Analysis of singular solutions for two nonlinear wave equations

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In memory of my brother Pasquale

Contents

Introduction

This thesis deals with two strongly nonlinear evolution *Partial Differential Equation* (in the following named P.D.E.) arising from mathematical physics. The first one was introduced first by Fokas and Fuchssteiner [28] as a bi-Hamiltonian equation, and then was rediscovered by R. Camassa and D.D. Holmas [13] as an higher order level of approximation of the unidirectional shallow water wave equation than the *Korteweg-de Vries* equation [38]. It can be written as

$$
\begin{cases}
u(t,x) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \\
\\
u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},
\end{cases}
\tag{1}
$$

here the unknown $u(t,x)$ represents the water's free surface over a flat bed and κ is a constant related to the critical shallow-water wave speed (see also [37] for an alternative derivation as an hyperelastic-rod wave equation). We refer to this equation as to the *Camassa-Holm* equation, in honour to the first two authors which found a physical meaning stemming from the Euler equation.

The second PDE we want to study is a system of hyperbolic equations with quadratic source

$$
\begin{cases}
(u_1, \ldots, u_N)(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^N \\
\\
(u_i)_t + \mathbf{c}_i \cdot \nabla_x u_i = \displaystyle\sum_{jk} a_{ijk} u_j u_k, \quad \text{for all } i = 1 \ldots N
\end{cases}
\tag{2}
$$

which is a discretization of the velocities in the plane $\mathbb{R}^2$ for the Boltzmann equation

$$
\begin{cases}
f(t,x,\xi) : \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \\
\\
\partial_t f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) = Q(f,f)(t,x,\xi).
\end{cases}
$$

The nonlinear nature of these equations leads to the possibility of *blow up* in finite time either for the solution itself, or for the gradient of the solution. The typical situation of blow up in finite time is given by the following *Ordinary Differential Equation* (O.D.E.)

$$
\frac{d}{dt}v = -v^2, \qquad v(0) = v_0
$$

where v should be either the solution u or the gradient u_x, possibly computed along the characteristic curves of the equation. It is well known that the solution of this equation has the behaviour $\approx \frac{1}{t-T}$, where T depends on the initial data and it is the time of blow-up whenever $v_0 < 0$.

This situation can occur for the solutions of the Camassa-Holm equation in the limit case $\kappa = 0$. With this condition, the equation (1) may be rewritten in nonlocal form as

$$u_t + u\,u_x = -\frac{1}{2}\left[e^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right)\right]_x . \tag{3}$$

Since the H^1norm is conserved for regular solutions, the L^∞ norm of u is bounded, namely $\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq \sqrt{E}$. Arguing as in the *Steepening Lemma* (see [14]), in [20, theorem 4.1] the authors prove that smooth solution to (3) may not be globally defined. Let start from an odd initial data $\bar{u} \in H^3(\mathbb{R})$ which has an inflection point in 0, $\bar{u}(0) < 0$ and consider the evolution of the slope at the inflection point $s(t) \doteq u_x(t,0) < 0$. The computation of the function s gives the differential inequality

$$\frac{ds}{dt} \leq -\frac{1}{2}s^2 \qquad s(0) = \bar{u}_x(0).$$

If $\bar{u}_x(0)$ is sufficiently small, we get

$$\frac{1}{s(t)} \geq \frac{1}{s(0)} + \frac{t}{2}$$

and then the slope becomes vertical at finite time. However, the singularities thus can occur only in form of wave breaking (see also [39, 18]), in fact even if its slope can become unbounded at finite time, the solution remains bounded, because of the inequality $\|u(t)\|_{L^\infty} \leq \sqrt{E}$.

The aim of Part I of this thesis (see also [10, 29]) is to construct a continuous semigroup of global solutions in two main cases:

1. on the space H^1_{per} of spatially periodic functions, locally in $H^1(\mathbb{R})$;

2. on a domain of H^1 functions with a certain exponential decay at $x \to \pm\infty$.

Result of existence of solutions can be found in [48, 49], and [17, 16] where the authors added a small diffusion term to the right hand side of 3 and obtained solution of the original equation as a vanishing viscosity limit. On the other hand, in [7] was developed an alternative technique, which relies on a new set of dependent and independent variables with the specific purpose to resolve all the singularities. With this change of variable the solution can be obtained as the unique fixed point of a contractive transformation. In Chapter 2 we present yet another approach based on the Hamiltonian structure of the Camassa-Holm equation. We shall construct the semigroup of global solution starting from

explicit solutions of the Camassa-Holm equation with initial condition in form of *multipeakon* function

$$u_0(x) = \sum_{j=1}^{N} p_j e^{-|x-q_j|} \, .$$

The motivation of this choice is given by the form of traveling wave solution (see [13, 20, 21, 22]). Looking for solution of the equation (3) in the traveling wave form $u(t,x) = U(x - ct)$, with a function U that vanishes at infinity, one obtains the function $U = ce^{-|x-ct|}$, which is a peaked *soliton* (from this fact derives the shortened term *peakon*). The multipeakon functions are stable, in fact not only a single peakon subject to (3) evolves with this form, but also the evolution of a superposition of traveling wave (e.g. initial data like u_0) remains of the same shape

$$u(t,x) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|} \, .$$

The reader can see also [2] for a recursive reconstruction of the multipeakon solutions, and [24] which prove that multipeakon solutions are orbitally stable, i.e. stable under a general nature of perturbations.

In [33] the authors prove the existence of a global multipeakon solution when the strengths p_i are positive for all $i = 1 \ldots N$. In this case the crucial fact is that no interaction between the peakons occurs, and then the gradient remains bounded, which yields existence and uniqueness of the coefficients $p_1(t), \ldots, p_N(t)$ and $q_1(t), \ldots, q_N(t)$ that are solutions of the Hamiltonian system

$$\begin{cases} \dot{q}_i = \displaystyle\sum_{j=1}^{N} p_j e^{-|q_i-q_j|}, \\ \dot{p}_i = p_i \displaystyle\sum_{j=1}^{N} p_j \, \text{sign}(q_i - q_j) e^{-|q_i-q_j|}, \end{cases} \tag{4}$$

with Hamiltonian $H = \sum_{i,j} p_i p_j e^{-|q_i-q_j|}$.

However, a general initial data contains both positive and negative peakons, as in the example of the peakon-antipeakon interaction: one positive peakon with strength p, centered in $-q$, moves forward and one negative anti-peakon in q, with strength $-p$ moves backward. The evolution of the system produces the overlapping of the two peakons at finite time $t = \tau$, so that $q \to 0$ (see Figure 1).

There are infinitely many ways to extend the solution after the time τ of the interaction, for example the vanishing viscosity approach in [17, 16] singles out the *dissipative* solutions. As far as the example of peakon-antipeakon interaction is concerned, the vanishing viscosity approach selects the solution that, after the time τ, is $\equiv 0$: all the energy E is lost. In section 2.1 we shall construct a *conservative* solution, i.e. a solution for which the quantity E is constant for a.e. time t. At the interaction time the energy $E \doteq \|u\|_{H^1}^2$ is described

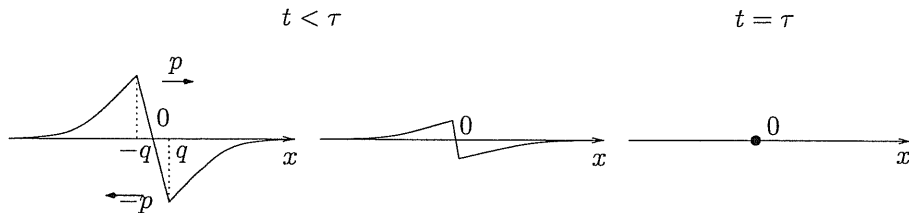$$t < \tau \qquad\qquad\qquad\qquad t = \tau$$

Figure 1: Peakon-antipeakon interaction.

by a Dirac measure entirely concentrated at the single interaction point. After the interaction, a positive and a negative peakon emerge, whose strengths are uniquely determined by imposing the conservation of the total energy.

In Chapter 3 we shall discuss the issue of the uniqueness and the stability.

Stability. The multipeakon solutions form a continuous semigroup whose domain is dense either in $H^1(\mathbb{R})$ or in H^1_{per}. The main novel feature in our approach is the construction of a metric $J(\cdot,\cdot)$ on the space H^1 (or H^1_{per}) determined by an optimal transportation problem. While the semigroup generated by (3) is not eve continuous w.r.t. the H^1 distance, we show that it is Lipschitz continuous w.r.t. our new distance functional J. The reader can see [47] for earlier applications of distances defined in term of optimal transportation problems, and [8], in which the authors recover a semigroup of dissipative solution for the Hunter-Saxton equation [34] (see also [11] for a fixed-point approach for both conservative and dissipative solutions).

The main well-posedness result is provided by a *Gronwall-type lemma*, stemming from the inequality

$$\frac{d}{dt} J(u,v) \le C(t) \cdot J(u,v)$$

whenever u and v are two multipeakon solutions (see Section 3.1.2).

Uniqueness. Example 3.1 in Section 3.5 shows that a solution of 3 need not be unique. Roughly speaking, every shifted antipeakon-peakon couple is also a conservative solution. A conservative solution can be characterized by an additional linear transport equation, accounting for the conservation of the total energy. It can be done by the following heuristic idea.

We can think that the absolutely continuous measure μ_t, which satisfies $d\mu_t = (u^2 + u_x^2)d\mathcal{L}$ tends, as $t \to \tau$, to a Dirac measure with support in 0. We introduce thus a further equation for the measure μ_t, whose absolutely continuous part is $u^2 + u_x^2$, in the following way. Since whenever $u^2 + u_x^2$ is regular it satisfies the equation

$$(u^2 + u_x^2)_t + \left[u(u^2 + u_x^2)\right]_x = \left[u^3 - 2ue^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right)\right]_x \doteq f(u)$$

it suggests that μ_t provides a measure-valued solution of

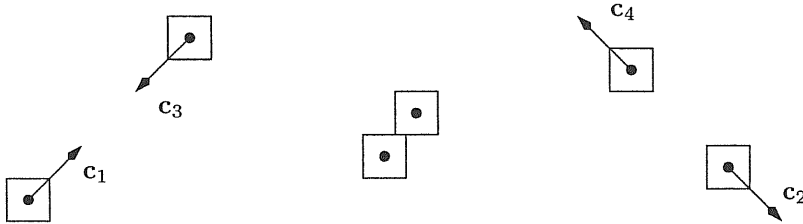$$\partial_t \mu + (u\mu)_x = f(u). \tag{5}$$

Figure 2: Two particle interaction

Our result (Theorem 1.3) shows that every solution (u, μ_t) of (3)-(5), such that the absolutely continuous part of μ_t has density $u^2 + u_x^2$, must coincide with the one provided by multipeakon approach.

Part II of the thesis is devoted to the analysis of blow-up for the discrete Boltzmann equation (2). Such a equation is obtained by considering a rarefied gas for which is supposed that the particles can move only along a finite number of direction characterized by the vectors $c_1, \ldots, c_N$. The unknowns u_i represent the densities of particles which travel at speed c_i. By a collision, a pair of incoming particles with speeds c_i, c_j is replaced by a new pair of particles say c_k, c_ℓ. The rate at which such collision occur is given by $a_{ijk} u_j u_k$ The concentration u_i is thus increasing (or at least is constant) when interact particles of speed different to c_i, decreasing when an i−particle collides with someone other. Then, the coefficients a_{ijk} are non negative when $j, k \neq i$ and negative when either $j = i$ or $k = i$.

If the initial data is suitably small, the solution remains uniformly bounded for all times [4]. For large initial data, on the other hand, the global existence and stability of solutions are known only in the one-dimensional case [3, 32, 45]. Since the right hand side has quadratic growth, it might happen that the solution blows up in finite time. Examples where the L^∞ norm of the solution becomes arbitrarily large as $t \to \infty$ are easy to construct [35].

In Chapter 5 (see [9]) we focus our analysis on the *two-dimensional Broadwell model* (see, for example, [12, 46, 15] for a description of the model) and examine the possibility that blow-up actually occurs in finite time. In this model the permitted direction are

$$c_1 = (1, 1), \quad c_2 = (1, -1), \quad c_3 = (-1, -1), \quad c_4 = (-1, 1)$$

and the particles have a diamond-shape (see Figure 2).

As we will show with the theory developed in Chapter 4, since the equations (2) admit a natural symmetry group (see Section 4.3, and [42] for a more general theory), one can perform an asymptotic rescaling of variables and ask whether there is a blow-up solution which, in the rescaled variables, converges to a steady state. This technique has been widely used to study blow-up singularities of reaction-diffusion equations with superlinear forcing terms [30, 31]. See also [36] for an example of self-similar blow-up for hyperbolic conservation laws.

Our main results show is an *a-priori* bound on the blow up rate in the L^∞ norm. Namely, if blow-up occurs at time T, then one has

$$\left\|u(t)\right\|_{\mathbf{L}^\infty} > \frac{1}{5}\frac{\ln\left|\ln(T-t)\right|}{T-t}.$$

This means that the blow-up rate must be different from the natural growth rate $\left\|u(t)\right\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\cdot(T-t)^{-1}$ which would be obtained in case of a quadratic equation $\dot{u} = C\,u^2$.

In the final section of Chapter 5 we discuss a possible scenario for blow-up. The analysis highlights how carefully chosen should be the initial data, if blow-up is ever to happen. This suggests that finite time blow-up is a highly non-generic phenomenon, something one would not expect to encounter in numerical simulations.

Part I

The Camassa-Holm equation

Chapter 1

The Camassa-Holm equation

The Camassa-Holm equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

arises from a higher order level of approximation of the asymptotic expansion of the Euler's equations for a shallow water wave theory. Here we do not enter into deep details of the interpretation of such a equation, for the physical motivations we refer to [13], [22], [23], [37].

In the following we focus our attention and we refer to *Camassa-Holm equation* the previous equation with $\kappa = 0$.

1.1   Non-local formulation

The Camassa-Holm equation can be written as a scalar conservation law with an additional integro-differential term:

$$u_t + (u^2/2)_x + P_x = 0 \,, \tag{1.1}$$

where P is defined as a convolution:

$$P(x) \doteq \frac{1}{2} e^{-|x|} * \left(u^2 + \frac{u_x^2}{2} \right) \,. \tag{1.2}$$

Earlier results on the existence and uniqueness of solutions can be found in [48], [49]. One can regard (1.1) as an evolution equation on a space of absolutely continuous functions with derivatives $u_x \in L^2$. In the smooth case, differentiating (1.1) w.r.t. x one obtains

$$u_{xt} + uu_{xx} + u_x^2 - \left(u^2 + \frac{u_x^2}{2} \right) + P = 0 \,. \tag{1.3}$$

Multiplying (1.1) by u and (1.3) by u_x we obtain the two conservation laws with source term

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + uP\right)_x = u_x P, \tag{1.4}$$

$$\left(\frac{u_x^2}{2}\right)_t + \left(\frac{uu_x^2}{2} - \frac{u^3}{3}\right)_x = -u_x P. \tag{1.5}$$

As a consequence, for regular solutions the total energy

$$E(t) \doteq \int \left[u^2(t,x) + u_x^2(t,x)\right] dx$$

remains constant in time.

As in the case of scalar conservation laws, by the strong nonlinearity of the equations, solutions with smooth initial data can lose regularity in finite time. For the Camassa-Holm equation (1.1), however, the uniform bound on $\|u_x\|_{L^2}$ guarantees that only the L^∞ norm of the gradient can blow up, while the solution u itself remains Hölder continuous at all times.

In order to construct global in time solutions, two main approaches have recently been introduced. On one hand, one can add a small diffusion term in the right hand side of (1.1), and recover solutions of the original equations as a vanishing viscosity limit [17, 16]. An alternative technique, developed in [7], relies on a new set of independent and dependent variables, specifically designed with the aim of "resolving" all singularities. In terms of these new variables, the solution to the Cauchy problem becomes regular for all times, and can be obtained as the unique fixed point of a contractive transformation.

In the present chapter, we implement yet another approach to the Camassa-Holm equation. As a starting point we consider all multi-peakon solutions, of the form

$$u(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x - q_i(t)|}. \tag{1.6}$$

These are obtained by solving the system of O.D.E's

$$\begin{cases} \dot{q}_i = \sum_j p_j \, e^{-|q_i - q_j|}, \\ \dot{p}_i = \sum_{j \neq i} p_i p_j \, \text{sign}(q_i - q_j) \, e^{-|q_i - q_j|}. \end{cases} \tag{1.7}$$

It is well known that this can be written in hamiltonian form:

$$\begin{cases} \dot{q}_i = \frac{\partial}{\partial p_i} H(p,q), \\ \dot{p}_i = -\frac{\partial}{\partial q_i} H(p,q), \end{cases} \qquad H(p,q) \doteq \frac{1}{2} \sum_{i,j} p_i p_j e^{-|q_i - q_j|}.$$

If all the coefficients p_i are initially positive, then they remain positive and bounded for all times. The solution $u = u(t,x)$ is thus uniformly Lipschitz

continuous. We stress, however, that here we are not making any assumption about the signs of the p_i. In a typical situation, two peakons can cross at a finite time τ. As $t \to \tau-$ their strengths p_i, p_j and positions q_i, q_j will satisfy

$$p_i(t) \to +\infty, \qquad p_j(t) \to -\infty, \qquad p_i(t) + p_j(t) \to \bar{p}, \qquad (1.8)$$

$$q_i(t) \to \bar{q}, \quad q_j(t) \to \bar{q}, \qquad q_i(t) < q_j(t) \text{ for } t < \tau, \qquad (1.9)$$

for some $\bar{p}, \bar{q} \in \mathbb{R}$. Moreover, $\left\| u_x(t) \right\|_{L^\infty} \to \infty$. In this case, we will show that there exists a unique way to extend the multi-peakon solution beyond the interaction time, so that the total energy is conserved.

Having constructed a set of "multi-peakon solutions", our main goal is to show that these solutions form a continuous semigroup, whose domain is dense in the space $H^1(\mathbb{R})$. Taking the unique continuous extension, we thus obtain a continuous semigroup of solutions of (1.1), defined on the entire space H^1.

One easily checks that the flow map $\Phi_t : u(0) \mapsto u(t)$ cannot be continuous as a map from H^1 into itself, or from L^2 into itself. Distances defined in terms of convex norms perform well in connection with linear problems, but occasionally fail when nonlinear features become dominant. In the present setting, we construct a new distance $J(u, v)$ between functions $u, v \in H^1$, defined by a problem of optimal transportation. Roughly speaking, $J(u, v)$ will be the minimum cost in order to transport the mass distribution with density $1 + u_x^2$ located on the graph of u onto the mass distribution with density $1 + v_x^2$ located on the graph of v. See Section 3.1 for details. With this definition of distance, our main result shows that

$$\left| \frac{d}{dt} J\big(u(t), v(t)\big) \right| \leq C \cdot J\big(u(t), v(t)\big)$$

for some constant C and any couple of multi-peakon solutions u, v. Moreover, $J(u_n, u) \to 0$ implies the uniform convergence $\|u_n - u\|_{L^\infty} \to 0$. The distance functional J thus provides the ideal tool to measure continuous dependence on the initial data for solutions to the Camassa-Holm equation. Earlier applications of distances defined in terms of optimal transportation problems can be found in the monograph [47]. The issue of uniqueness of solutions must here be discussed in greater detail. For a multi-peakon solution, as long as all coefficients p_i remain bounded, the solution to the system of ODE's (1.7) is clearly unique. For each time t, call μ_t the measure having density $u^2(t) + u_x^2(t)$ w.r.t. Lebesgue measure. Consider a time τ where a positive and a negative peakon collide, according to (1.8)-(1.9). As $t \to \tau-$, we have the weak convergence $\mu_t \rightharpoonup \mu_\tau$ for some positive measure μ_τ which typically contains a Dirac mass at the point $\bar{q}$. By energy conservation, we thus have

$$\int \left[u^2(\tau, x) + u_x^2(\tau, x) \right] dx + \mu_\tau\big(\{\bar{q}\}\big) = \lim_{t \to \tau-} \int \left[u^2(\tau, x) + u_x^2(\tau, x) \right] dx = E(\tau-).$$

There are now two natural ways to prolong the multi-peakon solution beyond time τ: a conservative solution, such that

$$E(t) = \int \left[u^2(t, x) + u_x^2(t, x) \right] dx = E(\tau-) \qquad t > \tau,$$

or a dissipative solution, where all the energy concentrated at the point $\bar{q}$ is lost. In this case

$$E(t) = \int \left[u^2(t,x) + u_x^2(t,x) \right] dx = E(\tau-) - \mu_\tau(\{\bar{q}\}) \qquad t > \tau.$$

For $t > \tau$, the dissipative solution is obtained by simply replacing the two peakons p_i, p_j with one single peakon of strength $\bar{p}$, located at $x = \bar{q}$. On the other hand, as we will show in Section 2.1, the conservative solution contains two peakons emerging from the point $\bar{q}$. As $t \to \tau+$, their strengths and positions satisfy again (1.8), while (1.9) is replaced by

$$q_i(t) \to \bar{q}, \quad q_j(t) \to \bar{q}, \qquad q_i(t) > q_j(t) \ \text{ for } \ t > \tau. \tag{1.10}$$

The vanishing viscosity approach in [17, 16] singles out the dissipative solutions. These can also be characterized by the Oleinik type estimate

$$u_x(t,x) \leq C(1 + t^{-1}),$$

valid for $t > 0$ at a.e. $x \in \mathbb{R}$. On the other hand, the coordinate transformation approach in [7] and the present one, based on optimal transport metrics, appear to be well suited for the study of both conservative and dissipative solutions.

In the following chapters we focus on conservative solutions to the Camassa-Holm equation. We start with the study of the spatially periodicity because it allows us to concentrate on the heart of the matter, i.e. the uniqueness and stability of solutions beyond the time of singularity formation. It will spare us some technicalities, such as the analysis of the tail decay of u, u_x as $x \to \pm\infty$. In this respect we shall discuss the decay analysis of solutions in Section 3.2 of Chapter 3.

The main ingredients can already be found in the paper [8], devoted to dissipative solutions of the Hunter-Saxton equation.

As initial data, we take

$$u(0,x) = \bar{u}(x), \tag{1.11}$$

with $\bar{u}$ in the space H^1 of absolutely continuous functions u with derivative $u_x \in L^2$. To fix the ideas, we assume that the period of a spatially periodic function in the space H^1_{per} is 1, so that

$$u(x+1) = u(x) \qquad x \in \mathbb{R}.$$

On H^1_{per} we shall use the norm

$$\|u\|_{H^1_{\text{per}}} \doteq \left(\int_0^1 |u(x)|^2 \, dx + \int_0^1 |u_x(x)|^2 \, dx \right)^{1/2}.$$

1.1.1   The main results

In this section we state the main results of Part I of this thesis. We shall write them for the spatially periodic case.

Theorem 1.1. *For each initial data $\bar{u} \in H^1_{\mathrm{per}}$, there exists a solution $u(\cdot)$ of the Cauchy problem (1.1), (1.11). Namely, the map $t \mapsto u(t)$ is Lipschitz continuous from $\mathbb{R}$ into L^2_{per}, satisfies (1.11) at time $t = 0$, and the identity*

$$\frac{d}{dt} u = -u u_x - P_x \tag{1.12}$$

is satisfied as an equality between elements in L^2_{per} at a.e. time $t \in \mathbb{R}$. This same map $t \mapsto u(t)$ is continuously differentiable from $\mathbb{R}$ into L^p_{per} and satisfies (1.12) at a.e. time $t \in \mathbb{R}$, for all $p \in [1, 2[\,$. The above solution is conservative in the sense that, for a.e. $t \in \mathbb{R}$,

$$E(t) = \int_0^1 \left[u^2(t, x) + u_x^2(t, x) \right] dx = E^{\bar{u}} \doteq \int_0^1 \left[\bar{u}^2(x) + \bar{u}_x^2(x) \right] dx. \tag{1.13}$$

Theorem 1.2. *Conservative solutions to (1.1) can be constructed so that they constitute a continuous flow Φ. Namely, there exists a distance functional J on H^1_{per} such that*

$$\frac{1}{C} \cdot \|u - v\|_{L^1_{\mathrm{per}}} \leq J(u, v) \leq C \cdot \|u - v\|_{H^1_{\mathrm{per}}} \tag{1.14}$$

for all $u, v \in H^1_{\mathrm{per}}$ and some constant C uniformly valid on bounded sets of H^1_{per}. Moreover, for any two solutions $u(t) = \Phi_t \bar{u}$, $v(t) = \Phi_t \bar{v}$ of (1.1), the map $t \mapsto J\big(u(t), v(t)\big)$ satisfies

$$J\big(u(t), \bar{u}\big) \leq C_1 \cdot |t|, \tag{1.15}$$

$$J\big(u(t), v(t)\big) \leq J(\bar{u}, \bar{v}) \cdot e^{C_2 |t|} \tag{1.16}$$

for a.e. $t \in \mathbb{R}$ and constants C_1, C_2, uniformly valid as u, v range on bounded sets of H^1_{per}.

The previous results can be extended to the following space of functions which exponential decay: let $\alpha \in]0, 1[$, then

$$X_\alpha \doteq \left\{ u \in H^{(}\mathbb{R}) \ \text{s.t.} \ \int_{\mathbb{R}} [u^2(x) + u_x^2(x)] e^{\alpha |x|} \right\}.$$

It is not so restrictive one can think, in fact the peakon functions (see Section 2.1), the natural solitary waves of the Camassa-Holm equation which have the *soliton* properties, belong to it.

Somewhat surprisingly, all the properties stated in Theorem 1.1 are still not strong enough to single out a unique solution. To achieve uniqueness, an additional condition is needed.

Theorem 1.3. *Conservative solutions $t \mapsto u(t)$ of (1.1) can be constructed with the following additional property:*

For each $t \in \mathbb{R}$, call μ_t the absolutely continuous measure having density $u^2 + u_x^2$ w.r.t. Lebesgue measure. Then, by possibly redefining μ_t on a set of times of measure zero, the map $t \mapsto \mu_t$ is continuous w.r.t. the topology of weak convergence of measures. It provides a measure-valued solution to the conservation law

$$w_t + (uw)_x = (u^3 - 2uP^u)_x \,.$$

The solution of the Cauchy problem (1.1), (1.11) satisfying the properties stated in Theorem 1.1 and this additional condition is unique.

In Section 2.1 we derive some elementary properties of multi-peakon solutions and show that any initial data can be approximated in H^1_{per} by a finite sum of peakons. In Section 3.1 we introduce our distance functional $J(u, v)$ and study its relations with other distances defined by Sobolev norms. The continuity of the flow (1.1), together with the key estimates (1.15)-(1.16) are then proved in the following two sections. The proofs of Theorems 1.1 and 1.2 are completed in Section 3.4. The uniqueness result stated in Theorem 1.3 is proved in Section 3.5. As a corollary, we also show that in a multi-peakon solution the only possible interactions involve exactly two peakons: one positive and one negative. In particular, no triple interactions can ever occur.

Now we dedicate the rest of this chapter to exhibit an example which will be the start point of the technique we shall develop in Chapter 3. In particular, we shall see how the *optimal transportation theory* can be useful nonlinear equation like Camassa-Holm equation. The key point is to define a *Monge-Kantorovich* like metric for a space of Radon measures. In Section 1.2 we start from the *Hunter-Saxton* equation [8] for give a brief heuristic idea for how this technique is involved in.

1.2   The Hunter-Saxton equation

The Hunter-Saxton equation describes the propagation of waves in a massive vector field of a nematic liquid crystal. Since the physical interpretation and its derivation are beyond to the description of this thesis, we refer to [34, 8]. It can be written in a non-local formulation as a conservation law with a source term:

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{1}{4}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right) u_x^2(t, y)\, dy \doteq Q^u(t, x) \qquad (1.17)$$

where

- $t \geq 0$ is the time variable,

- $x \in \mathbb{R}$ is the space variable in a reference frame,

- $u(t,x) \in \mathbb{R}$ is related to the orientation of the liquid crystal molecules in the position x at time t.

Suppose that there exists a smooth solution to (1.17). To the Hunter-Saxton equation we can associate the following two conservation laws

$$(u_x)_t + (uu_x)_x = \frac{u_x^2}{2}, \tag{1.18}$$

$$(u_x^2)_t + (uu_x^2)_x = 0, \tag{1.19}$$

which are obtained by computing the derivative of the equation (1.17) w.r.t. the x variable and then multiplying it by u_x to achieve the second one.

A further conservation law is satisfied by the source term $Q^u(t,x)$. Since for smooth solutions (1.19) yields the conservation of the energy

$$E_0 \equiv E(t) \doteq \int_{\mathbb{R}} u_x^2(t,x)\, dx$$

the function Q^u can be expressed in the following way

$$Q(t,x) = -\frac{E_0}{4} + \frac{1}{2} \int_{-\infty}^{x} u_x^2(t,y)\, dy.$$

By deriving w.r.t. t we obtain

$$Q_t + uQ_x = 0. \tag{1.20}$$

The function Q^u is constant along the characteristic curves $\xi_u(t,y)$ defined by

$$\frac{\partial}{\partial t}\xi_u(t,y) = u(t,\xi_u(t,y)), \qquad \xi_u(0,y) = y. \tag{1.21}$$

Let us remark that the previous equations holds for all the time t in which u is a classical solution. It can be seen by the method of characteristics that if $u_0 \neq 0$ is a smooth initial data and for some x_0 we have $u_{0x}(x_0) < 0$, along its outgoing characteristic the gradient blows up in finite time. Since the quantity $E(t)$ remains bounded also at the time of blow up, we can think that a finite amount of energy will be concentrated at the point of blow-up. In [8] the authors focus their attention on solutions that *dissipate* this quantity of energy. As far as the conservative solution is concerned, equation (1.19) will be satisfied in sense of measures, i.e. it means that thinking at a measure μ_t with absolutely continuous part which satisfies $d\mu_t = u_x^2(t,\cdot)d\mathcal{L}$ (here with $\mathcal{L}$ we indicate the Lebesgue measure), it satisfies

$$\partial_t \mu_t + \partial_x(u\mu_t) = 0 \qquad \text{in } \mathcal{D}'.$$

To find a *conservative solution* to the Cauchy problem (1.17) with finite energy smooth initial condition u_0 means then to find a couple $(u(t),\mu_t)$ which satisfies

the following system of conservation laws: let μ_0 be the absolutely continuous measure w.r.t. Lebesgue measure defined by $d\mu_0 = u_{0x}^2 d\mathcal{L}$, then

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2} \right) = -\dfrac{1}{4}\mu_t(\mathbb{R}) + \dfrac{1}{2}\mu_t(]-\infty, x]) & u(0,x) = \bar{u}(x), \\ \partial_t \mu_t + \partial_x(u\mu_t) = 0 & \mu_t|_{t=0} = \mu_0. \end{cases} \quad (1.22)$$

Due to the nonlinearity of the problem, as shows [8, Example 2] for the dissipative solution of the Hunter-Saxton equation, we can aspect that the usual "strong" distance stemming from convex norm is not useful in order to construct a continuous semigroup of solutions. In the following section, we give a sketch of the construction of a metric which yields continuity of solution with respect to the initial data.

1.2.1   A transportation map

Let u_0 and v_0 be two initial data whose associated measures μ_0^1 and μ_0^2 have the same total mass $\mu_0^1(\mathbb{R}) = \mu_0^2(\mathbb{R})$. Suppose that such a initial data are not constant in any interval of $\mathbb{R}$, so that the functions Q^{u_0} and Q^{v_0} are absolutely continuous and increasing.
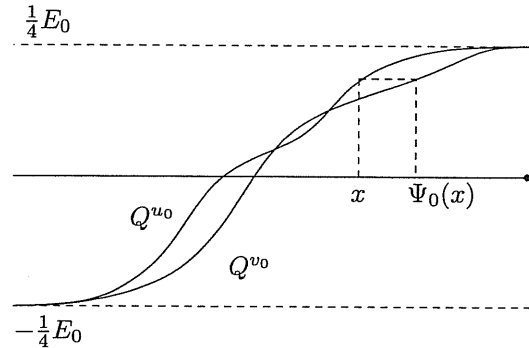
Figure 1.1: The map Ψ_0

We can thus define a continuous map Ψ_0 for which at every $x \in \mathbb{R}$ it associates the unique point $\Psi_0(x)$ such that

$$Q^{u_0}(x) = Q^{v_0}(\Psi_0(x)) \quad (1.23)$$

(see Figure 1.1). Let us Remark that the Ψ_0 is an increasing function, in fact since Q^{u_0} and Q^{v_0} are increasing, then $x < y$ implies

$$Q^{v_0}(\Psi_0(x)) = Q^{u_0}(x) < Q^{u_0}(y) = Q^{v_0}(\Psi_0(y)),$$

so $\Psi_0(x) < \Psi_0(y)$ holds.

Now we want to see how the map Ψ_0 evolves in time. Since by (1.20) Q^u, Q^v are conserved along the caracteristic curves (1.21), the equalities

$$Q^u(t, \xi_u(t,x)) = Q^{u_0}(x), \qquad Q^v(t, \xi_v(t,x)) = Q^{v_0}(x) \qquad \text{for all } x \in \mathbb{R}$$

yields the definition of the *transportation map*

$$\Psi_t(\xi_u(t,x)) \doteq \xi_v(t, \Psi_0(x)). \tag{1.24}$$

1.2.2 The stability issue of a system of ODE

As in [7, Section 3], we compute a change of variables in order to obtain a system of ODE with Lipschitz vector field. Let suppose that the initial data u_0 is in $H^1(\mathbb{R})$. Let set

$$\omega \doteq 2\arctan(u_x),$$

then ω belong to the unit circle $\mathbf{T} \doteq [0, 2\pi]$, with 0 and 2π identified. Computing the derivative of ω along the characteristics, having in mind the equation (1.18) for u_x we have

$$\frac{d}{dt}\omega(t, \xi(t,y)) = 2\frac{(u_x)_t + u(u_x)_x}{1 + u_x^2} = \frac{-u_x^2}{1 + u_x^2} = \frac{-\tan^2(\omega/2)}{1 + \tan^2(\omega/2)} = -\sin^2(\omega/2).$$

It is thus natural to consider the new unknowns which take values in the space $\mathbb{R}^2 \times \mathbf{T}$,

$$\mathbf{X}^u = \mathbf{X}^u(t,y) \doteq \begin{pmatrix} \xi_u(t,y) \\ u(t, \xi_u(t,y)) \\ \omega_u(t, \xi_u(t,y)) \end{pmatrix}$$

and write the corresponding Cauchy problem

$$\frac{d}{dt}\mathbf{X}^u(t,y) = \begin{pmatrix} u(t, \xi_u(t,y)) \\ Q^u(t, \xi_u(t,y)) \\ \omega_u(t, \xi_u(t,y)) \end{pmatrix} = f(\mathbf{X}^u(t,y)) \tag{1.25}$$

with initial data

$$\mathbf{X}^u(0,y) = \begin{pmatrix} y \\ u_0(y) \\ 2\arctan(u_{0x}(y)) \end{pmatrix}. \tag{1.26}$$

We remark that by definition, Q^u is far from to be Lipschitz continuous, then we cannot suppose that in the previous system of ODE the function f is a Lipschitz vector field. Hence, to overcome this lack of Lipschitz continuity, we shall make use of the function Ψ_t introduced in 1.2.1. Let $\mathbf{X}^u$ and $\mathbf{X}^v$ be two solution of the Cauchy problem (1.25)-(1.26), corresponding to the initial data u_0, v_0 respectively. We allow ourselves to make an abuse of notation by defining the map Ψ_t in the following way:

$$\Psi_t(\mathbf{X}^u(t,y)) \doteq \mathbf{X}^v(t, \Psi_0(y)).$$

We gain a sort of Lipschitz continuity for the function f if we restrict it on the manifold located by Ψ_t. Let us compute the difference of the vector field f evaluated in the points $\mathbf{X}^u(t,y)$ and $\Psi_t(\mathbf{X}^u(t,y))$.

$$|f(\mathbf{X}^u(t,y)) - f(\Psi_t(\mathbf{X}^u(t,y)))| = \left| \begin{pmatrix} u(t, \xi_u(t,y)) - v(t, \xi_v(t, \Psi_0(y))) \\ Q^u(t, \xi_u(t,y)) - Q^v(t, \xi_v(t, \Psi_0(y))) \\ \omega_u(t, \xi_u(t,y)) - \omega_v(t, \xi_v(t, \Psi_0(y))) \end{pmatrix} \right|$$

since by definition of the map Ψ_0 and by the equation (1.20) we have the identity

$$Q^u(t, \xi_u(t, y)) - Q^v(t, \xi_v(t, \Psi_0(y))) \equiv 0$$

we deduce the following estimate for the vector field f

$$|f(\mathbf{X}^u(t, y)) - f(\Psi_t(\mathbf{X}^u(t, y)))| \leq |\mathbf{X}^u(t, y) - \mathbf{X}^u(t, y)|.$$

From the previous inequality we can prove that the difference of the two solutions $X^u(t, y)$ and $X^v(t, \Psi_0(y))$ can be estimated by the initial data. In fact, Gronwall Lemma applied to the inequality

$$|\mathbf{X}^u(t, y) - \mathbf{X}^v(t, \Psi_0(y))| \leq \; |\mathbf{X}^u(0, y) - \mathbf{X}^v(0, \Psi_0(y))| \\ + \int_0^t |f(\mathbf{X}^u(s, y)) - f(\mathbf{X}^v(s, \Psi_0(y)))|\, ds$$

yields the estimate

$$|\mathbf{X}^u(t, y) - \mathbf{X}^v(t, \Psi_0(y))| \leq e^t |\mathbf{X}^{u_0}(y) - \mathbf{X}^{u_0}(\Psi_0(y))|.$$

The previous inequality suggests how to introduce a new distance in order to obtain a stability result for solutions of the Hunter-Saxton equation. For every $u \in H^1(\mathbb{R})$ let define the measure

$$\mu^u(A) \doteq \int_{\{x \in \mathbb{R}\,:\,(x, u(x), \omega(x)) \in A\}} u_x^2(x)\, dx$$

for every Borel set $A \subset \mathbb{R}^2 \times \mathbf{T}$. The function Ψ_0 can be regarded as a transportation map which transports the measure μ^u into the measure μ^v. The distance we shall introduce in Chapter 3 will be thus a sort of Wasserstein distance between measure [47].

Chapter 2

Conservative multi-peakon solution

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation then suddenly leaving it behind, rolled forward with great velocity, assuming the form of large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation..."

John Scott Russell, 1844

It was Scott Russell [44] who introduces the concept of *solitary waves* to indicate no more than wave which propagate without change of form and have some localized shape (see also, [26, 27]). His experiment is described in fig. 2.1. For more than sixty years it was only a pure scientific curiosity, until Korteweg and de Vries [38] derived the equation for the propagation of waves in one direction on the surface of a shallow canal. The profile of the travelling wave solution with permanent shape found by Miura [40] is precisely the shape of the wave which Scott Russel observed in his experiments. The term *soliton* is substantially different from solitary wave, it was introduced in 1965 by Zabusky and Kruskal [50] to indicate waves that whenever collide each other they do not
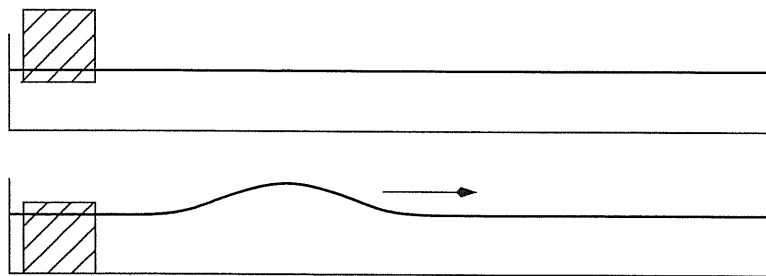
Figure 2.1: Diagram of Scott Russell's experiment to generate a solitary wave

break up and disperse, but remains almost identical to a solitary wave solution. In this chapter we investigate the shape of solitary waves for the Camassa-Holm equation (1.1).

2.1   Multi-peakon solution in the real line

In this section we shall construct a solution of the Camassa-Holm equation starting from an initial condition u_0 of the form

$$u_0(x) = \sum_{j=1}^{N} p_j e^{-|x-q_j|} .$$

The motivation of this choice is given by the shape of traveling wave solution (see [13] and [20, Example 5.2]). Looking for solution of the equation (1.1) in the traveling wave form $u(t,x) = U(x - ct)$, with a function U that vanishes at infinity, the limit of $\kappa \to 0$ leads to the function $U = ce^{-|x-ct|}$. This is not a solitary wave in the sense introduced by Scott Russell because of the presence of the cuspid at the position $x = ct$. However, the evolution of an initial data like u_0 remains of the same shape [13, 24]. It is a superposition of peaked solitary waves, which evolves as

$$u(t,x) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|} .$$

Hence we term *peakon* a peaked solitary wave to emphasize the soliton properties of such a function.

As long as the classical solution of the problem

$$
\begin{cases}
\dot{q}_i = \displaystyle\sum_{j=1}^{N} p_j e^{-|q_i-q_j|} , \\
\dot{p}_i = p_i \displaystyle\sum_{j=1}^{N} p_j \operatorname{sign}(q_i - q_j) e^{-|q_i-q_j|}
\end{cases}
\tag{2.1}
$$

exists, the solution of this system gives the coefficients $\mathbf{p}(t) = (p_1, \ldots, p_N)$ and $\mathbf{q}(t) = (q_1, \ldots, q_N)$ for the solution $u(t, x)$ to the Camassa-Holm equation. Let us observe that the previous system can be viewed as an Hamiltonian system with Hamiltonian function $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i,j} p_i p_j e^{-|q_i - q_j|}$.

In [33] the authors prove the existence of a global multipeakon solution when strengths p_i are positive for all $i = 1 \ldots N$ and the convergence of the sequence of multipeakon solution. If u_0 is an initial data such that the distribution $u_0 - u_{0xx}$ is a *positive* Radon measure, there exists a sequence of multipeakons that converges in $L^\infty(\mathbb{R}, H^1_{loc}(\mathbb{R}))$. In this case the crucial fact is that no interaction between the peakons occurs, and then the gradient remains bounded. However, a general initial data contains both positive and negative peakons, as in the so called peakon-antipeakon interaction: one positive peakon with strength p, centered in $-q$, moves forward and one negative anti-peakon in q, with strength p moves backward (fig. 2.2). The evolution of the system produces the overlapping of the two peakons at finite time $t = \tau$, so that $q \to 0$. The conservation of the energy $E = H(\mathbf{q}(t), \mathbf{p}(t))$ yields

$$E = \lim_{t \to \tau^-} p^2 (1 - e^{-2|q|}). \tag{2.2}$$

and then the quantity p blows up in finite time. At the point $(\tau, 0)$ occurs thus a singularity for the solution u. To extend the solution also after the interaction time with a solution which conserves the energy E we can think that at the interaction point an antipeakon/peakon couple emerge, the first, negative, moving backward and the second, positive, moving forward with coefficients $(-q, -p)$ and (q, p). According to the conservation of the energy, the choice of q and p must satisfy (2.2) as $t \to \tau^+$. It yields a change of variables which resolves the singularity at $(\tau, 0)$

$$\zeta \doteq p^2 q \qquad \omega \doteq \arctan(p)$$

with this choice, the Hamiltonian system leads to the ODE

$$\frac{d}{dt} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = f(\zeta, \omega), \qquad \begin{pmatrix} \zeta \\ \omega \end{pmatrix}(\tau) = \begin{pmatrix} \frac{E}{2} \\ \frac{\pi}{2} \end{pmatrix}$$

with

$$f(\zeta, \omega) = \begin{pmatrix} \left[1 - e^{-\zeta \cot^2(\omega)} - \zeta \cot^2(\omega) e^{-\zeta \cot^2(\omega)} \right] \tan^3(\omega) \\ \sin^2(\omega) e^{-\zeta \cot^2(\omega)} \end{pmatrix}$$

and f is a Lipschitz vector field in a neighborhood of the point $(\frac{E}{2}, \frac{\pi}{2})$. The solution $(\zeta(t), \omega(t))$ of this problem provides then the unique couple $(q(t), p(t))$ which coincides with the classical solution of the Hamiltonian system for $t < \tau$ and extends it for $t \geq \tau$.

This example suggests the way to construct the multipeakon solution whenever an interaction between peakons occurs (see also [10] for an "energetic"

$$t < \tau$$
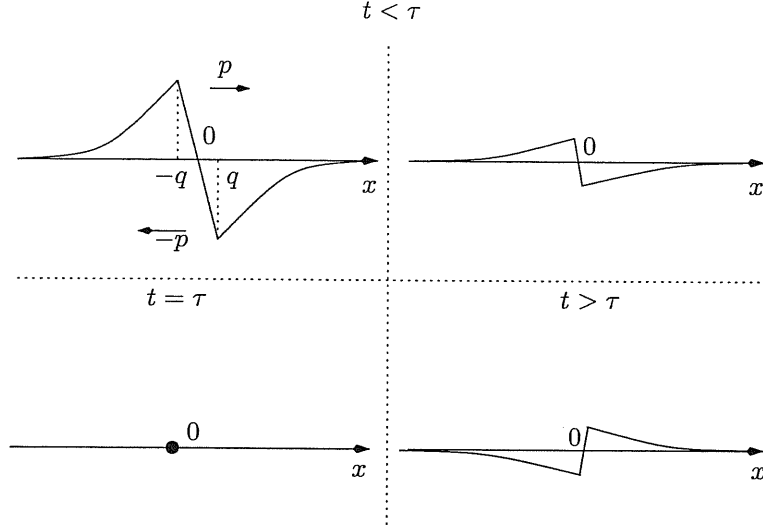
Figure 2.2: Peakon-antipeakon interaction

motivation). Suppose that two or more peakons with strengths $p_1, \ldots, p_k$ annihilate at the position $\bar{q}$ at time τ and produce a blow up of the gradient u_x. The conservation of the energy yields that there exists and is positive the limit

$$e_\tau \doteq \lim_{t \to \tau^-} \int_{\xi^-(t)}^{\xi^+(t)} u_x^2(t, x)\, dx$$

where ξ^- and ξ^+ are the smallest and the largest characteristic curve passing through the point $(\tau, \bar{q})$. Assume that after the interaction two peakons appear with strengths p_1, p_2 and placed at the position q_1, q_2. Let consider the change of variables

$$z = p_2 + p_1 \quad w = 2\arctan(p_2 - p_1) \quad \eta = q_2 + q_1 \quad \zeta = (p_2 - p_1)^2(q_2 - q_1),$$

then the system (2.1) turns out to be

$$\dot{w} = -\left[\sin(w)\cosh\left(\frac{\zeta}{2\tan^2(w/2)}\right) + 2z\sinh\left(\frac{\zeta}{2\tan^2(w/2)}\right)\right] \cdot \sum_{j \geq k+1} p_j e^{-q_j + \eta/2}$$

$$+ [z^2\cos^2(w/2) - \sin^2(w/2)]e^{-\frac{\zeta}{\tan^2(w/2)}}$$

$$\dot{z} = -\left[\tfrac{1}{2}\sin(w)\sinh\left(\frac{\zeta}{2\tan^2(w/2)}\right) + z\cosh\left(\frac{\zeta}{2\tan^2(w/2)}\right)\right] \cdot \sum_{j \geq k+1} p_j e^{-q_j}$$

$$\dot{\eta} = z[1 + e^{-\frac{\zeta}{\tan^2(w/2)}}] + 2\cosh\left(\frac{\zeta}{2\tan^2(w/2)}\right) \cdot \sum_{j \geq k+1} p_j e^{-q_j}$$

$$\dot{\zeta} = \frac{z^2\zeta}{\tan(w/2)}e^{-\frac{\zeta}{\tan^2(w/2)}} - \tan^3(w/2)\left(1 - e^{-\frac{\zeta}{\tan^2(w/2)}} - \frac{\zeta}{\tan^2(w/2)}\right) +$$

$$+ 2\zeta\left[\sinh\left(\frac{\zeta}{2\tan^2(w/2)}\right) \cdot \left(\frac{\tan^2(w/2)}{\zeta} - \frac{z}{\tan(w/2)}\right) + \right.$$

$$\left. - \cosh\left(\frac{\zeta}{2\tan^2(w/2)}\right)\right] \cdot \sum_{j \geq k+1} p_j e^{-q_j + \eta/2}$$

$$\dot{p}_i = p_i e^{-q_i + \eta/2} \left[z \cosh\left(\frac{\zeta}{2\tan^2(w/2)} \right) + \tan(w/2) \sinh\left(\frac{\zeta}{2\tan^2(w/2)} \right) \right] +$$
$$+ \sum_{j \geq k+1} p_i p_j \operatorname{sign}(q_i - q_j) e^{-|q_i - q_j|}$$

$$\dot{q}_i = e^{-q_i + \eta/2} \left[z \cosh\left(\frac{\zeta}{2\tan^2(w/2)} \right) + \tan(w/2) \sinh\left(\frac{\zeta}{2\tan^2(w/2)} \right) \right] +$$
$$+ \sum_{j \geq k+1} p_j e^{-|q_i - q_j|}$$

which is a system of ODE with locally Lipschitz continuous right hand side that can be extended smoothly also at the value $w = \pi$. The initial data become

$$z(\tau) = \lim_{t \to \tau^-} \sum_{i=1}^{k} p_i(t) \qquad w(\tau) = \pi \qquad \eta(\tau) = 2\bar{q} \qquad \zeta(\tau) = e_\tau$$

$$p_i(\tau) = \lim_{t \to \tau^-} p_i(t) \qquad q_i(\tau) = \lim_{t \to \tau^-} q_i(t) \qquad i = k+1, \ldots, N$$

Thus there exists a unique solution of such a system which provides a multipeakon solution defined on some interval $[\tau, \tau'[$, up to the next interaction time. As we will show in Corollary 3.1, since Camassa-Holm equation is time reversible, once we prove the uniqueness of the solution of a Cauchy problem, we have that maximal the number of peakons interaction is actually $k = 2$, one with positive strength the other with negative one.

2.2 Approximation of the initial data

In this section we shall construct an approximation with initial data with a multipeakon function. Our aim is to approximate it with a sequence u_ε which has an exponential decay at infinity uniformly w.r.t ε.

Lemma 2.1. *Let $f \in X_\alpha$. Then for every $\varepsilon > 0$ there exists a multipeakon function g of the form*

$$g(x) = \sum_{i=1}^{N} p_i e^{-|x - q_i|}$$

such that

$$\|f - g\|_{H^1(\mathbb{R})} < \varepsilon \tag{2.3}$$

$$\int_{\mathbb{R}} [g^2(x) + g_x^2(x)] e^{\alpha|x|} \, dx \leq C_0 \tag{2.4}$$

for some constant $C_0 > 0$ which does not depend on ε.

Proof. Let $\rho(x) \in C_0^\infty$ be a cut-off function such that

- $\rho(x) \geq 0$

- $\rho(x) = 1$ for every $|x| \leq 1$, $\rho(x) = 0$ for every $|x| > 2$

- $\int_{\mathbb{R}} \rho(x) \, dx = 1$

and $\rho_\varepsilon(x) \doteq \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ be a mollifiers sequence. Observe that for every $\varepsilon > 0$, $f_\varepsilon(x) \doteq \rho_\varepsilon * f(x)$ is a smooth function which approximates the function f in H^1−norm

$$\|f - f_\varepsilon\|_{H^1(\mathbb{R})} < C\varepsilon \qquad (2.5)$$

moreover it belongs to X_α, indeed

$$
\begin{aligned}
\int_{\mathbb{R}} [f_\varepsilon^2(x) + f_{\varepsilon x}^2(x)]e^{\alpha|x|}\, dx &\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} (f^2(y) + f_x^2(y))\rho_\varepsilon(x-y)\, dy \right] e^{\alpha|x|}\, dx \\
&\leq \int_{\mathbb{R}} [f^2(y) + f_x^2(y)] \int_{\mathbb{R}} \rho_\varepsilon(x-y)e^{\alpha|x|}\, dx\, dy \\
&\leq \int_{\mathbb{R}} [f^2(y) + f_x^2(y)]C_0 e^{\alpha|y|}\, dy = C_0 C^{\alpha,f} < \infty
\end{aligned}
$$

and C_0 is a constant which does not depend on ε. From the previous inequality we can assert that for every $R > 0$ one has $\|f_\varepsilon\|_{H^1(\mathbb{R}\setminus[-R,R])} \leq C_0 C_\alpha e^{-\alpha R}$ uniformly in $\varepsilon > 0$. We can choose thus R_ε big enough in order to have

$$\|f_\varepsilon\|_{H^1(\mathbb{R}\setminus[-R_\varepsilon,R_\varepsilon])} < \varepsilon/2. \qquad (2.6)$$

In the space $H^1([-R_\varepsilon, R_\varepsilon])$ we can approximate f_ε with a multipeakon function. By using the identity

$$\frac{1}{2}\left(I - \frac{\partial^2}{\partial x^2}\right)e^{-|x|} = \delta_0 \qquad in\, \mathcal{D}'$$

the function f_ε can be rewritten in convolution form

$$f_\varepsilon = e^{-|x|} * \left(\frac{f_\varepsilon - f_{\varepsilon xx}}{2}\right) = \int_{\mathbb{R}} e^{-|x-y|} \cdot \frac{f_\varepsilon(y) - f_{\varepsilon xx}(y)}{2}\, dy\,.$$

In the interval $[-R_\varepsilon, R_\varepsilon]$ the previous integral can now be approximated with a Riemann sum

$$g(x) = \sum_{i=-N}^{N} p_i e^{-|x-q_i|}, \qquad \begin{cases} q_i = \dfrac{i}{N}R_\varepsilon \\ p_i = \displaystyle\int_{q_{i-1}}^{q_i} \dfrac{f_\varepsilon(y) - f_{\varepsilon xx}(y)}{2}\, dy\,. \end{cases}$$

Choosing N sufficiently large we obtain $\|f_\varepsilon - g\|_{H^1([-R_\varepsilon,R_\varepsilon])} < \varepsilon$. Together with (2.5) and (2.6) this last estimate yields the result.   □

2.3   Periodic multi-peakon

By a periodic peakon we mean a function of the form

$$u(x) = p\,\chi(x - q)\,, \qquad \chi(x) \doteq \sum_{n\in\mathbb{Z}} e^{-|x-n|}\,. \qquad (2.7)$$

Observe that the periodic function χ satisfies

$$\chi(-x) = \chi(x) = \chi(x+1) \qquad x \in \mathbb{R},$$
$$\chi(x) = \frac{e^x + e^{1-x}}{e-1} \qquad x \in [0,1]. \qquad (2.8)$$

For future use, we observe that for every periodic function u, the convolution P, defined in (1.2), takes the form

$$P(x) = \frac{1}{2} \int_0^1 \chi(x-y) \left(u^2(y) + \frac{u_x^2(y)}{2} \right) dy \qquad (2.9)$$

We begin this section by observing that also any periodic initial data can be approximated by periodic multi-peakons.

Lemma 2.2. *Let $f \in H_{\mathrm{per}}^1$. Then for any $\varepsilon > 0$ there exists periodic multi-peakon g, of the form*

$$g(x) = \sum_{i=1}^N p_i \sum_{n \in \mathbb{Z}} e^{-|x - q_i - n|} = \sum_{i=1}^N p_i\, \chi(x - q_i)$$

such that

$$\|f - g\|_{H_{\mathrm{per}}^1} < \varepsilon.$$

Proof. By taking a suitable mollification, we can approximate f with a periodic function $\tilde{f} \in \mathcal{C}^\infty$, so that

$$\|f - \tilde{f}\|_{H_{\mathrm{per}}^1} < \varepsilon/2. \qquad (2.10)$$

Next, we observe that

$$\frac{1}{2} \left(e^{-|x|} - \frac{\partial^2}{\partial x^2} e^{-|x|} \right) = \delta_0,$$

where δ_0 denotes the Dirac distribution concentrating a unit mass at the origin. We can thus write $\tilde{f}$ as a convolution:

$$\tilde{f} = \delta_0 * \tilde{f} = \frac{1}{2} \left(e^{-|x|} - \frac{\partial^2}{\partial x^2} e^{-|x|} \right) * \tilde{f} = e^{-|x|} * \left(\frac{\tilde{f} - \tilde{f}''}{2} \right),$$

$$\tilde{f}(x) = \int_0^1 \chi(x-y) \cdot \frac{\tilde{f}(y) - \tilde{f}''(y)}{2}\, dy.$$

The above integral can now be approximated with a Riemann sum

$$g(x) = \sum_{i=1}^N p_i\, \chi(x - q_i), \qquad p_i = \int_{(i-1)/N}^{i/N} \frac{\tilde{f}(y) - \tilde{f}''(y)}{2}\, dy.$$

Choosing N sufficiently large we obtain $\|\tilde{f} - g\|_{H_{per}^1} < \varepsilon/2$. Together with (2.10) this yields the result. $\qquad \square$

Next, we show how to construct a unique conservative solution, for multi-peakon initial data. As long as the locations q_i of the peakons remain distinct, this can be obtained by solving the Hamiltonian system of O.D.E's (1.7).

However, at a time τ where two or more peakons interact, the corresponding strengths p_i become unbounded. A suitable transformation of variables is needed, in order to resolve the singularity and uniquely extend the solution beyond the interaction time.

Lemma 2.3. *Let $\bar{u}$ be any periodic, multi-peakon initial data. Then the Cauchy problem (1.1), (1.11) has a global, conservative multi-peakon solution defined for all $t \in \mathbb{R}$. The set $\mathcal{I}$ of times where two or more peakons interact is at most countable. Moreover, for all $t \notin \mathcal{I}$, the energy conservation (1.13) holds.*

Proof. The solution can be uniquely constructed by solving the hamiltonian system (1.7), up to the first time τ where two or more peakons interact. We now show that there exists a unique way to prolong the solution for $t > \tau$, in terms of two outgoing peakons. To fix the ideas, call

$$\bar{q} = \lim_{t \to \tau-} q_i(t) \qquad i = 1, \ldots, k,$$

the place where the interaction occurs, and let $p_1(t), \ldots, p_k(t)$ be the strengths of the interacting peakons. Later in Section 3.5 we will show that only the case $k = 2$ can actually occur, but at this stage we need to consider the more general case. We observe that the strengths $p_{k+1}, \ldots, p_N$ of the peakons not involved in the interaction remain continuous at time τ. Moreover, by (1.7) there exists the limit

$$\bar{p} = \lim_{t \to \tau-} \sum_{i=1}^{k} p_i(t).$$

We can thus write

$$u(\tau, x) = \lim_{t \to \tau-} \sum_{i=1}^{N} p_i(t)\, e^{-|x - q_i(t)|} = \bar{p}\, e^{-|x - \bar{q}|} + \sum_{i=k+1}^{N} p_i(\tau)\, e^{-|x - q_i(\tau)|}.$$

For $t > \tau$, we shall prolong the solution with two peakons emerging from the point $\bar{q}$. The strength of these two peakons will be uniquely determined by the requirement of energy conservation (1.13).

Call $\xi^-(t)$, $\xi^+(t)$ respectively the position of the smallest and largest characteristic curves passing through the point $(\tau, \bar{q})$, namely

$$\xi^-(t) \doteq \min \left\{ \xi(t); \; \xi(\tau) = \bar{q}, \; \dot{\xi}(s) = u\big(s, \xi(s)\big) \text{ for all } s \in [\tau - h, \tau + h] \right\},$$

$$\xi^+(t) \doteq \max \left\{ \xi(t); \; \xi(\tau) = \bar{q}, \; \dot{\xi}(s) = u\big(s, \xi(s)\big) \text{ for all } s \in [\tau - h, \tau + h] \right\}.$$

$$(2.11)$$

Moreover, define

$$e_{(\tau, \bar{q})} \doteq \lim_{t \to \tau-} \int_{\xi^-(t)}^{\xi^+(t)} u_x^2(t, x)\, dx.$$

The existence of this limit follows from the balance law (1.5). This describes how much energy is concentrated at the interaction point. For $t > \tau$ the solution will contain the peakons $p_{k+1}, \ldots, p_N$, located at $q_{k+1}, \ldots, q_N$, together with the two outgoing peakons p_1, p_2, located at $q_1 < q_2$. The behavior of p_i, q_i for $i \in \{k+1, \ldots, N\}$ is still described by a system of O.D.E's as in (1.7).

However, to describe the evolution of p_1, p_2, q_1, q_2 one has to use a different set of variables, resolving the singularity occurring at $(\tau, \bar{q})$. As $t \to \tau+$ we expect (1.8), (1.10) to hold. To devise a suitable set of rescaled variables, we observe that, by (1.3),

$$\frac{d}{dt}\, u_x\big(t, \xi(t)\big) = -\frac{1}{2}\, u_x^2\big(t, \xi(t)\big) + [u^2 - P]$$

along any characteristic curve $t \mapsto \xi(t)$ emerging from the point $\bar{q}$. Since u, P remain uniformly bounded, one has

$$u_x(t, x) \approx \frac{2}{t - \tau} \qquad t > \tau, \quad x \in \big[q_1(t),\, q_2(t)\big].$$

The total amount of energy concentrated in the interval between the two peakons is given by

$$
\begin{aligned}
\int_{q_1(t)}^{q_2(t)} \big[u^2(t,x) + u_x^2(t,x)\big]\, dx \;&\approx\; \left(\frac{u(q_2) - u(q_1)}{q_2 - q_1}\right)^2 \cdot (q_2 - q_1) \\
&\approx\; \frac{\big[(p_2 - p_1)\big(1 - e^{-|q_2 - q_1|}\big)\big]^2}{q_2 - q_1} \\
&\approx\; (p_2 - p_1)^2(q_2 - q_1) \approx e_{(\tau, \bar{q})}\,.
\end{aligned}
$$

The previous heuristic analysis suggests that, in order to resolve the singularities, we should work with the variables

$$z = p_1 + p_2\,, \quad w = 2\arctan(p_2 - p_1)\,, \quad \eta = q_2 + q_1\,, \quad \zeta = (p_2 - p_1)^2(q_2 - q_1)\,,$$

together with $p_{k+1}, \ldots, p_N\,, \quad q_{k+1}, \ldots, q_N$.

To simplify the following calculations we here assume $0 < q_1 < q_2 < q_{k+1} < \ldots < q_N < 1$, which is not restrictive.

Let χ defined in (2.7) and $\tilde{\chi}(x) \doteq \frac{-e^x + e^{1-x}}{e - 1}$, $x \in (0, 1)$. From the original

system of equations (1.7) it follows

$$\dot{z} = \cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) z\cos^2 w/2 \sum_{j=k+1}^{N} p_j\chi\left(q_j - \eta/2\right)$$
$$- \sinh\left(\frac{\zeta}{2}\cot^2 w/2\right)\tan w/2 \sum_{j=k+1}^{N} p_j\tilde{\chi}\left(q_j - \eta/2\right)$$

$$\dot{w} = \left(z^2\cos^2 w/2 - \sin^2 w/2\right)\chi\left(\zeta\cot^2 w/2\right)$$
$$+ 2\cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) z \sum_{j=k+1}^{N} p_j\chi\left(q_j - \eta/2\right)$$
$$+ 2\sinh\left(\frac{\zeta}{2}\cot^2 w/2\right)\sin w \sum_{j=k+1}^{N} p_j\tilde{\chi}\left(q_j - \eta/2\right)$$

$$\dot{\eta} = z\left[\chi(0) + \chi\left(\zeta\cot^2 w/2\right)\right] + \cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) \sum_{j=k+1}^{N} p_j\chi\left(q_j - \eta/2\right)$$

$$\dot{\zeta} = \left[\chi(0) - \chi\left(\zeta\cot^2 w/2\right)\right]\zeta\tan^2 w/2 + \chi\left(\frac{\zeta}{\cot}^2 w/2\right) z^2\zeta\cot w/2$$
$$- \sinh\left(\frac{\zeta}{2}\cot^2 w/2\right)\tan^2 w/2 \sum_{j=k+1}^{N} p_j\tilde{\chi}\left(q_j - \eta/2\right)$$
$$+ 2\zeta\cot w/2\left[\cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) z\cot w/2 \sum_{j=k+1}^{N} p_j\chi\left(q_j - \eta/2\right)\right.$$
$$\left. - \sinh\left(\frac{\zeta}{2}\cot^2 w/2\right) \sum_{j=k+1}^{N} p_j\tilde{\chi}\left(q_j - \eta/2\right)\right]$$

$$\dot{p}_i = p_i\left[\cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) z\chi\left(q_i - \eta/2\right) + \sinh\left(\frac{\zeta}{2}\cot^2 w/2\right)\tan w/2\tilde{\chi}\left(q_i - \eta/2\right)\right]$$
$$+ p_i \sum_{j=k+1}^{N} p_j\,\mathrm{sign}(q_i - q_j)\chi\left(|q_i - q_j|\right)$$

$$\dot{q}_i = \cosh\left(\frac{\zeta}{2}\cot^2 w/2\right) z\chi\left(q_i - \eta/2\right) + \sinh\left(\frac{\zeta}{2}\cot^2 w/2\right)\tan w/2\tilde{\chi}\left(q_i - \eta/2\right)$$
$$+ \sum_{j=k+1}^{N} p_j\chi\left(|q_i - q_j|\right)$$

with initial data

$$z(\tau) = \bar{p}, \qquad w(\tau) = \pi, \qquad \eta(\tau) = 2\bar{q}, \qquad \zeta(\tau) = e_{(\tau,\bar{q})},$$

$$p_i(\tau) = \lim_{t\to\tau-} p_i(t), \qquad q_i(\tau) = \lim_{t\to\tau-} q_i(t) \qquad i = k+1,\ldots,N.$$

For the above system of O.D.E's, a direct inspection reveals that the right hand side can be extended by continuity also at the value $w = \pi$, because all singularities are removable. This continuous extension is actually smooth, in a

neighborhood of the initial data. Therefore, our Cauchy problem has a unique local solution. This provides a multi-peakon solution defined on some interval of the form $[\tau, \tau'[$, up to the next interaction time.

The case where two or more groups of peakons interact exactly at the same time τ, but at different locations within the interval $[0, 1]$, can be treated in exactly the same way. Since the total number of peakons (on a unit interval in the x-variable) does not increase, it is clear that the number of interaction times is at most countable. The solution can thus be extended to all times $t > 0$, conserving its total energy. $\qquad\qquad\square$

Chapter 3

Distance defined by optimal transportation problem

3.1 A distance functional in the spatially periodic case

In this section we shall construct a functional $J(u,v)$ which controls the distance between two solutions of the equation (1.1). All functions and measures on $\mathbb{R}$ are assumed to be periodic with period 1. Let $\mathbf{T}$ be the unit circle, so that $\mathbf{T} = [0, 2\pi]$ with the endpoints 0 and 2π identified. The distance $|\theta - \theta'|_*$ between two points $\theta, \theta' \in \mathbf{T}$ is defined as the smaller between the lengths of the two arcs connecting θ with θ' (one clockwise, the other counterclockwise). We now consider the product space

$$X \doteq \mathbb{R} \times \mathbb{R} \times \mathbf{T}$$

with distance

$$d^\diamond\Big((x,u,w),\ (\tilde{x},\tilde{u},\tilde{w})\Big) \doteq \Big(|x-\tilde{x}| + |u-\tilde{u}| + |w-\tilde{w}|_*\Big) \wedge 1\,, \qquad (3.1)$$

where $a \wedge b \doteq \min\{a,b\}$. Let $\mathcal{M}(X)$ be the space of all Radon measures on X which are 1-periodic w.r.t. the x-variable. To each 1-periodic function $u \in H^1_{\mathrm{per}}$ we now associate the positive measure $\sigma^u \in \mathcal{M}(X)$ defined as

$$\sigma^u(A) \doteq \int_{\big\{x \in \mathbb{R} :\ (x, u(x), 2\arctan u_x(x)) \in A\big\}} \big(1 + u_x^2(x)\big)\, dx \qquad (3.2)$$

for every Borel set $A \subseteq \mathbb{R}^2 \times \mathbf{T}$. Notice that the total mass of $\sigma^{(u,\mu)}$ over one period is

$$\sigma^u\big([0,1] \times \mathbb{R} \times \mathbf{T}\big) = 1 + \int_0^1 u_x^2(x)\, dx\,.$$

On this family of positive, 1-periodic Radon measures, we now introduce a kind of Kantorovich distance, related to an optimal transportation problem. Given the two measures σ^u and $\sigma^{\tilde{u}}$, their distance $J(u, \tilde{u})$ is defined as follows.

Call $\mathcal{F}$ the family of all strictly increasing absolutely continuous maps $\psi :$ $\mathbb{R} \mapsto \mathbb{R}$ which have an absolutely continuous inverse and satisfy

$$\psi(x + n) = n + \psi(x) \qquad \text{for every } n \in \mathbb{Z}. \tag{3.3}$$

For a given $\psi \in \mathcal{F}$, we define the 1-periodic, measurable functions $\phi_1, \phi_2 : \mathbb{R} \mapsto [0, 1]$ by setting

$$\begin{aligned}
\phi_1(x) &\doteq \sup \left\{ \theta \in [0, 1]; \;\; \theta \cdot \left(1 + u_x^2(x)\right) \leq \left(1 + \tilde{u}_x^2(\psi(x))\right) \psi'(x) \right\}, \\
\phi_2(x) &\doteq \sup \left\{ \theta \in [0, 1]; \;\; 1 + u_x^2(x) \geq \theta \cdot \left(1 + \tilde{u}_x^2(\psi(x))\right) \psi'(x) \right\}.
\end{aligned} \tag{3.4}$$

Observe that the above definitions imply $\max \left\{ \phi_1(x), \; \phi_2(x) \right\} = 1$ together with

$$\phi_1(x) \left(1 + u_x^2(x)\right) = \phi_2(\psi(x)) \left(1 + \tilde{u}_x^2(\psi(x))\right) \psi'(x) \tag{3.5}$$

for a.e. $x \in \mathbb{R}$. We now define

$$\begin{aligned}
J^\psi(u, \tilde{u}) \doteq \int_0^1 d^\Diamond & \left(\left(x, \, u(x), 2 \arctan u_x(x)\right), \left(\psi(x), \, \tilde{u}(\psi(x)), 2 \arctan \tilde{u}_x(\psi(x))\right) \right) \\
& \qquad \qquad \cdot \phi_1(x) \left(1 + u_x^2(x)\right) dx \\
+ \int_0^1 & \left| \left(1 + u_x^2(x)\right) - \left(1 + \tilde{u}_x^2(\psi(x))\right) \psi'(x) \right| dx.
\end{aligned} \tag{3.6}$$

Of course, the integral is always computed over one period. Observe that the map $x \mapsto \psi(x)$ can be regarded as a *transportation plan*, in order to transport the measure σ^u onto the measure $\sigma^{\tilde{u}}$. Since these two positive measures need not have the same total mass, we allow the presence of some excess mass, not transferred from one place to the other. The penalty for this excess mass is given by the second integral in (3.6). The factor $\phi_1 \leq 1$ in the first integral indicates the percentage of the mass which is actually transported. Integrating (3.5) over one period, we find

$$\int_0^1 \phi_1(x) \left(1 + u_x^2(x)\right) dx = \int_0^1 \phi_2(y) \left(1 + \tilde{u}_x^2(y)\right) dy.$$

We can thus transport the measure $\phi_1 \, \sigma^u$ onto $\phi_2 \, \sigma^{\tilde{u}}$ by a map

$$\Psi : \left(x, \, u(x) \arctan u_x(x)\right) \mapsto \left(y, \, \tilde{u}(y), \arctan \tilde{u}_x(y)\right),$$

where $y = \psi(x)$. The associated cost is given by the first integral in (3.6). Notice that in this case the measure $\phi_2 \sigma^{\tilde{u}}$ is obtained as the push-forward of the measure $\phi_1 \sigma^u$. We recall that the *push-forward* of a measure σ by a

mapping Ψ is defined as $(\Psi \# \sigma)(A) \doteq \sigma(\Psi^{-1}(A))$ for every measurable set A. Here $\Psi^{-1}(A) \doteq \{z\,;\ \Psi(z) \in A\}$.

Our distance functional J is now obtained by optimizing over all transportation plans, namely

$$J(u, \tilde{u}) \doteq \inf_{\psi \in \mathcal{F}} J^{\psi}(u, \tilde{u})\,. \tag{3.7}$$

To check that (3.7) actually defines a distance, let $u, v, w \in H^1(\mathbb{R})$ be given.

1. Choosing $\psi(x) = x$, so that $\phi_1(x) = \phi_2(x) = 1$, we immediately see that $J(u, u) = 0$. Moreover, if $J(u, \tilde{u}) = 0$, then by the definition of $d^{\diamond}$ we have $\tilde{u} = u$.

2. Given $\psi \in \mathcal{F}$, define $\tilde{\psi} = \psi^{-1}$, so that $\tilde{\phi}_1 = \phi_2$, $\tilde{\phi}_2 = \phi_1$. This yields

$$J^{\tilde{\psi}}(\tilde{u}, u) = J^{\psi}(u, \tilde{u})\,.$$

Hence $J(\tilde{u}, u) = J(u, \tilde{u})$.

3. Finally, to prove the triangle inequality, let $\psi^{\flat}, \psi^{\sharp} : \mathbb{R} \mapsto \mathbb{R}$ be two increasing diffeomorphisms satisfying (3.3), and let $\phi_1^{\flat}, \phi_1^{\sharp}, \phi_2^{\flat}, \phi_2^{\sharp} : \mathbb{R} \mapsto [0,1]$ be the corresponding functions, defined as in (3.4). We now consider the composition $\psi \doteq \psi^{\sharp} \circ \psi^{\flat}$ and define the functions ϕ_1, ϕ_2 according to (3.4). Observing that

$$\phi_1(x) \geq \phi_1^{\flat}(x) \cdot \phi_1^{\sharp}\big(\psi^{\flat}(x)\big)\,,$$

$$\phi_2\big(\psi(x)\big) = \phi_2\Big(\psi^{\sharp}\big(\psi^{\flat}(x)\big)\Big) \geq \phi_2^{\flat}\big(\psi^{\flat}(x)\big) \cdot \phi_2^{\sharp}\Big(\psi^{\sharp}\big(\psi^{\flat}(x)\big)\Big)\,,$$

and recalling that the distance $d^{\diamond}$ at (3.1) is always ≤ 1, we conclude

$$J^{\psi}(u, w) \leq J^{\psi^{\flat}}(u, v) + J^{\psi^{\sharp}}(v, w)\,.$$

This implies the triangle inequality $J(u, v) + J(v, w) \geq J(u, w)$. $\qquad\square$

In the remainder of this section we study the relations between our distance functional J and the distances determined by various norms.
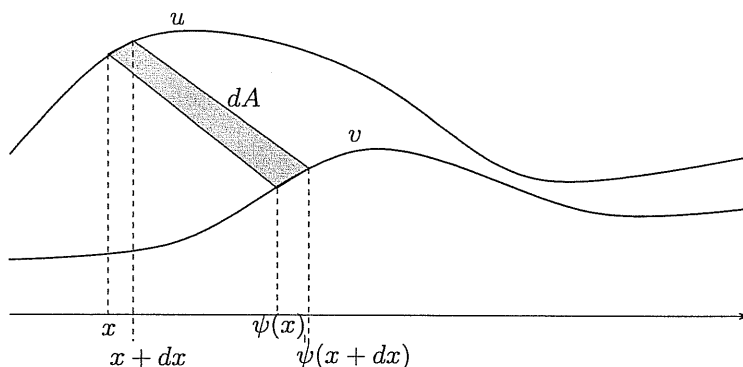
Lemma 3.1. *For any $u, v \in H^1_{\mathrm{per}}$ one has*

$$\frac{1}{C} \cdot \|u - v\|_{L^1_{\mathrm{per}}} \leq J(u, v) \leq C \cdot \|u - v\|_{H^1_{\mathrm{per}}}\,,$$

with a constant C uniformly valid on bounded subsets of H^1_{per}.

Proof. We shall use the elementary bound

$$\big|\arctan a - \arctan b\big| \cdot a^2 \leq 4\pi\big(|a| + |b|\big)\,|a - b|\,,$$

Figure 3.1: Infinitesimal area between u and v

valid for all $a, b \in \mathbb{R}$. In connection with the identity mapping $\psi(x) = x$ we now compute

$$
\begin{aligned}
J^{\psi}(u, v) &\leq \int_0^1 \left\{ |u(x) - v(x)| + 2 |\arctan u_x - \arctan v_x| \right\} (1 + u_x^2)\, dx \\
&\quad + \int_0^1 |u_x^2 - v_x^2|\, dx \\
&\leq \|u - v\|_{L^\infty} \|1 + u_x^2\|_{L^1} + (8\pi + 1) \int_0^1 |u_x + v_x|\, |u_x - v_x|\, dx \\
&\leq (8\pi + 3) \left(1 + \|u\|_{H^1} + \|v\|_{H^1} \right) \cdot \|u - v\|_{H^1},
\end{aligned}
$$

proving the second inequality in (3.8).

To achieve the first inequality, choose any $\psi \in \mathcal{F}$. For $x \in [0, 1]$, call γ^x the segment joining the point $P^x = \big(x, u(x)\big)$ with $Q^x = \big(\psi(x), v(\psi(x))\big)$. Clearly, the union of all these segments covers the region between the graphs of u and v. Moving the base point from x to $x + dx$, the corresponding segments sweep an infinitesimal area dA estimated by (fig.3.1)

$$
\begin{aligned}
|dA| &\leq |P^x - Q^x| \cdot \big(|dP^x| + |dQ^x| \big) \\
&\leq \left(\big|x - \psi(x)\big|^2 + \big|u(x) - v(\psi(x))\big|^2 \right)^{1/2} \cdot \left[(1 + u_x^2)^{1/2} + (1 + v_x^2)^{1/2} \psi'(x) \right] dx.
\end{aligned}
$$

Integrating over one period we obtain

$$
\begin{aligned}
\int_0^1 |u(x) - v(x)|\, dx &\leq \int_0^1 \left(\big|x - \psi_{(t)}(x)\big| + \big|u(x) - v(\psi_{(t)}(x))\big| \right) \cdot \\
&\qquad \cdot \left[\big(1 + u_x^2(x)\big)^{1/2} + \big(1 + v_x^2(\psi(x))\big)^{1/2} \psi'_{(t)}(x) \right] dx \\
&\leq \big(2 + \|u\|_{H^1} + \|v\|_{H^1} \big) \cdot \left[J^{\psi}(u, v) + J^{\psi^{-1}}(v, u) \right] \\
&\leq C \cdot J(u, v),
\end{aligned}
$$

completing the proof of (3.8).   □

Lemma 3.2. *Let $(u_n)_{n \geq 1}$ be a Cauchy sequence for the distance J, uniformly bounded in the H^1_{per} norm. Then*

(i) *There exists a limit function $u \in H^1_{\mathrm{per}}$ such that $u_n \to u$ in L^∞ and the sequence of derivatives $u_{n,x}$ converges to u_x in L^p_{per}, for $1 \leq p < 2$.*

(ii) *Let μ_n be the absolutely continuous measure having density $u^2_{n,x}$ with respect to Lebesgue measure. Then one has the weak convergence $\mu_n \rightharpoonup \mu$, for some measure μ whose absolutely continuous part has density u^2_x.*

Proof. **1.** By Lemma 3.1 we already know the convergence $u_n \to u$, for some limit function $u \in L^1_{\mathrm{per}}$. By a Sobolev embedding theorem, all functions u_n, u are uniformly Hölder continuous. This implies $\|u_n - u\|_{L^\infty} \to 0$. To establish the convergence of derivatives, we first show that the sequence of functions

$$v_n \doteq \exp\{2i \arctan u_{n,x}\}$$

is compact in L^1_{per}.

Indeed, fix $\varepsilon > 0$. Then there exists N such that $J(u_m, u_n) < \varepsilon$ for $m, n \geq N$. We can now approximate u_N in H^1_{per} with a piecewise affine function $\tilde{u}_N$ such that $J(\tilde{u}_N, u_N) \leq \varepsilon$. By assumption, choosing suitable transport maps ψ_n we obtain

$$\int_0^1 \left| \exp\left\{2i \arctan u_{n,x}(x)\right\} - \exp\left\{2i \arctan \tilde{u}_{N,x}(\psi_n(x))\right\} \right| dx \; \leq \; 2\,J(u_n, \tilde{u}_N)$$
$$\leq \; 4\varepsilon$$

for all $n \geq N$. We now observe that all functions

$$x \mapsto \exp\left\{2i \arctan \tilde{u}_{N,x}(\psi_n(x))\right\}$$

are uniformly bounded, piecewise constant with the same number of jumps: namely, the number of subintervals on which $\tilde{u}_N$ is affine. The set of all such functions is compact in L^1_{per}. This argument shows that the sequence $v_n \doteq \exp\{2i \arctan u_{n,x}\}$ eventually remains in an ε-neighborhood of a compact subset of L^1_{per}. Since $\varepsilon > 0$ can be taken arbitrarily small, by possibly choosing a subsequence we obtain the strong convergence $v_n \to v$ for some $v \in L^1_{\mathrm{per}}$.

2. From the uniform H^1 bounds and the L^1 convergence of the functions v_n, we now derive the L^p convergence of the derivatives. For a given $\varepsilon > 0$, define

$$M \doteq \sup_n \|u_n\|_{H^1_{\mathrm{per}}}, \qquad A_n \doteq \left\{ x \in [0,1]; \; |u_{n,x}(x)| > M/\varepsilon \right\}.$$

The above definitions imply

$$\mathrm{meas}(A_n) \leq \varepsilon^2$$

We now have

$$\|u_{m,x} - u_{n,x}\|_{L^p} \leq \left(\int_{A_n \cup A_m} |u_{m,x} - u_{n,x}|^p \, dx \right)^{1/p}$$
$$+ \left(\int_{[0,1] \setminus (A_n \cup A_m)} |u_{m,x} - u_{n,x}|^p \, dx \right)^{1/p}$$
$$\doteq I_1 + I_2 \, .$$

$$I_1 \leq \left[\int_{A_m \cup A_n} 1 \cdot dx \right]^{(2-p)/2p} \cdot \left[\int_{A_m \cup A_n} \left(|u_{m,x}| + |u_{n,x}| \right)^2 dx \right]^{1/2} \tag{3.8}$$
$$\leq \varepsilon^{(2-p)/p} \cdot 2M \, .$$

Next, choosing a constant C_ε such that

$$\left| e^{2i \arctan a} - e^{2i \arctan b} \right| \geq C_\varepsilon |a - b| \qquad \text{whenever } |a|, |b| \leq M/\varepsilon \, ,$$

we obtain

$$I_2 \leq C_\varepsilon \left[\int \left| e^{2i \arctan u_{m,x}} - e^{2i \arctan u_{n,x}} \right|^p dx \right]^{1/p} \, . \tag{3.9}$$

Taking $\varepsilon > 0$ small, we can make the right hand side of (3.8) as small as we like. On the other hand, choosing a subsequence such that $v_\nu = e^{2i \arctan u_{\nu,x}}$ converges in L^1_{per}, the right hand side of (3.9) approaches zero. Hence, for this subsequence,

$$\limsup_{m,n \to \infty} \|u_{m,x} - u_{n,x}\|_{L^p_{\text{per}}} = 0 \, .$$

Since $u_n \to u$ uniformly, in this case we must have

$$\|u_{n,x} - u_x\|_{L^p_{\text{per}}} \to 0 \, . \tag{3.10}$$

We now observe that from any subsequence we can extract a further subsequence for which (3.10) holds. Therefore, the whole sequence $(u_{n,x})_{n \geq 1}$ converges to u_x in L^p_{per}.

3. To establish (ii), we consider the sequence of measures having density $1 + u_{n,x}^2$ w.r.t. Lebesgue measure. This sequence converges weakly, because our distance functional is stronger than the Kantorovich-Waserstein metric which induces the topology of weak convergence on spaces of measures. Therefore, $\mu_n \rightharpoonup \mu$ for some positive measure μ.

Since the sequence $1 + u_{n,x}$ converges to $1 + u_x$ in L^1_{per}, by possibly choosing a subsequence we achieve the pointwise convergence $u_{n,x}(x) \to u_x(x)$, for a.e. $x \in [0,1]$. For any $\varepsilon > 0$, by Egorov's theorem we have the uniform convergence $u_{n,x}(x) \to u_x(x)$ for all $x \in [0,1] \setminus V_\varepsilon$, for some set with $\text{meas}(V_\varepsilon) < \varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small, this shows that the absolutely continuous part of the measure μ has density $u^2 + u_x^2$ w.r.t. Lebesgue measure. $\qquad \square$

3.1.1   Continuity in time of the distance functional

Here and in the next section we examine how the distance functional $J(\cdot,\cdot)$ evolves in time, in connection with multi-peakon solutions of the Camassa-Holm equation (1.1). We first provide estimates valid on a time interval where no peakon interactions occur. Then we show that the distance functional is continuous across times of interaction. Since the number of peakons is locally finite, this will suffice to derive the basic estimates (1.15)-(1.16), in the case of multi-peakon solutions.

Lemma 3.3. *Let $t \mapsto u(t) \in H^1_{\text{per}}$ be a multi-peakon solution of (1.1). Assume that no peakon interactions occur within the interval $[0,\tau]$. Then*

$$J\big(u(s),\, u(s')\big) \leq C \cdot |s - s'|, \qquad \text{for all } s, s' \in [0,\tau], \qquad (3.11)$$

for some constant C, uniformly valid as u ranges on bounded subsets of H^1_{per}.

Proof. Assume $0 \leq s < s' \leq \tau$. By the assumptions, the solution $u = u(t,x)$ remains uniformly Lipschitz continuous on the time interval $[0,\tau]$. Therefore, for each $s \in [0,\tau]$ and $x \in \mathbb{R}$, the Cauchy problem

$$\frac{d}{dt}\,\xi(t) = u\big(t, \xi(t)\big), \qquad \xi(s) = x, \qquad (3.12)$$

determines a unique characteristic curve $t \mapsto \xi(t; s, x)$ passing through the point (s,x). Given $s' \in [0,\tau]$, we can thus define a transportation plan by setting

$$\psi(x) \doteq \xi(s'; s, x). \qquad (3.13)$$

Of course, moving mass along the characteristics is the most natural thing to do. We then choose ϕ_1, ϕ_2 to be as large as possible, according to (3.4). Namely:

$$\phi_1(x) \doteq \sup\left\{\theta \in [0,1]; \ \theta \cdot \Big(1 + u_x^2(s,x)\Big) \leq \psi'(x) \cdot \Big(1 + u_x^2\big(s', \psi(x)\big)\Big)\right\},$$

$$\phi_2(x) \doteq \sup\left\{\theta \in [0,1]; \ \theta \cdot \Big(1 + u_x^2\big(s', \psi(x)\big)\Big)\psi'(x) \leq 1 + u_x^2(s,x)\right\}.$$

The cost of this plan is bounded by

$$J^\psi\big(u(s),\, u(s')\big) \leq \int_0^1 \left\{\big|x - \xi(s'; s, x)\big| + \big|u(s,x) - u(s',\xi(s'; s, x))\big|\right.$$

$$\left. + \big|2\arctan u_x(s,x) - 2\arctan u_x(s',\xi(s'; s, x))\big|_*\right\}\Big(1 + u_x^2(s,x)\Big)\,dx$$

$$+ \int_0^1 \big(1 - \phi_1(x)\big)\Big(1 + u_x^2(s,x)\Big)\,dx$$

$$+ \int_0^1 \big(1 - \phi_2(\psi(x))\big)\Big(1 + u_x^2\big(s',\psi(x)\big)\Big)\psi'(x)\,dx.$$

$$(3.14)$$

To estimate the right hand side of (3.14), we first observe that, for all $u \in H^1_{\mathrm{per}}$,

$$
\begin{aligned}
\|u\|_{L^\infty} &\leq \int_0^1 |u(x)|\, dx + \int_0^1 |u_x(x)|\, dx \\
&\leq \|u\|_{L^2} + \|u_x\|_{L^2} \leq 2\|u\|_{H^1_{\mathrm{per}}} = 2(E^u)^{1/2}.
\end{aligned}
\tag{3.15}
$$

Using (3.15) in (3.12) we obtain

$$
\big|\xi(s) - \xi(s')\big| \leq 2(E^{\bar{u}})^{1/2} \cdot |s - s'| .
\tag{3.16}
$$

Next, from the definition of the source term P at (1.2) it follows

$$
\|P\|_{L^\infty} \leq \frac{1}{2}\big\|e^{-|x|}\big\|_{L^\infty(\mathbb{R})} \cdot \left\|u^2 + \frac{u_x^2}{2}\right\|_{L^1([0,1])} \leq \|u\|^2_{H^1_{\mathrm{per}}} = E^u .
\tag{3.17}
$$

Similarly,

$$
\|P_x\|_{L^\infty} \leq \|u\|^2_{H^1_{\mathrm{per}}} = E^u .
\tag{3.18}
$$

Using (3.18) we obtain

$$
\begin{aligned}
\big|u\big(s', \xi(s')\big) - u\big(s, \xi(s)\big)\big| &\leq \int_s^{s'} \left|\frac{d}{dt} u\big(t, \xi(t)\big)\right| dt \\
&= \int_s^{s'} \big|P_x\big(t, \xi(t)\big)\big|\, dt \leq E^{\bar{u}} \cdot |s' - s| .
\end{aligned}
\tag{3.19}
$$

Concerning the term involving arctangents, recalling (1.3) we obtain

$$
\frac{d}{dt}\Big[2\arctan u_x\big(t, \xi(t,x)\big)\Big] = \frac{2}{1 + u_x^2}\left[u^2 - \frac{u_x^2}{2} - P\right].
$$

The bounds (3.17) and (3.19) thus yield

$$
\begin{aligned}
\Big|2\arctan u_x\big(s', \xi(s')\big) &- 2\arctan u_x\big(s,\, \xi(s)\big)\Big|_* \\
&\leq \Big(2\|u\|^2_{L^\infty} + 1 + 2\|P\|_{L^\infty}\Big) \cdot |s' - s| \\
&\leq \big(10\, E^{\bar{u}} + 1\big) \cdot |s' - s| .
\end{aligned}
\tag{3.20}
$$

This already provides a bound on the first integral on the right hand side of (4.4).

Next, call I_1, I_2 the last two integrals on the right hand side of (4.4). Notice that

$$
I_1 + I_2 = \int_0^1 \left|\Big(1 + u_x^2(s,y)\Big) - \xi_y(s'; s, y)\Big(1 + u_x^2\big(s', \xi(s'; s, y)\big)\Big)\right| dy
$$

Indeed, $I_1 + I_2$ measures the difference between the measure $\big(1 + u_x^2(s', y)\big)\, dy$ and the push-forward of the measure $\big(1 + u_x^2(s, x)\big)\, dx$ through the mapping $x \mapsto \xi(s'; s, x)$.

Since the push-forward of the measure $u_x^2 \, dy$ satisfies the linear conservation law

$$w_t + (uw)_x = 0 \,, \tag{3.21}$$

comparing (3.21) with (1.5) we deduce

$$\int_0^1 \left| u_x^2(s,y) - \xi_y(s,y) \, u_x^2\big(s', \xi(s';s,y)\big) \right| dy \;\leq\; \int_s^{s'} \int_0^1 2 \big| (u^2 - P) u_x \big| \, dx \, dt$$

$$\leq 2 \int_s^{s'} \big(\|u\|_{L^\infty}^2 + \|P\|_{L^\infty} \big) \|u_x\|_{L^1} \, dt \;\leq\; 2 \left(4 E^{\bar{u}} + E^{\bar{u}} \right) E^{\bar{u}} \cdot |s' - s| \,,$$

because of (3.15), (3.17) and (1.13). Finally, we need to estimate the remaining terms, describing by how much the Lebesgue measure fails to be conserved by the transformation $x \mapsto \xi(s'; s, x)$. Observing that

$$\frac{\partial}{\partial t} \xi_y(t,y) = u_x\big(t, \xi(t,y)\big) \, \xi_y(t,y) \,, \qquad\qquad \xi_y(0,y) = 1 \,,$$

we find

$$\int_0^1 \left| 1 - \xi_y(s';s,y) \right| dy \leq \int_s^{s'} \int_0^1 \left| \frac{\partial}{\partial t} \big[\xi_y(t,s,y) \big] \right| dy \, dt$$

$$\leq \int_s^{s'} \int_0^1 \xi_y(t;s,y) \left| u_x(t, \xi(t;s,y)) \right| dy \, dt \,. \tag{3.22}$$

To estimate the right hand side of (3.22), we use the decomposition $[0,1] = Y \cup Y' \cup Y''$, where

$$Y \doteq \left\{ y \,; \;\; \xi_y(t;s,y) \in \big[(1/2), \, 2 \big] \quad \text{for all } t \in [s,s'] \right\},$$

$$Y' \doteq \left\{ y \,; \;\; \xi_y(t;s,y) < 1/2 \quad \text{for some } t \in [s,s'] \right\},$$

$$Y'' \doteq \left\{ y \,; \;\; \xi_y(t;s,y) > 2 \quad \text{for some } t \in [s,s'] \right\}.$$

Integrating over Y one finds

$$\int_s^{s'} \int_Y \xi_y(t;s,y) \left| u_x(t, \xi(t;s,y)) \right| dy \, dt \;\leq\; 2 \int_s^{s'} \big\| u_x(t) \big\|_{L^1} \, dt \;\leq\; 2 E^{\bar{u}} \cdot |s' - s| \,.$$

Next, if $y \in Y'$ we define

$$\tau(y) \doteq \inf \big\{ t > s \,; \;\; \xi_y(t;s,y) < 1/2 \big\} \,.$$

Observe that $y \in Y'$ implies

$$\int_s^{\tau(y)} \left| u_x\big(t, \, \xi(t;s,y)\big) \right| dt \geq \ln 2 \,.$$

Therefore

$$\int_{Y'} dy \;\le\; \frac{1}{\ln 2} \int_{Y'} \left[\int_s^{\tau(y)} \left| u_x\big(t,\, \xi(t;s,y)\big) \right| dt \right] dy$$

$$\le\; \frac{2}{\ln 2} \int_s^{s'} \int_0^1 \left| u_x(t,x) \right| dx\, dt \;\le\; 4\, E^{\bar{u}} \cdot \left| s' - s \right|.$$

The estimate for the integral over Y'' is entirely analogous, Indeed, the push-forward of the Lebesgue measure along characteristic curves from $t = s$ to $t = s'$ satisfies exactly the same type of estimates as the pull-back of the Lebesgue measure from $t = s'$ to $t = s$. All together, these three estimates imply

$$\int_{Y \cup Y' \cup Y''} \left| 1 - \xi_y(s'; s, y) \right| dy \le 10\, E^{\bar{u}} \cdot \left| s' - s \right| \tag{3.23}$$

Putting together the estimates (3.16), (3.19), (3.20), (3.13) and (3.23), the distance in (4.4) can be estimated by

$$J^\psi\big(u(s),\, u(s')\big) \le \left[2(1 + E^{\bar{u}}) + E^{\bar{u}} + (10\, E^{\bar{u}} + 1) + 10\,(E^{\bar{u}})^2 + 10\, E^{\bar{u}} \right] \cdot \left| s' - s \right|.$$

This establishes (3.11).   □

According to Lemma 3.3, as long as no peakon interactions occur, the map $t \mapsto u(t)$ remains uniformly Lipschitz continuous w.r.t. our distance functional, with a Lipschitz constant that depends only on the total energy $E^{\bar{u}}$. Since interactions can occur only at isolated times, to obtain a global Lipschitz estimate it suffices to show that trajectories are continuous (w.r.t. the distance J) also at interaction times.

Lemma 3.4. *Assume that the multi-peakon solution $u(\cdot)$ contains two or more peakons which interact at a time τ. Then*

$$\lim_{h \to 0+} J\big(u(\tau - h),\, u(\tau + h)\big) = 0.$$

Proof. To fix the ideas, call $x = \bar{q}$ the place where the interaction occurs, and let $p_1, \ldots, p_k$ be the strengths of the peakons that interact at time τ. We here assume that $0 < \bar{q} < 1$. The case where two or more groups of peakons interact exactly at the same time τ, within the interval $[0,1]$, can be treated similarly.

For $|t - \tau| \le h$, call $\xi^-(t)$, $\xi^+(t)$ respectively the position of the smallest and largest characteristic curves passing through the point $(\tau, \bar{q})$, as in (2.11). We observe that u is Lipschitz continuous in a neighborhood of each point (τ, x), with $x \ne \bar{q}$. Hence, for $x \in [0,1] \setminus \{\bar{q}\}$ there exists a unique characteristic curve $t \mapsto \xi(t; \tau, x)$ passing through x at time τ. For a fixed $h > 0$, the transport map ψ is defined as follows. Consider the intervals $I_{-h} \doteq \left[\xi^-(\tau - h),\, \xi^+(\tau - h) \right]$ and $I_h \doteq \left[\xi^-(\tau + h),\, \xi^+(\tau + h) \right]$. On the complement $[0,1] \setminus I_{-h}$ we define

$$\psi\big(\xi(\tau - h; \tau, x)\big) = \xi(\tau + h; \tau, x),$$

so that transport is performed along characteristic curves. It now remains to extends ψ as a map from I_{-h} onto I_h. Toward this goal, we recall that our construction of multi-peakon solutions in Section 2.1 was specifically designed in order to achieve the identity

$$e_{(\tau,\bar{q})} = \lim_{h \to 0+} \int_{I_{-h}} u_x^2(\tau - h, x)\, dx = \lim_{h \to 0+} \int_{I_h} u_x^2(\tau + h, x)\, dx. \qquad (3.24)$$

For $h > 0$ we introduce the quantities

$$E(-h) \doteq \int_{I_{-h}} \left(1 + u_x^2(\tau - h,\, x)\right) dx, \qquad E(h) \doteq \int_{I_h} \left(1 + u_x^2(\tau + h,\, x)\right) dx,$$

$$e(h) \doteq 2 \min \left\{ E(-h),\, E(h) \right\} - \max \left\{ E(-h),\, E(h) \right\}.$$

Notice that (3.24) implies $e(h) > 0$ and

$$E(-h) \to e_{(\tau,\bar{q})}, \qquad E(h) \to e_{(\tau,\bar{q})}, \qquad e(h) \to e_{(\tau,\bar{q})}, \qquad (3.25)$$

as $h \to 0+$. Consider the point $x^* = x^*(h)$ inside the interval

$$I_{-h} = \left[\xi^-(\tau - h),\, \xi^+(\tau - h) \right],$$

implicitly defined by

$$\int_{\xi^-(\tau-h)}^{x^*} \left(1 + u_x^2(x)\right) dx = e(h).$$

For $x \in \left[\xi^-(\tau - h),\, x^* \right]$ we define $\psi(x)$ as the unique point such that

$$\int_{\xi^-(\tau+h)}^{\psi(x)} \left(1 + u_x^2(\tau + h,\, x)\right) dx = \int_{\xi^-(\tau-h)}^{x} \left(1 + u_x^2(\tau - h,\, x)\right) dx. \qquad (3.26)$$

We then extend ψ as an affine map from $\left[x^*,\, \xi^+(\tau - h)\right]$ onto $\left[\psi(x^*),\, \xi^+(\tau + h)\right]$, namely

$$\psi\left(\theta \cdot \xi^+(\tau - h) + (1 - \theta) \cdot x^*\right) = \theta \cdot \xi^+(\tau + h) + (1 - \theta) \cdot \psi(x^*) \qquad \theta \in [0, 1].$$

Finally, we prolong ψ to the whole real line according to (3.3).

As usual, the 1-periodic functions ϕ_1, ϕ_2 are then chosen to be as large as possible, according to (3.4). As $h \to 0+$, we claim that the following quantity approaches zero:

$$\begin{aligned}
J^\psi &\left(u(\tau - h),\, u(\tau + h)\right) \\
&= \int_0^1 d^\diamond \Big(\left(x,\, u(\tau - h, x),\, 2\arctan u_x(\tau - h, x)\right), \\
&\qquad\qquad \left(\psi(x),\, u(\tau + h,\, \psi(x)),\, 2\arctan \tilde{u}_x(\tau + h,\, \psi(x))\right) \Big) \cdot \\
&\qquad \cdot \phi_1(x) \left(1 + u_x^2(\tau - h,\, x)\right) dx \qquad\qquad\qquad\qquad (3.27) \\
&+ \int_0^1 \left(1 - \phi_1(x)\right) \left(1 + u_x^2(\tau - h,\, x)\right) dx \\
&+ \int_0^1 \left(1 - \phi_2(\psi(x))\right) \left(1 + u_x^2(\tau + h,\, \psi(x))\right) \psi'(x)\, dx.
\end{aligned}$$

It is clear that the restriction of all the above integrals to the complement $[0, 1] \setminus I_{-h}$ approaches zero as $h \to 0$. We now prove that their restriction to I_{-h} also vanishes in the limit. As $h \to 0+$, for $x \in I_{-h}$ we have

$$d^\diamond \Big(\big(x, u(\tau - h, x), 2 \arctan u_x(\tau - h, x)\big) \, ,$$
$$\big(\psi(x), u(\tau + h, \psi(x)), 2 \arctan u_x(\tau + h, \psi(x))\big) \Big) \; \to \; 0 \, ,$$

because all points approach the same limit $\big(\bar{q}, u(\tau, \bar{q}), \pi\big)$. The first integral in (3.27) thus approaches zero as $h \to 0+$.

Concerning the last two integrals, by (3.26) it follows

$$\phi_1(x) = \phi_2\big(\psi(x)\big) = 1 \qquad \text{for all } x \in \big[\xi^-(\tau - h), x^*\big] \, .$$

Moreover, our choice of x^* implies

$$\int_{x^*}^{\xi^+(\tau - h)} \big(1 + u_x^2(\tau - h, x)\big) \, dx + \int_{\psi(x^*)}^{\xi^+(\tau + h)} \Big(1 + u_x^2\big(\tau + h, \psi(x)\big)\Big) \psi'(x) \, dx$$
$$\leq 2 \max \big\{ E(-h), E(h) \big\} - 2 \min \big\{ E(-h), E(h) \big\} \, .$$
$$\tag{3.28}$$

By (3.25), as $h \to 0+$ the right hand side of (3.28) approaches zero. Hence the same holds for the last two integrals in (3.27). This completes the proof of the lemma. $\square$

3.1.2 Continuity w.r.t. the initial data

We now consider two distinct solutions and study how the distance $J\big(u(t) \; v(t)\big)$ evolves in time. To fix the ideas, let $t \mapsto u(t)$ and $t \mapsto v(t)$ be two multi-peakon solutions of (1.1), and assume that no interaction occurs within a given time interval $[0, T]$. In this case, the functions u, v remain Lipschitz continuous. We can thus define the characteristic curves $t \mapsto \xi(t, y)$ and $t \mapsto \zeta(t, \tilde{y})$ as the solutions to the Cauchy problems

$$\begin{aligned}
\dot{\xi} &= u(t, \xi), & \xi(0) &= y \, , \\
\dot{\zeta} &= v(t, \zeta), & \zeta(0) &= \tilde{y} \, ,
\end{aligned}$$

respectively. Let now $\psi_{(0)} \in \mathcal{F}$ be any transportation plan at time $t = 0$. For each $t \in [0, T]$ we can define a transportation plan $\psi_{(t)} \in \mathcal{F}$ by setting

$$\psi_{(t)}\big(\xi(t, y)\big) \doteq \zeta\big(t, \; \psi_{(0)}(y)\big)$$

The corresponding functions $\phi_1^{(t)}, \phi_2^{(t)}$ are then defined according to definitions in (3.4), namely

$$\phi_1^{(t)}(x) \doteq \sup \left\{ \theta \in [0, 1]; \quad \theta \cdot \Big(1 + u_x^2(t, x)\Big) \; \leq \; \Big(1 + v_x^2\big(t, \psi_{(t)}(x)\big)\Big) \psi_{(t)}'(x) \right\} \, ,$$

$$\phi_2^{(t)}(x) \doteq \sup \left\{ \theta \in [0,1]; \ 1 + u_x^2(t,x) \geq \theta \cdot \left(1 + v_x^2\big(t, \ \psi_{(t)}(x)\big)\right) \psi_{(t)}'(x) \right\}.$$

If initially the point y is mapped into $\tilde{y} \doteq \psi_{(0)}(y)$, then at a later time $t > 0$ the point $\xi(t,y)$ along the u-characteristic starting from y is sent to the point $\zeta(t,\tilde{y})$ along the v-characteristic starting from $\tilde{y}$. We thus transport mass from the point $\Big(\xi(t,y), \ u\big(t,\xi(t,y)\big), \ 2\arctan u_x\big(t,\xi(t,y)\big)\Big)$ to the corresponding point $\Big(\zeta(t,\tilde{y}), \ v\big(t,\zeta(t,\tilde{y})\big), \ 2\arctan v_x\big(t,\zeta(t,\tilde{y})\big)\Big),$

In the following, our main goal is to provide an upper bound on the time derivative of the function

$$J^{\psi(t)}\big(u(t), \ v(t)\big) \doteq \int_0^1 d^\diamond \Big(\big(x, \ u(t,x), \ 2\arctan u_x(t,x)\big),$$

$$\big(\psi_{(t)}(x), \ v(t, \ \psi_{(t)}(x)), \ 2\arctan v_x(t, \ \psi_{(t)}(x))\big)\Big) \cdot \phi_1^{(t)}(x)\left(1 + u_x^2(t,x)\right) dx$$

$$+ \int_0^1 \left(1 - \phi_1^{(t)}(x)\right)\left(1 + u_x^2(t,x)\right) dx$$

$$+ \int_0^1 \left(1 - \phi_2^{(t)}(\psi_{(t)}(x))\right)\left(1 + v_x^2\big(t, \ \psi_{(t)}(x)\big)\right) \psi_{(t)}'(x) \, dx.$$

$$(3.29)$$

Differentiating the right hand side of (3.29) one obtains several terms, due to

- Changes in the distance $d^\diamond$ between the points $(\xi, \ u, \ 2\arctan u_x)$ and $(\zeta, \ v, \ 2\arctan v_x)$.

- Changes in the base measures $(1 + u_x^2)\,dx$ and $(1 + v_x^2)\,dx$.

Throughout the following, by $\mathcal{O}(1)$ we denote a quantity which remains uniformly bounded as u, v range in bounded subsets of H_{per}^1. Using the elementary estimate

$$|u - v| \leq \big(1 + |u| + |v|\big) \min\big\{|u - v|, \ 1\big\},$$

we begin by deriving the bound

$$\begin{aligned} I_1 &\doteq \int_0^1 \frac{d}{dt} \left|x - \psi_{(t)}(x)\right| \cdot \phi_1^{(t)}(x)\left(1 + u_x^2(t,x)\right) dx \\ &\leq \int_0^1 \left|u(t,x) - v\big(t, \ \psi_{(t)}(x)\big)\right| \cdot \phi_1^{(t)}(x)\left(1 + u_x^2(t,x)\right) dx \\ &\leq \left(1 + \|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}\right) \cdot J^{\psi(t)}\big(u(t), \ v(t)\big) \\ &= \mathcal{O}(1) \cdot J^{\psi(t)}\big(u(t), \ v(t)\big). \end{aligned}$$

Here and in the sequel, the time derivative is computed along characteristics. Next, recalling the basic equation (1.1), we consider

$$\begin{aligned} I_2 &\doteq \int_0^1 \frac{d}{dt}\left|u(t,x) - v\big(t, \ \psi_{(t)}(x)\big)\right| \cdot \phi_1^{(t)}(x)\left(1 + u_x^2(t,x)\right) dx \\ &\leq \int_0^1 \left|P_x^u(t,x) - P_x^v\big(t, \ \psi_{(t)}(x)\big)\right| \cdot \left(1 + u_x^2(t,x)\right) dx. \end{aligned}$$

In the spatially periodic case, by (1.2) and (2.8) we can write the source terms P_x^u, P_x^v as

$$P_x^u(t, x) = \frac{1}{2} \int_{x-1}^x \chi'(x - y) \cdot \left[u^2(t, y) + \frac{u_x^2(t, y)}{2} \right] dy,$$

$$P_x^v\big(t, \psi_{(t)}(x)\big) = \frac{1}{2} \int_0^1 \chi'\big(\psi_{(t)}(x) - \tilde{y}\big) \cdot v^2(t, \tilde{y}) \, d\tilde{y}$$

$$+ \int_{x-1}^x \chi'\big(\psi_{(t)}(x) - \psi_{(t)}(y)\big) \cdot \frac{v_x^2\big(t, \psi_{(t)}(y)\big)}{4} \psi'_{(t)}(y) \, dy,$$

where, according to (2.8),

$$\chi'(x) = \frac{e^x - e^{1-x}}{e - 1} \qquad 0 < x < 1, \qquad \chi'(x) = \chi'(x + 1) \qquad x \in \mathbb{R}.$$

In the next computation, we use the estimate

$$\int_0^1 \left| \big(1 + u_x^2(y)\big) - \big(1 + v_x^2(\psi(y))\big)\psi'(y) \right| dy = \mathcal{O}(1) \cdot J^\psi(u, v).$$

which holds because of the last two terms in the definition (3.6). Observing that χ' is Lipschitz continuous on the open interval $]0, 1[$, we now compute (omitting explicit references to the time t)

$$\left| P_x^u(x) - P_x^v\big(\psi(x)\big) \right| \leq \frac{1}{2} \int_0^1 \left| \chi'(x - y) \cdot u^2(y) - \chi'(\psi(x) - y) \cdot v^2(y) \right| dy$$

$$+ \mathcal{O}(1) \cdot \int_{x-1}^x \left| x - y - (\psi(x) - \psi(y)) \right| \cdot \frac{v_x^2(\psi(y))}{2} \psi'(y) \, dy$$

$$+ \frac{1}{4} \left| \int_{x-1}^x \chi'(x - y) \left(u_x^2(y) - v_x^2(\psi(y)) \psi'(y) \right) dy \right|$$

$$= \mathcal{O}(1) \cdot \left(\left| x - \psi(x) \right| + \| u^2 - v^2 \|_{L^1} \right)$$

$$+ \mathcal{O}(1) \cdot \left(\left| x - \psi(x) \right| + \int_{x-1}^x \left| y - \psi(y) \right| \cdot \frac{v_x^2(\psi(y))}{2} \psi'(y) \, dy \right)$$

$$+ \mathcal{O}(1) \cdot \left(J^\psi(u, v) + \left| \int_{x-1}^x \chi'(x - y) \cdot [\psi'(y) - 1] \, dy \right| \right)$$

$$= \mathcal{O}(1) \cdot \left| x - \psi(x) \right| + \mathcal{O}(1) \cdot J^\psi(u, v)$$

$$+ \mathcal{O}(1) \cdot \left(\left| x - \psi(x) \right| + \int_{x-1}^x \chi''(x - y) \cdot [\psi(y) - y] \, dy \right)$$

$$= \mathcal{O}(1) \cdot \left| x - \psi(x) \right| + \mathcal{O}(1) \cdot J^\psi(u, v).$$

$$(3.30)$$

Integrating over one period we conclude

$$I_2 = \mathcal{O}(1) \cdot J^\psi\big(u(t), v(t)\big).$$

For future use, we observe that a computation entirely similar to (3.30) yields

$$\left| P^u(x) - P^v\big(\psi(x)\big) \right| = \mathcal{O}(1) \cdot \left| x - \psi(x) \right| + \mathcal{O}(1) \cdot J^\psi(u, v). \qquad (3.31)$$

Next, we look at the term

$$I_3 \doteq \int_0^1 \frac{d}{dt}\Big|2 \arctan u_x(t,x) - 2 \arctan v_x\big(t, \psi_{(t)}(x)\big)\Big| \cdot \phi_1^{(t)}(x) \left(1 + u_x^2(t,x)\right) dx \, .$$

Along a characteristic, according to (1.3) one has

$$\frac{d}{dt} 2 \arctan u_x(t, \xi(t)) = \frac{2}{1 + u_x^2}\Big[u^2 - \frac{u_x^2}{2} - P^u\Big] \, .$$

Call $\theta^u \doteq 2 \arctan u_x$, $\theta^v \doteq 2 \arctan v_x$, so that

$$\frac{1}{1 + u_x^2} = \cos^2 \frac{\theta^u}{2} \, , \qquad \frac{u_x}{1 + u_x^2} = \frac{1}{2} \sin \theta^u \, , \qquad \frac{u_x^2}{1 + u_x^2} = \sin^2 \frac{\theta^u}{2} \, .$$

We now have

$$\int_0^1 \left(1 + u_x^2(x)\right) \cdot \left|\frac{u_x^2(x)}{1 + u_x^2(x)} - \frac{v_x^2(\psi(x))}{1 + v_x^2(\psi(x))}\right| dx$$

$$= \int_0^1 \left(1 + u_x^2(x)\right) \cdot \left|\sin^2 \frac{\theta^u(x)}{2} - \sin^2 \frac{\theta^v(\psi(x))}{2}\right| dx \qquad (3.32)$$

$$\leq \int_0^1 \left(1 + u_x^2(x)\right) \cdot \left|\theta^u(x) - \theta^v(\psi(x))\right| dx$$

$$= \mathcal{O}(1) \cdot J(u,v) \, .$$

Next, using (3.31) we compute

$$\int_0^1 \left(1 + u_x^2(x)\right) \cdot \left|\frac{u^2(x) - P^u(x)}{1 + u_x^2(x)} - \frac{v^2(\psi(x)) - P^v(\psi(x))}{1 + v_x^2(\psi(x))}\right| dx$$

$$\leq \int_0^1 \left|u^2(x) - v^2(\psi(x))\right| dx + \int_0^1 \left|P^u(x) - P^v(\psi(x))\right| dx$$

$$+ \mathcal{O}(1) \cdot \int_0^1 \left|\frac{1}{1 + u_x^2(x)} - \frac{1}{1 + v_x^2(\psi(x))}\right| \cdot \left(1 + u_x^2(x)\right) dx$$

$$= \mathcal{O}(1) \cdot J^\psi(u,v) \, ,$$

where the last term was estimated by observing that

$$\left|\frac{1}{1 + u_x^2} - \frac{1}{1 + v_x^2}\right| \leq \left|2 \arctan u_x - 2 \arctan v_x\right|_* \, .$$

Putting together all previous estimates we conclude

$$I_1 + I_2 + I_3 = \mathcal{O}(1) \cdot J^\psi(u,v) \, .$$

To complete the analysis, we have to consider the terms due to the change in base measures. From (1.5) it follows that the production of new mass in the base measures is described by the balance laws

$$\begin{cases} (1 + u_x^2)_t + \big[u(1 + u_x^2)\big]_x = [2u^2 + 1 - 2P^u]u_x \doteq f^u \, , \\ (1 + v_x^2)_t + \big[v(1 + v_x^2)\big]_x = [2v^2 + 1 - 2P^v]v_x \doteq f^v \, . \end{cases}$$

This leads us to consider two further integrals I_4, I_5 :

$$
\begin{aligned}
I_4 &= \int_0^1 d^\diamond \Big(\big(x,\, u(x),\, 2\arctan u_x(x)\big),\, \big(\psi(x),\, v(\psi(x)),\, 2\arctan v_x(\psi(x))\big) \Big) \\
&\qquad\qquad \cdot \big| 2u^2(x) + 1 - 2P^u(x) \big| \, |u_x(x)| \, dx \\
&= \mathcal{O}(1) \cdot \int_0^1 d^\diamond \Big(\big(x,\, u(x),\, 2\arctan u_x(x)\big),\, \big(\psi(x),\, v(\psi(x)),\, 2\arctan v_x(\psi(x))\big) \Big) \\
&\qquad\qquad \cdot \big(1 + u_x^2(x)\big)\, dx \\
&= \mathcal{O}(1) \cdot J^\psi(u, v)\,.
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \int_0^1 \Big| \big[2u^2(x) + 1 - 2P^u(x) \big] u_x(x) - \big[2v^2(\psi(x)) + 1 - 2P^v(\psi(x)) \big] v_x(\psi(x))\, \psi'(x) \Big|\, dx \\
&\le 2 \int_0^1 \Big\{ \big| u^2(x) - v^2(\psi(x)) \big| + \big| P^u(x) - P^v(\psi(x)) \big| \Big\} \, |u_x(x)|\, dx \\
&\qquad + \int_0^1 \big| 2v^2(\psi(x)) + 1 - 2P^v(\psi(x)) \big| \cdot \big| u_x(x) - v_x(\psi(x))\psi'(x) \big|\, dx \\
&= I_5' + I_5''
\end{aligned}
$$

Using (3.31) we easily obtain

$$
\begin{aligned}
I_5' &\le \int_0^1 \Big\{ \big| u^2(x) - v^2(\psi(x)) \big| + \big| P^u(x) - P^v(\psi(x)) \big| \Big\} \big(1 + u_x^2(x) \big)\, dx \\
&= \mathcal{O}(1) \cdot J^\psi(u, v)\,.
\end{aligned}
$$

On the other hand, recalling (3.32) and using the change of variable $y = \psi(x)$, $x = \psi^{-1}(y)$, we find

$$
\begin{aligned}
I_5'' &= \mathcal{O}(1) \cdot \int_0^1 \big| u_x(x) - v_x(\psi(x))\psi'(x) \big|\, dx \\
&= \mathcal{O}(1) \cdot \int_0^1 \left| \frac{u_x(x)}{1 + u_x^2(x)} - \frac{v_x(\psi(x))}{1 + v_x^2(\psi(x))} \right| \big(1 + u_x^2(x) \big)\, dx \\
&\qquad + \mathcal{O}(1) \cdot \int_0^1 \big| v_x(\psi(x))\, \psi'(x) \big| \cdot \left| \frac{1 + u_x^2(x)}{\big(1 + v_x^2(\psi(x))\big)\psi'(x)} - 1 \right| \\
&\le \mathcal{O}(1) \cdot J^\psi(u, v) + \int_0^1 \Big| \big(1 + u_x^2(x)\big) - \big(1 + v_x^2(\psi(x))\big)\psi'(x) \Big|\, dx \\
&= \mathcal{O}(1) \cdot J(u, v)\,.
\end{aligned}
$$

All together, the previous estimates show that

$$
\frac{d}{dt} J^{\psi(t)}\big(u(t),\, v(t) \big) \le I_1 + I_2 + I_3 + I_4 + I_5' + I_5'' = \mathcal{O}(1) \cdot J^{\psi(t)}\big(u(t),\, v(t) \big)\,, \quad (3.33)
$$

where $\mathcal{O}(1)$ denotes a quantity which remains uniformly bounded as u, v range on bounded sets of H^1_{per}. As an immediate consequence we obtain

Lemma 3.5. *Let* $t \mapsto u(t)$, $t \mapsto v(t)$ *be two conservative, spatially periodic multipeakon solutions, as in Lemma 2. Then there exists a constant* κ, *depending only on* $\max\left\{\|u\|_{H^1_{\mathrm{per}}}, \|v\|_{H^1_{\mathrm{per}}}\right\}$, *such that*

$$J\big(u(t),\, v(t)\big) \leq e^{\kappa|t-s|} \cdot J\big(u(s),\, v(s)\big) \qquad s,t \in \mathbb{R}. \qquad (3.34)$$

Proof. For $t > s$ the estimate (3.34) follows from (3.33), taking the infimum among all transportation plans $\psi_{(s)}$ at time s. The case $t < s$ is obtained simply by observing that the Camassa-Holm equations are time-reversible. $\square$

3.2 A priori bounds

In [19] the author discusses the finite propagation speed property for the Camassa-Holm equation. Due to the nonlocal nature of the equation 1.1, it is not *a priori* clear that the evolution of an initial data with compact support will remains with compact support. On the contrary, [19] prove that the finite propagation speed property is valid only for the function $u - u_{xx}$ and not for u. In this section we start from this result, and we want to estabilish what is the "right" decay at the infinity of solutions to 1.1. For this purpose, we introduce the following functional space. Let $\alpha \in \,]0,1[$, then we set

$$X_\alpha \doteq \{u \in H^1(\mathbb{R}) \ \text{s.t.} \ C^{\alpha,u} \doteq \int_{\mathbb{R}} \left[u^2(x) + u_x^2(x)\right] e^{\alpha|x|}\, dx < +\infty\}. \qquad (3.35)$$

The present section is devoted to the study of some useful properties of the functions $u \in X_\alpha$. We start recalling an estimate for the L^∞−norm of the $H^1(\mathbb{R})$ functions. We have

$$\|f^2\|_{L^\infty} \leq \|f\|_{H^1(\mathbb{R})}^2. \qquad (3.36)$$

This estimate give us a bound on the L^∞−norm of the conservative solution u of (3), in fact the conservation of the energy yields

$$\|u(t)\|_{L^\infty} \leq \|u(t)\|_{H^1(\mathbb{R})} = \sqrt{E^{\bar{u}}} \qquad \text{for every } t \geq 0. \qquad (3.37)$$

Let us consider now the behaviour of the functions $u \in X_\alpha$ as $|x|$ goes to infinity. If we denote with $C^{\alpha,u}$ the constant $\int_{\mathbb{R}}(u^2 + u_x^2)e^{\alpha|x|}\, dx$, the following holds

$$\sup_{x \in \mathbb{R}} u^2(x)e^{\alpha|x|} \leq 2C^{\alpha,u}. \qquad (3.38)$$

Indeed, the function

$$f(x) \doteq u(t,x)e^{\frac{\alpha}{2}|x|}$$

belongs to $H^1(\mathbb{R})$, moreover

$$f_x = u_x e^{\frac{\alpha}{2}|x|} + \frac{\alpha}{2}\,\mathrm{sign}(x)u e^{\frac{\alpha}{2}|x|}$$

and then, by using (3.36), we have

$$|f(x)|^2 \leq \|f\|^2_{H^1(\mathbb{R})} \leq \int_{\mathbb{R}} [2u_x^2 + (1+\alpha)u^2]e^{\alpha|y|} \, dy \leq 2C^{\alpha,u} \,.$$

Now we study the behaviour at infinity of the multipeakon solutions of the Camassa-Holm equation.

Lemma 3.6. *(A-priori* bounds) *Let u be a multi-peakon solution to (1.1), with initial data $\bar{u}$ which belongs to the space X_α defined in (3.35). Then for every $t \in \mathbb{R}$ there exists a continuous function $C(t)$, which depends on $C^{\alpha,\bar{u}}$ and on the energy $E^{\bar{u}}$, such that*

$$\bullet \quad \int_{\mathbb{R}} [u^2(t,x) + u_x^2(t,x)]e^{\alpha|x|} \, dx \leq C(t) \,, \tag{3.39}$$

$$\bullet \quad \sup_{x \in \mathbb{R}} |P_x^u(t,x)|e^{\alpha|x|} \leq C(t) \,, \tag{3.40}$$

$$\bullet \quad \|u_x\|_{L^1(\mathbb{R})} \leq C(t). \tag{3.41}$$

Proof. Since $|P_x^u| = P^u$, it is sufficient to prove the second inequality with P_x^u replaced bu P_u. Setting

$$I(t) \doteq \int_{\mathbb{R}} [u^2(t,x) + u_x^2(t,x)]e^{\alpha|x|} \, dx \,,$$

we want to achieve a differential inequality of the form

$$\frac{d}{dt}I(t) \leq A + B \cdot I(t) \,,$$

for some constants A and B which depend on the initial data $\bar{u}$. We start the discussion proving a preliminary estimate for the function P^u. By applying the Fubini theorem to the identity

$$\int_{\mathbb{R}} P^u(t,x)e^{\alpha|x|} \, dx = \frac{1}{2} \int_{\mathbb{R}} e^{\alpha|x|} \, dx \int_{\mathbb{R}} e^{-|x-y|} \left[u^2(t,y) + \frac{u_x^2(t,y)}{2} \right] dy \tag{3.42}$$

we have to compute the following integral

$$\int_{\mathbb{R}} e^{\alpha|x|}e^{-|x-y|} \, dx = \frac{2\alpha}{1-\alpha^2}e^{-|y|} + \frac{2}{1-\alpha^2}e^{\alpha|y|} \qquad \text{for every } y \in \mathbb{R}. \tag{3.43}$$

For future use, we observe that the equality (3.43) holds for $\alpha \in (-1,1)$. Substituting (3.43) in (3.42) and using the definition of the energy $E^{\bar{u}}$ we have

$$\int_{\mathbb{R}} P^u(t,x)e^{\alpha|x|}dx \leq \frac{E^{\bar{u}}}{(1-\alpha^2)} + \frac{1}{1-\alpha^2} \, I(t) \,.$$

Having in mind the previous inequality, we are able to estimate the time derivative of the function I. From the equations (1.1) and (1.5) we have

$$
\begin{aligned}
\frac{d}{dt} I(t) &= \int_{\mathbb{R}} \left[2uu_t + \left(u_x^2 \right)_t \right] e^{\alpha|x|} \, dx \\
&= \int_{\mathbb{R}} \left[-2u(u_x + P_x^u) + \frac{2}{3}(u^3)_x - (uu_x^2)_x - 2u_x P^u \right] e^{\alpha|x|} \, dx \\
&\leq -2 \int_{\mathbb{R}} (uP^u + uu_x^2)_x \, e^{\alpha|x|} \, dx \leq -2u(P^u + u_x^2) e^{\alpha|x|} \Big|_{-\infty}^{\infty} \\
&\quad +2\alpha \int_{\mathbb{R}} |u|(P^u + u_x^2) \, e^{\alpha|x|} \, dx \\
&\leq \frac{2\alpha \|u\|_{L^\infty}}{1 - \alpha^2} \left(E^{\bar{u}} + 2I \right) \leq \frac{2\sqrt{E^{\bar{u}}}}{1 - \alpha^2} \left[E^{\bar{u}} + 2I(t) \right]
\end{aligned}
$$

the previous inequality gives then a bound on the function I, that is

$$
I(t) \leq (C^{\alpha,\bar{u}} + E^{\bar{u}}/2) \exp \left(\frac{4\sqrt{E^{\bar{u}}}}{1 - \alpha^2} \, t \right).
$$

To achieve the estimate (3.40), set

$$
K(t) \doteq \left\| P^u(t, \cdot) e^{\alpha|\cdot|} \right\|_{L^\infty}.
$$

Proceeding as before, fixed $x \in \mathbb{R}$ we compute the derivative w.r.t the time t of the function $e^{-|x|} * u_x^2$.

$$
\begin{aligned}
\frac{\partial}{\partial t} \left(e^{-|x|} * \frac{u_x^2}{4} \right) &= \frac{1}{4} \frac{\partial}{\partial t} \int_{\mathbb{R}} e^{-|x-y|} u_x^2(t, y) \, dy \\
&= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left[\left(\frac{u^3}{3} - \frac{uu_x^2}{2} - uP^u \right)_x + uP_x^u \right] dy \\
&\leq \|u\|_{L^\infty} P^u + \frac{\|u\|_{L^\infty}}{2} \int_{\mathbb{R}} e^{-|x-y|} P^u(t, y) \, dy \\
&\leq \|u\|_{L^\infty} P^u(t, x) + \frac{\|u\|_{L^\infty}}{2} K(t) \int_{\mathbb{R}} e^{-\alpha|x|} e^{-|x-y|} \, dy \\
&\leq \|u\|_{L^\infty} P^u(t, x) + \frac{1}{1-\alpha^2} e^{-\alpha|x|} \|u\|_{L^\infty} K(t)
\end{aligned}
$$

in the same way, the derivative of $e^{-|x|} * u^2$ is

$$
\begin{aligned}
\frac{\partial}{\partial t} \left(e^{-|x|} * \frac{u^2}{2} \right) &\leq \int_{\mathbb{R}} e^{-|x-y|} u \left(|P_x^u| + |uu_x| \right) dy \\
&\leq \|u\|_{L^\infty} \left(2P^u(t, x) + \int_{\mathbb{R}} e^{-|x-y|} P^u(t, y) \, dy \right) \\
&\leq \|u\|_{L^\infty} \left(2P^u(t, x) + \frac{1}{1-\alpha^2} e^{-\alpha|x|} K(t) \right).
\end{aligned}
$$

Multiplying the previous two inequalities with $e^{\alpha|x|}$ we get

$$
\frac{d}{dt} K(t) \leq \left(3 + \frac{2}{1 - \alpha^2} \right) \sqrt{E^{\bar{u}}} K(t)
$$

which yields (3.40).

To achieve the last inequality, we write

$$\int_{\mathbb{R}} |u_x(y)| \, dy \;=\; \int_{\{y:|u_x(y)|e^{\alpha|y|}<1\}} |u_x(y)| \, dy + \int_{\{y:|u_x(y)|e^{\alpha|y|}>1\}} |u_x(y)| \, dy$$

$$\leq \int_{\mathbb{R}} e^{-\alpha|y|} \, dy + \int_{\mathbb{R}} u_x^2(y) e^{\alpha|y|} \, dy \leq \frac{2}{\alpha} + I(t)$$

where the last estimate is given by (3.39). $\square$

3.3   Definition of the distance in the real line

In this section we define a metric in order to control the distance between two solutions of the equation (3) whenever we take initial data in the subspace X_α of $H^1(\mathbb{R})$. It is constructed as in the spatially periodic case by resolving an appropriate optimal transportation problem. Let $\mathbf{T} = [0, 2\pi]$ be the unit circle with the end points 0 and π identified.
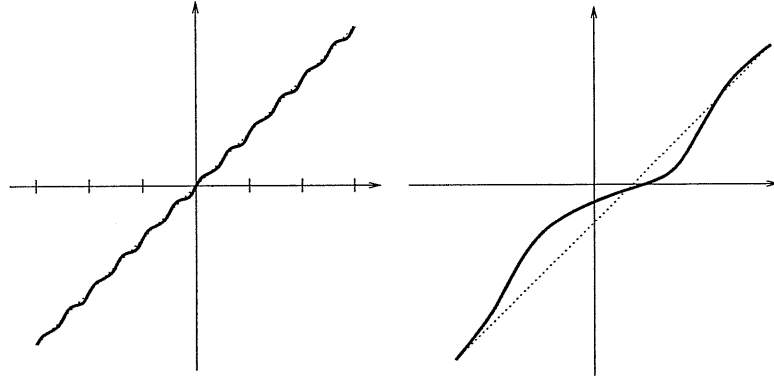

Figure 3.2: "Periodic" vs. "non-periodic" transportation map

Consider the metric space $(\mathbb{R}^2 \times \mathbf{T}, d^\lozenge)$, with distance

$$d^\lozenge((x, u, \omega), (x', u', \omega')) \doteq \min\{|x - x'| + |u - x'| + |\omega - \omega'|_*, 1\}$$

and for every function $u \in X_\alpha$, let us define the Radon measure on $\mathbb{R}^2 \times \mathbf{T}$

$$\sigma^u(A) \doteq \int_{\{x \in \mathbb{R}:(x, u(x), 2 \arctan u_x(x)) \in A\}} [1 + u_x^2(x)] \, dx$$

for every Borel set A of $\mathbb{R}^2 \times \mathbf{T}$.

Definition 3.1. The set $\mathcal{F}$ of transportation plans consists of the functions ψ with the following properties:

1. ψ is absolutely continuous, is increasing with its inverse;

2. $\sup\limits_{x \in \mathbb{R}} |x - \psi(x)| e^{\alpha/2|x|} < \infty$;

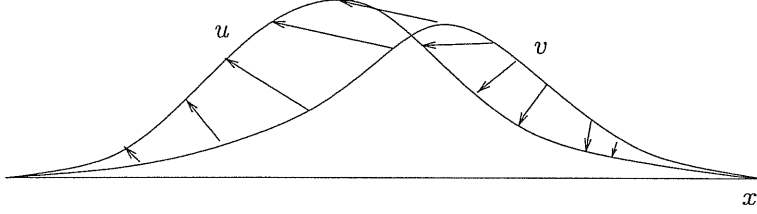3. $\int_{\mathbb{R}} |1 - \psi'(x)|\, dx < \infty$.

Figure 3.3: Transportation plan.

The conditions 2 and 3 are not restrictive. Indeed, thanks to the exponential decay of functions $u, v \in X_\alpha$, the measures σ^u and σ^v located on the graph of u and v respectively, have small mass at the infinity, and then a transportation plan which transports mass from one to the other can be almost the identity $\psi(x) \approx x$ (see fig. 3.3). In order to define a distance in the space X_α, we consider an optimization problem over all possible transportation plans. Given two functions u, v in X_α, we introduce two further measurable functions, related to a transportation plan ψ:

$$\phi_1(x) \doteq \sup\left\{\theta \in [0,1] \ \text{ s.t. } \ \theta \cdot (1 + u_x^2(x)) \le \left(1 + v_x^2(\psi(x))\right)\psi'(x)\right\}, \quad (3.44)$$

$$\phi_2(x) \doteq \sup\left\{\theta \in [0,1] \ \text{ s.t. } \ 1 + u_x^2(x) \le \theta \cdot \left(1 + v_x^2(\psi(x))\right)\psi'(x)\right\}. \quad (3.45)$$

The functions ϕ_1, ϕ_2 can be seen as weights that take into account the difference of the masses of the measure σ^u and σ^v. In fact, from the definitions (3.44)-(3.45) one has

$$\phi_1(x)(1 + u_x^2(x)) = \phi_2(\psi(x))(1 + v_x^2(\psi(x)))\psi'(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

According to the definitions, the identity $\max\{\phi_1(x), \phi_2(x)\} \equiv 1$ holds. Altough the two measures $\phi_1 \sigma^u$ and $\phi_2 \sigma^v$ have not finite mass, they satisfy $\phi_1 \sigma^u(A) = \phi_2 \sigma^v(A)$ for every bounded Borel set $A \subset \mathbb{R}^2 \times \mathbb{T}$. Thus, the functions ϕ_1 and ϕ_2 represent the percentage of mass actually transported from one measure to the other. A distance between the two functions u, v in X_α can be characterized in the following way.

For every transport map $\psi \in \mathcal{F}$, let define $\mathbf{X}^u = (x, u(x), 2\arctan u_x(x))$ and $\mathbf{X}^v = (\psi(x), v(\psi(x)), 2\arctan v_x(\psi(x)))$ and consider the functional

$$J^\psi(u,v) = \int_{\mathbb{R}} d^\diamond(\mathbf{X}^u, \mathbf{X}^v)\phi_1(x)(1 + u_x^2(x))\, dx + \int_{\mathbb{R}} \left|1 + u_x^2(x) - (1 + v_x^2(\psi(x)))\psi'(x)\right|\, dx.$$

Since the above functional is well defined for every $\psi \in \mathcal{F}$, we can define

$$J(u,v) \doteq \inf_{\psi \in \mathcal{F}} J^\psi(u,v).$$

The functional J here defined is thus a metric on the space X_α (see the previous Section 3.1).

3.3.1   Comparison with other topologies

Lemma 3.7. *For every u, $v \in X_\alpha$ one has*

$$\frac{1}{C} \cdot \|u - v\|_{L^1(\mathbb{R})} \leq J(u, v) \leq C \cdot \|u - v\|_{H^1(\mathbb{R})}. \tag{3.46}$$

Let (u_n) be a Cauchy sequence for the distance J such that $C^{\alpha, u_n} \leq C_0$ for every $n \in \mathbb{N}$. Then

   i) There exists a limit function $u \in X_\alpha$ such that $u_n \to u$ in L^∞ and the sequence of derivatives u_{n_x} converges to u_x in $L^p(\mathbb{R})$ for $p \in [1, 2[$.

   ii) Let μ_n be the absolutely continuous measure having density $u_{n_x}^2$ with respect to Lebesgue measure. Then there exists a measure μ whose absolutely continuous part has density u_x^2 such that $\mu_n \rightharpoonup \mu$.

Proof. The first inequality of (3.46) can be achieved by estimating the area between the two functions u and v. For every $\psi \in \mathcal{F}$ we can write

$$\int_\mathbb{R} |u - v| \, dx = \int_{S_1} |u - v| \, dx + \int_{S_2} |u - v| \, dx$$

where the two subsets S_1 and S_2 are

- $S_1 = \{x : |x - \psi(x)| \leq 1\} = \cup_j [x_{2j-1}, x_{2j}]$, where in this union we have to take into account that these intervals may be either finite or infinite, possibly having $x_j = \pm\infty$ for some j,

- $S_2 = \{x : |x - \psi(x)| > 1\}$.

The integral over S_2 can be estimate in the following way:

$$\int_{S_2} |u(x) - v(x)| \, dx \leq (\|u\|_{L^\infty} + \|v\|_{L^\infty}) \int_\mathbb{R} |x - \psi(x)| \, dx \leq (E^{\bar{u}} + E^{\bar{v}}) J(u, v). \tag{3.47}$$

The last inequality is given by the definition of the functional J.

As far as the integral over S_1 is concerned, the integral over S_1 can be viewed as a sum of the area of the regions A_j in the plane $\mathbb{R}^2$, bounded by the graph of the curves u, v and by the segments with slope ± 1 that join the points $Q_u(x_{2j-1}) = (x_{2j-1}, u(x_{2j-1}))$ and $Q_v(x_{2j}) = (\psi(x_{2j}), v(\psi(x_{2j})))$, where $\{x_i\} = \partial S_1$. We have

$$\int_{S_1} |u(x) - v(x)| \, dx \leq \sum_j \text{meas}(A_j).$$
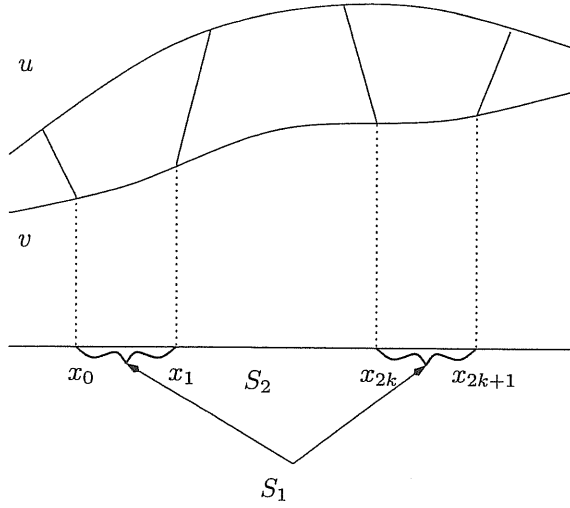
Figure 3.4: L^1—distance between two functions.

The measure of the subset A_j is the area swept by the segment $\overline{Q_u(x)\, Q_v(x)}$. Recalling that in every set A_j the function ψ satisfies $|x| - 1 \le |\psi(x)| \le |x| + 1$, a bound on this area is given by

$$\text{meas}(A_j) \le \int_{x_{2j-1}}^{x_{2j}} (|x - \psi(x)| + |u(x) - v(\psi)|)[1 + u_x^2 + (1 + v_x^2(\psi))\psi']\, dx$$

and then

$$\int_{S_1} |u(x) - v(x)|\, dx \quad \le \int_{S_1} (|x - \psi(x)| + |u(x) - v(\psi)|)[1 + u_x^2 + (1 + v_x^2(\psi))\psi']\, dx$$
$$\le J^\psi(u, v) + J^{\psi^{-1}}(u, v)$$

this inequality, together with (3.47), yields to

$$\|u - v\|_{L^1(\mathbb{R})} \le C(\bar{u}, \bar{v})\, J(u, v).$$

Concerning the second part of the lemma, let us observe that even if the embedding of $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$ is not compact, the uniform exponential decay of the function u_n ($C^{\alpha, u_n} \le C_0$ uniformly in n) allows us to extract a subsequence which converges to a function u in L^1-norm and, by the unformly Höder continuity od such functions, $\|u_n - u\|_{L^\infty} \to 0$. This allow us to prove the property ii) by dealing with analogous arguments to those developed in the periodic case. The proof of the second part of the lemma is thus perfectly similar to the one of the periodic case, once we take into account the exponential decay of the sequence u_n. $\qquad \square$

3.3.2 Stability of solutions w.r.t initial data

Let u_0 and v_0 be two multipeakon initial data. The technique developed in Section 2.1 ensures the existence of two multipeakon solutions $u(t), v(t)$ for (1.1) which conserve the energy unless interaction of peakon occurs. Suppose then that within a given interval $[0, T]$ no interaction occurs neither for $u(t)$ nor for $v(t)$. The aim of this section is to prove the continuity of the functional J w.r.t. the initial data, namely we prove that there exists a continuous, positive function $C(t)$ such that for $t \in [0, T]$ one has

$$J(u(t), v(t)) \leq C(t)J(u_0, v_0).$$

Lemma 3.8. *If $u(t)$ and $v(t)$ are two multipeakon solutions defined in the interval $[0, T]$ in which no interaction occurs, then there exists a positive, continuous function $c(t)$ which depends only on the energies E^u, E^v of the two solutions, such that*

$$\frac{d}{dt}J(u(t), v(t)) \leq c(t)J(u(t), v(t)) \qquad \text{for all } t \in [0, T]. \qquad (3.48)$$

Proof. We compute the time derivative of the function $J^\psi(u(t), v(t))$ with a particular choice of the transportation plan $\psi = \psi_{(t)}$. Given any $\psi_0 \in \mathcal{F}$, at every time $t \in [0, T]$ we construct $\psi_{(t)}$ by transporting the function ψ_0 along the characteristic curves. More precisely, since no interaction between peakon occurs in the interval $[0, T]$, the functions $u(t, \cdot)$, $v(t, \cdot)$ are Lipschitz continuous, then the flows φ_u^t, φ_v^t solutions of the Cauchy problems

$$\frac{d}{dt}\varphi_u^t(x) = u(t, \varphi_u^t(x)) \qquad \varphi_u^0(x) = x,$$

$$\frac{d}{dt}\varphi_v^t(y) = v(t, \varphi_v^t(y)) \qquad \varphi_v^0(y) = y,$$

which are the characteristics curves associated to the equation (1.1), are well defined. Now, let $x \in \mathbb{R}$. $\psi_{(t)}$ is defined as the composition

$$\psi_{(t)}(x) \doteq \varphi_v^t \circ \psi_0 \circ \left(\varphi_u^t\right)^{-1}(x), \qquad (3.49)$$

that is

$$\psi_{(t)}(\varphi_u^t(y)) = \varphi_v^t(\psi_0(y)).$$

The function $\psi_{(t)}$ belongs to $\mathcal{F}$, and hence $J^{\psi_{(t)}}$ is well defined, in fact

1. By the Property 1 of Definition 3.1 for the function ψ_0, and uniqueness of solution of ODE, the function $\psi_{(t)}$ is an increasing function.

2. Let $x \in \mathbb{R}$ and $\varphi_u^t(y)$ be the characteristic curve passing through x at time t. Evaluating $|x - \psi_{(t)}(x)|e^{\alpha/2|x|}$ along this characteristic curve, and computing the derivative w.r.t. t we obtain

$$\frac{d}{dt}|\varphi_u^t(y) - \varphi_v^t(\psi_0(y))|e^{\alpha/2|\varphi_u^t(y)|}$$

$$\leq \left[|u(t, x) - v(t, \psi_{(t)}(x))| + \frac{\alpha}{2}|u(t, x)| \cdot |x - \psi_{(t)}(x)|\right]e^{\alpha/2|x|}$$

by properties (3.38), (3.39), and since u, v are Lipschitz continuous in $[0, T]$, there exists two L^∞ functions $c_1(t)$, $c_2(t)$ such that

$$\frac{d}{dt}|x - \psi_{(t)}(x)|e^{\alpha/2|x|} \leq c_1(t)|x - \psi_{(t)}(x)|e^{\alpha/2|x|} + c_2(t)$$

by Gronwall Lemma and the hypothesis $|x - \psi_0(x)|e^{\alpha/2|x|} \leq C_0$, the previous inequality gives the Property 2 of Definition 3.1 for $\psi_{(t)}$

$$|x - \psi_{(t)}(x)|e^{\alpha/2|x|} \leq C_1(t) \doteq \left(C_0 + \int_0^t c_2(s)\, ds \right) e^{\int_0^t c_1(s)\, ds} \qquad (3.50)$$

3. The last property can be achieved by choosing the change of integration variable $x = \varphi_u^t\ (y)$

$$\begin{aligned}
\int_{\mathbb{R}} |1 - \psi_{(t)}(x)|\, dx &= \int_{\mathbb{R}} |1 - \psi_{(t)}(x)|(\varphi_u^t)'(y)\, dy \\
&= \int_{\mathbb{R}} |(\varphi_u^t)'(y) - (\varphi_v^t)'(\psi_0(y))\psi_0'(y)|\, dy \\
&\leq \int_{\mathbb{R}} |(\varphi_u^t)'(y) - 1|\, dy + \int_{\mathbb{R}} |(\varphi_v^t)'(y) - 1|\, dy \\
&\quad + \int_{\mathbb{R}} |1 - \psi_0'(y)|\, dy.
\end{aligned}$$

Since

$$|(\varphi_u^t)'(y) - 1| \leq \int_0^t |u_x(s, x)| \cdot |(\varphi_u^s)'(y) - 1|\, ds + \int_0^t |u_x(s, x)|\, ds$$

(and a similar estimate for φ_v^t) and $u_x, v_x \in L^\infty$, by the Gronwall lemma the first two integrals of the previous formula are bounded by an absolutely continuous function $C(t)$ in the interval $[0, T]$ and then also Property 3 of Definition 3.1 holds.

At the transportation plan $\psi_{(t)}$ we associate the functions $\phi_1^{(t)}$, $\phi_2^{(t)}$ defined according to (3.44), (3.45), the functional $J^{\psi_{(t)}}$ is thus

$$\begin{aligned}
J^{\psi_{(t)}}(u(t), v(t)) = &\int_{\mathbb{R}} d^\diamond(\mathbf{X}^u(t), \mathbf{X}^v(t))\phi_1^{(t)}(x)(1 + u_x^2(x))dx \\
&+ \int_{\mathbb{R}} \left| 1 + u_x^2(x) - (1 + v_x^2(\psi_{(t)}(x)))\psi_{(t)}'(x) \right| dx.
\end{aligned}$$

By deriving $J^{\psi_{(t)}}(u(t), v(t))$ w.r.t. t and computing the change of variables along the characteristics, the previous derivative can be estimate by the sum of the following terms (we leave out the dependence on the integrable variable when

it is not essential)

$$I_1 = \int_{\mathbb{R}} |u(t,x) - v(t, \psi_{(t)}(x))| \phi_1^{(t)}(x)(1 + u_x^2(t,x))\, dx$$

$$\leq (1 + \|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}) J^{\psi_{(t)}}(u(t), v(t)),$$

$$I_2 = \int_{\mathbb{R}} |P_x^u(t,x) - P_x^v(t, \psi_{(t)}(x))| \phi_1^{(t)}(x)(1 + u_x^2(t,x))\, dx,$$

$$I_3 = \int_{\mathbb{R}} \left| \frac{2u^2(t) - u_x^2(t) - 2P^u(t)}{1 + u_x^2(t)} - \frac{2v^2(t, \psi_{(t)}) - v_x^2(t, \psi_{(t)}) - 2P^v(t, \psi_{(t)})}{1 + v_x^2(t, \psi_{(t)})} \right| \cdot$$
$$\cdot \phi_1^{(t)}(1 + u_x^2(t))\, dx,$$

the term due to the variation of the base measure

$$I_4 = 2 \int_{\mathbb{R}} d^{\Diamond}(\mathbf{X}^u(t), \mathbf{X}^v(t)) \cdot u_x(t)(u^2(t) - P^u(t))\, dx,$$

and the terms due to the variation of the excess mass

$$I_5 = \frac{d}{dt} \int_{\mathbb{R}} \left| 1 + u_x^2(t) - (1 + v_x^2(t, \psi_{(t)}))\psi_{(t)}' \right| dx.$$

Let us start to estimate the term I_2. By definition, the difference of P^u and P^v is written in convolution form

$$\int_{\mathbb{R}} \left\{ e^{-|x-y|} \operatorname{sign}(x-y) \left[u^2(t,y) + \frac{u_x^2(t,y)}{2} \right] dy \right.$$
$$\left. - e^{-|\psi_{(t)}(x) - \psi_{(t)}(y)|} \operatorname{sign}(\psi_{(t)}(x) - \psi_{(t)}(y)) \left[v^2(t, \psi_{(t)}(y)) + \frac{v_x^2(t, \psi_{(t)}(y))}{2} \right] \psi_{(t)}'(y) \right\} dy$$

by this inequality, we can estimate the term I_2 by the sum of the following integrals

$$A = \int_{\mathbb{R}} (1 + u_x^2(t,x)) \int_{\mathbb{R}} e^{-|x-y|} |u^2(t,y) - v^2(t,y)|\, dy\, dx$$

$$B = \int_{\mathbb{R}} (1 + u_x^2(t,x)) \left| \int_{\mathbb{R}} e^{-|x-y|} \operatorname{sign}(x-y) [v^2(t,y) - v^2(t, \psi_{(t)}(y))\psi_{(t)}'(y)]\, dy \right| dx$$

$$C = \int_{\mathbb{R}} (1 + u_x^2(t,x)) \int_{\mathbb{R}} \left[v^2(t, \psi_{(t)}(y)) + \frac{v_x^2(t, \psi_{(t)}(y))}{2} \right] \cdot$$
$$\cdot \left| e^{-|x-y|} \operatorname{sign}(x-y) - e^{-|\psi_{(t)}(x) - \psi_{(t)}(y)|} \operatorname{sign}(\psi_{(t)}(x) - \psi_{(t)}(y)) \right| \cdot \psi_{(t)}'(y)\, dy\, dx$$

$$D = \frac{1}{2} \int_{\mathbb{R}} (1 + u_x^2(t,x)) \left| \int_{\mathbb{R}} e^{-|x-y|} \operatorname{sign}(x-y) [u_x^2(t,y) - v_x^2(t, \psi_{(t)}(y))\psi_{(t)}'(y)]\, dy \right| dx$$

A. Switching the order of the two integrals, the term A is bounded by the

L^1-norm of the difference between u and v:

$$A \leq (\|u\|_{L^\infty} + \|v\|_{L^\infty}) \int_{\mathbb{R}} |u(t,y) - v(t,y)| \int_{\mathbb{R}} e^{-|x-y|}(1 + u_x^2(t,x))\, dx\, dy$$
$$\leq (2 + E^u)(\|u\|_{L^\infty} + \|v\|_{L^\infty}) \|u(t) - v(t)\|_{L^1}$$

and then, by Lemma 3.7, $A \leq C(\bar{u}, \bar{v}) J(u(t), v(t))$.

B. Define

$$F(y) \doteq \int_{-\infty}^{y} (v^2(z) - v^2(\psi_{(t)}(z))\psi'_{(t)}(z))\,dz = \int_{\psi_{(t)}(y)}^{y} v^2(z)\,dz$$

we have, integrating by parts

$$\left| \int_{\mathbb{R}} e^{-|x-y|} \operatorname{sign}(x-y) F'(y)\,dy \right| \leq 2|F(x)| + \int_{R} e^{-|x-y|} |F(y)|\,dy$$

$$\leq \|v\|_{L^\infty}^2 |x - \psi(x)| + \int_{\mathbb{R}} e^{-|x-y|} |y - \psi(y)|\,dy$$

moreover, substituting the previous expression into the term B we obtain

$$B \leq 2E^{\bar{v}} J^{\psi_{(t)}}(u(t), v(t)) + \int_{\mathbb{R}} (1 + u_x^2(t,x)) \int_{\mathbb{R}} e^{-|x-y|} |y - \psi_{(t)}(y)|\,dy\,dx$$

$$= 2E^{\bar{v}} \left\{ J^{\psi_{(t)}}(u(t), v(t)) + \int_{\mathbb{R}} |y - \psi_{(t)}(y)| \int_{\mathbb{R}} (1 + u_x^2(t,x)) e^{-|x-y|}\,dx\,dy \right\}$$

$$\leq 2E^{\bar{v}}(3 + E^{\bar{u}}) \cdot J^{\psi(t)}(u(t), v(t)).$$

C. Observe that since the function $y \mapsto \psi_{(t)}(y)$ is non decreasing, the quantities $x - y$ and $\psi_{(t)}(x) - \psi_{(t)}(y)$ have the same sign, and since the function $t \mapsto e^{-|t|}$ is Lipschitz continuous either in $(-\infty, 0)$ or in $(0, +\infty)$ we have

$$\left| e^{-|x-y|} - e^{-|\psi_{(t)}(x)-\psi_{(t)}(y)|} \right| \leq e^{-\min\{|x-y|, |\psi_{(t)}(x)-\psi_{(t)}(y)|\}} \left| |x - y| - |\psi_{(t)}(x) - \psi_{(t)}(y)| \right|$$

$$\leq e^{-\min\{|x-y|, |\psi_{(t)}(x)-\psi_{(t)}(y)|\}} \left(|x - \psi_{(t)}(x)| + |y - \psi_{(t)}(y)| \right)$$

now

$$-\min\{|x - y|, |\psi_{(t)}(x) - \psi_{(t)}(y)|\} \leq -|x - y| + 2C_1(t),$$

where $C_1(t)$ is the function (3.50), related to the Property 2 of Definition 3.1 $\psi_{(t)}$, then

$$C \leq e^{C_1(t)} \int_{\mathbb{R}} |y - \psi(y)| \left[v^2(t, \psi_{(t)}(y)) + \frac{v_x^2(t, \psi_{(t)}(y))}{2} \right] \psi'_{(t)}(y) \cdot$$

$$\cdot \int_{\mathbb{R}} (1 + u_x^2(t,x)) e^{-|x-y|}\,dx\,dy + +2E^{\bar{v}} \int_{\mathbb{R}} (1 + u_x^2(t,x)) |x - \psi_{(t)}(x)|\,dx$$

$$\leq \left[(2 + E^u) e^{C_1(t)} (1 + \|u\|_{L^\infty}) + 2E^{\bar{v}} \right] J^{\psi_{(t)}}(u(t), v(t))$$

D. Here we can use the estimate given by the change in base measure. Since

$$\int_{\mathbb{R}} \left| 1 + u_x^2(t,x) - (1 + v_x^2(t, \psi_{(t)}(x))) \psi'_{(t)}(x) \right| dx \leq J^{\psi_{(t)}}(u(t), v(t)),$$

we obtain

$$
\begin{aligned}
D &\le \frac{1}{2}\int_{\mathbb{R}}(1+u_x^2(t,x))\int_{\mathbb{R}}e^{-|x-y|}\left|(1+u_x^2(t,y))-(1+v_x^2(t,\psi_{(t)}(y)))\psi'_{(t)}(y)\right|\,dy\,dx \\
&\quad +\frac{1}{2}\int_{\mathbb{R}}(1+u_x^2(t,x))\cdot\left|\int_{\mathbb{R}}e^{-|x-y|}\,\mathrm{sign}(x-y)\left[\psi'_{(t)}(y)-1\right]dy\right|dx \\
&\le \int_{\mathbb{R}}(1+u_x^2(t,x))\left|(\psi_{(t)}(x)-x)-\frac{e^{-|x-y|}}{2}\int_{\mathbb{R}}(\psi_{(t)}(y)-y)\,dy\right|dx \\
&\quad +(1+E^{\bar{u}})J^{\psi_{(t)}}(u(t),v(t)) \\
&\le 2(2+E^{\bar{u}})J^{\psi_{(t)}}(u(t),v(t))
\end{aligned}
$$

where in the last estimate we integrated by part as in the term B.

The control for the terms I_3, I_4 and I_5 can be obtained exactly as the ones in [10], whom we refer the reader to. The previous estimates implies that there exists a smooth function $C=C^{\bar{u},\bar{v}}(t)$ which depends only to the variable t and to the initial data $\bar{u}$, $\bar{v}$ such that

$$
\frac{d}{dt}J^{\psi}(u,v)\le C^{\bar{u},\bar{v}}(t)J^{\psi}(u,v)
$$

which yields

$$
J(u(t),v(t))\le J(u(s),v(s))e^{\left|\int_s^t C^{\bar{u},\bar{v}}(\sigma)\,d\sigma\right|}\qquad\text{for every }s,t\in\mathbb{R}.
$$

$\square$

3.4 Proof of the main theorems

Thanks to the analysis in the previous sections (3.1 for the spatially periodic case, 3.2 and 3.3 in the whole real line), we now all the ingredients toward a proof of Theorems 1.1 and 1.2. Since the two cases are analogous, we prove them in the periodic case. The estimates in (1.14) follow from Lemma 3.1. Given an initial data $\bar{u}\in H_{\mathrm{per}}^1$, to construct the solution of the Camassa-Holm equation we consider a sequence of multi-peakons $\bar{u}_n$, converging to $\bar{u}$ in H_{per}^1. Then we consider the corresponding solutions $t\mapsto u_n(t)$, defined for all $n\ge 1$ and $t\in\mathbb{R}$. This is possible because of Lemmas 2.2 and 2.3.

We claim that the sequence $u_n(t)$ is Cauchy in L_{per}^2. Indeed, by Lemma 3.1 and Lemma 3.3,

$$
\begin{aligned}
\left\|u_m(t)-u_n(t)\right\|_{L_{\mathrm{per}}^1} &\le C\cdot J\big(u_m(t),u_n(t)\big) \\
&\le C\cdot e^{\kappa|t|}J\big(u_m(0),u_n(0)\big)\le C^2\cdot e^{\kappa|t|}\left\|u_m(t)-u_n(t)\right\|_{H_{\mathrm{per}}^1}.
\end{aligned}
$$

Therefore, $u_n(t)\to u(t)$ in L_{per}^1, for some function $u:\mathbb{R}\mapsto H_{\mathrm{per}}^1$. By interpolation, the convergence $u_n\to u$ also holds in all spaces L_{per}^p, $1\le p\le\infty$. The continuity estimates (1.15)-(1.16) now follow by passing to the limit in Lemma 3.3 and 3.4.

It remains to show that the limit function $u(\cdot)$ is actually a solution to the Camassa-Holm equation and its energy $E(t)$ in (1.13) is a.e. constant. Toward these goals, we observe that all solutions u_n are Lipschitz continuous with the same Lipschitz constant, as maps from $\mathbb{R}$ into L^2_{per}. Indeed

$$\|u_{n,t}\|_{L^2_{\mathrm{per}}} \leq \|u_n\|_{L^\infty} \cdot \|u_{n,x}\|_{L^2_{\mathrm{per}}} + \left\| \frac{1}{2} e^{-|x|} \right\|_{L^2} \cdot \left\| u_n^2 + \frac{u_{n,x}^2}{2} \right\|_{L^1_{\mathrm{per}}}.$$

As a consequence, the map $t \mapsto u(t)$ has uniformly bounded H^1_{per} norm, and is Lipschitz continuous with values in L^2_{per}. In particular, u is uniformly Hölder continuous as a function of t, x and the convergence $u_m(t,x) \to u(t,x)$ holds uniformly for t in bounded sets. Moreover, since L^2_{per} is a reflexive space, the time derivative $u_t(t) \in L^2_{\mathrm{per}}$ is well defined for a.e. $t \in \mathbb{R}$.

We now observe that, for each $n \geq 1$, both sides of the equality

$$\frac{d}{dt} u_n = -u_n\, u_{n,x} - P_x^{u_n} \tag{3.51}$$

are continuous as functions from $\mathbb{R}$ into L^1_{per}, and the identity holds at every time $t \in \mathbb{R}$, with the exception of the isolated times where a peakon interaction occurs.

At any time t where no peakon interaction occur in the solution u_n, we define $\mu_t^{(n)}$ to be the measure with density $u_n^2(t,\cdot) + \frac{1}{2} u_{n,x}(t,\cdot)$ w.r.t. Lebesgue measure. By Lemmas 3.3 and 3.2, the map $t \mapsto \mu_t^{(n)}$ can be extended by weak continuity to all times $t \in \mathbb{R}$. We can now redefine

$$P^{u_n}(t,x) \doteq \int \frac{1}{2} e^{-|x-y|}\, d\mu_t^{(n)}(y)\,, \qquad P^u(t,x) \doteq \int \frac{1}{2} e^{-|x-y|}\, d\mu_t(y)\,. \tag{3.52}$$

where μ_t is the weak limit of the measures $\mu_t^{(n)}$. Because of the convergence $J\big(u_n(t),\, u(t)\big) \to 0$, by Lemma 3.2 the map $t \mapsto \mu_t$ is well defined and continuous w.r.t. the weak topology of measures. Using again Lemma 3.2, we can take the limit of (3.51) as $n \to \infty$, and obtain the identity (1.12), for every $t \in \mathbb{R}$ and $P = P^u$ defined by (3.52).

For each n, the total energy $\mu_t^{(n)}\big(]0,1]\big) = E^{\bar u_n}$ is constant in time and converges to $E^{\bar u}$ as $n \to \infty$. Therefore we also have

$$\mu_t\big(]0,1]\big) = E^{\bar u} \doteq \int_0^1 \left[\bar u^2(x) + \bar u_x^2(x) \right] dx \qquad t \in \mathbb{R}\,.$$

To complete the proof of Theorem 1.1, it now only remains to prove that the measure μ_t is absolutely continuous with density

$$u^2(t,\cdot) + \frac{1}{2} u_x^2(t,\cdot)$$

w.r.t. Lebesgue measure, for a.e. time $t \in \mathbb{R}$.

In this direction, we recall that, by (1.5), the function $w \doteq u_{n,x}^2$ satisfies the linear transport equation with source

$$w_t + (uw)_x = (u_n^2 - P^{u_n})u_{n,x}.$$

Moreover, along any characteristic curve $t \mapsto \xi(t)$ by (1.3) one has

$$\frac{d}{dt}\left[2\arctan u_{n,x}\big(t, \xi(t,x)\big)\right] \ = \ \frac{2}{1 + u_{n,x}^2}\left[u_n^2 - \frac{u_{n,x}^2}{2} - P^{u_n}\right] \ \leq \ -\frac{1}{2}\,, \quad (3.53)$$

whenever $u_{n,x}^2$ is sufficiently large. For $\varepsilon > 0$ small, consider the piecewise affine, 2π-periodic function (see fig. 3.5)

$$\varphi(\theta) = \begin{cases} \theta & \text{if } 0 \leq \theta \leq 1 \\ 1 & \text{if } 1 \leq \theta \leq \pi - \varepsilon\,, \\ (\pi - \theta)/\varepsilon & \text{if } \pi - \varepsilon \leq \theta \leq \pi + \varepsilon\,, \\ -1 & \text{if } \pi + \varepsilon \leq \theta \leq 2\pi - 1\,, \\ \theta - 2\pi & \text{if } 2\pi - 1 \leq \theta \leq 2\pi\,. \end{cases}$$
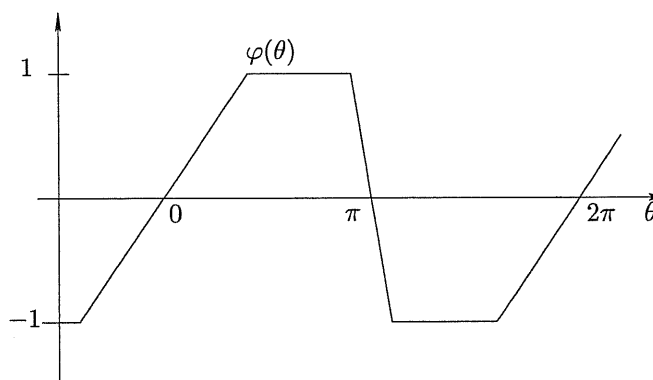

Figure 3.5: Definition of function φ

and define

$$\beta_n(t) = \int_0^1 \varphi\big(2\arctan u_{n,x}(t,x)\big)\, u_{n,x}^2(t,x)\, dx\,.$$

By (1.3) and (3.53) we now have

$$\frac{d}{dt}\beta_n(t) \geq \frac{1}{4\varepsilon}\int_{\{2\arctan u_{n,x} \in [\pi - \varepsilon, \pi] \cup [-\pi, -\pi + \varepsilon]\}} u_{n,x}^2(t,x)\, dx - C\cdot\int_{\{2\arctan u_{n,x} \in [-1,1]\}} u_{n,x}^2\, dx$$
$$(3.54)$$

for some constant C, independent of ε, n. Since all functions β_n remain uniformly bounded, by (3.54) for any time interval $[\tau, \tau']$ we obtain

$$\int_\tau^{\tau'} \int_{\{2\arctan u_{n,x} \in [\pi - \varepsilon, \pi] \cup [-\pi, -\pi + \varepsilon]\}} u_{n,x}^2(t,x)\, dx dt \leq \varepsilon C' \cdot \big(1 + \tau' - \tau\big)\,, \quad (3.55)$$

where the constant C' depends only on the H^1_{per} norm of the functions u_n, hence is uniformly valid for all n, ε. Because of (3.55), the sequence of functions $u^2_{n,x}$ is equi-integrable on any domain of the form $[\tau, \tau'] \times [0, 1]$. Namely

$$\lim_{\kappa \to \infty} \int_\tau^{\tau'} \int_{\{x \in [0,1],\ u^2_{n,x} > \kappa\}} u^2_{n,x}(t, x)\, dx dt = 0\,, \qquad (3.56)$$

uniformly w.r.t. n. By Lemma 3.2 we already know that $\left\| u^p_{n,x}(t) - u^p_x(t) \right\|_{L^1_{\text{per}}} \to 0$ for every fixed time t and $1 \le p < 2$. Thanks to the equi-integrability condition (3.56) we now have

$$u^2_{n,x} \to u^2_x \qquad \text{in} \quad L^1\big([\tau, \tau'] \times [0, 1]\big)\,.$$

By Fubini's theorem, this implies

$$\lim_{n \to \infty} \int_0^1 u^2_{n,x}(t, x)\, dx = \int_0^1 u^2_x(t, x)\, dx$$

for a.e. $t \in [\tau, \tau']$. At every such time t, the measure μ_t is absolutely continuous and the definition (3.52) coincides with (1.2). This completes the proof of Theorem 1.1. $\qquad \square$

3.5 Uniqueness

Before proving Theorem 1.3, we remark that the solution satisfying all conditions in Theorem 1.1 need not be unique.

Example 3.1. Let $u = u(t, x)$ be a solution containing exactly two peakons of opposite strengths $p_1(t) = -p_2(t)$, located at points $q_1(t) = -q_2(t)$ (see fig. 3.6). We assume that initially $p_1(0) > 0\ q_1(0) < 0$. At a finite time $T > 0$, the two peakons interact at the origin. In particular, as $t \to T^-$ there holds

$$p_1(t) \to \infty, \qquad p_2(t) \to -\infty, \qquad q_1(t) \to 0, \qquad q_2(t) \to 0\,.$$

Moreover, $\left\| u(t) \right\|_{L^\infty} \to 0$, while the measure μ_t approaches a Dirac mass at the origin. We now have various ways to extend the solution beyond time T:

$$\tilde{u}(\tau, x) \doteq 0\,,$$

$$u(\tau, x) = -u(\tau - T, -x)\,, \qquad (3.57)$$

Clearly, $\tilde{u}$ dissipates all the energy, and does not satisfy the identity (1.13). The function u in (3.57) is the one constructed by our algorithm in Section 2.1. However, there are infinitely many other solutions that still satisfy (1.13), for example

$$\tilde{u}(\tau, x) = u(\tau, x - b)$$

where u is as in (3.57) and $b \ne 0$. The additional condition in Theorem 1.3 rules out all of them, because as $\tau \to T+$, the corresponding measures $\tilde{\mu}_\tau$ approach a Dirac mass at the point $x = b$, not at the origin.
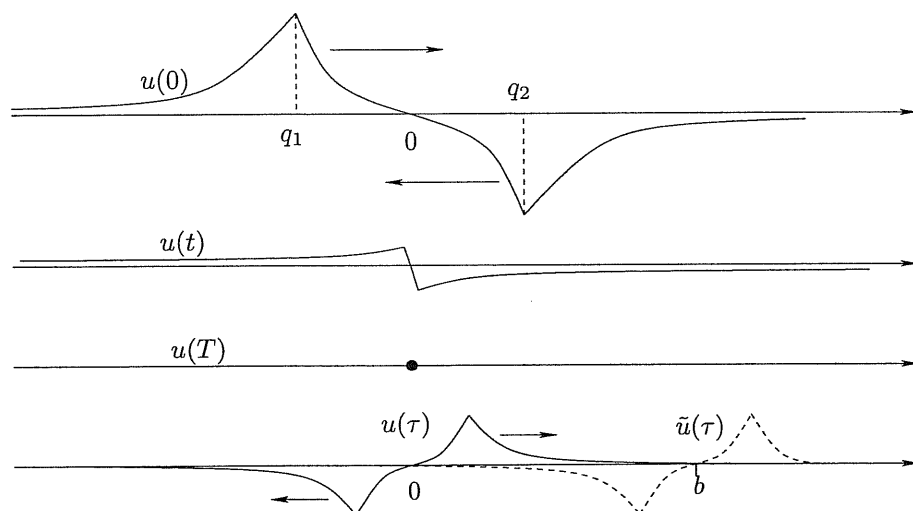
Figure 3.6: Two solutions of the peakon-antipeakon interaction

We can now give a proof of Theorem 1.3. As a first step, we extend our distance J to a larger domain $\mathcal{D}$, consisting of couples (u, μ), where $u \in H^1_{\text{per}}$ and μ is a positive (spatially periodic) measure whose absolutely continuous part has density $u^2 + u_x^2$ w.r.t. Lebesgue measure. This extension is achieved by continuity:

$$J\big((u, \mu),\, (\tilde{u}, \tilde{\mu})\big) \doteq \liminf_{n \to \infty} J(u_n, \tilde{u}_n)$$

where the infimum is taken over all couple of sequences $(u_n, \tilde{u}_n)_{n \geq 1}$ such that

$$\|u_n - u\|_{L^\infty} \to 0, \qquad \|\tilde{u}_n - \tilde{u}\|_{L^\infty} \to 0,$$

$$u_{n,x}^2 \rightharpoonup \mu, \qquad \tilde{u}_{n,x}^2 \rightharpoonup \tilde{\mu}.$$

We observe that the flow Φ constructed in Theorem 1.2 can be continuously extended to a locally Lipschitz continuous group of transformations on the domain $\mathcal{D}$.

Now let $t \mapsto \tilde{u}(t)$ be a solution of the Cauchy problem (1.1), (1.11), satisfying all the required conditions. In particular, the map $t \mapsto \big(\tilde{u}(t), \tilde{\mu}_t\big)$ is Lipschitz continuous w.r.t. the distance J, with values in the domain $\mathcal{D}$.

Calling $t \mapsto \big(\tilde{u}(t), \tilde{\mu}_t\big) \doteq \Phi_t(\bar{u}, \bar{u}_x^2)$ the unique solution of the Cauchy problem obtained as limit of multi-peakon approximations, we need to show that $\tilde{u}(t) = u(t)$ for all t. To fix the ideas, let $t > 0$. By the Lipschitz continuity of the flow, we can use the error estimate

$$J\Big(\big(\tilde{u}(t), \tilde{\mu}_t\big),\, \big(u(t), \mu_t\big)\Big) \leq e^{C_2 t} \int_0^t \liminf_{h \to 0} \frac{1}{h} \cdot J\Big(\big(\tilde{u}(\tau{+}h), \tilde{\mu}_{\tau+h}\big),\, \Phi_h\big(\tilde{u}(\tau), \tilde{\mu}_\tau\big)\Big)\, d\tau$$

$$(3.58)$$

For a proof of (3.58), see [6, pp. 25–27]. The conditions stated in Theorem 1.1 now imply that, at almost every time τ, the measure $\tilde{\mu}_t$ is absolutely continuous and the integrand in (3.58) vanishes. Therefore $\tilde{u}(t) = u(t)$ for all t. □

We can now prove that, in multi-peakon solutions, interactions involving exactly two peakons are the only possible ones.

Corollary 3.1. *Let $t \mapsto u(t, \cdot)$ be a multi-peakon solution of the form (1.6), which remains regular on the open interval $]0, T[$. Assume that at time $T > 0$ an interaction occurs, say among the first k peakons, so that*

$$\lim_{t \to T-} q_i(t) = \bar{q} \qquad i = 1, \ldots, k.$$

Then $k = 2$.

Proof. We first observe that the Camassa-Holm equations (1.1) are time reversible. In particular, our proof of Theorem 1.3 shows that the solution to a Cauchy problem is unique both forward and backward in time.

Now consider the data $\big(u(T), \mu_T\big) \in \mathcal{D}$, where μ_T is the weak limit of the measures μ_t having density $u^2(t) + u_x^2(t)$ w.r.t. Lebesgue measure, as $t \to T-$. By the analysis in Section 2.1, we can construct a backward solution of this Cauchy problem in terms of exactly two incoming peakons. By uniqueness, this must coincide with the given solution $u(\cdot)$ for all $t \in [0, T]$. □

Part II

The discrete Boltzmann equation

Chapter 4

Symmetry groups of differential equations

In this chapter we introduce the theory of the symmetry groups applied to differential equation, which is a tool that will fits in the study of evolutionary equations. The goal of this Chapter is to develop a useful method that will explicitly determine the symmetry group for the system of discrete Boltzmann equation, which will be the starting point of the discussion of Chapter 5 for the blow-up issue. The key point is to transform the equation which has an asymptotic blow-up at a time T into an equation, related to a rescaling of the first equation, which approach a steady state as τ goes to infinitive (see Section 4.2).

4.1 Group and differential equations

The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables with the property that it transforms solution of the system to other solution. In the first part of this section we review a general computational method for (almost) any given system of differential equations. For more information about the application of group theory to the differential equation, we refer the reader to [42, 43].

We start recalling some useful definition in the abstract theory of Transformation Group.

Definition 4.1. *Let $\mathcal{M}$ be a smooth manifold. A local group of transformations acting on $\mathcal{M}$ given by a (local) Lie group G is the couple $(\mathcal{U}, \Psi)$ where*

- *$\mathcal{U}$ is an open subset $\{\iota\} \times \mathcal{M} \subset \mathcal{U} \subset G \times \mathcal{M}$*

- *Ψ a smooth map $\Psi : \mathcal{U} \to \mathcal{M}$*

satisfy the properties

(a) If $(h, x) \in \mathcal{U}$, $(g, \Psi(h, x)) \in \mathcal{U}$ and $(g \cdot h, x) \in \mathcal{U}$ then

$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x).$$

(b) For all $x \in \mathcal{M}$, $\Psi(\iota, x) = x$.

(c) If $(g, x) \in \mathcal{U}$ then $(g^{-1}, \Psi(g, x)) \in \mathcal{U}$ and $\Psi(g^{-1}, \Psi(g, x)) = x$.

For brevity, when it does not make confusion, we denote $\Psi(g, x)$ by $g \cdot x$.

Definition 4.2. *Let G be a local group of transformation acting on a manifold $\mathcal{M}$. A subset $\mathcal{S} \subset \mathcal{M}$ is called G—invariant, and G is called a symmetry group of $\mathcal{S}$ if whenever $x \in \mathcal{S}$ and $g \in G$ are such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.*

Remark 4.1. In our applications, as far as the differential equation is concerned, the subset $\mathcal{S}$ will be usually the graph of the solution of the differential equation

$$\Phi(x, u, \ldots, D^\alpha u, \ldots) = 0, \tag{4.1}$$

where i.e. set of solutions determined by the common zeros of collection of smooth functions $\Phi = (\Phi_1, \ldots, \Phi_l)$, where $\Phi_j = \Phi_j(x, u, \ldots, p^\alpha, \ldots)$ depends on the variables x and the unknowns and their derivatives $u, D^\alpha u, \ldots$, and where, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, D^α indicates the differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}$$

In this context, it is thus useful to introduce the graph of a function $u : \Omega \to R^n$ defined on a open set $\Omega \subset \mathbb{R}^m$

$$\Gamma_u = \{(x, u(x)) : x \in \Omega\}$$

which is a smooth submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. The action of a given transformation $g \in G$ maps the graph Γ_u into the subset $g \cdot \Gamma_u = \{(\tilde{x}, \tilde{u}) = g \cdot (x, u) : (x, u) \in \Gamma_u\}$ which is not necessarily the graph of a function $\tilde{u}$. However, since G acts smoothly and the identity $\iota \in G$ leaves Γ_u unchanged, by restricting the domain Ω for every $g \in G$ near the identity the transformation $g \cdot \Gamma_u$ is the graph $\Gamma_{\tilde{u}}$ of a function $\tilde{u}$.

As an example of action on the graph of a function, let consider a vector field $\mathbf{v}(\xi, \eta) : \mathcal{M} = \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ which can be seen in local coordinate $(x_1, \ldots, x_m, u_1, \ldots, u_n)$ as

$$\mathbf{v} = \sum_{i=1}^m \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^n \eta_j(x, u) \frac{\partial}{\partial u_j}$$

it acts on a smooth scalar function $\phi : \mathbb{R}^{m+n} \to \mathbb{R}$ as a derivation

$$\mathbf{v} \bullet \phi(x, u) = \lim_{\varepsilon \to 0} \frac{\phi(x + \varepsilon \xi, u + \varepsilon \eta) - \phi(x, u)}{\varepsilon}. \tag{4.2}$$

The most important operation on vector fields is their Lie bracket or commutator. Whenever we think that two vector field $\mathbf{v}, \mathbf{w}$ act as a derivation, their Lie bracket $[\mathbf{v}, \mathbf{w}]$ is the unique vector field satisfying

$$[\mathbf{v}, \mathbf{w}] \bullet \phi = \mathbf{v} \bullet (\mathbf{w} \bullet \phi) - \mathbf{w} \bullet (\mathbf{v} \bullet \phi) \qquad \text{for all smooth functions } \phi \qquad (4.3)$$

The integral curve of the vector field $\mathbf{v}$ is a smooth parametrized curve $P(\theta) = (x, u)$ whose tangent vector at any point coincides with the value of $\mathbf{v}$ at the same point:

$$\frac{d}{d\theta} P = \mathbf{v}(P).$$

Starting from a given initial data $P(0) = \bar{P} = (\bar{x}, \bar{u})$ the corresponding integral curve is often denoted by the suggestive exponential notation

$$\exp(\theta\mathbf{v})\bar{P}.$$

From the existence and uniqueness of solution to systems of ordinary differential equations we easily obtain the semigroup property for the *flow* generated by $\mathbf{v}$:

$$\exp(0\mathbf{v})\bar{P} = \bar{P} \qquad (4.4)$$

$$\exp[(\theta_1 + \theta_2)\mathbf{v}]\bar{P} = \exp(\theta_1\mathbf{v})[\exp(\theta_2\mathbf{v})\bar{P}]. \qquad (4.5)$$

From these formulas, compared with the property (a)-(b) of definition 4.1, we see that the flow generated by a vector field is the same as a local action of the Lie group $\mathbb{R}$ on the manifold $\mathcal{M}$ which is called a *one-parameter group of transformations*. The vector field $\mathbf{v}$ is called the *infinitesimal generator* of the action. $\mathbf{v}$ is also called an *infinitesimal symmetry generator* for (4.1) if the map $P \mapsto \exp(\theta\mathbf{v})P$ transforms the graph of a solution u into the graph of another solution.

Remark 4.2. Recall that if θ is sufficiently small, there exists a neighborhood V such that the set

$$\Gamma_\theta = \{\exp(\theta\mathbf{v})(x, u) : (x, u) \in \Gamma_u\}$$

coincides on V with the graph Γ_{u^θ} of a smooth function u^θ. By using a Taylor expansion, the flow $\exp(\theta\mathbf{v})$ maps the point (x, u) into the point

$$\exp(\theta\mathbf{v})(x, u) = (x + \theta\xi(x, u) + o(\theta), u + \theta\eta(x, u) + o(\theta))$$

therefore, the differentiation w.r.t. θ at the origin yields the useful formula

$$\left.\frac{d}{d\theta} u^\theta(x)\right|_{\theta=0} = \eta - \nabla u(x) \cdot \xi. \qquad (4.6)$$

The previous formula gives a necessary condition in order to prove that a particular vector field $\mathbf{v} = (\xi, \eta)$ is an infinitesimal symmetry generator for the

differential equation (4.1). If the function u^θ another solution to this equation, then $\Phi(x, u^\theta, \ldots, D^\alpha u^\theta, \ldots) = 0$. Differentiating w.r.t. θ in 0 we get

$$\sum_\alpha \frac{\partial \Phi}{\partial p^\alpha} \cdot \frac{d}{d\theta}(D^\alpha u^\theta)\bigg|_{\theta=0} = \sum_\alpha \frac{\partial \Phi}{\partial p^\alpha} \cdot D^\alpha \left(\eta - \nabla u(x) \cdot \xi\right) = 0 \qquad (4.7)$$

In the following we shall prove that the previous condition is sufficient in order to construct an infinitesimal symmetry generator $\mathbf{v}$.

Proposition 4.1. *Let G be a connected group of transformation acting on the manifold $\mathcal{M}$. A smooth real-valued function $\zeta : \mathcal{M} \to \mathbb{R}$ is an invariant function for G if and only if*

$$\mathbf{v} \bullet \zeta = 0 \qquad \text{for all } x \in \mathcal{M} \qquad (4.8)$$

and every infinitesimal generator $\mathbf{v}$ of G.

Proof. Suppose that ζ is an invariant function for G. According to (4.2), if $x \in \mathcal{M}$

$$\frac{d}{d\theta}\zeta(\exp(\theta\mathbf{v})x) = \mathbf{v} \bullet \zeta[\exp(\theta\mathbf{v})x]$$

since ζ is invariant, setting $\theta = 0$ it proves the necessity of (4.8). Conversely, if (4.8) holds then $\zeta(\exp(\theta\mathbf{v})x)$ is a constant for the connected subgroup $\{\exp(\theta\mathbf{v})\}$ of $G_x \doteq \{g \in G : g \cdot x \text{ is defined}\}$. But by the properties of the Lie group, every element of G_x can be written as a finite product $g = \exp(\theta^1 \mathbf{v}_{i_1}) \cdots \exp(\theta^k \mathbf{v}_{i_k})$ for some infinitesimal generator v_i of G, hence $\zeta(g \cdot x) = \zeta(x)$ for all $g \in G_x$. $\square$

In a similar way we can prove the following theorem which gives an infinitesimal criterion of invariance for a general equation

$$\Phi(x) = 0 \qquad x \in \mathcal{M}$$

that will be useful whenever we are concerning a differential equation

$$\Phi(x, u, \ldots, D^\alpha u, \ldots) = 0.$$

Theorem 4.1. *Let G be a connected local Lie group of transformations acting on a $p-$dimensional manifold $\mathcal{M}$. Let $\Phi : \mathcal{M} \to \mathbb{R}^l$, $l \leq p$, define a system of equations*

$$\Phi_\nu(x) = 0 \qquad \nu = 1, \ldots, l \qquad (4.9)$$

and assume that the system has maximal rank at every solution x of the system, namely

$$\text{rank} \begin{pmatrix} \frac{\partial \Phi_1}{\partial x^1}(x) & \cdots & \frac{\partial \Phi_1}{\partial x^m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_l}{\partial x^1}(x) & \cdots & \frac{\partial \Phi_l}{\partial x^m}(x) \end{pmatrix} = l \qquad \Phi(x) = 0.$$

Then G is a symmetry group of the system if and only if

$$\mathbf{v} \bullet \Phi_\nu(x) = 0 \qquad \nu = 1, \ldots, l, \quad \Phi(x) = 0 \qquad (4.10)$$

for every infinitesimal generator $\mathbf{v}$ of G.

Proof. Let x_0 be a solution of the system (4.9). As in Proposition 4.1, the necessary condition follows from differentiating w.r.t. ε the identities

$$\Phi_\nu(\exp(\varepsilon\mathbf{v})x_0) = 0$$

and setting $\varepsilon = 0$. Conversely, by using the maximal rank condition, we can choose local coordinates $y = (y^1, \ldots, y^m)$ such that $x_0 = 0$ and Φ has the simple form $\Phi(y) = (y_1, \ldots, y^l)$. Let $\mathbf{v}$ be an infinitesimal generator of G, which can be expressed in the new coordinates as

$$\mathbf{v} = \xi^1(y)\frac{\partial}{\partial y^1} + \cdots + \xi^m(y)\frac{\partial}{\partial y^m}.$$

The condition (4.10) turns to be

$$\mathbf{v}(y^\nu) = \xi^\nu(y) = 0 \qquad \text{for all } \nu = 1\ldots l$$

whenever $y^1 = \cdots = y^l = 0$. Since the flow $\phi(\theta) = \exp(\theta\mathbf{v})x_0$ satisfies the system of ODE

$$\begin{cases} \frac{d}{d\theta}\phi^i = \xi^i(\phi(\theta)) \\ \phi^i(0) = 0 \end{cases} \qquad i = 1\ldots m,$$

the uniqueness of the solution yields to conclude that $\phi^i(\theta) = 0$ for θ sufficiently small. $\exp(\theta\mathbf{v})x_0$ is thus again a solution to $\Phi(x) = 0$. As in Proposition 4.1, by the properties of the connected local Lie group G we gain the result.

$\square$

The previous theorem can be adapted for the differential equation in order to get sufficient condition for obtain an infinitesimal symmetry generator. The equation (4.1) contains not only the unknowns u but also its derivatives, so in order to use Theorem 4.1 we can think that the solution is a point which contains all of these functions. To do this we need to prolong the basic space representing the independent and dependent variables under consideration to a space which also represents the various partial derivatives occurring in the system.

If we consider function $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, the number of derivatives of order k is

$$p_k = n \cdot \binom{m+k-1}{k}$$

If N is the maximum order of the derivatives involved in the differential equation (4.1), we introduce thus the $N-th$ *jet space* $\Omega \times U^{[N]} \doteq \Omega \times \mathbb{R}^m \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_N}$, whose coordinates represent all the derivatives of the function u from 0 to N. If u is a function whose graph lies in a manifold $\mathcal{M} \subset \Omega \times \mathbb{R}^m$, we define its prolongation

$$\mathrm{pr}^{(N)}u = \left(u, (D^{\alpha_1}u)_{|\alpha_1|=p_1}, \ldots, (D^{\alpha_N}u)_{|\alpha_N|=p_N}\right)$$

whose graph lies in the N–th jet space $\mathcal{M}^{(N)} \doteq \mathcal{M} \times \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_N}$. From this point of view, a smooth solution of the given system (4.1) is a function $u(x)$ such that

$$\Phi_\nu(x, \, \mathrm{pr}^{(N)}u(x)) = 0 \qquad \nu = 1, \ldots, l,$$

whenever x lies in the domain of u. It means that the graph of the prolongation of u must lie entirely within the subvariety of the zeroes of the system.

Now suppose that G is a local group of transformations acting on $\mathcal{M}$. The prolongation of G as a local action group on the prolonged manifold $\mathcal{M}^{(N)}$ is defined so that if $g \in G$, $\mathrm{pr}^{(N)}g$ transforms the derivatives of u into the corresponding derivatives of the transformed $g \cdot (x, u)$. To evaluate the action of the prolonged $\mathrm{pr}^{(N)}g$ on a couple $(x_0, u_0^{(N)})$ we simply choose a particular function f whose derivatives agree, up to N–th order, to the point $(x_0, u_0^{(N)})$, apply the action g to f and then prolong $g \cdot f$. Last, we have to define also the prolongation of a vector field up to the order N. It follows by viewing it as the infinitesimal generator of the corresponding action group $\mathrm{pr}^{(N)}[\exp(\theta\mathbf{v})]$:

$$\mathrm{pr}^{(N)}\mathbf{v} \doteq \left.\frac{d}{d\theta}\right|_{\theta=0} \mathrm{pr}^{(N)}[\exp(\theta\mathbf{v})].$$

Writing

$$\mathbf{v} = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u},$$

the general formula of such a vector field is given by [42, Theorem 2.36] and it is the following formal expression

$$\mathrm{pr}^{(N)}\mathbf{v} = \mathbf{v} + \sum_{1 \leq |\alpha| \leq N} \phi^\alpha(x, u^{(N)})\frac{\partial}{\partial u^\alpha} \qquad (4.11)$$

where

$$u^\alpha = D^\alpha u,$$
$$\phi^\alpha = D^\alpha\left(\eta - \xi \cdot \nabla u\right) + \xi \cdot \nabla u^\alpha.$$

For future use, the prolongation of the Lie bracket vector field is

$$\mathrm{pr}^{(N)}[\mathbf{v}, \mathbf{w}] = [\mathrm{pr}^{(N)}\mathbf{v}, \, \mathrm{pr}^{(N)}\mathbf{w}]. \qquad (4.12)$$

By applying Theorem 4.1 to the equation $\Phi(x, \, \mathrm{pr}^{(N)}u(x)) = 0$ we obtain the following theorem, which agree with the formula (4.7)

Theorem 4.2. *Suppose*

$$\Phi_\nu(x, \, \mathrm{pr}^{(N)}u(x)) = 0 \qquad \nu = 1, \ldots, l,$$

is a system of differential equations of maximal rank defined over $\mathcal{M} \subset \omega \times \mathbb{R}^n$. If G is a local group of transformations acting on $\mathcal{M}$, and

$$\mathrm{pr}^{(N)}\mathbf{v} \bullet \Phi_\nu(x, \, \mathrm{pr}^{(N)}u) = 0, \qquad \textit{for all } \nu = 1, \ldots, l,$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system. $\square$

4.2   Symmetries and blow-up

In this section we enter in deep detail of the application of local symmetry groups to the blow-up issue. For a more complete description, the reader can see [5]. Our goal is to apply the theory previously developed in order to choose an appropriate infinitesimal symmetry generator which describes the asymptotic behaviour of a solution which has blow-up in finite time.

We shall consider an evolution problem on a Banach space $(\| \cdot \|, E)$

$$\dot{x} = f(x). \tag{4.13}$$

Assume that there exists trajectories that blow-up in finite time

$$\lim_{t \to T^-} \|x(t)\| = +\infty.$$

Suppose that there exists a second vector field g such that

- Trajectories of $\dot{y} = f(y) - g(y)$ do not blow-up. Instead, they approach a steady state $\bar{y}$ as time goes to infinity.

- There is an explicit computable transformation that maps a trajectory $s \mapsto y(s)$, $s \in [s_0, +\infty[$ into a trajectory $t \mapsto x(t)$, $t \in [t_0, T[$.

In this case we could first accurately study the asymptotic behaviour of $y(s)$ as $s \to +\infty$, and then recover information on the behaviour of the blowing-up solution $x(t)$. To implement this approach, it is clear that the auxiliary vector field g must be carefully selected.

According to the theory developed in the previous section, we shall look for condition on

$$\frac{d}{d\theta} x^\theta = g(x^\theta) \tag{4.14}$$

in order to have that $u^\theta \doteq \exp(\theta g)u$ gives another solution to the equation (4.13), provided that we have existence and uniqueness of solution to the Cauchy problem (4.14).

As the following analysis will show, the crucial assumption on g is the relation

$$f + [f, g] \equiv 0. \tag{4.15}$$

Lemma 4.1. *Let g be a vector field such that 4.15 holds. Then the vector field*

$$\mathbf{v} \doteq -t\frac{\partial}{\partial t} + g(x)\frac{\partial}{\partial x} \tag{4.16}$$

is an infinitesimal generator of symmetry group. In other words, if $t \mapsto x(t)$ is a solution of (4.13) then

$$\{\exp(\theta \mathbf{v})(t, x(t)) \ : \ t \in I\}$$

is the graph of another solution

$$x^\theta(t) \doteq \exp(\theta g)(x(e^\theta t)).$$

Proof. To check that x^θ is indeed a solution, we write

$$\frac{dx^\theta}{dt} = e^\theta \operatorname{Jac}\left[\exp(\theta g)\right] f\left(\exp(-\theta g)x^\theta\right).$$

We claim that the right hand side of the previous formula coincides with $f(x^\theta)$, and this can be done by proving

$$v(\theta) \doteq \operatorname{Jac}\left[\exp(\theta g)\right] f\left(\exp(-\theta g)y\right) = e^{-\theta} f(y) \qquad \text{for all } y \in E. \tag{4.17}$$

Trivially, $v(0) = f(y)$. As far as the derivative of v is concerned, let use the Lie bracket property (see [25])

$$[f, g] = \lim_{\varepsilon \to 0} \frac{\operatorname{Jac}\left[\exp(\varepsilon g)\right] g(\exp(\varepsilon f)) - g}{\varepsilon}.$$

and the hypothesis (4.15)

$$\begin{aligned}
\frac{dv(\theta)}{d\theta} &= \lim_{\varepsilon \to 0} \frac{\operatorname{Jac}[\exp[(\varepsilon+\theta)]g)]f(\exp[(-\theta-\varepsilon)g]y) - \operatorname{Jac}[\exp(\theta g)]f(\exp(-\theta g)y)}{\varepsilon} \\
&= -\operatorname{Jac}\left[\exp(\theta g)\right] \cdot [g, f] \exp(-\theta g)y = -v(\theta).
\end{aligned}$$

The function $v(\theta)$ is thus

$$v(\theta) = e^{-\theta} v(0) = e^{-\theta} f(y).$$

$\square$

Theorem 4.3. *(Blow-up rescaling) Consider two vector fields f, g satisfying (4.15). Let $y : [0, \infty[\mapsto E$ be a solution to*

$$\dot{y}(s) = f(y(s)) - g(y(s)).$$

Then the function $x : [0, 1[\mapsto E$ defined by

$$x(t) \doteq \exp(sg)(y(s)) \qquad s \doteq \ln\frac{1}{1-t} \tag{4.18}$$

is a solution of (4.13).

Proof. Notice that

$$g(\exp(sg)y) = \frac{d}{d\varepsilon} \exp(\varepsilon g)(\exp(sg)y)\Big|_{\varepsilon=0} = \operatorname{Jac}[\exp(sg)]g(y)$$
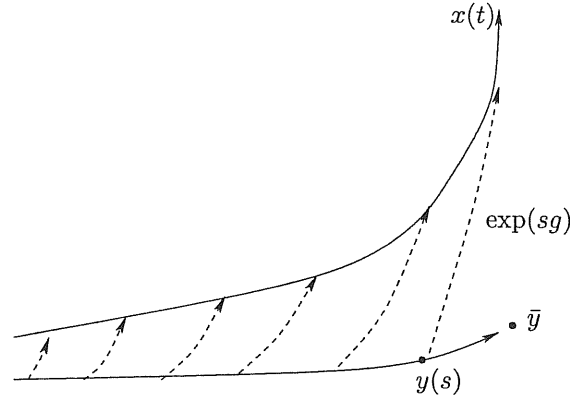
Let compute the derivative w.r.t. s of the function $\exp(sg)y(s)$. By the previous identity and by (4.17) we have

$$\begin{aligned}
\frac{d}{ds} \exp(sg)y(s) &= g(\exp(sg)y(s)) + \operatorname{Jac}[\exp(sg)]\dot{y}(s) \\
&= g(\exp(sg)y(s)) + \operatorname{Jac}[\exp(sg)]f(y(s)) - \operatorname{Jac}[\exp(sg)]g(y(s)) \\
&= \operatorname{Jac}[\exp(sg)]f(\exp(-sg)x) \\
&= e^{-s} f(x)
\end{aligned}$$

now we can check that (4.18) is a solution to (4.13).

$$\frac{d}{dt}x(t) = \frac{d}{ds}\exp(sg)y(s) \cdot \frac{ds}{dt} = e^{-s}f(x) \cdot e^s = f(x).$$

$\square$

Figure 4.1: Connection between $y(\cdot)$ and $x(\cdot)$

We shall apply the above theory to a special case of partial differential equations. Inside the equation (4.1) we highlight the time variable $x \rightsquigarrow (t, x)$, obtaining the evolution equation

$$F(x, \mathrm{pr}^{(N)}u) - u_t \doteq F(x, u, \ldots, D^\alpha u, \ldots) - u_t = 0 \qquad (4.19)$$

where the derivatives on D^α involves only derivatives with respect to the spatial variable x. In the following, we indicate with $[u]$ the prolonged function u in the space $\mathcal{M} \times U^{(N)}$. Thus, the total derivative of a function $F[u(t)]$ takes the form $\nabla F \cdot [u_t]$, where $[u_t] = (u_t, \ldots, D^\alpha u_t, \ldots)$.

The main tool is Theorem 4.2, which gives characterization on the symmetric vector field. From this theorem we obtain

Theorem 4.4. *An evolutionary vector field*

$$\mathbf{v} \doteq -t\frac{\partial}{\partial t} + \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u}$$

is an infinitesimal symmetry generator for (4.19) if and only if for every function u one has

$$(F + [F, G])[u] = 0 \qquad (4.20)$$

where

$$F[u] \doteq F(x, \mathrm{pr}^{(N)}u), \qquad G[u] \doteq -\xi(x, u)\nabla u + \phi(x, u),$$

$$[F, G][u] \doteq \lim_{\varepsilon \to 0} \frac{G[u + \varepsilon F[u]] - G[u] - F[u + \varepsilon G[u]] + F[u]}{\varepsilon}. \qquad (4.21)$$

Proof. Given any smooth solution u, let θ sufficiently small and u^θ be the transformed function by the vector field $\mathbf{v}$, according to Remark 4.2. As Theorem 4.2 states, the thesis follows once we prove (4.7). By formula (4.6), substituting (ξ, η) with $((-t, \xi), \phi)$ we have

$$\frac{du^\theta}{d\theta}\bigg|_{\theta=0} = \phi + tu_t - \xi \cdot \nabla u = tu_t + G[u].$$

Thus, formula (4.7) becomes

$$\left.\frac{d\Phi[u^\theta]}{d\theta}\right|_{\theta=0} = \frac{d\mathcal{F}[u]}{d\theta} - \frac{du_t^\theta}{d\theta} = \nabla F[u] \cdot [tu_t + G[u]] - u_t - tu_{tt} - \nabla G \cdot [u]_t$$
$$= \nabla F[u] \cdot [tu_t + G[u]] - F[u] - t\nabla F[u] \cdot [u_t] - \nabla G \cdot [u]_t$$
$$= -F[u] + \nabla F[u] \cdot [G[u]] - \nabla G[u] \cdot [F[u]].$$

thus

$$\left.\frac{d\Phi[u^\theta]}{d\theta}\right|_{\theta=0} = 0$$

if and only if $[F, G][u] = -F$. $\square$

4.3 A group of symmetry for the discrete Boltzmann equation

The work plan presented in the previous two sections fits in the study of the blow-up rate of discrete Boltzmann equation. Let consider the system of PDE (see [41])

$$\partial_t u_i + \mathbf{c}_i \nabla_x u_i = \sum_{jk} a_{ijk} u_j u_k \qquad i = 1, \ldots, l \qquad (4.22)$$

where

- $(t, x) \in \mathbb{R} \times \mathbb{R}^3$,

- $\mathbf{c}_i \in \mathbb{R}^3$ plays the role of velocity,

- $u_i(t, x)$ is the density of particles having speed $\mathbf{c}_i$,

- a_{ijk} are the coefficients of the quadratic collision term, with $a_{ijk} = 0$ if $j = k$.

We implement Theorem 4.4 in order to recover an evolutionary vector field which generates a group of symmetry for the system (4.22). Looking for a vector field of the form

$$-t\frac{\partial}{\partial t} + \xi(x) \cdot \nabla_x + \phi(u) \cdot \frac{\partial}{\partial u}$$

where

$$\nabla_x \doteq \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \qquad \frac{\partial}{\partial u} \doteq \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_l}\right)$$

we have to find condition for which (4.20) holds. Note that in this case the two maps F and G are

$$(F[u])_i \doteq -c_i \cdot \nabla_x + \sum_{j,k} a_{ijk} u_j u_k \qquad (G[u])_i \doteq -\xi(x) \cdot \nabla_x u_i + \phi_i(u) \qquad i = 1 \ldots l.$$

Theorem 4.5. *Setting*

$$\xi(x) \doteq -x \qquad \phi_i(u) \doteq u_i,$$

the identity

$$[F, G][u] + F[u] = 0$$

holds for every u solution to (4.22).

Proof. Let compute (4.21).

$$\frac{(G[u + \varepsilon F[u]] - G[u])_i}{\varepsilon} = \frac{x \cdot \nabla_x \left(u_i - \varepsilon \mathbf{c}_i \cdot \nabla_x u_i + \varepsilon \sum_{jk} a_{ijk} u_j u_k \right)}{\varepsilon}$$

$$+ \frac{u_i - \varepsilon \mathbf{c}_i \cdot \nabla_x u_i + \varepsilon \sum_{jk} a_{ijk} u_j u_k - x \cdot \nabla_x u_i - u_i}{\varepsilon}$$

$$= -x \cdot \nabla_x \left(\mathbf{c}_i \cdot \nabla_x u_i \right) + (F[u])_i$$

$$+ \sum_{jk} a_{ijk} \left(u_k x \cdot \nabla_x u_j + u_j x \cdot \nabla_x u_k \right)$$

$$(4.23)$$

$$\frac{(F[u + \varepsilon G[u]] - F[u])_i}{\varepsilon} = \frac{-\mathbf{c}_i \cdot \nabla_x u_i - \varepsilon \mathbf{c}_i \nabla_x (x \cdot \nabla_x u_i) - \varepsilon \mathbf{c}_i \nabla_x u_i}{\varepsilon}$$

$$+ \frac{\sum_{jk} a_{ijk}(u_j + \varepsilon x \cdot \nabla_x u_j + \varepsilon u_j)(u_k + \varepsilon x \cdot \nabla_x u_k + \varepsilon u_k)}{\varepsilon}$$

$$- \frac{-\mathbf{c}_i \nabla_x u_i + \sum_{jk} a_{ijk} u_j u_k}{\varepsilon}$$

$$= -\mathbf{c}_i \cdot \nabla_x u_i - \mathbf{c}_i \nabla_x (x \cdot \nabla_x u_i)$$

$$+ \sum_{jk} a_{ijk}(2 u_j u_k + u_j x \cdot \nabla_x u_k + u_k x \cdot \nabla_x u_j) + \mathcal{O}(\varepsilon)$$

$$(4.24)$$

Note that

$$x \cdot \nabla_x (\mathbf{c}_i \cdot \nabla_x u_i) = \sum_j x_j \sum_k (\mathbf{c}_i)_k u_{x_j x_k}$$

$$\mathbf{c}_i \cdot \nabla_x (x \cdot \nabla_x u_i) = \mathbf{c}_i \cdot \nabla_x u_i + \sum_k (\mathbf{c}_i)_k \sum_j x_j u_{x_j x_k}.$$

Hence, the limit of the difference of the formulas (4.23), (4.24) yields

$$([F, G][u])_i = -2(F[u])_i + (F[u])_i = -(F[u])_i.$$

$$\square$$

Corollary 4.1. *If $u(t, x) = (u_1(t, x), \dots, u_l(t, x))$ is a solution, then*

$$u^\theta(t, x) = e^\theta u(e^\theta t, e^\theta x)$$

is another solution to the system of PDE (4.22). Hence, by performing the change of variables

$$\begin{cases} \tau = \ln \dfrac{1}{1 - t} \\ \eta = e^\tau x = \dfrac{x}{1 - t} \\ w_i = e^{-\tau} u_i = (1 - t) u_i \end{cases} \qquad (4.25)$$

blow-up to (4.22) occurs if a solution to

$$(w_i)_\tau = -(\mathbf{c}_i + \eta) \cdot \nabla_\eta w_i + \sum_{jk} a_{ijk} w_j w_k - w_i$$

approach a steady state as τ goes to infinity.

Proof. By Theorem 4.5 the vector field

$$\mathbf{v} \doteq -t\frac{\partial}{\partial t} - x \cdot \nabla_x + u \cdot \frac{\partial}{\partial u}$$

is the generator of symmetry associated to G, hence fixed θ, the graph of the solution is mapped by $\mathbf{v}$ in the graph of another solution:

$$\Gamma^\theta = \{\exp(\theta\mathbf{v})(t,x,u) : u = u(t,x)\}$$

that is
$$\begin{cases} t(\theta) = e^{-\theta}t \\ x(\theta) = e^{-\theta}x \\ u(t(\theta), x(\theta)) = e^\theta u(t,x) \end{cases}$$

thus, the new solution is $u^\theta(t,x) = e^\theta u(e^\theta t, e^\theta x)$.

As far as the second part of the corollary is concerned, we shall use Theorem 4.3. Suppose that there exists a solution $w = (w_1(\tau, \eta), \ldots, w_l(\tau, \eta))$ to the system
$$\partial_\tau w = F[w] - G[w]$$

which approach a steady state as $\tau \to \infty$. Then the transformation by $\exp(\tau\mathbf{v})$ with $\tau = \ln\frac{1}{1-t}$ is the graph of the solution

$$u(t,x) = \exp(\tau G)w(\tau, \eta)$$

which corresponds to the change of variables

$$\begin{cases} x = e^{-\tau}\eta = (1-t)\eta \\ u_i = e^\tau w_i = \dfrac{w_i}{1-t} \end{cases}$$

which yields (4.25). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Chapter 5

The two dimensional Broadwell model

5.1 The discrete Boltzmann equation

Consider the simplified model of a gas whose particles can have only finitely many speeds, say $c_1, \ldots, c_N \in \mathbb{R}^n$. Call $u_i = u_i(t, x)$ the density of particles with speed c_i. The evolution of these densities can then be described by a semilinear system of the form

$$\partial_t u_i + c_i \cdot \nabla u_i = \sum_{j,k} a_{ijk}\, u_j u_k \qquad\qquad i = 1, \ldots, N. \tag{5.1}$$

Here the coefficient a_{ijk} measures the rate at which new i-particles are created, as a result of collisions between j- and k-particles. In a realistic model, these coefficients must satisfy a set of identities, accounting for the conservation of mass, momentum and energy.

Given a continuous, bounded initial data

$$u_i(0, x) = \bar{u}_i(x), \tag{5.2}$$

on a small time interval $t \in [0, T]$ a solution of the Cauchy problem can be constructed by the method of characteristics. Indeed, since the system is semilinear, this solution is obtained as the fixed point of the integral transformation

$$u_i(t, x) = \bar{u}_i(x - c_i t) + \int_0^t \sum_{j,k} a_{ijk}\, u_j u_k \big(s,\ x - c_i(t - s)\big)\, ds\,.$$

For sufficiently small time intervals, the existence of a unique fixed point follows from the contraction mapping principle, without any assumption on the constants a_{ijk}.

If the initial data is suitably small, the solution remains uniformly bounded for all times [4]. For large initial data, on the other hand, the global existence

and stability of solutions is known only in the one-dimensional case [3, 32, 45]. Since the right hand side has quadratic growth, it might happen that the solution blows up in finite time. Examples where the L^∞ norm of the solution becomes arbitrarily large as $t \to \infty$ are easy to construct [35]. In the present chapter we focus on the two-dimensional Broadwell model and examine the possibility that blow-up actually occurs in finite time.

Since the equations (5.1) admit a natural symmetry group, one can perform an asymptotic rescaling of variables and ask whether there is a blow-up solution which, in the rescaled variables, converges to a steady state. This technique has been widely used to study blow-up singularities of reaction-diffusion equations with superlinear forcing terms [30, 31]. See also [36] for an example of self-similar blow-up for hyperbolic conservation laws. Our results show, however, that for the two-dimensional Broadwell model no such self-similar blow-up solution exists.

If blow-up occurs at a time T, our results imply that for times $t \to T-$ one has

$$\left\| u(t) \right\|_{L^\infty} > \frac{1}{5} \frac{\ln \left| \ln(T-t) \right|}{T-t}.$$

This means that the blow-up rate must be different from the natural growth rate $\left\| u(t) \right\|_{L^\infty} = \mathcal{O}(1) \cdot (T-t)^{-1}$ which would be obtained in case of a quadratic equation $\dot{u} = C \, u^2$.

In the final section of this chapter we discuss a possible scenario for blow-up. The analysis highlights how carefully chosen should be the initial data, if blow-up is ever to happen. This suggests that finite time blow-up is a highly non-generic phenomenon, something one would not expect to encounter in numerical simulations.

5.2 Coordinate rescaling

In the following, we say that $P^* = (t^*, x^*)$ is a *blow-up point* if

$$\limsup_{x \to x^*, \, t \to t^*-} u_i(t, x) = \infty$$

for some $i \in \{1, \ldots, N\}$. Define the constant

$$C \doteq \max_i |\mathbf{c}_i|.$$

We say that (t^*, x^*) is a *primary blow-up point* if it is a blow-up point and the backward cone

$$\Gamma \doteq \left\{ (t, x) \, ; \; |x - x^*| < 2C \, (t^* - t) \right\}$$

does not contain any other blow-up point.

Lemma 5.1. *Let* $u = u(t, x)$ *be a solution of the Cauchy problem (5.1)-(5.2) with continuous initial data. If no primary blow-up point exist, then* u *is continuous on the whole domain* $[0, \infty[\times \mathbb{R}^n$.

Proof. If u is not continuous, it must be unbounded in the neighborhood of some point. Hence some blow-up point exists. Call $\mathcal{B}$ the set of such blow-up points. Define the function

$$\varphi(x) \doteq \inf_{(\tau,\xi)\in\mathcal{B}} \left\{ \tau + C\left|x - \xi\right| \right\}.$$

By Ekeland's variational principle (see [1], p.254), there exists a point x^* such that

$$\varphi(x) \geq \varphi(x^*) - \frac{C}{2}\left|x - x^*\right|$$

for all $x \in \mathbb{R}^2$. Then $P^* \doteq \left(\varphi(x^*),\, x^*\right)$ is a primary blow-up point. $\square$

Let now (t^*, x^*) be a primary blow-up point. One way to study the local asymptotic behavior of u is to rewrite the system in terms of the rescaled variables $w_i = w_i(\tau, \eta)$, defined by

$$\begin{cases} \tau &= -\ln(t^* - t), \\ \eta &= e^\tau x = \dfrac{x - x^*}{t^* - t}, \\ w_i &= e^{-\tau} u_i = (t^* - t)u_i. \end{cases} \tag{5.3}$$

The corresponding system of evolution equations is

$$\partial_\tau w_i + (\mathbf{c}_i + \eta)\cdot\nabla_\eta w_i = -w_i + \sum_{j,k} a_{ijk}\, w_j w_k \qquad i = 1,\ldots,n. \tag{5.4}$$

Any nontrivial stationary or periodic solution w of (5.4) would yield a solution u of (5.1) which blows up at (t^*, x^*). On the other hand, the non-existence of such solutions for (5.4) would suggest that finite time blow-up for (5.1) is unlikely.

5.3   The two-dimensional Broadwell model

Consider a system on $\mathbb{R}^2$ consisting of 4 types particles (fig. 5.1), with speeds

$$\mathbf{c}_1 = (1,1), \qquad \mathbf{c}_2 = (1,-1), \qquad \mathbf{c}_3 = (-1,-1), \qquad \mathbf{c}_4 = (-1,1).$$

The evolution equations are

$$\begin{cases} \partial_t u_1 + c_1\cdot\nabla u_1 = u_2 u_4 - u_1 u_3, \\ \partial_t u_3 + c_3\cdot\nabla u_3 = u_2 u_4 - u_1 u_3, \\ \partial_t u_2 + c_2\cdot\nabla u_2 = u_1 u_3 - u_2 u_4, \\ \partial_t u_4 + c_4\cdot\nabla u_4 = u_1 u_3 - u_2 u_4. \end{cases} \tag{5.5}$$

After renaming variables, the corresponding rescaled system (5.4) takes the form

$$\begin{cases} \partial_t w_1 + (x+1)\partial_x w_1 + (y+1)\partial_y w_1 = w_2 w_4 - w_1 w_3 - w_1, \\ \partial_t w_3 + (x-1)\partial_x w_3 + (y-1)\partial_y w_3 = w_2 w_4 - w_1 w_3 - w_3, \\ \partial_t w_2 + (x+1)\partial_x w_2 + (y-1)\partial_y w_2 = w_1 w_3 - w_2 w_4 - w_2, \\ \partial_t w_4 + (x-1)\partial_x w_4 + (y+1)\partial_y w_4 = w_1 w_3 - w_2 w_4 - w_4. \end{cases} \tag{5.6}$$
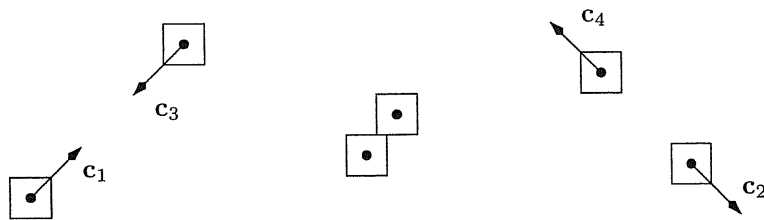
Figure 5.1: Moving particles with prescribed speeds

Our first result rules out the possibility of asymptotically self-similar blow-up solutions. A sharper estimate will be proved later.

Theorem 5.1. *The system (5.6) admits no nontrivial positive bounded solution which is constant or periodic in time.*

Proof. Assume

$$0 \leq w_i(t,x,y) \leq \kappa \tag{5.7}$$

for all t,x,y, $i = 1,2$. Choose $\varepsilon \doteq e^{-2\kappa}/2$, so that

$$\varepsilon < \frac{1}{\kappa}, \qquad \varepsilon e^{2\kappa x} \leq \frac{1}{2} \quad x \in [-1,1].$$

Define

$$Q_{14}(t,y) \doteq \int_{-1}^{1} \left[\left(1 - \varepsilon e^{2\kappa x}\right) w_1(t,x,y) + \left(1 - \varepsilon e^{-2\kappa x}\right) w_4(t,x,y) \right] dx.$$

$$Q_{14}(t) \doteq \sup_{|y| \leq 1} Q_{14}(t,y),$$

Restricted to any horizontal moving line $y = y(t)$ such that $\dot{y} = y+1$ (fig.5.2), the equations (5.6) become

$$\partial_t w_1 + (x+1)\partial_x w_1 = w_2 w_4 - w_1 w_3 - w_1,$$
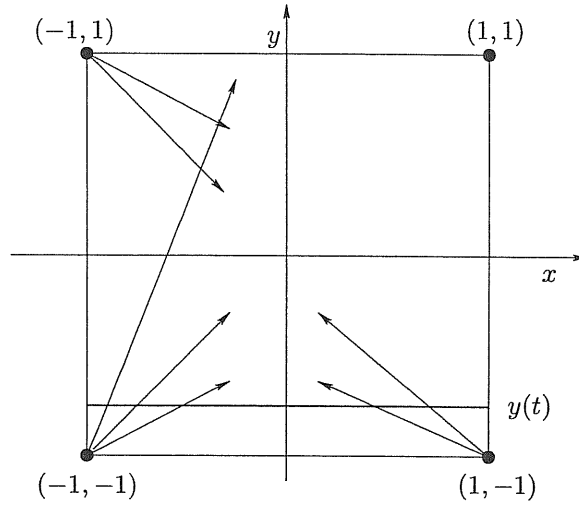$$\partial_t w_4 + (x-1)\partial_x w_4 = w_1 w_3 - w_2 w_4 - w_4.$$

Figure 5.2: Interaction on a moving line

A direct computation now yields

$$\frac{d}{dt}Q_{14}\big(t, y(t)\big)$$

$$\leq -2\varepsilon\kappa \int_{-1}^{1} \left[e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4\right] dx$$

$$+ \int_{-1}^{1} \left(\varepsilon e^{2\kappa x} - \varepsilon e^{-2\kappa x}\right)(w_1 w_3 - w_2 w_4)\, dx$$

$$\leq -\varepsilon\kappa \int_{-1}^{1} \left[e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4\right] dx - \varepsilon\kappa \int_{-1}^{0} e^{-2\kappa x}(1-x)w_4\, dx$$

$$-\varepsilon\kappa \int_{0}^{1} e^{2\kappa x}(1+x)w_1\, dx + \int_{-1}^{0} \varepsilon\kappa e^{-2\kappa x}w_4\, dx + \int_{0}^{1} \varepsilon\kappa e^{2\kappa x}w_1\, dx$$

$$\leq -\varepsilon\kappa \int_{-1}^{1} \left[e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4\right] dx\,.$$

Call

$$I(t,y) \doteq \int_{-1}^{1} w_1(t,x,y)\, dx\,.$$

The definition of ε and the bound (5.7) on w_1 imply

$$\int_{-1}^{1} e^{2\kappa x}(1+x)w_1\, dx \geq 2\varepsilon \int_{-1}^{1} (1+x)w_1\, dx$$

$$\geq 2\varepsilon \int_{-1}^{-1+I/\kappa} (1+x)\kappa\, dx$$

$$= \varepsilon\, I^2/\kappa$$

From this, and a similar estimate for w_4, we obtain

$$\int_{-1}^{1} \left[e^{2\kappa x}(1+x)w_1 + e^{-2\kappa x}(1-x)w_4 \right] dx \geq \frac{\varepsilon}{\kappa} \left(\int_{-1}^{1} w_1 \, dx \right)^2 + \frac{\varepsilon}{\kappa} \left(\int_{-1}^{1} w_4 \, dx \right)^2$$

$$\geq \frac{\varepsilon}{\kappa} \frac{Q_{14}^2}{2}.$$

Since $\varepsilon < \kappa^{-1}$, this yields

$$\frac{d}{dt} Q_{14}\big(t, y(t)\big) \leq -\frac{\varepsilon^2}{2} Q_{14}^2\big(t, y(t)\big). \tag{5.8}$$

Observing that the Cauchy problem

$$\dot{z} = -\frac{\varepsilon^2}{2} z^2, \qquad z(0) = 4\kappa$$

has the solution

$$z(t) = \left(\frac{1}{4\kappa} + \frac{\varepsilon^2}{2} t \right)^{-1},$$

by a comparison argument from (5.8) we deduce

$$Q_{14}(t) \leq \left(\frac{1}{4\kappa} + \frac{\varepsilon^2}{2} t \right)^{-1}.$$

Since

$$\int_{-1}^{1}\int_{-1}^{1} w_1(t, x, y) \, dx dy \leq 4Q_{14}(t),$$

and since a similar estimate can be performed for all components w_i, we conclude

$$\int_{-1}^{1}\int_{-1}^{1} w_i(t, x, y) \, dx dy \leq 4 \left(\frac{1}{4\kappa} + \frac{e^{-4\kappa}}{8} t \right)^{-1}. \tag{5.9}$$

The right hand side of (5.9) approaches zero as $t \to \infty$. Therefore, nontrivial constant or time-periodic L^∞ solutions of (5.6) cannot exist.   □

5.4   Refined blow-up estimates

If (t^*, x^*) is a blow-up point, our analysis has shown that in the rescaled coordinates τ, ξ the corresponding functions w_i must become unbounded as $\tau \to \infty$. In this section we refine the previous result, establishing a lower bound for the rate at which such explosion takes place.

Theorem 5.2. *Let u be a continuous solution of the Broadwell system (5.3). Fix any point (t^*, x^*) and consider the corresponding rescaled variables τ, ξ, w_i. If*

$$\max_{|\xi_1|, |\xi_2| \leq 1} w_i(\tau, \xi_1, \xi_2) \leq \theta \ln \tau \qquad i = 1, 2, 3, 4,$$

for some $\theta < 1/4$ and all τ sufficiently large, then

$$\lim_{\tau \to \infty} w_i(\tau, \xi) = 0 \qquad i = 1, 2, 3, 4 \,,$$

uniformly for $\xi \in \mathbb{R}^2$ in compact sets. Therefore (t^*, x^*) is not a blow up point.

Since $w_i = (t^* - t)u_i$ and $\tau \doteq \big| \ln(t^* - t) \big|$, the above implies

Corollary 5.1. *If (t^*, x^*) is a primary blow-up point, then*

$$\limsup_{x \to x^*, \ t \to t^*-} \ \ \big| u(t, x) \big| \cdot \frac{t^* - t}{\ln \big| \ln(t^* - t) \big|} \ \geq \ \frac{1}{4} \,.$$

Proof of Theorem 5.2.

Let $w_i = w_i(t, x, y)$ provide a solution to the system (5.6), with

$$0 \leq w_i(t, x, y) \leq \theta \ln t \doteq k(t) \tag{5.10}$$

for all $t \geq t_0$ and $x, y \in [-1, 1]$. The proof will be given in two steps. First we show that the L^1 norm of the components w_i approaches zero as $t \to \infty$. Then we refine the estimates, and prove that also the L^∞ norm asymptotically vanishes.

STEP 1: Integral estimates. Consider the function

$$Q_{14}(t, y) \doteq \int_{-1}^{1} \left[\left(1 - \frac{e^{2k(t)(x-1)}}{2} \right) w_1(t, x, y) + \left(1 - \frac{e^{-2k(t)(x+1)}}{2} \right) w_4(t, x, y) \right] dx$$

with $k(t)$ as in (5.10). As in the proof of Theorem 5.1, let $t \mapsto y(t)$ be a solution to $\dot{y} = y + 1$. Then

$$
\begin{aligned}
\frac{d}{dt} Q_{14}\big(t, y(t)\big) = \ &\int_{-1}^{1} \Big[-(x-1)k' e^{2k(t)(x-1)} w_1 + (x+1)k' e^{-2k(t)(x+1)} w_4 \Big] dx \\
&+ \int_{-1}^{1} \left(1 - \frac{e^{2k(t)(x-1)}}{2} \right) \Big[-(x+1)w_{1x} + w_2 w_4 - w_1 w_3 - w_1 \Big] dx \\
&+ \int_{-1}^{1} \left(1 - \frac{e^{-2k(t)(x+1)}}{2} \right) \Big[-(x-1)w_{4x} + w_1 w_3 - w_2 w_4 - w_4 \Big] dx
\end{aligned}
$$

To estimate the right hand side, we notice that

$$
\begin{aligned}
A &\doteq \int_{-1}^{1} \left(1 - \frac{e^{2k(t)(x-1)}}{2} \right) \big[(1+x)w_{1x} + w_1 \big] dx \geq k(t) \int_{-1}^{1} (x+1)e^{2k(x-1)} w_1 \, dx \\
B &\doteq \int_{-1}^{1} \left(1 - \frac{e^{-2k(t)(x+1)}}{2} \right) \big[(x-1)w_{4x} + w_4 \big] dx \geq k(t) \int_{-1}^{1} (1-x)e^{-2k(x+1)} w_4 \, dx \\
C &\doteq \int_{-1}^{1} (w_1 w_3 - w_2 w_4) \left(\frac{e^{2k(x-1)}}{2} - \frac{e^{-2k(x+1)}}{2} \right) dx \\
&\leq k(t) \int_{0}^{1} \frac{e^{2k(x-1)}}{2} w_1 \, dx + k(t) \int_{-1}^{0} \frac{e^{-2k(x+1)}}{2} w_4 \, dx \,.
\end{aligned}
$$

Therefore,

$$\frac{d}{dt}Q_{14}\big(t,\,y(t)\big) = \int_{-1}^{1}\Big[-(x-1)k'e^{2k(t)(x-1)}w_1 + (x+1)k'e^{-2k(t)(x+1)}w_4\Big]\,dx$$
$$-A-B+C$$
$$\le k'(t)\int_{-1}^{1}\Big[(1-x)e^{2k(t)(x-1)}w_1 + (1+x)e^{-2k(t)(x+1)}w_4\Big]\,dx$$
$$-\frac{k(t)}{2}\int_{-1}^{1}\Big[(1+x)e^{2k(t)(x-1)}w_1 + (1-x)e^{-2k(t)(x+1)}w_4\Big]\,dx.$$

If $k(t) \ge 1/2$, we claim that the following two inequalities hold:

$$(1-x)e^{2k(t)(x-1)} \le 1 - e^{2k(t)(x-1)},$$
$$\tag{5.11}$$
$$(1+x)e^{-2k(t)(x+1)} \le 1 - e^{-2k(t)(x+1)}.$$

To prove the first inequality we need to show that

$$h_k(s) \doteq 1 - e^{2ks} + se^{2ks} \ge 0 \qquad \text{for all} \ \ s \in [-2,0].$$

This is clear because $h_k(0) = 0$ and

$$h'_k(s) = e^{2ks}(1 - 2k + 2ks) \le 0 \qquad s \in [-2,0]$$

if $k \ge 1/2$. Hence $h_k(s)$ is positive for $s \in [-2,0]$, as claimed. The second inequality in (5.11) is proved similarly.

When $t \ge t_0 \doteq e^{1/(2\theta)}$ one has $k(t) \ge \frac{1}{2}$ and hence

$$\frac{d}{dt}Q_{14}\big(t,\,y(t)\big) \le k'(t)Q_{14} - \frac{k(t)}{2}\int_{-1}^{1}[(1+x)e^{2k(t)(x-1)}w_1 + (1-x)e^{-2k(t)(x+1)}w_4]\,dx.$$

Setting $I = \int_{-1}^{1}w_1\,dx$, we obtain

$$\int_{-1}^{1}(1+x)e^{2k(t)(x-1)}w_1\,dx \ge \int_{-1}^{-1+I/k(t)}(1+x)e^{-4k(t)}k(t)\,dx = e^{-4k(t)}\frac{I^2}{2k(t)}.$$

Using the above, and a similar estimate for the integral of w_4, we obtain

$$\frac{k(t)}{2}\int_{-1}^{1}[(1+x)e^{2k(t)(x-1)}w_1 + (1-x)e^{-2k(t)(x+1)}w_4]\,dx$$
$$\ge \frac{e^{-4k(t)}}{4}\left[\left(\int_{-1}^{1}w_1\,dx\right)^2 + \left(\int_{-1}^{1}w_4\,dx\right)^2\right] \tag{5.12}$$
$$\ge \frac{e^{-4k(t)}}{8}\,Q_{14}^2.$$

Calling

$$Q_{14}(t) = \max_{|y|\le 1}Q_{14}(t,y),$$

from (5.12) we deduce

$$\frac{d}{dt}Q_{14}(t) \leq k'(t)Q_{14}(t) - \frac{e^{-4k(t)}}{8}Q_{14}(t)^2.$$

Recalling that $k(t) = \theta \ln t$ for some $0 < \theta < 1/4$, the previous differential inequality can be written as

$$\frac{d}{dt}Q_{14} \leq \frac{\theta}{t}Q_{14} - \frac{1}{8t^{4\theta}}Q_{14}^2. \tag{5.13}$$

Notice that $Q_{14}(t_0, y(t_0)) \leq 2k(t_0)$, and define the constant

$$A_0 \doteq \max\left\{2k(t_0)t_0^{1-4\theta},\ 8(1-3\theta)\right\}.$$

Then the function

$$z(t) \doteq A_0\, t^{4\theta-1}$$

satisfies

$$\frac{d}{dt}z(t) \geq \frac{\theta}{t}z - \frac{1}{t^{4\theta}}z^2 \qquad z(t_0) \geq Q_{14}(t_0, y(t_0)). \tag{5.14}$$

Comparing (5.13) with (5.14) we conclude

$$Q_{14}(t) \leq z(t) \qquad t \geq t_0.$$

This implies the estimate

$$\int_{-1}^{1} w_i(t, x, y_0)\, dx \leq 2Q_{14}(t) \leq 2A_0\, t^{4\theta-1}$$

for $t \geq t_0$, $i \in \{1, 4\}$ and any $y_0 \in [-1, 1]$. An entirely similar argument applied to $Q_{12}, Q_{23}, \ldots$ yields the estimates

$$\int_{-1}^{1} w_i(t, x, y_0)dx \leq 2A_0 t^{4\theta-1}, \qquad \int_{-1}^{1} w_i(t, x_0, y)dy \leq 2A_0 t^{4\theta-1}. \tag{5.15}$$

for $i = 1, 2, 3, 4$, $x_0, y_0 \in [-1, 1]$ and $t \geq t_0$.

STEP 2: Pointwise estimates. Using the integral bounds (5.15), we now seek a uniform bound of the form

$$w_i(t, x, y) \leq C_0 \tag{5.16}$$

for some constant C_0 and all $x, y \in [-1, 1]$, $t > 0$.

To prove (5.16), let $t \mapsto x(t)$, $t \mapsto y(t)) \in [-1, 1]$ be solutions of

$$\dot{x} = x + 1, \qquad \dot{y} = y + 1.$$

Call

$$A(t) \doteq \int_{x(t)}^{1} (w_1 + w_4)(t, x, y(t))\, dx.$$

From our previous estimates (5.15) it trivially follows

$$A(t) \leq 4A_0 t^{4\theta-1}.$$

The time derivative of $A(t)$ is computed as

$$
\begin{aligned}
\frac{dA}{dt} =\ & -(x(t)+1)(w_1+w_4)(t,x(t),y(t)) \\
& + \int_{x(t)}^{1} \left[\partial_t w_1 + (y(t)+1)\partial_y w_1 + \partial_t w_4 + (y(t)+1)\partial_y w_4 \right] dx \\
=\ & -(x(t)+1)(w_1+w_4)\big(t,x(t),y(t)\big) \\
& - \int_{x(t)}^{1} \left[w_1 + (x+1)\partial_x w_1 + w_4 + (x-1)\partial_x w_4 \right] dx \\
=\ & -(x(t)+1)(w_1+w_4)\big(t,x(t),y(t)\big) - \int_{x(t)}^{1}(w_1+w_4)dx \\
& -2w_1\big(t,1,y(t)\big) + (x(t)+1)w_1\big(t,x(t),y(t)\big) + \int_{x(t)}^{1} w_1\, dx \\
& -2w_4\big(t,1,y(t)\big) + (x(t)-1)w_4\big(t,x(t),y(t)\big) + \int_{x(t)}^{1} w_4\, dx \\
\leq\ & \big[x(t) - 1 - (x(t)+1) \big] w_4\big(t,x(t),y(t)\big) = -2w_4\big(t,x(t),y(t)\big).
\end{aligned}
$$

This implies

$$w_4\big(t,x(t),y(t)\big) \leq -\frac{1}{2}\frac{dA}{dt}. \tag{5.17}$$

The total derivative of w_1 along a characteristic line is now given by

$$
\begin{aligned}
\frac{d}{dt} w_1\big(t,x(t),y(t)\big) =\ & w_2 w_4 - w_1 w_3 - w_1 \leq w_2 w_4 - w_1 \leq \frac{1}{2} w_2 \left(\frac{-dA}{dt} \right) - w_1 \\
\leq\ & -w_1 + \frac{k(t)}{2}\left(\frac{-dA}{dt} \right).
\end{aligned}
$$

In turn, for $t \geq t_0$ this yields the inequality

$$
\begin{aligned}
w_1\big(t,x(t),y(t)\big) \ & \leq e^{-(t-t_0)}\left[w_1(t_0) + \int_{t_0}^{t} e^{s-t_0}k(s)(-A'(s))ds \right] \\
& \leq e^{-(t-t_0)}\left[w_1(t_0) + A(t_0)k(t_0) \right] \\
& \quad + e^{-(t-t_0)} \int_{t_0}^{t} A(s)\big(e^{s-t_0}k(s)\big)' ds.
\end{aligned}
\tag{5.18}
$$

The first term on the right hand side of (5.18) approaches zero exponentially fast. Concerning the second, we have

$$e^{-(t-t_0)} \int_{t_0}^{t} A(s)\, e^{s-t_0}\big(k(s)+k'(s)\big)\, ds \leq \int_{t_0}^{t} e^{-(t-s)} 2A_0 s^{4\theta-1} \left(\theta \ln s + \frac{\theta}{s} \right) ds.$$

This also approaches zero as $t \to \infty$. Repeating the same computations for all components, we conclude that for some time t_1 sufficiently large there holds

$$w_i(t_1,x,y) < \frac{1}{2} \qquad \text{for all } x,y \in [-1,1]. \tag{5.19}$$

By continuity, the inequalities in (5.19) remain valid for all x, y in a slightly larger square, say $[-1 - \epsilon, \, 1 + \epsilon]$. For $t \geq t_1$ we now define

$$M(t) \doteq \max \left\{ w_i(t, x, y) ; \quad i = 1, 2, 3, 4, \quad x, y \in [-1 - e^{t-t_1}\epsilon, \, 1 + e^{t-t_1}\epsilon] \right\}.$$

From the equations (5.6) and (5.19) it now follows

$$\frac{d}{dt} M(t) \leq -M(t) + M^2(t) \leq \frac{M(t)}{2} \,, \qquad M(t_1) \leq \frac{1}{2} \,.$$

$$M(\tau) \leq \left[1 + e^{\tau - t_1} \left(\frac{1}{M_1} - 1 \right) \right]^{-1} \leq e^{-\tau} \cdot e^{t_1} \qquad \text{for all } \tau \geq t_1.$$

Returning to the original variables $u_i = e^{\tau} w_i$, this yields

$$u_i \leq e^{t_1}$$

in a whole neighborhood of the point $P^* = (t^*, x^*)$. Hence P^* is not a blow-up point. $\qquad \square$

5.5 A tentative blow-up scenario

For a solution of the rescaled equation (5.5), the total mass

$$m(t) \doteq \int_{-1}^{1} \int_{-1}^{1} \sum_{i=1}^{4} w_i(t, x, y) \, dx \, dy$$

may well become unbounded as $t \to \infty$. On the other hand, the one-dimensional integrals along horizontal or vertical segments decrease monotonically. Namely, if $t \mapsto y(t)$ satisfies $\dot{y} = y + 1$, then

$$\frac{d}{dt} \int_{-1}^{1} \left[w_1(t, x, y(t)) + w_4(t, x, y(t)) \right] dx \leq 0 \,.$$

Similarly, if $\dot{x} = x - 1$, then

$$\frac{d}{dt} \int_{-1}^{1} \left[w_3(t, x(t), y) + w_4(t, x(t), y) \right] dy \leq 0 \,.$$

Analogous estimates hold for the sums $w_1 + w_2$ and $w_2 + w_3$. Therefore, a bound on the initial data

$$w_i(0, x, y) \in [0, M] \qquad \text{for all } x, y \in [-1, 1] \,,$$

yields uniform integral bounds on the line integrals of all components:

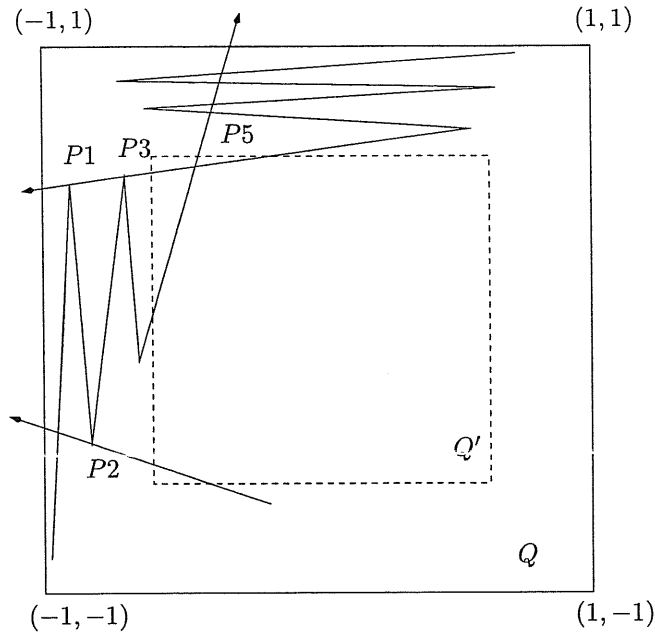$$\int_{-1}^{1} w_i(t, x, y) \, dx \leq 4M \,, \qquad \int_{-1}^{1} w_i(t, x, y) \, dy \leq 4M \,.$$

Figure 5.3: A possible interaction between particles

If finite time blow-up is to occur, the mass which is initially distributed along each horizontal or vertical segment must concentrate itself within a very small region, thus forming a narrow packet of particles with increasingly high density. A possible scenario is illustrated in fig. 5.3. A packet of 1-particles is initially located at P_1. In order to contribute to blow-up, this packet must remain within the unit square Q. At P_2 these 1-particles interact with 3-particles and produce a packet of 4-particles. In turn, at P_3 these interact with 2-particles and produce again a packet of 1-particles. After repeated interactions, the packet of alternatively 1- and 4-particles eventually enters within the smaller square Q'. After this time, it interacts with a packet of 2-particles at P_5 (transforming it into a packet of 1-particles) and eventually exits from the domain Q.

To help intuition, it is convenient to describe a packet as being "young" until it enters the smaller square Q', and "old" afterwards. To maintain a young packet inside Q, one needs the presence of old packets interacting with it near the points $P_2, P_3, P_4 \ldots$ On the other hand, after it enters Q', our packet can in turn be used to hit another young packet, say at P_5, and preventing it from leaving the domain Q.

As $t \to \infty$, the density of the packets must approach infinity. One thus expects that most of the mass will be concentrated along a finite number of one-dimensional curves. Say, the packet of alternatively 1- and 4-particles should be located along a moving curve $\gamma_{14}(t, \theta)$, where θ is a parameter along the curve. The time evolution of such a curve is of course governed by the equations

$$\frac{\partial}{\partial t}\gamma_{14} = c_1 \qquad \text{or} \qquad \frac{\partial}{\partial t}\gamma_{14} = c_4$$

depending on whether $\gamma_{14}(t,\theta)$ consists of 1- or 4-particles. The presence of interactions impose highly nonlinear constraints on these curves. For example, the interaction occurring in P_5 at time t implies the crossing of the two curves γ_{14} and γ_{12}, namely

$$\gamma_{14}(t,\theta) = \gamma_{12}(t,\tilde{\theta}) = P_5$$

for some parameter values $\theta, \tilde{\theta}$. The complicated geometry of these curves resulting from the above constraints has not been analyzed.

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