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Ghost models

in two-dimensional condensed matter physics

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Introduction

Since the late sixties, the field theoretic approach has been recognized as an invaluable tool in the study of critical phenomena [1, 2]. Particularly, during the last two decades, it has acquired an ever-increasing importance in the context of two-dimensional condensed matter physics ((2+0) classical and (1+1) quantum dimensions). For this peculiar value of the dimensionality, indeed, powerful techniques, such as bosonization (in its abelian [3] and non-abelian [4] forms) and conformal invariance [5], have provided the ideal background to approach one dimensional quantum spin chains [6–8], localization-delocalization transitions [9], integer [10,11] and fractional [12] quantum Hall effects and various models in the presence of disorder [11, 13–23]. Moreover, the field theoretic language has allowed to furnish such diverse problems with a unified description, in terms of their symmetry properties, relying on the so-called nonlinear sigma model (NLσM).

In general, a nonlinear sigma model is an effective field theory, describing the low-energy excitations of a system and their interactions, whose fields take values on a manifold G/H. Indeed, when some fields, transforming under the global symmetry G, acquire a non vanishing vacuum expectation value, invariant only under the subgroup H, the system develops low-energy modes, assuming values on the coset G/H. In dimensions higher than two, this results in a spontaneous breaking of the symmetry, with the consequent appearance of gapless Goldstone bosons. On the other hand, in two dimensions, the Mermin-Wagner-Coleman theorem [24] prevents this situation to occur, leading to a restoration of the initial symmetry and the development of a massive (gapped) regime. This mechanism goes by the name of *asymptotic freedom* or *dynamical mass transmutation*.

Albeit the nature of two-dimensional sigma models intrinsically excludes gapless modes, however it is still possible to find critical (massless) behaviors, which are associated to fixed

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points in the renormalization group space of local interactions. Basically, this situation can be achieved in two ways. On the one hand, terms, which change drastically the low-energy physics of a nonlinear sigma model, may be added to its action. Prominent examples are the Wess-Zumino-Witten (WZW) term and the so-called topological angle θ, which drive the system to a non-trivial massless phase. On the other hand, one can take advantage of the peculiar form assumed by the manifold G/H, for special values of the parameters which characterize it. Such critical behaviors are easily visible at the level of the perturbative beta function. Despite many advances in this last context have been made very recently [20,25–29], however a thorough understanding of the phase structures of such class of models (identification of critical points, exact computation of critical exponents and, above all, a detailed analysis of the possible off-critical scenarios) is still a fertile field of investigation. In this general framework, this thesis finds its main motivation. Its purpose, in particular, is addressed to the study of some specific off-critical systems, by means of non-perturbative approaches. However, before proceeding along this direction, it is worthwhile spending few words about the undisputed relevance of sigma models with WZW and θ terms, in the realm of condensed matter.

A privileged playground for the application of NLσM, supplemented by the action of WZW and θ terms, has been the context of quantum Heisenberg antiferromagnets. In a series of remarkable papers [6–8], Haldane and Affleck have inferred that the continuum (large-spin) limit, performed on one-dimensional quantum spin chains, is described by the $O(3)/O(2)$ sigma model. In the absence of the topological angle ($\theta = 0$), the system is gapped and reproduces the properties of integer-spin chains. On the other hand, when $\theta = \pi$ exactly, the model becomes gapless and the associated critical theory, describing the physics of half-integer-spin chains, turns out to correspond to the WZW model with $SU(2)$ symmetry, at the level $k = 1$ (briefly, $SU(2)_1$). In the same spirit, the transition between plateaux, in the integer Hall effect, has been attributed to the $N \to 0$ 'replica limit' of the $U(2N)/U(N) \times U(N)$ nonlinear sigma model, in the presence of the topological angle (the critical point corresponding to $\theta = \pi$) [10]. Further applications have been found in the study of the Anderson localization transition [9] and, in recent years, in the context of two-dimensional electronic systems in the presence of quenched disorder (see e.g. [20] and references therein).

Now, let us turn our attention to the other kind of systems, which inherently exhibit gapless behaviors. In order to illustrate the general mechanism behind the pattern of their phase diagrams, we consider one of the simplest, but by no means trivial, examples of nonlinear sigma models, namely the so-called *sphere sigma model*, defined on the manifold $S^{N-1} \cong O(N)/O(N-1)$, alias a $(N-1)$-dimensional sphere. (As we will see, the parameter N is destined to play a crucial role.) Its action

$$S = \frac{1}{g} \int d^2x \, \partial_\mu \varphi \cdot \partial_\mu \varphi \tag{1}$$

is expressed in terms of the N-component vector field $\varphi \equiv (\varphi_1, \ldots, \varphi_N)$, taking values on the aforementioned $(N-1)$-sphere and thus subject to the constraint $\varphi \cdot \varphi = 1$. The corresponding beta function, at leading order, assumes the form

$$\beta(g) \equiv \frac{dg}{d\ln L} \propto (N-2)\, g^2 \,, \tag{2}$$

where L is a length-scale. Some important observations soon descend from β. The value $g = 0$ represents the trivial fixed point, where the theory is described by free massless fields. Moreover, no other fixed points seem to emerge from (2). This, of course, does not imply that they do not exist, but simply that they cannot be detected by a purely perturbative expansion around the trivial fixed point. Afterwards, the analysis proceeds along two distinct lines. (In the following, we assume that the bare coupling constant is positive.) For $N > 2$ the conclusions previously derived for general sigma models in two-dimensions still hold. The trivial fixed point becomes unstable and the theory flows, at large distances, to a massive regime, restoring the initial $O(N)$ symmetry and opening a gap. For $N < 2$, instead, at long wavelengths the coupling renormalizes to zero and $g = 0$ becomes an attractive stable fixed point. $(N-1)$ gapless Goldstone excitations appear and the symmetry spontaneously breaks down to $O(N-1)$. (We would like to stress that no violation of the Mermin-Wagner-Coleman theorem occurs, because the hypothesis of its applicability do not apply in this range of N.) This picture holds for small enough g. Nevertheless, for large values of the coupling, a massive behavior is expected and therefore a non trivial fixed point at $g = g^*$ is believed to separate the gapless and the gapped phases. The nature of such critical point is extremely elusive and, being outside the range of validity of the perturbation theory, can only be tackled by non-perturbative methods.

It is worth emphasizing that the range of N we are interested in is not at all unrealistic. Indeed, it is well known that, for instance, the $N \to 0$ limit plays a crucial role, not only in the replica approach to disordered systems [30], but also in important statistical mechanics models of geometrical nature, such as self avoiding walks and polymers [31]. Moreover, very recently, a novel class of field theories has been proposed [25–29] (with the precise intention of confronting problems hardly solvable by standard techniques, which rely e.g. on current algebra symmetries), where the range of N spans also negative integer values. This has been achieved introducing new anticommuting variables nearby the usual commuting components of the vector field φ, in (1), and properly redefining N as the difference between these two kinds of components. In such a way, it has been obtained a 'supersymmetric' generalization, à la Parisi-Sourlas [32], of the ordinary nonlinear sigma models. A wealth of critical behaviors has been found [25], yielding, still, conformally invariant theories, but of a special kind: non-unitary and logarithmic ones [33]. Among them, a privileged position is occupied by the so-called *ghost* systems, whose fields violate the usual relation between spin and statistics [34]. In the context of condensed matter, originally noticed in the study of electronic systems with quenched disorder [22], such theories act as a sort of 'building blocks' of more complicated ones.

Ghost fields can have bosonic or fermionic nature and their properties at the critical point have been widely studied in the past years [34–36]. Being non-unitary theories, they exhibit many unusual characteristics, among which, negative values of the central charge, conformal operators with negative weights (which means that their correlators increase as power-laws of the distance, instead of decreasing) and logarithmic behaviors in some correlation functions. However, a remarkable relation to their corresponding ordinary counterparts (i.e. complex bosonic an fermionic fields) has been established, at the level of the space of states [37].

This thesis is devoted to the study of ghost theories out of the critical point, in two dimensions.

The first chapter offers a bird's eye view of the most important applications of ghosts to condensed matter physics. After a brief exposition of the basic (and less basic) facts concerning ghosts at the critical point, an outline of the non-perturbative methods, used in Part I and Part II, is furnished. Essentially, they rely on the so-called integrable approach, which is based on the possibility of describing all the states of an *integrable* quantum field

theory in terms of pseudo-particles in a Hilbert space. The scattering properties of such excitations are encoded into a matrix (the S-matrix), which can be exactly determined by imposing a set of stringent constraints, and which allows to specify completely the particle content of the theory (masses, multiplicities, bound states) [38, 39]. The knowledge of the scattering amplitude, then, permits to extract all the thermodynamic quantities of the system (free energy) by means of the *Thermodynamic Bethe ansatz* (TBA) technique [40] and, at least in principle, to determine the off-critical correlation functions of the local operators of the theory, thanks to the so-called *Form Factor bootstrap* approach [41–43].

Part I contains the simplest examples of off-critical ghost theories, namely the massive versions of the conformal free bosonic and fermionic ones [44]. Despite their non-interacting nature, still there are non-local sectors of the models, which exhibit a highly interacting behavior. Correlation functions of operators belonging to these sectors are computed exactly and a comparison with the massive ordinary counterparts is performed. Afterwards, the effects produced by the introduction of impurities are considered [45]. At the moment, such models lack a physical realization, but they are important as 'prototype' systems, shedding light on some crucial basic aspects (e.g. the choice of the most convenient basis for the space of states).

Part II deals with a deceptively simple representative of the aforementioned nonlinear sigma models defined on supersymmetric manifolds, where the vector field, with one commuting component and two anticommuting ones, transforms under the global symmetry $OSP(1|2)$. This system has a simple physical realization in terms of a dense loop model, where crossings of loops are allowed [28, 46]. At long wavelength, the theory is gapless and the Goldstone excitations are nothing but free fermionic ghosts [25, 28]. We propose the exact S-matrix for this system and present TBA calculations, supporting such conjecture. The bootstrap form factor approach is outlined, including a detailed discussion about the symmetry properties of the model and the explicit derivation of some basic objects, such as the minimal form factors. Moreover, we compute explicitly the two-point correlation function of a suitably chosen operator of the theory, comparing its large distance limit with the result expected on the basis of conformal field theory considerations. Since the work is still in progress [47], we conclude sketching the main goals and the route we intend to take, in order to pursue them.

Chapter 1

Ghost systems in two dimensions

1.1 The role of ghosts in condensed matter

Two-dimensional electronic systems in the presence of quenched disorder have been the object of a wide attention in the last decade [11, 13–23]. Many efforts have been devoted to understand the pattern of their phase transitions, encouraged by the ambitious goal to identify non trivial critical points and possibly to compute exactly critical exponents. Besides fruitful applications ranging from d-wave superconductivity [13] and the spin quantum Hall effect [21] to the random XY model [19] and delocalization transitions, they proved to be a privileged playground to study the crucial, and still elusive, role of impurities in the transition between Hall plateaux [11].

Despite the non-perturbative nature of many of these fixed points (of which the last one is only a famous example), in two dimensions powerful techniques, such as conformal field theory (CFT) [5], Bethe ansatz [48] and (non-)abelian bosonization [3,4], are available. Key tools rely on the replica approach [30] and the supersymmetric formalism [49]. Both these methods naturally lead to effective field theories governed by a non-trivial fixed point (or a line of) characterized by central charge[1] $c = 0$ and trivial partition function $Z = 1$, whose peculiarity resides in logarithmic contributions multiplying the leading power law

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[1]Naively, the central charge in non-interacting theories is a measure of the massless degrees of freedom. By construction, in the replica limit the number of fields vanishes and in the supersymmetric scenario an equal number of bosonic and fermionic variables exactly cancels.

behaviors in multi-point correlation functions [50, 51]. This last property characterizes a wider class of conformal field theories, known as logarithmic ones (LCFTs). Besides their possible applications, they bear an interest of their own and have been the object of a broad investigation in the past ten years [33, 52]. Only quite recently [16, 17, 50, 51, 53], they have been recognized as the suitable candidate for describing critical points in disordered systems, leading, for instance, to a satisfactory explanation of the multifractal behavior displayed by wavefunctions near a localization-delocalization transition [18].

Significant progress has been made in the study of non-interacting massless Dirac fermions in the presence of different kinds of randomness, relying on the current-algebra approach, in addition to the replica and the supersymmetric ones. Usually three types of disorder are studied: random mass, random scalar potential and random gauge potential [11]. In the simplest cases exhibiting only one kind of impurities, the effective field theories have been described by a Wess-Zumino-Novikov-Witten (WZNW) critical point perturbed by some marginal current-current perturbation [15]. Despite the highly non-trivial character of such points, some models of interest have been found to allow for special simplifications. In other words, the current algebra and its primary fields admit a description in terms of free unitary field theories supplemented by a non-unitary counterpart, in which the constituent fields violate the usual relation between spin and statistics. Such CFTs are known as *ghost* systems. Their role is of crucial importance, since they are directly responsible for all the weird properties characterizing the aforementioned critical points. A remarkable example of this decomposition occurs for massless Dirac fermions subject to a random $SU(2)$ gauge potential, where the peculiar features of the effective supersymmetric $OSP(2|2)_{-2}$ critical theory are simply inherited from the twist sector of the $c = -2$ fermionic ghost system [22]. A comment is in order. At first sight, it could seem that the appearance of ghost variables is quite natural in the supersymmetric approach, since the method itself requires from the very beginning the introduction of these fictitious fields (bosonic ghosts coupled to Dirac fermions), but it could be far from obvious in the replica formalism. In this regard, it has been shown [23] (in the specific case of a random $SU(2)$ potential) that the $c = -2$ theory is ubiquitous and emerges independently of the methods used to average over disorder.

Though current-algebra techniques have shed new light on very interesting problems, many issues still remain open and need different ideas to be tackled. In particular, the

simultaneous presence of different disorder interactions causes the system to flow to a strong coupling regime, inaccessible by perturbation theory [11]. On the other hand, to gain insight exploiting non-perturbative methods is highly non-trivial, because of the non-unitary nature of the bosonic (ghost) sector of the theory. In this light, a careful study of ghost systems (also departing from the critical point) could be very useful in the difficult task to figure out more complicated theories. A special example, which admits an exact solution, has been proposed to deal with the random XY model [19]. In this case, averaging over disorder results in current-current perturbations of a conformal supercurrent algebra, whose non-vanishing renormalization group β function can be computed exactly. Thanks to peculiar properties of the underlying algebra, the effective field theory can be rephrased as a 'nearly conformal' theory, in the sense that it possesses a scale invariant sub-sector which is not a current algebra. In field theoretic language, it is represented by a $PSL(N|N)$ non linear sigma model[2] (NLσM), exhibiting a line of fixed points with $c = -2$, as the sigma model coupling constant is varied [54,55].

Proceeding along this direction, quite recently, field theoretic models (NLσMs and other related quantum field theories QFTs) involving superalgebras as internal symmetries have attracted a great deal of attention [25–27,29]. A rich pattern of critical behaviors has been found [25], where fixed points (or lines of fixed points) display still conformal invariance but an intrinsic non-rational character. Even though the spectrum of conformal operators (derived by means of Coulomb gas techniques) consists only of rational conformal weights, an infinite number of them cannot be reorganized by any chiral algebra into a finite number of representations. Prominent examples of such critical theories are the ghost systems[3] themselves. The range of physical applications is rather wide, including, not only disordered models, but also percolation, polymers, lattice models, Hall transitions, strings in anti-de Sitter space...

Before concluding, it is worthwhile spending few words about other condensed matter systems in which ghosts play a significant role.

The connection between polymers and ghost theories has been known since the early

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[2]$PSL(N|N) = GL(N|N)/(U(1) \otimes U(1))$.

[3]Interacting ghosts represent non-rational CFTs. As regards their free counterparts the story is more subtle and the definition itself of 'rationality' must be formulated properly. See e.g. [36,56].

90s [57]. Starting from the premises that the low-temperature phase of the $O(N)$ model, in the limit $N \to 0$, describes non-intersecting dense polymers [58], a fermionic ghost theory with $c = -2$ has been argued to capture the physics of the infra-red (IR) fixed point [57]. Actually, the phase structure of dense polymers is an extremely subtle issue and has become the object of a very recent debate [28], in connection with more general loop models. A deeper investigation of such systems, which require an accurate study of the $O(N)$ model phases (with $-2 < N < 2$), will be the purpose of Part II of the present thesis.

Finally, ghost theories (both fermionic and bosonic) have been proposed to describe the edge excitations of (unusual) 'paired' states in the fractional quantum Hall (QH) effect [12,59]. This last application puts special emphasis on the topic of unitarity. Edge states, due to their highly correlated nature, propagate as density fluctuations and are deeply connected to their underlying bulk state. They can be constructed gluing together QFTs of the usual scalar bosons, describing the charge fluctuations, and of some gapless fermions. The main goal consists in determining a well defined theory for the fermionic fields. The key idea is that the bulk wavefunctions can be interpreted as conformal blocks of some CFT in two-dimensional space-time[4]. A sound theory of the edge excitations requires a positive definite Hilbert space (in order to provide a *physical* description of QH systems) and, at the same time, must reproduce properly the bulk properties, dictated by CFT. If the 'paired' states are of the Haldane-Rezayi (HR) form, the resulting CFT exhibits central charge $c = -2$. Hence, the requirement of a positive definite space of states on the edge becomes a delicate issue, which opens a discussion on the non-unitary nature of ghost systems. In particular, the choice itself of the Hilbert space of states in the non-unitary $c < 0$ theories merits special attention. This last topic will be at the heart of the next section.

1.2   Ghosts at criticality

As emerged from the previous section, free ghost systems at the critical point belong to the vast class of 'non-unitary' CFTs, which includes in particular all conformal field theories with negative central charge. In order to make as clear as possible the discussion about

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[4]Actually, the original idea (see Moore and Read in reference [12]) is more general and applies to holomorphic wavefunctions of particle systems in two spatial dimensions.

their space of states, it is worthwhile introducing briefly the basic concepts of conformal field theory. The first setting of conformal invariance in two-dimensional QFT appeared in the fundamental paper by Belavin, Polyakov and Zamolodchikov [5]. (A more pedagogical approach can be found in standard textbooks and lecture notes [60–62].)

In simple words, conformal transformations are coordinate mappings which leave unchanged the angle between two vectors, at a given point. They form a group which, in two dimensions, possesses the remarkable property of being infinite-dimensional and describes analytical substitutions of the complex variables

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}), \tag{1.1}$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ are independent combinations of space (or spacetime) coordinates. The corresponding infinite-dimensional algebra of generators, alias the modes L_n and $\bar{L}_n$ of the stress-energy tensor (after expansion in Laurent series), is the so-called Virasoro algebra, which extends the concept of Lie algebras, including a 'central term' parameterized in terms of the central charge c.

A key ingredient of CFT is the hypothesis of a complete algebra of local fields $A_k(x)$, endowed with the operator product expansion (OPE),

$$A_i(x)A_j(0) = \sum_k C_{ij}^k(x)A_k(0), \tag{1.2}$$

where the coefficients $C_{ij}^k(x)$ are supposed to be powers of the distance and can be used to construct all the correlation functions. The very calculation of such correlators is the main pursuit of field theory and, in principle, can be attained implementing the dynamical 'bootstrap' principle, i.e. imposing the associativity of the operator algebra. Though this task turns out to be hopeless in dimensions $d > 2$ and, in general, is hardly solvable, however, in two dimensions and under peculiar conditions, an infinite set of solutions can be found, in correspondence of the so-called minimal models. The crucial idea is to relate local fields to 'vectors' in a representation of the Virasoro algebra, in order to rephrase the bootstrap constraints on the coefficients $C_{ij}^k(x)$ in terms of some (solvable) linear differential equations.

Among the fields $A_k(x)$, the so-called 'primary' ones, $\phi_k(x)$, play a central role. Their transformation law under the mapping (1.1) reads

$$\phi_k(z, \bar{z}) \to \left(\frac{df}{dz}\right)^{h_k} \left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{h}_k} \phi_k(f, \bar{f}). \tag{1.3}$$

The parameters h_k and $\bar{h}_k$ are called, respectively, the holomorphic and the anti-holomorphic conformal weight and their combination yields the anomalous scale dimension $\Delta_k = h_k + \bar{h}_k$ and the spin[5] $s = h_k - \bar{h}_k$, associated to the field ϕ_k. The weights h and $\bar{h}$ are also related to the Hamiltonian of the system

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0),$$
(1.4)

being eigenvalues of the operators L_0 and $\bar{L}_0$, which generate dilatation transformations.

However, primary fields themselves cannot close under the operator algebra. For each of them, there exist infinitely many other fields (the secondary ones), whose scale dimensions form integer spaced series, starting from the anomalous dimension of the corresponding ancestor (primary) field. These towers of fields are called 'conformal families' and, since their components transform into each others under conformal transformations, they correspond to some representations of the Virasoro algebra. Each representation admits a convenient description in terms of vectors (primary states) on which the negative modes of the stress-energy tensor (L_m and $\bar{L}_m$, with $m < 0$) act, generating a whole space of states. The holomorphic (anti-holomorphic) sector of such space is known as Verma module.

Turning to correlation functions, an important result of [5] proves that those involving secondary fields can be entirely expressed in terms of correlators of the corresponding ancestors. Therefore, the problem at the heart of QFT reduces to the evaluation of multi-point functions concerning solely primary fields. Conformal invariance strictly constraints the form they are allowed to assume. For example, the two-point functions are given by

$$\langle \phi_k(z_1)\phi_k(z_2) \rangle \sim \frac{1}{z_{12}^{2h_k}},$$
(1.5)

where $z_{12} = z_1 - z_2$ and, for simplicity of notation, from now on we restrict only to the holomorphic sector. The three-point correlators are also fixed up to a constant, while the four-point ones

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4) \rangle = F(z) \prod_{j<k} z_{jk}^{h/3-h_j-h_k},$$
(1.6)

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[5]Such spin is defined by rotations of the Euclidean two-dimensional space-time, and must not be confused with what is usually called spin, which describes the transformation properties under $SU(2)$ rotations, leaving the spatial coordinate unchanged.

with $h = \sum_{j=1}^{4} h_j$, are defined up to an arbitrary function $F(z)$ (conformal block) of the so-called anharmonic ratio $z = z_{12}z_{34}/z_{13}z_{24}$. The expansion of $F(z)$ in powers of the anharmonic ratio corresponds to the previously mentioned OPE (1.2). Therefore, the bootstrap condition translates into a set of constraints for the conformal blocks. In general, the exact computation of $F(z)$ is a hard task. However, in the cases of the so-called 'degenerate' conformal families (which admit a natural description in the Verma modules' formalism), such constraints reduce to linear differential equations, whose solutions turn out to be generalized hypergeometric functions. Minimal models contain a finite number of conformal families, all of them being degenerate. (Minimal models can be conveniently parameterized in terms of two (co-prime) integers p and p', assuming the notation $\mathcal{M}_{p,p'}$. The corresponding central charge can be written as

$$c = 1 - 6\,\frac{(p-p')^2}{pp'}\,, \tag{1.7}$$

while the conformal weights, belonging to the associated primary fields, read

$$h_{r,s} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}\,, \tag{1.8}$$

where $1 \leq r < p$ and $1 \leq s < p'$. Such finite number of conformal families can be formally arranged in a rectangle in the (r,s) plane, called the Kac table, endowed with the symmetry $h_{r,s} = h_{p'-r,p-s}$.)

The 'unitarity' issue is strictly connected to the absence/presence of negative-norm (actually the norm-squared) states in the space of states created by the Virasoro algebra. Unitary representations are free of such states. Given the hermiticity condition for the generators of the conformal algebra $L_n^\dagger = L_{-n}$, all representations characterized by negative central charge are non-unitary, along with those containing fields, which exhibit negative dimension.

The simplest example of a non-unitary CFT is the Yang-Lee model, which has been identified with the minimal model $\mathcal{M}_{2,5}$. It differs little from the unitary minimal CFTs, since only a finite number of conformal families occurs (the theory is rational) and the spectrum of conformal weights is bounded from below. However, the free ghost systems introduced previously are not so innocuous, presenting logarithms in the leading behaviors of correlation functions and a huge (possibly infinite) number of operators with negative dimension. This

last topic is a peculiarity of the bosonic ghost system and has been extensively discussed in [36]. On the other hand, since logarithms are a common feature shared by both models, a brief mention of their origin and of their consequences will be given below.

1.2.1 Logarithmic conformal field theory

Logarithmic singularities in correlation functions originally appeared in relation with WZNW models, namely the $SU(2)$ [63] and the $GL(1,1)$ [64] ones, but the first robust setting of a logarithmic conformal field theory (LCFT) must be traced back to the important paper by Gurarie [33]. Its main result establishes a connection between such singularities and the existence of additional operators in the theory, which, together with the primary ones, form a non-diagonal basis for the operator L_0. In particular, the Hamiltonian operator is not diagonalizable anymore and assumes a Jordan cell-like form with respect with this new basis of fields.

Starting from correlation functions, the conformal blocks $F(z)$ in most cases, including for example all unitary minimal models, admit a Laurent series expansion, regular around $z = 0$. However, there exist special situations in which logarithmic divergencies can be observed near the origin, the most general solution to the hypergeometric differential equations being

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + \ln z \sum_{n=0}^{\infty} b_n z^n \,. \tag{1.9}$$

This is in explicit disagreement with the power law behavior predicted by (1.2).

The simplest example of LCFTs is the $c = -2$ fermionic ghost system, which can be formally identified with the $\mathcal{M}_{1,2}$ minimal model. The four-point function, involving the operator $\mu(z)$ with conformal weight $h_\mu = -1/8$ (alias the $\phi_{1,2}$ operator in the Kac table), reads

$$\langle \mu(z_1) \ldots \mu(z_4) \rangle = (z_{13} z_{24})^{1/4} [z(1-z)]^{1/4} F(z) \,, \tag{1.10}$$

where F(z) solves the linear differential equation

$$z(1-z) \frac{d^2}{dz^2} F(z) + (1-2z) \frac{d}{dz} F(z) - \frac{1}{4} F(z) = 0 \,. \tag{1.11}$$

Substituting $F(z) \sim z^\alpha$ near the origin, the indicial equation yields $\alpha^2 = 0$. Thus, the two independent solutions seem to share the same asymptotic behavior z^0, as $z \to 0$. But a

deeper analysis reveals that one of them actually scales as $F(z) \sim \ln z$. To take into account such logarithmic singularity, a new OPE must be introduced

$$\mu(z)\mu(0) \sim z^{1/4}(I \ln z + \omega + \ldots) \,, \tag{1.12}$$

where I is the identity operator and $\omega(z, \bar{z})$ is a new 'pseudo-operator', with the same dimension as I and unusual properties. Using the formal mode expansion $I(z)|0\rangle = \sum_{n=0}^{\infty} z^n |I, n\rangle$ for the conformal family of the Identity [33], where, for convenience, n indicates all the descendent states and $|0\rangle$ the $sl(2)$-invariant vacuum (along with an analogous expression for the conformal family of ω) and performing a dilatation transformation, it follows

$$L_0|I, n\rangle = n|I, n\rangle \tag{1.13}$$
$$L_0|\omega, n\rangle = |I, n\rangle + n|\omega, n\rangle \,. \tag{1.14}$$

Hence, the identity operator and ω form a basis for the Jordan cell of L_0 and the Hamiltonian is not diagonalizable anymore. Such couple of fields is called *logarithmic pair*.

This situation can be straightforwardly extended to general solutions of the kind (1.9). Every time two operators in the theory give rise to an OPE containing (primary) fields with the same conformal dimension (or, differing by integer numbers), L_0 is not fully reducible. Moreover, fields belonging to the same logarithmic pair, characterized by conformal weight h_C, exhibit the following two-point correlation functions

$$\langle D(z_1)D(z_2)\rangle = \frac{1}{z_{12}^{2h_C}}[\ln z_{12} + \lambda] \,,$$
$$\langle C(z_1)D(z_2)\rangle = \frac{1}{z_{12}^{2h_C}} \,,$$
$$\langle C(z_1)C(z_2)\rangle = 0 \,, \tag{1.15}$$

where, in this notation, C is the ordinary primary field and D plays the role of its logarithmic partner (pseudo-operator).

1.2.2 More about the space of states

Until now, the discussion about the space of states concerning non-unitary theories has been lead along the tracks of standard CFT, introducing Verma modules and negative norm states,

preserving the hermiticity of the modes of the Virasoro algebra. Though the conformal basis appears convenient for the computation of correlation functions, however it can create difficulties in the interpretation of some physical problems, the previously mentioned fractional Hall theory for the edge states of Haldane Rezayi-type being only one example [12,59]. Anyway, an alternative choice of the Hilbert space of states is possible. In the case of free ghosts, a suitably chosen basis, in addition, helps bringing to the light some aspects of the theory that, otherwise, would have remained hidden.

A remarkable result in this direction is due to Guruswamy and Ludwig [37], who could establish a 'canonical mapping' between (i) the $c = -2$ fermionic ghost theory and the $c = 1$ Dirac CFT, and (ii) the $c = -1$ bosonic ghost system and the $c = 2$ complex scalar theory, relying on a Fock-type basis of states. The $c < 0$ theories and their counterparts with positive central charge turn out to share the same space of states (which is now positive-definite) and the spectrum of the respective Hamiltonians. On the other hand, their spaces of fields differ and cannot be simply related in any local way.

The crucial observation made in [37], and which suggests the idea of a possible mapping, realizes the identity of the chiral partition functions, corresponding to the theories grouped in (i) (and analogously for those collected in (ii)), under the *same* boundary conditions along both the euclidean space and time directions. These prescriptions allow to rephrase the theories, originally defined on the infinite plane, on a torus of sizes β and L, along respectively the 'time' and 'space' directions. In a generic sector in the toroidal geometry, associated to arbitrary boundary conditions, the chiral partition functions turn out to coincide with the so-called characters

$$\chi(\beta/L) = Tr \, q^{L_0 - c/24} \,, \qquad q = e^{-2\pi\beta/L} \,. \tag{1.16}$$

Such fundamental objects encode all the information about the spectrum of the (chiral) Hamiltonian operator ($H = \frac{2\pi}{L}(L_0 - c/24)$ on the cylinder of circumference L), including also the space of states.

Free (ghost and ordinary) theories under the most general boundary conditions have been exhaustively analyzed in [37]. Here, only the basic information about the models in (i) and (ii) will be given, along with the expressions of the respective chiral partition functions, in some selected sectors[6], which can be significant for future discussions.

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[6]Antiperiodic (periodic) boundary conditions in the infinite plane correspond to the so-called twisted

(i) Consider a free *Dirac fermion (c = +1)*, on a torus, whose action, in terms of its chiral left- (ψ_L) and right-moving (ψ_R) components, reads

$$S = \int d\tau \int dx \, \{\psi_L^\dagger [\partial_\tau + i\partial_x]\psi_L + \psi_R^\dagger[\partial_\tau - i\partial_x]\psi_R\} \,. \tag{1.17}$$

The corresponding chiral partition function, obtained taking the trace over the Hilbert space of the fermion modes, in the twisted sector yields

$$\chi_{c=1}(q, \mu) = q^{-1/24} \prod_{n=0}^{\infty} \left(1 + e^{2\pi i\mu}q^{n+1/2}\right) \left(1 + e^{-2\pi i\mu}q^{n+1/2}\right) \,, \tag{1.18}$$

where $e^{-2\pi i\mu}$ is a 'fugacity', keeping trace of the U(1) charge of the fermion state with respect to the filled Fermi sea. The lowest conformal weight, relative to the ground state (the so-called Neveu-Schwarz vacuum), corresponds to $h = 0$.

An analogous analysis can be performed on the *symplectic fermion* model with $c = -2$, described by the action

$$S = \frac{1}{4\pi} \int d\tau \int dx \, J_{\alpha\beta} \, \partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta \,, \tag{1.19}$$

where the antisymmetric tensor satisfies $J_{-+} = -J_{+-} = 1$ and $J_{\alpha\gamma}J^{\gamma\beta} = \delta_\alpha^\beta$. The symplectic fermion field, whose components are the zero-dimensional anti-commuting fields Φ^α ($\alpha = \pm$), transforms as a doublet under symplectic transformations. The character in the twisted sector is

$$\chi_{c=-2}(q, \mu) = q^{-(-2/24)}q^{-1/8} \prod_{n=0}^{\infty} \left(1 + e^{2\pi i\mu}q^{n+1/2}\right) \left(1 + e^{-2\pi i\mu}q^{n+1/2}\right) \,, \tag{1.20}$$

where, in this case, the lowest conformal weight reads $h = -1/8$.

Comparing eq.s (1.18) and (1.20), the identity of the chiral partition functions automatically follows.

(ii) The *complex scalar field (c=2)* is described, in terms of the complex components φ and $\varphi^\dagger$, by the second derivative action

$$S = \frac{1}{8\pi} \int d\tau \int dx \, \partial_\mu \varphi^\dagger \partial^\mu \varphi \,. \tag{1.21}$$

(untwisted) sector on the torus.

Periodic boundary conditions in the spatial direction (untwisted sector) yield the character

$$\chi_{c=2}(q, \mu) = q^{-2/24} \frac{1}{(1 + e^{2\pi i \mu})} \prod_{n=1}^{\infty} \frac{1}{(1 + e^{2\pi i \mu} q^n)} \frac{1}{(1 + e^{-2\pi i \mu} q^n)} , \qquad (1.22)$$

where the first factor takes into account the contribution from the bosonic zero mode. The lowest conformal weight describing the ground state, in the so-called Ramond vacuum, is $h = 0$.

Finally, the action corresponding to the *bosonic ghost (c=-1)* theory (also called the $\beta\gamma$ system) can be written as

$$S = \frac{-i}{2} \int d\tau \int dx J^{ab} \beta_a [\partial_\tau + i\partial_x] \beta_b . \qquad (1.23)$$

J^{ab} is the previously mentioned antisymmetric tensor and, for convenience of notation, the two real variables $\beta_1(\tau, x)$, $\beta_2(\tau, x)$ have been used instead of the usual β, γ ones [34], to which they are simply related by the substitutions $\beta_1 \to \gamma$ and $i\beta_2 \to \beta$. They bear conformal weight (and spin) 1/2, in explicit violation of the spin statistic theorem. In this case, the character in the untwisted sector is

$$\chi_{c=-1}(q, \mu) = q^{-(-1/24)} q^{-1/8} \frac{1}{(1 + e^{2\pi i \mu})} \prod_{n=1}^{\infty} \frac{1}{(1 + e^{2\pi i \mu} q^n)} \frac{1}{(1 + e^{-2\pi i \mu} q^n)} , \qquad (1.24)$$

with the lowest conformal weight $h = -1/8$.

Again, the identity of the characters descends, from (1.22) and (1.24).

By construction, the chiral partition functions have been evaluated performing the trace over Fock-type Hilbert spaces of states. Remarkably, very different systems turn out to share exactly the same characters. Since the trace is defined up to similarity transformations on the basis of states, such coincidence suggests a deeper connection between the CFTs contained in (i) (and also in (ii)), at the level of the operators which create the space of states (see eq. (1.16)). In [37] the explicit expression of a mapping relating the modes for $c < 0$ and $c > 0$ theories has been found. However, this holds only for the generator of dilatation transformations L_0 (i.e. the Hamiltonian), the other Virasoro modes L_n being related in non-local ways, hinting at the fact that the spaces of fields, instead, are different.

Moreover, the mapping is crucial in revealing the existence of (otherwise) *hidden non-abelian symmetries* in the $c > 0$ theories, which are simply inherited from the corresponding

$c < 0$ CFTs. Indeed, since their generators are non-locally expressed in terms of the basic fields appearing in the $c > 0$ Lagrangians, such symmetries would not be visible, without the knowledge of the mapping.

This long discussion about the space of states in non-unitary theories, actually, is only part of a more general and old debate concerning the choice of a basis for an arbitrary conformal field theory. It is known, for example, that some strongly interacting problems may look simpler, using the proper basis, which permits to tackle them analytically. Fruitful applications have been found in the context of various Kondo models [65] and in the study of Quantum Hall point contact devices [66]. In this case, the criterion suggesting the choice of the space of states relies on the integrability of the model[7]. The resulting basis, which in general can describe not only CFTs, but especially systems out of the critical point, is of Fock-type and exhibits a completely different structure with respect to the Verma module one, isomorphic to the space of (conformal) fields.

The longstanding (and not yet solved) problem of how to reconcile the basis descending from integrability, out of the critical point, with the Virasoro one, in the conformal limit [67], is even more puzzling in the ghost systems' case. In fact, the occurrence of degenerate (Virasoro) states, associated to operators belonging to the same logarithmic pair (e.g. the identity and ω), is at best confusing, in view of performing a limiting procedure on the off-critical Fock-type basis, in which, by construction, the degeneracy is absent.

Nevertheless, it will be shown, in Part I of this thesis, that perturbations of ghost CFTs, which break conformal invariance, but still remain free, preserve a deep connection with their respective (unitary) massive counterparts.

1.3   Out of the critical point

The main object of this section is the description of a theory away from its critical regime. Before focusing the attention on off-critical ghost systems, which represent our principal interest, we introduce the concept of integrable field theory and the necessary tools to proceed

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[7]Such concept will be at the heart of the next section. Here we only give a flavor, anticipating some delicate observations.

further. An exhaustive review on this subject may be found in [68].

Conformal field theories, due to the assumptions of scale-invariance and locality, provide an efficient description of critical phenomena. In the Renormalization Group (RG) language, they are known to correspond to *fixed points* in the infinite space of local interactions. However, the main object of QFT does not restrict to critical points only, but aims at the understanding of the so-called *scaling region*, in their proximity, which determines the universal behaviors.

A convenient way to describe off-critical theories, alias RG trajectories departing from a fixed point, consists in considering them as perturbations of CFTs, by means of a proper combination of relevant (or asymptotically-free marginal) conformal operators[8] ϕ_k [39, 69]. The resulting action reads

$$S = S_{CFT} + \sum_k \lambda_k \int d^2x \, \phi_k(x) \,, \tag{1.25}$$

where the coupling constants λ_k are characterized by mass dimension $2(1 - h_k)$. The corresponding fixed point governs the ultra-violet (UV) physics, while, at large distances (IR), two different behaviors can be expected. Either the RG trajectories may develop a finite correlation length ξ, associated to the opening of a mass gap $m \sim \xi^{-1}$ and to the occurrence of a massive spectrum of excitations, or they may flow to another critical point, signalled by the divergence of the correlation length. Theories of the first kind are called *massive* field theories and can be uniquely characterized by the so-called S-matrix, ruling the scattering processes among the massive excitations of their spectrum. The second ones, known as *interpolating* flows, still admit a description in terms of the S-matrix, but the corresponding interpretation is much more subtle, due to their gapless nature. Examples of massive ghost theories will be given in Part I of the present thesis, while a particular massless flow will be the object of Part II.

Among all the trajectories flowing away from a critical point, the so-called *integrable* ones, characterized by an infinite set of conserved charges, play a central role [38, 39]. Indeed, the existence of infinitely many (commuting) integrals of motion allows for an exact solution of

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[8] Relevant operators are characterized by scaling dimension $\Delta_k = 2h_k < 2$, and marginal ones by $\Delta_k = 2$.

these theories, non-perturbatively. The occurrence of such structure of conserved currents is common at the fixed point, but in general is destroyed as soon as one departs from criticality.

However, for systems originating from perturbations of CFTs, a sufficient criterion, which goes under the name of *counting argument*, is available in order to establish the integrability of a model. The details, which involve the computation of the conformal characters, can be found in [39]. In such paper, Zamolodchikov could prove that (unitary) minimal models, perturbed by the action of the operators $\phi_{1,3}$, $\phi_{1,2}$ and $\phi_{2,1}$ in the Kac table, always define an integrable deformation. We have extended the same reasoning to the non-unitary minimal model $\mathcal{M}_{1,2}$, identified with the $c = -2$ fermionic ghost system. In this case, the operator $\phi_{1,3}$, which turns out to coincide with the logarithmic partner ω of the identity, yields an integrable free massive theory, which will be extensively discussed in Part I.

Further examples of integrable systems are not necessarily related to deformations of CFTs, many of them being pre-existent to such interpretation of the off-critical RG flows [38]. They correspond to certain well studied and longstanding models of quantum field theory, like for instance those exhibiting $O(N)$ invariance [38]. In general, they are conformally invariant at the classical level (the coupling constants being dimensionless quantities), but such symmetry breaks down in the quantum case, due to renormalization of the couplings. A manifestation of this phenomenon (also known as dynamical mass transmutation) leads to a drastic change in the nature of the quantum vacuum with respect to the classical one, hinting at the fact that classical conservation laws cannot be trivially extended to the quantum case. Fortunately, much work has been done in the past years, and, for the $O(N)$ NLσM, the presence of an infinite set of local conservation laws at the quantum level[9] has been proven by Polyakov [70]. The same is true for the Sine-Gordon model [72] and other two-dimensional quantum field theories [73]. Our interest in such class of QFTs is motivated by the fact that the model studied in Part II, in connection with dense polymers, can be equivalently described in terms of an $O(N)$ NLσM or as an appropriate perturbation of the $c = -2$ CFT.

In the following, we outline the main aspects of integrable flows. First, we introduce the on-shell scattering properties of the excitations, present in their spectrum. In a successive

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[9]Classical conservation laws have been discovered by Pohlmeyer [71].

step, we turn to the analysis of their off-shell features. The literature about these issues is wide and well consolidated. Therefore, here, we adopt the policy of conveying only the basic facts, which turn out to be useful for our future purposes, addressing to the original papers for a thorough discussion. As well, we consider for simplicity *massive integrable* models, postponing to Part II a deeper analysis of the massless case.

1.3.1 Scattering theory

A massive integrable field theory, after continuation to the Minkowski space-time, is equivalent to a relativistic scattering theory, uniquely determined by its S-matrix [38]. The existence of an infinite structure of local conservation laws has two remarkable consequences. The scattering is purely elastic, i.e. no particles' production occurs and the initial and final sets of momenta coincide, and the S-matrix describing an n-particle collision factorizes into $n(n-1)/2$ two-body processes. Hence, the building blocks of such a theory turns out to be the two-particle elastic S-matrices.

Therefore, consider a scattering process involving two incoming, A_a and A_b, and two outgoing, A_c and A_d, particles (fig. 5.23).
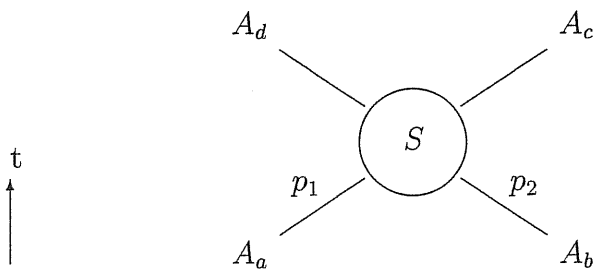

Figure 1.1: Two-particle S-matrix.

As a consequence of Lorentz invariance, the corresponding S-matrix depends on scalar combinations of the particles' momenta, through the so-called Mandelstam variables (s, t and u). Since in (1+1) dimensions and for elastic scattering only one of such variables is independent (e.g. $s = (p_1 + p_2)^2$), it is convenient to introduce a parameterization in terms of the rapidity variable θ. The dispersion relation of a massive excitation, e.g. A_a, reads

$$e_a \equiv p_1^0 = m_a \cosh \theta_1 \,, \qquad p_a \equiv p_1^1 = m_a \sinh \theta_1 \,, \qquad (1.26)$$

where e_a and p_a are its energy and momentum, and m_a indicates the mass. In this new parameterization, the S-matrix, describing the process in fig. 5.23, depends only on the difference between the rapidities, $\theta_{12} = \theta_1 - \theta_2$, according to

$$|A_a\left(\theta_1\right) A_b\left(\theta_2\right)\rangle_{in} = S_{ab}^{cd}\left(\theta_{12}\right) |A_d\left(\theta_2\right) A_c\left(\theta_1\right)\rangle_{out}, \tag{1.27}$$

with *in* and *out* labels identifying, respectively, the in- and out-states, associated to the incoming and outgoing particles. The convention $\theta_1 \geq \theta_2$ is assumed, in order to characterize in-states by decreasing values of their rapidities (and vice-versa for the out-states).

The unitarity condition for the two-particle amplitude reads

$$\sum_{c,d} S_{ab}^{cd}\left(\theta\right) S_{dc}^{ef}\left(-\theta\right) = \delta_a^f \delta_b^e, \tag{1.28}$$

while the analytic continuation of the scattering process from the s-channel to the t-channel, corresponding to the substitution $\theta \to i\pi - \theta$ in the rapidity variable, gives the crossing relation

$$S_{ab}^{cd}\left(\theta\right) = S_{a\bar{d}}^{c\bar{b}}\left(i\pi - \theta\right), \tag{1.29}$$

where the bar symbol indicates charge conjugation. Finally, the aforementioned factorization property translates into a set of constraints, known as *Yang Baxter* or *star-triangle* equations

$$\sum_{d,e,f} [S_{ab}^{de}\left(\theta_{12}\right)][S_{dc}^{a'f}\left(\theta_{13}\right)][S_{ef}^{b'c'}\left(\theta_{23}\right)] = \sum_{d,e,f} [S_{bc}^{de}\left(\theta_{23}\right)][S_{ae}^{fc'}\left(\theta_{13}\right)][S_{fd}^{a'b'}\left(\theta_{12}\right)], \tag{1.30}$$

which can be easily visualized in the following diagram

Hence, eq.s (1.28-1.30) allow to determine the form of a given S-matrix, up to the so-called CDD ambiguities, alias multiplicative factors which, by themselves, satisfy the same equations.

The analytic structure of the scattering matrix encodes all the information about the particle content of the theory. In the complex s-plane, the elastic two-particle amplitude is an analytic function, characterized by two square-root branch cuts, on the real axis, in correspondence of the two-particle thresholds. Bound states are associated to poles on the real axis. Turning to the θ-plane[10], the S-matrix becomes a meromorphic, $2\pi i$-periodic,

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[10]Upon the same mapping, the physical sheet in the variable s transforms into the the strip $0 \leq \Im m\theta \leq \pi$.
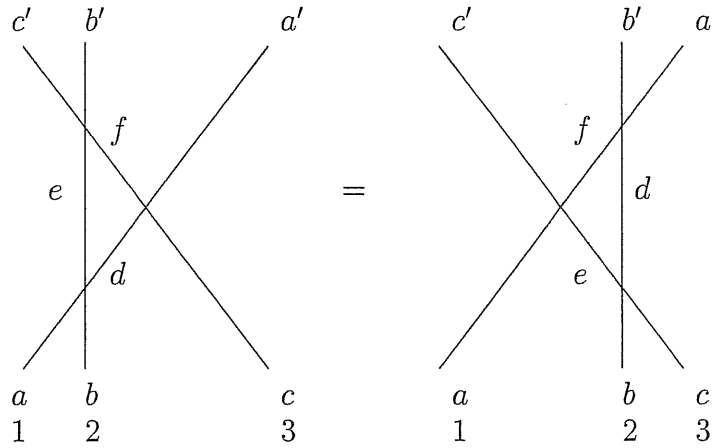
Figure 1.2: Yang-Baxter equations.

function, real at $\Re e\, \theta = 0$, whose stable bound state poles lie on the imaginary axis. In general, poles in the s-channel (or direct channel) correspond to positive residues. The occurrence of negative (or imaginary) values for such quantities has been interpreted as a manifestation of the non-unitary nature of the underlying CFT, whose deformation describes the massive flow. For an intermediate bound state A_e, appearing in the s-channel of the scattering process depicted in fig. 5.23 , at $\theta = iu_{ab}^e$, it follows

$$S_{ab}^{cd}(\theta) \sim \frac{i\, R_{ab}^e}{(\theta - iu_{ab}^e)}\,,\tag{1.31}$$

where R_{ab}^e indicates the residue and the formula is meant in the vicinity of the singularity. (R_{ab}^e is related to the on mass-shell coupling constants of the underlying quantum field theory by $R_{ab}^e = f_{ab}^e f_{cd}^e$.)
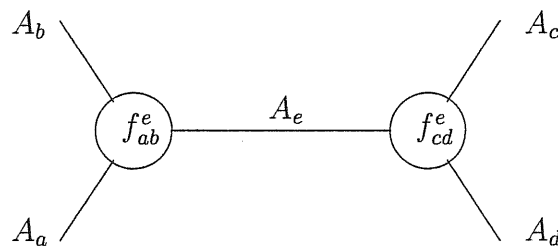
Figure 1.3: First-order pole

In order to capture thoroughly the dynamics of the theory, the key idea is to rely on the

'bootstrap' approach. It consists in dealing with bound states as if they were asymptotic particles, yielding a set of equations for the two-particle amplitudes. Consistently with the conservation laws and the symmetry properties of the model, the spectrum of excitations, including the knowledge of the particles' masses and of their fusion rules, can be completely reconstructed. A prominent example of this procedure is represented by the Ising model in a magnetic field, whose mass spectrum has been conjectured to be composed of eight particles [39].

Exact S-matrices characterize the on-shell properties of integrable field theories. However, the most intriguing pursuit aims at extending the investigation to their off-shell aspects. Exploiting the knowledge of the scattering amplitudes, it is possible both to gain insight into the finite size effects of such theories, by means of the thermodynamic Bethe ansatz technique, and to compute exactly off-critical correlation functions, thanks to the form-factors approach. In the following we will give a flavor of these two methods.

1.3.2   Thermodynamic Bethe ansatz

Thermodynamic Bethe ansatz (TBA) generalizes to purely elastic, factorized, scattering theories an old idea, according to which, the infinite volume thermodynamics of a massive QFT[11] is completely encoded in its S-matrix [74]. For a system characterized by n different species of particles A_a ($a = 1, \ldots n$), living on an infinite line at temperature T, it allows to extract the free energy per unit length, $f(T)$, in terms of the corresponding one-particle excitation energies $\epsilon_a(\theta)$. These 'pseudoenergies', related to the rapidity distributions of the particles, crucially satisfy a set of coupled non-linear integral equations, the so-called TBA equations, which originally appeared in a non-relativistic scattering problem [75].

The main breakthrough, associated to the relativistic version of the TBA technique, consists in the possibility of relating thermodynamic properties of the QFT under consideration, to finite size effects on the *same* system. This is a natural consequence of the very structure of the Euclidean spacetime. Indeed, consider such theory defined on a torus of periodicity (L,R). There are two equivalent ways of formulating it: either choosing the R-direction as space and the L-direction as euclidean time or the other way round. Therefore, the partition

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[11]Such massive QFT can be thought of as an infinite-volume gas of particles, in its low density phase. In one-dimensional cases, this usually represents the only phase.

function can be written both as

$$Z(R, L) = Tr \, e^{-LH_R} \,,\tag{1.32}$$

and

$$Z(R, L) = Tr \, e^{-RH_L} \,,\tag{1.33}$$

where $H_{L(R)}$ denotes the Hamiltonian of the theory on a (periodic) space of length $L(R)$. (This property is known as modular invariance and puts stringent constraints on the operator content of the theory.) In the limit $L \to \infty$, only the lowest-energy state contributes to the partition function. Therefore, it is possible to extract the *ground state* energy (alias the lowest eigenvalue of H_R) from (1.32)

$$E_0(R) = \lim_{L \to \infty} \frac{1}{L} \ln \left[e^{-LH_R} \right] \,.\tag{1.34}$$

On the other hand, since eq (1.33) describes a system periodic along the time direction R (i.e. at finite temperature $T \equiv 1/R$), the limit $L \to \infty$ coincides with the thermodynamic limit and the free energy per unit length reads $f(R) = -\frac{1}{LR} \ln Z$. Hence, the remarkable relation

$$E_0(R) = R f(R)\tag{1.35}$$

follows.

Thus, the Casimir energy $E_0(R)$ can be determined, requiring as unique input the exact S-matrix [40]. Though its evaluation, as a function of the volume of the space, can be achieved only numerically, nevertheless insights into its ultra-violet and infra-red asymptotics can be obtained analytically. After dimensional considerations, the ground state energy can be expressed in terms of a scaling function c_{eff}

$$E_0(R) = -\frac{\pi \, c_{eff}(r)}{6R} \,,\tag{1.36}$$

where the dimensionless parameter $r \equiv R/\xi$ yields a comparison between the two length scales into the game, alias the circumference of the cylinder R and the correlation length ξ. In a massive field theory, the low energy limit of c_{eff} vanishes, while its short distance behavior is ruled by the corresponding UV critical point. In the case of interpolating massless flows, a different asymptotics is expected for the IR regime, whose physics is governed by

another (IR) fixed point. However, in correspondence of a CFT, the scaling function assumes the form $c_{eff} = c - 12\Delta_0$, where c is the central charge and Δ_0 the lowest scaling dimension present in the theory. It is important to emphasize that the agreement between the TBA results and the CFT data may provide, a posteriori, a strong evidence that the conjectured S-matrix is correct and that the underlying perturbed CFT represents, indeed, a consistent QFT.

We now sketch the derivation of the TBA equations, which allow to recover all the thermodynamic quantities. A thorough treatment can be found in [40, 76].

Consider, for simplicity, a diagonal factorized scattering theory, where n different species of particles[12] A_a ($a = 1, \ldots n$) live on a (periodic) space of length L. The total number of particles is N, N_a of which are of species a. (L and N are supposed to be large, the TBA system being exact in the thermodynamic limit). When two particles are exchanged, the wavefunction, describing the system, turns out to be multiplied by the S-matrix element, governing their collision. Imposing convenient periodicity conditions at the boundary (for a detailed discussion, see ref. [76]), the following quantization relation holds

$$e^{ip_iL} \prod_{j\neq i} S_{ij}(\theta_i - \theta_j) = 1 \,, \tag{1.37}$$

with $p_i = m_i \sinh \theta_i$. Taking the logarithm of such expression, a set of coupled equations follow, which go by the name of *Bethe ansatz equations*

$$Lm_i \sinh \theta_i + \sum_{j\neq i} \delta_{ij}(\theta_i - \theta_j) = 2\pi i \,, \tag{1.38}$$

where delta denotes the phase shift, $\delta_{ij}(\theta_i - \theta_j) = -i \ln S_{ij}(\theta_i - \theta_j)$, and $i = 1, \ldots, N$. In the thermodynamic limit, eq. (1.38), assumes the form

$$\rho_a(\theta) = \frac{m_a}{2\pi} \cosh \theta + \frac{1}{2\pi} \sum_{b=1}^{n} (\varphi_{ab} * \rho_b^{(r)})(\theta) \,, \tag{1.39}$$

in terms of the rapidity density $\rho_a^{(r)}$ (i.e. the number of particles of species a with rapidities between θ and $\theta + \Delta\theta$, divided by $L\Delta\theta$) and the density of levels $\rho_a(\theta)$, associated to the

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[12]In this thesis, we deal with particles which are supposed to satisfy an exclusion principle. Such kind of excitations are called of fermionic 'type'. A detailed analysis about the concept of 'type', including its relationship with the scattering matrix, can be found in ref.s [40], [76]

species a. The $*$ symbol denotes the convolution and $\varphi_{ab}(\theta) = -i\frac{d}{t\theta}\ln S_{ab}(\theta)$ is a matrix kernel.

In order to derive the TBA equations, we exploit thermodynamics. First we write the free energy for such system at temperature T, and then we implement a minimization procedure. The entropy per unit length can be easily written in terms of the densities

$$s[\rho, \rho^{(r)}] = \sum_{a=1}^{n} \int d\theta \left[\rho_a \ln \rho_a - \rho_a^{(r)} \ln \rho_a^{(r)} - \left(\rho_a - \rho_a^{(r)} \right) \ln \left(\rho_a - \rho_a^{(r)} \right) \right] , \qquad (1.40)$$

while, the energy per unit length reads

$$h[\rho^{(r)}] = \sum_{a=1}^{n} \int d\theta \, \rho_a^{(r)} m_a \cosh \theta . \qquad (1.41)$$

Therefore, taking into account the definition of the pseudoenergy $\epsilon_a(\theta)$

$$\frac{\rho_a^{(r)}}{\rho_a} = \frac{1}{e^{\epsilon_a(\theta)} + 1} , \qquad (1.42)$$

the extremum condition for the free energy $f[\rho] = h[\rho^{(r)}] - Ts[\rho, \rho^{(r)}]$, gives the following TBA equations (at zero chemical potential[13])

$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi} \sum_{b=1}^{n} (\varphi_{ab} * L_b)(\theta) , \qquad (1.43)$$

where $L_a(\theta) \equiv \ln(1 + e^{-\epsilon_a(\theta)})$. The functions $\nu_a(\theta)$ are called 'energy terms' and turn out to be proportional to the masses of the corresponding particles, e.g. $\nu_a(\theta) = m_a R \cosh(\theta)$ in the massive case. (Generalizations to massless flows will be discussed in Part II.) The extremal free energy per unit length assumes the form

$$f(R) = -\frac{1}{2\pi R} \sum_a \int d\theta \, m_a \cosh \theta L_a(\theta) , \qquad (1.44)$$

thus yielding the Casimir energy

$$E_0(R) = -\frac{1}{2\pi R} \sum_a \int d\theta \, \nu_a(\theta) L_a(\theta) . \qquad (1.45)$$

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[13]For a complete discussion taking into account the effects of a non-vanishing chemical potential, see [76,77].

Turning to the case of non-diagonal scattering theories (like the one we are going consider in Part II), the so-called higher level Bethe ansatz technique [78] must be applied. Eq.s (1.43) still hold, but, due to the introduction of new pseudoparticles (or 'magnons') and of their bound states, which both carry no energy and momentum, some of the energy terms vanish. In a similar way, the Casimir energy is not affected by such unphysical particles. Therefore, the most delicate point concerns the identification of the spectrum of magnons and of their bound states, in the thermodynamic limit, which in principle requires a separate analysis for each single scattering model. Fortunately, in the most studied examples of non-diagonal theories [79–81], including the massless case in Part II, the corresponding TBA systems exhibit a rather universal structure, which allows to evaluate straightforwardly the asymptotic behaviors. Indeed, the matrix kernel can be recast in the simple form $\varphi_{ab}(\theta) = \varphi(\theta)l_{ab}$, with φ some universal function, and the set of equations (1.43) results entirely specified in terms of a diagram (actually the Dynkin diagram of some simply laced Lie algebra, characterized by the incidence matrix l_{ab}) and of the energy terms attached to its nodes. Explicit calculations will be shown in Part II.

1.3.3 Correlation functions

Another important problem, which takes advantage of the knowledge of the scattering amplitude, concerns the evaluation of the off-critical correlation functions of local fields. Among the possible techniques designed for this purpose, we investigate here the method involving the so-called form factors (FF), alias matrix elements of local operators between asymptotic states, which can be determined by solving a self-consistent set of functional equations. Such approach is based on the possibility of decomposing correlation functions into an infinite sum, over asymptotic multi-particle states, whose single contributions are just given in terms of the corresponding n-particle FF. This description appears very convenient for manifold reasons: it involves directly physical quantities, with no need of renormalization and takes into account the coupling constant dependence at all orders; though it can be considered as a large distance expansion for the correlators, it encodes thoroughly the UV CFT data and, finally, it exhibits a very fast convergence rate (at least for massive flows) due to the peculiar character of the two-dimensional phase-space, even at small distances.

However, despite such spectral series can be written down explicitly for any integrable

quantum field theory, the free case is up to now the only one in which all the infinite terms can be exactly resummed, yielding closed expressions for the correlators. Prominent examples are represented by theories whose spectra are composed of neutral fermionic particles (alias the Ising model away from its critical temperature) [82], and charged fermionic [83] and bosonic [84] excitations.

It is worthwhile emphasizing that, at first sight, the free character of the massive excitations might disguise the real complexity of the underlying theories. However, highly non trivial interacting behaviors manifest every time that correlation functions of operators, non-local with respect to the particles in the spectrum, are considered. Proceeding along this line, in Part I of this thesis, we study in detail the role played by *non-locality* in, deceptively simple, free ghost theories.

In the following, we shortly sketch the main features concerning the space of physical states, essential for the foundation of the FF approach.

Asymptotic states

Physical asymptotic states can be efficiently described, relying on the so-called Zamolodchikov-Faddev operators [43], which generate the following non-commutative, associative algebra

$$Z_a(\theta_1)Z_b(\theta_2) = S_{ab}^{cd}(\theta_{12})Z_d(\theta_2)Z_c(\theta_1)$$

$$Z_a^\dagger(\theta_1)Z_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_{12})Z_d^\dagger(\theta_2)Z_c^\dagger(\theta_1)$$

$$Z_a(\theta_1)Z_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_{12})Z_d^\dagger(\theta_2)Z_c(\theta_1) + 2\pi\delta_{ab}\,\delta(\theta_1 - \theta_2)\,, \qquad (1.46)$$

where the scattering amplitude plays the role of braiding operator.

Define the physical vacuum $|0\rangle$ as the state annihilated by the operators $Z_a(\theta)$, i.e. $Z_a(\theta)|0\rangle = 0$. Therefore, the excitations, entering the spectrum of the scattering theory, are created by the action of the operators $Z_b^\dagger(\theta)$ on the same vacuum and the space of states is composed by the vectors

$$|Z_{a_1}(\theta_1)\ldots Z_{a_n}(\theta_n)\rangle \equiv Z_{a_1}^\dagger(\theta_1)\ldots Z_{a_n}^\dagger(\theta_n)|0\rangle\,, \qquad (1.47)$$

normalized according to $\langle Z_a(\theta_1)|Z_b(\theta_2)\rangle = 2\pi\delta_{ab}\,\delta(\theta_1 - \theta_2)$.

Asymptotic (*in-*) *out*-states correspond to (decreasing) increasing order of the rapidities in (1.47), thus forming a complete (Fock-type) basis for the Hilbert space of states.

Form factors

Consider the simplest theory as possible, characterized by a single self-conjugate excitation A of mass m. Correlation functions of local operators (e.g. $\mathcal{O}_1$ and $\mathcal{O}_2$) can be written in terms of the spectral sum

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \sum_n \frac{1}{n!}\int \frac{d\theta_1}{2\pi}\cdots\frac{d\theta_n}{2\pi}\, F_n^{\mathcal{O}_1}(\theta_1,\ldots,\theta_n)\left[F_n^{\mathcal{O}_2}(\theta_1,\ldots,\theta_n)\right]^* e^{ip_n^\mu x_\mu} \qquad (1.48)$$

where p_n^μ is the total energy-momentum of the n-particle intermediate state and the functions

$$F_n^{\mathcal{O}}(\theta_1,\ldots\theta_n) \equiv \langle 0|\mathcal{O}|A(\theta_1)\ldots A(\theta_n)\rangle_{in} \qquad (1.49)$$

denote the corresponding n-particle form factors.

Lorentz invariance fixes F_n to be a function depending only on the differences $\theta_{ij} = \theta_i - \theta_j$ and, if $\mathcal{O}$ exhibits spin s, a simultaneous shift Λ in the rapidity variables, results in an overall multiplicative factor $e^{s\Lambda}$. Crossing symmetry, instead, allows to express the most general n-particle matrix elements, in terms of F_n, as follows

$$_{out}\langle A(\theta_1)\ldots A(\theta_m)|\mathcal{O}|A(\theta_{m+1})\ldots A(\theta_n)\rangle_{in} = F_n^{\mathcal{O}}(\theta_1 + i\pi,\ldots,\theta_m + i\pi, \theta_{m+1},\ldots,\theta_n)\,. \qquad (1.50)$$

The monodromy properties of the FF are encoded into a set of functional equations, known as Watson's equations,

$$F_n(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots\theta_n) = F_n(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots\theta_n)S(\theta_i - \theta_{i+1}) \qquad (1.51)$$

$$F_n(\theta_1 + 2\pi i,\ldots,\theta_{n-1},\theta_n) = F_n(\theta_2,\ldots,\theta_n,\theta_1) = \prod_{i=2}^{n} S(\theta_i - \theta_1)F_n(\theta_1,\ldots,\theta_n)\,.$$

The first relation identifies the exchange between two rapidities with the scattering process (1.46). The second describes the behavior of F_n under the analytic continuation $\theta_1 \to \theta_1 + 2\pi i$.

The general solution to the above equations reads [41]

$$F_n(\theta_1,\ldots,\theta_n) = K_n(\theta_1,\ldots,\theta_n)\prod_{i<j} F_{min}(\theta_{ij})\,, \qquad (1.52)$$

where the *minimal* form factor $F_{min}(\theta)$ satisfies (1.51) with $n = 2$. It is an analytic function in the strip $0 \leq \Im m\theta \leq \pi$, free of zeroes and converging to a constant value for large θ, whose form is uniquely fixed by all these requirements, up to a normalization constant. As concern

K_n, it fulfills the Watson's equations with $S_2 = 1$, and therefore it is a completely symmetric $2\pi i$-periodic function[14] of the θ_i. Moreover, all the physical poles' structure, inherent to the FF under consideration, is encoded into K_n.

Poles appear every time that a cluster of m particles goes on a mass-shell. Due to factorization, two kinds of singularity, leading to two different recursive relations for F_n, are important. The so-called 'kinematical' ones, alias one-particle poles in any three-particle channel, occurring at $\theta_{ij} = i\pi$, yield a relation between F_n and F_{n+2} [42]

$$\text{Res}_{\tilde{\theta}=0} F_{n+2}(\tilde{\theta} + i\pi, \theta, \theta_1, \ldots, \theta_n) = i \left(1 - \prod_{i=1}^{n} S(\theta - \theta_i) \right) F_n(\theta_1, \ldots, \theta_n). \tag{1.53}$$

The other kind of singularity, associated to one-particle poles in the two-particle channels, establishes a connection between F_n and F_{n+1}. They appears only when bound states are present in the theory. Since, this is not the case for the models we are going to study, we refer to the original papers for the corresponding residue formulas [42, 43].

At this stage, no information about the specific nature of the operator $\mathcal{O}$ has been conveyed. Therefore, the results obtained from the form factor bootstrap equations pertain to the whole operator content of the theory. Nevertheless, a classification of the FF solutions, particularly of those competing to the scaling operators, is highly desirable.

The anomalous dimensions Δ_Φ of the scaling operators Φ, provide an upper bound to the asymptotic behavior of the corresponding FF,

$$\lim_{|\theta_i| \to \infty} F_n^\Phi(\theta_1, \ldots, \theta_n) \leq const \, e^{\frac{\Delta_\Phi}{2} |\theta_i|}. \tag{1.54}$$

This condition works efficiently in models displaying some internal symmetry. On the other hand, more stringent constraints (the so-called 'cluster' equations) must be considered for generic systems, with no peculiar symmetry

$$\lim_{\alpha \to \infty} F_{r+l}^\Phi(\theta_1 + \alpha, \ldots, \theta_r + \alpha, \theta_{r+1}, \ldots, \theta_{r+l}) = \frac{1}{\langle \Phi \rangle} F_r^\Phi(\theta_1, \ldots, \theta_r) F_l^\Phi(\theta_{r+1}, \ldots, \theta_{r+l}). \tag{1.55}$$

Once the FF solutions corresponding to scaling operators have been identified, it is extremely important to find a criterion to associate each of them to a specific field. The

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[14]It should be noted, however, that, as a function of the rapidities, K_n contains a certain degree of arbitrariness.

evaluation of the scaling dimensions, starting from the FF expressions, is the key tool. A remarkable 'sum' rule has been derived for this purpose by Delfino, Simonetti and Cardy [85]

$$\Delta_\Phi^{UV} - \Delta_\Phi^{IR} = \frac{-1}{2\pi\langle\Phi\rangle} \int d^2x \langle\Theta(x)\Phi(0)\rangle_{connected}, \tag{1.56}$$

for scaling operators which do not mix under the renormalization group.

We conclude this analysis quoting another significant 'sum' rule, which has been proposed long time ago [69] (in the following integral form, explicitly by Cardy)

$$c_{UV} - c_{IR} = \frac{3}{4\pi} \int d^2x \, |x|^2 \, \langle\Theta(x)\Theta(0)\rangle_{connected}, \tag{1.57}$$

and which allows to recover conformal data (the variation of the central charge along the flow), starting from the knowledge of the off-critical theory. We would like to stress that the trace of the stress-energy tensor $\Theta(x)$ turns out to play a crucial role in both the sum rules.

1.3.4 Off-critical ghosts

After this long detour into the non-perturbative methods, concerning the integrable approach, we are able to tackle the topic at the heart of this thesis, i.e. ghost systems out of the critical point. The general flavor of such issue has been essentially conveyed in the Introduction, where, nevertheless, we did not enter any detail. However, before addressing the peculiar problems which aroused our attention (Part I and II), it is worth providing a more accurate (and, sometimes, technical) view on the kind of models we are going to consider.

As already mentioned in paragraph (1.1), due to the studies lead in the context of disordered systems, in the past few years, it has become clear that critical ghost theories are related to RG fixed points of certain field theoretic models (e.g. NLσ models, Gross Neveu models, Principal Chiral models...), displaying superalgebras as internal symmetries. Despite the huge interest about this topic, only two years ago an exhaustive classification of the possible critical behaviors has been achieved [25], and a complete analysis of the off-critical regimes is still lacking, demanding for further work.

In principle, the QFTs under investigation, being a generalization of their respective ordinary counterparts, present the same integrability properties as the latter. Therefore,

a priori, one should expect to extend straightforwardly the non-perturbative techniques (S-matrix, TBA and FF) to the supersymmetric cases, along the lines traced in the different, but strictly related, context of the $O(N \to 0)$ Gross Neveu model, some years ago [86]. However, a general lack of unitarity complicates tremendously the situation, both at the technical level and from the conceptual point of view, yielding many difficulties in the interpretation of the associated physical phenomena [26]. For instance, the first attempt to characterize a supersymmetric S-matrix (namely, an $OSP(2|2)$-invariant scattering amplitude) failed to provide any justification in support of its conjecture [87].

Important advances in this field have been achieved only very recently [26, 27, 29]. References [26, 29], which belong to the same series of papers, deal with various field theoretic models, displaying $OSP(m|2n)$ symmetry (m bosonic components and $2n$ fermionic ones). In [26], general expressions for the exact S-matrices are proposed with the corresponding TBA calculations, for many massive flows. Despite the non-unitary character of the conjectured amplitudes, still, formal thermodynamical calculations do make sense. The second paper [29] is devoted to the study of the special case $OSP(1|2)$. (In Part II of this thesis, we will concentrate on a peculiar system, exhibiting exactly this symmetry. Such model is particularly interesting because, though the simplest representative of a hierarchy of non-linear sigma models with $OSP(m|2n)$ symmetry, still it presents new non-trivial features with respect to the ordinary cases. Moreover, it admits a simple physical realization, in terms of lattice models, which sheds new light on the low-temperature phase of dense loops.) As concern reference [27], instead, it provides an algebraic formulation of exact scattering amplitudes for many two-dimensional theories, ranging from Potts models, restricted solid on solid (RSOS) models to NLσ models, invariant under the Lie superalgebras $sl(m+m|n)$ (technically, they are called $CP^{m+n-1|n}$ models and may be supplemented by the action of a topological θ-term). This last class of QFTs is rather general and can describe, as well, the $O(3)$ NLσM ($CP^{1|0}$), percolation-related problems and the spin quantum Hall transition ($CP^{1|1}$). Boundary amplitudes are also considered, in this algebraic formalism, since they turn out be very useful in the description of the Hall edge states.

The strong interest in such kind of QFTs is also motivated by the recent achievements in another branch of physics, which has developed almost in parallel and is devoted to the study of integrable lattice models and quantum spin chains, based on superalgebras. A re-

markable result in this field [46] establishes a connection between *intersecting* loop models, in a two-dimensional square lattice, and solvable super-spin chains with $OSP(m|2n)$ symmetry. Moreover, for the peculiar systems characterized by $OSP(1|2n)$ invariance, it has been possible to predict critical behaviors, ruled by $c = -2n$ CFTs. Analogously, other lattice models have attracted much attention, including the one describing bond percolation in two dimensions [88], and generalizations thereof [25,27], for their possible implications on the spin quantum Hall effect. The associated supersymmetric vertex models exhibit respectively $sl(2|1)$ and $sl(n + m|n)$ invariance. Finally, very recently [89], a peculiar representative of a further class of solvable lattice systems has been identified with the $su(2|1)$-symmetric tJ model, in the ferromagnetic regime.

A delicate point, which constitutes a bridge between integrable lattice models and supersymmetric QFTs, obviously concerns the continuum limit of the related quantum spin chains. Particularly, it is of fundamental importance to establish a relation between the critical theories derived in the framework of lattice models (e.g. by means of exact Bethe ansatz solutions) and those obtained, by integrability or renormalization group considerations, in the associated continuum field theories. In the case of ordinary algebras, such conformal field theories are related, in both contexts, to WZNW models on the corresponding groups. As concern the supersymmetric examples, the situation is less transparent and certainly richer. For instance, in the case of $OSP(1|2)$ symmetry (for the time being, a detailed discussion is available solely for the series $OSP(1|2n)$), only very recently [29], it has been argued that, starting from the corresponding spin chain, two possible results follows. For integer valued spins, the continuum limit is described by a supersymmetric generalization (actually the first) of the ordinary WZNW models [29]. For half-integer spins, instead, the continuum low-energy limit is not of WZNW type, but corresponds to the weak coupling regime of the supersphere sigma model $S^{(0|2)}$, defined on the target space $OSP(1|2)/SP(2)$ [28]. Such QFT, in the IR, renormalizes just to free symplectic fermions, yielding $c = -2$, consistently with the exact Bethe ansatz outcome, previously obtained in [46]. However, despite this agreement at the level of the critical theories, the interpretation of the results differs completely from references [28,29] to [46]. Indeed, in the former case, the underlying field theory is the aforementioned nonlinear sigma model $S^{(0|2)}$, while, in the latter, it is argued to be a WZNW-type theory.

In such general framework, we propose two distinct lines of research, which, nevertheless, share the common idea of focusing on off-critical correlation functions, choosing as chief tool the form factor approach. The motivation is twofold. Correlators are the most suited instruments to explore the scaling region, allowing not only to extract critical exponents (which are characteristic of the critical point), but also to derive the so-called *universal amplitude ratios* (generalized susceptibilities...). In second place, since the off-critical regime of such supersymmetric models is an extremely young field of investigation, the FF approach has never been considered before.

Therefore, in Part I we consider the simplest examples of massive ghost theories, where the fields are still *free*, while, in Part II, we deal with the nonlinear sigma model, with $OSP(1|2)$ symmetry.

Part I

Massive perturbation

Chapter 2

Free ghost theories: the bulk case

We consider *free massive* bosonic and fermionic ghost systems and concentrate on their non-trivial sectors containing the disorder operators, non-local with respect to the excitations constituting the mass spectrum. A unified analysis of the correlation functions of such operators can be performed for ghosts and ordinary complex bosons and fermions [44].

2.1 'Disorder' operators

'Disorder' operators first appeared in Kadanoff and Ceva [90] in the early seventies, in the study of the two-dimensional Ising model. Their introduction was motivated by the occurrence of an exact symmetry (the Kramers-Wannier duality [91]) relating the high- and low-temperature phases of the model: the role of the spin operator σ as order parameter in the broken phase was played by the disorder operator μ in the unbroken one. In other words, as σ was meant to describe the order in the lattice, μ was responsible for the disorder.

This initial idea turned out to be very fruitful, beyond statistical mechanics. Few years later it was translated into the QFT language by 't Hooft [92], with the purpose to explain the quark confinement in non-abelian gauge theories. Exploiting the competition between ordering and disordering effects, naturally arising in the theory of phase transitions, he focused the attention on disorder fields, as operators interpolating between different vacua in the broken phase. In this respect, non-locality was found to play a central role. It is worth recalling that two operators $A(x)$ and $B(y)$ are said to be mutually non-local with

non-locality phase $e^{2i\pi\alpha}$ if correlation functions, containing both of them, pick up such a phase when they are taken around each others in the Euclidean plane. Such correlators in two-dimensional free massive theories were extensively studied from the point of view of the deformation theory of differential equations in reference [93] and a series of related papers. Several works, which include [41,42,82,83,94], have been devoted afterwards to deal with this problem through more direct and general methods of quantum field theory. For instance, Marino et al. [94] introduced a euclidean functional integral approach for the construction of operators interpolating topological excitations and the corresponding disorder operators, in order to generalize the ones previously defined by Kadanoff and Ceva. In particular, exact matrix elements were derived.

However, among the different methods, the form factor approach, previously exposed in Section 1.3.3, proved to be not only a powerful tool, but also simpler. In this framework, correlators are expressed as sums over multi-particle asymptotic states. While spectral series can be written down explicitly for any integrable quantum field theory, the free case is up to now the only one in which they can be exactly resummed. The use of this technique to express the correlators through solutions of non-linear differential equations was illustrated in [82,83], for the neutral and charged fermionic cases, respectively.

In this chapter we adopt such approach, in order to deal with the free massive theories describing bosonic and fermionic ghost systems, along with the corresponding ordinary counterparts (complex bosons and fermions).

2.2 Free massive theories: ordinary *vs* ghost systems

In the massless limit, the fermionic (anticommuting scalars) and bosonic (commuting spinors) ghost systems bear central charges $c = -2$ and $c = -1$, respectively [34], and differ only for the sign from the central charges of their counterparts with the 'right' statistics, respectively the commuting complex scalar field and the anticommuting complex spinor field. Detailed studies of the $c = -2$ and $c = -1$ ghost conformal field theories can be found in [35] and [36], respectively. A comparison between the ghost systems and their counterparts with positive central charge has been performed in [37].

It is noteworthy that ghost systems and ordinary bosons and fermions are intimately related

also in the free massive case. Actually, we point out that the form factor approach allows for a simple unified treatment of the four cases in terms of the two parameters

$$S = \begin{cases} 1 & \text{for bosons} \\ -1 & \text{for fermions} \end{cases} \tag{2.1}$$

$$\varepsilon = \begin{cases} 1 & \text{for ordinary fields} \\ -1 & \text{for ghosts.} \end{cases} \tag{2.2}$$

In all cases the mass spectrum consists of a doublet of free particles A and $\bar{A}$ with mass m.

The Lagrangians of the free theories we are considering contain a kinetic and a mass term, each of them linear in the fields which interpolate the particles A and $\bar{A}$. (At the moment, we need only this information. However, in Section 2.5.1 we will provide further details, concerning the fermionic ghost case, since they will be useful for future developments.) In the case of ordinary spin-statistics, hermitian conjugation interchanges these two fields leaving the Lagrangian invariant. In the ghost case the operation made in the same way would change the sign of the Lagrangian because the terms are reordered with the 'wrong' statistics. Hence, a real Lagrangian requires that the two ghost fields are not exactly the hermitian conjugate of each other. A suitable choice of the conjugation matrix for all cases is given by

$$C = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}. \tag{2.3}$$

2.3   Correlation functions

In this section, we compute explicitly the two-point correlation function

$$G_{\alpha,\alpha'}^{(S,\varepsilon)}(t) \equiv \langle \tilde{\Phi}_\alpha(x) \tilde{\Phi}_{\alpha'}(0) \rangle, \tag{2.4}$$

where $\Phi_\alpha(x)$ denotes[1] the disorder operator, exhibiting a non-locality phase $e^{2i\pi\alpha}$ $(e^{-2i\pi\alpha})$ with respect to (the field which interpolates) the particle A ($\bar{A}$).

We work within the form factor approach in which the correlation functions are expressed as spectral series over intermediate multiparticle states after the computation of the form

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[1]We will use the notation $\tilde{\Phi}(x) \equiv \Phi(x)/\langle\Phi\rangle$ throughout Part I.

factors

$$f_n^\alpha(\theta_1,\ldots,\theta_n,\beta_1,\ldots,\beta_n) = \langle 0|\tilde{\Phi}_\alpha(0)|A(\theta_1)\ldots A(\theta_n)\bar{A}(\beta_1)\ldots\bar{A}(\beta_n)\rangle. \tag{2.5}$$

Here rapidity variables are used to parameterize the energy-momentum of a particle as $(e,p) = (m\cosh\theta, m\sinh\theta)$. The form factors can be determined in integrable quantum field theories solving a set of functional equations which, in the standard cases (see e.g. [42]), require as input the exact S-matrix (quite trivial in the free case we are dealing with) and the non-locality phases between the operators and the particles.

We are now in the position to write the form factor equations which read

$$f_n^\alpha(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n,\beta_1,\ldots,\beta_n) = S\,f_n^\alpha(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n,\beta_1,\ldots,\beta_n), \tag{2.6}$$

$$f_n^\alpha(\theta_1+2i\pi,\theta_2,\ldots,\theta_n,\beta_1,\ldots,\beta_n) = \varepsilon\,S\,e^{2i\pi\alpha}f_n^\alpha(\theta_1,\ldots,\theta_n,\beta_1,\ldots,\beta_n), \tag{2.7}$$

$$\text{Res}_{\theta_1-\beta_1=i\pi}f_n^\alpha(\theta_1,\ldots,\theta_n,\beta_1,\ldots,\beta_n) = iS^{n-1}(1-e^{2i\pi\alpha})f_{n-1}^\alpha(\theta_2,..,\theta_n,\beta_2,..,\beta_n). \tag{2.8}$$

We work with $0 < \alpha < 1$. Notice that only the presence of the factor ε in the second equation distinguish between ghosts and ordinary particles. The origin of this factor is the following. Shifting the rapidity of a particle by $i\pi$ means inverting the sign of its energy and momentum. This inversion, together with charge conjugation, amounts to cross the particle from the initial to the final state. Hence, the $2i\pi$ analytic continuation in eq. (2.7) corresponds to a double crossing from the initial to the final state and then again to the initial state, a process which produces the factor $C^2 = \varepsilon$.

The solution to the above equations can be written as

$$f_n^\alpha(\theta_1,\ldots,\theta_n,\beta_1,\ldots,\beta_n) = (-i)^{n\,\delta_{S,-\varepsilon}}S^{n(n-1)/2}(-\sin\pi\alpha)^n\,e^{\left(\alpha-\frac{1}{2}\delta_{S,\varepsilon}\right)\sum_{i=1}^n(\theta_i-\beta_i)}\,|A_n|_{(S)}, \tag{2.9}$$

where A_n is a $n\times n$ matrix ($A_0 \equiv 1$) with entries

$$A_{ij} = \frac{1}{\cosh\frac{\theta_i-\beta_j}{2}}, \tag{2.10}$$

and $|A_n|_{(S)}$ denotes the permanent[2] of A_n for $S = 1$ and the determinant of A_n for $S = -1$.

---

[2]The permanent of a matrix differs from the determinant by the omission of the alternating sign factors $(-1)^{i+j}$.

Correlation functions are obtained inserting in between the operators a resolution of the identity in the form

$$1 = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{d\theta_1 \ldots d\beta_n}{(n!)^2 (2\pi)^{2n}} |A(\theta_1) \ldots A(\theta_n) \bar{A}(\beta_1) \ldots \bar{A}(\beta_n)\rangle \langle \bar{A}(\beta_n) \ldots \bar{A}(\beta_1) A(\theta_n) \ldots A(\theta_1)|.$$

(2.11)

Since by crossing and Lorentz invariance we have

$$\langle \bar{A}(\beta_n) \ldots \bar{A}(\beta_1) A(\theta_n) \ldots A(\theta_1) | \tilde{\Phi}_\alpha(0)|0\rangle = \varepsilon^n f_n^\alpha(\beta_n + i\pi, \ldots, \beta_1 + i\pi, \theta_n + i\pi, \ldots, \theta_1 + i\pi)$$

$$= \varepsilon^n f_n^\alpha(\beta_n, \ldots, \beta_1, \theta_n, \ldots, \theta_1), \qquad (2.12)$$

the two-point functions take the form

$$G_{\alpha,\alpha'}^{(S,\varepsilon)}(t) = \langle \tilde{\Phi}_\alpha(x)\tilde{\Phi}_{\alpha'}(0)\rangle = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{(n!)^2 (2\pi)^{2n}} \int d\theta_1 \ldots d\theta_n d\beta_1 \ldots d\beta_n \, g_n^{(\alpha,\alpha')}(t \,|\theta_1, \ldots, \beta_n) \,,$$

(2.13)

where

$$g_n^{(\alpha,\alpha')}(t \,|\theta_1, \ldots, \beta_n) = f_n^\alpha(\theta_1, \ldots, \beta_n) f_n^{\alpha'}(\beta_n, \ldots, \theta_1) \, e^{-te_n}$$

$$= (\varepsilon S \sin \pi\alpha \, \sin \pi\alpha')^n \, e^{(\alpha-\alpha')\sum_{i=1}^{n}(\theta_i-\beta_i)} \, |A_n|_{(S)}^2 \, e^{-te_n} \,, \quad (2.14)$$

$$t = m|x| \,, \qquad e_n = \sum_{k=1}^{n} (\cosh \theta_k + \cosh \beta_k) \,.$$

A first important observation immediately descends from eq.s (3.26) and (2.14). The correlators are independent of the parameter ε, as a consequence of the cancellation occurring between the factor ε^n, contained in $g_n^{(\alpha,\alpha')}$, and the one explicitely appearing in (3.26). Therefore,

$$G_{\alpha,\alpha'}^{(S,\varepsilon)}(t) = G_{\alpha,\alpha'}^{(S)}(t) \,. \qquad (2.15)$$

Moreover, the spectral series for $G_{\alpha,\alpha'}^{(S)}(t)$ can be now resummed in a Fredholm determinant form. Indeed, following the lines traced in [83, 84] for the ordinary statistics' cases, it is possible to define a $n \times n$ matrix M_n with entries

$$M_{ij} \equiv M(\theta_i, \beta_j) = (\sin \pi\alpha \, \sin \pi\alpha')^{1/2} \, e^{-\frac{t}{2}\cosh \theta_i} \frac{h(\theta_i)h^{-1}(\beta_j)}{\cosh \frac{\theta_i-\beta_j}{2}} \, e^{-\frac{t}{2}\cosh \beta_j} \,, \qquad (2.16)$$

$$h(\theta) = e^{(\alpha-\alpha')\theta/2} \,,$$

and rewrite eq. (2.14) as

$$g_n^{(\alpha,\alpha')} = (\varepsilon S)^n \, |M_n|_{(S)}^2 = \varepsilon^n \left| \begin{array}{cc} 0 & M_n \\ M_n^T & 0 \end{array} \right|_{(S)} . \tag{2.17}$$

Symmetrizing with respect to the two sets of rapidities, by introducing a charge index ϵ which is 1 for a particle A and -1 for $\bar{A}$, (the ε-dependence disappears) eq. (3.26) assumes the form

$$G_{\alpha,\alpha'}^{(S)}(t) = \sum_{L=0}^{\infty} \frac{1}{L! \, (2\pi)^L} \sum_{\epsilon_1 \ldots \epsilon_L} \int d\theta_1 \ldots d\theta_L \, \left| K_{\epsilon_i \epsilon_j}(\theta_i, \theta_j) \right|_{(S)} , \tag{2.18}$$

where the $L \times L$ matrices K posses the following entries

$$\begin{aligned} K_{+-}(\theta, \beta) &= M(\theta, \beta) , \\ K_{-+}(\theta, \beta) &= M(\beta, \theta) , \\ K_{++}(\theta, \beta) &= K_{--}(\theta, \beta) = 0 . \end{aligned} \tag{2.19}$$

The last equation in (2.19) ensures that only the terms with $L = 2n$ occur in (2.18). Notice that the dependence on the statistics in (2.18) only reduces to taking the permanent or the determinant of the same matrix. According to the theory of Fredholm integral operators (see e.g. [95]) this leads to the result

$$G_{\alpha,\alpha'}^{(S)}(t) = \mathrm{Det} \left(1 + \frac{1}{2\pi} \mathbf{K} \right)^{-S} , \tag{2.20}$$

so that the correlators of operators having the same non-locality phase in the free fermion and free boson theories are the inverse of each other [84].

An alternative, convenient, way to express the solution (2.20) is the following

$$G_{\alpha,\alpha'}^{(S)}(t) \equiv \langle \tilde{\Phi}_\alpha(x) \tilde{\Phi}_{\alpha'}(0) \rangle = e^{S \, \Upsilon_{\alpha,\alpha'}(m|x|)} . \tag{2.21}$$

The function $\Upsilon_{\alpha,\alpha'}(t)$ is given by [83,93]

$$\Upsilon_{\alpha,\alpha'}(t) = \frac{1}{2} \int_{t/2}^{\infty} \rho \, d\rho \left[(\partial_\rho \chi)^2 - 4 \sinh^2 \chi - \frac{(\alpha - \alpha')^2}{\rho^2} \tanh \chi \right] , \tag{2.22}$$

where $\chi(\rho)$ satisfies the differential equation

$$\partial_\rho^2 \chi + \frac{1}{\rho}\partial_\rho\chi = 2\sinh 2\chi + \frac{(\alpha-\alpha')^2}{\rho^2}\tanh\chi\left(1-\tanh^2\chi\right), \qquad (2.23)$$

subject to asymptotic conditions such that for $\alpha + \alpha' < 1$ one obtains

$$\lim_{t\to 0} G_{\alpha,\alpha'}^{(S)}(t) = \left(C_{\alpha,\alpha'}\, t^{2\alpha\alpha'}\right)^{-S}. \qquad (2.24)$$

The amplitude follows from the work of reference [96] and reads

$$C_{\alpha,\alpha'} = 2^{-2\alpha\alpha'}\exp\left\{2\int_0^\infty \frac{dt}{t}\left[\frac{\sinh\alpha t\,\cosh(\alpha+\alpha')t\,\sinh\alpha' t}{\sinh^2 t} - \alpha\alpha' e^{-2t}\right]\right\}. \qquad (2.25)$$

2.4 Short distance limit

As already mentioned in the first chapter, some important information can be gained, once the form factors and the correlation functions are known, about the ultraviolet limit.

The central charge and the scaling dimensions of the operators, related to the ultraviolet theory, can be obtained in our off-critical framework through the sum rules [69, 85]

$$c = \frac{3}{4\pi}\int d^2x\,|x|^2\,\langle\Theta(x)\Theta(0)\rangle_{connected}, \qquad (2.26)$$

$$\Delta_\alpha = -\frac{1}{2\pi}\int d^2x\,\langle\Theta(x)\tilde{\Phi}_\alpha(0)\rangle_{connected}, \qquad (2.27)$$

where $\Theta(x)$ denotes the trace of the energy-momentum tensor. Since the only non-zero form factor of this operator in the free theories we are dealing with is

$$\langle 0|\Theta(0)|A(\theta)\bar{A}(\beta)\rangle = 2\pi m^2\left[-i\sinh\frac{\theta-\beta}{2}\right]^{\delta_{S,-\varepsilon}}, \qquad (2.28)$$

it easy to check that the sum rules yield, indeed, the expected results

$$c = 2^{\delta_{S,\varepsilon}}\varepsilon, \qquad \Delta_\alpha = S\alpha(\delta_{S,\varepsilon} - \alpha). \qquad (2.29)$$

(For instance, in the fermionic ghost case, the values $c = -2$ and $\Delta_{1/2} = -1/4$ are recovered, where the non-locality index $\alpha = 1/2$ pertains to the disorder operator usually denoted by μ, whose weights are known from conformal invariance to be $h_\mu = \bar{h}_\mu = -1/8$.)

For the discussion of the short distance behaviour of the correlators define the exponent $\Gamma_{\alpha,\alpha'}$ through the relation

$$\langle \Phi_\alpha(x)\Phi_{\alpha'}(0)\rangle \sim |x|^{-\Gamma_{\alpha,\alpha'}}, \qquad |x| \to 0. \tag{2.30}$$

The result (3.59) for $\alpha + \alpha' < 1$ follows from the operator product expansion

$$\langle \Phi_\alpha(x)\Phi_{\alpha'}(0)\rangle \sim |x|^{\Delta_{\alpha+\alpha'}-\Delta_\alpha-\Delta_{\alpha'}}\langle\Phi_{\alpha+\alpha'}\rangle + \dots. \tag{2.31}$$

The ε–dependence of the scaling dimensions in (2.51) affects only the term linear in α and cancels out in the above combination leaving

$$\Gamma_{\alpha,\alpha'} = 2S\,\alpha\alpha', \qquad 0 < \alpha + \alpha' < 1. \tag{2.32}$$

It seems more difficult to give a unified description for the range $1 < \alpha + \alpha' < 2$. On the basis of the discussion of reference [84] we expect that for $S = \varepsilon$ the short distance behaviour (2.31) still holds provided $\alpha + \alpha'$ is taken modulo 1. Then one finds

$$\Gamma_{\alpha,\alpha'} = 2S\left[\alpha\alpha' + 1 - (\alpha + \alpha')\right], \qquad 1 < \alpha + \alpha' < 2. \tag{2.33}$$

This result is recovered in the case $S = -1$, $\varepsilon = 1$ due to the fact that the first order off-critical correction becomes leading in this range of $\alpha + \alpha'$ [84]. The mechanism that should lead to (2.33) in the remaining case of the bosonic ghost is not clear to us at present.

At the border value $\alpha + \alpha' = 1$ the correlators develop a logarithmic correction that is most easily evaluated for the well studied case of ordinary complex fermions [96]. One concludes

$$\lim_{t\to 0} G_{\alpha,1-\alpha}^{(S)}(t) = [\mathcal{B}_\alpha\, t^{2\alpha(1-\alpha)}\ln(1/t)]^{-S}, \tag{2.34}$$

with

$$\mathcal{B}_\alpha = 2^{1-2\alpha(1-\alpha)}e^{-(I_\alpha + I_{1-\alpha})}, \tag{2.35}$$

$$I_\alpha = \int_0^\infty \frac{dt}{t}\left(\frac{\sinh^2\alpha t}{\sinh^2 t} - \alpha^2 e^{-2t}\right). \tag{2.36}$$

2.5 The fermionic ghost case

In this section, some useful facts about the fermionic ghost system are collected. In particular, the information concerning the lagrangian description of the model (action, mode expansion etc.) will be extensively used in the next chapter.

2.5.1 Lagrangian description

The Euclidean action is given by the expression

$$\mathcal{A} = \frac{1}{2} \int d^2x \, J_{\alpha\beta} \left(\partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta + m^2 \, \Phi^\alpha \Phi^\beta \right) , \tag{2.37}$$

where the symplectic form $J_{\alpha\beta}$ reads explicitly

$$J_{-+} = -J_{+-} = 1 \ , \quad J_{\alpha\gamma} J^{\gamma\beta} = \delta^\beta_\alpha , \tag{2.38}$$

and the ghost fields $\Phi^\pm$, for later convenience, are redefined according to

$$\Phi^+ \to \Phi$$

$$\Phi^- \to \bar{\Phi} .$$

They are the components of a massive doublet (the so-called *symplectic fermion*), transforming under the symplectic group $SP(2)$. The corresponding mode expansions are

$$\Phi_{(\pm)}(x,t) = \int d\beta \left[\bar{a}_{(\pm)}(\beta) e^{-im(t\cosh\beta - x\sinh\beta)} + a^\dagger_{(\pm)}(\beta) e^{im(t\cosh\beta - x\sinh\beta)} \right] ,$$

$$\bar{\Phi}_{(\pm)}(x,t) = \int d\beta \left[-a_{(\pm)}(\beta) e^{-im(t\cosh\beta - x\sinh\beta)} + \bar{a}^\dagger_{(\pm)}(\beta) e^{im(t\cosh\beta - x\sinh\beta)} \right] , \tag{2.39}$$

where the creation and annihilation operators satisfy the anti-commutation relations

$$\{a_{(\pm)}(\beta), a^\dagger_{(\pm)}(\beta')\} = 2\pi\delta(\beta - \beta') \ , \quad \{a_{(\pm)}(\beta), a_{(\pm)}(\beta')\} = 0 = \{a^\dagger_{(\pm)}(\beta), a^\dagger_{(\pm)}(\beta')\};$$

$$\{\bar{a}_{(\pm)}(\beta), \bar{a}^\dagger_{(\pm)}(\beta')\} = 2\pi\delta(\beta - \beta') \ , \quad \{\bar{a}_{(\pm)}(\beta), \bar{a}_{(\pm)}(\beta')\} = 0 = \{\bar{a}^\dagger_{(\pm)}(\beta), \bar{a}^\dagger_{(\pm)}(\beta')\} . \tag{2.40}$$

Finally, charge conjugation implemented on the Fock operators

$$\mathcal{C}a(\beta)\mathcal{C}^{-1} = \bar{a}(\beta) \ , \quad \mathcal{C}a^\dagger(\beta)\mathcal{C}^{-1} = \bar{a}^\dagger(\beta);$$

$$\mathcal{C}\bar{a}(\beta)\mathcal{C}^{-1} = -a(\beta) \ , \quad \mathcal{C}\bar{a}^\dagger(\beta)\mathcal{C}^{-1} = -a^\dagger(\beta) , \tag{2.41}$$

induces the following transformations on the ghost fields $\Phi \to \bar{\Phi}$ and $\bar{\Phi} \to -\Phi$.

2.5.2 An explicit example: the twist field μ

An interesting check of our results for the ghost correlation functions can be performed for the operator $\Phi_{1/2}$ in the fermionic ghost theory. In fact, the free massive fermionic ghost can

formally be regarded as a limit of the $\varphi_{1,3}$ perturbation of the minimal conformal models with central charge [5]

$$c = 1 - \frac{6}{p(p+1)}, \tag{2.42}$$

possessing the spectrum of scalar primary fields $\varphi_{l,k}$ with scaling dimensions

$$\Delta_{l,k} = \frac{((p+1)l - pk)^2 - 1}{2p(p+1)}. \tag{2.43}$$

The required values $c = -2$ and $\Phi_{1,3} = 0$ are found as[3] $p \to 1$. Our operator $\Phi_{1/2}$ with scaling dimension $-1/4$ is identified with $\varphi_{1,2}$. From the operator product expansion of the $\varphi_{l,k}$ we have for $p \to 1$

$$\langle \tilde{\varphi}_{1,2}(x) \tilde{\varphi}_{1,2}(0) \rangle \simeq \frac{|x|^{-2\Delta_{1,2}}}{\langle \varphi_{1,2} \rangle^2} \left(1 + C \langle \varphi_{1,3} \rangle |x|^{\Delta_{1,3}} \right)$$

$$\simeq \frac{|x|^{1/2}}{\langle \varphi_{1,2} \rangle^2} \left\{ 1 + C \langle \varphi_{1,3} \rangle \left[1 + (p-1) \ln |x| \right] \right\}. \tag{2.44}$$

It can be checked from the known values of the structure constant C [97] and of the vacuum expectation values in $\varphi_{1,3}$–perturbed minimal models [96] that $C \langle \varphi_{1,3} \rangle = -1$ and $(p - 1)/\langle \varphi_{1,2} \rangle^2 = \mathcal{B}_{1/2} \, m^{1/2}$ as $p \to 1$, so that the result (3.63) with $S = -1$ and $\alpha = 1/2$ is indeed recovered.

2.5.3 Results from the 'cluster' expansion

Finally, we present some results concerning the leading short distance behavior of $G_{\alpha,\alpha'}^{(S)}(t)$, obtained by means of the 'cluster' expansion [98, 99] in the fermionic ghost case. The two-point function (3.26) can be rewritten as

$$G_{\alpha,\alpha'}^{(-)}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} d\theta_1 ... d\beta_n \ h_n^{(\alpha,\alpha')}(t|\theta_1, ..., \beta_n) \, e^{-te_n}, \tag{2.45}$$

where

$$h_n^{(\alpha,\alpha')}(t|\theta_1, ..., \beta_n) = \frac{(-)^n}{n! \, (2\pi)^{2n}} \left(\sin \pi\alpha \, \sin \pi\alpha' \right)^n e^{(\alpha-\alpha') \sum_{i=1}^{n} (\theta_i - \beta_i)} \, |A_n|_{(-)}^2. \tag{2.46}$$

---

[3]The genuine minimal models of the series (5.49) have $p = 3, 4, \ldots$. It is known, however, that many results can be extended to continous values of p.

The logarithm of (2.45) reads

$$\ln G_{\alpha,\alpha'}^{(-)}(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} d\theta_1...d\beta_n \; \tilde{h}_n^{(\alpha,\alpha')}(t|\theta_1,...,\beta_n) \, e^{-te_n} \,, \tag{2.47}$$

with the first few terms given by

$$
\begin{aligned}
h_1^{(\alpha,\alpha')} &= \tilde{h}_1^{(\alpha,\alpha')} \\
h_{12}^{(\alpha,\alpha')} &= \tilde{h}_{12}^{(\alpha,\alpha')} + \tilde{h}_1^{(\alpha,\alpha')}\tilde{h}_2^{(\alpha,\alpha')} \\
h_{123}^{(\alpha,\alpha')} &= \tilde{h}_{123}^{(\alpha,\alpha')} + \tilde{h}_{12}^{(\alpha,\alpha')}\tilde{h}_3^{(\alpha,\alpha')} + \tilde{h}_{23}^{(\alpha,\alpha')}\tilde{h}_1^{(\alpha,\alpha')} + \tilde{h}_{31}^{(\alpha,\alpha')}\tilde{h}_2^{(\alpha,\alpha')} + \tilde{h}_1^{(\alpha,\alpha')}\tilde{h}_2^{(\alpha,\alpha')}\tilde{h}_3^{(\alpha,\alpha')} ...
\end{aligned}
$$

Since the functions $\tilde{h}_n^{(\alpha,\alpha')}$ depend on rapidity differences only, it is possible to extract from (2.47) a term decreasing logarithmically as $\ln\frac{1}{t}$, at any order. If the 'cluster' condition

$$\tilde{h}_n(\theta_1,...,\beta_n) = \mathcal{O}(e^{-|\beta_i|}) \tag{2.48}$$

is satisfied (for $\Re e\beta_i \to \pm\infty$), then the remaining multiple integrals converge and contribute to the value of the critical exponent. In the present case, for $\alpha, \alpha' \ll 1$ the first order term alone is sufficient to capture the leading short-distance behavior, according to

$$\ln G_{\alpha,\alpha'}^{(-)}(t) \simeq \left(\frac{2}{\pi} \frac{\sin\pi\alpha \, \sin\pi\alpha'}{\sin\pi(\alpha-\alpha')} (\alpha-\alpha') \right) \ln t \,. \tag{2.49}$$

2.6 Conclusions

Denoting $\Phi_\alpha(x)$ the disorder operator exhibiting a non-locality phase $e^{2i\pi\alpha}$ ($e^{-2i\pi\alpha}$) with respect to (the field which interpolates) the particle A ($\bar{A}$), we showed that

$$G_{\alpha,\alpha'}^{(S)}(t) \equiv \langle \tilde{\Phi}_\alpha(x)\tilde{\Phi}_{\alpha'}(0) \rangle = e^{S\,\Upsilon_{\alpha,\alpha'}(m|x|)} \,, \tag{2.50}$$

where $\Upsilon_{\alpha,\alpha'}(t)$ is a function expressed in terms of the solution of a non-linear differential equation of Painlevé type.

There are two main points to be remarked in eq. (2.50). Firstly, the r.h.s. depends on S but not on ε, which implies that the correlation functions of the disorder operators in the bosonic and fermionic ghost systems coincide with those for the ordinary complex bosons and fermions, respectively [44]. The ε-independence of the r.h.s. of (2.50) has to be

contrasted with the fact that the nature of the operators on the l.h.s. strongly depends on ε. Indeed, the values of the scaling dimensions Δ_α of the operators Φ_α and of the central charge in the ultraviolet limit can be summarized as follows:

$$c = 2^{\delta_{S,\varepsilon}}\varepsilon\,, \qquad \Delta_\alpha = S\alpha(\delta_{S,\varepsilon} - \alpha)\,. \tag{2.51}$$

Moreover, we recovered a remarkable inversion property, already known in literature [83, 84, 93], which involves the correlators and reads

$$G^{(+)}_{\alpha,\alpha'}(t) = \frac{1}{G^{(-)}_{\alpha,\alpha'}(t)}\,. \tag{2.52}$$

Chapter 3

Ghost theories with impurities

In this chapter, we consider free massive fermionic ghosts, in the presence of an extended line of impurities, relying on the Lagrangian formalism. We propose two distinct defect interactions, respectively, of relevant and marginal nature. The analysis includes the study of the corresponding scattering theories and the exact computation of the correlation functions, associated to the thermal and the disorder operators.

3.1 Introduction

After the seminal work by Ghoshal and Zamolodchikov [100] on integrable field theories in the presence of a boundary, a great deal of attention has been devoted to study finite size effects, due especially to their numerous applications to real physical problems. Quantum field theories with extended line of defects[1] generalize these boundary models, introducing new and original features [110–113].

The presence of impurities can be mimicked by the action of a 'defect' operator, placed along an infinite line in the Euclidean space. In the continuum limit and away from criticality, massive excitations can either participate to bulk scattering processes or interact with the defect, being reflected and/or transmitted. Such information can be encoded into a scattering theory enriched by adding to the bulk S-matrix the amplitudes relative to these two

---

[1]Actually, the critical behavior of statistical systems with lines of defects has been widely studied in the past years [101–109].

new processes. The integrability of the model, originally studied in [110], is guaranteed by imposing the factorization condition which translates into a set of cubic relations called the Reflection-Transmission equations. In particular, it has been showed that, for diagonal bulk scattering, non-trivial solutions for both the reflection and transmission amplitudes can be found only in non interacting bulk systems. In this light, free field theories play a prominent role.

Therefore, in the following, we generalize the model of free massive fermionic ghosts, previously studied in Chapter 2, in order to include the effects of inhomogeneities. In particular, the knowledge of the scattering amplitudes (and the spectrum of bulk excitations), along with general analyticity properties and relativistic invariance, allows to reconstruct thoroughly the off-shell dynamics, by computing exactly correlation functions.

The first step towards the realization of this program involves the derivation of the transmission and reflection amplitudes. One way to compute them consists in solving a bootstrap system of equations (unitarity, crossing and factorization). However, in this peculiar case, the absence of stringent constraints leaves a broad arbitrariness in the choice of the solutions. Fortunately, an alternative description is possible, in terms of the Lagrangian formalism

$$\mathcal{A} = \mathcal{A}_B + g \int d^2x \, \delta(x) \mathcal{L}_D(\varphi_i, \partial_y \varphi_i) \,, \tag{3.1}$$

where the bulk Euclidean action $\mathcal{A}_B$ (given by formula (2.37)) and the lagrangian density $\mathcal{L}_D$, encoding all the information relative to the scattering processes on the defect line, both depend on the local fields of the theory. According to the strength of the coupling constant g, the line of inhomogeneities can interpolate between a bulk and a surface statistical behavior. If the defect interaction is relevant (irrelevant), a bulk (surface) behavior is expected in the short distance limit, while the marginal case shares both regimes. In the following, a relevant and a marginal interactions are proposed and exact expressions for the correlators of the most significant operators in the theory are derived, by using the bulk form factors and the matrix elements corresponding to the defect operator. In the former case, resonance phenomena occur in the spectrum of excitations, while the latter perturbation is responsible for non-universal power-laws in the correlation functions of operators, non-local in the ghost fields.

3.2   Bootstrap approach

We focus on the model of free massive fermionic ghosts, studied in the previous chapter, in the presence of an infinite[2] line of impurities placed at $x = 0$, which, after a rotation in the Minkowski plane, will be identified with the time axis.

Let us recall here that the bulk spectrum of the theory is composed of a doublet of free particles A and $\bar{A}$ with mass m, bearing respectively U(1) charges ± 1. Their scattering is ruled, in the bulk, by the S-matrix $S = -1$. Due to the energy conservation, when a particle hits the defect it can be either reflected or transmitted. All the processes involved in the theory can be recast as a set of algebraic equations [110], relying on the algebra of the Faddeev-Zamolodchikov operators. After the usual parameterization of the particle's energy-momentum in terms of the rapidity variable $(e, p) = (m \cosh \theta, m \sinh \theta)$, we associate to excitations of type 'a' the formal operator $A_a(\theta)$ and to the defect line an operator $\mathcal{D}$, playing the role of a zero rapidity particle, during the whole time evolution of the system. The commutation relations, associated to the defect algebra, read

$$
\begin{aligned}
A_a(\theta)\mathcal{D} &= R_a^b(\theta)A_b(-\theta)\mathcal{D} + T_a^b(\theta)\mathcal{D}A_b(\theta)\,, \\
\mathcal{D}A_a(\theta) &= R_a^b(-\theta)\mathcal{D}A_b(-\theta) + T_a^b(-\theta)A_b(\theta)\mathcal{D}\,,
\end{aligned}
\tag{3.2}
$$

where, in the first equation, $R_a^b(\theta)$ and $T_a^b(\theta)$ denote, respectively, the reflection and transmission amplitudes of an asymptotic particle 'a' entering the defect with rapidity θ, from the left. The second equation, describing the scattering of a particle hitting the defect from the right, is obtained from the first one, after an analytic continuation $\theta \to -\theta$ in the rapidity variable. Consistency of (3.2) implies the unitarity conditions

$$
\begin{aligned}
R_a^b(\theta)R_b^c(-\theta) + T_a^b(\theta)T_b^c(-\theta) &= \delta_a^c\,, \\
R_a^b(\theta)T_b^c(-\theta) + T_a^b(\theta)R_b^c(-\theta) &= 0\,.
\end{aligned}
\tag{3.3}
$$

---

[2]In $(2 + 0)$-dimensional statistical mechanics models, the inhomogeneities are placed along a line in the two-dimensional space (for a detailed discussion about this point, see [110]). Actually, in $(1 + 1)$ dimensions, the defect is a point in the space which evolves as time goes by, describing an infinite line along the time direction.

Crossing relations read

$$C^{aa''}R^b_{a''}\left(i\frac{\pi}{2} - \theta\right) = S^{ab}_{a'b'}(2\theta)C^{b'b''}R^{a'}_{b''}\left(i\frac{\pi}{2} + \theta\right),$$

$$T^b_a(\theta) = C^{bb'}T^{a'}_{b'}(i\pi - \theta)C_{a'a}, \tag{3.4}$$

with an antisymmetric charge conjugation matrix, such that $C^2 = -1$. As regards factorization conditions, the main result of [110] guarantees that, for free theories diagonal in the bulk, the Reflection-Transmission equations, descending from integrability, are automatically satisfied.

At this point, solving the bootstrap system of equations (3.2)-(3.4), we are able in principle to determine the scattering amplitudes R^b_a and T^b_a. However, a proliferation of solutions occurs, due to the lack of constraints strong enough to fix the reflection and transmission matrices in a closed form. A simplified version of this model (i.e. a purely reflecting theory which coincides with a boundary problem [100]) helps visualizing the situation. Introduce the following parameterization of the reflection matrix components:

$$\begin{aligned} R^A_A(\theta) &= f(\theta)R(\theta) & R^{\bar A}_A(\theta) &= g(\theta)R(\theta) \\ R^A_{\bar A}(\theta) &= f'(\theta)R(\theta) & R^A_{\bar A}(\theta) &= g'(\theta)R(\theta). \end{aligned} \tag{3.5}$$

Consistency of the bootstrap system gives rise to the conditions

$$R(\theta)R(-\theta) = [f(\theta)f(-\theta) + g(\theta)g'(-\theta)]^{-1}$$

$$R(\theta)R(-\theta) = [f'(\theta)f'(-\theta) + g'(\theta)g(-\theta)]^{-1} \tag{3.6}$$

$$f(\theta)g(-\theta) + g(\theta)f'(-\theta) = 0$$

$$f'(\theta)g'(-\theta) + g'(\theta)f(-\theta) = 0 \tag{3.7}$$

$$-\frac{g'\left(\frac{i\pi}{2} + \theta\right)}{g'\left(\frac{i\pi}{2} - \theta\right)} = \frac{f'\left(\frac{i\pi}{2} + \theta\right)}{f\left(\frac{i\pi}{2} - \theta\right)} = \frac{f\left(\frac{i\pi}{2} + \theta\right)}{f'\left(\frac{i\pi}{2} - \theta\right)} = -\frac{g\left(\frac{i\pi}{2} + \theta\right)}{g\left(\frac{i\pi}{2} - \theta\right)}, \tag{3.8}$$

which allow a richness of solutions. A comparison with the well established theory of free massive Dirac fermions [100, 114], showing strong analogies with our ghost system, is in order. Such model, obtained as a particular limit of the Sine-Gordon one, admits non trivial boundary Yang-Baxter equations, which provide a solution for the reflection amplitude in terms of two parameters. In our case, starting directly from a free theory, it is impossible to exploit factorization constraints, in order to fix the form of the R-matrix.

3.3   Lagrangian description

To overcome the ambiguities, intrinsically concerned with the bootstrap scenario, the lagrangian approach proves to be an alternative route.

A detailed analysis of the bulk system, including mode expansions of the basic fields, commutation relations and charge conjugation properties, can be found in Section 2.5.1. Only, let us recall here the expression of the Euclidean action, describing the bulk dynamics,

$$\mathcal{A}_B = \frac{1}{2} \int d^2x \, J_{\alpha\beta} \left(\partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta + m^2 \, \Phi^\alpha \Phi^\beta \right) . \tag{3.9}$$

where Φ^α are zero dimensional anti-commuting fields (Φ and $\bar{\Phi}$), belonging to the same doublet, characterized by mass m, and $J_{\alpha\beta}$ is the antisymmetric tensor.

Inhomogeneities affect the bulk physics introducing a Lagrangian density along the impurity line, according to (3.1). A relevant and a marginal interactions will be the object of our study in order to derive explicit expressions for the reflection and transmission amplitudes.

3.3.1   Relevant perturbation

Consider the system described by

$$\mathcal{A} = \mathcal{A}_B + \frac{g}{2} \int d^2x \, \delta(x) \, J_{\alpha\beta} \Phi^\alpha \Phi^\beta , \tag{3.10}$$

where the dimension of the coupling constant g is [*mass*]. The equations of motion read

$$(\partial_\mu \partial^\mu - m^2)\Phi = g\delta(x)\Phi$$
$$(\partial_\mu \partial^\mu - m^2)\bar{\Phi} = g\delta(x)\bar{\Phi} . \tag{3.11}$$

It is useful to split the fields into components belonging to the two intervals $x < 0$ and $x > 0$ (after rotation to the Minkowski space)

$$\Phi(x,t) = \theta(x)\Phi_+(x,t) + \theta(-x)\Phi_-(x,t)$$
$$\bar{\Phi}(x,t) = \theta(x)\bar{\Phi}_+(x,t) + \theta(-x)\bar{\Phi}_-(x,t) , \tag{3.12}$$

in order to derive the boundary conditions at $x = 0$, given by

$$\Phi_+(0,t) - \Phi_-(0,t) = 0 ;$$
$$\partial_x(\Phi_+(0,t) - \Phi_-(0,t)) = \frac{g}{4}(\Phi_+(0,t) + \Phi_-(0,t)) \tag{3.13}$$

$$\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t) = 0;$$

$$\partial_x(\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t)) = \frac{g}{4}(\bar{\Phi}_+(0,t) + \bar{\Phi}_-(0,t)). \tag{3.14}$$

The mode expansions (2.39), in terms of the operators A and $\bar{A}$ which interpolate the bulk excitations, allow us to extract explicitly from (3.13)-(3.14) the reflection and transmission amplitudes

$$\begin{pmatrix} A_-(\beta) \\ \bar{A}_-(\beta) \\ A_+(-\beta) \\ \bar{A}_+(-\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,\kappa) & T(\beta,\kappa) \\ T(\beta,\kappa) & R(\beta,\kappa) \end{pmatrix} \begin{pmatrix} A_-(-\beta) \\ \bar{A}_-(-\beta) \\ A_+(\beta) \\ \bar{A}_+(\beta) \end{pmatrix}, \tag{3.15}$$

with

$$R(\beta,\kappa) = \frac{1}{\sinh\beta + i\kappa} \begin{pmatrix} -i\kappa & 0 \\ 0 & -i\kappa \end{pmatrix},$$

$$T(\beta,\kappa) = \frac{1}{\sinh\beta + i\kappa} \begin{pmatrix} \sinh\beta & 0 \\ 0 & \sinh\beta \end{pmatrix} \tag{3.16}$$

and $\kappa = g/4m$. R and T, thus obtained, satisfy crossing and unitarity conditions.

A strong analogy with the free bosonic theory, extensively treated in [110], emerges. A part from a doubling of the matrix elements, the scattering amplitudes coincide. The main features are the occurrence of resonances (i.e. unstable bound states possessing a real part in the unphysical sheet, which do not appear as asymptotic particles of the theory) for $\kappa > 1$ and phenomena of instabilities for $\kappa < -1$, characterized by poles with imaginary part fixed at the value $i\pi/2$, acquiring an increasing real part as κ is further depleted.

In the limit $g \to \infty$ ($\kappa \to \infty$), corresponding to the fixed boundary conditions $\Phi(0,t) = 0$ and $\bar{\Phi}(0,t) = 0$, the defect line acts as a purely reflecting surface. On the contrary, in the high-energy limit $\beta \to \infty$, due to the relevant character of the perturbation, the theory renormalizes to a bulk regime, the impurity line becoming transparent.

3.3.2 Marginal perturbation

The Euclidean action

$$\mathcal{A} = \mathcal{A}_B - ig \int d^2x\, \delta(x)\, (\Phi\partial_y\Phi + \bar{\Phi}\partial_y\bar{\Phi}), \tag{3.17}$$

where g is a dimensionless coupling constant, describes the effects of a marginal interaction on the defect line. The equations of motion

$$(\partial_\mu \partial^\mu - m^2)\bar{\Phi} - 2ig\delta(x)\partial\Phi = 0 \tag{3.18}$$

$$(\partial_\mu \partial^\mu - m^2)\Phi + 2ig\delta(x)\partial\bar{\Phi} = 0 \tag{3.19}$$

lead to the following boundary conditions in the Minkowski plane

$$\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t) = 0\,;$$
$$\partial_x(\bar{\Phi}_+(0,t) - \bar{\Phi}_-(0,t)) = g\,\partial_t\Phi(0,t) \tag{3.20}$$

$$\Phi_+(0,t) - \Phi_-(0,t) = 0\,;$$
$$\partial_x(\Phi_+(0,t) - \Phi_-(0,t)) = -g\,\partial_t\bar{\Phi}(0,t)\,. \tag{3.21}$$

Exploiting again the mode expansions in terms of the operators A and $\bar{A}$, the reflection and transmission matrices assume the form

$$R(\beta,\chi) = \frac{\sin\chi\cosh\beta}{\cosh^2\beta - \cos^2\chi}\begin{pmatrix} -\sin\chi\cosh\beta & -\cos\chi\sinh\beta \\ \cos\chi\sinh\beta & -\sin\chi\cosh\beta \end{pmatrix},$$

$$T(\beta,\chi) = \frac{\cos\chi\sinh\beta}{\cosh^2\beta - \cos^2\chi}\begin{pmatrix} \cos\chi\sinh\beta & -\sin\chi\cosh\beta \\ \sin\chi\cosh\beta & \cos\chi\sinh\beta \end{pmatrix}, \tag{3.22}$$

$$\sin^2\chi = \frac{g^2}{4+g^2}\,.$$

Some remarks are in order. The action (3.17) is invariant under charge conjugation, implemented by the transformations $\Phi \to \bar{\Phi}$ and $\bar{\Phi} \to -\Phi$. Therefore, the relations $R_A^A = R_{\bar{A}}^{\bar{A}}$ and $R_A^{\bar{A}} = -R_{\bar{A}}^A$, along with their analogous counterparts for the transmission matrix, hold. On the other hand, the U(1) symmetry, manifestly displayed by the bulk action, is broken by the defect interaction. As a consequence, scattering processes, which violate the conservation of U(1) charges on the impurity line, can occur, allowing for non-zero off-diagonal contributions. Exceptions to this behavior concern the fixed ($g \to \infty$, $\cos\chi \to 0$) and the free ($g \to 0$, $\sin\chi \to 0$) boundary conditions, where a restoration of the symmetry takes place.

Let us turn the attention on the analytic structure of the reflection and transmission matrices. Since the theory is non-unitary, a mechanism, akin to the one occurring in the scaling Lee-Yang model [115], is expected to take place. In other words, residues, corresponding to poles in the scattering amplitudes, are not supposed to be, a priori, real and positive. This phenomenon is reminiscent of the non-hermitian nature of the Hamiltonian associated to the system and does not contrast with the unitarity requirement (3.3), preserving the meaning of probability densities[3].

Poles appear both in the reflection and the transmission amplitudes at $\beta = i\chi$ and $\beta = i(\pi - \chi)$, with $\chi \in [0, \pi/2]$. In the case of diagonal matrix elements, the corresponding residues give

$$R_A^A \simeq R_{\bar{A}}^{\bar{A}} \simeq T_A^A \simeq T_{\bar{A}}^{\bar{A}} \simeq \frac{i}{2} \cdot \frac{\sin \chi \cos \chi}{\beta - i\chi}$$

$$R_A^A \simeq R_{\bar{A}}^{\bar{A}} \simeq T_A^A \simeq T_{\bar{A}}^{\bar{A}} \simeq \frac{i}{2} \cdot \frac{-\sin \chi \cos \chi}{\beta - i(\pi - \chi)} . \tag{3.24}$$

Therefore, the pole at $\beta = i\chi$ is associated to a boundary bound state in the direct channel, with positive binding energy $e_b = m \cos \chi$, while the other one lives in the crossed channel. Since $e_b < m$ for every value of the coupling constant, the boundary bound states are always stable and the theory is free of resonances and instabilities of other nature. As regards off-diagonal processes, the residues calculated at $\beta = i\chi$ assume the form

$$R_A^{\bar{A}} \simeq T_A^{\bar{A}} \simeq \frac{i}{2} \cdot \frac{i \sin \chi \cos \chi}{\beta - i\chi} \qquad R_{\bar{A}}^A \simeq T_{\bar{A}}^A \simeq \frac{i}{2} \cdot \frac{-i \sin \chi \cos \chi}{\beta - i\chi} , \tag{3.25}$$

while residues computed in the crossed channel display an overall minus sign. As mentioned before, the additional factor $\pm i$, appearing in the numerator, is a consequence of the anomalous charge conjugation properties of the ghost fields.

Finally, a comment on the marginal nature of the interaction: performing the ultraviolet limit, except for peculiar values of the coupling constant, all the scattering matrices' components remain simultaneously finite.

---

[3]Non-hermiticity of the Hamiltonian implies, in particular, its left eigenstates $\langle n_L|$ are not simply the adjoints of the right ones $|n_R\rangle$. Since, in addition, the Fock space states are also eigenstates of the charge-conjugation operator with eigenvalues $(\pm i)^N$, N being the particles' number, the relation $\langle n_L| = \langle n_R| C$ leads to the completeness condition $\sum_n |n_R\rangle\langle n_L| = \sum_n |n_R\rangle\langle n_R|(\pm i)^n$. On the other hand, eq. (3.3), relying only on the assumption that in and out-kets, constructed in terms of the asymptotic particles A and $\bar{A}$, form a basis in the Hilbert space, is insensitive to hermiticity properties of the Hamiltonian.

3.4 Correlation functions

The problem at the heart of this paper concerns the computation of correlation functions of the local fields $\phi_i(x, t)$, present in the theory.

To realize this idea, in order to fully exploit the knowledge of the bulk physics, it is worth performing a rotation in the Minkowsi plane ($x \to -it$, $t \to ix$), moving the defect line at $t = 0$. In this new picture, the Hilbert space of states is the same as in the bulk and the effects of impurities are taken into account by an operator $\mathcal{D}$, placed at $t = 0$, which acts on the bulk states. Therefore, correlation functions assume the form [110]

$$\langle \Phi_1(x_1, t_1)...\Phi_n(x_n, t_n) \rangle = \frac{\langle 0|T[\phi_1(x_1, t_1)...\mathcal{D}...\phi_n(x_n, t_n)]|0\rangle}{\langle 0|\mathcal{D}|0\rangle}, \qquad (3.26)$$

$\Phi_i(x_i, t_i)$ and $\phi_i(x_i, t_i)$ being the fields in the Heisenberg representation, whose time evolutions are ruled, respectively, by the exact Hamiltonian of the problem (bulk and defect interactions) and the bulk Hamiltonian alone. As it appears clearly, after inserting the completeness condition of the bulk states in the right-hand side of (3.26), the above equation can be computed only in terms of the Form Factors of the bulk fields and the matrix elements of the defect operator on the asymptotic states. Another consequence of the axis-rotation in the Minkowsi plane is the interchange of roles between energy and momentum. This affects the rapidity dependence of the scattering amplitudes according to $\theta \to (i\pi/2 - \theta)$. In compact notation it reads

$$\hat{R}^{ab}(\theta) = \mathcal{C}^{aa'} R^b_{a'}\left(i\frac{\pi}{2} - \theta\right),$$
$$\hat{T}^{ab}(\theta) = \mathcal{C}^{aa'} T^{b'}_{a'}\left(i\frac{\pi}{2} - \theta\right) \mathcal{C}_{b'b}. \qquad (3.27)$$

Let us recall here that asymptotic states are composed of neutral pairs $A(\theta)\bar{A}(\beta)$, obtained by acting with the corresponding operators A and $\bar{A}$ on the vacuum $|0\rangle$. Explicit expressions for the bulk Form Factors have been previously derived in Section 2.3, while the simplest matrix elements of the defect operator on the bulk states are

$$\langle A(\theta)|\mathcal{D}|A(\theta')\rangle = 2\pi \hat{T}^{AA}(\theta)\,\delta(\theta - \theta'),$$
$$\langle \bar{A}(\beta)A(\theta)|\mathcal{D}|0\rangle = 2\pi \hat{R}^{A\bar{A}}(\theta)\,\delta(\theta + \beta),$$
$$\langle 0|\mathcal{D}|A(\theta)\bar{A}(\beta)\rangle = -2\pi \hat{R}^{A\bar{A}}(\theta - i\pi)\,\delta(\beta + \theta - 2\pi i). \qquad (3.28)$$

In the remaining part of this section, we are going to study correlators of the operator

$$\omega(x,t) = \frac{J_{\alpha\beta}}{2}\,\Phi^\alpha\Phi^\beta(x,t)\,, \tag{3.29}$$

associated to the massive perturbation of the critical bulk theory, and the one-point function of the 'disorder' operator μ.

3.4.1 Thermal operator

The simplest correlation function involving ω is the one-point function, defined as

$$\omega_0(t,g) \equiv \langle\omega(x,t)\rangle = \sum_{n=0}^{\infty} \langle 0|\omega(x,t)|n\rangle\langle n|\mathcal{D}|0\rangle\,, \tag{3.30}$$

the resolution of the identity being given by (2.11)

Since ω is the operator perturbing the critical theory in the bulk, it turns out to be proportional to the trace of the stress-energy tensor [69]. This implies, for free theories, the remarkable property that only two-particle states can be coupled to the vacuum

$$\langle 0|\omega(x,t)|A(\theta_1)\bar{A}(\beta_1)\rangle = 2\pi\; e^{-mt(\cosh\beta_1+\cosh\theta_1)+imx(\sinh\beta_1+\sinh\theta_1)}\,. \tag{3.31}$$

Thus, exploiting (3.28), ω_0 can be recast as

$$\omega_0(t,g) = 2\int_0^\infty d\theta\, \hat{R}^{A\bar{A}}(\theta)\, e^{-2mt\cosh\theta}\,. \tag{3.32}$$

Such formula is amenable to discuss the different defect interactions.

For free boundary conditions, the reflection matrix is trivially zero and the one-point function vanishes. In the case of fixed boundary conditions, instead, $\hat{R}^{A\bar{A}}(\theta) = -1$ and the short distance limit is easily derived

$$\omega_0(t) = -2\int_0^\infty d\theta\, e^{-2mt\cosh\theta} = -2K_0(2mt) \to 2\ln(mt)\,, \qquad mt\to 0\,. \tag{3.33}$$

Concerning the relevant perturbation, (3.32) assumes the form

$$\omega_0(t,\kappa) = -\kappa\int_{-\infty}^{+\infty} d\theta\, \frac{\exp[-2mt\cosh\theta]}{\cosh\theta+\kappa}\,. \tag{3.34}$$

In the limit of fixed boundary conditions ($\kappa \to \infty$) the previous result (3.33) naturally follows while, in order to study the large and short-distance regimes for arbitrary κ, it could be meaningful to manipulate a little bit the expression of ω_0. The differential equation

$$\frac{\partial \omega_0(t, \kappa)}{\partial(2mt)} - \kappa \omega_0(2mt, \kappa) = 2\kappa K_0(2mt) \tag{3.35}$$

descending from (3.34), helps deducing the large distance limit. Substituting the trial expansion $\omega_0(t, \kappa) \sim e^{-2mt}(2mt)^{-\gamma} \sum a_l (2mt)^{-l}$ into it, the asymptotic behavior $\omega_0 \sim \frac{-2\kappa}{1+\kappa} K_0(2mt)$ is recovered as $mt \to \infty$. On the other hand, the ultra-violet limit emerges more clearly if we look at the expression

$$\omega_0(t, \kappa) = -2\kappa \, e^{(2mt)\kappa} \int_{2mt}^{\infty} d\eta \, e^{-\eta\kappa} K_0(\eta) \, . \tag{3.36}$$

As far as $mt \to 0$, ω_0 always assumes finite values. Summarizing, in the infra-red regime ω_0 follows the asymptotic behavior typical of the fixed boundary conditions, while for small distances it remains finite, approaching zero as the coupling constant vanishes.

An analogous analysis can be performed for the marginal interaction. The one-point function (3.32) specializes to

$$\omega_0(t, \chi) = -\sin^2 \chi \int_{-\infty}^{+\infty} d\theta \, \frac{\sinh^2 \theta}{\cosh^2 \theta - \sin^2 \chi} \, e^{-2mt \cosh \theta} \, . \tag{3.37}$$

The corresponding differential equation

$$\frac{\partial^2 \omega_0(t, \chi)}{\partial(2mt)^2} - \sin^2 \chi \, \omega_0(t, \chi) = -2 \sin^2 \chi \, \frac{K_1(2mt)}{2mt} \tag{3.38}$$

allows to derive both the asymptotic limits. Exploiting a series expansion, as we did in the relevant case, the low-energy regime leads to two different behaviors (see Fig. (3.78))

$$\begin{cases} \omega_0(t, \chi) \to e^{-(2mt)}(2mt)\sqrt{\frac{\pi}{2}} \frac{2\sin^2 \chi}{\sin^2 \chi - 1} & \sin^2 \chi \neq 1, \\ \omega_0(t, \chi) \to -2K_0(2mt) & \sin^2 \chi = 1. \end{cases} \tag{3.39}$$

As regards the ultra-violet limit, $\omega_0(t, \chi) \sim 2 \sin^2 \chi \ln(2mt)$.
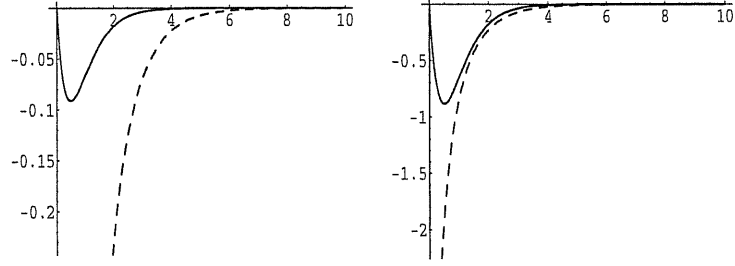
Figure 3.1: Infrared asymptotics of $\omega_0(t, \chi)$ as a function of mt. The dashed line corresponds to $\sin^2 \chi = 1$, while, the continuous ones are associated (from left to right) to $\sin^2 \chi = 0.3$ and $\sin^2 \chi = 0.7$.

We turn now the attention to the two-point functions involving the operator ω. Two different situations can occur.

Consider the case in which the operators lie on opposite sides of the defect line, i.e. $t_1 < 0$ and $t_2 > 0$. The correlator is given by

$$G_1(\rho_1, \rho_2; g) = \sum_{i,j} \langle 0|\omega(\rho_2)|i\rangle \langle i|\mathcal{D}|j\rangle \langle j|\omega(\rho_1)|0\rangle \,, \tag{3.40}$$

with the collective variable $\rho = (x, t)$. As before, the series contains only a finite number of terms. In order to perform the calculations, we need the expression of the 'defect' matrix element involving four particles

$$\langle \bar{A}(\beta_1)A(\theta_1)|\mathcal{D}|A(\theta_1')\bar{A}(\beta_1')\rangle = (2\pi)^2 \,[\hat{T}^{AA}(\theta_1)\hat{T}^{\bar{A}\bar{A}}(\beta_1)\,\delta(\theta_1 - \theta_1')\delta(\beta_1 - \beta_1') +$$
$$- \hat{R}^{A\bar{A}}(\theta_1)\hat{R}^{A\bar{A}}(\theta_1' - i\pi)\,\delta(\beta_1 + \theta_1)\delta(\beta_1' + \theta_1' - 2\pi i)]\,\,(3.41)$$

Introducing a redefinition of variables in terms of $t \equiv t_2 - t_1$ and $x \equiv x_2 - x_1$, we finally obtain

$$G_1(\rho_1, \rho_2; \kappa) = -\left[\frac{\partial F(mx, mt; \kappa)}{\partial(mt)}\right]^2 + \omega_0(t_1, \kappa)\omega_0(t_2, \kappa) \,, \tag{3.42}$$

$$F(x, t) = \int_{-\infty}^{+\infty} d\theta \, \frac{\exp[-t\cosh\theta + ix\sinh\theta]}{\cosh\theta + \kappa} \tag{3.43}$$

for the relevant perturbation and

$$G_1(\rho_1, \rho_2; \chi) = -\cos^4\chi \,\left[\frac{\partial^2 F(mx, mt; \chi)}{\partial(mt)^2}\right]^2 + \omega_0(t_1, \chi)\omega_0(t_2, \chi) \,, \tag{3.44}$$

$$F(x,t) = \int_{-\infty}^{+\infty} d\theta \, \frac{\exp[-t\cosh\theta + ix\sinh\theta]}{\cosh^2\theta - \sin^2\chi}, \tag{3.45}$$

for the marginal one. In the limit of an infinitely reflecting surface ($\kappa \to \infty$ and $\cos^2\chi \to 0$), only the vacuum expectation values of the two ω operators survive.

Another situation can happen, in which the two ω operators reside on the same half of the Minkowsi plane. Let us consider, for convenience, $t_2 \geq t_1 > 0$ and define $t \equiv t_2 - t_1$, $\bar{t} \equiv t_2 + t_1$, $x \equiv x_2 - x_1$, $r \equiv \sqrt{x^2 + t^2}$. The general expression for the two-point function is

$$G_2(\rho_1, \rho_2; g) = \sum_{i,j} \langle 0|\omega(\rho_2)|i\rangle\langle i|\omega(\rho_1)|j\rangle\langle j|\mathcal{D}|0\rangle. \tag{3.46}$$

Following the lines traced in [110], after straightforward calculations, we end up with

$$G_2(\rho_1, \rho_2; \kappa) = -[2K_0(mr) + \kappa F(m\bar{t}, mx)]^2 + \omega_0(t_1, \kappa)\omega_0(t_2, \kappa), \tag{3.47}$$

in the relevant case and

$$G_2(\rho_1, \rho_2; \chi) = -\left[2K_0(mr) + \sin^2\chi \, \frac{\partial^2 F(m\bar{t}, mx)}{\partial(mx)^2}\right]^2 + \omega_0(t_1, \chi)\omega_0(t_2, \chi), \tag{3.48}$$

for the marginal perturbation. As it appears clearly, the solutions found are invariant under translations along the x-axis, consistently with the picture adopted, which preserves momentum.

3.4.2 Disorder operators

Finally, we examine the one-point function of the operators Φ_α, non-local with respect to the ghost fields. First, we focus on the so-called twist operator μ, characterized by non-locality index $\alpha = 1/2$. In a second step, we consider disorder fields with a generic index α, extracting from their correlators the leading short-distance behavior, thanks to a 'cluster' expansion.

Twist operator μ

The one-point correlator can be written as follows

$$\mu_0(t, g) \equiv \langle\mu(x, t)\rangle = \sum_n \langle 0|\mu(x, t)|n\rangle\langle n|\mathcal{D}|0\rangle. \tag{3.49}$$

In this case, μ couples the vacuum to neutral states, composed of an even number of excitations. As a consequence, the sum does not truncate and, to explicitly evaluate (3.49), matrix elements involving an arbitrary (even) number of particles

$$\langle \bar{A}(\beta_n)...\bar{A}(\beta_1), A(\theta_n)...A(\theta_1)|\mathcal{D}|0\rangle = (-)^{\frac{n(n-1)}{2}} n! \, (2\pi)^n \prod_{k=0}^{n} \hat{R}^{A\bar{A}}(\beta_k) \, \delta(\beta_k + \theta_k) \qquad (3.50)$$

are required. In addition, since the defect operator $\mathcal{D}$ is responsible for processes involving only absorption or emission of couples of particles with opposite rapidities, μ_0 finally assumes the form

$$\mu_0(t,g) = \sum_{n=0}^{\infty} \frac{(-)^{n(n-1)/2}}{n!} \int \frac{d\beta_1}{2\pi} ... \frac{d\beta_n}{2\pi} \prod_{k=1}^{n} \left[\hat{R}^{A\bar{A}}(\beta_k) e^{-2mt\cosh\beta_k} \right] \cdot$$
$$\cdot f_n^{1/2}(-\beta_1, ..., -\beta_n, \beta_1, ..., \beta_n) \,. \qquad (3.51)$$

Exact expressions for the bulk Form Factors are given by

$$f_n^{1/2}(\theta_1, ..., \theta_n, \beta_1, ..., \beta_n) = \langle 0|\mu_{1/2}(0)|A(\theta_1)...A(\theta_n)\bar{A}(\beta_1)...\bar{A}(\beta_n)\rangle = (-)^{n(n+1)/2}|A_n| \,, \qquad (3.52)$$

where $|A_n|$ denotes the determinant of a matrix whose components read

$$A_{ij} = \frac{1}{\cosh \frac{\theta_i - \beta_j}{2}} \,.$$

In order to discuss the effects due to the different interactions localized along the defect line, (3.51) proves to be a good starting-point.

Again, free boundary conditions lead to the trivial solution $\mu_0 = 0$.

In the case of fixed boundary conditions, it is possible to recover the leading short-distance behavior of the one-point function, in an exact way. Since the reflection matrix component $\hat{R}^{A\bar{A}}$ is trivially -1, exploiting the theory of Fredholm determinants [95], μ_0 can be recast as

$$\mu_0(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \frac{d\theta_1}{2\pi} ... \frac{d\theta_n}{2\pi} \, e^{-2mt \sum_1^n \cosh\theta_k} \, |A_n| = \det\left(1 + \frac{1}{\pi} V(t) \right) \,, \qquad (3.54)$$

where the kernel is given explicitly by

$$V(\theta_i, \theta_j, t) = \frac{e(\theta_i, t) \, e(\theta_j, t)}{2\cosh \frac{\theta_i + \theta_j}{2}} \,,$$
$$e(\theta, t) = e^{-mt\cosh\theta} \,. \qquad (3.55)$$

Alternatively, μ_0 can be expressed as

$$\mu_0(t) = \prod_{i=1}^{\infty} \left(1 + \frac{1}{\pi} \lambda^{(i)}(t)\right)^{a_i(t)} , \tag{3.56}$$

where λ_i are the eigenvalues of the integral operator $V(t)$, distributed with multiplicity $a_i(t)$. As far as mt is finite, $V(t)$ is a square integrable operator possessing a discrete spectrum. However, in the short-distance limit, $mt \to 0$, this condition ceases to hold and the eigenvalues become dense in the interval $(-\infty, +\infty)$. Assuming their multiplicity to be uniform, thanks to the Mercer's theorem it is possible to derive exactly its behavior as a function of mt, alias $a(t) \sim \frac{1}{2\pi} \ln \frac{1}{mt}$. Given this information and considering the logarithm of μ_0

$$\ln \mu_0(t) = \sum_{i=1}^{\infty} a_i(t) \ln \left(1 + \frac{1}{\pi} \lambda^{(i)}(t)\right) , \tag{3.57}$$

it can be easily seen that the leading power-law behavior of the one-point function can be determined, once the eigenvalue problem is solved.

Therefore, the problem reduces to finding the exact solution to the eigenvalue equation

$$\int_{-\infty}^{+\infty} d\theta_2 \, \frac{1}{2 \cosh \frac{\theta_1 + \theta_2}{2}} \, \phi(\theta_2) = \lambda \, \phi(\theta_1) , \tag{3.58}$$

which, after proper changes of variables, assumes definitely the form

$$\int_0^{\infty} du \, \frac{1}{uv + 1} \, \xi(u) = \lambda \, \xi(v) . \tag{3.59}$$

The peculiar expression of the new kernel $K(u, v) = \frac{1}{uv+1}$ suggests to consider the Mellin transform of both sides of (3.59) [116, 117]. We finally end up with a simpler eigenvalue equation for the transformed quantities

$$(\lambda^2 - \tilde{K}(s)\tilde{K}(1 - s)) \, \tilde{\xi}(s) = 0 , \tag{3.60}$$

where

$$\tilde{K}(s) = \frac{\pi}{\sin \pi s} \quad , \quad 0 < \Re e \, s < 1 . \tag{3.61}$$

Some comments could be useful to evaluate the spectrum. Since the kernel is a symmetric function of its arguments and it is bounded, the spectrum has to be real and limited. Hence

$$\lambda_{\pm}(\tau) = \frac{\pm \pi}{\cosh \pi \tau} \quad , \quad \tau \in (-\infty, +\infty) . \tag{3.62}$$

Therefore, eq. (3.57) assumes the form

$$\ln \mu_0(t) = a(t) \int_{-\infty}^{\infty} d\tau \left[\ln \left(1 + \frac{1}{\pi} \lambda_+(\tau) \right) + \ln \left(1 + \frac{1}{\pi} \lambda_-(\tau) \right) \right], \qquad (3.63)$$

with $a(t)$ previously defined and the the critical exponent results [118]

$$\Delta_\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \left[\ln \left(1 + \frac{1}{\cosh \pi\tau} \right) + \ln \left(1 - \frac{1}{\cosh \pi\tau} \right) \right] =$$

$$= \frac{1}{\pi^2} \left[\frac{\pi^2}{4} - \frac{1}{2} \left(\arccos^2(1) + \arccos^2(-1) \right) \right] = -\frac{1}{4}. \qquad (3.64)$$

Concluding, the disorder operator μ follows the leading power-low behavior

$$\mu_0(t) \sim \frac{C}{(2t)^{\Delta_\mu}}, \qquad (3.65)$$

with $\Delta_\mu = -1/4$. This result is consistent with the intuitive idea that, upon approaching the impurity line in the ultra-violet limit, the operator μ, characterized by the conformal weight $(-\frac{1}{8}, -\frac{1}{8})$, starts interacting with its mirror image on the other side of the defect, along the identity channel. As a final remark, we hint at the possibility of sub-leading logarithmic corrections.

As regards the effects produced by the relevant perturbation, (3.51) behaves as

$$\mu_0(t; \kappa) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\pi} \right)^n \int_{-\infty}^{+\infty} d\theta_1 ... d\theta_n \left[\prod_{k=1}^{n} \frac{\kappa\, e^{-2mt \cosh \theta_k}}{2 \left(\cosh \theta_k + \kappa \right)} \right] |A_n| = \det \left(1 + \frac{1}{\pi} V(t; \kappa) \right) (3.66)$$

with the kernel

$$V(\theta_i, \theta_j, t; \kappa) = \frac{e(\theta_i, t; \kappa)\, e(\theta_j, t; \kappa)}{2 \cosh \frac{\theta_i + \theta_j}{2}},$$

$$e(\theta, t; \kappa) = \sqrt{\frac{\kappa}{\cosh \theta + \kappa}} \cdot e^{-mt \cosh \theta}. \qquad (3.67)$$

In the short-distance limit, $|V|^2$ becomes unbounded, the leading singularity being dictated by the fixed boundary conditions' one. Thus we find the same critical exponent as in the previous case.

More interesting is the marginal situation. From general considerations extrapolated from the Ising model [103,104], the non-universal nature of the marginal interaction is expected to

affect the non-local sector of the theory, inducing a critical exponent continuously dependent on the coupling constant. Indeed, μ_0 assumes the form

$$
\mu_0(t;\chi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\sin^2 \chi}{\pi} \right)^n \int_{-\infty}^{+\infty} d\theta_1 ... d\theta_n \left[\prod_{k=1}^{n} \frac{\sinh^2 \theta_k \; e^{-2mt \cosh \theta_k}}{2 \left(\cosh^2 \theta_k - \sin^2 \chi \right)} \right] |A_n| =
$$

$$
= \det \left(1 + \frac{\sin^2 \chi}{\pi} V(t;\chi) \right) , \tag{3.68}
$$

where

$$
V(\theta_i, \theta_j, t; \chi) = \frac{e(\theta_i, t; \chi) \, e(\theta_j, t; \chi)}{2 \cosh \frac{\theta_i + \theta_j}{2}} ,
$$

$$
e(\theta, t; \chi) = \sqrt{\frac{\cosh^2 \theta - 1}{\cosh^2 \theta - \sin^2 \chi}} \cdot e^{-mt \cosh \theta} . \tag{3.69}
$$

Repeating an analysis similar to the one carried out for the fixed boundary condition, but this time with a parameter depending on the coupling constant, in front of the kernel in (3.68), we finally obtain the critical exponent

$$
\Delta_\mu = \frac{1}{4} - \frac{1}{2\pi^2} \left[\arccos^2 (\sin^2 \chi) + \arccos^2 (-\sin^2 \chi) \right] . \tag{3.70}
$$

Disorder operators Φ_α

In this last paragraph, we discuss generic 'disorder' operators Φ_α, which pick up a non-locality phase $e^{\pm 2\pi i \alpha}$, when they are taken around the ghost fields in the Euclidean plane

$$
\Phi(z \, e^{2\pi i}, \bar{z} \, e^{-2\pi i}) \Phi_\alpha(0,0) = e^{2\pi i \alpha} \Phi(z, \bar{z}) \Phi_\alpha(0,0) ,
$$

$$
\bar{\Phi}(z \, e^{2\pi i}, \bar{z} \, e^{-2\pi i}) \Phi_\alpha(0,0) = e^{-2\pi i \alpha} \bar{\Phi}(z, \bar{z}) \Phi_\alpha(0,0) . \tag{3.71}
$$

In particular, we are interested in deriving the leading short-distance behavior of their one-point function in the case of fixed boundary conditions, in order to perform a comparison with the exact result previously obtained for the specific value $\alpha = \frac{1}{2}$.

The starting point is Eq. (3.51), where the Form Factors $f_n^{1/2}(-\beta_1, ..., \beta_n)$ must be replaced by the expression (2.9)

$$
f_n^\alpha(-\beta_1, ... - \beta_n, \beta_1, ..., \beta_n) = (-)^{n(n+1)/2} \left(\sin \pi \alpha \right)^n e^{-\left(\alpha - \frac{1}{2} \right) \sum_i^n 2\beta_i} |A_n| , \tag{3.72}
$$

with $|A_n|$ the determinant of the $n \times n$ matrix, whose components satisfy

$$A_{ij} = \frac{1}{\cosh \frac{\beta_i + \beta_j}{2}} \,. \tag{3.73}$$

In a compact form, we can rewrite

$$\Phi_0^\alpha(t) \equiv \langle \Phi_\alpha(x,t) \rangle = \sum_{n=0}^\infty \frac{1}{n!} \int_{-\infty}^{+\infty} d\beta_1 ... d\beta_n \, e^{-\rho \sum_j^n \cosh \beta_j} \, g_n^\alpha(\beta_1, ..., \beta_n) \,, \tag{3.74}$$

where $\rho = 2mt$ and

$$g_n^\alpha(\beta_1, ..., \beta_n) \equiv \frac{1}{(2\pi)^n} (\sin \pi\alpha)^n \, e^{-\left(\alpha - \frac{1}{2}\right) \sum_j^n 2\beta_j} \, |A_n| \,. \tag{3.75}$$

These last two relations appear suitable to perform a 'cluster' expansion, according to the technique exposed, for instance, in [98, 99] and already used in the bulk case (Chapter 2). Therefore, the logarithm of (3.74) assumes the form

$$\ln \Phi_0^\alpha(t) = \sum_{n=1}^\infty \frac{1}{n!} \int_{-\infty}^{+\infty} d\beta_1 ... d\beta_n \, e^{-\rho \sum_j^n \cosh \beta_j} \, h_n^\alpha(\beta_1, ..., \beta_n) \,, \tag{3.76}$$

where the functions h_n^α are proper combinations of the g_n^α. For our purposes, we need only the first few relations, which read explicitly [99]

$$\begin{aligned}
g_1^\alpha &= h_1^\alpha \\
g_{12}^\alpha &= h_{12}^\alpha + h_1^\alpha h_2^\alpha \\
g_{123}^\alpha &= h_{123}^\alpha + h_{12}^\alpha h_3^\alpha + h_{23}^\alpha h_1^\alpha + h_{31}^\alpha h_2^\alpha + h_1^\alpha h_2^\alpha h_3^\alpha \,.
\end{aligned} \tag{3.77}$$

The key point of the standard 'cluster' expansion is that, since the functions h_n depend only on rapidity differences, they contain a redundant variable. Thus, it is possible, at all orders, to extract the integral

$$\int_0^{+\infty} d\beta \, e^{-\rho \cosh \beta} = K_0(\rho) \,, \tag{3.78}$$

which is responsible for the logarithmic behavior $\ln \frac{1}{\rho}$, as $\rho \to 0$. The remaining integrals multiplying such result,

$$2 \sum_{n=1}^\infty \frac{1}{n!} \int_{-\infty}^{+\infty} d\beta_1 ... d\beta_{n-1} \, h_n^\alpha(\beta_1, ..., \beta_{n-1}, 0) \,, \tag{3.79}$$

give the approximate value of the critical exponent, provided that the 'cluster' condition

$$h_n(\beta_1, ..., \beta_n) = \mathcal{O}(e^{-|\beta_i|}) \tag{3.80}$$

is fulfilled, for $\Re e \beta_i \to \pm\infty$.

On the other hand, the fermionic ghost model displays a substantial difference. The functions h_n^α depend, by construction, on the sum of rapidities. Thus, only contributions of even order in the series (3.76) admit a redundant variable, finally leading to a logarithmic behavior. The remaining terms, of odd order, provide convergent pieces, useful to reconstruct the normalization constant of the one-point function.

In order to study explicitly the short-distance behavior of Φ_0^α, we focus the attention on the second order contribution. All we need to know is

$$h_{12}^\alpha(\beta_1, \beta_2) = -\left(\frac{\sin \pi\alpha}{2\pi}\right)^2 \frac{e^{-(2\alpha-1)(\beta_1+\beta_2)}}{\left(\cosh \frac{\beta_1+\beta_2}{2}\right)^2}. \tag{3.81}$$

Hence, substituting in (3.76), after straightforward calculations, we finally end up with

$$\ln \Phi_0^\alpha(t) = \Delta_\alpha \ln \frac{1}{2mt}, \tag{3.82}$$

where the critical exponent reads

$$\Delta_\alpha = -\frac{1-2\alpha}{2\pi} \tan(\pi\alpha). \tag{3.83}$$

For small values of the non-locality index, $\Delta_\alpha \to -\alpha/2$. However, we are mainly interested in the limit $\alpha \to 1/2$, where a comparison with the exact value $\Delta_{1/2} \equiv \Delta = -1/4 = -0.25$, previously derived, is possible. Eq. (3.83) leads to the result $\Delta_{1/2} \sim -1/\pi^2 \sim -0.10$, independent of α. This large discrepancy suggests that the 'cluster' approximation, for this particular non-locality index, fails to reproduce the exact critical exponent with accuracy, but, nevertheless, hints at its correct sign. Finally, Fig. 3.2 displays the ratio $\frac{-\Delta_\alpha}{\alpha/2}$, in order to make visible deviations from the small-α behavior.
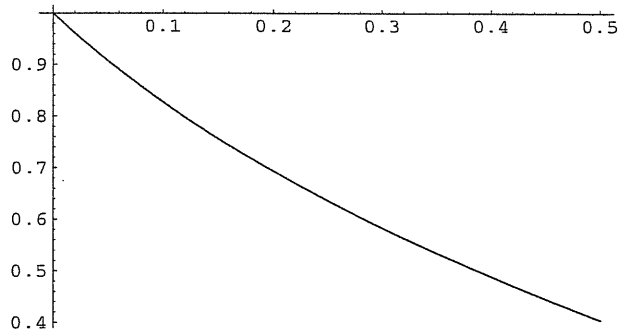
Figure 3.2: $-\frac{\Delta_\alpha}{\alpha/2}$ as a function of the non-locality index α, for $\alpha \in \left[0, \frac{1}{2}\right]$.

3.5 Conclusions and open questions

In this chapter we have studied the effects induced by a defect interaction on the free theory of massive fermionic ghosts.

Working in the Lagrangian approach, we have dealt with two defect perturbations, respectively, of relevant and marginal nature. Explicit expressions for the reflection and transmission matrices have been derived. A careful analysis of their excitation spectra has pointed out the possibility of resonances and instabilities in the former case, and the occurrence of imaginary residues, relative to poles in the scattering amplitudes, in the latter one. Successively, we turned our attention to the exact computation of correlation functions, involving the most interesting operators in the theory, i.e. ω, local in the ghost fields, and μ, belonging to one of the non-trivial sectors of the model. In the marginal situation, a non-universal behavior in the one-point function of the 'disorder' operator μ has clearly emerged. Finally, we have concluded with the analysis of the most general 'disorder' fields Φ_α, characterized by non-locality index α. The leading short-distance behavior of their one-point function has been investigated by means of the 'cluster' expansion [98, 99].

It is worth noticing that a delicate point of the present discussion concerns the comparison between the bootstrap approach and the Lagrangian description, in order to derive explicit expressions for the reflection and transmission amplitudes. In the former case, a richness of solutions descends but their physical explanation and 'classification', in terms of a fixed number of parameters related to the bulk S-matrix, results problematic. On the other hand, the Lagrangian approach, though subjected to the strong restriction of dealing only with

local interactions, allows for a limited number of solutions, amenable of an easiest control. For instance, besides the defect perturbations already introduced, analyzing other kind of interactions could help identifying new boundary conditions and, possibly, the operator content of the boundary theory.

Finally, we conclude with a remark on the simplified problem of a pure reflecting surface. As hinted at the end of the second section in relation to the free Dirac massive fermions, free theories, derived as limit of interacting ones, admit a richer structure, as it appears clearly in the bootstrap approach. It would be tempting, in this boundary case, to find an interacting theory, if any, behind the fermionic ghost model.

Chapter 4

Discussion and outlook

In this chapter, we briefly summarize some results obtained in Part I, with no intention of repeating the same discussions at the end of Chapter 2 and 3, but with the idea of putting the emphasis on the relations existing among the four free theories (bosonic and fermionic ghosts and their respective ordinary counterparts). Only, let us recall here a notation which will be useful in the following discussion of the massive case

$$S = \begin{cases} 1 & \text{for bosons} \\ -1 & \text{for fermions} \end{cases} \tag{4.1}$$

$$\varepsilon = \begin{cases} 1 & \text{for ordinary fields} \\ -1 & \text{for ghosts} . \end{cases} \tag{4.2}$$

At the conformal level, commuting fields share the same space of states and the spectrum of the corresponding Hamiltonians [37] (the same property holds for anti-commuting fields).

Out of the critical point, we have focused our attention on the non-local sectors of the massive free theories, containing the so-called disorder operators Φ_α, characterized by non-locality index α.

In the infinite plane, we have established a relation among such systems, at the level of the (two-point) correlation functions of their disorder fields. In particular, such objects *do not depend* on the ghost/ordinary character of the theory (ε), but only on the commutation rules of the associated free massive fields (S). In addition, we have recovered an already

known inversion property equating the correlators in the bosonic theories to the inverse of the same[1] correlators in the fermionic models (2.52)). These results reinforce the conviction that some 'symmetry' among the four different free models becomes apparent when a Fock-type basis for the space of states is considered, as it occurs *by construction* in the off-critical case.

The same analysis can be repeated, after the introduction of a boundary (here, for simplicity, we have in mind the fixed boundary conditions), at the level of the *one-point* function of the disorder fields Φ_α. In this case, the new geometry modifies substantially the aforementioned symmetry. Indeed, it reinstates the dependence on the variable ε, which discriminates the ghost/ordinary character of a theory, but still preserves some inversion relations. Specifically, the one-point function, involving the generic operator Φ_α, in the complex bosonic theory is the inverse of the same correlator in the fermionic ghost model, and an analogous relation holds in the remaining two cases. This can be seen, by looking at the explicit expressions of the form-factors (2.9) and of the matrix elements of the defect operator (3.50).

We conclude with an observation concerning the kernel (3.55), i.e.

$$V(\theta_1, \theta_2, t) = \frac{e(\theta_1, t)\, e(\theta_2, t)}{2 \cosh \frac{\theta_1 + \theta_2}{2}} \, ,$$
$$e(\theta, t) = e^{-mt \cosh \theta} \, . \tag{4.3}$$

In an analogous way, the kernel $\tilde{V}$ associated to the Painlevé III theory [83, 119, 120], is defined by

$$\tilde{V}(\theta_1, \theta_2, t) = \frac{e(\theta_1, t)\, e(\theta_2, t)}{2 \cosh \frac{\theta_1 - \theta_2}{2}} \, ,$$
$$e(\theta, t) = e^{-mt \cosh \theta} \, . \tag{4.4}$$

In the limit of short distances $mt \to 0$, in both cases, the corresponding eigenvalues become dense, yielding $\lambda_\pm(p) = \frac{\pm \pi}{\cosh \pi p}$ in the first situation and $\lambda(p) = \frac{\pi}{\cosh \pi p}$, $p \in (-\infty, +\infty)$, in the latter. It is then possible to argue, that, in such limit, the one-point function of the disorder field μ, with non-locality index $\alpha = 1/2$ satisfies

$$\langle \mu(t) \rangle = \det\left(1 + \frac{1}{\pi} V(t)\right) \sim \det\left(1 + \frac{1}{\pi} \tilde{V}(t)\right) \cdot \det\left(1 - \frac{1}{\pi} \tilde{V}(t)\right) \, , \qquad t \to 0 \, . \tag{4.5}$$

---

[1]i.e. containing disorder fields with the same non-locality indices.

This relation is particularly important, because it connects in a simple way a kernel (V), which appears typically in boundary systems (the plus sign is due to the reflection properties of the boundary), with a kernel $(\tilde{V})$, which turns out to play a crucial role in many significant bulk problems (see e.g. the expressions of the bulk form factors (2.9)).

Part II

Massless perturbation

Chapter 5

OSP(1|2) flow

In this chapter, we consider a nonlinear sigma model, whose vector field possesses one *commuting* component and two *anticommuting* ones, transforming under the global orthosymplectic symmetry $OSP(1|2)$. It is the simplest representative of a hierarchy of sigma models, based on the superalgebras $OSP(m|2n)$, and it is nothing but the 'supersymmetric' generalization of an ordinary $O(N)$ model, in a way that will become clear soon. In a rather technical, but compact, way, it is known as the $S^{(0|2)} \cong OSP(1|2)/SP(2)$ (sphere) sigma model. According to the value of the bare coupling constant, such system may flow, at large distances, to a gapped ($g_\sigma < 0$) or to a gapless ($g_\sigma > 0$) regime. Hereafter, we focus on the latter situation, where the initial $OSP(1|2)$ symmetry spontaneously breaks down to $SP(2)$.

Before entering the problem at the heart of our discussion, concerning the analysis of the off-critical features of such theory, we outline its possible physical realizations with particular emphasis on the recent developments about the strictly related loop models.

5.1 Introduction

The connection between loop models and theories of an N-component scalar field ϕ, with $O(N)$ symmetry in a d-dimensional Euclidean space, is a well consolidated issue [31,32]. In the early seventies, the continuum Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{1}{2} m^2 \phi \cdot \phi + \frac{1}{8} \lambda (\phi \cdot \phi)^2 \qquad (5.1)$$

was first introduced in the context of loop models, by de Gennes [31], who realized that the statistics of long polymer chains (namely self-avoiding walks) could have been efficiently reproduced, by taking the $N \to 0$ limit in (5.1) . The parameter λ takes into account the possibility of crossings among loops. Even though the original model [31] was focused on a weakly repulsive regime ($\lambda > 0$), where such events were disfavored, nevertheless, in our future discussion, λ is going to play a key role .

An analysis lead in the framework of the Landau Ginzburg theory reveals that the system (5.1) develops a finite correlation length as long as $m^2 > 0$ (high-temperature phase), while displaying a critical behavior as m^2 reaches the value zero. For m^2 becoming negative (low-temperature phase), the field ϕ assumes a non-zero vacuum expectation value, indicating that the initial $O(N)$ symmetry spontaneously breaks down to $O(N-1)$, with the consequent appearance of $(N-1)$ long-range massless excitations (the so-called Goldstone modes). Such Goldstone phase is effectively described by a nonlinear sigma model with target manifold $O(N)/O(N-1) \cong S^{N-1}$, alias a sphere.

On the other hand, starting from renormalization group (RG) considerations, it clearly emerges that the occurrence of a phase transition depends strongly on the dimensionality of the system and, in the case we are interested in, also on the value of N. For $d > 2$ the theory always exhibits a second-order phase transition, while for $d = 2$ such phenomenon is crucially related to the peculiar value of N. Indeed, for $N > 2$ only the massive high-temperature phase, where the symmetry is unbroken, sets in. However, for $N < -2$ a first order phase transition is expected [28] and, in the range $-2 \leq N \leq 2$, a second order one occurs. Let us remark here that, for N not a positive integer, the theory is supposed to be defined by analytic continuation of its perturbative expansion [28]. For N integer, however, the use of a 'supersymmetric' version of (5.1), à la Parisi Sourlas [32], allows for a rigorous treatment, which we will discuss in detail in relation to the $OSP(1|2)$ flow. In reference [32], the introduction of such approach was motivated by the necessity to overcome the difficulties intrinsically related to the $N \to 0$ limit, particularly, those concerning the physical understanding of the nature itself of the phase transition, where a negative number of spin-waves (Goldstone modes) appeared.

A crucial problem, soon arisen in this context [58, 121–123] and which has been revisited only very recently [28], is connected to the study of the various phases and transitions of the

$O(N \leq 2)$ model in $d = 2$. Particular attention is addressed to the low-temperature regime where, from our previous discussion, a Goldstone phase is expected, whose identification, nevertheless, is far from obvious.

The first attempt [122,123], in order to extract exact quantities (e.g. critical exponents) from (5.1), was based on the idea of performing a Coulomb gas analysis on lattice models of strictly non-intersecting loops, which were argued to exhibit the same universality properties as (5.1), being however simpler. This has been realized [121], considering a lattice version of (5.1) (defined for convenience on a honeycomb lattice) and implementing a truncation procedure on its high-temperature expansion. The resulting partition function has been expressed in the very simple form

$$Z = \sum K^{\# \, \text{monomers}} N^{\# \, \text{loops}} \,, \tag{5.2}$$

where the sum is taken over all configurations of self-avoiding and mutually avoiding loops on the hexagonal lattice, and $K \sim 1/T$. The main advantage of such formula consists precisely in the fact that, now, N acts simply as a parameter, assuming any value, with no restriction to positive integers. The critical point (also called *dilute*) is known exactly, $K_c = \left(2 + \sqrt{2 - N}\right)^{-1/2}$ [122], and for $-2 \leq N \leq 2$ the critical exponents of such second order phase transition have been computed explicitly in [122,123]. Moreover, parameterizing $N = -2 \cos \pi g$, $g \in [1, 2]$, the associated central charge reads

$$c = 1 - 6\frac{(g-1)^2}{g} \,, \tag{5.3}$$

leading to $c = 0$ in the case of dilute polymers ($N \to 0$). Turning to the low-temperature region $K_c < K < \infty$, a unique massless phase, called *dense phase*, which varies with N ($-2 < N < 2$) but is independent of K, sets in. Again, the exact critical exponents can be found in [122,123] and the corresponding central charge is given by (5.3), but now with the parameter $g \in [0, 1]$. In the case of dense polymers ($N \to 0$) the value $c = -2$ follows.

Shortly, after this analysis, it seemed natural to identify this low-temperature phase, in which loops never intersect, with the Goldstone phase originating from (5.1), which analogously occurs in a low-temperature region [58]. However, some difficulties were soon recognized [57,58], concerning a mysterious discrepancy between the non-interacting nature of the broken phase, predicting $c = N - 1$, and the conformally invariant theory of the dense

loop phase, characterized by the central charge (5.3), $g \in [0, 1]$. This could be immediately seen, in the polymers' case ($N \to 0$), where the values of the central charge were given, respectively, by $c = -1$ in the former situation, and $c = -2$, in the latter. Despite this initial perplexity, only very recently such problem has been tackled again, but from a different point of view [25, 28].

The crucial idea explaining the failure of the former interpretation is basically exposed in [25], while a more rigorous treatment, supported by a numerical analysis and a physical realization, in terms of an appropriate lattice model [46], can be found in [28]. The weak point of the previous discussion is embodied in the truncation procedure of the high-temperature expansion, as outlined in [121]. Indeed, the implicit assumption that the universality class does not change, going from (5.1) to (5.2), is still true in the high-temperature and critical regions, but fails in the low-temperature regime. Therefore, starting conceptually from the de Gennes' model of weak repulsion, where crossing of loops are disfavored, but allowed, the critical (dilute/dense) theories must be modified, by adding to their corresponding fixed point actions, the most relevant operator, responsible for the crossing of lines. Such (scalar) field is the so-called four-leg operator, characterized by conformal weight [123]

$$h_4 = g - \frac{(g-1)^2}{4g}.$$

(5.4)

In the high-temperature region, it is marginal at $N = 2$ and irrelevant for $-2 \leq N < 2$, thus confirming that the dilute critical point is indeed robust, remaining in the same universality class as (5.1). However, in the low-temperature regime it becomes relevant in the whole range of N, inducing a flow to a stable (non-interacting) fixed point. It is exactly *this* critical point the one which describes the massless Goldstone phase of (5.1).

Physical realizations of such loop models with self-crossing have been extensively studied in the context of Lorentz lattice gases [46].

5.1.1 The continuum limit field theory

We now analyze the main features of the field theory describing the low-temperature phase of the $O(N)$ model. As noticed in [25, 32], a convenient way of describing the peculiar range of N we are interested in, namely $-2 \leq N \leq 2$, relies on the introduction of m bosonic and

n fermionic variables, such that $N \equiv m - 2n$. Moreover, the loop expansion of the $O(N)$ model can be reproduced as well, by considering a system with $OSP(m|2n)$ symmetry. The interactions among the Goldstone excitations are thus governed by a nonlinear sigma model with target space $OSP(m|2n)/OSP(m-1|2n) \cong S^{m-1|2n}$, a supersphere, which is characterized by a single coupling constant g_σ [25,28]. The coordinates on the target manifold can be explicitly parameterized in terms of m commuting variables x_i, and $2n$ anticommuting ones η_j, satisfying $\sum_{i=1}^{m} x_i^2 + \sum_{j=1}^{n} \eta_{2j-1}\eta_{2j} = 1$. The nonlinear sigma model action reads

$$\mathcal{A} = \frac{1}{g_\sigma} \int d^2r \left[\sum_{i=1}^{m} (\partial_\mu x_i)^2 + \sum_{j=1}^{n} \partial_\mu \eta_{2j-1} \partial_\mu \eta_{2j} \right] , \tag{5.5}$$

where we adopt the convention $e^{-\mathcal{A}}$ for the Boltzmann weight. The corresponding perturbative beta function for the coupling g_σ, to the leading order, is simply given by

$$\beta(g_\sigma) \equiv \frac{dg_\sigma}{d\ln L} \propto (N-2)\, g_\sigma^2 = (m - 2n - 2)\, g_\sigma^2 . \tag{5.6}$$

At first sight, it emerges that, in the range of N we are considering, if g_σ is positive the systems renormalizes in the infra-red to a weak coupling regime, where the symmetry is spontaneously broken. The resulting critical theory is described by free massless Goldstone excitations, namely, $m - 1$ bosons and $2n$ fermions, yielding the conformal central charge $c = m - 1 - 2n = N - 1$. On the other hand, if the sign of g_σ is reverted, at large distances, a dynamical mass generation is expected, along with the restoration of the initial $OSP(m|2n)$ symmetry. It should be noted that the first situation, which is exactly the one we are going to discuss in detail, is perfectly well-defined, because the Mermin-Wagner theorem, which seems apparently violated by it, actually does not hold for theories where the symmetry cannot be realized as unitary operations on the vector field.

In particular, the object of our interest is the simplest example of systems with $OSP(m|2n)$ symmetry and positive coupling, where $n = m = 1$ $(N = -1)$. It is the $S^{(0|2)}$ supersphere sigma model, invariant under global $OSP(1|2)$ symmetry. Resolving the constraint, using the parameterization

$$x_1 = x = 1 - \phi\bar{\phi}$$

$$\eta_1 = \phi$$

$$\eta_2 = \bar{\phi} , \tag{5.7}$$

and rescaling the fields according to $\phi \to \phi \left(\frac{2}{g_\sigma}\right)^{1/2}$ and $\bar{\phi} \to \bar{\phi} \left(\frac{2}{g_\sigma}\right)^{1/2}$, the action (5.5) can be recast in the more transparent form

$$\mathcal{A} = \int d^2 r \left[\partial_\mu \phi \partial_\mu \bar{\phi} - \frac{g_\sigma}{2} \phi \bar{\phi} \, \partial_\mu \phi \partial_\mu \bar{\phi}\right] , \qquad (5.8)$$

where only the $SP(2)$ symmetry is manifest, the $OSP(1|2)$ being realized non-linearly in terms of the fields ϕ, $\bar{\phi}$. At large distances, g_σ vanishes, yielding a theory of free symplectic fermions, with central charge $c = -2$. Therefore, the IR fixed point is just free massless fermionic ghosts. The operator which perturbs such critical point, $\phi \bar{\phi} \, \partial_\mu \phi \, \partial_\mu \bar{\phi}$, has been conjectured to correspond to the $\varphi_{1,5}$ field in the associated Kac table[1], with conformal weights $h = \bar{h} = 1$ [26]. Hence, it is a marginal operator, or more precisely a marginally irrelevant one. Moreover, $\phi \bar{\phi} \, \partial_\mu \phi \, \partial_\mu \bar{\phi}$ has been identified with the logarithmic partner of the operator $\partial_\mu \phi \, \partial_\mu \bar{\phi}$, which, translated into the language of the Kac table, is equivalent to the primary field $i \, \varphi_{2,1}$ [26].

Summarizing, as concern the low-temperature phase of dense loops, the usual dense loop phase, in which loops never intersect [122], is characterized by central charge $c = -7$ (see eq. (5.3) with $g = 1/3$) and turns out to be unstable with respect to crossing of lines (the most relevant four-leg operator possesses conformal weights $h_4 = \bar{h}_4 = 0$). Therefore, this phase flows, at long wavelength, to an attractive stable fixed point, which is identified with the generic Goldstone phase of the supersphere sigma model $S^{(0|2)}$, with $SP(2)$ broken symmetry. The knowledge of the perturbative beta function indicates that such IR non-interacting critical point occurs at zero coupling constant g_σ. Despite no further information can be extracted from a mere perturbative analysis, nevertheless, it is believed [25,29] that a transition, occurring at some value g^*, may separate the weak coupling region from the strong coupling regime, which is supposed to be massive. The nature of this presumed critical point is extremely elusive and has not been fully understood yet. Some hypothesis [29] tentatively ascribe it to the dilute point introduced by Nienhuis [122], but, at the moment, no evidence of an integrable flow connecting such critical point to the generic Goldstone phase exists.

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[1]Let us recall here, that the CFT with $c = -2$ can be *formally* regarded as the minimal model $\mathcal{M}_{1,2}$, allowing for an interpretation of its operator content in terms of an 'extended' Kac table [52].

5.2   An outline of the OSP(1|2) massless flow

In this section we delineate the main features of the massless flow encoded in the action (5.8). The operator which perturbs the critical IR $c = -2$ CFT is a marginally irrelevant logarithmic field. First, we propose a consistent scattering theory, giving the exact expression of the S-matrix, which rules the scattering processes among the massless excitations, composing the spectrum. In a second step, by means of thermodynamic Bethe ansatz calculations, we present some results supporting our conjectured S-matrix.

5.2.1   The scattering theory

The most convenient way to derive the scattering properties of a theory, describing a massless flow, consists in looking directly at its large distance limit. The main motivation resides in the observation that the only stable massless excitations are those which remain at the IR fixed point[2]. A thorough study on this subject can be found in [125], where it originally appeared, and in [81, 86, 126], where it was extended and deepened.

In the specific case at the heart of our attention, the IR theory is simply expressed in terms of one free symplectic fermion, whose components ϕ and $\bar{\phi}$ transform as a doublet under $SP(2)$ symmetry. No other stable excitations are expected. At the classical level, the dimensionless nature of the coupling constant makes the theory (5.8) conformally invariant. However, in the quantum case, the renormalization of the running coupling constant g_σ, yields a non-vanishing beta function. As a consequence, the model still remains massless, but conformal invariance breaks and a mass scale M appears, marking a cross-over between the short- and large-distance regimes. The Goldstone excitations, almost free in the IR, interact in correspondence of such a mass scale, where a non-trivial scattering is expected to occur. Due to their massless character, the components of the symplectic fermion split into right-moving ($R(\theta)$ and $\bar{R}(\theta)$) and left-moving ($L(\theta)$ and $\bar{L}(\theta)$) parts, whose dispersion relations, in terms of the rapidity variable $-\infty \leq \theta \leq \infty$, read

$$\begin{cases} E = p = \frac{M}{2}\, e^{\theta} & \text{for the right-movers,} \\ E = -p = \frac{M}{2}\, e^{-\theta} & \text{for the left-movers.} \end{cases} \tag{5.9}$$

---

[2]Massless theories can also be regarded as an appropriate limit of massive ones. For this approach, see for instance reference [124].

Since we are dealing with a supersymmetric generalization of the well know $O(N)$ nonlinear sigma model [38], whose integrability has been proven in [70], our model is expected, as well, to be integrable [26]. Therefore, the numbers of left and right particles (along with their sets of momenta) result separately conserved and, if we ignore the conceptual difficulties intrinsically concerned with the very definition of a massless scattering[3], the n-body collision processes turn out to enjoy the factorization property. As in the massive situation, the basic objects are the two-body S-matrices.

Two distinct scattering configurations can be recognized, namely, the one between two particles of the same nature (both right- (RR) or left-movers (LL)) and the interaction involving, simultaneously, a right-mover and a left-mover (RL) or (LR). In the former case, Lorentz invariance fixes the scattering matrices S_{LL} and S_{RR} to depend uniquely on the ratio of the two momenta, the Mandelstam variable $s = (p_1 + p_2)^2$ vanishing. Therefore, no tracks of the mass scale M remain and, in principle, such amplitudes follow solely from the properties of the IR (free) fixed point, yielding the trivial results $S_{RR}(\theta) = S_{LL}(\theta) = -1$. Actually, our situation is more subtle and this earlier expectation will prove to be misleading. We will come back on this point during the discussion about the (RL) case, where, on the contrary, a genuine scattering process takes place, in correspondence of the mass scale M. Indeed, since the Lorentz-invariant variable s assumes the form $M^2 e^{\theta_1 - \theta_2}$, the scattering amplitude $S_{RL}(s)$ *does* depend on the mass scale.

Let us concentrate on the (RL) channel. A key tool, in order to derive the exact S-matrix and the corresponding spectrum, consists in implementing the action of the symmetry on the space of particles. Since the scattering of particles in the vector representation of $O(N)$ has been extensively studied in the massive case [38], here, we adopt the strategy to take full advantage of such results, extending them to the massless situation, and introducing, in a second step, the supersymmetric generalization, finally specializing to our peculiar flow. The formal commutation relation for right- and left-movers in the defining representation of $O(N)$ reads

$$R_a(\theta_1) L_b(\theta_2) = S_{ab}^{cd}(\theta_1 - \theta_2) \, L_d(\theta_2) R_c(\theta_1) \,, \tag{5.10}$$

where the indices range from 1 to N and the corresponding S-matrix is given by

$$S_{ab}^{cd}(\theta) = E \, \sigma_1(\theta) + P \, \sigma_2(\theta) + I \, \sigma_3(\theta) \,, \tag{5.11}$$

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[3]See for example [81, 125] and for a very simple review [127]

in terms of the projectors of the $O(N)$ group

$$E_{ab}^{cd} = \delta_{ab}\delta^{cd},$$ (5.12)

$$P_{ab}^{cd} = \delta_a^c\delta_b^d,$$

$$I_{ab}^{cd} = \delta_a^d\delta_b^c.$$

The amplitudes σ_1, σ_2 and σ_3 may be derived imposing the unitarity, crossing and factorization relations [38, 86]. For the massless case, unitarity and crossing symmetry lead the following constraints [86]

$$\sigma_2(\theta)\sigma_2(\theta)^* + \sigma_3(\theta)\sigma_3(\theta)^* = 1,$$

$$N\sigma_1(\theta)\sigma_1(\theta)^* + \sigma_1(\theta)\sigma_2(\theta)^* + \sigma_1(\theta)\sigma_3(\theta)^* + \sigma_2(\theta)\sigma_1(\theta)^*\sigma_3(\theta)\sigma_1(\theta)^* = 0,$$ (5.13)

$$\sigma_2(\theta)\sigma_3(\theta)^* + \sigma_3(\theta)\sigma_2(\theta)^* = 0,$$

and

$$\sigma_1^*(\theta) = \sigma_3(i\pi + \theta),$$

$$\sigma_2^*(\theta) = \sigma_2(i\pi + \theta).$$ (5.14)

As concern the Yang-Baxter equations (1.30), the massless situation is more delicate with respect to the massive one. If particles of the same kind participate to the three body scattering process, only two-body S-matrices in the same channels (RR) (or (LL)) will appear. On the other hand, collisions involving, for instance, two right-movers and one left-mover will yield *one* S_{RR} and *two* scattering amplitudes of type S_{RL} or S_{LR}. Hence, implementing the Yang-Baxter equations on such configuration (RRL), where according to symmetry reasons, also S_{RR} may be expressed in terms of the $O(N)$ invariant tensors (5.12) (the eventual trivial solution is included in the projectors' expansion), it is possible to derive the following fundamental relation

$$\sigma_2(\theta_{12})\sigma_3(\theta_{13})\sigma_3(\theta_{23}) + \sigma_3(\theta_{12})\sigma_3(\theta_{13})\sigma_2(\theta_{23}) = \sigma_3(\theta_{23})\sigma_2(\theta_{13})\sigma_3(\theta_{12}).$$ (5.15)

It can be easily realized that a diagonal interaction in the (RR) configuration (i.e. $\sigma_3 = \sigma_1 = 0$) turns out to be inconsistent with the requirement of non-trivial scattering amplitudes in the channels (RL), (LR) ($\sigma_1, \sigma_2, \sigma_3 \neq 0$). Therefore, the simple solution $S_{RR} = S_{LL} =$

-1 must be discarded, in favor of an $O(N)$-invariant one. Solving the other constraints descending from (1.30), we end up with

$$\sigma_1(\theta) = -\frac{2\pi i}{(N-2)(i\pi - \theta)}\,\sigma_2(\theta)\,,$$

$$\sigma_3(\theta) = -\frac{2\pi i}{(N-2)\theta}\,\sigma_2(\theta)\,, \qquad (5.16)$$

$$\sigma_2(\theta) = \frac{\Gamma(1 - \frac{\theta}{2\pi i})\Gamma(\frac{1}{2} + \frac{\theta}{2\pi i})\Gamma(-\frac{1}{N-2} + \frac{\theta}{2\pi i})\Gamma(\frac{1}{2} - \frac{1}{N-2} - \frac{\theta}{2\pi i})}{\Gamma(\frac{\theta}{2\pi i})\Gamma(\frac{1}{2} - \frac{\theta}{2\pi i})\Gamma(1 - \frac{1}{N-2} + \frac{\theta}{2\pi i})\Gamma(\frac{1}{2} - \frac{1}{N-2} + \frac{\theta}{2\pi i})}\,.$$

This outcome coincides with the formula derived in the case of the Gross Neveu model, defined for $N > 2$. It is important to notice that in the range of N we are considering, $-2 \leq N \leq 2$, no stable bound state poles appear in the physical strip, in agreement with the nature of the massless scattering, in which the physical strip itself shrinks to a point.

So far, our discussion has been lead along the lines traced in [38], for massive flows with $N \geq 2$, and in [86], where an analytic continuation of (5.16) to $N \to 0$ has been proposed to explain the on-shell properties of the random bond Ising model. Now, it is important to extend our result to $N < 1$. Such pursuit has been successfully achieved for scattering processes involving impenetrable particles, $\sigma_2 = 0$ [128], relying on an algebraic formulation in terms of the the Temperly Lieb algebra [129], at least formally[4], for arbitrary values of N.

An analogous generalization, in the case of $\sigma_2 \neq 0$, has been performed, at the moment for massive flows [26], only for negative *integer* values of N. The main idea, which consists in substituting the defining representation of $O(N)$ with the vector representation of the orthosymplectic algebra $OSP(m|2n)$ ($N = m - 2n$), can be applied as well to the massless case [47]. The projectors arising from the tensor product of such representation with itself may be recast according to [26]

$$E_{ab}^{cd} = \delta_{a\bar{b}}\,\delta^{c\bar{d}}\,(-)^{x(a)}(-)^{x(c)}$$

$$P_{ab}^{cd} = (-)^{p(a)p(b)}\,\delta_a^c\,\delta_b^d\,, \qquad (5.17)$$

---

[4]For comments on this subtle point, see [27].

where the following convention on the indices has been adopted

$$\begin{cases} \bar{a} = a & a = 1, ..., m, \\ \bar{a} = n + a & a = m + 1, ..., m + 2n \end{cases}, \quad \bar{\bar{a}} = a;$$

$$\begin{cases} x(a) = 1 & a = m + 1, ..., m + n, \\ x(a) = 0 & \text{otherwise}; \end{cases}$$

$$p(a) = x(a) + x(\bar{a}) = \begin{cases} 1 & \text{fermions}, \\ 0 & \text{bosons}. \end{cases} \tag{5.18}$$

In this supersymmetric generalization, the identity I remains unchanged, P is the graded permutation operator and the matrix elements of E are obtained by contracting the ingoing and outgoing indices, using the matrix

$$J = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}. \tag{5.19}$$

In the specific example we are interested in, the initial $OSP(1|2)$ symmetry breaks down to $OSP(0|2) \equiv SP(2)$. As a matter of fact, no grading is actually present (contrary to the massive trajectory obtained by setting $g_\sigma < 0$ [26]), and the tensor J turns out to coincide with the symplectic form introduced in Section 1.2.2. Moreover, the conceptual difficulties connected, in the graded cases [26], to the issue of non-unitarity[5] do not seem to arise in this context, a part from the obvious observation that the ghost components are not exactly the Hermitian conjugate of each others.

Therefore, the commutation relations for the particles in the vector representation of $SP(2)$ translate into

$$\begin{aligned} R_a(\theta_1) L_b(\theta_2) &= \sigma_1 (1 - \delta_{ab}) [\sum_d 2 L_d(\theta_2) R_a(\theta_1) - 2 L_a(\theta_2) R_a(\theta_1) + \\ &+ \sum_c L_c(\theta_2) R_c(\theta_1) - \sum_{c,d} L_d(\theta_2) R_c(\theta_1)] + \\ &+ \sigma_2 L_b(\theta_2) R_a(\theta_1) + \sigma_3 L_a(\theta_2) R_b(\theta_1). \end{aligned} \tag{5.20}$$

---

[5]For a detailed discussion concerning the related topics of unitarity, reality, Hermitian analiticity and 'one-particle unitarity', see reference [130].

Considering the combinations

$$R = R_1 + i R_2 \,,$$
$$\bar{R} = R_1 - i R_2 \,, \tag{5.21}$$

(and their analogous counterpart in the left sector) it is possible to pass from the (neutral) excitations, in the defining representation, to the usual charged particles, which constitute the IR massless spectrum and correspond to the components of the symplectic fermion. In terms of these latter fields, eq. (5.20) assume in the (RL) channel the form

$$R(\theta_1)L(\theta_2) = (\sigma_3 - \sigma_2)\, L(\theta_2)R(\theta_1)$$
$$R(\theta_1)\bar{L}(\theta_2) = (\sigma_3 - \sigma_1)\, L(\theta_2)\bar{R}(\theta_1) + (\sigma_1 - \sigma_2)\, \bar{L}(\theta_2)R(\theta_1) \,, \tag{5.22}$$

along with similar relations in which the role of the particles with and without the bar sign is interchanged. Finally, exploiting the expressions previously derived for the functions $\sigma_{1,2,3}$ (5.16) and taking the limit $N = (m - 2n) \to -2$, the scattering amplitude in the (RL) sector follows

$$S(\theta) = i \tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) S_v(\theta)
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{i\pi}{\theta - i\pi} & \frac{-\theta}{\theta - i\pi} & 0 \\
0 & \frac{-\theta}{\theta - i\pi} & \frac{i\pi}{\theta - i\pi} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \tag{5.23}$$

where

$$S_v(\theta) = \frac{\Gamma(-\frac{\theta}{2\pi i})\Gamma(\frac{1}{2} + \frac{\theta}{2\pi i})}{\Gamma(\frac{\theta}{2\pi i})\Gamma(\frac{1}{2} - \frac{\theta}{2\pi i})} \,. \tag{5.24}$$

Some comments are in order. The solution thus found actually coincides, apart from an overall factor $\left(-\tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right)\right)$, with the (RL) amplitude associated to the massless flow, described by the $O(3)$ nonlinear sigma model at the value of the topological term $\theta = \pi$. Strictly speaking, such trajectory flows from an UV theory of three interacting bosons ($c_{UV} = 2$) to an IR fixed point governed by the $SU(2)_1$ CFT ($c_{IR} = 1$), exhibiting a massless doublet of free fermions [81]. This observation will be useful, in the following, to simplify the thermodynamic Bethe ansatz calculations. As concern the analysis of the short- and large distance asymptotics, both in the low-energy limit $s \to 0$ ($\theta \to -\infty$) and at high energies $s \to \infty$ ($\theta \to \infty$), the trivial behavior $S_{ab}^{cd}(\theta) \to -\delta_a^c \delta_b^d$ is recovered, while

a non-trivial off-diagonal scattering occurs in the crossover region $s \sim M^2$. Such mass scale may be interpreted as a highly unstable resonance pole, appearing in unphysical sheet $(-\pi < \Im m\theta < 0)$ at $\theta = -\frac{i\pi}{2}$.

5.2.2 Thermodynamic Bethe ansatz (TBA) results

In this Section, we aim at verifying the conjecture about the S-matrix, with particular attention to the infrared limit. Therefore, we concentrate directly on this goal, without entering the details of a thorough TBA analysis. The key idea is to rely on the already mentioned similarity with the most studied $O(3)$ sphere sigma model at $\theta = \pi$ [81] (following the convention of [81], we call it SM_π), in order to extend, as far as possible, the results obtained in that system, to our case.

Both theories are off-diagonal, the precise form assumed by the corresponding S-matrices reflecting the lack of right-left factorization at the level of the symmetry, along the flow (in the SM_π model, the infrared $SU(2)_R \times SU(2)_L$ breaks down to $SU(2)$, when departing from the $SU(2)_1$ fixed point while, in the $OSP(1|2)$ case, the situation is more subtle and will be considered later on). Therefore, in order to extract thermodynamical quantities, it is necessary to apply the higher level Bethe ansatz technique, which allows to describe the isotopic structure of the Bethe wavefunction. This implies, in particular, that the spectrum of excitations turns out to contain some fictitious particles (the magnons, and their bound states), which bear no energy and do not contribute to the ground state energy. The most delicate point consists in determining exactly the pattern of interactions of these pseudoparticles, which can be entirely encoded into a so-called 'incidence' diagram, with energy terms attached to its nodes.

Luckily enough, the analogy between the two aforementioned models, extends straightforwardly to their magnonic structure, giving the same incidence diagram.

The explicit form of the diagram reflects the peculiar isotopic structure of the two theories. Indeed, the fact that the symmetry, along the flow, does not distinguish between right- and left-movers, automatically implies that *only one* common magnonic structure occurs, contrary to the cases in which the scattering is diagonal [81]. The corresponding TBA system
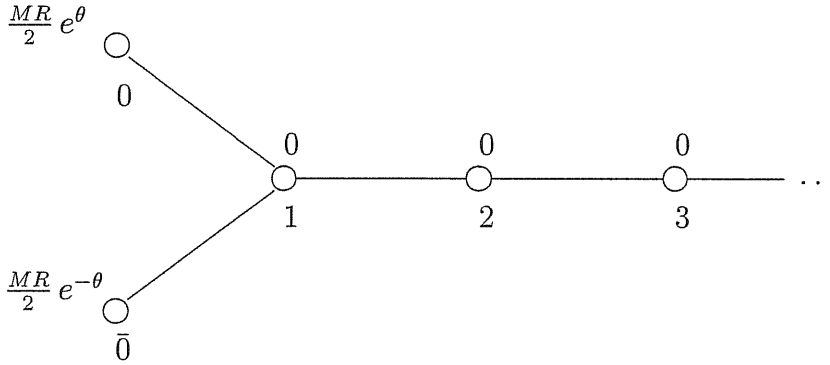
Figure 5.1: The TBA system.

of equations reads

$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi} \sum l_{ab} \left(\varphi * L_b \right)(\theta), \tag{5.25}$$

where l_{ab} is the incidence matrix associated to the diagram (actually the Dynkin diagram of a simply laced algebra), ν_a are the energy terms attached to each node, $L_b(\theta) = \ln(1+e^{-\epsilon_b(\theta)})$ and φ is a universal function, which differs in the two models.

Since we are interested the infrared limit ($R \to \infty$), the nodes labelled by $l = 1, \ldots \infty$ do not affect the thermodynamical properties of the systems, the only relevant ones, being 0 and $\bar{0}$. Therefore, for our purposes we do not need the explicit expression of the kernel function φ. In this case the TBA equations simply boil down to

$$\epsilon_0 \simeq \frac{MR}{2} e^{\theta},$$

$$\epsilon_{\bar{0}} \simeq \frac{MR}{2} e^{-\theta}. \tag{5.26}$$

Assuming that our excitations ($R, \bar{R}$ and $L, \bar{L}$) obey an exclusion principle, namely they are of 'fermionic' type, the ground state energy (1.45) reads

$$E_0(R) = -\frac{2}{2\pi R} \int d\theta \, \frac{MR}{2} \left[e^{\theta} \ln \left(1 + e^{-\frac{MR}{2} e^{\theta}} \right) + e^{-\theta} \ln \left(1 + e^{-\frac{MR}{2} e^{-\theta}} \right) \right]. \tag{5.27}$$

The integral may be computed exactly, yielding

$$E_0(R) = -\frac{2}{\pi R} \sum_{k=1}^{\infty} \frac{(-)^{k-1}}{k^2} = -\frac{\pi}{6R}. \tag{5.28}$$

Therefore, from the relation (1.36), connecting the ground state energy to the scaling function c_{eff}, we recover the infrared value $c_{eff} = 1$. Since we are dealing with fermions, the boundary conditions imposed on the Bethe wavefunction are expected to be antiperiodic. Thus, taking into account the lowest scaling dimensions in the twisted sector $(h_0 = \bar{h}_0 = -1/8)$, the IR central charge $c = c_{eff} + 12\Delta_0 = -2$ follows. Such outcome confirms the expectation of a low-energy theory, described in terms of free fermionc ghosts, and provides an evidence that the conjectured S-matrix, indeed, reproduces sensible results.

5.3   The form factor approach

Before addressing the computation of FF in our specific example, let us recall here the basic features about the form factor approach in massless theories [124]. To simplify the notations, a diagonal scattering matrix will be considered here, with the implicit understanding that all the formulas shown must be extended to our off-diagonal case.

Analogously to the massive flows, massless FF are defined as the matrix element of local operators between asymptotic states

$$F_{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_n) = \langle 0|\mathcal{O}(0)|A_{\alpha_1}(\theta_1) \ldots A_{\alpha_n}(\theta_n)\rangle, \tag{5.29}$$

where $\alpha_i = R, L$ and only one self-conjugate particle, split into massless right- (R) and left-mover (L) parts, is present in the spectrum. The massive Watson's equations (1.51) translate into

$$F_{\alpha_1 \ldots \alpha_i \alpha_{i+1} \ldots \alpha_n}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) =$$
$$= S_{\alpha_i, \alpha_{i+1}}(\theta_i - \theta_{i+1}) F_{\alpha_1 \ldots \alpha_{i+1} \alpha_i \ldots \alpha_n}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n), \tag{5.30}$$

and

$$F_{\alpha_1 \alpha_2 \ldots \alpha_n}(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) = F_{\alpha_2 \ldots \alpha_n \alpha_1}(\theta_2, \ldots, \theta_n, \theta_1), \tag{5.31}$$

where the form factors are meromorphic functions of the rapidities. As concern the poles' structure, in the (RL) channel, no singularity of any kind appears. Indeed, due to the vanishing of the threshold, bound state poles do not occur and, by construction, the kinematical ones are absent. On the other hand, in the (RR) and (LL) channels, which formally resemble the massive situation, the former singularities in principle may not be excluded and the

annihilation poles are perfectly well defined, inducing the recursive equation

$$\text{Res}_{\theta'=\theta+i\pi} F_{\alpha\alpha_1\ldots\alpha_n}(\theta',\theta,\theta_1,\ldots,\theta_n) = i\left(1 - \prod_{i=1}^{n} S_{\alpha_i\alpha}(\theta_i - \theta)\right) F_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n). \quad (5.32)$$

In the next paragraph we turn the attention to the computation of the *minimal* form factors. They are defined as the minimal solutions to the cross-unitarity equations

$$F_{\alpha_1\alpha_2}(\theta) = S_{\alpha_1\alpha_2}(\theta) \, F_{\alpha_1\alpha_2}(\theta + 2\pi i), \quad (5.33)$$

with neither zeroes nor poles in the strip $0 < \Im m\theta < 2\pi$. Moreover, it is useful to quote the following recipe, relating them to the S-matrix, both expressed in terms of a convenient integral representation. Given

$$S(\theta) = -\exp\left[\int_0^\infty \frac{dx}{x} f(x) \, \sinh\left(\frac{x\theta}{i\pi}\right)\right], \quad (5.34)$$

then

$$F_{min}(\theta) = \mathcal{N} \sinh\frac{\theta}{2} \exp\left[\int_0^\infty \frac{dx}{x} \frac{f(x)}{\sinh x} \, \sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right], \quad (5.35)$$

where $\hat{\theta} = i\pi - \theta$, $\mathcal{N}$ is an appropriate normalization and $f(x)$ is a function to be determined, in correspondence of each model, by imposing the Watson's equations.

5.3.1 Minimal form factors

The massless flow (5.8) admits a non-diagonal scattering theory, whose S-matrix is given by (5.23). For our future purposes it is convenient to introduce the notations

$$S_R(\theta) = i \tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) S_v(\theta) \frac{i\pi}{\theta - i\pi}$$

$$S_T(\theta) = i \tanh\left(\frac{\theta}{2} - i\frac{\pi}{4}\right) S_v(\theta) \frac{-\theta}{\theta - i\pi}, \quad (5.36)$$

indicating, respectively, the reflection and transmission processes, encoded into (5.23). (Moreover, it should be noted that $S^{(LR)}(\theta) \equiv [S^{(RL)}(-\theta)]^{-1}$, where $S^{(LR)}$ and $S^{(RL)}$ are the S-matrices in the (LR) and in the (RL) channels.) Starting from the basis of states on which

we have previously constructed our scattering amplitude, the general equations (5.33) cannot be written. Therefore, it is necessary to perform a proper change of basis, in order to recover expressions similar to them.

Let us generalize to the off-diagonal case the Watson's equations for the two-particle form factors. Combining the first monodromy equations (5.30)

$$\begin{aligned} F_{R\bar{L}}(\theta) &= S_T(\theta)F_{\bar{L}R}(-\theta) + S_R(\theta)F_{L\bar{R}}(-\theta)\,, \\ F_{\bar{R}L}(\theta) &= S_T(\theta)F_{L\bar{R}}(-\theta) + S_R(\theta)F_{\bar{L}R}(-\theta) \end{aligned}$$
(5.37)

with (5.31)

$$\begin{aligned} F_{R\bar{L}}(\theta + 2\pi i) &= -F_{\bar{L}R}(-\theta) \\ F_{\bar{R}L}(\theta + 2\pi i) &= -F_{L\bar{R}}(-\theta)\,, \end{aligned}$$
(5.38)

finally yield

$$\begin{aligned} F_{R\bar{L}}(\theta) &= -S_T(\theta)F_{R\bar{L}}(\theta + 2\pi i) - S_R(\theta)F_{\bar{R}L}(\theta + 2\pi i)\,, \\ F_{\bar{R}L}(\theta) &= -S_T(\theta)F_{\bar{R}L}(\theta + 2\pi i) - S_R(\theta)F_{R\bar{L}}(\theta + 2\pi i)\,. \end{aligned}$$
(5.39)

(Note that the second pair of equations has been derived from (5.31), implementing the charge conjugation properties, dictated by the matrix (2.3), previously defined in Section 2.2.) Considering the following combinations of FF

$$F_{\pm}^{(RL)}(\theta) = F_{R\bar{L}}(\theta) \pm F_{\bar{R}L}(\theta)$$
(5.40)

the resulting cross-unitarity relations (in the form (5.33)) may be obtained, for the new functions $F^{\pm}$,

$$\begin{aligned} F_{+}^{(RL)}(\theta) &= -(S_R(\theta) + S_T(\theta))F_{+}^{(RL)}(\theta + 2\pi i) \\ F_{-}^{(RL)}(\theta) &= (S_R(\theta) - S_T(\theta))F_{-}^{(RL)}(\theta + 2\pi i)\,. \end{aligned}$$
(5.41)

The scattering amplitudes which appear on the right-hand side can be recast in the more convenient way (5.34), exploiting the integral representation of the logarithm of the Euler Gamma function, $\Gamma(z)$ (see, e.g. formula 8.3417 in [118]). The final result is

$$\begin{aligned} S_R(\theta) + S_T(\theta) &= \left\{ \exp \int_0^\infty \frac{dx}{x} \left(-2\frac{\cosh(x/4)}{\cosh(x/2)} e^{-x/4} \right) \sinh\left(\frac{x\theta}{i\pi} \right) \right\}\,, \\ S_R(\theta) - S_T(\theta) &= \left\{ \exp \int_0^\infty \frac{dx}{x} \left(-2\frac{\sinh(3x/4)}{\cosh(x/2)} e^{-3x/4} \right) \sinh\left(\frac{x\theta}{i\pi} \right) \right\}\,. \end{aligned}$$
(5.42)

In order to derive explicitly the functions $F_{\pm}^{(RL)}$ it is useful to rewrite them according to

$$F_{\pm}^{(RL)}(\theta) = f_{\pm}(\theta)g_{\pm}(\theta),\qquad(5.43)$$

where $g_{\pm}(\theta)$ satisfy

$$g_{\pm}(\theta) = \mp g_{\pm}(\theta + 2\pi i),\qquad(5.44)$$

and

$$f_{+}(\theta) = \left\{\exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\cosh(x/4)}{\cosh(x/2)}e^{-x/4}\right)\sinh\left(\frac{x\theta}{i\pi}\right)\right\}f_{+}(\theta + 2\pi i),$$

$$f_{-}(\theta) = \left\{\exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\sinh(3x/4)}{\cosh(x/2)}e^{-3x/4}\right)\sinh\left(\frac{x\theta}{i\pi}\right)\right\}f_{-}(\theta + 2\pi i).\qquad(5.45)$$

As concern $f_{\pm}(\theta)$ the solution may be easily found, employing (5.35), namely,

$$f_{+}(\theta) = \exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\cosh(x/4)}{\cosh(x/2)}e^{-x/4}\right)\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right),$$

$$f_{-}(\theta) = \exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\sinh(3x/4)}{\cosh(x/2)}e^{-3x/4}\right)\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right).\qquad(5.46)$$

On the other hand, in order to fix uniquely the expressions of $g_{\pm}(\theta)$, it is necessary to rely on the IR asymptotic behavior of the full form factors. Indeed, equations (5.44) only say that $g_{+}(\theta)$ is a combination of $e^{\theta/2}$ and $e^{-\theta/2}$, while $g_{-}(\theta)$ involves e^{θ} and $e^{-\theta}$. Therefore, at low energies, (5.41) still hold, yielding

$$F_{\pm}^{(RL)}(\theta) \sim F_{\pm}^{(RL)}(\theta + 2\pi i),\qquad \theta \to -\infty,\qquad(5.47)$$

and

$$f_{+}(\theta) \sim e^{\theta/2},$$
$$f_{-}(\theta) \sim O(1),\qquad \theta \to -\infty.\qquad(5.48)$$

The final expression of the minimal form factors thus follows

$$F_{+}^{(RL)}(\theta) = \frac{1}{2\cosh\frac{\theta}{2}}\exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\cosh(x/4)}{\cosh(x/2)}e^{-x/4}\right)\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right),$$

$$F_{-}^{(RL)}(\theta) = \frac{1}{2\cosh\theta}\exp\int_0^\infty \frac{dx}{x}\left(-2\frac{\sinh(3x/4)}{\cosh(x/2)}e^{-3x/4}\right)\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right).\qquad(5.49)$$

Analogous relations may be found in the other channels.

So far, only the minimal form factors have been computed. Despite the off-diagonal nature of the initial S-matrix (5.23), it has been possible to derive straightforwardly the results (5.49), relying on a diagonalization procedure on the basis of states. Nevertheless, considering higher particle FF, the extension of the solution (1.52), corresponding to diagonal massive theories, to the off-diagonal massless cases, is far from obvious. In literature, massless reflectionless examples [124, 131, 132] along with the most important non-diagonal massive systems [43] have already been tackled, exhibiting, especially from the technical point of view, a rapidly increasing level of difficulty. However, to the best of our knowledge, this has not been pushed ahead with massless non-diagonal scattering theories, one of the simplest examples being the $O(3)$ nonlinear sigma model at $\theta = \pi$.

Anyway, the purpose of our investigation is quite modest with respect to the ambitious project of computing arbitrary n-particle FF. Indeed, our main interest consists in selecting a set of operators significant to our theory and calculating the corresponding correlation functions. In this regard, hopefully, only the lowest-particle FF are necessary, as we are going to show immediately with a simple example.

5.3.2 Two-point function of the operator $\partial_\mu \phi \, \partial^\mu \bar{\phi}$

In this section, we give an estimate of the infrared limit of the two-point function

$$G_{\mathcal{O}}(Mr) = \langle \mathcal{O}(x)\mathcal{O}(0) \rangle, \tag{5.50}$$

involving the operator $\mathcal{O}$, whose expression, in terms of the fermionic ghost fields, is $\mathcal{O} = \partial_\mu \phi \, \partial^\mu \bar{\phi} = \bar{\partial}\phi\partial\bar{\phi} + \partial\phi\bar{\partial}\bar{\phi}$. As already mentioned, such operator is supposed to correspond to the $i\varphi_{2,1}$ field in the Kac table, possessing scaling dimension $\Delta_{2,1} = 2$ (as it emerges, as well, from a purely dimensional analysis). We have chosen to start from this peculiar operator because it is the simplest one (it is not of logarithmic nature) which couples also to two-particle states (mixing right- and left-movers), possessing, in general, non-vanishing form factors with an even number of particles. Therefore, here, instead of considering the whole spectral series of the correlator $G_{\mathcal{O}}$, involving the associated n-particle form factors, we concentrate only on the two-particle contribution (namely, $G_{\mathcal{O}}^{(2)}$). We will show that

the result, though an approximation, reproduces correctly the leading power-law behavior, expected in the infrared limit, on the basis of CFT considerations.

Since $\mathcal{O}$ is a neutral operator, the only two-particle FF which actually enter the spectral expansion are those associated to neutral states (the same observation holds for higher particle FF). Therefore, states involving only excitations with or without the bar sign must be excluded. In principle, two types of possible combinations remains: namely $|-\rangle_{(RL)} \equiv |R(\theta_1)\bar{L}(\theta_2) - \bar{R}(\theta_1)L(\theta_2)\rangle$ $(R \leftrightarrow L)$ and an analogous expression with the plus sign. However, only $|-\rangle_{(RL)}$ remains unchanged under charge conjugation, the other one acquiring a minus sign. Hence, the matrix elements which contribute to $G_{\mathcal{O}}^{(2)}$ are of the kind $F_-^{(RL)}$ (see (5.49)).

The correlation function $G_{\mathcal{O}}^{(2)}$ (up to an overall normalization factor) is given by

$$G_{\mathcal{O}}^{(2)}(Mr) \equiv \langle \mathcal{O}(x)\mathcal{O}(0)\rangle^{(2)} \sim -\int \frac{d\theta_1}{2\pi} \int \frac{d\theta_2}{2\pi} \, e^{-\frac{Mr}{2}\left(e^{\theta_1}+e^{-\theta_2}\right)} \, |F_-^{(RL)}(\theta_1 - \theta_2)|^2 \,, \qquad (5.51)$$

where, the minus sign descends from the crossing properties of the states, $x = (ir, 0)$, and the integrals range from $-\infty$ to ∞. (An identical contribution comes from $F_-^{(LR)}$). Implementing the change of variables $\gamma = \theta_1 + \theta_2$ and $\theta = \theta_1 - \theta_2$, it follows

$$\begin{aligned} G_{\mathcal{O}}^{(2)}(Mr) \quad &\sim \quad -\frac{1}{2}\int \frac{d\theta}{2\pi} \int \frac{d\gamma}{2\pi} \, |F_-^{(RL)}(\theta)|^2 \, e^{-Mre^{\theta/2}\cosh\frac{\gamma}{2}} \\ &= \quad -\int \frac{d\theta}{2\pi^2} \, K_0(Mr\,e^{\frac{\theta}{2}}) \, |F_-^{(RL)}(\theta)|^2 \,. \end{aligned} \qquad (5.52)$$

The term containing explicitly the FF can be recast in the more convenient way

$$|F_-^{(RL)}(\theta)|^2 = \frac{\Omega(\theta)}{\cosh^2\theta \, \sqrt{\theta \coth\theta \left[\left(\frac{\theta}{\pi}\right)^2 + 1\right]}} \,, \qquad (5.53)$$

where

$$\Omega(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t} \, \tanh^2\left(\frac{t}{2}\right) \, \frac{\sinh\frac{3t}{4}\,e^{-\frac{3t}{4}}}{\sinh\frac{t}{2}} \, \cos^2\left(\frac{\theta t}{2\pi}\right)\right\} \,. \qquad (5.54)$$

The function Ω can be evaluated numerically and turns out to be

$$1.49 \lesssim \Omega(\theta) \lesssim 2.39 \,, \qquad (5.55)$$

for every value of θ. In particular, it does not affect the limits $\theta \to \pm\infty$. Since we are interested in the large distance behavior of eq. (5.52), we can safely replace it with a

constant belonging to the interval $[1.5, 2.4]$. Finally, expanding the Bessel function in (5.52) for $Mr \gg 1$, we arrive at the expression

$$G_{\mathcal{O}}^{(2)}(Mr) \rightarrow -\frac{const}{2\pi^2} \int d\theta \; \frac{e^{-Mr\,e^{\theta/2}}}{\cosh^2 \theta} \; \sqrt{\frac{\pi e^{-\theta/2}}{2Mr}} \; \frac{1}{\sqrt{\theta \coth \theta \left[\left(\frac{\theta}{\pi}\right)^2 + 1 \right]}} , \qquad (5.56)$$

which can be analyzed numerically. The result is shown in the following series of figures, where, instead of $G_{\mathcal{O}}^{(2)}(Mr)$, actually we have considered $|G_{\mathcal{O}}^{(2)}(Mr)|$. Fig. (5.2) displays a
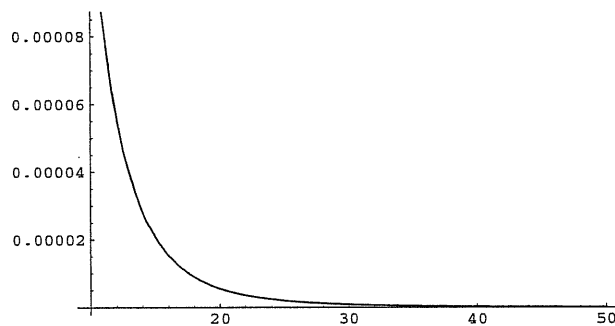

Figure 5.2: $|G_{\mathcal{O}}^{(2)}(Mr)|$ as a function of Mr, for $Mr \gg 1$

plot of $|G_{\mathcal{O}}^{(2)}(Mr)|$, for large enough values of Mr, evidencing a power-law decay

$$|G_{\mathcal{O}}^{(2)}(Mr)| \sim \frac{1}{(Mr)^\alpha} . \qquad (5.57)$$

Such behavior can be easily seen in the logarithmic plot (Fig. (5.3)). The value of the
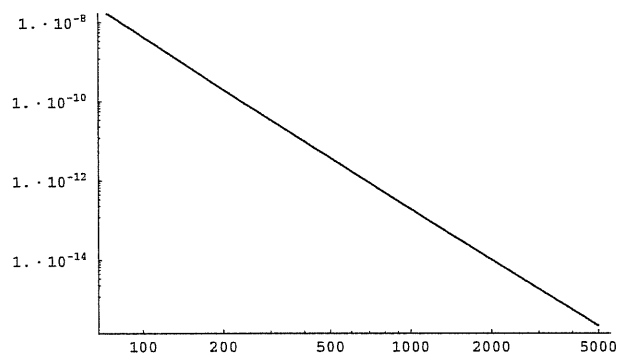

Figure 5.3: $|G_{\mathcal{O}}^{(2)}(Mr)|$ as a function of Mr, in logarithmic scale, for $10 \leq Mr \leq 5000$.

exponent α can be either read directly from the slope of the logarithmic plot or guessed observing Fig. (5.4), where the ratio

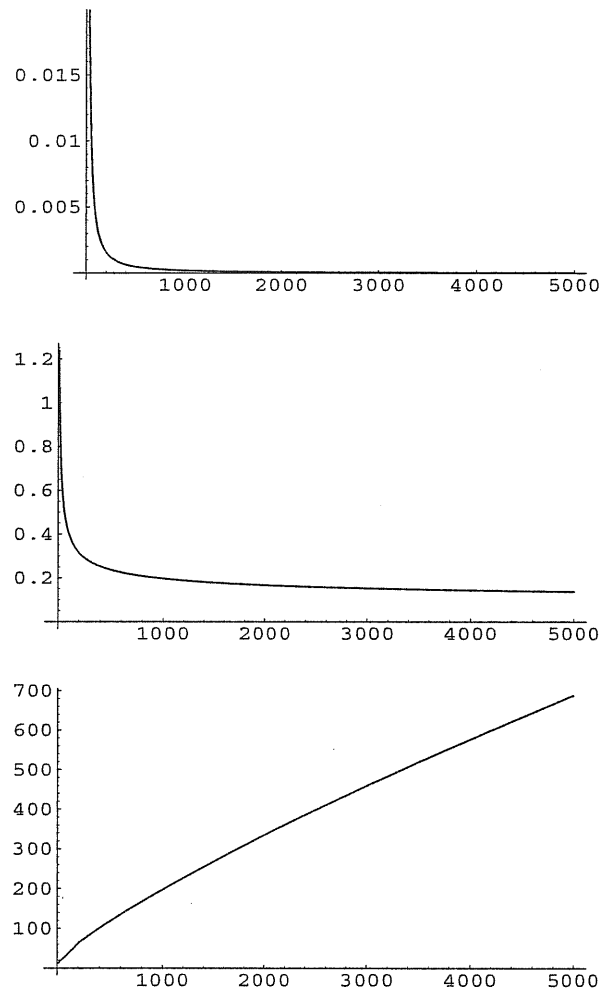$$\frac{|G_{\mathcal{O}}^{(2)}(Mr)|}{(Mr)^{-\alpha}} \tag{5.58}$$

is displayed.

Figure 5.4: Ratios $|G_{\mathcal{O}}^{(2)}(Mr)|/(Mr)^{-\alpha}$, for $Mr \gg 1$, corresponding to the values of $\alpha = 3, 4, 5$, starting from the top.

The first picture shows that $|G_{\mathcal{O}}^{(2)}(Mr)|$ vanishes more quickly than $1/(Mr)^3$, at large distances, while in the third one, the two-point correlation function is compared to $1/(Mr)^5$,

where the ratio diverges. For $\alpha = 4$, instead, the ratio stabilizes to a constant, yielding the exponent we were looking for. Such outcome is in perfect agreement with the predictions based on the conformal field theory analysis, the conformal dimension of $\mathcal{O}$ being $\Delta_{2,1} = 2$.

Before concluding, a last remark is in order. For the two-point function $G_{\mathcal{O}}^{(2)}(Mr)$ we have obtained an overall minus sign, coming from the crossing properties of the ghost states. This is consistent with the identification $\mathcal{O} \leftrightarrow i\varphi_{2,1}$, leading to a positive result for the two-point function of the primary operator $\varphi_{2,1}$.

5.3.3 The stress-energy tensor trace

The analysis now proceeds with a detailed discussion of the trace of the stress-energy tensor Θ. Such operator, which plays a prominent role in CFT, being the generator of dilatations, from the view-point of the off-critical correlators, is extremely important because it enters some fundamental sum rules (1.56), (1.57), which allow to extract information on the conformal points' data.

We anticipate here, that the lowest-particle FF of Θ involve four particles. At the moment an explicit calculation lacks, but the two-particle ones are supposed to act as 'building' blocks.

Anomalous breaking of scale invariance

The stress-energy tensor trace is deeply connected to the invariance properties of a system, under scale transformations, vanishing if the theory is scale-invariant. Therefore, RG critical points, being, in addition, conformal, posses a vanishing stress-energy tensor trace (or better its vacuum expectation value, VEV) in the plane[6]. But, as soon as one departs from a fixed point, Θ develops a non trivial behavior, which in the cases of the perturbed conformal field theories (1.25) assumes the very simple form

$$\Theta(z, \bar{z}) = -4\pi\lambda \, (1 - h)\phi(z, \bar{z}) \,, \tag{5.59}$$

where ϕ is the perturbing field, of conformal weight h, and λ the associated coupling.

However, as concern our nonlinear sigma model (with $g_\sigma > 0$), the marginally irrelevant $(h = 1)$ character of the perturbation entering the $c = -2$ fixed point prevents us to use the

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[6]A pedagogical exposition of these topics can be found in [62].

above relation (5.59), as a definition of the stress-energy tensor trace. At the classical level, the absence of a dimensionful coupling means that the theory is scale-invariant. Nevertheless, as soon as quantum corrections are considered, g_σ renormalizes and scale invariance is spoilt. Therefore, an alternative definition of the trace may be recovered

$$\Theta = \beta(g_\sigma) = \frac{\partial}{\partial g_\sigma}\mathcal{L}, \tag{5.60}$$

exploiting the effects produced on the Lagrangian $\mathcal{L}$, by a shift of the renormalized coupling. Explicitly, using the expression (5.8) for the Lagrangian, Θ reads

$$\Theta = -\frac{\beta(g_\sigma)}{2}\,\phi\bar{\phi}\,\partial_\mu\phi\partial_\mu\bar{\phi}, \tag{5.61}$$

where $\beta(g_\sigma) \propto -3\,g_\sigma^2$, at leading order.

Relation between the coupling and the mass scale

Before proceeding with the problem concerning the calculation of the FF associated to Θ and with the inherent discussion regarding the symmetries of the flow, it is worth spending few words about the relation between the running coupling constant g_σ and the mass scale M. In a UV conformal field theory, perturbed by a relevant operator (1.25), the result $\lambda \sim m^{2(1-h)}$ simply descends from dimensional considerations. A straightforward generalization to the interpolating flows can be obtained for irrelevant fields entering the IR fixed point. Complications seem to arise when marginal operators are involved, in which case, logarithmic behaviors are expected. Particularly, our perturbing field, $\phi\bar{\phi}\,\partial_\mu\phi\partial_\mu\bar{\phi}$, not only is marginal, but also logarithmic and this adds further subtleties. However, it is possible to derive the dependence of g_σ on the mass scale M, extending a result found in [96], for genuine minimal models perturbed by some relevant operators, to our case. We are interested in the conformal minimal model $\mathcal{M}_{p,p'}$, with $p = 1$ and $p' = 2$, deformed by the field $\varphi_{1,5}$. In this regard, the following formula [96]

$$M = \frac{2\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{3-5\xi}{3-3\xi}\right)\Gamma\left(\frac{1+\xi}{3-3\xi}\right)}\left[\frac{4\pi^2\,g_\sigma^2\,(1-4\xi)^2(1-2\xi)^2\,\Gamma^2\left(\frac{3-\xi}{1+\xi}\right)\Gamma\left(\frac{\xi}{1+\xi}\right)\Gamma\left(\frac{1-4\xi}{1+\xi}\right)}{(1+\xi)^4\,\Gamma^2\left(\frac{4\xi}{1+\xi}\right)\Gamma\left(\frac{1}{1+\xi}\right)\Gamma\left(\frac{5\xi}{1+\xi}\right)}\right]^{\frac{1+\xi}{12(1-\xi)}}, \tag{5.62}$$

originally defined for strictly relevant perturbations, i.e. for values of the parameters $2p < p'$, where $\xi = p/(p'-p)$, after proper analytic continuations can give us an answer. Since, in

our case, the limit $\xi \to 1$ must be taken, finally equation (5.62) assumes the form

$$g_\sigma^2 \to \frac{1}{4\pi^2}\,[1 - 6(1 - \xi)\ln M]\,\left[\frac{(1 - \xi)}{\sin\frac{2\pi}{3(1-\xi)}}\right]^{6(1-\xi)}\,, \qquad (5.63)$$

where, the marginal nature of the perturbing operator is, indeed, responsible for the logarithmic dependence on M and its logarithmic character yields an infinitely oscillating, highly non trivial term. We would like to stress that the same oscillatory behavior persists also at the level of the specific free energy, as has been noticed in [26], in connection with TBA calculations for the corresponding massive flown ($g_\sigma < 0$).

Symmetry properties

Coming back to the FF problem, a first fundamental step consists in analyzing the symmetry properties of the flow (5.8) and of the stress-energy tensor trace (proportional to the perturbing field $\phi\bar{\phi}\,\partial_\mu\phi\partial_\mu\bar{\phi}$), in order to establish *how many* and *which* states couples to Θ.

Starting from the observation that the trace (5.61) is a neutral operator, only neutral states can couple to it. Therefore, we could repeat an analysis analogous to the one performed in the case of the operator $\partial_\mu\phi\,\partial^\mu\bar{\phi}$. Looking at the two-particle form factors, the unique difference, with respect to the previous example, is that the generic combinations $|-\rangle_{\alpha_1,\alpha_2} \equiv |A_{\alpha_1}(\theta_1)\bar{A}_{\alpha_2}(\theta_2) - \bar{A}_{\alpha_1}(\theta_1)A_{\alpha_2}(\theta_2)\rangle$ (where we use the compact notation $\alpha_i = R, L$) must be considered, and not only those which mix right- and left-movers. Hence, considering higher particle states, it follows that only an odd number of combinations $|+\rangle_{\alpha_1,\alpha_2}$ is forbidden.

Next, we turn to the symmetries of the Lagrangian in order to gain further insights. As concern the continuous ones, no obvious factorization into the left and right sectors occurs, not even at the IR fixed point. Therefore, the only continuous symmetry along the flow is $SP(2)$ and no particular selection rule follows. On the other hand, writing the action (5.8) explicitly in terms of the right- and left-movers, some discrete symmetries can be easily identified (analogously to the case of the tricritical Ising model flowing into the Ising

one [124]). The low-energy critical point exhibits two symmetries, S_1 ($\mathbb{Z}_2$) and S_2

$$S_1 : \quad \begin{array}{ll} \phi_R \to -\phi_R & \phi_L \to -\phi_L \\ \bar{\phi}_R \to -\bar{\phi}_R & \bar{\phi}_L \to -\bar{\phi}_L \end{array} \qquad (5.64)$$

$$S_2 : \quad \begin{array}{ll} \phi_R \to -\phi_R & \phi_L \to \phi_L \\ \bar{\phi}_R \to \bar{\phi}_R & \bar{\phi}_L \to -\bar{\phi}_L \,. \end{array} \qquad (5.65)$$

In terms of the states, they select the combinations previously mentioned, with the additional requirement to allow only for the simultaneous presence of right- and left-movers. However, out of the critical point, S_2 breaks and this supplementary property ceases to hold.

At the end of the day, symmetry puts some limitations on the possible configurations of states, but, concerning the number of particles involved, in principle, all even number of them is allowed, starting from two. Since it is extremely important to determine exactly the lowest-particle states which couple to Θ, we will insist on this point, showing that more stringent constraints may be imposed, by demanding the conservation of the stress-energy tensor.

Conservation of the energy-momentum tensor

The discussion we are going to present here applies to all massless flows.

Since for our future purposes, it is convenient to work in the Minkowski space-time (x^0, x^1), the conservation of the stress-energy tensor ($\partial_\mu T^{\mu\nu} = 0$, where $T^{\mu,\nu}$ is symmetric in the indices) may be encoded into the general relations

$$\partial_0 T^{00} = \partial_1 T^{01} \,,$$
$$\partial_0 T^{01} = \partial_1 T^{11} \,, \qquad (5.66)$$

which hold also in the massive cases. Therefore, in this framework, the trace $\Theta = T^\mu_\mu$ assumes the form

$$\Theta = T^{00} - T^{11} = \left(1 - \partial_1^{-2}\partial_0^2\right) T^{00} \,, \qquad (5.67)$$

where the symbol ∂^{-1} formally indicates integration.

Turning to the massless case and considering a generic two-particle channel (without specifying, for the time being, the character of the single excitations, characterized by energy

and momentum (e_i, p_i), $i = 1, 2$) it follows

$$\Theta = \frac{-s}{(p_1 + p_2)^2} \, T^{00} \,, \tag{5.68}$$

with $s = 2(p_1 p_2 - e_1 e_2)$, the massless Lorentz invariant Mandelstam variable. From this formula it emerges clearly that, only when the (RR) or (LL) configurations are considered, $s = 0$ and the trace vanishes. Therefore, a first constraint, imposing the mixing of right- and left-movers, descends from the conservation laws (5.66).

But a more important restriction, which concerns the minimum number of particles coupling to Θ, may be derived exploiting the definition of the energy operator

$$E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 \, T^{00}(x) \,, \tag{5.69}$$

which, in the massive case, gives directly the normalization of the two-particle FF [133]. The main idea consists in evaluating the matrix element of both terms of this equation on the asymptotic states $\langle R(\theta')|$ and $|L(\theta)\rangle$, and comparing the results. On the left-hand side, we obtain

$$\frac{M}{2} \, e^{-\theta} \langle R(\theta')|L(\theta)\rangle \,, \tag{5.70}$$

while, the right-hand one yields

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 \, \langle R(\theta')|T^{00}(x)|L(\theta)\rangle \, \propto F_{RL}^{\Theta}(\theta' + i\pi, \theta) \, (e^{\theta'} + e^{-\theta})^2 \delta(\theta' + \theta - i\pi) \,. \tag{5.71}$$

Therefore, the equality between the two terms is never satisfied, with the crucial implication that the lowest-particle states which couple to the stress-energy tensor trace are those involving *four* particles.

Collecting the results so far obtained, we can say that FF of Θ can be computed, employing states composed of an even number of particles, combined in the neutral configurations $|-\rangle_{\alpha_1,\alpha_2}$ and $|+\rangle_{\alpha_1,\alpha_2}$ (where, nevertheless, an odd number of $|+\rangle_{\alpha_1,\alpha_2}$ is forbidden), starting from the four-particle ones. Once, this task has been achieved (the Watson's equation become extremely complicated, at the technical level, and the work is still in progress), the residue equations (5.32), related to the annihilation poles in the (RR) and (LL) subchannels, are expected to yield recursive relations useful to reconstruct higher-particle FF.

5.4   Results and perspectives

In this chapter, we have studied a model which is believed to describe the generic low-temperature phase of dense loops [28]. Indeed, in reference [28], it has been shown, using a supersymmetric formulation, that such system possesses a broken-symmetry Goldstone phase, which is nothing but free fermionic ghosts ($c = -2$). The associated field theory turns out to be the weak-coupling limit of a nonlinear sigma model (5.5), whose vector field, with one bosonic and two fermionic components, lives on a supersphere (technically, the $S^{0|2} \equiv OSP(1|2)/SP(2)$ target manifold). Therefore, for positive values of the sigma model coupling, the initial $OSP(1|2)$ symmetry breaks down spontaneously to $SP(2)$, flowing, at large distances, to the aforementioned critical phase. In terms of the infrared fields, such theory can be seen as a conformal $c = -2$ QFT, perturbed by a marginally irrelevant (logarithmic) operator (5.8).

In this general framework, we have proposed a scattering theory describing the interactions among the Goldstone fermions. Given the assumption that the only stable excitations along the flow are actually those which remain in the infrared spectrum, we have employed, for our calculations, a basis of massless states, splitting the symplectic fermions into their left- and right-moving components. Therefore, for the scattering amplitude in the (RL) sector (where a genuine interaction process is expected to occur at the mass scale M), we have obtained the expression (5.23). Remarkably, this result turns out to coincide, a part from a CDD factor, with the S-matrix, corresponding to the massless flow described by the $O(3)$ nonlinear sigma model with topological term $\theta = \pi$. Both models exhibit an off-diagonal scattering amplitude, indicating a lack of right-left factorization, at the level of the symmetry, along the flow. Moreover, from the solution (5.23), the origin of the mass scale M is naturally associated to a highly unstable resonance state.

In a second step we have verified the previous conjecture, investigating the finite size properties of our scattering theory. In order to prove our claim, we have deduced the value of the infrared central charge, from the knowledge of the ground state energy (5.28), derived by means the Bethe ansatz technique. Then, a comparison between the above result, $c = -2$, with the low-energy central charge, already known, has provided us with a confirmation of our proposal (5.23). An interesting direction for future developments concerns the analysis of the ultra-violet aspects of the flow.

Finally, we have tackled the problem of implementing the so-called form factor bootstrap approach, on the model under consideration. Actually, we have not completed this program yet, being the task complicated both by technical difficulties (the scattering is off-diagonal and massless) and by conceptual subtleties (related to the marginal and logarithmic nature of the perturbing operator). In this regard, we have computed the so-called 'minimal' form factors (5.49), which turn out to play a crucial role, representing, in a sense, the building blocks of the higher-particle ones. Then, instead of trying to solve directly the FF bootstrap equations (5.30), (5.31) for an arbitrary number of particles, we have decided to select a set of physically motivated operators and to concentrate on it, this choice being dictated by our main interest in the off-critical correlation functions. For this purpose, hopefully, only the lowest-particle FF are sufficient to capture the main features of the model (e.g. sum rules) and, symmetry properties can reduce the number of states, effectively entering the FF of a given operator. In this regard, we have started the analysis from the most innocuous operator of the theory (namely, the non-logarithmic field $\partial_\mu \phi \, \partial^\mu \phi = i\varphi_{2,1}$), considering its two-point function. Using the explicit expressions of the two-particle form factors (5.49), we have evaluated their contribution to such correlator. Then, extracting the infrared limit, we have obtained the leading power-law behavior, in perfect agreement with the decay predicted by conformal invariance considerations ($\sim 1/r^4$). Afterwards, we have turned our attention to the stress-energy tensor's trace Θ, which is the crucial ingredient of some important sum rules, yielding information about the CFT's data (variations of the central charge and of the anomalous dimensions from the short- to the large-distance regime). In order to understand how many and which states could couple to Θ (and, therefore, enter the definition of the corresponding form factors), we have analyzed all the symmetry properties of the trace and discussed in detail the implications of the stress-energy conservation on the massless flows. The next step would be to evaluate explicitly the four-particle FF and then try to implement the sum rules (1.57) and (1.56). In this way, important insights into the ultraviolet data, hardly accessible by other means, could be gained.

We conclude, pointing out other stimulating directions of investigation, concerning further physically interesting operators, such as, for instance, the (logarithmic) thermal field $\phi\bar{\phi}$, which is supposed to couple to two-particle states, including also those which do not mix right- and left-movers, and the disorder operators, which could give information on the

non-local sectors of the ultra-violet theory.

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