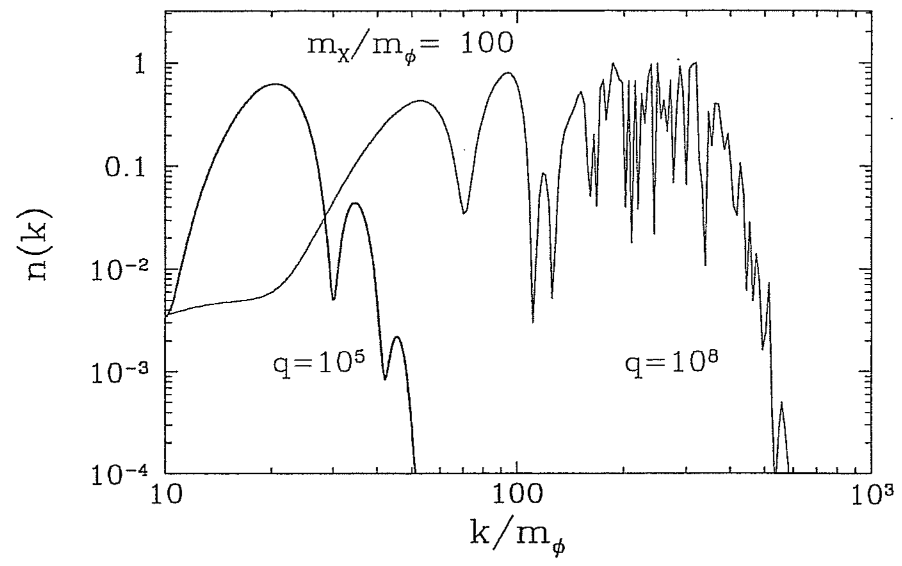

Figure 1.4: The final phase-space density of produced particles for two values of the parameter q. The X-fermions are taken to be 100 times heavier than the inflaton. At $q = 10^5$ the freeze-out of the particle production was reached after the first inflaton oscillation, while for $q = 10^8$ it required twenty oscillations.

in the figure already coincides with the final one. In fig. 1.3 we take instead $q = 10^8$ and we show the production after 1 and 20 oscillations of the inflaton field. In this case, only after 20 oscillations the fermionic production freezes-out and this agrees with the fact that $(m_X)_{\rm th}$ increases with increasing q.

Again looking at fig. 1.1 we notice that the energy density ρ_X of the produced fermions (i) scales almost linearly with q and (ii) depends very weakly on m_x (at fixed q) at least for m_x not too smaller than $(m_X)_{\rm th}$. These behaviors will be explained in the next chapter, where we will give an analytical expression for ρ_X as a function of q and m_X. The key point is that the whole process produces particles up to a maximum momentum $k_{\rm max}$ and that this quantity completely determines the final value of ρ_X. As can be already seen from figs. 1.2 and 1.3, the Pauli limit on the occupation number $n_k \leq 1$ is very well reached and almost saturated in the whole range $k < k_{\rm max}$. With a very good approximation, we can thus estimate the energy density of the produced fermions as

$$\rho_X = m_X\, n_X = m_X \int d^3k n(k) \propto m_X\, k_{\rm max}^3\,. \qquad (1.24)$$

Fig. 1.4 confirms the scaling given by eq. (1.24). In this figure we fix again $m_X = 100\, m_\phi$ and we compare the final distribution of the produced fermions for the two cases $q = 10^5$ and $q = 10^8$. Fig. 1.4 shows the ratio of about 100 between the maxima momenta $k_{\rm max}$ of the two cases. This result, together with eq. (1.24), indicates a nearly linear scaling of ρ_X with q, as it was clearly indicated in fig. 1.1.

1.3 Backreaction

When the energy density in created particles is comparable to the initial inflaton energy density, the issue of back reaction becomes relevant. From the results of the previous section, we gather that for $q \gtrsim 10^8$ the back reaction has to be considered. In the case of bosons this can be done numerically on a lattice with full account of all non-linear effects [59, 58]. In the fermionic case, we restrict ourselves to an approximate treatment of the back reaction in the Hartree approximation (similar calculations in the Bose case were performed in refs. [57, 16]).

In the Hartree approximation, the inflaton field is assumed to be homogeneous (all of its spatial fluctuations are neglected). Correspondingly, only the average value of the product $\bar{X}X$ is left in the equations of motion of the inflaton field, which takes the form

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi + g\langle\bar{X}X\rangle = 0\,. \tag{1.25}$$

The product $\langle\bar{X}X\rangle$ can be readily expressed through the momentum integration of the Bogolyubov coefficients using the field decomposition in eq. (1.5). However, a straightforward averaging leads to ultraviolet divergences, i.e. to extra powers of k at large k in the momentum integration compared to the integral of the particle number density, eq. (1.18). Therefore, the quantity $\langle\bar{X}X\rangle$ needs to be regularized. Similarly to the case of Minkowski space-time, the regularization amounts to the normal ordering or, equivalently, to the subtraction of vacuum zero-point fluctuations. To obtain a finite result, it is necessary to express the operator $\bar{X}X$ in normal form and subtract the part due to the vacuum fluctuations. Since the vacuum defined in eq. (1.17) is different at different times, the vacuum fluctuations subtracted during the reduction of the operator to normal form depend on time. The normally ordered $\bar{X}X$ operator has the form

$$N_\eta\left(\bar{X}X\right) \equiv \bar{X}X - \langle 0_\eta|\bar{X}X|0_\eta\rangle\,. \tag{1.26}$$

By its very definition, the operation of normal ordering gives $\bar{X}X$ only for the created particles. The vacuum averaging in eq. (1.25) is defined as the averaging with respect to the original vacuum state (we remind the reader that we are working in the Heisenberg representation)

$$\langle\bar{X}X\rangle \equiv \langle 0|N_\eta\left(\bar{X}X\right)|0\rangle\,. \tag{1.27}$$

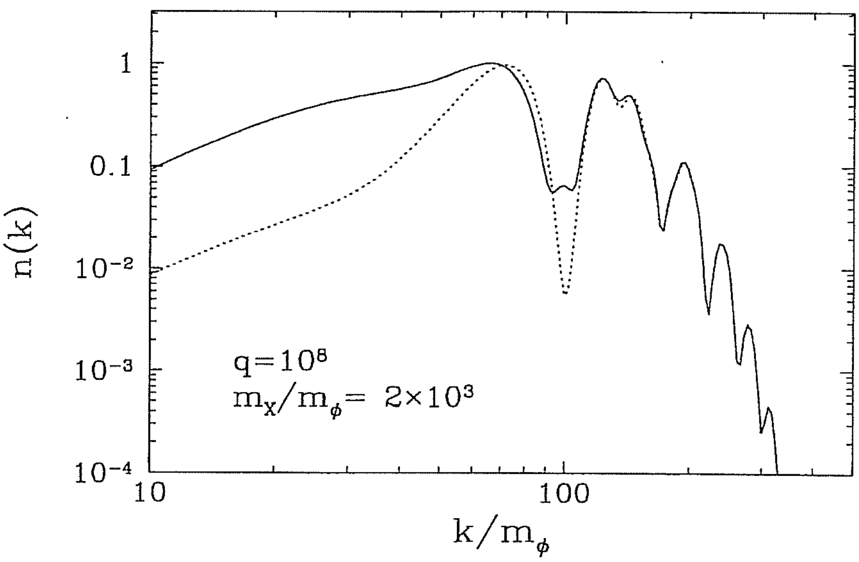

Figure 1.5: The solid line shows the final phase-space density of produced particles for $q = 10^8$ with back-reaction effects included in the Hartree approximation. The dotted line shows the phase-space density if the back-reaction effects are neglected.

An explicit calculation leads to

$$\langle \bar{X}X \rangle = \frac{2}{(2\pi a)^3} \int d^3k \left(|u_-|^2 + \frac{ma}{\omega} - 1 \right) . \tag{1.28}$$

In the case of Majorana spinors the numerical factor is twice smaller. We have used expression (1.28) when integrating the equation of motion (1.25) for the inflaton field . We have also consistently included the contribution from $\langle \bar{X}X \rangle$ in the equation of state when integrating Einstein equations for the scale factor.

The resulting phase-space density, with and without back-reaction, is shown in figs. 1.5 and 1.6 for $q = 10^8$ and $q = 10^{10}$, respectively. We observe that at $q = 10^8$ the spectra with and without back-reaction are identical at large k, the difference being appreciable only at small k. Since the total number density is saturated at large momenta, n_X turns out to be only slightly different when back reaction effects are included. For $q = 10^8$, n_X is *l*arger by 5% compared to the case without back reaction. At smaller q, the difference is even smaller. Therefore, the results of our calculations shown in the previous section can be trusted.

At larger q the back reaction effects become more significant. In fig. 1.6 we present the results for $q = 10^{10}$ and $m_X/m_\phi = 10^4$. First, we note that without back reaction $n_X \propto q$ in accordance with the scaling law presented in fig. 1.1. Integrating the phase-space over momenta, we find that the ratio ρ_X/ρ is approximately

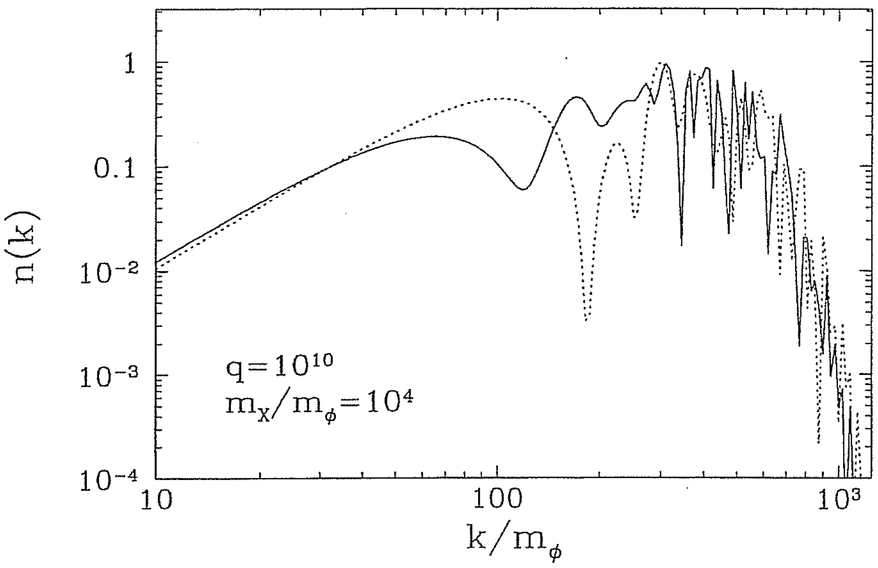

Figure 1.6: The same as fig. 1.5, but for $q = 10^{10}$.

50% *larger* than the value of the same ratio when back-reaction is not included. Therefore, the back reaction effects change the ratio of ρ_X/ρ by about factor of two, increasing it.

The time dependence of the inflaton field with and without back reaction is shown in fig. 1.7. We see that when back reaction is included, the field ϕ does not pass through the point at which $m(t) = 0$ because energy is very efficiently extracted from the inflaton field even at the first crossing. The non-zero value of $\langle \bar{X}X \rangle$ causes a change of the potential of the inflaton field, shifting the minimum around which the inflaton field oscillates. In particular, it changes the form of potential in such a way that ϕ oscillates around the point $m(t) = 0$.[4] Later on, $\langle \bar{X}X \rangle$ decreases because of the expansion of the Universe. Consequently the difference between the effective potential and the tree-level potential becomes small again and the inflaton field starts oscillations around the original minimum at $\phi = 0$, but with an amplitude that is smaller than in the case in which back-reaction is neglected.

To summarize we can say that, up to $q \sim 10^{10}$, the simplified calculations without back-reaction effects provides us with a lower bound for ρ_X/ρ. Other effects like the scattering of X particles, or their decay, will only help production since they remove particles from the already occupied Fermi levels.

[4]This effect will be explained in more details in the next chapter.

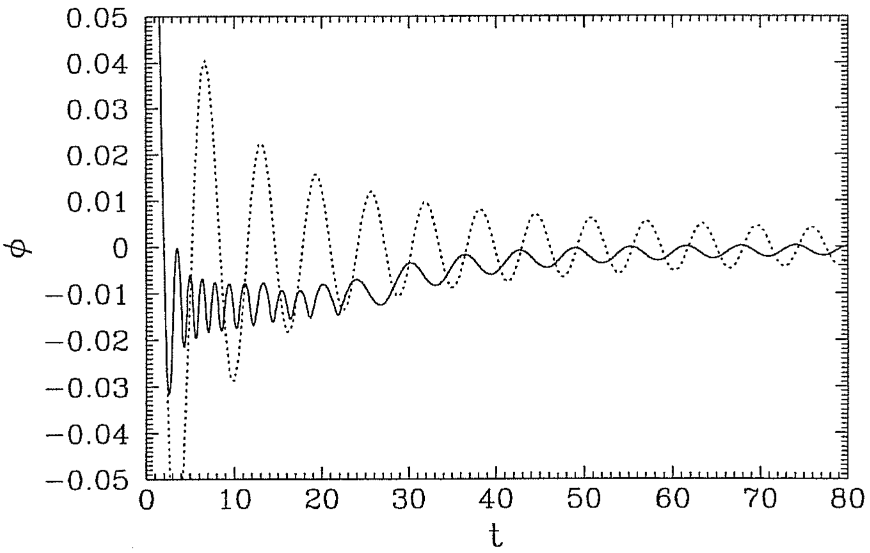

Figure 1.7: The time dependence of the inflaton field for $q = 10^{10}$ with back-reaction effects included in the Hartree approximation (solid line) and without back reaction (dotted line).

Chapter 2

Analytical approach

The aim of this chapter is to find an analytical expression for the occupation number and the energy density of the produced fermions, as a function of their bare mass and their coupling to the inflaton. This investigation is particularly welcome, since it shows up to which point we can trust the numerical results of the previous chapter. As we already anticipated, the numerical and the analytical results manifest an excellent agreement for m_X close to the cut-off value $(m_X)_{\rm th}$ (see the previous chapter). For smaller m_X the numerical evaluation becomes too long and the analytical formulae that we provide here remain the unique way to get a reliable estimate for the production.

As the starting point, in section 2.1 we derive the analytical formulae for the spectra of the fermions at any generic time. Our derivation follows the one developed in refs. [13, 16] for preheating of bosons. It exploits the fact that the production occurs only at short discrete intervals around the zeros of the total fermionic mass: the calculation is made possible from the fact that the occupation number can be considered as constant outside these small regions, and that the expansion of the Universe can be neglected inside them. As a result, the only physical quantities relevant for the creation are the time derivative ϕ' of the inflaton field and the value of the scale factor a at each production.

In section 2.2 we consider the production in a non-expanding Universe. In this case the analytical formulae considerably simplify, showing the presence of resonance bands which are anyhow limited by Pauli blocking.

In section 2.3 we study the more interesting case of production in an expanding Universe. As it occurs also for bosons, the expansion removes the resonance bands and the production (almost) saturates a Fermi sphere up to a maximal momentum. In section 2.3 we finally calculate the total energy density ρ_X of produced fermions, which may be the quantity of most physical relevance. To do this, a proper average of the analytical formulae must be done, exploiting the fact that the expansion of the Universe gives the production a stochastic character. In this way one can get a "mean" function that interpolates very well between the maxima and the minima of the spectra of the produced particles. Again we derived it in close analogy with what is done in the bosonic case, and again the results that we get are in very good agreement with the numerical ones of the previous chapter in the region of validity of the latter.

All this analysis neglects the backreaction of the produced fermions on the evolution of the inflaton field and of the scale factor. As we already remarked, backreaction effects can be studied in the Hartree approximation. In section 2.4 we see that the analytical formulae here provided allow to understand the effects of backreaction observed in the numerical simulations.

2.1 Analytical evaluation of the occupation number

In this section we calculate analytically the evolution of the Bogolyubov coefficients (1.14) during the oscillations of the inflaton field after chaotic inflation.

In this derivation, we exploit the fact that, in the regime of very massive fermions we are interested in, the creation occurs only for very short intervals about the points $\phi_* \equiv -m_X/g$ where the total fermionic mass (cfr. eq. (1.2)) vanishes. As the perfect agreement with the numerical results will confirm, this consideration allows one to treat the fermionic production with the same formalism adopted[1] in the bosonic case [13, 16]. While far from the zeros of the total mass m the Bogolyubov coefficients can be treated as constant, whenever ϕ crosses ϕ_* a sudden variation occurs. Since the interval of production is very narrow, one can safely neglect the expansion of the Universe during the production and also linearize the function $\phi(\eta) \simeq \phi_* + \phi'(\eta_*)\,(\eta-\eta_*)$. As a consequence, the frequency ω defined in eq. (1.10) acquires the form

$$\omega^2 \simeq k^2 + a^2\left(\eta_*\right)\phi'^{\,2}\left(\eta_*\right)\ \left(\eta-\eta_*\right)^2\,, \tag{2.1}$$

and the whole calculation strongly resembles the one for scattering through a quadratic potential.

The first step for the derivation of the analytical formulae is to consider asymptotic solutions of eqs. (1.8) and (1.9). We look for solutions valid for ϕ not very close to ϕ_*, where the adiabaticity condition

$$\omega' \ll \omega^2 \tag{2.2}$$

[1]The derivation of the "fermionic counterpart" of the formulae obtained for preheating of bosons in refs. [13, 16] has also been performed to a certain extent in the work [54], where the results of a single production during inflation is given. However, when one is interested in the successive productions, a more detailed study is necessary, as the more detailed study [27, 28] here reported show.

holds.

In this regime, the most general solutions of eqs. (1.8) and (1.9) are

$$\begin{aligned} u_+ &= A\left(1-\frac{am}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta} + B\left(1+\frac{am}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta}\,, \\ u_- &= -B\left(1-\frac{am}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta} + A\left(1+\frac{am}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta}\,, \end{aligned} \tag{2.3}$$

with $|A|^2+|B|^2=1$ following from the condition $|u_+|^2+|u_-|^2=2$.

We can always put solutions of eqs. (1.8) and (1.9) into the form (2.3), with A and B in general functions of η. However, in most of the evolution (whenever the adiabaticity condition holds) it is a very good approximation to treat the coefficients A and B as constant.

Substituting the expressions (2.3) into eqs. (1.12), one finds

$$F = 2\omega AB\,, \qquad E = \omega\left(|A|^2-|B|^2\right), \tag{2.4}$$

from which it follows

$$|\beta|^2 = \frac{|F|^2}{2\omega\,(E+\omega)} = |B|^2\,. \tag{2.5}$$

One can thus choose (up to an irrelevant global phase)

$$A \equiv \alpha \longrightarrow B \equiv \beta^*\,, \tag{2.6}$$

where α and β are nothing but the Bogolyubov coefficients we are interested in.

Notice that the initial choice $A(\eta_0)=1$, $B(\eta_0)=0$ corresponds to the initial condition (1.13) and to the zero particles state chosen in the previous section.

As we have said, these coefficients undergo a sudden change whenever ϕ crosses ϕ_0 and then they stabilize to new (almost) constant values. Our aim is to find the values at the end of the variation in terms of the ones prior to it.

We have not specified the lower limit of the integrals appearing in eqs. (2.3). For present convenience we choose it to be the time η_{*1} of the first production (that is when $\phi=\phi_*$ for the first time).

Let us consider the evolution equations (1.9) near the point η_{*1}. Since for high mass m_X the fermionic production is limited to a very short interval, one can neglect the expansion of the Universe during it and write the equation for $\phi(\eta)$ in a linearized form. We can thus write

$$a\,m(\eta) \simeq a_{*1}\, g\, \phi'_{*1}\,(\eta-\eta_{*1})\,, \qquad a_{*1} \equiv a\,(\eta_{*1})\,, \qquad \phi_{*1} \equiv \left.\frac{d\,\phi}{d\,\eta}\right|_{\eta_{*1}}\,. \tag{2.7}$$

We define (with the notation of [54])

$$p \equiv \frac{k}{\sqrt{g|\phi'_{*1}|a_{*1}}}, \qquad \tau = \sqrt{g|\phi'_{*1}|a_{*1}}\,(\eta - \eta_{*1}) . \tag{2.8}$$

In terms of these new quantities, eqs. (1.9) rewrite (dot denotes derivative with respect to τ)[2]

$$\ddot{u}_\pm + \left(p^2 \mp i + \tau^2\right) u_\pm = 0 . \tag{2.10}$$

The point η_{*1} is thus mapped into the origin of τ and the region of asymptotic adiabaticity is at large $|\tau|$.

In the asymptotic solutions (2.3) we can see the behaviors

$$\begin{aligned} \left(1+\frac{am}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{\sqrt{2}\tau}, \\ \left(1-\frac{am}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}, \\ e^{\pm i\int^\eta \omega_k d\eta} &\longrightarrow \left(\frac{2\tau}{p}\right)^{\pm ip^2/2} e^{\pm i\tau^2/2} e^{\pm ip^2/4}, \end{aligned} \tag{2.11}$$

for $\tau \to +\infty$, and

$$\begin{aligned} \left(1+\frac{am}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}, \\ \left(1-\frac{am}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{-\sqrt{2}\tau}, \\ e^{\pm i\int^\eta \omega_k d\eta} &\longrightarrow \left(\frac{p}{-2\tau}\right)^{\pm ip^2/2} e^{\mp i\tau^2/2} e^{\mp ip^2/4}, \end{aligned} \tag{2.12}$$

for $\tau \to -\infty$.

Equations (2.10) are solved by parabolic cylinder functions $D_\lambda(z)$ [60]. More precisely, the combination that matches the asymptotic solution (2.3) at $\tau \to -\infty$ is[3]

$$\begin{aligned} u_+(\tau) = {} & A^-\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{1+ip^2/2} e^{i\left(\frac{\pi}{4}-\frac{p^2}{4}\right)} e^{-\pi p^2/8} D_{-1-ip^2/2}\left(-\left(1+i\right)\tau\right) + \\ & + B^-\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}(-(1-i)\tau) . \end{aligned} \tag{2.13}$$

[2]For the production at the moment η_{*n}, when the total mass vanishes for the n-th time, eq. (2.10) must be replaced by

$$\ddot{u}_\pm + \left(p^2 \pm i\, sign\,(\phi'_{*n}) + \tau^2\right) u_\pm = 0 . \tag{2.9}$$

The effect of this replacement on the final results is reported below.

[3]We deal only with the function u_+, since the study of u_- leads to the same results.

In the above expression, A^- and B^- denote the values of the Bogolyubov coefficients before ϕ crosses ϕ_*, while the functions which multiply them are *exact* solutions of the *linearized* equation (2.10). The analytical approximation consists in considering them as solutions of the true evolution equation (1.9). For $\tau \to +\infty$ it is convenient to rewrite the solution (2.13) in terms of two different parabolic cylinder functions[4] [60]

$$
\begin{aligned}
u_+(\tau) = & \left[A^- \frac{\sqrt{\pi} p e^{-\pi p^2/4}}{\Gamma(1+ip^2/2)} e^{i(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2})} + B^- e^{-\pi p^2/2} \right] \times \\
& \times \left\{ \sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}((1-i)\tau) \right\} + \\
& + \left[-A^- e^{-\pi p^2/2} + B^- \frac{\sqrt{\pi}\, p\, e^{-\pi p^2/4}}{\Gamma(1-i\, p^2/2)} e^{-i(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2})} \right] \times \\
& \times \left\{ \sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{1+ip^2/2} e^{i(\frac{\pi}{4}-\frac{p^2}{4})} e^{-\pi p^2/8} D_{-1-ip^2/2}((1+i)\tau) \right\}. \qquad (2.14)
\end{aligned}
$$

In this new expression, the functions within curly brackets correspond to the asymptotic forms at $\tau = +\infty$ of the two terms of the solution (2.3). The coefficients in front of them give thus the new Bogolyubov coefficients in terms of the old ones.

All this derivation can be easily generalized when productions at successive zeros of the total mass m are considered. The only important points are

(i) a difference in the values of the scale factor a and of the derivative ϕ' at different η_{*i}'s,

(ii) a change of sign in the transfer matrix (the one which gives the new coefficients in terms of the old ones) whenever ϕ crosses ϕ_* from below to above (cfr. the footnote just before eq. (2.10)), and

(iii) the phase $e^{\pm i\int \omega_k d\eta}$ which accumulates between η_{*1} and the η_{*i} considered.

Putting all this together, one has

$$
\begin{aligned}
\begin{pmatrix} \alpha_n \\ \beta_n^* \end{pmatrix} &= \begin{pmatrix} F_n & H_n \\ -H_n^* & F_n^* \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1}^* \end{pmatrix} \qquad \text{for } n \text{ odd}, \\
H_n &\longleftrightarrow -H_n \qquad \text{for } n \text{ even}, \qquad (2.15)
\end{aligned}
$$

[4] This rewriting is always possible since eq. (2.10) has only two linearly independent solutions.

where a_n, β_n are the values of the Bogolyubov coefficients after the n-th production, and where

$$
\begin{aligned}
F_n &= \sqrt{1-e^{-\pi p_n^2}}\, e^{i\left(-\frac{\pi}{4}-\arg\Gamma(ip_n^2/2)+\frac{p_n^2}{2}\ln(p_n^2/2)-p_n^2/2\right)}\,, \\
H_n &= e^{-\pi p_n^2/2} e^{-2i\int_{\eta_{*1}}^{\eta_{*n}} \omega_k\, d\eta}\,, \qquad |F_n|^2+|H_n|^2=1\,. \qquad (2.16)
\end{aligned}
$$

We remind that $p_n = k/\sqrt{g|\phi'_{*n}|a_{*n}}$.

If one starts with no fermions at the beginning, one may choose $\alpha_0 = 1, \beta_0 = 0$. Then, applying successive times the "transfer" matrix (2.15), one can get the spectrum of fermions produced after every η_{*n}.

Our calculation reproduces the result

$$N_1 = |\beta_1|^2 = e^{-\pi p_1^2} \qquad (2.17)$$

found in ref. [54].

The results obtained with the analytical expression (2.15) are in very good agreement with the numerical ones. Just to give a couple of examples, we present here two cases at different regimes (we show them only with illustrative purpose, and the values of the parameters chosen have no particular importance). In figure 2.1 we present the spectrum of the fermions after two productions, that is after one complete oscillation of the inflaton field. In analogy with the bosonic case, we measure the strength of the coupling inflaton-fermions with the quantity $q \equiv g^2\phi_0^2/(4m_\phi^2)$, where $\phi_0 \simeq 0.28 M_p$ is the value of the inflaton at the beginning of reheating. In figure 2.1 we choose $q = 10^8$, while we fix the bare fermion mass to be $m_X = 100 m_\phi$. In figure 2.2 we show instead the resulting spectrum after 7 productions in the case $q = 10^4$, $m_X = 4m_\phi$.

2.2 Production in a non-expanding Universe

In the bosonic case, the study of the non-perturbative production in a non-expanding Universe has proven very useful in understanding the effects of the expansion. It is shown in refs. [13, 16] that the bosonic wave function satisfies the Mathieu equations, whose solutions are characterized by resonance bands (in momentum space) of very "explosive" and efficient production. It is then shown that, due to the expansion of the Universe, modes of a given comoving momentum k cross several resonance bands during the evolution. This gives the creation the stochastic character described in the works [13, 16]. In ref. [25] it is understood that an analogous behavior occurs also for fermions. The expansion is expected also in this case to spoil the clear picture of distinct resonance bands. This fact may help the transfer of energy to fermions, since the resonance bands in the fermionic case are anyhow limited by the Pauli principle. The expansion allows thus new modes to be occupied, and the

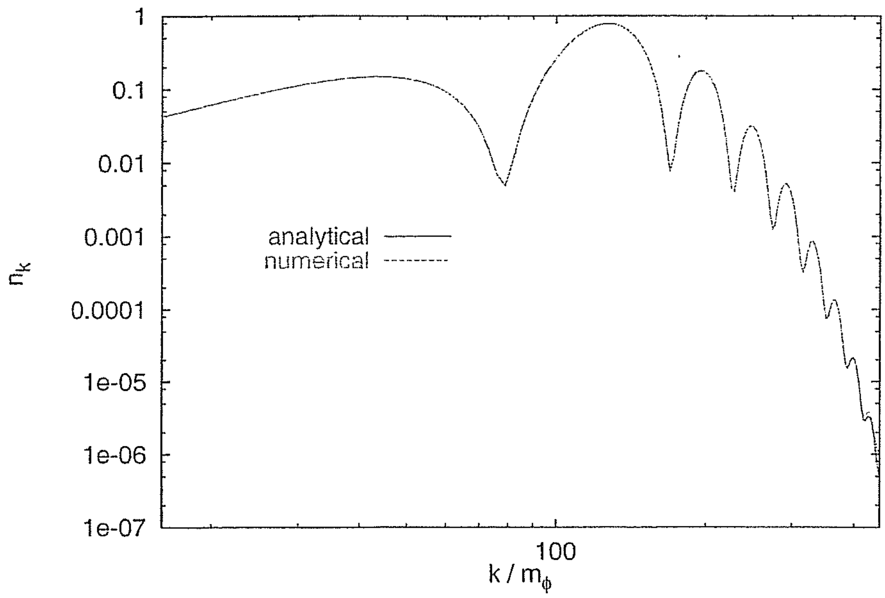

Figure 2.1: Spectrum of the fermions after two productions for $q = 10^8$ and $m_X = 100m_\phi$. The expansion of the Universe is taken into account.

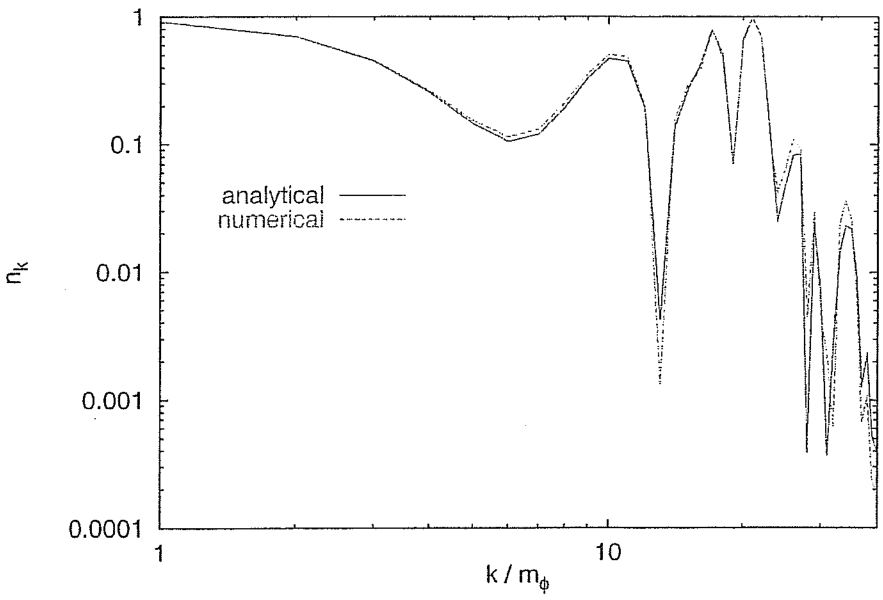

Figure 2.2: Spectrum of the fermions after seven productions for $q = 10^4$ and $m_X = 4m_\phi$. The expansion of the Universe is taken into account.

production is no longer limited to the regions of resonance. In ref. [25] it is stated that the production should then almost completely fill the whole Fermi sphere up to a maximal momentum $k_{\max}$. This behavior is confirmed by the numerical results of the work [17], which we have presented in the previous chapter. In this section we will see that our analytical formulae can reproduce the resonance bands, while in the next one we will discuss the effects of the expansion of the Universe.

Let us consider the matrices (we drop the index n in the matrix elements since all the p_n's have now the same value p)

$$M = \begin{pmatrix} F & G \\ -G & F^* \end{pmatrix}, \qquad T_1 = \begin{pmatrix} e^{i\vartheta_1^2} & 0 \\ 0 & e^{-i\vartheta_1^2} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} e^{i\vartheta_2^3} & 0 \\ 0 & e^{-i\vartheta_2^3} \end{pmatrix} \tag{2.18}$$

with $G \equiv e^{-\pi p^2/2}$ and $\vartheta_i^j \equiv \int_{\eta_{*i}}^{\eta_{*j}} \omega_k d\eta$ (F was defined in eq. (2.16)).

Without the expansion of the Universe, the inflaton field has the periodic evolution

$$\phi(\eta \equiv t) = \phi_0 \cos(m_\phi \eta), \tag{2.19}$$

and all the ϑ_i^j (2.15) are hence sums of ϑ_1^2 and ϑ_2^3 (remember the η_{*i} are the moments at which the total fermionic mass vanishes).

After the generic n-th complete oscillation one thus has

$$\begin{pmatrix} \alpha_{2n} \\ \beta_{2n}^* \end{pmatrix} = \begin{pmatrix} e^{-i\vartheta_1^{2n+1}} & 0 \\ 0 & e^{i\vartheta_1^{2n+1}} \end{pmatrix} T_2 M^T T_1 M \cdots T_2 M^T T_1 M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.20}$$

with the combination $\hat{O} \equiv T_2 M^T T_1 M$ repeated n times.

One has thus simply to study the eigenvalue problem for $\hat{O}$ (notice $\det\hat{O} = 1$). This operator has the form

$$\begin{aligned} \hat{O} &= \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \\ A &= F^2 e^{i(\vartheta_1^2+\vartheta_2^3)} + G^2 e^{-i(\vartheta_1^2-\vartheta_2^3)}, \\ B &= FGe^{i(\vartheta_1^2+\vartheta_2^3)} - F^* G e^{-i(\vartheta_1^2-\vartheta_2^3)} \end{aligned} \tag{2.21}$$

and its eigenvalues are $\lambda_{1,2} = e^{\pm i\Lambda}$ with $\cos\Lambda = \mathrm{Re}\,A$.

Rewriting the initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of the eigenvectors of $\hat{O}$ and substituting in formula (2.20), one gets the number of produced fermions

$$N_n = |\beta_n|^2 = \frac{|B|^2}{1-(\mathrm{Re}\,A)^2} \sin^2(n\Lambda) = \frac{|B|^2}{\sin^2\Lambda} \sin^2(n\Lambda) \tag{2.22}$$

after the complete n-th oscillation.[5]

We notice the presence of the envelope function

$$\tilde{E} \equiv \frac{|B|^2}{1-(\mathrm{Re}\,A)^2} = \frac{1-|A|^2}{1-(\mathrm{Re}\,A)^2} \qquad (2.23)$$

which modulates the oscillating function $\sin^2(n\Lambda)$.

With increasing n this last function oscillates very rapidly and can be at all effects averaged to 1/2. One is thus left with the envelope function which shows the presence of resonance bands. The resonance bands occur where A is real and $\tilde{E} \to 1$. It is easy to understand that their width exponentially decreases with increasing momenta k. To see this, let us consider the behavior of $\tilde{E}$ at high momenta. In this regime, the function A is given by

$$A \simeq (1-e^{-\pi p^2})e^{i\phi_A}\,, \qquad p = \frac{k}{\sqrt{g|\phi'_*|}}\,, \qquad (2.24)$$

where the phase ϕ_A can be read from eq. (2.21). Near to the points where $\cos\phi_A = 1$ the envelope function behaves like[6]

$$\tilde{E} \simeq \frac{1-(1-e^{-\pi p^2})^2}{1-(1-e^{-\pi p^2})^2\cos^2\phi_A} \simeq \frac{1}{e^{\pi p^2}(1-\cos\phi_A)+1}\,. \qquad (2.25)$$

The width of the band can be defined as the distance between the two successive points at which $\tilde{E} = 1/2$. From the last expression it follows that the difference between the phases of A in these two points is given by $\Delta\alpha = 2\sqrt{2}e^{-\pi p^2}$. Since the most rapidly varying term which contributes to the phase of A is $(p^2/2)\log(p^2/2)$, the width of the band can be thus estimated to be

$$\Delta p \simeq \frac{2\sqrt{2}}{p\log(p^2/2)}\,e^{-\pi p^2}\,. \qquad (2.26)$$

We show in figure 2.3 the envelope of the produced fermions in a static Universe for the parameters $q = 10^6$ and $m_X = 100\,m_\phi$. The peaks occur where A is real and it is confirmed that their width decreases very rapidly at increasing momenta. Due to the fact that the last peaks plotted are indeed very sharp, the resolution of figure 2.3 does not allow to see their top at $n_k = 1$.

[5] In ref. [25] it is said that it is possible to extract the average over one period of the occupation number from the knowledge of the solution of eq. (1.9) with initial conditions $u_+(\eta_0) = 1, u'_+(\eta_0) = 0$. The matching of this procedure with our formulae (2.3) and (2.21) gives the average time evolution (in our notation) $\bar{N}(t) = |B|^2\sin^2[(\pi-\Lambda)t/T]/\sin^2[\pi-\Lambda]$, where T is the period of one oscillation. Clearly, this result coincides with our eq. (2.22) at any given complete oscillation.

[6] A completely analogous behavior occurs where $\cos\phi_A = -1$.

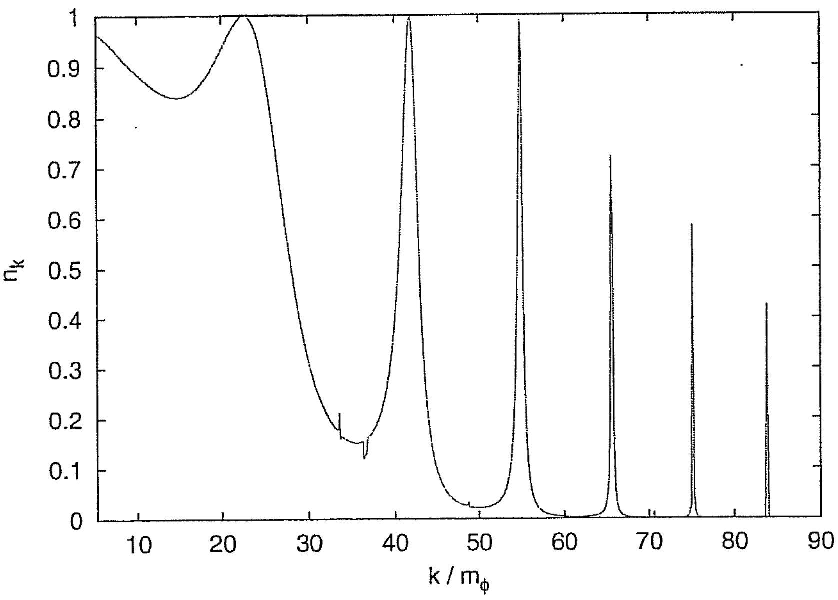

Figure 2.3: Envelope of the spectrum of the produced fermions in a static Universe. The physical parameters are $q = 10^6$ and $m_X = 100\, m_\phi$.

2.3 Expansion taken into account

As stated in the previous section, the resonance bands disappear when the expansion of the Universe is taken into account. In this section we will show how the occupation number varies when a non-vanishing Hubble parameter is considered. As we have seen in section 2.1, eqs. (2.15) and (2.16) give a very good agreement with the numerical results. On the other hand, the presence of phases in eq. (2.16) makes the exact analytical treatment of the occupation number impossible. Now, the same observation made in the bosonic case [13, 16] turns very useful also to us: the phases in eq. (2.16), when the expansion of the Universe is taken into account, are not correlated among themselves. As a result, the final spectra present several high frequency oscillations about some average function. The positions of the peaks of these oscillations depend on the details of the phases. However, the "mean" function can be understood in a surprisingly easy way. Our problem can be treated as one customary does when dealing with the "random walk". Imagine one has to calculate the quantity

$$S \equiv |A_1 + A_2 + A_3 + \cdots + A_n|^2 , \tag{2.27}$$

where the A_i are complex numbers with random phases. The "random walk recipe" indicates that the best estimation of the above quantity is achieved by summing the squares of the terms A_i, since the mixed products average to zero. The chaoticity of the final spectra for n_k suggests that this may also be true in our case, and this is confirmed by comparison with the numerical results.

With this method, eqs. (2.15) and (2.16) turn into the much simpler relations

$$\begin{pmatrix} |\alpha_n|^2 \\ |\beta_n^*|^2 \end{pmatrix} = \begin{pmatrix} |F_n|^2 & |H_n|^2 \\ |H_n|^2 & |F_n|^2 \end{pmatrix} \begin{pmatrix} |\alpha_{n-1}|^2 \\ |\beta_{n-1}^*|^2 \end{pmatrix}, \qquad (2.28)$$

where we remember

$$|F_n|^2 = 1 - e^{-\pi p_n^2}, \qquad |H_n|^2 = e^{-\pi p_n^2}, \qquad |F_n|^2 + |H_n|^2 = 1. \qquad (2.29)$$

Assuming no fermions in the initial state, and applying n times this formula, it is easy to see that the occupation number after the n-th production is given by

$$N_n(k) = \frac{1}{2} - \frac{1}{2} \prod_{i=1}^{n} \left(1 - 2e^{-\pi p_i^2}\right). \qquad (2.30)$$

A similar result holds for preheating of bosons, cfr. [13, 16] where the idea of averaging on almost random phases is first introduced. In the bosonic case, one can exploit the fact that, due to the high efficiency of the production, the occupation number after the $(n+1)$-th creation is (almost) proportional to the occupation number after the n-th one:

$$N_{n+1}(k) \simeq \left(1 + 2e^{\pi\kappa_n^2}\right) N_n(k), \qquad (2.31)$$

where the quantity κ_n is analogous to our parameter p_n. This simplification is not possible in our case, since the Pauli principle forbids N_n to be sufficiently high. However our final result, eq. (2.30), is also cast in a very simple and immediate form.

The validity of eq. (2.30) is confirmed by our numerical investigations, as we show here in one particular case. In figure 2.4 we compare the behavior of the "mean" function for the spectra with respect to the numerical one. We choose the physical parameters to be $q = 10^4, m_X = 4m_\phi$, and we look at the results after the 7-th production (this corresponds to the choices made in figure 2.2). We see that the "mean" function interpolates very well between the maxima and the minima of the numerical spectrum, and that it can indeed be considered as a very good approximation of the actual result.

This is confirmed by figure 2.5, where we plot (for the same values adopted in figure 2.4) the quantity $n_k k^2$ rather than the occupation number alone. This quantity is of more physical relevance when one is interested in the total energy transferred to the fermions, since the total number density of produced fermions is

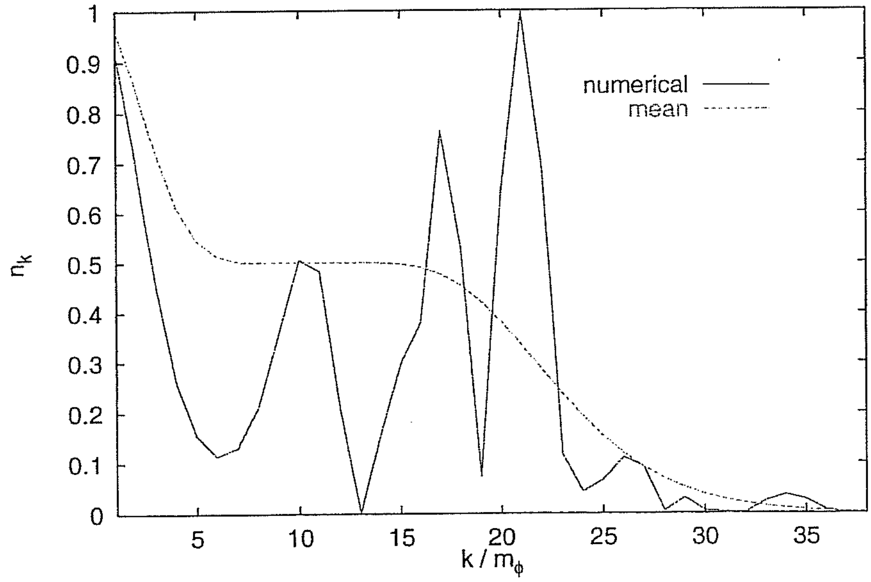

Figure 2.4: Comparison between the numerical spectrum and the "mean" function after seven productions for $q = 10^4$ and $m_X = 4m_\phi$.

(apart from the dilution due to the expansion of the Universe, that will be considered only in the final result)

$$N_X = \frac{2}{\pi^2} \int dk\, k^2\, n_k \,. \tag{2.32}$$

As shown in the plot, the result for N_X in the numerical and in the approximated case are in very good agreement.

After checking the validity of the approximation given by the "mean" function, we adopt it to understand how the production scales when different values of the parameters q and m_X are considered.

Equation (2.30) allows us to give an analytical estimate of the total amount of energy stored in the fermions after the n-th production, and in particular after that the whole process of non-perturbative production has been completed. Notice that all the dependence on the physical parameters is in the coefficients

$$z_i \equiv \frac{k}{\left(\pi^{1/2} p_i\right)} \,, \tag{2.33}$$

where k is the comoving momentum, and the only part we have to determine are the numbers

$$z_i^2 = \frac{2\sqrt{q}}{\pi} a\left(\eta_{*i}\right) \frac{\left|\phi'\left(\eta_{*i}\right)\right|}{\phi_0} \,. \tag{2.34}$$

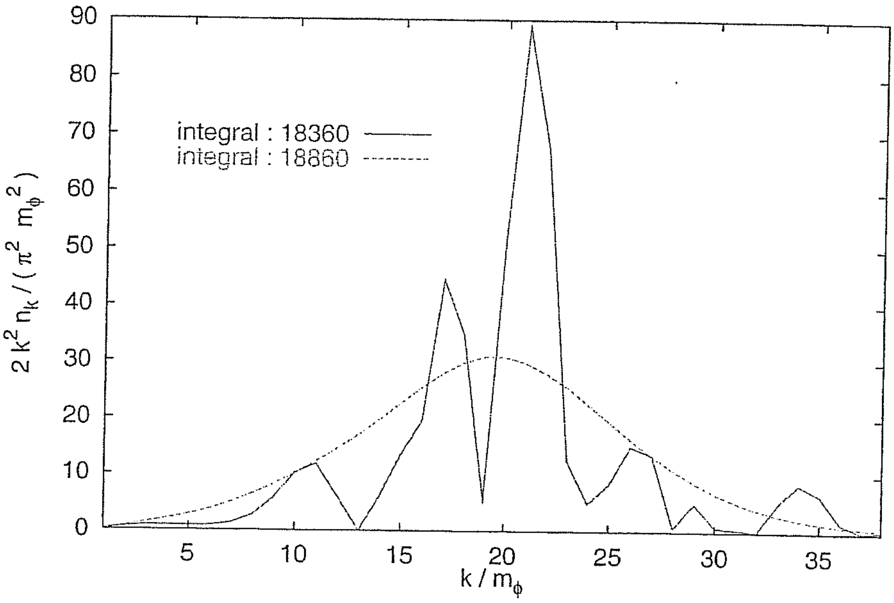

Figure 2.5: As in figure 2.4, but with the quantity $k^2 n_k$ plotted.

Here, and in what follows, we express the dimensionful quantities in units of m_ϕ, the inflaton mass, apart from the inflaton field ϕ that is given in units of M_p.

We also introduce the ratio $R \equiv 2q^{1/2}/m_X$. This quantity is the most relevant, since it determines the zeros of the total fermionic mass and thus the values of the z_i's. Indeed, from eq. (1.2) we see that the total mass vanishes for $\phi_* = -\phi_0/R$. It is convenient to study the production in terms of the two independent parameters q and R (rather than q and m_X) since, at fixed R, all the spectra are the same provided we rescale $k \propto q^{1/4}$ (cfr. eq. (2.34)).

One can now proceed in two different ways, and we devote the next two subsections to each of them. First, one can evolve the equations for the inflaton field alone and find numerically the values z_i for given q and R. Inserting these values in eq. (2.30) one can get final values for the production which, as we have reported, are very close to the numerical ones. This method allows to get results which average the actual ones, and it has the advantage of being much more rapid than a full numerical evolution.

Alternatively, one can proceed with a full analytical study in order to understand the results given by the first semi-analytical method.

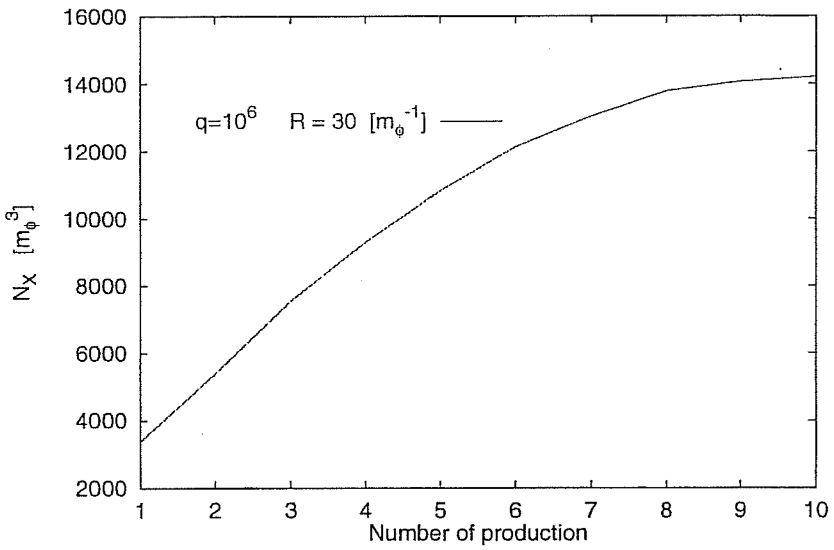

Figure 2.6: Growth of N_X with the number of productions for $q = 10^6$ and $R = 30$.

2.3.1 Semi-analytical results

In this subsection we evaluate the analytical formula (2.30), taking the coefficients z_i from a numerical evolution of the inflaton field. As we have said, this method gives results which are very close to the numerical ones, due to the fact that the "mean" function interpolates very well the actual spectra of the produced fermions.

The first thing worth noticing is that, for each choice of q and R, the maximum z_i occurs at about half of the whole process of non-perturbative creation. It thus follows that fermions of maximal comoving momenta will be mainly produced at the half of the process. Our semi-analytical evaluations support this idea: the total energy stored in the created fermions increases slowly during the second part of the preheating. To see this, we show in figure 2.6 the numerical results for the quantity $N_X = 2 \int dk\, k^2 n_k / \pi^2$ as function of the number of production. We fix the physical parameters to be $q = 10^6$ and $R = 30$, that is $m_X = 67\, m_\phi$. With this choice, the total mass vanishes 10 times, and figure 2.5 shows the results after each step of this production. We see indeed that the final productions are less efficient than the previous ones.

From eq. (2.30) it is also possible to show the evolution of the spectra with the number of productions. We do this in figure 2.7. We choose the same parameters as in figure 2.6, and we show the results after each complete oscillation of the inflaton field (that is after each two productions). We observe that the production rapidly approaches a step function in the momentum space, i.e. there exists a maximum

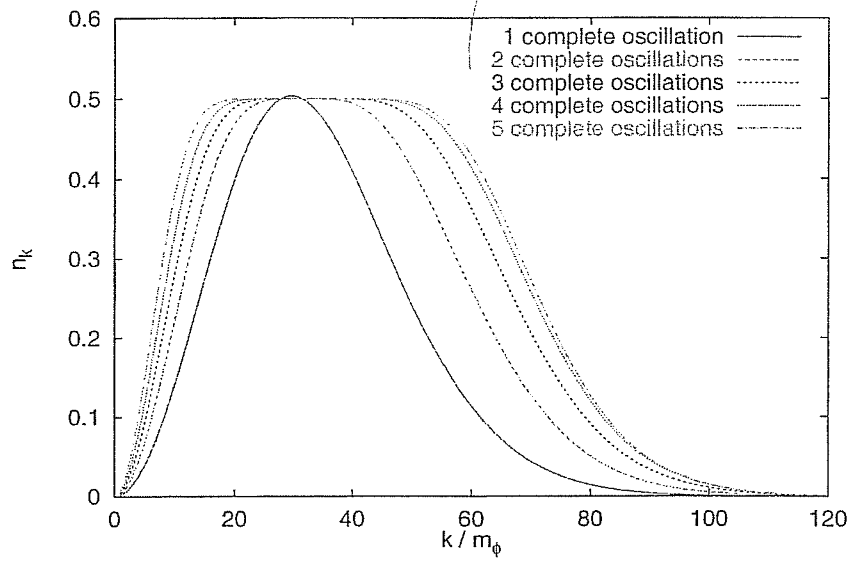

Figure 2.7: Evolution of the spectrum of produced fermions with the number of productions for $q = 10^6$ and $R = 30$.

momentum below which Pauli blocking is saturated (notice that the value 1/2 follows from the average understood in the "mean" function), and above which $n_k \simeq 0$. The fact that the last productions do not contribute much to the total energy is also confirmed.[7]

We now turn our attention to the total energy transferred to fermions after the whole preheating process is completed. We fix the parameter q to the value 10^6 and we investigate how the total integral N_X changes with different values for the parameter R.[8] The results are shown in figure 2.8, for R ranging from 5 to 10000. For the last value the total fermionic mass changes sign more than 3700 times and a full numerical evaluation would appear very problematic. This can be done in our case, thanks to the analytical expression (2.30) found, and our results extend the validity region of the numerical study of the previous chapter.

In figure 2.8, the results of our semi-analytical method are also compared to the full analytical ones of the next subsection. This comparison will be discussed below.

From the scaling of N_X with R just reported, it is now easy to estimate the total energy transferred to fermions for generic values of q and R. We are interested in comparing these results to the numerical ones of the previous chapter. To do so, we consider the ratio between the energy density given to fermions and the one in the

[7] The behavior at small k is inessential, since this region does not significantly contribute to the total energy.

[8] As reported, the scaling of the final result with q at fixed R is simply understood from eq. (2.34).

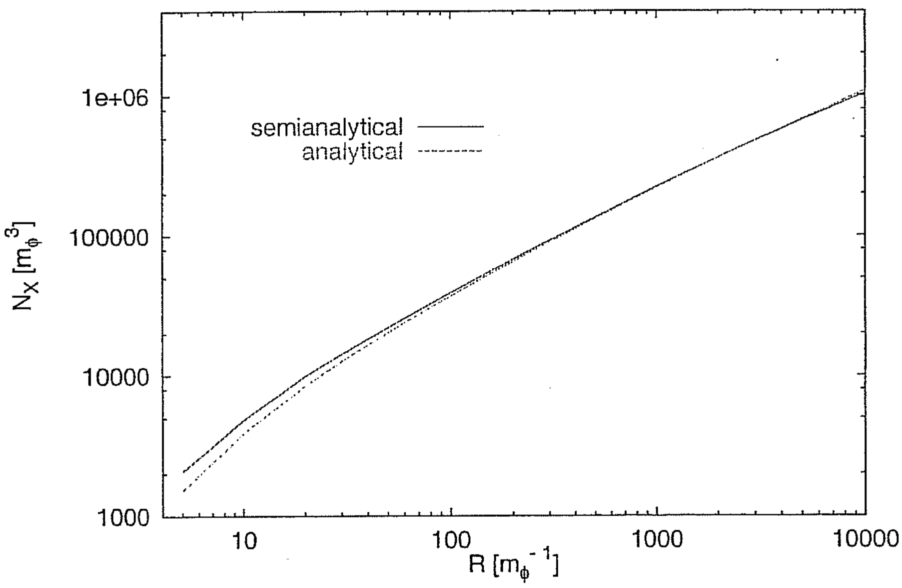

Figure 2.8: Comparison between the analytical and the semi-analytical results for N_X, at fixed $q = 10^6$.

inflaton field[9]

$$\frac{\rho_X}{\rho_\phi} = \frac{2m_X}{\pi^2}\int dk\, k^2 n_k \cdot \frac{2}{m_\phi^2\, \phi_0^2}\,. \tag{2.35}$$

We present our results in figure 2.9. We report them in terms of q and m_X, in order to have an immediate comparison to fig. 1.1 of the previous chapter.

In figure 2.9, for any fixed q, the greatest plotted value for m_X corresponds to the choice $R = 5$. We are not interested in extending this limit since we know that for greater m_X (actually for values greater than the bound $m_X \sim \sqrt{q}/2$) the production suddenly stops. The smallest value plotted for m_X (at any fixed value q) corresponds instead to $R = 10000$, that is to considering more than 3700 productions in the numerical evaluation of eq. (2.30).

Our final values are in good agreement with the ones of fig. 1.1 in the regime of validity of the latter. The numerical results reported in that figure exhibit small fluctuations about an average function $\rho_X(m_X)$. Our results give this average function.

[9]This ratio should be calculated at a time $t_{\rm end}$ at the end of preheating, when the total fermionic mass stops vanishing. Thus in the denominator of eq. (2.35) the comoving momentum k should be replaced by the physical one $p = k/a$, with a scale factor of the Universe at $t_{\rm end}$. Analogously, the value ϕ_0 of the beginning of reheating should be replaced by the one at $t_{\rm end}$. However, both the replacements cancel out in the ratio, since both ρ_X and ρ_ϕ redshift as energy densities of matter.

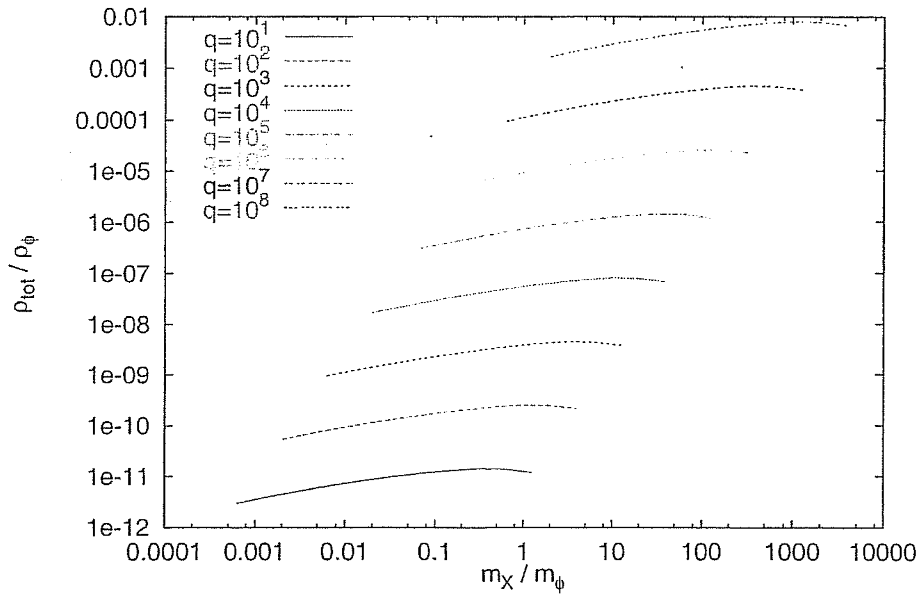

Figure 2.9: Total final energy density produced (normalized to the inflaton one) for different values of q and m_X.

This was expected, since the expression that we integrated, eq. (2.30), interpolates between the maxima and the minima of the numerical spectra.

The numerical results of the previous chapter have a smaller range of validity than the ones here presented. This occurs because the numerical evolution of that work is limited to the first 20 oscillations of the inflaton field, and so fermionic production has not come to its end for small values of m_X. We confirm that at high values of m_X (actually at small values of R for any given q) the production depends very weakly on m_X. In addition, our results show a decrease of the energy transferred to fermions for smaller values of m_X. This behavior will be explained in the next subsection.

2.3.2 Analytical results

We want now to show that all the results presented in the previous subsection can be also achieved with a full analytical study of eq. (2.30).

First of all, we have to estimate the quantities z_i given in eq. (2.34). To do this, it is more convenient to work in terms of the physical time t: after the first few oscillations, the inflaton evolution is very well approximated by the expression (remember t is expressed in units of m_ϕ^{-1}, while ϕ in units of M_p)

$$\phi(t) \simeq \frac{1}{\sqrt{3\pi}} \frac{\cos(t)}{t} . \tag{2.36}$$

The scale factor of the Universe follows the "matter-domination" law, and it is well approximated by $a(t) = t^{2/3}$.

The values t_{*i} are determined by the condition of vanishing of the total mass of the fermions, that is, by making use of eq. (2.36),

$$\cos(t_{*i}) = -\frac{t_{*i}}{RA} , \tag{2.37}$$

where we remind $R \equiv (2\sqrt{q})/m_X$. The parameter $A \equiv (\sqrt{3\pi}\,\phi_0)^{-1}$ is of order one and will not play any special role in what follows. Notice that the last production occurs at $t \simeq RA$.

Hence, keeping only the dominant contribution to the derivative of ϕ with respect to the physical time, we get the expression

$$z_i^2 \cong \frac{2\sqrt{q}}{\pi} R^{1/3} A^{4/3} \left(\frac{t_{*i}}{RA}\right)^{1/3} \sqrt{1 - \left(\frac{t_{*i}}{RA}\right)^2} , \tag{2.38}$$

where we can assume $t_{*i} \simeq i\pi$.

Equation (2.38) exhibits a very good agreement with the numerical evaluation of the same quantity. It also shows that the maximal value for z_i is reached at $t_{*i} = AR/2$, that is, at half of the whole process of non-perturbative creation. This was anticipated in the previous section, where we showed that the most of the fermionic production occurs in the first part of preheating.

Starting from eq. (2.38) we can also calculate the number density of produced particles

$$N_X(q, R) = \frac{2}{\pi^2} \int dk\, k^2 N(k) , \tag{2.39}$$

where $N(k)$ is obtained from eq. (2.30) with $n = n_{\max} = (RA)/\pi$.

For R big enough, the product in eq. (2.30) can be written as the exponential of an integral. Thus, we obtain

$$\begin{aligned} N_X &= \frac{m_X}{4\pi^3} \left(\frac{2A^{4/3}}{\pi}\right)^{3/2} q^{3/4} R^{1/2} \times \\ &\times \int_0^\infty dt\, t^2 \left\{ 1 - \exp\left[\frac{RA}{\pi} \int_0^1 dy \log\left| 1 - 2\exp\left(-\frac{t^2}{y^{1/3}\sqrt{1-y^2}} \right) \right| \right] \right\} , \end{aligned} \tag{2.40}$$

with the substitution $t = k \cdot \sqrt{\pi/\left(2A^{4/3}q^{1/2}R^{1/3}\right)}$.

The integral in dy which appears in eq. (2.40) cannot be calculated analytically. Anyhow, we can approximate it by

$$\int_0^1 dy \log\left|1 - 2\exp\left(-\frac{t^2}{y^{1/3}\sqrt{1-y^2}}\right)\right| \simeq g\left(t\right)\log\left|1 - 2e^{-1.5t^2}\right|, \tag{2.41}$$

where $g(t)$ is a function of order one that, for our purposes, can be approximated by a constant c in the range $0.5 \lesssim c \lesssim 1$.

Hence, the integrand within curly brackets in eq. (2.40) rewrites

$$1 - \left|1 - 2e^{-1.5t^2}\right|^{c\frac{RA}{\pi}}. \tag{2.42}$$

In the large-R limit this function approximates a step function, which evaluates to one for

$$\sqrt{\frac{\pi \log 2}{2RAc}} \lesssim t \lesssim \sqrt{\log\left(\frac{2RAc}{\pi \log 2}\right)} \tag{2.43}$$

and to zero for the remaining values of t.

Since the quantity (2.42) is proportional to the occupation number n_k, our analytical calculation confirms the usual assumption that, after few oscillations, the fermionic production saturates the Fermi sphere up to a given maximum momentum k_{max}. This is also shown in the previous figure 2.7. From eq. (2.43) it follows (apart from proportionality factors) [10]

$$k_{\rm max} \propto \frac{q^{1/3}}{m_X^{1/6}} \sqrt{\log\left(\frac{q^{1/2}}{m_X}\right)}. \qquad (2.46)$$

From eqs. (2.40) and (2.43), we obtain the final expression for the number density of the fermions created during the whole process

$$N_X\left(q,R\right) = \frac{1}{3\pi^2}\left(\frac{2A^{4/3}}{1.5\pi}\right)^{3/2} q^{3/4}R^{1/2}\left[\log\left(\frac{4Ac}{\pi\log 2}\frac{q^{1/2}}{m_X}\right)\right]^{3/2}. \qquad (2.47)$$

We can now go back to figure 2.8, where this last equation (called "analytical" in the figure) is compared to the result with the semi-analytical method of the previous subsection. In plotting eq. (2.47) we chose $c = 0.78$ for the numerical factor involved. As we can see, the final results achieved with the two methods are in very good agreement with each other, thus confirming the validity of the formula (2.47).[11]

Rewriting eq. (2.47) in terms of q and m_X we can draw some conclusions. First, apart from a logarithmic correction, the scaling of the total energy

$$\rho_X \propto m_X N_X \propto q m_X^{1/2}\left[\log\left(\frac{4Ac}{\pi\log 2}\frac{q^{1/2}}{m_X}\right)\right]^{3/2} \qquad (2.48)$$

[10]The scaling (2.46) can be also achieved from very immediate considerations. From the analytical formula (2.28), we notice that at high momenta k the occupation number is well approximated by

$$N_n\left(k\right) \simeq \sum_{i=1}^{n} e^{-k^2/z_i^2}\,, \qquad (2.44)$$

where we remember $z_i \propto q^{1/4}a^{1/2}(\eta_{*i})|\phi'(\eta_{*i})|^{1/2}$. In this last equation, we replace all the parameters z_i with a mean value $\bar{z}$, so that $N_n \sim n\exp(-k^2/\bar{z}^2)$. The scaling of $\bar{z}$ with the physical parameters q and m_X follows from the scaling of all the z_i. The maximal momentum $k_{\rm max}$ is thus expected to scale as the quantity $z_i(\log n)^{1/2}$. Considering now the evolution of the inflaton field in physical time t, we notice that both the number n of productions and the times t_{*i} at which they occur are proportional to the parameter $R = q^{1/2}/(2m_X)$. Moreover, we see that the z_i's scale as

$$z_i \propto q^{1/4}a_{*i}\left[\frac{d\phi}{dt}|_{*i}\right]^{1/2} \propto q^{1/4}t_{*i}^{2/3}\frac{1}{t_{*i}^{1/2}} \propto q^{1/4}R^{1/6}\,. \qquad (2.45)$$

The result (2.46) then immediately follows.

[11]The small discrepancy between the two curves can be attributed to the fact that c is not exactly constant.

is linear in q, as expected from the numerical results (see the previous chapter). The dependence of ρ_X on m_X requires some more care: the threshold value for m_X is given by the condition $R \sim 4$, that is,[12]

$$(m_X)_{\text{th}} \sim \frac{\sqrt{q}}{2}\,. \tag{2.49}$$

For values of m_X not much smaller than $(m_X)_{\text{th}}$, the total energy depends very weakly on m_X, this result being in agreement with the numerical evaluation of the previous chapter. On the other hand, for values of m_X much smaller than $(m_X)_{\text{th}}$, the factor $m_X^{1/2}$ starts to dominate, and we expect it to determine the scaling of the total energy when $m_X \to 0$.

2.4 Backreaction

The analytical results presented so far have been achieved neglecting the backreaction of the produced fermions on the evolution of the inflaton field and on the scale factor. As in the previous chapter, we will estimate now the effects of the backreaction in the Hartree approximation.

We remind that for what concerns preheating of fermions, the Hartree approximation consists in taking into account the term

$$g\langle \bar{X}X\rangle \tag{2.50}$$

into the evolution equations for the inflaton and the scale factor. The equation for the field ϕ thus rewrites (in physical time)

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi + g\langle \bar{X}X\rangle = 0\,. \tag{2.51}$$

The numerical study of the previous chapter indicates that backreaction starts to be important for $q > q_b \sim 10^8 - 10^{10}$. Figure 1.7 shows how the evolution of the inflaton field ϕ is modified when backreaction is considered and q is sufficiently high. First, one observes that the amplitude of the oscillations of the inflaton is very damped already after the first production. This effect is the most obvious one, since the term (2.50) takes into account the decay of the inflaton into fermion-antifermion pairs, while in its absence the equation for ϕ considers only the damp due to the expansion of the Universe. The second feature that emerges from the numerical simulations is that at the beginning the field ϕ does not oscillate about the minimum of the potential $V = m_\phi^2\phi^2/2$, but about the point ϕ_* where the total

[12]The number 4 comes from the fact that the value of the inflaton at its first minimum is $\phi \simeq -0.07M_p$, while at beginning $\phi_0 = 0.28M_p$.

fermionic mass vanishes. Moreover, the frequency of these oscillations is higher than m_ϕ.

These last two effects are due to the change in the effective potential for ϕ induced by the term (2.50), and disappear when the quantity $\langle\bar{X}X\rangle$ is decreased by the expansion of the Universe. Their net effect is to render the whole mechanism of preheating more efficient, since the rise in the frequency of the oscillations of ϕ increases the number of productions of fermions. In the numerical examples of the previous chapter it is indeed shown that for $q = 10^8$ the total production is about 5% larger than the one without backreaction, while for $q = 10^{10}$ the increase is about 50%.

We now briefly study the evolution of the inflaton ϕ under eq. (2.51) by means of the analytical results presented above. We show that even a very approximate analysis confirms the numerical result that indicates in $q \simeq 10^8 - 10^{10}$ the threshold above which backreaction should be considered.

To begin on, the term (2.50) needs to be normal ordered. This gives eq. (1.28), which, in terms of the Bogolyubov coefficients, evaluates to

$$\langle\bar{X}X\rangle = \frac{4}{(2\pi a)^3}\int d^3\mathbf{k}\left[|\beta|^2\frac{am}{\omega} + \mathrm{Re}\left(\alpha\beta\frac{k}{\omega}\,e^{2i\int\omega\,d\eta}\right)\right]. \tag{2.52}$$

We see that $\langle\bar{X}X\rangle$ vanishes for $\beta = 0$. This is obvious, since backreaction starts only after fermions are produced. Some approximations can render eq. (2.52) more manageable. First, we notice that the oscillating term in the exponential averages to very small values the integral of the second term in square brackets. Second, we see that $k \ll |am|$ where β_k is significantly different from zero. From both these considerations, the integrand in eq. (2.52) can be approximated (up to the sign of m) by the occupation number $|\beta|^2$, so that the whole effect is (approximatively) proportional to the number of produced fermions.

The previous numerical analysis show that, for the values of q for which the backreaction is to be considered, its effects can be seen already in the first oscillation of the inflaton field. Since we are only interested in estimating the order of magnitude of q_b, we thus concentrate on the first oscillation of ϕ, neglecting the expansion of the Universe in this short interval.

With all these approximations, eq. (2.51) rewrites

$$y'' + y + 10^{-12}q^{5/4}\frac{m}{|m|} = 0\,, \tag{2.53}$$

where we have rescaled $y \equiv \phi/\phi_0$ and we remind that the time is given in units of m_ϕ^{-1}.

This equation is very similar to the one obtained in the bosonic case [16]. The last term changes sign each time $m = 0$ and, when sufficiently high, forces the inflaton field to oscillate about the point ϕ_* at which the total fermionic mass vanishes. It is also responsible for the increase of the frequency of the oscillations. To see this, we assume that this term dominates over the second one of eq. (2.53) and we solve it right after the first fermionic production at the time t_{*1}. In the absence of the second term, eq. (2.53) is obviously solved by a segment of parabola until the time $t_{*2} \simeq t_{*1} + 2|y'(t_{*1})|/(q^{5/4}10^{-12})$ at which m vanishes again and the last term of eq. (2.53) changes sign. As long as the second term of eq. (2.53) can be neglected, the inflaton evolution proceeds along segments of parabola among successive zeros of the mass m. The "time" duration of these segments is expected to be of the same order of the first one, since the successive fermionic productions balance the decrease of $\langle \bar{X}X \rangle$ due to the expansion of the Universe.[13]

The period of these oscillations can thus be roughly estimated to be $T \sim 2(t_{*2} - t_{*1})$. We see that, for $q \gtrsim 10^9$, this period is smaller than the one that the inflaton oscillations would have neglecting backreaction. Since this increase of the frequency is the main responsible for the higher fermionic production, the result $q_b \sim 10^9$ can be considered our analytical estimate for the value of q above which backreaction should be taken into account.

This result, although obtained with several approximations, is in agreement with the numerical one of the previous chapter.

[13]However, after the first part of the process, the production looses its efficiency and the expansion of the Universe dominates. As we have said, the term $\langle \bar{X}X \rangle$ can then be neglected and the inflaton starts oscillating about the minimum of the tree level potential.

Chapter 3

Leptogenesis and preheating

After the inflationary expansion, the Universe is practically empty of matter and therefore it looks baryon symmetric. However, considerations about how the light element abundances were formed when the Universe was about 1 MeV hot lead us to conclude that a very tiny asymmetry between baryon and antibaryon existed at that time. This asymmetry can be quantified to $n_B/s = (2\text{–}9) \times 10^{-11}$, where n_B/s is the difference between the number density of baryons and that of antibaryons, normalized to the entropy density of the Universe.

Until now, several mechanisms for the generation of the baryon (B) asymmetry have been proposed (for a review, see for example [61]). Grand Unified Theories (GUTs) unify the strong with the electroweak forces and predict baryon-number violating interactions at tree level. In these theories, the out-of-equilibrium decay of heavy Higgs particles can indeed explain the observed baryon asymmetry. In the theory of electroweak baryogenesis, baryon number violation takes place at the quantum level, caused by unsuppressed baryon-number violating sphaleron transitions in the hot plasma [62].

Since B and L – where L is the lepton number – are reprocessed by sphaleron transitions, while the anomaly-free linear combination $B - L$ is left unchanged, the baryon asymmetry may be generated from a lepton asymmetry [50]. Indeed, once the lepton number is produced, thermal scatterings redistribute the charges and convert (a fraction of) L into baryon number. In the high-temperature phase of the standard model, the asymmetries of baryon number B and of $B - L$ are therefore proportional [63, 64]:

$$B = a(B - L), \qquad a \equiv \left(\frac{8n_g + 4n_H}{22n_g + 13n_H}\right), \tag{3.1}$$

where n_H is the number of Higgs doublets and n_g is the number of fermion generations.

In the standard model as well as in its unified extension based on the group $SU(5)$, $B - L$ is conserved and no asymmetry in $B - L$ can be generated. However, adding massive right-handed Majorana neutrinos to the standard model breaks $B-L$ and the primordial lepton asymmetry may be generated by their out-of-equilibrium decay. This simple extension of the standard model can be embedded into GUTs, as in the case of $SO(10)$. Heavy right-handed Majorana neutrinos are the key ingredient to explain the smallness of the light neutrino masses via the see-saw mechanism [65]. The presence of neutrino masses and mixings seems to be the most natural explanation of the recent reports from the Super-Kamiokande and other collaborations indicating the existence of neutrino oscillations. In light of these considerations, the generation of the baryon asymmetry through leptogenesis looks particularly attractive.

The leptogenesis scenario depends crucially on the mechanism that was responsible for populating the early Universe with right-handed neutrinos, and consequently on the thermal history of the Universe, and on the fine details of the reheating process after inflation. The goal of this chapter is to discuss several production mechanisms of heavy right-handed neutrinos in the Universe, to compare them and to identify the regions of the appropriate parameter space where the production mechanism is efficient enough to explain the observed baryon asymmetry.

The simplest way to envision the reheating process after inflation is to assume that the comoving energy density in the zero mode of the inflaton decays *perturbatively* into ordinary particles, which then scatter to form a thermal background [66, 67]. It is usually assumed that the decay width of this process is the same as the decay width of a free inflaton field. Of particular interest is a quantity known as the reheat temperature, denoted as T_{RH}. This is calculated by assuming an instantaneous conversion of the energy density in the inflaton field into radiation when the decay width of the inflaton Γ_ϕ is equal to H, the expansion rate of the Universe. This yields

$$T_{RH} \simeq \sqrt{\Gamma_\phi M_{\rm p}}, \tag{3.2}$$

where $M_{\rm p}$ is the Planck mass.

The commonly-accepted assumption is that the heavy right-handed neutrinos with mass M_N were as abundant as photons at very high temperatures. This assumption requires not only that $T_{RH} \gtrsim M_N$, but also that the heavy neutrinos are abundantly produced by thermal scatterings during the reheating stage. This condition, as we will discuss in the first section, significantly limits the allowed range of neutrino masses compatible with leptogenesis.

There might be one more problem associated with the hypothesis that $T_{RH} \gtrsim M_N$ in the old theory of reheating, and that is the problem of relic gravitinos [33, 34, 35, 36]. If one has to invoke supersymmetry to preserve the flatness of the inflaton potential, it is mandatory to consider the cosmological implications of the gravitino – the spin-3/2 partner of the graviton which appears in the extension of global supersymmetry to supergravity. The slow gravitino decay rate leads to a cosmological problem because the decay products of the gravitino destroy light nuclei by photodissociation and hadronic showers,thus ruining the successful predictions of nucleosynthesis. The requirement that not too many gravitinos are produced after inflation provides an upper bound[1] on the reheating temperature T_{RH} of about 10^8–10^{10} GeV, depending on the value of the gravitino mass (for a review, see [68]). In the following, T_{RH} will be therefore intended as the largest temperature allowed after inflation from considerations of the gravitino problem.

In order to relax the limit on M_N imposed by the gravitino problem, we will consider the possibility that the heavy neutrinos are produced directly through the inflaton decay process. This is kinematically accessible whenever

$$M_N < m_\phi, \tag{3.3}$$

where m_ϕ is the inflaton mass. In the case of chaotic inflation with quadratic potential, the density and temperature fluctuations observed in the present Universe determine m_ϕ and require M_N to be smaller than about 10^{13} GeV.

However, what is most interesting for us is the possibility that the right-handed neutrinos are produced nonthermally at preheating.[2] As we have seen in the previous chapters, fermions can be produced at preheating up to very high masses, even as high as 10^{18} GeV. This can offer in principle an unique way to evade the limit $M_N < 10^{13}$ GeV which was mandatory with perturbative production. We want also to stress that the out-of-equilibrium condition is naturally achieved in this scenario, since the distribution function of the fermionic quanta generated at the resonance is far from a thermal distribution. We will show that the observed baryon asymmetry may be explained by the phenomenon of leptogenesis after preheating, with a reheating temperature compatible with the gravitino problem. As we will see, this scenario can be successful even for to N masses as high as 10^{15} GeV.

This chapter is organized as follows. In section 3.1 we briefly review the leptogenesis scenario. In sections 3.2, 3.3, and 3.4 we discuss leptogenesis with right-handed neutrinos produced with the three different mechanisms listed above. A comparison of the three different cases is finally performed in section 3.5.

3.1 Leptogenesis

The Lagrangian terms relevant for leptogenesis describe the interactions between the massive right-handed neutrinos N, the lepton doublet ℓ_L, and the Higgs doublet

[1] With this value, we do not consider bounds coming from the nonperturbative production [29, 30, 31, 32] since they are more model dependent.

[2] Preheating in connection with baryogenesis was already considered in ref. [19], where the nonperturbative production of very heavy Higgs bosons inducing GUT baryogenesis is studied.

H,

$$\mathcal{L} = -\bar{N} h_\nu H \ell_L - \frac{1}{2} \bar{N}^c M N + \text{h.c.} \tag{3.4}$$

Here the Yukawa couplings h_ν and the Majorana mass M are 3×3 matrices and generation indices are understood. We choose to work in a field basis in which M is diagonal with real and positive eigenvalues ordered increasingly ($M_1 < M_2 < M_3$). The mass matrix of the light, nearly left-handed, neutrinos is given by

$$m_\nu = -h_\nu^T M^{-1} h_\nu \langle H \rangle^2. \tag{3.5}$$

The decays of the heavy neutrinos N into leptons and Higgs bosons violate lepton number

$$\begin{aligned} N &\to \bar{H}\ell, \\ N &\to H\bar{\ell}. \end{aligned} \tag{3.6}$$

The interference between the tree-level decay amplitude and the absorptive part of the one-loop diagram can lead to a lepton asymmetry of the right order of magnitude to explain the observed baryon asymmetry, as has been extensively discussed in the literature (for reviews, see refs. [69, 70]).

The interference with the one-loop vertex amplitude yields a CP-violating decay asymmetry for N_1 equal to

$$\epsilon_V = \frac{1}{8\pi \left(h_\nu h_\nu^\dagger\right)_{11}} \sum_{j=2,3} \text{Im}\left[\left(h_\nu h_\nu^\dagger\right)_{1j}\right]^2 f\left(\frac{M_j^2}{M_1^2}\right), \tag{3.7}$$

$$f(x) = \sqrt{x}\left[1 - (1+x)\ln\left(\frac{1+x}{x}\right)\right]. \tag{3.8}$$

where, as customary,

$$\epsilon \equiv \frac{\Gamma\left(N \to \bar{H}\ell\right) - \Gamma\left(N \to H\bar{\ell}\right)}{\Gamma\left(N \to \bar{H}\ell\right) + \Gamma\left(N \to H\bar{\ell}\right)}. \tag{3.9}$$

The absorptive part of the one-loop self-energy gives a contribution to the N_1 asymmetry which, in the case of only two-generation mixing, is given by

$$\epsilon_S = \frac{\text{Im}\left[\left(h_\nu h_\nu^\dagger\right)_{1j}\right]^2}{\left(h_\nu h_\nu^\dagger\right)_{11}\left(h_\nu h_\nu^\dagger\right)_{22}} \left[\frac{(M_1^2 - M_2^2) M_1 \Gamma_{N_2}}{(M_1^2 - M_2^2)^2 + M_1^2 \Gamma_{N_2}^2}\right]. \tag{3.10}$$

Here Γ_{N_i} is the total decay rate of the right-handed neutrino N_i,

$$\Gamma_{N_i} = \frac{\left(h_\nu h_\nu^\dagger\right)_{ii}}{8\pi} M_i. \tag{3.11}$$

The CP-violating asymmetry ϵ_S is enhanced when the mass difference between two heavy right-handed neutrinos is small, although not smaller than the decay width.

The total CP asymmetry ϵ has an involved dependence on the complete structure of the neutrino matrices h_ν and M. However, let us assume that the lepton asymmetry is generated only at the decay of the lightest right-handed neutrino N_1. This hypothesis is satisfied if the N_1 interactions are in equilibrium at the time of the $N_{2,3}$ decay (erasing any produced asymmetry), or if $N_{2,3}$ are too heavy to be produced after inflation. As will be made clear in the following, this is a very plausible working assumption. In this case, the dynamics of leptogenesis can be described in terms of only 3 parameters [3] : ϵ, M_1, and

$$\tilde{m}_1 \equiv \left(h_\nu h_\nu^\dagger\right)_{11} \langle H \rangle^2 / M_1. \tag{3.12}$$

The parameter $\tilde{m}_1$, which determines the relevant interactions of N_1, coincides with the light neutrino mass m_{ν_1} only in the limit of small mixing angles, see eq. (3.5).

In the following we will be mostly concerned with the production mechanisms of N_1 in the early cosmology. We will discuss a variety of these mechanisms and identify, in the different ranges of M_1 and $\tilde{m}_1$, the size of ϵ required to generate the appropriate baryon asymmetry, $n_B/s \sim (2\text{–}9) \times 10^{-11}$. Our results can be used to check if specific particle-physics models for neutrino mass matrices are compatible with the various leptogenesis mechanisms.

3.2 Thermal Production of the right-handed neutrinos

We start by considering the case in which the right-handed neutrino N_1 reaches thermal equilibrium by scattering with the bath after the inflaton decay. The amount of lepton asymmetry generated by the N decay can then be computed by integrating the appropriate Boltzmann equations [71, 72].

A measure of the efficiency for producing the asymmetry is given by the ratio K of the thermal average of the N_1 decay rate and the Hubble parameter at the temperature $T = M_1$,

$$K \equiv \left.\frac{\Gamma_{N_1}}{2H}\right|_{T=M_1} = \frac{\tilde{m}_1}{2 \times 10^{-3}\ \mathrm{eV}}. \tag{3.13}$$

[3] The other neutrino mass parameters come into play only for very large values of $\tilde{m}_i \equiv \left(h_\nu h_\nu^\dagger\right)_{ii} \langle H \rangle^2 / M_i$ ($i = 2, 3$), when the lepton-number violating interactions mediated by N_2 or N_3 can partially erase the lepton asymmetry.

Here we have expressed the N_1 decay width, see eq. (3.11), in terms of the parameters m_1 and M_1 as

$$\Gamma_{N_1} = \frac{G_F}{2\sqrt{2}\pi} m_1 M_1^2. \tag{3.14}$$

For $m_1 \lesssim 2 \times 10^{-3}$ eV, K is less than unity and the decay process is out of equilibrium when N_1 becomes non-relativistic. Under these conditions, the leptogenesis becomes very efficient. Indeed, the produced baryon asymmetry approaches its theoretical maximum value obtained by assuming that each N_1 in thermal equilibrium eventually generates ϵ baryons,

$$\left(\frac{n_B}{s}\right)_{\max} = \frac{135\ \zeta(3)\ a}{4\pi^4 g_*}\epsilon = 1 \times 10^{-3}\epsilon. \tag{3.15}$$

Here a is defined in eq. (3.1) and g_* counts the number of degrees of freedom (for the standard model particle content, $a = 28/79$ and $g_* = 427/4$).

For very small m_1, $K \ll 1$ and N_1 decouples when it is still relativistic. At temperatures T below M_1, the N_1 contribution to the energy density red-shifts like matter and therefore $\rho_{N_1}/\rho_{total} = (7M_1)/(4g_*T)$. Eventually N_1 matter-dominates the Universe at a temperature

$$T_{dom} = \frac{7\ M_1}{4\ g_*} \simeq 2 \times 10^{-2} M_1. \tag{3.16}$$

However this can happen only if N_1 does not decay beforehand. Since the decay temperature is

$$T_* = 0.8 g_*^{-1/4}\sqrt{\Gamma_{N_1} M_p} = \left(\frac{m_1}{10^{-6}\ \text{eV}}\right)^{1/2}\left(\frac{M_1}{10^{10}\ \text{GeV}}\right)\ 3 \times 10^8\ \text{GeV}, \tag{3.17}$$

neutrino matter-domination occurs when

$$m_1 < 4 \times 10^{-7}\ \text{eV}. \tag{3.18}$$

Under these conditions, the bulk of the energy of the Universe is stored in the non-relativistic N_1. At the time of decay, such energy density is converted into relativistic degrees of freedom whose temperature coincides with T_* given in eq. (3.17),

$$\rho_{N_1} = M_1 n_{N_1} = \frac{\pi^2}{30} g_* T_*^4. \tag{3.19}$$

This yields the following baryon asymmetry

$$\frac{n_B}{s} = \epsilon a \frac{n_{N_1}}{s} = \epsilon a \frac{3T_*}{4M_1} = \left(\frac{m_1}{10^{-6}\ \text{eV}}\right)^{1/2} 8 \times 10^{-3}\epsilon. \tag{3.20}$$

In order not to reintroduce the cosmological gravitino problem one has to require that the temperature T_* after the right-handed neutrino decay is less than the maximum value allowed T_{RH}. This implies

$$m_1 < \left(\frac{T_{RH}}{M_1}\right)^2 10^{-3} \text{ eV}. \quad (3.21)$$

Therefore, the leptogenesis process is very efficient also when T_{RH} is low enough to suppress gravitino production.

For $K \gg 1$, the departure from thermal equilibrium is reduced, and leptogenesis is less efficient. Larger values of ϵ are now required. However, for $m_1 \sim 10^{-2}$ eV, values of ϵ of about 10^{-5} are sufficient to generate the appropriate baryon asymmetry. Realistic neutrino mass matrices can comfortably reproduce such values of ϵ. Notice that the prediction for the baryon asymmetry depends weakly on M_1, as long as m_1 is not too large [72]. This is because both the production and decay thermal rates of N_1 at $T = M_1$ depend only on m_1, while the M_1 dependence arises from lepton-violating H–ℓ scattering.

Let us now turn to discuss how primordial thermal equilibrium of N_1 can be achieved. The first necessary condition is

$$M_1 < T_{RH}. \quad (3.22)$$

This can be quite constraining, especially in view of the bound derived from the disruptive gravitino effects on nucleosynthesis mentioned at the beginning of the chapter. When the inequality (3.22) is not satisfied, one expects the number density of the N_1-particles generated during the reheating stage to be quite small, making this case marginal, as far as the generation of baryon number is concerned. We would only like to mention here that such a number density depends upon the fine details of the dynamics of the reheating stage itself. In particular, the reheat temperature T_{RH} is not the maximum temperature obtained after inflation; the maximum temperature is, in fact, much larger than T_{RH} [73, 74]. As a result, the abundance of massive particles may be suppressed only by powers of the mass over the temperature, and not exponentially.

A second condition for N_1 thermalization is derived from the requirement that inverse decay or production processes of the kind $\bar{\ell} q_{(3)} \to N_1 t$ (mediated by Higgs-boson exchange) are in thermal equilibrium before N_1 becomes non-relativistic. This implies

$$m_1 \gtrsim 10^{-3} \text{ eV}. \quad (3.23)$$

This condition excludes the possibility of the most efficient leptogenesis with $K < 1$. However, even if m_1 is somewhat smaller than the value indicated by eq. (3.23), a sufficient number of N_1 can be produced. Indeed, for $m_1 \gtrsim 10^{-5}$ eV, values of $\epsilon \gtrsim$

10^{-5} can give rise to the observed baryon asymmetry. In the case of supersymmetric models, the constraint can be even less stringent [75], and values $\epsilon \gtrsim 10^{-5}$ are sufficient for $m_1 \gtrsim 10^{-6}$ eV.

The constraint on m_1 from thermalization can be evaded if new interactions, different from the ordinary Yukawa forces, bring N_1 in thermal equilibrium at high temperatures. For instance, one could use the extra $U(1)$ gauge interactions included in $SO(10)$ GUTs. These interactions can produce a thermal population of N_1 if, at $T = T_{RH}$,

$$\Gamma(\bar{f}f \to Z' \to NN) = \frac{169\ \alpha_{GUT}^2\ T^5}{3\pi\ M_{Z'}^4} > H. \tag{3.24}$$

This requires that the mass of the extra gauge boson $M_{Z'}$ should be close to T_{RH} and significantly lower than the GUT scale,

$$M_{Z'} < \left(\frac{T_{RH}}{10^{10}\ \text{GeV}}\right)^{3/4} 4 \times 10^{11}\ \text{GeV}. \tag{3.25}$$

3.3 Production at Reheating

Since it is very likely that the short period of preheating does not fully extract all of the energy density from the inflaton field, the Universe will enter a long period of matter domination after preheating where the dominant contribution to the energy density of the Universe is provided by the residual small amplitude oscillations of the classical inflaton field and/or by the inflaton quanta produced during the back-reaction processes. This period will end when the age of the Universe becomes of the order of the perturbative lifetime of the inflaton field. At this point the Universe will go through a period of reheating with a reheat temperature T_{RH} given by the perturbative result in eq. (3.2).

Let us suppose that the inflaton couples to N_1, either directly or through exchange of other particles. In this case, the inflaton decay process can generate a right-handed neutrino primordial population. The condition in eq. (3.22) is replaced by the weaker constraint

$$M_1 < m_\phi, \tag{3.26}$$

where m_ϕ is the inflaton mass.

The fate of the right-handed neutrinos produced by the inflaton decay depends on the parameter choice. If $M_1 < T_{RH}$ and $m_1 \gtrsim 10^{-3}$ eV, the Yukawa couplings are strong enough to bring N_1 into thermal equilibrium, and leptogenesis can proceed as in the usual scenario described in the previous section.

Let us now assume that the Yukawa couplings are much smaller, and that the right-handed neutrino decay temperature in eq. (3.17) satisfies $T_* < T_{RH}$, *i.e.* $m_1 <$

$(T_{RH}/M_1)^2 10^{-3}$ eV. After reheating, the N_1 behave like frozen-out, non-thermal, relativistic particles with typical energy $E_{N_1} \simeq m_\phi/2$. The N_1 population will become non-relativistic at a temperature $T_{NR} = T_{RH} M_1/E_{N_1}$. At this moment, the energy of the Universe is shared between the radiation and the N_1 component, with a ratio of the corresponding energy densities which has remained constant between T_{RH} and T_{NR},

$$\left.\frac{\rho_{N_1}}{\rho_R}\right|_{T=T_{NR}} = \left.\frac{\rho_{N_1}}{\rho_R}\right|_{T=T_{RH}} \tag{3.27}$$

$$\rho_R|_{T=T_{RH}} = \frac{\pi^2}{30} g_* T_{RH}^4 \qquad \rho_{N_1}|_{T=T_{RH}} = E_{N_1} n_{N_1} = \frac{m_\phi}{2} B_\phi n_\phi. \tag{3.28}$$

Here n_ϕ is the inflaton number density just before decay, obtained by requiring energy conservation

$$n_\phi|_{T=T_{RH}} = \frac{\pi^2 g_* T_{RH}^4}{30 m_\phi (1 - B_\phi/2)}, \tag{3.29}$$

and B_ϕ describes the average number of N_1 produced in a ϕ decay. Below T_{NR}, the N_1 density red-shifts like matter and eventually dominates the Universe at a temperature

$$T_{dom} = \frac{B_\phi}{(1 - B_\phi/2)} \left(\frac{M_1}{m_\phi}\right) T_{RH}. \tag{3.30}$$

Therefore, if

$$m_1 > \left(\frac{B_\phi}{1 - B_\phi/2}\right)^2 \left(\frac{T_{RH}}{10^{10}\ \text{GeV}}\right)^2 \left(\frac{10^{13}\ \text{GeV}}{m_\phi}\right)^2 1 \times 10^{-9}\ \text{eV}, \tag{3.31}$$

then $T_* > T_{dom}$ and N_1 decays before dominating. In this case, the baryon asymmetry is determined to be

$$\frac{n_B}{s} = \epsilon a \frac{n_{N_1}}{s} = \frac{3\ \epsilon a\ B_\phi\ T_{RH}}{4(1 - B_\phi/2) m_\phi}. \tag{3.32}$$

If the inequality (3.31) is not satisfied, N_1 matter-dominates the Universe and we recover the baryon asymmetry result in eq. (3.20).

A necessary condition to be satisfied is that lepton-number violating interactions mediated by N_i ($i = 1, 2, 3$) exchange are out of equilibrium at the temperature of N_1 decay,

$$\Gamma_{\Delta L} = \frac{4}{\pi^3} G_F^2 m_i^2 T^3 < H \quad \text{at } T = T_*, \tag{3.33}$$

where T_* is given in eq. (3.17). This implies

$$m_1 < \left(\frac{10^{12}\ \mathrm{GeV}}{M_1}\right)^{2/5} 0.1\ \mathrm{eV} \quad \text{and} \quad m_1 < \left(\frac{10^{12}\ \mathrm{GeV}}{M_1}\right)^2 \left(\frac{2\times 10^{-3}\ \mathrm{eV}^2}{\sum_{i=2,3} m_i^2}\right)^2 2\ \mathrm{eV}. \tag{3.34}$$

Finally, we discuss the case $T_* > T_{RH}$, *i.e.* $m_1 > (T_{RH}/M_1)^2 10^{-3}$ eV, in which N_1 decays immediately after it is produced. In this case, the baryon asymmetry is still given by eq. (3.32), but the out-of-equilibrium condition of lepton-violating interactions has to be imposed at $T = T_{RH}$. Therefore, eq. (3.34) is replaced by

$$m_i < \left(\frac{10^{10}\ \mathrm{GeV}}{T_{RH}}\right)^{1/2} 3\ \mathrm{eV}, \qquad i = 1,2,3. \tag{3.35}$$

The combination of the bounds shows that lepton-violating interactions do not give severe constraints on the parameters.

3.4 Production at Preheating

As we have seen in the previous chapters, heavy fermions are efficiently produced in a non-thermal state during the preheating stage. We can now apply those results to the case in which the produced fermions are the right-handed neutrinos from which leptogenesis originate[4]. In the previous investigations, we have tacitly assumed that the heavy fermions were stable. Of course, the parametric resonance is affected by a nonvanishing decay width of the N_1. However, contrary to what happens for bosons where the presence of a large decay width removes the particles from the resonance bands rendering the preheating less efficient [20], for fermions the presence of a decay width might be even beneficial. Indeed, for stable right-handed neutrinos the distribution function $n(k)$ is rapidly saturated to unity and further particle production is Pauli-blocked. However, if the decay width is large enough, the right-handed neutrinos may be produced at each inflaton oscillation when $m(t) \simeq 0$ and then decay right away. This will give rise to a certain amount of lepton asymmetry at each inflaton oscillation until the total mass (1.2) stops vanishing; the lepton asymmetry would be generated in a cumulative way. Strictly speaking, however, the calculation of fermion preheating presented in the previous chapters applies only to the case in which the right-handed neutrinos have a decay lifetime larger than the typical time-scale of the inflaton oscillation m_ϕ^{-1}

$$\Gamma_{N_1} \lesssim m_\phi \quad \Rightarrow \quad m_1 < \left(\frac{10^{15}\ \mathrm{GeV}}{M_1}\right)^2 \left(\frac{m_\phi}{10^{13}\ \mathrm{GeV}}\right) 8\times 10^{-3}\ \mathrm{eV}. \tag{3.36}$$

[4]To be exact, with respect to the previous calculation the production is reduced of a factor of 2 since right-handed neutrinos are Majorana fermions.

The right-handed neutrinos produced during preheating may annihilate into inflaton quanta. This back-reaction will render the final right-handed neutrino abundance smaller and therefore leptogenesis more difficult. Imposing that the back-reaction is negligible requires $\Gamma_A \sim n_{N_1}\sigma_A \lesssim m_\phi$, where $\sigma_A \sim g^4/(4\pi M_1)^2$. Since the energy spectrum of the right-handed neutrinos is dominated by the maximum momentum generated at the last inflaton oscillation, we assume that the number density of right-handed neutrinos is equal to the freeze-out value

$$n_{N_1} \sim \frac{10^{-2} q m_\phi^{7/2}}{m_{N_1}} L\,, \tag{3.37}$$

with $L \equiv \left[\ln\left(q^{1/2} m_\phi / m_{N_1}\right)\right]$ as follows from eq. (2.47), and we obtain

$$\Gamma_A < m_\phi \quad \Rightarrow \quad q < 10^{10}\left(\frac{M_1}{10^{15}\ \mathrm{GeV}}\right)^{5/6} L^{-1/3}\,. \tag{3.38}$$

One should also be sure that the number density of the right-handed neutrinos is not depleted by self-annihilations before they decay. This requires

$$\Gamma_A < \Gamma_{N_1} \quad \Rightarrow q < 5\cdot 10^{10}\left(\frac{m_1}{10^{-4}\ \mathrm{eV}}\right)^{1/3}\left(\frac{M_1}{10^{15}\ \mathrm{GeV}}\right)^{3/2} L^{-1/3}\,. \tag{3.39}$$

Suppose now that the right-handed neutrinos N_1 decay before the inflaton energy density is transformed into radiation by perturbative processes. This occurs when $\Gamma_{N_1} > \Gamma_\phi$, which implies

$$m_1 > \left(\frac{T_{RH}}{M_1}\right)^2 1\times 10^{-3}\ \mathrm{eV}. \tag{3.40}$$

A crucial point is that, after the generation of non-thermal right-handed neutrinos at the preheating stage, the ratio of the energy densities of N_1 and inflaton quanta remains constant: $\rho_{N_1}/\rho_\phi = (\rho_{N_1}/\rho_\phi)_{ph}$, where $(\rho_{N_1}/\rho_\phi)_{ph}$ is the ratio generated at the preheating stage. Since the energy density of the Universe is dominated by inflaton oscillations, after preheating we obtain

$$\rho_{N_1} = \left(\frac{\rho_{N_1}}{\rho_\phi}\right)_{ph}\frac{3H^2 M_{\mathrm{p}}^2}{8\pi}. \tag{3.41}$$

The analytical results of the previous chapter (2.47) indicate that

$$\left(\frac{\rho_{N_1}}{\rho_\phi}\right)_{ph} \sim 10^{-12} q\left(\frac{m_X}{m_\phi}\right)^{1/2} L \quad \text{for} \quad M_1 \lesssim \frac{q^{1/2}}{2} m_\phi. \tag{3.42}$$

At $t_{N_1} \sim \Gamma_{N_1}^{-1}$ the right-handed neutrinos decay and the energy density ρ_{N_1} is converted into a thermal bath with temperature

$$\frac{\pi^2}{30} g_* \widetilde{T}^4 = \rho_{N_1}, \tag{3.43}$$

where ρ_{N_1} is computed at $H = \Gamma_{N_1}$. Using eqs. (3.41) and (3.42), we find

$$\widetilde{T} = 2 \cdot 10^{14}\ \text{GeV} \left(\frac{q}{10^{10}}\right)^{1/4} \left(\frac{m_1}{10^{-4}\ \text{eV}}\right)^{1/2} \left(\frac{M_1}{10^{15}\ \text{GeV}}\right)^{9/8} L^{1/4} . \tag{3.44}$$

Before the inflaton decays, this thermal bath never dominates the energy density of the Universe since $\rho_{N_1} \ll \rho_\phi$ at $H \simeq \Gamma_{N_1}$ and the energy density in the inflaton field ρ_ϕ is red-shifted away more slowly than the radiation. Notice that at this time the asymmetry is still in the form of lepton number since the sphalerons which are responsible for converting the lepton asymmetry into baryon asymmetry are still out-of-equilibrium at $T = \widetilde{T}$. One might be worried that, since $\widetilde{T}$ is usually larger than T_{RH}, too many gravitinos are produced at the stage of thermalization of the decay products of the right-handed neutrino. However, one can estimate the ratio $n_{3/2}/s$ after reheating to be of the order of $10^{-15}(q/10^{10})^{3/2}\,(T_{RH}/10^{10}\ \text{GeV})$, which is quite safe. However, we have to require that the lepton-number violating processes within the thermal bath at temperature $\widetilde{T}$ are out-of-equilibrium in order not to wash out the lepton asymmetry generated by the right-handed neutrino decays. Therefore, we demand that

$$\Gamma_{\Delta L} = \frac{4}{\pi^3} G_F^2 m_i^2 T^3 < H \quad \text{at } T = \widetilde{T} \text{ and at } H = \Gamma_N . \tag{3.45}$$

This implies

$$m_1 \ < \ 10^{-2}\ \text{eV} \left(\frac{10^{10}}{q}\right)^{3/10} \left(\frac{10^{15}\ \text{GeV}}{M_1}\right)^{11/20} L^{-3/10} , \tag{3.46}$$

$$m_1 \ < \ 6 \times 10^{-5}\ \text{eV} \left(\frac{10^{10}}{q}\right)^{3/2} \left(\frac{10^{15}\ \text{GeV}}{M_1}\right)^{11/4} \left(\frac{2 \times 10^{-3}\ \text{eV}^2}{\sum_{i=2,3} m_i^2}\right)^2 L^{-3/2} \tag{3.47}$$

At $H \lesssim \Gamma_{N_1}$, the ratio n_L/n_ϕ keeps constant until the time $t_\phi \sim \Gamma_\phi^{-1}$ when the inflaton decays and the energy density in the inflaton field $\rho_\phi(t_\phi)$ is transferred to radiation. After reheating we obtain the following lepton asymmetry to entropy density ratio

$$\frac{n_L}{s} = \frac{3 n_L T_{RH}}{4 \rho_\phi(t_\phi)} = \frac{3\ \epsilon\ T_{RH}}{4\ M_1} \left(\frac{\rho_{N_1}}{\rho_\phi}\right)_{ph} . \tag{3.48}$$

Using eq. (3.42), the baryon asymmetry can be expressed as

$$\frac{n_B}{s} = 3 \cdot 10^{-7} \epsilon \left(\frac{T_{RH}}{10^{10}\,\mathrm{GeV}} \right) \left(\frac{10^{15}\,\mathrm{GeV}}{M_1} \right)^{1/2} L \left(\frac{q}{10^{10}} \right) . \tag{3.49}$$

The result in eq. (3.49) shows that the preheating production mechanism can lead to a successful leptogenesis even for M_1 as large as 10^{15} GeV, if $\epsilon q \sim 10^6$. Values of q as large as 10^{10} correspond to a perturbative coupling between N_1 and the inflaton $g^2 \simeq 0.4$ and are compatible with the constraints in eqs. (3.38) and (3.39). Values of ϵ of the order of 10^{-4} are quite large, but can be obtained with realistic neutrino mass matrices. For values of M_1 so close to the GUT scale, we expect that all the right-handed neutrino Majorana masses are comparable in size. Moreover, the atmospheric neutrino results, together with the requirement of perturbative Yukawa couplings, indicate that at least one Majorana mass is less than about 8×10^{15} GeV. This situation of comparable Majorana masses and some large Yukawa couplings naturally leads to large values of ϵ. Also, notice that the out-of-equilibrium condition is automatically satisfied in the preheating scenario for all 3 right-handed neutrinos, in a large range of parameters.

Let us finally consider the case in which N_1 decays after the reheating process and the inequality (3.40) is not satisfied. Again, it is important to establish whether N_1 dominates the Universe before decaying. Since the inflaton energy density is converted into radiation and the N_1 are non-relativistic, the temperature at which ρ_{N_1} dominates is given by

$$T_{dom} = \left(\frac{\rho_{N_1}}{\rho_\phi} \right)_{ph} T_{RH}. \tag{3.50}$$

Therefore, for $T_* > T_{dom}$, *i.e.* for

$$m_1 > 10^{-12}\ \mathrm{e}V \left(\frac{10^{15}\ \mathrm{GeV}}{M_1} \right) \left(\frac{T_{RH}}{10^10\ GeV} \right)^2 \left(\frac{q}{10^{10}} \right)^2 L^2 , \tag{3.51}$$

the estimate for the baryon asymmetry in eq. (3.49) is still valid. Otherwise, for $T_* < T_{dom}$, we obtain the result in eq. (3.20). Notice that, in this case, the constraint $T_* < T_{RH}$ is automatically satisfied.

3.5 Comparison of the Different Production Mechanisms

We want to compare here the different mechanisms discussed in this section for leptogenesis from N_1 decay. We summarize their most important features and estimate the size of the CP-violating parameter ϵ necessary to reproduce the observed baryon asymmetry.

Thermal production. This is the right-handed neutrino production mechanism usually considered in the literature for conventional leptogenesis. In the range 10^{-5} eV $< m_1 < 10^{-2}$ eV (or 10^{-6} eV $< m_1 < 10^{-2}$ eV in the case of the minimal supersymmetric model) and for $M_1 < T_{RH}$, the thermal production of unstable N_1 is efficient, and values of ϵ in the range 10^{-7}–10^{-5} can account for the present baryon asymmetry. The boundaries of the allowed region of neutrino mass parameters are determined as follows. For small values of m_1, the N_1 production rate is suppressed and larger values of ϵ are required. For large m_1, the Yukawa couplings maintain the relevant processes in thermal equilibrium for longer times, partially erasing the produced asymmetry. The values of M_1 are limited by the reheat temperature after inflation T_{RH}, which in turn is bounded by cosmological gravitino considerations to be below 10^8–10^{10} GeV.

Production at reheating. If N_1 is directly or indirectly coupled to the inflaton, the decay of the small amplitude oscillations of the classical inflaton field at the time of reheating can produce a right-handed neutrino population. This production mechanism enables us to extend the leptogenesis-allowed region to neutrino mass parameters which correspond to non-thermal N_1 populations. In particular, M_1 can be as large as $m_\phi \simeq 10^{13}$ GeV. The observed baryon asymmetry is reproduced for $\epsilon \sim 10^{-6}(10^{10}\ \mathrm{GeV}/T_{RH})(10^{-1}/B_\phi)$, where B_ϕ is the average number of N_1 produced by a single inflaton decay.

Production at preheating. The non-perturbative decay of large inflaton oscillations during the preheating stage can produce a non-thermal population of very massive right-handed neutrinos. In this case, the range of M_1 can be extended to values close to the GUT scale, while m_1 is bounded by the condition that lepton-number violating interactions are out of equilibrium after the N_1 decay, see eq. (3.47). A successful leptogenesis requires $\epsilon \sim 10^{-4}(10^{10}\ \mathrm{GeV}/T_{RH})(M_1/10^{15}\ \mathrm{GeV})(10^{10}/q)$, where q is related to the initial inflaton configuration and is defined in eq. (1.19).

In conclusion, leptogenesis provides an interesting and simple way to explain the present cosmic baryon asymmetry. The study of the different mechanisms in which it can be realized provides us with precious information on the neutrino mass parameters and the early history of the Universe.

Part II

Extra dimensions

The idea of extra dimensions is not new. It dates back at least to the works by Kaluza [76] and Klein [77] in the early twenties, in an attempt to unify gravity and electromagnetism. More recently, physicists have got accustomed to it from string theory. As a standard lore, these extra dimensions have been assumed compactified on some manifold with the size of the order the inverse Planck mass, i.e. the most natural scale for a theory of gravity. Such small dimensions are of course unaccessible at low energy and do not play any relevant phenomenological role apart maybe at Planck time.

The issue of extra dimensions entered a rich new phase about two years ago since the work [78] by Arkani-Hamed, Dimopoulos, and Dvali, where it is pointed out that, having no test of gravity below the millimeter scale, we do not really need such small compactification radii. Of course, compactification manifolds as large as fractions of millimeter can exist provided the whole space (usually called *bulk*) is accessible only to gravity, while Standard Model fields are confined on 3 dimensional walls. The possibility of matter and gauge fields constrained on a submanifold is again very common in strings. This situation naturally arises from (Dirichlet) boundary conditions which force the ends of open strings to live on lower dimensional subspaces (*branes* in string terminology) [5]. In addition, some field theoretical mechanisms for localizing fermions [80, 81], scalar, and gauge fields [82, 83] are also known.

Both string and quantum field theory thus provide this class of models (we will sometimes simply refer to them as to *brane models*) of some fundamental motivations. However, the great attention that brane models have gained in the last two years is due not so much to possible motivations from particle physics, but rather to the new phenomenological implications of the scenario proposed in [78]. In this work, the fundamental scale M of gravity is assumed to be very close to the electroweak one. This choice seems to offer a very natural solution to the hierarchy problem. The different strength of electroweak and gravitational interactions is then justified by the existence of extra dimensions, where only the latter propagate. This fact can be easily explained from flux considerations. Denoting by R the size of the extra space (assuming all the extra dimensions of comparable size), we have

$$M_{\rm P}^2 \sim M^{2+n}\, R^n\,. \tag{3.52}$$

The measured weakness of gravity is thus achieved at price of very large (large compared to the "more orthodox" expectation $M_{\rm P}^{-1}$, but also compared to M^{-1}) extra dimensions. For $M \sim 1-10$ TeV, even $n=2$ is in principle possible, since it requires R to be some fraction of millimeter; the size R then gets smaller [6] with

[5] For a review, see [79].

[6] Just to be quantitative, inserting the measured value $M_{\rm P} \simeq 10^{19}$ GeV in eq. (3.52) yields to

$$R \sim 10^{-17+30/n}\;{\rm cm}\;\left(\frac{1\;{\rm TeV}}{M}\right)^{1+2/n}\,. \tag{3.53}$$

increasing n.

The first and most obvious consequence of this scenario is a modification of the Newton law at distances smaller and comparable to R. This has lead to the proposal of several experiments to lower the distances at which gravity has been probed to date [84, 85, 86]. There are, however, several other possible signatures. Most of them are due to the Kaluza-Klein modes of the graviton. The possible detection of these modes is not due to the strength of their interaction – being modes of the graviton they interact only gravitationally – but to their large multiplicity. As an example, $R \sim 0.1$ mm corresponds to a Kaluza-Klein splitting of 10^{-2} eV. Present and future accelerator searches can thus pose limits on M or support the validity of this scenario, as it is widely discussed in the recent literature (for a short review, see [87]). In addition, brane models can have consequences of great relevance also for astrophysics and cosmology. In most of the standard cosmological scenarios, very high energy densities are considered. Loosely speaking, it is sometimes said that cosmology may be our unique possibility to discuss and "probe" physics close to the Planck scale. More prudently, an accepted (however evasible) bound for the energy density after inflation is given by the gravitino problem, but it is still as high as $T_{\rm RH} \lesssim 10^8 - 10^9$ GeV. [33, 34, 35, 36]

The scenario proposed in [78] drastically changes this picture, since it considerably lowers the bound ($M \sim$ TeV rather than $M_{\rm P}$) above which quantum gravity manifests and our present description of spacetime becomes meaningless. The limit $T_{\rm RH} < M$ which follows from this consideration is further lowered by the phenomenological ones [88], again related to the Kaluza-Klein modes of the graviton. These modes are massive from the four dimensional point of view, and hence their energy density redshifts only as the one of matter. Moreover, they are very long leaving, since - due to momentum conservation in the bulk - they can decay (again with a gravitational width) only in proximity of the branes. Both these facts render them very serious obstacles for a successful cosmology. Not too overproduce them, stringent bounds must be imposed on the reheating temperature. These limits can accommodate the minimal requirement of standard primordial nucleosynthesis (typically $T_{\rm RH} \lesssim$ few MeV), but pose several problems for baryogenesis.

However, the strongest objection moved to the scenario [78] does not come from the phenomenological bounds, but it is linked to a possible new appearance of the hierarchy problem. This is due to the very large values required for the size R of the extra dimension. While naturalness would have suggested $R \sim M^{-1}$, the measured $M_{\rm P}$ can be achieved in [78] only for $R\,M \gg 1$. As it is clear from eq. (3.53), for $n = 2$ we need $M\,R \sim 10^{16}$, which is the same distance between $M_{\rm P}$ and the Tev scale. Only for $n = 30$ we can have $M\,R \sim 10$. The hierarchy problem thus translates into the question of how the extra space can be compactified on a manifold with such an unnaturally large size. The situation is considerably improved in the proposal [89] by Randall and Sundrum, which drove still more attention towards brane models. The RS model [89] has only one extra dimension and, due to the presence of (accurately chosen) vacuum energies in the bulk and on two branes, a nonfactorizable geometry: the metric is again Poincaré invariant in the 4 usual dimensions, but it has a crucial exponential dependence – the so called *warp factor* – on the extra coordinate. The only fundamental scale of the model is assumed to be the observed Planck mass. However, the warp factor present in the metric redefines the physical scales on one of the two branes. Since this redefinition has an exponential dependence on the size R of the extra-dimension, the TeV scale can be achieved even for $R \sim 70\, M_{\rm P}^{-1}$.

The first chapter of this second part is introductory to brane models, and we analyze there in more details what we have discussed so far. Chapters 5 and 6 are then devoted to some more specific features of these scenarios. In chapter 5 we discuss some of the physics of the branes. From a field theoretical point of view, this discussion must start from the mechanisms which render brane themselves possible, i.e. which force matter and gauge fields to live only in a limited portion of the total space. We review the first and most known of these mechanism, due to Rubakov and Shaposhnikov [80]. We discuss some implications that this mechanism could have, when more matter fields are considered [90] and when a temperature is switched on [91]. This may have interesting consequences for baryogenesis which, as we already remarked, is a very difficult challenge for brane models. We stress that, although baryogenesis is a very standard topic in cosmology, the proposal described in chapter 5 is "intrinsically extra dimensional", i.e. it relies on physics peculiar to models with extra dimensions. Since brane models present more phenomenological problems (at least in cosmology) than the standard ones, it is very interesting to investigate if they can also provide some solutions which cannot be formulate in the ordinary four dimensions. The analysis of the baryogenesis scenario [91] that we present in chapter 5 should be considered only one example in this regard.

In chapter 6 we finally discuss the cosmological evolution of brane models, particularly referring to their expansion law in presence of matter on the branes. Contrarily to chapter 5, the specific shape of the branes is here unimportant, and they are assumed as idealized (infinitely thin) objects. Following the chronological order, we review some of the works which have lead to a gradual understanding of this issue. The first of them, in agreement with the first analysis of [92], concluded that both the scenarios [78] (with two walls and only one extra dimension) and [89] meet serious problems in reproducing standard cosmological evolution. This is due to a particular fine-tuning which must be imposed on the energy densities and pressures of the matter on the two branes. The origin of this fine-tuning was fully understood in the works [93, 94], where it was shown that it comes form the requirement of a static extra-dimension in absence of any stabilization mechanism. Indeed, matter on the two branes generates an effective potential for the field describing the dynamics of the extra space (the so called "radion"). Apart from accidental cancellations (i.e. apart from the fine-tuning first considered in [92]), this potential is minimized when the edges of the extra dimensions (where the two branes are fixed) move apart at infinity. It is therefore clear that standard four dimensional evolution cannot be achieved unless the extra space is stabilized by an additional mechanism.

As a first approach, one can perform a model independent analysis simply assuming that the equation which governs the dynamics of the radion is satisfied for a static configuration. This assumption is justified for matter energy densities below the scale of radion stabilization, which can be taken - at least in principle - very close to the fundamental scale of the system. For what concerns the Randall-Sundrum scenario [89], this has been done in ref. [95]. It is there shown that there exists a value for the energy densities below which the physical quantities are clearly identified and the system evolves as a standard (Friedmann-Robertson-Walker) four dimensional one. This value turns out very close to the physical TeV cut-off of our brane [95]. Alternatively, one can consider some specific stabilization mechanism and discuss the (more complicated) full set of Einstein equations of the system. This analysis would be relevant for scales comparable to the one of the stabilization mechanism and should give quite model dependent results. In our presentation we choose the first approach and we discuss the results of the work [95]. Just as an "existence proof", we finally review the simple Goldberger and Wise mechanism [96], which can ensure the stability of the radion in the Randall-Sundrum model.

Chapter 4

Large *vs.* warped extra dimensions

We devolve this chapter to an introduction to brane models. One of its main goals is to make a clear distinction between models with large extra dimensions and factorizable geometry [78], and models a là Randall and Sundrum [89], where the extra space has a small size and cannot be factorized in the metric of the system. These two scenarios, although introduced with the same aim to provide a solution to the hierarchy problem, have indeed a very different geometry, which results in distinct and clear signatures. Also the ways in which they solve the hierarchy problem are quite different. Following the chronological order of their proposal, we first discuss models with large extra dimensions. In these scenarios only the electroweak scale is assumed to be the fundamental one, and the observed weakness of gravity is due to the fact that gravity – contrarily to the other interactions which are confined to a four dimensional subspace – can propagate in the whole space. In this way the physical cut-off of the Standard Model is assumed not too far from its natural scale, and this automatically solves the hierarchy problem. This is discussed in the first section of this chapter.

For the scenario [78] to sufficiently suppress gravity, the extra dimensions have to be quite large, especially if they are few in number. In the extreme case of two large extra dimensions, their size has to be slightly smaller than one millimeter. Since the main consequence of this scenario is a modification of gravity at distances comparable to the size of the extra space, one would first guess that they are easily phenomenologically excluded. However, as it was pointed out in the original work [78], we do not have any knowledge of gravity at distances smaller than fractions of one millimeter and this leaves open even the case $n = 2$. There are anyhow several other phenomenological bounds that one has to consider, coming from accelerators, astrophysics, and cosmology. Almost all of them are due to the possibility to excite the Kaluza-Klein modes of the graviton. Although these modes are coupled only gravitationally, their Kaluza-Klein splitting can be even as small as fractions of eV and this large molteplicity may in principle allow us to observe them. We will discuss these bounds in section 2. We will particularly concentrate on possible cosmological consequences. The main result is that standard cosmology can be reproduced, provided the extra space is stabilized by some (unspecified) mechanism and provided the reheating temperature does not exceed very stringent bounds. These strong limits can be made compatible with nucleosynthesis, but very hardly allow successful baryogenesis scenarios. This will be also discuss in the next chapter.

The main objection moved to the scenario [78] is that the size of the extra dimensions must be much higher than the inverse of the fundamental scale. This can open a new hierarchy problem, since stabilizing the radii of the extra space at such high values may be unnatural. The Randall-Sundrum scenario [89] offers a better solution in this respect. We will discuss it in section 3. In this model, the only fundamental scale is assumed to be the observed Planck mass, but the presence of vacuum energies strongly modifies the geometry of the system. Usual Poincaré invariance is recovered in the observed space, but the physical scales result to be *exponentially* dependent on their position on the extra dimension (in this case, only one extra dimension is sufficient). This exponential dependence allows to get the observed Tev scale from the fundamental $M_{\rm P}$ for a size of the extra dimension not too larger than $M_{\rm P}^{-1}$. The scenario [89] does not involve any light Kaluza-Klein mode and it meets much less severe constraints then the previous one. However, also here there is a serious problem related to the fact that we are thought to be confined on a negative tension wall. To overcome this difficulty, several other models with a very similar geometry to [89] have been proposed. This new class of models has gained strong interest by itself, and some of these proposals do not have any connection to the hierarchy problem.

4.1 The hierarchy problem and large extra dimensions

The scalar sector of the Standard Model suffers from the well known hierarchy problem [97]. It arises when one calculates the corrections to the Higgs mass. Already at one loop, one gets

$$\delta m_H^2 = \frac{1}{8\,\pi^2}\,\left(\lambda_H^2 - \lambda_t^2\right)\Lambda^2 + \text{log. div.} + \text{finite terms.} \tag{4.1}$$

In the above expression λ_H^2 and λ_t^2 are, respectively, the self-coupling of the Higgs and its coupling to the top quark (the minus sign arising from the fermionic loop). Λ has instead to be understood as the physical cut-off of the theory, which is expected to be the Planck scale (10^{19} GeV) or the GUT scale (10^{15} GeV).

Then the theory contains two very different scales, namely Λ and the electroweak one about which (from unitarity reasons) the Higgs mass is forced to be. Thus, in the expression for the total Higgs mass (tree level value plus loops plus counterterm) a strong fine-tuning has to occur in order to cancel the quadratic divergence appearing in the above eq. (4.1) and to leave as a result the electroweak scale. What is worse, this adjustment must be made at each order in perturbation theory.

The most common solution to the hierarchy problem is offered by supersymmetry (for a review, see for example [98]). The presence of superpartners in the loop diagrams cancel indeed the quadratic divergences, and one is left only with logarithmic divergences proportional to the susy breaking scale (that is the difference of the masses of the superpartners). The conventional alternative to supersymmetry is offered by technicolor (for a recent review, see [99]), where fundamental scalars do not exist.

The underlying idea common to both supersymmetry and technicolor is the presence of an effective theory beyond the standard model which reveals itself at about the TeV scale. It is then assumed that either a "big" desert exists up to the Planck or the GUT scale, or that some new theories emerge at intermediate energies, maybe explaining some other problems as the origin of flavors and the different Yukawa couplings.

A completely new approach is suggested in the work [78], where the TeV scale is assumed to be the only fundamental one and the weakness of gravity (or, in other worlds, the greatness of M_{P}) is explained with the presence of large extra dimensions. Let us suppose that n extra dimensions are compactified on a $n-$torus with radii of comparable size $\sim R$. If gravity can propagate in the whole bulk, two test masses m_1 and m_2 placed within a distance $r \ll R$ will feel a gravitational potential dictated by the Gauss law in $(4+n)$ dimensions

$$V(r) \sim \frac{m_1\, m_2}{M_{(4+n)}^{n+2}} \frac{1}{r^{n+1}} \quad , \quad \text{for } r \ll R \, , \tag{4.2}$$

where $M_{(4+n)}$ is the fundamental scale of gravity.

On the other hand, if the masses are placed at distances $r \gg R$, their gravitational flux lines cannot continue to penetrate in the extra dimensions, and the usual $1/r$ potential is obtained,

$$V(r) \sim \frac{m_1\, m_2}{M_{(4+n)}^{n+2}\, R^n} \frac{1}{r} \quad , \quad \text{for } r \gg R \, . \tag{4.3}$$

Thus, one observer which probes gravity at distances larger than R would consider the quantity

$$M_{\mathrm{P}}^2 \sim M_{(4+n)}^{2+n}\, R^n \tag{4.4}$$

as the fundamental scale of gravity.

As we said, the hierarchy problem is automatically solved if the fundamental $M_{(4+n)}$ is assumed to be the electroweak scale and R is chosen to reproduce the observed $M_{\rm P}$. This yields

$$R \sim 10^{-17+30/n}\ {\rm cm}\ \left(\frac{1\,{\rm TeV}}{M_{(4+n)}}\right)^{1+2/n} . \qquad (4.5)$$

It is remarkable that, for $M_{(4+n)}$ not too much larger than TeV, even the case $n = 2$ is in principle acceptable, since it can lead to $R \lesssim 0.1$ mm. In the next section we will review some cosmological bounds that apply on this scenario.

4.2 Cosmological bounds

The framework described in the previous section drastically modifies physics above the TeV scale, where it presents a new and rich phenomenology. Because of this, the fact that we do not have so far any evidence of extra dimensions can be translated into bounds on this class of models, as several analyses performed in the last two years show.

The main new feature of this scenario is the presence of the Kaluza-Klein modes of the graviton. From the four dimensional point of view, these modes interact gravitationally with matter but present a very strong molteplicity, especially when the number n of extra dimensions is small (for example, for $M \simeq 10$ TeV the Kaluza-Klein splitting is about 10^{-2} eV for $n = 2$, 0.3 MeV for $n = 4$, and 0.1 GeV in the case $n = 6$). The virtual exchange of these new particles is expected to strongly modify the Standard Model cross sections [100]. The amplitude for the exchange of the infinite tower of gravitons naively diverges, and it is typically regularized considering only the ones with mass below $M_{(4+n)}$. For what concerns the production of real gravitons [100], the main signature would be missing energy (from the point of view of 4 dimensional observes) carried away into the bulk.

These new features can be used to get several bounds on this scenario, from accelerator physics, astrophysics and cosmology. For what concerns accelerators, current experiments already force $M_{(4+n)}$ to be greater than about 1 TeV, while an improvement of about one order of magnitude is expected at LHC, depending on the number of extra dimensions. We do not discuss these bounds here, and we rather refer the interested reader to the review [87] and to references therein. The most stringent bounds of astrophysical origin come instead from the supernova SN1987A. As discussed in ref. [88], nucleon-nucleon brehmstrahlung in the core of supernovae leads to graviton overproduction unless the limit

$$M_{(4+n)} \gtrsim 10^{(-4.5n+15)/(n+2)}\ {\rm TeV} \qquad (4.6)$$

is respected. This bound is relevant only in the $n = 2$ case, where it reads $M_{(6)} \gtrsim$ 30 TeV.

Let us now discuss in more details the bounds of cosmological origin [88]. Since the fundamental scale of gravity is now $M_{(4+n)}$, rather than the Planck mass, it is reasonable to assume that a field theoretical description of the Universe is meaningful only below this temperature. Going at lower temperatures, one finds that the evolution of the Universe strictly depends on the mechanism responsible for the stability of the extra dimensions. We will discuss this issue in chapter 6 . For the moment we only assume that there exist a (so called) "normalcy" temperature T_* below which the sizes of the extra dimensions are fixed and the Universe evolves as a four dimensional one. In the rest of the section we show that T_* cannot exceed some upper bounds, to avoid overproduction of the Kaluza-Klein modes of the gravitons.

The rate of production of gravitons from matter on the brane can be easily estimated. First of all, the production must be suppressed by the usual factor $1/M_{(4+n)}^{n+2}$. Then, from dimensional considerations, the rate of production of $(4+n)$ dimensional gravitons produced per relativistic species ("photons") on the wall is given by

$$\frac{d}{dt}\frac{n_{grav}}{n_\gamma} = \langle n_\gamma \sigma_{\gamma\gamma\to\text{to grav}} v \rangle \sim \frac{T^{n+3}}{M_{(4+n)}^{n+2}} \,. \qquad (4.7)$$

It thus follows that the total number density of gravitons produced during a Hubble time starting at temperature T_* is

$$\frac{n_{grav}}{n_\gamma} \sim \frac{T_*^{n+1} M_\mathrm{P}}{M_{(4+n)}^{n+2}} \,. \qquad (4.8)$$

On the contrary, gravitons can be very long-living, since they cannot decay in the empty bulk. This is because, as long as the momentum in the extra dimensions is conserved, the graviton (which is massless from the (4+n) dimensional point of view) cannot decay into two other massless particles. Of course, interaction with the wall breaks translational invariance and allows momentum non-conservation in the extra dimensions, but this requires that the decay takes place on the wall. Being more quantitative, this interaction can only take place if the graviton is within its Compton wavelength $\sim E^{-1}$ from the wall. The probability that this occurs in extra dimensions of volume R_n^n is

$$P_{\text{grav. near wall}} \sim (E\, R_n)^{-n} \,. \qquad (4.9)$$

On the other hand, when the graviton is sufficiently close to the wall, the decay into photons has again a width suppressed by $1/M_{(4+n)}^{(n+2)}$,

$$\Gamma_{\text{near wall}} \sim \frac{E^{n+3}}{M_{(4+n)}^{n+2}} \,. \qquad (4.10)$$

The total width Γ is given by the product of these two factors and it simply reads [1]

$$\Gamma \simeq \frac{E^3}{M_{\rm P}^2} \,. \tag{4.11}$$

Let us discuss the phenomenological bounds implied by these results. First of all we must require $n_{grav} < n_\gamma$ at $T = T_*$. Doing so, one assures that most of the energy of the brane does not escape into the bulk and that the expansion rate is indeed the standard one. From eq. (4.8), this requirement rewrites $T_*^{n+1} M_{\rm P} < M_{(4+n)}^{n+2}$. A stronger related bound applies requiring that the produced gravitons do not significantly affect the expansion rate of the universe during BBN. The energy density in gravitons red-shifts away as R^{-3} rather than R^{-4}. This is because, from the four dimensional point of view, the gravitons produced at temperature T are massive KK modes with mass $\sim T$. Alternately, from the $(4+n)$ dimensional point of view, while the graviton is massless, the extra radii are frozen and not expanding, so the component of the graviton momentum in the extra dimensions is not red-shifting. The ratio of the energy density in gravitons versus photons by the time of BBN is then

$$\frac{\rho_{grav.}}{\rho_\gamma}|_{\rm BBN} \sim \frac{T_*}{1\,{\rm MeV}} \cdot \frac{T_*^{n+1} M_{\rm P}}{M_{(4+n)}^{n+2}} \,, \tag{4.12}$$

Therefore, to insure normal expansion rate during BBN, we require

$$T_* \lesssim 10^{(6n-9)/(n+2)}\,{\rm MeV} \cdot \frac{M_{(4+n)}}{1\,{\rm TeV}} \,. \tag{4.13}$$

A stronger limit comes from the possible over-closure of the Universe by gravitons. As follows from eq. (4.11), the lifetime of a graviton of energy E is

$$\tau(E) \sim \frac{M_{\rm P}^2}{E^3} \sim 10^{10}{\rm yr} \cdot \left(\frac{100{\rm MeV}}{E}\right)^3 . \tag{4.14}$$

The gravitons produced at temperatures beneath $\sim$ 100 MeV have lifetimes of at least the present age of the Universe. The ratio n_{grav}/n_γ which was constrained to be $\lesssim 1$ in the above analysis must be in fact much smaller in order for the gravitons not to overclose the Universe. As we have mentioned, most of the gravitons are "massive" with mass $\sim T_*$ from the 4-d point, and they dramatically over-close the

[1] This simple result could have also been understood directly from the four dimensional point of view: the coupling of any Kaluza-Klein mode is suppressed by $1/M_{(4)}$, so the width for any individual Kaluza-Klein mode to go into Standard Model fields is suppressed by $1/M_{(4)}^2$. The result (4.11) then follows from dimensional analysis.

Universe if their abundance is comparable to the photon abundance at early times. The energy density stored in the gravitons produced at temperature T_* is

$$\rho_{grav} \sim T_* \cdot n_{\rm grav} \simeq \frac{T_*^{n+5} M_{\rm P}}{M_{(4+n)}^{n+2}} , \tag{4.15}$$

which then red-shifts mostly as R^{-3}. The ratio ρ_{grav}/T^3 is invariant. The critical density of the universe today corresponds to to $(\rho_{crit}/T^3) \sim 3 \cdot 10^{-9}$ GeV. For the gravitons not to over-close the universe, we therefore require for critical density at the present age of the universe. This gives

$$T_* \lesssim 10^{(6n-15)/(n+2)} \, {\rm MeV} \cdot \frac{M_{(4+n)}}{\rm TeV} . \tag{4.16}$$

This limit is quite strong. Assuming $M \simeq 10$ TeV, the temperature T_* is bounded by about 1.7 MeV, 0.3 GeV, or 2 GeV in the presence of, respectively, 2, 4, or 6 extra dimensions. In particular, we notice that the case $n = 2$ cannot be compatible with BBN for a fundamental scale lower than the one chosen in this example.

As we see from eq. (4.14), not all the created gravitons survive until today, since the one with higher energy have a longer lifetime. The strongest cosmological bound on T_* comes indeed from possible distortions of the Cosmic Microwave Background Radiation (CMBR) caused by the photon generated in the decay of those gravitons. Photons produced between nucleosynthesis and recombination easily photodissociates the light elements produced in the former. For some period before recombination, photon number changing interactions in the thermal plasma are out of equilibrium, so photons from graviton decay produced at this time would generate a chemical potential for the microwave background. Finally, also photons produced after recombination would appear as a clear distortion of the CMBR black-body spectrum. We do not review the calculation of the bounds in the different cases, but we refer the reader to the careful analysis of ref. [101]. We just include here a figure taken from that work, where the final results are summarized. Although slightly stronger than the ones given by eq. (4.16), these bounds exhibit the same qualitative behavior than the ones obtained from the over-closure of the Universe.

In conclusion, we can say that at least the minimal requirement of a successful nucleosynthesis (namely $T_* >$ MeV) can be satisfied by this class of brane models, provided $M_{(n+4)}$ is above the phenomenological and astrophysical bounds. We remark that a crucial point for this to occur is that the size of the compact dimensions is stabilized, as we will discuss in chapter 6. Due to the low upper bounds on T_*, baryogenesis appears instead as a much more challenging issue. Unless the scale $M_{(n+4)}$ is taken very high, sphalerons are out of equilibrium and electroweak baryogenesis and leptogenesis [2] are not viable options. This problem is discussed in

[2] Leptogenesis in these models is discuss in details in ref. [44], where it is shown that in principle

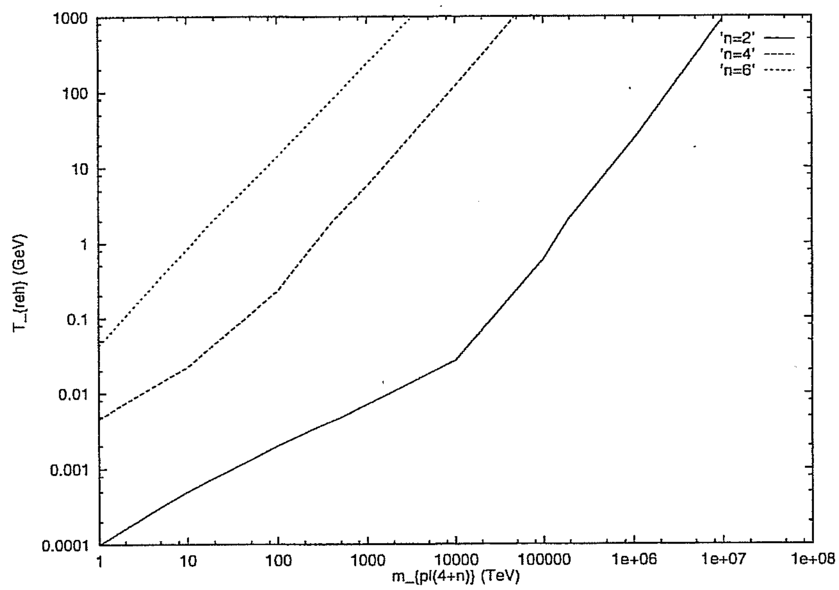

Figure 4.1: Maximum allowed normalcy temperature $T_{\rm reh}$ as a function of the fundamental scale $m_{{\rm pl}(n+4)}$ for different numbers n of large extra dimensions, taken from the work [101].

ref. [101], where it is concluded that baryogenesis models in this scenario are generically quite contrived. This is essentially due to two distinct problems. The first is that, due to the low allowed temperatures (and the consequent slow expansion rate), having an out of equilibrium decay from which baryogenesis could originate is a difficult task. Second, allowing baryon number violating processes, one has to face the problem of proton instability, mainly due to the absence of an high mass scale which suppresses these interactions. There is maybe a possible way out, which relies on considering some baryogenesis models which are *intrinsically* extra dimensional, that is which make use of some features peculiar to this scenario and do not have them only as phenomenological bounds. In the next chapter we will present one attempt in this direction [91], which considers the possible changes at finite temperature of the solitonic background which localizes fermions. As we will show, these changes may strongly modify the rates of baryon number violating interactions, making baryogenesis in the early Universe compatible with proton stability

the leptonic asymmetry can be converted into baryonic one from out of equilibrium sphaleron transitions. This requires T_* to be greater than at least 5 GeV, which means for example $M_{(n+4)}$ greater than about 10^6 TeV, 10^4 TeV, or 100 TeV for, respectively, 2, 4, or 6 extra dimensions. In this regime, the sphaleronic interactions are of course exponentially suppressed and only a tiny fraction of the lepton asymmetry can be reprocessed into a baryonic one. This requires the generation of a large lepton asymmetry, which makes the scenario quite complicated.

now.

4.3 The Randall–Sundrum model

In the previous section we have discussed the viability of models with large extra dimensions versus several bounds of various origin. There is however another difficulty that it is worth stressing. As we have seen, one of the main proposals of this scenario is to solve the hierarchy problem. To do so, the compactification radii (assuming them of comparable size) must be sufficiently large to give rise to the observed M_P. In particular, from eq. (4.4) it must be

$$R\,M = (M_P/M)^{2/n} \simeq 10^{32/n} \left(\frac{\text{TeV}}{M}\right)^{2/n} \gg 1 \,. \tag{4.17}$$

From naturalness reasons, R should have been expected of the size M^{-1}. However from the above equation we see that for $n = 2$ we must require $R\,M \sim 10^{16}$, that is the same distance between the Planck and the electroweak scale. A value $R\,M \sim 10$ can be recovered only for more than 30 extra dimensions. Hence, one is typically forced to compactify on very large extra dimensions (see [102] for some proposals in this direction) and this – at least in principle – could reintroduce the hierarchy problem in a new guise.

A consistent improvement can be achieved if the whole space has a non factorizable geometry, as the model [89] by Randall and Sundrum shows. In this model only one compact extra dimension with two four dimensional branes at its edges is considered. The two branes are placed at positions $y = 0$ and $y = 1/2$ in the extra coordinate, and the orbifold symmetry $y \leftrightarrow -y$ is imposed [3]. Finally, vacuum energies Λ_b, V_0 and V_i are included, respectively, in the bulk and on the two branes. The total action is thus composed on the gravitational action in the five dimensional bulk

$$S_{\text{gravity}} = \int d^4x \int_{-1/2}^{1/2} d\,y\,\sqrt{G}\,\left\{-\frac{R}{2\,\kappa^2} - \Lambda_b\right\} \tag{4.18}$$

plus the action of the two branes

$$S_i = \int d^4\text{x}\,\sqrt{-G_{|i}}\,\{\mathcal{L}_i - V_i\} \qquad i = 0, 1/2\,. \tag{4.19}$$

[3]This topology is strongly inspired by string theory. In the Hořava–Witten picture [103, 104] of the nonperturbative regime of the $E_8 \times E_8$ string theory, the string coupling is interpreted as an eleventh compact dimension with a $\mathbb{Z}_2$ symmetry that truncates the spectrum in order achieve an $\mathcal{N} = 1$ supersymmetry in 4D after compactification on a Calabi-Yau manifold. There is good evidence [105, 106] that over a wide range of energies the theory behaves like a 5D theory compactified on a $\mathbb{Z}_2$ orbifold with two 3-branes, viewed as the remnants of the 10D hypersurfaces where the E_8 gauge groups were living.

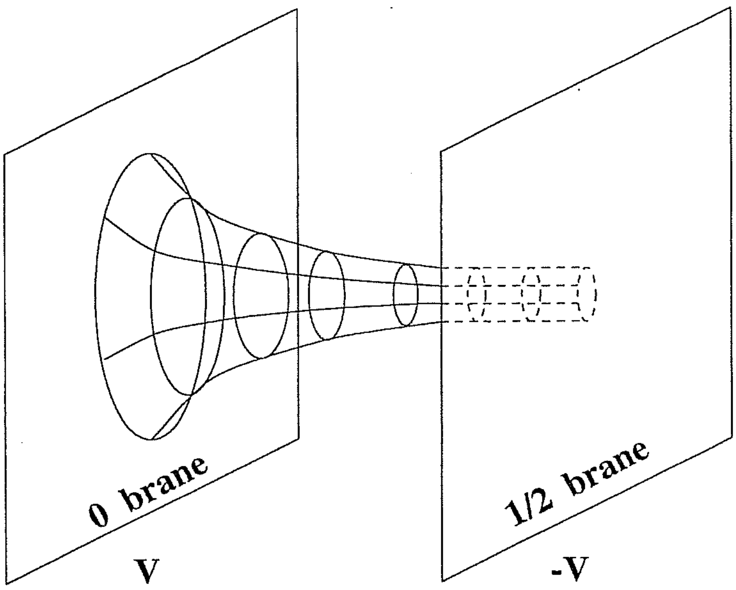

Figure 4.2: Schematic representation of the Randall-Sundrum static configuration.

In the above equations, R is the five dimensional Ricci scalar, $\kappa^{-3/2}$ is the fundamental scale of gravity, while $\mathcal{L}_i$ denotes the matter lagrangians on the two branes.

In the work [89], the Einstein equations of the model are solved when the contribution of matter on the two branes is negligible with respect to the vacuum energies (matter becomes nonnegligible when the cosmological expansion of the model is considered. We will turn back to this point in chapter 6). It is then shown that a static configuration is possible, provided the vacuum energy satisfy the relation [4]

$$V_0 = -V_{1/2} = -\frac{\Lambda_b}{m_0} = \frac{6\,m_0}{\kappa^2} \;, \tag{4.20}$$

with m_0 mass parameter.

This system admits indeed the metric

$$d\,s^2 = e^{-m_0\,b_0\,|y|}\eta_{\mu\nu}dx^\mu dx^\nu - b_0^2 dy^2 \;, \tag{4.21}$$

which is static and Poincaré invariant in the four coordinates x^μ. Notice however the presence of the exponential factor (the so called "warp factor") which renders the metric nonfactorizable. This is quite different from the metric $d\,s^2 = \eta_{\mu\nu}dx^\mu dx^\nu - b_0^2 dy^2$ of the scenario [78], where the ordinary and the extra

[4] As we will discuss in chapter 6, the relation (4.20) sets to zero the effective cosmological constant measured by observer on the two branes and it is quite analogous to the usual fine tuning $\Lambda = 0$ that one has to do in standard cosmology.

dimensions are well separated. As we now show, this exponential warp factor allows the model we are describing to provide a better solution to the hierarchy problem than the one discussed in the previous section.

The first step to see this is to to identify the gravitons with the massless gravitational fluctuations about the metric (4.21). The latter are given by

$$d\,s^2 = e^{-m_0 T(x)|y|} \left[\eta_{\mu\nu} + \overline{h}_{\mu\nu}\,(x)\right] dx^\mu dx^\nu + T^2\,(x)\,dy^2\;. \tag{4.22}$$

Here, $\overline{h}_{\mu\nu}$ represents tensor fluctuations about Minkowski space and is the physical graviton of the four-dimensional effective theory. The compactification radius b_0 appearing in the solution (4.21), is instead the vacuum expectation value of the modulus field, $T(x)$. At this stage this value is arbitrary and for this reason also $T\,(x)$ is a massless fluctuation of the system. However, as we will discuss in details in chapter 6, it is crucial that the radius is kept fixed by some stabilization mechanism. We thus fix $T\,(x) \equiv b_0$ and we postpone to chapter 6 any consideration in this regard. Finally, notice that in compactifying extra dimensions, one frequently encounters vector zero modes from $A_\mu\,dx^\mu\,dy$ fluctuations of the metric (that is the original Kaluza-Klein idea), corresponding to the continuous isometries of the higher dimensions. In the case there are no such isometries due to the presence of the two branes. So all such off-diagonal fluctuations of the metric are massive and excluded from the low-energy effective theory.

The four-dimensional effective theory now follows by substituting eq. (4.22) into the original action. In particular, the curvature term now reads

$$\int d^4x \int_{-1/2}^{1/2} dy\,\frac{b_0}{\kappa^2}\,e^{-2\,m_0\,b_0\,|y|}\,\sqrt{-\overline{g}}\,\overline{R}\;, \tag{4.23}$$

where $\overline{R}$ denotes the four-dimensional Ricci scalar made out of

$$\overline{g}_{\mu\nu}\,(x) = \eta_{\mu\nu} + \overline{h}_{\mu\nu}\,(x)\;. \tag{4.24}$$

Integrating the expression (4.23) over the fifth coordinate we get the effective four dimensional Planck mass

$$M_{\rm P}^2 = \frac{1}{\kappa^2\,m_0}\left[1 - e^{-m_0\,b_0}\right]\;. \tag{4.25}$$

This result shows that $M_{\rm P}$ depends very weakly on b_0 in the large $m_0\,b_0$ limit. Contrary to the framework [78], where the fundamental scale was the TeV one, the most natural choice suggested by eq. (4.25) is to assume both m_0 and $\kappa^{-1/2}$ of the order the observed Planck mass, about 10^{19} GeV. Although the exponential in eq. (4.25) has very little effect in determining the Planck scale, we now show that it plays a crucial role in the determination of the visible sector masses.

Let us consider the lagrangian for a field in one of the two branes. In order to gain a physical interpretation, we need to know [5] the coupling of the 3-brane fields to the low-energy gravitational fields, in particular to the metric, $\overline{g}_{\mu\nu}(x)$. Everything is transparent for fields on the zero brane, since there the warped factor evaluates to one and we immediately recover Minkowski space–time. This is not the case on the brane at $y = 1/2$. Let us indeed consider for simplicity the action for a scalar field on the $1/2-$brane

$$\begin{aligned} S_{1/2} \supset & \int d^4x \sqrt{-g\,(x\,,y=1/2)}\ \{g^{\mu\nu}(x\,,y=1/2)\ \partial_\mu\phi\,\partial_\nu\phi - m^2\,\phi^2\} = \\ = & \int d^4x\, e^{-2m_0 b_0}\ \{e^{m_0 b_0}\,\overline{g}_{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - m^2\,\phi^2\}\ . \end{aligned} \tag{4.26}$$

Minkowski space–time is also recovered on the $1/2$ brane, but the fields and the masses appearing in the original action (4.26) have to be redefined according to

$$\begin{aligned} \phi \to \tilde{\phi} &\equiv e^{-m_0 b_0/2}\,\phi\ , \\ m \to \tilde{m} &\equiv e^{-m_0 b_0/2}\,m\ . \end{aligned} \tag{4.27}$$

As it is clear form the above discussion, only the redefined fields and masses have to be considered the physical ones. In particular, the fundamental mass m of the matter field can also be assumed of order the gravitational ones, $\sim 10^{19}$ GeV. Then, only a very small hierarchy between b_0^{-1} and the other scales ($m_0\,b_0 \sim 70$) allows to produce Tev physical masses from the Planck scale. This situation considerably improves the one we had with factorized large extra dimensions [78]. We remark that, although we explicitly showed it only for the simple case of a scalar field, this fact is completely general, as it should be clear also from dimensional arguments.

Notice that on the zero brane the physical masses are not redefined and that their natural value is therefore $M_{\rm P}$. This mean that the hierarchy problem can be solved only if we assume that we are leaving on the $1/2$–brane. The fact that this brane has a negative tension (see eq. (4.20)) opens however serious problems when one wants to give a concrete realization of this scenario [6]. For this reason, several other models with nonfactorizable geometry have been considered, which do not aim to solve the hierarchy problem [107], and/or include more than three branes [108, 109].

Let us conclude this section with some phenomenological remarks about the gravitational modes of this model. A detailed analysis of their effective couplings to matter and masses requires an explicit Kaluza-Klein decomposition, which has performed in ref. [107]. The result is that both masses and couplings are determined

[5] In other words, we need to canonically normalize these fields; see the next eqs. (4.26) and (4.27).

[6] In particular, both the stability of a negative tension object and the localization of matter on it remain unsolved questions.

by the TeV scale. These features are quite different from the ones we encountered in the scenario [78], where the Kaluza-Klein splittings are much smaller than the weak scale, possibly smaller than an eV.

For what concerns cosmology and astrophysics, the situation is in principle considerably improved, since most of the bounds discussed in section 4.2 came from very light and long leaving modes, which are absent in the present model. Also for what concerns collider physics, the two scenarios present very distinctive signatures. For a product spacetime [78], each excited state couples with gravitational strength, and the key for observing these states in accelerator experiments is their large multiplicity due to their fine splittings. In the model [89], only a relatively small number of excitations can be kinematically accessible at accelerators, being the Kaluza-Klein mass splitting at the TeV scale. However, also the couplings of these modes to matter are set by the weak scale rather than the Planck one. Instead of gravitational strength couplings $\sim$ Energy/$M_{\rm P}$, each excited state coupling is indeed of order Energy/TeV [107], and therefore each can be *individually* detected.

Chapter 5

Localization of fermions and baryogenesis

In this chapter we discuss some of the physics of the branes. While we can expect branes to naturally arise in string theory from boundary conditions for open strings, one can look at them also from the field theoretical point of view. In this regard, the most crucial thing worth studying is maybe the mechanism which renders branes possible, that is which forces fermions and gauge fields to live only on a limited portion of the whole space. The knowledge of some specific example is very important, since it can allow to discuss the physics of the branes, and not to consider them only as (infinitely thin) idealized objects. In this way brane models can become a completely new context where to study particle physics, since many new options offered by the presence of the extra space may be explored.

The first and most known of these mechanisms is due to Rubakov and Shaposhnikov [80], and it consists in localizing fermions on a solitonic background given by an additional scalar field. The underlying idea is very simple. Let us consider only one extra dimension and an Yukawa interaction between the fermions and the scalar field. This interaction gives the fermions a five dimensional mass $m\left(x_5\right)$ which varies along the fifth dimension. As a consequence, the fermionic wave function is enhanced where the total fermionic mass (given by $m\left(x_5\right)$ plus a constant term which we refer to as "bare mass") vanishes. This mechanism allows the localization of only one of the modes coming from the Kaluza-Klein decomposition of the five dimensional initial spinor. From the four dimensional point of view, this localized mode is a massless and chiral spinor, which is very welcome in building realistic scenarios. We will discuss all these features in section 5.1.

In the Rubakov-Shaposhnikov mechanism the wall is therefore not an idealized object, but its width is identify with the portion of extra space where most of the energy density of the solitonic background is concentrated. This width - given by the parameters of the model - is typically greater than the inverse of the fundamental scale of the theory. Due to the finite size, it is possible to localize different fermions at slightly different positions in the extra space, just giving them different bare masses or different couplings to the background. As a consequence, we have some freedom in choosing the couplings between two different fermionic fields, since localizing two fermions at a bigger distance results in suppressing the overlapping of their wave functions and thus their mutual interaction. This can have interesting consequences, as emphasized in ref. [90] and as we show in section 5.2.

For example, let us assume that fermions acquire mass via the usual Higgs mechanism. One may reproduce the entries of the Cabibbo–Kobayashi–Maskawa matrix just placing quarks of different generations at different and appropriately chosen positions (this requires the presence of at least two extra dimensions, as shown in [110]) without assuming any hierarchy in the original Yukawa couplings of the fermions to the Higgs field. Even more interesting, we can suppress proton decay just placing leptons and quarks at different positions in the extra space [90]. This may be a mandatory choice: indeed the brane models we are considering have a physical fundamental scale close to the electroweak one. Thus, allowing baryon (B) and lepton (L) number violation, one easily meets serious problems with proton decay, due to the absence of any large scale which could suppress it. One possibility [1] is to suppress (up to a sufficiently high dimension, see ref. [101]) operators which render the proton unstable by appropriate symmetries. There are examples that show that this option is indeed viable, although usually they are quite contrived.

In the last two sections we explore another possibility [91], which is based on the works [80, 90] and it is thus "intrinsically extra dimensional". We start from the proposal for proton stability given in [90]. That is, we assume that quarks and leptons are now (i.e. at zero temperature) sufficiently apart in the extra space. However, we wonder if finite temperature effects could change this picture increasing the interactions between quarks and leptons at early times. This would render proton stability *now* compatible with baryogenesis in the *first instances* of the Universe. An exact computation of the corrections on a solitonic background presents some technical difficulties. Moreover, relying on a perturbative analysis at a scale close to the cut-off of the theory may be unsafe. For this reason, we do not give a precise final value for the temperature necessary to achieve the observed baryon asymmetry. We limit ourselves to a perturbative analysis made on dimensional arguments, which anyhow indicates that finite temperature effects should indeed increase the interactions between baryons and leptons.

As it is well known, baryon number violation alone is not sufficient for baryogenesis. To present a more complete analysis, in section 5.4 we discuss a particular

[1] As an alternative possibility, one could be tempted to assume B conservation. In this case, however, our Universe would be matter-antimatter symmetric. This poses several difficulties in explaining the existence of domains of matter as large as our cluster of galaxies, and the successes of standard nucleosynthesis.

model reminiscent of GUT baryogenesis. In doing so, we meet another problem typical of these brane scenarios. Due to the low energy densities involved, the expansion rate of the Universe is always very small. If baryogenesis originates from the decay of a boson X , the out of equilibrium condition requires m_X much higher than the physical cut-off of brane models. This problem can be overcome for example if the temperature of the Universe never exceeds m_X and if these bosons are created nonthermally (for instance at preheating). This and some other options are briefly discussed in the last section.

5.1 Localization of fermions on a solitonic background

In this section we review the mechanism for localizing fermions proposed in 1983 by Rubakov and Shaposnikov [80]. We follow the presentation given in the works [90, 91].

We start from a five dimensional Universe and we want to localize chiral (from the four dimensional point of view) fermionic fields on a 4 dimensional subspace [2]. The localization necessitates breaking of higher dimensional translation invariance. This is accomplished in our construction of a thick wall by a spatially varying expectation value for a five-dimensional scalar. We postpone to the next section an explicit realization of this scenario and for the moment we just suppose that the scalar field has a kink profile as in fig. 5.1.

The lagrangian for a fermionic field Ψ with an Yukawa interaction with the scalar Φ reads[3]

$$\mathcal{L}_{\Phi\Psi} = \bar{\Psi}\left(i\partial\!\!\!/_5 + \frac{1}{\widetilde{M}_0^{1/2}}\,\Phi\left(x_5\right) + m_0\right)\Psi \ , \tag{5.2}$$

where x_5 denotes the fifth coordinate, the fields and the parameters have the fol-

[2] Considering more extra dimensions does not change the description of the mechanism. We comment more on this option in section 5.3, where we discuss the thermal corrections to the scenario that we present here.

[3] A convenient representation for the 4×4 gamma matrices in five dimensions is

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \overline{\sigma}^i & 0 \end{pmatrix} \quad i=0,..,3, \quad \gamma^5 = -i\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{5.1}$$

where $\sigma^0 = \overline{\sigma}^0 = \mathbf{1}$, while the other $\sigma^i = -\overline{\sigma}^i$ are the three Pauli matrices. Notice the we indicate with a capital letter the five dimensional fields, to distinguish them from the four dimensional ones that we introduce below.

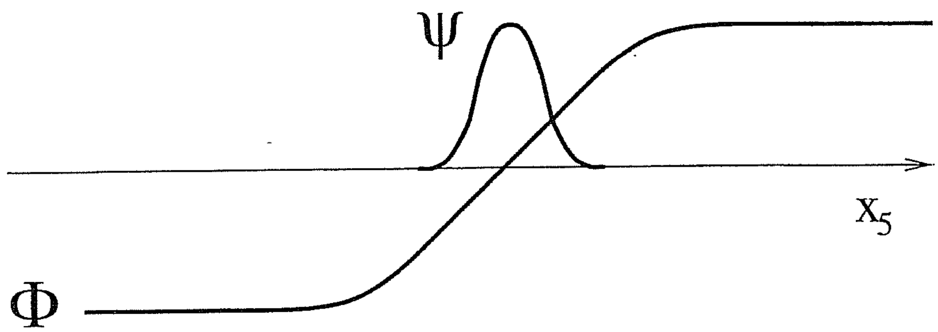

Figure 5.1: Profile of the scalar field Φ and of the fermionic field Ψ along the fifth dimension. The fermionic field, here assumed with vanishing bare five dimensional mass, is localized where its total mass (5.4) vanishes. The figure is taken from ref. [90].

lowing mass dimensions

$$[\Phi] = 3/2 \,,\; [\Psi] = 2 \,,\; [m_0] = \left[\widetilde{M_0}\right] = 1 \;, \tag{5.3}$$

and where the suffix 0 indicates that these parameters are here considered at zero temperature.

To simplify notation, we also denote with $M_{\rm T}$ the total five dimensional fermion mass

$$M_{\rm T} = \frac{1}{\widetilde{M}_0^{1/2}}\, \phi\,(x_5) + m_0 \;, \tag{5.4}$$

given by the coupling to the scalar field and by the "bare" mass m_0.

The Dirac equation following from the lagrangian (5.2)

$$\left(i\,\gamma^\mu\,\partial_\mu + \gamma^5\,\partial_5 + M_{\rm T}\right)\psi = 0 \quad , \quad \mu = 1, \ldots, 4 \,, \tag{5.5}$$

is separable in a part depending only on x_5 and in a second part depending on the other coordinates (we denote them generically with x). To see this, we perform the expansion

$$\begin{aligned} \Psi\,(x\,,\,x_5) &= \sum_n L_n\,(x_5)\;P_L\,\psi_n\,(x) + \sum_n R_n\,(x_5)\;P_R\,\psi_n\,(x) \;, \\ \bar{\Psi}\,(x\,,\,x_5) &= \sum_n \bar{\psi}_n\,(x)\;P_R\,L_n^*\,(x_5) + \sum_n \bar{\psi}_n\,(x)\;P_L\,R_n^*\,(x_5) \;, \end{aligned} \tag{5.6}$$

with $P_{L,R} = (1 \pm i\gamma^5)/2$. Let us "square" eq (5.5) multiplying it by $(-\,i\,\gamma^\mu\,\partial_\mu - \gamma^5\,\partial_5 + M_{\rm T})$. This gives

$$\left(\Box - {\partial_5}^2 - i\,\gamma^5\,\dot{M}_{\rm T} + M_{\rm T}^2\right)\Psi = 0 \ , \tag{5.7}$$

where $\Box$ is the Dalambertian operation in 3+1 dimensions and dot denotes derivative with respect to x_5 . If we now perform the expansion (5.6), we can rewrite the above eq. (5.7) as two independent equations multiplied, respectively, by P_L and P_R . In each of them, the dependences on x and x_5 can be factorized, leaving as a final result

$$\left(-\,\partial_5^2 + M_{\rm T}^2 - \dot{M}_{\rm T}\right) L_n = \mu_{n\,L}^2\, L_n \quad , \quad -\,\Box\,(P_L\psi_n) = \mu_{n\,L}^2\,(P_L\psi_n) \tag{5.8}$$

and

$$\left(-\,\partial_5^2 + M_{\rm T}^2 + \dot{M}_{\rm T}\right) R_n = \mu_{n\,R}^2\, R_n \quad , \quad -\,\Box\,(P_R\psi_n) = \mu_{n\,R}^2\,(P_R\psi_n) \ . \tag{5.9}$$

Let us define

$$A \equiv \partial_5 + M_{\rm T} \quad , \quad A^\dagger \equiv -\,\partial_5 + M_{\rm T} \ . \tag{5.10}$$

In terms of these operators we have

$$\begin{aligned} A^\dagger\,A &= -\,\partial_5^2 + M_{\rm T}^2 - \dot{M}_{\rm T} \ , \\ A\,A^\dagger &= -\,\partial_5^2 + M_{\rm T}^2 + \dot{M}_{\rm T} \ , \end{aligned} \tag{5.11}$$

so that eqs (5.8) and (5.9) acquire the more compact form

$$A^\dagger\,A\,L_n = \mu_{n\,L}^2\,L_n \ , \tag{5.12}$$

$$A\,A^\dagger\,R_n = \mu_{n\,R}^2\,R_n \ . \tag{5.13}$$

We normalize the (orthogonal) eigenvectors L_n and R_n to form two sets of orthonormal functions.

Multiplying eq. (5.12) by A, we notice that the state$A\,L_n$ is an eigenfunction of the operator $A\,A^\dagger$ with eigenvalue $\mu_{n\,L}^2$. This shows that the two sets $\{\mu_{n\,L}\}$ and $\{\mu_{n\,R}\}$ coincide. Posing $\mu_{n\,L} = \mu_{n\,R} \equiv \mu_n$ we see that the eigenstates L_n and R_n are related by

$$R_n = \frac{1}{\mu_n}\,A\,L_n \quad , \quad L_n = \frac{1}{\mu_n}\,A^\dagger\,R_n \ . \tag{5.14}$$

Notice that this relation does not involve the zero modes L_0 and R_0 corresponding to $\mu_0 = 0$. [4]

[4]The choice of this notation recalls the one of the simple harmonic oscillator (SHO). Indeed, the operators A and $A^\dagger$ become the usual SHO creation and annihilation operators (up to a normalization factor) for a mass of the form $M_{\rm T} \propto x_5$. The system we are describing can be reconducted to an analogous supersymmetric quantum mechanical one, where the functions $P_L\,L_n$ and $P_R\,R_n$ play the role of "boson" and "fermion" eigenstates. The equality $\mu_{n\,L} = \mu_{n\,R}$ is thus interpreted as the usual boson–fermion degeneracy of supersymmetric theories. This is discussed by Witten in refs. [111, 112].

We can now express the fermionic action coming from the lagrangian (5.2) in terms of the four dimensional fields ψ_n. From the orthonormal relation on the sets $\{L_n\}$ and $\{R_n\}$ and from eq. (5.14) it follows

$$S = \int d^4x \left[\bar{\psi}_{0\,L}\, i\,\gamma^\mu\, \partial_\mu \psi_{0\,L} + \bar{\psi}_{0\,R}\, i\,\gamma^\mu\, \partial_\mu \psi_{0\,R} + \sum_{n=1}^{\infty} \bar{\psi}_n \left(i\,\gamma^\mu \partial_\mu + \mu_n \right) \psi_n \right] \quad . \tag{5.15}$$

The first two terms correspond to four dimensional two-component massless chiral fermions, and arise from the zero modes of eqs. (5.12) and (5.13). The third term describes an infinite tower of Dirac fermions corresponding to the (Kaluza-Klein) modes with non-zero μ_n in the expansion (5.6). If the extra space is sufficiently small, these modes decouple from the low energy theory.

The zero mode wave functions are found by integrating the two equations $A\,L_0 = 0$ and $A^\dagger R_0 = 0$. The solutions

$$\begin{aligned} L_0 &\sim \exp\left[- \int^{x_5} dy\, M_T\,(y) \right] \quad , \\ R_0 &\sim \exp\left[\int^{x_5} dy\, M_T\,(y) \right] \quad , \end{aligned} \tag{5.16}$$

are exponentials with support near the zeros of Φ. If the field Φ has a kink configuration as in fig. 5.1 and if the extra dimension is infinite, only the left-handed mode L_0 is normalizable and localized about the zero of its total mass M_T. The theory describing the four dimensional wall at low energy thus contains only one massless chiral fermionic field. This is true even if the extra dimension is compact, since the mode R_0 (of course now normalizable) is in this case localized at the other end of the extra dimension.

5.2 Many chiral fermions and proton decay suppression

We can give a concrete realization of the localization mechanism described in the previous section by introducing the following $\mathbf{Z}_2$ symmetric lagrangian for the field Φ

$$\mathcal{L}_\Phi = \frac{1}{2} \partial_\mu \Phi\, \partial^\mu \Phi - \left(-\,\mu_0^2\, \Phi^2 + \lambda_0\, \Phi^4 \right) \quad . \tag{5.17}$$

We remind that the field Φ has mass dimension $3/2$. The parameters μ_0 and λ_0 have instead mass dimension 1 and -1, respectively.

This scalar lagrangian admits the kink solution

$$\phi = \frac{\mu_0}{\sqrt{2\,\lambda_0}}\, \tanh\left(\mu_0\, y\right) \quad , \tag{5.18}$$

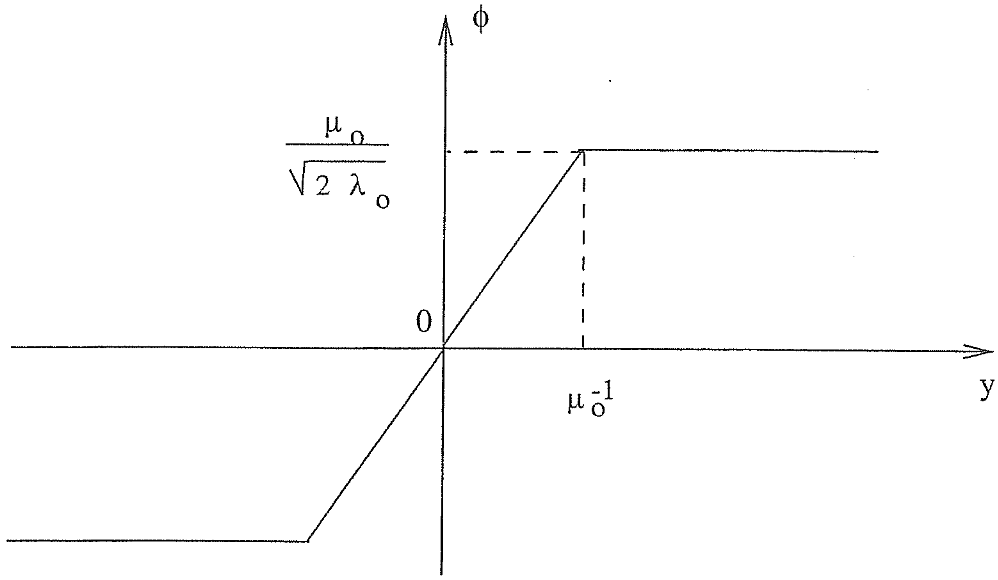

Figure 5.2: Linearized approximation of the soliton configuration arising form the lagrangian (5.17).

that we approximate with a straight line interpolating between the two vacua (see figure 5.2)

$$\begin{aligned} \phi(y) &\simeq \frac{\mu_0^2}{\sqrt{2\lambda_0}}\, y \ , \quad |y| < \frac{1}{\mu_0} \\ \phi(y) &\simeq \pm\frac{\mu_0^2}{\sqrt{2\lambda_0}} \ , \quad |y| > \frac{1}{\mu_0} \ . \end{aligned} \tag{5.19}$$

As we have seen in the previous section, from the four dimensional point of view a left handed chiral massless fermionic field results from the localization mechanism, if the scalar Φ has the above configuration (5.18). Notice that the localization can only occur if the fermionic (five dimensional) bare mass satisfies

$$m_0 < \frac{\mu_0}{\sqrt{2\lambda_0 \widetilde{M}_0}} \ , \tag{5.20}$$

since otherwise the total mass $M_{\rm T}$ defined in eq. (5.4) never vanishes.

The fermionic zero mode with opposite chirality remains instead delocalized. This does not prevent the possibility to build realistic scenarios, since it is customary to limit the Standard Model and the Minimal Supersymmetric Standard Model fermionic content only to fields of a given chirality. A "more symmetric" situation can occur if a kink–antikink solution is assumed for the scalar Φ. As a result, the left fields continue to be localized on the kink, while the right ones are confined to the antikink. If the kink and the antikink are sufficiently far apart, the left handed and right handed fermions do not interact and again the model reproducing our four dimensional world can be built by fermions of a defined chirality. The fermionic content of the full dimensional theory is in this case doubled with respect to the usual one, and observers on one of the two walls will refer to the other as to a "mirror world". The presence of a kink–antikink configuration may be required by stability consideration if thermal effects are considered, as it is the case in the next sections. However most of the physics in one brane is not affected by the presence of the mirror one, and we can almost always safely concentrate only on the single kink (5.18) configuration.

In a realistic scenario, at least another scalar field acting as a Higgs in the four dimensional theory must be considered, in order to give mass to the (four dimensional) fermions. As it is shown in ref. [90], the mechanism described above could give an explanation to the hierarchy among the Yukawa couplings responsible for the fermionic mass matrix. If indeed one chooses different five dimensional bare masses for the different fermionic fields, the latter are localized at different positions in the fifth direction. As a consequence, the wave functions of different fermions do only partially overlap, and increasing the difference between the five dimensional bare masses of two fermions results in suppressing their mutual interactions.

The same idea can be adopted to guarantee proton stability [90]. Let us give, respectively, leptons and baryons the "masses"

$$(m_0)_l = 0 \ , \ (m_0)_b = m_0 \ , \tag{5.21}$$

which correspond to the localizations [5]

$$y_l = 0 \ , \ y_b = \frac{m_0 \sqrt{2\,\lambda_0\,\widetilde{M}_0}}{\mu_0^2} < \frac{1}{\mu_0} \ . \tag{5.22}$$

The shape of the fermion wave functions along the fifth dimension has been derived (up to the normalization factor that we add now) in eq. (5.16). It can be cast in an explicit and simple form if we consider the limit $y_b \ll 1/\mu_0$, in which the effect of

[5] The last inequality in the next expression comes from (5.20). We assume quarks of different generations to be located in the same y position in order to avoid dangerous FCNC mediated by the Kaluza-Klein modes of the gluons [113].

the plateau for $y > 1/\mu_0$ can be neglected: [6]

$$
\begin{aligned}
f_l(y) &= \left(\frac{\mu_0^2}{\sqrt{2\,\lambda_0\,\widetilde{M_0}\pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2\,y^2}{2\sqrt{2\,\lambda_0\,\widetilde{M_0}}}\right\} \\
f_b(y) &= \left(\frac{\mu_0^2}{\sqrt{2\,\lambda_0\,\widetilde{M_0}\pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2\,(y-y_b)^2}{2\sqrt{2\,\lambda_0\,\widetilde{M_0}}}\right\} \,. \qquad (5.23)
\end{aligned}
$$

We assume the Standard Model to be embedded in some theory which, in general, contains some additional bosons X whose interactions violate baryon number conservation. If it is the case, the four fermion interaction $qq \longleftrightarrow ql$ can be effectively described by

$$
\int d^4x\,dy\,\frac{q\,q\,q\,l}{\Lambda\,m_X^2}\,, \qquad (5.24)
$$

where m_X is the mass of the intermediate boson X and Λ is a parameter of mass dimension one related to the five-dimensional coupling of the X-particle to quarks and leptons.

This scattering is thus suppressed by [7]

$$
\begin{aligned}
I &= \frac{1}{\Lambda\,m_X^2}\int dy \frac{\mu_0^2}{\pi\sqrt{2\,\lambda_0\,\widetilde{M_0}}} \exp\left\{-\frac{\mu_0^2/2}{\sqrt{2\,\lambda_0\,\widetilde{M_0}}}\left[y^2+3\,(y-y_b)^2\right]\right\} = \\
\ldots &= \frac{\mu_0}{\Lambda\,m_X^2\,\sqrt{2\pi}\,\left(2\,\lambda_0\,\widetilde{M_0}\right)^{1/4}} \exp\left\{-\frac{3\,\left(2\,\lambda_0\,\widetilde{M_0}\right)^{1/2}}{8}\,\frac{m_0^2}{\mu_0^2}\right\} \,. \qquad (5.25)
\end{aligned}
$$

Current proton stability [114] requires $I \lesssim (10^{16}\,\mathrm{GeV})^{-2}$, that is

$$
\frac{m_0}{\mu_0} \gtrsim \frac{\sqrt{200 - 6\,\mathrm{Log}_{10}\left(\frac{\Lambda\,m_X^2}{\mu_0}\Big/\mathrm{GeV}^2\right)}}{\left(2\,\lambda_0\,\widetilde{M_0}\right)^{1/4}} \,. \qquad (5.26)
$$

The numerator in the last equation is quite insensitive to the mass scales of the model, and – due to the logarithmic mild dependence – can be safely assumed to be

[6] This is also the limit in which the approximation (5.19) is valid.

[7] From the approximation (5.19), only the squared difference of the five dimensional masses affects the suppression factor. For this reason, the above choice $(m_0)_l = 0$ was only done in order to simplify notation and does not have any physical meaning.

of order 10. For definiteness, we will thus fix it at the value of 10 in the rest of this chapter.

Conditions (5.20) and (5.26) give altogether

$$\frac{10\,\mu_0}{\left(2\,\lambda_0\,\widetilde{M_0}\right)^{1/4}} \lesssim m_0 \lesssim \frac{\mu_0}{\left(2\,\lambda_0\,\widetilde{M_0}\right)^{1/2}} \,, \tag{5.27}$$

that we can rewrite

$$\begin{aligned} 2\,\lambda_0\,\widetilde{M_0} &\lesssim 10^{-4} \\ \frac{m_0}{\mu_0} &\gtrsim 10^2 \quad . \end{aligned} \tag{5.28}$$

Notice that the last limit in eqs. (5.28) is stronger than the one given in ref. [90] where proton stability is achieved if the ratio of the massive scales of the model is of order 10. However, in ref. [90] the field Φ simply scales linearly as a function of y, while we expect that whenever a specific model is assumed, conditions analogous to our (5.20) and (5.28) should be imposed [91].

5.3 Thermal correction to the coefficients

Once the localization mechanism is incorporated in a low energy effective theory – as the system described above may be considered –, one can legitimately ask if thermal effects could play any significant role. We are mainly interested in any possible change in the argument of the exponential in eq. (5.25), that will be the most relevant for the purpose of baryogenesis. For this reason, we introduce the dimensionless quantity

$$a(T) = \frac{m(T)^2}{\mu(T)^2}\sqrt{2\,\lambda(T)\widetilde{M}(T)} \quad . \tag{5.29}$$

From eqs. (5.26) and (5.28), we can set $a(0) \gtrsim 100$ at zero temperature. Thermal effects will modify this value. There are however some obstacles that one meets in evaluating the finite temperature result. Apart from some technical difficulties arising from the fact that the scalar background is not constant, the main problem is that nonperturbative effects may play a very relevant role at high temperature. As it is customary in theories with extra dimensions, the model described by eqs. (5.2) and (5.17) is nonrenormalizable and one expects that there is a cut-off (generally related to the fundamental scale of gravity) above which it stops holding. Our considerations will thus be valid only for low temperature effects, and may be assumed only as an indication for what is expected to happen at higher temperature.

Being aware of these problems, by looking at the dominant finite-temperature one-loop effects, we estimate the first corrections to the relevant parameters to be

$$\begin{aligned} \lambda\,(T) &= \lambda_0 + c_\lambda \frac{T}{\widetilde{M}_0^2} \\ \widetilde{M}\,(T) &= \widetilde{M}_0 + c_{\widetilde{M}}\, T \\ m\,(T) &= m_0 + c_m \frac{T^2}{\widetilde{M}_0} \\ \mu^2\,(T) &= \mu_0^2 + c_\mu \frac{T^3}{\widetilde{M}_0} \quad , \end{aligned} \tag{5.30}$$

where the c's are dimensionless coefficients whose values are related to the exact particle content of the theory.

In writing the above equations, the first of conditions (5.28) has also been taken into account. For example, both a scalar and a fermionic loop contribute to the thermal correction to the parameter λ_0. While the contribution from the former is of order $\lambda_0^2 T$, the one of the latter is of order $T/\widetilde{M}_0^2$ and thus dominates. [8]

Substituting eqs. (5.30) into eq. (5.29), we get, in the limit of low temperature,

$$a\,(T) \simeq a\,(0) \cdot \left[1 + \frac{T}{\widetilde{M}_0}\left(\frac{c_\lambda}{2\,\lambda_0\,\widetilde{M}_0} + \frac{c_{\widetilde{M}}}{2} + \frac{2\,c_m\,T}{m_0} - \frac{c_\mu\,T^2}{\mu_0^2}\right)\right] \quad . \tag{5.31}$$

From the smallness of the quantity $\lambda_0\,\widetilde{M}_0$ (see cond. (5.28)) we can safely assume (apart from high hierarchy between the c's coefficients that we do not expect to hold) that the dominant contribution in the above expression comes from the term proportional to c_λ.

We thus simply have

$$a\,(T) \simeq a\,(0)\left(1 + c_\lambda \frac{T}{2\,\lambda_0\,\widetilde{M}_0^2}\right) \quad . \tag{5.32}$$

We notice that the parameter c_λ, being related to the thermal corrections to the ϕ^4 coefficient due to a fermion loop, is expected to be *negative* [115]: the first thermal effect is to decrease the value of the parameter $a(T)$, making hence the baryon number violating reactions more efficient at finite rather than at zero temperature.

There is another effect which may be very crucial at finite temperature, linked to the stability of the Z_2 symmetry. When a temperature is turned on, we generally

[8] Notice also that with our choice (5.21) loops with internal leptons dominate over loops with internal quarks, since the former have vanishing five dimensional bare mass and thus are not Boltzmann suppressed. However, although this choice is the simplest one, one may equally consider the most general case where all the fermions have a nonvanishing five dimensional mass.

expect the formation of a fermion–antifermion condensate $\langle\bar{\psi}\,\psi\rangle \neq 0$. If it is the case, the Yukawa coupling $\Phi\,\bar{\psi}\,\psi$ in the lagrangian (5.2) renders one of the two vacua unstable. While this leads to an instantaneous decay of the kink configuration, a kink–antikink system could have a sufficiently long lifetime provided the two objects are enough far apart.

Let us conclude this section with an important remark. While in this discussion we have considered only the localization of fermions along one extra dimension, almost everything we have said can be generalized if two or more additional dimensions are present [9]. The index theorem guarantees [116, 117] indeed the possibility of localizing fermions on a topological defect of an arbitrary dimension. Just to give an example, let us consider the localization on a Nielsen–Olesen [118] vortex in the case of two extra–dimensions, as it is discussed in ref. [78]. Also in this case, one can localize quarks and leptons at two different positions (actually along different circles about the center of the vortex). Once again, proton stability requires conditions completely analogous to conds. (5.28) here discussed. Of course the calculation of thermal corrections gives different results, since the dimension of the couplings of the model changes according to the number of spatial dimensions. However, also in the case of the two dimensional vortex, the qualitative result turns out to be identical to what has been derived in the one dimensional case: the most significant effect comes from the variation of the coefficient λ of the ϕ^4 interaction, and it is in the direction of enhancing the quark–lepton interaction with increasing temperature.

5.4 Baryogenesis

We saw in the previous section that thermal effects may increase the rate of baryon number violating interactions of the system. This is very welcome, since a theory which never violates baryon number cannot lead to baryogenesis and thus can hardly reproduce the observed Universe. Anyhow baryon number violation is only one of the ingredients for baryogenesis, and the aim of this section is to investigate how the above mechanism can be embedded in a more general context.

A particular scheme which may be adopted is baryogenesis through the decay of massive bosons X. [10] This scheme closely resembles GUT baryogenesis, but there are some important peculiarities due to the different scales of energy involved. In GUT baryogenesis the massive boson X , coupled to matter by the interaction

[9] This is mandatory in the scenario [78], since the presence of only one large extra dimension is phenomenologically excluded.

[10] We may think of these bosons as the intermediate particles which mediate the four fermion interaction described by the term (5.24).

$g\, X\, \psi\, \bar{\psi}$, has the decay rate

$$\Gamma \simeq \alpha\, m_x \quad , \quad \alpha = \frac{g^2}{4\,\pi} \quad . \tag{5.33}$$

An important condition is that the X boson decays when the temperature of the Universe is below its mass (out of equilibrium decay), in order to avoid thermal regeneration. From the standard equation for the expansion of the Universe,

$$H \simeq g_*^{1/2}\, \frac{T^2}{M_{\mathrm{P}}} \tag{5.34}$$

(where g_* is the number of relativistic degrees of freedom at the temperature T), this condition rewrites

$$m_X \gtrsim g_*^{-1/2}\, \alpha\, M_{\mathrm{P}} \quad . \tag{5.35}$$

If X is a Higgs particle, α can be as low as 10^{-6}. Even in this case however the X boson must be very massive. In principle this may be problematic in the theories with extra dimensions we are interested in, which have the main goal of having a very low fundamental scale.

There are some possibilities to overcome this problem. One is related to a possible deviation of the expansion of the system from the standard Friedmann law. This is a concrete possibility, since the exact expansion law is very dependent on the particular brane model one is considering and on the fact that the size of the compact dimension is or is not stabilized. For example, we will show in the next chapter that the Randall-Sundrum model [89] with a stabilized radion can have (depending on the energy density on the zero brane) an expansion rate higher than the standard one for temperatures close to the cut-off scale of the system (that is TeV). This accelerated expansion could in principle favor the out of equilibrium condition for the X bosons.

However this issue is very dependent on the specific cosmological scenario adopted, and one may be interested in more general solutions for the out of equilibrium problem. One very natural possibility is to create the X particles non thermally and to require the temperature of the Universe to have been always smaller than their mass m_X. In this way, one kinematically forbids regeneration of the X particles after their decay. In addition, although interactions among these bosons can bring them to thermal equilibrium, chemical equilibrium cannot be achieved.

Nonthermal creation of matter has raised a considerable interest in the last years. In particular, the mechanism of preheating has proven quite successful, as we have discuss in the first part of this work. The efficiency of preheating has been exploited in the work [19] to revive GUT baryogenesis in the context of standard four dimensional theories. Here, we will not go into the details of the processes that could have lead to the production of the X bosons. Rather, we will simply assume that, after inflation, their number density is n_X. To simplify our computations, we will also suppose that their energy density dominates over the thermal bath produced by the perturbative decay of the inflaton field. [11]

Just for definiteness, let us consider a very simple model where there are two species of X boson which can decay into quarks and leptons, according to the four dimensional effective interactions

$$g\,X\,\bar{q}\,\bar{q} \quad , \quad g\,e^{-a/4}\,X\,l\,q \ , \tag{5.36}$$

where (remember the suppression given by the different localization of quarks and leptons) the quantity a is defined in eq. (5.29). Again for definiteness we will consider the minimal model where no extra fermionic degrees of freedom are added to the ones present in the Standard Model. Moreover we will assume $B-L$ to be conserved, even though the extension to a more general scheme can be easily performed.

The decay of the X bosons will reheat the Universe to a temperature that can be evaluated to be

$$T_{\rm rh} \simeq \left(\frac{30}{\pi^2}\,\frac{m_X\,n_X}{g_*}\right)^{1/4} \ . \tag{5.37}$$

Since we do not want the X particles to be thermally regenerated after their decay, we require $T_{\rm rh} \lesssim m_X$, that can be rewritten as an upper bound on n_X

$$n_X \lesssim 30\,\left(\frac{g_*}{100}\right)\,m_X^3 \ . \tag{5.38}$$

Another limit comes from the necessity to forbid the B violating four fermion interaction (5.24) to erase the B asymmetry that has been just created by the decay of the X bosons. We thus require the interaction (5.24) to be out of equilibrium at temperatures lower than $T_{\rm rh}$. From eq. (5.25) we see that we can parameterize the four fermion interaction with a coupling $g^2\,e^{-3\,a/8}/m_X^2$. Hence, the out of equilibrium condition reads

$$g^4\,e^{-3\,a/4} \lesssim g_*\,\frac{m_X}{M_{\rm P}}\,\left(\frac{m_X}{T_{\rm rh}}\right)^3 \ . \tag{5.39}$$

[11] An alternative way to overcome the bound (5.35) relies on the fact that, as observed in the works [73, 74], the maximal temperature reached by the thermal bath during reheating can indeed be much higher than the final reheating temperature. In this case, even if $T_{\rm rh}$ is considerably lower than m_X, X particles can be produced in a significant amount, and the out of equilibrium condition is easily achieved. However, the treatment of this mechanism is in our case somewhat different from the one given in ref. [73]: due to the slowness of the expansion of the Universe, the X bosons will decay before the freeze out of their production. The final baryon asymmetry cannot be estimated with the use of the formulae of [73], which are valid only if the decay of the X particles occurs well after their freeze out.

One more upper bound on the reheating temperature comes from the out of equilibrium condition for the sphalerons. This requirement is necessary only if one chooses the theory to be $B-L$ invariant, while it does not hold for $B-L$ violating schemes. We can approximately consider the sphalerons to be in thermal equilibrium at temperatures above the electroweak scale. Thus, if $B-L$ is a conserved quantity, we will require the reheat temperature to be smaller than about 100 GeV.

If one neglects the presence of the thermal bath prior to the decay of the X bosons, the very first decays will be only into couples of quarks, since the channel into one quark and one lepton is strongly suppressed by the $e^{-a(T=0)}$ factor due to the fact that the kink is not modified by any thermal correction. However, the decay process is not an instantaneous event. It is shown in ref. [73] that the particles produced in the very first decays are generally expected to thermalize very rapidly, so to create a thermal bath even when most of the energy density is still stored in the decaying particles. [12] The temperature of this bath can even be considerably higher than the final reheating temperature. The presence of the heat bath modifies in turn the shape of the kink, as shown in the previous section, and we can naturally expect that this modification enhances the B violating interactions.

If the energy density of the Universe is dominated by the X bosons before they decay, one has

$$\eta_B \simeq 0.1\ (N_X\, T_{\rm rh}/m_X)\ \langle r-\bar{r}\rangle\ , \tag{5.40}$$

where N_X is the number of degrees of freedom associated to the X particles and $\langle r-\bar{r}\rangle$ is the difference between the rates of the decays $X \to q\, l$ and $\bar{X} \to \bar{q}\, \bar{l}$.

We denote with X_1 and X_2 the two species of bosons whose interactions (5.36) lead to baryon number violation, and parameterize by ϵ the strength of CP-violation in these interactions. Considering that e^{-2a} is always much smaller than one, we get [119]

$$\langle r-\bar{r}\rangle \sim 3\, g^2\, e^{-a/2}\, \epsilon\, {\rm Im}\, {\rm I}_{SS}\, (M_{X_1}/M_{X_2})\ , \tag{5.41}$$

where the function ${Im}\, {\rm I}_{SS}(\rho) = [\, \rho^2\ {Log}(1+1/\rho^2) - 1\,]\, /\, (16\,\pi)$ can be estimated to be of order $10^{-3} - 10^{-2}$. It is also reasonable to assume $\epsilon \sim 10^{-2} - 1$.

Collecting all the above estimates, and assuming N_X to be of order 10, we get

$$\eta_B \simeq \left(10^{-5}\ -\ 10^{-2}\right)\, g^2\, \frac{T_{\rm rh}}{m_X}\, e^{-a(T_{\rm rh})/2}\ . \tag{5.42}$$

[12] As shown in ref. [73], what is called the reheating temperature is indeed the temperature of the thermal bath when it starts to dominate. After the first decays, the temperature of the light degrees of freedom can be even much higher than $T_{\rm rh}$.

From the requirement $T_{\rm rh} \lesssim m_X$ we get an upper limit on the baryon asymmetry

$$\eta_B \lesssim \left(10^{-5} - 10^{-2}\right) g^2 \, e^{-a/2} \,, \tag{5.43}$$

where the factor $a\,(T)$ has to be calculated for a value of T of the order of the reheating temperature.

We get a different limit on η_B from the bound (5.39): assuming $m_X \sim$ TeV and $g_* \sim 100$ indeed one obtains

$$\eta_B \lesssim \left(10^{-6} - 10^{-10}\right) g^{2/3} \, e^{-a/4} \,. \tag{5.44}$$

Since the observed amount of baryon asymmetry is of order 10^{-10}, even in the case of maximum efficiency of the process (that is, assuming maximal CP violation and $g \sim 1$), the bounds (5.43) and (5.44) imply that $a\,(T_{\rm rh}) \lesssim 40$. Unfortunately, the temperature at which the condition $a\,(T) \lesssim 40$ occurs cannot be evaluated by means of the expansion of eq. (5.32), that have been obtained under the assumption $|a\,(T) - a\,(0)| \ll a\,(0)$. On the other hand, it is remarkable that our mechanism may work with a ratio $a\,(T_d)\,/a\,(0)$ of order one. We thus expect that a successful baryogenesis may be realized for a range of the parameters of this model which – although not evaluable through a perturbative analysis – should be quite wide and reasonable.

As we have discussed in the previous chapter, in scenarios with large extra dimensions and low scale gravity, the maximal temperature reached by the Universe after inflation is strongly bounded from above in order to avoid overproducing Kaluza-Klein graviton modes, which may eventually contradict cosmological observations [88]. For instance, in models with two large extra dimensions the reheating temperature cannot exceed much 1 MeV (unless the fundamental scale M is unnaturally high, see fig. 4.1). This value is too low for the scenario we are describing since η_B is proportional to the ratio $T_{\rm rh}/m_X$, and hence the observed amount of baryons would be reproduced at the price of an unnaturally small value of $a\,(T_{\rm rh})$. However, other schemes with extra dimensions exist where the bounds on $T_{\rm rh}$ are less severe. For example, in the proposals [89, 120] the mass of the first graviton KK mode is expected to be of order TeV. The reheating temperature can thus safely be taken to be of order $10 - 100\,$GeV.

There are of course several possible baryogenesis schemes alternative to the one just presented. A possible option which also requires a minimal extension to the Standard Model could be to achieve the baryon asymmetry directly through the 4 fermions interactions $q + q \leftrightarrow q + l$ in the thermal primordial bath. The out of equilibrium condition may be provided by the change of the kink as the temperature of the bath decreases. [13] What may be problematic is the source of CP violation

[13]This condition may be easily achieved due of the exponential dependence of the rate of this process on the temperature, see eqs. (5.25) and (5.32).

which may lead the creation of the baryon asymmetry. A possibility in this regard may be provided by considering a second Higgs doublet, but the whole mechanism certainly deserves a deep analysis by itself.

Chapter 6

Expansion of the Universe in brane models

In the last decades we have gained precise and firm knowledge of many aspects of our Universe. The observation of the Cosmic Microwave Background Radiation (CMBR) acoustic peaks shows [121, 122] that it is highly flat and homogeneous. The CMBR temperature, the observed abundances of the light elements, and the average speed of the nearby galaxies allow us to trace back its evolution up to temperatures as high as 1 MeV, when nucleosynthesis started. It is remarkable that all these (and several other) observations strongly favor the simplest expanding four dimensional (model of) Universe that we can think of. The requirement of four dimensional standard evolution from nucleosynthesis on is by itself a very difficult challenge for brane models. While 1 MeV is a very low energy if the scale of gravity is M_P, this is certainly not true for models where the extra dimensions and possibly more fundamental physics show up close to the electroweak scale.

We do not have such a clear knowledge of what happened before nucleosynthesis, and much more room is there left for theoretical speculations. Brane models may be seen in this regard as a completely new and atypical context, with maybe new possibilities – as well as new problems – for several cosmological issues as for example inflation (see for instance [123, 124, 125]) and baryogenesis [91]. We have seen in the previous chapter the difficulties in building successful baryogenesis scenarios with only low scales at disposal. Among the other problems, the out of equilibrium condition is typically hardly achieved, due to the slow expansion of the Universe at low energy densities. Wile this last statement is certainly true for a standard expansion law, there is the possibility that brane models have a faster then standard expansion at temperatures above 1 MeV, which may result in a consistent improvement for the out of equilibrium problem.

Motivated by these considerations, the cosmological evolution of brane models has been the object of several works over the last two years. These works came to conclusions which in some cases drastically differ the one from the others, bringing to a gradual understanding of the matter. The first relevant paper on this subject is due to Binétruy, Deffayet, and Langlois [92], and we review it in section 6.1. These authors studied the expansion law of models with factorizable geometry. They solved the Einstein equations of a model with only one extra dimension and two walls on its edges, showing that consistency of the model (or, as better realized later in [93, 94], the requirement of a static extra dimension in absence of any form of energy in the bulk) forces a precise fine-tuning between the energy densities and pressures on the two branes. As a consequence of this fine-tuning, the expansion rate of the system turns out always different from the standard one ($H \propto \rho$ rather than $H \propto \sqrt{\rho}$), completely at odd with phenomenology. In ref. [92] it was concluded that this fine-tuning is of topological origin and that brane models cannot in general reproduce standard cosmological evolution. These results were also soon extended to the Randall-Sundrum model [89] with two walls. By a similar analysis (but only in an expansion series on the matter energy density) it was shown that also in this scenario matter on the two branes must be correlated, with a consequent wrong sign of the term leading the expansion of the system ($H^2 \propto -\rho$ rather than $H^2 \propto +\rho$).

The situation became clearer after the two works [93, 94], where it was realized that the fine-tuning originates from the requirement of a static extra dimension in absence of any stabilization mechanism. Matter on the branes generates an effective potential for the field governing the size of the extra space (the so called "radion"). Apart from accidental cancellations (i.e. apart from the fine-tuning first considered in [92]), this potential is minimized when the edges of the extra dimensions (where the two branes are fixed) move apart at infinity. It is therefore clear that standard four dimensional evolution cannot be achieved unless the extra space is stabilized by an additional mechanism. One can now proceed in two ways. Firstly, propose a particular mechanism and study the Einstein equations of the system (maybe perturbatively in the energy densities on the branes). Alternatively, one can try a model independent analysis simply assuming that the equation which governs the dynamics of the radion is satisfied for a static configuration. This assumption is justified for matter energy densities below the scale of radion stabilization, which can be taken - at least in principle - very close to the fundamental scale of the system.

In the work [95] this second choice is made, showing that it allows to exactly determine the evolution of the RS model with arbitrary matter on the two branes. With this approach, one can define and compute a value for the energy densities below which the physical quantities are clearly identified and the system evolves as a standard (Friedmann-Robertson-Walker) four dimensional one. The analysis of [95] indicates that this value is not too far from the physical TeV cut-off of our brane. It also shows that at higher energies the expansion law of the system differs from the standard one, although in this regime the identification of the physical relevant quantities results more problematic. The results of this work, as well as those of [93], are presented in section 6.2.

We do not discuss here the problem of the stabilization of the extra space, since it would require a detailed analysis by itself [1]. Only as an "existence proof", we report in section 6.3 a simple mechanism, due to Goldberger and Wise [96], which can ensure the stability of the radion in the Randall-Sundrum model. From this mechanism, we deduce that considering the radion fixed up to (relatively) high energy densities may indeed be a justified assumption.

6.1 Expansion law without radion stabilization

Let us consider a system with two four dimensional branes and a large transverse dimension. It is described (cfr. eqs. (4.18) and (4.19)) by the action

$$S_{(5)} = -\frac{1}{2\kappa^2}\int d^5x\,\sqrt{-\tilde{g}}\,\tilde{R} + \int d^5x\,\sqrt{-\tilde{g}}\mathcal{L}_m\,, \tag{6.1}$$

with $\tilde{g}$ denoting the five dimensional metric (Minkowski space is here taken with metric $\tilde{g} = \mathrm{diag}\,(-1\,,1\,,1\,,1\,,1)$), $\tilde{R}$ the Ricci scalar associated to it, $k^{-3/2}$ the fundamental scale of gravity, and $\mathcal{L}_m$ the matter lagrangian.

We are interested in the expansion law of the this model in absence of vacuum energies, with only matter on the branes acting as a source for the metric $\tilde{g}$. Following the analysis of [92], we assume the fifth dimension compact in the interval $-1/2 \leq y \leq 1/2$ and impose the orbifold symmetry $y \leftrightarrow -y$. We place the two branes at the endpoints ot the relevant interval, that is at $y=0$ and $y=1/2$. We also assume an idealized situation where the branes are infinitely thin. Rigorously, a physical brane can be assumed to have a width in the fifth dimension close to the inverse of the fundamental scale of the underlying theory. The thin-brane approximation will thus be valid when the energy scales at which we consider the theory are smaller than the fundamental scale. With this assumptions, the stress-energy tensor evaluates to

$$T^A_B = \frac{\delta\,(y)}{b}\mathrm{diag}\,(-\,\rho_0\,,\,p_0\,,\,p_0\,,\,p_0\,,\,0) \tag{6.2}$$

and to

$$T^A_B = \frac{\delta\,(y-1/2)}{b}\mathrm{diag}\left(-\,\rho_{1/2}\,,\,p_{1/2}\,,\,p_{1/2}\,,\,p_{1/2}\,,\,0\right) \tag{6.3}$$

on the two branes, and to zero in the bulk.

[1]For some considerations about this problem in superstring and supergravity and for some proposals for its solution, see for instance [126, 127, 128] and references therein.

We are interested in the expansion law of one of the two branes, say the one at $y = 0$. We take for the metric the ansatz

$$ds^2 = -\,n^2\,(t\,,\,y)\;dt^2 + a^2\,(t\,,\,y)\;\delta_{ij}\,dx^i\,dx^j + b^2\,(t\,,\,y)\;dy^2\,. \tag{6.4}$$

where it is assumed for simplicity flatness in the ordinary spatial dimensions (in addition of course to homogeneity and isotropy).

The dynamics of the system is governed by the five-dimensional Einstein equations

$$\tilde{G}_{AB} = \kappa^2\,\tilde{T}_{AB}\,. \tag{6.5}$$

Inserting the ansatz (6.4) for the metric, the non-vanishing components of the Einstein tensor $\tilde{G}_{A\,B}$ are found to be [92]

$$\begin{aligned}
\tilde{G}_{00} &= 3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)-\frac{n^2}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)\right\}, &(6.6)\\
\tilde{G}_{ij} &= \frac{a^2}{b^2}\delta_{ij}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right)-\frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right)+2\frac{a''}{a}+\frac{n''}{n}\right\}+\\
&+\frac{a^2}{n^2}\delta_{ij}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a}+2\frac{\dot{n}}{n}\right)-2\frac{\ddot{a}}{a}+\frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a}+\frac{\dot{n}}{n}\right)-\frac{\ddot{b}}{b}\right\}, &(6.7)\\
\tilde{G}_{05} &= 3\left(\frac{n'}{n}\frac{\dot{a}}{a}+\frac{a'}{a}\frac{\dot{b}}{b}-\frac{\dot{a}'}{a}\right), &(6.8)\\
\tilde{G}_{55} &= 3\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right)-\frac{b^2}{n^2}\left(\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)+\frac{\ddot{a}}{a}\right)\right\}. &(6.9)
\end{aligned}$$

In this notation, prime denotes derivative with respect to y, and dot with respect to t.

In order to satisfy eqs. (6.5), some terms of the Einstein tensor G_{AB} have to be non regular on the two branes, if these are assumed infinitely thin. The metric (6.4) is of course required to be continuous everywhere to have a well defined geometry. However its derivatives with respect to y can be discontinuous across the two branes, and this entails the existence of a Dirac delta function in the second derivatives.

To take into account these singularities, let us define [92] the *jump* and the *mean value* of a given function f across a point y by, respectively,

$$[f]_y = f\,(y^+) - f\,(y^-) \tag{6.10}$$

and

$$\sharp f \sharp_y = \frac{f\,(y^+) + f\,(y^-)}{2}\,. \tag{6.11}$$

Concerning the metric (6.4), for the function [2] $a\,(t\,,\,y)$, one has for example

$$a'' = \widehat{a''} + [a']_0\,\delta\,(y) + [a']_{1/2}\,\delta\,(y - 1/2)\ , \tag{6.12}$$

with $\widehat{a''}$ denoting the non-distributional part of the double derivative of a (i.e. the standard derivative).

Matching the distributional part of the $(0\,0)$ and of the $(i\,i)$ components of the Einstein equations (6.5), one obtains the following Israel [129] junction conditions

$$\begin{aligned} \frac{[a']_0}{a_0\,b_0} &= -\,\frac{\kappa^2}{3}\,\rho_0 \quad , \quad \frac{[n']_0}{n_0\,b_0} = \frac{\kappa^2}{3}\,(3\,p_0 + 2\,\rho_0)\ , \\ \frac{[a']_{1/2}}{a_{1/2}\,b_{1/2}} &= -\,\frac{\kappa^2}{3}\,\rho_{1/2} \quad , \quad \frac{[n']_{1/2}}{n_{1/2}\,b_{1/2}} = \frac{\kappa^2}{3}\left(3\,p_{1/2} + 2\,\rho_{1/2}\right)\ , \end{aligned} \tag{6.13}$$

where the subscript 0 (1/2) for $a\,,\,b\,,\,n$ means that these functions are taken at $y = 0\ (1/2)$.

Let us now consider the regular part of eqs. (6.5). For what concerns a'', we make the simplest possible ansatz for the regular part, so that the whole expression reads

$$a'' = [a']_0\,(\delta\,(y) - 1) + [a']_{1/2}\,(\delta\,(y - 1/2) - 1)\ . \tag{6.14}$$

This is the simplest choice which does not preclude the possibility of having a global solution of eqs. (6.5), since it ensures flux conservation in the compact y direction [92]

.

Integrating this last expression, we obtain [3]

$$\begin{aligned} a' &= \frac{1}{2}\,\mathrm{sign}\,(y)\,[a']_0 - y\left([a']_0 + [a']_{1/2}\right) \\ a &= a_0 + \left(\frac{1}{2}|y| - \frac{1}{2}y^2\right)[a']_0 - \frac{1}{2}\,y^2\,[a']_{1/2}\ . \end{aligned} \tag{6.15}$$

We assume the same ansatzs for the function $n\,(t\,,\,y)$, with the additional requirement $n_0 \equiv 1$, so that t is the physical time measured by observers on the zero brane. For what concerns instead the function b, we assume a linear dependence on y but we require it to be *static*. That is

$$b = b_0 + 2|y|\,\left(b_{1/2} - b_0\right)\ . \tag{6.16}$$

[2]The same occurs also for the function $n\,(t\,,\,y)$.

[3]The integration constants are chosen so to satisfy $\int dy\,a\,(t\,,\,y) = 0$, as required from the identification of the two points $y = \pm 1/2$ and from the continuity of a, and $a\,(y = 0) = a_0$.

With all the above choices and taking conditions (6.13) into account, the regular part of the (0 0) component of eq. (6.5) acquires in $y = 0$ the form

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa^2\, a_{1/2}\, b_{1/2}}{3\, a_0\, b_0^2}\, \rho_{1/2} + \frac{\kappa^2\, b_{1/2}}{3\, b_0^2}\, \rho_0 + \frac{\kappa^4}{36}\, \rho_0^2\,. \tag{6.17}$$

Eq. (6.17) is the expansion law of the zero brane. We notice the presence of two terms linear in the energy densities on the two branes plus a third term proportional to ρ_0^2. Let us go on with the analysis repeating the same procedure with the $(i\, i)$ component. Doing so, and summing the result with the above expression, we get

$$\begin{aligned} \frac{\dot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0} &= \frac{\kappa^4}{36}\, \rho_0\, (-3\, p_0 - \rho_0) + \frac{\kappa^2\, b_{1/2}}{2\, b_0^2} \left(\rho_0 + \frac{a_{1/2}\, \rho 1/2}{a_0} \right) + \\ &- \frac{\kappa^2\, b_{1/2}}{6\, b_0^2} \left[3\, p_0 + 2\, \rho_0 + \frac{n_{1/2}}{n_0} \left(3\, p_{1/2} + 2\, \rho_{1/2} \right) \right]\,. \end{aligned} \tag{6.18}$$

Let us now look at the (5 5) component. Its *mean value* gives (notice that $\sharp f\, g \sharp = \sharp f \sharp\, \sharp g \sharp + \frac{1}{2}\, [f]\, [g]$)

$$\frac{\dot{a_0}^2}{a_0^2} + \frac{\ddot{a_0}}{a_0} = -\frac{\kappa^4}{36}\, \rho_0\, (\rho_0 + 3\, p_0)\ . \tag{6.19}$$

By the comparison of these two last expressions, one is lead to deduce [92] that the (5 5) component is compatible with the others only if the relations

$$\rho_0 a_0 = -\rho_{1/2}\, a_{1/2}\ , \tag{6.20}$$
$$(2\, \rho_0 + 3\, p_0)\, n_0 = -\left(2\, \rho_{1/2} + 3\, p_{1/2}\right) n_{1/2} \tag{6.21}$$

hold. Hence, these results indicate that – in absence of any other sources for the Einstein equations – the matter energy density and pressure on the two branes cannot be chosen arbitrary, but have to be fine tuned the ones to the others. Because of the fine-tuning (6.20), in eq. (6.17) the two terms linear in ρ cancel, and one is left with an expansion law $H_0^2 \equiv (\dot{a}_0/a_0)^2 \propto \rho^2$, rather than the standard $H^2 \propto \rho$. This deviation from the standard behavior does not disappear with decreasing ρ, rendering the scenario here described phenomenologically excluded [92].

The only possibility not to conclude that brane models cannot generically reproduce standard cosmology [92] is to investigate the physical origin of eqs. (6.20) and (6.21), to see how they can evaded. As first noticed in the works [93, 94], the origin of these fine tunings is strictly connected to the problem of dilaton stabilization. We discuss this problem in details in the next section, which, although dealing with the expansion law of the Randall-Sundrum model [89], contains several considerations about brane models in general.

6.2 Expansion law of the RS model

Right after the work [92] and the proposals of the two RS models [89, 107], the cosmological evolution of scenarios with nonfactorizable geometry was also investigated [130, 131, 132, 133, 134]. The conclusions were that, while the model [107] could reproduce standard cosmology, the scenario [89] with two walls was plagued with some problems concerning the expansion law.

Indeed, the results were qualitatively very similar to the ones found for a factorizable geometry. Also in this case the problem comes from the necessity to fine-tune the energy densities and the pressures of the two walls. In the RS [89] model this fine-tuning acquires the form

$$\begin{aligned} \rho_{1/2} &= -\,\Omega_0^2\, p_0 \,, \\ p_{1/2} &= -\,\Omega_0^2\, p_{1/2} \quad , \quad \Omega_0 \equiv \mathrm{e}^{-m_0\, b_0/2} \quad , \end{aligned} \tag{6.22}$$

where ρ_i and p_i are, respectively, the matter [4] density and the pressure on the $i-th$ brane ($i = 0\,,\, 1/2$), m_0 a "measure" of the strength of the vacuum energies (see eq. (4.20)), and b_0 the size of the compactification radius (the so called "radion") of the extra dimension.

In the above works, the expansion law of the model was investigated with a perturbation expansion in ρ_i and p_i. The square of the Hubble parameter can also be written as an expansion series in the energy densities. However, after the constraint $\rho_0 = -\,\Omega_0^2\,\rho_{1/2}$ is imposed, again the linear term cancels and one is left with

$$\begin{aligned} H_{1/2}^2 &= \frac{\Omega_0^4}{1-\Omega_0^2}\left[\frac{\kappa^2\,\Lambda_b^2}{6} + \frac{\kappa^4\,\left(\rho_{1/2}+V_{1/2}\right)^2}{36}\right] + \text{higher orders} \\ &= -\,\frac{1}{3\,M_{\mathrm{P}}}\,\Omega_0^4\,\rho_{1/2} + \text{higher orders}\,. \end{aligned} \tag{6.23}$$

There is a slight improvement with respect to the previous case, due to the presence of the vacuum energies Λ_b and V_i. From the fine-tuning they are subjected to even at zero temperature, see eq (4.20), the terms proportional to the vacuum energies cancel (as we already remarked, relation (4.20) results in a zero effective cosmological constant from the point of view of observers on the two branes), while the mixed term $\propto V_{1/2}\,\rho_{1/2}$ introduces a new linear term in $\rho_{1/2}$ [5]. However (although the absolute value is correct) there is a crucial minus sign in the coefficient,

[4] With matter we generically indicate any possible component which adds to the vacuum energies V_0, $V_{1/2}$ of the original static configuration [89].

[5] Notice also the presence of the Ω_0^4 factor left, which indicates that the expansion rate is indeed proportional to the physical (i.e. redefined, see eq. (4.27)) energy density on the 1/2 brane.

which requires an unphysical negative energy density on our brane to reproduce the observed expansion of the Universe.

We see that the problem again originates from the necessity of fine-tuning the energies of the two branes. In turn, this is related to the absence of any mechanism responsible for stabilizing the compact dimension, as we discuss in the following subsection.

6.2.1 The role of dilaton stabilization

One can obtain eqs. (6.22) and (6.23) in a way very similar to what we did in section 6.1, that is trying to find a sufficiently generic ansatz for the metric, and looking at the Einstein equations of the system as well as the junction conditions across the two branes.

In this subsection we mostly refer to the detailed analysis performed in the work [93], where the cosmological evolution of the RS model [89] is studied at first order in the energy density and pressure on the two branes. In this study, the metric is assumed to be of the form [6]

$$ds^2 = n\,(t\,,\,y)^2\,dt^2 - a\,(t\,,\,y)^2\,\left(dx_1^2 + dx_2^2 + dx_3^2\right) - b\,(t)^2\,dy^2\,. \tag{6.24}$$

The study of the Einstein and the jump conditions of the system leads to the following ansatz (at first order in $\rho_i\,,\,p_i$) [93]

$$\begin{aligned} n\,(t\,,\,y) &= \mathrm{e}^{-\,|y| b_0\, m_0}\,\left(1 + \frac{\kappa^2\,(2\,\rho_0 + 3\,p_0)}{12\,m_0}\,\left(\mathrm{e}^{2\,|y| m_0\, b_0} - 1\right)\right)\,, \\ a\,(t\,,\,y) &= a_0\,(t)\;\mathrm{e}^{-\,|y| b_0\, m_0}\,\left(1 - \frac{\kappa^2 \rho_0}{12\,m_0}\,\left(\mathrm{e}^{2|y| m_0\, b_0} - 1\right)\right)\,. \end{aligned} \tag{6.25}$$

Inserting this ansatz back into the action of the model, eqs. (4.18) and (4.19), and integrating over the fifth dimension, one gets the effective action

$$\begin{aligned} S_{eff} &= -\frac{1}{2\,\kappa^2\,m_0}\int dt\,a_0^3\,\left(1 - \Omega_b^2\right)\,\mathcal{R}_{(4)}\,(a_0) + S_{\mathrm{P}}^M + S_{\mathrm{TeV}}^M + \\ &+ \int dt\,a_0^3\,\left(\frac{3}{4}\,\frac{1}{\kappa^2\,m_0}\,(m_0\,b)^2\,\Omega_b^2\,\frac{\dot{b}^2}{b^2} - V_r\,(b)\right)\,, \end{aligned} \tag{6.26}$$

where

$$S_{\mathrm{P}}^M \equiv \int dt\,a_0^3\,\mathcal{L}_{\mathrm{P}} \quad \text{and} \quad S_{\mathrm{TeV}}^M \equiv \int dt\,a_0^3\,\Omega_b^4\,\mathcal{L}_{\mathrm{TeV}} \tag{6.27}$$

[6]Notice that a time-dependence for the radion is here allowed.

come from the matter action for the two branes, and where

$$\mathcal{R}_{(4)} \equiv -6\,\frac{\ddot{a}_0}{a_0} - 6\left(\frac{\dot{a}_0}{a_0}\right)^2 \quad , \quad \Omega_b \equiv \mathrm{e}^{-m_0\,b/2}\,. \tag{6.28}$$

If b is assumed to be static at the value required to solve the hierarchy problem, $b\,(t) \equiv b_0$, the gravitational part of the above expression reduces to the standard Friedmann-Robertson-Walker action [7], and the factor Ω_b^4 appearing inside S^M_{TeV} exactly reproduces TeV physical energy densities and pressures starting from the original ones of order M_{P}^4 .

To start discussing the physics of the radion, let us consider the last term of eq. (6.26). The kinetic part is due to the fact that we are now allowing b to be a function of time. The other term is not included in the action of the system that we have written so far (eqs. (4.18) and (4.19)), and it comes from a possible (here unspecified) potential for the radion. We now show that indeed this term is necessary to obtain a viable cosmological evolution of the model. To see this, let us for the moment require for a static radion, $\dot{b} = 0$, in absence of any potential, $V_r\,(b) = 0$. With this choices, the gravitational part of the action is the standard FRW one, and the standard equations

$$\begin{aligned} \frac{\dot{a}_0^2}{a^2} &= \frac{1}{3\,M_{\mathrm{P}}}\left(\rho_0 + \Omega_0^4\,\rho_{1/2}\right)\,, \\ \frac{\dot{a}_0^2}{a^2} + 2\,\frac{\ddot{a}_0}{a_0} &= -\frac{1}{M_{\mathrm{P}}}\left(p_0 + \Omega_0^4\,p_{1/2}\right)\,, \end{aligned} \tag{6.29}$$

hold.

More interesting is the study of the equation of motion for b . The point is that, due to the dependence of Ω_b on b , the presence of the matter generates a potential for the radion. To see this, we compute the variation of the above action with respect to b, noting that (we assume) S^M depends on the radion only through the warp factor Ω_b. Thus, the contribution of the matter fields to the equation of motion for b is

$$\frac{\delta S^M}{\delta b} = \frac{\delta S^M}{\delta \tilde{g}^{\mu\nu}}\,\frac{\delta \tilde{g}^{\mu\nu}}{\delta b} = -\sqrt{\tilde{g}}\,\tilde{T}_{\mu\nu}\,\tilde{g}^{\mu\nu}\,\frac{\Omega_b'}{\Omega_b} = -\sqrt{g}\,\frac{\partial}{\partial b}\,\frac{1}{4}\,\tilde{T}\,\Omega_b^4\,, \tag{6.30}$$

where $\tilde{g}_{\mu\nu} = \Omega_b^2\,\mathrm{diag}\,(1\,,\,-a^2\,,\,-a^2\,,\,-a^2)$, and $\tilde{T}$ is the trace of the stress tensor [8] in terms of the bare fields and bare masses, and is equal to $\rho - 3\,p$ for a perfect

[7] Notice that eq. (6.26) was obtained with an expansion in ρ_i , and so higher order corrections are expected to modify this picture at higher temperature. See the next subsection.

[8] Thus we see that b couples to the trace of the stress tensor, and has "dilaton"-like couplings.

fluid. Thus, the matter fields on the TeV brane generate an effective potential for the radion that is

$$V_{eff}(b) = \frac{1}{4}\Omega_b^4 \left(\rho_{1/2} - 3\,p_{1/2}\right) \; . \tag{6.31}$$

Taking this into account, the equation of motion for the radion is (again with $\dot{b} = 0$ and $V_r = 0$)

$$\left(3\,p_{1/2} - \rho_{1/2}\right)\Omega_b = \frac{6\,\Omega_b^2}{\kappa^2\,m_0}\left(\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2}\right) \; . \tag{6.32}$$

Using eqs. (6.29), this relation simplifies to

$$\frac{\Omega_b^4}{1-\Omega_b^2}\left(3\,p_{1/2} - \rho_{1/2}\right) = -\frac{\Omega_b^2}{1-\Omega_b^2}\left(3\,p_0 - \rho_0\right) \; . \tag{6.33}$$

For generic ρ_0 and $\rho_{1/2}$ the solution to this equation is $\Omega_b \rightarrow 0$. Since $\Omega_b = e^{-m_0 b/2}$, $b \rightarrow \infty$. That is, the branes want to blow apart. There is another solution, however, and that is to allow a fine-tuning between the matter on the two branes. This fine-tuning is precisely given by eqs. (6.22). [9]

So, this demonstrates that the constraints (6.22) directly follow from requiring that the radion modulus is static even when there is no stabilizing potential. From this perspective it is clear that, with a radion potential, ρ_i and p_i will not need any longer to be correlated. For uncorrelated ρ_i and p_i, the branes tend to go off to infinity; this, however, will be balanced by the restoring force from the radion potential. What was once a constraint equation for ρ_i and p_i, in the presence of the radion potential V_r becomes an equation determining the new equilibrium point.

Had we reproduced here the study of the system in a similar way as what we did in section 6.1, we would have found that the above relation (6.32) is the average of the (5 5) component of the Einstein equations across the TeV brane. This is exactly what happened for models with nonfactorizable geometry. Indeed, we remark that also in that case the fine-tuning on the energy densities and on the pressures of the two branes were forced by an apparent inconsistency of the (5 5) component with respect to the other Einstein equations of the model. Also there the problem originated from the fact that we were asking for a static extra-dimension in absence of a mechanism which stabilizes it.

6.2.2 Exact solutions in presence of matter

We now solve the Einstein equations of the RS model. We assume (without entering into the details of the problem [10]), that a five dimensional potential $U(b)$ has been

[9] In principle, one can ask for the whole combination $3\,p_{1/2} - \rho_{1/2} = -\,\Omega_0^2\left(3\,p_0 - \rho_0\right)$ to vanish. However, if one also considers the energy conservation laws on the two branes, see the next eqs. (6.45), the relations (6.22) hold.

[10] See however the last section for a specific example.

generated in the five dimensional theory by some mechanism. Then, starting from the usual ansatz

$$ds^2 = n^2(t, y)\, dt^2 - a^2(t, y)\, \delta_{ij}\, dx^i\, dx^j - b^2(t, y)\, dy^2 , \tag{6.34}$$

the equations of motion in the bulk are given by

$$\begin{aligned} G_{00} &= \kappa^2 n^2 [\Lambda + U(b)] \quad , \quad G_{ii} = -\kappa^2 a^2 [\Lambda + U(b)] , \\ G_{05} &= 0 \quad , \quad G_{55} = -\kappa^2 b^2 [\Lambda + U(b) + b\, U'(b)] , \end{aligned} \tag{6.35}$$

with G_{AB} given in the section 6.1.

We would like to present some general solutions, which do not depend on the particular mechanism of localization of the radion, i. e. on $U(b)$. We just assume that the mass of the radion is very heavy, and that near its minimum U is expanded into

$$U(b) = M_b^5 \left(\frac{b - b_0}{b_0} \right)^2 , \tag{6.36}$$

where b_0 is the stabilized value of the radius. If M_b is much higher than any other mass in the theory, then the solution to the (5 5) Einstein equation (6.35) is simply given by $b = b_0$, with no other constraint on a and n. With this solution we also find that $U(b_0)$=0. Thus in the presence of a very heavy radion field the relevant Einstein equations are the (0 0), the $(i\, i)$, and the (0 5) components, with the radius fixed to be at the stable value b_0. This already shows how the constraint is eliminated. One of the equations of motion, which played a vital role in establishing the correlation between the matter on the two branes, is simply not appearing and it is automatically satisfied in the presence of the stable radius. [11] While this is strictly true only for zero energy density on the two branes, we expect it to be sufficiently accurate also for ρ_0 and $\rho_{1/2}$ not too close to he scale of radion stabilization, i.e. for ρ_0 , $\rho_{1/2} \lesssim M^4$.

Surprisingly enough, if one fixes $b \equiv b_0$ and neglects the (5 5) component, the system of the other equations (plus the two junction conditions) can be solved *exactly* [95] [12]. For a static radion, the nontrivial components of the Einstein tensor

[11] More accurately, from this equation one can now understand which potentials $U(b)$ are suitable for stabilizing the radion at the values needed to solve the hierarchy problem.

[12] The derivation of ref. [95] is analogous to the one performed in the work [94], limited to the case of a single brane.

of the system take the form

$$
\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}}{a}\right)^2 - 3\,\frac{n^2}{b_0^2}\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right], && (6.37)\\
G_{ii} &= \frac{a^2}{n^2}\left[-\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{\dot{a}}{a}\frac{\dot{n}}{n} - 2\,\frac{\ddot{a}}{a}\right] + \frac{a^2}{b_0^2}\left[\left(\frac{a'}{a}\right)^2 + 2\,\frac{a'}{a}\frac{n'}{n} + 2\,\frac{\ddot{a}}{a} + \frac{n''}{n}\right] && (6.38)\\
G_{05} &= 3\left[\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a}\right], && (6.39)
\end{aligned}
$$

where again dot denotes differentiation with respect to t and prime with respect to y.

First of all, we integrate the Einstein equation for (6.39). This is solved either for $\dot{a} = 0$ (in this case one recovers a class of static solutions including the RS solution (4.21)), or for

$$n\,(t\,,\,y) = \lambda\,(t)\,\dot{a}\,(t\,,\,y)\;. \tag{6.40}$$

This relation introduces an unknown function of time only, and considerably simplifies the remaining equations. Note that we have a complete freedom in the choice of λ, since different λ's correspond to different definitions of the time variable.

By inserting eq. (6.40) into eq. (6.37), we can eliminate the time-derivatives, and we obtain a simple second-order differential equation for a^2

$$\left(a^2\,(t\,,\,y)\right)'' - 4\,m_0^2\,b_0^2\,a^2\,(t\,,\,y) = \frac{2\,b_0^2}{\lambda\,(t)^2}\;. \tag{6.41}$$

This equation admits the solution

$$a^2\,(t\,,\,y) = a_0^2\,(t)\;\omega_0^2\,(y) + a_{1/2}^2\,(t)\;\omega_{1/2}^2\,(y) + \frac{\omega_0^2\,(y) + \omega_{1/2}^2\,(y) - 1}{2\,m_0^2\,\lambda\,(t)^2}\;, \tag{6.42}$$

where

$$
\begin{aligned}
\omega_0^2\,(y) &= \cosh\left(2\,m_0\,b_0\,|y|\right) - \frac{C_0}{S_0}\,\sinh\left(2\,m_0\,b_0\,|y|\right)\;,\\
\omega_{1/2}^2\,(y) &= \frac{\sinh\left(2\,m_0\,b_0\,|y|\right)}{S_0}\;, && (6.43)
\end{aligned}
$$

with $C_0 \equiv \cosh\,(m_0\,b_0)$ and $S_0 \equiv \sinh\,(m_0\,b_0)$. Eq. (6.42) relates the value of $a\,(t\,,\,y)$ in the whole space to the values on the two branes $a_0\,(t) \equiv a\,(t\,,\,0)$ and $a_{1/2}\,(t) \equiv a\,(t\,,\,1/2)$. These two unknown time–dependent functions are determined below.

Rather than the last remaining non–trivial equation associated to the component (6.38) of the Einstein tensor, we consider – in strict analogy to what is customarily done in conventional FRW cosmology – the equation associated to the identity [13]

$$G_{ii} = -\frac{a}{\dot{a}}\frac{d}{dt}\left\{\frac{a^2}{3\,n^2}G_{00}\right\} - \frac{a^2}{3n^2}G_{00}\ . \qquad (6.44)$$

Substituting in this relation the components of the Einstein tensor with the corresponding components of the energy–momentum tensor, we get an expression which is trivially satisfied in the bulk, while on the two branes it reduces to

$$\dot{\rho}_i + 3\frac{\dot{a}_i}{a_i}\left(\rho_i + p_i\right) = 0\ ,\quad i = 0\,,\, 1/2\ . \qquad (6.45)$$

These two equations are nothing but the energy–conservation law in the two branes and they are identical to the energy–conservation law of standard FRW cosmology.

We finally have to determine the functions $a_0\,(t)$ and $a_{1/2}\,(t)$ appearing in expression (6.42). This can be done by solving eq. (6.37) across the two branes. Analogously to what done in the work [92], we put this last step in form of junction conditions which relate the discontinuity of n' and a' to the delta–like source $\left(T_A^B\right)_{brane}$. From the symmetry $y \leftrightarrow -y$, we can write the junction conditions in the form (cfr. eqs. (6.13))

$$\begin{aligned}
\frac{a'\,(t\,,\,0)}{a\,(t\,,\,0)} &= -\frac{\kappa^2}{6}\,b_0\,(V_0+\rho_0)\ ,\\
\frac{n'\,(t\,,\,0)}{n\,(t\,,\,0)} &= \frac{\kappa^2}{6}\,b_0\,[2\,\left(V_0+\rho_0\right)+3\,\left(-V_0+p_0\right)]\ ,\\
\frac{a'\,(t\,,\,1/2)}{a\,(t\,,\,1/2)} &= \frac{\kappa^2}{6}\,b_0\left(V_{1/2}+\rho_{1/2}\right)\ ,\\
\frac{n'\,(t\,,\,1/2)}{n\,(t\,,\,1/2)} &= -\frac{\kappa^2}{6}\,b_0\left[2\,\left(V_{1/2}+\rho_{1/2}\right)+3\,\left(-V_{1/2}+p_{1/2}\right)\right]\ .
\end{aligned} \qquad (6.46)$$

These equations lead to the following system for a_0, $a_{1/2}$

$$\begin{aligned}
\left[1+\frac{\kappa^2\,\rho_0}{6\,m_0}-\frac{C_0}{S_0}\right]a_0^2+\frac{a_{1/2}^2}{S_0} &= \frac{C_0-1}{2\,m_0^2\,\lambda^2\,S_0}\ ,\\
\frac{a_0^2}{S_0}+\left[-1+\frac{\kappa^2\,\rho_{1/2}}{6\,m_0}-\frac{C_0}{S_0}\right]a_{1/2}^2 &= \frac{C_0-1}{2\,m_0^2\,\lambda^2\,S_0}\ .
\end{aligned} \qquad (6.47)$$

As expected, the system admits no solution in absence of matter on the two branes, $\rho_0 = \rho_{1/2} = 0$. Indeed, for this choice one recovers the static RS solution, which

[13] To achieve this identity, the expression (6.40) must be used.

is not accounted for by the relation (6.40). When matter is instead included, the system (6.47) gives the solutions

$$a_0^{-2}\lambda^{-2} = \frac{\kappa^2\, m_0}{3\,(1-\Omega_0^2)} \frac{\rho_0+\Omega_0^4\rho_{1/2}-\frac{\kappa^2}{12\,m_0}\left(1-\Omega_0^4\right)\rho_0\rho_{1/2}}{1-\left(1-\Omega_0^2\right)\frac{\kappa^2\,\rho_{1/2}}{12\,m_0}} \,, \tag{6.48}$$

$$a_{1/2}^{-2}\lambda^{-2} = \frac{\kappa^2\, m_0}{3\,(1-\Omega_0^2)}\,\frac{1}{\Omega_0^2}\, \frac{\rho_0+\Omega_0^4\rho_{1/2}-\frac{\kappa^2}{12\,m_0}\left(1-\Omega_0^4\right)\rho_0\rho_{1/2}}{1-\left(\Omega_0^{-2}-1\right)\frac{\kappa^2\,\rho_0}{12\,m_0}} \,. \tag{6.49}$$

Since $\lambda = n_0/\dot{a}_0 = n_{1/2}/\dot{a}_{1/2}$, we can interpret these equations as the expansion laws of the two branes. As we will see in subsection 6.2.4, eqs. (6.45), (6.48), and (6.49) give standard FRW evolution on both branes at low energy.

6.2.3 The effective action

In this subsection we calculate the effective action of the system in terms of the scale factors $a_0\,(t)$, $a_{1/2}\,(t)$ on the two branes. To do this, we integrate the whole action over the extra dimension y , making use of the result (6.42).

We first focus on the "purely gravitational" five dimensional action, that is we integrate the RS action in the absence of matter on the two branes. The latter will be considered eventually when we deal with the equations of motion. Our starting point is thus

$$\begin{aligned} S &= -\int d^4x\, dy\, \sqrt{g}\left[\frac{R}{2\,\kappa^2}+\Lambda+\frac{\delta\,(y)}{b_0}V_0+\frac{\delta\,(y-1/2)}{b_0}V_{1/2}\right] \\ &= -\frac{1}{2\,\kappa^2}\int d^4x\, dy\sqrt{g}\,\left[R-12\,m_0^2+12\,m_0\left(\frac{\delta\,(y)}{b_0}-\frac{\delta\,(y-1/2)}{b_0}\right)\right] \,, \end{aligned} \tag{6.50}$$

with the full (five dimensional) curvature scalar given by

$$R = 6n^{-2}\left[\frac{\dot{n}}{n}\frac{\dot{a}}{a}-\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2\right]+2b_0^{-2}\left[\frac{n''}{n}+3\frac{n'}{n}\frac{a'}{a}+3\frac{a''}{a}+3\left(\frac{a'}{a}\right)^2\right] \,. \tag{6.51}$$

Since we are interested in the evolution of the two four dimensional branes, we rewrite $n\,(t\,,\,y)$ and $a\,(t\,,\,y)$ by making use of the above results, eqs. (6.40) and (6.42). It is then convenient to write $\sqrt{g}R$ and $\sqrt{g}$ in terms of a^2 and λ

$$\begin{aligned} \sqrt{g}R &= \frac{6b_0}{\lambda}\left(\frac{\dot{\lambda}}{\lambda}a^2-\frac{1}{2}\frac{d\,a^2}{dt}\right)+\frac{\lambda}{2b_0}\left[\frac{d\,(a^4)''}{dt}-3\left(a^2\right)'\frac{d\,(a^2)'}{dt}\right] \,, \\ \sqrt{g} &= \frac{\lambda\, b_0}{4}\frac{d\,a^4}{dt} \,. \end{aligned} \tag{6.52}$$

With all these considerations, [14]the integral over y of the action (6.50) gives

$$S = -\frac{1}{2\kappa^2}\int d^4x \left\{ \frac{1-\Omega_0^2}{m_0}\frac{1}{1+\Omega_0^2}\left[\frac{6}{\lambda}\left(\frac{\dot\lambda}{\lambda}\left(a_0^2+a_{1/2}^2\right) - \left(a_0\,\dot a_0 + a_{1/2}\,\dot a_{1/2}\right)\right)\right] + \right.$$
$$\left. + \frac{24\,m_0}{1-\Omega_0^4}\lambda\left[\Omega_0^2\left(a_0\,\dot a_0\,a_{1/2}^2 + a_{1/2}\,\dot a_{1/2}\,a_0^2\right) - \left(\Omega_0^4\,a_0^3\,\dot a_0 + a_{1/2}^3\,\dot a_{1/2}\right)\right]\right\}. \quad (6.53)$$

By substituting $\lambda = n_0/\dot a_0 = n_{1/2}/\dot a_{1/2}$ in the last expression[15] we get

$$S = -\frac{M_p^2}{2(1+\Omega_0^2)}\int d^4x \left\{ n_0\,a_0^3\,\frac{6}{n_0^2}\left[\frac{\dot n_0\,\dot a_0}{n_0\,a_0} - \frac{\ddot a_0}{a_0} - \left(\frac{\dot a_0}{a_0}\right)^2\right] + \right. \quad (6.54)$$
$$+ n_{1/2}\,a_{1/2}^3\,\frac{6}{n_{1/2}^2}\left[\frac{\dot n_{1/2}\,\dot a_{1/2}}{n_{1/2}\,a_{1/2}} - \frac{\ddot a_{1/2}}{a_{1/2}} - \left(\frac{\dot a_{1/2}}{a_{1/2}}\right)^2\right] +$$
$$\left. + \frac{24\,m_0^2}{(1-\Omega_0^2)^2}\left[\Omega_0^2\left(a_0\,n_0\,a_{1/2}^2 + a_{1/2}\,n_{1/2}\,a_0^2\right) - \left(\Omega_0^4\,a_0^3\,n_0 + a_{1/2}^3\,n_{1/2}\right)\right]\right\}.$$

As we will discuss in more detail in the next subsection, in the low energy limit the equality $a_{1/2}(t) = \Omega_0\,a_0(t)$ and the related one $n_{1/2}(t) = \Omega_0\,n_0(t)$ hold. As a consequence, the expansion rates of the two branes are identical and the above action rewrites in the standard FRW form

$$S = -\frac{M_p^2}{2}\int d^4x\,\bar n\,\bar a^3\,\frac{6}{\bar n^2}\left[\frac{\dot{\bar n}}{\bar n}\frac{\dot{\bar a}}{\bar a} - \frac{\ddot{\bar a}}{\bar a} - \left(\frac{\dot{\bar a}}{\bar a}\right)^2\right], \quad (6.55)$$

where $\bar a \equiv a_0 = \Omega_0^{-1}\,a_{1/2}$, $\bar n \equiv n_0 = \Omega_0^{-1}\,n_{1/2}$.

From the effective action (6.54) we notice that the entire five dimensional system can be ("holographically") expressed in terms of the physics that takes place on the boundaries at $y = 0$ and $y = 1/2$ of the extra space. Notice also that the last term in the action (6.54) couples the metrics of the two walls.

Going back to the effective action (6.54), and including also matter on the two walls, we obtain the equations of motion

$$\frac{\dot a_0^2}{n_0^2\,a_0^2} = \frac{1+\Omega_0^2}{3\,M_p^2}\rho_0 + \frac{4\,m_0^2}{(1-\Omega_0^2)^2}\Omega_0^2\left(\frac{a_{1/2}^2}{a_0^2} - \Omega_0^2\right),$$
$$\frac{\dot a_{1/2}^2}{n_{1/2}^2\,a_{1/2}^2} = \frac{1+\Omega_0^2}{3\,M_p^2}\rho_{1/2} + \frac{4\,m_0^2}{(1-\Omega_0^2)^2}\Omega_0^2\left(\frac{a_0^2}{a_{1/2}^2} - \frac{1}{\Omega_0^2}\right), \quad (6.56)$$

[14]The calculation can be further simplified by noticing that, from the periodicity imposed in the extra space, the integral of a derivative of any continuous function of y vanishes.

[15]In this way, we substitute $\lambda(t)$ with the two degrees of freedom $n_0(t)$ and $n_{1/2}(t)$. The equations of motion of the effective four dimensional theory have thus to be supported by the constraint $n_0/\dot a_0 = n_{1/2}/\dot a_{1/2}$. This relation cannot be obtained from the action (6.54), since it is linked to the equation $G_{05} = 0$ that has no counterpart in the four dimensional effective theory.

in addition to the relations which give energy conservation on the two branes, eqs. (6.45).

We notice that, in the limit $\rho_0 \to 0$, $\rho_{1/2} \to 0$, the only solution of the above equations is the static RS solution $a_{1/2} = \Omega_0 a_0$. Moreover, one can verify that eqs. (6.56) are equivalent to the equations (6.48) and (6.49) obtained in the previous subsection.

6.2.4 Standard evolution of the system at low energy

Before interpreting the results of the previous subsections, we come back to the static RS case. We recall that in [89] (see section 4.3) the four-dimensional metric $\bar{g}_{\mu\nu}$ on both branes is defined as

$$\bar{g}_{\mu\nu} = n\,(y)^{-2}\, g_{\mu\nu}\,. \tag{6.57}$$

The goal of this redefinition is to achieve Minkowski spacetime on both branes, in order to gain a simple physical interpretation of the system. An analogous procedure has to be applied also in the general case with matter on the two branes. Generally speaking, multiplying the metric by an overall function f is not equivalent to a change of the coordinate system. Thus, to have canonical normalization of the fields, the function f has to be absorbed by a redefinition of the fields themselves. In order to preserve the equations of motion of the fields, we see that we cannot choose f to depend on the coordinates t and x, but it can be at most a function of y.

In analogy with what was done in the static case, we now wonder whether it is possible to rewrite the first four components of the five-dimensional metric in the form

$$g_{\mu\nu}\,(t\,,\, y) = f\,(y)\; \bar{g}_{\mu\nu}\,(t)\;, \tag{6.58}$$

with $\bar{g}_{\mu\nu}$ of the standard FRW form $\mathrm{diag}(1, -\bar{a}^2, -\bar{a}^2, -\bar{a}^2)$. This requires the ratio n/a to be independent on y, that is $a'/a = n'/n$ for every value of y. From the junction conditions (6.46) we see that this implies $\rho + p = 0$ and, consequently, $\dot{\rho} = 0$ on the two branes. In other words, the above factorization is possible only if the two branes contain exclusively cosmological constants (in particular this is the case for the static RS solution).

Anyhow, it is natural to expect that condition (6.58) is approximately recovered when the matter on the two branes has a sufficiently low energy density. This can be understood from the results of the previous subsections. From eqs. (6.48) and (6.49) we have

$$\frac{a_{1/2}^2}{a_0^2} = \Omega_0^2\, \frac{1 - \frac{\kappa^2\left(1-\Omega_0^2\right)}{12\, m_0\, \Omega_0^2}\, \rho_0}{1 - \frac{\kappa^2\left(1-\Omega_0^2\right)}{12\, m_0}\, \rho_{1/2}}\,. \tag{6.59}$$

If ρ_0 and $\rho_{1/2}$ are sufficiently small, the scale factors of the two branes are (approximately) proportional [16] by the constant factor Ω_0. Since $n(y, t) = \lambda(t)\, \dot{a}(y, t)$, we have also $n_{1/2}(t) = \Omega_0 n_0(t)$. In particular, the ratio n/a is (approximately) independent on y. [17]

In this low-energy limit, we can thus define the four-dimensional effective theory just as in the static RS model. First, we can choose the time coordinate so that n_0 and $n_{1/2}$ are simultaneously time-independent. This is equivalent to setting $\lambda(t) = \lambda_0/\dot{a}_{1/2}(t)$, where λ_0 is an arbitrary factor. Then, we recover eq.(6.58) with

$$f(0) = \lambda_0^2\, \Omega_0^{-2}, \qquad f(1/2) = \lambda_0^2, \qquad \bar{a} = \lambda_0^{-1}\Omega_0 a_0 = \lambda_0^{-1} a_{1/2}\,. \tag{6.60}$$

We can now use the freedom to fix the time coordinate, and choose a particular value of λ_0. Choosing $\lambda_0 = \Omega_0$ we recover, in the limit $\rho_0\,,\, \rho_{1/2} \to 0$, the static RS solutions as presented in [89]. With this choice, the scale factor of the effective metric reads $\bar{a} = a_0 = \Omega_0^{-1}\, a_{1/2}$.

We can identify the five-dimensional quantities with those measured at low energy in our brane

- the fields must be redefined by a factor Ω_0. So, for instance, the observed density is $\bar{\rho}_{1/2} = \Omega_0^4\, \rho_{1/2}$. On the other brane the canonically normalized density reads $\bar{\rho}_0 = \rho_0$.

- the total four dimensional effective action (6.54) acquires the form of the standard FRW action in terms of the scale factor $\bar{a}$, see eq. (6.55).

- The Hubble parameter of the low energy theory is given by $\dot{\bar{a}}/\bar{a}$. From both eq. (6.48) and eq. (6.49) we get the standard evolution law

$$H^2 = \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 \simeq \frac{1}{3\, M_p^2}\, \left(\bar{\rho}_0 + \bar{\rho}_{1/2}\right) \quad , \tag{6.61}$$

while from eqs. (6.45) we recover

$$\begin{aligned} \dot{\bar{\rho}} &+ 3\frac{\dot{\bar{a}}}{\bar{a}}\,(\bar{\rho} + \bar{p}) = 0 \;, \\ \bar{\rho} &\equiv \bar{\rho}_0 + \bar{\rho}_{1/2} \;\; , \;\; \bar{p} \equiv \bar{p}_0 + \bar{p}_{1/2} \;. \end{aligned} \tag{6.62}$$

Some considerations are in order. First, we would like to emphasize that at low energy, from the point of view of observers on both branes, the effective theory leads exactly to a standard four-dimensional FRW Universe. This follows from the

[16] Notice that this relation holds exactly in the static RS case.

[17] From eqs. (6.42), (6.48), and (6.49), it is indeed possible to show that, in the low energy limit, the quantity $n'/n - a'/a$ is of the same order as $a_{1/2}/\left(\Omega_0 a_0\right) - 1$.

fact that the standard Friedmann law is recovered, and that the energy densities on both branes scale with the same Hubble parameter. In particular, for what concerns observers on our brane, the matter on the 0–brane is regarded as dark matter [93] that would completely escape any direct or indirect experimental detection (apart of course from its gravitational interactions). The gravitational effect of the matter on the 0–brane is not suppressed by powers of Ω_0, as it is the case for $\bar{\rho}_{1/2}$. Since the only natural mass scale of the model is the Planck scale, $\bar{\rho}_0$ must hence be fine–tuned to small values not to conflict with observations (see the next subsection).

Second, we remark that some care has to be paid in the interpretation of the physically observable quantities in the low energy effective theory. For instance, the alternative choice $\lambda_0 = 1$ [135] in eqs. (6.40) and (6.60) is not compatible with the identification of $\bar{\rho}_{1/2} = \Omega_0^4 \rho_{1/2}$ as the observed energy density on our brane. This would lead to a misinterpretation of the expansion laws of the two branes.

Then, in order to put quantitative limits on the validity of the low energy theory, we rewrite eq. (6.59) in terms of the observed matter densities [18]

$$\frac{a_{1/2}^2}{a_0^2} = \Omega_0^2 \, \frac{1 - \frac{\bar{\rho}_0}{10\, M_p^2\, \mathrm{Te}V^2}}{1 - \frac{\bar{\rho}_{1/2}}{10\, \mathrm{Te}V^4}} \quad . \tag{6.63}$$

We see that the low energy approximation is valid as long as the observed matter densities satisfy the bounds

$$\bar{\rho}_0 \ll 10\, M_p^2\, \mathrm{Te}V^2 \quad , \quad \bar{\rho}_{1/2} \ll 10\, \mathrm{Te}V^4 \, . \tag{6.64}$$

Finally, we would like to comment on Planck mass in the RS model. At low energy, there are two possible ways to define it, one related to the five–dimensional expansion, and one from the four dimensional effective action. These two definitions are called, respectively, *local* and *global* in ref. [136]. We see that indeed the values of M_p obtained with these two definitions coincide once all the quantities in the four dimensional action are properly identified.

6.2.5 Evolution at high energy

We now focus on the equations of motion when the low-energy conditions (6.64) are not fulfilled anymore. From what we said in the previous subsection, it is clear that in this regime it is not possible any longer to have a simple interpretation of the effective four dimensional action in terms of observable quantities. However, this is not important, because we make measurements only today, in the low-energy limit. So, it is legitimate to study the evolution of the system at high energy (eqs. (6.45),

[18] We use $m_0 \simeq \kappa^{-2/3} \simeq M_p$ and $M_{\mathrm{P}}\, \Omega_0 \simeq$ TeV.

(6.48), and (6.49)), and then make contact with the quantities that we observe today.[19]

We keep the previous definitions of $\bar{\rho}_i$, $\bar{p}_i$, $M_{\rm P}$, and the choice $\lambda = \Omega_0/\dot{a}_{1/2}$, so that eqs. (6.49) and (6.45) rewrite

$$\left(\frac{\dot{a}_{1/2}}{a_{1/2}}\right)^2 = \frac{1}{3\,M_p^2}\,\frac{\bar{\rho}_0+\bar{\rho}_{1/2}-\frac{\kappa^2}{12\,m_0}\left(\Omega_0^{-2}-\Omega_0^2\right)\bar{\rho}_0\bar{\rho}_{1/2}}{1-\left(\Omega_0^{-2}-1\right)\frac{\kappa^2\,\bar{\rho}_0}{12\,m_0}}\ , \tag{6.65}$$

$$\dot{\bar{\rho}}_{1/2} + 3\,\frac{\dot{a}_{1/2}}{a_{1/2}}\left(\bar{\rho}_{1/2}+\bar{p}_{1/2}\right) = 0\ . \tag{6.66}$$

With our ansatz for $\lambda(t)$, the warp factor on our brane is constant. So, all the Euler-Lagrange equations on our brane are the same at high and low energy (i.e., they remain exactly identical to the standard equations of physics in four dimensions)[20]. In order to close the differential system, we need an equation of evolution for $\bar{\rho}_0$. It is obtained from eqs. (6.45), (6.48), and (6.49)

$$\dot{\bar{\rho}}_0 = -3\frac{\dot{a}_{1/2}}{a_{1/2}}(\bar{\rho}_0+\bar{p}_0)\left(1-\frac{3(\bar{\rho}_0+\bar{p}_0)}{2(\frac{12m_0}{\kappa^2}\frac{\Omega_0^2}{1-\Omega_0^2}-\bar{\rho}_0)}\right)\left(1-\frac{3(\bar{\rho}_{1/2}+\bar{p}_{1/2})}{2(\frac{12m_0}{\kappa^2}\frac{\Omega_0^4}{1-\Omega_0^2}-\bar{\rho}_{1/2})}\right)^{-1}. \tag{6.67}$$

The differences between the evolution equations for $\bar{\rho}_0$ and $\bar{\rho}_{1/2}$ (i.e., the terms in the parentheses) show explicitly that, at high energy, $\bar{\rho}_0$ is not equivalent to dark matter in our brane.

Since it is assumed that $m_0 \simeq \kappa^{-2/3} \simeq M_{\rm P}$ and that $\Omega_0\,M_p \simeq$ TeV , the above equations can be cast in the more transparent form

$$\left(\frac{\dot{a}_{1/2}}{a_{1/2}}\right)^2 = \frac{\bar{\rho}_{1/2}}{3\,M_p^2}\left(1+\frac{\bar{\rho}_0}{\bar{\rho}_{1/2}}-\frac{\bar{\rho}_0}{10\,M_p^2\,{\rm Te}V^2}\right)\left(1-\frac{\bar{\rho}_0}{10\,M_p^2\,{\rm Te}V^2}\right)^{-1}\ , \tag{6.68}$$

$$\dot{\bar{\rho}}_0 = -3\frac{\dot{a}_{1/2}}{a_{1/2}}(\bar{\rho}_0+\bar{p}_0)\left(1-\frac{3(\bar{\rho}_0+\bar{p}_0)}{2(10M_p^2\,{\rm Te}V^2-\bar{\rho}_0)}\right)\left(1-\frac{3(\bar{\rho}_{1/2}+\bar{p}_{1/2})}{2(10\,{\rm Te}V^4-\bar{\rho}_{1/2})}\right)^{-1}.$$

[19]This remark should be important, for instance, when looking at cosmological perturbations in the early Universe. In this section, we derive only the evolution equations of the homogeneous background. When studying the perturbations, one should keep in mind that a full five-dimensional description is required at high energy.

[20]Since the freedom in choosing $\lambda(t)$ is equivalent to the freedom in fixing the time coordinate, it is obvious from general relativity principles that all physical results would not be affected by another choice of $\lambda(t)$, with the correct low-energy behavior $\lambda(t) \to \lambda_0/\dot{a}_{1/2}(t)$. It is meaningless to wonder which choice of $\lambda(t)$ makes sense physically at high energy, since contact with observations is only made at low energy. So, it is sufficient to give the set of equations that follows from the simplest choice for $\lambda(t)$.

We now discuss the implications of these equations for the cosmological evolution in the early Universe. First of all, it is worth noticing that the above equations (6.68) encounter a singularity when $\bar{\rho}_0 \simeq 10 M_{\rm P}^2\,{\rm Te}V^2$. It may be possible that the presence of such singularity puts a limit on the theory, at least as long as the dilaton is assumed to be stabilized. However, as we now show, phenomenological bounds from primordial nucleosynthesis indicate that this limit is hardly reached for $\bar{\rho}_{1/2} \leq {\rm Te}V^4$.

In the regime of validity of the low-energy effective theory, $\bar{\rho}_0$ behaves as ordinary dark matter in our brane. So, the constraints that we usually have for dark matter apply to it. Although in principle we cannot say much about the physics on the 0-brane (in particular "non–standard" equations of state may be expected), we assume for simplicity that $\bar{\rho}_0$ can be decomposed into a constant term $\bar{\rho}_0^\Lambda$ ($w_0 = -1$), plus matter $\bar{\rho}_0^m$ and radiation $\bar{\rho}_0^r$ components (with $w_0 = 0, 1/3$).

For what concerns the constant component, the sum of the cosmological terms $\bar{\rho}_0^\Lambda$ and $\bar{\rho}_{1/2}^\Lambda$ is bounded by the current value of the critical density, which is of order $10^{-123}\, M_{\rm P}^4$. So, the amount of fine-tuning required here is the same as in usual 4-dimensional theories:

$$\bar{\rho}_0^\Lambda + \bar{\rho}_{1/2}^\Lambda = \rho_0^\Lambda + \Omega_0^4\, \rho_{1/2}^\Lambda \leq 10^{-123}\, M_{\rm P}^4. \tag{6.69}$$

The matter and radiation components also have to be fine-tuned to small values. The best current constraint on the radiation density $\bar{\rho}_0^r$ comes from nucleosynthesis: since the observed abundances of light elements are only compatible with an effective number of neutrinos $N_{eff} = 3 \pm 1$, we see that $\bar{\rho}_0^r$ is bounded by the density of one family of relativistic neutrinos. The matter density $\bar{\rho}_0^m$ is obviously bounded by the value of the critical density today. So, in the five-dimensional theory, both ρ_0^r and ρ_0^m have to be fine-tuned to $\sim \Omega_0^4\; \rho_{1/2}^r$ and $\sim \Omega_0^4\; \rho_{1/2}^m$, while one may naively expect $\rho_0 \sim \rho_{1/2}$ in the early Universe.

Without the knowledge of the behavior of the RS model at high energy, one may have hoped that corrections to the standard Friedmann law could have solved this problem. For example, starting from $\rho_0 \sim \rho_{1/2}$ at high energy, the equations of motion of the system could have naturally lead to $\rho_0 \ll \rho_{1/2}$ at temperatures of the order of the one at which primordial nucleosynthesis occurred. Our analysis shows that this is not the case. Indeed, let us assume $\bar{\rho}_{1/2} \sim \bar{\rho}_0$ at the nucleosynthesis scale ($\bar{\rho}_i \sim {\rm Me}V^4$) and let us consider the behavior of the system when it was close to the natural cut–off $\bar{\rho}_{1/2} \sim {\rm Te}V^4$. Significant deviations from the standard evolution are expected if at that epoch the energy $\bar{\rho}_0$ was almost of order $M_p^2\,{\rm Te}V^2$ [see eq. (6.68)]. Going backwards in time, $\bar{\rho}_0$ can increase relatively to $\bar{\rho}_{1/2}$ if $w_0 > w_{1/2}$. However, assuming radiation domination on our brane above the nucleosynthesis scale, the above requirement can be met only for $w_0 \geq 2$, which does not seem to be a realistic possibility.

A possible solution of the problem of the fine–tuning of $\bar{\rho}_0$ may arise from the stabilization mechanism for the dilaton, especially if it occurs at (relatively) low energy. Other possibilities are briefly discussed in ref. [93].

6.3 An explicit stabilization mechanism

In this section we present a very simple mechanism for stabilizing the radion of the Randall-Sundrum model, due to Goldberger and Wise [96].

Let us add to the RS model [89] a scalar field Φ with the following bulk action

$$S_b = \frac{1}{2} \int d^5x \sqrt{G} \left(G^{A\,B}\, \partial_A \Phi\, \partial_B \Phi - m^2\, \Phi^2 \right) \tag{6.70}$$

and the following interaction terms

$$\begin{aligned} S_0 &= - \int d^4x\, \sqrt{-g_0}\, \lambda_0 \left(\Phi^2 - v_h^2 \right)^2 \,, \\ S_{1/2} &= - \int d^4x \sqrt{-g_{1/2}}\, \lambda_{1/2} \left(\Phi^2 - v_{1/2}^2 \right)^2 \,, \end{aligned} \tag{6.71}$$

on the two branes [21].

The underlying idea is very simple. The actions we have written give a solitonic configuration to the VEV of Φ along the fifth dimension. For sufficiently high couplings λ_i, we can force the boundary conditions

$$\Phi\,(0) \simeq v_0 \quad , \quad \Phi\,(1/2) \simeq v_{1/2} \,. \tag{6.72}$$

The first term in the action (6.70) gives the scalar a "kinetic energy" which increases as the variation of Φ becomes more pronounced, that as if the branes get closer. The mass term gives instead a potential energy which increases with increasing distance of the two branes. The competition between these two effects gives the radius of the extra dimensions a static value.

The solitonic configuration for the field Φ is [96]

$$\begin{aligned} \Phi\,(y) &= e^{2\, m_0\, b\, |y|} \left[A\, e^{\nu m_0\, b\, |y|} + B\, e^{-\,\nu m_0\, b\, |y|} \right] \,, \\ A &= v_0\, e^{-\,(2+\nu)\, m_0\, b/2} - v_{1/2}\, e^{-\,\nu\, m_0\, b} \,, \\ B &= v_{1/2} \left(1 + e^{\nu\, m_0\, b} \right) - v_0\, e^{-\,(2+\nu)\, m_0\, b/2} \,, \end{aligned} \tag{6.73}$$

where $\nu \equiv \sqrt{4 + m^2/m_0^2}$ and where subleading powers of $\mathrm{e}^{-\,m_0\, b\, |y|}$ have been neglected. We now suppose that $m/m_0 < 1$ so that $\nu = 2 + \epsilon$, with $\epsilon \simeq m^2/\left(4\, m_0^2\right)$ a small quantity.

[21] Notice that Φ and $v_{0\,,\,1/2}$ have mass dimension 3/2, while $\lambda_{0\,,\,1/2}$ have mass dimension -2.

Inserting the above solution for the field Φ into the original action and integrating over y yields an effective four dimensional potential for b which, in the large $m_0\, b$ limit, reads [96]

$$\begin{aligned} V_r\,(b) &= 4\,m_0 \mathrm{e}^{-2\,m_0\,b} \left(v_0 - v_{1/2}\mathrm{e}^{-\epsilon\,m_0\,b/2}\right)^2 \left(1+\frac{\epsilon}{4}\right) + \\ &- \epsilon\,m_0\,v_0 \mathrm{e}^{-(4+\epsilon)\,m_0\,b/2} \left(2v_0 - v_{1/2}\,e^{-\epsilon m_0\,b/2}\right)\,. \end{aligned} \tag{6.74}$$

In this expression, terms of order ϵ^2 are neglected. This potential has a minimum at finite value b_0. Ignoring terms proportional to ϵ, we find

$$m_0\,b_0 = \left(\frac{8\,m_0^2}{m^2}\right)\,\ln\left[\frac{v_{1/2}}{v_0}\right]\,. \tag{6.75}$$

With $\ln\left(v_0/v_{1/2}\right)$ of order unity, we only need m^2/m_0^2 of order $1/10$ to get the value $m_0\,b_0 \sim 70$ necessary to solve the hierarchy problem.

The physical mass of the radion is connected to the second derivative of V_r by the relation

$$m_r^2 = \frac{2}{3\,m_0^2}\frac{V_r''\,(b_0)}{M_{\mathrm{P}}^2\,\Omega_0^2}\,\left(1-\Omega_0^2\right)^2\,. \tag{6.76}$$

The factor multiplying V_r'' comes after the canonical normalization of b in the action (6.26). For V_r given by eq. (6.74) we have

$$V''\,(b_0) = 4\epsilon^{3/2}\,m_0^3\,v_{1/2}^2 \mathrm{e}^{-2\,m_0\,b_0} + \mathrm{O}\left(\epsilon^2\right)\,, \tag{6.77}$$

and so

$$m_r^2 = \frac{8}{3}\,\epsilon^{3/2}\,\frac{m_0\,v_{1/2}^2}{M_{\mathrm{P}}^2}\,\Omega^2\,. \tag{6.78}$$

Assuming $v_{1/2}^{2/3} \sim M_{\mathrm{P}}$ as all the other scales of the model, and taking $m^2/m_0^2 \sim 1/10$, we find that the mass of the radion is about one order of magnitude below the physical cut-off scale on our brane ($\sim$ TeV). It is not difficult to have m_r as the highest physical mass of the problem, just allowing $v_{1/2}$ to be slightly higher than the other original parameters. Although this is just a particular example, it justifies the analysis performed in refs. [93, 95] which assumes the radion to be stabilized up to energy densities on the two branes not too far from the physical cut-off of the system.

Acknowledgments

When I came to Trieste I wasn't quite sure on which topic I would have made my Ph.D. thesis. Among the possible, the one I did not have a clue about, was *astroparticles.* I'd heard about inflation because I like to read newspapers and mainly because things usually tend to become more expensive with time. I knew I was made of baryons, but I didn't care much where the antibaryons had gone. I thought that the Universe started somewhen (maybe also somewhere), and that it was very old, very big so that you don't have to worry to fall out, and very "long-living" so that there should be good chances even for my team (Verona) to win again the championship soon or later. When I came to Trieste I was also a bit concerned about the courses (which are quite high-level) and about the fact that maybe working as a Ph.D. student wouldn't have been as enjoying and pleasant as it was it when I was just a simple (i.e. undergraduate) student.

In this period I met several people (inside and outside SISSA, also inside and outside physics), who allowed me to learn something about the last $\sim$ 10 billion years, without regretting having spent the last 3 of them doing it. Actually, outside physics they are not so many, since most of my last holidays have been spent either with my Ph.D. "class-mates" (giving a perfect example of confinement) or at very nice (I add *also profitable* of course...) summer schools. They are not so many, but one of them is the first of the list, and she is Frida. I still don't understand how she can stand me since I keep complaining that "there are too many photons", or that "maybe only a fat brane can help" [22] or simply that "I should have go on making chips rather than gravitinos". I heard her once answering that her boyfriend makes *space particles* and it sounds a much better definition than astroparticles that I started saying it myself. Well, I'm grateful to her also for more serious reasons, but – as we usually say when we don't want to write them – they are certainly beyond the aim of the present discussion.

Ok, about physics now. I'd like to thank my supervisor, prof. Antonio Masiero. I think we are all grateful to him because while we work with him we really realize what it means doing with joy and enthusiasm what you are doing. I'm also grateful for the many things he taught me during these years of research, in particular the

[22] By the way, I couldn't force myself to discuss fat branes inside this thesis.

approach that he has with problems that I really wish I've been able to learn, at list in part. Finally, I thank him for the suggestions (also outside research) and the opportunities he gave me, both to travel and to know people with whom then I collaborated.

I'd like to thank these collaborators, who also gave (and are giving) me the opportunity to learn several things and also to produce some works. Even more important, I'm lucky since most of them are quite good friends, as well as collaborators. I thank Daniele Montanino, who supported me in the first months of research, Lorenzo Sorbo, with whom I had the luck to share almost everything in the last year (sorry for the time you were working while I was writing this thesis, now it should be over), Antonio Riotto, who keeps suggesting me new ideas, I only regret that I cannot catch up with all of them, Sergio Pastor and Julien Lesgourgues, who I used to call "my post-docs of reference", Hans Peter Nilles, who is the one who will have the problem to deal with me in the next two years, Francesca Rosati, Rula Tabbash, Gian Francesco Giudice, Igor Tkachev, Gennaro (Rino) Miele, Ofelia Pisanti, Salvatore Esposito, and last but surely not least (also because he's a very good cook) Gianpiero Mangano.

Let me conclude by thanking some other friends who shared with me these (or part of these) three years at SISSA. First the three "class-mates", i.e. Enrico (thanks for the several tennis lessons, I hope to find someone also in Bonn), Dario (who I also thank for the style and the LaTeX format of the thesis, unluckily all the rest I had to add it myself), and the already cited Lorenzo; it's mainly due to them if these years were enjoyable, especially the first one when we were really a class. On the same ground I'm happy to mention Stefano, Alessandro, Ugo, Cecilia, Annalisa, Ignazio, Mario, Marco, and Patrizia.

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