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Single-input control-affine systems: Local regularity of optimal trajectories and a geometric controllability problem

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Sunto

La tesi tratta di sistemi di controllo non lineari, sebbene affini rispetto al controllo, del tipo

$$\dot{q} = f(q) + u\, g(q), \qquad u \in [-1, 1], \tag{$\triangle$}$$

dove f e g sono due campi vettoriali definiti su una varietà differenziabile M. Si suppone che f, g ed M siano lisci ($\mathcal{C}^\infty$).

I primi tre capitoli sono rivolti allo studio di proprietà di regolarità locale del sistema ($\triangle$). I metodi usati coniugano strumenti classici della teoria del controllo ottimale, quali il principio di massimo di Pontryagin e le sue generalizzazioni note sotto il nome di *proprietà sufficienti del secondo ordine*, e tecniche più specificamente non lineari, che mettono in relazione proprietà geometriche locali di ($\triangle$) con strutture algebriche, come ad esempio la *configurazione delle parentesi di Lie* del sistema.

Le proprietà di regolarità locale di ($\triangle$) sono intese come segue: si vogliono trovare condizioni stabili sulla struttura differenziale del sistema ad un dato punto q di M, tali che ogni traiettoria di ($\triangle$) che non si discosti troppo da q e che minimizzi il tempo necessario a congiungerne gli estremi, sia liscia a tratti. Si vogliono inoltre stabilire delle limitazioni quantitative sul numero delle componenti lisce, dette *archi*, di una traiettoria di questo tipo, e altre, qualitative, sul tipo di controllo corrispondente a ciascun arco. In modo particolare, si cercano criteri che garantiscano l'assenza di fenomeni di tipo Fuller, ovvero di traiettorie minimizzanti il cui corrispondente controllo commuti infinite volte, in un tempo finito, da un estremo all'altro del segmento $[-1, 1]$.

L'ultimo capitolo della tesi è dedicato allo studio di un problema più specifco, che ammette una formulazione del tipo ($\triangle$). Si tratta di un'estensione del classico problema di Dubins al caso di superfici Riemanniane non omogenee. In termini concreti, si immagina di controllare, per mezzo del solo sterzo, un mezzo meccanico m che viaggi a velocità costante su una superficie N. Dati due punti di N ed assegnata a ciascuno di essi una direzione, si vuole stabilire se essi siano o meno congiungibili da una traiettoria percorribile da m e tangente alle direzioni scelte. Risultati negativi sono noti nel caso in cui N abbia curvatura negativa. Il nostro contributo consiste nel mostrare che, viceversa, se la curvatura di N è nonnegativa, per lo meno al di fuori di un compatto, allora, a prescindere dalle limitazioni sulla capacità sterzante di m, il problema ammette una soluzione. In assenza di un agente localizzante su N, quale ad esempio una struttura di gruppo di Lie, l'approccio proposto consiste nel coordinare, in un'ottica discreta, le descrizioni locali del comportamento del sistema. È nostro auspicio che il metodo si riveli adatto ad ulteriori applicazioni.

Contents

Introduction

Single-input control systems of the type

$$\dot{q} = f(q) + u\,g(q), \qquad u \in [-1,1], \tag{I.1}$$

where f and g are smooth vector fields on a manifold M, always attracted a special attention among nonlinear control theorists, since they represent a sort of laboratory, where nonlinear features appear in great purity.

Of course, systems in the form (I.1) are of far more than purely theoretical interest, as they offer, at least in first approximation, a convenient modeling of many 'real' single-input control systems.

Our attention is mainly devoted to the study of local regularity properties of trajectories of (I.1). For the moment, let us focus on trajectories of (I.1) which are time-optimal, that is, which minimize the time needed to join two given points. Our viewpoint is local in the following sense: Given $q_0 \in M$, we study whether there exist a neighborhood U of q_0 and a time $T > 0$, such that any control function corresponding to a time-optimal trajectory of the system (I.1), contained in U and defined on a time-interval of length less than T, is piecewise smooth. Moreover we are interested in finding an integer $k \geq 1$, if one exists, such that the number of smooth pieces of any such control function can be bounded by k.

Notice that control functions are defined up to modification on zero measure sets, and their smoothness should be understood accordingly. A piece of trajectory where the control is smooth is called an *arc*. *Bang arcs* correspond to constant control equal to $+1$ or -1. An arc which is not bang is called *singular*. If two bang arcs are concatenated, we call *switching time* the instant at which the control changes sign. A finite concatenation of bang arcs is called a *bang-bang trajectory*. If U, T and k, with the property described above, exist, then we say that *all short trajectories near* q_0 are concatenations of at most k arcs. The first tool which is used in order to restrict the family of candidate time-optimal trajectories is the Pontryagin maximum principle, which states that, if $q:[0,T] \to M$ is a time-optimal trajectory of (I.1), then there exists a nontrivial lift $\lambda(\cdot)$ of $q(\cdot)$ in T^*M such that, for almost every $t \in [0,T]$,

$$\dot{\lambda}(t) = -\lambda(t)(Df(q(t)) + u(t)Dg(q(t)))$$

and

$$\langle \lambda(t), (f + u(t)g)(q(t)) \rangle = \min_{v \in [-1,1]} \langle \lambda(t), (f + vg)(q(t)) \rangle \leq 0\,.$$

We call *extremal trajectory* any solution of (I.1) which satisfies the Pontryagin maximum principle.

As it is well-known (see Sussmann [57]), no irregularity of the optimal control can be apriori excluded. Fixed any measurable control function $u(\cdot)$, there always exists a control system of the type (I.1), a time-optimal trajectory of which corresponds to $u(\cdot)$. The correct question is: What kind of behavior can we expect for time-optimal trajectories of a generic system? Agrachev in [1] proved that, for a generic system, the union of bang and singular arcs is a dense set in the domain of definition of any extremal trajectory. Moreover, he showed that the set of discontinuities of a *boundary control*, that is, a control function with values in $\{-1, +1\}$, contains generically an isolated point (if it is not empty). A complementary result by Bonnard and Kupka [12], proves that, for a generic system, an extremal trajectory has only singular arcs of minimal order. Even without entering in the details of the definition of minimal order singular arcs, we can say that they are the easiest to compute, as they are projections on M of the trajectories in T^*M of an explicitly defined Hamiltonian system.

Many natural questions still have to be answered. Fuller first proposed [26] a control problem, which admits a polynomial formulation of the type (I.1), having a time-optimal solution corresponding to a chattering control function. We call *chattering* a boundary control whose switching times form a monotone convergent sequence. Since then, many efforts have been made in order to understand whether this phenomenon is typical – that is, stable with respect to small perturbations of the system – or not. When M has dimension two, for a generic system, chattering controls do not show up, as was first pointed out by Sussmann [55]. On the other hand, in big enough dimension the chattering phenomenon most likely is typical: stable chattering extremals were constructed by Kupka [32], though the optimality of these extremals is not proved. Other results of the same nature can be found in the monograph on chattering control by Zelikin and Borisov [65]. In particular, it is shown that stable chattering behavior appear when the dimension of M is greater than or equal to seven. As for the stability of chattering in dimension three, neither a positive nor a negative answer is known.

Another open problem is whether Fuller phenomenon is the worst possible typical behavior. Proposition 1.5 gives a partial answer to this question, proving that, generically, an extremal trajectory corresponding to a boundary control is either bang-bang or such that its restriction to a subinterval is chattering. Therefore, any possible bad behavior which is stable is built up, in some sense, by chattering modes. This result still allows very complicated behaviors of the control function, but it can be used to show that, if in a certain region chattering can be excluded, then a time-optimal trajectory passing through that region is bang-bang or contains at least one singular arc.

Beside these fascinating theoretical challenges, a finite bound on the number of arcs of time-optimal trajectories has a clear role in applications, since it restricts drastically – from infinite to finite dimension – the family of candidate open loop optimal strategies. Moreover, an apriori restriction on the local structure of optimal trajectories is a crucial step in order to prove the existence and to get a topological characterization of an optimal feedback flow. Such kind of results can be obtained by means of the so-called Boltyansky-Brunovsky synthesis, whose idea is that, under certain regularity properties, an extremal feedback flow is actually optimal. The first attempt in this direction was made by Boltyansky [11]; after then, his work has been considerably refined (Brunovsky [18]; Sussmann [54]). One of the regularity properties which plays an essential role in the construction of regular syntheses is precisely a local upper bound on the number of arcs of extremal trajectories, even if some recent results open the perspective of enlarging the analysis to chattering controls as well (Piccoli and Sussmann [42]).

Transversality theory gives us adequate instruments for investigating generic properties of the system. Let $k \geq 0$ and denote by $J^{2,k}M$ the vector bundle on M of all k-jets of pairs of vector fields on M. Let A be a stratified subset of $J^{2,k}M$ such that $J^{2,k}_q M \cap A$ is of codimension larger than m in $J^{2,k}_q M$, for every $q \in M$. Then, generically, the set of points $q \in M$ such that $J^{2,k}_q(f,g)$, the k-jet of (f,g) at q, belongs to A, has codimension larger than m in M.

In particular, if $m = \dim M$, then, generically, $J^{2,k}_q(f,g)$ does not belong to A, for every $q \in M$. Therefore, if we find A such that, for every f, g, and q such that $J^{2,k}_q(f,g) \notin A$, all short time-optimal trajectories near q are finite concatenations of bang and singular arcs, and if $m = \dim M$, then, generically, chattering does not occur. More generally, for any $m \leq \dim M$, if we can show that no chattering appears near points for which $J^{2,k}_q(f,g) \in A$, then we actually give a bound on the dimension of the set of points near which chattering cannot be excluded.

Subsets of $J^{2,k}M$ are conveniently defined in terms of so-called "non-resonance conditions" on the configuration of iterated Lie brackets between f and g, that is, by means of relations of linear independence of tangent vectors chosen between $f(q)$, $g(q)$, $[f,g](q)$, $[f,[f,g]](q)$, $[g,[f,g]](q)\ldots$ Indeed, Lie brackets bear the intrinsic relations between the derivatives of the jet. Moreover, they potentially contain all informations about the system. In fact, the family of all relations between iterated Lie brackets form a complete set of differential invariants of an analytic system (see Sussmann [54]).

The origin of the local analysis of (I.1) by means of Lie bracket relations dates back to the pioneering work by Lobry [34]. Lobry pointed out that, if f, g, and $[f,g]$ are linearly independent at a point q_0 of a three-dimensional manifold, then there exists a neighborhood U of q_0 such that the set R of points of U which are attainable from q_0 with trajectories staying in U, has the following structure. Its boundary is given by the union of $\partial U \cap \overline{R}$ and of two surfaces S_1 and S_2. Each S_j, $j = 1, 2$, is obtained as the support of the family of two-bang trajectories leaving q_0, staying in U, for which the control switches from $(-1)^j$ to $(-1)^{j+1}$.

Sussmann started his analysis of local regularity properties of generic planar systems [55] with the remark that, if a two-dimensional system is non-degenerate at a point, then the augmented system where the time is considered as a third component verifies the hypothesis under which Lobry's result holds. Even if the existence of an upper bound on the number of arcs of short time-optimal trajectories of a generic planar system is stated in [55], the announced sequel of the paper never appeared. Further analysis of smooth systems is contained in [58], while [59] provides a generic bound for analytic systems. In the final paper of the series [60], Sussmann proves the existence of a regular synthesis for analytic systems, under mild non-explosion assumptions. An upper bound for generic smooth systems in dimension two was given by Piccoli in [40]. For a new and shorter proof see [16], where many further advances in the classification of planar optimal syntheses are collected.

The three dimensional situation, as it is reasonable to expect, is more complicated, and our knowledge of its generic features is still considerably incomplete. Sussmann proves in [56] that any short time-optimal bang-bang trajectory, near a point at which

$$f \wedge g \wedge [f,g] \neq 0 \tag{I.2}$$

and

$$g \wedge [f,g] \wedge [f \pm g, [f,g]] \neq 0\,, \tag{I.3}$$

has at most two switchings. The result is based on the second order necessary condition for optimality know as *envelope theory*. The theory extends the classical envelope technique of calculus of variations, and is based on the idea of testing the optimality of an extremal trajectory by embedding it in a field of admissible ones. A subfamily is extracted among these trajectories, whose elements have the same length and the same initial and final point as the reference curve. Therefore, if we can show that at least one of the elements of the family is not optimal, then neither the reference trajectory is.

In a later work [17], Bressan shows that, if q is an equilibrium point for f and the two conditions (I.3) hold at q, then there exists a neighborhood U of q such that any time-optimal trajectory steering a point of U to q is the concatenation of at most three between bang and singular arcs. Schättler analyzes the local structure of time-optimal trajectories far from equilibria of f, under generic conditions [48, 49]. The computational techniques used in this thesis are similar in nature to the ones introduced by Schättler, although the two approaches differ substantially at the level of identification of obstructions to optimality.

In [31], Krener and Schättler return somehow to Lobry's intuition and to the spirit in which Sussmann applied it to planar problems. Indeed, they consider a system of the type (I.1) in dimension four, and they analyze the structure of a small-time attainable set from a non-degenerate point. As a consequence, they derive that, in dimension three, near a point where (I.3) holds, short time-optimal trajectories are concatenations of at most three arcs. The same result is proved, for bang-bang trajectories only, by Agrachev and Gamkrelidze [4], while developing another kind of second order optimality condition, the so-called *index theory*. A third proof of the result, valid for bang-bang trajectories only, is given by Sussmann in [61], in the framework of envelope theory. A generalization of the procedure adopted by Krener and Schättler is contained in a paper by Schättler and Jankovic [51], where a local optimal synthesis is derived for trajectories steering to an equilibrium point q of f at which only one of the two conditions (I.3) holds. A bound on the number of concatenated bang and singular arcs of a short time-optimal trajectory steering to q is given by four. In particular it is shown that *saturated singular controls* are a typical phenomenon in dimension three.

This thesis shares the viewpoint of [4], assigning to index theory a fundamental role, and, in some sense, developing its possible means of implementation. The idea behind the method shares the same spirit as the Pontryagin maximum principle, that is, non-optimality is proved through conditions which ensure the openness of the endpoint mapping. What is done in practice, is the evaluation of the asymptotics of the endpoint mapping at the reference control function. In the case of bang-bang controls, the computations take place in a suitable finite dimensional space, and become rather explicit. An important property of index theory is that it provides not only necessary, but also sufficient conditions for local optimality (see Sarychev [47] and Agrachev, Stefani and Zezza [9]). Chapter 1 discusses different formulations of this index optimality criterion. In the same chapter, it is also recalled another fundamental second order condition for optimality, the so-called *generalized Legendre condition*, which embodies the same kind of principle, for the case of singular controls.

Our first contribution to the understanding of generic properties of three dimensional systems is the analysis of the case in which one of the two conditions in (I.3) fails to hold. Let $g \wedge [f,g] \wedge [f+g,[f,g]] = 0$ and $g \wedge [f,g] \wedge [f-g,[f,g]] \neq 0$ at q. Assume in addition that $g \wedge [f,g] \wedge [f+g,[f+g,[f,g]]](q) \neq 0$. Then, we are able to prove that short time-optimal trajectories near q are concatenation of at most four arcs. As already remarked, a result of this kind can be read as an apriori bound on the dimension of the (possible) chattering phenomena of a generic three-dimensional system.

The study of order two singularities is started in chapter 3. Some of these singularities can be treated by refinements of the reasonings applied in lower order cases, and corresponding bounds are eventually given. The complete understanding of codimension two singularities is reduced to the case in which both conditions in (I.3) fail to hold. We propose an analysis of the local behavior of the system near points at which such conditions are violated, by means of some sort of nilpotent approximation, in the same spirit as [5] and [10]. The approximation which we adopt seems to acknowledge the fact – suggested by the previous results – that the direction of f has not a special role in determining local regularity properties of optimal trajectories.

When the dimension of M is equal to four, we give a local finite bound on the number of arcs of time-optimal trajectories in dimension four, near points at which

$$\begin{aligned} g \wedge [f,g] \wedge [f+g,[f,g]] \wedge [f-g,[f,g]] &\neq 0\,, \\ g \wedge [f,g] \wedge [f+g,[f,g]] \wedge [f+g,[f+g,[f,g]]] &\neq 0\,, \\ g \wedge [f,g] \wedge [f-g,[f,g]] \wedge [f-g,[f-g,[f,g]]] &\neq 0\,. \end{aligned}$$

Actually, this bound, as well as the ones which we discussed for the three-dimensional case, is formulated in a more general contest. Indeed, let S be a codimension s submanifold of M. We will call *manifold-to-point time-optimal problem* the control problem with dynamics (I.1), where the goal is to connect S with points of M in minimal time.

The results appeared in the literature dealing with the general manifold-to-point problem are, up to our knowledge, scarcer than the ones devoted to the *point-to-point problem*, where S reduces to a point. The main contributions are given by the works by Bonnard, Launay and Pelletier [13] and Launay and Pelletier [33]. The two articles furnish a classification of time-optimal syntheses for the manifold-to-point problem, when $s = 1$ and M has dimension two or three, for generic analytic systems. In [33] the analysis is extended to the case in which the dimension of M is any, still for $s = 1$. The synthesis is obtained near all points of S where the system has a singularity of codimension smaller than three. Another situation which is studied is the case of g everywhere tangent to S, as it is motivated by applications to control problems for batch reactors (see also Bonnard and de Morant [14]).

The results which are obtained in chapter 2 hold for source manifolds S of codimension less than or equal to four, with no restriction on the dimension of M. We are able to give a bound on the number of arcs of short time-optimal trajectories near all points of S at which the system has singularities of codimension smaller than $5 - s$. The results are partially new for codimension zero, one, and two singularities, when $s = 1$, since we require the system just to be smooth and not necessarily analytic. They are completely new in all the other cases.

Second order necessary conditions for optimality are conveniently reformulated, in chapter 1, in order to make them apply to the manifold-to-point problem.

Beyond time-minimality, another interesting optimality notion for systems with drift is given by time-maximality. If, for instance, S represents the boundary of a safety region outside which every trajectory is driven by the drift, and if the scope of the controller is to make the trajectories stay inside such region, then the strategy which should be implemented is the one which maximizes the time needed to steer the starting point to S. A class of trajectories which includes time-optimal and time-maximal ones, the class of *quasi optimal trajectories*, is introduced in chapter 1. All local regularity properties are stated in terms of this class.

Chapter 4 deals with a specific geometric problem, which can be described by a three-dimensional control system in the form (I.1). Let N be a connected Riemannian oriented two-dimensional complete manifold and $M = T^1N$ its unit tangent bundle. Denote by π the bundle projection from M to N. Let f be the geodesic vector field on M, i.e., the restriction on M of the vector field on TN whose flow, projected on N, gives the usual geodesic flow. Define g as the vector field whose flow is the fiberwise rotation of constant angular velocity equal to one. Remark that, since $\pi_*(f(q)) = q$ and $\pi_*(g(q)) = 0$, there is a one-to-one correspondence between admissible trajectories of (I.1) and curves in N obtained by their projection. The inverse operation is obtained just by pairing state and velocity of curves in N. If N is the Euclidean plane, then the control problem (I.1) is classically known as the *Dubins' problem.* Dubins' original formulation [25] is the following: Which is the shortest curve in the Euclidean plane, with geodesic curvature bounded by one, which joins two given points, being tangent to two prescribed directions at that points? In the robotics literature, the system obtained on M models the motion of a unicycle (or rolling penny) in the plane. Dubins himself determined the global structure of the solutions of the problem: he showed that length-minimizing curves are concatenations of at most three pieces made of circles of radius one or straight lines. Further restrictions on the length of the arcs of an optimal concatenation have been proved by Sussmann and Tang [63].

Even when N is not necessarily the Euclidean plane, trajectories of (I.1) correspond to curves in N with geodesic curvature bounded by one, and the time-optimal control problem can be reformulated exactly in Dubins' terms. For this reason, we will continue to call it *Dubins' problem.* Notice that, while the original Dubins' formulation extends directly to the case in which the dimension of N is any – and, in fact, it was stated for N being any Euclidean space $\mathbf{R}^n$ –, the same is not true for its expression in the form (I.1).

From the viewpoint of local regularity properties, the system is highly non-degenerate, since, at every point of M, both (I.2) and (I.3) hold, as it can be easily computed. Our idea is to exploit what we know about the local behavior of the system, in order to obtain results of global nature. An important step for this kind of approach is the estimate of the size of the subsets of M in which a description *à la Lobry* of small-time attainable sets applies. We will focus on controllability properties, and we will try, in particular, to address the following question: Which kind of geometric conditions on N guarantee that the system

$$\dot{q} = f(q) + u\,g(q), \qquad u \in [-\varepsilon, \varepsilon]\,, \tag{I.4}$$

is completely controllable for every $\varepsilon > 0$? If such controllability property holds, we say that the Dubins' problem on N is *unrestrictedly controllable.*

The first example of condition on N which ensures unrestricted controllability is of topological nature: If N is compact, then the Dubins' problem is unrestrictedly controllable. The result follows from some Poisson stability argument, or can be directly deduced from a much more general theorem by Lobry on control systems defined by conservative vector fields [35].

Let us assume that N is noncompact. A geometric quantity which seems to play a crucial role in the characterization of controllable Dubins' problems is the Gaussian curvature K of N. The curvature appears quite soon in the study of the Lie bracket configuration of the problem; indeed, $[f,[f,g]](q) = -K(\pi(q))g(q)$, for every point $q \in M$. This relation suggests the role of K, at least for what concerns the local behavior of the system. On the other hand, if K is identically equal to zero on N, then N is isometrically diffeomorphic to a cylinder, possibly degenerating into a plane. A controllability strategy on a N is therefore obtained by projecting the one valid on the plane, seen as the universal covering of N. This simple idea of applying strategies which are valid on local or global covering of the manifold plays a special role in our approach.

The unrestricted controllability property extends to manifolds N on which K tends to zero at infinity. A key remark, that we use in the proof, is that the uniform decay of K guarantees the existence of almost flat disks of any prescribed size, which are local covering of far enough subsets of N. On these disks one can approximate flat control strategies, and the controllability of (I.4) follows.

An example of Riemannian manifold for which the controllability property fails to hold is the Poincaré half-plane: If $K \equiv -1$ and $\varepsilon \leq 1$, then attainable sets of (I.4) are proper subsets of M, as it was first shown by Monroy-Perez in [38]. Roughly speaking, this happens because the negativeness of K, not only prevents the geodesics to have conjugate points, but actually is an obstacle for the controlled turning action to overcame the spreading of the geodesics.

It is reasonable to try to understand what happens when the curvature of N is non-negative. If $K \geq 0$ and K is not identically equal to 0, then M is homeomorphic to the plane $\mathbf{R}^2$ and, due to Cohn-Vossen theorem [23],

$$\int_M K dA \leq 2\pi ,$$

where dA is the surface element in M. In particular, K decays at infinity in integral sense. The same is true, in general, if $K \geq 0$ outside a compact subset of N, even if the topology of N can be more complicated. The integral decay of K may be exploited, under the assumption that K is bounded from above, in order to estimate the behavior of trajectories of the system, at a local level. After that, we glue together families of subsets of N on whose unit bundle local estimates apply, and we obtain covering domains of prescribed size. The design of approximated control strategies in these domains, as well as the proof of the existence of exact ones, are now possible, although more delicate than in the case in which K is uniformly small at infinity. Finally, we prove that, if both K and the subset of N where K is negative are bounded, then the Dubins' problem on N is unrestrictedly controllable.

Chapter 1

First and second order necessary conditions for optimality

The present chapter is mostly an introduction to the language employed in the thesis and a presentation of index theory, the framework in which regularity properties of optimal trajectories are discussed in chapters 2 and 3. Section 1.1 fixes some notations of chronological calculus, a tool first developed by Agrachev and Gamkrelidze in [3], which allows intrinsic description and manipulation of nonlinear objects and dynamics.

Section 1.2 discusses the application of Pontryagin maximum principle [44] to single-input control-affine systems. The notion of extremal pair is introduced, together with the one of transversality condition for the manifold-to-point problem.

Proposition 1.5 (first appeared in our paper [7]) is obtained from simple properties of differentiation of the switching function. It is an improvement of Proposition 2 in [1].

Section 1.4 contains the formulation of various second order necessary conditions for optimality. Theorems 1.6 and 1.8 present the results on single-input control-affine systems which are at the theoretical core of our approach. Theorem 1.6, in a much more general setting, was first proved in [4]. The proof which is provided here deals more specifically and with more details with single-input control-affine system. Theorem 1.8 applies index theory to the case of general initial value conditions. Its original formulation is contained in our work [8].

Finally, the generalized Legendre condition on optimal singular trajectories is recalled. Its first formulation and mathematical proof date back, respectively, to [29] and [2].

1.1 Chronological calculus

Fix a smooth finite dimensional manifold M and denote by $\mathrm{Vec}M$ the space of smooth vector fields on M. We will assume that all differential objects which are considered are C^∞, unless otherwise specified. A vector field $X \in \mathrm{Vec}M$ is said to be *complete* if, for every $q_0 \in M$, the solution of the Cauchy problem

$$\begin{cases} \dot{q}(t) = X(q(t)), \\ q(0) = q_0, \end{cases} \tag{1.1}$$

is defined for every $t \in \mathbf{R}$. If X is complete, then the map which associates with any $q_0 \in M$ the value of the solution of (1.1) evaluated at a fixed time t is a diffeomorphism from M into itself, denoted by

$$e^{tX} : q_0 \mapsto e^{tX}(q_0) \,,$$

and called the *flow of* X *at time* t. Consider now a non-autonomous vector field $t \mapsto X_t$, that is, a measurable function from $\mathbf{R}$ into the space $\mathrm{Vec}M$. It is well known that, for every $q_0 \in M$ and $t_0 \in \mathbf{R}$, there exists a local Caratheodory solution of the Cauchy problem

$$\begin{cases} \dot{q}(t) = X_t(q(t)) \,, \\ q(t_0) = q_0 \,, \end{cases} \tag{1.2}$$

that is, there exist a neighborhood I of t_0 and a unique absolutely continuous curve $q : I \to M$ such that $q(t_0) = q_0$ and, for almost every $t \in I$, $\dot{q}(t) = X_t(q(t))$. If $t \mapsto X_t$ is complete, that is, if, for every t_0 and q_0, (1.2) has a solution defined for all $t \in \mathbf{R}$, then we denote by

$$\overrightarrow{\exp} \int_{t_0}^{t_1} X_t dt : M \to M$$

the flow which associates with $q_0 \in M$ the solution of (1.2) at time t_1. Remark that, by definition,

$$\overrightarrow{\exp} \int_{t_0}^{t_1} X_t dt = \left(\overrightarrow{\exp} \int_{t_1}^{t_0} X_t dt \right)^{-1} .$$

Here and in the sequel, we assume that all the vector fields which are taken into account are complete. This is justified by the fact that our attention is mainly devoted to local results. In chapter 4, where a global viewpoint is adopted, completeness of the vector fields is explicitly discussed.

A vector field X can be identified in a natural way with an operator on $\mathcal{C}^\infty(M)$: Given a smooth function a on M and a point $q \in M$, we define $(Xa)(q)$ as the derivative of a in the direction $X(q)$ at the point q. From this viewpoint the *Lie bracket (or commutator)* of two vector fields X_1 and X_2 is given by

$$(\mathrm{ad}X_1)X_2 = [X_1, X_2] = X_1 \circ X_2 - X_2 \circ X_1 \,, \tag{1.3}$$

that is, for every $a \in \mathcal{C}^\infty(M)$,

$$[X_1, X_2]a = X_1(X_2 a) - X_2(X_1 a) \,.$$

This relation indeed defines a vector field, which can be represented – with respect to any fixed system of coordinates – as follows,

$$[X_1, X_2](q) = (DX_2)X_1(q) - (DX_1)X_2(q) \,,$$

where $[X_1, X_2](q)$, $X_1(q)$, and $X_2(q)$ are identified with column vectors, and DX_i denotes the Jacobian matrix of X_i evaluated at the point q, $i = 1, 2$. Remark that the Lie bracket operation equips $\mathrm{Vec}M$ with a Lie algebra structure.

Any diffeomorphism $P : M \to M$ acts on $\mathrm{Vec}M$, associating with $X \in \mathrm{Vec}M$ the vector field $\mathrm{Ad}\, P(X)$, according to the formula

$$\mathrm{Ad}\, P(X)(q) = \left(P^{-1}\right)_* (X(P(q))) \,,$$

where $(\ )_*$ denotes the standard push-forward operator. From (1.3), one easily gets

$$\operatorname{Ad} P[X_1, X_2] = [\operatorname{Ad} P(X_1), \operatorname{Ad} P(X_2)] . \tag{1.4}$$

The relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\operatorname{Ad}\ \overrightarrow{\exp} \int_{t_0}^{t} X_\tau d\tau\right) Y\right)(q) = \left(\left(\operatorname{Ad}\ \overrightarrow{\exp} \int_{t_0}^{t} X_\tau d\tau\right) [X_t, Y]\right)(q) ,$$

which holds, at q fixed, for almost every t, justifies the notation

$$\overrightarrow{\exp} \int_{t_0}^{t} \operatorname{ad} X_\tau d\tau = \operatorname{Ad}\ \overrightarrow{\exp} \int_{t_0}^{t} X_\tau d\tau . \tag{1.5}$$

For the particular case of autonomous vector fields, we write

$$e^{t \operatorname{ad} X} = \operatorname{Ad} e^{tX} .$$

Let $\phi, \gamma : \mathbf{R} \to \mathbf{R}$ be two smooth functions and fix $q_0 \in M$ and $X, Y \in \operatorname{Vec} M$. Consider the smooth curve in M defined by $q(t) = e^{\phi(t)X} \circ e^{\gamma(t)Y}(q_0)$. A consequence of all what has been said since now is that the following chain rule holds

$$\frac{d}{dt} q(t) = \dot\phi(t) X(q(t)) + \dot\gamma(t) \left(e^{-\phi(t) \operatorname{ad} X} Y\right)(q(t)) . \tag{1.6}$$

1.2 The Pontryagin maximum principle

Let f, g be chosen in $\operatorname{Vec} M$ and consider the associated control system

$$\dot q = f(q) + u\, g(q), \qquad u \in [-1, 1] . \tag{1.7}$$

A *control function* is a measurable map $t \mapsto u(t)$ with values in the interval $[-1,1]$. A Caratheodory solution of (1.7), corresponding to an admissible control function, is called a *trajectory* of the system.

For all $T > 0$ and $q_0 \in M$, the *attainable set from* q_0 *at time* T is defined as follows,

$$A(T, q_0) = \{q(T) \mid q : [0, T] \to M \text{ is a trajectory of (1.7) such that } q(0) = q_0\} .$$

From the classical Filippov's theorem, it follows that $A(T, q_0)$ is compact.

Given a point $q \in M$, the action on $T_q M$ of a covector $\lambda \in T_q^* M$ is denoted by $\langle \lambda, \cdot \rangle$. We say that λ is orthogonal to a linear subspace W of $T_q M$, and we write $\lambda \perp W$, when $\langle \lambda, w \rangle = 0$ for every $w \in W$.

Let $q : [0, T] \to M$ be a trajectory of (1.7) and $u(\cdot)$ the corresponding control function. Assume that $q(\cdot)$ minimizes the time needed to steer $q(0)$ to $q(T)$, i.e., $q(T) \notin A(t, q(0))$ for every $t < T$. In this case, we say that $q(\cdot)$ is *time-optimal* (or, equivalently, *time-minimal*). By the Pontryagin maximum principle, there exist $c \leq 0$ and an absolutely continuous covector trajectory $\lambda : [0, T] \to T^* M$ such that

$$\lambda(t) \in T^*_{q(t)} M \setminus \{0\} \qquad \text{for every } t \in [0, T] , \tag{1.8}$$

and which verify for almost every t the equation

$$\dot\lambda(t) = \vec{h}_{u(t)}(\lambda(t)) , \tag{1.9}$$

and the relation

$$h_{u(t)}(\lambda(t)) = \min_{v\in[-1,1]} h_v(\lambda(t)) \equiv c\,, \tag{1.10}$$

where the family of Hamiltonians h_v is defined by

$$h_v(\lambda(t)) = \langle \lambda(t), (f+vg)(q(t))\rangle\ , \tag{1.11}$$

and $\vec{h}_v \in \mathrm{Vec}(T^*M)$ denotes the Hamiltonian vector field associated with h_v. If a system of coordinates is fixed and covectors are identified with row vectors, then (1.9) can be written as

$$\dot\lambda(t) = -\lambda(t)(Df(q(t)) + u(t)Dg(q(t)))\,. \tag{1.12}$$

If $q(\cdot)$ is, instead of time-minimal, time-maximal, i.e., $q(T) \notin A(t, q(0))$ for every $t > T$, then $\lambda(\cdot)$ and c as above still exist, with the only difference that in this case c verifies the opposite inequality $c \geq 0$.

An *extremal pair for (1.7)* is a pair $(\lambda(\cdot), q(\cdot))$, where $q(\cdot)$ is a trajectory of (1.7) and $\lambda(\cdot)$ verifies (1.8 – 1.10). If $(\lambda(\cdot), q(\cdot))$ is an extremal pair, then we say that $q(\cdot)$ is an *extremal trajectory* and that $\lambda(\cdot)$ is an *extremal lift* of $q(\cdot)$.

Let S be a submanifold of M. We can associate with S the *attainable set from S at time T*

$$A(T,S) = \{q(T) \mid q\colon[0,T] \to M \text{ is a trajectory of (1.7) such that } q(0) \in S\}\,,$$

and, as above, we can define time-minimal and time-maximal trajectories for the control problem

$$\begin{cases} \dot q = f(q) + u\,g(q), & u \in [-1,1], \\ q(0) \in S\,. \end{cases} \tag{1.13}$$

It is clear that, if $q\colon[0,T] \to M$ is time-minimal or time-maximal in this sense, then it also minimizes or maximizes the time needed to steer $q(0)$ to $q(T)$, and, therefore, it admits an extremal lift $\lambda(\cdot)$. The Pontryagin maximum principle, applied to (1.13), states that $\lambda(\cdot)$ can be assumed to verify, in addition,

$$\lambda(0) \perp T_{q(0)}S\,. \tag{1.14}$$

We will call S the *source* of the control problem (1.13). A time-minimal or time-maximal trajectory $q\colon[0,T] \to M$ of (1.7) can be seen as a solution of a control problem of the type (1.13), where S reduces to the singleton $\{q(0)\}$. In this case we have $T_{q(0)}S = (0)$ and so, not surprisingly, (1.14) does not give any extra information on the extremal lifts of $q(\cdot)$.

Definiton 1.1 *Let S be a submanifold of M. An extremal pair $(\lambda(\cdot), q(\cdot))$ for (1.7) is called S-***extremal** *if $q(0) \in S$ and $\lambda(0)$ satisfies (1.14). If $(\lambda(\cdot), q(\cdot))$ is an S-extremal pair, then we say that $q(\cdot)$ is an S-***extremal trajectory** *and that $\lambda(\cdot)$ is an S-***extremal lift** *of $q(\cdot)$.*

The proof of Pontryagin maximum principle, as in [44], implies that not only time-minimizing and time-maximizing trajectories are extremal, but all the elements of a wider class, which we will call *quasi optimal trajectories*. It is easier to define them negatively:

Definiton 1.2 *Let $q:[0,T]\to M$ be a trajectory of (1.7) (respectively, (1.13)). We say that $q(\cdot)$ is* **essential** *(respectively, S-***essential***) if and only if there exists a neighborhood O of (f,g) in $\mathrm{Vec}M\times\mathrm{Vec}M$, with respect to the C^1-topology, such that, for every $(f',g')\in O$, $q(T)$ belongs to the interior of the attainable set from $q(0)$ (respectively, from S) at time T for the system*

$$\dot{q}=f'(q)+ug'(q)\,.$$

If $q(\cdot)$ is not essential (respectively, not S-essential), then it is called **quasi optimal** *(respectively, S-***quasi optimal***).*

Remark 1.3 Time-minimal and time-maximal trajectories of (1.13) are S-quasi optimal. Indeed, fix a trajectory $q:[0,T]\to M$ of (1.7), and notice that, on a compact neighborhood of $A(T,q(0))$, time-rescaled systems with scaling factor close to one are C^1-close to the original one. Therefore, if $q(\cdot)$ is S-essential, then $q(T)\in A(t,S)$ for t close to T, which means that $q(\cdot)$ is not the fastest trajectory connecting S with $q(T)$ neither the slowest.

Remark 1.4 A property which is shared by quasi optimality and time-optimality (and, in general, by any optimality defined through an integral cost) is the fact that the time-reversed of a quasi optimal trajectory is quasi optimal for the time-reversed system

$$\dot{q}=-f(q)-u\,g(q)\,. \tag{1.15}$$

Indeed, if $q:[0,T]\to M$ is essential for the control system (1.7), then there exist a neighborhood U of $q(0)$ and a neighborhood W of (f,g) in $\mathrm{Vec}M\times\mathrm{Vec}M$ such that, for any $(f',g')\in W$ and for any $q\in U$, $q(T)$ belongs to the attainable set from q at time T for the system $\dot{q}=f'(q)+ug'(q)$, as we can derive by a reparameterization argument similar to the one above. Thus, $q(0)$ belongs to the interior of the attainable set from $q(T)$ at time T for any system in a C^1-neighborhood of (1.15). In the case of control problems whose optimality notion gives asymmetric roles to the initial and the final condition, as it happens for S-quasi optimality, a property of this kind makes no sense. As we will see in remark 1.9, this lack of symmetry is partially recovered in the applications of index theory.

The notion of quasi optimality is wide and flexible and happens to be appropriate for getting necessary conditions which are even stronger than the Pontryagin maximum principle.

Let us now introduce the terminology in terms of which the regularity of a trajectory, or, equivalently, of a control function, is expressed. It is reasonable to consider every control function as defined up to modifications on a set of measure zero, since two control functions which are equal almost everywhere lead to equal solutions of (1.7). Given an admissible trajectory $q:[0,T]\to M$, its restriction $q|_{[\tau_1,\tau_2]}$ to a subinterval $[\tau_1,\tau_2]$ of $[0,T]$ is called an *arc* if the corresponding control function $u|_{[\tau_1,\tau_2]}$ is C^∞. We will use the word arc also to refer to the interval $[\tau_1,\tau_2]$ itself. We will assume that any arc is maximal, i.e., that, $[\tau_1,\tau_2]$ is an arc when, for every interval $[t_1,t_2]$ with $[\tau_1,\tau_2]\subset[t_1,t_2]\subset[0,T]$, $u|_{[t_1,t_2]}$ is smooth if and only if $[\tau_1,\tau_2]=[t_1,t_2]$. We say that two distinct arcs $[\tau_1,\tau_2]$ and $[t_1,t_2]$ are *concatenated* if $\tau_2=t_1$ or $\tau_1=t_2$. Special arcs are the so-called *bang arcs*, for which $u(\cdot)$ is constantly equal to -1 or $+1$. Depending on the sign of the control, a bang arc is called a $-$ arc or a $+$ arc. The time-instant between two concatenated bang arcs is called a *switching time*. A trajectory which is a finite concatenation of bang arcs is called a *bang-bang trajectory*. We say that a control function is a *boundary control* if it takes values in $\{-1,+1\}$. An arc which is not bang is said to be *singular*. A finite concatenation of arcs is described by juxtaposition of letters S and B, each S corresponding to a singular arc, each B to a bang one. A BSB trajectory, for instance, is the concatenation of a bang, a singular, and a bang arc. The letter B is sometimes replaced by a $+$ or a $-$, depending on the sign of the corresponding control.

With an extremal pair $(\lambda(\cdot), q(\cdot))$, it is classically associated the *switching function*

$$\varphi(t) = \langle \lambda(t), g(q(t)) \rangle , \tag{1.16}$$

which has the property, easily deducible from (1.10), that

$$u(t) = -\mathrm{sign}(\varphi(t)) ,$$

for every t such that $\varphi(t) \neq 0$.

Equation (1.12) implies that, given $X \in \mathrm{Vec}M$, for almost every t in $[0,T]$, the function $s \mapsto \langle \lambda(s), X(q(s)) \rangle$ is differentiable at t and

$$\begin{aligned} \frac{d}{dt} \langle \lambda(t), X(q(t)) \rangle &= \langle \lambda(t), DX(f(q(t)) + u(t)g(q(t))) \rangle \\ &\quad - \langle \lambda(t)(Df(q(t)) + u(t)Dg(q(t))), X(q(t)) \rangle \\ &= \langle \lambda(t), [f + u(t)g, X](q(t)) \rangle . \end{aligned} \tag{1.17}$$

In particular, for almost every $t \in [0,T]$ we have

$$\dot{\varphi}(t) = \langle \lambda(t), [f,g](q(t)) \rangle .$$

This equality holds, moreover, for every $t \in [0,T]$. Indeed, being φ absolutely continuous, it is everywhere equal to the primitive of its derivative. From the fundamental theorem of calculus, it follows that $\dot{\varphi}(t)$ is equal to $\langle \lambda(t), [f,g](q(t)) \rangle$ for every t. Thus, φ is a $\mathcal{C}^1$ function, its derivative is absolutely continuous and

$$\ddot{\varphi}(t) = \langle \lambda(t), [f + u(t)g, [f,g]](q(t)) \rangle \tag{1.18}$$

almost everywhere. In the interior of bang and singular arcs, the right hand side of (1.17) is absolutely continuous with respect to t. The higher order derivatives of φ can be computed iterating (1.17), showing that φ is $\mathcal{C}^\infty$ along any arc.

Let $X \in \mathrm{Vec}M$ and fix $t_1, t_2 \in [0,T]$. According to (1.5), we have

$$\begin{aligned} &\left\langle \lambda(t_1), \left(\overrightarrow{\exp} \int_{t_1}^{t_2} \mathrm{ad}\,(f + u(\tau)g) d\tau X \right) (q(t_1)) \right\rangle \\ &= \left\langle \lambda(t_1), \left(\overrightarrow{\exp} \int_{t_1}^{t_2} (f + u(\tau)g) d\tau \right)_* (X(q(t_2))) \right\rangle \\ &= \left\langle \left(\overrightarrow{\exp} \int_{t_2}^{t_1} (f + u(\tau)g) d\tau \right)^* \lambda(t_1), X(q(t_2)) \right\rangle . \end{aligned}$$

Since $\lambda(\cdot)$ is solution of (1.9), it turns out (see [6, Proposition 11.3]) that

$$\left(\overrightarrow{\exp} \int_{t_2}^{t_1} (f + u(\tau)g) d\tau \right)^* \lambda(t_1) = \lambda(t_2) .$$

Therefore,

$$\langle \lambda(t_2), X(q(t_2)) \rangle = \left\langle \lambda(t_1), \left(\overrightarrow{\exp} \int_{t_1}^{t_2} \mathrm{ad}\,(f + u(\tau)g) d\tau \, X \right) (q(t_1)) \right\rangle . \tag{1.19}$$

In particular,

$$\varphi(t_2) = \left\langle \lambda(t_1), \left(\overrightarrow{\exp} \int_{t_1}^{t_2} \mathrm{ad}\,(f + u(\tau)g) d\tau \, g \right) (q(t_1)) \right\rangle . \tag{1.20}$$

1.3 A regularity result on extremal trajectories of a generic system

Let Lie(f, g) be the subalgebra of Vec$M$ generated by f and g and denote by $I(g)$ the ideal of Lie(f, g) generated by g. It was proved by Agrachev in [1, Proposition 1] that, if

$$\{X(q) | \, X \in I(g)\} = T_q M \qquad \text{for every } q \in M \, ,$$

then the control function corresponding to an extremal trajectory $q : [0, T] \to M$ is smooth on an open dense set O of $[0, T]$. This section investigates, under suitable additional assumptions, what else can be said on O when it is known that it contains no singular arc.

For the sake of concision we set

$$X_{\pm} = [f \pm g, [f, g]] \, ,$$

and, in general, for every word w with letters in $\{+, -\}$, we define

$$X_{\pm w} = [f \pm g, X_w] \, . \tag{1.21}$$

We find also useful to write

$$X_{(m\pm)} = (\mathrm{ad}\,(f \pm g))^m \, [f, g] \, ,$$

and, for every $q_0 \in M$,

$$V_{\star}(q_0) = \mathrm{span}\{g(q_0), [f, g](q_0), X_{\star}(q_0), \ldots, X_{(m\star)}(q_0), \ldots\} \, .$$

Proposition 1.5 *Fix $f, g \in$ Vec$M$. Assume that, for every $q_0 \in M$, $V_+(q_0) = V_-(q_0) = T_{q_0}M$. Let $q : [0, T] \to M$ be an extremal trajectory of (1.7) such that the corresponding control function u verifies $|u| = 1$ on an open dense subset of $[0, T]$ and let Σ be the set of discontinuities of u (not avoidable by changing u on a set of measure zero). Then either Σ is discrete or it contains a monotone sequence $\{t_n\}_{n \in \mathbf{N}}$ of isolated points such that, for every $n \in \mathbf{N}$, the open interval identified by t_n and t_{n+1} does not intersect Σ.*

Proof. Let O be the maximal open dense subset of $[0, T]$ on which u is smooth. Thus, $\Sigma \cup \{0, T\} = \partial O$. By hypothesis, u has only bang arcs, which are exactly the connected components of O.

We say that two distinct points of $\Sigma \cup \{0, T\}$ are *subsequent* if the open interval which they identify does not intersect Σ (that is, as it follows from the density of O, if it is an arc). Assume by contradiction that neither Σ is discrete, nor it contains an infinite sequence of subsequent isolated points. With each bang arc $[\tau_1, \tau_2]$ we associate the smallest point of Σ which is larger than or equal to τ_2 and which is not isolated, unless there exists a finite sequence of subsequent points of $\Sigma \cup \{0, T\}$ including τ_2 and T. Denote by A the set of all points which can be associated with some arc following the described procedure. By our assumptions A is nonempty and preperfect, i.e., each point of A is a density point for A. We define a partition $A = A_+ \cup A_-$ by the following rule: $a \in A_\pm$ if and only if u is equal to ± 1 on a left neighborhood of a. It turns out that there exist $\star \in \{+, -\}$ and a subset B of $A_\star$ which is nonempty and preperfect. Indeed, if $a \in A_+$ is not a density point for A_+, then there exists a neighborhood U of a such that $(U \setminus \{a\}) \cap A$ is a preperfect nonempty subset of A_-. Thus, either A_+ is nonempty and preperfect or A_- has a preperfect nonempty subset.

Let $\lambda(\cdot)$ be an extremal lift of $q(\cdot)$. To complete the argument we want to prove that at each point $\tau \in B$ the covector $\lambda(\tau)$ annihilates $V_\star(q(\tau))$. It is clear that $\varphi(\tau) = \langle \lambda(\tau), g(q(\tau)) \rangle = 0$ for each $\tau \in \Sigma$. Within the interval identified by two subsequent points of Σ, $\varphi'(\cdot) = \langle \lambda(\cdot), [f, g](q(\cdot)) \rangle$ has at least one zero, and thus, by continuity, $\langle \lambda(\tau), [f, g](q(\tau)) \rangle = 0$ for every $\tau \in A$.

Remark that, given a smooth function $\psi : [\tau_1, \tau_2] \to \mathbf{R}$ such that

$$\psi(\tau_1) = \psi(\tau_2) = \dot{\psi}(\tau_2) = \cdots = \psi^{(m-1)}(\tau_2) = 0\,,$$

there exists $\tau \in (\tau_1, \tau_2)$ such that $\psi^{(m)}(\tau) = 0$. From the preperfectness of B, it follows that $\big\langle \lambda(\tau), X_{(m\star)}(q(\tau)) \big\rangle = 0$ for every $\tau \in B$. □

1.4 Second order optimality conditions

Let V be a linear finite dimensional space and consider a quadratic form $Q : V \to \mathbf{R}$. The *index of* Q is classically defined as the dimension of the maximal subspace W of V such that the restriction of Q to W is negative definite.

Theorem 1.6 *Let* $(\lambda(\cdot), q(\cdot))$ *be an extremal pair for (1.7) and let* $u : [0, T] \to [-1, 1]$ *be the corresponding control function. Assume that* $u(\cdot)$ *is bang-bang, with* $K+1$ *bang arcs. Denote by* $(0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< T)$ *its* K *switching times and by* ν *its value on* $[0, \tau_1]$. *Assume that* $\lambda(\cdot)$ *is the unique extremal lift of* $q(\cdot)$, *up to multiplication by a positive scalar. Fix* $\overline{\tau} \in [0, T]$ *and define*

$$h_i = \left(\overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_i} \operatorname{ad}\,(f + u(\tau) g) d\tau \right) (f + (-1)^i \nu g)\,, \qquad i = 0, \ldots, K\,. \tag{1.22}$$

Let Q *be the quadratic form*

$$Q(\alpha) = \sum_{0 \le i < j \le K} \alpha_i \alpha_j \,\langle \lambda(\overline{\tau}), [h_i, h_j](q(\overline{\tau})) \rangle\,, \tag{1.23}$$

defined on the space

$$H = \left\{ \alpha = (\alpha_0, \ldots, \alpha_K) \in \mathbf{R}^{K+1} \,\middle|\, \sum_{i=0}^{K} \alpha_i = 0,\ \sum_{i=0}^{K} \alpha_i h_i(q(\overline{\tau})) = 0 \right\}. \tag{1.24}$$

If $q(\cdot)$ *is quasi optimal, then the index of* Q *is equal to zero.*

Proof. Define, for every control function $v:[0,T] \to [-1,1]$,

$$G(v) = \left(\overrightarrow{\exp} \int_{\bar{\tau}}^{T} (f + u(\tau)g)d\tau \right)^{-1} \circ \left(\overrightarrow{\exp} \int_{0}^{T} (f + v(\tau)g)d\tau \right) (q(0)) .$$

Notice that, up to the composition with the diffeomorphism

$$P = \left(\overrightarrow{\exp} \int_{\bar{\tau}}^{T} (f + u(\tau)g)d\tau \right)^{-1} , \tag{1.25}$$

which is independent of v, the function $v \mapsto G(v)$ is the endpoint mapping from the point $q(0)$ of the system (1.7). Our aim is to prove that, if the index of Q is positive, then G is locally open at $u(\cdot)$ and, moreover, that this property is stable with respect to $\mathcal{C}^1$ perturbations of the system.

For any $\alpha = (\alpha_0, \dots, \alpha_K) \in \mathbf{R}^{K+1}$ such that $\sum_{i=0}^{K} \alpha_i = 0$ and for s small enough, we can define $v_s^\alpha(\cdot)$ as the bang-bang control with switching times

$$(0 <)\tau_1 + s\alpha_0 < \tau_2 + s(\alpha_0 + \alpha_1) < \cdots < \tau_K + s \sum_{i=0}^{K-1} \alpha_i (< T) ,$$

satisfying $v_s^\alpha|_{[0,\tau_1+s\alpha_0]} = \nu$. Remark that $s \mapsto G(v_s^\alpha)$, defined in a neighborhood of zero, is a curve contained in the set $P(A(T, q(0)))$. The tangent vector to this curve at zero is given by

$$V_1(\alpha) = \frac{\mathrm{d}}{\mathrm{d}s} G(v_s^\alpha) \Big|_{s=0} = \sum_{i=0}^{K} \alpha_i h_i(q(\bar{\tau})) ,$$

as it follows from the chain rule (1.6).

If α belongs to the space H defined in (1.24), that is, if $V_1(\alpha) = 0$, then the second order derivative of G with respect to s at $s = 0$ is an intrinsically defined element of $T_{q(\bar{\tau})}M$, given by

$$V_2(\alpha) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} G(v_s^\alpha) \Big|_{s=0} = \sum_{i=0}^{K} \sum_{j=i+1}^{K} \alpha_i \alpha_j [h_i, h_j](q(\bar{\tau})) .$$

The expression of $V_2(\alpha)$ can be computed, for instance, by fixing a local system of coordinates.

Roughly speaking, if $V_1(\alpha) = 0$, then $V_2(\alpha)$ is the direction of the curve of attainable points $s \mapsto G(v_s^\alpha)$. Assume that the index of Q is positive. Therefore, there exists $\alpha \in H$ such that

$$\langle \lambda(\bar{\tau}), V_2(\alpha) \rangle > 0 .$$

Define

$$w_0 = V_2(\alpha) .$$

Associate with every $t \in [0, T]$ the operator

$$F_t = \overrightarrow{\exp} \int_{\bar{\tau}}^{t} \mathrm{ad}(f + u(\tau)g)d\tau : \mathrm{Vec}M \longrightarrow \mathrm{Vec}M .$$

The extremality condition (1.10) implies that

$$\begin{aligned}\langle\lambda(\overline{\tau}), F_t(f+u(t)g)(q(\overline{\tau}))\rangle &= c\,,\\ \langle\lambda(\overline{\tau}), F_t(f-u(t)g)(q(\overline{\tau}))\rangle &\geq c\,,\end{aligned}$$

for every $t \in [0,T]$, as it follows from (1.19). Moreover, the uniqueness of $\lambda(\cdot)$ implies that the closed convex cone generated by $\{u(t)F_t(g)(q(\overline{\tau})) \mid t \in [0,T]\}$ is equal to $\{w \in T_{q(\overline{\tau})}M \mid \langle\lambda(\overline{\tau}), w\rangle \leq 0\}$. Therefore there exist $t_1,\ldots,t_n \in [0,T]$ such that the convex cone generated by w_0 and

$$w_i = u(t_i)F_{t_i}(g)(q(\overline{\tau})), \qquad i = 1,\ldots,n,$$

is equal to $T_{q_0}M$. Since $t \mapsto u(t)F_t(g)(q(\overline{\tau}))$ is continuous along the bang arcs, we can assume that $t_1,\ldots,t_n \in (0,T)\setminus\{\tau_1,\ldots,\tau_K\}$.

Introduce the family of admissible controls

$$v_{(s_0,s_1,\ldots,s_n)}(t) = \begin{cases} -u(t) & \text{if } t \in \cup_{i=1}^n [t_i, t_i+s_i]\,,\\ v_{s_0}^{\alpha}(t) & \text{otherwise}\,,\end{cases}$$

defined for $s_0, s_1,\ldots,s_n > 0$ small enough. Fix $a_0, a_1,\ldots,a_n > 0$ and remark that

$$\frac{\mathrm{d}}{\mathrm{d}s} G(v_{(a_0 s, a_1 s^2/2,\ldots,a_n s^2/2)})\Big|_{s=0+} = 0\,.$$

Moreover, it is intrinsically defined

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} G(v_{(a_0 s, a_1 s^2/2,\ldots,a_n s^2/2)})\Big|_{s=0+} = \sum_{i=0}^n a_i w_i\,.$$

A standard application of the Brower fixed point theorem implies that, for any continuous map which is C^0 close to $(s_0,s_1,\ldots,s_n) \mapsto G(v_{(\sqrt{s_0},s_1,\ldots,s_n)})$, the image of a small neighborhood of 0 in $[0,+\infty)^{n+1}$ contains a neighborhood of $q(\overline{\tau})$ in M. □

Remark 1.7 The second order necessary condition which is provided by Theorem 1.6 is actually independent of the choice of $\overline{\tau}$. Varying $\overline{\tau}$, we obtain a family of equivalent formulations of the same principle. In applications, we will choose $\overline{\tau}$ with the scope of making the statement computationally simpler to handle.

Let us see how Theorem 1.6 can be strengthened when quasi optimality is replaced by S-quasi optimality. Let $q{:}[0,T] \to M$ be an S-extremal trajectory of (1.13). Assume that $q(\cdot)$ has a unique S-extremal lift $\lambda(\cdot)$, up to multiplication by a positive scalar.

Suppose that u, the control function corresponding to $q(\cdot)$, is bang-bang on a subinterval $[\tau_0, \tau_{K+1}]$ of $[0,T]$, and let $(\tau_0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< \tau_{K+1})$ be its K switching times. Denote by ν the value of u on $[\tau_0, \tau_1]$.

Fix a time $\overline{\tau} \in [0,T]$ and define h_i as in (1.22). Choose a family of $n = \dim S$ vector fields $Y_1,\ldots,Y_n$ which span TS in a neighborhood of $q(0)$ on S. Choosing moving initial values on S, we produce new types of variations of the reference trajectory $q(\cdot)$. In our approach the starting point is perturbed following from $q(0)$ the flows of the vector fields $Y_1,\ldots,Y_n$. In this way, we obtain what can be seen as a new control problem, where the starting point is fixed, but the directions Y_i initially are admissible.

In detail, for every pair (α,β) in

$$W=\left\{(\alpha=(\alpha_0,\dots,\alpha_K),\beta=(\beta_1,\dots,\beta_n))\in\mathbf{R}^{K+1}\times\mathbf{R}^n\;\middle|\;\sum_{i=0}^{K}\alpha_i=0\right\}$$

and for every $s\in\mathbf{R}$, with $|s|$ small enough, let $v_s^\alpha(\cdot)$ be the control function which is equal to $u(\cdot)$ outside $[\tau_0,\tau_{K+1}]$, is bang-bang on $[\tau_0,\tau_{K+1}]$, with switching times

$$(\tau_0<)\tau_1+s\alpha_0<\tau_2+s(\alpha_0+\alpha_1)<\cdots<\tau_K+s\sum_{i=0}^{K-1}\alpha_i(<\tau_{K+1}),$$

and verifies $v_s^\alpha|_{[\tau_0,\tau_1+s\alpha_0]}=\nu$. Define, moreover,

$$\begin{aligned}G(\alpha,\beta,s) &= \left(\overrightarrow{\exp}\int_{\overline{\tau}}^{\tau_{K+1}}(f+u(\tau)g)d\tau\right)^{-1}\circ\left(\overrightarrow{\exp}\int_0^{\tau_{K+1}}(f+v_s^\alpha(\tau)g)d\tau\right)\\ &\quad\circ e^{s\beta_nY_n}\circ\cdots\circ e^{s\beta_1Y_1}(q(0)).\end{aligned}$$

Let P be defined as in (1.25). The derivative of $s\mapsto G(\alpha,\beta,s)$ at $s=0$ is given by

$$V_1(\alpha,\beta)=\sum_{j=1}^{n}\beta_j\mathrm{Ad}\,P(Y_j)(q(\overline{\tau}))+\sum_{i=0}^{K}\alpha_ih_i(q(\overline{\tau})).$$

If $(\alpha,\beta)\in W$ is such that $V_1(\alpha,\beta)=0$, then it is well defined

$$\begin{aligned}V_2(\alpha,\beta) &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}G(\alpha,\beta,s)\Big|_{s=0}\\ &= \sum_{1\le i<j\le n}\beta_i\beta_j\mathrm{Ad}\,P([Y_i,Y_j])(q(\overline{\tau}))+\sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j[\mathrm{Ad}\,P(Y_j),h_i](q(\overline{\tau}))\\ &\quad+\sum_{0\le i<j\le K}\alpha_i\alpha_j[h_i,h_j](q(\overline{\tau})).\end{aligned}$$

Geometrically, the kernel of V_1 can be described as follows: Let $\alpha\in\mathbf{R}^{K+1}$ be such that $\sum_{i=0}^{K}\alpha_i=0$; then, $V_1(\alpha,\beta)=0$ has a solution $\beta\in\mathbf{R}^n$ if and only if

$$\sum_{i=0}^{K}\alpha_ih_i(q(\overline{\tau}))\in\left(\overrightarrow{\exp}\int_0^{\overline{\tau}}(f+u(\tau)g)d\tau\right)_*(T_{q_0}S).$$

Moreover, if β exists, it is unique.

The same reasonings as in the proof of Theorem 1.6 imply that, if there exists $(\alpha,\beta)\in\ker V_1$ such that $\langle\lambda(\overline{\tau}),V_2(\alpha,\beta)\rangle<0$, then $q(\cdot)$ is S-essential.

Since S is an integral leaf for the distribution generated by the Y_i, then $[Y_i,Y_j](q(0))\in T_{q(0)}S$ for every i,j; therefore, due to the transversality condition (1.14),

$$\langle\lambda(\overline{\tau}),\mathrm{Ad}\,P([Y_i,Y_j])(q(\overline{\tau}))\rangle=0,\qquad\text{for every } i,j=1,\dots,n,$$

as it can be deduced from (1.19). Thus,

$$\begin{aligned}\langle\lambda(\overline{\tau}),V_2(\alpha,\beta)\rangle &= \sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j\,\langle\lambda(\overline{\tau}),[\operatorname{Ad}P(Y_j),h_i](q(\overline{\tau}))\rangle\\ &+\sum_{0\le i<j\le K}\alpha_i\alpha_j\,\langle\lambda(\overline{\tau}),[h_i,h_j](q(\overline{\tau}))\rangle\,.\end{aligned}$$

Let

$$\overline{Q}(\alpha)=\sum_{0\le i<j\le K}\alpha_i\alpha_j\,\langle\lambda(\overline{\tau}),[h_i,h_j](q(\overline{\tau}))\rangle\ , \tag{1.26}$$

and

$$R=\sum_{j=1}^{n}\sum_{i=0}^{K}\alpha_i\beta_j\,\langle\lambda(\overline{\tau}),[\operatorname{Ad}P(Y_j),h_i](q(\overline{\tau}))\rangle\ .$$

Fix a system of coordinates $(x_1,\ldots,x_{n+s})$ in a neighborhood of $q(\overline{\tau})$, in such a way that

$$\left(\overrightarrow{\exp}\int_0^{\overline{\tau}}(f+u(\tau)g)d\tau\right)(S)=\{(x_1,\ldots,x_{n+s})|\ x_{n+1}=0,\ldots,x_{n+s}=0\}\,.$$

Due to the freedom in the choice of Y_i, we can assume that $AdP(Y_i)=\partial_{x_i}$, $i=1,\ldots,n$. Therefore,

$$[\operatorname{Ad}P(Y_j),h_i](q(\overline{\tau}))=\partial_{x_j}h_i(q(\overline{\tau}))\,.$$

We can associate with every $X\in\operatorname{Vec}M$, its *horizontal* and *vertical part* X^{h} and X^{v}, verifying, at every point of the neighborhood,

$$X^{\mathrm{h}}\in\operatorname{span}\{\partial_{x_1},\ldots,\partial_{x_n}\},\qquad X^{\mathrm{v}}\in\operatorname{span}\{\partial_{x_{n+1}},\ldots,\partial_{x_{n+s}}\}. \tag{1.27}$$

Notice, in particular, that $\langle\lambda(\overline{\tau}),X(q(\overline{\tau}))\rangle=\langle\lambda(\overline{\tau}),X^{\mathrm{v}}(q(\overline{\tau}))\rangle$. For every $j=1,\ldots,n$ and every $i=0,\ldots,K$, let $H^{\mathrm{h}}_{ij}\in\mathbf{R}$ be defined by

$$h^{\mathrm{h}}_i(q(\overline{\tau}))=\sum_{j=1}^{n}H^{\mathrm{h}}_{ij}\partial_{x_j}\,.$$

If (α,β) is in the kernel of V_1, then

$$\sum_{j=1}^{n}\beta_j\operatorname{Ad}P(Y_j)(q(\overline{\tau}))=-\sum_{i=0}^{K}\alpha_ih_i(q(\overline{\tau}))=-\sum_{i=0}^{K}\alpha_ih^{\mathrm{h}}_i(q(\overline{\tau}))\,,$$

and so

$$\beta_j=-\sum_{i=0}^{K}\alpha_iH^{\mathrm{h}}_{ij}\,.$$

Finally

$$\begin{aligned}R=R(\alpha) &= \sum_{j=1}^{n}\beta_j\left(\sum_{i=0}^{K}\alpha_i\,\langle\lambda(\overline{\tau}),\partial_{x_j}h^{\mathrm{v}}_i(q(\overline{\tau}))\rangle\right)\\ &= -\sum_{j=1}^{n}\left(\sum_{i=0}^{K}\alpha_iH^{\mathrm{h}}_{ij}\right)\left(\sum_{i=0}^{K}\alpha_i\,\langle\lambda(\overline{\tau}),\partial_{x_j}h^{\mathrm{v}}_i(q(\overline{\tau}))\rangle\right)\,.\end{aligned} \tag{1.28}$$

We proved the following result,

Theorem 1.8 *Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair for (1.7) and let $u(\cdot)$ be the corresponding control function. Assume that $u(\cdot)$ is bang-bang on a subinterval $[\tau_0, \tau_{K+1}]$ of the domain of definition of $q(\cdot)$, with $K+1$ bang arcs. Denote by $(\tau_0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< \tau_{K+1})$ its K switching times and by ν its value on $[\tau_0, \tau_1]$. Assume that $\lambda(\cdot)$ is the unique S-extremal lift of $q(\cdot)$, up to multiplication by a positive scalar. Fix $\overline{\tau}$ in the domain of definition of $q(\cdot)$ and let h_i be defined as in (1.22), $i = 0, \ldots, K$. Let Q be the quadratic form*

$$Q(\alpha) = \sum_{0 \le i < j \le K} \alpha_i \alpha_j \langle \lambda(\overline{\tau}), [h_i, h_j](q(\overline{\tau})) \rangle - \sum_{j=1}^{n} \left(\sum_{i=0}^{K} \alpha_i H_{ij}^{\mathrm{h}} \right) \left(\sum_{i=0}^{K} \alpha_i \langle \lambda(\overline{\tau}), \partial_{x_j} h_i^{\mathrm{v}}(q(\overline{\tau})) \rangle \right) \tag{1.29}$$

defined on the space

$$H = \left\{ \alpha = (\alpha_0, \ldots, \alpha_K) \in \mathbf{R}^{K+1} \,\middle|\, \sum_{i=0}^{K} \alpha_i = 0, \quad \sum_{i=0}^{K} \alpha_i h_i^{\mathrm{v}}(q(\overline{\tau})) = 0 \right\}. \tag{1.30}$$

If $q(\cdot)$ is quasi optimal, then the index of Q is equal to zero.

Remark 1.9 Let $q : [0, T] \to M$ be an admissible trajectory of (1.7) such that $q(T) \in S$. If $q(\cdot)$ is a time-minimal (respectively, time-maximal) trajectory connecting $q(0)$ with S, then its time-reversed trajectory $q'(t) = q(T-t)$ is time-minimal (respectively, time-maximal) among the trajectories of the time-reversed system joining S to $q(0)$. As already noticed in remark 1.4, the time-reversed system, which we will denote by $(1.7)'$, is a control problem in the form (1.7), in which f and g are replaced by $-f$ and $-g$. It is possible to obtain second order optimality conditions verified by $q(\cdot)$ thanks to the S-quasi optimality of $q'(\cdot)$ for $(1.7)'$.

Assume that $q : [0, T] \to M$ is a trajectory of (1.7) and that $q'(\cdot)$ is S-extremal for $(1.7)'$. There is a one-to-one correspondence between extremal lifts of $q(\cdot)$ and of $q'(\cdot)$. Indeed, $t \mapsto (\lambda(t), q(t))$ is an extremal pair for (1.7) if and only if $t \mapsto (-\lambda(T-t), q'(t))$ is an extremal pair for $(1.7)'$. Therefore, there exists an extremal lift $\lambda(\cdot)$ of $q(\cdot)$ verifying

$$\lambda(T) \perp T_{q(T)} S. \tag{1.31}$$

Assume, as in the statement of Theorem 1.6, that $q(\cdot)$ is bang-bang on $[\tau_0, \tau_{K+1}] \subset [0, T]$, and denote by $(\tau_0 <)\tau_1 < \tau_2 < \cdots < \tau_K (< \tau_{K+1})$ its K switching times. Assume in addition that $q(\cdot)$ has a unique extremal lift $\lambda(\cdot)$ which verifies (1.31), up to multiplication by a positive scalar. Fix $\overline{\tau} \in [0, T]$ and define h_i as in (1.22), $i = 0, \ldots, K$.

The trajectory $q'(\cdot)$ of $(1.7)'$, restricted to the interval $[T - \tau_{K+1}, T - \tau_0,]$, is bang-bang. Associate with its $K+1$ bang arcs the corresponding vector fields h_i'. According to the definition (1.22), for every $i = 0, \ldots, K$,

$$\begin{aligned} h_i' &= \left(\overrightarrow{\exp} \int_{T-\overline{\tau}}^{T-\tau_{K-i}} \mathrm{ad}\,(-f - u(T-\tau) g) d\tau \right) (-f - (-1)^{K-i} \nu g) \\ &= \left(\overrightarrow{\exp} \int_{\overline{\tau}}^{\tau_{K-i}} \mathrm{ad}\,(f + u(\tau) g) d\tau \right) (-f - (-1)^{K-i} \nu g) \\ &= -h_{K-i}. \end{aligned} \tag{1.32}$$

Associate Q and H with $h_0, \ldots, h_K$ as in (1.29) and (1.30), where the horizontal-vertical splitting is given by a system of coordinates which rectifies

$$\left(\overrightarrow{\exp} \int_T^{\overline{\tau}} (f + u(\tau)g) d\tau \right) (S)$$

in a neighborhood of $q(\overline{\tau})$. In the same way associate Q' and H' with $h'_0, \ldots, h'_K$. It follows from (1.32) that $H = H'$ and $Q = Q'$. Therefore, if the index of Q is positive, then Theorem 1.8 implies that $q'(\cdot)$ is S-essential.

Let us recall the more classical second order necessary condition for optimal singular trajectories.

Theorem 1.10 (Generalized Legendre condition) *Let $(\lambda(\cdot), q(\cdot))$ be an extremal pair for (1.7). Assume that $\lambda(\cdot)$ is uniquely defined, up to multiplication by a positive scalar, and let I be a singular arc contained in the domain of definition of $q(\cdot)$, such that φ is identically equal to zero on I. Then $\langle \lambda(t), [g, [f, g]](q(t)) \rangle \leq 0$ for every $t \in I$.*

A proof of Theorem 1.10 can be found in [6, Chapter 20]. We point out that the sign condition in Theorem 1.10 is formulated in the opposite way than in [6]. This is due to the fact that, in the present statement of the Pontryagin maximum principle, condition (1.10) is given in terms of minimization of the Hamiltonian, whereas in [6] it has the more standard maximization form.

The proof of Theorem 1.10, as it is given in [6], extends to the case of S-extremal trajectories, where $\lambda(\cdot)$ is maybe not unique among all extremal lifts, but it is when also (1.14) is taken into account. For the sake of clarity we state such extension autonomously.

Theorem 1.11 *Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair for (1.7). Assume that $\lambda(\cdot)$ is uniquely defined, up to multiplication by a positive scalar, and let I be a singular arc contained in the domain of definition of $q(\cdot)$, such that $\varphi|_I \equiv 0$. Then $\langle \lambda(t), [g, [f, g]](q(t)) \rangle \leq 0$ for every $t \in I$.*

Chapter 2

Finite bounds for the manifold-to-point problem

In this chapter we shall be mainly concerned with the manifold-to-point problem. For analytic systems and codimension one sources, an analysis of small codimension singularities has been proposed in [13] and [33]. The methods and the perspective of these earlier works are different from ours, since they deal with the whole local synthesis of the problem, in whose regards an upper bound on the number of arcs of optimal trajectories represents a preliminary step. Our contribution extends the domain of application of upper bounds to the non analytic case, to higher codimension sources and singularities.

Section 2.1 gives a classification of Lie bracket configuration, which are studied in sections 2.2 – 2.7, mainly by means of general asymptotic considerations and second order optimality conditions. Section 2.8 collects the corresponding local regularity results, and gives their interpretation in terms of properties of generic systems. The original results are contained in [8]. Special attention is given to the special case of the point-to-point problem. The corresponding bounds obtained in dimension two are a slight improvement of the ones available in the literature [16, 40, 55]. The dimension three case was originally studied in a joint work with A. A. Agrachev [7].

Section 2.9 discusses the sharpness of the results. In particular, it is show that the bound obtained for the point-to-point three dimensional problem, is sharp. The proof is based on the sufficiency counterpart of Theorems 1.6 and 1.8, deduced form the results in [9]. The general picture of sharp bounds is still incomplete and can be the object of further analysis.

2.1 Classification of the singularities

Let M be a finite dimensional manifold and S an embedded submanifold of M of codimension s. Let $n + s$ be the dimension of M.

From now on, q_0 will denote a fixed point in S. Our purpose is to study (1.13) near q_0, under some nondegeneracy conditions on the relative positions of S and the pair of vector fields $(f, g) \in \text{Vec}M \times \text{Vec}M$. We will express these conditions as transversality properties between $T_{q_0}S$ and the iterated Lie brackets of f and g, evaluated at q_0.

The triples $(q_0, S, \text{Lie}(f, g))$ which are going to be covered by our analysis are identified by means of a classification of singularities up to order $4 - s$ of the relative positions of

$T_{q_0}S$ and $\mathrm{Lie}(f,g)(q_0)$. That is, exhibiting a stratified subset of codimension $5-s$ of the jet fiber $J^{2,k}_{q_0}M$, $k \geq 3$. For the sake of concision, we introduce the notation

$$V = \mathrm{span}\,\{g(q_0), [f,g](q_0)\} + T_{q_0}S\,.$$

Given two subspaces W_1 and W_2 of $T_{q_0}S$, we write $W_1 \pitchfork W_2$ to denote that they intersect transversally, i.e., that $W_1 + W_2 = T_{q_0}S$. With a slight abuse of notation, if $w_1 \in T_{q_0}S$ is such that $\mathrm{span}\,\{w_1\} \pitchfork W_2$, then we write $w_1 \pitchfork W_2$. The notation $X_\star$ introduced in (1.21) is also widely used.

We label the singularities of the classification by means of the codimension of the source and the codimension of the singularity: An (s,d)-point is a codimension d singularity on a codimension s source.

(1,0) $g(q_0) \pitchfork T_{q_0}S$;

(1,1) $g(q_0) \in T_{q_0}S$ and $[f,g](q_0) \pitchfork T_{q_0}S$;

(1,2) $g(q_0), [f,g](q_0) \in T_{q_0}S$ and $X_+(q_0), X_-(q_0) \pitchfork T_{q_0}S$;

(1,3) $g(q_0), [f,g](q_0), X_+(q_0) \in T_{q_0}S$ and $X_-(q_0), X_{++}(q_0) \pitchfork T_{q_0}S$;

(2,0) $\mathrm{span}\,\{g(q_0), [f,g](q_0)\} \pitchfork T_{q_0}S$;

(2,1) $\mathrm{codim}\, V = 1$ and $\mathrm{span}\,\{g(q_0), X_+(q_0)\}, \mathrm{span}\,\{g(q_0), X_-(q_0)\} \pitchfork T_{q_0}S$;

(2,2a) $\mathrm{codim}\, V = 1$, $X_+(q_0) \in V$, and the intersection of $\mathrm{span}\,\{g(q_0), X_-(q_0)\}$, $\mathrm{span}\,\{g(q_0), X_{++}(q_0)\}$ with $T_{q_0}S$ is transversal;

(2,2b) $g(q_0) \in T_{q_0}S$ and $\mathrm{span}\,\{[f,g](q_0), X_+(q_0)\}, \mathrm{span}\,\{[f,g](q_0), X_-(q_0)\} \pitchfork T_{q_0}S$;

(3,0) $\mathrm{span}\,\{g(q_0), [f,g](q_0), X_+(q_0)\}, \mathrm{span}\,\{g(q_0), [f,g](q_0), X_-(q_0)\} \pitchfork T_{q_0}S$;

(3,1) $\mathrm{codim}\, V = 1$, $X_+(q_0) \in V$, and $X_-(q_0), X_{++}(q_0) \pitchfork V$;

(4,0) $\mathrm{span}\{g(q_0), [f,g](q_0), X_+(q_0), X_-(q_0)\}, \mathrm{span}\{g(q_0), [f,g](q_0), X_+(q_0), X_{++}(q_0)\}$, and $\mathrm{span}\,\{g(q_0), [f,g](q_0), X_-(q_0), X_{--}(q_0)\}$ intersect $T_{q_0}S$ transversally.

The classification omits to consider singularities which can be obtained from the ones listed above by performing a transposition between $+$ and $-$, that is, by replacing each vector field X_w by $X_{\mathcal{T}(w)}$, with the agreement that $\mathcal{T}(\mp) = \pm$ and $\mathcal{T}(\pm w) = \mp\mathcal{T}(w)$. This is justified, since the substitution of g by $-g$ in (1.7) preserves the nature of the control system, reversing the formal roles of $+$ and $-$.

In order to shorten the formulation of local properties which hold near the fixed point q_0, we find it useful to introduce the following agreement: We say that all *short* trajectories of a certain class (for instance, S-extremal trajectories) have a given property $(\mathcal{P})$ if there exist $T > 0$ and a neighborhood U of q_0 such that all trajectories in the class, which are contained in U and have time-length smaller than T, satisfy $(\mathcal{P})$.

Section 2.8 provides a detailed picture of the bounds which we are able to give on the number of arcs of short S-quasi optimal trajectories near (s,d)-points.

2.2 Preliminary results

Lemma 2.1 *Fix an open, relatively compact, subset U of M and an Euclidean structure on the cotangent bundle $T^*\overline{U}$, that is, fix an Euclidean structure on each T_q^*M, $q \in \overline{U}$, smoothly varying with respect to q. Let X be a vector field on M. Then, for every $T > 0$ there exists a constant L such that, for every interval I of time-length smaller than T, for every extremal trajectory $q : I \to U$ and for every corresponding extremal lift $\lambda(\cdot)$, normalized in such a way that $|\lambda(t)| = 1$ at some $t \in I$, the function $t \mapsto \langle\lambda(t), X(q(t))\rangle$ is L-Lipschitz continuous.*

Proof. From Gronwall inequality applied to (1.12), it follows that, fixed $T > 0$, there exists $C > 0$ such that, for every $q(\cdot)$ and $\lambda(\cdot)$ as in the hypothesis of the lemma, $|\lambda(t)| \le C$ for every $t \in I$. Thus, the derivative of $t \mapsto \langle\lambda(t), X(q(t))\rangle$, whose expression is given in (1.17), is uniformly bounded. □

Corollary 2.2 *Let U be an open, relatively compact, subset of M and consider a family $Y_1, \dots, Y_{n+s}$ of vector fields on M, linearly independent at every point of $\overline{U}$. Let, for every $q \in \overline{U}$ and $\lambda \in T_q^*M$, the norm $|\lambda|$ be given by $\max\{|\langle\lambda, Y_i(q)\rangle| \mid i = 1, \dots, n+s\}$. Let $X \in \mathrm{Vec}M$ be linearly independent of $Y_1, \dots, Y_{n+s-1}$ at every point of $\overline{U}$. Then, there exist $\varepsilon_0 \in (0,1)$ and two nonincreasing functions $T, \delta : [0, \varepsilon_0] \to (0, +\infty)$ such that, for every $\varepsilon \in [0, \varepsilon_0]$ and every extremal pair $(\lambda(\cdot), q(\cdot))$ defined on a domain I of time-length smaller than $T(\varepsilon)$, normalized in such a way that $|\lambda(\overline{\tau})| = 1$ at some $\overline{\tau} \in I$, if each of the functions $t \mapsto |\langle\lambda(t), Y_i(q(t))\rangle|$, $i = 1, \dots, n+s-1$, attains at least one value smaller than or equal to ε in I, then $|\langle\lambda(t), X(q(t))\rangle| \ge \delta(\varepsilon)$ for every $t \in I$.*

Proof. The value of

$$\inf\{|\langle\lambda, X(q)\rangle| \mid q \in U,\ \lambda \in T_q^*M,\ |\lambda| = 1,\ \langle\lambda, Y_1(q)\rangle = \cdots = \langle\lambda, Y_{n+s-1}(q)\rangle = 0\}$$

is larger than zero, due to the compactness of $\overline{U}$. Fix $\varepsilon_1 > 0$ such that

$$\delta_1 = \inf\{|\langle\lambda, X(q)\rangle| \mid q \in U,\ \lambda \in T_q^*M,\ |\lambda| = 1,\ |\langle\lambda, Y_1(q)\rangle|, \dots, |\langle\lambda, Y_{n+s-1}(q)\rangle| \le \varepsilon_1\} \tag{2.1}$$

is positive as well.

Lemma 2.1 implies that there exists $L > 0$ such that, for every extremal pair $(\lambda(\cdot), q(\cdot))$ defined on a domain I of time-length smaller than one, normalized in such a way that $|\lambda(\overline{\tau})| = 1$ at some $\overline{\tau} \in I$, the functions $x(t) = \langle\lambda(t), X(q(t))\rangle$ and $y_i(t) = \langle\lambda(t), Y_i(q(t))\rangle$, $i = 1, \dots, n+s$, are L-Lipschitz continuous.

Fix $\varepsilon \ge 0$ and $0 < T \le 1$ such that $\varepsilon + TL < \varepsilon_1$. Assume that I has length smaller than T and that, for every $i = 1, \dots, n+s-1$, the function $t \mapsto |y_i(q(t))|$ attains at least one value smaller than or equal to ε in I. Then

$$|y_i(t)| < \varepsilon_1 < 1\,,$$

for every $t \in I$ and every $i = 1, \dots, n+s-1$. It follows from (2.1) that $|x(\overline{\tau})| \ge \delta_1$. The lemma is proved with $\varepsilon_0 = \min\left\{\frac{\delta_1}{2L}, \frac{\varepsilon_1}{2}\right\}$, $T(\varepsilon) = \min\left\{1, \frac{\varepsilon_1 - \varepsilon}{L}\right\}$ and $\delta(\varepsilon) = \delta_1 - LT(\varepsilon)$. □

2.3 The very small codimension cases: $s + d \leq 2$

Assume that q_0 is a (1,0)-point. Fix n vector fields $Y_1, \ldots, Y_n \in \mathrm{Vec}M$ such that, for every point q in a neighborhood of q_0 in S,

$$\mathrm{span}\{Y_1(q), \ldots, Y_n(q)\} = T_q S \,. \tag{2.2}$$

Let, in addition, $Y_{n+1} = X = g$. Due to the transversality condition (1.14), we can apply Corollary 2.2 on a conveniently small neighborhood of q_0, with $\varepsilon = 0$. We deduce that the switching function corresponding to a short S-extremal pair has constant sign. Therefore, a short S-extremal trajectory is made of a single bang arc. In particular, it does not contain singular arcs.

The same reasoning as above, applied to the case (1,1), with the choice $Y_{n+1} = X = [f, g]$, implies that, for every short S-extremal pair, the derivative of the corresponding switching function does not change sign. In the case (2,0), we fix $Y_{n+1} = g$ and $Y_{n+2} = X = [f, g]$, and the same conclusion holds for every short S-extremal pair such that the corresponding switching function has at least one zero. In both situations (1,1) and (2,0), along a short S-extremal trajectory which is not made of a single bang arc, the switching function is monotone. Therefore, a short S-extremal trajectory does not contain singular arcs and is the concatenation of at most two bang arcs.

2.4 Intermediate considerations for cases of higher codimension

We start the section with some general remarks which apply to all but one higher codimension cases. If $s \neq 4$ and $s + d = 3, 4$, then V has codimension one in $T_{q_0}M$ and $X_-(q_0) \pitchfork V$. From now on, let, as in section 2.3, $\{Y_1, \ldots, Y_n\}$ be a family of vector fields spanning $T_q S$ at every $q \in S$ close to q_0.

We want to complete $\{Y_1, \ldots, Y_n\}$ to a full rank distribution on a neighborhood of q_0. Choose $Y_{n+1}, \ldots, Y_{n+s-1}$ between $g, [f, g]$ in such a way that

$$\mathrm{span}\,\{Y_1(q_0), \ldots, Y_{n+s-1}(q_0)\} = V \,, \tag{2.3}$$

and set $Y_{n+s} = X_-$. Let U be a relatively compact neighborhood of q_0, such that $\{Y_i\}_{i=1}^{n+s}$ is a moving basis in $\overline{U}$. Associate with this moving basis the corresponding Euclidean structure $|\cdot|$ on $T^*\overline{U}$, as in the statement of Corollary 2.2. We can always assume that $T > 0$ is such that, for every admissible control function $u : [0, T] \to [-1, 1]$, for every $t \in [0, T]$ and $q \in S \cap \overline{U}$, we have

$$\left(\overrightarrow{\exp} \int_0^t (f + u(\tau)g)d\tau\right)_* (T_q S) \pitchfork \mathrm{span}\,\{Y_{n+1}(q'), \ldots, Y_{n+s}(q')\} \,, \tag{2.4}$$

where

$$q' = \left(\overrightarrow{\exp} \int_0^t (f + u(\tau)g)d\tau\right)(q) \,.$$

Once a moving basis and its corresponding Euclidean structure are fixed, we say that an extremal pair $(\lambda(\cdot), q(\cdot))$ is a *normalized extremal pair* if, at some point, $|\lambda(t)| = 1$.

If the switching function has at least two distinct zeros (this is true, for instance, if it contains a singular arc or a compactly contained bang one), then its derivative annihilates at least once. Due to (1.14), we can apply Corollary 2.2 (with $\varepsilon = 0$) and we obtain that for any vector field X which is transversal to V at q_0, there exists a positive constant δ_X, such that, for every short normalized S-extremal pair $(\lambda(\cdot), q(\cdot))$ whose corresponding switching function φ has at least two zeros, we have

$$|\langle \lambda(t), X(q(t)) \rangle| \geq \delta_X ,$$

for all t. A possible choice of X is given by $X = X_-$. Whenever $s + d = 3$, as well as in the case (2,2b), the role of X can also be played by X_+. In all the other cases studied here, the alternative choice of X_{++} is allowed.

Therefore, we can assume that the class Ξ of short normalized S-extremal pairs with at least two zeros of φ (a class which depends on the choice of T and U) satisfies one of the following conditions

(A) there exists $\delta > 0$ such that $|\langle \lambda(t), X_\pm(q(t)) \rangle| \geq \delta$ for every $(\lambda(\cdot), q(\cdot)) \in \Xi$, for all t;

(B) there exists $\delta > 0$ such that $|\langle \lambda(t), X_-(q(t)) \rangle|, |\langle \lambda(t), X_{++}(q(t)) \rangle| \geq \delta$ for every $(\lambda(\cdot), q(\cdot)) \in \Xi$, for all t.

Remark 2.3 In the case in which $\dim S = 0$, where the transversality condition (1.14) gives no information, the assumption can be further strengthened; indeed, we can suppose that (A) or (B) hold for the class of all normalized extremal pairs with at least two zeros of φ. That is, we can neglect the requirement that the initial point of the trajectory lies in S. We will omit to mention at each step this kind of extension, which applies to all regularity properties of S-extremal or S-quasi optimal trajectories which are going to be stated. We will come back to the consequences of this fact in section 2.8.

The case (4,0) is reduced to subproblems sharing one of the properties (A) or (B), as follows. Complete $\{Y_1, \ldots, Y_n\}$ to a local moving basis by taking $Y_{n+1} = g$, $Y_{n+2} = [f, g]$, $Y_{n+3} = X_+$, and $Y_{n+4} = X_-$. This choice defines, as above, an Euclidean structure in the cotangent bundle over a compact neighborhood of q_0. Keep on calling Ξ the class of short normalized S-extremal pairs whose switching function annihilates at least twice. By Lemma 2.1 we can assume that $|\langle \lambda(t), Y_i(q(t)) \rangle|$ is smaller than any prescribed positive constant, for every $i = 1, \ldots, n+2$ and every pair $(\lambda(\cdot), q(\cdot))$ in Ξ. Given any $\eta \in (0, 1)$, we split Ξ in three subclasses: the class Ξ^1_η of pairs $(\lambda(\cdot), q(\cdot))$ for which

$$|\langle \lambda(0), X_+(q(0)) \rangle| < \eta , \tag{2.5}$$

the class Ξ^2_η defined by

$$|\langle \lambda(0), X_-(q(0)) \rangle| < \eta , \tag{2.6}$$

and the complement Ξ^3_η of $\Xi^1_\eta \cup \Xi^2_\eta$ in Ξ. If η is fixed and small enough, then it follows from Lemma 2.1 and Corollary 2.2 that there exists a common $T = T(\eta)$ such that Ξ^1_η satisfies property (B), Ξ^3_η satisfies property (A), and Ξ^2_η satisfies the analogous of property (B), where the role of $+$ is played by $-$ and viceversa. Since the definition of (4,0)-point is symmetric in $+$ and $-$, the the regularity properties which are satisfied by Ξ^1_η apply to Ξ^2_η as well. Therefore, we will essentially neglect Ξ^2_η, and restrict our attention to Ξ^1_η and Ξ^3_η.

In order to fix η, we impose an additional requirement whose importance will be clear in section 2.7. Since $[g,[f,g]]$ is transversal to span $\{Y_1,\dots,Y_{n+s-1}\}$, we can again apply Corollary 2.2 and assume that all functions $|\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\rangle|$ are separated from zero, uniformly in Ξ^1_η. Moreover, due to the monotonicity of the function δ appearing in the statement of Corollary 2.2, we can choose η small enough, in such a way that the sign of

$$\langle\lambda(\cdot),X_-(q(\cdot))\rangle = \langle\lambda(\cdot),X_+(q(\cdot))\rangle - 2\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\rangle$$

is equal to the sign of $-\langle\lambda(\cdot),[g,[f,g]](q(\cdot))\rangle$ along the trajectory. Finally, we choose η such that Ξ^1_η satisfies (B) and

(B$'$) $\text{sign}(\langle\lambda(t),X_-(q(t))\rangle) = -\text{sign}(\langle\lambda(t),[g,[f,g]](q(t))\rangle)$ for every $(\lambda(\cdot),q(\cdot))\in\Xi^1_\eta$, for all t.

Notice that, for all (s,d)-points with $s\neq 4$, $s+d=3,4$, if Ξ does not satisfy (A), then the same reasoning as above shows that Ξ can be assumed to verify both (B) and (B$'$).

The crucial step toward a full understanding of the behavior of S-quasi optimal trajectories is given by the following result, which focuses on bang-bang regularity.

Proposition 2.4 *There exists an integer-valued function $k(s,d)$ such that, if q_0 is an (s,d)-point with $s+d\leq 4$, then there exist a neighborhood U of q_0 and a time $T>0$ for which a trajectory in U of (1.13), of time-length smaller than T, which contains more that $k(s,d)$ concatenated bang arcs, is S-essential.*

Notice that the proposition has already been proved for (s,d)-points such that $s+d\leq 2$. We showed that a possible choice of k is given by $k(1,0)=1, k(1,1)=k(2,0)=2$.

Next two sections are devoted to the proof of Proposition 2.4 in the remaining cases, that is, when $s+d$ is equal to three or four.

2.5 Bounds on the number of arcs of S-quasi optimal bang-bang trajectories when $s<4$

2.5.1 General facts

Throughout this section we assume that $s+d=3,4$ and $s\neq 4$. Fix $Y_1,\dots,Y_{n+s-1}$, $Y_{n+s}=X_-$, and the corresponding Euclidean structure on the cotangent bundle over a small enough neighborhood of q_0, as in section 2.4.

In order to apply Theorem 1.8, a corank one condition on S-extremal lifts must be recovered.

Lemma 2.5 *A short S-extremal trajectory which has at least one compactly contained $+$ arc and one compactly contained $-$ arc admits a unique covector lift, up to multiplication by a positive scalar.*

Proof. Let $(\lambda(\cdot),q(\cdot))$ be an S-extremal pair and assume that it has at least one compactly contained $+$ arc and one compactly contained $-$ arc. Denote the compactly contained $+$

arc by (t_0, t_0+t_1). The equations $\varphi(t_0)=0$ and $\varphi(t_0+t_1)=0$ can be written, according to (1.20), as

$$\langle \lambda(t_0), g(q(t_0))\rangle = 0\,, \tag{2.7}$$

$$\left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f+g)} g(q(t_0))\right\rangle = 0\,. \tag{2.8}$$

Let $\lambda(\cdot)$ be normalized in such a way that

$$|\lambda(t_0)| = 1. \tag{2.9}$$

Define

$$a_i = \langle \lambda(t_0), Y_i(q(t_0))\rangle \tag{2.10}$$

for every $i=1,\ldots,n+s$. We can reformulate (2.9) as follows

$$\max\{|a_i|\,|\, i=1,\ldots,n+s\}=1\,.$$

We also set

$$\pi_0 = \langle \lambda(t_0), [f,g](q(t_0))\rangle\,, \tag{2.11}$$

and, for every word w with letters in $\{-,+\}$,

$$\pi_w = \langle \lambda(t_0), X_w(q(t_0))\rangle\,. \tag{2.12}$$

Notice that $a_{n+s}=\pi_-$, while $a_{n+1},\ldots,a_{n+s-1}$ are taken among $\varphi(t_0)=0$ and π_0.

We want to describe the asymptotic behavior, as T goes to zero, of real valued functions of the trajectory and of the chosen $+$ arc. One example of this kind of functions is given by t_1, which associates with the trajectory the length of the chosen $+$ arc.

We say that a function of this type is a $O(1)$ if its absolute value can be bounded uniformly on the set of all $+$ arcs of trajectories which lift in Ξ. Clearly $t_1 = O(1)$. We write that a function is an $O(t_1^r)$ or an $O(T)$ to express that its quotient with, respectively, t_1^r or the total length of the trajectory is an $O(1)$.

From (1.14) we deduce that, for every $i=1,\ldots,n$,

$$0 = a_i + \langle \lambda(t_0), (\mathrm{Ad}\,P^{-1}-\mathrm{Id})Y_i(q(t_0))\rangle = a_i + \sum_{j=1}^{n+s} a_j O(T)\,, \tag{2.13}$$

where

$$P = \overrightarrow{\exp}\int_0^{t_0}(f+u(\tau)g)d\tau\,, \tag{2.14}$$

and Id denotes the identity operator on Vec$M$. Similarly, from (2.8) we get

$$\pi_0 = \sum_{j=1}^{n+s} a_j O(t_1)\,. \tag{2.15}$$

Thus,

$$\max\{|a_i|\,|\, i=1,\ldots,n+s-1\}\le O(T),$$

and, in particular, we can assume that $|a_{n+s}|=1$. Recall now that, since (A) or (B) holds, $\langle \lambda(t), X_-(q(t))\rangle$ does not change sign along the trajectory. Along the compactly contained

$-$ arc of $q(\cdot)$, $\varphi(t)$ is nonnegative and $\ddot{\varphi}(t) = \langle \lambda(t), X_-(q(t)) \rangle$. Thus, $\ddot{\phi}$ must be negative and so $a_{n+s} = \pi_- = -1$.

We can single out a system of $n+s-1$ linear equations for $a_1, \dots, a_{n+s-1}$, associating with any $i = 1, \dots, n$ the corresponding equation (2.13) and, eventually, adding some extra equations chosen between (2.7) and (2.15), depending on which vector fields, if any, have been chosen as $Y_{n+1}, \dots, Y_{n+s-1}$. The matrix of the coefficients of the system is of the type,

$$\begin{pmatrix}
1+O(T) & O(T) & \cdots & \cdots & O(T)\\
O(T) & \ddots & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \vdots\\
O(T) & \cdots & O(T) & 1+O(T) & O(T)\\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix},$$

and its determinant is equal to $1 + O(T)$. The system has a unique solution, provided that T is small enough. $\square$

Assume now that $(\lambda(\cdot), q(\cdot))$ is a short S-extremal pair and that $q(\cdot)$ contains a bang-bang concatenation of the type $-+-+$. Lemma (2.5) guarantees that we can apply Theorem 1.8 to the trajectory $q(\cdot)$.

Let t_0 be the second switching time and denote by t_1 and t_2 the length of, respectively, the second and the third bang arc. The switching times $t_0 - t_1$, t_0, and $t_0 + t_2$ are characterized by the equations

$$\left\langle \lambda(t_0), e^{-t_1 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle = 0, \tag{2.16}$$

$$\langle \lambda(t_0), g(q(t_0)) \rangle = 0, \tag{2.17}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0. \tag{2.18}$$

Renormalize, if necessary, $\lambda(\cdot)$, in order to have $|\lambda(t_0)| = 1$. Let a_i and $\pi_\star$ be defined as in (2.10), (2.11) and (2.12). Lemma 2.5 states that they can be considered functions of the trajectory and of the choice of the bang-bang concatenation. Moreover, we can assume that $\pi_- = -1$. Equalities (2.16) and (2.17) imply that

$$\pi_0 = \frac{t_1}{2}\pi_+ - \frac{t_1^2}{6}\pi_{++} + O\left(t_1^3\right). \tag{2.19}$$

From (2.18) and (2.19), analogously, we deduce that

$$\begin{aligned}
t_2 &= 2\pi_0 + O\left(t_2^2\right)\\
&= t_1\pi_+ - \frac{t_1^2\pi_{++}}{3} + \pi_+^2 O\left(t_1^2\right) + O\left(t_1^3\right).
\end{aligned} \tag{2.20}$$

Remark that $t_2 = O(t_1)$.

The role of the time $\overline{\tau}$ which appears in the statement of Theorem 1.8 will be played by t_0. According to (1.22), we have

$$\begin{aligned} h_0 &= e^{-t_1 \operatorname{ad}(f+g)}(f-g) &= f-g+2t_1[f,g]-t_1^2X_+ + O\left(t_1^3\right) , \\ h_1 &= f+g , \\ h_2 &= f-g , \\ h_3 &= e^{t_2 \operatorname{ad}(f-g)}(f+g) &= f+g+2t_2[f,g]+O\left(t_2^2\right) . \end{aligned}$$

Let

$$\sigma_{ij} = \langle \lambda(t_0), [h_i, h_j](q(t_0)) \rangle , \qquad 0 \le i < j \le 3 . \tag{2.21}$$

From the above asymptotic expressions for $h_1, \ldots, h_3$, we get

$$\begin{aligned} \sigma_{01} &= 2\pi_0 - 2t_1\pi_+ + t_1^2\pi_{++} + O\left(t_1^3\right) , \\ \sigma_{02} &= 2t_1 + O\left(t_1^2\right) , \\ \sigma_{12} &= -2\pi_0 , \\ \sigma_{03} &= \sigma_{01} + \sigma_{23} - 2\pi_0 + O\left(t_1^2 t_2\right) , \\ \sigma_{13} &= 2t_2\pi_+ + O\left(t_2^2\right) , \\ \sigma_{23} &= 2\pi_0 - 2t_2 + O\left(t_2^2\right) . \end{aligned}$$

A system of coordinates which rectify $P(S)$ can be obtained from the coordinate mapping

$$\mathcal{M}(x_1, \ldots, x_{n+s}) = e^{x_{n+s}Y_{n+s}} \circ \cdots \circ e^{x_{n+1}Y_{n+1}} \circ e^{x_n \operatorname{Ad} P^{-1}(Y_n)} \circ \cdots \circ e^{x_1 \operatorname{Ad} P^{-1}(Y_1)}(q(t_0)) , \tag{2.22}$$

which is non-degenerate at $(0, \ldots, 0)$, since we assume that (2.4) holds. Associate with $(x_1, \ldots, x_n)$ a horizontal-vertical splitting as in (1.27).

There is a point here which should be clarified, in order to avoid confusion. The important ingredient, in the definition of the horizontal-vertical splitting, is the rectification of S. The vector fields $Y_1, \ldots, Y_n$ appearing in (2.22) are not the same as the ones bearing the same name which are used in chapter 1, through the intermediate steps toward the recovery of (1.28). In particular, we do not claim, as it would be in general false, that $\operatorname{Ad} P^{-1}(Y_i) = \partial_{x_i}$, $i = 1, \ldots, n$.

From the definition of $\mathcal{M}$ it follows that

$$\partial_{x_j} Y_i^{\mathrm{v}}(q(t_0)) = 0 \qquad \text{and} \qquad Y_i^{\mathrm{h}}(q(t_0)) = 0, \tag{2.23}$$

for every $j = 1, \ldots, n$ and every $i = n+1, \ldots, n+s$. It is important to remark that, for every fixed vector field X, for every $j = 1, \ldots, n$, the j-th component of $X^{\mathrm{h}}(q(t_0))$, as well as $\left\langle \lambda(t_0), \partial_{x_j} X^{\mathrm{v}}(q(t_0)) \right\rangle$, are $O(1)$ functions of the trajectory and of the choice of t_0.

In section 2.5.2 we will treat separately the different (s, d) situations. When convenient, we will consider second order variations of the switching times on a shorter part of the bang-bang piece of $q(\cdot)$, that is, on the concatenation of three instead of four bang arcs. Denote by K the number of switching times which are involved in the variation.

Let H, $\overline{Q}(\alpha)$, and $R(\alpha)$ be defined as in (1.30), (1.26), and (1.28). Recall that H consists of all $(\alpha_0, \ldots, \alpha_K) \in \mathbf{R}^{K+1}$ such that

$$\sum_{i=0}^{K} \alpha_i = 0 , \tag{2.24}$$

and

$$\sum_{i=0}^{K} \alpha_i h_i(q(t_0)) \in \Sigma\,, \tag{2.25}$$

where

$$\Sigma = P_*\left(T_{q(0)}S\right)\,. \tag{2.26}$$

We will find it convenient, in most situations, to replace (2.25) by

$$\sum_{i=0}^{K} \alpha_i (h_i - f)(q(t_0)) \in \Sigma\,, \tag{2.27}$$

as it is justified by (2.24).

We claim that the codimension of H in $\mathbf{R}^{K+1}$ is equal to s, for T small enough. Indeed, let

$$A : \left\{ (\alpha_0, \dots, \alpha_K) \in \mathbf{R}^{K+1} \,\middle|\, \sum_{i=0}^{K} \alpha_i = 0 \right\} \longrightarrow T_{q(t_0)}M \big/ \Sigma$$

be the linear function which maps $(\alpha_0, \dots, \alpha_K)$ into the class $\sum_{i=0}^{K} \alpha_i h_i(q(0)) + \Sigma$. Since $q(\cdot)$ is an S-extremal trajectory, then there exists $\lambda \in T^*_{q(t_0))} \setminus \{0\}$ which is orthogonal to Σ and to $(h_i - h_{i-1})(q(t_0))$, for $i = 1, \dots, K$. The previous assertion is just a reformulation, obtained through (1.19), of (1.14) and of the fact that φ is equal to zero at the switching times.

In the proof of Lemma 2.5 it is shown that these orthogonality relations identify λ uniquely, up to multiplication by a scalar. Therefore, the codimension of the image of A in $T_{q(0)}M \big/ \Sigma$ is equal to one. Since H is equal to the kernel of A, its dimension is equal to $K - s + 1$. Finally, as claimed, H has codimension s in $\mathbf{R}^{K+1}$.

2.5.2 Case analysis

In this section the different types of (s, d)-points are considered separately. Each paragraph deals with one or two classes of points, specified by the opening framed declaration.

(1,2)-(1,3) We compute the second order variation of $q(\cdot)$ with respect to its $-+-$ concatenation. It means that $K = 2$ and that H is a codimension one subspace of $\mathbf{R}^3$. An explicit expression for H is given by (2.24), as follows,

$$H = \left\{ (\alpha_0, \alpha_1, \alpha_2) \in \mathbf{R}^3 \middle| \, \alpha_0 = -\alpha_1 - \alpha_2 \right\}\,.$$

The quadratic form $\overline{Q}$, defined in (1.26), is given by

$$\begin{aligned} \overline{Q}(\alpha_1, \alpha_2) &= \left(2\pi_0 - 2t_1\pi_+ + O\left(t_1^2\right)\right)(-\alpha_1 - \alpha_2)\alpha_1 \\ &\quad + \left(2t_1 + O\left(t_1^2\right)\right)(-\alpha_1 - \alpha_2)\alpha_2 - 2\pi_0\alpha_1\alpha_2 \\ &= \left(-t_1\pi_+ + O\left(t_1^2\right)\right)\alpha_1^2 + \left(-2t_1 + \pi_+ O(t_1) + O\left(t_1^2\right)\right)\alpha_1\alpha_2 \\ &\quad - \left(2t_1 + O\left(t_1^2\right)\right)\alpha_2^2\,. \end{aligned}$$

Let G_j be the j-th component of $g(q(t_0))$ and η_j be equal to $\langle \lambda(t_0), \partial_{x_j} g^{\mathrm{v}}(q(t_0)) \rangle$, for $j = 1, \dots, n$. Then

$$\begin{aligned} h_0^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) - g^{\mathrm{h}}(q(t_0)) + O(t_1) &= f^{\mathrm{h}}(q(t_0)) - \textstyle\sum_{j=1}^n (G_j + O(t_1))\partial_{x_j} , \\ h_1^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) + g^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) + \textstyle\sum_{j=1}^n G_j \partial_{x_j} , \\ h_2^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) - g^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) - \textstyle\sum_{j=1}^n G_j \partial_{x_j} , \end{aligned}$$

and

$$\begin{aligned} \langle \lambda(t_0), \partial_{x_j} h_0^{\mathrm{v}}(q(t_0)) \rangle &= \langle \lambda(t_0), \partial_{x_j} f^{\mathrm{v}}(q(t_0)) \rangle - \eta_j + O(t_1) , \\ \langle \lambda(t_0), \partial_{x_j} h_1^{\mathrm{v}}(q(t_0)) \rangle &= \langle \lambda(t_0), \partial_{x_j} f^{\mathrm{v}}(q(t_0)) \rangle + \eta_j , \\ \langle \lambda(t_0), \partial_{x_j} h_2^{\mathrm{v}}(q(t_0)) \rangle &= \langle \lambda(t_0), \partial_{x_j} f^{\mathrm{v}}(q(t_0)) \rangle - \eta_j , \end{aligned}$$

for $j = 1, \dots, n$. Thus

$$\begin{aligned} R(\alpha_1, \alpha_2) &= \sum_{j=1}^n ((2G_j + O(t_1))\alpha_1 + O(t_1)\alpha_2) ((2\eta_j + O(t_1))\alpha_1 + O(t_1)\alpha_2) \\ &= O(1)\alpha_1^2 + O(t_1)\alpha_1\alpha_2 + O(t_1^2)\, \alpha_2^2 . \end{aligned}$$

Whenever t_1 is small enough, the coefficient of α_2^2 of $Q = \overline{Q} + R$ is negative. Notice that, even if we computed the variation only on a smaller part of the bang-bang piece, our reasoning assumes that $\pi_- = -1$, which was justified by the presence of a compactly contained $-$ arc. It follows from Theorem 1.8 that a short trajectory which contains a $-+-+$ or a $+-+-$ concatenation cannot be S-quasi optimal. Proposition 2.4 is proved for $(s, d) = (1, 2), (1, 3)$, with $k(1, 2) = k(1, 3) = 3$.

(2,1)-(2,2a) Let, as above, $K = 2$. The space H has codimension two, and can be described by (2.24) and by another independent linear relation, deduced from (2.27). Notice, for instance, that the component of $\sum_{i=0}^K \alpha_i(h_i - f)(q(t_0))$ in the direction $g(q(t_0))$, with respect to the basis

$$P_* (Y_1(q(0))) , \dots, P_* (Y_n(q(0))) , g(q(t_0)), X_-(q(t_0)) ,$$

is equal to zero. Thus,

$$-(1 + O(t_1))\alpha_0 + \alpha_1 - \alpha_2 = 0 . \tag{2.28}$$

From (2.24) and (2.28) and we obtain

$$\begin{aligned} \alpha_1 &= O(t_1)\alpha_0 , \\ \alpha_2 &= -(1 + O(t_1))\alpha_0 . \end{aligned}$$

The transversality of g and Σ also implies, as follows from (2.23), that $R(\alpha_0) = O(t_1^2)\, \alpha_0^2$. The quadratic form $\overline{Q}$ is easily computed, and, finally, Q can be written as

$$Q(\alpha_0) = \overline{Q}(\alpha_0) + R(\alpha_0) = -\left(2t_1 + O\left(t_1^2\right)\right) \alpha_0^2 .$$

Thus, Q is negative definite for small t_1. We conclude, as above, that a short trajectory with four concatenated bang arcs is S-essential. This proves Proposition 2.4 in the cases (2,1) and (2,2a), with $k(2, 1) = k(2, 2\mathrm{a}) = 3$.

(2,2b) Let, here as in the next cases, $K=3$. Denote by γ the component of $g(q(t_0))$ in the direction $[f,g](q(t_0))$, with respect to the basis

$$P_*\left(Y_1(q(0))\right),\ldots,P_*\left(Y_n(q(0))\right),[f,g](q(t_0)),X_-(q(t_0)).$$

The space H is characterized by (2.24) and by the component of the relation

$$\sum_{i=0}^{3}\alpha_i(h_i-f+g)(q(t_0))\in\Sigma$$

in the direction $[f,g](q(t_0))$, that is, by the system

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3 = 0\,, \tag{2.29}$$
$$\left(2t_1+O\left(t_1^2\right)\right)\alpha_0+2\gamma\alpha_1+\left(2\gamma+2t_2+O\left(t_2^2\right)\right)\alpha_3 = 0\,, \tag{2.30}$$

from which we obtain

$$\alpha_0 = \left(-\frac{\gamma}{t_1}+\gamma O(1)\right)\alpha_1+\left(-\frac{\gamma+t_2}{t_1}+\gamma O(1)+O(t_1)\right)\alpha_3\,, \tag{2.31}$$
$$\alpha_2 = -\alpha_0-\alpha_1-\alpha_3\,. \tag{2.32}$$

Consider the linear change of variables on H,

$$\beta_1 = \alpha_1+\alpha_3\,,$$
$$\beta_2 = \alpha_1-\alpha_3\,.$$

The quadratic form $\overline{Q}(\beta_1,\beta_2)$ turns out to have the following expression

$$\begin{aligned}\overline{Q}(\beta_1,\beta_2) = -\frac{1}{2t_1}\Big[&\left(4\gamma^2+2\gamma(\pi_+-2)t_1-\pi_+(\pi_++4)t_1^2+\gamma^2O(t_1)+\gamma O\left(t_1^2\right)+O\left(t_1^3\right)\right)\beta_1^2\\ &+2\left(t_1\pi_+(t_1-\gamma)+\gamma^2O(t_1)+\gamma O\left(t_1^2\right)+O\left(t_1^3\right)\right)\beta_1\beta_2\\ &+\left(\pi_+^2t_1^2+\gamma O\left(t_1^2\right)+O\left(t_1^3\right)\right)\beta_2^2\Big]\,.\end{aligned}$$

Let G_j be the j-th component of $g(q(t_0))$ and η_j be equal to $\langle\lambda(t_0),\partial_{x_j}g^{\mathrm{v}}(q(t_0))\rangle$, for $j=1,\ldots,n$. Then

$$\begin{aligned}R(\beta_1,\beta_2) &= -\sum_{j=1}^{n}\left((2G_j+O(t_1))\beta_1+O(t_1)\beta_2\right)\left((2\eta_j+O(t_1))\beta_1+O(t_1)\beta_2\right)\\ &= O(1)\beta_1^2+O(t_1)\beta_1\beta_2+O\left(t_1^2\right)\beta_2^2\,.\end{aligned}$$

Finally, the coefficient of β_2^2 of the quadratic form

$$Q(\beta_1,\beta_2)=\overline{Q}(\beta_1,\beta_2)+R(\beta_1,\beta_2)$$

is given by $\pi_+^2t_1+\gamma O(t_1)+O\left(t_1^2\right)$. Since property (A) ensures that π_+^2 is uniformly separated from zero, then $q(\cdot)$ cannot be S-quasi optimal for T and U (consequently, t_1 and $|\gamma|$) small enough.

Due to the symmetry in $+$ and $-$, we conclude that Proposition 2.4 holds in the case (2,2b), with $k(2,2b) = 3$.

(3,0) We describe H, which has codimension equal to three, by (2.24) and by the components of (2.27) in the directions $g(q(t_0))$ and $[f,g](q(t_0))$,

$$\begin{array}{lllll} \alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & = 0\,, \\ -\left(1+O\left(t_1^2\right)\right)\alpha_0 & +\alpha_1 & -\alpha_2 & +\left(1+O\left(t_1^2\right)\right)\alpha_3 & = 0\,, \\ \left(2t_1+O\left(t_1^2\right)\right)\alpha_0 & & & +\left(2t_2+O\left(t_1^2\right)\right)\alpha_3 & = 0\,. \end{array}$$

From the last relation we obtain

$$\alpha_2 = \left(\frac{t_2}{t_1} + O(t_1)\right)\alpha_3\,, \tag{2.33}$$

while, replacing this last expression of α_2 in the first two, we have

$$\begin{aligned} \alpha_0 &= -\left(\frac{t_2}{t_1} + O(t_1)\right)\alpha_3\,, \\ \alpha_1 &= -\left(1+O\left(t_1^2\right)\right)\alpha_3\,. \end{aligned}$$

Therefore, computing $\overline{Q}$ according to its definition, one gets

$$\overline{Q}(\alpha_3) = -2\left(t_1\pi_+^2 + O\left(t_1^2\right)\right)\alpha_3^2.$$

It suffices to take into account (2.23) for what concerns $Y_{n+1} = g$ in order to get that $R(\alpha_3) = O\left(t_1^2\right)\alpha_3^2$. Since (A) holds, Q is negative definite, at least for short trajectories. The symmetry in $+$ and $-$ implies that Proposition 2.4 holds for the case (3,0), with $k(3,0) = 3$.

(3,1) The computations made for the case (3,0) are still valid, but, since property (A) fails to hold, we cannot conclude as above. We need to take into account higher order terms in the expansion of Q. In particular, it is no more true that $t_1 = O(t_2)$, while we find it convenient to replace, in the estimate of the remainders, the still valid relation $t_2 = O(t_1)$ by the more accurate one $t_2 = \pi_+ O(t_1) + O\left(t_1^2\right)$. We re-write the parameterization of H obtained for the case (3,0), distinguishing between the roles of t_1 and t_2,

$$\begin{aligned} \alpha_0 &= -\left(\tfrac{t_2}{t_1} + O(t_2)\right)\alpha_3\,, \\ \alpha_1 &= -(1+O(t_1 t_2))\alpha_3\,, \\ \alpha_2 &= \left(\tfrac{t_2}{t_1} + O(t_2)\right)\alpha_3\,. \end{aligned}$$

Recalling that (2.23) holds for $Y_{n+1} = g$ and $Y_{n+2} = [f,g]$, it follows that $R(\alpha_3) = O\left(t_1^2 t_2^2\right)$. On the other hand,

$$\overline{Q}(\alpha_3) = -2t_2\left(\pi_+ + \pi_+ O(t_1) + O\left(t_1^2\right)\right)\alpha_3^2\,.$$

Assume that Q is nonnegative definite. Notice that, since $\pi_0 = \dot\varphi(t_0) \geq 0$, the following inequality holds

$$\pi_+ - t_1\frac{\pi_{++}}{3} + O\left(t_1^2\right) \geq 0\,.$$

From this last relation and the sign condition on Q, we deduce that

$$t_1 \pi_{++} + \pi_+ O(t_1) + O\left(t_1^2\right) \leq 0 \, .$$

Since (B) holds, π_{++} is uniformly bounded away from 0. A necessary condition for Q to be nonnegative definite is that $\pi_{++} < 0$, provided that T and U are small. In particular, if $X_-(q_0)$ and $X_{++}(q_0)$ point on the opposite side of the hyperplane V, then a short trajectory is S-essential if it contains a $-+-+$ concatenation of arcs.

We already noticed that the time-reversed of a trajectory of (1.7) is admissible for the system $\dot{q} = -f(q) - u\,g(q)$. In remark 1.9 we pointed out a second order necessary condition for optimality which is tested on the time-reversed system. If we replace f and g by $-f$ and $-g$, then the roles of $X_\pm$ and $X_{++}(q_0)$ are played, respectively, by $-X_\pm(q_0)$ and $X_{++}(q_0)$. Notice, in particular, that q_0 is a (3,1)-point for the time-reversed system as well. Remark that, in obtaining all the asymptotic relations for (1.7), we never used the fact that the transversality condition (1.14) holds *at the starting point* of the trajectory. We used it only to get that $a_i = O(T)$ for $i = 1, \ldots, n$. The same relations can be recovered for trajectories attaining S at their final point T, replacing (1.14) by the symmetric transversality condition (1.31). It turns out that, if $X_-(q_0)$ and $X_{++}(q_0)$ point on the same side of V, then the quadratic form associated with a short trajectory of the time-reversed system which contains a $-+-+$ concatenation is negative definite, and thus a short trajectory of the original system which contains a $+-+-$ concatenation is S-essential.

Finally, a short S-quasi optimal trajectory of the original system has at most four concatenated bang arcs, i.e., we proved Proposition 2.4 in the case (3,1), with $k(3,1) = 4$.

2.6 A bound on the number of arcs of S-quasi optimal bang-bang trajectories in the case (4,0)

Throughout this section we assume that q_0 is a (4,0)-point. As for the cases treated in section 2.5, the preliminary step is to investigate the uniqueness of S-extremal lifts.

Lemma 2.6 *A short S-extremal trajectory which has at least two concatenated compactly contained bang arcs admits a unique S-extremal lift, up to multiplication by a positive scalar.*

Proof. Let $(\lambda(\cdot), q(\cdot))$ be an S-extremal pair and assume that it has two concatenated compactly contained bang arcs. Denote by t_0 the switching time between them. Let $Y_1, \ldots, Y_{n+4}$ be chosen as in section 2.4 and define

$$a_i = \langle \lambda(t_0), Y_i(q(t_0)) \rangle \, ,$$

for $i = 1, \ldots, n+4$. Normalize $\lambda(\cdot)$ in such a way that

$$\max\{|a_i| \mid i = 1, \ldots, n+4\} = 1 \, .$$

Due to the transversality condition (1.14), we get

$$a_i = \sum_{j=1}^{n+4} a_j O(T) \, , \qquad\qquad i = 1, \ldots, n \, . \tag{2.34}$$

Define $\pi_\star$ as in (2.11) and (2.12). Notice that

$$a_{n+1} = \varphi(t_0) = 0\,, \tag{2.35}$$

$a_{n+2} = \pi_0$, $a_{n+3} = \pi_+$, and $a_{n+4} = \pi_-$. Since $\langle \lambda(t), [f,g](q(t)) \rangle = \dot{\varphi}(t)$ has at least one zero along the trajectory, then we can assume that $|a_{n+2}|$ is smaller than one. The presence of compactly contained $+$ and $-$ arcs implies, for T small, that either $a_{n+3} = 1$ or $a_{n+4} = -1$.

Without loss of generality the control switches at t_0 from $+1$ to -1. Denote by t_1 and t_2 the lengths of the $+$ and the $-$ arc, respectively. From the relation $\varphi(t_0 - t_2) = \varphi(t_0) = \varphi(t_0 + t_1) = 0$ we get

$$\pi_0 - \frac{t_2}{2}\pi_- + \sum_{j=1}^{n+4} a_j O\left(t_2^2\right) \;=\; 0\,, \tag{2.36}$$

$$\pi_0 + \frac{t_1}{2}\pi_+ + \sum_{j=1}^{n+4} a_j O\left(t_1^2\right) \;=\; 0\,. \tag{2.37}$$

Collecting (2.34), (2.35), (2.36), and (2.37), we obtain a linear system of $n+3$ homogeneous linear equations satisfied by $a_1, \ldots, a_{n+4}$. The coefficient matrix of the system has the form

$$\begin{pmatrix}
1+O(T) & O(T) & \cdots & \cdots & \cdots & \cdots & O(T) \\
O(T) & \ddots & \ddots & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & \vdots \\
O(T) & \cdots & O(T) & 1+O(T) & O(T) & \cdots & O(T) \\
0 & \cdots & 0 & 1 & 0 & 0 & 0 \\
O(t_2^2) & \cdots & \cdots & O(t_2^2) & 1+O(t_2^2) & -\frac{t_2}{2}+O(t_2^2) & O(t_2^2) \\
O(t_1^2) & \cdots & \cdots & O(t_1^2) & 1+O(t_1^2) & O(t_1^2) & \frac{t_1}{2}+O(t_1^2)
\end{pmatrix}$$

and its rank, for T small, is equal to $n+3$. Therefore, the solutions of the linear system form a one-dimensional subspace of $\mathbf{R}^{n+4}$. Its intersection with

$$\{(b_1, \ldots, b_{n+4}) |\ |b_i| \leq 1 \text{ for every } i = 1, \ldots, n+4;\ b_{n+3} = 1 \text{ or } b_{n+4} = -1\}$$

has cardinality one. □

Fix an S-extremal pair $(\lambda(\cdot), q(\cdot))$ and assumes that it contains a $+-+-+$ concatenation. Let t_0 be the second switching time of the bang-bang concatenation and denote by t_1, t_2, and t_3 the length of, respectively, the second, the third, and the fourth bang arc.

Thus, the following equations are satisfied

$$\left\langle \lambda(t_0), e^{-t_1 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0, \tag{2.38}$$

$$\langle \lambda(t_0), g(q(t_0)) \rangle = 0, \tag{2.39}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle = 0, \tag{2.40}$$

$$\left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f+g)} e^{t_3 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle = 0. \tag{2.41}$$

Assume that $(\lambda(\cdot), q(\cdot))$ belongs to $\Xi^1_\eta \cup \Xi^3_\eta$, where Ξ^i_η and η are defined as in section 2.4. If T is small enough, then, as follows from Lemma 2.1, we have

$$|\langle \lambda(t), X_-(q(t)) \rangle| \geq \eta/2 \tag{2.42}$$

along the trajectory. The presence of a compactly contained $-$ arc implies that the sign of $\langle \lambda(t), X_-(q(t)) \rangle$ is negative. Possibly renormalizing $\lambda(\cdot)$, we may assume that

$$\langle \lambda(t_0), X_-(q(t_0)) \rangle = -1 .$$

Remark that, applying this renormalization, it is possible that we exit from the class of normalized pairs, as it was defined in section 2.4.

Let, for $i = 1, 3$, $\widetilde{\Xi}^i_\eta$ be the classes of S-extremal pairs containing a $+-+-+$ concatenation which are obtained from Ξ^i_η by means of this renormalization. Remark that, since (2.42) holds, the rescaling factor is bounded from below by $2/\eta$. Therefore, for any vector field X, $\langle \lambda(t_0), X(q(t_0)) \rangle = O(1)$ as a function of the pair chosen in $\widetilde{\Xi}^1_\eta \cup \widetilde{\Xi}^3_\eta$ and of the choice of the bang-bang concatenation.

Define $\pi_\star$, $\star = 0, +, ++$, as in (2.11) and (2.12) and notice that we can still assume that $\widetilde{\Xi}^1_\eta$ satisfies (B) and $\widetilde{\Xi}^3_\eta$ satisfies (A).

From equations (2.38 − 2.40) we obtain that

$$\begin{aligned} \pi_0 &= -\frac{t_2}{2}\pi_+ - \frac{t_2^2}{6}\pi_{++} + O\left(t_2^3\right) , \\ t_1 &= -2\pi_0 + O\left(t_1^2\right) \\ &= t_2\pi_+ + \frac{t_2^2}{3}\pi_{++} + \pi_+^2 O\left(t_2^2\right) + O\left(t_2^3\right) . \end{aligned}$$

Equation (2.41), in turns, implies

$$0 = \pi_0 + t_2\pi_+ - \frac{t_3}{2} + \frac{t_2^2}{3}\pi_{++} + O\left(t_2^3\right) + O(t_2 t_3) + O\left(t_3^2\right) ,$$

and so

$$t_3 = t_2\pi_+ + \frac{2}{3}t_2^2\pi_{++} + \pi_+ O\left(t_2^2\right) + O\left(t_2^3\right) . \tag{2.43}$$

We find it useful to introduce another agreement on how to express asymptotic relations. We say that a function is an $\Omega(t_2)$ if it can be expressed as a sum of the type $\pi_+ O(1) + O(t_2)$. In short,

$$\Omega(t_2) = \pi_+ O(1) + O(t_2) . \tag{2.44}$$

Notice that $t_1, t_3 = t_2\Omega(t_2)$. In order to recover Q, we compute

$$\begin{aligned}
h_0 &= f + g - 2t_1[f,g] + t_1^2 X_- + O\left(t_1^3\right) , \\
h_1 &= f - g , \\
h_2 &= f + g , \\
h_3 &= f - g - 2t_2[f,g] - t_2^2 X_+ + O\left(t_2^3\right) , \\
h_4 &= f + g + 2t_3[f,g] + 2t_2t_3X_+ + t_3^2 X_- + O\left(t_2^2 t_3\right) .
\end{aligned}$$

Asymptotic expansions for σ_{ij}, $0 \leq i < j \leq 4$, defined as in (2.21), are obtained from the above relations, as follows,

$$\begin{aligned}
\sigma_{01} &= -2\pi_0 + 2t_1\pi_- + O\left(t_1^2\right) , \\
\sigma_{02} &= 2t_1\pi_+ + O\left(t_1^2\right) , \\
\sigma_{12} &= 2\pi_0 , \\
\sigma_{03} &= \sigma_{01} + \sigma_{23} + 2\pi_0 + O\left(t_1 t_2^2\right) , \\
\sigma_{13} &= -2t_2\pi_- + O\left(t_2^2\right) , \\
\sigma_{23} &= -2\pi_0 - 2t_2\pi_+ - t_2^2\pi_{++} + O\left(t_2^3\right) , \\
\sigma_{04} &= \sigma_{02} + \sigma_{24} + O(t_1t_2t_3) , \\
\sigma_{14} &= 2\pi_0 + 2t_3\pi_- + O(t_2t_3) , \\
\sigma_{24} &= 2t_3\pi_+ + 2t_2t_3\pi_{++} + O\big(t_3^2\big) + O\big(t_2^2t_3\big) , \\
\sigma_{34} &= \sigma_{14} - \sigma_{23} - 2\pi_0 + O\big(t_2^2t_3\big) .
\end{aligned}$$

In analogy with what was done in section 2.5.1, one can derive from Lemma 2.6 that, for T small enough, the space H has codimension four in $\mathbf{R}^5$. A system of equations for H is given by (2.24) and the components of (2.27) in the directions g, $[f,g]$ and X_+, as follows,

$$\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 , \quad (2.45)\\
\left(1 + O\left(t_1^2\right)\right)\alpha_0 - \alpha_1 + \alpha_2 - \left(1 + O\left(t_2^3\right)\right)\alpha_3 + (1 + t_2t_3\Omega(t_2))\alpha_4 &= 0 , \quad (2.46)\\
-\left(2t_1 + O\left(t_1^2\right)\right)\alpha_0 - \left(2t_2 + O\left(t_2^3\right)\right)\alpha_3 + (2t_3 + t_2t_3\Omega(t_2))\alpha_4 &= 0 , \quad (2.47)\\
O\left(t_1^2\right)\alpha_0 - \left(t_2^2 + O\left(t_2^3\right)\right)\alpha_3 + (2t_2t_3 + t_2t_3\Omega(t_2))\alpha_4 &= 0 . \quad (2.48)
\end{aligned}$$

From (2.48) we deduce that

$$\alpha_3 = O\left(\frac{t_1^2}{t_2^2}\right)\alpha_0 + \left(2\frac{t_3}{t_2} + O\left(\frac{t_3}{t_2}\right)\Omega(t_2)\right)\alpha_4 . \quad (2.49)$$

Using this relation in (2.47) we get

$$\alpha_0 = \frac{t_3}{t_1}\left(1 + \Omega(t_2)\right)\alpha_4 ,$$

and from (2.49), in turns, we obtain

$$\alpha_3 = 2\frac{t_3}{t_2}(1 + \Omega(t_2))\alpha_4 .$$

Therefore, from (2.45) and (2.46) it follows that

$$\begin{aligned} \alpha_1 &= -\alpha_3 + t_2 t_3 \Omega(t_2)\alpha_4 \,, \\ \alpha_2 &= -\alpha_0 - (1 + t_2 t_3 \Omega(t_2))\alpha_4 \,. \end{aligned}$$

Finally, mixing together all the ingredients, we get

$$\overline{Q}(\alpha_4) = -\frac{2t_2}{3t_1}\left(t_3\left(6\pi_+^2 + 6t_2\pi_+\pi_{++} + t_2^2\pi_{++}^2\right) + t_2^2\Omega\left(t_2^3\right)\right)\alpha_4^2 \,. \tag{2.50}$$

Let us derive now an asymptotic expression for $R(\alpha_4)$. Due to (2.23), we have

$$\begin{aligned} h_0^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) + O\left(t_1^3\right) \,, \\ h_1^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) \,, \\ h_2^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) \,, \\ h_3^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) + O\left(t_2^3\right) \,, \\ h_4^{\mathrm{h}}(q(t_0)) &= f^{\mathrm{h}}(q(t_0)) + O\left(t_2^2 t_3\right) \,, \end{aligned}$$

and so, for every $0 \leq i \leq 4$,

$$h_i^{\mathrm{h}}(q(t_0))\alpha_i = f^{\mathrm{h}}(q(t_0))\alpha_i + O\left(t_2^2 t_3\right)\alpha_4 \,.$$

Similarly, for every $1 \leq j \leq n$, we have

$$\left\langle \lambda(t_0), \partial_{x_j} h_i^{\mathrm{v}}(q(t_0))\right\rangle \alpha_i = \left\langle \lambda(t_0), \partial_{x_j} f^{\mathrm{v}}(q(t_0))\right\rangle \alpha_i + O\left(t_2^2 t_3\right)\alpha_4 \,.$$

Thus, $R(\alpha_4) = O\left(t_2^4 t_3^2\right)\alpha_4^2$ and, finally, Q has the same asymptotic expression as $\overline{Q}$ in (2.50).

On the class $\widetilde{\Xi}_\eta^3$, the quantity π_+ is uniformly separated from zero, for T small. It follows from Theorem 1.8 that a short S-extremal pair in $\widetilde{\Xi}_\eta^3$ which contains a $+-+-+$ concatenation is S-essential. By symmetry in $+$ and $-$, we actually proved that a short S-extremal pair in $\widetilde{\Xi}_\eta^3$ with five concatenated arcs is S-essential.

Lemma 2.7 *If $q(\cdot)$ is short and S-quasi optimal, then the pair (π_+, π_{++}) lies in the interior of the second or of the fourth quadrant of $\mathbf{R}^2$.*

Proof. If $q(\cdot)$ is S-quasi optimal, then $Q \geq 0$. Taking into account the asymptotic expression for t_3 given in (2.43), we get from (2.50) that

$$\begin{aligned} 0 \;\geq\; & (6 + O(t_2))\pi_+^3 + (10 + O(t_2))\pi_+^2\pi_{++}t_2 + (5 + O(t_2))\pi_+\pi_{++}^2 t_2^2 \\ & + \left(\frac{2}{3} + O(t_2)\right)\pi_{++}^3 t_2^3 \,. \end{aligned} \tag{2.51}$$

Notice that the leading term of (2.51) is an homogeneous polynomial inequality in $(\pi_+, t_2\pi_{++})$. Its set of solutions in $\mathbf{R}^2$ is given by the cone

$$\begin{aligned} C &= \left\{(x,y) \in \mathbf{R}^2 \,\middle|\, 6x^3 + 10x^2 y + 5xy^2 + \frac{2}{3}y^3 \leq 0\right\} \\ &= \left\{(r\cos\theta, r\sin\theta) \,\middle|\, r \in [0,+\infty),\ \theta \in \mathcal{S}^1,\ P(\theta) \leq 0\right\}, \end{aligned}$$

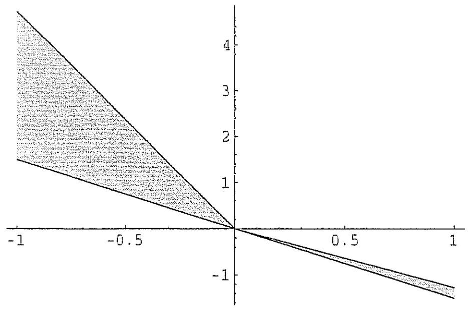

Figure 2.1: the set $C \cap D$.

where

$$P(\theta) = 6\cos^3(\theta) + 10\cos^2(\theta)\sin(\theta) + 5\cos(\theta)\sin^2(\theta) + \frac{2}{3}\sin^3(\theta)\,.$$

Fix a cone C' such that $C \subset C'$ and $\partial C \cap \partial C' = \{0\}$. Taking t_2 small enough we have that $(\pi_+, t_2\pi_{++}) \in C'$. Indeed, the trigonometric polynomial which defines C has six simple zeros on $\mathcal{S}^1$, which are stable by small perturbations of the coefficients.

Let

$$D = \left\{(x,y)\,\middle|\, x + \frac{y}{3} \geq 0\right\},$$

and consider a cone D' which contains the half-plane D and such that $\partial D \cap \partial D' = \{0\}$. Since the condition $\pi_0 \leq 0$ must also be satisfied, then $(\pi_+, t_2\pi_{++}) \in D'$ for t_2 small. If we choose C' and D' close enough to C and D, then $C' \cap D'$ is contained in the union of the second and the fourth quadrant and the lemma is proved. □

Let

$$\tilde{\pi}_+ = \langle \lambda(t_0 - t_1), X_+(q(t_0 - t_1))\rangle\,,$$

$$\tilde{\pi}_{++} = \langle \lambda(t_0 - t_1), X_{++}(q(t_0 - t_1))\rangle\,.$$

We have that

$$\tilde{\pi}_+ = \lim_{t \to t_1^-} \ddot{\varphi}(t_0 - t) = \ddot{\varphi}(t_0) - t_1\varphi^{(3)}(t_0 - \bar{t})\,,$$

where $\bar{t} \in [0, t_1]$. Since

$$\sup_{t\in[0,t_1]} |\varphi^{(3)}(t_0 - t)| = \sup_{t\in[0,t_1]} |\langle \lambda(t_0 - t), X_{++}(q(t_0 - t))\rangle| = O(1)\,,$$

then

$$\tilde{\pi}_+ = \pi_+ + t_2\Omega(t_2)\,.$$

For the same reason,

$$\tilde{\pi}_{++} = \pi_{++} + t_2\Omega(t_2)\,.$$

In particular, an inequality in the form (2.51) is still true if we replace π_+ and π_{++} by, respectively, $\tilde{\pi}_+$ and $\tilde{\pi}_{++}$. Analogously, the relation $t_3 > 0$ can be rewritten, in terms of $\tilde{\pi}_+$ and $\tilde{\pi}_{++}$, as

$$\tilde{\pi}_+ + \frac{2}{3}t_2\tilde{\pi}_{++} + t_2\Omega(t_2) > 0\,.$$

Thus, for T small enough, $(\tilde{\pi}_+, t_2\tilde{\pi}_{++}) \in C' \cap D'$, where C' and D' are chosen as in the proof of Lemma 2.7. In particular, for short pairs in $\tilde{\Xi}^1_\eta$, $(\tilde{\pi}_+, t_2\tilde{\pi}_{++})$ lies in the interior of the second or of the fourth quadrant.

Consider now a short S-extremal pair $(\lambda(\cdot), q(\cdot))$ which contains seven concatenated bang arcs, the first and the last one corresponding to control $+1$. Let $\tau_1 < \cdots < \tau_8$ be the boundary points of such arcs. Notice that $q|_{[\tau_1,\tau_6]}$ and $q|_{[\tau_3,\tau_8]}$ are both $+-+-+$ restrictions of $q(\cdot)$, and that $q(\tau_4)$ is both the first switching point of $q|_{[\tau_3,\tau_8]}$ and the second switching point of

$$[\tau_1, \tau_6] \ni t \mapsto q(\tau_1 + \tau_6 - t)\,, \tag{2.52}$$

which is a $+-+-+$ piece of trajectory for the time-reversed system (1.15).

Normalize $\lambda(\cdot)$ according to the choice of the $+-+-+$ concatenation $q|_{[\tau_3,\tau_8]}$. Denote by $q'(\cdot)$ the time-reversed trajectory of $q(\cdot)$ and by $\lambda'(\cdot)$ an extremal lift of $q'(\cdot)$ for (1.15), which satisfies (1.31). Assume that $\lambda'(\cdot)$ is normalized according to the choice of the $+-+-+$ concatenation given in (2.52).

Choosing T small enough (depending on η), we can assume that either one between these lifts is in $\tilde{\Xi}^3_\eta$ or they both belong to $\tilde{\Xi}^1_\eta$ or to $\tilde{\Xi}^2_\eta$. In the first case we already proved that the trajectory is S-essential.

Assume that they both are in $\tilde{\Xi}^1_\eta$. The role of $(\tilde{\pi}_+, \tilde{\pi}_{++})$ for $(\lambda(\cdot), q(\cdot))$ is played by a pair which is positively proportional to

$$p_1 = \left(\langle \lambda_0, X_+(q(\tau_4))\rangle\,, \langle \lambda_0, X_{++}(q(\tau_4))\rangle\right),$$

where $\lambda_0 = \lambda(\tau_4)$. Similarly, the role of (π_+, π_{++}) for $(\lambda'(\cdot), q'(\cdot))$ is played by a pair which is positively proportional to

$$p_2 = (\langle(-\lambda_0), -X_+(q(\tau_4))\rangle, \langle(-\lambda_0), X_{++}(q(\tau_4))\rangle) = (\langle\lambda_0, X_+(q(\tau_4))\rangle, -\langle\lambda_0, X_{++}(q(\tau_4))\rangle)\,.$$

If $q(\cdot)$ were S-quasi optimal, then p_1 and p_2 should both lie in the interior of the second or of the fourth quadrant of $\mathbf{R}^2$, which is, clearly, impossible.

Proposition 2.4 is proved, with $k(4,0) = 7$.

2.7 Regularity of non-bang-bang trajectories

Let q_0 be an (s,d)-point, $s + d \le 4$. Fix a short S-extremal pair $(\lambda(\cdot), q(\cdot))$. We can assume that $(\lambda(\cdot), q(\cdot))$ belongs to a class (Ξ or some Ξ^i_η) which satisfies (A) or (B-B'). In particular, $\lambda(\cdot)$ never annihilates $X_-(q(\cdot))$. The same is true for $X_+(q(\cdot))$ when (A) holds, and for $X_{--}(q(\cdot))$ in the situation (B).

Lemma 2.8 *Given a subinterval I of the domain of definition of $q(\cdot)$, if $q|_I$ does not contain bang arcs, then φ is identically equal to zero on I and $u|_I$ is smooth.*

Proof. If $\varphi(t) \neq 0$ at some $t \in I$, then the maximal neighborhood J of t in I on which $u(\cdot)$ is smooth is nonempty. It cannot be a bang arc, by hypothesis, and therefore it must be singular. The set

$$\tilde{J} = \operatorname{int}\{\tau \in J | \varphi(\tau) = 0\}$$

is a proper nonempty subset of J. Let $\bar{\tau}$ be in the boundary of $\tilde{J}$ and in the interior of J. By definition, $\bar{\tau}$ is both a density point for $\tilde{J}$, where $\varphi^{(n)} \equiv 0$ for every $n \ge 0$, and for $\{\tau \in J | \varphi(\tau) \neq 0\}$, where $|u| = 1$. Since u and φ are smooth on J, it follows that $|u(\overline{\tau})| = 1$ and $\varphi^{(n)}(\overline{\tau}) = 0$ for every $n \geq 0$. As already remarked in section 2.1, however, $\varphi^{(n)}(\overline{\tau})$ can be computed iterating (1.17). We reach a contradiction both with condition (A) and with condition (B). It follows that $\varphi|_I \equiv 0$.

Thus, also $\dot{\varphi}$ and the further derivatives of φ are identically equal to zero on I. In particular,

$$\lambda(t) \perp g(q(t)),\ [f,g](q(t)),$$

and, for almost every $t \in I$,

$$\langle \lambda(t), [f,[f,g]](q(t))\rangle + u(t)\,\langle \lambda(t), [g,[f,g]](q(t))\rangle = 0\,. \tag{2.53}$$

In both cases (A) and (B), $\langle \lambda(t), [g,[f,g]](q(t))\rangle \neq 0$ for every t for which (2.53) holds, otherwise $\langle \lambda(t), X_-(q(t))\rangle$ would be equal to zero. If, however,

$$\langle \lambda(\bar{t}), [g,[f,g]](q(\bar{t}))\rangle = 0$$

for some $\bar{t} \in I$, we would have that near $\bar{t}$ the function $t \mapsto |\langle \lambda(t), [f,[f,g]](q(t))\rangle|$ is uniformly separated zero and, consequently, $|u(t)| > 1$ for some t at which (2.53) holds. Thus, for every $t \in I$, $\langle \lambda(t), [g,[f,g]](q(t))\rangle \neq 0$ and

$$u(t) = -\frac{\langle \lambda(t), [f,[f,g]](q(t))\rangle}{\langle \lambda(t), [g,[f,g]](q(t))\rangle}\,. \tag{2.54}$$

Substituting (2.54) in (1.9), we find that $\lambda|_I$ is a solution of the smooth (autonomous) Hamiltonian system generated by the Hamiltonian

$$h(\lambda) = \langle \lambda, f\rangle - \frac{\langle \lambda, [f,[f,g]]\rangle}{\langle \lambda, [g,[f,g]]\rangle}\langle \lambda, g\rangle\,,$$

and, in particular, it is smooth. Thus, $q|_I$ is also smooth and, according to (2.54), the same is true for $u|_I$. □

Remark 2.9 The lemma implies, in particular, that the union of bang and singular arcs is dense in the domain of definition of $q(\cdot)$. A property of this kind turns out to be more general, and provides an extension of the one recalled at the beginning of section 1.3. Indeed, a straightforward generalization of the proof of Proposition 1 in [1] shows that, independently of the dimension of M and of S, if

$$\{X(q_0)|\ X \in I(g)\} + T_{q_0}S = T_{q_0}M\,,$$

then the control function corresponding to a short S-extremal trajectory is smooth on an open dense set of its domain of definition.

A consequence of Lemma 2.8 is that, along any singular arc, $\varphi \equiv 0$. In particular, if t is such that $\varphi(t) = 0$ and $\dot{\varphi}(t) \neq 0$, then it is the switching time between two concatenated bang arcs. In both cases (A) and (B), the second derivative of φ has constant sign along all $-$ arcs. Therefore, $\dot{\varphi}$ is different from zero at the boundary points of each compactly contained $-$ arc. It follows that each compactly contained $-$ arc is concatenated to two $+$ arcs. If (A) holds, a symmetric reasoning for $+$ arcs leads to the conclusion that if $q(\cdot)$

has at least one compactly contained bang arc, then it is purely bang-bang, because of Proposition 2.4. On the other hand, if $q(\cdot)$ does not have compactly contained bang arcs, then it follows from Lemma 2.8 that it is the concatenation of at most a bang, a singular, and a bang arc.

If (B) holds, the situation is slightly more complicated. Nevertheless, it is still true that a compactly contained $-$ arc cannot be concatenated to a singular arc. The condition $\langle \lambda(\cdot), X_{++}(\cdot)\rangle \neq 0$ implies that the third derivative of the switching function along $+$ arcs has constant sign. Therefore $\ddot{\varphi}$ can change sign only once along a $+$ arc, and always in the same direction (from negative to positive or the other way round). In particular, $q(\cdot)$ cannot have a bang, a singular, and a bang concatenated compactly contained arcs. In addition, a compactly contained bang arc is always concatenated to at least another bang arc.

We want to prove that, if $q(\cdot)$ has a singular arc, then it cannot have more than one compactly contained bang arc. Assume by contradiction that it has two. Without loss of generality they are concatenated; indeed, if they are not, they identify a bounded nonempty interval I, situated between the two. If I contains no bang arc, then, due to Lemma 2.8, we detected a BSB compactly contained concatenation, which is impossible. If it contains one, this one is concatenated to another bang arc, compactly contained as well. As it was proved in Lemma 2.5 and Lemma 2.6, the existence of two compactly contained concatenated bang arcs arc implies (at least for T small) the uniqueness of the corresponding covector trajectory. Thus, from the generalized Legendre condition, Theorem 1.11, it follows that

$$\langle \lambda(t), [g, [f, g]](q(t))\rangle \leq 0 \tag{2.55}$$

along the singular arcs of the trajectory. Since we assumed that also (B$'$) holds, we have that (2.55) is satisfied for every t and that $\ddot{\varphi}$ is positive along each $-$ arc, as φ is. It follows that $q(\cdot)$ cannot have compactly contained $-$ arcs. We reached a contradiction, and, therefore, we have proved that, if $q(\cdot)$ is not purely bang-bang, then it admits at most one compactly contained bang arc.

Finally, either a short S-quasi optimal trajectory is bang-bang or it is of the type $-+$S$\pm$ or $\pm$S$+-$ (allowing some arc to have length zero). In the cases in which property (A) holds, we further restricted the possible S-quasi optimal concatenations to bang-bang and $\pm$S$\pm$ trajectories.

2.8 Properties of generic quasi optimal control problems

The following table collects all upper bounds on the number of arcs of short S-quasi optimal trajectories, which were obtained in the previous sections.

Every row corresponds to one class of (s, d)-points. The second column associates the corresponding bound for short trajectories which have at least one singular arc of positive length, and accounts for the maximal non-bang-bang concatenations which are candidate to be S-quasi optimal. The third column contains the bound which applies to purely bang-bang trajectories, that is, the value of k at (s, d).

(s,d)	non-bang-bang bound		bang-bang bound	general bound
(1,0)	0	/	1	1
(1,1)	0	/	2	2
(1,2)	3	BSB	3	3
(1,3)	4	BSBB, BBSB	3	4
(2,0)	0	/	2	2
(2,1)	3	BSB	3	3
(2,2a)	4	BSBB, BBSB	3	4
(2,2b)	3	BSB	3	3
(3,0)	3	BSB	3	3
(3,1)	4	BSBB, BBSB	4	4
(4,0)	4	BSBB, BBSB	7	7

The whole of the bounds contained in the table, due to the transversality considerations recalled in the introduction, can be summarized as follows.

Theorem 2.10 *Let M be a finite dimensional manifold and S a submanifold of M. For a generic pair of vector fields $f, g \in \mathrm{Vec}M$, there exists a stratified set W of codimension five in M such that, for every point q_0 of $S \setminus W$, every short S-quasi optimal trajectory of (1.13), contained in a sufficiently small neighborhood of q_0, is the concatenation of at most seven bang and singular arcs.*

If the dimension of M is less than or equal to four, then W is fact empty.

As remark 2.3 pointed out, when the dimension of S is equal to zero the above regularity results admit a stronger formulation. Indeed, if $S = \{q_0\}$, the class of trajectories which verify the asymptotic relations found in sections 2.5 and 2.6 does not reduce to S-extremal ones, since the initial condition $q(0) \in S$ stops having any role in the expansion. The same is true for what concerns section 2.7, whose arguments apply to all quasi optimal trajectories contained in a small enough neighborhood of q_0. Taking into account these considerations, the following three propositions specify the general bounds to the point-to-point problem.

Proposition 2.11 *Let M be a two-dimensional manifold. Then, for a generic pair of vector fields $f, g \in \mathrm{Vec}M$, for every point $q_0 \in M$, there exist a neighborhood U of q_0 and a time $T > 0$, such that a quasi optimal trajectory of the system (1.7) contained in U and of time-length smaller than T is the concatenation of at most four bang and singular arcs. The only possible maximal concatenations are of the type BBB, BSBB, BBSB.*

Proposition 2.12 *Let M be a three-dimensional manifold. For a generic pair of vector fields $f, g \in \mathrm{Vec}M$, there exist a one-dimensional and a two-dimensional stratified sets W_1 and W_2 in M, such that,*

- *$W_1 \subset W_2 \subset M$;*
- *for every point q_0 in $M \setminus W_2$, there exist a neighborhood U of q_0 and a time $T > 0$, such that a quasi optimal trajectory of the system (1.7) contained in U and of time-length smaller than T is the concatenation of at most three bang and singular arcs, with possible maximal concatenations of the type BBB, BSB;*

- *for every point q_0 in $W_2 \setminus W_1$, there exist a neighborhood U of q_0 and a time $T > 0$, such that a quasi optimal trajectory of the system (1.7) contained in U and of time-length smaller than T is the concatenation of at most four bang and singular arcs, with possible maximal concatenations of the type BBBB, BSBB, BBSB.*

Proposition 2.13 *Let M be a four-dimensional manifold. Then, for a generic pair of vector fields $f, g \in \mathrm{Vec}M$, there exists a three-dimensional stratified set $W \subset M$, such that, for every point q_0 in $M \setminus W$, there exist a neighborhood U of q_0 and a time $T > 0$, such that a quasi optimal trajectory of the system (1.7) contained in U and of time-length smaller than T is the concatenation of at most seven bang and singular arcs. The only possible maximal concatenations including singular arcs are of the type BSBB, BBSB.*

2.9 Sharpness of the results

We already noticed that our results partially overlap the ones of [33] for the cases (1,0), (1,1) and (1,2). The restrictions given here, namely, that the maximal possible concatenations for a short S-quasi optimal trajectory are of the type BBB or BSB, are sharp, since in the classification of time-optimal syntheses given in [33] these kind of concatenations actually appear.

For what concerns the case $s = 2$, both BBB and BSBB short time-optimal trajectories can be observed in the regular syntheses of a generic time-optimal flow in the plane, classified in [16].

In order to prove the sharpness of the bounds given for $s = 3$, we apply the sufficiency condition for optimality proved in [9]. The same reasonings applied to the present case imply that, if the quadratic form Q is positive definite, then the corresponding trajectory $q : [0, T] \to M$ is (state,time)-locally time-optimal, that is, there exists a neighborhood W of the graph $\{(t, q(t)) \mid t \in [0, T]\}$ in $[0, T] \times M$ such that $q(\cdot)$ is time-optimal among all the admissible trajectories whose graph is contained in W (see also [43]).

We work in the point-to-point setting and we fix $M = \mathbf{R}^3$ and $S = \{0\}$.

Proposition 2.14 *Let $J_0^{2,4}\mathbf{R}^3$ be the space of 4-jets at 0 of elements $\mathrm{Vec}\mathbf{R}^3 \times \mathrm{Vec}\mathbf{R}^3$ and denote by $C_1 \subset J_0^{2,4}\mathbf{R}^3$ the set of all $J_0^{2,4}(f, g)$ such that 0 is a (3,1)-point for the system determined by f and g. Then, there exists an open nonempty subset A_1 of C_1 such that, if the 4-jet of (f, g) belongs to A_1, then, for every $T > 0$ there exists a trajectory made of four bang arcs, passing through 0 and of time-length smaller than T, which is (state,time)-locally time-optimal.*

Proof. Fix $f, g \in \mathrm{Vec}\mathbf{R}^3$ such that $J_0^{2,4}(f, g) \in C_1$. Let $q : [0, T] \to \mathbf{R}^3$ be a $+-+-$ extremal trajectory and denote by $t_0 < t_0 + t_1 < t_0 + t_1 + t_2$ its switching times. Assume, moreover, that $q(t_0) = 0$. We already know that, for T small enough, $q(\cdot)$ has a unique extremal lift $\lambda(\cdot)$ normalized is such way that

$$|\lambda(t_0)| = \max\{|\langle \lambda(t_0), [f, g](0) \rangle|, |\langle \lambda(t_0), X_-(0) \rangle|\} = 1\,.$$

The equations satisfied by the switching times are

$$0 = \langle \lambda(t_0), g(0) \rangle\,, \tag{2.56}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f-g)} g(0) \right\rangle, \tag{2.57}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f-g)} e^{t_2 \mathrm{ad}\,(f+g)} g(0) \right\rangle. \tag{2.58}$$

Define $\pi_\star$ as as in (2.11) and (2.12). Notice that $\pi_0 = \dot\varphi(t_0)$ is nonnegative, and, as follows from the usual considerations, $\pi_- = -1$. We can assume, moreover, that $|\pi_{++}|$ is uniformly bounded from below by a positive constant. Equation (2.57) implies that

$$\pi_0 = \frac{t_1}{2} - \frac{t_1^2}{6}\pi_{--} + O\big(t_1^3\big) . \tag{2.59}$$

For every word w with letters in $\{-,+\}$, denote by δ_w and η_w the components of $X_w(0)$ in the directions $[f,g](0)$ and $X_-(0)$, with respect to the basis $g(0), [f,g](0), X_-(0)$. Notice that $\pi_w = -\eta_w + \delta_w\pi_0 = -\eta_w + O(t_1)$. In particular, since $\eta_+ = 0$, $\pi_+ = O(t_1)$.

From (2.58) we get

$$\begin{aligned} 0 &= \pi_0 - t_1 + \frac{t_2^2}{6}\pi_{++} + O\big(t_1^2\big) + O(t_1t_2) + O\big(t_2^3\big) \\ &= -\frac{t_1}{2} + \frac{t_2^2}{6}\pi_{++} + O\big(t_1^2\big) + O(t_1t_2) + O\big(t_2^3\big) . \end{aligned}$$

Therefore,

$$t_2^2 = \frac{3}{\pi_{++}}t_1 + O\Big(t_1^{3/2}\Big) .$$

We have

$$\begin{aligned} h_0 &= f+g, \\ h_1 &= f-g, \\ h_2 &= f+g+2t_1[f,g] + t_1^2X_- + O\big(t_1^3\big) , \\ h_3 &= f-g-2t_2[f,g] - t_2^2X_+ - 2t_1t_2X_- - \frac{t_2^3}{3}X_{++} + O\big(t_1^2\big) , \end{aligned}$$

and

$$\begin{aligned} \sigma_{01} &= -2\pi_0 , \\ \sigma_{02} &= -2t_1 + t_1^2\pi_{+-} + O\big(t_1^3\big) , \\ \sigma_{12} &= 2\pi_0 - 2t_1 + t_1^2\pi_{--} + O\big(t_1^3\big) , \\ \sigma_{03} &= -2\pi_0 - 2t_2\pi_+ - t_2^2\pi_{++} - 2t_1t_2\pi_{+-} - \frac{t_2^3}{3}\pi_{+++} + O\big(t_1^2\big) , \\ \sigma_{13} &= 2t_2 - t_2^2\pi_{-+} - 2t_1t_2\pi_{--} - \frac{t_2^3}{3}\pi_{-++} + O\big(t_1^2\big) , \\ \sigma_{23} &= \sigma_{03} - \sigma_{12} + 2\pi_0 - 2t_1t_2^2\pi_\times + O\Big(t_1^{5/2}\Big) , \end{aligned}$$

where

$$\pi_\times = \langle \lambda(t_0), [[f,g],X_+](0)\rangle .$$

The space on which Q is defined turns out to be described by the system

$$\begin{aligned} \alpha_0 &= -\Big(1 + O\Big(t_1^{3/2}\Big)\Big)\,\alpha_2 , \\ \alpha_1 &= \left(-\frac{t_1}{t_2} + t_1\frac{\delta_+}{2} + \frac{t_1t_2}{2}\left(\frac{\delta_{++}}{3} - \frac{\delta_+^2}{2} - \gamma_+\right) + O\big(t_1^2\big)\right)\alpha_2 , \\ \alpha_3 &= \left(\frac{t_1}{t_2} - t_1\frac{\delta_+}{2} - \frac{t_1t_2}{2}\left(\frac{\delta_{++}}{3} - \frac{\delta_+^2}{2}\right) + O\big(t_1^2\big)\right)\alpha_2 , \end{aligned}$$

where γ_+ is the component of $X_+(0)$ in the direction $g(0)$. Finally, we get

$$Q(\alpha_2) = \left(t_1^2 t_2 \left(\frac{\pi_{-++} + \delta_{++}}{3} - \delta_+^2 - \pi_{-+}\delta_+ - \gamma_+ - 2\pi_\times + \frac{2}{9}\pi_{++}\pi_{--} \right) + O(t_1^3) \right) \alpha_2^2 .$$

To conclude the proof of Lemma 2.14 it suffices to exhibit a system on $\mathbf{R}^3$ for which the origin is a (3,1)-point and the following sign conditions hold

$$\begin{aligned} \eta_{++} &< 0, \\ \frac{\eta_{-++} - \delta_{++}}{3} + \delta_+^2 - \eta_{-+}\delta_+ + \gamma_+ - 2\eta_\times - \frac{2}{9}\eta_{++}\eta_{--} &< 0, \end{aligned}$$

where $\eta_\times$ denotes the component of $[[f,g],X_+](0)$ in the direction $X_-(0)$.

This is the case, for instance, of the control system

$$f(x,y,z) = \begin{pmatrix} 0 \\ -x \\ -\frac{x^2}{4} - \frac{x^3}{4} + \frac{y}{2} + y^2 - z \end{pmatrix}, \qquad g(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

for which we get $\eta_{++} = -2$ and

$$Q = Q(\alpha_2) = \left(\frac{17}{6} t_1^2 t_2 + O(t_1^3) \right) \alpha_2^2 . \qquad \square$$

Also the non-bang-bang part of the bound which we gave for $s = 3$ turns out to be sharp. Indeed, in [51] it is shown that short time-optimal concatenations of two bang, one singular, and one bang arc, exiting from a (3,1)-point are structurally stable ($\dim S = 0$). We stress that the optimality notion considered in [51] is global, and not only (time-state)-local.

The sharpness issue for the bounds given in the case $s = 4$ has not yet been investigated. Its study could be the object of some further research, linked with the classification of generic optimal syntheses in dimension four.

Chapter 3

Codimension two singularities for the three dimensional point-to-point problem

The positive regularity results of chapter 2 ask, in some sense, for a negative counterpart. As it was recalled in the introduction, chattering phenomena are stable for single-input control-affine systems in manifolds of large enough dimension. This chapter gets a little closer to the border between the chattering territory and the non-chattering one.

The first four sections present results which are contained in our work [53]. A classification of codimension two singularities for the three-dimensional point-to-point problem is proposed. For two of the three classes which have been introduced, a local bound on the numbers of arcs of quasi optimal trajectories is proved. The language and the methods are very much in the style of the ones used in chapter 2, although new technical problems arise.

The remaining part of the chapter deals with the singularity classified in section 3.1 and not treated in [53]. The aim is to show why the previous approach is not sufficient to give a complete understanding, and to propose some method which we hope could be useful for a further research in the subject. In particular, a non-standard nilpotent approximation of the system is discussed.

3.1 Classification of codimension two singularities

Let M be a three-dimensional manifold. In the previous chapter we obtained local regularity properties which apply, in particular, at codimension zero and one singularities of the point-to-point problem on M. Let us recall how (3,0)- and (3,1)-points were characterized, adapting the definitions to the case $S = \{q_0\}$,

(3,0) $g(q_0) \wedge [f,g](q_0) \wedge X_+(q_0) \neq 0$ and $g(q_0) \wedge [f,g](q_0) \wedge X_-(q_0) \neq 0$;

(3,1) $g(q_0) \wedge [f,g](q_0) \wedge X_+(q_0) = 0$, $g(q_0) \wedge [f,g](q_0) \wedge X_{++}(q_0) \neq 0$ and $g(q_0) \wedge [f,g](q_0) \wedge X_-(q_0) \neq 0$.

The definition of the vectors fields $X_\star$ is given in (1.21).

Define (3,2a)- and (3,2b)-points as follows,

(3,2a) $g(q_0) \wedge [f,g](q_0) \wedge X_+(q_0) = g(q_0) \wedge [f,g](q_0) \wedge X_{++}(q_0) = 0,$
$g(q_0) \wedge [f,g](q_0) \wedge X_{(3+)}(q_0) \neq 0$ and $g(q_0) \wedge [f,g](q_0) \wedge X_-(q_0) \neq 0;$

(3,2b) $g(q_0) \wedge [f,g](q_0) = 0, \;\; g(q_0) \wedge X_+(q_0) \wedge X_-(q_0) \neq 0,$
$g(q_0) \wedge X_+(q_0) \wedge X_{++}(q_0) \neq 0$ and $g(q_0) \wedge X_-(q_0) \wedge X_{--}(q_0) \neq 0.$

Local regularity results for (3,2a)- and (3,2b)-points are given in sections 3.2, 3.3 and 3.4. In order to give a complete picture of codimension two singularities, we should consider points at which

$$g \wedge [f,g] \wedge X_+ \;=\; 0 \tag{3.1}$$
$$g \wedge [f,g] \wedge X_- \;=\; 0, \tag{3.2}$$

and find appropriate non-degeneracy conditions, independent of (3.1) and (3.2), which allows to describe the behavior of (1.7) near such points. This will be the subject of sections 3.5 and 3.6.

Since we do not consider the general manifold-to-point situation, a more explicit notion of local regularity can be adopted, which expressly accounts for the size of the neighborhood in which the bound holds. That is, given a (3,2a)- (respectively, (3,2b)-) point q_0, we will say that a neighborhood U of q_0 is (3,2a)-*adapted* (respectively, (3,2b)-*adapted*) if it is relatively compact, connected and the relations in inequality form characterizing the (3,2a) (respectively, (3,2b)) configuration hold throughout $\overline{U}$. Once q_0 is fixed, we say that all *short* trajectories of a certain class have a given property $(\mathcal{P})$ if, for any adapted neighborhood U of q_0, there exist $T > 0$ such that all trajectories in the class, which are contained in U and have time-length smaller than T, satisfy $(\mathcal{P})$.

3.2 A bound on the number of arcs of quasi optimal bang-bang trajectories in the (3,2a) case

Let $q_0 \in M$ be a (3,2a)-point and fix a (3,2a)-adapted neighborhood U of q_0. A moving basis in U is given by the triple of vector fields $Y_1 = g$, $Y_2 = [f,g]$ and $Y_3 = X_-$. Associate with Y_1, Y_2, Y_3 an Euclidean structure on $T^*\overline{U}$ as in the statement of Corollary 2.2.

3.2.1 Short $-+-+$ trajectories

Let $(\lambda(\cdot), q(\cdot))$ be a short extremal $-+-+$ pair. Its switching times, denoted by $t_0 < t_0 + t_1 < t_0 + t_1 + t_2$, verify

$$0 \;=\; \langle \lambda(t_0), g(q(t_0)) \rangle \,, \tag{3.3}$$
$$0 \;=\; \left\langle \lambda(t_0), e^{t_1 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle , \tag{3.4}$$
$$0 \;=\; \left\langle \lambda(t_0), e^{t_1 \operatorname{ad}(f+g)} e^{t_2 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle . \tag{3.5}$$

In analogy with what was done in chapter 2, define, for every $i = 1,2,3$,

$$a_i = \langle \lambda(t_0), Y_i(q(t_0)) \rangle \;,$$

and normalize $\lambda(\cdot)$ in such a way that

$$|\lambda(t_0)| = \max\{|a_1|, |a_2|, |a_3|\} = 1\,.$$

Equality (3.3) implies that $a_1 = 0$, while from (3.4) we obtain

$$a_2 = a_2 O(T) + a_3 O(T)\,. \tag{3.6}$$

The normalization of $\lambda(\cdot)$ implies that $|a_3| = 1$. Moreover, since $q(\cdot)$ has a compactly contained $-$ arc, we can assume that $a_3 = -1$. Therefore, (3.6) defines uniquely a_2. This proves the uniqueness, up to normalization, of the extremal lift of $q(\cdot)$.

Following (2.11) and (2.12), we define $\pi_0 = a_2$ and $\pi_w = \langle \lambda(t_0), X_w(q(t_0))\rangle$, for any word w with letters in $\{-,+\}$. From (3.4) we get

$$\pi_0 = -\frac{t_1}{2}\pi_+ - \frac{t_1^2}{6}\pi_{++} - \frac{t_1^3}{24}\pi_{(3+)} + O\big(t_1^4\big)\ .$$

From (3.5), in turns, we obtain

$$\begin{aligned} t_2 &= 2\pi_0 + 2t_1\pi_+ + t_1^2\pi_{++} + \frac{t_1^3}{3}\pi_{(3+)} + O\big(t_1^4\big) + O(t_1t_2) + O\big(t_2^2\big) \\ &= t_1\pi_+ + \frac{2}{3}t_1^2\pi_{++} + \frac{t_1^3}{4}\pi_{(3+)} + \pi_+O\big(t_1^2\big) + \pi_{++}O\big(t_1^3\big) + O\big(t_1^4\big)\ . \end{aligned}$$

In order to express asymptotic relations concisely, we find it useful to introduce, as it was done for $\Omega(\cdot)$ in (2.44),

$$\omega(t_1) = \pi_+O(1) + \pi_{++}O(t_1) + O\big(t_1^2\big)\ .$$

For example, $\pi_0, t_2 = t_1\omega(t_1)$.

Define h_i as in (1.22), $0 \le i \le 3$, and let, as in (2.21),

$$\sigma_{ij} = \langle \lambda(t_0), [h_i, h_j](q(t_0))\rangle\,, \qquad 0 \le i < j \le 3\,.$$

We have

$$\begin{aligned} h_0 &= f - g\,, \\ h_1 &= f + g\,, \\ h_2 &= f - g - 2t_1[f,g] - t_1^2X_+ - \frac{t_1^3}{3}X_{++} + O\big(t_1^4\big)\ , \\ h_3 &= f + g + 2t_2[f,g] + 2t_1t_2X_+ + t_1^2t_2X_{++} + t_1t_2\omega(t_1)\,, \end{aligned}$$

and so

$$\begin{aligned} \sigma_{01} &= 2\pi_0\,, \\ \sigma_{02} &= 2t_1 + O\big(t_1^2\big)\ , \\ \sigma_{12} &= -2\pi_0 - 2t_1\pi_+ - t_1^2\pi_{++} - \frac{t_1^3}{3}\pi_{(3+)} + O\big(t_1^4\big)\ , \\ \sigma_{03} &= 2\pi_0 - 2t_2 + O(t_1t_2)\,, \\ \sigma_{13} &= 2t_2\pi_+ + 2t_1t_2\pi_{++} + t_1^2t_2\pi_{(3+)} + t_1t_2\omega(t_1)\,, \\ \sigma_{23} &= \sigma_{03} - \sigma_{12} - 2\pi_0 + O\big(t_1^2t_2\big)\ . \end{aligned}$$

The space H, defined in (1.24), can be described by

$$\sum_{i=0}^{3} \alpha_i = 0$$

and by the components of

$$\sum_{i=0}^{3} \alpha_i (h_i - f)(q(t_0)) = 0 \tag{3.7}$$

in the directions $g(q(t_0))$ and $[f,g](q(t_0))$, with respect to the basis

$$g(q(t_0)),\ [f,g](q(t_0)),\ X_-(q(t_0))\,.$$

That is, by the following system

$$\left\{\begin{array}{lllll} \alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & = 0\,, \\ -\alpha_0 & +\alpha_1 & -\left(1+O\left(t_1^2\right)\right)\alpha_2 & +(1+O(t_1t_2))\alpha_3 & = 0\,, \\ & & -\left(2t_1+O\left(t_1^2\right)\right)\alpha_2 & +(2t_2+O(t_1t_2))\alpha_3 & = 0\,. \end{array}\right.$$

from which we deduce

$$\begin{array}{rcl} \alpha_0 & = & -\left(\frac{t_2}{t_1}+O(t_2)\right)\alpha_3\,, \\ \alpha_1 & = & -(1+O(t_1t_2))\alpha_3\,, \\ \alpha_2 & = & \left(\frac{t_2}{t_1}+O(t_2)\right)\alpha_3\,. \end{array}$$

Thus,

$$Q(\alpha_3) = -2t_2\left(\pi_+ + t_1\pi_{++} + \frac{t_1^2}{2}\pi_{(3+)} + t_1\omega(t_1)\right)\alpha_3^2\,. \tag{3.8}$$

If Q is nonnegative definite, then

$$\pi_+ + t_1\pi_{++} + \frac{t_1^2}{2}\pi_{(3+)} + t_1\omega(t_1) \le 0\,.$$

Assume that $q(\cdot)$ is quasi optimal. We can associate with $q(\cdot)$ a system of inequalities which accounts for the sign conditions $Q \ge 0$, $\pi_0 \le 0$ and $t_2 > 0$, as follows

$$\left\{\begin{array}{lll} \pi_+ + t_1\pi_{++} + \frac{t_1^2}{2}\pi_{(3+)} + t_1\omega(t_1) & \le & 0\,, \\ \pi_+ + \frac{t_1}{3}\pi_{++} + \frac{t_1^2}{12}\pi_{(3+)} + t_1\omega(t_1) & \ge & 0\,, \\ \pi_+ + \frac{2}{3}t_1\pi_{++} + \frac{t_1^2}{4}\pi_{(3+)} + t_1\omega(t_1) & > & 0\,. \end{array}\right. \tag{3.9}$$

Reasoning as in Lemma 2.7, we get that, for T small enough, $(\pi_+, \pi_{++}, \pi_{(3+)})$ belongs to one of the octants $(++-)$, $(+-\pm)$ and $(-+-)$, with the agreement that the $(\nu_1\,\nu_2\,\nu_3)$ octant of $\mathbf{R}^3$ is the set $\{(x_1,x_2,x_3)|\ \nu_i x_i > 0,\ i=1,2,3\}$.

3.2.2 Short $+-+-+$ trajectories

Let us repeat the previous scheme of computations for a short quasi optimal $+-+-+$ pair $(\lambda(\cdot), q(\cdot))$. Denote by t_0 the first switching time of $q(\cdot)$ and by t_1, t_2, t_3 the length of, respectively, the second, the third, and the fourth bang arc. It follows that

$$0 = \langle \lambda(t_0), g(q(t_0)) \rangle , \tag{3.10}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f-g)} g(q(t_0)) \right\rangle , \tag{3.11}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f-g)} e^{t_2 \mathrm{ad}\,(f+g)} g(q(t_0)) \right\rangle , \tag{3.12}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \mathrm{ad}\,(f-g)} e^{t_2 \mathrm{ad}\,(f+g)} e^{t_3 \mathrm{ad}\,(f-g)} g(q(t_0)) \right\rangle . \tag{3.13}$$

Remark that $\lambda(\cdot)$ is unique, up to the normalization $|\lambda(t_0)| = 1$, as it has been proved in section 3.2.1. From (3.10 – 3.13), we get

$$\pi_0 = \frac{t_1}{2} + O\big(t_1^2\big) , \tag{3.14}$$

$$t_1 = t_2\pi_+ + \frac{t_2^2}{3}\pi_{++} + \frac{t_2^3}{12}\pi_{(3+)} + t_2^2\omega(t_2) , \tag{3.15}$$

$$t_3 = t_2\pi_+ + \frac{2}{3}t_2^2\pi_{++} + \frac{t_2^3}{4}\pi_{(3+)} + t_2^2\omega(t_2) . \tag{3.16}$$

Therefore,

$$\begin{aligned}
h_2 &= f + g + 2t_1[f,g] + O\big(t_1^2\big) , \\
h_3 &= f - g - 2t_2[f,g] - t_2^2 X_+ - \frac{t_2^3}{3} X_{++} + t_2^2\omega(t_2) , \\
h_4 &= f + g + 2(t_1 + t_3)[f,g] + 2t_2t_3X_+ + t_2^2t_3X_{++} + t_2^2\omega(t_2)\omega(t_2) ,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{01} &= -2\pi_0 , \\
\sigma_{02} &= 2t_1\pi_+ + O\big(t_1^2\big) , \\
\sigma_{12} &= 2\pi_0 - 2t_1 + O\big(t_1^2\big) , \\
\sigma_{03} &= -2\pi_0 - 2t_2\pi_+ - t_2^2\pi_{++} - \frac{t_2^3}{3}\pi_{(3+)} + t_2^2\omega(t_2) , \\
\sigma_{13} &= 2t_2 + O\big(t_2^2\big) , \\
\sigma_{23} &= \sigma_{03} - \sigma_{12} + 2\pi_0 + O\big(t_1t_2^2\big) , \\
\sigma_{04} &= 2(t_1 + t_3)\pi_+ + 2t_2t_3\pi_{++} + t_2^2t_3\pi_{(3+)} + t_2^2\omega(t_2)\omega(t_2) , \\
\sigma_{14} &= 2\pi_0 - 2(t_1 + t_3) + t_2^2\omega(t_2) , \\
\sigma_{24} &= \sigma_{04} - \sigma_{02} + t_2^3\omega(t_2)\omega(t_2) , \\
\sigma_{34} &= \sigma_{14} - \sigma_{03} - 2\pi_0 + t_2^2\omega(t_2) .
\end{aligned}$$

The space H is characterized by the system

$$0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 , \tag{3.17}$$

$$0 = \alpha_0 - \alpha_1 + \big(1 + O\big(t_1^2\big)\big)\,\alpha_2 - \big(1 + O\big(t_2^2\big)\big)\,\alpha_3 + \big(1 + t_2^2\omega(t_2)\big)\,\alpha_4 , \tag{3.18}$$

$$0 = \big(2t_1 + O\big(t_1^2\big)\big)\,\alpha_2 - \big(2t_2 + O\big(t_2^2\big)\big)\,\alpha_3 + \big(2(t_1 + t_3) + t_2^2\omega(t_2)\big)\,\alpha_4 . \tag{3.19}$$

From the (3.19) we obtain

$$\alpha_3 = \left(\frac{t_1}{t_2} + t_2\omega(t_2)\right)\alpha_2 + \left(\frac{t_1+t_3}{t_2} + t_2\omega(t_2)\right)\alpha_4\,,$$

which we can plug in (3.17) and (3.18), getting

$$\alpha_0 = -\left(1 + t_2^2\omega(t_2)\right)\alpha_2 - \left(1 + t_2^2\omega(t_2)\right)\alpha_4$$

and

$$\alpha_1 = -\left(\frac{t_1}{t_2} + t_2\omega(t_2)\right)\alpha_2 - \left(\frac{t_1+t_3}{t_2} + t_2\omega(t_2)\right)\alpha_4\,.$$

Thus, after some computations,

$$\begin{aligned} Q = Q(\alpha_2,\alpha_4) = &\left(-2\pi_+(t_1+t_3) - 2t_2t_3\pi_{++} - t_2^2t_3\pi_{(3+)} + t_2^2\omega(t_2)\omega(t_2)\right)\alpha_4^2 \\ &+ \left(-4t_1\pi_+ + t_2^2\omega(t_2)\omega(t_2)\right)\alpha_2\alpha_4 - \left(2t_1\pi_+ + t_2^2\omega(t_2)\omega(t_2)\right)\alpha_2^2\,. \end{aligned}$$

If Q is nonnegative definite then, in particular, the coefficient of α_4^2 must be nonnegative. Using the relations (3.15) and (3.16), it turns out that

$$4\pi_+^2 + 4t_2\pi_+\pi_{++} + \frac{4}{3}t_2^2\pi_{++}^2 + \frac{5}{3}t_2^2\pi_+\pi_{(3+)} + \frac{7}{6}t_2^3\pi_{++}\pi_{(3+)} + \frac{t_2^4\pi_{(3+)}^2}{4} + t_2\omega(t_2)\omega(t_2) \le 0. \quad (3.20)$$

Since $t_1, t_3 > 0$, then the following inequalities are also satisfied

$$\pi_+ + \frac{t_2\pi_{++}}{3} + \frac{t_2^2\pi_{(3+)}}{12} + t_2\omega(t_2) \;>\; 0\,, \quad (3.21)$$

$$\pi_+ + \frac{2}{3}t_2\pi_{++} + \frac{t_2^2\pi_{(3+)}}{4} + t_2\omega(t_2) \;>\; 0\,. \quad (3.22)$$

Again, we can interpret (3.20), (3.21), (3.22) as a necessary condition on $q(\cdot)$ to be quasi optimal, expressed as a set of constraints on the position of the triple $(\pi_+, \pi_{++}, \pi_{(3+)})$ in $\mathbf{R}^3$. The leading terms of the three inequalities can be put together to define the closed cone

$$C = \left\{ (x,y,z) \in \mathbf{R}^3 \;\middle|\; \begin{array}{l} -4x^2 - 4xy - \frac{4}{3}y^2 - \frac{5}{3}xz - \frac{7}{6}yz - \frac{z^2}{4} \ge 0 \\ x + \frac{y}{3} + \frac{z}{12} \ge 0 \\ x + \frac{2}{3}y + \frac{z}{4} \ge 0 \end{array} \right\}.$$

Fix a cone C' which contains C and such that $\partial C \cap \partial C' = \{0\}$. Taking T small enough, whenever $q(\cdot)$ is quasi optimal the triple $\left(\pi_+, t_2\pi_{++}, t_2^2\pi_{(3+)}\right)$ is contained in C'. Elementary calculations show that C' can be chosen in such a way that it intersects only the octants $(+-+)$ and $(\pm+-)$.

3.2.3 Conclusions for the bang-bang (3,2a) case

Let $(\lambda(\cdot), q(\cdot))$ be a short bang-bang extremal pair with seven arcs, the first and the last one corresponding to control $+1$. Let $\tau_1 < \cdots < \tau_8$ denote the boundary points of the seven arcs. Then, $q(\tau_4)$ is the first switching point of the $+-+-+$ trajectory $q|_{[\tau_3,\tau_8]}$ and, at the same time, the first switching point of

$$[\tau_1, \tau_5] \ni t \longmapsto q(\tau_1 + \tau_5 - t)\,,$$

which is an admissible $-+-+$ trajectory of the time-reversed system.

Define

$$\pi_\star^{(j)} = \langle \lambda(\tau_j), X_\star(q(\tau_j)) \rangle ,$$

for every $j = 1,\ldots,8$ and for $\star = +,++,(3+)$. Assume that $q(\cdot)$ is quasi optimal. Then the triple $(\pi_+^{(4)}, \pi_{++}^{(4)}, \pi_{(3+)}^{(4)})$ necessarily belongs to one of the three octants $(+-+)$, $(\pm+-)$, while the triple $(\pi_+^{(4)}, -\pi_{++}^{(4)}, \pi_{(3+)}^{(4)})$ must lie in one of the octants $(++-)$, $(+-\pm)$, $(-+-)$. The only possibility is that $(\pi_+^{(4)}, \pi_{++}^{(4)}, \pi_{(3+)}^{(4)})$ is in the $(++-)$ octant. By symmetry, $(\pi_+^{(5)}, \pi_{++}^{(5)}, \pi_{(3+)}^{(5)})$ must belong to the $(+--)$ octant.

According to Corollary 2.2, $\langle \lambda(t), X_{(3+)}(q(t)) \rangle$ can be bounded uniformly away from zero and, as follows from (1.17), it is the derivative of $\langle \lambda(t), X_{++}(q(t)) \rangle$ with respect to t along $+$ arcs. Since $\pi_{(3+)}^{(4)}$ is negative, then $t \mapsto \langle \lambda(t), X_{++}(q(t)) \rangle$ is decreasing along all $+$ arcs. In particular, $(\pi_+^{(3)}, \pi_{++}^{(3)}, \pi_{(3+)}^{(3)})$ belongs to one of the octants $(++-)$ or $(-+-)$. Therefore, $q(\tau_3)$ could not be the fourth switching point of a short quasi optimal bang-bang extension of $q(\cdot)$ with nine arcs. It means that a short bang-bang trajectory contained in U with nine arcs, the first and the last one corresponding to control $+1$, is essential. We proved the following result.

Proposition 3.1 *Let $q_0 \in M$ be a (3,2a)-point. Then, a short bang-bang quasi optimal trajectory of (1.7) has at most nine bang arcs.*

Remark 3.2 There is no evidence at all that this result is sharp. On the contrary, it is reasonable to expect the bound to be overestimated. Proposition 3.1 allows us, anyhow, to exclude the existence of chattering quasi optimal trajectories, a property which will be crucial, in section 3.4, for obtaining general regularity properties.

3.3 A bound on the number of arcs of quasi optimal bang-bang trajectories in the (3,2b) case

Let $q_0 \in M$ be a (3,2b)-point and fix a (3,2b)-adapted neighborhood U of q_0. Fix $Y_1 = g, Y_2 = X_+$, and $Y_3 = X_-$, and associate with this moving basis the corresponding Euclidean structure in $T^*\overline{U}$. For every $T > 0$, let Ξ be the class of extremal $-+-+$ pairs $(\lambda(\cdot), q(\cdot))$ of time-length smaller than T, normalized in such a way that $|\lambda(t_0)| = 1$. Choose $(\lambda(\cdot), q(\cdot)) \in \Xi$ and denote by $t_0 < t_0 + t_1 < t_0 + t_1 + t_2$ its switching times, which verify

$$0 = \langle \lambda(t_0), g(q(t_0)) \rangle , \tag{3.23}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle , \tag{3.24}$$

$$0 = \left\langle \lambda(t_0), e^{t_1 \operatorname{ad}(f+g)} e^{t_2 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle . \tag{3.25}$$

Let $\pi_\star$ be defined as in the previous section. Notice that, due to (3.23),

$$\max\{|\pi_+|, |\pi_-|\} = |\lambda(t_0)| = 1 .$$

Since $q(\cdot)$ has both a compactly contained $+$ arc and a compactly contained $-$ one, then either $\pi_+ = 1$ or $\pi_- = -1$. In our standard notation, $a_2 = \pi_+$ and $a_3 = \pi_-$. From

(3.23), (3.24), and (3.25), we get

$$
\begin{aligned}
0 &= \pi_0 + \frac{t_1}{2}a_2 + a_2\,O(t_1^2) + a_3\,O(t_1^2)\ , \\
0 &= \pi_0 + t_1 a_2 + \frac{t_2}{2}a_3 + a_2\left(O(t_1t_2) + O(t_2^2)\right) + a_3\left(O(t_1t_2) + O(t_2^2)\right)\ .
\end{aligned}
$$

Therefore,

$$
0 = a_2\left(t_1 + O(t_1^2) + O(t_1t_2) + O(t_2^2)\right) + a_3\left(t_2 + O(t_1^2) + O(t_1t_2) + O(t_2^2)\right)\ . \tag{3.26}
$$

For T small enough, (3.26) can be solved either in a_2 or in a_3, that is, at least one of the coefficients multiplying a_2 and a_3 is different from zero, as it can be seen, for instance, by considering separately the case in which $t_1 \leq t_2$ and the opposite one. Therefore, (a_2, a_3) is uniquely determined up to multiplication. It follows that $\lambda(\cdot)$ is the unique extremal lift of $q(\cdot)$, up to normalization.

For every η in the open interval $(0,1)$, we split Ξ in three subclasses, Ξ^1_η, Ξ^2_η, and Ξ^3_η, as follows: Ξ^1_η and Ξ^2_η are characterized, respectively, by $|\pi_+| < \eta$ and $|\pi_-| < \eta$, while Ξ^3_η is defined as $\Xi \setminus (\Xi^1_\eta \cup \Xi^2_\eta)$.

Lemma 3.3 *There exist $0 < \eta < 1$ and $T > 0$ such that, if $(\lambda(\cdot), q(\cdot)) \in \Xi^1_\eta$ and $q(\cdot)$ is quasi optimal, then $\pi_{++} < 0$.*

Proof. Assume that $(\lambda(\cdot), q(\cdot)) \in \Xi^1_\eta$ and notice that $\pi_- = -1$. From (3.4) we have

$$
\pi_0 = -\frac{t_1}{2}\pi_+ - \frac{t_1^2}{6}\pi_{++} + O(t_1^3)\ .
$$

Remark that, since π_0 is the derivative of the switching function at time t_0, then the following inequality holds,

$$
\pi_+ + \frac{t_1}{3}\pi_{++} + O(t_1^2) \geq 0\,. \tag{3.27}
$$

From (3.5) we get

$$
\begin{aligned}
t_2 &= 2\pi_0 + 2t_1\pi_+ + t_1^2\pi_{++} + O(t_1^3) + O(t_1t_2) + O(t_2^2) \\
&= t_1\pi_+ + \frac{2}{3}t_1^2\pi_{++} + \pi_+ O(t_1^2) + O(t_1^3)\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
h_2 &= f - g - 2t_1[f,g] - t_1^2 X_+ + O(t_1^3)\ , \\
h_3 &= f + g + 2t_2[f,g] + 2t_1t_2X_+ + O(t_1^2t_2) + O(t_2^2)\ ,
\end{aligned}
$$

and so

$$
\begin{aligned}
\sigma_{01} &= 2\pi_0\,, \\
\sigma_{02} &= 2t_1 + O(t_1^2)\ , \\
\sigma_{12} &= -2\pi_0 - 2t_1\pi_+ - t_1^2\pi_{++} + \pi_+ O(t_1^2) + O(t_1^3)\ , \\
\sigma_{03} &= 2\pi_0 - 2t_2 + O(t_1t_2)\,, \\
\sigma_{13} &= 2t_2\pi_+ + 2t_1t_2\pi_{++} + O(t_1^2t_2) + O(t_2^2)\ , \\
\sigma_{23} &= \sigma_{03} - \sigma_{12} - 2\pi_0 + O(t_1^2t_2)\ .
\end{aligned}
$$

The space H, defined in (1.24), is determined by a system of the form

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{3.28}$$
$$-\alpha_0 + \alpha_1 - (1 + O(t_1))\alpha_2 + (1 + O(t_2))\alpha_3 = 0, \tag{3.29}$$
$$-t_1 A_2 \alpha_2 + t_2 A_3 \alpha_3 = 0, \tag{3.30}$$

where

$$\begin{aligned} A_2 &= 2\delta + t_1 + O(t_1^2), \\ A_3 &= 2\delta + 2t_1 + O(t_1^2) + O(t_2), \end{aligned}$$

and δ is the component of $[f, g](q(t_0))$ in the direction $g(q(t_0))$, with respect to the basis $g(q(t_0)), X_+(q(t_0)), X_-(q(t_0))$.

If T and η are small, then $O(t_1^2)$ and $O(t_2)$ are small with respect to t_1. Choose T and η in such a way that $A_3 - A_2 \geq \frac{t_1}{2}$. Then, one of the following two condition holds,

$$|A_2| \geq \frac{t_1}{4}, \tag{3.31}$$
$$|A_3| \geq \frac{t_1}{4}. \tag{3.32}$$

Consider the covering of $\Xi_\eta^1 = \Omega_1 \cup \Omega_2$, defined as follows: An element of Ξ_η^1 belongs to Ω_1 (respectively, Ω_2) if (3.31) (respectively, (3.32)) holds.

The ratio $F = A_3/A_2$ is well defined on Ω_1 and, moreover, seen as a function from Ω_1 to $\mathbf{R}$,

$$F = 1 + \frac{t_1 + O(t_1^2) + O(t_2)}{A_2} = O(1).$$

Assume that $(\lambda(\cdot), q(\cdot))$ is in Ω_1. The space H is described by

$$\begin{aligned} \alpha_0 &= -\left(\frac{t_2}{t_1}F + O(t_2)\right)\alpha_3, \\ \alpha_1 &= -(1 + O(t_2))\,\alpha_3, \\ \alpha_2 &= \frac{t_2}{t_1}F\alpha_3. \end{aligned}$$

The asymptotic expression for Q turns out to be

$$Q(\alpha_3) = 2t_2\left(\pi_+(F - F^2 - 1) + \frac{2}{3}t_1\pi_{++}\left(F - F^2 - \frac{3}{2}\right) + \pi_+ O(t_1) + O(t_1^2)\right)\alpha_3^2.$$

If $q(\cdot)$ is quasi optimal, then

$$\pi_+(F - F^2 - 1) + \frac{2}{3}t_1\pi_{++}\left(F - F^2 - \frac{3}{2}\right) + \pi_+ O(t_1) + O(t_1^2) \geq 0. \tag{3.33}$$

Remark that, independently of the value of F, $F - F^2 - 1$ is negative and separated from zero. From (3.27) and (3.33) we obtain that

$$(F^2 - F + 2)t_1\pi_{++} + \pi_+ O(t_1) + O(t_1^2) \geq 0,$$

and we conclude, for π_+ and T small enough, that π_{++} is negative. In a similar way, the same property is established for pairs in Ω_2 as well. □

When $0 < \eta < 1$ is fixed, we can assume, by taking T small enough, that if $(\lambda(\cdot), q(\cdot)) \in \Xi^1_\eta$, then the time-reversed of $q(\cdot)$ has lift in the class $\Xi^1_{2\eta}$ corresponding to the time-reversed system. It follows from Lemma 3.3 that, if η is small enough and $q(\cdot)$ is quasi optimal, then $q(\cdot)$ cannot be extended to a short concatenation of five bang arcs, preserving quasi optimality. By symmetry, we can assume that the same is true when $(\lambda(\cdot), q(\cdot)) \in \Xi^2_\eta$.

From now on, let η be fixed so that these two properties hold, and assume that $(\lambda(\cdot), q(\cdot)) \in \Xi^3_\eta$. To prove that $q(\cdot)$ cannot be quasi optimal we can follow a procedure very similar to the one described in Lemma 3.3. We just give, for completeness, a sketch of the intermediate passages.

Remark that $|\pi_+|$ and $|\pi_-|$ can be assumed to be bounded from below by $\eta/2$. From (3.24) and (3.25) we obtain

$$\begin{aligned} \pi_0 &= -\frac{t_1}{2}\pi_+ + O(t_1^2)\,, \\ t_2\pi_- &= t_1\pi_+ + O(t_1^2)\,. \end{aligned}$$

The asymptotic expressions for h_0, h_1, h_2, h_3 can be easily computed and yield

$$\begin{aligned} \sigma_{01} &= 2\pi_0\,, \\ \sigma_{02} &= 2t_1\pi_- + O(t_1^2)\,, \\ \sigma_{12} &= -2\pi_0 - 2t_1\pi_+ + O(t_1^2)\,, \\ \sigma_{03} &= 2\pi_0 - 2t_2\pi_- + O(t_1^2)\,, \\ \sigma_{13} &= 2t_2\pi_+ + O(t_1^2)\,, \\ \sigma_{23} &= 2\pi_0 - 2t_2\pi_- + 2t_1\pi_+ + O(t_1^2)\,. \end{aligned}$$

The space H is characterized by the following system,

$$\begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 &= 0\,, \\ -\alpha_0 + \alpha_1 - (1 + O(t_1))\alpha_2 + (1 + O(t_1))\alpha_3 &= 0\,, \\ -t_1 A_2 \alpha_2 + t_2 A_3 \alpha_3 &= 0\,, \end{aligned}$$

where

$$\begin{aligned} A_2 &= 2\delta + t_1 + O(t_1^2)\,, \\ A_3 &= 2\delta + 2t_1 + O(t_1^2)\,. \end{aligned}$$

Let Ω_1 and Ω_2 be defined as in the proof of Lemma 3.3. On Ω_1, $F = A_3/A_2$ is well defined and it is an $O(1)$. The quadratic form Q has the expression

$$Q(\alpha_3) = 2t_2\pi_+ \left(F - F^2 - 1 + O(t_1)\right)\alpha_3^2\,,$$

and thus it is negative definite for T small enough. The analogue of these computations in Ω_2 allow us to conclude that $q(\cdot)$ is not quasi optimal.

Symmetry considerations extend the result to short $+-+-$ trajectories.

Proposition 3.4 *Let $q_0 \in M$ be a (3,2b)-point. Then, a short bang-bang quasi optimal trajectory of (1.7) has at most four bang arcs.*

3.4 Regularity of non-bang-bang trajectories in both (3,2a) and (3,2b) case

Let $q_0 \in M$ be a (3,2a)- or a (3,2b)-point and fix an adapted neighborhood U of q_0. With a slight abuse of notation with respect to section 3.3, denote by Ξ the class of short extremal pairs $(\lambda(\cdot), q(\cdot))$, normalized in order to have $|\lambda(\overline{\tau})| = 1$ at some time, and such that the corresponding switching function has at least two zeros. Recall the definition of properties (A), (B), and (B′) given in section 2.4. In the case in which q_0 is a (3,2a)-point, the class Ξ does not verify, in general, neither (A), nor (B), nor (B′). Anyhow, as we already remarked in section 3.2, we can assume that $|\langle\lambda(\cdot), X_-(q(\cdot))\rangle|$ and $|\langle\lambda(\cdot), X_{(3+)}(q(\cdot))\rangle|$ are uniformly separated from zero, as $(\lambda(\cdot), q(\cdot))$ varies in Ξ.

As for the (3,2b) case, we split Ξ, as we did in section 3.3, into three subclasses Ξ^1_η, Ξ^2_η, and Ξ^3_η. A pair $(\lambda(\cdot), q(\cdot)) \in \Xi$ belongs to Ξ^1_η (respectively, Ξ^2_η) if $|\langle\lambda(\overline{\tau}), X_+(q(\overline{\tau}))\rangle| < \eta$ (respectively, $|\langle\lambda(\overline{\tau}), X_-(q(\overline{\tau}))\rangle| < \eta$). The complement of $\Xi^1_\eta \cup \Xi^2_\eta$ in Ξ gives Ξ^3_η. Taking η small enough, we can assume that (B) and (B′) are verified by Ξ^1_η, (A) by Ξ^3_η, and that Ξ^2_η satisfies the symmetric properties to (B) and (B′), where $+$ is replaced by $-$ and viceversa.

Notice that the statement of Lemma 2.8 is valid for trajectories in Ξ. Indeed, the crucial property which makes the proof of Lemma 2.8 work, is that $T_{q_0}S$ and

$$\text{span}\{g(q_0), [f,g](q_0), X_+(q_0), X_-(q_0)\}$$

are transversal. Here, the condition is satisfied everywhere in U, in the sense that g, $[f,g]$, X_+, and X_- span the tangent space to M at every point of U.

When q_0 is a (3,2b)-point, all the reasoning which applied in section 2.7 to quasi optimal trajectories near (3,0)-points can be used for quasi optimal trajectories which lift in Ξ^3_η. Analogously, quasi optimal pairs in Ξ^1_η can be studied as it was done near (3,1)-points. A symmetric analysis applies to Ξ^2_η. We end up with the following result.

Theorem 3.5 *Let $q_0 \in M$ be a (3,2b)-point and fix a (3,2b)-adapted neighborhood U of q_0. Then, there exists $T > 0$ such that a quasi optimal trajectory of (1.7) contained in U, of time-length smaller than T, is the concatenation of at most four arcs. The only possible maximal concatenations are of the type BBBB, BSBB, BBSB.*

The case of (3,2a)-points presents some new phenomena. Let $(\lambda(\cdot), q(\cdot)) \in \Xi$ and denote by φ its corresponding switching function. Define, for every word w with letters in $\{-,+\}$, $\psi_w(\cdot) = \langle\lambda(\cdot), X_w(q(\cdot))\rangle$.

Assume that $q(\cdot)$ is quasi optimal. Therefore, due to propositions 1.5 and 3.1, either $q(\cdot)$ is bang-bang or it contains at least one singular arc. Assume that $q(\cdot)$ is not bang-bang and denote by $[0,T]$ its domain of definition. If no bang arc is compactly contained in $[0,T]$, then $q(\cdot)$ is the concatenation of at most a bang a singular and a bang arc. On the other hand, since $|\psi_-(\cdot)|$ is separated from zero, then each compactly contained $-$ arc is concatenated to two $+$ ones. Therefore, if a compactly contained bang arc exists, then we can assume that it is a $+$ arc and that at one of its boundary points $\dot\varphi = 0$. Thus, the restriction of $\psi_+(\cdot)$ to the $+$ arc has at least one zero. Define $\overline{\Xi}$ as the subclass of Ξ made of extremal pairs for which $\lambda(\cdot)$ annihilates $X_+(q(\cdot))$ at least once. It follows from Lemma 2.1 that $\overline{\Xi}$ can be assumed to satisfy (B′).

Since $(\lambda(\cdot), q(\cdot))$ is in $\overline{\Xi}$, then the generalized Legendre condition implies that $\ddot\varphi$ is positive along $-$ arcs. In particular, $q(\cdot)$ does not have compactly contained $-$ arcs. If

$q(\cdot)$ has only one singular arc, then it is a concatenation of the type $-+\text{S}+-$, where possibly some of the bang arcs have length zero.

Assume that it has two distinct singular arcs. Then, there exists at least one $+$ arc $[t_1, t_2]$ contained in the interval which they identify. Since $[t_1, t_2]$ cannot be concatenated to $-$ arcs, then $\dot{\varphi}(t_1) = \dot{\varphi}(t_2) = 0$. It follows that ψ_{++} is equal to zero at some point of (t_1, t_2).

Since $\psi_{(3+)}(\cdot)$ has constant sign on $[0, T]$ and it is equal to $\varphi^{(4)}(\cdot)$ on (t_1, t_2), then it follows that $\psi_{(3+)}(\cdot)$ is negative and that $\psi_{++}(t_1) > 0 > \psi_{++}(t_2)$.

According to Lemma 2.8, the control $u(\cdot)$ corresponding to $q(\cdot)$ is given by (2.54). Since $(\lambda(\cdot), q(\cdot)) \in \overline{\Xi}$, we can assume that $u(\cdot)$ is uniformly as close to $+1$ as required along singular arcs. In particular, we can assume that $\psi_{(3+)}(\cdot)$ is decreasing not only on $+$ arcs, but also on singular ones.

We want to show that $[t_1, t_2]$ is the only bang arc compactly contained in $[0, T]$. Assume that there is a second one. Without loss of generality, it is contained in $[t_2, T]$. Let $t_3 \geq t_2$ be the smallest density point of the union of bang arcs contained in $[t_2, T]$. Then (t_2, t_3) is a singular arc, possibly empty. Therefore, $\psi_{++}(t_3) < 0$. It follows that there exists a compactly contained $+$ arc (t_4, t_5) such that $t_3 \leq t_4$, $\dot{\varphi}(t_4) = 0$ and $\psi_{++}|_{(t_4,t_5)} < 0$. It follows that $\psi_{+}|_{(t_4,t_5)} = \ddot{\varphi}|_{(t_4,t_5)} < 0$ and thus $t_5 = T$. Therefore, (t_4, t_5) was not compactly contained in $[0, T]$ and we reached a contradiction.

We proved the following result.

Theorem 3.6 *Let $q_0 \in M$ be a (3,2a)-point and fix a (3,2a)-adapted neighborhood U of q_0. Then, there exists $T > 0$ such that a quasi optimal trajectory of (1.7) contained in U, of time-length smaller than T, is the finite concatenation of at most nine bang and singular arcs. The only possible maximal concatenations including singular arcs are of the type $-+S+-$, $\pm S+S\pm$.*

3.5 First attempt for the (3,2c) case

Assume that $q_0 \in M$ is such that

$$g(q_0) \wedge [f, g](q_0) \wedge X_+(q_0) = 0, \tag{3.34}$$
$$g(q_0) \wedge [f, g](q_0) \wedge X_-(q_0) = 0. \tag{3.35}$$

We look for appropriate conditions of linear independence between elements of $\text{Lie}(f, g)(q_0)$, which allow to investigate the behavior of the system near q_0. We call q_0 a (3,2c)-point if, in addition to (3.34) and (3.35), the following relations hold,

$$g(q_0) \wedge [f, g](q_0) \wedge X_{++}(q_0) \neq 0, \tag{3.36}$$
$$g(q_0) \wedge [f, g](q_0) \wedge X_{--}(q_0) \neq 0. \tag{3.37}$$

The first approach, described in this section, is to apply index theory. Fix a (3,2c)-adapted neighborhood U of q_0, that is, in analogy with the analogous definitions given in section 3.1, a relatively compact, connected neighborhood of q_0 such that (3.36) and (3.37) hold at every point of $\overline{U}$.

Fix $Y_1 = g, Y_2 = [f, g]$, and $Y_3 = X_{--}$, and associate with this moving basis the corresponding Euclidean structure in $T^*\overline{U}$.

Consider a short $-+-+$ extremal pair $(\lambda(\cdot), q(\cdot))$. Let t_1 and t_2 be the length of, respectively, the second and the third bang arc. Denote by t_0 the second switching time and define $a_i = \langle \lambda(t_0), Y_i(q(t_0)) \rangle$, $i = 1, 2, 3$. Assume that $\lambda(\cdot)$ is normalized in such a way that $|\lambda(t_0))| = 1$. The switching times verify,

$$\begin{aligned} \left\langle \lambda(t_0), e^{-t_1 \operatorname{ad}(f+g)} g(q(t_0)) \right\rangle &= 0, & (3.38)\\ \langle \lambda(t_0), g(q(t_0)) \rangle &= 0, & (3.39)\\ \left\langle \lambda(t_0), e^{t_2 \operatorname{ad}(f-g)} g(q(t_0)) \right\rangle &= 0. & (3.40) \end{aligned}$$

From (3.39) and (3.40) we obtain that $a_1 = 0$ and

$$a_2 = a_2 O(t_2) + a_3 O(t_2).$$

In particular, $|a_3| = 1$ and a_2 is a function of a_3. The sign of a_3 is uniquely determined by the remark that the first nonzero right derivative of φ at t_0 is positive. Therefore, $\lambda(\cdot)$ is unique up to multiplication by a positive scalar.

Let $\pi_\star$ be defined as in (2.11) and (2.12). Equalities (3.38) and (3.39) imply that

$$\pi_0 = \frac{t_1}{2}\pi_+ - \frac{t_1^2}{6}\pi_{++} + O\left(t_1^3\right). \tag{3.41}$$

From (3.40), analogously, we deduce that

$$\pi_0 = -\frac{t_2}{2}\pi_- - \frac{t_2^2}{6}\pi_{--} + O\left(t_2^3\right). \tag{3.42}$$

Remark that it is not true that $t_2 = O(t_1)$, nor that $t_1 = O(t_2)$. This lack of a priori asymptotic hierarchy between the length of the bang arcs is, as we will see below, the technical gap which makes index theory ineffective in the case studied here.

According to (1.22) we have

$$\begin{aligned} h_0 &= f - g + 2t_1[f,g] - t_1^2 X_+ + O\left(t_1^3\right), \\ h_1 &= f + g, \\ h_2 &= f - g, \\ h_3 &= f + g + 2t_2[f,g] + t_2^2 X_- + O\left(t_2^3\right), \end{aligned}$$

from which we get

$$\begin{aligned} \sigma_{01} &= 2\pi_0 - 2t_1\pi_+ + t_1^2\pi_{++} + O\left(t_1^3\right), \\ \sigma_{02} &= -2t_1\pi_- + t_1^2\pi_{-+} + O\left(t_1^3\right), \\ \sigma_{12} &= -2\pi_0, \\ \sigma_{03} &= \sigma_{01} + \sigma_{23} - 2\pi_0 + O\left(t_1^2 t_2\right) + O\left(t_1 t_2^2\right), \\ \sigma_{13} &= 2t_2\pi_+ + t_2^2\pi_{+-} + O\left(t_2^3\right), \\ \sigma_{23} &= 2\pi_0 + 2t_2\pi_- + t_2^2\pi_{--} + O\left(t_2^3\right). \end{aligned}$$

The space H is given by the solutions of the system

$$\begin{array}{lllll} \alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & = 0, \\ -\left(1+O(t_1^2)\right)\alpha_0 & +\alpha_1 & -\alpha_2 & +\left(1+O(t_2^2)\right)\alpha_3 & = 0, \\ \left(2t_1+O(t_1^2)\right)\alpha_0 & & & +\left(2t_2+O(t_2^2)\right)\alpha_3 & = 0, \end{array}$$

which leads to the following set of relations,

$$\begin{aligned}\alpha_0 &= \left(-\frac{t_2}{t_1} + O\left(\frac{t_2^2}{t_1}\right) + O(t_2)\right)\alpha_3\,,\\ \alpha_1 &= \left(-1 + O(t_1t_2) + O(t_2^2)\right)\alpha_3\,,\\ \alpha_2 &= \left(\frac{t_2}{t_1} + O\left(\frac{t_2^2}{t_1}\right) + O(t_2)\right)\alpha_3\,.\end{aligned}$$

Thus,

$$\begin{aligned}Q(\alpha_3) &= -2\frac{t_2}{t_1}\left[-2\pi_0 + t_1\pi_+ - t_2\pi_- + t_1t_2\pi_{+-}\right.\\ &\quad \left.+\pi_-O(t_1t_2) + \pi_-O(t_2^2) + O(t_1^2t_2) + O(t_1t_2^2) + O(t_2^3)\right]\alpha_3^2\\ &= -\frac{t_2}{t_1}\left[t_1\pi_+ - t_2\pi_- + \frac{t_1^2}{3}\pi_{++} + t_1t_2\pi_{+-} + \frac{t_2^2}{3}\pi_{--}\right.\\ &\quad \left.+O(t_1^3) + \pi_-O(t_1t_2) + \pi_-O(t_2^2) + O(t_1^2t_2) + O(t_1t_2^2) + O(t_2^3)\right]\alpha_3^2\,.\end{aligned}$$

The expression of Q does not allow to determine its sign. On the other hand, it is not clear how to compute the asymptotic behavior of the quadratic form corresponding to a trajectory with more than four arcs, because of the lack of hierarchy relations between the length of the arcs. We are enforced to look for a completely different approach.

3.6 An approximating system at (3,2c)-points

3.6.1 The approximation procedure

Time-rescaling gives a one-to-one correspondence between trajectories of (1.7) of time-length equal to $\varepsilon > 0$ and trajectories of

$$\dot q = \varepsilon(f(q) + ug(q))\,, \qquad |u| \le 1\,, \tag{3.43}$$

of time-length equal to one. In particular, quasi optimal trajectories of (1.7) of time-length smaller than T are in one-to-one correspondence with quasi optimal trajectories of (3.43) of time-length equal to one, for ε varying between zero and T.

We study now (3.43), thinking at $\varepsilon > 0$ as small but fixed. Let $q(\cdot)$ be the solution of (3.43) corresponding to control $u{:}[0,1] \to [-1,1]$. Then

$$q(1) = \left(\overrightarrow{\exp}\int_0^1 \varepsilon(f + u(t)g)dt\right)(q(0)) = \left(\overrightarrow{\exp}\int_0^1 \varepsilon u(t)e^{\varepsilon t\,\mathrm{ad}\, f} g\, dt\right)\circ e^{\varepsilon f}(q(0))\,.$$

Since $e^{\varepsilon f}$ is independent of $u(\cdot)$, it can be seen as a simple change of coordinates. It is worth to concentrate on the non-autonomous control system

$$\dot q = u\,\varepsilon\, e^{\varepsilon\, t\,\mathrm{ad}\, f} g(q)\,, \qquad |u| \le 1\,. \tag{3.44}$$

The expansion of the non-autonomous vector field $\varepsilon e^{\varepsilon t \mathrm{ad}\, f} g$ with respect to t and ε gives

$$\varepsilon e^{\varepsilon t \mathrm{ad}\, f} g = \varepsilon g + \varepsilon^2 t[f,g] + \varepsilon^3\frac{t^2}{2}[f,[f,g]] + \cdots + \varepsilon^{k+1}\frac{t^k}{k!}\mathrm{ad}^{\,k} f\, g + \cdots$$

In general, we will study control system of the type

$$\dot{q} = u g_{\varepsilon,t}(q)\,, \qquad |u| \le 1\,, \tag{3.45}$$

such that $g_{\varepsilon,t}$ has an expansion of the type

$$g_{\varepsilon,t} = \varepsilon g_1 + \varepsilon^2 t g_2 + \varepsilon^3 \frac{t^2}{2} g_3 + \cdots + \varepsilon^{k+1}\frac{t^k}{k!} g_{k+1} + \cdots$$

near a point q_0 at which the family of vector fields g_k satisfy some relations of mutual (in)dependence. Let, for every $k \ge 0$,

$$E_k = \text{span}\,\{\text{ad}\, g_{i_1} \cdots \text{ad}\, g_{i_m} g_j(q_0) |\, m \ge 0, i_1 + \cdots + i_m + j \le k\}\,.$$

Consider the flag $\mathcal{E} = \{E_1 \subset E_2 \subset \cdots E_k \subset \cdots\}$ associated with the system (3.45). We say that $\mathcal{E}$ has length k if $E_k = T_{q_0}M$ and $\dim E_{k-1} < 3$, and, such being the case, we call *growth vector of* $\mathcal{E}$ the k-tuple $(\dim E_1, \ldots, \dim E_k)$. It is easy to check that the flag corresponding to the system (3.44) has length three when q_0 is a (3,0)-, (3,1)-, (3,2a)-, or a (3,2b)-point. Assume now that the growth vector of $\mathcal{E}$ is equal to $(1, 2, 2, 3)$. This is the case, for instance, of the system (3.44) when q_0 is a (3,2c)-point.

We want to find a convenient coordinate description of (3.45) in a neighborhood U of q_0. Let $\psi : U \to \mathbf{R}^3$ be a local system of coordinates such that $\psi(q_0) = (0, 0, 0)$. Denote by (x, y, x) the coordinates of $\mathbf{R}^3$ and assume that ψ rectifies g, that is,

$$\psi_*(g_1) \equiv \frac{\partial}{\partial x}\,.$$

We can, moreover, suppose that $\frac{\partial}{\partial y} \in \psi_*(E_2)$, so that

$$\psi_*(E_2) \;=\; \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}, \tag{3.46}$$

$$\psi_*(E_3) \;=\; \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}, \tag{3.47}$$

$$\psi_*(E_4) \;=\; \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}. \tag{3.48}$$

Define, for every $s > 0$, the dilation

$$\begin{array}{rccc} \delta_s : & \mathbf{R}^3 & \longrightarrow & \mathbf{R}^3 \\ & (x, y, z) & \longmapsto & (sx, s^2 y, s^4 z)\,. \end{array} \tag{3.49}$$

Given $m \ge -4$, we say that $X \in \text{Vec}M$ is *m-homogeneous* if

$$\text{Ad}\,\delta_s(X) = s^{-m} X\,,$$

where we identify $\text{Vec}M$ and $\text{Vec}\mathbf{R}^3$ by ψ. The choice of coordinates induces a filtration on the algebra $\text{Vec}M$. Every vector field X can be written as $X = X^{(-4)} + X^{(-3)} + \cdots + X^{(m)} + \cdots$, where, for every $m \ge -4$, $X^{(m)}$ is m-homogeneous.

Notice that all terms of order -3 and -4 are, respectively, multiples of $x\frac{\partial}{\partial z}$ and of $\frac{\partial}{\partial z}$. Due to (3.46), we know that $g_2^{(-4)}$, the term of order -4 of g_2, is equal to zero. Moreover we can normalize the coordinates in such a way that

$$g_2(q_0) = g_2(0, 0, 0) = \frac{\partial}{\partial y}\,. \tag{3.50}$$

Therefore,

$$g_2 = \underbrace{\alpha x \frac{\partial}{\partial z}}_{g_2^{(-3)}} + \underbrace{\frac{\partial}{\partial y} + (ax^2 + by)\frac{\partial}{\partial z}}_{g_2^{(-2)}} + \sum_{m \geq -1} g_2^{(m)}.$$

Equality (3.47) implies that $[g_1, g_2]^{(-4)} = 0$ and $g_3^{(-4)} = 0$. Thus, $\alpha = 0$ and

$$g_1 = \frac{\partial}{\partial x}, \tag{3.51}$$

$$g_2 = \frac{\partial}{\partial y} + (ax^2 + by)\frac{\partial}{\partial z} + \sum_{m \geq -1} g_2^{(m)}, \tag{3.52}$$

$$g_3 = cx\frac{\partial}{\partial z} + \sum_{m \geq -2} g_3^{(m)}, \tag{3.53}$$

$$g_4 = d\frac{\partial}{\partial z} + \sum_{m \geq -3} g_4^{(m)}. \tag{3.54}$$

Notice that the change of coordinates $(x', y', z') = (x, y, z - bx^2/2)$, being such that $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \frac{\partial}{\partial y'} = \frac{\partial}{\partial y} + by\frac{\partial}{\partial z}, \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$, preserves the rectification of g_1 and the normalization (3.50). Therefore, without loss of generality, we can set $b = 0$.

If $\mathcal{E}$ is associated with (3.44) and q_0 is a (3,2c)-point, then (3.36) and (3.37) imply that,

$$\begin{aligned} 2a + 2c + d &\neq 0, \\ 2a - 2c + d &\neq 0. \end{aligned}$$

If we assume, in addition, that

$$g(q_0) \wedge [f, g](q_0) \wedge [f, [f, [f, g]]](q_0) \neq 0,$$

then we have $d \neq 0$ and, by renormalization, we can set $d = 1$.

Applying the change of coordinates $\delta_{\frac{1}{\eta}}$ to (3.45) we obtain

$$\begin{aligned} \dot{q} = u \Big[& \frac{\varepsilon}{\eta} g_1 + \frac{\varepsilon^2 t}{\eta^2}\left(g_2^{(-2)} + \eta g_2^{(-1)} + \cdots\right) + \frac{\varepsilon^3 t^2}{2\eta^3}\left(g_3^{(-3)} + \eta g_3^{(-2)} + \cdots\right) \\ & + \frac{\varepsilon^4 t^3}{6\eta^4}\left(g_4^{(-4)} + \eta g_4^{(-3)} + \cdots\right) + \cdots \Big]. \end{aligned}$$

In particular, taking $\varepsilon = \eta$ we get

$$\begin{aligned} \dot{q} = u \Big[& g_1 + t g_2^{(-2)} + \frac{t^2}{2} g_3^{(-3)} + \frac{t^3}{6} g_4^{(-4)} \\ & + \varepsilon\left(t g_2^{(-1)} + \frac{t^2}{2} g_3^{(-2)} + \frac{t^3}{6} g_4^{(-3)} + \frac{t^4}{24} g_5^{(-4)}\right) + O(\varepsilon^2) \Big]. \end{aligned} \tag{3.55}$$

3.6.2 General properties of the first order approximating system

Consider the first order approximation of (3.55) with respect to the rescaling factor ε. In the coordinates (x, y, z) the corresponding non-autonomous system is given by

$$\dot{x} = u, \tag{3.56}$$

$$\dot{y} = ut, \tag{3.57}$$

$$\dot{z} = ut\left(ax^2 + \frac{cxt}{2} + \frac{t^2}{6}\right). \tag{3.58}$$

Let, for every $p \in \mathbf{R}^3$ and for every t_0, $t \in \mathbf{R}$ such that $t_0 \leq t$, $A_{t,t_0}(p)$ be the set of points which are attainable from p by admissible trajectories of (3.56 − 3.58) defined on the time interval $[t_0, t]$. Let, moreover, for every $t \geq 0$, $A_t = A_{0,t}(0)$, .

The final part of the section collects some properties of the set A_1. We stress the study of (3.56 − 3.58) is, unfortunatly, still at the stage of work in progress.

If $s \mapsto q(s) = (x(s), y(s), z(s))$ is an admissible trajectory of (3.56 − 3.58) such that $(x(0), y(0), z(0)) = (0, 0, 0)$, then, for every $\alpha > 0$, the same is true for $s \mapsto \delta_\alpha q\left(\frac{s}{\alpha}\right)$. Therefore,

$$A_{\alpha t} = \delta_\alpha A_t,$$

for every $t, \alpha > 0$. Moreover, since $u \equiv 0$ is an admissible control, then $A_{t_1} \subset A_{t_2}$ for every $t_1 \leq t_2$.

Another remark which can be done, is that the control system (3.56 − 3.58) is *small time locally controllable*, that is, for every $p_0 \in \mathbf{R}^3$ and every $t_0 < t$, p_0 belongs to the interior of $A_{t_0,t}(p_0)$. Indeed, let G_s be the non-autonomous vector field

$$G_s(x, y, z) = \begin{pmatrix} 1 \\ s \\ s\left(ax^2 + \frac{cxs}{2} + \frac{s^2}{6}\right) \end{pmatrix}.$$

Then the differential of $u(\cdot) \mapsto \overrightarrow{\exp} \int_{t_0}^{t} u(s) G_s ds$ at $u \equiv 0$ is onto for every choice of t_0 and t.

Therefore,

$$\partial A_1 \subset A_1 \setminus (\cup_{0 \leq t < 1} A_t).$$

Let us apply Pontryagin maximum principle to the system (3.56 − 3.58). Denote covectors by $(\lambda, \mu, \nu)^T$. Define

$$H(x, y, z, \lambda, \mu, \nu, t) = \lambda + \mu t + \nu t\left(ax^2 + \frac{cxt}{2} + \frac{t^2}{6}\right).$$

Let $q: [t_0, t_1] \to \mathbf{R}^3$ be a trajectory of (3.56 − 3.58) and denote its components by $x(\cdot)$, $y(\cdot)$, and $z(\cdot)$). If $q(t_1) \in \partial A_{t_0,t_1}(q(t_0))$, then there exist $\lambda, \mu, \nu: [t_0, t_1] \to \mathbf{R}$ such that, for almost every $t \in [t_0, t_1]$,

$$u(t) H(x(t), y(t), z(t), \lambda(t), \mu(t), \nu(t), t) = -H(x(t), y(t), z(t), \lambda(t), \mu(t), \nu(t), t),$$

and

$$\begin{cases} \dot{\lambda} = -u\nu t\left(2ax + \frac{ct}{2}\right), \\ \dot{\mu} = 0, \\ \dot{\nu} = 0. \end{cases}$$

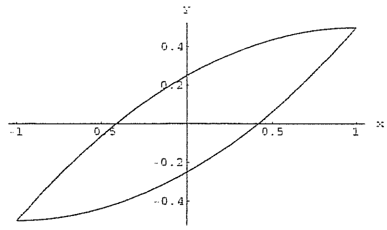

Figure 3.1: the set $\mathcal{A}$.

In particular, $\mu \equiv \mu_0$ and $\nu \equiv \nu_0$.

Remark that the evolution of the coordinates (x, y) in the system $(3.56-3.58)$ is independent of z. That is, $(3.56-3.57)$ is a well defined non autonomous control system on the (x, y) plane, independent of the parameters a and c. The attainable set $\mathcal{A}$ from the origin $(0,0)$ at time one can be explicitly described and gives the projection of A_1 on the (x, y) plane. Trajectories $t \mapsto (x(t), y(t))$ steering $(0,0)$ to $\partial\mathcal{A}$ have at most one bang. It follows that

$$\partial\mathcal{A} = \left\{ \left(1-2s, \frac{1}{2} - s^2\right) \middle| \; s \in [0,1] \right\} \cup \left\{ \left(2s-1, s^2 - \frac{1}{2}\right) \middle| \; s \in [0,1] \right\}.$$

Fix now $(x_1, y_1) \in \mathcal{A}$ and let

$$Z(x_1, y_1) = \{z \in \mathbf{R} |\; (x_1, y_1, z) \in A_1\}.$$

For every $z \in Z(x_1, y_1)$, for every control function $u : [0,1] \to [-1,1]$ such that the corresponding trajectory $(x(\cdot), y(\cdot), z(\cdot))$ steers the origin to (x_1, y_1, z), we have

$$x_1 = \int_0^1 u(t)dt, \qquad y_1 = \int_0^1 \int_0^t u(\tau)d\tau dt, \tag{3.59}$$

and

$$z = a\frac{x_1^3}{3} + c\frac{x_1^2}{4} + \frac{x_1}{6} - \int_0^1 \left(a\frac{\left(\int_0^t u(\tau)d\tau\right)^3}{3} + c\frac{t\left(\int_0^t u(\tau)d\tau\right)^2}{2} + \frac{t^2 \int_0^t u(\tau)d\tau}{2} \right) dt, \tag{3.60}$$

where (3.60) is obtained using integration by parts. Since the constraints in (3.59) are linear in $u(\cdot)$, we have that $Z(x_1, y_1)$ is connected, since it is the image of a connected set through a continuous functional. Finally, if $(x(\cdot), y(\cdot), z(\cdot))$ steers the origin to the boundary of A_1, then $u(\cdot)$ is solution of a maximization or minimization problem.

Let us finally discuss singular trajectories for (3.56-3.58). Associate with an extremal trajectory $q:[t_1, t_2] \to \mathbf{R}^3$ the function

$$\phi(t) = H(x(t), y(t), z(t), \lambda(t), \mu_0, \nu_0, t),$$

and assume that $\phi \equiv 0$. Then, for every $t \in (t_1, t_2)$,

$$0 = \dot{\phi} = \mu_0 + \nu_0 \left(ax^2 + cxt + \frac{t^2}{2}\right).$$

Moreover, from the relation $\ddot{\phi} \equiv 0$ we get that, for almost every $t \in (t_1, t_2)$,

$$u(t)(2ax(t) + ct) + (cx(t) + t) = 0\,. \tag{3.61}$$

and so

$$u(t) = -\frac{cx + t}{2ax + ct}\,, \tag{3.62}$$

unless $cx + t = 2ax + ct = 0$. Assume that $a, c^2 - 2a \neq 0$. Then u is given by (3.62) and equation (3.56) can be integrated. We obtain

$$x(t) = -\frac{ct \pm \sqrt{2aK + (c^2 - 2a)t^2}}{2a}\,,$$

with $K = t_1^2 + 2ct_1x(t_1) + 2ax(t_1)^2$. The explicit expressions of the other components of $q(\cdot)$ can be obtained from (3.62).

Chapter 4

Controllability of the Dubins' problem on Riemannian surfaces

Dubins' problem appears in many textbooks in optimal control theory, as a nontrivial example of how to design globally optimal strategies. Furthermore, it is a popular issue from the viewpoint of mechanical applications, which has inspired many extensions and generalizations. The scope of these extensions is usually to rectify, to some extent, the naivety of the so-called *Dubins' car*, the moving object modeled by Dubins' problem. (A paradigmatic example is given by the Reeds-Shepp car [45].) Our approach is different, since we rather focus on the terrain where the Dubins' car moves. Generalizations of the Dubins' problem to Riemannian manifolds different from the Euclidean plane have already been studied (see [15, 28, 36, 37, 38, 39, 64]). In all the cited papers, global results are obtained in the presence of a Lie group structure. The method proposed in this chapter, whose first formulation is contained in a joint work with Y. Chitour [22], has the advantage to apply to non-homogeneous situations.

Section 4.2 proves the controllability of the Dubins' problem for manifolds in which the Gaussian curvature K tends to zero at infinity. The result is not new, having been first proved by Chitour in [21]. Our presentation more precisely defines the covering domains' construction, which makes the proof work and which plays a fundamental role also in the latter part of the chapter, where new results are proved. Namely, in section 4.3, controllability of the Dubins' problem is shown to hold when K is bounded and nonnegative outside a compact subset of N.

The boundedness hypothesis on K seems to be unavoidable for the approach adopted here. Nevertheless, no counterexample or clear geometric motivation suggests the existence of obstacles to controllability in the unbounded case. Removing this apparent gap could lead to a more complete understanding of the Dubins' problem on surfaces, and seems to be a suitable subject for further research.

4.1 Basic notations and first results

4.1.1 Differential geometric notions

Let $(N, \mathfrak{g})$ be a complete, connected, oriented, two-dimensional Riemannian manifold. Denote by K its Gaussian curvature and by M the unit tangent bundle T^1N. Let π :

$M \to N$ be the canonical bundle projection of M onto N. We will usually denote by p a point in N and by $q = (p, v)$ one in M, where $p = \pi(q)$ and $v \in T_pN$, $\mathfrak{g}(v, v) = 1$. Given $v \in T_pN$, we write $v^\perp$ for the counterclockwise rotation in T_pN of angle $\pi/2$, i.e., $v^\perp$ is the unique element of T_pN such that $\mathfrak{g}(v^\perp, v^\perp) = \mathfrak{g}(v, v)$ and the oriented angle between v and $v^\perp$ is equal to $\pi/2$. For every $q = (p, v) \in M$, we set $q^\perp = (p, v^\perp)$ and $q^- = (p, -v)$.

Given $p_1, p_2 \in N$, $d(p_1, p_2)$ denotes the geodesic distance between p_1 and p_2. When no confusion is possible, we simply write $\|p\|$ (respectively, $\|q\|$) to denote the distance $d(p, p_0)$ (respectively, $d(\pi(q), p_0)$) from a fixed point $p_0 \in N$.

Notice that M is a three-dimensional Riemannian manifold, equipped with the Sasaki metric inherited from $\mathfrak{g}$ (see [46] for more details).

Let $f \in \text{Vec}M$ be the geodesic spray on TN, whose restriction to M (still denoted by f) is a well defined vector field on M. Recall that f is characterized by the following property: $p(\cdot)$ is a geodesic on N if and only if $(p(\cdot), \dot{p}(\cdot))$ is an integral curve of f. In particular, f satisfies the relation

$$\pi_\star(f(q)) = q\,. \tag{4.1}$$

Denote by g the smooth vector field on M, whose exponential flow at time t is the fiberwise rotation of angle t. In terms of the local horizontal-vertical splitting of $T(TN)$ described in [46, Chapter 4], f and g are, respectively, an horizontal and a vertical vector field, whose expressions are given by $f(p, v) = (v, 0)$ and $g(p, v) = \big(0, v^\perp\big)$.

In terms of the covariant derivative on N, the equations satisfied by the integral curves of f and g have, respectively, the following expressions,

$$\left\{ \begin{array}{l} \dot{p} = v\,, \\ \nabla_v v = 0\,, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} \dot{p} = 0\,, \\ \nabla_v v = v^\perp\,. \end{array} \right.$$

The construction of geodesic (or Fermi) coordinates introduced below will be used, in the sequel, to obtain a wide class of local covering domains of N.

Given $q \in M$, consider the map

$$\begin{array}{rccl} \phi_q: & \mathbf{R}^2 & \longrightarrow & N \\ & (x, y) & \longmapsto & \pi(e^{yf} e^{\frac{\pi}{2}g} e^{xf}(q))\,. \end{array}$$

Fix $R = [x_1, x_2] \times [y_1, y_2] \subset \mathbf{R}^2$ and assume that the origin $(0, 0)$ belongs to R. If ϕ_q is a local diffeomorphism at every point of R, then R can be endowed with the Riemannian structure lifted from N, in such a way that ϕ_q becomes a local isometry. If this happens, we denote by $R(q)$ the manifold with boundary which is obtained. The geodesic segment in $R(q)$ given by $[x_1, x_2] \times \{0\}$ is called the *base curve* of $R(q)$. The Gaussian curvature of $R(q)$ at a point (x, y) is given by $K(\phi_q(x, y))$, and, where no confusion is possible, will be denoted by $K(x, y)$. If R is a neighborhood of $(0, 0)$ and $\phi_q|_R$ is injective, then $\phi_q|_R$ is in fact a geodesic chart on N, and we have

$$\mathfrak{g}(x, y) = B^2(x, y)dx^2 + dy^2,$$

with

$$B(x, 0) \equiv 1, \qquad B_y(x, 0) \equiv 0, \qquad \text{and} \qquad B_{yy} + KB = 0\,, \tag{4.2}$$

where the index y in B_y, B_{yy} stands for the partial differentiation with respect to y. (See, for instance, [30].) For every point $q \in M$, for every small enough R of $(0, 0)$, $\phi_q|_R$ is a geodesic chart on N.

In general, if B is the solution of (4.2) on R, with $K = K \circ \phi_q$, then $R(q)$ is well defined as soon as B is everywhere positive in R.

The unit bundle $T^1R(q)$ can be identified with the following subset of $\mathbf{R}^4$,

$$\{(x, y, v_x, v_y)| \ (x, y) \in R, \ B^2(x, y)v_x^2 + v_y^2 = 1\} .$$

Equivalently, a set of coordinates in $T^1R(q)$ is given by $(x, y, \theta) \in R \times S^1$, with the identification

$$Bv_x = \cos\theta, \qquad v_y = \sin\theta . \tag{4.3}$$

Notice that, for all points in the unit bundle such that $y = 0$, the coordinate θ measures the Riemannian angle between the corresponding unit vector and the base curve of $R(q)$. Moreover, a unit vector of coordinates $(x, y, 0)$ or (x, y, π) is always $\mathfrak{g}$-orthogonal to the segment $\{x\} \times [y_1, y_2]$.

In geodesic coordinates, f and g are given by

$$f(x, y, \theta) = \left(\frac{\cos\theta}{B(x, y)}, \sin\theta, C(x, y)\cos\theta\right)^T, \qquad g(x, y, \theta) = (0, 0, 1)^T, \tag{4.4}$$

where

$$C(x, y) = \frac{B_y(x, y)}{B(x, y)}. \tag{4.5}$$

The Lie bracket configuration of the pair $(f, g) \in \mathrm{Vec}M \times \mathrm{Vec}M$ can be easily computed from (4.4), and is characterized by the relations

$$\text{(i)} \quad [f, g] = h , \qquad \text{(ii)} \quad [g, h] = f , \qquad \text{(iii)} \quad [h, f] = Kg , \tag{4.6}$$

where h is defined by $h(q) = f(q^\perp)$, that is, in geodesic coordinates,

$$h(x, y, \theta) = \left(\frac{\sin\theta}{B(x, y)}, -\cos\theta, C(x, y)\sin\theta\right)^T . \tag{4.7}$$

Equivalently, (4.6) could have been recovered, without passing through geodesic coordinates, directly from Cartan's formula (see [21]). Notice that, for every $q \in M$, $f(q), g(q), h(q)$ form an orthonormal basis of T_qM, with respect to the Sasaki metric. In particular, the pair of vector fields (f, g) defines a *contact distribution* on M.

4.1.2 The control system

For every $\varepsilon > 0$, let (D_ε) be the control system given by

$$(D_\varepsilon) \ : \qquad \dot{q} = f(q) + ug(q) , \qquad q \in M, \qquad u \in [-\varepsilon, \varepsilon] .$$

It follows form (4.1) and the definition of g that, for every admissible trajectory $q : [0, T] \to M$ of (D_ε), $d(\pi(q(0)), \pi(q(T))) \le T$. Therefore, being N complete, for every control function $u : \mathbf{R} \to [-\varepsilon, \varepsilon]$, the non-autonomous vector field $f + u(t)g$ is complete.

Relation **(i)** in (4.6) ensures that, for every $\varepsilon > 0$, the control system (D_ε) is *bracket generating*, i.e., such that the iterated Lie brackets of f and g span the tangent space to M at every point. An important consequence is that, for every $t > 0$ and $q \in M$, $e^{tf}(q)$ belongs to $\mathrm{Int}(\mathcal{A}_q)$, the interior of $\mathcal{A}_q$. This follows, for instance, from the description of small-time attainable sets for single-input non degenerate three dimensional control systems given by Lobry in [34] and recalled in the introduction.

Notice that, in the language of the previous chapters, $X_\pm = -Kg \pm f$, which implies that each point of M is a (3,0)-point.

For every $q \in M$, let $A_q = \cup_{T \geq 0} A(T, q)$ be the attainable set from q for (D_ε). The control system (D_ε) is called *completely controllable* if $A_q = M$ for every $q \in M$.

We say that *the Dubins' problem for N has the unrestricted controllability property* if, for every $\varepsilon > 0$, the control system (D_ε) is completely controllable.

In local geodesic coordinates, (D_ε) can be written as follows,

$$\dot{x} = \frac{\cos\theta}{B}, \tag{4.8}$$
$$\dot{y} = \sin\theta, \tag{4.9}$$
$$\dot{\theta} = u + C\cos\theta. \tag{4.10}$$

More intrinsically, we can rewrite system (4.8 − 4.10) in the form

$$\begin{cases} \dot{p} = v\,, \\ \nabla_v v = u v^\perp, \end{cases} \tag{4.11}$$

which accounts for a clear geometric interpretation of the unrestricted controllability property: The Dubins' problem on N is unrestrictedly controllable if and only if, for every $(p_1, v_1), (p_1, v_1) \in M$, for every $\varepsilon > 0$, there exists a curve $p : [T_1, T_2] \to N$ with geodesic curvature smaller than ε such that $p(T_i) = p_i$, $\dot{p}(T_i) = v_i$, $i = 1, 2$.

Remark 4.1 If $q : [0, T] \to M$ is an admissible trajectory of (D_ε) corresponding to some control function $u : [0, T] \to [-\varepsilon, \varepsilon]$, then the trajectory $q(T - \cdot)^-$ obtained from $q(\cdot)$ by reflection and time-reversion is itself an admissible trajectory of (D_ε) and steers $q(T)^-$ to $q(0)^-$. Its corresponding control function is equal to $-u(T - \cdot)$. Therefore, for every $q, q' \in M$, q' belongs to A_q if and only if q^- belongs to $A_{(q')^-}$.

Remark 4.2 Assume that, for every q in M, $q^- \in A_q$. Then, due to remark 4.1, $q' \in A_q$ if and only if $A_q = A_{q'}$. Since the system is bracket generating, then every attainable set A_q has a nonempty interior and one immediately recovers that A_q itself must be open. Therefore, $\{A_q\}_q \in M$ is an open partition of M. It follows from connectedness of N that (D_ε) is completely controllable.

Thanks again to remark 4.1, we have the following equivalence: (D_ε) is completely controllable if and only if, for every $q \in M$, there exists $q' \in A_q$ such that $(q')^- \in A_{q'}$.

Remark 4.3 If N is not oriented, then the vector field g is not well defined. Anyhow, the control problem still makes sense, since, locally, g can be defined fixing arbitrarily an orientation, and the system is independent of this choice. Formally, the Dubins' problem can be defined as a control-affine system with multiple controls, using a partition of unity on N in order to glue the local definitions of g together. Since N admits an oriented double covering, and since the hypothesis under which we get unrestricted controllability for the Dubins' problem are shared by any finite covering, the results of this chapter extend to non-oriented manifolds.

A condition which ensures unrestricted controllability is the compactness of N. This fact is a consequence of a more general result on controllability of bracket generating systems made of conservative vector fields due to Lobry [35]. We give below a stronger formulation of Lobry's result, adapted to the specific control system (D_ε), which implies also that every attainable set is unbounded when N is open. The proof is a variation on the classical one of Poincaré's theorem on volume preserving flows.

Lemma 4.4 *If there exists $q \in M$ such that A_q is relatively compact in M, then N is compact and $A_q = M$.*

Proof. Fix $q \in M$ and assume that $F = \overline{A_q}$ is compact in M. We already remarked that, for every $t > 0$ and $q' \in M$, $e^{tf}(q') \in \mathrm{Int}(A_{q'})$. The compactness of F and the continuous dependence of $A(T, q')$ on q' imply that there exists $\rho > 0$ such that, for every $q' \in F$,

$$B_\rho(e^f(q')) \subset A_{q'}\,, \tag{4.12}$$

where $B_\rho(e^f(q'))$ denotes the ball in M of center $e^f(q')$ and radius ρ, with respect to the Sasaki metric. We want to prove that ∂A_q is empty. Let, by contradiction, $r \in \partial A_q$. A well-known theorem by Krener states that any attainable set of a bracket generating system is contained in the closure of its interior. Therefore, $V = A_q \cap B_\rho(r)$ has nonempty interior and, in particular, its volume is strictly positive. Since e^f is a volume preserving diffeomorphism of M (see, for instance, [46]) and A_q has finite volume (it is bounded), then $\{e^{nf}(V)\}_{n\in\mathbf{N}}$ cannot be a disjoint family, being $e^{nf}(V) \subset A_q$ for every $n \in \mathbf{N}$. Therefore, there exist $n_1 < n_2$ such that $e^{n_1 f}(V) \cap e^{n_2 f}(V)$ is not empty. Equivalently, there exists a point in $e^{(n_2-n_1-1)f}(V)$ whose image by e^f lies in V. Due to (4.12), it follows that $r \in \mathrm{Int}(A_q)$ and the contradiction is reached. □

Corollary 4.5 *If N is compact, then Dubins' problem has the unrestricted controllability property property.*

In the rest of the chapter N is assumed to be non compact.

4.2 Asymptotic flat manifolds

Throughout this section, we assume that N is *asymptotically flat*, that is,

$$\lim_{\|p\|\to\infty} K(p) = 0\,. \tag{4.13}$$

For every $L > 0$, let $Q_L = [0, 2L] \times [-L, L]$. According to the notation introduced in section 4.1.1, if the map ϕ_{q_0}, $q_0 \in M$, is a local diffeomorphism at every point of $[0, 2L] \times [-L, L]$, then $Q_L(q_0)$ denotes the Riemannian manifold with boundary $Q_L(q_0)$, obtained endowing Q_L with the Riemannian structure lifted from N.

Let us characterize the values of L for which the construction of $Q_L(q_0)$ can be carried out. Let B be the solution of (4.2) on Q_L, with $K = K \circ \phi_{q_0}$. Set

$$\delta = \max_{Q_L} |K \circ \phi_{q_0}|\,. \tag{4.14}$$

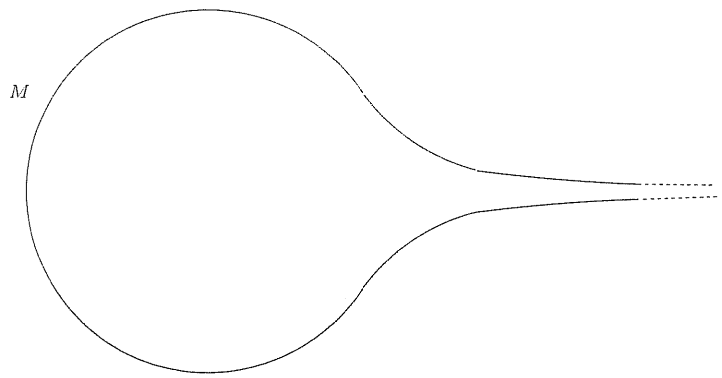

Figure 4.1

By Sturm-Liouville theory, we can compare B with the solution of (4.2) corresponding to K constantly equal to δ, obtaining that, if $\sqrt{\delta}|y| \leq \frac{\pi}{2}$, then $B(x,y) \geq \cos(\sqrt{\delta}y) \geq 0$ for every $x \in [0, 2L]$. Thus, $Q_L(q_0)$ is defined as soon as

$$L < \frac{\pi}{2\sqrt{\delta}}. \tag{4.15}$$

In particular, since N is asymptotically flat, then, for every $L > 0$, for every q_0 outside a compact set of M (depending, in general, on L), $Q_L(q_0)$ is defined.

We stress that no global finiteness property is stated (nor needed) for the projection ϕ_{q_0} from $Q_L(q_0)$ onto its image. In general, for L fixed, the cardinality of the set of preimages $\phi_{q_0}^{-1}(q_0)$ can be unbounded when q_0 varies in N, as illustrated by the example in figure 4.2. The situation will be different in section 4.3.

Together with $\mathfrak{g}$, the control problem (D_ε) as well can be lifted from M to $T^1Q_L(q_0)$. Let us stress the trivial, but crucial, property that every admissible trajectory of the lifted control system is projected by ϕ_{q_0} to an admissible trajectory of (D_ε). In the coordinates (x, y, θ) of $T^1Q_L(q_0)$, the dynamics of the lifted system is described by (4.8 – 4.10). Due to remark 4.2, the proof of the complete controllability of (D_ε) reduces to show that $q_0^- \in A_{q_0}$ if δ is small enough. This will be done by designing an admissible trajectory for the control problem lifted to $T^1Q_L(q_0)$ which steers $(0,0,0)$ to $(0,0,\pi)$.

Fix $q_0 \in M$, $L > 0$ and assume that

$$\sqrt{\delta} \leq \frac{\pi}{3L}, \tag{4.16}$$

where δ is defined as in (4.14). Sturm-Liouville theory implies not only that $Q_L(q_0)$ is well defined, but also that

$$\cos(\sqrt{\delta}y) \leq B(x,y) \leq \cosh(\sqrt{\delta}y), \tag{4.17}$$

and

$$|C(x,y)| = \left|\frac{B_y(x,y)}{B(x,y)}\right| \leq \sqrt{\delta}\frac{\sinh(\sqrt{\delta}|y|)}{\cos(\sqrt{\delta}y)}, \tag{4.18}$$

for every $(x,y) \in Q_L(q_0)$. An upper bound for $|C|$ in $Q_L(q_0)$ is given by $\sqrt{\delta}\frac{\sinh(\sqrt{\delta}L)}{\cos(\sqrt{\delta}L)}$. Then, we can assume that

$$\max_{Q_L(q_0)} |C| \leq \frac{\varepsilon}{2}, \tag{4.19}$$

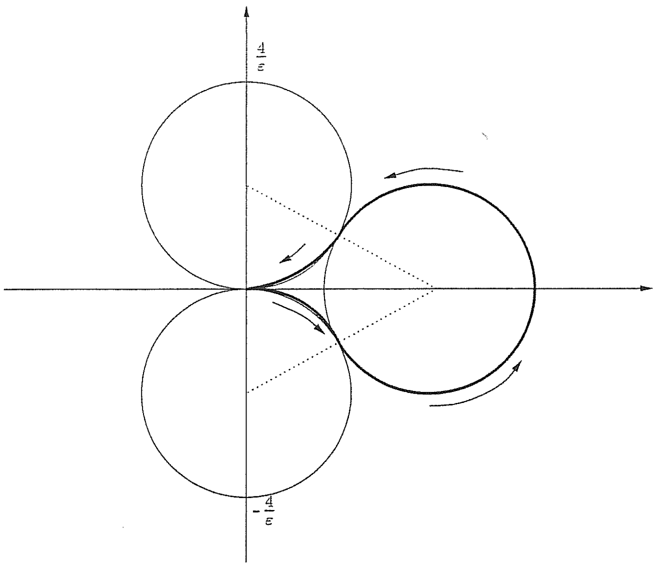

Figure 4.2

by taking

$$\sqrt{\delta} \leq \frac{\varepsilon}{4\sinh\left(\frac{\pi}{3}\right)} . \tag{4.20}$$

Consider now the control system $(D_{\varepsilon/2})$ on the unit bundle of the Euclidean plane. Let $\overline{u}(\cdot)$ be the control function corresponding to the trajectory whose projection on $\mathbf{R}^2$ is a teardrop of size $2/\varepsilon$ which leaves the origin horizontally and arrives at the origin with the opposite direction (see figure 4.2). Thus, $\overline{u}(\cdot)$, is piecewise constant, switching between $-\varepsilon/2$ and $\varepsilon/2$. Denote by $(\overline{x}(\cdot), \overline{y}(\cdot), \overline{\theta}(\cdot))$ the coordinates in $\mathbf{R}^2 \times S^1$ of the teardrop trajectory. It follows from straightforward computations that $(\overline{x}(\cdot), \overline{y}(\cdot))$ takes values in the rectangle $[0, 2(\sqrt{3}+1)/\varepsilon] \times [-2/\varepsilon, 2/\varepsilon]$ and that the length of the trajectory is equal to $\frac{14\pi}{3\varepsilon}$. Fix

$$L = \frac{3}{\varepsilon} .$$

The idea is to apply to the lifted system the time-variant feedback control

$$u(t) = \overline{u}(t) - C(x, y)\cos\theta . \tag{4.21}$$

The control strategy is admissible, since (4.19) holds.

Consider the solution $\gamma(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ of (4.8 – 4.10) corresponding to $u(\cdot)$, with initial condition $\gamma(0) = (x_0, 0, 0)$. As long as $(x(t), y(t))$ stays in Q_L, we have $y(t) = \overline{y}(t)$ and $\theta(t) = \overline{\theta}(t)$. Therefore,

$$\begin{aligned} |x(t) - \overline{x}(t) - x_0| &\leq \int_0^t |\cos(\theta(s))| \left| \frac{1}{B(x(t), y(t))} - 1 \right| ds \\ &\leq \frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right| . \end{aligned}$$

It follows from (4.17) that, for every $\alpha \in (0, \frac{\pi}{2})$, if

$$\sqrt{\delta} L \leq \alpha , \tag{4.22}$$

then

$$\max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right| \leq \frac{\cosh(\alpha) - 1}{\cos(\alpha)}.$$

Therefore, it is possible to fix α, independent of ε, such that, for every δ which satisfies (4.22),

$$\frac{14\pi}{3\varepsilon} \max_{Q_L(q_0)} \left| \frac{1}{B} - 1 \right| \leq \frac{1}{4\varepsilon}.$$

Assume that (4.22) is satisfied and fix $x_0 = \frac{1}{4\varepsilon}$. Then $\gamma(\cdot)$ is defined for the entire time duration of $\overline{u}(\cdot)$. At its final point it has coordinates of the type $(x_1, 0, \pi)$. Concatenating γ with two trajectories corresponding to control equal to zero, we obtain an admissible trajectory for the Dubins' problem lifted to $T^1 Q_L(q_0)$, steering $(0,0,0)$ to $(0,0,\pi)$. We proved the following theorem.

Theorem 4.6 *If N is asymptotically flat, then the Dubins' problem is unrestrictedly controllable.*

Actually, from the nature of the above argument, a stronger result follows:

Proposition 4.7 *There exists a constant $\mu > 0$, independent of N and ε, such that, if $\limsup_{\|p\| \to \infty} |K(p)| \leq \mu\varepsilon^2$, then (D_ε) is controllable.*

Indeed, Proposition 4.7 can be recovered from the smallness conditions (4.16), (4.20), and (4.22) imposed on δ, where L should be replaced by $3/\varepsilon$ and α can be given explicitly.

4.3 Manifolds with nonnegative curvature outside a compact set

4.3.1 Construction of the covering domain

In addition to the general assumptions on N made in section 4.1, let, from now on, K be nonnegative outside a compact subset of N. Since $\int_N K dA$, the so-called *total curvature of N*, is well defined (allowing extended values), and larger than $-\infty$, then it follows from a result by Huber [27], that N is finitely connected. Therefore, Cohn-Vossen theorem [23] applies, i.e.,

$$\int_N K dA \leq 2\pi\chi(N), \tag{4.23}$$

where $\chi(N)$ is the Euler characteristic of N. In particular, for every $R > 0$,

$$\lim_{\|p\| \to \infty} \int_{B_R(p)} K dA = 0. \tag{4.24}$$

In the sequel, we assume that K is bounded on N and we set

$$K_\infty = \sup_{p \in N} K(p).$$

For every $L, d > 0$, let $Q_{L,d} = [-L, L] \times [-d, d]$. As already remarked, if d is smaller than $\frac{\pi}{2\sqrt{K_\infty}}$, then the local covering $Q_{L,d}(q_0)$ is well defined. Fix

$$d = \frac{\pi}{3\sqrt{K_\infty}}. \tag{4.25}$$

Lemma 4.8 *For every $L > 0$, $\lim_{\|q\|\to\infty} \int_{Q_{L,d}(q)} K dA = 0$.*

Before starting the proof of Lemma (4.8), we need to recall some facts from the general theory of Riemannian manifolds. The *injectivity radius at a point p of N* is defined as the least upper bound of all $r > 0$ such that the exponential map $\exp_p$, restricted to the disk $B_r(p) \subset T_pN$, is injective. It is denoted by $i_p(N)$, while $i(N) = \inf_{p\in N} i_p(N)$ is called the *injectivity radius of N*. It is clear that the lemma holds whenever $i(N) > 0$, since we already remarked that N satisfies (4.24). In the case where $0 \leq K \leq K_\infty$, a theorem by Sharafutdinov [52] states that $i(N)$ is, indeed, positive. (It actually states that $i(N)$ is larger than or equal to the minimum between $\pi/\sqrt{K_\infty}$ and the injectivity radius of the Cheeger-Gromoll soul of N. It follows from the proof of Sharafutdinov's result, as in [24], that, in the case in which the soul reduces to a point, its injectivity radius can be replaced by $+\infty$.)

A general result due to Cheeger and Ebin [20, Lemma 5.6], which holds for any complete Riemannian manifold, states that, for every $p \in N$,

$$\begin{aligned} i_p(N) \quad = \quad & \min\{t > 0 \mid t \text{ is a conjugate time for a geodesic } \gamma\colon [0,+\infty) \to N, \gamma(0) = p, \\ & \text{or } 2t \text{ is the length of a closed geodesic passing through } p\}\,. \end{aligned} \tag{4.26}$$

Proof of Lemma 4.8. Fix a compact set N_0 of N such that $\{p \in N \mid K \leq 0\} \subset N_0$. We can assume that $N \setminus N_0$ is a finite union of *tubes of N*, that is, according to the definition of Busemann [19], subsets of N which are homeomorphic to half-cylinders with smooth boundary. Moreover, due to Cohn-Vossen theorem, we can suppose that the total curvature of each tube is strictly smaller than 2π.

Fix one of these tubes and denote it by T. We have to prove that

$$\lim_{q\in T,\ \|q\|\to\infty} \int_{Q_{L,d}(q)} K dA = 0\,. \tag{4.27}$$

If $\inf_{p\in T} i_p(N) > 0$, then we are done. Assume that $\inf_{p\in T} i_p(N) = 0$. Since $K \leq K_\infty$, the first conjugate time for any half geodesic contained in N is bounded from below by $\pi/\sqrt{K_\infty}$. Therefore, due to (4.26), there exists a sequence $\{\gamma_n \mid n \in \mathbf{N}\}$ of simple closed geodesics contained in T, whose length goes to zero as n tends to infinity. Each γ_n identifies two connected regions of T, a bounded and an unbounded one. The bounded region Ω_n must contain the boundary of T, otherwise Gauss-Bonnet theorem would constrain the total curvature of Ω_n to be equal to 2π, contradicting the assumptions made on T. Therefore, the unbounded region of $T \setminus \gamma_n$ is itself a tube, denoted by T_n. We can assume that $T_{n+1} \subset T_1$ for every n. Applying Gauss-Bonnet theorem, we obtain that K must be identically equal to zero on $T_n \setminus T_1$. Since γ_n goes to infinity with n, we have that $K \equiv 0$ on T_1, and (4.27) is proved. □

Let us show how the smallness of K, in integral sense, on the 'lifted rectangles' $Q_{L,d}(q)$ can be used to derive estimates on the behavior of geodesics. Let $p(\cdot)$ be a geodesic in $Q_{L,d}(q)$; then $\dot p(\cdot)$ is a curve in $T^1Q_{L,d}(q)$, whose coordinates satisfy the following system of equations (which is a particular case of (4.8 – 4.10)),

$$\dot x = \frac{\cos\theta}{B}\,, \tag{4.28}$$

$$\dot y = \sin\theta\,, \tag{4.29}$$

$$\dot\theta = C\cos\theta\,. \tag{4.30}$$

Consider L as a function of ε and K_∞, whose explicit expression will be given later. Let

$$M^0 = \{q \in M |\ d(\pi(q), K^{-1}((-\infty, 0))) > L + d\},$$

and remark that the complement of M^0 in M is compact. The set M^0 is defined in such a way that, for every $q_0 \in M^0$, K is nonnegative on $Q_{L,d}(q_0)$.

Denote by $\sigma_z(\cdot)$ the triple which is solution of the system (4.28 – 4.30), with initial condition $\sigma_z(0) = z \in T^1 Q_{L,d}(q)$, and let $[-T_{1,z}, T_{2,z}]$ be its maximal interval of definition.

Fix $\frac{\pi}{4} < \overline{\theta} < \frac{\pi}{3}$. The role of $\overline{\theta}$ is purely technical and its choice can be made independently of all the parameters of the problem; for instance we could take $\overline{\theta} = \frac{7}{24}\pi$.

Lemma 4.9 *There exists $\delta_0 > 0$, depending only on ε and K_∞, such that, for every $q_0 \in M^0$, if $Q = Q_{L,d}(q_0)$ and $\delta > 0$ verify $\int_Q K dA \leq \delta \leq \delta_0$, then, for every $z_0 = (0, y_0, \theta_0)$ with $y_0 \in [-\frac{d}{2}, \frac{d}{2}]$ and $\theta_0 \in [-\overline{\theta}, \overline{\theta}]$, the corresponding $\sigma_{z_0}(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ satisfies the following condition,*

(i) for every t in $[-T_{1,z_0}, T_{2,z_0}]$,

$$|x(t) - \cos(\theta_0)t| \leq (2L + d)\delta, \tag{4.31}$$
$$|y(t) - y_0 - \sin(\theta_0)t| \leq 2L\delta, \tag{4.32}$$
$$|\theta(t) - \theta_0| \leq \delta. \tag{4.33}$$

Moreover, if $|\theta_0| \leq \frac{d}{8L}$, then

(ii) $x(-T_{1,z_0}) = -L$ and $x(T_{2,z_0}) = L$.

Proof. As it was done for the estimate (4.17) of section 4.2, one obtains from (4.25) that, for every $(x, y) \in Q$,

$$\frac{1}{2} \leq B(x, y) \leq 1. \tag{4.34}$$

Fix y_0, θ_0 and $z_0 = (0, y_0, \theta_0)$ as in the statement of the lemma. Let $[-T_1, T_2] = [-T_{1,z_0}, T_{2,z_0}]$ be the maximal interval of definition of $\sigma_{z_0}(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ and $(-t_1, t_2)$ the maximal open neighborhood of zero (in $[-T_1, T_2]$) such that, for every $t \in (-t_1, t_2)$,

$$|\theta(t)| < \frac{\pi}{3}. \tag{4.35}$$

Using (4.34) and (4.35) in (4.28), one deduces that, for every $t \in (-t_1, t_2)$,

$$\frac{1}{2} \leq \dot{x}(t) \leq 2. \tag{4.36}$$

Thus, $t_1, t_2 \leq 2L$. Furthermore, we can define a map $\tau : (x(-t_1), x(t_2)) \to (-t_1, t_2)$ by means of the relation $x(\tau(\xi)) = \xi$. Notice that τ is continuous, as well as the function $\eta : (x(-t_1), x(t_2)) \to [-d, d]$ given by $\eta(\xi) = y(\tau(\xi))$.

We define an open region $G \subset Q$ by

$$G = \cup_{\xi \in (x(-t_1), x(t_2))} I(\eta(\xi)),$$

where $I(l)$ denotes the open interval with 0 and l as boundary points.

Using (4.28) and (4.30), we have, for every $t \in (-t_1, t_2)$,

$$\begin{aligned} |\theta(t) - \theta_0| &= |\theta(\tau(x(t))) - \theta_0| = \left| \int_{I(x(t))} \frac{\dot{\theta}(\tau(\xi))}{\dot{x}(\tau(\xi))} d\xi \right| \\ &= \left| \int_{I(x(t))} B_y(\xi, \eta(\xi)) d\xi \right| \leq \int_{I(x(t))} d\xi \left(\int_{I(\eta(\xi))} -B_{yy}(\xi, v) dv \right) \\ &= \int_{I(x(t))} \int_{I(\eta(\xi))} K(\xi, v) B(\xi, v) dv \, d\xi, \end{aligned}$$

where the last equality follows from (4.2). Notice that the above computations are justified by the assumption that $q_0 \in M^0$. Since the surface element of Q is given by $B(\xi, v) dv \, d\xi$, we have

$$|\theta(t) - \theta_0| \leq \int_G K dA \leq \delta. \tag{4.37}$$

If δ_0 is small enough, i.e., $\delta_0 < \frac{\pi}{3} - \overline{\theta}$, then $|\theta(\cdot)|$ is bounded by $\pi/3$ on $(-t_1, t_2)$ and, therefore, $t_1 = T_1$ and $t_2 = T_2$.

Integrating (4.29) leads to

$$|y(t) - y_0 - \sin(\theta_0) t| \leq \left| \int_{I(t)} (\theta(s) - \theta_0) ds \right| \leq |t| \delta \, .$$

Then, for every $t \in [-T_1, T_2]$,

$$|y(t)| < \frac{d}{2} + |t|(|\theta_0| + \delta) \leq \frac{d}{2} + 2L(|\theta_0| + \delta) \, ,$$

which implies that the endpoints of σ_{z_0} must be characterized by the relations $x(-T_1) = -L$ and $x(T_2) = L$, provided that $|\theta_0|, \delta \leq \frac{d}{8L}$. Point *(ii)* is thus proved. It remains to establish (4.31). Integrating (4.28) we get, for every $t \in [-T_1, T_2]$,

$$\begin{aligned} |x(t) - \cos(\theta_0) t| &= |x(t) - \cos(\theta_0) \tau(x(t))| = \left| \int_{I(x(t))} \left(1 - \cos(\theta_0) \frac{B(\xi, \eta(\xi))}{\cos(\theta(\tau(\xi)))} \right) d\xi \right| \\ &\leq \int_{I(x(t))} (1 - B(\xi, \eta(\xi))) d\xi + 2 \int_{I(x(t))} |\cos(\theta_0) - \cos(\theta(\tau(\xi)))| \, d\xi \, . \end{aligned}$$

The second integral is bounded by $2L \max_{\xi \in [x(-T_1), x(T_2)]} |\theta(\tau(\xi)) - \theta_0|$, itself being bounded by $2L\delta$. As for the first term

$$\int_{I(x(t))} (1 - B(\xi, \eta(\xi))) d\xi \leq \int_{I(x(t))} d\xi \left| \int_{I(\eta(\xi))} B_y(\xi, v) dv \right| \leq \delta d \, ,$$

where the last inequality can be recovered by performing the same computations as in the estimate of $\theta - \theta_0$ done previously. Gathering all the partial estimates, we obtain, for every $t \in [-T_1, T_2]$,

$$|x(t) - \cos(\theta_0) t| \leq (2L + d)\delta.$$

□

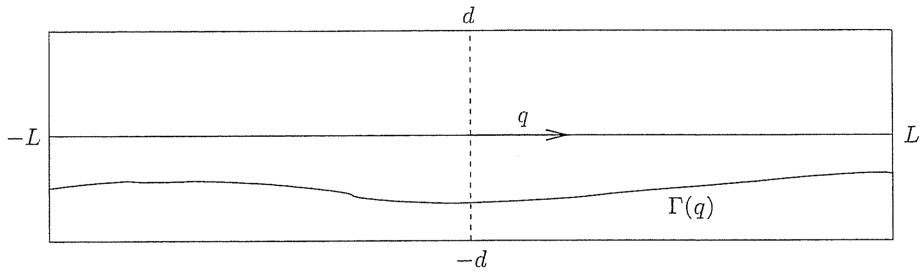

Figure 4.3

For every $\delta > 0$, for every q outside a compact subset of M,

$$K \geq 0 \text{ on } Q_{L,d}(q) \qquad \text{and} \qquad \int_{Q_{L,d}(q)} K dA \leq \delta . \tag{4.38}$$

We can choose $\delta \leq \delta_0$ (defined as in Lemma 4.9), and fix $q_0 \in M$ such that (4.38) holds for every q verifying $d(\pi(q), \pi(q_0)) \leq 2L$. We next build a Riemannian two-dimensional manifold, which will be a finite covering of a region of N close to q_0. The covering domain will glue together several rectangles of the type $Q_{L,d}(q)$. The purpose is to track a teardrop of size r/ε, $r > 1$ to be fixed, for the lifted Dubins' problem and to eventually obtain that $q_0^- \in A_{q_0}$. In this perspective, since L will measure the size of the covering domain, we fix

$$L = \frac{4r}{\varepsilon} . \tag{4.39}$$

The unrestricted controllability property will follow, as in the asymptotic flat case, from the property that the tracking operation can be performed for q_0 outside a big enough compact subset of M (depending, in general, on ε).

From now on, we assume, sometimes implicitly, that δ is as small as needed with respect to ε and K_∞. We will denote by $C(\varepsilon, K_\infty)$ any constant which is a function of ε and K_∞. Fix

$$\overline{y} = \frac{d}{16} .$$

For every q verifying (4.38), let $\sigma(q, \cdot) = (x(q, \cdot), y(q, \cdot), \theta(q, \cdot))$ be the solution in $T^1 Q_{L,d}(q)$ of the system (4.28 − 4.30), with initial condition

$$(x(q, 0), y(q, 0), \theta(q, 0)) = (0, -\overline{y}, 0) .$$

Denote by $[-T_1(q), T_2(q)]$ the maximal interval of definition of $\sigma(q, \cdot)$. If δ is small enough, then we can assume that $y(q, \cdot)$ takes only negative values. Let $W(q)$ be the region of $Q_{L,d}(q)$ defined by

$$W(q) = \{(x(q, s), t) | \;\; s \in [-T_1(q), T_2(q)], \; t \in [y(q, s), 0]\} , \tag{4.40}$$

whose boundary is given by $[-L, L] \times \{0\}$, $\{-L\} \times [y(q, -T_1(q)), 0]$, $\{L\} \times [y(q, T_2(q)), 0]$, and by $\Gamma(q)$, the support of the curve $s \mapsto (x(q, s), y(q, s))$.

Set

$$l = \left[\frac{L}{\overline{y}} \right] , \tag{4.41}$$

where by $[\cdot]$ we denote the integer part. For every $k = 1, \dots, l$, define

$$q_k = e^{\frac{\pi}{2}g}(e^{(k-1)\overline{y}f}(q_0)), \tag{4.42}$$

and, correspondingly, $Q_k = Q_{L,d}(q_k)$, $\sigma_k(\cdot) = \sigma(q_k, \cdot)$, $[-T_{1,k}, T_{2,k}] = [-T_1(q_k), T_2(q_k)]$, $W_k = W(q_k)$ and $\Gamma_k = \Gamma(q_k)$. Set, moreover, Γ_0 as the segment $[-L, L] \times \{0\}$ contained in W_1.

Let $k \in \{1, \dots, l-1\}$. For every $s \in (-L, L) \cap (-T_{1,k}, T_{2,k})$, we want to identify the points $(x_k(s), y_k(s)) \in \Gamma_k \subset W_k$ and $(s, 0) \in W_{k+1}$. By construction, there exist a neighborhood U_1^k of $(x_k(s), y_k(s))$ in Q_k, a neighborhood U_2^k of $(s,0)$ in Q_{k+1}, and an isometry $\iota^k : U_1^k \to U_2^k$ (with respect to the metric induced by N) such that $\iota^k(U_1^k \cap \partial W_k) = U_2^k \cap \partial W_{k+1}$. Consider the Riemannian two-dimensional manifold with boundary $\mathcal{D}(q_0)$, obtained from the abstract union $\cup_{1 \le k \le l} W_k$ by identification of $(x_k(s), y_k(s)) \in W_k$ with $(s, 0) \in W_{k+1}$, for every $k \in \{1, \dots, l-1\}$ and every $s \in [-L, L] \cap [-T_{1,k}, T_{2,k}]$. The metric on $\mathcal{D}(q_0)$ is the one induced by N on each W_k. The Riemannian structure obtained in this way is well defined, because the gluing of two adjacent strips W_k and W_{k+1} is rendered isometric by ι^k.

Let us assume, from now on, that

$$\int_{\mathcal{D}(q_0)} K dA \le \delta\,. \tag{4.43}$$

4.3.2 The fundamental tessellation

Our purpose is to define a tessellation on $\mathcal{D}(q_0)$, that is, to subdivide the domain by geodesic polygons, in a checked pattern. Any such subdivision can be seen as a discrete system of coordinates on $\mathcal{D}(q_0)$, assigning to any point the polygon which contains it.

A tessellation is determined by a *grid of vertical and horizontal lines*. We choose as vertical lines the curves Γ_k defined above. Next lemma provides the estimates which allow to complete the construction of the desired tessellation.

Lemma 4.10 *There exist $C(\varepsilon, K_\infty) > 0$ and a positive $\delta_0' \le \delta_0$ depending only on ε and K_∞ such that the following holds: Let $\overline{q} \in M$ verify (4.38) with $\delta \le \delta_0'$. Take $t_0 \in \mathbf{R}, \theta_0 \in S^1$ such that $|t_0| \le L - \overline{y}/2$ and $|\theta_0 - \frac{\pi}{2}| \le \delta$. Let $z_0 = (t_0, 0, -\theta_0)$, $\sigma_{z_0}(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ and denote by $[0, T]$ its maximal interval of definition in $W(\overline{q})$. Then $(x(T), y(T)) \in \Gamma(\overline{q})$ and $|T - \overline{y}| \le C(\varepsilon, K_\infty)\delta$. Moreover, for every $s \in [0, T]$, $|x(s)| + |y(s) + s| + \left|\theta(s) + \frac{\pi}{2}\right| \le C(\varepsilon, K_\infty)\delta$.*

Proof. As it was done for estimates (4.17) and (4.18) of section 4.2, we have, for every $(x, y) \in W(\overline{q})$,

$$\frac{1}{2} \le B(x,y) \le 1, \qquad |C(x,y)| \le \sqrt{K_\infty}\,.$$

Let $[0, T_0)$ be the largest interval (in $[0, T]$) so that $|\theta(s) + \frac{\pi}{2}| \le \frac{\pi}{3}$. Since $\theta(0) = -\theta_0$, then $T_0 > 0$. The function $v(s) = \theta(s) + \pi/2$ verifies the differential inequality

$$|\dot{v}| \le \sqrt{K_\infty}|v|, \tag{4.44}$$

which clearly implies that

$$\left|\theta(s) + \frac{\pi}{2}\right| \le \delta e^{\sqrt{K_\infty}s}\,. \tag{4.45}$$

Since $\dot{y} = \cos v$ and $|\dot{x}| \leq 2|\sin v|$, we have, for every $s \in [0, T_0)$,

$$|y(s) + s| \leq \int_0^s v(\tau)^2 d\tau \leq \delta^2 s e^{2\sqrt{K_\infty} s}, \tag{4.46}$$

and

$$|x(s)| \leq 2\delta s e^{\sqrt{K_\infty} s}. \tag{4.47}$$

It is clear from (4.45)–(4.47) that, if δ is small enough with respect to d, then $T_0 = T$ and $\sigma(T) \in \Gamma(\overline{q})$. Moreover, $|T - \overline{y}| \leq C(\varepsilon, K_\infty)\delta$. The estimates on $x(\cdot)$, $y(\cdot)$, and $\theta(\cdot)$ follow. □

Let, for every $j \in \{-l+1, \ldots, l-1\}$, Δ_j be the support of the geodesic in $\mathcal{D}(q_0)$ which starts at $(j\overline{y}, 0) \in \Gamma_0 \subset W_1$, making an oriented angle $-\pi/2$ with Γ_0. Assume that (4.43) holds, with $\delta \leq \delta_0'$. For every $j \in \{-l+1, \ldots, l-1\}$, a repeated application of Lemma 4.10 to Δ_j and the successive Γ_k shows that, for every $k \in \{0, \ldots, l\}$, Δ_j intersects Γ_k transversally. Indeed, at every step of the iteration, the angle determined by Δ_j and Γ_k at their point of intersection differs from $\pi/2$ by $\int_{D_{j,k}} K dA \leq \delta$, where $D_{j,k}$ is the geodesic quadrilateral of $\mathcal{D}(q_0)$ bounded by Δ_0, Δ_j, Γ_0 and Γ_k. For every $j \in \{-l+1, \ldots, l-1\}$ and $k \in \{0, \ldots, l\}$, denote by $z_{j,k}$ the point of intersection of Δ_j and Γ_k. Due to Lemma 4.10, the length of the portion of Δ_j connecting $z_{j,k}$ with $z_{j,k+1}$ differs from $\overline{y}$ by at most $C(\varepsilon, K_\infty)\delta$. For the same reason, applying the argument to horizontal, instead of vertical, strips, the length of the portion of Γ_k which joins $z_{j,k}$ and $z_{j+1,k}$ differs from $\overline{y}$ by at most $C(\varepsilon, K_\infty)\delta$.

Denote by $P_{j,k}$ the geodesic quadrilateral with vertices $z_{j,k}$, $z_{j,k+1}$, $z_{j+1,k+1}$, and $z_{j+1,k}$ for all pairs (j,k) in $\{-l+1, \ldots, l-2\} \times \{0, \ldots, l-1\}$. The edges of $P_{j,k}$ are portions of the horizontal and vertical lines Δ_j, Δ_{j+1}, Γ_k, and Γ_{k+1}. The family of all such $P_{j,k}$ is called a *tessellation* on $\mathcal{D}(q_0)$.

Consider the tessellation on the Euclidean plane which covers the rectangle $[0, l\overline{y}] \times [-(l-1)\overline{y}, (l-1)\overline{y}]$, given by the family of squares

$$C_{j,k} = [j\overline{y}, (j+1)\overline{y}] \times [k\overline{y}, (k+1)\overline{y}], \qquad (j,k) \in \{-l+1, \ldots, l-2\} \times \{0, \ldots, l-1\}.$$

Define $\mathcal{T}$ as the union of all $\partial C_{j,k} \subset \mathbf{R}^2$, that is,

$$\mathcal{T} = \left(\cup_{j=-l+1}^{l-1} [0, l\overline{y}] \times \{j\overline{y}\}\right) \cup \left(\cup_{k=0}^{l} \{k\overline{y}\} \times [-(l-1)\overline{y}, (l-1)\overline{y}]\right),$$

and, similarly, $\mathcal{T}'$ as the union of all $\partial P_{j,k} \subset \mathcal{D}(q_0)$.

Consider a teardrop of size $\frac{r}{\varepsilon}$, starting from the point $(0,0) \in \mathbf{R}^2$ in the direction $(1,0)$, contained in the Euclidean rectangle $[0, (l+1)\overline{y}] \times [-(l-1)\overline{y}, (l-1)\overline{y}]$. The trajectory designing the teardrop intersects $\mathcal{T}$ in a sequence of points $p_m = (x_m, y_m)$, $0 \leq m \leq V$, numbered according to the order of intersection (see figure 4.3.2). Notice that

$$p_0 = p_V = (0,0). \tag{4.48}$$

The idea is now to pullback the points p_m on $\mathcal{T}'$, to equip them with a direction, and to construct a control strategy steering q_0 to q_0^-, passing through all these intermediate states. Each elementary control problem corresponding to the passage between two subsequent states will be eventually stated and solved in a proper coordinate strip.

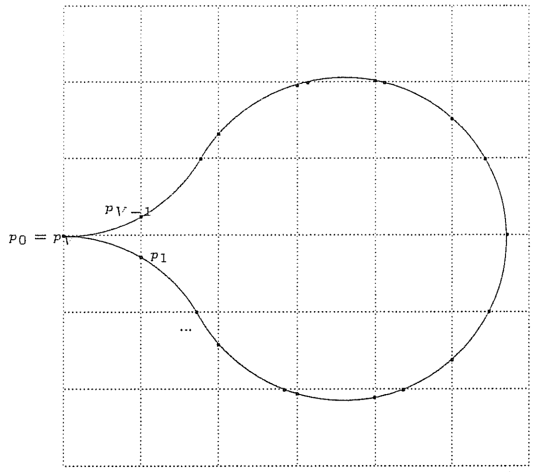

Figure 4.4

4.3.3 Reduction to the elementary problem

In the following discussion, it simplifies the presentation to assume that $\overline{y}$ is small with respect to the size r/ε of the teardrop. If this is the case, then each portion of teardrop which is contained in a square of the tessellation, considered as a planar curve, does not change direction much. To this extent, we will ask ε to be small with respect to $\overline{y}$. This is not restrictive, because d, and consequently $\overline{y}$, are fixed, and our final goal is to prove the unrestricted controllability property for the Dubins' problem.

In detail, take $\omega > 0$ and assume that, for every $r > 1$, the teardrop of size r/ε intersects the grid of step $\overline{y}$ in a sequence of points p_m, $0 \leq m \leq V$, such that

(a) the total variation of the angle component of the portion of teardrop connecting p_m with p_{m+1} is smaller than ω, for every $0 \leq m \leq V-1$.

Fix $\omega > 0$ small enough, such that, from every sequence which satisfies (a) we can extract a subsequence, still denoted by p_m, $0 \leq m \leq V$, such that

(b) (4.48) holds;

(c) the Euclidean distance between p_m and p_{m+1} is larger than $\overline{y}/2$, for every $0 \leq m \leq V-1$;

(d) the portion of teardrop connecting p_m with p_{m+1} intersects the interior of at most two squares of the tessellation, for every $0 \leq m \leq V-1$.

Notice that (d) implies

(a') the total variation of the angle component of the portion of teardrop connecting p_m with p_{m+1} is smaller than 2ω, for every $0 \leq m \leq V-1$.

With any p_m we associate a point p'_m in $\mathcal{T}'$, as follows: assume that p_m lies in the segment $[j\overline{y}, (j+1)\overline{y}] \times \{k\overline{y}\}$ (respectively $\{j\overline{y}\} \times [k\overline{y}, (k+1)\overline{y}]$). Then we choose p'_m in the portion of Γ_k (respectively Δ_j) joining $z_{j,k}$ and $z_{j+1,k}$ (respectively $z_{j,k}$ and $z_{j,k+1}$), in such a way that the proportion between the lengths of the two geodesic segments in which p_m and p'_m split the tessellation edges which they belong to is the same.

We want now to identify a geodesic segment in $\mathcal{D}(q_0)$ which joins p'_m and p'_{m+1}. Consider the segment S_m in $\mathbf{R}^2$ connecting p_m with p_{m+1}. Its slope can be measured by an angle $\alpha_m \in \mathcal{S}^1$, such that $p_{m+1} - p_m \in e^{i\alpha_m}\mathbf{R}_+$. Assume that $-\frac{\pi}{4} < \alpha_m \leq \frac{\pi}{4}$. (The other cases, when $(2k+1)\frac{\pi}{4} < \alpha_m \leq (2k+3)\frac{\pi}{4}$, $k = 0, 1, 2$, can be treated in an analogous way.) Choose j such that p_m and p_{m+1} belong to the strip $[0, (l+1)\bar{y}] \times [j\bar{y} - \frac{d}{8}, j\bar{y} + \frac{d}{8}]$. Then p'_m and p'_{m+1} are identifiable with two points of $Q_{L,\frac{d}{4}}(q_j)$, as it can be easily deduced from Lemma 4.10. Let (x'_m, y'_m) and (x'_{m+1}, y'_{m+1}) be the coordinates in $Q_{L,\frac{d}{4}}(q_j)$ of, respectively, p'_m and p'_{m+1}. Then, as follows from the estimates of Lemma 4.9 and Lemma 4.10,

$$|(x'_{m+1} - x'_m) - (x_{m+1} - x_m)| + |(y'_{m+1} - y'_m) - (y_{m+1} - y_m)| \leq C(\varepsilon, K_\infty)\delta \,.$$

Therefore,

$$\left| \frac{y'_{m+1} - y'_m}{x'_{m+1} - x'_m} - \tan \alpha_m \right| \leq C(\varepsilon, K_\infty)\delta \,,$$

and so

$$\left| \arctan\left(\frac{y'_{m+1} - y'_m}{x'_{m+1} - x'_m} \right) - \alpha_m \right| \leq C(\varepsilon, K_\infty)\delta \,.$$

Lemma 4.9 estimates the coordinate behavior of geodesics starting from p_m or, to be more precise, of solutions of the system (4.28 – 4.30) with initial conditions of the type (x'_m, y'_m, θ_0), with $|\theta_0|$ far from $\pi/2$. By standard continuity considerations, there exists a geodesic segment S'_m joining p'_m and p'_{m+1}, whose initial direction, is given by (x'_m, y'_m, α'_m), with $|\alpha'_m - \alpha_m| \leq C(\varepsilon, K_\infty)\delta$.

Denote by β_m be the oriented angle determined by the teardrop passing through p_m and the segment S_m (with the agreement that the teardrop is oriented in its running sense and the segment from p_m to p_{m+1}). The m-th intermediate state of the aimed teardrop in $\mathcal{D}(q_0)$ can be defined as the point p'_m equipped with the direction which makes an angle β_m with S'_m. This direction can be represented by a unit vector $v'_m \in T^1_{p'_m}N$. When $m = V$, let $(p'_V, v'_V) = q_0^-$. We call *elementary problem* the task of designing a control strategy which steers (p'_m, v'_m) to (p'_{m+1}, v'_{m+1}).

The elementary problem is conveniently formulated in the geodesic coordinates of the rectangle whose base curve is S'_m. In these coordinates $p'_m = (0,0)$ and $p'_{m+1} = (|S'_m|, 0)$, where $|S'_m|$ denotes the length of S'_m. Remember that, at the points of the base curve S'_m, the coordinate angle measures the true Riemannian angle between the corresponding unitary vector and S'_m. Therefore, what has to be solved is the control problem (4.8 – 4.10) with initial condition $(0, 0, \beta_m)$ and final condition $(|S'_m|, 0, \beta_{m+1} + \gamma'_{m+1})$, where γ'_{m+1} is the angle between S'_m and S'_{m+1} at p'_{m+1}, with the agreement that S'_m is oriented from p'_m to p'_{m+1} and S'_{m+1} from p'_{m+1} to p'_{m+2}. In order to estimate the value of γ'_{m+1}, let us go back to the coordinate strip $Q_{L,\frac{d}{4}}(q_j)$ which contains p'_m and p'_{m+1}. As follows from the assumptions (a')–(d), p'_{m+2} as well stays in the strip, and the coordinate slope of the geodesic segment S'_{m+1} can be estimated with the same technique as before. That is, there exists an angle α'_{m+1} such that $|\alpha'_{m+1} - \alpha_{m+1}| \leq C(\varepsilon, K_\infty)\delta$ and S'_{m+1} is the projection on N of the solution of (4.28 – 4.30) with initial condition $(x'_{m+1}, y'_{m+1}, \alpha'_{m+1})$.

The angle between S'_m and S'_{m+1}, measured in coordinates, is given by $\alpha'_{m+1} - \widetilde{\alpha}'_m$, where $\widetilde{\alpha}'_m$ is the angle coordinate of the tangent vector to S'_m at p'_{m+1}.

Let γ_{m+1} be the angle between S_m and S_{m+1}, with respect to the orientation agreement introduced above. Since $\gamma_{m+1} = \alpha_{m+1} - \alpha_m$, we have that

$$|\alpha'_{m+1} - \widetilde{\alpha}'_m - \gamma_{m+1}| \leq C(\varepsilon, K_\infty)\delta\,. \tag{4.49}$$

The angle γ'_{m+1} is not, in general, equal to $\alpha'_{m+1} - \widetilde{\alpha}'_m$. We need to take into account a correction due to the fact that α'_{m+1} and $\widetilde{\alpha}'_m$ are expressed in coordinates. However, we can prove the following fact:

Lemma 4.11 *The sign of* $\alpha'_{m+1} - \widetilde{\alpha}'_m$ *and the sign of* γ'_{m+1} *are equal, and* $\frac{1}{2}|\alpha'_{m+1} - \widetilde{\alpha}'_m| \leq |\gamma'_{m+1}| \leq 2|\alpha'_{m+1} - \widetilde{\alpha}'_m|$.

Proof. Remark that the angle $\overline{\alpha}$ made by a unitary vector with coordinates $(x'_{m+1}, y'_{m+1}, \alpha)$ with the horizontal line $\{(x,y)|\ y = y'_{m+1}\}$ verifies the relation

$$\tan\alpha = B(x'_{m+1}, y'_{m+1}) \tan\overline{\alpha}\,, \tag{4.50}$$

as follows from (4.3). In particular the sign of α and $\overline{\alpha}$ is the same. As already remarked in (4.34), we have $\frac{1}{2} \leq B(x'_{m+1}, y'_{m+1}) \leq 1$. The lemma follows from (4.50) and straightforward computations. □

Therefore, by taking (4.49) into account,

$$|\gamma'_{m+1} - \gamma_{m+1}| \leq |\gamma_{m+1}| + C(\varepsilon, K_\infty)\delta\,.$$

To complete our estimate of γ'_{m+1}, let us make a remark on the size of γ_{m+1}. The length $|S_m|$, due to (c), is bounded by $\sqrt{5}\overline{y}$. Thus, γ_{m+1} is maximized when S_m and S_{m+1} are two concatenated cords of length $\sqrt{5}\overline{y}$ of a circle of radius r/ε (see figure 4.3.3). By easy trigonometric considerations,

$$|\gamma_{m+1}| \leq \arcsin\left(\frac{\sqrt{5}}{r}\varepsilon\overline{y}\right) \leq C(K_\infty)\frac{\varepsilon}{r}\,. \tag{4.51}$$

Therefore, we can assume that

$$|\gamma'_{m+1} - \gamma_{m+1}| \leq C(K_\infty)\frac{\varepsilon}{r}\,.$$

Finally, we are left with a control problem of the type (4.8 – 4.10), with initial condition $(0, 0, \beta_m)$ and final condition $(|S_m| + e_1, 0, \beta_{m+1} + \gamma_{m+1} + e_2)$, where

$$|e_1| \leq C(\varepsilon, K_\infty)\delta\,, \tag{4.52}$$
$$|e_2| \leq C(K_\infty)\frac{\varepsilon}{r}\,. \tag{4.53}$$

Let us fix the constant appearing in (4.53) and denote it by C_0.

The triple $(\beta_m, |S_m|, \beta_{m+1} + \gamma_{m+1})$ characterizes the boundary conditions of the corresponding portion of teardrop in the Euclidean plane. Such curve is the concatenation of at most two pieces of circular trajectory, with geodesic curvature equal to r/ε. When

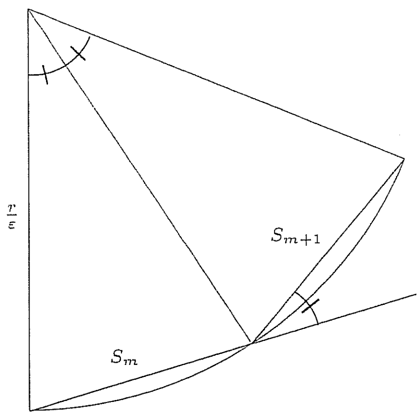

Figure 4.5

it reduces to a single circular arc – which is the case for all but at most two elementary problems – we have

$$\beta_{m+1} + \gamma_{m+1} = -\beta_m = \pm \arcsin\left(\frac{|S_m|\varepsilon}{2r}\right).$$

In any case, an estimate of the following type holds,

$$|\beta_m| \leq C(K_\infty)\frac{\varepsilon}{r}. \tag{4.54}$$

4.3.4 Solution of the elementary problem

Consider, in the Euclidean plane, the control problem

$$\begin{cases} \dot{x} = \cos\theta, \\ \dot{y} = \sin\theta, \\ \dot{\theta} = u, \\ (x, y, \theta)(0) = (0, 0, \beta_m), \end{cases} \qquad u \in [-\varepsilon, \varepsilon]. \tag{4.55}$$

Recall that $|S_m| \in \left[\frac{\overline{y}}{2}, \sqrt{5}\overline{y}\right]$. If ε is small enough with respect to d, then every solution of (4.55) intersects the surface

$$\{(x, y, \theta) \in \mathbf{R}^2 \times \mathcal{S}^1 | \, x = |S_m|\}$$

in a time close to $\overline{y}$. Fix $T > 0$ such that every admissible trajectory intersects transversally $\{x = |S_m|\}$ within time T. Let $E(\cdot)$ be the map which associates with an admissible control $u\colon [0, T] \to [-\varepsilon, \varepsilon]$ the coordinates (y, θ) of the trajectory corresponding to $u(\cdot)$, evaluated at the first point of intersection with $\{x = |S_m|\}$. Notice that E is a continuous map from the space of admissible control functions, endowed with the L^1 topology, into $\mathbf{R} \times \mathcal{S}^1$ or, equivalently, $\mathbf{R}^2$.

The family of bang-bang control functions which are concatenation of two arcs, the first one corresponding to control $+\varepsilon$ and the second one to control $-\varepsilon$, form a continuous curve in $L^1([0, T])$, joining the two constant control functions $u \equiv \varepsilon$ and $u \equiv -\varepsilon$. Taking into account also the two-bang concatenations where the controls are applied in the reversed order, it turns out that the family of all bang-bang control functions with at most two arcs forms a closed curve in the set of admissible controls. Choose a parametrization of such curve of the type $\{u_s : [0,T] \to [-\varepsilon,\varepsilon]\}_{s\in S^1}$. Then $\gamma : s \mapsto E(u_s)$ is a continuous closed curve in $\mathbf{R}^2$. Let, for every $\xi \leq |S_m|$ and for $\nu = -1, 1$,

$$u_\xi^\nu = \nu\varepsilon \left(1_{[0,\xi)} - 1_{[\xi,T]}\right) .$$

Integrating the system (4.55) we get

$$E(u_\xi^\nu) = \left(\beta_m|S_m| + \nu\varepsilon\left(\frac{|S_m|^2}{2} - (|S_m| - \xi)^2\right) + O(\varepsilon^2),\ \beta_m + \nu\varepsilon(2\xi - |S_m|) + O(\varepsilon^2)\right) . \tag{4.56}$$

Recall that γ_{m+1}, β_m, and β_{m+1} can be estimated using (4.51) and (4.54). It is easy to check that if r is large enough, then the distance from the support of γ to

$$\Sigma = \left\{(0, \beta_{m+1} + \gamma_{m+1} + \tilde{e}_2)\ \middle|\ |\tilde{e}_2| \leq C_0\frac{\varepsilon}{r}\right\}$$

can be bounded from below by a constant $C(\varepsilon, K_\infty) > 0$, uniformly in $|S_m|$. The expression of the leading term of (4.56) shows, moreover, that the curve $s \mapsto \gamma(s)$ from S^1 to $\mathbf{R}^2 \setminus \Sigma$ is not contractible.

Consider now the non-flat elementary problem

$$\begin{cases} \dot{x} = \frac{\cos\theta}{B}\,, \\ \dot{y} = \sin\theta\,, \\ \dot{\theta} = u + C\cos\theta\,, \\ (x,y,\theta)(0) = (0,0,\beta_m)\,, \end{cases} \qquad u \in [-\varepsilon,\varepsilon]\,. \tag{4.57}$$

Fix any admissible control function $u:[0,T] \to [-\varepsilon,\varepsilon]$. Denote by $(x(\cdot), y(\cdot), \theta(\cdot))$ (respectively, $(x'(\cdot), y'(\cdot), \theta'(\cdot))$) the solution of (4.55) (respectively, of (4.57)) corresponding to $u(\cdot)$. The same computations as in Lemma 4.9 imply that

$$|x(t) - x'(t)| + |y(t) - y'(t)| + |\theta(t) - \theta'(t)| \leq C(\varepsilon, K_\infty)\delta\,.$$

In particular we can assume that $(x'(\cdot), y'(\cdot), \theta'(\cdot))$ intersects transversally $\{x = |S_m| + e_1\}$ within time T. Define $E'(u)$ as the pair of coordinates $(y'(\cdot), \theta'(\cdot))$ evaluated at the first point of intersection. The map E' verifies

$$|E(u) - E'(u)| \leq C(\varepsilon, K_\infty)\delta\,.$$

Thus, the curve $\gamma' : s \mapsto E'(u_s)$, which is closed and continuous, encloses a region which contains Σ, at least for δ small with respect to ε and K_∞. By standard degree theory considerations, the image of E' contains Σ, and the elementary problem is solved.

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