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# Some problems in the Calculus of Variations

Thesis submitted for the degree of
"Doctor Philosophiæ"

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SUPERVISOR
Prof. Arrigo Cellina

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Scuola Internazionale Superiore di Studi Avanzati
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## Part II. An existence result (vectorial case)

# Introduction

In this thesis we study the minimum problem for integral functionals of the form

$$\mathcal{I}(u) = \int_\Omega f(x, u(x), \nabla u(x))dx,$$

under given boundary conditions, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$ and $u$ is a function defined on $\Omega$ with values in $\mathbb{R}^m$ $(n, m \geq 1)$.

Actually our work concerns various properties of the solutions of the minimum problem: existence, uniqueness, continuity with respect to boundary data and regularity for certain classes of functionals, and a particular care is devoted to functionals with nonconvex integrands since in this case, up to now, a general existence theory does not exist.

The content of Part I regards the scalar case (i.e. the case in which $n = 1$ and $m \geq 1$ or $n \geq 1$ and $m = 1$). In Part II it is given an existence result in the vectorial case $(n, m > 1)$.

The standard approach to the problem of existence of minimizers (Direct Method) relies on the following argument:

**Theorem.** *Let $X$ be a topological space and let $\mathcal{I} : X \to \bar{\mathbb{R}}$. If $\mathcal{I}$ is coercive and lower semicontinuous on $X$, then $\mathcal{I}$ has at least a minimum point in $X$.*

The idea of the proof is the following: the coercivity of $\mathcal{I}$ means that for any real $M$ the set $\{x \in X : \mathcal{I}(x) \leq M\}$ is (sequentially) relatively compact in $X$, hence any minimizing sequence $x_n$ admits a subsequence $x'_n$ converging to $\bar{x}$. The lower semicontinuity implies that

$$\mathcal{I}(x) \leq \liminf \mathcal{I}(x'_n) = \inf \mathcal{I}. \qquad (**)$$

This result finds a wide application in the Calculus of Variations since coercivity and lower semicontinuity of integral functionals $\mathcal{I}$ can be characterized in terms of properties of the integrand $f$. Indeed, by imposing suitable growth conditions on $f$ with respect to the last variable, the functional turns out to be coercive in the Sobolev spaces $W^{1,p}$ $(1 \leq p \leq \infty)$ endowed with the weak (weak* if $p = \infty$) topology. Hence, in order to apply the direct method it is necessary to ensure the weak lower semicontinuity of $\mathcal{I}$ in $W^{1,p}$, and this is possible, in the scalar case, according with the following

**Theorem.** *A necessary and sufficient condition for the weak lower semicontinuity of the functional $\mathcal{I}$, under suitable regularity and growth conditions, is that the map $\xi \rightarrow f(x,u,\xi)$ is convex for every $x, u$.*

However the weak lower semicontinuity of the functional is far from being a necessary conditions for the existence of a minimizer, since condition (**) is needed only on minimizing sequences (actually it is sufficient on *one* minimizing sequence) and not on all converging sequences. Hence in the last years several efforts have been produced in order to obtain existence results in problems with nonconvex integrands.

An important device in such setting is the so called relaxed problem, consisting in the definition of a weak lower semicontinuous functional $\bar{\mathcal{I}}$, the largest lower semicontinuous functional pointwise less or equal than $\mathcal{I}$, whose minimum coincides with the infimum of $\mathcal{I}$. In the scalar case, under suitable growth conditions, the relaxed functional $\bar{\mathcal{I}}$ can be represented as the integral functional:

$$\bar{\mathcal{I}}(u) = \int_\Omega f^{**}(x, u(x), \nabla u(x))dx,$$

where $f^{**}$ is the convex envelope of $f$ with respect to the last variable. By the direct method, the relaxed problem admits always at least a solution and necessary and sufficient condition for $\tilde{u}$, minimizer of $\bar{\mathcal{I}}$, to be a minimizer for $\mathcal{I}$ is

$$f(x, \tilde{u}(x), \nabla\tilde{u}(x)) = f^{**}(x, \tilde{u}(x), \nabla\tilde{u}(x))$$

almost everywhere. For this reason, in the literature on nonconvex integral functionals, two approaches are essentially used: to impose conditions such that any solution of the relaxed problem is also a solutions of the original problem, or to show that there exists at least one solution of the relaxed problem such that the above equality holds on $\Omega$. These methods require special assumptions on $f$ and many results in these directions have been given. A good guide in the literature on the subject is [Ma2], we refer to it for a detailed discussion of various results on nonconvex variational problems.

We stress that even when the direct method is not applied, and coercivity is not directly considered, one is still forced to impose growth condition on $f$.

As far as it concerns the scalar case $n = 1, m \geq 1$, among the various results mentioned above, we quote in particular Cellina Colombo's theorem ([CC]) which improves the first results on nonconvex variational problems ([Ce2], [O3])

**Theorem.** *Let $g, h : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be such that $f(\cdot, x, x') = g(\cdot, x) + h(\cdot, x')$ satisfies growth conditions. Assume further that $x \to g(\cdot, x)$ is concave a.e. Then problem*

$$\text{Minimize} \quad \int_0^T f(t, x(t), x'(t))dt, \quad x \in W^{1,p}(0,T,\mathbb{R}^n), \quad x(0) = a, x(T) = b$$

*admits at least a solution.*

The proof is based on Liapunov theorem on the range of vector measures and the technique used requires the notion of primitive, hence it cannot be transposed to cases in which $n$ is larger than one.

For the other scalar case we quote the results contained in [C1], [C2] which provide necessary and sufficient conditions for the existence and for the uniqueness of solutions of the problem

$$\text{Minimize} \quad \int_\Omega g(\nabla u(x))dx, \quad u \in u_0 + W_0^{1,1}(\Omega, \mathbb{R}^n)$$

where $u_0$ is an affine boudary datum. The proof consists essentially in the explicit construction of a scalar function whose gradient takes value in the set in which $g$ coincides with $g^{**}$.

In the vectorial case $(n, m > 1)$ the application of the direct method finds a new difficulty due to the fact that the weak lower semicontinuity cannot be characterized in a simple way as in the scalar case. Indeed even though convexity of the integrand is still a sufficient condition for the weak lower semicontinuity of the functional, it is not necessary, and the analogous of the above theorem is the following:

**Theorem.** *A necessary and sufficient condition for the weak lower semicontinuity of the functional $\mathcal{I}$, under suitable regularity and growth conditions, is that the map $\xi \to f(x, u, \xi)$ is quasiconvex for every $x, u$.*

A function $f$, defined on the spaces of matrices, is said to be quasiconvex if, for any matrix $A$,

$$f(A) \leq \int_D f(A + \nabla u(x))dx$$

for every bounded domain $D$ contained in $\mathbb{R}^n$ and for every $u \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$.

Actually convexity implies quasiconvexity but in the vectorial case the converse is false, and, in general, it is difficult to ascertain if a function is quasiconvex or not. Moreover the relaxed functional is expressed in terms of the quasiconvex envelope of $f$ (i.e. the largest quasiconvex function pointwise less than $f$) and a simple expression for the infimum of the functional is not available. For these reasons, new convexity-type concepts, and relative envelopes, have been introduced (policonvexity, rank-one convexity) in order to investigate nonconvex problems through relaxation (a good exposition of these topics can be found in [D1]). Actually, in some cases, one of which is considered in the present thesis (chapter 4), relaxation is obtained by convexification: by this way, once given a sufficiently simple expression for the infimum of the functional, one can try to prove existence of a minimizer.

## Presentation of our results

As we have remarked above, the direct method in the calculus of variations requires essentially two kinds of independent assumptions: growth conditions (coercivity) and convexity (weak lower semicontinuity).

Even dealing with nonconvex problems, i.e. when the direct method cannot be applied, growth conditions cannot in general be dropped. Since growth conditions impy relative weak compactness in Sobolev spaces, in chapter 1 we study relatively weakly compact subsets of $L^1$, i.e. sets of the form

$$\left\{ u \in L^1 : \int \phi(|u|) \leq M \right\}$$

where $M$ is a positive constant and $\phi$ is a real valued function satisfying the growth condition usually met when dealing with variational problems in $W^{1,1}$:

$$\lim_{t \to \infty} \frac{\phi(t)}{t} = +\infty.$$

We study such sets from a metric and topological point of view, extending results previously obtained in the context of multifunctions and differential inclusions.

In chapter 2 we consider a problem strictly connected to the problem of regularity of minimizers. Actually the regularity of minimum points is usually exploited by studying the regularity of the solutions of Euler-Lagrange equations which are well known necessary condition for a function to be a minimizer.

The derivation of Euler-Lagrange equations, which is a classical argument, requires regularity and some upper bounds on the growth of the integrand and of its derivatives. These bounds are not only unnecessary for the existence of the minimum point, but are in some sense in contrast with the standard growth assumptions used in existence theorems. The content of chapter 2 constitutes a contribution in the direction of weakening these conditions for functionals of the scalar case $n = 1$, $m \geq 1$, and, incidentally, it provides a class of functionals for which Lavrentiev Phenomenon does not occur.

In chapter 3 we consider the solutions of the problem

$$\text{Minimize} \quad \int_\Omega g(\nabla u(x))dx, \quad u \in u_0 + W_0^{1,1}(\Omega, \mathbb{R})$$

whose existence and uniqueness are characterized in the papers [C1], [C2] quoted above, and study their continuity with respect to the (affine) boundary data. We give a necessary and sufficient condition, which recalls Olech's lemma on the strong convergence of selections of a multifunction to extreme points, for such continuity. By this way Cellina's theorems and the present result provide a complete well posedness theory for this problem (relatively to affine data).

The second part (chapter 4) contains an existence result in a vectorial case in which the relaxed functional is expressed in terms of a convex envelope. More precisely we consider the functional

$$\mathcal{I}(T) = \int_\Omega g(\Phi(\nabla T(x)))dx$$

where $\Phi$ is a generic real valued quasiaffine function (in particular one can take as $\Phi$ the determinant), $g$ is lower semicontinuous and satisfies growth conditions and $T$ is a transformation from $\Omega$ to $\mathbb{R}^n$.

We prove that the functional under consideration admits at least a minimizer in the class $T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ where $T_B$ is an affine boundary datum.

The contents of the various chapters are almost independent between them, hence each chapter contains a short introduction explaining in a more precise way the aim of the work and the results, and a section of preliminaries and notations.

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# Part I

## Chapter 1.

## Weakly precompact subsets of $L^1$

## 1.1. INTRODUCTION

Convexity is a fundamental concept in the study of Differential Inclusions and of Calculus of Variations and in the last several years, in order to avoid convexity assumptions, an effort has been produced to find classes of sets that inherit some of the properties of convex sets from less geometric and more analytic considerations. The theory of decomposable sets is the product of such an effort. A decomposable set $D$ is a subset of $L^1 = L^1(T, E)$, where $T$ is a measure space and $E$ a Banach space, having the property that $u\chi_A + v\chi_{T-A} \in D$ for every $u, v \in D$ and for every measurable subset $A$ of $T$. One can find in [O2], [BC], [CCF] and in [F1] results in which decomposability is in some sense a substitute of convexity. In particular the analysis of selection problems, which is fundamental in the theory of differential inclusions, makes a large use of the concept of decomposability since, as it turned out, a decomposable, closed and bounded set is the set of mesurable selections of an integrably bounded multifunction (that could describe the constraint of the problem). This means that dealing with a bounded decomposable set, actually we deal with a set of functions pointwise a.e. bounded by some function in $L^1$. On the contrary in variational problems this limitation is not met in general, and the techniques we have mentioned above lead to consider bounds coming from growth conditions or coercivity conditions.

In variational problems one meets so called sublevel sets which in general can be defined as sets of functions $u \in L^1$ such that $\int \Phi \circ \|u\|$ is bounded by a constant $M$, possibly infinite, where the function $\Phi : [0, \infty[ \to [0, \infty[$ has the only property that $\Phi \circ \|u\|$ is measurable for any $u \in L^1$. Indeed when $\Phi$ is assumed to satisfy some growth condition such as

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = +\infty$$

the corresponding sublevels are weakly relatively compact subsets of $L^1$, i.e. the sets containing the sequences of the derivatives of minimizing sequences for integral functionals.

Since the assumptions on $\Phi$ are very weak, the class of sublevels contains a certain amount of subsets of $L^1$, such as Orlicz classes, $L^p$ spaces $(0 < p < \infty)$, balls in such spaces and so on. However these sets, denoted by $\Phi_M$, are *not decomposable*. Purpose of the work reported in this section is to extend the theory developed for decomposable sets to sets of the type $\Phi_M \cap D$ i. e. to sets which are the intersection of a sublevel and of a set of selections of a measurable multifunction (but not necessarily bounded). By way we cover the case of a weakly relatively compact family of functions in $L^1$, taking values in a closed subset $M$ of $E$, assuming neither the boundedness of $M$ nor the existence of an integral bound for the family itself, as it would be required within the framework of decomposable sets. It should be remarked that $\Phi_M$ or $D$ can be chosen to be the whole space, including, as a special case, the theory for decomposable sets and for sublevels.

We present a metric and a topological result: first we prove that Kuratowski index $\alpha$ of a set of the type specified above coincides with its diameter, then we give a version of Dugundji extension theorem. In [BC] the authors give an extension theorem in which decomposability in $L^1$ is used instead of convexity in a generic Banach space, obtaining, as consequences, that a closed decomposable set in a separable space $L^1$ is a retract, and that, in general, it has the compact fixed point property. We prove analogous statement for our case.

## 1.2. NOTATIONS AND PRELIMINARY RESULTS

We consider a measure space $(T, \mathcal{F}, \mu)$ where $\mathcal{F}$ is a $\sigma$-algebra of subsets of $T$ and $\mu$ is a bounded positive nonatomic measure on $T$. If $f : T \to \mathbb{R}$ is a $\mu$-measurable function we denote by $f \cdot \mu$ the measure having density $f$ with respect to $\mu$. E is a Banach space with norm $\|\cdot\|$ and $p \geq 1$ $L^p(T, E)$ is the Banach space of Bochner $\mu$-integrable E-valued functions defined on T endowed with the norm $\|f\|_p = (\int_T \|f\|^p d\mu)^{1/p}$. If A and B are two sets, $A - B$ is their difference and $A \triangle B$ is their symmetric difference $(A-B) \cup (B-A)$. By $\chi_A$ we denote the characteristic function of the set A and by $S^n$ the set $\{x = (x_0, ..., x_n) \in \mathbb{R}^{n+1}$ s. t. $\sum_{i=0}^{n} |x_i| = 1\}$. For a metric space X with distance $d$ and for $A \subset X$, we set $d(x, A) \equiv \inf\{d(x,y),\ y \in A\}$, and $\operatorname{diam}(A) \equiv \sup\{d(x,y),\ x, y \in A\}$.

**Definition 1.2.1.** A set $D \subseteq L^1(T, E)$ is called *decomposable* if for every $A \in \mathcal{F}$ and for every $u,\ v \in D$ it is

$$u\chi_A + v\chi_{T-A} \in D.$$

The decomposable hull of a subset $S$ of $L^1(T, E)$ is defined as the smallest decomposable set containing $S$ and is denoted by $\operatorname{dec}(S)$.

**Definition 1.2.2.** Let $F : T \to 2^E$ be a multifunction. $F$ is said measurable if for every closed $C \subseteq E$ it is $F^{-1}(C) \in \mathcal{F}$. The set of measurable selections of $F$ is denoted by $S_F$.

The following statement is a characterization of decomposable sets (see [HU]).

**Theorem 1.2.1.** *Let $S \subset L^1(T, E)$ be nonempty and closed. Then $S$ is decomposable if and only if there exists a measurable multifunction $F$ such that $S = S_F$. Moreover, if $S$ is bounded, $F$ is integrably bounded i.e. there exists $h \in L^1(T, \mathbb{R})$ such that $\|u(t)\| \leq h(t)$ $\mu - a.\ e.$ for all $u \in S_F$.*

We now list some well known results which in some sense justify our work.

**Definition 1.2.3.** A family $S$ of $L^1(T,E)$ is called absolutely equiintegrable if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta \in \mathcal{F}$ with $\mu(\Delta) < \delta$ it is $\int_\Delta \|u\| d\mu < \epsilon$ for all $u \in S$.

Notice that an absolutely equiintegrable family of $L^1(T,E)$ is bounded. A bounded decomposable subset of $L^1$ is equiintegrable.

The following theorem is well known (see for example [Cel]).

**Theorem 1.2.2.** *Let $S \subset L^1(T,E)$. The following statements are equivalent:*
*i) $S$ is sequentially weakly relatively compact in $L^1(T,E)$;*
*ii) $S$ is absolutely equiintegrable;*
*iii) there exists $\Phi : [0,+\infty[ \to \mathbb{R}$ nonnegative, increasing, satisfying $\Phi(t)/t \to +\infty$ as $t \to +\infty$, and a constant $L > 0$, such that $\int_T \Phi(\|u\|)d\mu \leq L$ for all $u \in S$.*

Since in the literature it is not easy to find the proof of the equivalence of *ii)* and of *iii)* we remind it.

*Proof.*

For $u \in S$ set

$$T(u,\alpha,\beta) = \{t \in T : \alpha \leq \|u(t)\| < \beta\}$$

$$T(u,\alpha) = \{t \in T : \|u(t)\| \geq \alpha\}.$$

*ii)⇒ iii).* For $y \geq 0$ define

$$\gamma(y) = \sup_{u \in M} \int_{T(u,y)} \|u\| d\mu,$$

$\gamma$ is a nonnegative, nonincreasing function and $\lim_{y \to +\infty} \gamma(y) = +\infty$. To prove the last assertion suppose that there exist $\epsilon > 0$, and $\{y_n\}_{n \in \mathbb{N}}$ increasing sequence in $\mathbb{R}$ with

$$\lim_{n \to \infty} y_n = +\infty$$

and $\{u_n\}_{n \in \mathbb{N}}$ in $S$ such that $\int_{T(u_n,y_n)} \|u_n\| d\mu \geq \epsilon$; since $\lim_{n \to \infty} \mu(T(u_n,y_n)) = 0$ (otherwise it would be $sup_{u \in S} \int_T \|u\| d\mu = +\infty$) we get a contradiction.

Choose now two real increasing sequences $\{a_n\}_{n \geq 0}$, $\{y_n\}_{n \geq 0}$ satisfying:

$$y_0 = 0, \qquad \lim_{n \to \infty} y_n = +\infty,$$

$$a_0 \geq 0, \qquad \lim_{n\to\infty} a_n = +\infty, \qquad \sum_{n=0}^{\infty} a_n \gamma(y_n) = L < \infty$$

and define

$$\phi(y) = \sum_{n=0}^{\infty} a_n y \chi_{[y_n, y_{n+1}[}.$$

$\phi$ has the properties stated above and

$$\int_T \phi(\|u\|) d\mu = \sum_{n=0}^{\infty} \int_{T(u,y_n,y_{n+1})} \phi(\|u\|) d\mu = \sum_{n=0}^{\infty} a_n \int_{T(u,y_n,y_{n+1})} \|u\| d\mu \leq$$

$$\leq \sum_{n=0}^{\infty} a_n \int_{T(u,y_n)} \|u\| d\mu \leq \sum_{n=0}^{\infty} a_n \gamma(y_n) = L.$$

*iii)*$\Rightarrow$ *ii)* (De La Valleé Poussin).

Set $\psi(y) = \phi(y)/y$; fix $\epsilon > 0$, choose $A > 0$ such that $\psi(A) > 2L/\epsilon$ and $\delta$ such that $0 < \delta < \epsilon/2A$. For $\Delta \in \mathcal{F}$ with $\mu(\Delta) < \delta$ and $u \in S$, define

$$\Delta_1 = \Delta \cap T(u, A), \qquad \Delta_2 = \Delta \cap T(u, 0, A),$$

it is

$$\int_\Delta \|u\| d\mu = \int_{\Delta_1} \|u\| d\mu + \int_{\Delta_2} \|u\| d\mu \leq$$

$$\leq \frac{1}{\psi(A)} \int_{\Delta_1} \phi(\|u\|) d\mu + \int_{\Delta_2} A d\mu \leq \frac{L}{\psi(A)} + A\delta < \epsilon.$$

This ends the proof. □

Functions satisfying the growth conditions specified in previous theorem play an important role in functional analysis, indeed the theory of Orlicz classes is based on such functions, the reader can found a wide exposition in [KJF].

These considerations justify the following

**Definition 1.2.4.** Let $\Phi : E \to \mathbb{R}^+ \cup \{0\}$ having the property that $\Phi \circ u$ is measurable for every $u \in L^1(T, E)$. Let $M \in \mathbb{R}^+ \cup \{+\infty\}$; we set

$$\Phi_M \equiv \Phi_M(T, E) \equiv \{u \in L^1(T, E) \text{ s.t. } \int_T \Phi \circ u d\mu < M\}$$

We call this set $\Phi$-sublevel. If $M < \infty$ the strict inequality in the definition of $\Phi_M$ can be replaced by the weak one without affecting our results.

The most interesting case is that one of a function $\Phi(u) = \Psi(\|u\|)$ where $\Psi$ satisfies some superlinear growth condition.

Among sets which can be described as sublevels in the sense specified above we mention, for example, balls in $L^p$-spaces and in Orlicz spaces (with radius possibly infinite).

We end this section reminding the main tool used in establishing our results, that is to say the following version of Liapunov theorem about vector measure.

**Theorem 1.2.3.** *Let $\{g_1, ..., g_N\}$ be a finite family of nonnegative functions in $L^1(T, \mathbb{R})$. Then there exists a family $\{A_\alpha,\ \alpha \in [0,1]\}$ in $\mathcal{F}$ with the properties:*

*a)* $A(0) = \emptyset,\ \ A(1) = T,\ \ A(\alpha) \subseteq A(\beta)\ \forall \alpha, \beta \in [0,1]\ \alpha \leq \beta$

*b)* $\int_{A(\alpha)} g_n d\mu = \alpha \int_T g_n d\mu\ \forall \alpha \in [0,1], \forall n = 1, ..., N.$

*c)* $\mu(A_\alpha \triangle A_\beta) \leq |\alpha - \beta|.$

Such family is called a refinement of $T$ with respect to the measures $g_n \cdot \mu$, see [F1].

In what follows we shall consider sets that are intersections of decomposable sets and of sublevel sets. A typical application is to families of functions taking values in a closed subset $M$ of $E$. To deal with such sets within the framework of bounded decomposable sets one had to impose that either $M$ itself is bounded or that there exists an integral bound. In this paper we extend the theory to cover the case of a generic weakly relatively compact set of $L^1$ of maps taking values in $M$.

## 1.3. KURATOWSKI INDEX

We now recall the definition of Kuratowski's index $\alpha$ and a theorem which will be used later.

**Definition 1.3.1.** Let X be a metric space, $A \subset X$ bounded.

$$\alpha(A) = \inf\{\epsilon > 0 :\ A = \bigcup_{i \in I} A_i \quad \text{where I is a finite set and } \operatorname{diam}(A_i) \leq \epsilon\}.$$

**Theorem 1.3.1.** (Lusternik, Schnirelman Borsuk) *Let $\{S_0, ..., S_n\}$ be a covering of closed sets of $S^n$. Then there is at least one set $S_i$ that contains a pair of antipodal points.*

For the proof see, for example, [DG].

In [CM] the following proposition is established.

**Theorem 1.3.2.** *Let $D \subset L^1(T,E)$ be decomposable and bounded. Then* $\alpha(D) = \text{diam}(D)$.

**Corollary 1.3.1.** *Let $S \subset L^1(T,E)$ be bounded. Consider two sequences $\{u_n\}_{n\in\mathrm{N}}$ and $\{v_n\}_{n\in\mathrm{N}}$ such that* $\lim_{n\to\infty} \|u_n - v_n\|_1 = \text{diam}(S)$, *and suppose* $\text{dec}(\{u_n, v_n\}) \subseteq S$ *for every $n \in \mathbf{N}$. Then* $\alpha(S) = \text{diam}(S)$.

*Proof.*
Take $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|u_n - v_n\| \geq \text{diam}(S) - \epsilon$; it follows:

$$\text{diam}(S) \geq \alpha(S) \geq \alpha(\text{dec}(\{u_n, v_n\}) = \text{diam}(\text{dec}\{u_n, v_n\}) =$$

$$= \|u_n - v_n\|_1 \geq \text{diam}(S) - \epsilon,$$

Since $\epsilon$ is arbitrary we obtain the result. □

The following is the first result of this chapter.

**Theorem 1.3.3.** *Let $D \subseteq L^1(T,E)$ be decomposable and $\Phi_M \subseteq L^1(T,E)$ be a $\Phi$-sublevel, suppose $D \cap \Phi_M$ nonempty and bounded. Then* $\alpha(D \cap \Phi_M) = \text{diam}(D \cap \Phi_M)$.

*Proof.* It is sufficient to prove that $\alpha(D \cap \Phi_M) \geq \text{diam}(D \cap \Phi_M)$ (the converse inequality is obvious); for this purpose take $u,\ v \in D \cap \Phi_M$ and, using Liapunov theorem, consider a family $\{A(\alpha),\ \alpha \in [0,1]\}$, a refinement of $T$ with respect to the measures $\|u\| \cdot \mu$, $\|v\| \cdot \mu$, $\Phi(u) \cdot \mu$, $\Phi(v) \cdot \mu$. Then, for $j \in \{0, 1, ..., n\}$ and $x \in S^n$, set

$$p_j(x) = \sum_{i=0}^{i=j} |x_i|,$$

$$N_0(x) = A(p_0(x)),$$

$$N_i(x) = A(p_i(x)) - A(p_{i-1}(x)), \quad i \geq 1,$$

$$I_x{}^+ = \{i : \ x_i > 0\}, \qquad I_x{}^- = \{i : \ x_i < 0\},$$

and define $\omega_n : S^n \longrightarrow L^1(T,E)$ by the formula:

$$\omega_n(x) = \sum_{i\in I_x+} u\chi_{N_i(x)} + \sum_{i\in I_x-} v\chi_{N_i(x)}.$$

Obviously $\omega(S^n) \subset D$ and

$$\int_T \Phi(\omega_n(x))d\mu = \sum_{i\in I_x+} \int_{N_i(x)} \Phi(u)d\mu + \sum_{i\in I_x-} \int_{N_i(x)} \Phi(v)d\mu < \sum_{i=0}^{n} p_i(x)\cdot M \ = \ M.$$

hence $\omega(S^n) \subset D \cap \Phi_M$. The continuity of $\omega_n$ follows easily from property *c)* in Theorem 2.3.

Now suppose that there exists a finite covering $\{K_0, ..., K_n\}$ of $D \cap \Phi_M$ where each $K_i$ is closed, it is

$$S^n = {\omega_n}^{-1}(K_0) \cup ... \cup {\omega_n}^{-1}(K_n),$$

each ${\omega_n}^{-1}(K_i)$ is closed and by Theorem 3 there exist $x$ and $i$ such that $\omega_n(x) \in K_i$ and $\omega_n(-x) \in K_i$, since $\|\omega_n(x) - \omega_n(-x)\|_1 = \|u - v\|_1$, $\mathrm{diam}(K_i) \geq \|u - v\|_1$ and this implies $\alpha(D \cap \Phi_M) \geq \|u - v\|_1$. Since $u$ and $v$ are arbitrary we obtain the result. □

## 1.4. AN EXTENSION THEOREM AND A CLASS OF ABSOLUTE RETRACTS

We recall Dugundji's extension theorem ([Du], p. 188).

**Theorem 1.4.1.** *Let $A$ be a closed subset of a metric space $X$ and $K$ be a convex subset of a Banach space $Z$. Then every continuous map $f : A \to K$ has a continuous extension $\tilde{f} : X \to K$.*

In [BC] the authors prove an analogous result for $Z = L^1(T,E)$ assuming $K$ decomposable instead of convex. Following their argument we state an extension theorem requiring neither convexity nor decomposability. To do this we need an extended version of Liapunov theorem due to Bressan and Colombo (see [BC]).

**Lemma 1.4.1.** *Let $\{g_k,\ k \geq 0\} \subset L^1(T,\mathbb{R})$ be a sequence of nonnegative functions with $g_0 \equiv 1$. Then there exists a map $A : \mathbb{R}^+ \times [0,1] \to \mathcal{F}$ with the properties:*

*i)* $A(\tau,\alpha_1) \subseteq A(\tau,\alpha_2)$, if $\alpha_1 \leq \alpha_2$,

*ii)* $\mu(A(\tau_1,\alpha_1) \triangle A(\tau_2,\alpha_2)) \leq |\alpha_1 - \alpha_2| + 2|\tau_1 - \tau_2|$,

*iii)* $\int_{A(\tau,\alpha)} g_k d\mu = \alpha \int_T g_k d\mu \quad \forall k \leq \tau$, *for all* $\alpha,\ \alpha_1,\ \alpha_2 \in [0,1],\ \tau,\ \tau_1,\ \tau_2 \geq 0$.

We are now ready to state our extension theorem whose proof is a slight modification of that one of theorem 1 in [BC].

**Theorem 1.4.1.** *Let $X$ be a metric space with distance $d$ and $A \subseteq X$ be closed. Consider $D \subseteq L^1(T,E)$ decomposable, $\Phi_M \subseteq L^1(T,E)$ a $\Phi$-sublevel and $D \cap \Phi_M \neq \emptyset$. If either $X$ or $L^1(T,E)$ is separable and $f$ is a continuous function from $A$ to $L^1(T,E)$ such that $f(A) \subseteq D \cap \Phi_M$, then $f$ has a continuos extension $\tilde{f} : X \longrightarrow L^1(T,E)$ with $\tilde{f}(X) \subseteq D \cap \Phi_M$.*

*Proof.* Assume $L^1(T,E)$ separable. For each $x \in X - A$ take an open ball $B(x,r_x)$ with radius $r_x < \frac{1}{2}d(x,A)$. The family $\{B(x,r_x),\ x \in X - A\}$ is an open covering of the paracompact space $X - A$ hence it admits an open neighbourhood-finite refinement $\{V_i,\ i \in I\}$ where I is possible uncountable index set. For any $i \in I$ select two points $x_i \in V_i$ and $y_i \in A$ such that $d(x_i,y_i) < 2d(x_i,A)$. Since $L^1(T,E)$ is separable it is possible to define a countable set $K = \{u_n,\ n \geq 1\} \subset f(A)$ dense in $f(A)$. For every $i \in I$ select $u_{\nu(i)} \in K$ such that $\|u_{\nu(i)} - f(y_i)\|_1 \leq d(x_i,y_i)$. Consider a continuous partition of unity $\{p_i, i \in I\}$ subordinate to $\{V_i\}$ and define, for every $n \geq 1$ the open set $W_n = \cup\{V_i,\ \nu(i) = n\}$ and the continuous partition of unity subordinate to the covering $\{W_n\}$ given by the formula $q_n(x) = \sum_{\nu(i)=n} p_i(x)$. Construct a sequence of continuous functions $\{h_n,\ n \geq 1\}$ such that $h_n \equiv 1$ on $supp(q_n)$ and $supp(h_n) \subseteq W_n$ and define on $X - A$ the continuous function

$$\lambda_n(x) = \sum_{m \leq n} q_m(x),\ n \geq 0$$

and

$$\tau(x) = \sum_{m,n \geq 1} h_m(x) h_n(x) 2^m 3^n.$$

Consider now the sequence $\{g_k,\ k \geq 0\}$ in $L^1(T,\mathbb{R})$ given by:

$$g_k(t) = \begin{array}{l} \{\|u_n(t) - u_m(t)\| \quad \text{if } k = 2^m 3^n, \\ \Phi(u_m(t)) \quad \text{if } k = 2^m 3^n - 1, \\ \Phi(u_n(t)) \quad \text{if } k = 2^m 3^n - 2, \\ 1 \quad \text{otherwise.}\} \end{array}$$

for $m, n \geq 1$.

Applying Lemma 4.1 to such sequence we obtain a family $A(\tau, \alpha)$ and define, for any $x \in X - A$

$$\chi_n(x) = \chi_{A(\tau(x),\lambda_n(x)) - A(\tau(x),\lambda_{n-1}(x))}.$$

Then we extend $f$ to the whole space $X$ by the formula:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{n\geq 0} u_n \cdot \chi_n(x) & \text{if } x \in X - A, \end{cases}$$

We remark that

$$\tau(x) \geq 2^m 3^n \quad \forall x \in supp(q_m) \cap supp(q_n) \tag{4.1}$$

and then it is easy to show that $\tilde{f}$ takes values in $\Phi_M$, indeed, for $x \in X - A$, it is

$$\int_T \Phi(\tilde{f}(x))d\mu \;=\; \sum_{n\geq 1} q_n(x) \int_T \Phi(u_n)d\mu \;<\; M$$

since the sum is taken over the finite set of indices $n$ such that $p_n(x) \neq 0$, and for such indices,

$$\int_{A(\tau(x),\alpha)} \Phi(u_n)d\mu \;=\; \alpha \int_T \Phi(u_n)d\mu \quad \forall \alpha \in [0,1],$$

by property *iii)* of Lemma 4.1. Hence, obviously, $\tilde{f}(X) \subseteq \Phi_M \cap D$. Continuity of $\tilde{f}$ in $X - A$ is a consequence of contunuity of $\lambda_n(\cdot)$ of $\tau(\cdot)$, of property *ii)* in Lemma 4.1 and of the fact that the summation defining $\tilde{f}$ is locally finite.

To prove continuity on A take $a \in A$ and $\epsilon > 0$. There exists $\delta > 0$ such that $\delta < \epsilon/12$ and $\|f(y) - f(a)\|_1 < \epsilon/2$ whenever $y \in A$ and $d(y, a) < 12\delta$. If $d(x, a) < \delta$ and $x \in V_i$ for some $i \in I$, then $\text{diam}(V_i) < 2\delta$, $d(x_i, A) < 3\delta$ and $d(x_i, y_i) < 6\delta$. Therefore $p_i(x) \neq 0$ implies $d(y_i, a) < 9\delta$, $\|f(y_i) - f(a)\|_1 < \epsilon/2$ and $\|u_{\nu(i)} - f(a)\|_1 < \epsilon$. Then

$$\|u_n - f(a)\|_1 < \epsilon \;\; \forall n \text{ such that } q_n(x) \neq 0. \tag{4.2}$$

For any $x \in X - A$ with $d(x, a) < \delta$, fix $j$ such that $q_j(x) \neq 0$. Using (4.1), (4.2) and property *iii)* of Lemma 4.1 we have:

$$\begin{aligned}\|f(a)-\tilde{f}(x)\|_1 &\leq \|f(a)-u_j\|_1+\|u_j-\tilde{f}(x)\|_1 \leq \\ &\leq \epsilon+\sum_{n\geq 1}\int_T \|u_j-u_n\|\chi_n(x)d\mu \\ &= \epsilon+\sum_{n\geq 1}\int_T g_{2^j3^n}\chi_n(x)d\mu \\ &= \epsilon+\sum_{n\geq 1} q_n(x)\int_T g_{2^j3^n}d\mu \\ &= \epsilon+\sum_{n\geq 1} q_n(x)\|u_n-u_j\|_1 \leq 3\epsilon.\end{aligned}$$

Since $\epsilon$ is arbitrary we obtain the result.

It is left to consider the case in which $L^1(T,E)$ is not separable. Since $X-A$ is separable the covering $\{V_i, i\in I\}$ defined as in previous case, is countable. For each $i\in I$ choose $x_i\in V_i$ and $y_i\in A$ such that $d(x_i,y_i)<2d(x_i,A)$. Then the analog of the set K of the previous case can be chosen to be $\{f(y_i),\ i\in I\}$ arranged in a sequence $\{u_n,\ n\geq 1\}$; then the proof proceeds in the same way. □

Then, as in [BC], we can state the following consequences.

**Corollary 1.4.1.** *Suppose that $L^1(T,E)$ is separable. Let $D\subseteq L^1(T,E)$ be decomposable and $\Phi_M\subseteq L^1(T,E)$ be a $\Phi$-sublevel, then if $D\cap\Phi_M$ is nonempty and closed, it is a retract of the whole space.*

The following is a generalization of results in [C3] and in [F2].

**Corollary 1.4.2.** *Let $D\subseteq L^1(T,E)$ be decomposable and $\Phi_M\subseteq L^1(T,E)$ be a $\Phi$-sublevel. If $K\equiv D\cap\Phi_M$ is nonempty and closed it has the compact fixed point property i. e. every continuous function $f:K\longrightarrow K$ with relatively compact image has a fixed point.*

*Proof.* Consider $f:K\to K$ with relatively compact image and continuous and call $X$ the closure of the convex hull of $f(K)$; $X$ is compact, hence it is separable. Using Theorem 4.2 extend the identity map $i:X\cap K\to X\cap K$ to a continuous function $\tilde{i}:X\to K$. The composed function $f\cdot\tilde{i}$ maps $X$ into $X\cap K$. By Schauder Theorem it has a fixed point $x_0$ which is also a fixed point of $f$. □

# Chapter 2.

## On Lavrentiev Phenomenon and Euler-Lagrange equations

## 2.1. INTRODUCTION

Euler-Lagrange Equations $(EL)$ are well known necessary conditions for a function $x$ to be a minimizer of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t))dt$$

under given boundary conditions.

In order to prove the validity of such Equations one imposes that the Gateaux derivative in $x$ of the functional along a certain class of directions is zero. More precisely, one imposes:

$$\frac{d}{d\theta}\mathcal{I}(x + \theta\xi)\Big|_{\theta=0} = 0,$$

for any $\xi$ with essentially bounded derivative and zero boundary conditions. This procedure requires the differentiability of $f$ with respect to the second and the third variable and the integrability of $f$ and of its derivatives along trajectories close to the minimizer $x$. This last requirements can be satisfied by imposing some integrable bound on the growth of $f$, $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of $x$. Actually, such assumptions are strong enough to ensure the continuity along a wider class of variations including, in particular, lipschitzian approximations of $x$. In other words, the hypotheses under which Euler-Lagrange Equations are usually derived, exclude Lavrentiev Phenomenon, which consists in the relevant fact that the infimum of $\mathcal{I}$ on the class of admissible trajectories with essentially bounded derivative can be strictly larger than the minimum on the class of all admissible trajectories.

If we consider Manià's functional, which constitutes a widely studied example in the framework of Lavrentiev Phenomenon, we see that even though the minimizer satisfies formally equations $(EL)$, the standard assumptions are not fulfilled. Our work is devoted to the study of more general hypotheses for the validity of equations $(EL)$ and to the problem of finding a class of functionals which do not exhibit the Lavrentiev Phenomenon. We apply our results to the particular case of Manià's functional.

The study of conditions excluding Lavrentiev Phenomenon was considered since the first works on the subject ([L], [M] and [T]) and, more recently, by many authors ([A], [BM], [CV], [CA] and [Lo]). In [T] Tonelli defined a kind of lipschitzian approximations of the minimizer and determined a class of functionals, characterized by some assumptions involving the differentiability properties of the integrand $f$, which are continuous along such approximations. In [A] the author, with a refinement of the idea of Tonelli, gives a very general condition on $f$ which excludes Lavrentiev Phenomenon. In our first result, with a proof similar to that one of Angell, we provide a class of functionals for which the Lavrentiev Phenomenon does not occur, that is strictly larger than the class of functionals singled out by Tonelli.

The approximation procedure that we use in the proof of such result suggests to consider a class of variations around the minimizer, depending on a continuous parameter, along which the continuity of the functional is ensured. Even though such variations are not taken along a fixed direction, we study the differentiability of $\mathcal{I}$ along these variations, in the aim of deriving Euler-Lagrange Equations under weaker assumptions than the standard ones. Indeed, since such variations are obtained by truncating the derivative of $x$, some of the classical requirements on the behaviour of $f$ near $x$ can be removed. In such a way, we obtain a result which enlarges the range of validity of Equations $EL$.

## 2.2. PRELIMINARIES AND NOTATIONS

We consider an open subset $A$ of $\mathbb{R} \times \mathbb{R}^n$ and a compact interval of $\mathbb{R}$, $I = [a,b]$, and assume that, for any $t \in I$, the set $\{x \in \mathbb{R}^n : (t,x) \in int(A)\}$ is nonempty. Let $f : A \times \mathbb{R}^n \longrightarrow \mathbb{R}$; we are interested in the study of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t))dt$$

defined on the class of admissible trajectories with given boundary conditions, i.e. on the set

$$\Omega = \{x \in W^{1,1}(I, \mathbb{R}^n) : \text{ for any } t \in I,\ (t, x(t)) \in A \text{ and } x(a) = x_a,\ x(b) = x_b\},$$

where $x_a,\ x_b \in \mathbb{R}^n$ are such that $(a, x_a), (b, x_b)$ belong to $A$ and $\Omega$ is nonempty.

Our work mainly concerns the problem

$$\mathcal{P} : \qquad \text{Minimize } \{\mathcal{I}(x);\ x \in \Omega\}.$$

Given $x \in \Omega$ we call graph of $x$ the set $\Gamma = \{(t, x(t))\ t \in I\}$ and given, $\sigma > 0$, we call $\sigma$-neighbourhood of the graph of $x$ the set $\Gamma_\sigma = \{(t,y) : t \in I, \text{ s. t. } |y - x(t)| \leq \sigma\}$. We say that the graph of $x$ lies in the interior of $A$ if there exists a $\sigma-$neighbourhood of the graph of $x$ contained in $A$. We say that $x \in \Omega$ gives a strong local minimum for $\mathcal{I}$ if there exists $\sigma > 0$ such that for any $y \in \Omega$, with graph contained in $\Gamma_\sigma$, it is $\mathcal{I}(y) \geq \mathcal{I}(x)$. We say that $x \in \Omega$ gives a weak local minimum for $\mathcal{I}$ if there exists $\sigma > 0$ and $\tau > 0$ such that, for any $y \in \Omega$ with graph contained in $\Gamma_\sigma$ and such that it is $|y'(t) - x'(t)| < \tau$ for a.e. $t \in I$, it is $\mathcal{I}(y) \geq \mathcal{I}(x)$.

We shall use the following standard notations. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^n$ and by $|\cdot|$ the associated norm; $E^c$ is the complement of the set $E$ and $\mu(\cdot)$ is the Lebesgue measure. We shall denote by $C(I)$, $L^p(I)$ and $W^{1,p}(I)$, the spaces $C(I, \mathbb{R}^n)$, $L^p(I, \mathbb{R}^n)$ and $W^{1,p}(I, \mathbb{R}^n)$, for $1 \leq p \leq \infty$, and by $\|\cdot\|_{C(I)}$, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ the respective norms. By $\nabla_x f$ and $\nabla_{x'} f$ we denote the gradients of $f$ with respect to the second and the third variable. We set also, for $p \geq 1$, $p' = p/(p-1)$.

**Definition 2.2.1.** Let $E \subseteq \mathbb{R}^n$ be measurable, $h : E \to \mathbb{R}$ be measurable and $\alpha, \beta \in \bar{\mathbb{R}}$. We set

$$E_{\alpha,\beta}(h) := \{t \in E : h(t) \in ]\alpha, \beta]\},$$

and $E_\alpha(h) := E_{\alpha,+\infty}(h)$ . We set also $\omega(h,\alpha) := \mu(E_\alpha(h))$.

We will make use of the following theorem (see [WZ] pp. 81-83).

**Theorem 2.2.1.** *Let $E \subseteq \mathbb{R}^n$ be measurable, $\mu(E) < \infty$, $h : E \to \mathbb{R}$ be measurable, $\alpha, \beta \in \bar{\mathbb{R}}$, $\alpha \leq \beta$ and $\phi : \mathbb{R} \to \mathbb{R}$ be continuous and such that $\phi \circ h \in L^1(E)$. Then*

$$\int_{E_{\alpha,\beta}(h)} \phi(h(x))dx = -\int_\alpha^\beta \phi(\sigma)d\omega(h,\sigma) \tag{2.1}$$

*(where the last is a Stieltjes integral). In particular*

$$\int_{E_{\alpha,\beta}(h)} h^p = -\int_\alpha^\beta \sigma^p d\omega(h,\sigma) = -\beta^p\omega(h,\beta) + \alpha^p\omega(h,\alpha) + p\int_\alpha^\beta \sigma^{p-1}\omega(h,\sigma)d\sigma. \tag{2.2}$$

We recall Tchebyshev inequality (see for instance [WZ] p.82).

**Theorem 2.2.2.** *Let $E \subseteq \mathbb{R}^n$ be measurable and $h$ belong to $L^p(E,\mathbb{R}^n)$. Then*

$$\omega(|h|,\sigma) \;\leq\; \frac{\|h\|_{L^p}^p}{\sigma^p} \text{ for any } \sigma > 0.$$

## 2.3. LAVRENTIEV PHENOMENON

We say that the functional $\mathcal{I}$ exhibits the so called Lavrentiev phenomenon if

$$\inf_{x\in\Omega\cap W^{1,\infty}(I)} \mathcal{I}(x) > \min_{x\in\Omega} \mathcal{I}(x).$$

In the study of such phenomenon it is of particular interest the following example due to Manià (see for instance [Lo], [Ce1] pp. 514-516, [D1] pp. 92-95, [BM] p.13), consisting in the minimum problem

$$\mathcal{P}_g : \text{ Minimize } \left\{\mathcal{I}_g(x) := \int_0^1 g(t,x(t),x'(t))dt;\; x \in W^{1,1}([0,1],\mathbb{R}),\; x(0) = 0,\; x(1) = 1\right\}$$

where $g(t,x,v) = (x^3 - t)^2|v|^q$. It is easy to see that the solution of $\mathcal{P}_g$ is $x_0(t) = t^{\frac{1}{3}}$ and $\mathcal{I}(x_0) = 0$. We have the following result.

**Proposition 2.3.1.**

*i)* *If* $0 \le q < \frac{9}{2}$ *then* $\inf\{\mathcal{I}_g(x);\ x \in W^{1,\infty}([0,1],\mathbb{R}),\ x(0) = 0,\ x(1) = 1\} = 0.$

*ii)* *If* $q \ge \frac{9}{2}$ *then* $\inf\{\mathcal{I}_g(x);\ x \in W^{1,\infty}([0,1],\mathbb{R}),\ x(0) = 0,\ x(1) = 1\} > 0.$

*Proof.* We prove statement *i)*, for *ii)* see [BM].
Let us define the following sequence $\{x_k\}_{k\in\mathbb{N}}$ in $W^{1,\infty}(I,\mathbb{R})$ of lipschitzian approximations of $x_0(t) = t^{\frac{1}{3}}$:

$$x_k(t) = \begin{cases} 3kt, & t \in [0,(3k)^{-\frac{3}{2}}[ \\ t^{\frac{1}{3}}, & t \in [(3k)^{-\frac{3}{2}},1] \end{cases}$$

It is, by easy computations,

$$\mathcal{I}_g(x_k) = \int_0^1 g(t,x_k(t),x_k'(t))dt = \frac{8}{105}(3k)^{q-\frac{9}{2}}.$$

Hence $\lim_{k\to\infty}\mathcal{I}_g(x_k) = 0$. □

The study of Manià example leads to the investigation of general properties of the integrand $f$ which prevent Lavrentiev phenomenon to occur; Theorem 3.1 below provides a result in this direction extending the original work of Tonelli [T] (see Remark 3.2 below).

We shall need the following technical lemma.

**Lemma 2.3.1.** *Let $E \subseteq \mathbb{R}^n$ be measurable, $\mu(E) < \infty$, and let $x$ belong to $L^p(E)$, $p > 0$. Let $q_1, q_2, \gamma_1, \gamma_2$ be positive numbers such that $q_2 \le p$ and $\gamma_1(p - q_1) = (q_2 - p)\gamma_2$. Then, for any $\delta \ge 0$,*

$$\left(\int_{(E_\delta(|x|))^c} |x(t)|^{q_1}\,dt\right)^{\gamma_1} \left(\int_{E_\delta(|x|)} |x(t)|^{q_2}\,dt\right)^{\gamma_2} \le$$
$$\left(\int_{(E_\delta(|x|))^c} |x(t)|^{p}\right)^{\gamma_1} \left(\int_{E_\delta(|x|)} |x(t)|^{p}\,dt\right)^{\gamma_2}.$$

*Proof.* First of all the integrals in the l.h.s. do exist. For any $\sigma > 0$ we set $\psi(\sigma) = -\omega(|x|, \sigma)$; $\psi$ is non decreasing, hence, if $f$ and $g$ are real valued continuous functions defined on $[0, +\infty[$ such that $f \leq g$, it is

$$\int_\alpha^\beta f(\sigma) d\psi(\sigma) \leq \int_\alpha^\beta g(\sigma) d\psi(\sigma)$$

for any $\alpha, \beta \in \bar{\mathbb{R}}^+$. Now using formula (2.1) we have

$$\left( \int_{(E_\delta(|x|))^c} |x(t)|^{q_1} dt \right)^{\gamma_1} \left( \int_{E_\delta(|x|)} |x(t)|^{q_2} dt \right)^{\gamma_2} = \left( \int_0^\delta \sigma^{q_1} d\psi(\sigma) \right)^{\gamma_1} \left( \int_\delta^\infty \tau^p \tau^{q_2 - p} d\psi(\tau) \right)^{\gamma_2}. \tag{3.1}$$

Since for $\tau \geq \delta$ it is $\tau^{q_2 - p} \leq \delta^{q_2 - p}$, and for $0 < \sigma \leq \delta$ it is $\delta^{(q_2 - p)\frac{\gamma_2}{\gamma_1}} \leq \sigma^{(q_2 - p)\frac{\gamma_2}{\gamma_1}}$, the right hand side of (3.1) can be estimated as

$$\left( \int_0^\delta \sigma^{q_1} d\psi(\sigma) \right)^{\gamma_1} \left( \int_\delta^\infty \tau^p \delta^{q_2 - p} d\psi(\tau) \right)^{\gamma_2} =$$
$$\left( \int_0^\delta \sigma^{q_1} \delta^{(q_2 - p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma) \right)^{\gamma_1} \left( \int_\delta^\infty \tau^p d\psi(\tau) \right)^{\gamma_2} \leq$$
$$\left( \int_0^\delta \sigma^{q_1 + (q_2 - p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma) \right)^{\gamma_1} \left( \int_\delta^\infty \tau^p d\psi(\tau) \right)^{\gamma_2}.$$

Since $q_1 + (q_2 - p)\frac{\gamma_2}{\gamma_1} = p$, this ends the proof. □

Following Angell and Cesari ([A], [Ce1] and [CA]) we give the following

**Definition 2.3.1.** We say that $f : A \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies Caratheodory condition (C) provided that given $\epsilon > 0$, there is a compact subset $K_\epsilon \subset I$ such that $\mu(I \setminus K_\epsilon) < \epsilon$, $A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbb{R}^n)$ is closed, and the function $f$ is continuous on $A_{K_\epsilon} \times \mathbb{R}^n$.

In the main result of this section we shall assume that $f$ satisfies one of the following conditions.

$H_1$: *$f$ satisfies condition (C) and maps bounded subsets of its domain into bounded subsets of* $\mathbb{R}$.

*$H_2$: $f$ is continuous on its domain.*

**Theorem 2.3.1.** *Let $f$ satisfy either $H_1$ or $H_2$ and let $x$ be an element of $\Omega \cap W^{1,p}(I)$ whose graph is contained in the interior of $A$ (i.e. there exists a $\sigma$-neighbourhood $\Gamma_\sigma$ of the graph of $x$ contained in $A$) and such that $f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I)$. Assume that*

*$H_3$: there exist $m, M \geq 0$ and $\gamma > 0$ such that, for any $(t,y) \in \Gamma_\sigma$ it is*

$$|f(t,y,x'(t)) - f(t,x(t),x'(t))| \leq (m + M|x'(t)|^q)\,|x(t)-y|^\gamma, \text{ where } q = p(\gamma+1)-\gamma.$$

*Then, given $\epsilon > 0$, there exists $y \in W^{1,\infty}(I) \cap \Omega$ such that*

$$\|y - x\|_{W^{1,p}} \leq \epsilon$$

$$|\mathcal{I}(y) - \mathcal{I}(x)| \leq \epsilon.$$

**Corollary 2.3.1.** *Under the hypotheses of theorem 3.1, if $x$ is a solution of $\mathcal{P}$, then*

$$\inf_{x \in \Omega \cap W^{1,\infty}(I)} \mathcal{I}(x) = \min_{x \in \Omega} \mathcal{I}(x);$$

*that is to say, the hypotheses of theorem 3.1 exclude Lavrentiev phenomenon.*

**Remark2. 3.1.** Hypothesis $H_3$ in theorem 3.1 includes as a special case, ($\gamma = 1$, $q = 2p-1$), the following

*$H_4$: $f$ is continuously differentiable with respect to the second variable and there exist positive constants $m$, $M$ such that $|\nabla_x f(t,y,x'(t))| \leq m + M|x'(t)|^{2p-1}$ for any $(t,y) \in \Gamma_\sigma$.*

**Remark 2.3.2.** Hypothesis $H_4$ provides an extension of condition ($\beta$) in [T] (see also [Ce1] Remark, p. 512):

*($\beta$): $f$ is continuously differentiable with respect to the second variable and there exist positive constants $m$,$M$ such that $|\nabla_x f(t,y,v)| \leq m + M|v|$ for any $(t,y,v) \in \Gamma_\sigma \times \mathbb{R}^n$.*

We emphasize that, in the case in which $x$ is in $W^{1,p}(I)$, the proof of Tonelli can be easily reproduced under the weaker assumption that there exist positive constants $m$, $M$ such that $|\nabla_x f(t,y,v)| \leq m + M|v|^p$ for any $(t,y,v) \in \Gamma_\sigma \times \mathbb{R}^n$.

**Remark 2.3.3.** Example.
Let us consider the following Manià type functionals: $\mathcal{I}_{g_m}(x) = \int_0^1 g_m(t, x(t), x'(t))dt$; where $g_m(t,x,v) = (x^3 - t)^{2m}|v|^q$, $m \in \mathbb{N}$. The hypotheses of Theorem 3.1 are satisfied for $q < \frac{3}{2} + m$, while Tonelli's condition $(\beta)$ holds only for $q < \frac{3}{2}$.

*Proof of Theorem* 3.1. We may assume $x' \in L^p(I) \setminus L^\infty(I)$ since when $x'$ is essentially bounded there is nothing to prove. For any positive number $\rho$ define the sets $I_\rho = \{t \in I : |x'(t)| > \rho\}$. Take $R > 0$ such that the complement of $I_R$, $I_R^c$, has positive measure, and for any $\delta \in \mathbb{R}$, $\delta \geq R$, we set

$$\beta_\delta = \frac{1}{\mu(I_R^c)} \int_{I_\delta} x'(\tau)d\tau.$$

Since $x'$ belongs to $L^p(I)$ we have

$$\lim_{\delta \to \infty} \mu(I_\delta) = 0 \tag{3.2}$$

and, obviously,

$$\lim_{\delta \to \infty} \beta_\delta = 0. \tag{3.3}$$

Consider, for any $\delta \geq R$, the function $y_\delta$ defined by setting

$$y'_\delta(t) = \begin{cases} 0, & t \in I_\delta \\ x'(t), & t \in I_R \setminus I_\delta \\ x'(t) + \beta_\delta, & t \in I_R^c \end{cases}$$

and

$$y_\delta(t) = x_a + \int_0^t y'_\delta(\tau)d\tau.$$

Since $y'_\delta$ is bounded by $\delta$, $y_\delta$ is in $W^{1,\infty}(I)$. We have $y_\delta(a) = x_a$ and

$$y_\delta(b) = x_a + \int_{I_\delta^c} x'_\delta(\tau)d\tau + \beta_\delta \mu(I_R^c) = x(a) + \int_I x'_\delta(\tau)d\tau = x(b) = x_b.$$

Moreover

$$\int_I |y_\delta'(t) - x'(t)|^p \, dt \leq \int_{I_\delta} |x'(t)|^p \, dt + |\beta_\delta|^p \mu(I_R^c).$$

Hence, by (3.2) and (3.3), $y_\delta$ is arbitrarily close to $x$ in $W^{1,p}(I)$, and also in $C(I)$, when $\delta$ is sufficiently large; in particular we have the estimate

$$\|y_\delta - x\|_{C(I)} \leq 2 \int_{I_\delta} |x'(\tau)| d\tau. \tag{3.4}$$

Inequality (3.4) ensures that there exists $\delta_0$ such that the set $\{(t, y_\delta(t)),\ t \in I\}$ is contained in $\Gamma_\sigma \subset A$ for every $\delta > \delta_0$. In particular, for any $\delta > \delta_0$, $y_\delta$ belongs to $W^{1,\infty}(I) \cap \Omega$.

To prove the theorem we shall show that $\mathcal{I}(y_\delta)$ is arbitrarily close to $\mathcal{I}(x)$ when $\delta$ is sufficiently large.

Let us write

$$\begin{aligned} |\mathcal{I}(y_\delta) - \mathcal{I}(x)| = &\left| \int_I (f(t, y_\delta(t), y_\delta'(t)) - f(t, x(t), x'(t)))\, dt \right| \leq \\ &\int_{I_\delta} |f(t, y_\delta(t), 0) - f(t, x(t), x'(t))|\, dt + \\ &\int_{I_R^c} |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))|\, dt + \\ &\int_{I_R \setminus I_\delta} |f(t, y_\delta(t), x'(t)) - f(t, x(t), x'(t))|\, dt = \\ &\Lambda_1(\delta) + \Lambda_2(\delta) + \Lambda_3(\delta). \end{aligned}$$

We claim that $\lim_{\delta \to \infty} \Lambda_i(\delta) = 0$, $i = 1, 2, 3$.

1.)

$$\Lambda_1(\delta) \leq \int_{I_\delta} |f(t, y_\delta(t), 0)|\, dt + \int_{I_\delta} |f(t, x(t), x'(t))|\, dt. \tag{3.5}$$

For any $\delta > \delta_0$, $|y_\delta(t)| \leq |x(t)| + \sigma$, $t \in I$, hence, since $f$ maps bounded subsets of its domain into bounded subsets of $\mathbb{R}$, the first integrand in (3.5) is bounded by a constant. By hypothesis $f(\cdot, x(\cdot), x'(\cdot))$ belongs to $L^1(I)$; hence, by (3.2), absolute continuity of the integral implies that $\lim_{\delta \to \infty} \Lambda_1(\delta) = 0$.

2.) On the set $I_R^c$ the family $\{x'(\cdot) + \beta_\delta, \delta \geq \delta_0\}$ is uniformly bounded by a constant. Hence the family

$$\{h_\delta(t) = |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))|\,, \quad \delta \geq \delta_0\}$$

is integrably bounded on $I_R^c$.

Assume first that $f$ satisfies $H_2$ (i.e. $f$ is continuous on its domain). By the pointwise convergence of $y_\delta$ to $x$ on $I_R^c$, by (3.3) and by dominated convergence we have $\lim_{\delta\to\infty} \Lambda_2(\delta) = 0$.

Assume that $f$ satisfies hypothesis $H_1$. Given $\epsilon > 0$ we can take a compact set $K_\epsilon$ contained in $I$ such that $f$ is continuous on $A_{K_\epsilon} \times \mathbb{R}^n$ ($A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbb{R}^n)$) and such that the measure of $I \setminus K_\epsilon$ is small enough so that, by the absolute equiintegrabilty of the family $\{h_\delta\}$,

$$\int_{(I\setminus K_\epsilon)\cap I_R^c} h_\delta(t)dt < \frac{\epsilon}{2} \qquad \text{for any } \delta > \delta_0.$$

By the continuity of $f$ on $A_{K_\epsilon}$, the pointwise convergence of $y_\delta$ to $x$ and by (3.3), there exists $\delta_\epsilon$ such that, by dominated convergence,

$$\int_{K_\epsilon\cap I_R^c} h_\delta(t)dt < \frac{\epsilon}{2} \qquad \text{for any } \delta > \delta_\epsilon.$$

Hence $\int_{I_R^c} h_\delta(t)dt < \epsilon$ for any $\delta > \delta_\epsilon$, and, also in this case, $\lim_{\delta\to\infty} \Lambda_2(\delta) = 0$.

3.) Hypothesis $H_3$ and (3.4) imply that

$$\begin{aligned}
\Lambda_3(\delta) \le &\int_{I_\delta^c} |f(t,y_\delta(t),x'(t)) - f(t,x(t),x'(t))|\, dt \le \\
&\int_{I_\delta^c} (m + M|x'(t)|^q)\, |y_\delta(t) - x(t)|^\gamma dt \le \\
&2^\gamma \int_{I_\delta^c} (m + M|x'(t)|^q)\, dt \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma \le \\
&2^\gamma m \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma + 2^\gamma M \left(\int_{I_\delta^c} |x'(\tau)|^q d\tau\right) \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma .
\end{aligned} \tag{3.6}$$

Applying Lemma 3.1 with $q_1 = q,\ q_2 = 1,\ \gamma_1 = 1,\ \gamma_2 = \gamma$ to the second term in the r.h.s. of (3.6), we have

$$\Lambda_3(\delta) \le 2^\gamma m \left(\int_{I_\delta} |x'(\tau)|d\tau\right)^\gamma + 2^\gamma M \left(\int_I |x'(\tau)|^p d\tau\right) \left(\int_{I_\delta} |x'(\tau)|^p d\tau\right)^\gamma .$$

Hence, by (3.2), $\lim_{\delta\to\infty} \Lambda_3(\delta) = 0$. □

## 2.4. EULER-LAGRANGE EQUATIONS

Euler-Lagrange equations are well known necessary conditions for a function $x$ in $\Omega$ to be a local minimum for the functional $\mathcal{I}$, when $f$ is assumed to be of class $C^1$ on its domain and to satisfy some growth conditions in a neighbourhood of the graph of $x$.

Our aim is to weaken these growth assumption. We begin by stating the classical theorem (see for instance [Ce1], Th. 2.2.i p. 30, Remark 2 pp. 40-41, and Remark 1 p. 44) in order to compare it with our result (Theorem 4.2).

**Theorem 2.4.1.** *Let $f$ belong to $C^1(A \times \mathbb{R}^n, \mathbb{R})$ and let $x$ belong to $\Omega \cap W^{1,p}(I)$, $1 \leq p < \infty$. Assume that the graph of $x$ lies in the interior of $A$, (i.e. there exist $\sigma$ positive and a $\sigma$-neighbourhood $\Gamma_\sigma$ of the graph of $x$ contained in $A$), that $x$ gives a weak local minimum for $\mathcal{I}$ and that there exist positive constants $m$, $M$ such that $f$ satisfies the following conditions:*

$C_1$: $|f(t,y,v)| \leq m + M|v|^p$,

$C_2$: $|\nabla_x f(t,y,v)| \leq m + M|v|^p$,

$C_3$: $|\nabla_{x'} f(t,y,v)| \leq m + M|v|^p$

*for any* $(t,y,v) \in \Gamma_\sigma \times \mathbb{R}^n$.

*Then*

$$\frac{d}{dt}\nabla_{x'} f(t,x(t),x'(t)) = \nabla_x f(t,x(t),x'(t)) \quad \text{a.e. } t \in I. \qquad (EL)$$

*Or*

$$\frac{d}{dt}\Big(\frac{\partial}{\partial x_i'} f(t,x(t),x'(t))\Big) = \frac{\partial}{\partial x_i} f(t,x(t),x'(t)) \quad \text{a.e. } t \in I \quad i = 1,...,n.$$

Let us consider Manià example introduced at the beginning of section 3. The assumptions $C_1$-$C_3$ of theorem 4.1 are satisfied only for $q < \frac{3}{2}$, since the solution $x_0(t) = t^{\frac{1}{3}}$ belongs to $W^{1,p}(I)$ for $p < \frac{3}{2}$. On the other hand it is easy to check, by direct inspection, that $x_0$ satisfies equations *(EL)* for any $q$. This simple example shows that conditions $C_1$-$C_3$ are far from being optimal, hence it is worth to make an effort in order to enlarge the range of validity of equations $(EL)$. The following theorem goes in this direction.

**Theorem 2.4.2.** *Let $f$ belong to $C^1(A \times \mathbb{R}^n, \mathbb{R})$ and let $x$ belong to $\Omega \cap W^{1,p}(I)$, $1 < p < \infty$. Assume that the graph of $x$ lies in the interior of $A$, (i.e. there exist $\sigma$ positive and a $\sigma$-neighbourhood $\Gamma_\sigma$ of the graph of $x$ contained in $A$), that $x$ gives a strong local minimum for $\mathcal{I}$ and that $f$ satisfies the following conditions:*

$E_1$ : $f(\cdot, x(\cdot), x'(\cdot)) \in L^p(I)$;

$E_2$ : $\nabla_x f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I)$;

$E_3$ : *there exist* $m_1, M_1 \geq 0$ *such that, for any* $(t, y, v) \in \Gamma_\sigma \times \mathbb{R}^n$,

$$|\nabla_{x'} f(t, y, v)| \leq m_1 + M_1 |v|^p$$

$E_4$ : *there exist* $m_2, M_2 \geq 0$, $\gamma \geq 1$ *such that for any* $(t, z) \in \Gamma_\sigma$

$$|\nabla_x f(t, z, x'(t)) - \nabla_x f(t, x(t), x'(t))| \leq (m_2 + M_2 |x'(t)|^q)\, |z - x(t)|^\gamma$$

*where* $p < q < p(\gamma + 1) - \gamma$.

*Then equations (EL) hold true.*

**Remark 2.4.1.** Comparison between hypotheses $C_1$-$C_3$ and $E_1$-$E_4$.
The proof of Theorem 4.1 is performed by taking the Gateaux derivative of the functional along directions determined by elements of $W_0^{1,\infty}(I)$. To do this one needs integrability of $f$, $\nabla_x f$, $\nabla_{x'} f$ along trajectories whose graph is contained in a neighbourhood of the graph of the solution: conditions $C_1$-$C_3$ ensure such property since guarantee that near the solutions $f$, $\nabla_x f$ and $\nabla_{x'} f$ are bounded by an integrable power of the derivative. If we consider Manià's functional, we notice that the integrand $g = g(t, x, v)$ and its derivatives with respect to $x$ and $v$ are zero along the solution, but, if $q \geq \frac{3}{2}$, they are not integrable along trajectories contained in a neighbourhood of its graph. Hypotheses $E_1$ and $E_2$ are intended to take into account integrands which behaves "well" along the solution $x$, disregarding the behaviour in a neighbourhood of the graph of $x$.
While $E_3$ is analogous to $C_3$, we replace $C_2$ by $E_2$ and $E_4$, where $E_4$ involves some continuity of $\nabla_x f$ and $E_2$ guarantees its integrability along the solution.
As far as it concerns Manià example it is easy to see that:

$C_1$, $C_2$, $C_3$ are satisfied for $q < \frac{3}{2}$;

$E_1$, $E_2$ are satisfied for any $q$;

$E_3$ is satisfied for $q < \frac{5}{2}$;

$E_4$ is satisfied for $q < 2$.

*Proof of Theorem* 4.2. Our aim is to show that, given any $\xi \in W_0^{1,\infty}(I)$, it is

$$\int_I [\langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle + \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle] \, dt = 0.$$

If this is so, by integration by parts, and by a standard argument (see for example [Cel] p. 42), it follows that

$$-\int_a^t \nabla_x f(s, x(s), x'(s)) ds + \nabla_{x'} f(t, x(t), x'(t)) = const. \qquad t \in [a, b]$$

and then, by differentiation, one obtains equations $(EL)$.

In the following we set, for the sake of brevity,

$$G(t) = \langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle + \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle \, .$$

Hypotheses $E_2$ and $E_3$ imply that $G \in L^1(I)$.

In the proof of Theorem 4.1 one considers variations around the solution $x$ of the form

$$x' \longrightarrow x' + \theta \xi', \qquad \text{and} \qquad x \longrightarrow x + \theta \xi,$$

for a real $\theta$ belonging to a neighbourhood of the origin. As we have already remarked, this requires some bounds on the growth of $f$, $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of $x$ (see hypotheses $C_1$-$C_3$ in Theorem 4.1). Since hypotheses $E_1$-$E_4$ do not guarantee such properties, we perform a different kind of variations which involve, as in the proof of Theorem 3.1, truncation of the derivative of $x$; this choice weakens the requirements on $f$ and on $\nabla_x f$, and forces us to assume that $x$ is strong local minimum.

1.) Take $\xi \in W_0^{1,\infty}(I)$ and $\alpha \in ]0, \gamma[$ such that $q = p + \frac{\gamma - \alpha}{1 + \alpha}(p - 1)$. We consider, as in previous section, the family of subsets of $I$, $I_\rho = \{t \in I : |x'(t)| > \rho, \ \rho \geq 0\}$, and define $\delta : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^+$ by setting $\delta(\theta) = |\theta|^{\frac{1+\alpha}{1-p}}$. Take $R > 0$ such that $\mu(I_R^c) > 0$ and $\theta_0$ such that $\delta(\theta_0) \geq R$. For any $\theta \in [-\theta_0, \theta_0]$ we define the function $\eta_\theta$ by setting:

$$\beta_\theta = \frac{1}{\mu(I_R^c)} \left[ \int_{I_{\delta(\theta)}} x'(\tau) d\tau + \theta \int_{I_{\delta(\theta)}} \xi'(\tau) d\tau \right]$$

$$\eta_0'(t) = 0 \qquad\qquad \text{for } \theta = 0$$

$$\eta_\theta'(t) = \begin{cases} -x'(t), & t \in I_{\delta(\theta)} \\ \theta\xi'(t) + \beta_\theta, & t \in I_R^c \\ \theta\xi'(t), & t \in I_R \setminus I_{\delta(\theta)} \end{cases} \qquad\qquad \text{for } \theta \in [-\theta_0, \theta_0] \setminus \{0\}$$

and

$$\eta_\theta(t) = \int_a^t \eta_\theta'(\tau)d\tau \qquad t \in I = [a, b].$$

For any $\theta \in [-\theta_0, \theta_0]$, $\eta_\theta$ is in $W^{1,p}(I)$ and, remarking that $\int_I \xi'(\tau)d\tau = 0$, we have

$$\eta_\theta(b) = -\int_{I_{\delta(\theta)}} x'(\tau)d\tau + \theta\int_{I^c_{\delta(\theta)}} \xi'(\tau)d\tau + \mu(I_R^c)\beta_\theta = 0 = \eta_\theta(a).$$

Hence $\eta_\theta \in W_0^{1,p}(I)$.

We now list some properties useful in the following, denoting by $c_1$, $c_2$, $c_3$ suitable positive constants depending on $\theta_0$, $\|x\|_{W^{1,p}}$, $\|\xi\|_{W^{1,\infty}}$, $R$ and $\mu(I)$.

*i)* Using Tchebishev inequality:

$$\mu(I_{\delta(\theta)}) \leq \|x'\|_{L^p}^p \delta(\theta)^{-p} \leq \|x'\|_{L^p}^p |\theta|^{(1+\alpha)p'}. \tag{4.1}$$

*ii)* By Hölder inequality, (4.1) implies that, for any $h \in L^p(I)$:

$$\int_{I_{\delta(\theta)}} |h(\tau)|d\tau \leq \mu(I_{\delta(\theta)})^{\frac{1}{p'}} \left(\int_{I_{\delta(\theta)}} |h(\tau)|^p d\tau\right)^{\frac{1}{p}} \leq \|x'\|_{L^p}^{p-1} \|h\|_{L^p} |\theta|^{1+\alpha}. \tag{4.2}$$

*iii)* By (4.2)

$$|\beta_\theta| \leq \frac{1}{\mu(I_R^c)} \left(\|x'\|_{L^p}^p |\theta|^{1+\alpha} + \|\xi'\|_{L^p} \|x'\|_{L^p}^{p-1} |\theta|^{2+\alpha}\right) \leq c_1 |\theta|^{1+\alpha}. \tag{4.3}$$

*iv)* It is

$$\frac{\eta_\theta'(t)}{\theta} - \xi'(t) = \begin{cases} -\frac{1}{\theta}x'(t) - \xi'(t), & t \in I_{\delta(\theta)} \\ \frac{\beta_\theta}{\theta}, & t \in I_R^c \\ 0, & t \in I_R \setminus I_{\delta(\theta)}. \end{cases}$$

By (4.1) and (4.3) we have that

$$\lim_{\theta\to 0}\left|\frac{\eta_\theta'(t)}{\theta}-\xi'(t)\right|=0 \qquad \text{a.e. } t\in I. \tag{4.4}$$

*v)* It is, by (4.2) and (4.3),

$$\left|\frac{\eta_\theta(t)}{\theta}-\xi(t)\right| \;\le\; \int_a^t\left|\frac{\eta_\theta'(\tau)}{\theta}-\xi'(\tau)\right|d\tau \;\le$$

$$\frac{1}{\theta}\int_{I_{\delta(\theta)}}|x'(\tau)|d\tau+\int_{I_{\delta(\theta)}}|\xi'(\tau)|d\tau+\frac{\beta_\theta}{\theta}\mu(I_R^c)\;\le\; c_2|\theta|^\alpha .$$

Hence

$$\lim_{\theta\to 0}\|\frac{\eta_\theta}{\theta}-\xi\|_{L^\infty}=0 \tag{4.5}$$

and, in particular,

$$\|\eta_\theta\|_{L^\infty}\le c_3|\theta| \tag{4.6}$$

2.) Consider now $\sigma_1$ such that $0<\sigma_1\le\sigma$ and for any $y\in W^{1,p}(I)\cap\Omega$ with graph contained in $\Gamma_{\sigma_1}$ ($\subseteq\Gamma_\sigma$) it is $\mathcal{I}(x)\le\mathcal{I}(y)$. By (4.6) there exists $\theta_1$, $0<\theta_1\le\theta_0$, such that for any $\theta\in[-\theta_1,\theta_1]$ the graph of $x+\eta_\theta$ is contained in $\Gamma_{\sigma_1}$. Hence $x+\eta_\theta$ belongs to $\Omega$ and $\mathcal{I}(x+\eta_\theta)\ge\mathcal{I}(x)$ for any $\theta\in[-\theta_1,\theta_1]$, and the function $\varphi:[-\theta_1,\theta_1]\to\mathbb{R}$, defined by $\varphi(\theta)=\mathcal{I}(x+\eta_\theta)$ has a minimum in $\theta=0$. Our aim is to show that such a function is differentiable in zero and that $\varphi'(0)$ coincide with $\int_I G(t)dt$. This would prove the theorem.

Let us write

$$\begin{aligned}
\left|\frac{\varphi(\theta)-\varphi(0)}{\theta}-\int_I G(t)dt\right| &= \left|\frac{\mathcal{I}(x+\eta_\theta)-\mathcal{I}(x)}{\theta}-\int_I G(t)dt\right|\le\\
&\int_{I_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),0)-f(t,x(t),x'(t))}{\theta}-G(t)\right|dt+\\
&\int_{I_R^c}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t)+\beta_\theta)-f(t,x(t),x'(t))}{\theta}-G(t)\right|dt+\\
&\int_{I_R\setminus I_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t))-f(t,x(t),x'(t))}{\theta}-G(t)\right|dt=\\
&\Lambda_1(\theta)+\Lambda_2(\theta)+\Lambda_3(\theta).
\end{aligned}$$

We claim that $\lim_{\theta\to 0}\Lambda_i(\theta)=0$ for $i=1,2,3$.

3.) Estimate of $\Lambda_1(\theta)$.

$$\Lambda_1(\theta) \le \frac{1}{\theta}\int_{I_{\delta(\theta)}} |f(t,x(t)+\eta_\theta(t),0)|\,dt + \frac{1}{\theta}\int_{I_{\delta(\theta)}} |f(t,x(t),x'(t))| + \int_{I_{\delta(\theta)}} |G(t)|\,dt$$

Recalling (4.6), the set $\{(t,x(t)+\eta_\theta(t),0),\ t\in I,\ \theta\in[-\theta_1,\theta_1]\}$ is contained in a fixed compact subset of $A\times \mathbb{R}^n$, and since $f$ is continuous, there exists a positive constant $M$ such that

$$|f(t,x(t)+\eta_\theta(t),0)| \le M \quad \text{for any } t\in I_{\delta(\theta)}.$$

Hence, recalling (4.1), (4.2) and $E_1$

$$\begin{aligned}\Lambda_1(\theta) \le & \frac{1}{\theta}M\mu(I_{\delta(\theta)}) + \frac{1}{\theta}\|f(\cdot,x(\cdot),x'(\cdot))\|_{L^p}\|x'\|_{L^p}^{p-1}|\theta|^{1+\alpha} + \int_{I_{\delta(\theta)}} |G(t)|\,dt \le \\ & M\|x'\|^p|\theta|^{(1+\alpha)p'-1} + \|f(\cdot,x(\cdot),x'(\cdot))\|_{L^p}\|x'\|^{p-1}|\theta|^{\alpha} + \int_{I_{\delta(\theta)}} |G(t)|\,dt.\end{aligned}$$

Since $G$ is in $L^1(I)$ we have that $\lim_{\theta\to 0}\Lambda_1(\theta)=0$.

4.) Estimate of $\Lambda_2(\theta)$.

$$\begin{aligned}\Lambda_2(\theta) \le & \\ & \int_{I_R^c}\Bigg|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t)+\beta_\theta) - f(t,x(t)+\eta_\theta(t),x'(t))}{\theta} - \\ & \qquad\qquad \langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle\Bigg|\,dt + \\ & \int_{I_R^c}\Bigg|\frac{f(t,x(t)+\eta_\theta(t),x'(t)) - f(t,x(t),x'(t))}{\theta} - \\ & \qquad\qquad \langle\nabla_{x}f(t,x(t),x'(t)),\xi(t)\rangle\Bigg|\,dt.\end{aligned} \tag{4.7}$$

By mean value theorem there exist two functions, $y_\theta$, $z_\theta$, defined on $I_R^c$, such that: $y_\theta(t)$ lies in the line segment joining $x'(t)$ and $x'(t)+\theta\xi'(t)+\beta_\theta$, for a.e. $t\in I_R^c$, $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t)+\eta_\theta(t)$ for $t\in I_R^c$, and the right hand side of (4.7) is equal to

$$\int_{I_R^c}\left|\left\langle\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t)),\xi'(t)+\frac{\beta_\theta}{\theta}\right\rangle-\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle\right|dt+$$
$$\int_{I_R^c}\left|\left\langle\nabla_x f(t,z_\theta(t),x'(t)),\frac{\eta_\theta(t)}{\theta}\right\rangle-\langle\nabla_x f(t,x(t),x'(t)),\xi(t)\rangle\right|dt. \tag{4.8}$$

We remark that both integrands in (4.8) equal a.e. measurable functions and then are measurable. On $I_R^c$ $|x'|$ is bounded by $R$, hence, recalling (4.3) and (4.6) the sets $\{(t,x(t)+\eta_\theta(t),y_\theta(t)),\ t\in I,\ \theta\in[-\theta_1,\theta_1]\}$ and $\{(t,z_\theta(t),x'(t)),\ t\in I,\ \theta\in[-\theta_1,\theta_1]\}$ are contained in a fixed compact subset of $A\times\mathbb{R}^n$ and, since $f$ is of class $C^1(A\times\mathbb{R}^n)$, there exists a positive constant $L$ such that

$$|\nabla_x f(t,z_\theta(t),x'(t))|\leq L,\qquad |\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t))|\leq L\quad \text{for } t\in I_R^c.$$

These inequalities and hypotheses $E_2$, $E_3$ imply that both integrands in (4.8) are uniformly bounded by an integrable function. Moreover, (4.3) and (4.6) imply that they tend to zero a.e. on $I_R^c$ and, by dominated convergence, we have $\lim_{\theta\to 0}\Lambda_2(\theta)=0$.

5.) Estimate of $\Lambda_3(\theta)$.

$$\Lambda_3(\theta)\leq$$
$$\int_{I^c_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t)+\theta\xi'(t))-f(t,x(t)+\eta_\theta(t),x'(t))}{\theta}-\right.$$
$$\left.\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle\right|dt+$$
$$\int_{I^c_{\delta(\theta)}}\left|\frac{f(t,x(t)+\eta_\theta(t),x'(t))-f(t,x(t),x'(t))}{\theta}-\right.$$
$$\left.\langle\nabla_x f(t,x(t),x'(t)),\xi(t)\rangle\right|dt=$$
$$\Lambda_{3.1}(\theta)+\Lambda_{3.2}(\theta).$$

As in point 4., we can find $y_\theta(t)$ belonging, for a.e. $t\in I^c_{\delta(\theta)}$, to the line segment joining $x'(t)$ and $x'(t)+\theta\xi'(t)$ such that

$$\Lambda_{3.1}(\theta)=\int_{I^c_{\delta(\theta)}}|\langle\nabla_{x'}f(t,x(t)+\eta_\theta(t),y_\theta(t)),\xi'(t)\rangle-\langle\nabla_{x'}f(t,x(t),x'(t)),\xi'(t)\rangle|\,dt.$$

By $E_3$, we have

$$\left|\langle \nabla_{x'} f(t, x(t) + \eta_\theta(t), y_\theta(t)), \xi'(t)\rangle\right| \leq$$
$$(m_1 + M_1|x'(t) + \theta\xi'(t)|^p)\,|\xi'(t)| \leq m_1' + M_1'|x'(t)|^p, \tag{4.9}$$

where $m_1'$ and $M_1'$ are positive constants depending on $\|\xi'\|_{L^\infty}$. Since $x'$ belongs to $L^p(I)$, (4.5), (4.9) and dominated convergence imply that $\lim_{\theta\to 0}\Lambda_{3.1}(\theta) = 0$.

Let $z_\theta$, defined on $I^c_{\delta(\theta)}$, be such that $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t) + \eta_\theta(t)$ for any $t \in I^c_{\delta(\theta)}$. It is

$$\Lambda_{3.2}(\theta) \leq \int_{I^c_{\delta(\theta)}} \left|\left\langle \nabla_x f(t, z_\theta(t), x'(t)), \frac{\eta_\theta(t)}{\theta}\right\rangle - \left\langle \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta}\right\rangle\right| dt +$$

$$\int_{I^c_{\delta(\theta)}} \left|\left\langle \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta} - \xi(t)\right\rangle\right| dt =$$

$$\Lambda'_{3.2}(\theta) + \Lambda''_{3.2}(\theta).$$

Recalling $E_2$ and (4.5), we have $\lim_{\theta\to 0}\Lambda''_{3.2}(\theta) = 0$.

Using $E_4$ we have

$$\Lambda'_{3.2}(\theta) = \int_{I^c_{\delta(\theta)}} \left|\left\langle \nabla_x f(t, z_\theta(t), x'(t)) - \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta}\right\rangle\right| dt \leq$$

$$\int_{I^c_{\delta(\theta)}} \left|\frac{\eta_\theta(t)}{\theta}\right| (m_2 + M_2|x'(t)|^q)\,|z_\theta(t) - x(t)|^\gamma\, dt.$$

Now, by (4.6), $\left|\frac{\eta_\theta(t)}{\theta}\right| \leq c_3$ and $|z_\theta(t) - x(t)| \leq |\eta_\theta(t)| \leq c_3|\theta|$ for any $t \in I^c_{\delta(\theta)}$. Hence

$$\Lambda'_{3.2}(\theta) \leq c_3^{1+\gamma} m_2 \mu(I)|\theta|^\gamma + c_3^{1+\gamma} M_2|\theta|^\gamma \int_{I^c_{\delta(\theta)}} |x'(t)|^q\, dt. \tag{4.10}$$

Recalling formula (2.2) and Tchebishev inequality, we have

$$\int_{I^c_{\delta(\theta)}} |x'(t)|^q dt = -(\delta(\theta))^q \omega(|x'|, \delta(\theta)) + q \int_0^{\delta(\theta)} \sigma^{q-1} \omega(|x'|, \sigma) d\sigma \leq$$

$$q\|x'\|^p_{L^p} \int_0^{\delta(\theta)} \sigma^{q-p-1} d\sigma = q\|x'\|^p_{L^p} |\theta|^{(q-p)\frac{1+\alpha}{1-p}}. \tag{4.11}$$

Inserting (4.11) in (4.10) and denoting by $c'$ and $c''$ suitable positive constants, we have:

$$\Lambda'_{3.2}(\theta) \leq c'|\theta|^\gamma + c''|\theta|^{\gamma + (q-p)\frac{1+\alpha}{1-p}} = c'|\theta|^\gamma + c''|\theta|^\alpha.$$

Hence $\lim_{\theta\to 0} \Lambda'_{3.2}(\theta) = 0$ and, finally, $\lim_{\theta \to 0} \Lambda_3(\theta) = 0$.

Collecting the results of points 3.), 4.) and 5.) we have the proof. □

# Chapter 3.

## Functional of the gradient

# 3.1. INTRODUCTION

A well known result in the framework of integrals of multifunctions, Olech's Lemma, gives a condition implying strong convergence out of a very weak form of convergence to extreme points ([O1], [AR]). Namely, if $e$ is an extreme point of the integral of a multifunction there is a unique integrand in the multifunction that gives $e$; moreover if $u$ and $v$ are arbitrary selections and $\int u$ and $\int v$ are sufficiently close to $e$, then $u$ and $v$ are close to each other in $L^1$. Hence this result exhibits a condition (extremality) which implies both uniqueness and continuous dependence.

Purpose of the work reported in this section is to investigate a similar property in the context of the calculus of variations. More precisely we consider the problem of minimizing a functional of the gradient under linear boundary conditions:

$$\mathcal{P}_a: \qquad \text{Minimize} \int_\Omega g(\nabla u(x))dx; \qquad u \in \langle a, \cdot \rangle + W_0^{1,1}(\Omega); \quad (\Omega \subset \mathbb{R}^n)$$

and study the dependence on $a \in \mathbb{R}^n$ of the solutions to $\mathcal{P}_a$.

Analogously to the case of Olech's Lemma, which infers strong convergence of the selections from the convergence of their integrals to the extreme points of the integral of the multifunction, here we have a vector parameter $a$ playing the role of the integral, in the sense that the location of $(a, g^{**}(a))$ with respect to the facial structure of the epigraph of $g^{**}$ (the bipolar of $g$) determines whether or not continuous dependence of the solutions of $\mathcal{P}_a$ with respect to boundary data holds.

As shown in [C1] and in [C2] uniqueness for problem $\mathcal{P}_a$ holds if and only if the dimension $d$ of the face of the epigraph of $g^{**}$ to whose relative interior $(a, g^{**}(a))$ belongs is strictly less than $n$, the dimension of the space, and in this case the solution is $u_a = \langle a, \cdot \rangle$. Hence we might ask the following question: given a point $a$ such that the previous uniqueness condition holds, is it true that whenever a point $a'$ is sufficiently close to $a$, solutions of $\mathcal{P}_{a'}$ are close to $u_a$ in $W^{1,1}$? This is certainly true in a special case: assume indeed that, given $a$, there exists a neighbourhood $U$ of $a$ such that: for any point $a'$ in $U$, $\mathcal{P}_{a'}$ has the unique solution $u_{a'}$; in this case continuous dependence follows from the explicit form of the solutions.

Hence the problem arises whenever the point $a$ is such that $\mathcal{P}_a$ admits the unique solution $u_a$ and there are points $a_k$, arbitrarily close to $a$, for which the corresponding problem $\mathcal{P}_{a_k}$ has infinitely many solutions. This happens when $(a_k, g^{**}(a_k))$ belongs to an $n$-dimensional face $F$ of $epi(g^{**})$ and $(a, g^{**}(a))$ belongs to a face $F_1$ of dimension less than $n$ contained in the relative boundary of $F$. It is to this case we shall refer in our main result, according to which the following conditions are equivalent:

i) all the solutions $u^k$ of $\mathcal{P}_{a_k}$ are close to $u_a$ in $W^{1,1}$ whenever $a_k$ is close to $a$;

ii) $(a, g^{**}(a))$ is an extreme point of the epigraph of $g^{**}$.

As such our result is the exact *replica* to Olech's Lemma, but it is not true, in general, that uniqueness implies always continuous dependence. Indeed uniqueness holds whenever the dimension $d$ of the face $F_1$ is in $\{0, 1, ..., n-1\}$ while for $d = 1, ..., n-1$ continuous dependence does not hold. Hence our result provides a characterization of extreme points in the sense that whenever $\mathcal{P}_{a_k}$ admits solutions different from the affine one (i.e. when $(a_k, g^{**}(a_k))$ belongs to an $n$-dimensional face) and $a_k \to a$, then a sequence $\{u^k\}_{k \in \mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ converges strongly to $u_a$ if and only if $(a, g^{**}(a))$ is extremal. Moreover our result provides a precise definition of the type of convergence (partially weak, partially strong) that occurs for $1 \leq d \leq n-1$.

The previous analysis applies in particular to the special case of a rotationally symmetric function $g$. In Remark 4.2 we present a detailed description of this case.

## 3.2. PRELIMINARIES AND NOTATIONS

In this section we study the solutions of the problems:

$$\mathcal{P}_a : \qquad \text{Minimize} \int_\Omega g(\nabla u(x))dx; \qquad u \in u_a + W_0^{1,1}(\Omega);$$

$$\mathcal{P}_a^{**} : \qquad \text{Minimize} \int_\Omega g^{**}(\nabla u(x))dx; \qquad u \in u_a + W_0^{1,1}(\Omega);$$

where $g$ is a lower semicontinuous (l.s.c.), not necessarily convex, function defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ bounded from below and $g^{**}$ is its bipolar (see [ET] for a definition). $\Omega$ is an open, bounded subset of $\mathbb{R}^n$ with piecewise $C^1$ boundary and $u_a \equiv \langle a, x\rangle$ $(a \in \mathbb{R}^n)$. Here and in the following $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$ and $|\cdot|$ the associated norm. A point in $\mathbb{R}^n \times \mathbb{R}$ is denoted as a pair $(x, z)$ with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. We use the spaces $L^1(\Omega)$ and $W_0^{1,1}(\Omega)$ endowed with the usual norms $\|\cdot\|_{L^1(\Omega)}$, $\|u\|_{W_0^{1,1}(\Omega)} = \|\nabla u\|_{L^1(\Omega)}$. The weak convergence in such spaces is denoted with the half arrow $\rightharpoonup$.

For $S$ subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, $dist(x, S)$ is the distance of $x$ from $S$, $S^c$ is the complement, $co(S)$ is the convex hull and $\mu(\cdot)$ is the Lebesgue measure . When 0 belongs to $S$, the smallest linear manifold containing $S$ is denoted with $span(S)$; the dimension of an affine set is the dimension of the subspace parallel to it, and we say that a subset $S$ of $\mathbb{R}^n$ has dimension $p$ if the dimension of the affine hull of $S$ is $p$, and write $dim(S) = p$. For a scalar function $f$ we define the negative and the positive parts $f^- \equiv \max(-f, 0)$ and $f^+ \equiv \max(f, 0)$.

We make use in this paper of basic elements of convex analysis such as the notions of face, extreme point of a convex set, relative boundary (*r.b.*), relative interior (*r.i.*) and polytope, following the notations contained in [R]; we call $extr(C)$ the set of extreme points of a convex set $C$.

Given a subset $S$ of $\mathbb{R}^n \times \mathbb{R}$ we denote by $\hat{S}$ the projection of $S$ on $\mathbb{R}^n$, i.e.: $\hat{S} = \{x \in \mathbb{R}^n : \exists z \in \mathbb{R} : \ (x, z) \in S\}$.

The study of problems $\mathcal{P}_a$ and $\mathcal{P}_a^{**}$ involves the properties of the epigraph of $g^{**}$, $epi(g^{**})$, which is a convex subset of $\mathbb{R}^n \times \mathbb{R}$; we recall now some properties of the epigraph of a convex function.

**Proposition 3.2.1.** (see [C1], [C2]) *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function, then:*

*i) the collection of the relative interior of the faces of epi(h) is a partition of epi(h).*

*ii) If F is a face of epi(h) containing a point (x,h(x)) in its relative interior, F is a proper face and dim(F)$\leq$ n; moreover dim(F)= dim($\hat{F}$).*

*iii) If $F_1$ is a proper face of a proper face $F$ of epi(h) and r.i.$(F_1)$ contains a point $(x, h(x))$, then $\hat{F}_1$ is a proper face of $\hat{F}$. Moreover a point (x,h(x)) is an extreme point of epi(h) if and only if x is an extreme point of all the projections of the faces that contain (x,h(x)).*

*Proof.* Statement *i)* is a particular case of Theorem 18.2 of [R]. To prove *ii)* we simply notice that $(x, h(x))$ cannot belong to the relative interior of $epi(h)$, then $dim(F) \leq n$; moreover $F$ cannot contain a point $(x, z)$ with $z > h(x)$ hence $dim(F) = dim(\hat{F})$. Statement *iii)* is trivial. □

We shall need the following characterization of faces of a convex set, see [O1].

**Lemma 3.2.1.** *Let $F$ be a convex subset of $\mathbb{R}^n$ and $F_d$ a d-dimensional face of $F$ such that 0 belongs to r.i.$(F_d)$. Then there exist n-d orthonormal vectors $h_1, ..., h_{n-d}$ such that $F$ is contained in the cone*

$$C := \{x : \langle h_1, x\rangle > 0\} \bigcup \{x : \langle h_1, x\rangle = 0,\ \langle h_2, x\rangle > 0\} \bigcup ...$$

$$... \bigcup \{x : \langle h_1, x\rangle = \langle h_2, x\rangle = ... = \langle h_{n-d-1}, x\rangle = 0,\ \langle h_{n-d}, x\rangle > 0\} \bigcup$$

$$\bigcup \{x : \langle h_1, x\rangle = \langle h_2, x\rangle = ... = \langle h_{n-d}, x\rangle = 0\}$$

*and*

$$F_d = F \bigcap \{x : \langle h_1, x\rangle = \langle h_2, x\rangle = ... = \langle h_{n-d}, x\rangle = 0\}.$$

We remind now well known criteria of weak convergence in $L^1(\Omega)$ and $W_0^{1,1}(\Omega)$.

**Theorem 3.2.1.** ([D1], p. 19) *Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, and $\{f_k\}_{k\in\mathbb{N}}$ a sequence in $L^1(\Omega)$, then*

$$f_k \rightharpoonup f \quad \text{in } L^1(\Omega)$$

*if and only if*
*i)* $\|f_k\|_{L^1(\Omega)} \leq M$,
*ii) $f_k$ is absolutely equiintegrable,*
*iii)* $\lim_{k\to\infty} \int_D [f_k(x) - f(x)]dx = 0$ *for any cube $D \subset \Omega$.*

**Theorem 3.2.2.** ([B], p. 175) *Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, and $\{f_k\}_{k\in\mathbb{N}}$ a sequence in $W_0^{1,1}(\Omega)$, then*

$$f_k \rightharpoonup f \quad \text{in } W_0^{1,1}(\Omega)$$

*if and only if*

$$D_i f_k \rightharpoonup D_i f \quad \text{in } L^1(\Omega).$$

*for $i = 1, ..., n$.*

We end this section with a

**Definition 2.1** We say that a function $g : \mathbb{R}^n \to \mathbb{R}$ satisfies the growth condition (C) if there exists a function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty$$

and

$$g(y) \geq \phi(|y|) \quad \text{for any } y \in \mathbb{R}^n.$$

## 3.3. EXISTENCE AND UNIQUENESS

In [C1] and [C2] the author gives sufficient and necessary conditions on the affine boundary datum $u_a$ for the existence and the uniqueness of solutions of $\mathcal{P}_a$ and $\mathcal{P}_a^{**}$ investigating the facial structure of the epigraph of $g^{**}$. The main results stated in the quoted papers can be summarized as follows. We emphasize that a solution of $\mathcal{P}_a$ is a solution of $\mathcal{P}_a^{**}$ too.

**Theorem 3.3.1.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be l.s.c. (not necessarily convex), bounded from below, satifying growth condition (C); let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with piecewise $C^1$ boundary.*

*i) If $\mathcal{P}_a$ admits a solution, then*

(1.) *either $g(a) = g^{**}(a)$ or the face of epi($g^{**}$) to whose relative interior $(a, g^{**}(a))$ belongs has dimension $n$.*

*ii) Conversely, if condition* (1.) *holds then $\mathcal{P}_a$ admits at least one solution.*

**Theorem 3.3.2.** *Assume the hypotheses of Theorem* 3.1. *Then*

*i) $\mathcal{P}_a^{**}$ admits the unique solution $u_a$ if and only if $(a, g^{**}(a))$ belongs to the relative interior of a face of epi($g^{**}$) of dimension strictly less than $n$.*

*ii) $\mathcal{P}_a$ admits the unique solution $u_a$ if and only if $g^{**}(a) = g(a)$ and $(a, g^{**}(a))$ belongs to the relative interior of a face of epi($g^{**}$) of dimension strictly less than $n$.*

The proof of the second part of theorem 3.1 consists essentially of the explicit construction of the solution of $\mathcal{P}_a$ in the case in which $(a, g^{**}(a))$ belongs to the relative interior of an $n$-dimensional face of *epi*($g^{**}$). Since in the proof of our main result we need this construction, we recall it in its main steps and refer to [C2] for details.

We begin with a

**Lemma 3.3.1.** *Let $\{y_i,\ i = 1, ..., m\}$ be a set of vectors in $\mathbb{R}^n$, and let $S$ be the set $co\{y_i,\ i = 1, ..., m\}$. Suppose dim(S)=n, $0 \in int(S)$ and call $S^*$ the polar set of $S$. Then there exists a finite partition $\{S_i^*,\ i = 1, ..., m\}$ of $S^*$ and a Lipschitz continuous function $w$, defined on $\mathbb{R}^n$, such that:*

*i) $w=0$ on $(S^*)^c$;*

*ii) $\nabla w = y_i$ a.e. in $S_i^*$, i=1,...,m;*

*iii) there exists an index set I contained in $\{1, ..., m\}$ such that the set $\{y_i, i \in I\}$ contains a system of $n$ linearly independent vectors and $\mu(S_i^*) > 0$ for $i \in I$.*

*Proof.* The proof of *i)* and of *ii)* can be found in [C2]. To prove statement *iii)* we recall that since 0 belongs to the interior of $S$, $m > n$, the polar $S^*$ is bounded and it can be written as

$$S^* = \bigcap_{i=1}^{m} \{x : \langle y, x \rangle \leq 1\}$$

and that the sets $S_i^*$ are defined by

$$S_i^* := co\{F_i^*, 0\},$$

where $F_i^* = S^* \cap \{x : \langle y_i, x \rangle = 1\}$. Since $dist(0, F_i^*) > 0$ for any index $i$, $\mu(S_i^*) > 0$ if and only if $dim(F_i^*) = n - 1$. $S^*$ has at least $n + 1$ faces of dimension $n - 1$, hence we may assume, renaming the indices, that there exists $p \geq n+1$ such that $dim(F_i^*) = n-1$ for $i = 1, ..., p$ and $dim(F_i^*) < n-1$ for $i = p+1, ..., m$. A face $F_j^*$ with $j > p$ is a proper face of a face $F_i^*$ with $i < p$, hence $S_j^* \subset S_i^*$, and we can write

$$S^* = \bigcap_{i=1}^{p} \{x : \langle y_, x\rangle \leq 1\} .$$

Then the set $\{y_i, i = 1, ..., p\}$ contains a system of $n$ linearly independent vectors since otherwise $S^*$ would be unbounded. □

*Proof of Theorem* 3.1 *ii).* Assume that $(a, g^{**}(a))$ belongs to r.i.$(F)$, the relative interior of $F$, where $F$ is an $n$-dimensional face of $epi(g^{**})$. Our aim is to construct a solution of $\mathcal{P}_a$ (different from $u_a$). Since $g$ satisfies the growth condition (C), $F$ is bounded and is contained in a hyperplane $H$ separating it from $epi(g^{**})$. According Proposition 2.1 $H$ cannot be vertical i.e. $H = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : \ z = \langle h, x\rangle + k\}$ ($h \in \mathbb{R}^n$, $k \in \mathbb{R}$) and, since the extreme points of $F$ are of the form $(y, g(y))$,

$$extr(\hat{F}) = \{y \in \mathbb{R}^n : \ (y, g(y)) \in extr(F)\} .$$

Consider a subset $\{y_i, \ i = 1, ..., m\}$ of $extr(\hat{F})$ such that $dim(co\{y_i, \ i = 1, ..., m\}) = n$ and $a \in r.i.(co\{y_i, \ i = 1, ..., m\})$; we remark that whenever

$$a = \sum_{i=1}^{m} \lambda_i y_i, \ \ 0 < \lambda_i < 1, \ \ \sum_{i=1}^{m} \lambda_i = 1, \tag{3.1}$$

it is

$$g^{**}(a) = \langle h, a\rangle + k = \sum_{i=1}^{m} \lambda_i(\langle h, y_i\rangle + k) = \sum_{i=1}^{m} \lambda_i g^{**}(y_i) = \sum_{i=1}^{m} \lambda_i g(y_i); \tag{3.2}$$

and define the polytope $S(a) := co\{y_i - a, \ i = 1, ..., m\}$. We can apply Lemma 3.1, defining a partition $\{S_i^*(a), \ i = 1, ..., m\}$ of $S^*(a)$ and a Lipschitz function $w^a$ such that $w^a = 0$ on $(S^*(a))^c$ and $\nabla w^a = y_i - a$ a.e. on $S_i^*(a)$.

We now consider the collection of subsets of $\Omega$

$$\mathcal{U} = \{z + rS^*(a), \ z \in \Omega, \ r \in \mathbb{R}, \ r < dist(z, \Omega^c)\} ,$$

$\mathcal{U}$ is a Vitali covering of $\Omega$ and we can select a countable subcovering $\{\Omega_j(a), \ j \in \mathbb{N}\}$ such that:

1) $\Omega_j(a) = z_j + r_j S^*(a) \subset \Omega, \quad \forall j \in \mathbb{N}$;
2) $\Omega_j(a) \bigcap \Omega_k(a) = \emptyset, \quad \text{if } j \neq k$;
3) $\Omega = N \bigcup (\bigcup_{j=1}^{\infty} \Omega_j(a))$ where $\mu(N) = 0$;
4) $\Omega_j(a) = \bigcup_{i=1}^{m} \Omega_j^i(a)$ (disjoint union); where $\Omega_j^i(a) = z_j + r_j S_i^*(a)$.

We set also $\Omega^i(a) = \bigcup_{j=1}^{\infty} \Omega_j^i(a)$, obtaining $\Omega = \bigcup_{i=1}^{m} \Omega^i(a)$, and define

$$w_j^a(x) = r_j w^a(\frac{x - z_j}{r_j}) \qquad \text{a.e. } x \in \Omega_j(a), \ j \in \mathbb{N}$$

and

$$v^a(x) = \sum_{j=1}^{\infty} w_j^a(x), \quad \text{a.e. } x \in \Omega.$$

$v^a$ belongs to $W_0^{1,1}(\Omega)$ and it is

$$\nabla v^a = y_i - a \ \text{ a.e. on } \Omega^i(a), \ i = 1, ..., m \tag{3.3}$$

then

$$0 = \int_{\Omega} \nabla v^a = \sum_{i=1}^{m} \int_{\Omega^i(a)} (y_i - a),$$

i.e.

$$a = \sum_{i=1}^{m} \frac{\mu(\Omega^i(a))}{\mu(\Omega)} y_i. \tag{3.4}$$

We set $u(x) := v^a(x) + \langle a, x \rangle = v^a(x) + u_a(x)$; first of all (3.3) implies.

$$\nabla u = y_i \ \text{ a.e. on } \Omega^i(a), \tag{3.5}$$

and by virtue of (3.1), (3.2) and (3.4) $u$ is a solution of $\mathcal{P}_a$ □

We are interested in the following question: consider a sequence $\{a_k\}_{k \in \mathbb{N}}$ converging to a point $a$ such that $\mathcal{P}_a$ admits the unique solution $u_a$, and a sequence $\{u^k\}_{k \in \mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ ($\mathcal{P}_{a_k}^{**}$) (in the sense that for any $k \in \mathbb{N}$, $u^k$ is a solutions of $\mathcal{P}_{a_k}$ ($\mathcal{P}_{a_k}^{**}$)): we ask whether $\{u^k\}_{k \in \mathbb{N}}$ converges (in some topology) to $u_a$ as $k \to \infty$. When it happens that, for any $k \in \mathbb{N}$, $(a_k, g^{**}(a_k))$ belongs to the relative interior of a face of dimension strictly less than $n$, the question is trivial because $u^k \equiv u_{a_k}$, and converges to $u_a$ strongly in $W_0^{1,1}(\Omega)$. The interesting case is when $(a, g^{**}(a))$ belongs to the relative interior of a face $F_1$ of $epi(g^{**})$ of dimension strictly less than $n$, and, for an infinite number of indices $k \in \mathbb{N}$, $(a_k, g^{**}(a_k))$ belongs to the relative interior of at least one $n$-dimensional face of $epi(g^{**})$ containing $(a, g^{**}(a))$ (and also $F_1$) in its relative boundary. According to our main result $(a, g^{**}(a))$ is an extreme point of $epi(g^{**})$ if and only if any sequence $\{u^k\}_{k\in\mathbb{N}}$ converges strongly to $u_a$ in $W_0^{1,1}(\Omega)$.

## 3.4. CONTINUITY WITH RESPECT TO BOUNDARY DATA

We shall need the following technical lemmas.

**Lemma 3.4.1.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and $\{v^k\}_{k\in\mathbb{N}}$ a sequence in $W_0^{1,1}(\Omega)$. Suppose that $v^k \to 0$ in $L^1(\Omega)$ and that, for some $i \in \{1, ...n\}$ $|D_i v^k| \leq M$ a.e. in $\Omega$, where $M$ is a positive constant. Then*

$$D_i v^k \rightharpoonup 0 \quad \text{in } L^1(\Omega)$$

*Proof.* We can suppose $i = 1$ and $\Omega = I \times S$, where $I$ is an open bounded interval of $\mathbb{R}$ and $S$ an open bounded subset of $\mathbb{R}^{n-1}$, since $v^k$ can be extended as zero out of $\Omega$. Let us write $v^k = v^k(x_1, x')$ with $x_1 \in I$ and $x' \in S$; the uniform boundedness of $|D_1 v^k|$ implies that the sequence $\{D_1 v^k\}_{k\in\mathbb{N}}$ is bounded in $L^1$-norm and is absolutely equiintegrable. According to Theorem 2.1 it is sufficient to prove that for any cube $D \subset \Omega$

$$\lim_{k\to\infty} \int_D D_1 v^k \to 0.$$

(1). Suppose first $v^k \in C_0^1(\Omega)$ and define

$$\varphi^k(x_1) = \int_S |v^k(x_1, x')| dx', \quad k \in \mathbb{N}, \ x_1 \in I,$$

$\{\varphi^k\}_{k\in\mathbb{N}}$ is a sequence of nonnegative continuous functions on $I$; They are differentiable a.e. for any $k$ and the sequence of derivatives is uniformly bounded, hence they are equicontinuous and equibounded. Moreover $\lim_{k\to\infty} \|\varphi^k\|_{L^1(I)} = \lim_{k\to\infty} \|v^k\|_{L^1(\Omega)} = 0$, then $\varphi^k \to 0$ uniformly on $I$.

Consider a cube $D \subset \Omega$, $D = (\xi, \eta) \times Q$ where $\xi, \eta \in I$ and $Q$ is an $(n-1)$-dimensional cube contained in $S$. It is

$$\left| \int_D D_1 v^k(x_1, x') dx_1 dx' \right| = \left| \int_Q \left( \int_\xi^\eta D_1 v^k(x_1, x') dx_1 \right) dx' \right| \leq$$

$$\leq 2 \sup_{x_1 \in I} \left( \int_S |v^k(x_1, x')| dx' \right).$$

Hence $D_1 v^k \rightharpoonup 0$ in $L^1(\Omega)$.

(2). Consider now the general case $v^k \in W_0^{1,1}(\Omega)$.

By density there exists a sequence $w^k \in C_0^1(\Omega)$ such that $|D_1 w^k|$ is uniformly bounded in $\Omega$ and

$$\|v^k - w^k\|_{W_0^{1,1}} \leq \frac{1}{k} \quad \forall k \in \mathbb{N}.$$

Obviously $w^k \rightarrow 0$ in $L^1(\Omega)$ and the previous arguments show that $D_1 w^k \rightharpoonup 0$ in $L^1(\Omega)$; hence $D_1 v^k \rightharpoonup 0$ in $W_0^{1,1}(\Omega)$. □

**Lemma 3.4.2.** *Let $P$ be a polytope in $\mathbb{R}^n$ and $F$ a proper face of $P$. Then $F$ is exposed, i.e. there exists a supporting hyperplane $\pi$ of $P$ such that $F = P \cap \pi$.*

*Proof.* We can as well assume $0 \in F$. Set $P = co\{v_1, ..., v_m\}$ and $V = \max\{|v_1|, ..., |v_m|\}$. Consider the collection of all nontrivial hyperplanes $H_\alpha$ separating $F$ from $P$. Let $\nu_\alpha$ be the number of vectors $v_1, ..., v_m$ contained in $H_\alpha$ but not belonging to $F$ and call $\nu_0$ the minimum, attained for some hyperplane $H_0$ defined by $H_0 = \{x : \langle h_0, x \rangle = 0\}$; we wish to show that $\nu_0 = 0$. Assume, by contradiction, that it is positive. Set $P_0$ to be $P \cap H_0$. Notice that there is $\eta > 0$ such that for every $v_i$ in $P$ but not in $P_0$, $\langle h_0, v_i \rangle \geq \eta$. Also, $F \cap H_0$ is a proper face of $P \cap H_0$, so that there is a unit vector $k$ in $H_0$ separating $F \cap H_0$ from $P \cap H_0$ i.e. $\langle k, x \rangle = 0$ for $x \in F \cap H_0$ and for some $y$ in $P \cap H_0$, $\langle k, y \rangle > 0$. Since $y \in co\{v_1, ..., v_m\}$, there is $v_j$ in $P \cap H_0$ such that $\langle k, v_j \rangle > 0$. Consider $h_1 = h_0 + \frac{\eta}{2V} k$. We have that $\langle h_1, x \rangle = 0$ for $x \in F$, that for $v_i$ in $P$ but not in $P \cap H_0$, $\langle h_1, v_i \rangle \geq \eta - \frac{\eta}{2V} |v_i| \geq \frac{\eta}{2}$ and that $\langle h_1, v_j \rangle > 0$, contadicting the definition of $\nu_0$. □

The following is our main result; it is convenient to introduce the following definition.

**Definition 3.4.1.** Let $\{v^k\}_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,1}(\Omega)$ and $v \in W_0^{1,1}(\Omega)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$. Let $d$ be the largest integer such that there exists a $d$-dimensional subspace $L$ of $\mathbb{R}^n$ such that, given any vector $e$ in $L$, it is

$$\langle \nabla v^k - \nabla v, e \rangle \overset{k\to\infty}{\longrightarrow} 0 \quad \text{in} \quad L^1(\Omega).$$

We say that $\{v^k\}_{k\in\mathbb{N}}$ *converges d-strongly* to $v$ in $W_0^{1,1}(\Omega)$.

**Remark 3.4.1.** In order to prove that a sequence converges $d$-strongly, it is sufficient to find a system $E$ of $d$ independent vectors such that the condition expressed in definition 4.1 holds and that for any vector $e$ in the orthogonal complement of $E$, $\langle \nabla v^k - \nabla v, e \rangle$ does not converge to zero in $L^1(\Omega)$.
We should also notice that $\{v^k\}_{k\in\mathbb{N}}$ is a $d$-strongly converging sequence in $W_0^{1,1}(\Omega)$ if and only if there exists a nonsingular change of coordinates $U$ such that, setting $w^k(x) = v^k(Ux)$, $D_j w^k$ converges strongly in $L^1(U\Omega)$ for $j = 1, ..., d$ and does not converge in $L^1(U\Omega)$ for $j = d+1, ..., n$. Obviously $d$-strong convergence in $W_0^{1,1}(\Omega)$ implies strong convergence in $L^1(\Omega)$ for any $d \geq 1$ (Poincaré inequality) and it is equivalent to strong convergence in $W_0^{1,1}(\Omega)$ when $d = n$.

The following is the main result of this chapter

**Theorem 3.4.1.** *Let $g$ be l.s.c. satisfying the growth condition* (C), *let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with piecewise $C^1$ boundary. Suppose that $g(a) = g^{**}(a)$ and that $(a, g^{**}(a))$ belongs to the relative interior of a proper face $F_1$ of an $n$-dimensional face $F$ of epi$(g^{**})$, and let $\{a_k\}_{k\in\mathbb{N}}$ be any sequence such that $(a_k, g^{**}(a_k))$ belongs to the relative interior of $F$ for any $k \in \mathbb{N}$ and* $\lim_{k\to\infty} a_k = a$.

- *i) If any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ converges $(n-r)$-strongly to $u_a$ in $W_0^{1,1}(\Omega)$, then dim$(F_1) = r$. In particular, if any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ converges strongly to $u_a$ in $W_0^{1,1}(\Omega)$, then $(a, g^{**}(a))$ is an extreme point of epi$(g^{**})$.*
- *ii) If dim$(F_1) = r$ then any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}^{**}$ converges $(n-r)$-strongly to $u_a$ in $W_0^{1,1}(\Omega)$. In particular, if $(a, g^{**}(a))$ is an extreme point of epi$(g^{**})$, then any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}^{**}$ converges strongly to $u_a$ in $W_0^{1,1}(\Omega)$.*

*Proof.* First of all, since $dim(F_1) < n$ and $g(a) = g^{**}(a)$, $\mathcal{P}_a$, as well as $\mathcal{P}_a^{**}$, admits the unique solution $u_a$; moreover growth condition (C) implies that $\hat{F}$ is bounded and we set

$$L = \sup_{y \in \hat{F}} |y|.$$

In the proof we assume, without loosing generality, $a = 0$.

*i)* Suppose that any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ converges $(n-r)$-strongly to zero. We proceed by contradiction: we assume that $dim(\hat{F}_1)$ is greater than $r$ and show that there exists a sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}$ that does not converge $(n-r)$-strongly to zero in $W_0^{1,1}(\Omega)$.

(1.) Set $d = r + 1$ and consider $y_1, ..., y_p \in extr(\hat{F}_1)$ $(p \geq d+1)$ such that $a = 0 \in r.i.(co\{y_1, ..., y_p\})$ and $dim(span\{y_1, ..., y_p\}) = d$. Since $a_k \in r.i.(\hat{F})$, for any $k$ there are $n+1$ extreme points of $\hat{F}$, $v_{p+1}^k, ..., v_{p+n+1}^k$ such that, setting $m = p+n+1$, the polytope $P_k := co\{y_1, ..., y_p, v_{p+1}^k, ..., v_m^k\}$ has dimension $n$ and $a_k \in int(P_k)$. Setting $y_i^k = y_i - a_k$ for $i = 1, ..., p$ and $y_i^k = v_i^k - a_k$ for $i = p+1, ..., m$, and considering the polytope $P_k - a_k := co\{y_i^k, i = 1, ..., m\}$, we can define the (bounded) polar $S^*(a_k)$ of $P_k - a_k$ and a solution $u^k$ of $\mathcal{P}_{a_k}$, defined as in section 3, whose gradient, recalling (3.5), takes the values $y_i$ or $v_i^k$ on the sets $\Omega^i(a_k)$.

(2.) Extracting if necessary subsequences, we may assume that $y_i^k$ converges to a vector $y_i \in \partial\hat{F}$ for $i = p+1, ..., m$ and $y_i \in \hat{F}_1$ for $i = p+1, ..., s$, $y_i \in \partial\hat{F} \setminus \hat{F}_1$ for $i = s+1, ..., m$ where $p \leq s \leq m$. We define the limit polytope $P := co\{y_i, i = 1, ..., m\}$ and its polar $S^*$ written as

$$S^* = \bigcap_{i=1}^{m} \{x : \langle y_i, x\rangle \leq 1\}.$$

We define also

$$C^* = \bigcap_{i=1}^{s} \{x : \langle y_i, x\rangle \leq 1\}, \quad T^* = \bigcap_{i=s+1}^{m} \{x : \langle y_i, x\rangle \leq 1\},$$

so that $S^* = C^* \bigcap T^*$, and remark that when $s = m$, $T^* = \mathbb{R}^n$ and $S^* = C^*$; in the following we shall consider $s < m$ since otherwise the proof proceeds in a similar and simpler way. Recalling the definition of the partition of a polar set (Lemma 3.1) we can apply the same definition to $C^*$ obtaining $C^* = \bigcup_{i=1}^{s} C_i^*$ and $S_i^* \subseteq C_i^*$.

Set $L_d$ to be $span\{y_i, i = 1, ..., s\}$ and write $\mathbb{R}^n = L_d \oplus L_d^{\perp}$; the cylinder $C^*$ can be written $C^* = S^{d*} \oplus L_d^{\perp}$, and, analogously, $C_i^* = S_i^{d*} \oplus L_d^{\perp}$; where we have set $S^{d*} := C^* \bigcap L_d$ and $S_i^{d*} := C_i^* \bigcap L_d$. We remark that $S^{d*}$ is the polar of the $d$-dimensional set

$co\{y_1,...,y_s\} \cap L_d$ and $\{S_i^{d*}, i = 1,...,s\}$ is the relative partition; since $0 \in r.i.(co\{y_i, i = 1,...,s\})$, $S^{d*}$ is bounded and by point *iii)* of Lemma 3.1 there exists a set $I$ of $d$ indices such that $\{y_i, i \in I\}$ are linearly independent and, calling $\mu_d$ the $d$-dimensional measure in $L_d$,

$$\frac{\mu_d(S_i^{d*})}{\mu_d(S^{d*})} \geq \lambda > 0 \qquad \text{for } i \in I. \tag{4.1}$$

Moreover, boundedness of $S^{d*}$ implies that there exists $M$ $(M \geq -1)$ such that

$$C^* = \bigcap_{i=1}^{s} \left[ \bigcup_{\alpha_i \in [-M,1]} \{x : \langle y_i, x \rangle = \alpha_i\} \right]. \tag{4.2}$$

(3.) Consider now $P_d := co\{y_i, i = 1,...,s\}$: $P_d$ is a face of $P$ and by Lemma 4.2 it is exposed. Let $w$ be a unit vector in $L_d^\perp$ such that $P_d = P \bigcap \{x : \langle w, x \rangle = 0\}$ and $\langle w, x \rangle > 0$ for all $x \in P$. Let $z_{d+1},...,z_n$ be orthonormal vectors in $L_d^\perp$ such that $w = (n-d)^{-\frac{1}{2}} \sum_{j=d+1}^n z_j$. For every $y \in \{y_{s+1},...,y_m\}$, it is $\sum_{j=d+1}^n \langle z_j, y \rangle = (n-d)^{\frac{1}{2}} \langle w, y \rangle > 0$.

(4.) Let us define the family of sets

$$Q_{[R_1,R_2]} := \bigcap_{j=d+1}^{m} \{x : \langle z_j, x \rangle \in [R_1, R_2]\} \quad R_1, R_2 \in \mathbb{R};$$

it is our purpose to show that there exist $R_0 > 0$ and $\alpha > 1$ such that for any $R > R_0$ it is

$$S^* \bigcap Q_{[-\alpha R, -R]} = C^* \bigcap Q_{[-\alpha R, -R]}. \tag{4.3}$$

Since $S^* = C^* \bigcap T^*$ it is enough to prove that the r.h.s. is contained in the l.h.s. (when $s = m$, i.e. $S^* = C^*$, this is obvious). We show that in general $C^* \bigcap Q_{[-\alpha R,-R]}$ is contained in $T^*$: it follows that a point in $C^* \bigcap Q_{[-\alpha R,-R]}$ is in $C^* \bigcap T^* = S^*$, hence in $S^* \bigcap Q_{[-\alpha R,-R]}$. So, let $x$ be any point in $C^* \bigcap Q_{[-\alpha R,-R]}$; recalling (4.2), we have

$$\langle y_i, x \rangle \in [-M, 1], \quad i = 1,...,s$$

$$\langle z_j, x \rangle \in [-\alpha R, -R], \quad j = s+1,...,m.$$

Take $y \in \{y_i, i = s+1,...,m\}$; since $\{y_1,...,y_s, z_{d+1},...,z_n\}$ contains a system of $n$ independent vectors, $y$ can be written as: $y = \sum_{i=1}^s \nu_i y_i + \sum_{j=d+1}^n \mu_j z_j$, where, by point (3.), $\sum_{j=d+1}^n \mu_j > 0$. Writing $\mu_j^+ = \max(\mu_j, 0)$ and $\mu_j^- = \max(-\mu_j, 0)$ we have, recallig (4.2),

$$\langle y, x \rangle = \sum_{i=1}^s \nu_i \langle y_i, x \rangle + \sum_{j=d+1}^n \mu_j \langle z_j, x \rangle \leq (|M|+1) \sum_{i=1}^s |\nu_i| + \left( -\sum_{j=d+1}^n \mu_j^+ + \alpha \sum_{j=d+1}^n \mu_j^- \right) R.$$

By choosing $\alpha$ satisfying $1 < \alpha < (\sum_{j=d+1}^{n} \mu_j^+)(\sum_{j=d+1}^{n} \mu_j^-)^{-1}$, the term in parenthesis becomes negative, hence, if $R$ is greater than some $R_0$ sufficiently large, it turns out that $\langle y, x\rangle \leq 1$. Repeat this choice for every $y \in \{y_i, i = s+1, ..., m\}$; take $R_0$ as the largest value and $\alpha$ as the smallest value so obtained, and have $x \in T^*$. By an analogous procedure we have also

$$S_i^* \bigcap Q_{[-\alpha R, -R]} = C_i^* \bigcap Q_{[-\alpha R, -R]}. \tag{4.4}$$

(5.) We take now $R \geq R_0$ and consider the sets $S^* \bigcap Q_{[-\alpha R, R]}$ and $S_i^* \bigcap Q_{[-\alpha R, R]}$ for $i = 1, ..., s$. Such sets are bounded; by (4.3) and (4.4), we have

$$\mu\left(S^* \bigcap Q_{[-\alpha R, R]}\right) \leq \mu\left(C^* \bigcap Q_{[-\alpha R, R]}\right) = \mu_d(S^{d*})((\alpha+1)R)^{n-d},$$

and

$$\mu\left(S_i^* \bigcap Q_{[-\alpha R, R]}\right) = \mu\left(S_i^* \bigcap Q_{[-\alpha R, -R]}\right) + \mu\left(S_i^* \bigcap Q_{[-R, R]}\right) =$$
$$\mu\left(C_i^* \bigcap Q_{[-\alpha R, -R]}\right) + \mu\left(S_i^* \bigcap Q_{[-R, R]}\right) \geq \mu_d(S_i^{d*})((\alpha-1)R)^{n-d}.$$

Hence, recalling (4.1),

$$\frac{\mu\left(S_i^* \bigcap Q_{[-\alpha R, R]}\right)}{\mu\left(S^* \bigcap Q_{[-\alpha R, R]}\right)} \geq \frac{\mu_d(S_i^{d*})}{\mu_d(S^{d*})}\left(\frac{\alpha-1}{\alpha+1}\right)^{n-d} \geq \gamma > 0, \quad i \in I, \tag{4.5}$$

for some positive $\gamma$.

(6.) Consider now the sets $S^*(a_k) = \bigcup_{i=1}^{m} S_i^*(a_k)$, polars of $P_k - a_k$ and their decompositions. Given $Q = Q_{[R_1, R_2]}$ the sets $S^*(a_k) \cap Q$ and $S_i^*(a_k) \cap Q$ are bounded polytopes whose vertices converge to the vertices of $S^* \cap Q$ and $S_i^* \cap Q$ respectively, since the vertices of $P_k$ converges to the vertices of $P$. In particular the measures of $S^*(a_k) \cap Q$ and $S_i^*(a_k) \cap Q$ converge to the measures of $S^* \cap Q$ and $S_i^* \cap Q$. Setting

$$\gamma_i^k(R) := \frac{\mu\left(S_i^*(a_k) \bigcap Q_{[-\alpha R, R]}\right)}{\mu\left(S^*(a_k) \bigcap Q_{[-\alpha R, R]}\right)},$$

we have, by (4.5),

$$\lim_{k\to\infty} \gamma_i^k(R) = \frac{\mu\left(S_i^* \bigcap Q_{[-\alpha R, R]}\right)}{\mu\left(S^* \bigcap Q_{[-\alpha R, R]}\right)}, \quad i \in I,\ R \geq R_0. \tag{4.6}$$

The sets $S^*(a_k)$ are bounded for any $k$, hence there exists a sequence $R_k$ in $\mathbb{R}^+$ such that $R_k \nearrow +\infty$ as $k \to \infty$ and

$$\gamma_i^k(R_k) = \frac{\mu(S_i^*(a_k))}{\mu(S^*(a_k))};$$

this last equality and (4.6) imply that

$$\liminf_{k\to\infty} \frac{\mu\left(S_i^*(a_k)\right)}{\mu\left(S^*(a_k)\right)} \geq \gamma, \quad i \in I. \tag{4.7}$$

(7.) Take $i \in I$. When $i \leq p$, $y_i^k = y_i - a_k$ and $\nabla u^k = y_i$ a.e. on $\Omega^i(a_k)$, then (4.7) implies that, for $k$ sufficiently large,

$$\int_\Omega \left|\left\langle \nabla u^k(x), y_i \right\rangle\right| dx \geq \int_{\Omega^i(a_k)} |y_i|^2 \, dx =$$

$$\mu(\Omega^i(a_k))\, |y_i|^2 \geq \mu(\Omega) \frac{\mu(S_i^*(a_k))}{\mu(S^*(a_k))} |y_i|^2 \geq \mu(\Omega) \frac{\gamma}{2} |y_i|^2 .$$

When $i > p$, $y_i^k = v_i^k - a_k$ and $\nabla u^k = v_i^k$ a.e. on $\Omega^i(a_k)$; remarking that $v_i^k \to y_i$ and that $|v_i^k| \leq L$, we have, through similar computations, for $k$ sufficiently large,

$$\int_{\Omega^i(a_k)} \left\langle \nabla u^k(x), v_i^k \right\rangle dx = \int_{\Omega^i(a_k)} |v_i^k|^2 dx \geq \frac{\gamma}{4} \mu(\Omega) |y_i|^2,$$

hence

$$\int_{\Omega^i(a_k)} \left\langle \nabla u^k(x), y_i \right\rangle dx = \int_{\Omega^i(a_k)} \left\langle \nabla u^k(x), v_i^k \right\rangle dx + \int_{\Omega^i(a_k)} \left\langle \nabla u^k(x), y_i - v_i^k \right\rangle dx \geq$$

$$\frac{\gamma}{4} \mu(\Omega) |y_i|^2 - L \mu(\Omega) |y_i - v_i^k| \geq \frac{\gamma}{8} \mu(\Omega) |y_i|^2 .$$

We have shown that

$$\int_\Omega \left|\left\langle \nabla u^k(x), y_i \right\rangle\right| dx \geq \int_{\Omega^i(a_k)} \left\langle \nabla u^k(x), y_i \right\rangle dx \geq \frac{\gamma}{8} \mu(\Omega) |y_i|^2,$$

hence for any $y_i$, $i \in I$, $\left\langle \nabla u^k, y_i \right\rangle$ does not converge in $L^1(\Omega)$. Since $\{y_i, i \in I\}$ is a system of $d$ linearly independent vectors, $u^k$ cannot converge $n - d + 1 = n - r$ strongly to zero in $W_0^{1,1}(\Omega)$ and part *i)* of the theorem is proved.

*ii)* Suppose now $dim(F_1) = r$ and consider a sequence $\left\{u^k\right\}_{k \in \mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}^{**}$. We wish to show that there exist $n - r$ orthonormal vectors $h_i$ in $\mathbb{R}^n$ such that $\left\langle h_i, \nabla u^k \right\rangle$ goes to zero in $L^1(\Omega)$ as $k$ goes to infinity.

(1.) We begin by remarking that $a = 0 \in r.i.(\hat{F}_1)$ and $a_k \in r.i.(\hat{F})$; by Theorem 1 in [C1] $(\nabla u^k(x), g^{**}(\nabla u^k(x)) \in F$ and $\nabla u^k(x) \in \hat{F}$ a.e. on $\Omega$, hence $|\nabla u^k(x)| \leq L$ a.e. on $\Omega$. Recalling Lemma 2.1, let $h_1, ..., h_{n-r}$ be the vectors defining the cone $C$ such that $\hat{F} \subset C$.

It is $\langle h_1, \nabla u^k \rangle \geq 0$ a.e. in $\Omega$; writing $\nabla u^k = a_k + \nabla v^k$ with $v^k \in W_0^{1,1}(\Omega)$ it turns out that a.e. in $\Omega$

$$\langle h_1, \nabla v^k \rangle \geq - \langle h_1, a_k \rangle . \tag{4.8}$$

We extend $v^k$ setting $\tilde{v}^k = v^k$ on $\Omega$ and $\tilde{v}^k = 0$ on $\Omega^c$; $\tilde{v}^k$ is in $W^{1,1}(\mathbb{R}^n)$ with compact support. Take a basis $\{e_1, ..., e_n\}$ in $\mathbb{R}^n$ such that $e_i = h_i$, $i = 1, ..., n-r$, and write a point $\xi$ of $\mathbb{R}^n$ as $\xi = (\xi_1, ..., \xi_n) = (\xi_1, \xi')$ where $\xi_i$ is the component with respect to $e_i$. Define the functions

$$\varphi_{\xi'}(\xi_1) = \tilde{v}^k(\xi_1, \xi');$$

$\varphi_{\xi'}(\cdot)$ is a function of $W^{1,1}(\mathbb{R})$ with compact support for almost every $\xi'$ (see [Z] p. 44) and this implies that the integral of its derivative is equal to zero; since

$$\frac{d}{d\xi_1} \varphi_{\xi'}(\xi_1) = \langle h_1, \nabla \tilde{v}^k(\xi_1, \xi') \rangle$$

this means that

$$\int_{\mathbb{R}} \left( \langle h_1, \nabla \tilde{v}^k(\xi_1, \xi') \rangle \right)^- d\xi_1 = \int_{\mathbb{R}} \left( \langle h_1, \nabla \tilde{v}^k(\xi_1, \xi') \rangle \right)^+ d\xi_1 .$$

By repeated integration and by a unitary change of variables, we obtain

$$\int_{\Omega} \left( \langle h_1, \nabla v^k(x) \rangle \right)^- dx = \int_{\Omega} \left( \langle h_1, \nabla v^k(x) \rangle \right)^+ dx .$$

Remarking that the r.h.s. of (4.8) is negative, we then have

$$\int_{\Omega} \left| \langle h_1, \nabla v^k(x) \rangle \right| dx = 2 \int_{\Omega} \left( \langle h_1, \nabla v^k(x) \rangle \right)^- dx \leq 2 |a_k| \mu(\Omega),$$

and

$$\int_{\Omega} \left| \langle h_1, \nabla u^k(x) \rangle \right| dx \leq 3 |a_k| \mu(\Omega).$$

Hence $\langle h_1, \nabla u^k \rangle \overset{k \to \infty}{\to} 0$ in $L^1(\Omega)$.

(2.) Let now $\epsilon > 0$; by Egorov's Theorem there exists a compact subset $\Omega_\epsilon$ of $\Omega$ such that $\mu(\Omega \backslash \Omega_\epsilon) \leq \epsilon$ and $\langle h_1, \nabla u^k \rangle \overset{k \to \infty}{\longrightarrow} 0$ uniformly on $\Omega_\epsilon$. Let $k_\epsilon \in \mathbb{N}$ such that $|a_k| \leq \epsilon$ and $\sup_{\Omega_\epsilon} | \langle h_1, \nabla u^k \rangle | \leq \epsilon$ for any $k \geq k_\epsilon$. For $x \in \Omega_\epsilon$ and $k \geq k_\epsilon$, $\nabla u^k(x)$ belongs to an $\epsilon$-neighbourhood of $\hat{F} \cap H_1$, where $H_1 = \{x : \langle h_1, x \rangle = 0\}$. A point $y$ in an $\epsilon$-neighbourhood of $\hat{F} \cap H_1$ can be written as $y = y_1 + y_\epsilon$ where $y_1 \in \hat{F} \cap H_1$ and $|y_\epsilon| \leq \epsilon$; it is $\langle h_2, y_1 \rangle \geq 0$ and $\langle h_2, y \rangle = \langle h_2, y_1 \rangle + \langle h_2, y_\epsilon \rangle \geq \langle h_2, y_\epsilon \rangle \geq -\epsilon$. Hence $\langle h_2, \nabla u^k(x) \rangle \geq -\epsilon$

and $\langle h_2, \nabla v^k(x)\rangle \geq -\epsilon - |a_k| \geq -2\epsilon$ for any $x \in \Omega_\epsilon$. By computations analogous to those of point (1.), we obtain, for any $k \geq k_\epsilon$

$$\int_{\Omega_\epsilon} |\langle h_2, \nabla v^k(x)\rangle| \, dx \leq 4\epsilon\mu(\Omega);$$

then,

$$\int_\Omega |\langle h_2, \nabla u^k(x)\rangle| \, dx = \int_{\Omega \setminus \Omega_\epsilon} |\langle h_2, \nabla u^k(x)\rangle| \, dx + \int_{\Omega_\epsilon} |\langle h_2, \nabla u^k(x)\rangle| \, dx \leq \epsilon M + 5\epsilon\mu(\Omega).$$

Hence $\langle h_2, \nabla u^k\rangle \stackrel{k\to\infty}{\longrightarrow} 0$ in $L^1(\Omega)$.

This process can be iterated in order to show that $\langle h_i, \nabla u^k\rangle \stackrel{k\to\infty}{\longrightarrow} 0$ in $L^1(\Omega)$ for $i = 1, ..., n-r$ and this proves that $\{u^k\}_{k\in\mathbb{N}}$ converges $d$-strongly to $u_a$ in $W_0^{1,1}(\Omega)$ for some $d \geq n-r$. By point $i$), if $d > n-r$ it would be $dim(F_1) < r$, a contradiction. Hence $\{u^k\}_{k\in\mathbb{N}}$ converges $(n-r)$-strongly in $W_0^{1,1}(\Omega)$. □

**Corollary 3.4.1.** *Assume the hypotheses of Theorem 4.1. Then any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}^{**}$ converges weakly to $u_a$.*

*Proof.* Take any sequence $\{u^k\}_{k\in\mathbb{N}}$ of solutions of $\mathcal{P}_{a_k}^{**}$. The derivatives of $u^k$ are uniformly bounded a.e., and by point *ii* of Theorem 4.1, $u^k \to 0$ in $L^1(\Omega)$. Then the proof is a straightforward application of Lemma 4.1. □

**Remark 3.4.2.** (Rotational symmetry) Theorem 4.2 states that in general continuous dependence of the solutions from boundary data does not hold if the point $(a, g^{**}(a))$ is not extremal. However in one special case continuous dependence holds whenever the solution is unique: assume indeed that the function $g$ is rotationally symmetric i.e. there exists $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(\nabla u) = h(|\nabla u|)$. Two cases are possible: either $a$) $h(0) < h(r)$ for all $r > 0$ or $b$) there exists $R > 0$ such that $h(R) = h(0)$ (assume that $R$ is the largest such point). In case $a$) there are no extremal faces of $epi(g^{**})$ having dimension $n$; hence, by the previous results, uniqueness and continuous dependence hold for every boundary datum. In case $b$) there is a unique $n$-dimensional face, the ball of radius $R$, whose relative boundary consists of its extreme points; hence both uniqueness and continuous dependence hold if and only if $|a| \geq R$.

For a more general $g$ one has the following result:

**Corollary 3.4.2.** *Assume the hypotheses of Theorem 3.2.*

*i) If the point $\{a\}$ is such that $\mathcal{P}_a$ ($\mathcal{P}_a^{**}$) admits the unique solution $u_a$ and there exists a neighbourhood $U$ of $\{a\}$ such that for any $b \in U$ $(b, g^{**}(b))$ belongs to a face of epi($g^{**}$) of dimension strictly less than $n$, then continuous dependence holds in $U$.*

*ii) If the point $\{a\}$ is such that $\mathcal{P}_a$ ($\mathcal{P}_a^{**}$) admits the unique solution $u_a$ and $(a, g^{**}(a))$ belongs to the relative bondary of an $n$-dimensional face of epi($g^{**}$), then continuous dependence holds if and only if $(a, g^{**}(a))$ is an extreme point of epi($g^{**}$).*

**Remark 3.4.3.** In Theorem 4.1 we have assumed that the sequence $\{(a_k, g^{**}(a_k))\}_{k\in\mathbb{N}}$ is entirely contained in a *fixed* $n$-dimensional face $F$. We can suppose otherwise that, as $k$ goes to infinity, the sequence touches different faces of the epigraph of $g^{**}$.
In this case the proof of statement $i)$ of theorem 4.1 does not need any modification since to prove that $dim(F_1) = r$ it is sufficient to consider a subsequence of $\{(a_k, g^{**}(a_k))\}_{k\in\mathbb{N}}$ entirely contained in the relative interior of one $n$-dimensional face.
Conversely we may assume $dim(F_1) = r$ and study the behaviour of a sequence of solutions $\{u^k\}_{k\in\mathbb{N}}$ of $\mathcal{P}_{a_k}^{**}$ when $a^k \to a$ and $(a_k, g^{**}(a_k))$ belongs to more than one face. In general we may assume that there exists a finite collection $\{F_1, ..., F_q\}$ of $n$-dimensional faces that contain $(a, g^{**}(a))$ in their respective relative boundaries and such that $(a_k, g^{**}(a_k))$ belongs to to the relative interior of each of the $F_i$ for an infinite number of indices $k$. The sequence $\{u^k\}_{k\in\mathbb{N}}$ can be decomposed in the disjoint union of $q+1$ subsequences $\{u^{k_i}\}_{k_i\in\mathbb{N}}$, $i = 1, ..., q+1$ where the indices $k_1, ..., k_q$ are those ones for which $(a_{k_i}, g^{**}(a_{k_i}) \in r.i.(F_i)$ while the values $(a_{k_{q+1}}, g^{**}(a_{k_{q+1}}))$ belong to faces of dimension strictly less than $n$. Since $u^{k_{q+1}} \equiv \langle a_{k_{q+1}}, \cdot \rangle$ it converges strongly to $u_a$ in $W_0^{1,1}(\Omega)$, while the sequences $\{u^{k_i}\}_{k_i\in\mathbb{N}}$, $i = 1, ..., q$ converge $(n-r)$-strongly to $u_a$ in $W_0^{1,1}(\Omega)$ in the sense that $\langle \nabla u^{k_i}, e_j^i \rangle \stackrel{k_i \to \infty}{\longrightarrow} \langle \nabla u_a, e_j^i \rangle$ in $L^1(\Omega)$, where $E^i = \{e_1^i, ..., e_{(n-r)}^i\}$ is an orthonormal system in $(span(\hat{F}_1 - a))^\perp$ for any $i = 1, ..., q$. Hence the whole sequence $\{u^k\}_{k\in\mathbb{N}}$ converges $(n-r)$-strongly (and also weakly) to $u_a$ in $W_0^{1,1}(\Omega)$. Hence statement *ii)* of Theorem 4.1 and Corollary 4.2 remain true.

# Part II

## Chapter 4.

## An existence result in the vectorial case of the Calculus of Variation

## 4.1. INTRODUCTION

In this section we prove existence of solutions for the problem

$$\mathcal{P}: \quad \text{Minimize} \int_{\Omega} g(\Phi(\nabla T(x))dx; \quad T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$$

where $g$ is a lower semicontinuous real valued function defined on $\mathbb{R}$, $\Phi$ is a real valued quasiaffine function defined on the space $\mathcal{M}_n$ of $n \times n$ matrices, $T$ a transformation from $\Omega$ to $\mathbb{R}^n$ and $T_B$ an affine boundary datum.

When $\Phi(A) = \det(A)$ (we recall that the determinant is the simplest example of non affine quasiaffine function) the problem, which arises in the study of equilibrium of gases, and constitutes a typical nonconvex problem in the vectorial case of the calculus of variations, has been considered in [D2] and in [MS]. In this case, the application of the direct method finds a difficulty due to the fact that, although the convexity of $g$ is still sufficient for the weak lower semicontinuity of the integral functional, growth conditions on $g$ do not guarantee that the functional is coercive on $W^{1,p}$. However, in [D2] it is proved that the relaxed problem

$$\text{Minimize} \int_{\Omega} g^{**}(\det(\nabla u(x))dx; \quad u \in C^{\infty}(\Omega, \mathbb{R}^3), \quad u = u_0 \ \text{ on } \partial\Omega$$

admits at least a smooth solution provided that the boudary datum $u_0$ has positive jacobian determinant, and satisfies some regularity conditions.

In [MS] it is given a proof, based on Moser's Theorem on volume preserving diffeomorfisms (see [Mo], [DM]), of existence of a solution for the problem

$$\text{Minimize} \int_{\Omega} g(\det(\nabla u(x))dx; \quad u \in u_0 + W^{1,\infty}(\Omega, \mathbb{R}^n),$$

for a $C^2$ homeomorfism $u_0$ with positive jacobian determinant in $\bar{\Omega}$.

We show that $\mathcal{P}$ admits at least a solution for any quasiaffine function $\Phi$ and for any affine boundary datum $T_B$.

Actually we treat separately the simpler case $\Phi = \det$ (section 4) and the general case, (section 5), since in the first one the proof can be performed by easier algebraic arguments and provides the leading ideas for the further developement. Indeed the procedure used for the determinant can be refined and applied to a generic $\Phi$ by virtue of a representation theorem, due to Ball [Ba], according to which a real valued function $\Phi(A)$ defined on $\mathcal{M}_n$

is quasiaffine if and only if it can be expressed as an affine function of all the minors of $A$. We stress that the result of section 5 includes that one of section 4, hence reading section 4 is not strictly necessary to understand the work.

We stress that the proof is easier whenever the datum $T_B$ is such that $\nabla T_B$ is not a critical point for $\Phi$ (in the case of the jacobian determinant this means that the rank of $\nabla T_B$ is larger or equal than $n-1$); otherwise the result is obtained by solving the equivalent problem

$$\text{Minimize} \int_\Omega g(\Phi(\nabla T(x))dx; \quad T \in T_r + W_0^{1,1}(\Omega, \mathbb{R}^n)$$

where $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ is a piecewise affine transformation such that $\Phi(\nabla T_r) = \Phi(\nabla T_B)$ and $d\Phi(\nabla T_r) \neq 0$ almost everywhere in $\Omega$.

## 4.2. PRELIMINARIES AND NOTATIONS

In this section we use the following notations: vectors (of $\mathbb{R}^n$) are meant to be columns, given $b \in \mathbb{R}^n$, $b^t$ is the transpose of $b$ and $(b)^\perp$ is the orthogonal complement of $span(b)$, $a \cdot b$ is the inner product of $a$ and $b$ vectors of $\mathbb{R}^n$ and $|\cdot|$ is the associated norm. The canonical base in $\mathbb{R}^n$ is denoted by $\{e_i, i = 1, \cdots, n\}$. A subset of $\mathbb{R}^n$ is said $n$-dimensional if its linear span is the whole space; for a convex set $K$ $extr(K)$ is the set of its extreme points.

A $n \times n$ matrix $A$ is written as

$$A = (a_1, ..., a_n) = \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} = \begin{pmatrix} a_1^1 & \cdots & a_1^n \\ \vdots & & \vdots \\ a_n^1 & \cdots & a_n^n \end{pmatrix}$$

where the $a_i$ are its columns and $a^i$ are its rows. We denote by $\mathcal{M}_n$ the space of $n \times n$ matrices endowed with the inner product

$$\langle\langle A, B \rangle\rangle_n = \sum_{i,j=1}^n a_i^j b_i^j = \sum_{i=1}^n a_i \cdot b_i = \sum_{j=1}^n a^j \cdot b^j.$$

Given two vectors $v, w \in \mathbb{R}^n$ we denote by $v \otimes w$ the matrix of rank one obtained taking the usual row-times-column product of matrices of $v$ and $w^t$, i.e., writing $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n)$,

$$v \otimes w = \begin{pmatrix} v_1 w_1 & \cdots & v_1 w_n \\ \vdots & & \vdots \\ v_n w_1 & \cdots & v_n w_n \end{pmatrix} = (w_1 v, \cdots, w_n v) = \begin{pmatrix} v_1 w \\ \vdots \\ v_n w \end{pmatrix}$$

For $T$, a regular tranformation from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, $\nabla T$ is the jacobian matrix; for $v$, scalar valued function, $v_{x_i}$ is the derivative with respect to the $i$-th variable and $\nabla v$ is its gradient, seen as a row vector. By this way, given a vector $b$, $b \otimes \nabla v$ is a $n \times n$ matrix, while $\nabla v \cdot b$ is a scalar (inner product).

We use the Sobolev spaces $W_0^{1,1}(\Omega, \mathbb{R}^n)$ and $W_0^{1,1}(\Omega, \mathbb{R})$ endowed with the usual norms, and adopt the convention that an element of $W_0^{1,1}(\Omega, \mathbb{R})$ or $W_0^{1,1}(\Omega, \mathbb{R}^n)$ is said to be continuous if it admits a continuous representative.

An open bounded subset $E$ of $\mathbb{R}^n$ is called regular if the divergence theorem can be applied to $E$ and to $\partial E$; the complement of a subset $E$ of $\mathbb{R}^n$ is $E^c$. The Lebesgue measure is denoted by $\mu(\cdot)$.

For a smooth function $\Phi$ defined on $\mathcal{M}_n$ we denote by $d^l\Phi(A)$ the $l$-th differential of $\Phi$ at $A$ and, abusing the notation, by

$$\left(d^l\Phi(A)\right)_{(i_1,j_1),\cdots,(i_l,j_l)}; \quad i_k, j_k = 1, \cdots n$$

the tensor representing the $l$-th differential of $\Phi$ at $A$ with respect to the canonical base in $\mathcal{M}_n$. In particular

$$(d\Phi(A))_{(i_1,j_1)}; \quad i_1, j_1 = 1, \cdots n$$

is the $n \times n$ matrix representing the first differential, in the sense that, for any $B \in \mathcal{M}_n$,

$$(d\Phi(A))(B) = \langle\langle d\Phi(A), B\rangle\rangle_n.$$

We recall, from [D1] p. 99, the following

**Definition 4.2.1.** A Borel measurable and locally integrable function $\Phi : \mathcal{M}_n \to \mathbb{R}$ is said to be *quasiaffine* if

$$\Phi(A) = \frac{1}{\mu(D)} \int_D \Phi(A + \nabla u(x))dx$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $A \in \mathcal{M}_n$ and for every $u \in W_0^{1,\infty}(D, \mathbb{R}^n)$.

We recall also that there exists also a representation theorem for quasiaffine functions (see [Ba] or [D1] p. 117) expressed in terms of the map

$$T : \mathcal{M}_n \to \mathcal{M}_{\nu(1)} \times \mathcal{M}_{\nu(2)} \times \cdots \times \mathcal{M}_{\nu(n-1)} \times \mathcal{M}_{\nu(n)};$$

where $\nu(s) = \binom{n}{s} = \frac{n!}{s!(n-s)!}$, given by

$$T(A) = (A, adj_2(A), \cdots, adj_{n-1}(A), \det A)$$

where $adj_s(A)$ stands for the $\nu(s) \times \nu(s)$ matrix of $s \times s$ minors of $A$. Roughly speaking $T(A)$ is a "vector" whose "components" are square matrices of order $\nu(s)$.

**Theorem 4.2.1.** *Let $\Phi : \mathcal{M}_n \to \mathbb{R}$. Then the following conditions are equivalent:*

*i) $\Phi$ is quasiaffine.*

*ii) There exists*

$$\beta = (\beta_1, \cdots, \beta_n) \in \mathcal{M}_{\nu(1)} \times \mathcal{M}_{\nu(2)} \times \cdots \times \mathcal{M}_{\nu(n-1)} \times \mathcal{M}_{\nu(n)}$$

*such that*

$$\Phi(A) = \Phi(0) + \sum_{s=1}^{n} \langle\!\langle \beta_s, adj_s(A) \rangle\!\rangle_{\nu(s)}.$$

*iii) For any* $a, b \in \mathbb{R}^n$

$$\Phi(A + a \otimes b) = \Phi(A) + \langle\!\langle d\Phi(A), a \otimes b \rangle\!\rangle_n.$$

**Remarks.**

1. Point *ii)* implies that $\Phi$ is a polynomial of degree less or equal to $n$; hence, in particular, if $\Phi$ is nonconstant, for every $A \in \mathcal{M}_n$ there exists $l \in \{1, \cdots n\}$ such that $d^l\Phi(A) \neq 0$.
2. Point *iii)* states that Taylor developement of $\Phi$ with respect to a rank-one increment stops at the first order.
3. When $\Phi(A) = \det A$, identifying as usual the differential with the representing matrix, it is $d\Phi(A) = adj_{n-1}(A)$ (see [D1] p. 191). Hence a matrix $A$ has rank $k$ if and only if $d^l\Phi(A) = 0$ for $l = 1, \cdots, n-k-1$ and $d^l\Phi(A) \neq 0$ for $l = n-k, \cdots, n$.
4. Since the matrix representing the differential of the function determinant is the matrix of its maximal minors, point *ii)* implies that each entry of the tensor representing a differential of some order of $\Phi$ at $A$ is still a quasiaffine real valued function of $A$.

**Lemma 4.2.1.** *Let $E$ be an open bounded subset of $\mathbb{R}^n$, and $V = \{v_i, i = 1, ..., m\}$ a set of vectors of $\mathbb{R}^n$ such that $0 \in int(co(V))$. Then there exist open regular subsets of $E$: $\{E_i, i = 1, ..., m\}$ and a continuous function $w \in W_0^{1,\infty}(E, \mathbb{R})$ such that:*

*i)* $E_i \bigcap E_j = \emptyset, \quad i \neq j;$

*ii)* $E = (\bigcup_{i=1}^m E_i) \bigcup N, \quad$ *N null set;*

*iii)* $\nabla w = \sum_{i=1}^m v_i \chi_{E_i}$ a.e. on $\Omega$;

*iv)* $\sum_{i=1}^m \mu(E_i) v_i = 0.$

The proof of this lemma is also contained in the proof of Theorem 3.3.1 part *ii)* of chapter 3.

*Proof.* Let $V^*$ be the polar set of $co\{v_i, i = 1, ..., m\}$. By Lemma 3.3.1 there exist a collection of $m$ polytopes $V_1^*, ..., V_m^*$ contained in $V^*$ and a Lipschitz continuous function $u$, defined on $\mathbb{R}^n$ such that $V^* = \bigcup_{i=1}^m V_i^*$, $int(V_i^*) \bigcap int(V_j^*) = \emptyset$ for $i = j$, $u\big|_{(V^*)^c} = 0$, $\nabla u = \sum_{i=1}^m v_i \chi_{V_i^*}$ and

$$\sum_{i=1}^{m} \mu(V_i^*) v_i = 0. \tag{2.1}$$

Consider the following Vitali covering of $E$:

$$\{x + rV^*, \quad x \in E, \quad 0 < r < dist(x, E^c)\}$$

and select a denumerable subcovering $\{S^j\}_{j \in \mathbb{N}}$,

$$S^j = \{x_j + r_j V^*, \quad x_j \in E, \quad r_j > 0\}$$

such that:

*a)* $int(S^j) \bigcap int(S^k) = \emptyset$ for $j \neq k$,

*b)* $E = \left(\bigcup_{j=1}^{\infty} S^j\right) \bigcup N$, $N$ null set,

*c)* $\mu(E) = \mu(V^*) \sum_{j=1}^{\infty} r_j^n$.

For any $j \in \mathbb{N}$ we define the subsets of $S^j$

$$S_i^j = \{x_j + r_j V_i^*\}, \quad i = 1, ..., m,$$

for any $x \in \mathbb{R}^n$, we set

$$u_j(x) \equiv r_j u \left(\frac{x - x_j}{r_j}\right)$$

and, for $k \in \mathbb{N}$:

$$U_k = \left(\sum_{j=1}^{k} u_j\right)\Bigg|_E.$$

Since $u_j$ has the same regularity of $u$ and $u_j\big|_{(S^j)^c} \equiv 0$, $U_k$ belongs to $W_0^{1,\infty}(E, \mathbb{R})$ and moreover, for any $l \in \mathbb{N}$,

$$U_{k+l}(x) - U_k(x) = \begin{cases} 0, & \text{for } x \in \left(\bigcup_{j=1}^{k} S^j\right) \bigcup \left(\bigcup_{j=k+l}^{\infty} S^j\right) \\ u_j(x), & \text{for } x \in S^j, \quad j = k, ..., k+l. \end{cases}$$

By $b)$, it is

$$\lim_{k\to\infty} \mu(\{x \in E : U_{k+l}(x) - U_k(x) \neq 0\}) = 0,$$

hence the sequence $\{U_k\}_{k\in\mathbb{N}}$ is fundamental in $W^{1,1}(E,\mathbb{R})$.

Now set

$$E_i \equiv \bigcup_{j=1}^{\infty} int(S_i^j)$$

and

$$w \equiv W^{1,1}(E,\mathbb{R})- \lim_{k\to\infty} U_k$$

First of all we remark that, since each $E_i$ is the union of a countable family of interior of polytopes, it is regular in the sense specified above, moreover properties $i)$ and $ii)$ are trivially satisfied. Obviously $w$ belongs to $W_0^{1,1}(E,\mathbb{R})$, and, given $j \in \mathbb{N}$ it is

$$U_k(x) = U_j(x) = u_j(x), \quad \text{for any } x \in S^j, \quad \text{and for any } k \geq j,$$

hence, by pointwise convergence, $w(x) = u_j(x)$ for a.e. $x$ in $S^j$; hence $w$ is continuous, belongs to $W_0^{1,1}(\Omega,\mathbb{R})$ and

$$\nabla w(x) = \nabla u\left(\frac{x - x_j}{r_j}\right) = v_i \quad \text{for a.e. } x \in S_i^j.$$

This implies $iii)$. Statement $iv)$ is a trivial consequence of (2.1), of $c)$ and of the fact that

$$\mu(E_i) = \mu(V_i^*)\sum_{j=1}^{\infty} r_j^n.$$

□

## 4.3. STATEMENT OF THE MAIN RESULT

We anticipate now the main result of this chapter.

**Theorem 4.3.1.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function satisfying*

$$\lim_{|t| \to \infty} \frac{g(t)}{|t|} = +\infty.$$

*Then for any affine transformation $T_B : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(\nabla T_B)$ is in co(dom(g)), the problem*

$$\mathcal{P}: \text{ Minimize } \int_\Omega g(\Phi(\nabla T(x)))dx; \qquad T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$$

*admits at least a solution.*

This theorem will be proved at the end of this chapter (Theorem 5.2). As will be clear, whenever $g(\Phi(\nabla T_B) = g^{**}(\Phi(\nabla T_B)$, (here and in the following $g^{**}$ denotes the bipolar of $g$) $T_B$ is a solution. When $g(\Phi(\nabla T_B)$ is strictly larger than $g^{**}(\Phi(\nabla T_B)$ growth condition on $g$ implies that the infimum of the functional is $\mu(\Omega)g^{**}(\Phi(\nabla T_B)) = \mu(\Omega)(\lambda g(\alpha)+(1-\lambda)g(\beta))$ where $\alpha$ and $\beta$ are real numbers such that $\Phi(\nabla T_B) = \lambda\alpha + (1-\lambda)\beta$. Hence the proof of the main result consists in showing that for any affine transformation $T_B$ there exists a transformation $T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that $\Phi(\nabla T)$ takes the values $\alpha$ and $\beta$ on two disjoint subsets of $\Omega$ whose relative measures are the coefficient $\lambda$ and $1-\lambda$ of the above convex combination.

We shall prove the existence of such a transformation assuming first $\Phi = \det$ (Theorem 4.1) and then in the general case (Theorem 5.1).

## 4.4. CASE OF THE JACOBIAN DETERMINANT

In this section we treat the case of a functional of the jacobian determinant showing that problem $\mathcal{P}$ admits solutions for any affine boundary datum when $\Phi$ coincides with the determinant. The proof is divided in two steps. We begin with a lemma in which we define a piecewise affine transformation of rank $n-1$ coinciding with $T_B$ on the boundary of $\Omega$.

**Lemma 4.4.1.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$. Let $T_B : \mathbb{R}^n \to \mathbb{R}^n$ to be an affine transformation such that* $\text{rank}(\nabla T_B) \leq n-1$. *Then there exists a collection $\{\Omega_i, i = 1, ..., m\}$ of open regular subsets of $\Omega$, a null set $N$ with $\Omega = (\bigcup_{i=1}^m \Omega_i) \bigcup N$ and a continuous transformation $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that the restriction of $T_r$ to each $\Omega_i$ is affine and* $\text{rank}(\nabla T_r) = n-1$ *on* $\bigcup_{i=1}^m \Omega_i$.

*Proof.* Set $k$ to be $\text{rank}(\nabla T_B)$, if $k = n-1$ set $T_r = T_B$ (the collection of open subsets of $\Omega$ consists of $\Omega$ itself). If $k < n-1$ let $b_{i_1}, ..., b_{i_k}$ be $k$ linearly independent columns of $B$. Given $j$ not in $\{i_1, ..., i_k\}$ there exist $\lambda_1^{(j)}, ..., \lambda_k^{(j)} \in \mathbb{R}$ such that

$$b_j = \sum_{h=1}^k \lambda_h^{(j)} b_{i_h}.$$

Let $c \neq 0$, $c$ in $(span\{b_{i_1}, ..., b_{i_k}\})^\perp$. We are going to consider a transformation $F_v$ defined by means of a continuous scalar function $v \in W_0^{1,1}(\Omega, \mathbb{R})$ by setting

$$F_v(x) = T_B(x) + v(x)c, \tag{4.1}$$

so that, a.e. in $\Omega$,

$$\nabla F_v(x) = B + c \otimes \nabla v(x).$$

For $j \in \{i_1, ..., i_k\}$ the $i$-th column of $\nabla F_v$ is $b_j + v_{x_j} c$, while given $i \notin \{i_1, ..., i_k\}$ is

$$\sum_{h=1}^k \lambda_h^{(i)} b_{i_h} + v_{x_i} c = \sum_{h=1}^k \lambda_h^{(i)} \left(b_{i_h} + v_{x_{i_h}} c\right) + \left(v_{x_i} - \sum_{h=1}^k \lambda_h^{(i)} v_{x_{i_h}}\right) c,$$

hence the rank of $\nabla F_v$ does not change if we replace the $i$-th column by

$$\left(v_{x_i} - \sum_{h=1}^k \lambda_h^{(i)} v_{x_{i_h}}\right) c,$$

i.e. it is $k+1$ whenever

$$\left( v_{x_i} - \sum_{h=1}^{k} \lambda_h^{(i)} v_{x_{i_h}} \right) = \nabla v \cdot f \neq 0,$$

where we have introduced the vector $f$ whose components are defined by

$$f_j = \begin{cases} 1, & \text{if } j = i; \\ -\lambda_j^{(i)}, & \text{if } j = i_h,\ h = 1, ..., k; \\ 0, & \text{otherwise.} \end{cases}$$

It is our purpose to define a continuous piecewise affine scalar function $v$ such that the previous condition holds a.e. in $\Omega$.

Let $\{s^1, ..., s^n\}$ be the vertices of a $n$-symplex in $(e)^\perp$ containing 0 in its relative interior. Consider the set of $2n$ vectors

$$\{ s^i - f, \quad s^i + f, \quad i = 1, ..., n \};$$

applying Lemma 2.1 to this set of vectors and to the open set $\Omega$, we infer the existence of a finite collection $\{\Omega_j, j = 1, ..., 2n\}$ of open regular subsets of $\Omega$ and of a continuous piecewise affine function $v \in W_0^{1,1}(\Omega, \mathbb{R})$ such that

$$\nabla v = \sum_{j=1}^{n} (s^j - f)\, \chi_{\Omega_j} + \sum_{j=1}^{n} (s^j + f)\, \chi_{\Omega_{n+j}}.$$

From the above,

$$\nabla v \cdot f = (-|f|^2)\, \chi_{\cup_{j=1}^n \Omega_j} + (|f|^2)\, \chi_{\cup_{j=1}^n \Omega_{n+j}}.$$

The transformation $F_1$ defined by (4.1) through this $v$ is affine on each $\Omega_j$ and we have, a.e. in $\Omega$,

$$\text{rank}(\nabla F_1) = k + 1.$$

If $k = n-2$ we set $T_r = F_1$ and are done, otherwise for each $j \in \{1, ..., 2n\}$, repeating the previous procedure, we can define a finite collection $\{\Omega_{j,l}, l = 1, ..., 2n\}$ of regular open subsets of $\Omega_j$, a continuous piecewise affine function $v_j \in W_0^{1,\infty}(\Omega_j, \mathbb{R})$ and a vector $c_j$ such that the rank of the jacobian of the transformation, defined on $\Omega_j$,

$$F_{v_j}(x) = F_1(x) + v_j(x) c_j$$

(which is affine on each $\Omega_{j,l}$, $l = 1, ..., 2n$) is $k+2$. Then we extend $v_j$ by 0 on $\Omega \setminus \Omega_j$ and, for $x \in \Omega$, set

$$F_2(x) = F_1(x) + \sum_{j=1}^{2n} v_j(x) c_j.$$

Since the last term in the r.h.s. is in $W_0^{1,1}(\Omega, \mathbb{R}^n)$, $F_2$ belongs to $T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$; moreover it is affine on each $\Omega_{j,l}$, and the rank of $\nabla F_2$ is $k+2$ a.e. on $\Omega$.

If $k = n-3$ we set $T_r = F_2$ and are done, otherwise the above steps can be repeated until the desired result is reached. □

**Theorem 4.4.1.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $T_B : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation. Let $\alpha, \beta \in \mathbb{R}$ ($\alpha < \beta$), $\lambda \in ]0,1[$ be such that*

$$\det(\nabla T_B) = \lambda\alpha + (1-\lambda)\beta.$$

*Then there exist two open disjoint subsets $\Omega^\alpha$ and $\Omega^\beta$ of $\Omega$, a null set $N$, with $\Omega = \Omega^\alpha \bigcup \Omega^\beta \bigcup N$, and a transformation $T \in T_B + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that:*

*i)* $\det(\nabla T) = \alpha\chi_{\Omega^\alpha} + \beta\chi_{\Omega^\beta}$;

*ii)* $\frac{\mu(\Omega^\alpha)}{\mu(\Omega)} = \lambda$; $\frac{\mu(\Omega^\beta)}{\mu(\Omega)} = 1 - \lambda$.

*Proof.*

*a)* Assume first $\det(\nabla T_B) \neq 0$. Let $b_1$ be the first column of $\nabla T_B$ and, for a given $v \in W_0^{1,1}(\Omega, \mathbb{R})$, consider the transformation

$$T_v(x) = T_B(x) + v(x) b_1,$$

so that $T_v$ belongs to $W_0^{1,1}(\Omega, \mathbb{R}^n)$, a.e. in $\Omega$, it is

$$\nabla T_v = B + b_1 \otimes (\nabla v)$$

and

$$\det(\nabla T_v) = \det\left(b_1(1+v_{x_1}), b_2 + b_1 v_{x_2}, ..., b_n + b_1 v_{x_n}\right) = \det(\nabla T_B)(1 + v_{x_1}).$$

Hence

$$\det(\nabla T_v) = \begin{cases} \alpha \\ \beta \end{cases} \quad \text{if and only if} \quad v_{x_1} = \begin{cases} \frac{\alpha}{\det(\nabla T_B)} - 1 \\ \frac{\beta}{\det(\nabla T_B)} - 1. \end{cases} \tag{4.2}$$

Set

$$v_1^\alpha = \frac{\alpha}{\det(\nabla T_B)} - 1 \ \ (v_1^\alpha < 0) \quad \text{and} \quad v_1^\beta = \frac{\alpha}{\det(\nabla T_B)} - 1 \ \ (v_1^\beta > 0).$$

Let $S_{n-1} = \{s_i, i = 1, ..., n\}$ be a $(n-1)$-symplex in $span\{e_i, i = 2, ..., n\}$ containing 0 in its relative interior. Recall Lemma 2.1 with $E = \Omega$ and $V$ the set of $2n$ vectors $S = \{s_i + v_1^\alpha e_1, s_i + v_1^\beta e_1\}$. By this Lemma there exist open disjoint subsets of $\Omega$, $\{\Omega_i^\alpha, \Omega_i^\beta, i = 1, ..., n\}$ and $w \in W_0^{1,1}(\Omega, \mathrm{IR})$ such that, a.e. in $\Omega$,

$$\nabla w = \sum_{j=1}^{n} (s_i + v_1^\alpha e_1)\, \chi_{\Omega_i^\alpha} + \sum_{j=1}^{n} \left(s_i + v_1^\beta e_1\right) \chi_{\Omega_i^\beta}$$

and

$$\sum_{j=1}^{n} (s_i + v_1^\alpha e_1)\, \mu(\Omega_i^\alpha) + \sum_{j=1}^{n} \left(s_i + v_1^\beta e_1\right) \mu(\Omega_i^\beta) = 0,$$

in particular the equality of the first component of the previous (vectorial) equality, yields:

$$v_1^\alpha \sum_{j=1}^{n} \mu(\Omega_i^\alpha) + v_1^\beta \sum_{j=1}^{n} \mu(\Omega_i^\beta) = 0. \tag{4.3}$$

Now set $\Omega^\alpha = \bigcup_{j=1}^{n}(\Omega_i^\alpha)$, $\Omega^\beta = \bigcup_{j=1}^{n}(\Omega_i^\beta)$ and $T = T_w$. Recalling (4.2), we have

$$\det(\nabla T) = \alpha \chi_{\Omega^\alpha} + \beta \chi_{\Omega^\beta}.$$

By (4.3) it is

$$\mu(\Omega^\alpha)\left(\frac{\alpha}{\det(\nabla T_B)} - 1\right) + \mu(\Omega^\beta)\left(\frac{\beta}{\det(\nabla T_B)} - 1\right) = 0,$$

that is to say

$$\alpha\mu(\Omega^\alpha) + \beta\mu(\Omega^\beta) = \det(\nabla T_B)\mu(\Omega),$$

recalling that $\det(\nabla T_B) = \lambda\alpha + (1-\lambda)\beta$ we have *ii)*.

*b)* Assume $\text{rank}(\nabla T_B) = n - 1$, so that $\alpha < \det(\nabla T_B) = 0 < \beta$. Hence there exist $j \in \{1, .., n\}$ and coefficients $\{\gamma_i, i \neq j\}$ such that

$$b_j = \sum_{i \neq j} \gamma_i b_i.$$

Let $c$ in $(span\{b_i, i = 1, ..., n\})^\perp$ be such that $\det(b_1, ..., b_{j-1}, c, b_{j+1}, ..., b_n) = 1$. As before, for a continuous $v \in W_0^{1,1}(\Omega, \mathbb{R})$, consider the transformation $T_v \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$, defined by

$$T_v(x) = T_B(x) + v(x)c$$

so that, a.e. in $\Omega$,

$$\nabla T_v = \nabla T_B + c \otimes \nabla v =$$

$$\left( b_1 + v_{x_1}c, ..., b_{j-1} + v_{x_{j-1}}c, \sum_{i\neq j} \gamma_i b_i + v_{x_j}c, ..., b_{j+1} + v_{x_{j+1}}c, ..., b_n + v_{x_n}c \right) =$$

$$\left( b_1 + v_{x_1}c, ..., \sum_{i\neq j} \gamma_i (b_i + v_{x_i}c) + \left( v_{x_j} - \sum_{i\neq j} \gamma_i v_{x_i} \right) c, ..., b_n + v_{x_n}c \right).$$

Hence

$$\det(\nabla T_v) = \det \left( b_1 + v_{x_1}c, ..., \left( v_{x_j} - \sum_{i\neq j} \gamma_i v_{x_i} \right) c, ..., b_n + v_{x_n}c \right) =$$

$$\left( v_{x_j} - \sum_{i\neq j} \gamma_i v_{x_i} \right) \det(b_1, ..., b_{j-1}, c, b_{j+1}, ..., b_n) = \left( v_{x_j} - \sum_{i\neq j} \gamma_i v_{x_i} \right) = \nabla v \cdot e$$

where $e$ is the vector $(-\gamma_1, ..., -\gamma_{j-1}, 1, -\gamma_{j+1}, ..., -\gamma_n)^t$. Let $S_{n-1} = \{s_i, i = 1, ..., n\}$ be a $(n-1)$-symplex in $(e)^\perp$ containing 0 in its relative interior. Recall Lemma 2.1 with $E = \Omega$ and $V$ the set of $2n$ vectors

$$\left\{ s_i + \frac{\alpha}{|e|^2} e, s_i + \frac{\beta}{|e|^2} e, i = 1, ..., n \right\}.$$

By this lemma there exist open subsets of $\Omega$, $\Omega_i^\alpha$ and $\Omega_i^\beta$ and $w \in W_0^{1,1}(\Omega, \mathbb{R})$ such that, a.e. in $\Omega$,

$$\nabla w = \sum_{i=1}^n \left( s_i + \frac{\alpha}{|e|^2} e \right) \chi_{\Omega_i^\alpha} + \sum_{i=1}^n \left( s_i + \frac{\beta}{|e|^2} e \right) \chi_{\Omega_i^\beta}$$

so that

$$\nabla w \cdot e = \alpha \sum_{i=1}^n \chi_{\Omega_i^\alpha} + \beta \sum_{i=1}^n \chi_{\Omega_i^\beta}$$

and

$$\sum_{i=1}^n \left( s_i + \frac{\alpha}{|e|^2} e \right) \mu(\Omega_i^\alpha) + \sum_{i=1}^n \left( s_i + \frac{\beta}{|e|^2} e \right) \mu(\Omega_i^\beta) = 0.$$

In particular, taking the inner product of both sides with $e$, we have

$$\alpha \sum_{i=1}^{n} \mu(\Omega_i^\alpha) + \beta \sum_{i=1}^{n} \mu(\Omega_i^\beta) = 0 \tag{4.4}$$

Set $\Omega^\alpha = \bigcup_{i=1}^n \Omega_i^\alpha$, $\Omega^\beta = \bigcup_{i=1}^n \Omega_i^\beta$ and $T = T_w$. $T$ is in $W_0^{1,1}(\Omega, \mathbb{R}^n)$, by the above,

$$\det(\nabla T) = \nabla w \cdot e = \alpha\chi_{\Omega^\alpha} + \beta\chi_{\Omega^\beta},$$

and, by (4.3),

$$\frac{\mu(\Omega^\alpha)}{\mu(\Omega)}\alpha + \frac{\mu(\Omega^\beta)}{\mu(\Omega)}\beta = 0.$$

*c)* Assume $\operatorname{rank}(\nabla T_B) < n - 1$. By Lemma 4.1 there exist a finite collection of regular open subsets of $\Omega$, $\{\Omega_j, j = 1, ..., k\}$, a null set $N$ such that $\Omega = N \bigcup \left(\bigcup_{j=1}^k \Omega_j\right)$ and a continuous transformation $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that

$$T_r = \sum_{j=1}^{k} T_j \chi_{\Omega_j}$$

where $T_j$ is affine and $\operatorname{rank}(\nabla T_j) = n - 1$. Consider the procedure of step *b)* replacing $\Omega$ by $\Omega_j$ and $T_B$ by $T_j$. For every $j$ we infer the existence of open disjoint subset $(\Omega_j)^\alpha$, $(\Omega_j)^\beta$ of $\Omega$ and of a transformation $T'_j$ in $T_j + W_0^{1,\infty}(\Omega_j, \mathbb{R}^n)$ such that, a.e. in $\Omega_j$

$$\det(\nabla T'_j) = \alpha\chi_{(\Omega_j)^\alpha} + \beta\chi_{(\Omega_j)^\beta} \tag{4.5}$$

and

$$\frac{\mu((\Omega_j)^\alpha)}{\mu(\Omega_j)}\alpha + \frac{\mu((\Omega_j)^\beta)}{\mu(\Omega_j)}\beta = 0.$$

We extend $T'_j$ by setting $T_j$ on $\Omega \setminus \Omega_j$ and define

$$T = \sum_{j=1}^{k} T'_j \chi_{\Omega_j}.$$

It is

$$T = T_r + \sum_{j=1}^{k} \left(T'_j - T_j\right) \chi_{\Omega_j}$$

and since the last term in the r.h.s. is in $W_0^{1,1}(\Omega, \mathbb{R}^n)$, $T$ belongs to $T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$. Moreover, by (4.5), it is, a.e. in $\Omega$,

$$\det(\nabla T) = \alpha \sum_{j=1}^{k} \chi_{(\Omega_j)^\alpha} + \beta \sum_{j=1}^{k} \chi_{(\Omega_j)^\beta}.$$

Setting $\Omega^\alpha = \bigcup_{j=1}^k (\Omega_j)^\alpha$ and $\Omega^\beta = \bigcup_{j=1}^k (\Omega_j)^\beta$, we conclude the proof. □

This last result is sufficient to prove the main result in the case of the jacobian determinant. The reader interested only to the this case can skip over next section going directly to Theorem 5.2 setting $\Phi = \det$.

## 4.5. GENERAL CASE

We consider now the case of a generic quasiaffine function $\Phi$. The proof follows the ideas of previous section with the difference that the role played by the rank is now assumed by the differentials of $\Phi$. More precisely the proof can be performed directly when $\nabla T_B$ is not a critical point for $\Phi$ (this corresponds to the case of rank equal $n$ or $n-1$), otherwise the auxiliary piecewise transformation $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ is now defined in such a way that $d\Phi(\nabla T_r)$ is different from zero almost everywhere and $\Phi(\nabla T_r) = \Phi(\nabla T_B)$.

**Lemma 4.5.1.** *Let $A \in \mathcal{M}_n$ be a nonzero matrix. Let $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 < 0 < \gamma_2$. Then there exist a vector $b \in \mathbb{R}^n$ and a n-dimensional polytope $P \subset \mathbb{R}^n$ with vertices $\{v_i^1, v_i^2, i = 1, \cdots, n\} = extr(P)$, containing 0 in its relative interior, such that*

$$\langle\langle A, b \otimes v_i^j \rangle\rangle_n = \gamma_j \quad i = 1, \cdots, n; \ j = 1, 2.$$

*Proof.* Write

$$A = \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix},$$

take a row $a^j$ different from zero and choose $b = e_j$. Let $S$ be a symplex in $(a^j)^\perp$ with vertices $\{s_i, i = 1, \cdots, n\}$ containing zero in its relative interior, define

$$v_i^1 = s_i + \frac{\gamma_1}{|a^j|^2} a^j, \quad v_i^2 = s_i + \frac{\gamma_2}{|a^j|^2} a^j \quad i = 1, \cdots, n.$$

Notice that for any vector $v$, $e_j \otimes v$ is the matrix whose rows are all zero except the $j$-th one which coincides with the row vector $v^t$. Hence

$$\langle\langle A, e_j \otimes v_i^j \rangle\rangle_n = a^j \cdot v_i^j,$$

and, by the choice of $v_i^j$, we have the result. □

**Lemma 4.5.2.** *Let $\Phi$ be a nonconstant real valued quasiaffine function defined on $\mathcal{M}_n$, and let $A$ be a critical point for $\Phi$. Let $k \in \{0, \cdots n-2\}$ such that $d^{n-k}\Phi(A) \neq 0$ and $d^l\Phi(A) = 0$ for any $l = 1, \cdots, n-k-1$. Then there exist a vector $b \in \mathbb{R}^n$ and a n-dimensional polytope $P \subset \mathbb{R}^n$ with vertices $\{v_i, i = 1, \cdots, 2n\} = extr(P)$, containing zero in its interior, such that*

$$d^{n-k-1}\Phi(A + b \otimes v_i) \neq 0 \quad i = 1, \cdots, 2n,$$

$$d^l\Phi(A + b \otimes v_i) = 0 \quad i = 1, \cdots, 2n; \; l = 1, \cdots, n-k-2.$$

*Hence, as a consequence of Theorem 2.1 iii), $\Phi(A + b \otimes v_i) = 0$ for $i = 1, \cdots, 2n$.*

*Proof.* Consider the tensor representing the $(n-k-1)$-th differential of $\Phi$ at $A$:

$$\left(d^{n-k-1}\Phi(A)\right)_{(i_1,j_1),\cdots,(i_{n-k-1},j_{n-k-1})}.$$

By assumption this tensor is zero and there exists a multiindex

$$\bar{J}_{n-k-1} = (\bar{i}_1, \bar{j}_1), \cdots, (\bar{i}_{n-k-1}, \bar{j}_{n-k-1})$$

such that

$$d\left(d^{n-k-1}\Phi(A)\right)_{\bar{J}_{n-k-1}}$$

is a nonzero matrix. By Lemma 5.1 there exist a vector $b$ and a polytope $P$ such that

$$\langle\langle d\left(d^{n-k-1}\Phi(A)\right)_{\bar{J}_{n-k-1}}, b \otimes v_i\rangle\rangle_n \neq 0$$

for every $v_i \in extr(P)$. Now we recall that the map $A \to \left(d^{n-k-1}\Phi(A)\right)_{\bar{J}_{n-k-1}}$ is real valued and quasiaffine; hence by point *iii)* of Theorem 2.1

$$\begin{aligned}(d^{n-k-1}(\Phi(A + b \otimes v_i))_{\bar{J}_{n-k-1}} &= (d^{n-k-1}(\Phi(A))_{\bar{J}_{n-k-1}} + \\ &\langle\langle d\left(d^{n-k-1}\Phi(A)\right)_{\bar{J}_{n-k-1}}, b \otimes v_i\rangle\rangle_n \neq 0\end{aligned}$$

for every $v_i \in extr(P)$.

Moreover, for any $l \in \{1, \cdots n-k-2\}$, if we denote by

$$d\left(d^l\Phi(A)\right)_{J_l},$$

where $J_l = (i_1, j_1), \cdots, (i_l, j_l)$, a generic entry of the $l$-th differential of $\Phi$ in $A$, we have, as before:

$$(d^l(\Phi(A + b \otimes v_i))_{J_l} = (d^l(\Phi(A))_{J_l} + \langle\langle d\left(d^l\Phi(A)\right)_{J_l}, b \otimes v_i\rangle\rangle_n = 0$$

for any $v_i \in extr(P)$. Hence all the differential of $\Phi$ at $A$ are zero until the $(n-k-2)$-th one while the $(n-k-1)$-th one is different from zero. □

The following lemma is the analogous of Lemma 4.1.

**Lemma 4.5.3.** *Let $T_B : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $d\Phi(\nabla T_B) = 0$. Then there exists a continuous transformation $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$, a finite collection of open regular subsets of $\Omega$, $\{\Omega_i, i = 1, \cdots, m\}$ a null set $N$ with $\Omega = N \bigcup (\bigcup_{i=1}^m \Omega_i)$, such that*

$$T_r = \sum_{i=1}^{m} T_i \chi_{\Omega_i},$$

*where each $T_i$ is affine, $d\Phi(\nabla T_i) \neq 0$ a.e. in $\Omega$ and $\Phi(\nabla T_i) = \Phi(\nabla T_B)$.*

*Proof.* Let $k$ $(k \geq n-2)$ be such that $d^{n-k}\Phi(\nabla T_B) \neq 0$ and $d^l\Phi(\nabla T_B) = 0$ for $l = 1, \cdots, n-k-1$. By Lemma 5.2 there exist $b \in \mathbb{R}^n$ and a $n$-dimensional polytope $P$ containing zero in its interior such that

$$\begin{aligned} &d^{n-k-1}\Phi(\nabla T_B + b \otimes v_i) \neq 0 \\ &d^l\Phi(\nabla T_B + b \otimes v_i) = 0 \quad l = 1, \cdots, n-k-2. \end{aligned}$$

for any $v_i \in extr(P)$. Let $u \in W_0^{1,1}(\Omega, \mathbb{R})$ be defined as in Lemma 2.1 with $V = P$ and $E = \Omega$. Consider the transformation

$$T_1(x) = T_B(x) + u(x)b$$

so that

$$\nabla T_1(x) = \nabla T_B + b \otimes \nabla u(x), \text{ a.e. in } \Omega.$$

$T_1$ is continuous, belongs to $W_0^{1,1}(\Omega, \mathbb{R}^n)$ and there exist a finite collection of open regular subsets of $\Omega$: $\{\Omega_i^1, i = 1, \cdots, 2n\}$ such that $\Omega = N \bigcup \left(\bigcup_{i=1}^{2n} \Omega_i^1\right)$ and

$$T_1 = \sum_{i=1}^{2n} T_1^i \chi_{\Omega_i^1},$$

where each $T_1^i$ is affine and, more precisely,

$$\nabla T_1^i(x) = \nabla T_B + b \otimes v_i, \quad i = 1, \cdots, 2n.$$

Hence, for any $i \in \{1, \cdots, 2n\}$, $d^{n-k-1}\Phi(\nabla T_1^i) \neq 0$, $d^l\Phi(\nabla T_1^i) = 0$ for $l = 1, \cdots, n-k-2$ and $\Phi(\nabla T_1^i) = \Phi(\nabla T_B)$. If $k = n-2$ we set $T_r = T_1$ and $\Omega_i = \Omega_i^1$.

Otherwise, repeating the previous procedure, for each $i \in \{1, \cdots, 2n\}$, we can define a continuous piecewise affine transformation $T_2^i \in T_1^i + W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that

$$d^{n-k-2}\Phi(\nabla T_2^i) \neq 0,$$

$$d^l\Phi(\nabla T_2^i) = 0, \quad l = 1, \cdots, n-k-3,.$$

$$\Phi(\nabla T_2^i) = \Phi(\nabla T_B)$$

We extend $T_2^i$ on $\Omega$ by setting $T_2^i = T_1^i$ on $\Omega \setminus \Omega_i$ and define

$$T_2 = \sum_{i=1}^{2n} T_2^i \chi_{\Omega_i^1}.$$

It is

$$T_2 = T_1 + \sum_{i=1}^{2n} (T_2^i - T_1^i)\chi_{\Omega_i^1}.$$

Since the second term in the right hand side is in $W_0^{1,1}(\Omega, \mathbb{R}^n)$, $T_2$ belongs to $T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ and has the same properties of $T_1$. This procedure can be iterated $n-k-1$ times to obtain the required transformation $T_r = T_{n-k-1}$. □

**Theorem 5.5.1.** *Let $\Phi : \mathcal{M}_n \to \mathbb{R}$ be nonconstant and quasiaffine. Let $T_B : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation. Let $\alpha, \beta \in \mathbb{R}$ $(\alpha < \beta)$ $\lambda \in ]0,1[$ be such that*

$$\Phi(\nabla T_B) = \lambda\alpha + (1-\lambda)\beta.$$

*Then there exist two open regular disjoint subsets of $\Omega$, $\Omega^\alpha$ and $\Omega^\beta$, a null set $N$ and a continuous piecewise transformation $T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$, such that $\Omega = N \bigcup \Omega^\alpha \bigcup \Omega^\beta$ and*

*i)* $\Phi(\nabla T) = \alpha\chi_{\Omega^\alpha} + \beta\chi_{\Omega^\beta}$.

*ii)* $\int_\Omega \Phi(\nabla T(x))dx = \mu(\Omega)\Phi(\nabla T_B)$,

*or, in other words,*

*ii)'* $\frac{\mu(\Omega^\alpha)}{\mu(\Omega)} = \lambda, \quad \frac{\mu(\Omega^\beta)}{\mu(\Omega)} = 1 - \lambda.$

*Proof.*

*a)* Suppose $d\Phi(\nabla T_B) \neq 0$. Apply Lemma 5.1 with $A = \nabla T_B$, $\gamma_1 = \alpha - \Phi(\nabla T_B)$, $\gamma_2 = \beta - \Phi(\nabla T_B)$ to obtain a vector $b$ and a $n$-dimensional polytope $P \subset \mathbb{R}^n$ with vertices $\{v_i^\alpha, v_i^\beta\}$ containing zero in its interior such that

$$\langle\langle d\Phi(\nabla T_B), b \otimes v_i^\alpha \rangle\rangle_n = \alpha - \Phi(\nabla T_B)$$

and

$$\langle\langle d\Phi(\nabla T_B), b \otimes v_i^\beta \rangle\rangle_n = \beta - \Phi(\nabla T_B).$$

Then we define $u \in W_0^{1,1}(\Omega, \mathbb{R})$ as in Lemma 2.1 with $V = P$ and $E = \Omega$. Consider the transformation

$$T(x) = T_B(x) + u(x)b.$$

$T$ is continuous, piecewise affine and belongs to $T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$, moreover, by Lemma 2.1 and by Theorem 2.1 *iii)*, there exist two open regular disjoint subsets of $\Omega$, $\Omega^\alpha$ and $\Omega^\beta$, such that $\Omega = N \bigcup \Omega^\alpha \bigcup \Omega^\beta$ and

$$\Phi(\nabla T) = \Phi(\nabla T_B + b \otimes \nabla u) =$$

$$\Phi(\nabla T_B) + \langle\langle d\Phi(\nabla T_B), b \otimes \nabla u \rangle\rangle_n = \alpha \chi_{\Omega^\alpha} + \beta \chi_{\Omega^\beta}.$$

Since $\Phi$ is quasiaffine, we have

$$\int_\Omega \Phi(\nabla T(x))dx = \alpha\mu(\Omega^\alpha) + \beta\mu(\Omega^\beta) = \mu(\Omega)\Phi(\nabla T_B),$$

i.e. point *ii)*.

*b)* Suppose

$$d\Phi(\nabla T_B) = 0,\ d^2\Phi(\nabla T_B) = 0, \cdots, d^{n-k-1}\Phi(\nabla T_B) = 0,$$

and

$$d^{n-k}\Phi(\nabla T_B) \neq 0$$

$(k \geq n-2)$. By Lemma 5.3 there exist an integer $m$ and a continuous transformation $T_r \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that

$$\nabla T_r = \sum_{i=1}^{m} \nabla T_r^i \chi_{\Omega_i^r}$$

where $\{\Omega_i^r, i = 1, \cdots, m\}$ is a collection of open regular subsets of $\Omega$ such that $\Omega = N \bigcup (\bigcup_{i=1}^m \Omega_i^r)$ ($N$ null set), each $T_r^i$ is affine, and, for any $i = 1, \cdots, m$

$$d\Phi(\nabla T_r^i) \neq 0,$$

$$\Phi(\nabla T_r^i) = \Phi(\nabla T_B).$$

Repeating the procedure of point *a)*, for each index $i$ we can define a continuous piecewise affine transformation $\tilde{T}^i \in T_r^i + W_0^{1,\infty}(\Omega_i^r, \mathrm{I\!R}^n)$ such that

$$\Phi(\nabla \tilde{T}^i) = \alpha \chi_{(\Omega_i^r)^\alpha} + \beta \chi_{(\Omega_i^r)^\beta}.$$

where $(\Omega_i^r)^\alpha$ and $(\Omega_i^r)^\beta$ are open regular subsets of $\Omega_i^r$ such that $\Omega_i^r = N \bigcup (\Omega_i^r)^\alpha \bigcup (\Omega_i^r)^\beta$ ($N$ null set). We extend $\tilde{T}^i$ on $\Omega$ by setting $\tilde{T}^i = T_r^i$ on $\Omega \setminus \Omega_r^i$ and define

$$T = \sum_{i=1}^m \tilde{T}^i \chi_{\Omega_r^i}.$$

Obviously $T$ is continuous, piecewise affine and

$$T = T_r + \sum_{i=1}^m \left(\tilde{T}^i - T_r^i\right) \chi_{\Omega_r^i}.$$

Since the second term in the right hand side is in $W_0^{1,1}(\Omega, \mathrm{I\!R}^n)$, $T$ belongs to $T_B + W_0^{1,1}(\Omega, \mathrm{I\!R}^n)$ and

$$\Phi(\nabla T) = \alpha \chi_{\Omega^\alpha} + \beta \chi_{\Omega^\beta},$$

where $\Omega^\alpha = \bigcup_{i=1}^m (\Omega_i^r)^\alpha$ and $\Omega^\beta = \bigcup_{i=1}^m (\Omega_i^r)^\beta$. As before

$$\int_\Omega \Phi(\nabla T(x))dx = \alpha\mu(\Omega^\alpha) + \beta\mu(\Omega^\beta) = \mu(\Omega)\Phi(\nabla T_B).$$

This ends the proof. $\square$

We are ready to prove the main result of this chapter.

**Theorem 5.5.2.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function satisfying*

$$\lim_{|t|\to\infty} \frac{g(t)}{|t|} = +\infty.$$

*Then for any affine transformation $T_B : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi(\nabla T_B)$ is in co(dom(g)), the problem*

$$\mathcal{P}: \text{ Minimize } \int_\Omega g(\Phi(\nabla T(x)))dx; \qquad T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$$

*admits at least a solution.*

*Proof.* Consider $g^{**}$, the bipolar of $g$, as defined in [ET].

*a)* Assume $co(dom(g)) \neq \mathbb{R}$ ($dom(g)$ is the set in which $g$ is strictly less than infinity) and consider first the case in which $\Phi(\nabla T_B) \in \partial(co(dom(g)))$. We claim that $T_B$ is a solution of $\mathcal{P}$. In the case $\Phi(\nabla T_B) = \sup(co(dom(g)))$ we remark that for any $T \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$ we have, since the map $A \to \Phi(A)$ is quasiaffine,

$$\int_\Omega (\Phi(\nabla T(x)) - \Phi(\nabla T_B(x)))\, dx = 0.$$

If

$$\int_\Omega g\left(\Phi(\nabla T(x))\right) dx < +\infty,$$

for a.e. $x \in \Omega$ it must be $\Phi(\nabla T(x)) \leq \Phi(\nabla T_B(x))$, hence $\Phi(\nabla T(x)) = \Phi(\nabla T_B(x))$ for a.e. $x \in \Omega$, so that $T_B$ is a solution and any other solution $T_1$ of $\mathcal{P}$ must be such that $\Phi(\nabla T_1) = \Phi(\nabla T_B)$. The case $\Phi(\nabla T_B) = \inf(co(dom(g)))$ is analogous.

*b)* Consider now the case $\Phi(\nabla T_B) \in int(co(dom(g)))$ (where this set can be the whole $\mathbb{R}$). There exist a line $\rho$ separating the point $(\Phi(\nabla T_B), g^{**}(\Phi(\nabla T_B)))$ from the closed convex set $epi(g^{**})$. Since $\Phi(\nabla T_B)$ is in the interior of $dom(g^{**})$, $\rho$ cannot be vertical, i.e. there exist $\gamma, \delta \in \mathbb{R}$ such that, for $t \in dom(g^{**})$:

$$g(t) \geq g^{**}(t) \geq \gamma t + \delta$$

and

$$g^{**}(\phi(\nabla T_B)) = \gamma\Phi(\nabla T_B) + \delta.$$

Let $\tilde{T} \in T_B + W_0^{1,1}(\Omega, \mathbb{R}^n)$, it is, since $\Phi$ is quasiaffine,

$$\int_\Omega g\left(\Phi\left(\nabla\tilde{T}(x)\right)\right) dx \geq \int_\Omega g^{**}\left(\Phi\left(\nabla\tilde{T}(x)\right)\right) dx \geq$$

$$\int_\Omega \left(\gamma\Phi\left(\nabla\tilde{T}(x)\right) + \delta\right) dx = \int_\Omega \left(\gamma\Phi\left(\nabla T_B(x)\right) + \delta\right) dx = \int_\Omega g^{**}\left(\Phi\left(\nabla T_B(x)\right)\right) dx.$$

When $g\left(\Phi\left(\nabla T_B\right)\right) = g^{**}\left(\Phi\left(\nabla T_B\right)\right)$ the above argument shows that $T_B$ is a solution. Otherwise, by a slight modification of IX.3.3. of [ET], taking into account the superlinear growth condition on $g$, we can say that there exist $\alpha, \beta \in \mathbb{R}$, $\lambda \in ]0,1[$ such that

$$\Phi\left(\nabla T_B\right) = \lambda\alpha + (1-\lambda)\beta.$$

and

$$g^{**}\left(\Phi\left(\nabla T_B\right)\right) = \lambda g(\alpha) + (1-\lambda)g(\beta).$$

In this case, it is

$$\int_\Omega g^{**}\left(\Phi\left(\nabla T_B(x)\right)\right) dx = \lambda\mu(\Omega)g(\alpha) + (1-\lambda)\mu(\Omega)g(\beta).$$

Hence the transformation $T$ given by Theorem 5.1 (Theorem 4.1) is a solution of $\mathcal{P}$.
□

**Remarks.**

1. Since the function $\Phi$ is real valued, the problem of finding a solution is underdetermined, and, in general, one cannot expect uniqueness of the solution. Actually, in the case of the jacobian determinant it is easy to see that the problem admits infinitely many solutions. Indeed when $T_B$ is not a solution, the assertion follows easily from the construction of the solution defined in Theorem 4.1 and in Theorem 5.2, since it depends on a scalar function $v$ which can be defined in infinite ways (depending on the choice of the set of vectors which constitute the range of the gradient of $v$). When $T_B$ is a solution of $\mathcal{P}$ we simply notice that, given a regular transformation $J : \Omega \to \Omega$, different from the identity, such that $\det(\nabla J) = 1$ on $\Omega$ and $J\big|_{\partial\Omega} = I\big|_{\partial\Omega}$ ($I$ denotes the identity), it is

$$\det\left(\nabla\left(T_B \circ J\right)\right) = \det(\nabla T_B)$$

and

$$T_B \circ J\big|_{\partial\Omega} = T_B\big|_{\partial\Omega}.$$

Hence $T \equiv T_B \circ J$ is a solution. Since there exist infinitely many transformations $J$ with such properties (see [DM]) the assertion is proved.

2. The assumptions on the boundary data can be slightly relaxed. Indeed the proof of existence can be easily reproduced for a piecewise affine bondary datum $T_B$ provided $\Phi(\nabla T_B)$ is constant.

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