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The Konishi Anomaly Approach to Effective Superpotentials

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Introduction

The understanding of the low energy dynamics of Yang–Mills gauge theories is one of the most interesting open problems in theoretical physics. Their strongly coupled infrared dynamics exhibits a variety of interesting phenomena: external sources are confined by strings of gauge field strength, a mass gap arises, thus forbidding massless excitations, and so on. What makes this subject even more fascinating is that, while the physical picture is (supposed to be) quite clear, and confirmed by experiment and numerical analysis, the gauge theory itself is strongly coupled and thus completely outside of any analytic control. The best one can do is to take QCD, the physical realization of Yang–Mills theory, make some physically motivated *guess* for its low energy behavior and check it against experiment.

However, there is another point of view, that has demonstrated to be extremely useful in physics: *change* the theory in such a way that it is more tractable, without spoiling the interesting features of the original theory. At first sight, even this can be a difficult problem. However, usually it can be accomplished by following an old tradition of theoretical physics: to increase the symmetries of the problem. And here is the point where Supersymmetry comes in the play. This character, was actually introduced a long time ago in a completely different context. Supersymmetry is supposed to play a key role in the understanding of the high energy properties of our world, where it is conjectured to be an exact symmetry. Its origin is deeply connected with String Theory. The relevant point is that the minimal ($\mathcal{N} = 1$) supersymmetric extension of Yang–Mills theory, is expected to share some of the non perturbative features of the original Yang–Mills theory. Due to these facts, it is worth to study the low energy dynamics of supersymmetric theories by themselves. Moreover, the fact that a theory is supersymmetric, usually makes it easier to control it. Unfortunately, a complete solution of the theory is still far from reach.

An important step towards the solution of supersymmetric Yang–Mills theory, was recently made by Dijkgraaf and Vafa in [1] summarizing their previous works [2], [3]. These authors conjectured a precise rule to obtain the low energy *exact* effective superpotential for a wide range of $\mathcal{N} = 1$ gauge theories in four dimensions, in terms of the relevant degrees of freedom in the infrared. These are conjectured to be the so called glueballs and not the gluons arising from the usual quantization of the gauge field (precisely in the same way as in QCD, where quarks are the relevant variables in the high energy theory while mesons and pions describe the low energy dynamics), and the effective superpotential constrains their dynamics.

Originally the conjecture regarded an $\mathcal{N} = 1$ $U(N)$ gauge theory with matter in the adjoint representation of the gauge group and arose from arguments coming from

String Theory. Namely the gauge theory was embedded in String Theory due to a particular D–brane configuration and its low energy dynamics was related, through a chain of dualities, to an auxiliary matrix model. Then the effective superpotential is extracted from the planar limit of the free energy of the auxiliary matrix model. The surprising point is that, the perturbative (diagrammatic) analysis of this matrix model, gives information on the gauge theory effective superpotential, that being an infrared concept, is non perturbative by its own nature. The effective superpotential is given by two contributions: the first is the Veneziano–Yankielowicz superpotential [4] (see [5] for a complete discussion), the second is a sum of instanton corrections to the former that arises when integrating out the matter fields.

The conjecture was then proved in a purely gauge theoretical context in [6], by means of super–Feynman diagrams techniques, and in [8] by using the concepts of the chiral ring, which encodes the information about the holomorphic sector of the theory, and of the generalized Konishi anomaly, to fix the form of the low energy superpotential. This is the approach we will follow along this thesis.

The basic idea of [8] is to identify the subsector of the theory, the chiral ring, that is relevant to the computation of the effective superpotential. Then, by considering a generalized form of the Konishi anomaly, they proved the conjecture and introduced a practical and purely gauge theoretical method to compute the effective superpotential, without relying on String Theory and on the auxiliary matrix model. This method was further generalized by the author in collaboration with L.F. Alday, in [9] to theories with other gauge (classical Lie) groups and matter in various representations, not necessarily the adjoint. There, the method of the Konishi anomaly, showed to be more suited for practical computations, while the diagrammatic evaluation of the related matrix model increases in complexity in going to higher orders.

The String Theory insight relates the effective superpotential of the gauge theory to the planar limit of the auxiliary matrix model. This leads immediately to wonder what non planar contributions correspond to. This problem was already addressed in [1] where the authors conjectured that the non planar corrections to the free energy are related in the gauge theory side to gravitational corrections to the effective superpotential. This conjecture was proved by [10, 11] using diagrammatic techniques, extending to the gravitational case the super-Feynman diagram techniques of [6]. Crucial ingredient in the proof was the modification of the chiral ring relations due to the coupling of the gauge theory to supergravity. In particular, if one restricts to the first non-trivial gravitational F-term contribution, corresponding to the genus one correction in the related matrix model, one needs to take into account just the modification which follows from standard $\mathcal{N} = 1$ supergravity tensor calculus. Further works on the genus one corrections include [13, 14, 15, 17, 18].

The problem of understanding F-terms which describe the coupling of $\mathcal{N} = 1$ gauge theories to $\mathcal{N} = 1$ supergravity from a purely gauge theoretical point of view, was solved in [12], extending the Konishi anomaly method of [8] for $U(N)$ gauge theories with adjoint matter, to include gravity. This method is modified in the presence of gravity by three ingredients, namely the modification of the chiral ring relations, already pointed out in [10, 11], a direct gravitational contribution to the Konishi anomaly and finally, the lack of factorization of the chiral ring correlators. In the following this method was extended by the author in collaboration with L.F. Alday in [16] to include a generic classical Lie group and other representations for the matter field.

The coupling of the gauge theory to standard $\mathcal{N} = 1$ supergravity, corresponds just to the genus one correction in the auxiliary matrix model. The problem of how to implement the higher genera corrections to the matrix model in the gauge theory, is a difficult one and requires insights from the stringy point of view. In the language of the dual closed string theory side [19], one needs to introduce a more drastic modification in the chiral ring relation, to account for a non-trivial vacuum expectation value of the (self-dual) graviphoton field strength $F_{\alpha\beta}$ of the parent $\mathcal{N} = 2$ string theory. This requires a non standard analysis since the graviphoton is part of the $\mathcal{N} = 2$ gauge multiplet. The theory has to be understood as rigid $\mathcal{N} = 1$ supersymmetric gauge theory coupled to a non dynamical $\mathcal{N} = 2$ gravitational background. The graviphoton field strength F^2 plays the role of the genus counting parameter on the gauge theory side and thus needs to be non zero in order to compare with the matrix model genus expansion. In [20] L.F. Alday, J.R. David, E. Gava, K.S. Narain and the author, showed how the Konishi anomaly method can be extended to take into account the matrix model corrections at all genera, by properly using the modified chiral ring and its properties. The gauge theory genus expansion was showed to match the matrix model expansion, provided the graviphoton field strength F^2 is identified with the genus counting parameter. Finally in [21] the needed modification of the chiral ring was proved without any reference to String Theory, but by analyzing the spontaneous partial breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supergravity coupled to a vector multiplet and then taking a rigid limit which results in the above mentioned non trivial gravitational background.

This thesis is organized as follows. In the first Chapter we will give a basic introduction to supersymmetric gauge theories maily focusing on the superspace formalism for $\mathcal{N} = 1$ theories. Within this formalism we will introduce the concept of Wilsoninan Effective Action, that will be constantly used throughout this thesis. This action allows one to get information on the low energy limit of a known ultraviolet theory, by means of the renormalization group analysis. Moreover, for supersymmetric theories, the low energy effective action is further restricted by important tools that display the full power of supersymmetry. Namely, by using the ultraviolet symmetries of the theory, as well as the holomorphic dependence on the parameters that is manifest in the superspace formalism, one can constrain the form of the low energy action. Other important tools that can give us a better understanding on the gauge theory dynamics are anomalies, namely classical symmetries of the action that are broken by quantum effects. One of these anomalies, the Konishi anomaly, will play a prominent role in this thesis. Finally we add some qualitative comments on the expected low energy dynamics of Yang–Mills theories and their supersymmetric extensions, namely confinement and chiral symmetry breaking.

In the second Chapter, we will enter in the main body of the thesis by stating the Dijkgraaf–Vafa conjecture. This conjecture allows one to compute the effective superpotential for a $U(N)$ supersymmetric gauge theory with adjoint matter by means of an auxiliary bosonic matrix model. We re–formulate the conjecture in purely gauge theoretical terms, by introducing the ring generated by the chiral operators in the gauge theory and a generalized form of the Konishi anomaly. When restricted to the chiral ring, the Konishi anomaly gives a set of Ward identities that can be solved due to the factorization property of operators in the chiral ring: this allows one to determine the effective superpotential. Finally we show how all this can be generalized to other $\mathcal{N} = 1$ supersymmetric gauge theories by introducing the appropriate generalization of the Konishi anomaly. We also show explicitly some computations.

The third Chapter is devoted to the introduction of the topic of gravitational corrections to the low energy effective action. We begin to study the coupling of the $\mathcal{N} = 1$ theory to $\mathcal{N} = 1$ supergravity. This can be understood by generalizing the chiral ring relations and the Konishi anomaly to include also gravity. We show that this analysis encodes the information of the genus one contribution to the matrix model free energy. A key point is that correlators in the chiral ring do not factorize anymore due to the presence of the gravitational background as well as matrix model correlators do not factorize anymore if genus one corrections to the planar limit are included. This analysis is then extended to theories based on other gauge groups and with matter in various representation; we comment on the comparison with matrix model result.

Finally in the last Chapter we study the $\mathcal{N} = 1$ supersymmetric gauge theory coupled to a non dynamical graviphoton background. In presence of this background, the chiral ring identities are modified non trivially and this modification reflects in the Konishi anomaly analysis. First of all, we explain how to compute the genus one corrections. Then, we give a full treatment of the all genera solution by introducing appropriate generating functionals for the connected part of the correlators. Finally we compare our result with the matrix model.

Chapter 1

Supersymmetric Gauge Theories

The aim of this Chapter is to briefly review the dynamical properties of supersymmetric $\mathcal{N} = 1$ gauge theories. We will mainly follow [25] and [22]. We further collect argument explained in [31, 23, 32, 24] that are nicely summarized in [33]. To begin with, we will outline the construction of four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories in the superspace formalism, assuming the reader is familiar with the supersymmetry algebra in four dimensions and its representations. We include some comment on the $\mathcal{N} = 2$ theories and on supergravity. We will then introduce the concept of Wilsonian effective action that is a basic ingredient in the study of the low energy dynamics of gauge theories. The Wilsonian effective action is the proper tool to describe a physics at a determined energy scale. It is basically obtained by averaging over the short distance fluctuations of the fields. The study of the Wilsonian effective action for supersymmetric theories is simplified by the non–renormalization theorems. These are selection rules derived from the symmetries of the high energy theory, that can be used to constrain the form of the low energy action. Finally, some insight on the non perturbative dynamics of gauge theories can come from anomalies. In particular, we will introduce the Konishi anomaly that we will extensively use throughout this thesis. Finally we end this Chapter with some general comment on the expected low energy dynamics of gauge theory.

1.1 Supersymmetry in Superspace

In this section we will introduce $\mathcal{N} = 1$ theories and the superspace formalism. This presentation follows [22] and [25]; we refer the reader to these works and to [26] for a more complete treatment of the subject. Our conventions, those of [25] are summarized in the Appendix A.

As usual symmetries have a natural interpretation in terms of the action of a group on local fields defined on the space–time manifold, supersymmetry can be defined through its action on an extension of usual space–time, namely the superspace. Supersymmetry transformations are interpreted as translation in superspace. To clarify this concept, let us consider ordinary translation of an ordinary scalar field ϕ. They are induced by an operator P_μ such that

$$\phi(x+\varepsilon) = e^{-i\varepsilon^\mu P_\mu}\phi(x)e^{i\varepsilon^\mu P_\mu}. \tag{1.1}$$

For ε infinitesimal, this relation can be expanded giving

$$[P_\mu, \phi(x)] = i\partial_\mu \phi(x); \tag{1.2}$$

this relation can be rephrased by saying that $i\partial_\mu$ is the representation of the operator P_μ in field space.

Let us follow the same idea in the $\mathcal{N}=1$ superspace. This is obtained by adding four Grassman coordinates θ^α and $\bar{\theta}_{\dot{\alpha}}$ to the usual space–time coordinate x^μ. Then a superfield is defined as a generic function on the superspace $F\left(x,\theta,\bar{\theta}\right)$. Given the anti-commuting nature of the Grassman coordinate, the field F can be always be expanded as

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= f(x)+\theta\phi(x)+\bar{\theta}\bar{\chi}(x)+\theta\theta m(x)+\bar{\theta}\bar{\theta}n(x)+\theta\sigma^\mu\bar{\theta}A_\mu(x)+ \\ &\quad \theta\theta\bar{\theta}\bar{\lambda}(x)+\bar{\theta}\bar{\theta}\theta\psi(x)+\theta\theta\bar{\theta}\bar{\theta}d(x). \end{aligned} \tag{1.3}$$

Then the natural generalization of the concept of translation is given by

$$\begin{aligned} x^\mu &\rightarrow x^\mu+\varepsilon^\mu+i\theta\sigma^\mu\bar{\xi}-i\xi\sigma^\mu\bar{\theta} \\ \theta_\alpha &\rightarrow \theta_\alpha+\xi_\alpha \\ \bar{\theta}_{\dot{\alpha}} &\rightarrow \bar{\theta}_{\dot{\alpha}}+\bar{\xi}_{\dot{\alpha}}, \end{aligned} \tag{1.4}$$

where the additional terms in the transformation law for x are needed so that the composition of two fermionic transformations is a translation in x. These transformations are induced by operators P_μ, Q_α and $\bar{Q}^{\dot{\alpha}}$ such that

$$\begin{aligned} &F(x^\mu+\varepsilon^\mu+i\theta\sigma^\mu\bar{\xi}-i\xi\sigma^\mu\bar{\theta},\theta_\alpha+\xi_\alpha,\bar{\theta}_{\dot{\alpha}}+\bar{\xi}_{\dot{\alpha}}) = \\ &e^{-i\varepsilon P+\xi Q+\bar{\xi}\bar{Q}}F(x^\mu,\theta_\alpha,\bar{\theta}_{\dot{\alpha}})e^{i\varepsilon P-\xi Q-\bar{\xi}\bar{Q}}. \end{aligned} \tag{1.5}$$

thus extending (1.1) to the full superspace. One can check that the operators P_μ, Q_α and $\bar{Q}^{\dot{\alpha}}$ are given by

$$\begin{aligned} P_\mu &= i\partial_\mu \\ Q_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \\ \bar{Q}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \end{aligned} \tag{1.6}$$

and obey the $\mathcal{N}=1$ supersymmetry algebra[1]

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{1.7}$$

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{1.8}$$

[1]It's worth mentioning that this algebra can be extended by including central charges, giving thus rise to theories with extended supersymmetry.

A good candidate for a manifestly invariant Lagrangian is the integral of a superfield over the full superspace

$$L = \int \mathrm{d}^4 \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} F(x,\theta,\bar{\theta}). \tag{1.9}$$

This follows from the fact that supersymmetry transformations are interpreted as translations in the superspace (one can easily check that the integration measure is translation invariant); thus Lagrangian densities can be constructed by taking the highest dimension component, called $d(x)$ in (1.3), of a superfield or even a generic function of it.

Usually it is useful to introduce superfields with less degrees of freedom than (1.3); this can be accomplished by imposing a constraint on the form of the superfield itself leading to the *chiral* and *vector* superfields.

To define the chiral superfield let us introduce the super–covariant derivatives

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{1.10}$$

They satisfy $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$ and anticommute with the supercharges. Then Φ is a chiral superfield if

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{1.11}$$

This equation is solved by

$$\Phi = \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{2}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta f(x) \tag{1.12}$$

The field content of Φ includes a complex scalar $\phi(x)$ and a Weyl spinor $\psi(x)$, giving a local representation of the so called $\mathcal{N} = 1$ scalar multiplet ($f(x)$ is an auxiliary field required for the off–shell closure of the algebra). Again, one can obtain a manifestly supersymmetric invariant Lagrangian by integrating a chiral superfield over the chiral superspace

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\ \Phi \tag{1.13}$$

thus Lagrangian densities can be constructed by taking the highest component, called $f(x)$ in (1.12), of a chiral superfield . A term like (1.13) that cannot be written as an integral over the full superspace, is called an *F–term*. All other possible terms are named *D–terms* [2]. Moreover, one can easily show that if Φ is a chiral superfield, then a generic *holomorphic* function of it $W(\Phi)$ will be a chiral superfield. Then the general F–term

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\ W(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\ \bar{W}(\bar{\Phi}) \tag{1.14}$$

is a supersymmetric interaction and the function $W(\Phi)$ is called superpotential. Finally, the most general $\mathcal{N} = 1$ Lagrangian for the scalar multiplet is given by

$$L = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ K(\Phi,\bar{\Phi}) + \int \mathrm{d}^4x\mathrm{d}^2\theta\ W(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\ \bar{W}(\bar{\Phi}), \tag{1.15}$$

[2]Note that, if F is generic superfield, a term like $\int \mathrm{d}^4x\mathrm{d}^2\theta\ \bar{D}^2F$ is not an F–term as it can be rewritten as $\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ F$.

where the non holomorphic function K is called the Kähler potential, since the metric on the scalar field space is given by $g^{i\bar{j}} = \frac{\partial^2 K}{\partial\phi_i \partial\phi_{\bar{j}}}$. These terms can also be restricted by requiring their invariance under the R–symmetry group $U(1)_R$ (for $\mathcal{N}=1$ theories, R–symmetry is only a phase rotation of the supersymmetry generators). By normalizing the R–symmetry generator R such that the supercharge generator Q has charge -1 (*i.e.* $R(Q)=-1$) it can be shown that the overall supercharge of the superpotential must be $+2$ while K should be R–neutral.

Vector superfield are used to introduce gauge fields and are defined by a superfield satisfying $V=V^\dagger$. In components,

$$\begin{aligned} V^a &= C^a(x) + i\theta\chi^a(x) - i\bar{\theta}\bar{\chi}^a(x) + \frac{i}{2}\theta\theta\left(M^a(x)+iN^a(x)\right) \\ &\quad -\frac{i}{2}\bar{\theta}\bar{\theta}\left(M^a(x)-iN^a(x)\right) - \theta\sigma^\mu\bar{\theta}A^a_\mu(x) \\ &\quad +i\theta\theta\bar{\theta}\left(\bar{\lambda}^a(x)+\frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi^a(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda^a(x)+\frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}^a(x)\right) \\ &\quad +\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D^a(x)+\frac{1}{2}\Box C^a(x)\right). \end{aligned} \tag{1.16}$$

where a is a Lie algebra index ($V=V^aT^a$ belongs to the adjoint representation of a gauge group). This superfield can be put in a simpler form by using a gauge transformation $e^V \to e^{V'} = e^{-i\bar{\Lambda}}e^V e^{i\Lambda}$, where Λ is a chiral superfield, to set $C=M=N=\chi=0$ (the so called Wess–Zumino gauge), resulting in

$$V^a = -\theta\sigma^\mu\bar{\theta}A^a_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}^a(x) - i\bar{\theta}\bar{\theta}\theta\lambda^a(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D^a(x). \tag{1.17}$$

Note that this gauge breaks supersymmetry but not the gauge symmetry of A^a_μ. The vector superfield is a local representation of the $\mathcal{N}=1$ vector multiplet, given by a gauge massless vector boson $A_\mu(x)$ and a Weyl spinor $\lambda(x)$ (as for the chiral superfield, $D(x)$ is an auxiliary field necessary for the supersymmetry algebra to close off-shell). The gauge field strength is defined by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}(e^{-V}(D_\alpha e^V)), \tag{1.18}$$

and transforms as

$$W_\alpha \to W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}. \tag{1.19}$$

In components W_α reads

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - i\sigma^{\mu\nu}{}_\alpha{}^\beta\theta_\beta F_{\mu\nu}(y) + \theta\theta\sigma^\mu_{\alpha\dot{\beta}}D_\mu\bar{\lambda}^{\dot{\beta}}(y), \tag{1.20}$$

where $F_{\mu\nu}$ is the usual gauge field strength and D_μ the covariant derivative. With W_α one can write the pure gauge theory action

$$\frac{\tau}{16\pi}\int d^4x d^2\theta\, \mathrm{Tr}\, W^\alpha W_\alpha + \frac{\bar{\tau}}{16\pi}\int d^4x d^2\bar{\theta}\, \mathrm{Tr}\, \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}, \tag{1.21}$$

where $\tau = \frac{4\pi}{g^2} + i\frac{\Theta}{2\pi}$ contains the gauge coupling g and the Θ–angle. In component it reads

$$\frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - 2i\bar{\lambda}\bar{\sigma}^{\mu}D_{\mu}\lambda + D^2\right) + \frac{\Theta}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\lambda}, \tag{1.22}$$

as expected. It is useful to introduce new derivative operators

$$\begin{aligned} \bar{\nabla}_{\dot{\alpha}} &= \bar{D}_{\dot{\alpha}} \\ \nabla_{\alpha} &= e^{-V}D_{\alpha}e^{V}. \end{aligned} \tag{1.23}$$

By using (1.24) we define

$$\{\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}\} \equiv i\nabla_{\alpha\dot{\alpha}}. \tag{1.24}$$

that can be used to rewrite the gauge field strength (1.18) as

$$W_{\alpha} = -\frac{i}{4}[\bar{\nabla}^{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}] \tag{1.25}$$

Finally, the most general action[3] including both the scalar and the vector $\mathcal{N} = 1$ multiplet, is given by

$$\begin{aligned} S &= \frac{\tau}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathrm{Tr}\,W^{\alpha}W_{\alpha} + \frac{\bar{\tau}}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\,\mathrm{Tr}\,\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \\ &+ \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,K(e^V\Phi, \bar{\Phi}) + \int \mathrm{d}^4x\mathrm{d}^2\theta\,W(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\,\bar{W}(\bar{\Phi}) \end{aligned} \tag{1.26}$$

where the form of the Kähler potential is dictated by gauge invariance. Let us consider the classical scalar potential for the field $\phi(x)$; this is the only possible contribution to the vacuum energy, since in a relativistic quantum field theory, fields that transform non trivially under Poincaré group cannot have a vacuum expectation value. By expanding (1.26) in components, one finds

$$\begin{aligned} U &= \frac{g^2}{2}\sum_a\left(\frac{\partial K(\phi,\bar{\phi})}{\partial\phi^i}(T^a)^i_j\phi^j + c.c.\right)^2 + g^{i\bar{\imath}}\frac{\partial W(\phi)}{\partial\phi^i}\frac{\partial\bar{W}(\bar{\phi})}{\partial\bar{\phi}^{\bar{\imath}}} \\ &= \frac{1}{2g^2}D^aD^a + g^{i\bar{\imath}f_i f_{\bar{\imath}}} \end{aligned} \tag{1.27}$$

where the second line is evaluated on the equations of motion for the auxiliary fields D and f. Supersymmetric vacua are characterized by the vanishing of the vacuum energy, *i.e.* we must have $U = 0$, that requires simultaneously the vanishing of the D (D–flatness) and f (F–flatness) term. In particular, one can show that (if the Fayet–Iliopoulos term is absent) the F–flatness condition is necessary and sufficient in order to find supersymmetric vacua. In other words, to gain information on the supersymmetric vacua, one only needs to know the *holomorphic* sector of the theory (governed by the superpotential).

[3]Strictly speaking one could also introduce the *Fayet-Iliopoulos* term, if the gauge group has an abelian factor. We will not consider these kind of terms.

The space of all supersymmtric inequivalent vacua of the theory is know as classical moduli space. This is a complex manifold parametrized by inequivalent solutions of the flatness equations and endowed with the (pull–back of the) Kähler metric. More precisely, one has to solve the D–flatness equation modulo gauge transformations. It can be proven that the solutions of the D–flatness equation modulo gauge transformations span the space $\{\phi^i\}/G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the complexified gauge group (*i.e.* the gauge transformations parameters are complex numbers). In other words, the orbit of the complexified gauge group through any point ϕ^i contains a solution to the D–flatness equations. This space can be parametrized by a set of independent holomorphic gauge invariants $X_r(\phi)$. Finally, the complete moduli space is simply obtained by restricting the moduli $X_r(\phi^i)$ by the conditions coming from F–flatness. Of course, this picture will be modified once taking into account quantum effects. The problem of determining explicitly the quantum moduli space of a theory can be very hard.

1.2 Extended Supersymmetry: $\mathcal{N} = 2$

Theories with extended supersymmetry are a fascinating subject of theoretical physics. Since this work is mainly concerned with $\mathcal{N} = 1$ theories, we will only give some small comment on $\mathcal{N} = 2$ theories; a complete introduction can be found in [28] and in the original works of Seiberg and Witten [30, 29]. The only role played by extended supersymmetry in this thesis, is that some of the theories we will encounter can be thought of as arising from an $\mathcal{N} = 2$ softly broken to $\mathcal{N} = 1$ by the presence of a tree level superpotential. We will simply show the most general renormalisable lagrangian for the $\mathcal{N} = 2$ massless gauge multiplet. This consists of a gauge field A_μ, two Weyl spinors, λ and ψ, and a complex scalar ϕ, all in the adjoint representation of a gauge group G. Note that, we have precisely the field content of an $\mathcal{N} = 1$ theory with a gauge multiplet W_α and a chiral multiplet Φ, but with the constraint that they have to be in the same representation of the gauge group. The $\mathcal{N} = 2$ lagrangian for these fields can be obtained by taking the $\mathcal{N} = 1$ lagrangian and impose the extra supersymmetry. We simply state the result and then add some comment:

$$S = \frac{\tau}{16\pi}\mathrm{Tr}\left(\int \mathrm{d}^4x\mathrm{d}^2\theta\, W^\alpha W_\alpha + 2\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, \bar{\Phi}e^V\Phi\right) \tag{1.28}$$

First of all we note that no superpotential is allowed. In fact, the $\mathcal{N} = 2$ theory as an R–symmetry that exchanges the two fermions

$$\psi \to \lambda \qquad \lambda \to -\psi. \tag{1.29}$$

A superpotential term, containing only one fermion, is not invariant under this symmetry. The addition of a tree level superpotential, in fact breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ However note that in the $\mathcal{N} = 1$ theory obtained in this way, the gauge superfield and the chiral matter superfield are in the same representation of the gauge group. Moreover, the relative normalization between the gauge and the chiral superfield terms is *fixed* by supersymmetry.

1.3 Elements of Supergravity

In this section we want to introduce some basic elements of supergravity. A full treatment of supergravity is completely outside the scope of this thesis and can be found, for example, in [25]. However, in the following we will encounter some non trivial gravitational *non dynamical* backgrounds. More precisely, we will study theories with a rigid supersymmetry, but coupled to an $\mathcal{N}=1$ or $\mathcal{N}=2$ gravitational background. Therefore we just want to comment on the supergravity field content in four dimensions. By no means we intend this section to be exhaustive; since all gravitational fields that we will mention will be intended as background fields, we don't need to develop the full formulation of supergravity in superspace. We refer the interested reader to [25, 27].

The basic idea is to promote the rigid transformations (1.5) to local transformations, where the parameter ξ depends on the space–time coordinates. As is known from General Relativity, invariance under local space–time transformation, requires the presence of gravity. Then one builds the usual formalism of General Relativity by defining the vielbein, the connection, the curvature tensor and so on. However, since the local version of (1.5) involves also Grassmann variables, these concepts need the appropriate generalization to superspace. The requirement of invariance under $\mathcal{N}=1$ supersymmetry implies that the spin 2 graviton, has a spin $\frac{3}{2}$ superpartner, the gravitino. Then all the General Relativity concepts have to be covariantized to superspace. All the ideas of last section, can be extended to a theory with gravity with simple (but quite technical) modifications: covariant derivatives will include the spin connection, etc. In order to construct invariant actions, one can again extract the highest component of superfields. The graviton and the gravitino determine the $\mathcal{N}=1$ supergravity multiplet and can be used to construct the $\mathcal{N}=1$ Weyl superfield $G_{\alpha\beta\gamma}$, whose lowest component is the gravitino field strength

$$\psi^{\alpha}_{ab} = \hat{D}_a \psi^{\alpha}_b - \hat{D}_b \psi^{\alpha}_a, \tag{1.30}$$

where

$$\hat{D}_a \psi^{\alpha}_b = \partial_a \psi^{\alpha}_b + \psi^{\beta}_b \omega^{\alpha}_{a\beta}. \tag{1.31}$$

and $\omega^{\alpha}_{a\beta}$ is the spin connection. Strictly speaking, one should include also auxiliary fields. Since we are interested only in the basic concepts of supergravity, we will assume to be on shell and set all auxiliary fields to zero.

Since the main results showed in this thesis can be traced back to an $\mathcal{N}=2$ theory broken down to $\mathcal{N}=1$, we will comment also on the $\mathcal{N}=2$ supergravity theory. This can be basically constructed using an appropriate extension of the formalism outlined above. The main difference is that the $\mathcal{N}=2$ gravity multiplet, containing the graviton, will enter in the game. Since the theory has extended $\mathcal{N}=2$ supersymmetry, this multiplet contains two gravitinos ψ^A_μ, with $A=1,2$. In order to have a supersymmetric multiplet, we then must add to the bosonic sector another $U(1)$ gauge field, called the graviphoton. This field will play a fundamental role in Chapter 4.

1.4 Wilsonian Effective Actions

One of the most intriguing open problem in theoretical physics is to determine the low energy physics resulting from an asymptotically free gauge theory. In general, this is an hard problem, since the infrared (IR) regime is strongly coupled. This means that the relevant degrees of freedom that describe the ultraviolet (UV) theory, can be completely different from the IR degrees of freedom. This is the case, for example, in (massless) QCD, where, while the UV theory is a theory of quarks and gluons, the IR theory is described in terms of pions and massive glueballs. The emergence of new degrees of freedom can be easily understood, since in the strongly coupled regime, it is often impossible to determine asymptotic states, that characterize the UV degrees of freedom in the perturbative quantization of the theory. Thus, the study of the low energy dynamics, is usually based on physically motivated but formally arbitrary assumptions. The basic strategy to understand the IR physics is to begin with a *guess* for the low energy degrees of freedom and their symmetries and write down an effective action, consistent with the symmetries and the weak coupling (perturbative) limits. Of course, predictions of the effective action, should be used to check the original guess. We shall introduce these ideas following [31].

The starting point is the Wilsonian effective action. Let us consider a quantum field theory described by a Lagrangian $\mathcal{L}(\phi)$. Then, the Wilsonian effective action

$$S_\mu = \int \mathrm{d}^4 x \mathcal{L}_\mu(\phi) \tag{1.32}$$

is obtained by integrating out the fields with energy above the mass scale μ. Since high-energy degrees of freedom have been integrated out, the tree level action S_μ is an accurate description of the physics at energies $E \sim \mu$. Of course physical processes at energies $E \ll \mu$ will receive quantum corrections due to propagating degrees of freedom with energy between E and μ, but these corrections can be absorbed in the couplings to define a new effective action at the energy scale E. The two effective theories are related by the action of the renormalization group. Let us assume for the action at the energy scale μ the general form

$$S_\mu = \int \mathrm{d}^4 x \sum_i g_i(\mu) \mathcal{O}_i \tag{1.33}$$

The evolution of the couplings is given by the renormalization group equations

$$\mu \frac{\partial g_i(\mu)}{\partial \mu} = \beta_i(g(\mu), \mu) \tag{1.34}$$

that describe the flow of the theory in the space of all couplings. One can show that, in going from a scale μ_0 to a scale $\mu < \mu_0$ the operators $\mathcal{O}_i$ scale as $\left(\frac{\mu}{\mu_0}\right)^{\Delta_i}$, where Δ_i is the classical scaling dimension of the operator. Then an operator is called[4] *relevant* if

[4]The factor of 4 is coming from the integration measure d^4x. This number changes when considering superfields in superspace, depending on the scaling properties of the superspace integration measure. In particular it is equal to 3 for chiral superfield interactions and to 2 for a generic superfield interaction. This follows from the fact that for a superfield the scaling dimension is defined as the scaling of its lowest component.

$\Delta_i < 4$, corresponding to a growing of the coupling along the flow, *irrelevant* if $\Delta_i > 4$, corresponding to a damping of the coupling, and *marginal* if $\Delta_i = 4$. Of course, quantum effects can modify this picture and one as to identify the correct quantum scaling of an operator.

The same ideas translate in a simple way to the supersymmetric case: the supersymmetric effective action at some scale μ will be composed by an effective Kähler term K_{eff}, an effective superpotential W_{eff} and an effective gauge coupling τ_{eff}. Suppose that the effective theory is the low energy limit of an UV theory which depends on the UV couplings $\{\lambda_i, \tau\}$ and a given superpotential $W_{\text{UV}} = W_{\text{UV}}(\Phi, \tau)$. Then one can use the so-called *localization* trick: this consists in promoting the parameters of the theory to background fields such that their expectation value exactly match the original coupling. This trick is familiar from elementary quantum field theory when proving Noether theorem where constant parameters of global symmetries are treated as position dependend, *i.e.* classical fields. In the supersymmetric case the only difference is that we have to promote the parameters to superfields and not to ordinary fields. Thus, we simply promote λ_i to the chiral superfield (since it enters in the same place as chiral superfields) $\lambda_i(x, \theta)$ such that the vacuum expectation value of its scalar component is exactly λ_i (and the vacuum expectation value of all the other components is zero); and similarly for τ. But, since chiral superfields can only enter holomorphically in W_{eff} and τ_{eff}, we conclude that the bare coupling constants can only enter holomorphically as well.

1.5 Non–renormalization theorems

Non–renormalization theorems in supersymmetric theories are basically selection rules that can be used to constraint the form of low energy effective actions [23]. In particular in this section, we want to show that the effective superpotential is not renormalized perturbatively, following [31].

There are several techniques developed to constraint as much as possible the form of the effective superpotential. Of course, one has to propose a precise guess for the relevant IR degrees of freedom. Once this guess is given, one can use the power of holomorphy (in the fields and in the bare couplings) to constraint the effective superpotential, as well as the weak coupling limits (where the theory should match the known perturbative description). Another powerful method is the use of the localization trick explained above; assigning to the bare parameters of the UV lagrangian, promoted to fields, transformation properties under global symmetries, one can enlarge the global symmetry group of the theory. If these symmetries are not anomalous, they should be shared even by the effective action, thus giving further constraints on its form. However, is important to remember that the Kähler potential is *not* an holomorphic function and thus these techniques are not of any help in computing it.

Let us give a concrete example. Let us suppose we have the following UV theory at the scale μ_0

$$\mathcal{S}_{\mu_0} = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ \bar{\Phi}\Phi + \int \mathrm{d}^4x\mathrm{d}^2\theta\ \frac{1}{2}\lambda_2\Phi^2 + \frac{1}{3}\lambda_3\Phi^3 \tag{1.35}$$

and let us try to determine the effective superpotential at some lower scale $\mu < \mu_0$.

Symmetry	$U(1)$	$U(1)_R$
Φ	+1	+1
θ	0	+1
$d\theta$	0	−1
W_{UV}	0	+2
λ_2	−2	0
λ_3	−3	−1

Table 1.1: Charge assignment of fields and parameters

We assign to the fields and couplings some charges under the symmetries of the theory as explained in Table 1.1. There $U(1)_R$ is the R-symmetry group of the theory, that rotates the supercharges and consequently the superspace coordinate θ (as is immediate from the form of the supercharges (1.7)). The auxiliary $U(1)$ is the symmetry of the kinetic term that is enlarged to a symmetry of the full theory by assigning charges to the bare couplings $\lambda_{2,3}$. Then, assuming the $U(1) \otimes U(1)_R$ global symmetry is not anomalous, the effective superpotential at the scale μ, $W_{\text{eff}}(\Phi, \lambda_2, \lambda_3)$, is constrained to be of the form

$$W_{\text{eff}} = \lambda_2 \Phi g\left(\frac{\lambda_3 \Phi}{\lambda_2}\right) \tag{1.36}$$

where, by now, g is an arbitrary function. Now let us take the $\lambda_3 \to 0$ limit keeping λ_2 fixed. This limit should match with the perturbative expansion in λ_3, *i.e.* only non negative powers of λ_3 can appear:

$$W_{\text{eff}} = \sum_{n \geq 0} g_n \lambda_2^{1-n} \lambda_3^n \Phi^{n+2} \tag{1.37}$$

Similarly, by appropriately taking the limit where the theory is perturbative in λ_2, we find that negative powers of λ_2 cannot appear. Thus the effective superpotential assume the final form

$$W_{\text{eff}} = g_0 \lambda_2 \Phi^2 + g_1 \lambda_3 \Phi^3 \tag{1.38}$$

A more precise matching with perturbation theory actually shows that the coefficents g_0 and g_1 should be equal to the classical coefficient appearing in the UV superpotential, multiplied by the appropriate power of the classical scaling factor $\left(\frac{\mu_0}{\mu}\right)$ in going from the UV scale μ_0 to the IR scale μ.

Let us give now a more general argument on the perturbative non–renormalization of the effective superpotential. Let $W_{\mu_0} = W_{\mu_0}(\Phi_n)$ be the UV superpotential with an arbitrary number of chiral superfields Φ_n. For the moment let us suppose no vector superfield is present. Now the idea is to localize ... 1! That is to say, replace the superpotential with YW_{μ_0} where Y is a chiral superfield such that the vacuum expectation value of its lowest component is precisely 1. The theory is now invariant under the $U(1)_R$ R–symmetry given by the charge assignment $R(Y) = +2$ and $R(\Phi_n) = 0$. This symmetry and holomorphy constrain the effective superpotential at a lower scale μ to assume the form

$$W_\mu = Y g(\Phi_n) \tag{1.39}$$

where g is a, by now, arbitrary but holomorphic function. Finally the condition that the IR action should match the UV action in the weak coupling limit (that is to say for $Y \to 0$ appropriately, since this makes the theory free) implies that

$$g(\Phi_n) = W_{\mu_0}(\Phi_n) \tag{1.40}$$

In the end, we set $Y = 1$ to recover the original theory and conclude that $W_{\mu_0} = W_\mu$.

One could wander what happens if we include also vector superfields, or, more precisely, if the effective superpotential could depend also on their coupling constant τ. This is not the case as the theory have to be invariant under shiftings of τ corresponding to the periodicity of the ϑ–angle. This shift cannot be absorbed by a rigid translation of the fields Φ, these being not gauge fields (we will return on the ϑ–angle in the next section).

In a similar way, one can also prove a more general non–renormalization theorem for the generalized effective superpotential, given by the effective superpotential and the gauge field kinetic term with the effective coupling. We will not enter in the details, but give only the conclusion: the effective superpotential is not perturbatively renormalized, as above, while the effective gauge coupling gets only perturbative contributions at one–loop.

Let us conclude this section with a few more comments. Since the effective superpotential of the theory still depends on the bare coupling, one could take the (wrong) conclusion that the couplings do not run and the theory is trivial. However this is not the case. In fact, the kinetic term, being not protected by these arguments, will get quantum corrections. In particular, the fields will get a wave function renormalization

$$\Phi_n \to \sqrt{Z_n(\mu)}\Phi. \tag{1.41}$$

Thus, to get a canonically normalized kinetic term, one has to rescale all the fields and consequently to define new canonically normalized couplings that will depend on the wave function renormalization. Clearly, these new couplings are running. More precisely, their beta functions will depend on the anomalous dimension of the fields Φ_n

$$\gamma_n(\mu) = \frac{\mathrm{d}\ln Z_n(\mu)}{\mathrm{d}\ln\mu}. \tag{1.42}$$

Unfortunately, the anomalous dimensions are not under control since they come from the renormalization of the Kähler term, for which, as explained before, wa cannot use the powerful machinery introduced in this section.

1.6 Anomalies

In this section we will briefly summarize some known fact about anomalies and their role in the dynamics of quantum gauge theories, as explained in [32, 31, 23, 33]. This will lead us to the Konishi anomaly that will play a fundamental role in this thesis.

As is well known, anomalies are symmetries of the classical theory that are broken by quantum effects. This is usually observed in perturbation theory, where observable are computed in some regularization scheme. It may happen that the regularization procedure breaks some of the symmetries of the classical theory; but an anomaly appears when the symmetry remains broken even if the regulator is removed.

An important example of an anomaly, is the *trace anomaly*. It is associated with scale invariance: classically the dilation current $\mathcal{D}_\mu$ is conserverd

$$\partial^\mu \mathcal{D}_\mu = T^\mu_\mu \equiv 0 \tag{1.43}$$

since the trace of the energy–momentum tensor T^μ_μ vanishes in classically scale invariant theories. But in Yang–Mills theories, as well as their supersymmetric extensions, the scale symmetry is violated quantum mechanically. This is reflected in the emergence of a strong–coupling scale

$$|\Lambda| = \mu e^{\frac{-8\pi^2}{bg^2(\mu)}} \tag{1.44}$$

(to one loop order; this phenomenon is also called dimensional transmutation). The coefficient b is given by

$$b = \frac{11}{6}T(\text{adj}) - \frac{1}{3}\sum_i T(R_i) - \frac{1}{6}\sum_a T(R_a) \tag{1.45}$$

where the index i runs over the Weyl fermions in the representation R_i of the gauge group and the index a runs over the complex scalars in the representation R_a of the gauge group and $T(R)$ is the index of the representation defined by

$$T(R) = \frac{C(R)}{C(\text{fund})} \tag{1.46}$$

where $C(R)$ is the quadratic Casimir of the representation R

$$\text{Tr}_\text{R}\left(T^a T^b\right) = C(R)\delta^{ab} \tag{1.47}$$

In terms of $|\Lambda|$, the RG equations for the running of the coupling can be solved by

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}\ln\left(\frac{|\Lambda|}{\mu}\right) \tag{1.48}$$

From (1.48) we see that the effective gauge coupling $g(\mu)$ diverges as the scale μ approaches the strong coupling scale $|\Lambda|$, hence its name. Moreover, we easily see that the sign of b determines if the theory is strongly or weakly coupled in the IR and in the UV.

By considering also the θ–angle term in the theory, it is customary to define the complexified gauge coupling

$$\tau(\mu) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(\mu)} \tag{1.49}$$

that can be used to generalize (1.48) as

$$\tau(\mu) = \frac{b}{2\pi i}\ln\left(\frac{\Lambda}{\mu}\right) \tag{1.50}$$

where Λ is the complex RG invariant scale

$$\Lambda = |\Lambda| e^{\frac{i\theta}{b}} = \mu e^{\frac{2\pi i \tau(\mu)}{b}} \tag{1.51}$$

All the above discussion can be immediately applied to the supersymmetric extensions of Yang–Mills theories, since these can be simply understood as usual gauge theories with a particular prescription for the matter content.

Other important symmetries that can be anomalous are chiral symmetries, in which the left–handed fermions transform differently than the right–handed fermions. In four dimensions, left and right Majorana–Weyl fermions transform in complex conjugate representations and thus they can give rise to chiral anomalies. This means that, given a chiral symmetry, the associated current is non conserved quantum mechanically,

$$\langle \partial_\mu J^\mu \rangle \neq 0 \tag{1.52}$$

The basic strategy to compute the anomaly is to think of the chiral symmetry as a gauge symmetry, that is to couple the chiral current J^μ to a background gauge field (this can be thought of using the localization trick to promote the chiral symmetry transformation parameter to a gauge field). The anomaly can be computer perturbatively via the one loop triangle diagrams of figure (1.1) obtaining

$$\langle \partial_\mu J^\mu_a \rangle \propto \sum_i \mathrm{Tr}_{R_i} \left(T_a \{ T_b, T_c \} \right) F^{\mu\nu}_b \widetilde{F}_{\mu\nu c} \tag{1.53}$$

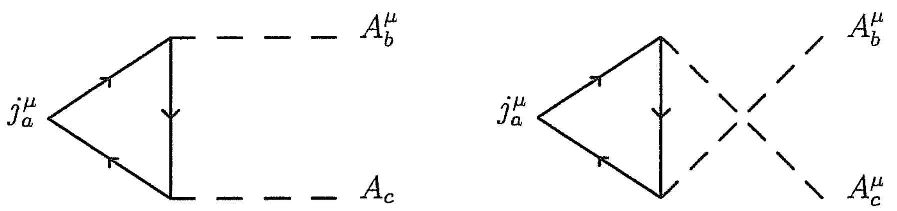

Figure 1.1: Triangle diagrams contributing to the chiral anomaly.

This result is perturbatively exact, since higher loop do not contribute (by the Adler–Bardeen theorem). Now, let us suppose that the symmetry group of the theory consists in the product of a global (chiral) group $\mathcal{G}$ and a gauge group $\mathcal{H}$. Then, the symmetry generators split as $T^a = \{T^a_{\mathcal{G}}, T^a_{\mathcal{H}}\}$. Let us suppose that the chiral current generators are in $\mathcal{G}$ while the external background fields are in $\mathcal{H}$, then

$$\begin{aligned} \langle \partial_\mu J^\mu_{\mathcal{G}} \rangle &= \frac{1}{32\pi^2} \sum_i \mathrm{Tr}_{\mathcal{G}_i \otimes \mathcal{H}_i} \left(T_{\mathcal{G}_i} \{ T^a_{\mathcal{H}_i}, T^b_{\mathcal{H}_i} \} \right) F^{\mu\nu}_a \widetilde{F}_{\mu\nu b} \\ &= \frac{1}{16\pi^2} \sum_i \mathrm{Tr}_{\mathcal{G}_i} \left(T_{\mathcal{G}_i} \right) \mathrm{Tr}_{\mathcal{H}_i} \left(T^a_{\mathcal{H}_i} T^b_{\mathcal{H}_i} \right) F^{\mu\nu}_a \widetilde{F}_{\mu\nu b} \\ &= \frac{1}{16\pi^2} \sum_i q_i C \left(R_{\mathcal{H}} \right) F^{\mu\nu}_a \widetilde{F}_{\mu\nu a} \end{aligned} \tag{1.54}$$

where the index i runs over the (massless) fermions. Since the anomaly depends on $\mathrm{Tr}_{\mathcal{G}_i}(T_{\mathcal{G}_i})$, it can only be in the abelian factors of the chiral symmetry group. Without any lack of generality we can thus assume that $\mathcal{G} = U(1)$. In the last line of (1.55) we have introduced the charge of the i^{th}–fermion under this $U(1)$: $q_i = \mathrm{Tr}_{\mathcal{G}_i}(T_{\mathcal{G}_i})$. The anomaly is proportional to the term $F_a^{\mu\nu}\widetilde{F}_{\mu\nu a}$ that in the gauge theory action is precisely the ϑ–term:

$$S_\vartheta = -\frac{\vartheta}{16\pi^2}\int \mathrm{d}^4x \mathrm{Tr}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right) = n\vartheta \tag{1.55}$$

The ϑ–term is quantized in units of ϑ and n is an integer that label the winding number of the gauge configuration (seen as a Lie algebra valued function). This term contributes as e^{iS_ϑ} to the path integral and thus the theory is invariant for $\vartheta \to \vartheta + 2\pi$, hence the name ϑ–angle. In terms of the complex gauge coupling (1.49), this corresponds to the shift $\tau \longrightarrow \tau + 1$. The interpretation of the ϑ–term is that the theory as infinite homotopically inequivalent vacua. These are related by gauge field configurations that correspond to instantons (in the Euclidean theory).

From these results, we can see that the chiral symmetry group is not completely broken, but a $\mathbb{Z}_n$ symmetry survives, corresponding to the transformation

$$\begin{aligned} \psi^i &\longrightarrow e^{iq_i\varepsilon}\psi^i \\ \vartheta &\longrightarrow \vartheta + 2\varepsilon\sum_i q_i C(R_i) \end{aligned} \tag{1.56}$$

for values of ε such that $2\varepsilon\sum_i q_i C(R_i) \in 2\pi\mathbb{Z}$.

It is now easy to extend these result to the $\mathcal{N} = 1$ supersymmetric gauge theory. Its field content is given by the superfield Φ in some representation R of a gauge group, that encodes a complex scalar and a Weyl spinor, and the gauge field W_α in the adjoint representation of the gauge group, that encodes the gauge boson A_μ and a Weyl fermion, the gaugino λ_α. The classical action

$$\begin{aligned} S &= \frac{\tau}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathrm{Tr}\, W^\alpha W_\alpha + \frac{\bar{\tau}}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \mathrm{Tr}\, \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \\ &+ \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, \bar{\Phi}e^V\Phi + \int \mathrm{d}^4x\mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{W}(\bar{\Phi}) \end{aligned} \tag{1.57}$$

has a chiral symmetry

$$\Phi \longrightarrow e^{i\varepsilon}\Phi \simeq \Phi + i\varepsilon\Phi \tag{1.58}$$

broken classically by the superpotential. By rewriting the kinetic term as

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, \bar{\Phi}e^V\Phi = \int \mathrm{d}^4x\mathrm{d}^2\theta\, \bar{D}^2(\bar{\Phi}e^V\Phi) \tag{1.59}$$

and using the superspace version of Noether theorem, we get the classical equation

$$-\bar{D}^2(\bar{\Phi}e^V\Phi) = \Phi\frac{\partial W}{\partial\Phi} \tag{1.60}$$

where the LHS is the supersymmetric version of the divergence of the current $\partial_\mu J^\mu$, that can be obtained by taking the θ^2 component, and the RHS is the classical symmetry breaking caused by the superpotential. But, quantum mechanically, an anomaly is expected, precisely given by (the superspace version of) (1.55)

$$-\bar{D}^2(\bar{\Phi}e^V\Phi) = \Phi\frac{\partial W}{\partial\Phi} + \frac{1}{32\pi^2}\mathrm{Tr}\,(W^\alpha W_\alpha) \tag{1.61}$$

This equation is known as the *Konishi anomaly* [34, 35].

1.7 The Dynamics of Gauge Theories

We will conclude this chapter by briefly reviewing the dynamical properties of gauge theories, and in particular of $\mathcal{N}=1$ theories. When giving explicit formulas, we will generically refer to $SU(N)$ gauge theories, in order to simplify the notation. However the results explained are expected to be valid also for other gauge groups. A more general and complete discussion can be found in [23, 24].

The dynamical properties of a gauge theory are strictly dependent on the *phase* the theory is in. The main tool to study the different phases of gauge theories, at least for supersymmetric theories, the low energy effective action introduced in the previous sections. As the energy scale is modified, gauge theories can undergo phase transition and thus exhibit drastically differen behaviors. The phases of gauge theories can be characterized by the potential $V(r)$ between electric test charges separated by a large distance r. The main phases as well as the conjectured behavior of their potential are summarized in table 1.2. The theories can be given both an electric and

Phase	Electric potential	Magnetic potential
Coulomb	$\frac{1}{r}$	$\frac{1}{r}$
Free electric	$\frac{1}{r\ln(r\Lambda)}$	$\frac{\ln(r\Lambda)}{r}$
Free magnetic	$\frac{\ln(r\Lambda)}{r}$	$\frac{1}{r\ln(r\Lambda)}$
Higgs	constant	ρr
Confining	σr	constant

Table 1.2: Charge assignment of fields and parameters

a magnetic description, as well as free quantum electrodynamics. This property is conjectured to be shared by non–Abelian gauge theories with sources and is usually called Electric–Magnetic duality. As far as we know, there is not any convincing prove of this statement, but an increasing amount of evidence. In terms of the electric potential, the first three phases have massless gauge fields and potentials of the form $e(r)/r$ where $e(r)$ is the renormalized electric charge. This charge is constant for the Coulomb phase (hence its name) while in the free electric phase it is renormalized to zero as $r \to \infty$ by the massless charged particles and in the free magnetic phase it is renormalized to infinity as $r \to \infty$ by massless monopoles. In terms of the magnetic potential, this behavior can be understood as a consequence of Dirac quantization condition $e(r)g(r) \sim 1$ that relates the electric to the magnetic (renormalized) charge $g(r)$. The Higgs phase is pretty similar to the Meissner effect in superconductivity. Electrically charged particles condense giving a mass gap to the gauge field by the Higgs mechanism, while the flux between two magnetic sources is confined into a thin flux–tube with constant tension ρ, resulting in a linear potential. This reminds us of condensed matter physics where charged electrons can condense in Cooper pairs on top of which magnetic flux is confined.

Finally, the most intriguing situation is the confining phase. This is expected to be a description of real world QCD[5]. Roughly speaking, given two electric test charges, the potential between them is a linear function of the distance and consequently the force is a constant, independent on the distance. Thus the particles are confined. This can be equivalently stated by looking at the properties of the electric field; this is confined in a flux tube, or string, between the two sources. Actually, this is a more precise definition of confinement, that now is stated relying only on the properties of the Yang–Mills fields. It is conjectured that this picture of confinement can be given a dual description in terms of magnetic variables. Just like the electric Higgs phase, where condensation of electrically charged particles imply that magnetic field is confined in flux tubes, we may conjecture that electric confinement can be described by the condensation of monopoles. Again, there is no proof of this statement but only some evidence .

Consider pure Yang–Mills theory. We know it is a strongly coupled theory in the infrared, at the dynamically generated scale Λ. More precisely, this theory in the IR develops a *mass gap* and becomes confining. Let us explain better what these statements mean. Physically, the existence of a mass gap, implies that there are no massless fields in the spectrum, but only a discrete set of states with mass of order Λ. This may appear a bit weird, since the UV theory is a theory of massless fields, the gluons. The key point is that gluons are not the correct variables to describe the IR physics. Actually, they emerge in the perturbative quantization of the theory as asymptotic states; this is possible since the theory is asymptotically free and thus weakly coupled in the UV. But in the IR, simply we cannot apply this quantization scheme and new (massive) degrees of freedom emerge, the glueballs. These do not consist of a bound state of gluons (gluon number is not a conserved quantity); they are simply the *real* objects that characterize the low energy physics. Moreover the theory becomes confining, meaning, as we have already said, that the electric field itself is confined in flux tubes: it cannot spread out in space over regions larger than about Λ^{-1} in radius. There is a deep relation between the confinement of the electric field and the generation of a mass gap. This picture is perfectly fitting in the "dual" description we have given above, where condensation of magnetically charged object implies the confinement of electric field. In fact, as is well known from condensed matter physics, particle condensation can be associated with the generation of a mass gap via the Higgs mechanism.

[5]Note however that QCD is not formally a real confining theory, since quarks can escape from hadrons as seen in deep inelastic scattering processes. This is because there exist quark lighter than the energy scale at which confinement occurs and pair production can overcome the confining potential. However this mechanism is in some sense "accidental" and the low energy dynamics is captured by confinement.

Most of the statements made until know, do not have a formal proof. They come mainly from physical intuition and from direct (computational) evidence in some cases. Actually the dynamics of gauge theories is an hard subject. A lot of insight has come with the aid of computer simulations. The idea is to *discretize* space time, so that the theory is defined on a lattice. Then gauge fields are not anymore continuous functions but can only take values in a finite set (this can be interpreted as a regularization of the theory). Then one can compute correlators analytically or with the aid of a calculator (with an high precision). In this way, most of the conjectures we have encountered can be confirmed, pointing out that the physical picture given is probably correct. However, this cannot be regarded as a proof. In fact, in defining the theory on a lattice, one is actually *changing* the theory, simply because one needs to introduce another scale, *i.e.* the lattice spacing. It turns out that all the correlators will now depend on this new scale. To give a formal proof of the above stated conjectures, one finally needs to show that these properties survive when the lattice spacing is taken to zero, or in other words that the theory does not undergo a phase transition. But this task is as difficult as the direct approach to gauge theories. However, lattice computations agree with all analytical computations one can do in the continuum theory. Thus it is widely believed that the lattice description is correct and that the lattice answers can be trusted. Actually lattice gauge theories provide the most powerful method we have to get physical insight on the dynamics of gauge theories.

Another possible technique to study gauge theories is to consider their supersymmetric version. This is in a sense quite similar to the lattice approach, in that you change the theory in a "controlled" way to make it easier[6]. Let us now consider pure $\mathcal{N} = 1$ Yang–Mills. This theory is very interesting since it exhibits the same properties of pure Yang–Mills, confinement and mass gap generation, but is easier to study since it is supersymmetric. The strategy is then to "solve" this theory and then break supersymmetry to recover ordinary Yang–Mills theory and get a clear understanding of its dynamics. But even $\mathcal{N} = 1$ Yang–Mills is too difficult to be solved exactly; however the Dijkgraaf–Vafa conjecture has definitely been a progress in this direction. The main difference between pure and supersymmetric Yang–Mills is the presence of the gluino (or gaugino) λ_α; however to ensure supersymmetry, the gaugino has to be in the adjoint representation of the gauge group as well as the gauge field, and fields in the adjoint representation cannot break electric flux tubes. Thus we expect that the confinement mechanism is the same as in pure Yang–Mills (and this is true even if we add other adjoint matter, for example a chiral superfield in the adjoint representation, like in (1.26)). But this theory exhibit also another remarkable phenomenon, gaugino condensation. The theory has an $U(1)_R$ chiral R–symmetry (phase rotations of the gaugino); as explained in the previous section, chiral symmetries are broken by quantum effects. However, as seen in eq. (1.56) a discrete symmetry survives the anomaly. Since $C(\text{adj}) = h$ where h is the dual Coxeter number

$$\text{Tr}_{\,\text{adj}}(T_a^{adj} T_b^{adj}) = h\delta_{ab} \tag{1.62}$$

[6]Again, if one could show some dynamical property of the supersymmetric theory, then one would need to break supersymmetry in order to extend it to the original theory. But this introduces a new energy scale, the one at which supersymmetry is broken. This is the same problem encountered in lattice gauge theories

we see that $U(1)_R$ is broken down to $\mathbb{Z}_{2h}$ or, explicitly, the remaining symmetry is

$$\lambda_\alpha \to e^{i\varepsilon}\lambda_\alpha \qquad \text{with} \quad \varepsilon = \frac{2\pi}{2h}n \quad , \quad n \in \mathbb{Z}_{2h} \tag{1.63}$$

This fact has a clear interpretation in terms of instanton calculus. In fact the first non zero correlator in an instanton background is obtained by soaking the zero modes of the Dirac operator in an instanton background (of course, a Wick rotation to Euclidean space is understood when talking about instantons)

$$\langle (\lambda\lambda)^h \rangle \sim \Lambda^{3h} \tag{1.64}$$

where the RHS can be understood by dimensional analysis and by the fact that being a non perturbative evaluation it has to depend on Λ.

At strong coupling, this discrete symmetry is further broken down to $\mathbb{Z}_2$. Actually this means that gaugino bilinear get a vacuum expectation value

$$\langle \lambda\lambda \rangle \sim \Lambda^3 e^{2\pi i k/h} \qquad k = 0, \dots, h-1 \tag{1.65}$$

together with h inequivalent (confining) vacua, related by a rotation in the ϑ–angle $\vartheta \to \vartheta + 2\pi k$. This phenomenon is called gaugino condensation and is deeply related to the generation of the mass gap (intuitively chiral symmetry breaking allows a mass term for the gaugino).

Now it is possible to write down an effective action for the low energy theory that captures the phenomenon of gaugino condensation Let us introduce the glueball superfield

$$S = -\frac{1}{32\pi^2}\mathrm{Tr}\,(W^\alpha W_\alpha) = \frac{1}{16\pi^2}\mathrm{Tr}\,(\lambda^\alpha\lambda_\alpha + \dots). \tag{1.66}$$

this is the chiral superfield whose lowest component is the gaugino bilinear; because of this gaugino condensation can be described as S getting a non zero expectation value. The dynamic of the glueball superfield is captured by the Veneziano–Yankielowicz effective superpotential [4]

$$W_{\mathrm{VY}} = NS\left(1 - \ln\frac{S}{\Lambda}\right) \tag{1.67}$$

This effective action was simply derived as the most general potential compatible with the symmetries of the theory and that comprises the gaugino condensation (this in fact can be easily obtained by taking the equation of motion for the field S in (1.67)).

Let us pause a moment to clarify the implications of all this. Actually, the Veneziano–Yankielowicz effective superpotential is based on the highly non trivial assumption that S is the correct low energy degree of freedom. This guess is not in contradiction with any known property of the theory and moreover seems to capture correctly the low energy dynamics of the theory. Then, it is reasonable to conclude that it is true. However, it would be very interesting, and probably could shed some light on the mechanism through which confinement arises, to understand exactly how S comes in the game. Unfortunately, this task is presently out of reach.

Chapter 2

The Dijkgraaf–Vafa conjecture and its field theoretical derivation

The Dijkgraaf–Vafa conjecture [1] allows one to compute the exact effective superpotential for an $\mathcal{N} = 1$ gauge theory with chiral matter in some representation of a given gauge group. It is based on an auxiliary matrix model associated with the original gauge theory. The outstanding feature of the conjecture is that it gives a non perturbative information (the resulting superpotential is exact, meaning that it includes instantonic contributions) relying only on perturbative techniques (the derivation is entirely diagrammatic). The original motivation for the conjecture is arising from String Theory [1, 2, 3]. However it can be given also a purely gauge theoretical derivation [6, 8]. To explain this derivation, we will introduce the concept of the chiral ring, namely a ring structure constructed by all chiral operators of the theory. The restriction to the chiral ring of a generalized form of the Konishi anomaly will allow us to compute the effective superpotential. This way of reasoning can be extended to other supersymmetric theories based on other gauge groups and with chiral matter in other representations than the adjoint [9]. We will give some explicit examples and comment on some subtleties that arise when computing superpotentials for $Sp(N)$ gauge theories.

2.1 A Perturbative Window into Non Perturbative Physics

The Dijkgraaf–Vafa conjecture is a striking result that allows one to compute the exact effective superpotential for $\mathcal{N} = 1$ supersymmetric gauge theories with chiral matter in some representation of the gauge group and a tree level superpotential that is a polynomial in the chiral superfield. The obtained superpotential is the sum of the Veneziano–Yankielowicz superpotential and instanton correction arising from integrating out the chiral matter. The basic strategy is to associate to the gauge theory a *matrix model* whose action is given by the tree level superpotential of the gauge theory. Then, one can extract the effective superpotential for the gauge theory from the partition function of the matrix model, given a correct identification between the gauge theory and the matrix model parameters. More precisely, the measure of the matrix model partition function gives the Veneziano–Yankielowicz superpotential while the instanton corrections can be extracted from the planar limit of the matrix model. The remarkable fact is that the planar evaluation of the matrix model is entirely perturbative.

The original formulation of the conjecture is based on a chain of dualities based on String Theory [1, 2, 3], that have their origin in [36, 37, 38]. We will not enter in the details of how String Theory suggest the emergence of the matrix model, but only give a general idea. Basically one can engineer the gauge theory on a $D5$ branes configuration wrapping a 2–cycle of a Calabi–Yau manifold. The four dimensional part of the branes where the gauge theory lives is outside of the Calabi–Yau manifold. The key point is that the topological string theory living inside the Calabi–Yau manifold is controlling the holomorphic sector of the gauge theory. Finally, topological string theory correlators can be computed by an auxiliary matrix model (that basically arises as the topological string theory localizes to its zero modes). At the end, one can simply forget the String Theory behind all this ending up with a prescription to associate a matrix model to the original gauge theory. For related work inspired from String Theory see [39, 40, 41, 42, 43, 44, 45, 46]

There exists a purely gauge theoretical derivation of this result, that we will describe in the next section [8]. In this section we will describe the matrix model approach, following [1, 22]. For a more accurate discussion, see [7]. A complete introduction to matrix models can be found in [47, 48, 49].

Since [1] has appeared, a lot of literature followed. The interested reader can consult some work on the application of the conjecture to various theories, namely theories with extended supersymmetry such $\mathcal{N} = 2$ [50] or $\mathcal{N} = 4$ [51] and their breaking to $\mathcal{N} = 2^*$ and $\mathcal{N} = 1^*$ introduced in [52]. Some attempts to understand the Argyres–Douglas points [54] were made in [53], while Seiberg duality [56] was studied in [55], both in the framework of the matrix model techniques and of the purely gauge theoretical approach of [8]. Furthermore, the original conjecture was generalized to include flavor [57] and baryons [58] and even multi–trace deformations [60]. Some works on chiral theories can be found in [61, 82]. Some authors have used these techniques to study field theories arising from orbifolds and quiver theories in [59]. Finally, there have been some attempts to generalize the conjecture to lower or higher dimensions [62]. We will not introduce any of these subjects; to explain the material exposed in this thesis, a simpler setup will be enough. We refer the interested reader to the original literature

Let us state in a more precise way the conjecture in the case of a $U(N)$ gauge group with chiral matter in the adjoint representation. The action is given by

$$\begin{aligned} S &= \frac{\tau}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathrm{Tr}\, W^\alpha W_\alpha + \frac{\bar\tau}{16\pi}\int \mathrm{d}^4x\mathrm{d}^2\bar\theta\, \mathrm{Tr}\, \bar W_{\dot\alpha}\bar W^{\dot\alpha} \\ &+ \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, \bar\Phi e^V\Phi + \int \mathrm{d}^4x\mathrm{d}^2\theta\, W_{\text{tree}}(\Phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar\theta\, \bar W_{\text{tree}}(\bar\Phi) \end{aligned} \tag{2.1}$$

with

$$W_{\text{tree}} = \sum_{k=0}^{n} \frac{g_k}{k+1}\mathrm{Tr}\, \Phi^{k+1} \tag{2.2}$$

Classical vacua are determined by the extrema of W_{tree} (seen as a holomorphic function of a complex variable). In a classical vacuum, the field Φ is a diagonal matrix whose eigenvalues are in the set of critical point a_i with multeplicity N_i. One can describe a generic vacuum by picking out a partition

$$N = N_1 + \cdots + N_n \tag{2.3}$$

that corresponds to distributing N_i eigenvalues at the critical point a_i. This configuration corresponds to the symmetry breaking pattern

$$U(N) \to U(N_1) \times \cdots \times U(N_n) \tag{2.4}$$

The low energy degrees of freedom are the glueball superfield

$$S = -\frac{1}{32\pi^2}\mathrm{Tr}\,(W^\alpha W_\alpha) = \frac{1}{16\pi^2}\mathrm{Tr}\,(\lambda^\alpha \lambda_\alpha + \dots) \tag{2.5}$$

corresponding to the symmetry breaking pattern. For later use, we also introduce the abelian (thus IR free) fields

$$w_{i\alpha} = -\frac{1}{4\pi^2}\mathrm{Tr}\,(W_{i\alpha}) \tag{2.6}$$

Now, we associate with the gauge theory, the bosonic one matrix model given by the partition function

$$Z_{\mathrm{matrix}} = \int \mathcal{D}M e^{-\frac{N'}{g_m}W_{\mathrm{tree}}(M)} \tag{2.7}$$

where M is an $N' \times N'$ hermitian matrix and N' has *no relation* with N; $\frac{N'}{g_m} = g_s$ can be seen as a parameter (actually it is the string coupling, the only remnant of String Theory acting behind the conjecture). The vacua of the matrix model are chosen, as for the gauge theory, by picking out a partition

$$N' = N'_1 + \cdots + N'_n \tag{2.8}$$

where we demand that the symmetry breaking pattern of the matrix model and the one of the gauge theory are the same. Now we can evaluate the matrix model partition function by taking the 't Hooft large N' limit that gives an expansion in terms of double line diagrams. The partition function admits the topological expansion in the genus g

$$\begin{aligned} Z_{\mathrm{matrix}} &= e^{-\frac{N'^2}{g_m^2}\mathcal{F}} \\ &= e^{\sum_{g\geq 0} g_s^{2g-2}\mathcal{F}_g(g_s N'_i)} \end{aligned} \tag{2.9}$$

where $\mathcal{F}$ is called the free energy of the matrix model and $\mathcal{F}_g$ are the coefficients of its genus expansion. The main contribution to the path integral is given by the planar limit $\mathcal{F}_{g=0}$. The connection with the gauge theory arises when identifying $S_i = g_s N'_i$, as we will explain later. Note that this identification is non trivial, since S_i are dynamical fields while g_s and N'_i are simply parameters of the matrix model. Now we can finally state the conjecture. The effective superpotential of the gauge theory is given by two contributions. The first one is the Veneziano–Yankielowicz superpotential, that can be derived from the measure of the matrix model partition function; the second is given by the perturbative evaluation of the matrix model[1]

$$W_{\text{pert}}\left(S_i, w_{i\alpha}, g_k\right) = \sum_i N_i \frac{\partial \mathcal{F}_{g=0}\left(S_i, g_k\right)}{\partial S_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 \mathcal{F}_{g=0}\left(S_i, g_k\right)}{\partial S_i \partial S_j} w_i^\alpha w_{j\alpha} \tag{2.10}$$

Note that it is only the planar limit of the matrix model that contributes to the superpotential. The rank of the matrix N' is disappeared in S_i while the ranks of the gauge groups N_i simply appear linearly. One could ask what is the physical interpretation of the non planar contribution; we will show in the next chapters that they correspond to gravitational corrections to the effective superpotential.

The planar free energy of the matrix model can be explicitly worked out by using the *loop equations* of the matrix model. These are simply Ward identities following from the variations

$$\delta M = \epsilon M^{n+1}. \tag{2.11}$$

These variations give the identities

$$0 = \int \mathrm{d}M \mathrm{Tr}\left(\frac{\partial}{\partial M} M^n\right) e^{-\frac{N'}{g_m} W_{\text{tree}}(M)} \tag{2.12}$$

that can be summarized by the following relation

$$0 = \int \mathrm{d}M \mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{1}{z-M}\right) e^{-\frac{N'}{g_m} W_{\text{tree}}(M)} \tag{2.13}$$

Evaluating this expression, one gets

$$\left(\frac{g_m}{N'}\right)^2 \left\langle \left(\mathrm{Tr}\,\frac{1}{z-M}\right)^2 \right\rangle = \frac{g_m}{N'} \left\langle \frac{W'_{\text{tree}}(M)}{z-M} \right\rangle \tag{2.14}$$

Let us now define the matrix model resolvent

$$R_m(z) = \frac{g_m}{N'} \left\langle \mathrm{Tr}\,\frac{1}{z-M} \right\rangle \tag{2.15}$$

Now, by substituting (2.15) in (2.14), we find

$$\left\langle R_m(z)^2 \right\rangle = \left\langle W'(z) R_m(z) \right\rangle + \frac{1}{4} f_m(z) \tag{2.16}$$

where

$$f_m(z) = 4\frac{g_m}{N'} \left\langle \mathrm{Tr}\,\frac{W'_{\text{tree}}(M) - W'_{\text{tree}}(z)}{z-M} \right\rangle \tag{2.17}$$

is an unknown polynomial of degree $n-1$. Let us now take the large N' limit of equation (2.16): correlators in the matrix model factorize as

$$\left\langle R_m(z)^2 \right\rangle = \left\langle R_m(z) \right\rangle^2 . \tag{2.18}$$

[1]This rather peculiar form of the superpotential is actually quite natural from the String Theory point of view, where it is motivated by special geometry.

Note that this is exactly the point where the planar limit enters; if we had not taken the large N' 't Hooft limit, the correlators would have not factorized. Finally, after the limit, equation (2.16) reads

$$R_m(z)^2 = W'_{\text{tree}}(z)R_m(z) + \frac{1}{4}f_m(z) \tag{2.19}$$

where the expectation value of $R_m(z)$ is understood. In the matrix model, the choice of the function $f_m(z)$ corresponds to the choice of how to distribute N' eigenvalues of the matrix M among the n critical points of W_{tree}. By taking the derivative of the matrix model free energy with respect to the coupling constants of the tree level potential, one learns that

$$\frac{\partial \mathcal{F}}{\partial g_k} = \left\langle \frac{1}{k+1} \text{Tr}\, M^{k+1} \right\rangle \tag{2.20}$$

that is, the matrix model resolvent (2.15) is the generating functional for $\frac{\partial \mathcal{F}}{\partial g_k}$. Note that this relation hold order by order in the genus expansion of the matrix model free energy, *i.e.* the planar limit of the free energy is obtained by the planar limit of the resolvent. The function $f_m(z)$ in the matrix model has a well known effect on the analytic structure of $R(z)$: it opens n cuts over the z–plane. By now $R(z)$ can be solved in terms of the coefficients of the polynomial $f_m(z)$. Let us now perform a change of variables that will give us a more deep physical insight on what is happening. If $\mathcal{C}_i$ is a contour in the complex z–plane surrounding the i^{th} cut, we define

$$S_i = \frac{1}{2\pi i} \oint \mathrm{d}z R_m(z) \tag{2.21}$$

If we plug in (2.15) we get

$$S_i = \frac{1}{2\pi i} \oint \mathrm{d}z \frac{g_m}{N'} \left\langle \text{Tr}\, \frac{1}{z-M} \right\rangle = \frac{g_m N'_i}{N'} = g_s N'_i \tag{2.22}$$

where N'_i is the number of eigenvalues of M near the i^{th} critical point. Now $R(z)$ and hence the free energy $\mathcal{F}$ can be expressed in term of the variables S_i. Now, the key point is that in writing (2.10), we interpret S_i not simply as matrix model variables, but as the physical glueball superfield.

2.2 Gauge Theory Derivation

In this section we will explain how the Dijkgraaf–Vafa conjecture can be derived from the gauge theory point of view, without any reference to String Theory. Actually it is worth mentioning that there exist two ways to prove the conjecture: one is based on the use of anomalies [8] and the other is purely diagrammatic [6]. Here we will only discuss the first one, following [8, 22]; the diagrammatic derivation was nicely reviewed in [33]. The work in [8] was further generalized in a series of subsequent papers by the same authors where fundamental matter was added [65] and a detailed study of the phases of gauge theories was begun [63, 64]. For further extensions along this line, see [66].

The key ingredients are the concepts of the chiral ring, that encodes the information on the holomorphic sector of the theory, and the already mentioned Konishi anomaly. Once these two character come into the play, one can obtain Ward identities descending from the Konishi anomaly relating operators of the chiral ring. Remarkably, these Ward identities are enough to constraint the form of the effective superpotential and to write down an explicit formula for it.

2.2.1 The Chiral Ring

The chiral ring is a key object in the study of supersymmetric gauge theories. It is a mathematical structure that encodes all the chiral operators of the theory. But, it also has a nice physical interpretation. In referring to supersymmetric gauge theory, we always assume to be in a supersymmetric vacuum. This means that the vacuum state is annihilated by the supersymmetry generators. It is natural, from a physical point of view to identify operators that have the same expectation value in a supersymmetric vacuum.

Chiral operators are defined as being gauge invariant operators annihilated by the supercharge $\bar{Q}_{\dot{\alpha}}$

$$[\bar{Q}_{\dot{\alpha}}, \mathcal{O}\} = 0 \tag{2.23}$$

The chiral ring is then the set of all gauge invariant chiral operator modulo $\bar{Q}_{\dot{\alpha}}$ commutators. That is

$$\mathcal{O}_1(x) = \mathcal{O}_2(x) + [\bar{Q}^{\dot{\alpha}}, X_{\dot{\alpha}}(x)\}, \tag{2.24}$$

where $X_{\dot{\alpha}}(x)$ is a local gauge invariant operator. Clearly, since we are assuming the vacuum state to be supersymmetric, $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ have the same vacuum expectation value. Eq. (2.24) can be easily translated in superspace language (since the lowest component of a chiral superfield is a chiral operator): two chiral field are in the same equivalence class if their difference is of the form $\bar{D}Y$ where Y is a local and gauge invariant superfield.

Let us assume that the operators $\mathcal{O}$ are bosonic to simplify the notation (actually all the statements we will show are completely general). Chiral operators are independent of position x in the chiral ring, that is, discarding (vacuum expectation values of) $\bar{Q}_{\dot{\alpha}}$ commutators:

$$\langle \frac{\partial}{\partial x^\mu}\mathcal{O}(x)\rangle \simeq \langle [P^\mu, \mathcal{O}(x)]\rangle \simeq \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \langle \{\bar{Q}_{\dot{\alpha}}, [\bar{Q}_\alpha, \mathcal{O}]\}\rangle = 0 \tag{2.25}$$

up to numerical factors. Actually one can prove in a similar way the more general relation

$$\langle \frac{\partial}{\partial x^\mu}\mathcal{O}(x)\mathcal{O}(y_1)\dots\mathcal{O}(y_n)\rangle = 0 \tag{2.26}$$

The idea behind this relation is to use the supersymmetry algebra to write the space–time derivative in terms of supersymmetry generators and then commute these until they act on the invariant vacuum, using the fact that the operators are chiral. Using this result, we can separate the operators by an arbitrary large distance in space–time and then use the cluster decomposition principle to factorize any correlator

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1(x_1)\rangle \dots \langle \mathcal{O}_n(x_n)\rangle \tag{2.27}$$

As we will see in the next chapters, this property does not hold anymore when the theory is considered in a gravitational background. So far, all this can sound a bit abstract. To be more concrete, let us specialize to the chiral ring of the $\mathcal{N}=1$ theory (2.1). The basic element of the theory are the gauge field strength W_α and the chiral matter superfield Φ. Let us begin by considering the gauge field. By an explicit computation, one can show that

$$\{W_\alpha, W_\beta\} = -\frac{i}{4}\bar{\nabla}^{\dot\alpha}\nabla_{\alpha\dot\alpha}W_\beta = -\frac{1}{4}\bar{\nabla}^{\dot\alpha}\{\bar{\nabla}_{\dot\alpha},\nabla_\alpha\}W_\beta = -\frac{1}{4}\bar{\nabla}^{\dot\alpha}\bar{\nabla}_{\dot\alpha}\nabla_\alpha W_\beta \tag{2.28}$$

where we have used the fact that W_α is a chiral superfield (*i.e.* $\bar{\nabla}W=0$) and the definition of W_α (1.25) as well as the definition of $\nabla_{\dot\alpha\alpha}$ (1.24). This relation means that gauge field strength anticommute in the chiral ring,

$$\mathrm{Tr}\,(W_\alpha W_\beta W_{\gamma_1}\dots W_{\gamma_n}) = -\mathrm{Tr}\,(W_\beta W_\alpha W_{\gamma_1}\dots W_{\gamma_n}) - \frac{1}{4}\bar{D}^2\mathrm{Tr}\,(\nabla_\alpha W_\beta W_{\gamma_1}\dots W_{\gamma_n}) \tag{2.29}$$

Hence, the only non trivial object in the chiral ring that can be made by taking trace of a product of gauge field strength is the glueball superfield S. In fact, since $W's$ anticommute in the chiral ring, we may assume that the trace of any string of W' is antisymmetric in the exchange of the gauge fields. But, since supersymmetry indices $\alpha's$ can only assume two values, traces of three or more $W's$ vanish in the chiral ring. By following the same strategy as for equation (2.28), one can also show

$$[W_\alpha, \Phi] = 0 \mod \bar{D} \tag{2.30}$$

Hence a complete list of independent (single–trace) chiral operators is

$$\mathrm{Tr}\,\Phi^k \qquad \mathrm{Tr}\,\Phi^k W_\alpha \qquad \mathrm{Tr}\,\Phi^k W^\alpha W_\alpha \tag{2.31}$$

Strictly speaking, these argument should apply only to an $SU(N)$ gauge theory; one actually can prove they are true for all classical Lie groups (by now the extension to exceptional groups is a conjecture, that has been proven only for G_2). Let us note that, due to algebraic relations, Φ being an $N\times N$ matrix, operators like $\mathrm{Tr}\,\Phi^k$ with $k>N$ can be expressed in terms of operators $\mathrm{Tr}\,\Phi^l$ with $l\le N$.

The chiral ring as defined in (2.24) is a classical object; it could (and actually does) receive quantum corrections. The following relation holds in the classical chiral ring

$$S^h = 0 \tag{2.32}$$

where h is the dual Coxeter number of the gauge group. This relation was proven in [67] by means of group theory relations for all classical groups and in [68] for G_2 and is conjectured to hold even for the others exceptional groups. But if this were an exact quantum statement, it would follow that the vacuum expectation value of S^h, and hence of S by factorization, would vanish in any supersymmetric vacuum. The only possible correction to this relation arises non perturbatively, since the instanton factor Λ^{3h} has the same chiral properties of S^h. Moreover, we know that instantons lead to an expectation value $\langle S^h \rangle = \Lambda^{3h}$. Therefore, we conclude that the quantum chiral ring generalization of (2.32) is

$$S^h = \Lambda^{3h} \tag{2.33}$$

Let us stress once more that all these relations hold in the chiral ring, that is up to $\bar{Q}$ commutators.

We note here, and we will comment on this later, that, since chiral ring operators get quantum corrections, they should be handed carefully. In particular, all operators like S^k with $k \geq h$ will get corrections; this will imply some subtle modification to the original Dijkgraaf–Vafa conjecture.

Since we have observed that the chiral ring has an important physical interpretation, it would be useful to have an easy way to handle all chiral ring operators. This can be accomplished by defining the following three operators:

$$\begin{aligned}
\mathcal{R}(z)_{ij} &= -\frac{1}{32\pi^2}\left(\frac{W^2}{z-\Phi}\right)_{ij}, \qquad & R(z) &= \mathrm{Tr}\,\mathcal{R}(z) \\
\rho_{\alpha\ ij}(z) &= \frac{1}{4\pi}\left(\frac{W_\alpha}{z-\Phi}\right)_{ij}, \qquad & w_\alpha(z) &= \mathrm{Tr}\,\rho_\alpha(z) \\
\mathcal{T}(z)_{ij} &= \left(\frac{1}{z-\Phi}\right)_{ij}, \qquad & T(z) &= \mathrm{Tr}\,\mathcal{T}(z)
\end{aligned} \tag{2.34}$$

The idea behind this definition is that the coefficients of the expansion of the operators $R(z)$, $w_\alpha(z)$ and $T(z)$ in powers of z, are precisely the operators of the chiral ring. Then, *all* the information on the chiral ring is encoded in (2.34). As a remark, the function $R(z)$ is also called resolvent, in analogy with the matrix model resolvent (2.15).

2.2.2 Planar Diagrams

In this section we will give some argument to show that only planar diagrams in 't Hooft's double line notation can contribute to the effective superpotential in the gauge theory. Following the ideas outlined in the Chapter 1, let us try to constrain the form of the quantum effective action by means of symmetries. For simplicity, we will restrict ourselves to the case of unbroken gauge group, but our conclusions hold for the more general case of a symmetry breaking as well. The free $\mathcal{N}=1$ lagrangian has two UV symmetries, the R–symmetry $U(1)_R$ and the chiral $U(1)$

$$\Phi \to e^{i\varepsilon}\Phi \simeq \Phi + i\varepsilon\Phi \tag{2.35}$$

already encountered in the last Chapter. These symmetries are also symmetries of the interacting theory with a superpotential (2.2), if we allow the couplings to transform non trivially, by the localization trick. Let us assign to the fields and couplings charges as in table 2.1. These symmetries are anomalous. In particular, the chiral $U(1)$ anomaly is the Konishi anomaly already mentioned in Chapter 1. However, the anomaly is a one loop effect, that leaves invariant higher loop contributions. If we demand these

	$U(1)_R$	$U(1)$
Φ	$\frac{2}{3}$	1
W_α	1	0
g_l	$\frac{2}{3}(2-l)$	$-(l+1)$

Table 2.1: Charge assignment of fields and couplings under the R–symmetry $U(1)_R$ and the chiral $U(1)$

symmetries to be symmetries of the quantum superpotential, we see that it must depend on the couplings and on the gauge field as

$$W_{\text{eff}} = W_\alpha^2 F\left(\frac{g_k W_\alpha^{k-1}}{g_1^{\frac{(k+1)}{2}}}\right) \tag{2.36}$$

except at one loop. We have used a schematic notation, just to indicate the overall power of W_α, but actually W^2 could mean S as well as w^2.

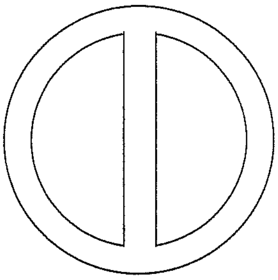

Figure 2.1: This diagram has $h = 3$ index loops and two ordinary loops.

If we consider a planar Feynman diagram in double line 't Hooft notation with vertices of degree $k_i + 1$, then the number of index loops is

$$h = 2 + \frac{1}{2}\sum_i (k_i - 1) \tag{2.37}$$

To convince ourselves that this formula is true, let us look at the diagram in figure (2.1): the diagram has two vertices, each with $k_i = 2$ and three index loops, as predicted by (2.37). Now, if we add a propagator, we add one index loop and increase $k_i - 1$ by one at each end of the propagator. Then, (2.37) is still valid. In this way one can prove by induction that (2.37) holds for any planar diagram.

Equation (2.37) together with (2.36) means that in the effective superpotential the power of W_α coming from an h–index loop planar diagram is $2h-2$. Since $S \sim W^2$, this implies a term like S^{h-1} if $h-1$ index loops each have two W_α insertions and one has none, or a term like $S^{h-1}w^2$ if $h-2$ index loops each have two W_α insertions and the remaining two loops have one W_α insertion each. Actually, any index loop in a Feynman diagram can contribute with a factor of S, w_α or N depending on how many gauge field insertions are one the loop. In fact no other operator can enter in the game since more insertions of the gauge field are trivial in the chiral ring and thus cannot contribute to F–terms. The generalization of (2.37) to a non planar diagrams of genus g is

$$h = 2 + \frac{1}{2}\sum_i (k_i - 1) \tag{2.38}$$

Then, the power of W_α coming from an h–index loop non planar diagram is $2h+4g-2$. But this means that for an index loop with $g \neq 0$ would have more that two W_α insertions. Since this contributions are trivial in the chiral ring, they cannot contribute to the superpotential. Thus, only the planar diagrams are relevant for the computation of F–terms.

2.2.3 The Generalized Konishi Anomaly

In this section we will introduce the basic tool to prove the Dijkgraaf–Vafa conjecture, that is a generalized form of the Konishi anomaly (1.61). As we already seen, the Konishi anomaly is the relation

$$\bar{D}^2(\bar{\Phi}e^V\Phi) = -\Phi\frac{\partial W}{\partial \Phi} - \frac{1}{32\pi^2}\mathrm{Tr}\,(W^\alpha W_\alpha) \tag{2.39}$$

Clearly, the LHS vanish in the chiral ring. Thus by taking the vacuum expectation value, we have a quantum relation between operators in the chiral ring.

Now let us consider the most general variation of Φ in the chiral ring

$$\delta\Phi = f(\Phi, W_\alpha) \tag{2.40}$$

where f is a general holomorphic function of the chiral superfields Φ and W_α. This leads to a generalized form of the Konishi anomaly[2]

$$\bar{D}^2\mathrm{Tr}\,\bar{\Phi}e^{\mathrm{Ad}V}f(\Phi, W_\alpha) = \mathrm{Tr}\,f(\Phi, W_\alpha)\frac{\partial W(\Phi)}{\partial \Phi} + \sum_{ijkl} A_{ij,kl}\frac{\partial f(\Phi, W_\alpha)_{ji}}{\partial \Phi_{kl}} \tag{2.41}$$

where we have emphasized that V is in the adjoint representation:

$$(\mathrm{Ad}V\ \Phi)^i{}_j = V^i{}_k\Phi^k{}_j - \Phi^i{}_k V^k{}_j. \tag{2.42}$$

and

$$A_{ij,kl} = \frac{1}{32\pi^2}\,[W^\alpha, [W_\alpha, T_{lk}]]_{ij} \tag{2.43}$$

T_{lk} being the generators of the gauge group ($U(N)$ in this case). Note that at this point this is quite general and a change in the gauge group will reflect only in the explicit form of the generators T_{lk}. We will use this fact later. For $U(N)$ we have $(T_{lk})_{ij} = (e_{lk})_{ij} = \delta_{il}\delta_{jk}$ and

$$\bar{D}^2\mathrm{Tr}\,\bar{\Phi}e^{\mathrm{Ad}V}f(\Phi, W_\alpha) = \mathrm{Tr}\,f(\Phi, W_\alpha)\frac{\partial W(\Phi)}{\partial \Phi} + \frac{1}{32\pi^2}\sum_{i,j}\left[W^\alpha, \left[W_\alpha, \frac{\partial f}{\partial \Phi_{ij}}\right]\right]_{ji}. \tag{2.44}$$

[2]Here and in the following we will consider $W(\Phi)$ as a matrix every time it appears inside a trace.

Finally, taking the vacuum expectation value, we find

$$\left\langle \mathrm{Tr} f(\Phi, W_\alpha)\frac{\partial W}{\partial \Phi}\right\rangle = -\frac{1}{32\pi^2}\left\langle \sum_{i,j}\left(\left[W^\alpha, \left[W_\alpha, \frac{\partial f(\Phi, W_\alpha)}{\partial \Phi_{ij}}\right]\right]\right)_{ji}\right\rangle. \tag{2.45}$$

where the have used the fact that terms of the form $\bar{D}$ of something gauge invariant annihilate a supersymmetric vacuum. Then the equation (2.45) is an *exact* relation between quantum operators. Let us anticipate that, in the next section, we will take for the function $f(\Phi, W_\alpha)$ the generators of the chiral ring (2.34). This will give us equations containing all the information of the chiral ring; solving these equations will allow us to find the effective superpotential.

2.2.4 The Effective Superpotential

In this section we will finally explain how to derive the effective superpotential for an $U(N)$ gauge theory [8, 22]. In the following sections we will generalize this to a general classical Lie group and give some explicit examples.

To begin with, let us consider the classical variation of the field Φ in the chiral ring

$$\delta\Phi_{ij} = -\frac{1}{32\pi^2}\left(\frac{W^\alpha W_\alpha}{z-\Phi}\right)_{ij} = \mathcal{R}_{ij}(z) \tag{2.46}$$

Then, by looking at the generalized form of the Konishi anomaly (2.44), we see that the RHS contains

$$\begin{aligned}\frac{\partial f_{kl}(\Phi, W_\alpha)}{\partial \Phi_{ij}} &= -\frac{1}{32\pi^2}\frac{\partial}{\partial \Phi_{ij}}\left(\frac{W^\alpha W_\alpha}{z-\Phi}\right)_{kl} \\ &= -\frac{1}{32\pi^2}\left(\frac{W^\alpha W_\alpha}{z-\Phi}\right)_{ki}\left(\frac{1}{z-\Phi}\right)_{jl}\end{aligned} \tag{2.47}$$

Next we note the algebraic relation

$$\sum_{i,j}\left[\chi_1, \left[\chi_2, \frac{\partial}{\partial \Phi_{ij}}\frac{\chi_1\chi_2}{z-\Phi}\right]\right]_{ij} = \left(\mathrm{Tr}\,\frac{\chi_1\chi_2}{z-\Phi}\right)^2 \tag{2.48}$$

which holds if $\chi_1^2 = \chi_2^2 = 0$ and $[\Phi, \chi_\alpha] = 0$. Then, by applying (2.48) with $\chi_\alpha = W_\alpha$, (2.44) looks like

$$-\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\left(W'(\Phi)\frac{W^\alpha W_\alpha}{z-\Phi}\right)\right\rangle = \frac{1}{(32\pi^2)^2}\left\langle \left(\mathrm{Tr}\,\frac{W^\alpha W_\alpha}{z-\Phi}\right)^2\right\rangle. \tag{2.49}$$

Let us now rewrite this expression in a more useful form. First of all by adding and subtracting $W'(z)$ inside the trace, the LHS looks like

$$-\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\left(W'(\Phi)\frac{W^\alpha W_\alpha}{z-\Phi}\right)\right\rangle = W'(z)\left\langle R(z)\right\rangle + \frac{1}{4}\left\langle f(z)\right\rangle \tag{2.50}$$

where

$$f(z) = \frac{1}{8\pi^2}\mathrm{Tr}\left((W'(z) - W'(\Phi))\frac{W^\alpha W_\alpha}{z-\Phi}\right). \tag{2.51}$$

Let us note that $f(z)$ is a polynomial of degree $n-1$

$$f(z) = \sum_{k=0}^{n-1} f_k z^k; \tag{2.52}$$

moreover it cancels all the non negative powers in z of $W'(z)R(z)$ such that the RHS of (2.50) has the same large z behavior of the LHS, *i.e.* $\frac{1}{z}$. Up to know, the coefficients f_k are only unknown parameters; we will comment later on their physical significance.

On the RHS, we can use the factorization properties of the chiral ring to write

$$\langle R^2(z)\rangle = \langle R(z)\rangle^2. \tag{2.53}$$

We note that the same relation was used in the matrix model setup, but there it was justified by the large N' limit; here it is simply a property of the chiral ring. Finally, we can write the generalized Konishi anomaly as

$$R(z)^2 = W'(z)R(z) + \frac{1}{4}f(z) \tag{2.54}$$

(equation (2.54) has to be intended as a vacuum expectation value). Note that this equation is identical to the matrix model loop equation (2.19). Equation (2.54) can easily be solved to yield

$$R(z) = \frac{1}{2}\left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right), \tag{2.55}$$

where the sign of the square root has been chosen in order to get the right $\frac{1}{z}$ behavior at infinity.

From (2.55) we see that the function $R(z)$ (better, its vacuum expectation value), has, in principle, n branch cuts in the complex z–plane. This is because of the presence of the function $f(z)$ in the square root. Its effect is to split some of the n zeros of $W'(z)$ (eventually all) into branch cuts; these cuts can be seen as a sort of quantum resolution of the zeros of the tree level superpotential. If C_i is a contour going around the i^{th} branch cut, we have

$$S_i = \frac{1}{2\pi i}\oint_{C_i} \mathrm{d}z R(z) = -\frac{1}{4\pi i}\oint_{C_i} \mathrm{d}z\sqrt{W'(z)^2 + f(z)} \tag{2.56}$$

where we have used the solution for $R(z)$ (2.55). In the classical limit this equation gives precisely the glueball superfield. Quantum mechanically, it has to be interpreted as a quantum definition of the glueball.

Equation (2.56) has to be understood as follows. Semiclassically, to evaluate the integral, we set Φ to its vacuum value, that is a diagonal matrix with diagonal entries

$\phi_1, \ldots, \phi_N$ (which are equal to the a_i with multiplicity N_i). Then, for any matrix M, we have

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_i} \mathrm{d}z \,\mathrm{Tr}\, \frac{M}{z-\Phi} &= \frac{1}{2\pi i}\oint_{C_i} \mathrm{d}z \sum_{m=1}^{N} \frac{M_{mm}}{z-\phi_m} \\ &= \sum_{\lambda_m \in C_i} M_{mm} = \mathrm{Tr}\, P_i M \equiv \mathrm{Tr}\, M_i. \end{aligned} \tag{2.57}$$

Here $\lambda_m \in C_i$ means that λ_m is inside the contour C_i, and P_i is the projector onto eigenspaces of Φ corresponding to eigenvalues that are inside this contour. Since projectors should not receive quantum corrections, this relations hold even at the quantum level. Then we see that (2.56) is proper quantum definition of the glueball superfield.

In general, the choice of the function $f(z)$ determines the gauge symmetry breaking pattern and selects the vacuum: as we have seen, it is in fact a statement on the low energy degrees of freedom S_i of the theory. Moreover we see that, if we set $f(z)$ to zero, we get the classical expression (1.66) for S.

Let us now take the variation in the chiral ring

$$\delta\Phi_{ij} = \left(\frac{1}{z-\Phi}\right)_{ij} = \mathcal{T}_{ij}(z) \tag{2.58}$$

By repeating the same steps that led to (2.54), we find

$$2R(z)T(z) = W'(z)T(z) + \frac{1}{4}c(z) \tag{2.59}$$

where $c(z)$, like $f(z)$, is a polynomial of degree $n-1$, defined by

$$c(z) = 4\mathrm{Tr}\left((W'(z) - W'(\Phi))\frac{1}{z-\Phi}\right). \tag{2.60}$$

Equation (2.59) can be used together with (2.55) in order to derive a closed equation for $T(z)$

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} \tag{2.61}$$

where again, the vacuum expectation value is intended everywhere. Equation (2.61) is a quantum relation that relates the operator $T(z)$ to two unknown polynomials of degree $n-1$. However, as we have already explained, the function $f(z)$ is completely determined by the choice of the vacuum and of the gauge symmetry breaking pattern. Moreover, also the function $c(z)$ can be fixed by some "boundary condition". In fact, if we take equation (2.57) with the matrix M equal to the identity, we get

$$N_i = \frac{1}{2\pi i}\oint_{C_i} \mathrm{d}z T(z) = -\frac{1}{8\pi i}\oint_{C_i} \mathrm{d}z \frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} \tag{2.62}$$

that can be used to fix $c(z) = \sum_{k=0}^{n-1} c_k z^k$. Then, the (quantum expectation value) of the function $T(z)$ is completely determined once the vacuum and the symmetry breaking pattern have been chosen. This is enough to derive the effective superpotential for the theory. In fact, the tree level superpotential is assumed to be of the form

$$W = \sum_{k=0}^{n} \frac{g_k}{k+1} \mathrm{Tr}\, \Phi^{k+1} \tag{2.63}$$

It can be shown, using the power of holomorphy and the localization trick, that this relation extends to the quantum level to the set of relations

$$\frac{\partial W_{\text{eff}}}{\partial g_k} = \left\langle \frac{1}{k+1} \mathrm{Tr}\, \Phi^{k+1} \right\rangle . \tag{2.64}$$

But the RHS of this equation contains precisely the coefficients of the expansion of $T(z)$ in powers of z

$$T(z) = \mathrm{Tr}\, \frac{1}{z-\Phi} = \sum_{k\geq 0} z^{-1-k} \mathrm{Tr}\, \Phi^k . \tag{2.65}$$

From this we can derive a simple rule. Given a tree level superpotential of the form (2.2), choose a vacuum, hence the function $f(z)$, and use (2.61) to find the quantum value of the function $T(z)$ in that vacuum. Then, expand $T(z)$ in powers of z to get a set of partial differential equations that contain the derivative of the quantum superpotential with respect to the coupling constants. Finally, simply integrate this set of equations to get the quantum superpotential itself, up to an integration constant, *independent* of the coupling constants. But we already know this constant, it is the pure gauge part, the Veneziano–Yankielowicz superpotential, that depends only on the glueball superfield S and on the dynamically generated scale Λ. Actually, the Veneziano–Yankielowicz superpotential cannot be derived in the framework of the Konishi anomaly. This is only equivalent to the diagrammatic part of the matrix model planar free energy. The Veneziano–Yankielowicz term, that in the matrix model setup arises from the measure, is not contained in the Konishi anomaly approach to the superpotential. It can be derived, by the heuristical method outlined in the previous chapter, or by the more formal method of integrating in. But actually, all these methods rely on unjustified, even if physically reasonable, assumptions. By now, there does not exists a purely gauge theoretical derivation of the Veneziano–Yankielowicz superpotential from first principles.

We have just seen how to use practically the generalized Konishi anomaly equations to compute the intanton corrections to the Veneziano–Yankielowicz superpotential generated by integrating out the chiral matter. Now let us turn to a more formal derivation of this statement, for a $U(N)$ gauge group and matter in the adjoint representation. First of all, we remark that the overall $U(1)$ factor of $U(N)$ is free. This is reflected in an exact symmetry of shifting W_α by a constant Weyl spinor ψ_α. In the low energy theory with gauge group $\prod_i U(N_i)$, this symmetry still acts as a simultaneous shift of all the gaugino superfields as $w_\alpha \to W_\alpha - 4\pi\psi_\alpha$. The low energy fields transform as

$$\begin{aligned} S_i &\to S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2} N_i\, \psi^\alpha \psi_\alpha , \\ w_i^\alpha &\to w_i^\alpha - N_i\, \psi^\alpha . \end{aligned} \tag{2.66}$$

The idea behind all this is that the low energy superpotential, whatever its form is, has to be invariant under this shift symmetry, since it cannot depend on the decoupled overall $U(1)$; hence we can use this shift symmetry to constraint its form. Now, we promote the symmetry parameter ψ_α to an auxiliary Grassmann coordinate and introduce the ψ–superfield

$$\begin{aligned} \mathcal{S}_i &= -\frac{1}{2}\mathrm{Tr}\left(\frac{W_i^\alpha}{4\pi} - \psi^\alpha\right)\left(\frac{W_{\alpha i}}{4\pi} - \psi_\alpha\right) \\ &= S_i + w_i^\alpha \psi_\alpha - \frac{1}{2}N_i \psi^\alpha \psi_\alpha . \end{aligned} \tag{2.67}$$

Translations of the Grassmann variable ψ induce the transformations (2.66). Invariance under this transformation implies that the effective action has to be of the form[3]

$$W_{\text{eff}} = -\int \mathrm{d}^2\psi \mathcal{F}(\mathcal{S}_i, g_k), \tag{2.68}$$

Then, by doing the ψ integral, we find that

$$W_{\text{eff}} = \sum_i N_i \frac{\partial \mathcal{F}}{\partial S_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 \mathcal{F}}{\partial S_i \partial S_j} w_i^\alpha w_{\alpha j}. \tag{2.69}$$

that has the general structure (2.10) claimed by Dijkgraaf and Vafa, with $\mathcal{F}$ still to be determined. We can easily see that equation (2.64) in this formalism reads

$$\int \mathrm{d}^2\psi \frac{\partial}{\partial g_k}\mathcal{F}(\mathcal{S}_i, g_l) = -\left\langle \frac{1}{k+1}\mathrm{Tr}\,\Phi^{k+1}\right\rangle \tag{2.70}$$

but since

$$\frac{1}{k+1}\left\langle \mathrm{Tr}\,\Phi^{k+1}\right\rangle = -\frac{1}{2(k+1)}\int \mathrm{d}^2\psi \left\langle \mathrm{Tr}\left(\frac{1}{4\pi}W^\alpha - \psi^\alpha\right)^2 \Phi^{k+1}\right\rangle, \tag{2.71}$$

we can conclude that

$$\frac{\partial}{\partial g_k}\mathcal{F}(\mathcal{S}_i, g_l) = -\frac{1}{2(k+1)}\left\langle \mathrm{Tr}\left(\frac{1}{4\pi}W^\alpha - \psi^\alpha\right)^2 \Phi^{k+1}\right\rangle. \tag{2.72}$$

Let us make some comments on this equation. First of all, it is not restrictive to take the $\psi = 0$ component, since in the superspace formalism all the superfield components can be derived from the lowest one by applying a supersymmetry transformation. So, no information is lost. This means that the full information about the function $\mathcal{F}$ is encoded in the gauge theory resolvent $R(z)$, that is obtained from the RHS of (2.72) by taking the lowest component.

[3] We have already mentioned that the kind of theories we are studying can be thought of an $\mathcal{N}=2$ theory broken down to $\mathcal{N}=1$ by turning on the tree level superpotential. In this interpretation, ψ can be seen as a remnant of the $\mathcal{N}=2$ superspace.

Finally, to complete the prove, we show that the matrix model free energy is equal to the $\psi = 0$ component of the function $\mathcal{F}$. Since the matrix model resolvent $R_m(z)$ and the gauge theory resolvent $R(z)$ obey the same loop equation, it is natural to identify them. This identification is obtained if the two functions $f_m(z)$ an $f(z)$ entering the loop equations, are the same. But, finally, this identification, implies, by (2.21) and (2.56) that the identification between the matrix model parameters $g_s N_i'$ and the gauge superfield S_i is indeed correct. However, note that the identification of the resolvents, implies the identifications of the *derivatives* of the matrix model free energy and the gauge function $\mathcal{F}$, with respect to the couplings. This means that we still have the freedom of adding a function of S independent of the couplings. By demanding this equivalence to be a full equivalence, we finally have to add, as expected, to the gauge theory superpotential (2.69), the Veneziano–Yankielowicz term.

2.3 The Konishi Anomaly for other Gauge Groups

In this section, we will derive the Konishi anomaly and the equation for $T(z)$ for $SO(N)$ (in some detail) and $Sp(N)$ with matter in the adjoint and symmetric (antisymmetric for $Sp(N)$), both traceful and traceless, representations and finally for $SU(N)$ with matter in the adjoint representation. We will then use these results in the next section following the outlined approach to compute effective superpotentials in some specific cases for $SU(N)$, $SO(N)$ and $Sp(N)$[4] with matter in various representations. Some superpotentials for $SO(N)$ and $Sp(N)$ have already been computed in the framework of [1, 6], see for example [69, 70, 71, 75, 77, 78, 79, 80]. For adjoint matter, the results obtained reflect the charge of the orientifold plane used in the geometric engineering of the gauge theory. For discussions on $SU(N)$ see for example [82]. For related works on $SO(N)/Sp(N)$ gauge theories, see for example [72, 73, 74, 76]

Again, the main idea is to use the generalized Konishi anomaly to write down Ward identities that allow us to write closed expressions for the generating functions of correlators. These Ward identities are the analog of (2.61) for other gauge groups.

Let us begin with the case of an $SO(N)$ gauge theory with adjoint matter and evaluate explicitly (2.41). We take the generators of $SO(N)$ to be $T_{lk} = (e_{lk} - e_{kl})$ with $(e_{lk})_{ij} = \delta_{il}\delta_{jk}$. First of all, we note that the identity (2.48) holds due to the spinorial properties of χ_α and is independent of the generators up to numerical factors. As can be easily checked the equation for $R(z)$ (2.54) then becomes

$$\frac{1}{2}R^2(z) = W'(z)R(z) + \frac{1}{4}f(z) \qquad (2.73)$$

whose solution is

$$2R(z) = 2W'(z) - 2\sqrt{W'(z)^2 + \frac{f(z)}{2}} \qquad (2.74)$$

Now let us focus on the equation for $T(z)$ (2.59) and restrict ourselves to variations of

[4] Here we use conventions such that N is an even number, i.e. the rank of the group is $\frac{N}{2}$.

the form[5]

$$\delta\Phi = f(\Phi) = -\frac{1}{32\pi^2}\frac{1}{z-\Phi} \tag{2.75}$$

Then the equation for the anomaly gives

$$\begin{aligned}\bar{D}^2 \quad \left(\mathrm{Tr}\,\bar{\Phi}e^{\mathrm{Ad}V} f(\Phi, W_\alpha)\right) &= -\frac{1}{32\pi^2}\mathrm{Tr}\,\frac{1}{z-\Phi}\frac{\partial W(\Phi)}{\partial\Phi} \\ + \quad \frac{1}{32\pi^2}\frac{1}{4}\sum_{ijkl}[W^\alpha,[W_\alpha,(e_{lk}-e_{kl})]]_{ij} & \left(\frac{1}{z-\Phi}(e_{kl}-e_{lk})\frac{1}{z-\Phi}\right)_{ji}\end{aligned} \tag{2.76}$$

Let us focus on the second term on the right hand side

$$\begin{aligned}&\frac{1}{32\pi^2}\frac{1}{4}\sum_{ijkl}[W^\alpha,[W_\alpha,(e_{lk}-e_{kl})]]_{ij}\left(\frac{1}{z-\Phi}(e_{kl}-e_{lk})\frac{1}{z-\Phi}\right)_{ji} = \\ &\frac{1}{32\pi^2}\frac{1}{4}\left(4\mathrm{Tr}\,\frac{W^\alpha W_\alpha}{z-\Phi}\mathrm{Tr}\,\frac{1}{z-\Phi} - 8\mathrm{Tr}\left(W^\alpha W_\alpha\frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T\right) - 4\mathrm{Tr}\,\frac{W^\alpha}{z-\Phi}\mathrm{Tr}\,\frac{W_\alpha}{z-\Phi}\right)\end{aligned} \tag{2.77}$$

where we have used the commutation properties of the operators in the chiral ring. Now, being Φ an antisymmetric matrix, we have

$$\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z+\Phi} \tag{2.78}$$

Next we use the identity

$$\frac{1}{z-\Phi}\frac{1}{z+\Phi} = \frac{1}{2z}\left(\frac{1}{z-\Phi}+\frac{1}{z+\Phi}\right) \tag{2.79}$$

in order to write

$$\mathrm{Tr}\,W^\alpha W_\alpha\frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z}\mathrm{Tr}\,W^\alpha W_\alpha\frac{1}{z-\Phi} \tag{2.80}$$

Taking expectation values of (2.76) and using the definitions (2.34) we have

$$W'(z)T(z) + \frac{1}{4}c(z) = R(z)T(z) - 2\frac{R(z)}{z} \tag{2.81}$$

Using the relation (2.74) we finally obtain the equation for $T(z)$

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}} - \frac{2}{z}\frac{W'(z)-\sqrt{W'(z)^2+f(z)}}{\sqrt{W'(z)^2+f(z)}} \tag{2.82}$$

Here we absorbed a factor of $\frac{1}{2}$ in a redefinition of $f(z)$ (we will always use this convention when speaking about $SO(N)$ and $Sp(N)$). As previously explained, from

[5]Properly speaking one should add also the term $\frac{1}{z+\Phi}$ since $\delta\Phi$ has to be an element of $SO(N)$ in the adjoint representation, that is to say an antisymmetric matrix. However it can be checked that it will contribute exactly as the previous, giving only an overall factor of 2. Because of this it will be omitted in the following analysis.

(2.82) we can obtain the effective superpotential for an $SO(N)$ gauge theory with adjoint matter and tree level superpotential (2.2).

Now let us consider the same gauge theory but with matter in the symmetric representation (that is, Φ is now a symmetric matrix and we use a symmetric representation for the $SO(N)$ basis). In this case (2.78) becomes

$$\left(\frac{1}{z-\Phi}\right)^T = \frac{1}{z-\Phi} \tag{2.83}$$

and

$$\begin{aligned} \mathrm{Tr}\, W^\alpha W_\alpha \frac{1}{z-\Phi}\left(\frac{1}{z-\Phi}\right)^T &= \mathrm{Tr}\, W^\alpha W_\alpha \left(\frac{1}{z-\Phi}\right)^2 \\ &= -\frac{d}{dz}\left(\mathrm{Tr}\, W^\alpha W_\alpha \frac{1}{z-\Phi}\right) \end{aligned} \tag{2.84}$$

Again, from (2.44), (2.34) and using now (2.84) one finds

$$W'(z)T(z) + \frac{1}{4}c(z) = R(z)T(z) - 2R'(z) \tag{2.85}$$

and the equation for $T(z)$ becomes

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}} - 2\frac{\frac{d}{dz}\left(W'(z)-\sqrt{W'(z)^2+f(z)}\right)}{\sqrt{W'(z)^2+f(z)}} \tag{2.86}$$

To complete our discussion about $SO(N)$, let us consider now Φ in the traceless symmetric representation. All we have to do is to take the previous results and subtract the trace of Φ. For instance, (2.75) will now become

$$\delta\Phi = f(\Phi) = -\frac{1}{32\pi^2}\left(\frac{1}{z-\Phi} - \frac{1}{N}\mathrm{Tr}\,\frac{1}{z-\Phi}\right) \tag{2.87}$$

This will not produce any change in (2.78) (since the trace part is proportional to the identity matrix and it is entering in the commutator); the only modifications will arise in the left hand side of (2.45) which now becomes

$$-\frac{1}{32\pi^2}\mathrm{Tr}\left(\frac{1}{z-\Phi} - \frac{1}{N}\mathrm{Tr}\left(\frac{1}{z-\Phi}\right)\right)\frac{\partial W(\Phi)}{\partial\Phi} \tag{2.88}$$

and in the equation for $R(z)$ (2.54) which now reads[6]

$$R^2(z) = \left(W'(z) - \frac{1}{N}W'(\Phi)\right)R(z) + \frac{1}{4}f(z) \tag{2.89}$$

[6]Remember that we are taking vacuum expectation values; properly speaking $W'(\Phi)$ has to be understood as $\langle W'(\Phi)\rangle$.

Now equation (2.85) becomes

$$T(z)\left(W'(z)-\frac{1}{N}W'(\Phi)\right)+\frac{1}{4}c(z)=R(z)T(z)-2R'(z) \tag{2.90}$$

Finally we can write the equation for $T(z)$ for matter in the symmetric traceless representation

$$\begin{aligned}T(z)= & -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}}\\ & -2\frac{\frac{d}{dz}\left(\left(W'(z)-\frac{1}{N}W'(\Phi)\right)-\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}\right)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}}\end{aligned} \tag{2.91}$$

Now we will focus on an $Sp(N)$ gauge theory with matter in the adjoint (symmetric) and in the antisymmetric (both traceful and traceless) representations. With symmetric (antisymmetric) we mean that Φ has to be considered as a matrix MJ where M is a symmetric (antisymmetric) matrix and J is the invariant antisymmetric tensor of $Sp(N)$. We take the generators of $Sp(N)$ as $(e_{lk}+e_{kl})$ with $(e_{lk})_{ij}=\delta_{il}\delta_{jk}$. The analysis for the $Sp(N)$ case is almost identical to the one for the $SO(N)$ case, the only change being the sign in the generators (and of course the different properties of the matrices representing the field Φ, since the antisymmetric invariant J will enter in the intermediate steps). Because of this we will only state our results. For matter in the symmetric representation the equation for $T(z)$ becomes

$$T(z)=-\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}}+\frac{2}{z}\frac{W'(z)-\sqrt{W'(z)^2+f(z)}}{\sqrt{W'(z)^2+f(z)}} \tag{2.92}$$

for matter in the antisymmetric traceful representation

$$T(z)=-\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2+f(z)}}+2\frac{\frac{d}{dz}\left(W'(z)-\sqrt{W'(z)^2+f(z)}\right)}{\sqrt{W'(z)^2+f(z)}} \tag{2.93}$$

and finally for matter in the antisymmetric traceless representation

$$
\begin{aligned}
T(z) = & -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} \\
& + 2\frac{\frac{d}{dz}\left(\left(W'(z)-\frac{1}{N}W'(\Phi)\right)-\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}\right)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}}
\end{aligned}
\tag{2.94}
$$

As a last example, let us consider the $SU(N)$ gauge group with matter in the adjoint representation. This is basically equivalent to consider an $U(N)$ gauge theory subtracting the trace as in (2.87) (remember that the term containing the trace will not produce any modification when entering in a commutator). Then, one can easily find

$$
T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} \tag{2.95}
$$

2.4 The Effective Superpotential

In this section we apply the previous results in order to find the effective superpotential for $SO(N)$, $Sp(N)$ and $SU(N)$ gauge theories with quartic and cubic superpotential and matter in various representations. This section will show how to use in practice the set up built in the previous sections. Moreover, the derivation of the effective superpotentials, will appear as a completely straightforward, even if technical, computation. In fact, computing effective superpotentials by the Konishi anomaly method, is simpler that by using the diagrammatic method. The key point is that the Konishi anomaly method is completely algorithmic. On the other hand, when drawing Feynman diagrams for the matrix model, one has to compute the combinatorial factor; this, going to higher orders in perturbation theory, can become a very difficult task.

As already mentioned, the general strategy is to write down the equation for $T(z)$ for every particular case, expand it in powers of $\frac{1}{z}$ and extract the vacuum expectation values of the operators $\langle \mathrm{Tr}\,\Phi^k \rangle$, from where the effective superpotential can be obtained by using equation (2.64). The result contained in these sections are in agreement with, and extend, previous literature on the Dijkgraaf–Vafa conjecture. However, in some cases, there is a mismatch with results obtained by means of other methods [83, 84]. This mismatch, already noted in [80] and subsequently confirmed in [85], led to a refinement of the conjecture in [86, 87, 88]. We will comment on this in the next section.

2.4.1 Quartic Superpotential

$Sp(N)/SO(N)$ with Matter in the Antisymmetric/Symmetric Representation

Let us suppose the following tree level superpotential:

$$
W(\Phi) = \frac{m}{2}\mathrm{Tr}\,\Phi^2 + \frac{g}{4}\mathrm{Tr}\,\Phi^4 \tag{2.96}
$$

As seen in the previous section we obtain the following equation for $T(z)$

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} + 2\epsilon\frac{W''(z) - (\sqrt{W'(z)^2 + f(z)})'}{\sqrt{W'(z)^2 + f(z)}} \tag{2.97}$$

Where $\epsilon = \pm 1$ for $Sp(N)/SO(N)$ and $c(z)$ and $f(z)$ are polynomials of degree 2

$$f(z) = \sum_{i=0}^{2} f_i z^i \tag{2.98}$$

and

$$c(z) = \sum_{i=0}^{2} c_i z^i \tag{2.99}$$

The denominator of both terms in equation (2.97) can be factorized as explained previously. We impose (see for example [79]):

$$W'(z)^2 + f(z) = g^2(z^2 - k^2)^2(z^2 - 4\mu^2) \tag{2.100}$$

From this condition we arrive to the following expressions:

$$k = \sqrt{-\frac{m + 2g\mu^2}{g}} \tag{2.101}$$

$$\mu = \sqrt{-\frac{gm + \sqrt{g^2(-3f_2 + m^2)}}{6g^2}} \tag{2.102}$$

Imposing condition (2.100) gives a system of equations for k and μ with a set of solutions. Note that we choose the particular k and μ tending to $\sqrt{-\frac{m}{g}}$ and 0 for f_2 going to zero (that is the classical limit). This means we place the branch cut around zero. In order to have the correct asymptotic behavior of $R(z)$ for large z ($R(z) \sim \frac{S}{z}$, see eq. (2.34)) it can be shown that[7] $f_2 = -2gS$. Similarly the correct asymptotic behavior of $T(z)$ for large z ($T(z) \sim \frac{N}{z}$) sets $c_2 = -4gN$. c_0 and c_1 can be found by asking the condition that $T(z)$ has no poles in k and $-k$ (i.e. we choose our vacuum around $\Phi = 0$ and the gauge group remains unbroken) or, equivalently:

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_k} dz\, T(z) = 0 \\ \frac{1}{2\pi i}\oint_{C_{-k}} dz\, T(z) = 0 \end{aligned} \tag{2.103}$$

For the present case we obtain:

$$\begin{aligned} c_0 &= 4\left(2\epsilon\left(m + gk\left(3k - 2\sqrt{k^2 - 4\mu^2}\right)\right) + gk^2N\right) \\ c_1 &= 0 \end{aligned} \tag{2.104}$$

[7]The difference with [8] is due to our redefinition of $f(z)$.

Next, we expand $T(z)$ in powers of $\frac{1}{z}$ and obtain:

$$\frac{\partial W_{eff}}{\partial m} = \frac{2\epsilon}{g}\left(m + 3g\mu^2 + \sqrt{(m+6g\mu^2)\,(m+2g\mu^2)}\right) + \mu^2 N \tag{2.105}$$

This expression can be expanded in powers of S and integrated in order to obtain the effective superpotential up to any given order; for instance up to fourth order, it reads

$$\begin{aligned} W_{eff} &= \frac{1}{2}\,(-2\,\epsilon + N)\,S\log m + \frac{g}{8m^2}\,(-10\,\epsilon + 3N)\,S^2 \\ &- \frac{g^2}{16m^4}\,(-38\,\epsilon + 9N)\,S^3 + \frac{g^3}{96m^6}\,(-662\,\epsilon + 135N)\,S^4 + \ldots \end{aligned} \tag{2.106}$$

Several comments are in order. Having obtained W_{eff} integrating with respect to m, the result is correct up to a function of g and S (a part of which is the Veneziano-Yankielowicz superpotential); we could have chosen the coefficient of the term $\frac{1}{z^4}$ and integrated with respect to g. As the perturbative part of the potential depends only on the ratio $\frac{g}{m^2}$ as seen in (2.36), a function of only one of the coupling constants cannot contribute. For the same reason, from now on, we will only consider the $\frac{1}{z^3}$ term.

$Sp(N)/SO(N)$ with Matter in the Adjoint Representation

This case is completely analogous to the case studied before. Now the equation for $T(z)$ reads:

$$T(z) = -\frac{1}{4}\frac{c(z)}{\sqrt{W'(z)^2 + f(z)}} + \epsilon\frac{2}{z}\frac{W'(z) - \sqrt{W'(z)^2 + f(z)}}{\sqrt{W'(z)^2 + f(z)}} \tag{2.107}$$

Where again $c(z)$ and $f(z)$ are polynomials of degree 2. The denominator of both terms can be factorized as before (since it is the same) and we obtain the same values for the parameters k and μ. Again $f_2 = -2gS$ and $c_2 = -4gN$ and the conditions (2.103) must be imposed. The values obtained for c_0 and c_1 are:

$$\begin{aligned} c_0 &= 4\left(2\epsilon(gk^2 + m) + gk^2 N\right) \\ c_1 &= 0 \end{aligned} \tag{2.108}$$

Again, expanding $T(z)$ in powers of $\frac{1}{z}$ and and extracting the coefficient of $\frac{1}{z^3}$ we obtain:

$$\frac{\partial W_{eff}}{\partial m} = \mu^2(N + 2\epsilon) \tag{2.109}$$

that can be expanded in powers of S and integrated with respect to m, to give:

$$W_{eff} = (N+2\epsilon)\left(\frac{S}{2}\log m + \frac{3gS^2}{8m^2} - \frac{9g^2S^3}{16m^4} + \frac{45g^3S^4}{32m^6} - \frac{567g^4S^5}{128m^8} + \frac{5103g^5S^6}{320m^{10}} + \ldots\right) \tag{2.110}$$

This result agrees with the one of [81], where the effective superpotentials were evaluated using both matrix model techinques and in terms of closed strings on Calabi-Yau geometry with fluxes.

2.4.2 Cubic Superpotential

$SO(N)/Sp(N)$ with Traceful Symmetric/Antisymmetric Matter

Now the superpotential under consideration takes the form:

$$W(\Phi) = \frac{m}{2}\mathrm{Tr}\,\Phi^2 + \frac{g}{3}\mathrm{Tr}\,\Phi^3 \tag{2.111}$$

The equation for $T(z)$ reads exactly as in (2.97) but now $c(z)$ and $f(z)$ are polynomials of degree 1. Again we factorize the denominator of both terms, now as follows:

$$W'(z)^2 + f(z) = g^2(z-k)^2(z+a+b)(z+a-b) \tag{2.112}$$

In this case, the parameters a, b and k are complicated functions of m, g and f_1; because of this we will only write their expansion in powers of S:

$$\begin{aligned} k &= -\frac{m}{g} + a \\ a &= \frac{g}{m^2}S + 3\frac{g^3}{m^5}S^2 + 16\frac{g^5}{m^8}S^3 + 105\frac{g^7}{m^{11}}S^4 + 768\frac{g^9}{m^{14}}S^5 + 6006\frac{g^{11}}{m^{17}}S^6 + 49152\frac{g^{13}}{m^{20}}S^7 + \dots \\ b &= \sqrt{\frac{S}{2m}}\left(2 + 2\frac{Sg^2}{m^3} + 9\frac{S^2g^4}{m^6} + 55\frac{S^3g^6}{m^9} + \frac{1547}{4}\frac{S^4g^8}{m^{12}} + \frac{11799}{4}\frac{S^5g^{10}}{m^{15}} + \frac{189805}{8}\frac{S^6g^{12}}{m^{18}} + \dots\right) \end{aligned} \tag{2.113}$$

Note that in the classical limit (that is $S \to 0$) the parameters a and b go to zero, while k tends to its classical value $-\frac{m}{g}$. Again the asymptotic behavior of $R(z)$ and $T(z)$ imposes $f_1 = -2gS$ and $c_1 = -4gN$, and, as before, c_0 is set by the condition that $T(z)$ does not have a pole at $z = k$:

$$\frac{1}{2\pi i}\oint_{C_k} dz\, T(z) = 0 \tag{2.114}$$

and from this

$$c_0 = 8\epsilon\left(2gk + g\sqrt{(a-b+k)\,(a+b+k)} + m\right) + 4gkN \tag{2.115}$$

As before, $T(z)$ can be expanded in powers of $\frac{1}{z}$ and we can integrate the coefficient of $\frac{1}{z^3}$ with respect to m in order to obtain the effective superpotential.

We stress that without too much difficult one can obtain the result up to the desired order. For instance, up to seventh order:

$$\begin{aligned} W_{eff} &= \frac{1}{2}(-2\epsilon + N)\,S\log m - \frac{g^2}{2m^3}(-3\epsilon + N)\,S^2 - \frac{1}{12}\frac{g^4}{m^6}(-59\epsilon + 16N)\,S^3 \\ &- \frac{1}{24}\frac{g^6}{m^9}(-591\epsilon + 140N)\,S^4 - \frac{1}{16}\frac{g^8}{m^{12}}\left(-\frac{4775}{2}\epsilon + 512N\right)S^5 \\ &- \frac{1}{80}\frac{g^{10}}{m^{15}}(-80763\epsilon + 16016N)\,S^6 - \frac{1}{96}\frac{g^{12}}{m^{18}}(-704809\epsilon + 131072N)\,S^7 + \dots \end{aligned} \tag{2.116}$$

Note that our results are in perfect agreement, up to S^5, with the ones of [80] found using the matrix model perturbative approach of [6].

$SO(N)/Sp(N)$ with Traceless Symmetric/Antisymmetric Matter

In the case of matter in the traceless representation, the equation of $T(z)$ reads:

$$\begin{aligned} T(z) &= -\frac{1}{4}\frac{c(z)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} \\ &+ 2\epsilon\frac{\frac{d}{dz}\left(\left(W'(z)-\frac{1}{N}W'(\Phi)\right)-\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}\right)}{\sqrt{\left(W'(z)-\frac{1}{N}W'(\Phi)\right)^2+f(z)}} \end{aligned} \tag{2.117}$$

Here, as we will see, the strategy we follow is different, due to the fact that we are considering the traceless representation and that $\mathrm{Tr}\,(\Phi^2)$ appears explicitly in the denominator of $T(z)$. First we factorize the denominator in the usual way:

$$(W'(z)-\frac{1}{N}W'(\Phi))^2+f(z)=g^2(z-k)^2(z^2+az+b) \tag{2.118}$$

As we are in the traceless representation we have:

$$\begin{aligned} \mathrm{Tr}\,\Phi &= 0 \\ W'(\Phi) &= g\mathrm{Tr}\,\Phi^2 \end{aligned} \tag{2.119}$$

The polynomial $c(z)$ can be fixed as before, and again $f_1=-2gS$. The condition of $\mathrm{Tr}\,\Phi=0$ implies that the coefficient of $\frac{1}{z^2}$ in the expansion of $T(z)$ must be zero. We can use this condition together with the conditions of factorization in order to obtain a system of equations from where $\mathrm{Tr}\,\Phi^2$ can be evaluated. Equivalently the traceless condition can be used to determine c_0 and equation (2.114) together with the conditions from factorization can be used to determine $\mathrm{Tr}\,\Phi^2$. As before we obtain from this the effective superpotential. It should be stressed that such evaluation can be done at any desired number of loops, without many technical complications. For the effective superpotential one finds:

$$\begin{aligned} W_{eff} &= (N-2\epsilon)\frac{S}{2}\log m+\frac{g^2(-\epsilon N+4)S^2}{2Nm^3} \\ &+ \frac{g^4(160\,\epsilon-24N-N^2\,\epsilon)S^3}{12m^6N^2}+\frac{g^6(3584-256\,\epsilon N-36N^2-\epsilon\,N^3)S^4}{24m^9N^3} \\ &+ \frac{g^8\,(67584\,\epsilon-704N^2\,\epsilon-48N^3-N^4\,\epsilon)\,S^5}{32m^{12}N^4} \\ &+ \frac{7g^{10}\,(1171456+79872\,\epsilon N-8320N^2-1280\epsilon N^3-60N^4-\epsilon N^5)\,S^6}{240m^{15}N^5}+\ldots \end{aligned} \tag{2.120}$$

Note that these results agree with the ones of [80]; however using this method is easier to compute higher loop corrections.

$SU(N)$ with Adjoint Matter

The authors of [80] showed that for a cubic potential, like (2.111), the perturbative part of the effective superpotential is zero up to terms of order S^4, due to cancelations in the diagrammatic evaluation. In this paragraph we will show that the generalized Konishi anomaly implies that the perturbative part of W_{eff} is exactly vanishing to all orders. Let us consider equation (2.95) and expand it in powers of $\frac{1}{z}$

$$\begin{aligned} T(z) &= -\frac{1}{4}\frac{c_0 + c_1 z}{\sqrt{\left(W'(z) - \frac{1}{N}W'(\Phi)\right)^2 + f(z)}} \\ &= -\frac{c_1}{4g}\frac{1}{z} + \frac{1}{4g}\left(-c_0 + \frac{c_1 m}{g}\right)\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \\ &= \frac{N}{z} + \frac{\langle \mathrm{Tr}\,\Phi\rangle}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \tag{2.121}$$

From the terms of order $\frac{1}{z}$ we find the familiar condition $c_1 = -4gN$. Considering the term $\frac{1}{z^2}$ and imposing the tracelessness of Φ in $SU(N)$ we obtain the relation $c_0 = \frac{c_1 m}{g}$. Again the denominator of (2.121) can be factorized as in (2.118)

$$(W'(z) - \frac{1}{N}W'(\Phi))^2 + f(z) = g^2(z-k)^2(z^2 + az + b) \tag{2.122}$$

then the condition (2.114) gives the following relation

$$a = 0 \tag{2.123}$$

With this condition only odd powers of $\frac{1}{z}$ will be present in the expansion of $T(z)$; in particular

$$\frac{\partial W_{eff}}{\partial g} = 0 \tag{2.124}$$

from which we see that the perturbative part of the effective superpotential is identically zero (remember that the perturbative part depends only on a specific ratio of m and g, in this case $\frac{g^2}{m^3}$).

We stress that the vanishing of the superpotential is a particular characteristic of the cubic superpotential. One can easily check that for a quartic tree level superpotential, a non zero result is obtained.

2.4.3 On Supersymmetric $Sp(N)$ theories

We have just shown how to compute effective superpotential for gauge theories with various gauge groups, namely $SO(N)$, $Sp(N)$ and $SU(N)$, with matter in various representations. We have already mentioned these result agree with and extend previous literature based on the Dijkgraaf–Vafa diagrammatic approach.

In this section we want to add some comment on supersymmetric $Sp(N)$ theories with matter in the antisymmetric representation. These theories were studied in the 90's; by means of holomorphy several effective superpotentials were determined in [83, 84]. It is natural to aspect that in the IR, where all fields have been integrated out, the two approaches give the same answer. However, in [80] a discrepancy was found for several examples, namely $Sp(4)$, $Sp(6)$ and $Sp(8)$ in the unbroken classical vacuum with a cubic superpotential.The difference always sets in at order Λ^{3h}, where the dual

Coxeter number h is $N/2+1$ for $Sp(N)$. The results in [80] were obtained using matrix model techniques, and were confirmed and extended in [9] by using the Konishi anomaly method as reported in the previous section, and subsequently in [85]. This result, was puzzling; the Konishi anomaly method is completely derived in a gauge theory setup, how can then give a different result from other gauge theoretical calculations?

A possible explanation to this discrepancy was suggested in [86]. There it was discovered an ambiguity in the terms of the superpotential of order h or higher, if one allows for supergroups. A prescription to "F-term complete" a theory was given by thinking about the gauge group $G(N)$ as embedded in a supergroup $G(N+k|k)$ with $k \to \infty$. Moreover, the matrix model, glueball superpotential, or Konishi anomaly computations, were claimed to be computing the superpotential of the F-completion $G(N+k|k)$. This completion can give different result compared with the standard gauge completion. In fact there may be residual instanton effects in the broken part of the group in the Higgsing $G(N+k|k) \to G(N)$. This possibility depends on the geometric structure of the gauge group[8]. Therefore, the analysis of [80, 9, 85], which was compared to the standard $Sp(N)$ field theory calculations give in [83, 84], had actually being done for $Sp(N) \times Sp(0)$. Here $Sp(0)$ is a remnant of the F-completion $Sp(N+k_1+k_2|k_1+k_2)$ broken down to $Sp(N+k_1|k_1) \times Sp(k_2|k_2)$. This completion produces residual instantons that are not present in the standard UV completion of $Sp(N)$.

In [63] a purely gauge theoretical derivation of the correct gauge superpotential was given. This derivation is based on the properties of the generating function $T(z)$. In fact, there it is shown that quantum mechanically $T(z)\mathrm{d}z$ becomes a meromorphic differential on a (hyperelliptic) Riemann surface. Then, remarkably, the simple condition of the integrality of its periods (on–shell) on the Riemann surface, accounts for the full IR dynamics. In the process of imposing the integrality condition of the periods of $T(z)dz$ in the $Sp(N)$ theory the author of [63] found a very precise map to a $U(N+2n)$ theory, where $n+1$ is the degree of the tree level superpotential. This map may hint to a new duality.

[8] One can show that a necessary condition for instantons to be present is the non vanishing of the third homotopy group associated with the breaking $\pi_3\left(G(N+k|k)/G(N)\right)$.

Chapter 3

Gravitational Corrections

In the Dijkgraaf–Vafa conjecture, the quantum effective superpotential is computed with a simple formula, whose basic ingredient is the planar limit of an associated matrix model. In this Chapter we will face the problem of the physical interpretation of the non planar diagrams, following mainly [12]. It was argued already in [1] that these non planar diagrams describe the coupling of $\mathcal{N} = 1$ gauge theories to $\mathcal{N} = 1$ supergravity. This statement was subsequently proved by [10] using diagrammatic techniques, extending to the gravitational case the super-Feynman diagram techniques of [6]. Crucial ingredient in the proof was the modification of the chiral ring relations due to the coupling of the gauge theory to supergravity. In particular, if one restricts to the first non-trivial gravitational F-term contribution, corresponding to the genus one correction in the related matrix model, one needs to take into account just the modification which follows from standard $\mathcal{N} = 1$ supergravity tensor calculus, namely taking into account also the $\mathcal{N} = 1$ Weyl multiplet $G_{\alpha\beta\gamma}$. The corrections corresponding to higher genera in the matrix model, are more subtle, and will be the subject of next Chapter. For all this Chapter, we will restrict ourselves only to the genus one corrections. For related work on the genus one corrections, see [13, 14, 15].

The main problem we want to face is to understand the genus one corrections from the viewpoint of generalized Konishi anomaly relations in the chiral ring. This problem was solved in [12] by extending to the case of $\mathcal{N} = 1$ gauge theories coupled to $\mathcal{N} = 1$ supergravity the strategy of [8], explained in the previous Chapter. Since it is relevant to the following discussion, we will follow the procedure outlined in [12] in detail. The key point in our analysis will be, together with the modification of the chiral ring mentioned above, the observation that in the presence of a non-trivial supergravity background the usual factorization property of chiral correlators does not hold. In particular connected two point functions are generically non-vanishing, much like in matrix models, where connected correlators receive a subleading $1/N^2$ contribution, in the $1/N$ expansion. Then, the basic idea is to derive generalized loop equations that contain the information on the connected correlators. Once these are known, one can derive and solve equations constraining the form of the effective superpotential. Finally, we will generalize these results to $SO(N)$ and $Sp(N)$ gauge groups and comment on the differences. We will work out the $SO(N)$ case in some detail, showing how the Konishi anomaly equations have to be generalized for adjoint and symmetric matter. Then we will compare our results with the matrix model. For a treatment of the $SU(N)$ case, see [17, 18].

3.1 The Chiral Ring

As we have seen, a basic tool in the Konishi anomaly approach to effective superpotentials, is the chiral ring, since $\bar{D}_{\dot{\alpha}}$ exact terms do not contribute to F terms. For the $U(N)$ $\mathcal{N}=1$ gauge theory with adjoint matter considered in the previous Chapter, the chiral ring relations are given by equations (2.28) and (2.30)

$$[W_\alpha, \Phi] = 0 \mod \bar{D}, \qquad \{W_\alpha, W_\beta\} = 0 \mod \bar{D}. \tag{3.1}$$

The chiral ring relations given above are modified in the presence of a curved background [10]. We review here the derivation of the modified chiral ring in presence of background $\mathcal{N}=1$ gravity. Consider the following $\bar{D}$ exact quantity

$$\{\bar{D}^{\dot{\alpha}}, [D_{\alpha\dot{\alpha}}, W_\beta]\} = \{[\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}], W_\beta\} \tag{3.2}$$

where$D_{\alpha\dot{\alpha}}$ is the full covariant derivative containing the gauge field and the spin connection. We used the Jacobi identity and the fact that W_β is chiral to obtain the second term in the above equality. From the Bianchi identity [25] for covariant derivatives one has

$$[\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}] = 4iW_\alpha - 8iG_{\alpha\beta\gamma}M^{\beta\gamma} \tag{3.3}$$

where $G_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ Weyl multiplet. From [25] the lowest component of the $\mathcal{N}=1$ Weyl multiplet starts off as the gravitino field strength and is given by

$$G_{\alpha\beta\gamma} = \frac{1}{12}(\sigma^{ab}_{\alpha\beta}\psi_{ab\gamma} + \sigma^{ab}_{\beta\gamma}\psi_{ab\alpha} + \sigma^{ab}_{\gamma\alpha}\psi_{ab\beta}), \tag{3.4}$$

where we have set the auxiliary field in the above formula to zero, as we are working on shell and a, b refer to the local Lorentz indices. The gravitino field strength is defined by

$$\psi^\alpha_{ab} = \hat{D}_a\psi^\alpha_b - \hat{D}_b\psi^\alpha_a, \qquad \hat{D}_a\psi^\alpha_b = \partial_a\psi^\alpha_b + \psi^\beta_b\omega^\alpha_{a\beta}, \tag{3.5}$$

where $\omega^\alpha_{a\beta}$ is the spin connection. As we are working with an on shell background, equations of motion for the gravitino imply $\sigma^{ab}_{\alpha\beta}\psi_{ab\gamma} = \sigma^{ab}_{\alpha\gamma}\psi_{ab\beta}$. Therefore on shell, the lowest component of the Weyl multiplet is given by

$$G_{\alpha\beta\gamma} = \frac{1}{4}\sigma^{ab}_{\alpha\beta}\psi_{ab\gamma} \tag{3.6}$$

In (3.3), $M^{\alpha\beta}$ refers to the Lorentz generator on spinors whose action on a spinor is given by

$$[M^{\alpha\beta}, \psi_\gamma] = \frac{1}{2}(\delta^\alpha_\gamma\psi^\beta + \delta^\beta_\gamma\psi^\alpha) \tag{3.7}$$

Substituting the Bianchi identity (3.3) in (3.2) and using the action of the Lorentz generator we find the deformed chiral ring given below

$$\{W_\alpha, W_\beta\} = 2G_{\alpha\beta\gamma}W^\gamma \mod \bar{D}, \qquad \{W^\alpha, \Phi\} = 0 \mod \bar{D}. \tag{3.8}$$

The second equation is obtained by replacing the W_β in (3.2) by Φ. For the conventional $\mathcal{N}=1$ supergravity theory, in the first equation above, only the $SU(N)$ part of the gauge field W^γ appears in the right hand side. In [10], the modification of the ring involved the $U(1)$ part of the gauge field as well, which corresponds to a non-standard $\mathcal{N}=1$ supergravity theory relevant to D-brane gauge theories. In this Chapter we will always be restricting ourselves to the standard $\mathcal{N}=1$ supergravity. In order to avoid explicitly including the $SU(N)$ projectors in all the formulae below, we shall always take gauge field backgrounds to be in the $SU(N)$ part of $U(N)$.

Using the deformed chiral ring we can derive many identities valid in the ring which are used crucially in the next sections. From the definition of W^2 and (3.8) we have

$$\begin{aligned} W_\alpha W_\beta &= \frac{\epsilon_{\alpha\beta}}{2} W^2 + G_{\alpha\beta\gamma} W^\gamma, \\ W_\alpha W_\beta W_\gamma &= \frac{1}{2}\epsilon_{\alpha\beta} W^2 W_\gamma + \frac{1}{2} G_{\alpha\beta\gamma} W^2 + G_{\alpha\beta\delta} G^\delta{}_{\gamma\sigma} W^\sigma, \end{aligned} \tag{3.9}$$

using the above identities we are led to the following relations

$$\begin{aligned} W_\alpha W^2 &= -\frac{1}{2} W^2 W_\beta - \frac{1}{2} G^2 W_\alpha, \\ W^2 W_\alpha &= -\frac{1}{2} W_\alpha W^2 - \frac{1}{2} G^2 W_\alpha, \\ [W^2, W_\alpha] = 0, \qquad & W^2 W_\alpha = -\frac{1}{3} G^2 W_\alpha, \quad W^2 W^2 = -\frac{1}{3} G^2 W^2, \\ (G^2)^2 &= G^{\alpha\beta\gamma} G_{\alpha\delta\sigma} G^\sigma{}_{\beta\rho} G^{\delta\rho}{}_\gamma. \end{aligned} \tag{3.10}$$

These identities imply that the gauge invariant combination of certain chiral operators vanish in the chiral ring. The following chiral operator vanishes in the chiral ring.

$$G_{\alpha\beta\gamma} \mathrm{Tr}\,(W^\gamma \Phi\Phi \ldots) = 0 \quad \mathrm{mod}\ \bar{D}. \tag{3.11}$$

It is clear that if there are no Φ's in the trace, the above equation is true for the gauge group $SU(N)$. To proof the above identity for arbitrary number of Φ we use the following equation

$$\begin{aligned} \mathrm{Tr}\,(W_\alpha W_\beta \Phi\Phi \ldots) &= -\mathrm{Tr}\,(W_\beta W_\alpha \Phi\Phi \ldots), \\ &= \frac{1}{2}\epsilon_{\alpha\beta} \mathrm{Tr}\,(W^2 \Phi\Phi \ldots). \end{aligned} \tag{3.12}$$

To obtain the first equation above we have used the cyclic property of trace and (3.8). Now multiplying the first equation of (3.10) with arbitrary number of Φ's and using (3.12) we obtain (3.11). Multiplying (3.11) by $G^{\alpha\beta\gamma}$ and using (A.10) we obtain

$$G^2 \mathrm{Tr}\,(W_\alpha \Phi\Phi \ldots) = 0, \quad \mathrm{mod}\ \bar{D}. \tag{3.13}$$

Another important identity in the chiral ring is

$$G^4 = (G^2)^2 = 0, \quad \mathrm{mod}\ \bar{D} \tag{3.14}$$

The proof goes along the same lines as the derivation of the deformed chiral ring. Consider the following $\bar{D}$ exact quantity

$$\{\bar{D}^{\dot{\alpha}}, [D_{\alpha\dot{\alpha}}, G_{\beta\gamma\delta}]\} = \{[\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}], G_{\beta\gamma\delta}\} \tag{3.15}$$

As $G_{\beta\gamma\delta}$ is uncharged with respect to gauge field the covariant derivative $D_{\alpha,\dot{\alpha}}$ contains only the spin connection. The Bianchi identity for covariant derivatives now implies

$$[\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}] = -8iG_{\alpha\beta\gamma}M^{\beta\gamma} \tag{3.16}$$

Substituting the above equation in (3.15) we obtain the following equation in the chiral ring

$$G_{\alpha\beta\sigma}G^{\sigma}{}_{\gamma\delta} + G_{\alpha\gamma\sigma}G^{\sigma}{}_{\beta\delta} + G_{\alpha\delta\sigma}G^{\sigma}{}_{\beta\gamma} = 0 \tag{3.17}$$

Multiplying this equation by $G^{\alpha\gamma\rho}G^{\beta\delta}{}_{\rho}$ so that all the free indices are contracted and using the last equation in (3.11) and (A.10) we obtain (3.14). As a result the gravitational corrections to the F-terms truncate at order G^2.

3.2 Anomaly Equations and Matrix Model Loop Equations

In this section we will use the generalized Konishi anomaly to extract the gravitational corrections to the effective superpotential. We recall that the anomaly equations obtained in [8] in the absence of gravitational fields are as follows

$$\begin{aligned}\langle R(z)R(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0\\ 2\langle R(z)w_\alpha(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)\rho_\alpha(z))\rangle = 0\\ 2\langle R(z)T(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle = 0\end{aligned} \tag{3.18}$$

Here W denotes the classical superpotential of degree $n+1$. We have indicated above the full two point functions that include the disconnected and connected two point functions. The latter vanish in the absence of gravitational field but as we will show in the following do not vanish in the presence of the gravitational field. These equations were obtained by the generalized Konishi anomalies upon transforming the adjoint chiral field Φ as $\delta\Phi_{ij}$ equal to $\mathcal{R}_{ij}(z)$, $\eta^\alpha\rho_{\alpha\ ij}(z)$ and $\mathcal{T}_{ij}(z)$ respectively, with η^α being an arbitrary field independent spinor. In general for the infinitesimal transformation $\delta\Phi_{ij} = f_{ij}$, the generalized Konishi anomaly is given by

$$\frac{\delta f_{ji}}{\delta\Phi_{k\ell}}A_{ij,k\ell} \tag{3.19}$$

In the absence of gravitation

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^{\alpha}_{kj}W_{\alpha i\ell} \tag{3.20}$$

Using the above anomaly and the equation $\{W_\alpha W_\beta\} = 0$ in the chiral ring, one obtains the equation (3.18). This was explained in detail in Chapter 2.

In the presence of the gravitational field $G_{\alpha\beta\gamma}$, these equations are modified for two reasons: firstly there is a direct gravitational anomaly (ie. even in the absence of gauge fields, in other words when chiral multiplets couple only to gravitational fields) and secondly due to the modification of the ring (3.8).

The Konishi anomaly equation including the pure gravitational contribution in superspace is given by [94, 93]

$$\bar{D}^2(\bar{\Phi}e^V\Phi) = \frac{1}{32\pi^2}\mathrm{Tr}_{\,\mathrm{Ad}}(W^2) + \alpha\frac{1}{32\pi^2}G^2 \tag{3.21}$$

Here α is an unknown normalization constant which we will fix below. The θ^2 component of the above equation together with its anti-holomophic counterpart should reduce to the familiar equation of the chiral anomaly including the gravitational contribution given below:

$$\partial_\mu(\bar{\psi}\bar{\sigma}^\mu\psi) - \partial_\mu(\psi\sigma^\mu\bar{\psi}) = \frac{1}{32\pi^2}\frac{i}{2}\epsilon^{mnlk}\mathrm{Tr}_{\,\mathrm{Ad}}(F_{mn}F_{lk}) + \frac{1}{32\pi^2}\frac{1}{24}\frac{i}{2}\epsilon^{mnlk}R_{mnst}R^{st}{}_{lk} \tag{3.22}$$

The coefficients in the above equation have been obtained from [95] Our strategy to fix the normalization constant α will be to extract the contribution of $R\wedge R$ from the superspace equation in (3.21) and require it to agree with the coefficient in (3.22).

The supersymmetric transformation on the gravitino field strength is given by

$$\delta\psi^\alpha_{mn} = -\xi^\beta R_{mnab}\sigma^{ab\alpha}_\beta + \cdots \tag{3.23}$$

The dots in the above equation all refer to terms that involve the fermions, which are not of interest for the present purpose of determining the coefficient of $R\wedge R$. Substituting this variation in (3.6) we obtain

$$\delta G_{\alpha\beta\gamma} = \frac{1}{4}\sigma^{ab}_{\alpha\beta}\sigma^{cd}_{\delta\gamma}\xi^\delta R_{abcd} \tag{3.24}$$

The θ^2 component of G^2 contains the $R\wedge R$ term, which is imaginary and thus contributes to the anomaly. This is given by

$$G^2|_{\theta^2} = \frac{i}{2}\frac{1}{16}\epsilon^{a'b'ab}R_{a'b'cd}R^{cd}_{ab} \tag{3.25}$$

The total contribution to the anomalous current is obtained by subtracting this out with the $G_{\dot\alpha\dot\beta\dot\gamma}$, the anti-holomorphic contributions. Therefore the coefficient of the $R\wedge R$ term from W^2 is 1/8. Comparing with the coefficient of $R\wedge R$ in (3.22) we see that $\alpha = 1/3$ in (3.21) in order to reproduce the chiral anomaly, including the gravitational contribution.

Then, the infinitesimal transformation $\delta\Phi_{ij} = f_{ij}$ the pure gravitational contribution to the anomaly is

$$\frac{\delta f_{ji}}{\delta\Phi_{k\ell}}\frac{1}{3}G^2\delta_{kj}\delta_{i\ell} \tag{3.26}$$

where from now on G^2 will denote $\frac{1}{32\pi^2}G^{\alpha\beta\gamma}G_{\alpha\beta\gamma}$. This changes

$$A_{ij,k\ell} \longrightarrow A_{ij,k\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell} \tag{3.27}$$

It is easy to see that the pure gravitational anomaly and the modification of the chiral ring (3.8) together with the consequent identities given in the equations (3.11),(3.13),(3.14), give rise to the following modification of the equations (3.18):

$$\begin{aligned} \langle R(z)R(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0 \\ 2\langle R(z)w_\alpha(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)W_\alpha(z))\rangle = 0 \\ 2\langle R(z)T(z)\rangle &- \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle \\ &+ \langle w^\alpha(z)w_\alpha(z)\rangle - \frac{1}{3}G^2\langle T(z)T(z)\rangle = 0 \end{aligned} \tag{3.28}$$

Note that in the first two equations above the pure gravitational anomaly cancels with the contributions coming from the modification of the chiral ring via eq(3.11). This happens due to the remarkable fact that the pure gravitational anomaly (3.27) comes with a factor of $\frac{1}{3}$!! We have also used the fact that $G^2 w_\alpha$ vanishes in the chiral ring. In the last equation however there is no contribution due to the modification of the ring and hence the last two terms on the left hand side arise solely from the pure gravitational anomaly. Here again, a priori, the two point functions are the sum of connected and disconnected parts.

For later purposes let us write the first equation in (3.28) more explicitly

$$\langle R(z)\rangle^2 - W'(z)\langle R(z)\rangle - \frac{1}{4}f(z) = -\langle R(z)R(z)\rangle_c \tag{3.29}$$

where $f(z)$ is a polynomial of degree $n-1$ defined by

$$\langle \mathrm{Tr}\,(W'(\Phi) - W'(z)\mathcal{R}(z)\rangle = \frac{1}{4}f(z) \tag{3.30}$$

and the subscript c denotes the connected part of the correlation function. In fact, as shown in [10, 11], the gravitational corrections enter at genus one in the related matrix model; thus correlators in the matrix model do not factorize anymore. This implies that even in the gauge theory side correlators do not factorize as a consequence of the gravitational background. Indeed this is shown in Appendix B, where we collect the estimates of the connected correlators as performed in [12]. This analysis shows that the RHS of (3.29) goes as G^4 and therefore is trivial in the chiral ring. As a result the equation for R is unmodified by the gravitational field. Since the finite polynomial f is determined completely by the periods of R, i.e. the contour integrals around the various branch cuts C_i, $i = 1, ..., n$,

$$\frac{1}{2\pi i}\int_{C_i} dz R(z) = S_i \tag{3.31}$$

we conclude that R does not receive any gravitational corrections.

The strategy now is to expand all of the quantities appearing above in a perturbation series in G^2. Of course, this series ends at order G^2 since G^4 is trivial. Thus for example we write

$$\langle T\rangle = T^{(0)} + G^2 T^{(1)} \tag{3.32}$$

and similarly for the connected parts of the 2-point functions appearing in the above equations. As discussed in the last section, the latter start from order G^2, and therefore the equations for $T^{(0)}$ and $w_\alpha^{(0)}$ are the same as in [8]. To go beyond this we need to solve for the connected two point functions.

3.2.1 Equations for the Connected Two Point Functions in the Presence of Gravitational Fields

We will now derive equations for the connected two point functions that appear in eq.(3.28). Although, in eq.(3.28) the connected 2-point functions are of the form $\langle R(z)T(z)\rangle$, i.e. both the operators are at same z, it turns out to be more convenient to consider the two operators at different points in the complex plane (say z and w). The reason is that we can impose conditions on a connected 2-point function, like $\langle R(z)T(w)\rangle$ that their integrals around various branch cuts in z and w plane vanish separately. As a result, we will be able to solve completely the corresponding generalized Konishi anomaly equations.

We illustrate the general method of obtaining the generalized Konishi anomaly equations for the connected 2-point functions in one example and then give the complete set of equations which can easily be derived following the methods given below.

Consider the infinitesimal transformation (local in superspace coordinates $(x^\mu, \theta, \bar{\theta})$)

$$\delta\Phi_{ij} = \mathcal{R}_{ij}(z)T(w) \tag{3.33}$$

The Jacobian of this transformation has two parts

$$\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} = \frac{\delta R_{ji}(z)}{\delta\Phi_{kl}}T(w) + \sum_m R_{ji}(z)T_{mk}(w)T_{lm}(w) \tag{3.34}$$

The first term in the equation above together with the variation of the classical superpotential gives rise to

$$\begin{aligned}\langle(R(z)R(z) \;-\; & \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))T(w)\rangle \\ &= \langle(R(z)R(z) - \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))\rangle\langle T(w)\rangle + 2\langle R(z)\rangle\langle R(z)T(w)\rangle_c \\ &\;- \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c \end{aligned} \tag{3.35}$$

where the subscript c denotes completely connected 2 or 3 point functions as indicated. The first term on the right hand side vanishes by virtue of the first equation of (3.28).

The second term in the Jacobian when combined with the anomaly (3.20,3.27) gives rise to a single trace contribution

$$-\frac{1}{3}G^2\langle \mathrm{Tr}\,(\mathcal{R}(z)\mathcal{T}(w)\mathcal{T}(w))\rangle = -\frac{1}{3}G^2\partial_w\frac{\langle R(z)\rangle - \langle R(w)\rangle}{z-w} \tag{3.36}$$

Combining eqs.(3.35),(3.36) and the first equation of (3.28), one obtains the following equation for the connected correlation functions:

$$(2\langle R(z)\rangle - I(z))\langle R(z)T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c - \frac{1}{3}G^2\partial_w\frac{\langle R(z)\rangle - \langle R(w)\rangle}{z-w} = 0 \tag{3.37}$$

Here the integral operator $I(z)$ denotes the following:

$$I(z)A(z) = \frac{1}{2\pi i}\int_{C_z} dy\frac{W'(y)A(y)}{y-z} \tag{3.38}$$

with the contour C_z encircling z and ∞. It is clear that for A equal to $\mathcal{R}$, ρ_α or $\mathcal{T}$, the integral operator reduces to

$$I(z)A(z) = W'(\Phi)A(z) \tag{3.39}$$

as is familiar in the matrix model works.

Since the last term in the eq.(3.37) is of order G^2 and involves one point function of R which is certainly not zero, we conclude that the sum of the remaining terms which involve connected correlations functions cannot all vanish at order G^2. This proves our basic assertion. In fact the connected 3-pt. function $\langle R(z)R(z)T(w)\rangle_c$ vanishes, as argued in eq.(B.13) in the Appendix B, so eq.(3.37) implies that the connected 2-pt function $\langle R(z)T(w)\rangle_c$ does not vanish at order G^2.

In order to completely solve the relevant connected correlation functions, we need to consider all transformations of the form

$$\delta\Phi_{ij} = A_{ij}(z)B(w) \tag{3.40}$$

where A is $\mathcal{R}$, ρ_α or $\mathcal{T}$ and B is R, w_β or T. The resulting generalized Konishi anomaly equations can be derived in the same way as above and can be summarized in the following matrix equation:

$$\begin{bmatrix} \hat{M}(z) & 2\langle T(z)\rangle & 0 \\ 0 & M(z) & 0 \\ 0 & 0 & M(z) \end{bmatrix} \begin{bmatrix} \langle T(z)T(w)\rangle_c & \langle T(z)R(w)\rangle_c & \langle T(z)w_\beta(w)\rangle_c \\ \langle R(z)T(w)\rangle_c & \langle R(z)R(w)\rangle_c & \langle R(z)w_\beta(w)\rangle_c \\ \langle w_\alpha(z)T(w)\rangle_c & \langle w_\alpha(z)R(w)\rangle_c & \langle w_\alpha(z)w_\beta(w)\rangle_c \end{bmatrix} =$$
$$= \frac{1}{3}G^2\partial_w \begin{bmatrix} T(z,w) & R(z,w) & 0 \\ R(z,w) & 0 & 0 \\ 0 & 0 & 5\epsilon_{\alpha\beta}R(z,w) \end{bmatrix} \tag{3.41}$$

We have dropped various connected 3-pt functions that vanish via eq.(B.13). $M(z)$ denotes the integral operator $(2\langle R(z)\rangle - I(z))$, $\hat{M}(z)$ denotes $M(z) - \frac{2}{3}G^2\langle T(z)\rangle$ and finally $R(z,w)$ and $T(z,w)$ denote $(\langle R(z)\rangle - \langle R(w)\rangle)/(z-w)$ and $(\langle T(z)\rangle - \langle T(w)\rangle)/(z-w)$ respectively.

There are a few points to note about these equations:

1) Chiral ring equations are consistent with the above matrix equation. For example, if one takes the equation for $\langle w_\alpha(z)B(w)\rangle_c$ and multiplies by G^2 one finds that the equation identically vanishes in the chiral ring.

2) We have used the estimates given in the Appendix B only to drop all the completely connected 3-pt. functions as they were shown to go as G^4. The estimates for the connected 2-pt. functions given in the Table B, says that $\langle R(z)R(w)\rangle_c$, $\langle R(z)w_\alpha(w)\rangle_c$, and $\langle T(z)w_\alpha(w)\rangle_c$ all vanish in the chiral ring. We have not used these estimates in the above equation but we note that the (1,3), (2,2), (2,3), (3,1) and (3,2) matrix elements on the right hand side vanish. Thus the estimates given in the table are indeed consistent with the matrix equation above. In fact, in the next subsection, we will show that the solutions to this equation are unique thereby proving that these connected 2-pt. functions vanish. Similarly had we used the estimate for the 2-pt. function $\langle T(z)T(w)\rangle_c$ which was shown in the last section to go as G^2, we could have

replaced $\hat{M}$ by M in the above equation.
3) The integrability condition is satisfied: the above equation is of the form

$$\mathcal{M}(z)N(z,w) = \partial_w K(z,w) \tag{3.42}$$

where $\mathcal{M}(z)$ is the first matrix operator appearing on the left hand side of eq(3.41), $N(z,w)$ and $K(z,w)$ satisfy $N(z,w) = N^t(w,z)$ and $K(z,w) = K^t(w,z)$. The non-trivial consistency condition then is

$$(\partial_w K(z,w))\mathcal{M}^t(w) = \mathcal{M}(z)\partial_z K(z,w) \tag{3.43}$$

The crucial identity needed for this is the one involving the integral operator $I(z)$ and is as follows:

$$(I(z)\partial_z - I(w)\partial_w)\frac{A(z)-A(w)}{z-w} = (\partial_z - \partial_w)\frac{I(z)A(z)-I(w)A(w)}{z-w} \tag{3.44}$$

for any function A which is smooth at z and w. This can be proved by using the definition of the contours involved in $I(z)$ and $I(w)$. It follows that the (1,2), (2,1) and (3,3) components of the integrability condition (3.43) implies the following equation:

$$G^2(\langle R(z)\rangle^2 - I(z)\langle R(z)\rangle) = 0 \tag{3.45}$$

This equation is just G^2 times the first equation of (3.28) if one takes into account the fact that the connected part of the correlation function appearing in the latter already is of order $(G^2)^2$. The only other non-trivial part of the integrability condition is its (1,1) component:

$$G^2[(2\langle R(z)\rangle - I(z))\langle T(z)\rangle - \frac{1}{3}G^2\langle T(z)\rangle^2] = 0 \tag{3.46}$$

which is just G^2 times the disconnected part of the third equation of (3.28) thereby proving the integrability condition for the (1,1) component. This is because all the connected parts appearing in that equation will be trivial when multiplied by G^2. Note also that the last term in eq.(3.46) could have been dropped as it is trivial. Its origin comes from the extra term in $\hat{M}$ appearing in the (1,1) component of $\mathcal{M}$ which as argued in the point 2) above could have been dropped.

3.2.2 Uniqueness of the Solutions for the Connected Two Point Functions

Since the integrability conditions are satisfied, solution to eq.(3.41) exists. However the solution has a finite ambiguity which will be fixed by the physical requirement that the contour integrals around all the branch cuts of the connected two point functions in the the z and w planes must vanish separately. The reason for this is that the following operator equations hold:

$$\frac{1}{2\pi i}\int_{C_i} dz R(z) = S_i, \quad \frac{1}{2\pi i}\int_{C_i} dw T(w) = N_i, \quad \frac{1}{2\pi i}\int_{C_i} dz w_\alpha(z) = w_{\alpha\, i}. \tag{3.47}$$

where S_i is the chiral superfield whose lowest component is the gaugino bilinear in the i^{th} gauge group factor in the broken phase $U(N) \longrightarrow \prod_{i=1}^{n} U(N_i)$ and $w_{\alpha\ i}$ is the $U(1)$ chiral gauge superfield of the $U(N_i)$ subgroup. Since these fields are background fields, in the connected correlation functions the contour integrals around the branch cuts must vanish. Similarly since these background fields are independent of the gravitational fields, order G^2 corrections to the one point functions of R, w_α and T must also have vanishing contour integrals around the branch cuts.

For later use, we can write a complete set of normalized differentials using eq. (3.29) up to order G^2 as

$$\omega_j = \frac{1}{4}\frac{dz}{[2\langle R(z)\rangle - W'(z)]}\frac{\partial}{\partial S_j}f(z), \qquad \frac{1}{2\pi i}\int_{C_i}\omega_j = \delta_{ij} \tag{3.48}$$

To illustrate the method, we will again focus on $\langle R(z)T(w)\rangle_c$. The action of the operator $I(z)$ is given as:

$$I(z)\langle R(z)T(w)\rangle_c = W'(z)\langle R(z)T(w)\rangle_c + \sum_{k=0}^{n-1} c_k(w)z^k \tag{3.49}$$

where the first term on the right hand side comes from the contour integral around z and the second term from that around ∞, with n being the order of $W'(z)$. Here we have used the fact that $R(z)$ asymptotically vanishes as $1/z$ [1]. The coefficients $c_k(w)$ are arbitrary functions of w which asymptotically vanish as $1/w^2$. Similarly we have

$$I(w)\langle R(z)T(w)\rangle_c = W'(w)\langle R(z)T(w)\rangle_c + \sum_{k=0}^{n-1} \tilde{c}_k(z)w^k \tag{3.50}$$

with $\tilde{c}_k(z)$ being arbitrary functions of z that vanish asymptotically as $1/z^2$.

From (3.41), the two equations that this two point function satisfies are as follows:

$$\begin{aligned}(2\langle R(z)\rangle - I(z))\langle R(z)T(w)\rangle_c &= \frac{1}{3}G^2\partial_w R(z,w)\\ (2\langle R(w)\rangle - I(w))\langle R(z)T(w)\rangle_c &= \frac{1}{3}G^2\partial_z R(z,w)\end{aligned} \tag{3.51}$$

The solutions to these two equations are

$$\begin{aligned}\langle R(z)T(w)\rangle_c &= \frac{1}{2\langle R(z)\rangle - W'(z)}[\frac{1}{3}G^2\partial_w R(z,w) + \sum_{k=0}^{n-1} c_k(w)z^k]\\ &= \frac{1}{2\langle R(w)\rangle - W'(w)}[\frac{1}{3}G^2\partial_z R(z,w) + \sum_{k=0}^{n-1} \tilde{c}_k(z)w^k]\end{aligned} \tag{3.52}$$

[1] Actually the coefficient of $1/z$ is $\mathrm{Tr}\, W^2$ and hence in the connected 2-pt function it vanishes. As a result the connected 2-pt function goes as $1/z^2$ (and for similar reasons $1/w^2$ asymptotically in w) which means that the sum over k is between 0 and $n-2$. However in the above expression we have kept the sum up to $n-1$ since, as it will turn out, the condition of vanishing contour integrals around all the branch cuts will in particular imply that $c_{n-1} = 0$

Equating the two right hand sides , we see that c_k and $\tilde{c}_k$ are not arbitrary functions of the respective arguments but are fixed up to a finite polynomial ambiguity in z as well as w. They must be of the form

$$\begin{aligned}\sum_{k=0}^{n-1} c_k(w)z^k &= \frac{1}{(2\langle R(w)\rangle - W'(w))}\frac{G^2}{3}[\langle R(w)\rangle\frac{(W'(w)-W'(z)+(z-w)W''(w))}{(z-w)^2} \\ &+\frac{1}{4}\frac{(f(w)-f(z)+(z-w)f'(w))}{(z-w)^2}+\sum_{k,\ell=0}^{n-1} c_{k\ell}z^k w^\ell] \\ \tilde{c}_k(z) &= c_k(z)+\sum_\ell (c_{k\ell}-c_{\ell k})z^\ell \end{aligned} \tag{3.53}$$

where $c_{k\ell}$ are arbitrary coefficients to be determined later. In deriving the above equation we have used the equation (3.29) with the right hand side set to zero. This is because the correction coming from the right hand side is of order G^4 and hence trivial.

Substituting this expression in eq.(3.52) and repeatedly using eq.(3.29), we obtain after some algebra:

$$\begin{aligned}\langle R(z)T(w)\rangle_c &= \frac{G^2}{3}\frac{1}{[2\langle R(z)\rangle - W'(z)][2\langle R(w)\rangle - W'(w)]}[\sum_{k,\ell=0}^{n-1} c_{k\ell}z^k w^\ell + \\ &\frac{[\langle R(z)\rangle\langle R(w)\rangle - W'(z)\langle R(w)\rangle - \frac{1}{4}f(z)+(z\leftrightarrow w)]}{(z-w)^2}]\end{aligned} \tag{3.54}$$

Note that the second term in the bracket is symmetric in $z \leftrightarrow w$.

As mentioned earlier, the connected two point function must obey the following conditions:

$$\int_{C_i} dz\langle R(z)T(w)\rangle_c = \int_{C_i} dw\langle R(z)T(w)\rangle_c = 0 \tag{3.55}$$

for all i, where C_i is the contour around the i-th branch cut.

Let us first consider contour integrals around the branch cuts in w-plane. To this end we can use the first equation in (3.52). The first term on the right hand side is a total derivative in w and therefore its contribution to the contour integral vanishes. Thus we arrive at the condition:

$$\int_{C_i} c_k(w) = 0 \tag{3.56}$$

Using the expression (3.53) for c_k, and the fact that $w^\ell/(2\langle R(w)\rangle + W'(w))$ for $\ell = 0, ..., n-1$ form a complete basis of holomorphic 1-forms in the present case, these equations determine $c_{k\ell}$ completely. Very explicitly if

$$\begin{aligned}\sum_{k=0}^{n-1} t_k^{(i)} z^k &= \int_{C_i} dw\frac{1}{(2\langle R(w)\rangle - W'(w))}[\langle R(w)\rangle\frac{(W'(w)-W'(z)+(z-w)W''(w))}{(z-w)^2} \\ &+\frac{1}{4}\frac{(f(w)-f(z)+(z-w)f'(w))}{(z-w)^2}]\end{aligned} \tag{3.57}$$

then using the basis of normalized differentials eq.(3.48),

$$\sum_{k,\ell=0}^{n-1} c_{k\ell} z^k w^\ell = -\frac{1}{4}\sum_{i=1}^{n}\sum_{k=0}^{n-1} t_k^{(i)} z^k \frac{\partial}{\partial S_i} f(w) \tag{3.58}$$

We will now show that $c_{k\ell}$ are symmetric in k and ℓ exchange. Let us define a matrix G_{ij} by the following equation

$$G_{ij} = \frac{1}{2\pi i}\int_{C_j} dz \frac{1}{2\langle R(z)\rangle - W'(z)} \sum_{k=0}^{n-1} t_k^{(i)} z^k \tag{3.59}$$

We first simplify eq.(3.57) by using (3.29) as

$$\sum_{k=0}^{n-1} t_k^{(i)} z^k = -\int_{C_i} dw \frac{1}{(2\langle R(w)\rangle - W'(w))} \frac{2W'(z)W'(w) + f(z) + f(w)}{4(z-w)^2} \tag{3.60}$$

where we have omitted a total derivative term with respect to w since it does not contribute to the contour integral. Substituting this in eq.(3.59) and noting that the residue of the first order pole $1/(z-w)$ vanishes due to eq.(3.29), we find that G_{ij} is symmetric. Eq.(3.59) can be solved explicitly for $t_k^{(i)}$ as

$$\sum_{k=0}^{n-1} t_k^{(i)} z^k = \frac{1}{4}\sum_{j=1}^{n} G_{ij} \frac{\partial}{\partial S_j} f(z) \tag{3.61}$$

Plugging this equation in eq.(3.58) and using the fact that G_{ij} is symmetric, we find that the coefficients $c_{k\ell}$ are symmetric in k and ℓ exchange as claimed above.

Finally note that the symmetry of $c_{k\ell}$ implies that $\langle R(z)T(w)\rangle_c$ is symmetric in z and w. This will be crucial in the following. In particular this also implies that the contour integrals around the branch cuts in the z-plane vanish.

To summarize this subsection, although we have discussed in detail the example of $\langle R(z)T(w)\rangle_c$, it is easily seen that the eq.(3.41) and the conditions like eq.(3.55) fix the solutions for all the connected two point functions uniquely.

3.2.3 Solutions for the Connected Two Point Functions

Note that the right hand side of equation (3.41) vanishes for all components except $(1,1),(1,2),(2,1)$ and (3,3) (the last being proportional to $\epsilon_{\alpha\beta}$). The uniqueness of the solution then implies

$$\langle R(z)R(w)\rangle_c = \langle w_\alpha(z)T(w)\rangle_c = \langle w_\alpha(z)R(w)\rangle_c = \langle w_{(\alpha}(z)w_{\beta)}(w)\rangle_c = 0 \tag{3.62}$$

This is in accordance with the estimates given in the Table B. We have already obtained the solution for $\langle R(z)T(w)\rangle_c$ in equations (3.54,3.57,3.58). Let us denote this solution as $G^2H(z,w)$.

The remaining two equations are:

$$M(z)\langle w^\alpha(z)w_\alpha(w)\rangle_c = \frac{10}{3}G^2\partial_w R(z,w)$$
$$M(z)\langle T(z)T(w)\rangle_c + 2\langle T(z)\rangle\langle R(z)T(w)\rangle_c = \frac{1}{3}G^2\partial_w T(z,w) \tag{3.63}$$

Comparing the first equation with that of $\langle R(z)T(w)\rangle_c$ namely eq.(3.51) we conclude that

$$\langle w^\alpha(z)w_\alpha(w)\rangle_c = 10G^2H(z,w) \tag{3.64}$$

Finally taking the derivative of eq.(3.51) with respect to $N_i\frac{\partial}{\partial S_i}$, and using the fact that to the leading order (i.e. order(1)) $\langle T(z)\rangle = (N_i\frac{\partial}{\partial S_i} + \frac{1}{2}w_i^\alpha w_{\alpha\ j}\frac{\partial^2}{\partial S_i\partial S_j})\langle R(z)\rangle$, we obtain the following equation:

$$M(z)N_i\frac{\partial}{\partial S_i}\langle R(z)T(w)\rangle_c + 2\langle T(z)\rangle\langle R(z)T(w)\rangle_c = \frac{1}{3}G^2\partial_w T(z,w) \tag{3.65}$$

where we have used the fact that $G^2w_{\alpha\ i}$ is trivial. This equation is the same as the second equation of (3.63). Uniqueness of the solution then implies that up to order G^2,

$$\langle T(z)T(w)\rangle_c = G^2N_i\frac{\partial}{\partial S_i}H(z,w) \tag{3.66}$$

We are now in a position to compute the gravitational corrections to the one point functions of $R(z)$, $w_\alpha(z)$ and $T(z)$ from eq.(3.28). Note that in this equation the two point functions contain both the disconnected and connected pieces. Since $\langle R(z)R(z)\rangle_c$ and $\langle R(z)w_\alpha(z)\rangle_c$ vanish, the first two equations do not contain any connected parts. Uniqueness then implies that one point function of R and $w_\alpha = w^i_\alpha\frac{\partial}{\partial S_i}R$ do not get any gravitational correction. The non-trivial equation is the third one. Using the results of this subsection we get

$$(M(z) - \frac{1}{3}G^2\langle T(z)\rangle)\langle T(z)\rangle + 12G^2H(z,z) = 0 \tag{3.67}$$

Expanding $\langle T(z)\rangle = T^{(0)} + G^2T^{(1)}$, with $T^{(0)} = N_i\frac{\partial}{\partial S_i}R + \frac{1}{2}w_i^\alpha w_{\alpha j}\frac{\partial}{\partial S_i}\frac{\partial}{\partial S_j}R$ we obtain

$$M(z)T^{(1)}(z) + [\frac{1}{3}(T^{(0)}(z))^2] + 12H(z,z) = 0 \tag{3.68}$$

Here the term indicated in the square bracket goes as N^2 (note that only the N_i dependent term in $T^{(0)}$ contribute since G^2w_α is trivial) and therefore represents genus 0 contribution. On the other hand the term proportional to $H(z,z)$ does not come with any factors of N_i as is seen from the explicit solution given in (3.54), (3.57) and (3.58). This contribution therefore comes from genus 1. Writing $T^{(1)} = T_0^{(1)} + T_1^{(1)}$ where the subscript denotes the genus, we have the following solution to the above equation:

$$T_0^{(1)}(z) = -\frac{1}{6}N_i\frac{\partial}{\partial S_i}T^{(0)}(z)$$
$$T_1^{(1)}(z) = -\frac{12}{(2R^{(0)}(z) - W'(z))}[H(z,z) + c^{(1)}(z)] \tag{3.69}$$

where $c^{(1)}(z)$ is a polynomial of degree $n-2$ and is uniquely determined by the requirement that the contour integrals of $T_1^{(1)}(z)$ around every branch cut vanishes. In the next subsection we will show that this is exactly the answer the Matrix model provides.

Let us note that the genus 0 contribution $T_0^{(1)}$ above can be absorbed in $T^{(0)}$ by a field redefinition

$$S_i \to S_i + \frac{1}{6}G^2 N_i \tag{3.70}$$

In particular this means that the contribution of $T_0^{(1)}$ to the effective superpotential can be absorbed by the above field redefinition into the original genus 0 effective superpotential in the absence of the gravitational field. This also implies that this term does not contribute to the superpotential when evaluated at the classical solution of S_i in agreement with the statement made in [10, 11].

3.2.4 Comparison with the Matrix Model Results

In the Matrix model a systematic approach to computing higher genus contributions has been developed in [96, 97], however in the following we will rederive their results in a way parallel to the gauge theory discussion above. This will make the comparison between the two very transparent. Consider a hermitian matrix model with action given by $S = \frac{\hat{N}}{g_m}\sum_k \frac{g_k}{k}\mathrm{Tr}\, M^k \equiv \frac{\hat{N}}{g_m}W$, where M is a hermitian $\hat{N}\times\hat{N}$ matrix. In Matrix model the resolvent $\Omega(z) \equiv \frac{g_m}{\hat{N}}\mathrm{Tr}\,\frac{1}{z-M}$ satisfies a loop equation similar to gauge theory $R(z)$:

$$\langle\Omega(z)\rangle^2 - I(z)\langle\Omega(z)\rangle + \langle\Omega(z)\Omega(z)\rangle_c = 0 \tag{3.71}$$

Here $I(z)$ is the same integral operator as in the gauge theory discussion above. In the large $\hat{N}$ limit, the two point function factorizes. However in the subleading order in $1/\hat{N}^2$ the connected part of the two point function (in fact the planar connected graph) contributes and yields the genus 1 contribution to the resolvent via the above equation. By definition

$$\begin{aligned}\langle\Omega(z)\Omega(w)\rangle_c &= \frac{1}{\hat{N}^2}\sum_{k=0}^{\infty}\frac{k}{z^{k+1}}\frac{\partial}{\partial g_k}\langle\Omega(w)\rangle \\ &\equiv \frac{1}{\hat{N}^2}\mathcal{O}(z)\langle\Omega(w)\rangle\end{aligned} \tag{3.72}$$

Let us expand the 1-point function as :

$$\langle\Omega(z)\rangle = \Omega_{(0)}(z) + \frac{1}{\hat{N}^2}\Omega_{(1)}(z) + \ldots \tag{3.73}$$

where dots represent terms of higher order in $1/\hat{N}^2$. Inserting these expansions in the above equations we get:

$$\begin{aligned}\Omega_{(0)}(z)^2 - I(z)\Omega_{(0)}(z) &= 0 \\ (2\Omega_{(0)}(z) - I(z))\Omega_{(1)}(z) &+ \mathcal{O}(z)\Omega_{(0)}(z) = 0\end{aligned} \tag{3.74}$$

Now we need to solve for $\mathcal{O}(w)\Omega_{(0)}(z)$. This can be done by applying the differential operator $\mathcal{O}(w)$ on the first equation of (3.74). To this end we need the following identity:

$$\mathcal{O}(w)W'(y) = \sum_{k=1}^{\infty} \frac{k}{w}(\frac{y}{w})^k = \frac{1}{(w-y)^2} \tag{3.75}$$

which is valid for $|w| > |y|$. It follows that

$$\begin{aligned} \int_{C_z} dy[\mathcal{O}(w)W'(y)]\frac{\Omega_{(0)}(y)}{y-z} &= \int_{C_z,|y|<|w|} dy\frac{1}{(w-y)^2}\frac{\Omega_{(0)}(y)}{y-z} \\ &= \partial_w \frac{\Omega_{(0)}(z)-\Omega_{(0)}(w)}{z-w}. \end{aligned} \tag{3.76}$$

Using this, we obtain the following equation by applying $\mathcal{O}(w)$ on the first equation of (3.74)

$$(2\Omega_{(0)}(z) - I(z))\mathcal{O}(w)\Omega_{(0)}(z) - \partial_w \frac{\Omega_{(0)}(z)-\Omega_{(0)}(w)}{z-w} = 0. \tag{3.77}$$

Since $\Omega_{(0)}$ of the matrix model is the same as the $R^{(0)}$ for the gauge theory, we see that $\mathcal{O}(w)\Omega_{(0)}(z)$ satisfies the same equation (3.51) as $3\langle R(z)T(w)\rangle_c$. We now impose the conditions

$$\int_{C_i} dz\mathcal{O}(w)\Omega_{(0)}(z) = \int_{C_i} dw\mathcal{O}(w)\Omega_{(0)}(w) = 0 \tag{3.78}$$

which are the analogues of the equations (3.55). It follows from the discussion of uniqueness that $\mathcal{O}(w)\Omega_{(0)}(z)$ is equal to $3H(z,w)$. Note that as we showed in the last subsection, $H(z,w)$ is symmetric in z and w. This is consistent with the fact that $\mathcal{O}(w)\Omega_{(0)}(z)$ is symmetric in z and w. Finally, substitution of $\mathcal{O}(z)\Omega_{(0)}(z)$ in the second equation of (3.74), results in an equation for $\Omega_{(1)}(z)$ which is identical to that for the genus 1 part of the gauge theory equation (3.68) for $T_1^{(1)}$. Using the fact that the integral of $\Omega_{(1)}(z)$ around every branch cut is zero, we conclude, from the uniqueness of the solution, that

$$\Omega_{(1)}(z) = \frac{1}{4}T_1^{(1)}(z), \tag{3.79}$$

with the right hand side being the genus 1 part of the solution given in (3.69). While in the matrix model the $\frac{1}{N^2}$ correction to the effective potential is obtained by integrating the asymptotic expansion of $\Omega_{(1)}(z)$ with respect to the couplings g_k, the order G^2 correction to the effective superpotential in gauge theory is obtained by integrating the asymptotic expansion of $T_1^{(1)}(z)$ with respect to the coupling constants g_k (we already argued in the last subsection that the genus 0 contribution coming from $T_0^{(1)}$ can be absorbed by a field redefinition of S_i). Eq.(3.79) implies therefore that the genus 1 contribution to the effective potential in matrix model is equal to the genus 1 contribution to the order G^2 term in the gauge theory effective superpotential. In fact, the relative coefficient 4 in eq.(3.79) is exactly reproduced if one follows the numerical factors in the diagrammatic computations given in [10].

3.3 Genus One Solution for Other Gauge Groups

In this section we will generalize to other gauge groups the previous results, exactly as we have done in the last Chapter. We will derive the generalized loop equations in the case of $SO(N)$ and $Sp(N)$ gauge groups. From these we will derive the gravitational corrections to the effective superpotential, given by the $\mathcal{N} = 1$ Weyl multiplet. We will present the derivation for an $SO(N)$ gauge theory with matter in the adjoint representation in some detail and then simply state the results for other groups and representations.

3.3.1 $SO(N)$ with Matter in the Adjoint Representation

Let us begin with the $SO(N)$ theory with for matter in the adjoint representation; in presence of the gravitational background field $G_{\alpha\beta\gamma}$ the anomaly equation is modified as [93] [94]

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^{\alpha}_{kj}W_{\alpha i\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell} - (k \longleftrightarrow \ell)\,. \tag{3.80}$$

Now, following a familiar strategy, we take variations in the chiral ring of the form $\delta\Phi_{ij} = \mathcal{R}(z)_{ij}$, $\delta\Phi_{ij} = \eta^\alpha\rho_\alpha(z)_{ij}$ and $\delta\Phi_{ij} = \mathcal{T}(z)_{ij}$, to derive the following Ward identities

$$\begin{aligned}
&\frac{1}{2}\langle R(z)R(z)\rangle - \frac{1}{2}G^2\frac{\langle R(z)\rangle}{z} - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0,\\
&\langle R(z)w_\alpha(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\rho_\alpha(z))\rangle = 0\\
&\langle R(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle - \frac{1}{6}G^2\langle T(z)T(z)\rangle\\
&\qquad -2\frac{\langle R(z)\rangle}{z} + \frac{1}{6}G^2\frac{\langle T(z)\rangle}{z} + \frac{1}{2}\langle w_\alpha(z)w^\alpha(z)\rangle = 0.
\end{aligned} \tag{3.81}$$

where $W(\Phi)$ is the tree level superpotential. As for the $U(N)$ case, two point functions do not factorize because of the non trivial gravitational background, but in general contain a non vanishing connected piece. Then we need a set of equations for the connected two point functions, that again can be obtained by considering variations of the form

$$\delta\Phi_{ij} = \mathcal{A}_{ij}(z)B(w) \tag{3.82}$$

where $\mathcal{A}$ can be $\mathcal{R}$, $\eta^\alpha\rho_\alpha$ or $\mathcal{T}$ and B can be R, $\eta^\alpha w_\alpha$ or T. The reason for considering two point functions in different points w and z is that we can impose conditions on them that their integrals around various branch cuts in w and z planes vanish separately. The Jacobian of this transformation has two pieces

$$\begin{aligned}
\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} &= \frac{\delta\mathcal{A}_{ji}(z)}{\delta\Phi_{kl}}B(w) + \sum_m \mathcal{A}_{ji}(z)\left(\mathcal{B}_{mk}(w)\mathcal{T}_{lm}(w) - \mathcal{B}_{ml}(w)\mathcal{T}_{km}(w)\right) \qquad (3.83)\\
&= \frac{\delta\mathcal{A}_{ji}(z)}{\delta\Phi_{kl}}B(w) + \sum_m \mathcal{A}_{ji}(z)\left(\mathcal{B}_{mk}(w)\mathcal{T}_{lm}(w) \mp \mathcal{B}_{mk}(-w)\mathcal{T}_{lm}(-w)\right)
\end{aligned}$$

where the plus sign holds when $\mathcal{B}_{mk}(w) = \eta^\alpha \rho_\alpha(w)_{mk}$ and the minus sign in all the other cases. This is a consequence of the symmetry properties of the field Φ, so that

$$T(w)^t = -T(-w), \qquad w_\alpha(w)^t = w_\alpha(-w), \qquad R(w)^t = -R(-w) \tag{3.84}$$

Where to simplify the notation we have indicated with $T(z)^t$ the trace of the transpose $\mathrm{Tr}\,\mathcal{T}^t(z)$, etc. Considering these variations together with the classical variation of the superpotential, one obtains a set of equations for the two point connected functions that can be summarized in the following matrix equation

$$\begin{bmatrix} M(z) & \langle T(z)\rangle - \frac{2}{z} & 0 \\ 0 & M(z) & 0 \\ 0 & 0 & M(z) \end{bmatrix} \begin{bmatrix} \langle T(z)T(w)\rangle_c & \langle T(z)R(w)\rangle_c & \langle T(z)w_\beta(w)\rangle_c \\ \langle R(z)T(w)\rangle_c & \langle R(z)R(w)\rangle_c & \langle R(z)w_\beta(w)\rangle_c \\ \langle w_\alpha(z)T(w)\rangle_c & \langle w_\alpha(z)R(w)\rangle_c & \langle w_\alpha(z)w_\beta(w)\rangle_c \end{bmatrix} =$$
$$= \frac{G^2}{6}\partial_w \begin{bmatrix} \tilde{T}(z,w) & \tilde{R}(z,w) & 0 \\ \tilde{R}(z,w) & 0 & 0 \\ 0 & 0 & 5\epsilon_{\alpha\beta}\tilde{R}(z,w) \end{bmatrix} \tag{3.85}$$

where we have introduced the operator $M(z) = \langle R(z)\rangle - I(z)$ and $\tilde{B}(z,w) = B(z,w) + B(z,-w)$ where

$$B(z,w) = \frac{\langle B(z)\rangle - \langle B(w)\rangle}{z-w} \tag{3.86}$$

The integral operator $I(z)$ was defined in equation (3.38).

The equation in (3.85) is of the form

$$\mathcal{M}(z)N(z,w) = \partial_w K(z,w) \tag{3.87}$$

with $\mathcal{M}(z)$ the matrix operator appearing on the left hand side of eq. (3.85), and $N(z,w)$ and $K(z,w)$ satisfying $N(z,w) = N^t(w,z)$ and $K(z,w) = K^t(w,z)$. The non-trivial consistency condition (integrability condition) is then given by

$$(\partial_w K(z,w))\mathcal{M}^t(w) = \mathcal{M}(z)\partial_z K(z,w). \tag{3.88}$$

as in (3.43). Using the methods of [12], as explained in section 3.2, it can be shown that the above integrability condition is satisfied, and this fact guarantees the existence of solutions for the connected two point functions, equation (3.85); all this was explained in detail in the previous sections. However these solutions suffer from ambiguities, in the form of a finite set of parameters. These ambiguities will be fixed by the physical requirement that the contour integrals around the branch cuts of the connected two point-functions, both in the z and w planes, vanish separately. The reason for this is that the following operator equations hold:

$$\frac{1}{2\pi i}\oint_{C_i} dz R(z) = S_i, \qquad \frac{1}{2\pi i}\oint_{C_i} dw T(w) = N_i, \qquad \frac{1}{2\pi i}\oint_{C_i} dz w_\alpha(z) = w_{\alpha\ i}. \tag{3.89}$$

where S_i is the chiral superfield whose lowest component is the gaugino bilinear in the i-th gauge group factor in the broken phase $SO(N) \to SO(N_0) \times \prod_{i=1}^n U(N_i)$ and $w_{\alpha\ i}$ is the $U(1)$ chiral gauge superfield of the $U(N_i)$ subgroup. Since these fields are background fields, in the connected correlation functions the contour integrals around the branch cuts must vanish:

$$\oint_{C_i} dz\langle A(z)B(w)\rangle_c = \oint_{C_j} dw\langle A(z)B(w)\rangle_c = 0 \tag{3.90}$$

This requirement makes the solution of the equations of (3.85) unique (see section 3.2). We can express such solutions in term of one function $H(z,w)$, whose explicit form will not be needed for the analysis done here. The two point functions we will need are

$$\begin{aligned} \langle R(z)R(w)\rangle_c &= 0 \\ \langle R(z)T(w)\rangle_c &= \frac{1}{6}G^2 H(z,w) \\ \langle w(z)_\alpha w_\beta(w)\rangle_c &= \frac{5}{6}G^2 H(z,w)\epsilon_{\alpha\beta} \end{aligned} \tag{3.91}$$

Now we could plug these solutions into (3.81) and solve for the one point functions. However note that we can instead perform the following rescaling

$$R(z) \to R(z) + \frac{G^2}{6}T(z) + \frac{G^2}{6}\frac{1}{z} \tag{3.92}$$

Note that such rescaling changes the boundary conditions (3.89) in a non trivial way. This rescaling simplifies equations (3.81) to

$$\frac{1}{2}\langle R(z)R(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0, \tag{3.93}$$
$$\langle R(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle - 2\frac{\langle R(z)\rangle}{z} - \frac{1}{3}\frac{G^2}{z^2} + \frac{1}{2}\langle w_\alpha(z)w^\alpha(z)\rangle = 0.$$

To derive equations (3.93), we have used the chiral ring relations and the equations of motion. We have not included the equation for $w_\alpha(z)$ since it can be set consistently to zero. In order to find the corrections to the zero order one point functions, let us expand the functions $R(z)$ and $T(z)$ into a zeroth order and a first order term in G^2

$$\langle R(z)\rangle = R^{(0)}(z) + R^{(1)}(z), \quad \langle T(z)\rangle = T^{(0)}(z) + T^{(1)}(z). \tag{3.94}$$

Note that, since G^4 terms are trivial in the chiral ring, this expansion is exact. Substituting the above expansions in (3.93) we obtain for the zeroth order

$$\begin{aligned} &\frac{1}{2}R^{(0)}(z)R^{(0)}(z) - I(z)R^{(0)}(z) = 0 \\ &R^{(0)}(z)T^{(0)}(z) - I(z)T^{(0)}(z) - \frac{2}{z}R^{(0)}(z) = 0 \end{aligned} \tag{3.95}$$

Note that these equations are consistent with (3.29) and (2.82). More interesting is the result at order one

$$\begin{aligned} M(z)R^{(1)}(z) &= 0 \\ M(z)T^{(1)}(z) &= \frac{1}{3}G^2\frac{1}{z^2} - G^2 H(z,z) \end{aligned} \tag{3.96}$$

Note that, as expected, the genus zero contribution (*i.e.* going with N^2) has been adsorbed with the redefinition (3.92).

3.3.2 $SO(N)$ with Matter in the Symmetric Representation

If now we add matter in the symmetric representation to the $SO(N)$ gauge theory, the anomaly equation gets modified to

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^{\alpha}_{kj}W_{\alpha i\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell} + (k \longleftrightarrow \ell) \qquad (3.97)$$

while the loop equations for the generating functions of the chiral ring become

$$\begin{aligned} &\frac{1}{2}\langle R(z)R(z)\rangle - \frac{1}{2}G^2\langle\frac{\partial R(z)}{\partial z}\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0, \\ &\langle R(z)w_\alpha(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\rho_\alpha(z))\rangle = 0 \\ &\langle R(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle - \frac{1}{6}G^2\langle T(z)T(z)\rangle \\ &\qquad -2\langle\frac{\partial R(z)}{\partial z}\rangle + \frac{1}{6}G^2\langle\frac{\partial T(z)}{\partial z}\rangle + \frac{1}{2}\langle w_\alpha(z)w^\alpha(z)\rangle = 0. \end{aligned} \qquad (3.98)$$

However note that, unlike as for the previous case, the contribution at genus zero cannot be reabsorbed by a field redefinition, the obstruction being the differential operators appearing in (3.98). The existence of such redefinition is expected for theories that can be obtained by softly breaking $\mathcal{N}=2$ to $\mathcal{N}=1$, and this is not our case since in the $\mathcal{N}=2$ theory matter has to be in the same representation of the gauge field.

Again the two point functions will contain a non vanishing connected piece, and we need equations for such pieces. This can be obtained as before by taking variations in the chiral ring of the form

$$\delta\Phi_{ij} = \mathcal{A}_{ij}(z)B(w) \qquad (3.99)$$

The result is

$$\begin{bmatrix} M(z) & \langle T(z)\rangle - 2\frac{\partial}{\partial z} & 0 \\ 0 & M(z) & 0 \\ 0 & 0 & M(z) \end{bmatrix} \begin{bmatrix} \langle T(z)T(w)\rangle_c & \langle T(z)R(w)\rangle_c & \langle T(z)w_\beta(w)\rangle_c \\ \langle R(z)T(w)\rangle_c & \langle R(z)R(w)\rangle_c & \langle R(z)w_\beta(w)\rangle_c \\ \langle w_\alpha(z)T(w)\rangle_c & \langle w_\alpha(z)R(w)\rangle_c & \langle w_\alpha(z)w_\beta(w)\rangle_c \end{bmatrix} =$$
$$= \frac{1}{3}G^2\partial_w \begin{bmatrix} T(z,w) & R(z,w) & 0 \\ R(z,w) & 0 & 0 \\ 0 & 0 & 5R(z,w)\epsilon_{\alpha\beta} \end{bmatrix} \qquad (3.100)$$

As before, the solution to such equations is unique, and can be expressed in terms of one function $J(z,w)$. The relevant two point functions are finally

$$\begin{aligned} \langle R(z)R(w)\rangle_c &= 0 \\ \langle R(z)T(w)\rangle_c &= \frac{1}{3}G^2J(z,w) \\ \langle w_\alpha(z)w_\beta(w)\rangle_c &= \frac{5}{3}G^2J(z,w)\epsilon_{\alpha\beta} \end{aligned} \qquad (3.101)$$

Now, following the same ideas of the last section, we insert this results into (3.98) and expand in a genus zero and a genus one contribution. The zeroth order equations hold precisely the loop equations found in the previous Chapter without gravity, while the first order equations are

$$\begin{aligned} M(z)R^{(1)}(z) &= \frac{1}{2}G^2\frac{\partial R^{(0)}}{\partial z} \\ M(z)T^{(1)}(z) &= T^{(0)}(z)R^{(1)}(z)+\frac{G^2}{6}(T^{(0)})^2+ \\ 2 &\ \frac{\partial R^{(1)}(z)}{\partial z}-\frac{G^2}{6}\frac{\partial T^{(0)}(z)}{\partial z}-\frac{1}{3}G^2J(z,w) \end{aligned} \tag{3.102}$$

Note that in these equations, there is a genus zero contribution that cannot be eliminated by a field redefinition.

3.3.3 $Sp(N)$ Gauge Theory

Now we will focus on an $Sp(N)$ gauge theory with matter in the adjoint (symmetric) and in the antisymmetric representations. We use the same conventions of the second Chapter, that is: in the symmetric (antisymmetric) representation, Φ has to be considered as a matrix MJ where M is a symmetric (antisymmetric) matrix and J is the invariant antisymmetric tensor of $Sp(N)$. We take the generators of $Sp(N)$ to be $(e_{lk}+e_{kl})$ with $(e_{lk})_{ij}=\delta_{il}\delta_{jk}$. The analysis for the $Sp(N)$ case is almost identical to the one for the $SO(N)$ case, the only change being the sign in the generators (and of course the different properties of the matrices representing the field Φ, that are relevant since the antisymmetric invariant J will enter in the intermediate steps). Because of this the only difference with the case of $SO(N)$ will be a change of sign in front of the terms $\frac{1}{z}$ for matter in the adjoint representation and of the terms $\frac{\partial R(z)}{\partial z}$ and $\frac{\partial T(z)}{\partial z}$ for antisymmetric matter, as we have already seen in the second Chapter. The results obtained can be expressed in terms of a single function H^{Sp} when matter is in the adjoint representation

$$\begin{aligned} M(z)R^{(1)}(z) &= 0 \\ M(z)T^{(1)}(z) &= \frac{1}{3}G^2\frac{1}{z^2}-G^2H^{Sp}(z,z) \end{aligned} \tag{3.103}$$

(note that the genus zero contributions have been absorbed) and a function J^{Sp} when matter is in the antisymmetric representation

$$\begin{aligned} M(z)R^{(1)}(z) &= -\frac{1}{2}G^2\frac{\partial R^{(0)}}{\partial z} \\ M(z)T^{(1)}(z) &= T^{(0)}(z)R^{(1)}(z)+\frac{G^2}{6}(T^{(0)})^2 \\ &- 2\frac{\partial R^{(1)}(z)}{\partial z}+\frac{G^2}{6}\frac{\partial T^{(0)}(z)}{\partial z}-\frac{1}{3}G^2J^{Sp}(z,z) \end{aligned} \tag{3.104}$$

We remark that, while in the case of adjoint matter, the genus zero contribution to the genus one loop equation has been absorbed with a field redefinition, this has not been done for antisymmetric matter. However, we don't aspect this to be possible, since this theory does not arise as a soft breaking of an $\mathcal{N}=2$ theory to $\mathcal{N}=1$.

3.4 Comparison with the Matrix Model

In this section we will finally show that our loop equations agree with the matrix model results. In particular, we will focus on the case of $SO(N)$ with adjoint and symmetric matter (an analogous treatment can also be done for $Sp(N)$).The planar limit of the matrix model for $SO(N)$ was studied in [77, 70, 71, 81].

3.4.1 $SO(N)$ with Adjoint Matter

Let us consider the following one–matrix model with action

$$S = \frac{M}{g_s} \sum_k \frac{g_k}{M} \mathrm{Tr}\, \Phi^k \tag{3.105}$$

where Φ is now an $M \times M$ matrix in the adjoint representation of $SO(M)$. The matrix model resolvent is defined as

$$\Omega(z) = \frac{g_s}{M} \mathrm{Tr} \frac{1}{z - \Phi} \tag{3.106}$$

and satisfies the loop equation

$$\frac{1}{2}\langle\Omega(z)\rangle^2 - I(z)\langle\Omega(z)\rangle - \frac{1}{M}\frac{1}{2z}\langle\Omega(z)\rangle + \frac{1}{2}\langle\Omega(z)\Omega(z)\rangle_c = 0 \tag{3.107}$$

The connected part of the correlator $\langle\Omega(z)\Omega(z)\rangle_c$ goes like $\frac{1}{M^2}$ and is usually neglected in the planar large M limit; it corresponds to non planar contributions that capture the gravitational background. In order to show the M dependence explicitly, we will write it as $\frac{1}{2}\langle\Omega(z)\Omega(z)\rangle_c = \frac{1}{M^2}\Omega(z,z)$. Note that this loop equation is analogous to the equation fulfilled by the generator $R(z)$ in the corresponding gauge theory, equation (3.81), provided we identify G^2 with $\frac{1}{M}$.

Next we expand the resolvent in powers of $\frac{1}{M}$ to identify the higher genus corrections to the planar limit

$$\langle\Omega(z)\rangle = \Omega^{(0)}(z) + \frac{1}{M}\Omega^{(1)}(z) + \frac{1}{M^2}\Omega^{(2)}(z) + \cdots \tag{3.108}$$

Plugging (3.108) into the loop equation, we find

$$\begin{aligned}
&\left(\frac{1}{2}\Omega^{(0)}(z) - I(z)\right)\Omega^{(0)}(z) = 0 \\
&\left(\Omega^{(0)}(z) - I(z)\right)\Omega^{(1)}(z) - \frac{1}{2z}\Omega^{(0)}(z) = 0 \\
&\left(\Omega^{(0)}(z) - I(z)\right)\Omega^{(2)}(z) - \frac{1}{2z}\Omega^{(1)}(z) + \frac{1}{2}(\Omega^{(1)}(z))^2 + \Omega(z,z) = 0
\end{aligned} \tag{3.109}$$

In order to compare these equations with their gauge theory counterpart let us perform the following rescaling on the gauge theory side

$$R(z) \rightarrow R(z) + \frac{G^2}{6}T(z) \tag{3.110}$$

After this, equations (3.81) become

$$\begin{aligned}
&\frac{1}{2}\langle R(z)R(z)\rangle - \frac{1}{6}G^2\frac{\langle R(z)\rangle}{z} - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0, \\
&\langle R(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle \\
&\quad -2\frac{\langle R(z)\rangle}{z} - \frac{1}{6}G^2\frac{\langle T(z)\rangle}{z} + \frac{1}{2}\langle w_\alpha(z)w^\alpha(z)\rangle = 0.
\end{aligned} \tag{3.111}$$

We can see that $\Omega^{(0)}$ and $R^{(0)}$, as well as $\Omega^{(1)}$ and $R^{(1)}$ follow the same equations. in order to make the identifications more precise, let us now introduce the operator

$$D = \sum_i N_i \frac{\partial}{\partial S_i} \tag{3.112}$$

We can split the function $T(z)$ as

$$T^{(0)}(z) = DR^{(0)}(z) + \hat{T}^{(0)}(z), \qquad T^{(1)}(z) = DR^{(1)}(z) + \hat{T}^{(1)}(z) \tag{3.113}$$

By plugging these in the loop equation, we obtain equations for $\hat{T}^{(0)}(z)$ and $\hat{T}^{(1)}(z)$

$$\begin{aligned}
&M(z)\hat{T}^{(0)}(z) - \frac{2}{z}R^{(0)}(z) = 0 \\
&M(z)\hat{T}^{(1)}(z) - \frac{2}{z}R^{(1)}(z) + \hat{T}^{(0)}(z)R^{(1)}(z) - \frac{G^2}{6}\frac{1}{z}\hat{T}^{(0)}(z) + \text{connected} = 0
\end{aligned} \tag{3.114}$$

With "connected" we mean the connected two point functions. Comparing with (3.109) we see that $\hat{T}^{(0)}(z)$ and $\hat{T}^{(1)}(z)$ satisfy (up to some factors) the same equations that $\Omega^{(1)}$ and $\Omega^{(2)}$. Note that, for a precise identification, not only the equations but also the boundary conditions (3.89) should be the same; this is the reason for using $\hat{T}^{(0)}(z)$ and $\hat{T}^{(1)}(z)$ instead of $T^{(0)}(z)$ and $T^{(1)}(z)$.

The results obtained here are consistent with the diagrammatic analysis. In calculating the effective superpotential a given diagram of L quantum loops, contributes with a power of W_α equal to $2h + 4g + 2c - 2$ with h the number of index loops, g the genus of the diagram and c the number of crosscaps.[2]. On the other hand, the maximun number of W_α depends on the chiral ring. In absence of gravitation

$$2h + 4g + 2c - 2 \leq 2h \Rightarrow 4g + 2c - 2 \leq 0, \tag{3.115}$$

implying that only diagrams with $g = c = 0$ and $g = 0$, $c = 1$ will contribute to the effective superpotential. In presence of the gravitational background under consideration, since for non trivial gravitational backgrounds we can put more than two W_α in one index loop,

$$2h + 4g + 2c - 2 \leq 2h + 2 \Rightarrow 4g + 2c \leq 4 \tag{3.116}$$

Now diagrams with $g = 1$, $c = 0$ and $g = 0$, $c = 2$ will contribute. In order to study which diagrams will contribute to $R(z)$ one should take into account that it contains already two W_α.

[2]The relation $L = h + 2g + c - 1$ holds

3.4.2 $SO(N)$ with Symmetric Matter

In this case, the analysis is completely analogous to the latter section. The only difference is that now in the matrix model Φ is in the symmetric representation of $SO(M)$. The loop equation for the resolvent turns out to be

$$\frac{1}{2}\langle\Omega(z)\rangle^2 - I(z)\langle\Omega(z)\rangle - \frac{1}{M}\langle\frac{\partial}{\partial z}\Omega(z)\rangle + \frac{1}{2}\langle\Omega(z)\Omega(z)\rangle_c = 0 \qquad (3.117)$$

Again, note that this loop equation is analogous to the equation fulfilled by the generator $R(z)$ in the corresponding gauge theory, equation (3.98), provided we identify G^2 with $\frac{1}{M}$.

Expanding as in (3.108) and plugging the expansion in the loop equation, we finally obtain

$$\left(\frac{1}{2}\Omega^{(0)}(z) - I(z)\right)\Omega^{(0)}(z) = 0 \qquad (3.118)$$

$$\left(\Omega^{(0)}(z) - I(z)\right)\Omega^{(1)}(z) - \frac{\partial}{\partial z}\Omega^{(0)}(z) = 0 \qquad (3.119)$$

$$\left(\Omega^{(0)}(z) - I(z)\right)\Omega^{(2)}(z) - \frac{\partial}{\partial z}\Omega^{(1)}(z) + \frac{1}{2}(\Omega^{(1)}(z))^2 + \Omega(z,z) = 0 \qquad (3.120)$$

The same rescaling (3.110) can be done here and again the same analysis following equations (3.109) holds.

Chapter 4

Gravitational F–terms at All Genera

In this Chapter we will complete the treatment of the gravitational corrections to the effective superpotential, by including all the non planar contribution to the matrix model free energy in the gauge theory setup. We have seen in the previous Chapter how this task is accomplished when the theory is coupled to standard $\mathcal{N}=1$ supergravity: namely loop equations get a direct gravitational contribution and the chiral ring relations have to be modified. However, as explained in [11, 10], higher genera in the matrix model are more subtle. They showed that in order to capture genus $g \geq 2$ contributions, in the language of the dual closed String Theory side [19], one needs to introduce a more drastic modification in the chiral ring relation, to account for a non-trivial vacuum expectation value of the (self-dual) graviphoton field strength $F_{\alpha\beta}$ of the parent $\mathcal{N}=2$ String Theory. In this case the relation reads $\{W_\alpha, W_\beta\} = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma$. In particular, one is modifying the Grassmann nature of the fermionic superfield W_α. In this case one has to face ordering ambiguities in manipulating W's in the generalized Konishi anomaly equations, somewhat similar to the base point dependence in path ordered exponentials found in [11, 10]. We fix the ambiguities by requiring that traces of (graded) commutators be trivial in the chiral ring. This requirement will lead us to get identities involving gauge invariant and gravitational operators, which will be very important when analyzing the anomaly equations. We will then first, following the strategy outlined in Chapter 3, reconsider the $g=1$ case in the presence of both $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$ and find that the order F^2 and G^2 superpotential terms have the expected structure and agree with the $g=1$ matrix model result.

In order to proceed with arbitrary genus analysis, for which a non-trivial $F_{\alpha\beta}$ is essential, we will need to further generalize the strategy of [12], explained in the last Chapter, to derive anomaly equations for the generating functional of (connected) correlators, by coupling the relevant chiral gauge invariant operators to external sources. From these equations we will extract the correlators which are required to determine the one-point functions of $R(z)$, $T(z)$ and $w_\alpha(z)$ to all orders in F^2. From these one can determine the effective superpotential. The connection with the matrix model will be proved by generalizing the loop equation of the latter to a full generating functional of connected correlators of resolvents, by coupling the matrix model resolvent $\Omega_m(z)$

to an external source. We will prove that, in fact, the gauge theory $\langle R(z)\rangle$ coincides with the matrix model $\langle \Omega_m(z)\rangle$, to all orders, if we identify F^2 with the matrix model genus-expansion parameter $1/\hat{N}^2$. More precisely, there is a full class of gauge theory correlators, $\langle \mathrm{Tr}\, W^2\Phi^k\rangle$, whose expansion in powers of F^2 coincides with the $1/\hat{N}^2$ expansion of $\langle \mathrm{Tr}\, M^k\rangle$, where M is a $\hat{N}\times\hat{N}$ hermitian random matrix, whose expectation value is computed with the measure $\exp(-\frac{\hat{N}}{g_m}\mathrm{Tr}\, W(M))$.

It is also worth mentioning that lower genus contributions to a given genus term can be gotten rid of by an operator redefinition, $R \to R + \frac{1}{6}G^2T$, thereby generalizing the shift $S \to S + \frac{1}{6}G^2N$ needed to remove the genus zero contribution to the order G^2 term.

It is interesting to note that, by following [8] and introducing an auxiliary Grassman coordinate ψ_α, which can be thought of as a second supercoordinate of the broken $\mathcal{N} = 2$ parent theory, we can rewrite the anomaly equations in a shift-invariant way, if, in addition to assembling R, w_α and T in $\mathcal{R}(\psi)$ like we have done in Chapter 2, we assemble also F and G as $\mathcal{H} = F_{\alpha\beta} - \frac{1}{2}\psi^\gamma G_{\alpha\beta\gamma}$. This suggests that the effective superpotential can be formally written in a manifest shift-invariant way, $\int d^2\psi \mathcal{H}^{2g}\mathcal{F}_g(S + \psi^\alpha w_\alpha - \frac{1}{2}\psi^2 N)$. However, it will turn out that, due to the modified chiral ring relations the $g \geq 2$ terms are all trivial from the $\mathcal{N} = 1$ point of view. We should stress, nevertheless once again, that the all-orders identification of the gauge theory $\langle R(z)\rangle$ with the matrix model $\langle \Omega_m(z)\rangle$, implying the exact identity of an infinite family of correlators on the two sides, survives the chiral ring relations, since on the gauge theory side it involves powers of F^2 only.

4.1 The Chiral Ring

The chiral ring in $\mathcal{N} = 1$ gauge theories consists of all operators which are annihilated by the covariant derivative $\bar{D}_{\dot\alpha}$, which is conjugate to the supercharge $\bar{Q}_{\dot\alpha}$, modulo $\bar{D}_{\dot\alpha}$ exact terms, where the exact terms should be gauge invariant and local operators. All relations in the chiral ring are therefore defined modulo $\bar{D}_{\dot\alpha}$ exact terms. The chiral ring and the various relations in it play an important role in deriving the effective F-terms in $\mathcal{N} = 1$ gauge theory obtained by integrating out the adjoint matter [8]. This is because the contributions to the effective action of all $\bar{D}_{\dot\alpha}$ exact terms can be written as an integral involving both holomorphic and anti-holomorphic integrations in superspace, $\int d^2x d^2\theta d^2\bar\theta S(\theta, \bar\theta)$. Thus they do not contribute to the F-terms, which involve only either a holomorphic or anti-holomorphic integrations in the superspace co-ordinates.

For the $\mathcal{N} = 1$ gauge theory with a single adjoint field on R^4 the chiral ring relations are given by [8] [1]

$$[W_\alpha, \Phi] = 0 \bmod \bar{D}, \quad \{W_\alpha, W_\beta\} = 0 \bmod \bar{D}, \tag{4.1}$$

where W_α is the $\mathcal{N} = 1$ gauge chiral superfield and Φ is the chiral matter superfield in the adjoint representation of the gauge group. We are interested in studying the F-

[1] All relations in the chiral ring are modulo $\bar{D}$, for the rest of paper this will not be explicitly mentioned, but understood wherever necessary.

terms in $\mathcal{N}=1$ $U(N)$ gauge theory obtained by integrating out a single adjoint scalar in presence of gravity as well as the self-dual graviphoton field strength $F_{\alpha\beta}$, using the generalized anomaly equations approach. The chiral ring relations in the presence of these backgrounds are given by [11, 10]

$$[W_\alpha, \Phi] = 0, \qquad \{W_\alpha, W_\beta\} = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma. \tag{4.2}$$

These relations were proved in [21]. There, the partial supersymmetry breaking of an $\mathcal{N}=2$ to an $\mathcal{N}=1$ theory was considered and relations (4.2) were showed to be a consequence of the standard $\mathcal{N}=2$ supergravity Bianchi identities. It was pointed out in [11] that the modification of ring relations in the presence of $F_{\alpha\beta}$ seems to require a non-traditional interpretation: the classical Grassmanian nature of the $\mathcal{N}=1$ gauge multiplet apprently no longer holds. However, as discussed in [21] within the framework of $\mathcal{N}=1$ theories obtained from spontaneously broken $\mathcal{N}=2$ theories, the presence of the graviphoton in (4.2) follows from traditional supergravity tensor calculus. This implies that there is no drastic change in the Grassmann nature of the gaugino superfield. The basic idea developed in [21] to prove equations (4.2) is to begin with $\mathcal{N}=2$ supergravity, that encodes the graviphoton in the supergravity multiplet. The fields in the gravity multiplet can be splitted in a background term and a fluctuation. Then, the authors of [21] rescaled the gravity fields by the appropriate Planck mass m_p power so as to retain a non dynamical gravity background, while the fluctuations were scaled in order to decouple from the matter sector in the $m_p \to \infty$ limit. In this way one obtains a theory with only rigid supersymmetry but coupled to a supergravity background. Finally in [21] they found a vacuum which partially breaks supersymmetry and performed a tensor calculus analysis of the Bianchi identities expanded around that vacuum. This analysis reproduces (4.2) up to a field redefinition. It was then showed that this field redefinition does not spoil the usual Konishi anomaly treatment of F–terms.

All the gauge invariant operators constructed out of the basic fields of the $\mathcal{N}=1$ gauge multiplet and the chiral multiplet can be arranged into the operators (2.34) that here we recall:

$$\begin{aligned} \mathcal{R}(z)_{ij} &= -\frac{1}{32\pi^2}\left(\frac{W^2}{z-\Phi}\right)_{ij}, & R(z) &= \mathrm{Tr}\,\mathcal{R}(z), \\ \rho_\alpha(z)_{ij} &= \frac{1}{4\pi}\left(\frac{W_\alpha}{z-\Phi}\right)_{ij}, & w_\alpha(z) &= \mathrm{Tr}\,\rho_\alpha(z), \\ \mathcal{T}(z)_{ij} &= \left(\frac{1}{z-\Phi}\right)_{ij}, & T(z) &= \mathrm{Tr}\,\mathcal{T}(z). \end{aligned} \tag{4.3}$$

As we have explained in the second Chapter, the above set of operators is exhaustive in the chiral ring: placing more W's in the trace does not yield any more gauge invariant operators, as they can be converted to one of the above operators by the ring relations in (4.2). In the next sub-section we will derive various identities from the relations (4.2), with a motivated ansatz that the adjoint action with W_α on gauge invariant operators vanishes. In the subsequent sub-section we will re–derive these identities from the closed string dual using the $\mathcal{N}=2$ Bianchi identities.

4.1.1 Chiral Ring Identities from Gauge Theory

From the definition of W^2 and the basic ring equations in (4.2) we can derive the following identities using simple algebraic manipulations,

$$\begin{aligned} [W_\alpha, W^2] &= -2F_{\alpha\beta}W^\beta, \\ \{W_\alpha, W^2\} &= -\frac{2}{3}\left(G^2 W_\alpha + G_{\alpha\beta\gamma}F^{\beta\gamma}\right), \end{aligned} \tag{4.4}$$

adding the above equations we find

$$W_\alpha W^2 = -F_{\alpha\beta}W^\beta - \frac{1}{3}G^2 W_\alpha - \frac{1}{3}G_{\alpha\beta\gamma}F^{\beta\gamma}. \tag{4.5}$$

Considering the equation (4.2) with a product of arbitrary number of scalars Φ and taking trace on both sides of the equation, we obtain

$$\begin{aligned} \mathrm{Tr}\,(\{W_\alpha, W_\beta\}\Phi\Phi\ldots) &= \mathrm{Tr}\,((F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma)\Phi\Phi\ldots), \\ &= \mathrm{Tr}\,(\mathrm{Ad}_{W_\alpha} W_\beta \Phi\Phi\ldots). \end{aligned} \tag{4.6}$$

We see that on the left hand side of the equation we have the adjoint action on the operator $W_\beta\Phi\Phi\ldots$. For ordinary Grassman W_α, the trace of the adjoint action is zero, but here, from the algebra in (4.2), it is not clear that this will still hold. However if the trace of the adjoint action is not zero, then in any gauge invariant operator, like the one considered in (4.6), there will be an ambiguity in the ordering of W_α, such that the cyclic property of the trace will not be obeyed. This is the base point ambiguity noted in [11]. To remove such ambiguities we demand that the trace of the adjoint action is trivial in the chiral ring. This leads us to the following equation

$$\mathrm{Tr}\,((F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma)\Phi\ldots) = 0. \tag{4.7}$$

We can write the above equation compactly in terms of the operators in (4.3). After performing the following convenient redefinitions, $F_{\alpha\beta} \to 32\pi^2 2\sqrt{2}F_{\alpha\beta}$ and $G_{\alpha\beta\gamma} \to \sqrt{32\pi^2}\, G_{\alpha\beta\gamma}$, (4.7) can be written as [2]

$$2F_{\alpha\beta}T + G_{\alpha\beta\gamma}w^\gamma = 0. \tag{4.8}$$

This equation, introduced here to remove the above mentioned ambiguity, was rigorously proven in [21] in the framework of $\mathcal{N}=2$ theories spontaneously broken down to $\mathcal{N}=1$ by using the generalized Konishi anomaly and the precise relation between the superpotential and the gauge function τ that hold for $\mathcal{N}=2$ theories. This relation excludes any dependence on the chiral field Φ of the gauge function, dependence that would modify the Konishi anomaly relation.

A similar equation can be obtained by considering the first equation in (4.4) with products of Φ's and a trace on both sides, we obtain

$$\mathrm{Tr}\,([W_\alpha, W^2]\Phi\ldots) = -2F_{\alpha\beta}\mathrm{Tr}\,(W^\beta\Phi\ldots). \tag{4.9}$$

[2]From now on we will use these scaled variables for the rest of the paper.

Again demanding that the adjoint action is trivial in the chiral ring we get

$$F_{\alpha\beta}\mathrm{Tr}\left(W^{\beta}\Phi\ldots\right)=0, \tag{4.10}$$

that written in terms of the operators in (4.3) becomes

$$F_{\alpha\beta}w^{\beta}=0. \tag{4.11}$$

Using Bianchi identities of $\mathcal{N}=1$ supergravity it was shown in [12], that the spin 2 combination of a product of two $\mathcal{N}=1$ Weyl multiplet was trivial in the ring. This combination is given by

$$G_{\alpha\beta\sigma}G^{\sigma}{}_{\gamma\delta}+G_{\alpha\gamma\sigma}G^{\sigma}{}_{\beta\delta}+G_{\alpha\delta\sigma}G^{\sigma}{}_{\beta\gamma}=0. \tag{4.12}$$

This ensures that the gravitational corrections to the F-terms truncate at order G^2. If a similar Bianchi identity were applied on the symmetric tensor $F_{\alpha\beta}$ we would obtain the following product of the graviphoton and the $\mathcal{N}=1$ Weyl multiplet

$$G_{\alpha\beta\gamma}F^{\gamma}{}_{\sigma}+G_{\alpha\sigma\gamma}F^{\gamma}{}_{\beta}, \tag{4.13}$$

which should therefore vanish in the chiral ring. The product of G and F in (4.13) contains both a spin 3/2 and a spin 1/2 part, but we make the minimal ansatz that the spin 3/2 combination in the tensor product is trivial in the chiral ring as given below

$$G_{\alpha\beta\gamma}F^{\gamma}_{\sigma}+G_{\alpha\sigma\gamma}F^{\gamma}_{\beta}+G_{\sigma\beta\gamma}F^{\gamma}_{\alpha}=0 \tag{4.14}$$

From the basic identities in the chiral ring, (4.8), (4.11), (4.12) and (4.14), we derive other identities which are used at several instances in this Chapter. The first identity is obtained by multiplying the equation (4.11) by $F_{\alpha'}{}^{\alpha}$, which gives

$$F_{\alpha'}{}^{\alpha}F_{\alpha\beta}w^{\beta}=\frac{1}{2}\epsilon_{\alpha'\beta}F^{\gamma\delta}F_{\gamma\delta}w^{\beta}=\frac{1}{2}F^{2}w_{\alpha'}=0 \tag{4.15}$$

Another important identity is given by

$$G^{2}F_{\alpha\beta}=0. \tag{4.16}$$

In order to prove this identity, we first multiply (4.14) by $G_{\delta}^{\alpha\beta}$ to obtain

$$\frac{1}{2}G^{2}F_{\delta\sigma}-2G_{\delta\beta\alpha}G^{\alpha}_{\sigma\gamma}F^{\gamma\beta}=0 \tag{4.17}$$

The product $G_{\delta\beta\alpha}G^{\alpha}_{\sigma\gamma}$ contains both the spin 0 and the spin 2 part. The spin 2 part vanishes by (4.12), therefore this product contains only the spin 0 part, which is given by

$$G_{\delta\beta\alpha}G^{\alpha}_{\sigma\gamma}=-\frac{1}{6}\left(\epsilon_{\delta\sigma}\epsilon_{\beta\gamma}+\epsilon_{\beta\sigma}\epsilon_{\delta\gamma}\right)G^{2} \tag{4.18}$$

Substituting the above equation in (4.17) gives (4.16). We also have the identity

$$F_{\alpha\beta}G^{\alpha\beta}_{\gamma}F^{\gamma\sigma}\equiv(F\cdot G)_{\gamma}F^{\gamma\sigma}=0. \tag{4.19}$$

This equation is obtained by simply multiplying (4.14) by $F_{\alpha\beta}$, the last two terms vanish due to the symmetry of $G_{\alpha\beta\gamma}$ in all the indices. Finally, using (4.8) and (4.11) we can show that

$$(F \cdot G)_\alpha w_\beta = -(F \cdot G)_\beta w_\alpha \tag{4.20}$$

The identities (4.15), (4.16), (4.18) and (4.19) imply that terms containing G^2 do not admit any expansions in higher powers of either F or G. And the terms containing $(F \cdot G)$ also do not admit any higher powers of either F or G. From these considerations we can conclude that the only expansion which admits arbitrary powers is an expansion purely in F.

4.1.2 Chiral Ring Identities from Closed String Dual

It is instructive to understand the previous analysis from the point of view of the open/closed string duality conjectured in [19]. The basic conjecture of [19] is that $\mathcal{N} = 1$, $U(N)$ gauge theory with a single adjoint chiral superfield Φ, realized, for instance, by N D5-branes wrapped on a two-cycle of a local Calabi Yau threefold, is dual to closed String Theory on a CY threefold which is related by a conifold transition to the previous one. On the closed string side $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = 1$ by the presence of three-form fluxes on the CY space [90]. The tree level superpotential for Φ is related to the geometry of the CY manifold. More precisely, the duality identifies the lowest moment of the gauge theory operators of (4.3) with the components of an $U(1)$ $\mathcal{N} = 2$ vector multiplet on the closed string side

$$V(\hat{\theta}, \theta) = S(\theta) + \hat{\theta}^\alpha w_\alpha(\theta) + \hat{\theta}^2 N. \tag{4.21}$$

S is the closed string field dual to $\mathrm{Tr}\,(W^\alpha W_\alpha)/32\pi^2$, and it corresponds to the complex structure modulus of the CY threefold, w_α is dual to $\mathrm{Tr}\,(W_\alpha)/4\pi$ and N, the auxiliary field corresponding to the three-form flux on the closed string side, is dual to $\mathrm{Tr}\,(1)$ of the gauge theory. θ is the usual $\mathcal{N} = 1$ superspace coordinate and $\hat{\theta}$ is the additional superspace coordinate for $\mathcal{N} = 2$ superspace. The duality is expected to still hold after coupling the gauge theory on one side and the vector multiplet on the other side, to the supergravity background given by $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$. In particular, a class of gravitational F-terms, usually called $\mathcal{F}_g$ [91, 92] are expected to match on the two sides, with the above identification of fields.

We can also organize the background fields, the $\mathcal{N} = 1$ Weyl multiplet and the graviphoton field strength as an $\mathcal{N} = 2$ Weyl multiplet as follows.

$$H_{\alpha\beta}(\hat{\theta}, \theta) = F_{\alpha\beta}(\theta) + \hat{\theta}^\gamma G_{\alpha\beta\gamma}(\theta) \tag{4.22}$$

In the above equation $F_{\alpha\beta}$ stands for the $\mathcal{N} = 1$ self-dual graviphoton multiplet and $G_{\alpha\beta\gamma}$ refers to the $\mathcal{N} = 1$ Weyl multiplet which contains the self-dual part of the Riemann curvature. We have set the auxiliary field of the Weyl multiplet to be zero as it does not play any role in deriving the ring relations discussed in the previous sub-section.

Our strategy to prove the basic ring relations in (4.8), (4.11) and (4.14) would be to use the $\mathcal{N} = 2$ Bianchi identities to show that these equations are $\bar{D}$ exact. Here $\bar{D}$

refers to the derivative of the $\mathcal{N}=1$ superspace coordinate. This is the same method used to obtain the ring relations in (4.1). Consider the following $\bar{D}$ exact quantity

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}V) = [\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}]V \tag{4.23}$$

In writing the equality we have used the fact that V, the $\mathcal{N}=2$ vector multiplet, is annihilated by $\bar{D}$. From the definition of covariant derivatives in superspace we have the following (see for instance [25])

$$\begin{aligned}(\mathcal{D}_C\mathcal{D}_B \ &- \ (-1)^{bc}\mathcal{D}_B\mathcal{D}_C)V^{(A_1A_2\ldots)} \\ &= \ -R^{A_1}_{CBD}V^{(DA_2\ldots)} - R^{A_2}_{CBD}V^{(A_1D\ldots)} - \cdots - T^{D}_{CB}\mathcal{D}_D V^{(A_1A_2\ldots)}\end{aligned} \tag{4.24}$$

Here A, B, C, D, etc. refer either to bosonic or fermionic coordinates in superspace, b, c refer to their grading, a bosonic coordinate having grade 0 and a fermionic one grade 1. R and T stand for the curvature and the torsion in superspace coordinates respectively. For $\mathcal{N}=2$ superspace, the complete solution for the Bianchi identities has been given in [27] and one can read out the required curvature and torsion symbols. The equation in (4.23) can then be written as

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}V) = T^{\dot{\alpha}}_{\alpha\dot{\alpha}\gamma}\hat{D}^{\gamma}V = H_{\alpha\gamma}\hat{D}^{\gamma}V \tag{4.25}$$

There are no curvature contributions as V is a scalar in $\mathcal{N}=2$ superspace, there are other contributions to this Bianchi identity, but they vanish on shell [3]. The zeroth and the first component in $\hat{\theta}$ reduces to

$$\begin{aligned}F_{\alpha\gamma}w^{\gamma} &= 0, \\ 2F_{\alpha\beta}N + G_{\alpha\beta\gamma}w^{\gamma} &= 0\end{aligned} \tag{4.26}$$

The $\hat{\theta}^2$ component is identically zero. These operator equations verify the lowest moment of the ring relations in (4.8) and (4.11) from the closed string side. To verify all the moments of these relations we need map which relates all the gauge invariant operators of the gauge theory to closed string fields; at present such a detail map is lacking, though it is obvious they will all be mapped to vector multiplets on the closed string side. To prove the relation in (4.14) consider the following $\bar{D}$ exact quantity

$$\begin{aligned}\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}H^{\beta\gamma}) &= [\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}]H^{\beta\gamma}, \\ &= R^{\dot{\alpha}\beta}{}_{\alpha\dot{\alpha}\rho}H^{\rho\gamma} + R^{\dot{\alpha}\gamma}{}_{\alpha\dot{\alpha}\rho}H^{\beta\rho} + T^{\dot{\alpha}}_{\alpha\dot{\alpha}\rho}\hat{D}^{\rho}H^{\beta\gamma}.\end{aligned} \tag{4.27}$$

Substituting the required curvature and torsion symbols we get [4]

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}H^{\beta\gamma}) \ = \ G^{\beta}{}_{\alpha\rho}H^{\rho\gamma} + G^{\gamma}{}_{\alpha\rho}H^{\rho\beta} + H_{\alpha\rho}\hat{D}^{\rho}H^{\beta\gamma} \tag{4.28}$$

There are other terms in this Bianchi identity, but they all vanish on shell. From the lowest component in $\hat{\theta}$ of the above equation we obtain the relation (4.14). Note that on lowering the β and γ indices, the combination is entirely symmetric, thus containing only the spin 3/2 part of the tensor product of G and F. This completes the proof of (4.14) from the closed string side. The linear term in $\hat{\theta}$ of (4.28) reduces to (4.12)

[3] This is the holomorphic counterpart of equation (7.6) in [27].

[4] We have used the holomorphic counterpart of equations (4.25) and (7.6) of [27].

4.2 Genus One Analysis

The chiral ring relations (4.1) ensures that only planar graphs contribute to the computation of the superpotential in the absence of gravity or the graviphoton field strength. The diagrammatic analysis of [11] and [10] show that higher genus diagrams contribute when either gravity or the graviphoton background is turned on. In fact the contribution of gravity alone enters at genus one in the superpotential, and as we have shown in the previous Chapter the genus one correction to the loop equation in the corresponding matrix model agrees with the gravitational corrected anomaly equations in the gauge theory [12]. In this Chapter we extend this to the situation when the graviphoton field strength is also turned on. The graviphoton affects the gauge theory loop equations at all genera. In this section as an important preliminary step to the analysis at all genera and we will analyze the anomaly equations of the gauge theory with the graviphoton also turned on at genus one.

To derive the Ward identities constraining the gauge invariant generating functions of (4.3) in the presence of gravity and the graviphoton field strength we need three ingredients. Firstly, the background modifies the ring to (4.2) and the associated ring equations discussed in the last section play a crucial role. The generalized Konishi anomaly [34] forms the second ingredient: one can derive the Ward identities constraining the functions $R(z)$, $w_\alpha(z)$ and $T(z)$ by considering an infinitesimal variation $\delta\Phi_{ij} = f_{ij}$ where f_{ij} is the matrix elements of the operators given in (4.3). This variation is anomalous and, in absence of gravity, the anomaly is given by

$$\frac{\delta f_{ji}}{\delta\Phi_{k\ell}} A_{ij,k\ell}, \tag{4.29}$$

with

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^\alpha_{kj}W_{\alpha i\ell}. \tag{4.30}$$

When a gravitational background is turned on, there is a direct contribution to the Konishi anomaly This is just the generalized gravitational contribution of the chiral anomaly [94, 93]. To include this contribution we replace $A_{ij,kl}$ in (4.30) with

$$A_{ij,k\ell} \longrightarrow A_{ij,k\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell} \tag{4.31}$$

We expect that the presence of a graviphoton background will not affect the Konishi anomaly. The graviphoton field strength is of dimension 3 and all terms in the Konishi anomaly equation are Lorentz scalars and of dimension 3. Thus there is no Lorentz invariant term which can be constructed out of the graviphoton field strength which is of dimension 3. Therefore, the effect of the graviphoton in the anomaly equations can be seen only through the ring (4.2). Using these two ingredients, the equations determining the gauge invariant operators of (4.3) are given by

$$\begin{aligned}
&\langle R(z)R(z)\rangle + \frac{1}{6}G^2\langle w^\alpha(z)w_\alpha(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z))\rangle = 0,\\
&2\langle R(z)w_\alpha(z)\rangle - \frac{1}{3}G^2\langle w_\alpha(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\rho_\alpha(z))\rangle = 0,\\
&2\langle R(z)T(z)\rangle - \langle \mathrm{Tr}\,(W'(\Phi)\mathcal{T}(z))\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle - \frac{1}{3}G^2\langle T(z)T(z)\rangle = 0.
\end{aligned} \tag{4.32}$$

To arrive at these equations we have repeatedly used the identities in the chiral ring derived in the previous section . Had we not used those identities, we would have found ambiguities in various terms, due to the fact that cyclic property of the trace is not obeyed. Note that the above equations reduce to the same equations derived in the last Chapter (and in [12]) in absence of the graviphoton fields strength. To see this, we have to use the chiral ring equation $G^2 w_\alpha = 0$ in the first equation of (4.32). Finally, the third ingredient in solving for the gauge invariant operators is that the above Ward identities involve two point functions of the gauge invariant operators. In absence of either gravity or the graviphoton field strength these operators factorize in the chiral ring [8]. However, in the presence of these background there is no apriori reason for factorization; in fact the correspondence of the gauge theory with the matrix model and the diagrammatic calculations of [11, 10] imply that these operators do not factorize. Therefore, we need a further set of Ward identities determining the connected two-point functions. For the case of the gravitational background alone, this was done in [12] and explained in the last Chapter, and it was shown there that the corrections to the gauge invariant operator T is precisely that of genus one correction to the resolvent of the matrix model. We will repeat the same analysis for the case where the graviphoton field strength is turned on.

Before we proceed, we note that the equations of (4.32) simplify if we perform the following field redefinition

$$R(z) \to R(z) + \frac{1}{6} G^2 T(z) \tag{4.33}$$

This field redefinition shifts all moments in the generating functional, and it is therefore a generalization of the field redefinition (3.70) noted in [12], which removed the genus zero contribution of the gravitational correction to the superpotential. In fact we will see that this field redefinition does the same job in the general case. Using this redefinition in (4.32) we obtain

$$\begin{aligned} \langle R(z)R(z)\rangle - I(z)\langle R(z)\rangle\rangle &= 0 \\ 2\langle R(z)w_\alpha(z)\rangle - I(z)\langle w_\alpha(z)\rangle &= 0 \\ 2\langle R(z)T(z)\rangle - I(z)\langle T(z)\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle &= 0 \end{aligned} \tag{4.34}$$

In obtaining these equations we have used the chiral ring identities as well as (4.32). The integral operator $I(z)$ was introduced in equation (3.38) as

$$I(z)A(z) = \frac{1}{2\pi i} \oint_{C_z} dy \frac{W'(y)A(y)}{y-z}, \tag{4.35}$$

with the contour C_z encircling z and ∞. We recall that if A is equal to $\mathcal{R}$, ρ_α or $\mathcal{T}$, the integral operator reduces to

$$I(z)A(z) = W'(\Phi)A(z) \tag{4.36}$$

With the field redefinition above, the Ward identities reduce exactly to those with no gravitational or graviphoton background derived in Chapter 2; in fact the first equation of (4.34) is identical to the equation for the matrix model resolvent.

4.2.1 Connected Two Point Functions

Now we will derive a set of equations for the connected two point functions. We see that, to solve for the one point functions from (4.34), it is sufficient to determine the connected two-point functions evaluated at coincident points in the z plane. However, it is more convenient to determine the connected two point functions at two different points in the complex plane say z and w, for example $\langle R(z)T(w)\rangle$. This turns out be useful as we can impose conditions on the connected two point function, such as their contour integrals around various branch cuts in z and w plane vanish separately, which enable us to solve the corresponding generalized Konishi anomaly equations completely.

We will briefly review the method used in [12] using the example of the two point function $\langle R(z)T(w)\rangle$. Consider the infinitesimal transformation which is local in superspace coordinates $(x^\mu, \theta, \bar{\theta})$,

$$\delta\Phi_{ij} = \mathcal{R}_{ij}(z)T(w) \tag{4.37}$$

The Jacobian of this transformation has two pieces

$$\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} = \frac{\delta\mathcal{R}_{ji}(z)}{\delta\Phi_{kl}}T(w) + \sum_m \mathcal{R}_{ji}(z)\mathcal{T}_{mk}(w)\mathcal{T}_{lm}(w) \tag{4.38}$$

The first term in the equation above together with the variation of the classical superpotential gives rise to

$$\begin{aligned}\langle(R(z)R(z) &\quad -\mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))T(w)\rangle \\ &= \langle(R(z)R(z) - \mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))\rangle\langle T(w)\rangle + 2\langle R(z)\rangle\langle R(z)T(w)\rangle_c \\ &\quad -\langle\mathrm{Tr}\,(W'(\Phi)\mathcal{R}(z)))T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c\end{aligned} \tag{4.39}$$

where the subscript c denotes completely connected 2- or 3-point functions. The first term on the right hand side vanishes when we use the first equation of (4.34). From the second term in the Jacobian, when combined with the anomaly (3.20,4.31), we obtain the following single trace contribution

$$-\frac{1}{3}G^2\langle tr(\mathcal{R}(z)\mathcal{T}(w)\mathcal{T}(w))\rangle = -\frac{1}{3}G^2\partial_w\frac{\langle R(z)\rangle - \langle R(w)\rangle}{z-w} \equiv -\frac{1}{3}G^2\partial_w R(z,w) \tag{4.40}$$

Here we have introduced the notation $A(z,w) = \langle A(z) - A(w)\rangle/(z-w)$ for A equal to R, w_α and T. Note that the field redefinition of (4.33) does not affect the single trace contribution, as it already comes with order G^2. The field redefinition introduces a correction of order G^4 for the above single trace quantity, which vanishes in the chiral ring. Combining (4.39), (4.40) and the first equation of (4.34), one obtains the following equation for the connected correlation functions:

$$(2\langle R(z)\rangle - I(z))\langle R(z)T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c - \frac{1}{3}G^2\partial_w R(z,w) = 0 \tag{4.41}$$

Using estimates as the ones performed in [12] and discussed in Appendix B, shows that the completely connected three-point functions vanish. We will not need these estimates when we discussing the solution for all genera, as will be seen in the next section. The method to obtain the other relevant connected correlation functions is similar. We need to consider a general variation of the form

$$\delta\Phi_{ij} = A_{ij}(z)B(w) \tag{4.42}$$

where A can be $\mathcal{R}$, ρ_α or $\mathcal{T}$ and B can be R, w_α or T. The resulting generalized Konishi anomaly equations can be derived in the same way as above and are written as the following matrix equation:

$$\begin{bmatrix} M(z) & 2\langle T(z)\rangle & 2\langle w^\alpha(z)\rangle \\ 0 & M(z) & 0 \\ 0 & 2\langle w_\alpha(z)\rangle & M(z) \end{bmatrix} \begin{bmatrix} \langle T(z)T(w)\rangle_c & \langle T(z)R(w)\rangle_c & \langle T(z)w_\beta(w)\rangle_c \\ \langle R(z)T(w)\rangle_c & \langle R(z)R(w)\rangle_c & \langle R(z)w_\beta(w)\rangle_c \\ \langle w_\alpha(z)T(w)\rangle_c & \langle w_\alpha(z)R(w)\rangle_c & \langle w_\alpha(z)w_\beta(w)\rangle_c \end{bmatrix} =$$
$$= \partial_w \begin{bmatrix} \frac{1}{3}G^2T(z,w) & \frac{1}{3}G^2R(z,w) & \frac{1}{3}G^2w_\beta(z,w) \\ \frac{1}{3}G^2R(z,w) & 16F^2R(z,w) & -8\left(F\cdot G\right)_\beta R(z,w) \\ \frac{1}{3}G^2w_\alpha(z,w) & -8\left(F\cdot G\right)_\alpha R(z,w) & Q_{\alpha\beta} \end{bmatrix} \tag{4.43}$$

Where we have introduced the operators $M(z) = (2\langle R(z)\rangle - I(z))$ and $Q_{\alpha\beta} = \frac{5}{3}G^2\epsilon_{\alpha\beta}R(z,w) - 8\left(F\cdot G\right)_\alpha w_\beta(z,w)$, here the second term is also proportional to $\epsilon_{\alpha\beta}$ using the chiral ring equation (4.20). To obtain such equations we have used the chiral ring relations extensively. We have also dropped all connected three-point functions. The order at which they occur can be inferred using the estimates of [12]: they either vanish in the chiral ring or occur at a higher order. For genus one we are interested in the solution at order G^2, F^2 or $(F\cdot G)$. The equation in (4.43) is of the form

$$\mathcal{M}(z)N(z,w) = \partial_w K(z,w) \tag{4.44}$$

with $\mathcal{M}(z)$ the first matrix operator appearing on the left hand side of eq. (4.43), $N(z,w)$ and $K(z,w)$ satisfy $N(z,w) = N^t(w,z)$ and $K(z,w) = K^t(w,z)$. The non-trivial consistency condition (integrability condition) is then given by

$$(\partial_w K(z,w))\mathcal{M}^t(w) = \mathcal{M}(z)\partial_z K(z,w). \tag{4.45}$$

By using the same ideas outlined in the last Chapter, it can be shown that the above integrability condition is satisfied. The existence of solutions for the connected two point functions, equation (4.43), is guaranteed by the fulfillment of the integrability conditions (4.45). However, as familiar also in matrix models, these solutions suffer from ambiguities, in the form of a finite set of parameters. These ambiguities will be fixed by the physical requirement that the contour integrals around the branch cuts of the connected two point-functions, both in the z and w planes, vanish separately. The reason for this is that the following operator equations hold:

$$\frac{1}{2\pi i}\int_{C_i} dz R(z) = S_i, \quad \frac{1}{2\pi i}\int_{C_i} dw T(w) = N_i, \quad \frac{1}{2\pi i}\int_{C_i} dz w_\alpha(z) = w_{\alpha\ i}. \tag{4.46}$$

where S_i is the chiral superfield whose lowest component is the gaugino bilinear in the i-th gauge group factor in the broken phase $U(N) \to \prod_{i=1}^n U(N_i)$ and $w_{\alpha\ i}$ is the $U(1)$

chiral gauge superfield of the $U(N_i)$ subgroup. Since these fields are background fields, in the connected correlation functions the contour integrals around the branch cuts must vanish. This requirement makes the solutions of the equations of (4.43) unique.

We now write the solution of all connected two-point functions in terms of a single function. Consider the equation for the correlation function $\langle R(z)R(w)\rangle_c$ from the matrix equation (4.43)

$$M\langle R(z)R(w)\rangle_c - 16F^2\partial_w R(z,w) = 0. \tag{4.47}$$

The above equation is a linear equation with an inhomogeneous term which is proportional to F^2. Let the solution be given by

$$\langle R(z)R(w)\rangle_c = -16F^2 H^{(1)}(z,w) \tag{4.48}$$

where $H^{(1)}(z,w)$ solves the following equation [5]

$$MH^{(1)}(z,w) + \partial_w R(z,w) = 0, \tag{4.49}$$

here the superscript in H refers to the fact that we are working at genus one. It is possible to define such a function, as the inversion of operator M is unambiguous. In [12] it was shown that the function $H(z,w)$ is symmetric in z and w. We now illustrate how the connected two point function $\langle T(z)T(w)\rangle_c$, can be expressed in terms of the function $H^{(1)}(z,w)$. From (4.43), the equation satisfied by this correlator is given by

$$M(z)\langle T(z)T(w)\rangle_c + 2\langle T(z)\rangle\langle R(z)T(w)\rangle_c + 2\langle w^\alpha(z)\rangle\langle w_\alpha(z)T(w)\rangle_c = \frac{G^2}{3}\partial_w T(z,w) \tag{4.50}$$

Now let us define the following operators:

$$D = N_i\frac{\partial}{\partial S_i}, \qquad \delta_\alpha = w_{\alpha\, i}\frac{\partial}{\partial S_i} \tag{4.51}$$

Applying the operator $(D+\delta^2/2)$ on equation (4.49) we see that it reduces to (4.50) if one uses the relations $\langle T(z)\rangle = (D+\frac{1}{2}\delta^2)\langle R(z)\rangle$, and $\langle w_\alpha(z)\rangle = \delta_\alpha\langle R(z)\rangle$. Though these relations are valid only to the zeroth order, it is possible to use them here since corrections occur at higher order than G^2 in (4.50). Therefore, using the uniqueness of solutions of the equations involving the operator M, we find

$$\langle T(z)T(w)\rangle_c = -\frac{G^2}{3}(D+\frac{1}{2}\delta^2)H^{(1)} \tag{4.52}$$

We can find all other two point functions in a similar manner, at genus one order they are given by

$$\begin{aligned} \langle R(z)R(w)\rangle_c &= -16F^2H^{(1)}, & \langle R(z)T(w)\rangle_c &= -\frac{1}{3}G^2H^{(1)}, \\ \langle w_\alpha(z)w_\beta(w)\rangle_c &= (-\frac{5}{3}G^2H^{(1)} + 8F^2DH^{(1)})\epsilon_{\alpha\beta}, & \langle R(z)w_\alpha(w)\rangle_c &= 8(F\cdot G)_\alpha H^{(1)}, \\ \langle T(z)w_\alpha(w)\rangle_c &= -\frac{1}{3}G^2\delta_\alpha H^{(1)}, & \langle T(z)T(w)\rangle_c &= -\frac{1}{3}G^2(D+\frac{1}{2}\delta^2)H^{(1)}. \end{aligned} \tag{4.53}$$

[5]This definition of H differs from the one given in Chapter 3 by a sign.

4.2.2 One Point Functions at Genus One

To solve for the corrections to the one point functions we first expand the one point functions as

$$\langle R(z)\rangle = R^{(0)}(z) + R^{(1)}(z),\quad \langle w_\alpha(z)\rangle = w_\alpha^{(0)}(z) + w_\alpha^{(1)}(z),\quad \langle T(z)\rangle = T^{(0)}(z) + T^{(1)}(z), \tag{4.54}$$

where the terms with the superscript 0 denote the zeroth order contribution in F^2, G^2 and $(F \cdot G)$ and terms with the superscript 1 denote the first order contribution. Substituting the above expansions in (4.34) we obtain the following equations for the genus one contributions

$$\begin{aligned}
&(2R^{(0)}(z) - I(z))R^{(1)}(z) + 16F^2 H^{(1)}(z,z) = 0\\
&(2R^{(0)}(z) - I(z))w_\alpha^{(1)}(z) - 16(F\cdot G)_\alpha H^{(1)}(z,z) + 2R^{(1)}(z)w_\alpha^{(0)}(z) = 0\\
&(2R^{(0)}(z) - I(z))T^{(1)}(z) + 4G^2 H^{(1)}(z,z) -\\
&\qquad -16F^2 D H^{(1)}(z,z) + 2w^{(0)\,\alpha}(z)w_\alpha^{(1)}(z) + 2R^{(1)}(z)T^{(0)}(z) = 0
\end{aligned} \tag{4.55}$$

All the above equations are linear in the one point functions with different inhomogeneous terms. The linear operator is $M = (2R^{(0)} - I(z))$, therefore we consider the equation

$$(2R^{(0)}(z) - I(z))\Omega^{(1)}(z) = H^{(1)}(z,z) \tag{4.56}$$

From the definition of the operator $I(z)$ the solution of this equation is given by

$$\Omega^{(1)}(z) = \frac{H^{(1)}(z,z) + c^{(1)}(z)}{2R_0(z) - W'(z)}, \tag{4.57}$$

here $c^{(1)}$ is the finite ambiguity in the solution, a polynomial of degree $n-2$. This ambiguity is again fixed by the physical requirement that the contour integral of $\Omega^{(1)}$, which is proportional to the genus one correction to anyone one-point function of interest, vanishes around branch cuts. This requirement ensures that operator equations (4.46) are valid and the background fields S_i, N_i and $w_{\alpha i}$ do not receive any G^2, F^2 or $(F \cdot G)$ corrections. Using these inputs, the genus one corrections to the one-point functions of interest are given by

$$\begin{aligned}
R^{(1)}(z) &= 16F^2\Omega^{(1)}(z)\\
w_\alpha^{(1)}(z) &= -16(F\cdot G)_\alpha \Omega^{(1)}(z)\\
T^{(1)}(z) &= 4G^2\Omega^{(1)}(z) - 16F^2 D\Omega^{(1)}(z)
\end{aligned} \tag{4.58}$$

At this juncture it is worthwhile to point out the difference in the results had we not used the field redefinition in (4.33). We would, in fact, be left with genus zero contributions, in addition to the corrections found in (4.58). This is seen as follows: without the field redefinition there will be additional terms proportional to G^2, multiplying products of one point functions at the zeroth order. For example, in the last equation of (4.32) there is a term proportional to $G^2(T^{(0)}(z))^2$ which, since it goes like N^2 , it represents a genuine genus 0 contribution,

4.3 Solution at All Genera

In this section we obtain the complete solution for the one point functions R, w_α and T for all genera. From (4.34) we see that we need the connected two point functions $\langle RR\rangle_c, \langle Rw_\alpha\rangle_c, \langle RT\rangle_c$ and $\langle w^\alpha w_\alpha\rangle_c$ to solve for the one point functions. Our strategy in this section is to first obtain equations for generating functionals for any arbitrary correlator using the generalized Konishi anomaly. This can be done by introducing sources coupled to the operators of interest. The equations constraining the generating functionals turn out to be a set of integro-differential equations. From these we solve for the relevant two-point functions which in turn enables us to obtain the one point-functions of interest. We then compare the results with the matrix model. In the appendix we demonstrate that the integro-partial differential equations constraining the generating functional are consistent and provide the details of the solution for all connected two point functions.

4.3.1 Generating Functionals for Connected Correlators.

To obtain equations for the generating functionals for connected correlators we extend the method used to obtain equations for the connected two point functions. Consider the following generating functional for the operators of our interest

$$\langle Z\rangle = \langle \exp\left[\int dw\, (j_R(w)R(w) + j_w^\alpha(w)w_\alpha(w) + j_T(w)T(w))\right]\rangle \qquad (4.59)$$

The generating functional is a function of three variables j_R, j_w^α, j_T. There are three equations which constrain Z which are obtained by considering the following three variations

$$\delta\Phi_{ji} = \mathcal{R}_{ji}(z)Z, \quad \delta\Phi_{ji} = \eta^\alpha\rho_{\alpha ji}(z)Z, \quad \delta\Phi_{ji} = \mathcal{T}_{ji}(z)Z, \qquad (4.60)$$

here η^α is an arbitrary spinor. We will now derive the constraint imposed by the Konishi anomaly for the first variation. The derivation proceeds along the lines followed to derive the equations for the connected two-point functions. The Jacobian of the first variation in (4.60) is given by

$$\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} = \frac{\delta\mathcal{R}_{ji}}{\delta\Phi_{kl}}Z + \mathcal{R}_{ji}(z)\frac{\delta Z}{\delta\Phi_{kl}} \qquad (4.61)$$

where

$$\begin{aligned}\frac{1}{Z}\frac{\delta Z}{\delta\Phi_{kl}} &= \sum_m \int dw\, (j_R(w)\mathcal{R}_{mk}(w)\mathcal{T}_{lm}(w) + j_w^\alpha(w)w_{\alpha mk}(w)\mathcal{T}_{lm}(w) \\ &+ \quad j_T(w)\mathcal{T}_{mk}(w)\mathcal{T}_{lm}(w))\end{aligned} \qquad (4.62)$$

Note that the expression contains terms similar to the Jacobians in (4.38). Therefore, we can use the anomaly equations constraining the two-point functions in order to obtain the equations constraining the generating functional. The first term of the Jacobian in (4.62) gives rise to a term which is the product of the first anomaly equation in (4.34) times Z, while the second term of the Jacobian in (4.62) gives rise to inhomogeneous single trace terms similar to those in (4.40). Using these considerations, the anomaly equation for the above variation can be shown to reduce to

$$\begin{aligned} &\langle R(z)R(z)Z\rangle - I(z)\langle R(z)Z\rangle \\ &- \left\langle \int dw \left(16F^2 j_R \partial_w R(z,w) + \frac{G^2}{3} j_T \partial_w R(z,w) - 8 j_w^\alpha (F\cdot G)_\alpha \partial_w R(z,w)\right) Z \right\rangle = 0 \end{aligned} \tag{4.63}$$

where $R(z,w) = (R(z) - R(w))/(z-w)$, $T(z,w) = (T(z) - T(w))/(z-w)$. We can write the above equation as an integro-differential equation by introducing functional derivatives with respect to the sources on the generating functional. We define

$$\begin{aligned} \partial_R^z &= \frac{\delta}{\delta j_R(z)}, \quad \partial_T^z = \frac{\delta}{\delta j_T(z)}, \quad \partial_\alpha^z = \frac{\delta}{\delta j_w^\alpha(z)}, \\ O_R^{(z,w)} &= \partial_w \left(\frac{1}{z-w}\left[\frac{\delta}{\delta j_R(z)} - \frac{\delta}{\delta j_R(w)}\right]\right), \\ O_T^{(z,w)} &= \partial_w \left(\frac{1}{z-w}\left[\frac{\delta}{\delta j_T(z)} - \frac{\delta}{\delta j_T(w)}\right]\right), \\ O_\alpha^{(z,w)} &= \partial_w \left(\frac{1}{z-w}\left[\frac{\delta}{\delta j_w^\alpha(z)} - \frac{\delta}{\delta j_w^\alpha(w)}\right]\right). \end{aligned} \tag{4.64}$$

Now writing (4.63) in terms of these derivatives gives

$$\left[\partial_R^2 - I(z)\partial_R - \int dw \left(16F^2 j_R O_R + \frac{G^2}{3} j_T O_R - 8 j_w^\alpha (F\cdot G)_\alpha O_R\right)\right] Z = 0 \tag{4.65}$$

here we have suppressed the superscripts z, w in the derivatives for clarity of notation. The generalized Konishi anomaly constraints from the other two variations in (4.60) are given by

$$\begin{aligned} &[2\partial_R\partial_\alpha - I(z)\partial_\alpha \\ &+ \int dw \left(\left(8 j_R (F\cdot G)_\alpha + \frac{5}{3}G^2 j_{w\alpha}\right)O_R - \left(\frac{G^2}{3} j_T + 8 j_w^\beta (F\cdot G)_\beta\right)O_\alpha\right)\Bigg] Z = 0 \\ &\left[2\partial_R\partial_T - I(z)\partial_T + \partial^\alpha\partial_\alpha - \frac{G^2}{3}\int dw\,(j_R O_R + j_T O_T + j_w^\alpha O_\alpha)\right] Z = 0 \end{aligned} \tag{4.66}$$

The above equations form a set of closed integro-partial differential equations which determine Z

For the connected two point functions of our interest it is sufficient to consider the following ansatz for the generating functional

$$Z = \exp(M) = \exp\left(M_R(j_R) + j_T M_T(j_R) + j_w^\alpha M_\alpha(j_R) + \frac{1}{2} j_w^\alpha j_w^\beta M_{\alpha\beta}(j_R)\right). \tag{4.67}$$

Here the product $j_T M_T$ is understood to mean $\int dz j_T(z) M_T(z)$, similarly for other products in the above expression. In a following section we will generalize this to include all two point functions. Note that for the various cumulants in the ansatz we have allowed an arbitrary dependence of j_R while we allow only a finite set of moments in j_T and j^α_w. As we will see subsequently, it is possible that higher moments in j_R are non vanishing, but higher moments in the other currents truncate in the chiral ring. An indication of this is clear form the nature of the two point functions at genus one in (4.53). Only the $\langle RR\rangle_c$ correlator is a function of F^2 alone, all the others involve powers of G, thus they will truncate at some order in the chiral ring. With this ansatz the connected two-point functions of interest are given by

$$\begin{aligned} \langle R(z)R(w)\rangle_c &= \partial^z_R\partial^w_R M_R(j_R), \qquad & \langle R(z)w_\alpha(w)\rangle_c &= \partial^z_R M_\alpha(w, j_R), \\ \langle R(z)T(w)\rangle_c &= \partial^z_R M_T(w, j_R), \qquad & \langle w_\alpha(z)w_\beta(w)\rangle_c &= M_{\alpha\beta}(z, w, j_R). \end{aligned} \tag{4.68}$$

The three equations of (4.65) and (4.66) in terms of the cumulant generating functional M become

$$\begin{aligned} &(\partial_R M)^2 + \big[\partial_R^2 - I(z)\partial_R \\ &- \int dw \left(16F^2 j_R O_R + \frac{G^2}{3} j_T O_R - 8j^\alpha_w (F\cdot G)_\alpha O_R\right)\bigg] M = 0 \end{aligned} \tag{4.69}$$

$$\begin{aligned} &2\partial_R M\partial_\alpha M + [2\partial_R\partial_\alpha - I(z)\partial_\alpha \\ &+ \int dw \left((8j_R(F\cdot G)_\alpha + \frac{5}{3}G^2 j_{w\alpha})O_R - (\frac{G^2}{3}j_T + 8j^\beta_w (F\cdot G)_\beta)O_\alpha\right)\bigg] M = 0 \end{aligned} \tag{4.70}$$

$$\begin{aligned} &2\partial_R M\partial_T M + \partial^\alpha M\partial_\alpha M \\ &+ \left[2\partial_R\partial_T - I(z)\partial_T + \hat\partial^\alpha\hat\partial_\alpha - \frac{G^2}{3}\int dw\,(j_R O_R + j_T O_T + j^\alpha_w O_\alpha)\right] M = 0 \end{aligned} \tag{4.71}$$

As M_R is only a function of j_R, the equation determining M_R can be obtained from (4.69), by setting $j_T = j_\alpha = 0$. This is given by

$$(\partial_R M_R)^2 + \left[\partial_R^2 - I(z)\partial_R - 16F^2\int dw j_R O_R\right] M_R = 0 \tag{4.72}$$

We will see in the following that the above equation is identical to the generating functional equation for the resolvent of the matrix model. In fact the F^2 expansion of the above equation can be identified with the $1/\hat N^2$ expansion of the equation of the resolvent of the matrix model. We now find the solutions of the connected two-point functions of interest, by performing a similar analysis to that of the genus one case. In fact, the solution is a direct generalization of that case. To obtain the connected two point function $\langle RR\rangle_c$ differentiate (4.69) by ∂^w_R, where w refers to another point in the complex plane and set $j_T = j_\alpha = 0$. We obtain the following equation

$$\begin{aligned} (2\partial^z_R M_R - I(z))\partial^z_R\partial^w_R M_R + (\partial^z_R)^2\partial^w_R M_R - 16F^2 O_R^{(z,w)} M_R \\ -16F^2\int dw' j_R O_R^{(z,w')}\partial^w_R M_R = 0 \end{aligned} \tag{4.73}$$

This forms a basic equation out of which the solutions for the other cumulant generating functions in (4.67) will be constructed. The equation is linear in $\partial_R^z \partial_R^w M_R$ with an inhomogeneous proportional to F^2, therefore the solution is proportional to F^2. Let the solution of (4.73) be given by

$$\langle R(z)R(w)\rangle_c = \partial_R^z \partial_R^w M_R = -16F^2 H(z,w), \tag{4.74}$$

where $H(z,w)$ is the solution of the following equation

$$\begin{aligned} &(2\partial_R^z M_R - I(z))H(z,w) + \partial_R^z H(z,w) \\ +O_R^{(z,w)} M_R - 16F^2 &\int dw' j_R \partial_{w'} \left(\frac{H(z,w) - H(w',w)}{z-w'}\right) = 0 \end{aligned} \tag{4.75}$$

By this definition $H(z,w)$ is symmetric in z and w. We have assumed that the solutions of these equations are unique. An argument in favour of this is as follows. For the lowest order in the genus expansion (setting $j_R = 0$, and $\partial_R^z H(z,w) = 0$) these equations reduce to the ones of section 3.3, which were studied in [12]. It was shown there that, by demanding the vanishing of the integrals of the the various connected two-point functions around the branch cuts both in the z and w plane, the solution is unique. The equation in (4.75) is a generalization of those equations. One can envisage a generalization of those arguments for these equations, proving that the solution of (4.75) is unique. To obtain the connected two point function $\langle R(z)w_\alpha(w)\rangle_c$ we differentiate (4.69) with respect to $\hat{\partial}_\alpha^w$ and then set $j_T = j_\alpha = 0$, to obtain

$$\begin{aligned} (2\partial_R^z M_R - I(z))\partial_R^z M_\alpha(w) + \partial_R^z \partial_R^z M_\alpha(w) + 8(F\cdot G)_\alpha O_R^{(z,w)} M_R \\ -16F^2 \int dw' j_R O_R^{(z,w')} M_\alpha(w) = 0 \end{aligned} \tag{4.76}$$

Again comparing (4.75) and (4.76) we obtain

$$\langle R(z)w_\alpha(w)\rangle_c = \partial_R^z M_\alpha(w) = 8(F\cdot G)_\alpha H(z,w) \tag{4.77}$$

At this point one might wonder if the equation for the above correlator obtained by differentiating (4.70) by ∂_R^z will reduce to (4.76). In Appendix C we show that this is indeed the case and that the set of integro-differential equations in (4.69), (4.70) and (4.71) is in fact consistent. In order to obtain the correlator $\langle R(z)T(w)\rangle_c$ we differentiate (4.69) by ∂_T^w and then set $j_T = j_\alpha = 0$, obtaining

$$\begin{aligned} (2\partial_R^z M_R - I(z))\partial_R^z M_T(w) + \partial_R^z \partial_R^z M_T(w) - \frac{G^2}{3} O_R^{(z,w)} M_T(w) \\ -16F^2 \int dw' j_R O_R^{(z,w')} M_T(w) = 0 \end{aligned} \tag{4.78}$$

Comparing (4.78) and (4.75) yields

$$\langle R(z)T(w)\rangle_c = \partial_R^z M_T(w) = -\frac{G^2}{3} H(z,w) \tag{4.79}$$

Now we differentiate (4.70) with respect to $\partial_\beta^{\bar{z}}$ and then set $j_\alpha, j_T = 0$ to obtain the connected correlator $\langle w_\alpha(z) w_\beta(w)\rangle_c$. We get

$$\begin{aligned}&(2\partial_R^{\bar{z}} M_R - I(z))M_{\alpha\beta}(z,w) + 2\partial_R^{\bar{z}} M_\beta(w) M_\alpha(z) \\ +\ & 2\partial_R^{\bar{z}} M_{\alpha\beta}(z,w) - 8(F\cdot G)_\alpha \int dw' j_R O_R^{(z,w')} M_\beta(w) - \frac{5}{3} G^2 \epsilon_{\alpha\beta} O_R(w,z) M_R \\ -\ & 8(F\cdot G)_\beta \partial_w \left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) = 0\end{aligned} \tag{4.80}$$

The last term in the above equation can be written as

$$\begin{aligned}-8(F\cdot G)_\beta \partial_w \left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) &= 8\epsilon_{\alpha\beta} F^2 \partial_w \left(\frac{M_T(z) - M_T(w)}{z-w}\right), \\ &= 8\epsilon_{\alpha\beta} F^2 D O_R^{(z,w)} M_R\end{aligned} \tag{4.81}$$

In the last equality we have used the relation, $T = (D + 1/2\delta^2)R$ which is valid at the zeroth order and the chiral ring relation (4.11) and (4.20). The second term in (4.80) contains $\partial_R^{\bar{z}} M_\beta(w)$. Substituting (4.77) for this we and using the equation (4.11) we see that this term vanishes. Using (4.77) we see that term containing the integral in (4.80) is proportional to $(F\cdot G)_\alpha (F\cdot G)_\beta$, which also vanishes using (4.12). For clarity we write down (4.80) after dropping these terms

$$\begin{aligned}&(2\partial_R^{\bar{z}} M_R - I(z))M_{\alpha\beta}(z,w) + 2\partial_R^{\bar{z}} M_{\alpha\beta}(z,w) \\ -\ & \epsilon_{\alpha\beta}\frac{5}{3} G^2 O_R(w,z) M_R + 8\epsilon_{\alpha\beta} F^2 D O_R^{(z,w)} M_R = 0\end{aligned} \tag{4.82}$$

There are two inhomogeneous terms in the above equation which motivates the following ansatz

$$M_{\alpha\beta}(z,w) = \epsilon_{\alpha\beta}\left(-\frac{5}{3} G^2 H(z,w) + 8F^2 D H(z,w)\right) \tag{4.83}$$

With this ansatz the term containing the connected three-point function vanishes, because it contains the derivative $\partial_R^{\bar{z}}$ which contains an extra factor of F^2. This is seen as follows: (4.75) is identical to the corresponding matrix model equation for the connected two point function of the resolvent, and from a t'Hooft counting analysis, the connected three point function is down by a factor of $1/\hat{N}^2$, which in the gauge theory implies that there is an extra factor of F^2 since the F^2 expansion of (4.72) and (4.75) is identical to the $1/\hat{N}^2$ expansion of the matrix model. Comparing (4.75), we see that (4.83) solves equation (4.82), as with this ansatz the connected three point function in (4.75) also vanishes, since it occurs with an extra factor of F^2.

4.3.2 Solutions for the One Point Functions.

Having obtained the required connected two point functions we can now solve for the corrections to the one point functions. The analysis is identical to the genus one case. We first expand the one point functions about the zeroth order as

$$\langle R(z)\rangle = R^{(0)}(z) + \tilde{R}(z), \quad \langle w_\alpha(z)\rangle = w_\alpha^{(0)}(z) + \tilde{w}_\alpha(z), \quad \langle T(z)\rangle = T^{(0)}(z) + \tilde{T}(z). \tag{4.84}$$

here $\tilde{R}$, $\tilde{w}$ and $\tilde{T}$ denote corrections to the zeroth order solution. Now substituting the above expansion in (4.34) we obtain the same equations as (4.55) but with $H^{(1)}(z,z)$ replaced with the full connected two point function $H(z,z)$, which is the solution of (4.75). These are given below

$$\begin{aligned}&(2R^{(0)} - I(z))\tilde{R}(z) - 16F^2H(z,z) = 0,\\&(2R^{(0)} - I(z))\tilde{w}_\alpha(z) + 16(F\cdot G)_\alpha H(z,z) + 2\tilde{R}w^{(0)}_\alpha = 0,\\&(2R^{(0)} - I(z))\tilde{T}(z) - 4G^2H(z,z) + 16F^2DH(z,z) + 2\tilde{w}^\alpha w^{(0)}_\alpha + 2T^{(0)}\tilde{R} = 0\end{aligned} \tag{4.85}$$

The corrections are given by

$$\begin{aligned}\tilde{R}(z) = 16F^2\Omega(z), \quad \tilde{w}_\alpha = -16(F\cdot G)_\alpha\Omega(z),\\ \tilde{T}(z) = 4G^2\Omega(z) - 16F^2D\Omega(z)\end{aligned} \tag{4.86}$$

where

$$\Omega(z) = \frac{1}{2R^{(0)} - W'(z)}(H(z,z) + c(z)) \tag{4.87}$$

where $c(z)$ is a polynomial of degree $n-2$ and is uniquely determined by the requirement that the contour integrals of $\Omega(z)$ around each branch cut vanishes. Note that the last term of the second equation in (4.85) vanishes, because after substituting the solution for $\tilde{R}$, we see that that term is proportional to $F^2 w_\alpha$, which is trivial in the chiral ring. In the section 4.3.4 we show that this is exactly the answer obtained from the matrix model.

4.3.3 Shift Invariance of the Anomaly Equations

In this section we show that we can assemble all the equations for the generating functions given in (4.69), (4.70) and (4.71) into one superfield equation by introducing the auxiliary fermionic coordinate ψ_α. We first assemble the loop variables as

$$\mathcal{R}(z,\psi) = R(z) + \psi^\alpha w_\alpha(z) - \frac{1}{2}\psi^2 T(z) \tag{4.88}$$

that generalize equation (2.67) Note that with this notation the generalized Konishi anomaly equations (4.34) is given by

$$\langle\mathcal{R}^2(z,\psi)\rangle - I(z)\langle\mathcal{R}(z,\psi)\rangle = 0 \tag{4.89}$$

These generalized Konishi anomaly equations are the same as in the case of $\mathcal{N}=1$ gauge theories in flat space with no graviphoton background. In that case, the connected two-point functions in (4.89) vanish, and this shift symmetry of the equations implied that the superpotenial could be written as the integral [8]

$$W_{\text{eff}} = \int d^2\psi\mathcal{F}_p\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right) \tag{4.90}$$

We would like to demonstrate that this is also true for the case of $\mathcal{N}=1$ theories studied here.

We first show that the equations for the connected correlators can be written as a supermultiplet in the ψ space. The background fields, the $\mathcal{N}=1$ Weyl multiplet and the $\mathcal{N}=1$ spin 3/2 multiplet can be assembled as

$$\mathcal{H}_{\alpha\beta}(\psi) = F_{\alpha\beta} - \frac{1}{2}\psi^\gamma G_{\alpha\beta\gamma} \tag{4.91}$$

To write the generating functional in the superspace ψ we introduce the multiplet of the sources as follows

$$\mathcal{J}(z,\psi) = J_T + \psi^\alpha j_\alpha - \frac{\psi^2}{2} j_R. \tag{4.92}$$

Then the generating functional can be written as

$$Z = \exp\left(\int d^2\psi dw \mathcal{J}(w,\psi)\mathcal{R}(w,\psi)\right), \tag{4.93}$$

here we have normalized $\int d^2\psi\psi^2 = -2$. To obtain the various correlators from the generating functional we introduce the following derivative in the supermultiplet space

$$\partial^z_{\mathcal{R}} = \partial^z_R + \psi^\alpha\partial^z_\alpha - \frac{\psi^2}{2}\partial^z_T \tag{4.94}$$

With this notation the set of equations (4.69), (4.70) and (4.71) reduce to

$$\begin{aligned}\Big[(\partial^z_{\mathcal{R}})^2 - I(z)\partial^z_{\mathcal{R}} - \frac{2}{3}\hat{D}^2(\mathcal{H}^2)\int dw \mathcal{J}(w,\psi) O^{(z,w)}_{\mathcal{R}} \\ +8\int dw \hat{D}^2(\mathcal{H}^{\alpha\beta}\mathcal{J}(w,\psi))\mathcal{H}_{\alpha\beta}O^{(z,w)}_{\mathcal{R}}\Big] Z = 0\end{aligned} \tag{4.95}$$

where

$$\hat{D}^2 = \epsilon^{\alpha\beta}\frac{\partial}{\partial\psi^\beta}\frac{\partial}{\partial\psi^\alpha} \tag{4.96}$$

In (4.95) $O_{\mathcal{R}}$ is defined using the super derivative of (4.94). In deriving (4.95) we have used the identities (4.8), (4.11) and (4.20). This implies that the solutions of the loop equations can be written in a shift invariant way. It is easy to verify this from the solutions we have found in the previous subsection. The corrections to the one point functions can be written as

$$\begin{aligned}\tilde{\mathcal{R}}(\psi,z) &= 16\mathcal{H}^2\Omega(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i, \mathcal{H}^2), \\ &= 16F^2\Omega(S_i,F^2) - 16\psi^\alpha(F\cdot G)_\alpha\Omega(S_i,0) \\ &- \frac{1}{2}\psi^2\left(4G^2\Omega(S_i,0) - 16F^2 D\Omega(S_i,0)\right)\end{aligned} \tag{4.97}$$

To obtain the expansion in the second line we have used the chiral ring relation $F^2 w_{\alpha i} = 0$ and $2F_{\alpha\beta}N_i + G_{\alpha\beta\gamma}w^\alpha_i = 0$. Here the dependence of F^2 in the last two terms are set to zero as the expansion in F truncates in the chiral ring for these terms. Since the solutions to the correlators in $\mathcal{R}$ can be written as a supermulitplet in ψ, the corrections to the super potential must be written as

$$\tilde{W}_{\text{eff}} = \int d^2\psi \mathcal{H}^2 \mathcal{F}\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right) \tag{4.98}$$

This motivates the completion of the gravitational corrections to the superpotential to as

$$\tilde{W}_{\text{eff}} = \int d^2\psi \mathcal{H}^{2g} \mathcal{F}_g\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right) \tag{4.99}$$

which is in agreement with what is obtained from the closed string duality with ψ playing the role of the second $\mathcal{N} = 2$ superspace coordinate. However because of the identities in the chiral ring all terms for $g > 1$ are trivial form $\mathcal{N} = 1$ point of view. Nevertheless, the generating functional $\langle R(z)\rangle$ sees the complete genus expansion and is identified with the matrix model genus expansion as we will see in the next subsection.

4.3.4 Comparison with the Matrix Model Results.

Consider a hermitian matrix model with an action given by

$$S = \frac{\hat{N}}{g_m}\sum_k \frac{g_k}{k}\operatorname{Tr} M^k \equiv \frac{\hat{N}}{g_m}W, \tag{4.100}$$

where M is a hermitian $\hat{N}\times\hat{N}$ matrix. The basic loop equations for the resolvent is given by

$$\langle \Omega_m(z)\Omega_m(z)\rangle - I(z)\langle\Omega_m(z)\rangle = 0 \tag{4.101}$$

here Ω_m is the matrix model resolvent given by

$$\Omega_m(z) = \frac{g_m}{\hat{N}}\operatorname{Tr}\left(\frac{1}{z-M}\right) \tag{4.102}$$

We now obtain the loop equations satisfied by the variation

$$\delta M_{ji} = \Omega_{m\,ji}\exp\left(\int dw J(w)\Omega_m(w)\right) = \Omega_{m\,ji}Z_m \tag{4.103}$$

here Z_m is the generating functional for the n-point functions for the resolvent of the matrix model. The loop equations for the variation in (4.103) is given by

$$\langle\left[\Omega_m^2(z) - I(z)\Omega_m(z) + (\frac{g_m}{\hat{N}})^2\int dw J\partial_w\left(\frac{\Omega_m(z)-\Omega_m(w)}{z-w}\right)\right]Z_m\rangle = 0 \tag{4.104}$$

Writing this loop equation by introduction functional derivatives in the current J we get

$$\left[\partial_J^2 - I(z)\partial_J + (\frac{g_m}{\hat{N}})^2\int dw J O_J\right]Z_m = 0 \tag{4.105}$$

To obtain equations for the connected correlators we introduce the cumulant generating functional $Z_m = \exp(M_m)$. The cumulant generating functional satisfies the following equation

$$(\partial_J M_m)^2 + \left[\partial_J^2 - I(z)\partial_J + (\frac{g_m}{\hat{N}})^2 \int dw J O_J\right] M_m = 0 \tag{4.106}$$

This equation is identical to the equation (4.72) which is satisfied by cumulant generating functional M_R of the gauge theory. The F^2 expansion in the gauge theory is analogous to the $1/\hat{N}^2$ expansion in the matrix model. The equation for the connected two point function is obtained by differentiating the above equation with ∂_J^w, which is given by

$$\begin{aligned}(2\partial_J^z M_m - I(z))\partial_J^z\partial_J^w M_m + \partial_J^z\partial_J^z\partial_J^w M_m + (\frac{g_m}{\hat{N}})^2 O_J^{(z,w)} \\ +(\frac{g_m}{\hat{N}})^2 \int dw' J O_J^{(z,w')}\partial_J^w M_m = 0\end{aligned} \tag{4.107}$$

By comparison with (4.75) we see that the solution of the connected two point function of the matrix model resolvent is given by

$$\langle \Omega_m(z)\Omega_m(w)\rangle = \partial_J^z\partial_J^w M_m = (\frac{g_m}{\hat{N}})^2 H(z,w). \tag{4.108}$$

Here again we have used the fact that the inversion of the operator $(2\partial_J^z M_m - I(z))$ is unambiguous if the ambiguity is fixed by demanding that contour integrals of the two point function around the branch cuts vanish. Note that in (4.107) using a t'Hooft counting analysis one finds that the contribution of a l loop planar Feynman graph to the three point function $\partial_J^z\partial_J^z\partial_J^w M_m$ is proportional to $(g_m/\hat{N})^{(4+l)}$. Therefore its contribution is subleading compared to the other two terms; this and the fact that (4.106) and (4.72) are identical was used in the gauge theory analysis to drop certain three point functions in the chiral ring. To find the the solution of the one point function of the resolvent we expand $\Omega_m(z) = \Omega_m^{(0)}(z) + \tilde{\Omega}_m(z)$, and substitute it in (4.104). $\Omega_m^{(0)}$ is the contribution of the planar graphs to the resolvent. The correction $\tilde{\Omega}_m(z)$ satisfies the following equation

$$((2\Omega_m^{(0)}(z) - I(z))\tilde{\Omega}_m(z) + (\frac{g_m}{\hat{N}})^2 H(z,z) = 0 \tag{4.109}$$

Demanding that the contour integral of $\tilde{\Omega}_m(z)$ around the branch cuts are vanishing ensures that the correction is identical to (4.86), therefore proving the all-orders matching between gauge theory and matrix model correlators claimed in the beginning.

Conclusions

In this thesis we have described the Konishi anomaly approach to the study of the exact effective superpotential that characterizes the low energy behavior of an $\mathcal{N} = 1$ supersymmetric gauge theory coupled with chiral matter. Moreover, we have explained how this effective superpotential is modified if the theory is defined on a gravitational, non–dynamical background. In particular, we have considered an $\mathcal{N} = 1$ background, encoding the information about the $\mathcal{N} = 1$ gravity multiplet, that is the graviton and its superpartner, the gravitino, as well as a graviphoton background, whose origin lies in the $\mathcal{N} = 2$ gravity multiplet.

The Konishi anomaly method, is completely equivalent to the Dijkgraaf–Vafa matrix model derived from String Theory, apart from the Veneziano–Yankielowicz superpotential, that, however, can be consistently added by hand, or, more formally "integrated in". Moreover, it provides a more satisfying approach to gauge theories, since it does not rely on any auxiliary concept, as the matrix model. Finally, it is simpler, since it is completely algorithmic while in the matrix model perturbative evaluation, in going up with the order of the diagrams, one has to compute the combinatorial factors by which the planar diagrams are weighted. Practically, one simply needs the Konishi anomaly that express the operator $T(z)$ in terms of the polynomial $f(z)$. The factorization properties of $f(z)$ determine the symmetry breaking pattern, since it is related by a change of variables to the glueball superfield. Then all the physical information enters in $T(z)$ through $f(z)$ and in some sense, holomorphy does the rest of the job (and that is perfectly consistent with the fact that we are computing F–terms). While the original conjecture referred to a $U(N)$ gauge theory coupled to chiral matter in the adjoint representation, that can be understood as an $\mathcal{N} = 2$ theory broken down to $\mathcal{N} = 1$ by a tree level superpotential, this procedure can be easily extended to all the classical Lie groups. We have explicitly computed effective superpotentials for $SO(N)$, $Sp(N)$ and $SU(N)$ given a tree level superpotential and some specific representation for the chiral matter. To fully exhibit the power of the Konishi anomaly techniques, we have easily computed some superpotentials up to six or seven loops. These results first appeared in [16] done in collaboration with L.F. Alday. Moreover, we showed that even from the gauge theory side it is present the ambiguity already noted in [80]; these results lead to a refinement of the conjecture in [86, 87, 88].

The second part of this thesis consists in a complete treatment of the gravitational corrections to the effective superpotential within the Konishi anomaly method. These corrections arise from two sources: the $\mathcal{N} = 1$ Weyl multiplet, including the graviton and its superpartner, and corresponding on the matrix model side to genus one corrections to the planar free energy, and the graviphoton field, that is a remnant of the

$\mathcal{N}=2$ origin of the theory, and corresponds to all genera corrections to the planar free energy of the matrix model. The first problem was already solved in [21] in purely gauge theoretical terms: in this analysis gravity enters only through a modification of the chiral ring relations that follow from standard supergravity tensor calculus and a direct gravitational contribution to the Konishi anomaly. The main characteristic of the gravitational background is the lack of factorization of correlators of the chiral ring operators, the connected part being proportional to a gravitational term. This is in perfect agreement with the matrix model expectations since there, the correlators do not factorize anymore due to non planar contributions. These ideas were further generalized in [9] in collaboration with L.F. Alday to include also the cases where the gauge theory is based on a $SO(N)$ or $Sp(N)$ gauge group and coupled to chiral matter non necessarily in the adjoint representation. These theories showed a slightly different behavior depending on their having an $\mathcal{N}=2$ origin or not.

The problem of encoding the all genera non planar corrections to the matrix model free energy in the gauge theory, was solved by L.F. Alday, J.R. David, E. Gava, K.S. Narain and the author in [20], where it was showed how to take into account the effect of a non trivial graviphoton background, that basically plays the role of the genus counting parameter in the gauge theory (more precisely the matrix model genus counting parameter gets identified with the square of the graviphoton field strength F^2). In the gauge theory side, the graviphoton enters only through a non trivial modification of the chiral ring relations, that was proved in [12]. This is reflected in the graviphoton corrections to the connected part of the correlators, which hold at every genus. We have determined the one point functions $R(z)$, $T(z)$ and $w_\alpha(z)$ to all orders in F^2 that can be used to determine the superpotential. The gauge theory resolvent $\langle R(z)\rangle$ coincides with the matrix model resolvent $\langle\Omega_m(z)\rangle$, provided we identify F^2 with the matrix model genus expansion parameter $1/\hat{N}^2$. This shows that the gauge theory in a non trivial graviphoton background encodes all the information of the full matrix model free energy.

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Appendix A

Conventions

In this thesis we have used the Wess and Bagger conventions [25]. Here we collect some useful formula. The reader is referred to [25] for a more complete treatment.

We use a "mostly plus" space–time metric, $\eta_{\mu\nu} = \text{diag}\,(-1,+1,+1,+1)$.

Spinors with dotted indices transform under the $\left(0,\frac{1}{2}\right)$ representation of the Lorentz group while spinors with undotted indices transform under $\left(\frac{1}{2},0\right)$. Raising and lowering of spinor indices are done by the ϵ tensor as follows

$$W^\alpha = \epsilon^{\alpha\beta}W_\beta, \quad W_\alpha = \epsilon_{\alpha\beta}W^\beta, \qquad \epsilon^{\alpha\beta}\epsilon_{\beta\alpha} = 2, \quad \epsilon^{\alpha\beta}\epsilon_{\beta\alpha'} = \delta^\alpha_{\alpha'}. \tag{A.1}$$

Gamma matrices γ^μ satisfy the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ and are always intended in the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{A.2}$$

where the σ–matrices

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.3}$$

connect the Lorentz group with $SL(2,\mathbb{C})$. σ^μ has the index structure $\sigma^\mu_{\alpha\dot\alpha}$. Its indices are raised with the ϵ–tensor

$$\bar\sigma^{\mu\alpha\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}.$$

The generators of the Lorentz group in the spinor representation are given by

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\right) \tag{A.4}$$

$$\bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\left(\bar\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\beta} - \bar\sigma^{\nu\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\beta}\right). \tag{A.5}$$

We use the following spinor summation convention:

$$\begin{aligned} \psi\chi &= \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \bar\psi\bar\chi &= \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi. \end{aligned} \tag{A.6}$$

We define complex conjugation on Weyl spinors as

$$(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}} \qquad (\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}, \tag{A.7}$$

that implies

$$(\chi\psi)^* = (\chi^\alpha\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}. \tag{A.8}$$

Integration in superspace is defined as follows

$$\begin{aligned} \int \mathrm{d}\theta &= 0 \\ \int \mathrm{d}\theta\theta &= -1 \end{aligned} \tag{A.9}$$

We also define the following products of the $\mathcal{N}=1$ gauge multiplet and the $\mathcal{N}=1$ Weyl multiplet

$$\begin{aligned} W^2 &= W^\alpha W_\alpha = -W_\alpha W^\alpha, \\ G^2 = G^{\alpha\beta\gamma}G_{\alpha\beta\gamma}, &\qquad G_{\delta\gamma\alpha}G^{\delta\gamma}{}_\beta = -G_{\delta\gamma\beta}G^{\delta\gamma}{}_\alpha = \frac{\epsilon_{\alpha\beta}}{2}G^2. \end{aligned} \tag{A.10}$$

In the main text, the fields have been sometimes rescaled by factors such as $\sqrt{32\pi^2}$; this is explained throughout the text.

Appendix B

Estimates of Connected Part of Correlators

In this Appendix we give, following [12] an estimate of the connected part of the gauge theory correlators in an $\mathcal{N}=1$ gravitational background. This analysis complete the study of correlators at genus one performed in Chapter 3.

In the absence of gravity correlators of gauge invariant operators factorize in the chiral ring [8]. This factorization enabled one to show that the loop equations satisfied by the resolvent on the gauge theory agreed with the loop equations of the matrix model in the large N limit. On the matrix model side factorization in the loop equations was due to the the large N limit. However the correspondence of the gauge theory with the matrix model proposed by Dijkgraaf and Vafa in [1] goes beyond the large N approximation. If the $\mathcal{N}=1$ gauge theory is placed in a background gravitational field then the gravitational corrections to the F terms of the gauge theory is of the form

$$\Gamma_1 = \int d^4x d^2\theta G^2 F_1(S). \tag{B.1}$$

The Dijkgraaf and Vafa proposal states that the one can calculate F_1 from the genus one diagrams of the corresponding matrix model. It has been shown in [10] that the modification of the chiral ring in the presence of gravity allows the computation of the gravitational corrections to the F-term. The gravitational corrections enter at genus one on the gauge theory side and they reduce to the genus one diagrams of the corresponding matrix model. This implies that the loop equations of the gauge theory would not factorize in presence of gravity, as the loop equations of the matrix model do not factorize at genus one. Therefore a priori we expect that gauge invariant operators do not factorize in the presence of gravity. In this section we estimate the gravitational corrections to the connected parts of various correlators that can appear in the loop, equations using the deformation of the chiral ring in presence of gravity (3.8).

To estimate the gravitational corrections to the F-term obtained by integrating out the chiral multiplet we require the two point function the chiral scalars $\langle\Phi\Phi\rangle$. This is obtained after integrating out the antichiral scalar and it is given by

$$\langle\Phi(x,\theta)\Phi(x',\theta')\rangle = \frac{\bar{m}}{-\Box + \frac{i}{2}D^\alpha W_\alpha + iW^\alpha D_\alpha + m\bar{m}}\delta^4(x-x')\delta^2(\theta-\theta') \tag{B.2}$$

This propagator in the presence of a gravitational background was derived in [26] and in writing the above equation we have assumed that the gravitational background is on shell which allows one to set the other terms which occur in the propagator to zero, $\Box$ in (B.2) stands of the full gauge and gravitational covariant derivative. The action of W_α is dictated by the representation of Φ, in this paper we will restrict ourselves to the adjoint action. The delta functions in (B.2) refer to the full covariant delta function in curved superspace. In order to obtain the leading estimates for the connected component of various correlators due to the modification of the chiral ring it is sufficient to use the free d' Alembertian operator and a constant gaugino background. We argue this as follows, we can expand the propagator of (B.2) in a weak field as

$$\begin{aligned}\frac{\bar{m}}{-\Box + iD^\alpha W_\alpha + m\bar{m}} &= \frac{\bar{m}}{-\Box_0 + iW^\alpha D_\alpha + m\bar{m}} \\ &+ \frac{\bar{m}}{-\Box_0 + iW^\alpha D_\alpha + m\bar{m}}(\Box - \Box_0)\frac{1}{-\Box_0 + iW^\alpha D_\alpha + m\bar{m}} + \dots\end{aligned} \tag{B.3}$$

here $\Box_0$ refers to the free d'Alembertian operator. We have also dropped the terms $D^\alpha W_\alpha$ in (B.2) as we have considered a covariantly constant gaugino background. From the expansion we see that the corrections in using the free d' Alembertian operator in the propagator always occur with the factor $(\Box - \Box_0)$, which is proportional to the gravitational background and therefore subleading to the estimate obtained using the free d' Alembertian operator in the first term. One has to make a similar expansion for the covariant delta function in (B.2) and again one can see corrections in using the flat space delta function are subleading. However we will see later that if there is no gravitational contribution form the deformed chiral ring the leading estimate for the connect part of correlators arise from corrections in the propagator due to the presence of the full covariant $\Box$ and the covariant delta function. For the free d'Alembertian operator in the propagator it is possible to go over to momentum space and to write a Schwinger parameterization of the propagator as follows.

$$\langle \Phi(x,\theta)\Phi(x',\theta)\rangle = \int ds d^4p d^2\pi e^{ip(x-x')} e^{-\frac{s}{\bar{m}}(p^2 + W^\alpha \pi_\alpha + m\bar{m})} \tag{B.4}$$

where $\pi^\alpha = iD^\alpha$. In the above equation we have restricted to the superspace variable θ to be the same at x and x' as we will be interested in correlators at the same point in the superspace variable θ. We will now use this propagator and the modified chiral ring (3.8) to make estimates for the connected part of various correlators. The modified ring allows more than two insertions of W_α in a given index loop, using the identities in (3.11) such contributions can be converted to gravitational corrections. At this point one might wonder if contributions to the connected diagrams of gauge invariant operators in presence of a gravitational background are in contradiction with the result found in [89]. There it was found that on an arbitrary Kähler manifold gauge invariant operators of the $\mathcal{N} = 1$ theory factorize. The background considered in [89] was entirely bosonic, and we find that the estimates of contribution to the connected diagram indeed vanish for a purely bosonic background; thus there is no contradiction with [89]. We will indicate this as we evaluate the estimates of various correlators.

The various operators involved in the correlators of interest are those of equation (2.34) that we recall here

$$\begin{aligned}
\mathcal{R}(z)_{ij} &= -\frac{1}{32\pi^2}\left(\frac{W^2}{z-\Phi}\right)_{ij}, \qquad & R(z) &= \mathrm{Tr}\,\mathcal{R}(z) \\
\rho_{\alpha\ ij}(z) &= \frac{1}{4\pi}\left(\frac{W_\alpha}{z-\Phi}\right)_{ij}, \qquad & w_\alpha(z) &= \mathrm{Tr}\,\rho_\alpha(z) \\
\mathcal{T}(z)_{ij} &= \left(\frac{1}{z-\Phi}\right)_{ij}, \qquad & T(z) &= \mathrm{Tr}\,\mathcal{T}(z)
\end{aligned} \tag{B.5}$$

The contour integrals of R, w_α and T around i-th branch cut define the gaugino bilinear S_i, the $U(1)$ gauge field $w_{\alpha i}$ and N_i respectively in $U(N_i)$ subgroup of $U(N)$ as in [8]. The fact that we are here restricting the background gauge field to be in $SU(N)$ rather than $U(N)$ implies that $\sum_i w_{\alpha i} = 0$. The chiral ring relation $G^2 w_\alpha(z) = 0$ implies that $G^2 w_{\alpha i} = 0$ for all i.

We first consider estimates of the connected part of two point function, we will discuss in detail the estimate for the follow correlator

$$\langle R(z,x,\theta), R(w,y,\theta)\rangle_c, \tag{B.6}$$

where the subscript stands for the connected part and out line the derivation of the estimates for the other two point functions. The various contribution to this correlator in (4.74) can be found by expanding in z and w, by definition of the connected correlator the expansion starts of with the power $1/z^2w^2$. Let us focus on a Feynman diagram consisting of l loops, there will be $l+1$ bosonic and fermionic momentum integrations in this diagram. The extra momentum integral comes from the final Fourier transform to convert to the position space representation for the above correlator. We can organize this diagram into index loops due to the adjoint action of W_α. The fermionic momentum integral forces us to bring down $2(l+1)$ powers of W from the propagator in (B.4). To obtain the leading gravitational correction we would like the number of index loops to be as large as possible so that we can avoid having more than two W's in a given index loop. The number of index loops h and the number of loops are related by $l = h - 1 + 2g$, where g is the genus of the diagram. For a given number of Feynman loops the number of index loops is largest for genus zero, thus the leading estimate to the connected graph arises from the planar diagram. For a planar diagram we need to saturate the fermionic momentum integrals by bringing down $2h$ W's. This can be done by inserting W^2 in h index loops. We still have two more external W^2 in (B.6). This can at best be inserted in two different index loops. Thus we have two index loops with $(W^2)^2$ insertions. Using the identities in (3.11) we see that each of them reduces to G^2W^2. Thus there is a term proportional to G^4W^2 on one of the index loops. Note that if one had a purely bosonic background both G and W would start at θ in the superspace expansion, thus G^4W^2 would vanish in agreement with [89]. In fact G^4W^2, as is trivial in the chiral ring by (3.14), the leading estimate for the correlator in (B.6) vanishes. In the next section it is shown that the two point function in (B.6) in fact vanishes. Next we consider the following two point function

$$\langle R(z,x,\theta) w_\alpha(w,x',\theta)\rangle_c, \tag{B.7}$$

we can apply the same counting again, finally in the planar diagram we will be left with at best with one index loop with $(W^2)^2$ and one with $W^2 W_\alpha$ insertions. This reduces to a $G^2 W^2$ insertion and a $G^2 W_\alpha$ insertions, which implies that the leading gravitational contribution to (B.7) is proportional to G^4. Now we have seen in (3.13) that $G^2 W_\alpha$ is zero in the chiral ring, thus this leading estimate in fact vanishes in the chiral ring. Similarly, consider the correlator

$$\langle R(z,x,\theta)T(w,x',\theta)\rangle_c. \tag{B.8}$$

Here we will be left with $(W^2)^2$ in a single index loop, which reduces to G^2W^2. Thus the above correlator is proportional to G^2W^2. For a purely bosonic background we see that this contribution again vanishes, consistently with [89]. For the case of

$$\langle w^\alpha(z,x,\theta)w_\alpha(w,x',\theta)\rangle_c, \tag{B.9}$$

we will be left with either $W^\alpha W^2$ insertion in two different index loops or a $(W^2)^2$ insertion in a single index loop. The former case vanishes in the chiral ring, but the latter case survives, with a contribution proportional to G^2W^2. For the following two point function

$$\langle w_\alpha(z,x,\theta)T(w,x',\theta)\rangle_c, \tag{B.10}$$

there is a $W^\alpha W^2$ insertion in a single index loop, which is proportional to $G^2 W_\alpha$. Note that this leading contribution vanishes in the chiral ring due to (3.13) and also for a purely bosonic background. Finally, we have the two point function

$$\langle T(z,x,\theta)T(w,x',\theta)\rangle \tag{B.11}$$

For this case all the h index loops are saturated with one W^2 and there are extra insertions of W^2 for any of the h index loops, as there is no external W. Thus we have no contribution for the connected part of this correlator from the modified ring. However we will see later that there is a direct gravitational contribution to the above correlator. This can be seen roughly as follows: the d' Alembertian in (B.2) carries the covariant derivatives which can possibly contribute to the connected two point function, as seen in the expansion of the propagator in (B.4). This fact can be further justified by the evaluation of the 1-loop effective action obtained by integrating out the chiral multiplet in the absence of the gauge field background, which gives a term proportional to $G^2 \ln(m)$ [26]. Therefore we expect the leading term in the correlator in (B.11) to be proportional to G^2 and this will be shown explicitly in the next section. Again we see that for a purely bosonic background G^2 is proportional to θ^2, which implies that the lowest components of the superfields in (B.11) factorize consistently with [89] [1]. We summarize the estimates of the various connected two point correlators in the table B for future reference where S represents schematically any of the S_i's, and we have used the chiral ring relations $G^4 = 0$ and $G^2 w_{\alpha i} = 0$.

Now we provide estimates for the fully connected part of various three point functions. All the fully connected part of the three point functions are proportional to

[1] Note that the fact that the lowest component factorize cannot be used to promote it to a superfield equation as the is no Q_α which preserves the background.

Correlator	Estimate
$\langle RR\rangle_c$	$G^4 = 0$
$\langle Rw_\alpha\rangle_c$	$G^4 = 0$
$\langle RT\rangle_c$	G^2S^h
$\langle w^\alpha w_\alpha\rangle_c$	G^2S^h
$\langle w_\alpha T\rangle_c$	$G^2S^{h-1}w_{\alpha\ i} = 0$
$\langle TT\rangle_c$	G^2S^{h-1}

Table B.1: Estimate of connected two point functions

at least G^4, therefore using (3.14) they all vanish in the chiral ring. We discuss the method of arriving at the estimates for one case in detail and just outline the results for the others. Consider the following fully connected three point function.

$$\langle R(z,x,\theta)R(w,x',\theta)R(u,x'',\theta)\rangle_c. \tag{B.12}$$

By the definition of the full connected three point function, the first possibly non zero term of the expansion in z, w, u starts at $1/(zwu)^2$. Consider a contribution to any of the correlators appearing in this expansion. A Feynman diagram consisting of l loops will now have $l+2$ bosonic and fermionic momentum integrations. This is because a three point function in momentum space will in general have two external independent momenta, and then converting that to position space will involve these additional momentum integrals. As we have argued earlier for the case of the two point function, the leading contribution will be from the genus zero graphs. For a planar graph then there are $2(h+1)$ fermionic momentum integrals to be done. Therefore in addition to inserting h index loops by W^2, at best four different index loops will have $(W^2)^2$ insertions (we assume l is large enough). Using the identities in (3.11) we see that each $(W^2)^2$ insertion is proportional to G^2W^2. Thus the above three point function is proportional to G^8, but G is fermionic and has 4 independent components, thus G^8 vanishes due to Fermi statistics. Similar arguments show that the following correlators vanish

$$\begin{aligned}\langle R(z,x,\theta)R(w,x',\theta)w_\alpha(u,x'',\theta)\rangle_c &= 0,\\ \langle R(z,x,\theta)R(w,x',\theta)T(u,x'',\theta)\rangle_c &= 0,\\ \langle R(z,x,\theta)w^\alpha(w,x',\theta)w_\alpha(u,x'',\theta)\rangle_c &= 0,\\ \langle R(z,x,\theta)w^\alpha(w,x',\theta)T(u,x'',\theta)\rangle_c &= 0,\end{aligned} \tag{B.13}$$

The first correlator in (B.13) is proportional to G^8 and the rest are proportional to G^6, thus they vanish due to Fermi statistics. Now consider the following three point function

$$\langle w^\alpha(z,x,\theta)T(w,x',\theta)T(u,x'',\theta)\rangle_c. \tag{B.14}$$

We have seen that in a planar graph the fermionic momentum integrations force one to insert at least one factor of W^2 in all of the h index loops and there is at least one index loop with a $(W^2)^2$ insertion. Using the identities in (3.11) this can be manipulated to a gravitational contribution proportional to G^2W^2. For the above correlator there is one more index loop with an insertion of W^2W^α and again using the identities in (3.11), this term is proportional to G^2W^α. Thus the leading gravitational contribution to the three point function in (B.14) is proportional to G^4W_α and thus it is zero in the chiral ring using (3.13). Next we consider the following three point function

$$\langle R(z,x,\theta)T(w,x',\theta)T(u,x'',\theta)\rangle_c,$$

As discussed above, since there are $2(h+1)$ momentum integration, all the h index loops have at least a W^2 insertion with one having a $(W^2)^2$ insertions, the above correlator has an external W^2. Therefore the leading gravitational contribution arises with two different index loops each with a $(W^2)^2$ insertion and using the identities in (3.11) this can be manipulated in the chiral ring to give a factor of G^4. The following three point function is also proportional to G^4

$$\langle W^\alpha(z,x,\theta)W_\alpha(w,x',\theta)T(u,x'',\theta)\rangle_c,$$

Here again due to the momentum integrations there is already a factor of G^2, the two external W's can be inserted in another index loop, but this loop already has an insertion of W^2, which gives rise to a contribution proportional to G^2. Thus the three point function in (B.15) is proportional to G^4. Therefore the correlators in (B.15) and (B.15) vanish in the chiral ring due to (3.14). Finally, let us consider the following three point function

$$\langle T(z,x,\theta)T(w,x',\theta)T(u,x'',\theta)\rangle_c.$$

As in the case for (B.11) we can not estimate the G dependence of this correlator solely using the chiral ring. But from the fact that the correlator in (B.11) is proportional to G^2 and since there is at least one index loop with $(W^2)^2$ insertion, we can arrive at the conclusion that the above three point function will be proportional to G^4. To sum up we have examined all possibly non-vanishing full connected three point fully and found them to be least proportional to G^4, and therefore using (3.14) they vanish in the chiral ring.

Appendix C

Other two Point Correlators and Integrability Conditions

For completeness in this Appendix we derive the connected two point functions $\langle T(z)T(w)\rangle_c$ and $\langle w_\alpha(z)T(w)\rangle_c$. These are not used in evaluating the full one-point functions, but serve to demonstrate the consistency of the constraints on the generating functional given by the equations (4.69), (4.70) and (4.71). For convenience we define the following moments

$$\begin{aligned} \partial_T^w \partial_\alpha^z M|_{j_T=j_w^\alpha=0} &= M_{\alpha T}(z,w) = \langle w_\alpha(z)T(w)\rangle_c, \\ \partial_T^w \partial_T^z M|_{j_T=j_w^\alpha=0} &= M_{TT}(z,w) = \langle T(z)T(w)\rangle_c \end{aligned} \tag{C.1}$$

We proceed as in section 4.2, to determine the correlator $\langle w_\alpha(z)T(w)\rangle$. We differentiate (4.70) by ∂_T^w and then set $j_T = j_\alpha = 0$ which gives

$$\begin{aligned} &(2\partial_R^z M_R - I(z))M_{\alpha T}(z,w) + 2\partial_R^z M_T(w)M_\alpha(z) + 2\partial_R^z M_{\alpha T}(z,w) \\ &+8(F\cdot G)_\alpha \int dw' j_R O_R^{(z,w')} M_T(w) - \frac{G^2}{3}\partial_z\left(\frac{M_\alpha(z)-M_\alpha(w)}{z-w}\right) = 0 \end{aligned} \tag{C.2}$$

First note that the term containing the integral drops out as it involves a $\partial_R^z M_T(w)$ which, using (4.79), is proportional to $G^2(F\cdot G)$ and thus is trivial in the chiral ring. The solution of the above equation can also be related to the function $H(z,w)$, and this can be seen as follows: differentiate (4.75) by δ_α, to obtain

$$\begin{aligned} &(2\partial_R^z M_R - I(z))\delta_\alpha H(z,w) + 2\delta_\alpha \partial_R^z M_R H(z,w) + \partial_R^z \delta_\alpha H(z,w) \\ &-16F^2 \int dw' j_R \partial_z \left(\frac{\delta_\alpha H(z,w) - \delta_\alpha H(w',w)}{z-w'}\right) + \delta_\alpha O_R^{(z,w)} M_R = 0 \end{aligned} \tag{C.3}$$

Now we note that for the one point functions we have the following relations

$$\delta_\alpha \partial_R^z M_R = M_\alpha(z), \quad \delta_\alpha O_R^{(z,w)} M_R = M_\alpha(z,w) \tag{C.4}$$

These relations are valid at the zeroth order, but it is sufficient for our purpose as we will consider a solution which is proportional to G^2 . Therefore the higher order corrections are trivial in the chiral ring. By comparing (C.2) and (C.3) we see that the ansatz

$$M_{T\alpha}(w,z) = -\frac{G^2}{3}\delta_\alpha H(z,w) \tag{C.5}$$

solves (C.2), as it reduces to (C.3). To show this one uses (C.4), the fact that the term containing the integral vanishes in the chiral ring for both the equations and that the three point functions in (C.2) and (C.3) vanish with the ansatz of (C.5), as they come with a higher power of F^2. Finally, to obtain the correlator $\langle T(z)T(w)\rangle_c$ we take the derivative of (4.71) with respect to ∂_T^w and set $j_T = j_\alpha = 0$, giving the following equation

$$\begin{aligned}(2\partial_R^z M_R - I(z))M_{TT}(w,z) + 2\partial_T^w\partial_R^z M_R M_T(z) + 2M^\alpha(z)M_{\alpha T}(z,w) + 2\partial_R^z M_{TT}(z,w) \\ +\partial_T^w\hat{\partial}^{z\alpha}\hat{\partial}_\alpha^z M|_{j_T,j_\alpha=0} - \frac{G^2}{3}\int dw j_R O_R^{(w',z)} M_T(z) - \frac{G^2}{3}\partial_w\left(\frac{M_T(z)-M_T(w)}{z-w}\right) = 0\end{aligned} \tag{C.6}$$

On substituting the value of $O_R^{(w',z)}M_T(z)$ from (4.79) in the term containing the integral, we see that it is proportional to G^4 and therefore trivial in the chiral ring. We can also substitute the solutions of $\partial_T^w\partial_R^z M_R$ and M_{TT} obtained in (4.79) and (C.5) in the above equations. With this substitution it is easy to see that the solution of (C.9) equation can be written in terms of $H(z,w)$, by differentiating (4.75) by the operator $D+\delta^2/2$. We obtain

$$\begin{aligned}(2\partial_R^z M_R - I(z))(D+\frac{1}{2}\delta^2)H(z,w) + 2(D+\frac{1}{2}\delta^2)\partial_R^z M_R H(z,w) \\ +2\delta^\alpha\partial_R^z M_R\delta_\alpha H(z,w) + (D+\frac{1}{2}\delta^2)\partial_R^z H(z,w) + (D+\frac{1}{2}\delta^2)O_R^{(z,w)}M_R \\ -16F^2(D+\frac{1}{2}\delta^2)\int dw' j_R\partial_{w'}\left(\frac{H(z,w)-H(w',w)}{z-w'}\right) = 0\end{aligned} \tag{C.7}$$

We also have the following relation at the zeroth order

$$(D+\frac{1}{2}\delta^2)\partial_R^z M_R = M_T(z), \tag{C.8}$$

Again it is sufficient to use the relation at the zeroth order, since we will be interested in a solution which is proportional to G^2, therefore higher order corrections to the above relation vanish in the chiral ring. From comparing (C.6) and (C.7) and using the relations (C.4) and (C.8), we see that the following ansatz satisfies (C.6)

$$M_{TT}(w,z) = -\frac{G^2}{3}(D+\frac{1}{2}\delta^2)H(z,w). \tag{C.9}$$

Note that, with this ansatz, the terms containing the integral and the three-point functions in both (C.6) and (C.7) vanish, so that they reduce to the same equation.

To verify that the equations constraining the generating functionals are consistent, we obtain the correlators $\langle R(z)w_\alpha(w)\rangle_c$, $\langle R(z)T(w)\rangle_c$, $\langle w_\alpha(z)T(w)\rangle_c$ by a different route and show that they lead to the same results as discussed earlier. It is possible to obtain these correlators by a different route, as partial derivatives commute and it is not obvious that the results for the correlators (4.69), (4.70) and (4.71) will be the same.

First consider the correlator $\langle R(z) w_\alpha(w)\rangle_c$, in (4.76) we had obtained it by differentiating (4.69) by ∂^w_α; we could also obtain the same correlator by differentiating (4.70) by ∂^w_R. Here we verify that we get the same result. On performing the differentiation and setting $j_T = j_\alpha = 0$ we obtain

$$\begin{aligned}&(2\partial_R M - I(z))\partial^w_R M_\alpha(z) + 2\partial^w_R\partial^z_R M_R M_\alpha(z) \\ &+2\partial^w_R\partial^z_R M_\alpha(z) + \int dw' 8 j_R (F\cdot G)_\alpha O_R^{(z,w')} M_\alpha(w) + 8(F\cdot G)_\alpha O_R^{(z,w)} M_R = 0\end{aligned} \tag{C.10}$$

The following ansatz solves the equation (4.77)

$$\partial^w_R M_\alpha(z) = 8(F.G)_\alpha H(z,w). \tag{C.11}$$

To see this, note that with this ansatz the term containing $\partial^w_R\partial^z_R M_R M_\alpha(z)$ is trivial in the chiral ring as it is proportional to $F^2 w_\alpha$, and that the term containing the integral also vanishes. Therefore comparing (4.75) and (C.10) we see that the above ansatz satisfies the latter equation. The solution is consistent with the one obtained in (4.77) as $H(z,w)$ is a symmetric function in z and w. Consider the two point function $\langle RT\rangle_c$: we now obtain this correlator by differentiating (4.71) with ∂^w_R and verify that the result is consistent with (4.79). On performing the differentiation and setting $j_T = j_\alpha = 0$ we obtain

$$\begin{aligned}&(2\partial_R - I(z))\partial^w_R M_T(z) + 2\partial^w_R\partial^z_R M_R M_T(z) + 2\partial^w_R M^\alpha(z) M_\alpha(z) \\ &+2\partial^w_R\partial^z_R M^z_T + \partial^w_R M^\alpha_\alpha(z,z) - \frac{G^2}{3}\int dw' j_R O_R^{(z,w')}\partial_w M_R - \frac{G^2}{3} O_R^{(z,w)} M_R = 0\end{aligned} \tag{C.12}$$

The combination of the second and third terms in the above equation is trivial in the chiral ring as shown below

$$\begin{aligned}2\partial^w_R\partial^z_R M_R M_T(z) \;&+\; 2\partial^w_R M^\alpha(z) M_\alpha(z) \\ &=\; 2(16F^2 M_T(z) - 8(F\cdot G)_\alpha M^\alpha(z))H(z,w) = 0\end{aligned} \tag{C.13}$$

Here we have used (4.8). Now comparing (C.12) and (4.75), it is clear that the following ansatz solves (C.12)

$$\partial^w_R M_T(z) = -\frac{G^2}{3} H(z,w) \tag{C.14}$$

The terms containing the integral and the connected three point functions are trivial in the chiral ring, with the ansatz in (C.14) for both (C.12) and (4.75) as they come with a higher power of F^2. This solution is consistent with the one obtained in (4.79) because $H(z,w)$ is a symmetric function. Finally we consider the two-point function $\langle T w_\alpha\rangle$, we obtain this by differentiating (4.71) by ∂^w_α and setting $j_T = j_\alpha = 0$

$$\begin{aligned}&(2\partial^z_R M_R - I(z))M_{\alpha T}(w,z) + 2\partial^z_R M_\alpha(w) M_T(z) + 2M_\alpha^{\ \beta}(z,w) M_\beta(z) \\ &+2\partial_R M_{T\alpha}(z,w) + \partial^w_\alpha \partial^{\beta z}\partial_\beta M|_{j_T=j_\alpha=0} - \frac{G^2}{3}\int dw' j_R O_R^{(z,w')} M_\alpha(w) \\ &\qquad -\frac{G^2}{3}\partial_w\left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) = 0\end{aligned} \tag{C.15}$$

Again the second and third term of the above equation can be further simplified in the chiral ring, by substituting the correlators from (4.77) and (4.83)

$$\begin{aligned} & 2\partial_R^z M_\alpha(w) M_T(z) + 2M_\alpha^{\ \beta}(z,w) M_\beta(z) \\ = \;& 2\left(8(F\cdot G)_\alpha H(z,w) M_T(z) + \frac{5}{3}G^2 H(z,w) M_\alpha(z) - 8F^2 D H(z,w) M_\alpha(z)\right) \\ = \;& -\frac{2}{3}G^2 M_\alpha(z) H(z,w) \end{aligned} \tag{C.16}$$

Here we have used the chiral ring equations (4.8) and (4.11). Now comparing (C.15) and (C.3) we see that the the solution can be written as

$$M_{\alpha T}(z,w) = -\frac{G^2}{3}\delta_\alpha H(w,z) \tag{C.17}$$

The terms containing the three point functions and the integrals in (C.15) and (C.3) vanish with the above solution. This concludes the proof of the integrability of the constraints on the generating functional.

Bibliography

[1] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[2] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644**, 3 (2002) [arXiv:hep-th/0206255].

[3] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644**, 21 (2002) [arXiv:hep-th/0207106].

[4] G. Veneziano and S. Yankielowicz, "An Effective Lagrangian For The Pure N=1 Supersymmetric Yang-Mills Theory," Phys. Lett. B **113**, 231 (1982).

[5] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, "Nonperturbative Aspects In Supersymmetric Gauge Theories," Phys. Rept. **162** (1988) 169.

[6] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, "Perturbative computation of glueball superpotentials," Phys. Lett. B **573**, 138 (2003) [arXiv:hep-th/0211017].

[7] R. Dijkgraaf, S. Gukov, V. A. Kazakov and C. Vafa, "Perturbative analysis of gauged matrix models," Phys. Rev. D **68** (2003) 045007 [arXiv:hep-th/0210238].

[8] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP **0212**, 071 (2002) [arXiv:hep-th/0211170].

[9] L. F. Alday and M. Cirafici, "Effective superpotentials via Konishi anomaly," JHEP **0305** (2003) 041 arXiv:hep-th/0304119.

[10] H. Ooguri and C. Vafa, "Gravity induced C-deformation," arXiv:hep-th/0303063.

[11] H. Ooguri and C. Vafa, "The C-deformation of gluino and non-planar diagrams," arXiv:hep-th/0302109.

[12] J. R. David, E. Gava and K. S. Narain, "Konishi anomaly approach to gravitational F-terms," arXiv:hep-th/0304227.

[13] A. Klemm, M. Marino and S. Theisen, "Gravitational corrections in supersymmetric gauge theory and matrix models," JHEP **0303**, 051 (2003) [arXiv:hep-th/0211216].

[14] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Matrix models and gravitational corrections," Adv. Theor. Math. Phys. **7**, 1155 (2004) [arXiv:hep-th/0211241].

[15] R. Dijkgraaf, M. T. Grisaru, H. Ooguri, C. Vafa and D. Zanon, "Planar gravitational corrections for supersymmetric gauge theories," JHEP **0404**, 028 (2004) [arXiv:hep-th/0310061].

[16] L. F. Alday and M. Cirafici, "Gravitational F-terms of SO/Sp gauge theories and anomalies," JHEP **0309** (2003) 031 [arXiv:hep-th/0306299].

[17] H. Ita, H. Nieder and Y. Oz, "Gravitational F-terms of N = 1 supersymmetric SU(N) gauge theories," JHEP **0312**, 046 (2003) [arXiv:hep-th/0309041].

[18] I. Adam and Y. Oz, "Gravitational F-terms of N = 1 supersymmetric gauge theories," arXiv:hep-th/0407200.

[19] C. Vafa, "Superstrings and topological strings at large N," *J. Math. Phys.* **42** (2001) 2798–2817, `hep-th/0008142`.

[20] L. F. Alday, M. Cirafici, J. R. David, E. Gava and K. S. Narain, "Gravitational F-terms through anomaly equations and deformed chiral rings,"arXiv:hep-th/0305217.

[21] J. R. David, E. Gava and K. S. Narain, "Partial N = 2 → N = 1 supersymmetry breaking and gravity deformed chiral rings," JHEP **0406** (2004) 041 [arXiv:hep-th/0311086].

[22] R. Argurio, G. Ferretti and R. Heise, "An introduction to supersymmetric gauge theories and matrix models," Int. J. Mod. Phys. A **19** (2004) 2015 [arXiv:hep-th/0311066].

[23] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," Nucl. Phys. Proc. Suppl. **45BC** (1996) 1 [arXiv:hep-th/9509066].

[24] M. J. Strassler, "On confinement and duality," *Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001*

[25] J. Wess and J. Bagger, "Supersymmetry and supergravity,". Princeton, USA: Univ. Pr. (1992) 259 p.

[26] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace, or one thousand and one lessons in supersymmetry," *Front. Phys.* **58** (1983) 1–548, `hep-th/0108200`.

[27] P. S. Howe, "Supergravity in superspace," *Nucl. Phys.* **B199** (1982) 309.

[28] L. Alvarez-Gaume and S. F. Hassan, "Introduction to S-duality in N = 2 supersymmetric gauge theories: A pedagogical review of the work of Seiberg and Witten," Fortsch. Phys. **45** (1997) 159 [arXiv:hep-th/9701069].

[29] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485] [arXiv:hep-th/9407087].

[30] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431** (1994) 484 [arXiv:hep-th/9408099].

[31] P. Argyres "Non–renormalization theorems in susy gauge theories," *Prepared for ICTP Introductory School on Recent Developments in Supersymmetric Gauge Theories, Trieste, Italy, 14-25 Jun 2004*

[32] N. Dorey "Instanton Physics," *Prepared for ICTP Introductory School on Recent Developments in Supersymmetric Gauge Theories, Trieste, Italy, 14-25 Jun 2004*

N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The calculus of many instantons," Phys. Rept. **371** (2002) 231 [arXiv:hep-th/0206063].

[33] P. B. Ronne, "On the Dijkgraaf-Vafa conjecture," arXiv:hep-th/0408103.

[34] K. i. Konishi, "Anomalous Supersymmetry Transformation Of Some Composite Operators In Sqcd," Phys. Lett. B **135** (1984) 439.

[35] K. i. Konishi and K. i. Shizuya, "Functional Integral Approach To Chiral Anomalies In Supersymmetric Gauge Theories," Nuovo Cim. A **90** (1985) 111.

[36] F. Cachazo and C. Vafa, "N = 1 and N = 2 geometry from fluxes," arXiv:hep-th/0206017.

[37] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, "A geometric unification of dualities," Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[38] F. Cachazo, K. A. Intriligator and C. Vafa, "A large N duality via a geometric transition," Nucl. Phys. B **603**, 3 (2001) [arXiv:hep-th/0103067].

[39] M. Aganagic, A. Klemm, M. Marino and C. Vafa, "Matrix model as a mirror of Chern-Simons theory," JHEP **0402** (2004) 010 [arXiv:hep-th/0211098].

[40] R. Roiban, R. Tatar and J. Walcher, "Massless flavor in geometry and matrix models," Nucl. Phys. B **665** (2003) 211 [arXiv:hep-th/0301217].

[41] C. I. Lazaroiu, "Holomorphic matrix models," JHEP **0305** (2003) 044 [arXiv:hep-th/0303008].

[42] A. Klemm, K. Landsteiner, C. I. Lazaroiu and I. Runkel, "Constructing gauge theory geometries from matrix models," JHEP **0305** (2003) 066 [arXiv:hep-th/0303032].

[43] D. Berenstein, "Solving matrix models using holomorphy," JHEP **0306** (2003) 019 [arXiv:hep-th/0303033].

[44] N. Halmagyi and V. Yasnov, "Chern-Simons matrix models and unoriented strings," JHEP **0402** (2004) 002 [arXiv:hep-th/0305134].

[45] K. Landsteiner, C. I. Lazaroiu and R. Tatar, "(Anti)symmetric matter and superpotentials from IIB orientifolds," JHEP **0311** (2003) 044 [arXiv:hep-th/0306236].

[46] F. Ferrari, "Planar diagrams and Calabi-Yau spaces," Adv. Theor. Math. Phys. **7** (2004) 619 [arXiv:hep-th/0309151].

[47] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].

[48] I. R. Klebanov, "String theory in two-dimensions," arXiv:hep-th/9108019.

[49] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[50] S. G. Naculich, H. J. Schnitzer and N. Wyllard, "The N = 2 U(N) gauge theory prepotential and periods from a perturbative matrix model calculation," Nucl. Phys. B **651** (2003) 106 [arXiv:hep-th/0211123].

H. Itoyama and A. Morozov, "The Dijkgraaf-Vafa prepotential in the context of general Seiberg-Witten theory," Nucl. Phys. B **657** (2003) 53 [arXiv:hep-th/0211245].

S. G. Naculich, H. J. Schnitzer and N. Wyllard, "Matrix model approach to the N = 2 U(N) gauge theory with matter in the fundamental representation," JHEP **0301** (2003) 015 [arXiv:hep-th/0211254].

Y. Demasure and R. A. Janik, "Explicit factorization of Seiberg-Witten curves with matter from random matrix models," Nucl. Phys. B **661** (2003) 153 [arXiv:hep-th/0212212].

M. Matone, "Seiberg-Witten duality in Dijkgraaf-Vafa theory," Nucl. Phys. B **656** (2003) 78 [arXiv:hep-th/0212253].

C. h. Ahn and S. k. Nam, "N = 2 supersymmetric SO(N)/Sp(N) gauge theories from matrix model," Phys. Rev. D **67** (2003) 105022 [arXiv:hep-th/0301203].

S. G. Naculich, H. J. Schnitzer and N. Wyllard, "Matrix-model description of N = 2 gauge theories with non-hyperelliptic Seiberg-Witten curves," arXiv:hep-th/0305263.

[51] N. Dorey, T. J. Hollowood, S. Prem Kumar and A. Sinkovics, "Exact superpotentials from matrix models," JHEP **0211** (2002) 039 [arXiv:hep-th/0209089].

N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, "Massive vacua of N = 1* theory and S-duality from matrix models," JHEP **0211** (2002) 040 [arXiv:hep-th/0209099].

N. Dorey, T. J. Hollowood and S. P. Kumar, "S-duality of the Leigh-Strassler deformation via matrix models," JHEP **0212** (2002) 003 [arXiv:hep-th/0210239].

T. J. Hollowood, "New results from glueball superpotentials and matrix models: The Leigh-Strassler deformation," JHEP **0304** (2003) 025 [arXiv:hep-th/0212065].

T. Mansson, "Another Leigh-Strassler deformation through the matrix model," JHEP **0303** (2003) 055 [arXiv:hep-th/0302077].

M. Petrini, A. Tomasiello and A. Zaffaroni, "On the geometry of matrix models for N = 1," JHEP **0308** (2003) 004 [arXiv:hep-th/0304251].

T. Itoh, "Chiral rings, anomalies and loop equations in N = 1* gauge theories," JHEP **0307** (2003) 005 [arXiv:hep-th/0306035].

[52] J. Polchinski and M. J. Strassler, "The string dual of a confining four-dimensional gauge theory," arXiv:hep-th/0003136.

[53] T. Eguchi and Y. Sugawara, "Branches of N = 1 vacua and Argyres-Douglas points," JHEP **0305** (2003) 063 [arXiv:hep-th/0305050].

G. Bertoldi, "Matrix models, Argyres-Douglas singularities and double scaling limits," JHEP **0306** (2003) 027 [arXiv:hep-th/0305058].

D. Shih, "Singularities of N = 1 supersymmetric gauge theory and matrix models," arXiv:hep-th/0308001.

M. A. Ganjali, "Glueball superfield and Argyres Douglas points," arXiv:hep-th/0309097.

[54] P. C. Argyres and M. R. Douglas, "New phenomena in SU(3) supersymmetric gauge theory," Nucl. Phys. B **448** (1995) 93 [arXiv:hep-th/9505062].

[55] L. Mazzucato, "Chiral rings, anomalies and electric-magnetic duality," arXiv:hep-th/0408240.

B. Feng and Y. H. He, "Seiberg duality in matrix models. II," Phys. Lett. B **562** (2003) 339 [arXiv:hep-th/0211234].

B. Feng, "Seiberg duality in matrix model," arXiv:hep-th/0211202.

B. Feng, "Note on Seiberg duality in matrix model," Phys. Lett. B **572** (2003) 68 [arXiv:hep-th/0303144].

[56] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," Nucl. Phys. B **435** (1995) 129 [arXiv:hep-th/9411149].

[57] R. Argurio, V. L. Campos, G. Ferretti and R. Heise, "Exact superpotentials for theories with flavors via a matrix integral," Phys. Rev. D **67**, 065005 (2003) [arXiv:hep-th/0210291].

J. McGreevy, "Adding flavor to Dijkgraaf-Vafa," JHEP **0301** (2003) 047 [arXiv:hep-th/0211009].

H. Suzuki, "Perturbative derivation of exact superpotential for meson fields from matrix theories with one flavour," JHEP **0303** (2003) 005 [arXiv:hep-th/0211052].

I. Bena and R. Roiban, "Exact superpotentials in N = 1 theories with flavor and their matrix model formulation," Phys. Lett. B **555** (2003) 117 [arXiv:hep-th/0211075].

Y. Demasure and R. A. Janik, "Effective matter superpotentials from Wishart random matrices," Phys. Lett. B **553** (2003) 105 [arXiv:hep-th/0211082].

Y. Tachikawa, "Derivation of the Konishi anomaly relation from Dijkgraaf-Vafa with (bi-)fundamental matters," arXiv:hep-th/0211189.

I. Bena, S. de Haro and R. Roiban, "Generalized Yukawa couplings and matrix models," Nucl. Phys. B **664** (2003) 45 [arXiv:hep-th/0212083].

C. Hofman, "Super Yang-Mills with flavors from large N(f) matrix models," arXiv:hep-th/0212095.

B. Feng, "Note on matrix model with massless flavors," Phys. Rev. D **68** (2003) 025010 [arXiv:hep-th/0212274].

Y. Nakayama, "Effective gauge degrees of freedom and the (non)existence of the glueball superpotential," JHEP **0308** (2003) 049 [arXiv:hep-th/0306007].

Y. Demasure, "Affleck-Dine-Seiberg from Seiberg-Witten," arXiv:hep-th/0307082.

P. Merlatti, "Gaugino condensate and phases of N = 1 super Yang-Mills theories," arXiv:hep-th/0307115.

B. M. Gripaios and J. F. Wheater, "Veneziano-Yankielowicz superpotential terms in N = 1 SUSY gauge theories," arXiv:hep-th/0307176.

R. A. Janik, "The Dijkgraaf-Vafa correspondence for theories with fundamental matter fields," arXiv:hep-th/0309084.

B. M. Gripaios, "Superpotentials for Gauge and Conformal Supergravity Backgrounds," arXiv:hep-th/0311025.

[58] R. Argurio, V. L. Campos, G. Ferretti and R. Heise, "Baryonic corrections to superpotentials from perturbation theory," Phys. Lett. B **553** (2003) 332 [arXiv:hep-th/0211249].

I. Bena, R. Roiban and R. Tatar, "Baryons, boundaries and matrix models," arXiv:hep-th/0211271.

H. Suzuki, "Mean-field approach to the derivation of baryon superpotential from matrix model," JHEP **0303** (2003) 036 [arXiv:hep-th/0212121].

I. Bena, H. Murayama, R. Roiban and R. Tatar, "Matrix model description of baryonic deformations," JHEP **0305** (2003) 049 [arXiv:hep-th/0303115].

S. Corley, "Notes on anomalies, baryons, and Seiberg duality," arXiv:hep-th/0305096.

[59] T. J. Hollowood and T. Kingaby, "The phase structure of mass-deformed SU(2) x SU(2) quiver theory," JHEP **0301** (2003) 005 [arXiv:hep-th/0210096].

R. Dijkgraaf, A. Neitzke and C. Vafa, "Large N strong coupling dynamics in non-supersymmetric orbifold field theories," arXiv:hep-th/0211194.

S. Seki, "Comments on quiver gauge theories and matrix models," Nucl. Phys. B **661** (2003) 257 [arXiv:hep-th/0212079].

S. G. Naculich, H. J. Schnitzer and N. Wyllard, "Cubic curves from matrix models and generalized Konishi anomalies," JHEP **0308** (2003) 021 [arXiv:hep-th/0303268].

R. Casero and E. Trincherini, "Quivers via anomaly chains," arXiv:hep-th/0304123.

R. Casero and E. Trincherini, "Phases and geometry of the N = 1 A(2) quiver gauge theory and matrix models," arXiv:hep-th/0307054.

[60] V. Balasubramanian, J. de Boer, B. Feng, Y. H. He, M. x. Huang, V. Jejjala and A. Naqvi, "Multi-trace superpotentials vs. Matrix models," arXiv:hep-th/0212082.

M. Alishahiha and H. Yavartanoo, "On the multi trace superpotential and corresponding matrix model," arXiv:hep-th/0303074.

[61] R. Argurio, G. Ferretti and R. Heise, "Chiral SU(N) gauge theories and the Konishi anomaly," JHEP **0307** (2003) 044 [arXiv:hep-th/0306125].

K. Landsteiner, C. I. Lazaroiu and R. Tatar, "Chiral field theories, Konishi anomalies and matrix models," arXiv:hep-th/0307182.

K. Landsteiner, C. I. Lazaroiu and R. Tatar, "Chiral field theories from conifolds," arXiv:hep-th/0310052.

[62] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, "Nonperturbative superpotentials and compactification to three dimensions," arXiv:hep-th/0304061.

M. Alishahiha and A. E. Mosaffa, "On effective superpotentials and compactification to three dimensions," JHEP **0305** (2003) 064 [arXiv:hep-th/0304247].

R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, "Factorization of Seiberg-Witten curves and compactification to three dimensions," arXiv:hep-th/0305189.

M. Alishahiha, J. de Boer, A. E. Mosaffa and J. Wijnhout, "N = 1 G(2) SYM theory and compactification to three dimensions," arXiv:hep-th/0308120.

A. Ritz, "A note on domain walls and the parameter space of N = 1 gauge theories," arXiv:hep-th/0308144.

R. Dijkgraaf and C. Vafa, "N = 1 supersymmetry, deconstruction, and bosonic gauge theories," arXiv:hep-th/0302011.

T. J. Hollowood, "Five-dimensional gauge theories and quantum mechanical matrix models," JHEP **0303** (2003) 039 [arXiv:hep-th/0302165].

I. Bena and R. Roiban, "N = 1* in 5 dimensions: Dijkgraaf-Vafa meets Polchinski-Strassler," arXiv:hep-th/0308013.

T. J. Hollowood, A. Iqbal and C. Vafa, "Matrix models, geometric engineering and elliptic genera," arXiv:hep-th/0310272.

S. Ganguli, O. J. Ganor and J. A. Gill, "Twisted Six Dimensional Gauge Theories on Tori, Matrix Models,and Integrable Systems," arXiv:hep-th/0311042.

[63] F. Cachazo, N. Seiberg and E. Witten, "Chiral Rings and Phases of Supersymmetric Gauge Theories," JHEP **0304**, 018 (2003) [arXiv:hep-th/0303207].

[64] F. Cachazo, N. Seiberg and E. Witten, "Phases of N = 1 supersymmetric gauge theories and matrices," JHEP **0302**, 042 (2003) [arXiv:hep-th/0301006].

[65] N. Seiberg, "Adding fundamental matter to 'Chiral rings and anomalies in supersymmetric gauge theory'," JHEP **0301**, 061 (2003) [arXiv:hep-th/0212225].

[66] F. Ferrari, "Quantum parameter space and double scaling limits in N = 1 super Yang-Mills theory," Phys. Rev. D **67** (2003) 085013 [arXiv:hep-th/0211069].

F. Ferrari, "Quantum parameter space in super Yang-Mills. II," Phys. Lett. B **557** (2003) 290 [arXiv:hep-th/0301157].

T. Friedmann, "On the quantum moduli space of M theory compactifications," Nucl. Phys. B **635** (2002) 384 [arXiv:hep-th/0203256].

D. Berenstein, "Quantum moduli spaces from matrix models," Phys. Lett. B **552** (2003) 255 [arXiv:hep-th/0210183].

R. Gopakumar, "N = 1 theories and a geometric master field," JHEP **0305** (2003) 033 [arXiv:hep-th/0211100].

K. Ohta, "Exact mesonic vacua from matrix models," JHEP **0302** (2003) 057 [arXiv:hep-th/0212025].

V. Balasubramanian, B. Feng, M. x. Huang and A. Naqvi, "Phases of N = 1 supersymmetric gauge theories with flavors," arXiv:hep-th/0303065.

T. J. Hollowood, "Critical points of glueball superpotentials and equilibria of integrable systems," arXiv:hep-th/0305023.

M. Matone and L. Mazzucato, "Branched matrix models and the scales of supersymmetric gauge theories," JHEP **0307** (2003) 015 [arXiv:hep-th/0305225].

M. Matone and L. Mazzucato, "On the chiral ring of N = 1 supersymmetric gauge theories," arXiv:hep-th/0307130.

P. Svrcek, "Chiral rings, vacua and gaugino condensation of supersymmetric gauge theories," arXiv:hep-th/0308037.

M. Klein and S. J. Sin, "On effective superpotentials and Kutasov duality," arXiv:hep-th/0309044.

H. Fuji and S. Mizoguchi, "Remarks on phase transitions in matrix models and N = 1 supersymmetric gauge theory," arXiv:hep-th/0309049.

C. Ahn and Y. Ookouchi, "The matrix model curve near the singularities," arXiv:hep-th/0309156.

M. Klein and S. J. Sin, "Matrix model, Kutasov duality and factorization of Seiberg-Witten curves," arXiv:hep-th/0310078.

[67] E. Witten, "Chiral ring of Sp(N) and SO(N) supersymmetric gauge theory in four dimensions," arXiv:hep-th/0302194.

[68] P. Etingof and V. Kac, "On the Cachazo-Douglas-Seiberg-Witten conjecture for simple Lie algebras," arXiv:math.qa/0305175.

[69] R. Abbaspur, A. Imaanpur and S. Parvizi, "N = 2 SO(N) SYM theory from matrix model," JHEP **0307**, 043 (2003) [arXiv:hep-th/0302083].

[70] R. A. Janik and N. A. Obers, "SO(N) superpotential, Seiberg-Witten curves and loop equations," Phys. Lett. B **553**, 309 (2003) [arXiv:hep-th/0212069].

[71] S. K. Ashok, R. Corrado, N. Halmagyi, K. D. Kennaway and C. Romelsberger, "Unoriented strings, loop equations, and N = 1 superpotentials from matrix models," Phys. Rev. D **67**, 086004 (2003) [arXiv:hep-th/0211291].

[72] C. h. Ahn, B. Feng, Y. Ookouchi and M. Shigemori, "Supersymmetric gauge theories with flavors and matrix models," arXiv:hep-th/0405101.

[73] C. h. Ahn, B. Feng and Y. Ookouchi, "Phases of N = 1 USp(2N(c)) gauge theories with flavors," Phys. Rev. D **69**, 026006 (2004) [arXiv:hep-th/0307190].

[74] C. h. Ahn, B. Feng and Y. Ookouchi, "Phases of N = 1 SO(N(c)) gauge theories with flavors," Nucl. Phys. B **675**, 3 (2003) [arXiv:hep-th/0306068].

[75] C. h. Ahn and Y. Ookouchi, "Phases of N = 1 supersymmetric SO / Sp gauge theories via matrix model," JHEP **0303**, 010 (2003) [arXiv:hep-th/0302150].

[76] C. h. Ahn, "Supersymmetric SO(N)/Sp(N) gauge theory from matrix model: Exact mesonic vacua," Phys. Lett. B **560**, 116 (2003) [arXiv:hep-th/0301011].

[77] B. Feng, "Geometric dual and matrix theory for SO/Sp gauge theories," Nucl. Phys. B **661**, 113 (2003) [arXiv:hep-th/0212010].

[78] H. Ita, H. Nieder and Y. Oz, "Perturbative computation of glueball superpotentials for SO(N) and USp(N)," JHEP **0301**, 018 (2003) [arXiv:hep-th/0211261].

[79] Y. Ookouchi and Y. Watabiki, "Effective superpotentials for SO/Sp with flavor from matrix models," Mod. Phys. Lett. A **18**, 1113 (2003) [arXiv:hep-th/0301226].

[80] P. Kraus and M. Shigemori, "On the matter of the Dijkgraaf-Vafa conjecture," JHEP **0304**, 052 (2003) [arXiv:hep-th/0303104].

[81] H. Fuji and Y. Ookouchi, "Comments on effective superpotentials via matrix models," JHEP **0212**, 067 (2002) [arXiv:hep-th/0210148].

[82] A. Brandhuber, H. Ita, H. Nieder, Y. Oz and C. Romelsberger, "Chiral rings, superpotentials and the vacuum structure of N = 1 supersymmetric gauge theories," Adv. Theor. Math. Phys. **7** (2003) 269 [arXiv:hep-th/0303001].

[83] P. L. Cho and P. Kraus, "Symplectic SUSY gauge theories with antisymmetric matter," Phys. Rev. D **54**, 7640 (1996) [arXiv:hep-th/9607200].

[84] C. Csaki, W. Skiba and M. Schmaltz, "Exact results and duality for Sp(2N) SUSY gauge theories with an antisymmetric tensor," Nucl. Phys. B **487**, 128 (1997) [arXiv:hep-th/9607210].

[85] P. Kraus, A. V. Ryzhov and M. Shigemori, "Loop equations, matrix models, and N = 1 supersymmetric gauge theories," JHEP **0305**, 059 (2003) [arXiv:hep-th/0304138].

[86] M. Aganagic, K. Intriligator, C. Vafa and N. P. Warner, "The glueball superpotential," Adv. Theor. Math. Phys. **7** (2004) 1045 [arXiv:hep-th/0304271].

[87] K. Intriligator, P. Kraus, A. V. Ryzhov, M. Shigemori and C. Vafa, "On low rank classical groups in String Theory, gauge theory and matrix models," Nucl. Phys. B **682**, 45 (2004) [arXiv:hep-th/0311181].

[88] F. Cachazo, "Notes on supersymmetric Sp(N) theories with an antisymmetric tensor," arXiv:hep-th/0307063.

[89] E. Witten, "Supersymmetric Yang-Mills theory on a four manifold," *J. Math. Phys.* **35** (1994) 5101–5135, hep-th/9403195.

[90] T. R. Taylor and C. Vafa, "RR flux on Calabi-Yau and partial supersymmetry breaking," *Phys. Lett.* **B474** (2000) 130–137, hep-th/9912152.

[91] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Topological amplitudes in String Theory," *Nucl. Phys.* **B413** (1994) 162–184, hep-th/9307158.

[92] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," *Nucl. Phys.* **B405** (1993) 279–304, hep-th/9302103.

[93] N. Magnoli, "An Anomalous Transformation Law In Supergravity Models," Mod. Phys. Lett. A **5** (1990) 1773.

[94] K. Konishi, N. Magnoli and H. Panagopoulos, "Spontaneous Breaking Of Local Supersymmetry By Gravitational Instantons," Nucl. Phys. B **309** (1988) 201.

[95] L. Alvarez-Gaume and P. Ginsparg, "The structure of gauge and gravitational anomalies," *Ann. Phys.* **161** (1985) 423.

[96] J. Ambjorn, L. Chekhov, C. F. Kristjansen, and Y. Makeenko, "Matrix model calculations beyond the spherical limit," *Nucl. Phys.* **B404** (1993) 127–172, hep-th/9302014.

[97] G. Akemann, "Higher genus correlators for the hermitian matrix model with multiple cuts," *Nucl. Phys.* **B482** (1996) 403–430, hep-th/9606004.